Oscillations and Waves

Dr Andrew McKinley

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Preface

Welcome to the course notes for the Oscillations and Waves component of PHYS10012, Core Physics I.

The most recent version of these notes can be found at https://awmckinley1.github.io/oscandwaves. Please ensure you are always working from the most up-to-date version of these notes and compare the change-log with that on the live website.

About these notes

These notes have been prepared in a format which maintains compatibility with screen-readers, while also allowing the facility for both PDF and EPUB downloads for those who wish to use them.

The HTML also has themes to facilitate easier reading (font colours, serif/sans serif fonts etc.). Please explore the top bar of the web environment to explore this.

The HTML notes allow for embedding of rich content (videos, animations, etc.), and for this reason I recommend accessing the course using the web-links provided, as these cannot be included in static PDF or EPUB documents. However, I will ensure links are included in these formats as far as possible.

These notes are a "live document", and as such they can (and will) be updated should any erroneous explanations be found or additional explanations be needed. For this reason, please do email me if you have any questions or if anything is not clear.

The most recent version can always be found at the link at the top of this section, and any changes are reflected in the change-log.

Please note that as this is a "live document", a downloaded version can become obsolete. Please refer back regularly and refer to the changelog for any updates.

Error reporting

As much as we try, sometimes errors creep in. At best, it is little more than a careless typo which makes the reader tut with disappointment, at worst it risks seriously misleading the learner. For this reason I have created a feedback form to report errors; this is accessible at the bottom of this page.

Course history

This iteration of the course has been written by Andrew McKinley for 2024; this is extensively based on his delivery of the previous course from 2021-23, which in turn was adapted from previous course notes by Ben Maughan, Simon Hanna and Massimo Antognozzi.

Changelog

2024-01-24 Fixing PDF and image 1.7

2024-01-11 Intitial creation of the Oscillations and Waves content for PHYS10012.

Report an error

Direct link to error reporting form (opens in new window)

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Chapter 1

Simple Harmonic Motion

Simple harmonic motion (SHM) is a simple and common type of oscillatory motion. It is a model which is widely used in modelling systems due to its simplicity.

In general, an object will move under SHM where its acceleration is:

1. proportional to its displacement, but
2. in the opposite direction.

The force causing this acceleration is often termed a *restoring force* as it acts to push the object back to its starting point.

1.1 A simple example of SHM

Consider a block on a spring (Figure 1.1)

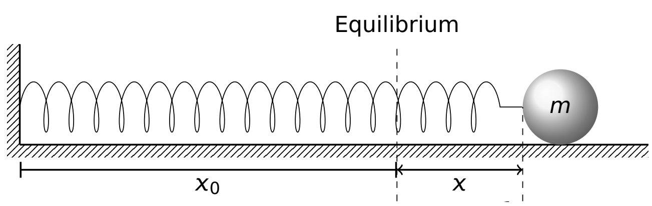

Figure 1.1: A mass on a spring, stretched distance x past its equilibrium length x_0

By Hooke's law, the spring exerts a force on the block proportional to its displacement x, but in the opposite direction, pushing the block back to its equilbrium position, shown mathematically in Equation 1.1:

$$F_x = -kx \tag{1.1}$$

In this example, F_x is considered a **restoring force**, while k is the force constant of the spring.

Applying Newton's Second Law to this problem, we can obtain the mathematical description of the system (Equation 1.2):

$$\begin{aligned} F_x &= ma_x \\ kx &= m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \end{aligned} \tag{1.2}$$

... and through rearrangement and combination with Equation 1.1 we obtain the description of how this mass will move (Equation 1.3):

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{k}{m}x \tag{1.3}$$

The general form of this expression for any system can be considered as shown in Equation 1.4:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -Cx \quad \text{or} \quad \ddot{x} = -Cx \tag{1.4}$$

...where C is a positive constant which depends on the system and represents a ratio of the elastic (k) and inertial (m) contributions within the system.

Key Terms

- **Period**: The time T for one complete oscilation back and forth (units s)
- **Frequency**: The reciprocal of the period; $f = \frac{1}{T}$, units s^{-1} or Hz.

1.2 Positioning in SHM

SHM can be described by a general equation of motion, defining the position (x) of the oscillating mass using a cosine function (Equation 1.5):

$$x = A\cos(\omega t + \delta) \tag{1.5}$$

The parameters in this equation are:

- A: The amplitude of the oscillation
- $\omega t + \delta$: Phase of motion
- δ: Phase constant

For any single oscillator, the time origin can always be chosen so that $\delta = 0$. For two or more oscillators there will generally be a phase difference between them *i.e.* they will not always be at the same 'zero' position at time zero in Figure 1.2:

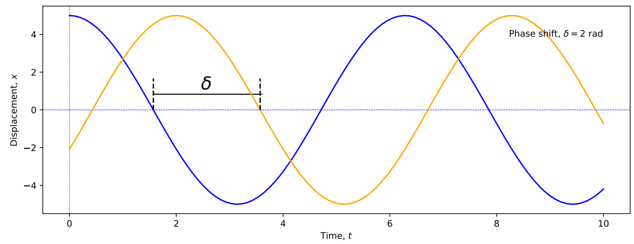

Figure 1.2: The changing position of two oscillators with respect to time, with a relative phase shift of 2 radians.

1.3 Velocity in SHM

To find the velocity of the oscillating mass, we can simply find the first derivative of its position with respect to time (Equation 1.6):

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -A\omega \sin(\omega t + \delta) \end{aligned} \tag{1.6}$$

A quick inspection of this shows that the velocity v is maximised when x is at a minimum; *i.e.* as the object passes through its equilibirum position.

1.4 Acceleration in SHM

Again, to find the acceleration, we find the second derivative of its position with respect to time (Equation 1.7):

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} \\ &= -A\omega^2 \cos(\omega t + \delta) \end{aligned} \tag{1.7}$$

...or, to use the Newtonian "dot" notation (Equation 1.8):

$$a = \ddot{x} = -\omega^2 x \tag{1.8}$$

If we now compare this with Equation 1.3 we can see that we have an expression for ω for the oscillating mass m on a spring of force constant k (Equation 1.9):

$$
\begin{aligned}
\omega^2 &= \frac{k}{m} \\
\omega &= \sqrt{\frac{k}{m}}
\end{aligned}
\tag{1.9}
$$

1.5 Comparing displacement, velocity and acceleration

When we now compare the displacement, velocity and acceleration we make a number of observations. Firstly, they are all sinusoidal functions; variously sine and cosine functions. However, when we overlay these we have a better indication of how they interrelate (Figure 1.3)

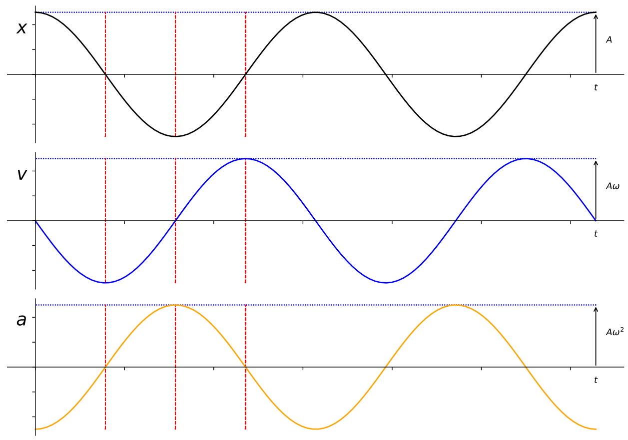

Figure 1.3: Comparing the changes of position, velocity and acceleration with time for a harmonic oscillator. Note that when x is at zero, v is maximised, while a is at a maximum when v is zero. The relative amplitudes of each of the waves is given.

Some key observations

- When the displacement x is at a maximum (x_{max}), the velocity v is zero while the acceleration is at its maximum **but negative with respect to displacement** ($a = -a_{\text{max}}$)
- When the displacement x is zero, the velocity v is at its maximum value ($v = \pm v_{\text{max}}$) and the acceleration is zero.
- The pattern repeats with each period; namely x_0 (displacement at time $t = 0$) is equal to the displacement x_T (displacement after one period of oscillation, T), and the same for the acceleration and velocity.
- In general, $x_t = x_{t+T}$; the displacement at time t is equal to the displacement at the time t plus one period of oscillation, T.

We can directly compare the displacement, velocity and acceleration at four points in the oscillation (Table 1.1):

Table 1.1: Relating the displacement, velocity and acceleration at different times in the oscillation for a simple harmonic oscillator.

Time	Displacement, x	Velocity, v	Acceleration, a
$t=0$	$x_0 = A$	$v_0 = 0$	$a_0 = -a_{\max}$
$t=\frac{T}{4}$	$x_{\frac{T}{4}} = 0$	$v_{\frac{T}{4}} = -v_{\max}$	$a_{\frac{T}{4}} = 0$
$t=\frac{T}{2}$	$x_{\frac{T}{2}} = -A$	$v_{\frac{T}{2}} = 0$	$a_{\frac{T}{2}} = a_{\max}$
$t=\frac{3T}{4}$	$x_{\frac{3T}{4}} = 0$	$v_{\frac{3T}{4}} = v_{\max}$	$a_{\frac{3T}{4}} = 0$
$t=T$	$x_T = x_0 = A$	$v_T = v_0 = 0$	$a_T = a_0 = -a_{\max}$

1.6 Initial conditions

We mentioned in Section 1.5 that the displacement, velocity and acceleration expressions were based on sinusoial functions, and each function had a scaling factor A (the **amplitude** of the oscillation) and a **phase** component δ. In most problems, we wish to determine the value of these constants. In order to determine these, we establish the initial conditions of the oscillation.

In Figure 1.3 we defined our displacement at $+A$ which set up the rest of the problem. However, we will not always be so fortunate. For a general case, we then need to be more discerning.

We can establish expressions for both the amplitude and the phase component by setting $t = 0$ in our general expressions (Equation 1.10):

$$\begin{array}{rclcrcl} x_t &=& A\cos(\omega t+\delta) & \rightarrow & x_0 &=& A\cos(\delta) \\ v_t &=& -A\omega\sin(\omega t+\delta) & \rightarrow & v_0 &=& -A\omega\sin(\delta) \end{array} \tag{1.10}$$

We now treat these as simultaneous equations to find δ and A (Equation 1.11)[1]:

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = -\frac{v_0}{\omega x_0} \quad \text{and} \quad A^2 = x_0^2 + \frac{v_0^2}{\omega^2} \tag{1.11}$$

1.7 Frequency and angular frequency

In Section 1.5 we stated that the nature of the oscillation meant that it repeats after every oscillation; mathematically $x(t) = x(t+T)$; the position x at time t is equal to the position at time $(t+T)$.

[1]Note that we use the trigonometric identity $\cos^2\alpha + \sin^2\alpha = 1$ to find A

When we apply this to the velocity, we obtain the following expression:

$$\begin{aligned} v(t) &= v(t+T) \\ A\cos(\omega t+\delta) &= A\cos(\omega(t+T)+\delta) \\ &= A\cos([\omega t+\delta]+\omega T) \end{aligned}$$

Due to the cyclic nature of a cosine function, $\cos(\alpha) = \cos(\alpha + 2\pi)$, this must therefore mean (Equation 1.12):

$$\omega T = 2\pi \quad \text{or} \quad \omega = \frac{2\pi}{T} \tag{1.12}$$

This gives us a way to think about ω; its connection to circular motion (the clue is the 2π). It can be thought of as the **angular frequency**, with units radians s^{-1}, and an oscillation of 2π radians corresponds to one period of oscillation.

Additionally, since the frequency of the oscillation f is the reciprocal of the period of oscillation ($f = \frac{1}{T}$), the angular frequency can be rewritten as $\omega = 2\pi f$, and $f = \frac{\omega}{2\pi}$.

For the spring system we discussed in Section 1.1, we stated that the angular frequency $\omega = \sqrt{\frac{k}{m}}$. Therefore we can obtain an expression for the frequency of our oscillator (Equation 1.13):

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1.13}$$

Inspection of this equation reveals the behaviour of our oscillator:

- If we have a stiffer spring (larger k), we expect the frequency f to increase,
- If we use an oscillator with larger mass (larger m), we would expect the frequency (f) to decrease.
- The frequency (and therefore period) of simple harmonic oscillation is independent of amplitude.[2]

1.8 SHM and circular motion

We mentioned an "angular frequency" for SHM; this would appear to suggest behaviour akin to circular motion. It is therefore worth exploring our descriptions of circular motion.

Imagine a point mass moving in a circle (Figure 1.4). For convenience, we imagine this using Cartesian $x - y$ axes, shown in Figure 1.4.

[2] A caveat to this is for large amplitudes where other factors start to affect the behaviour. But this is then no longer *simple* harmonic motion!

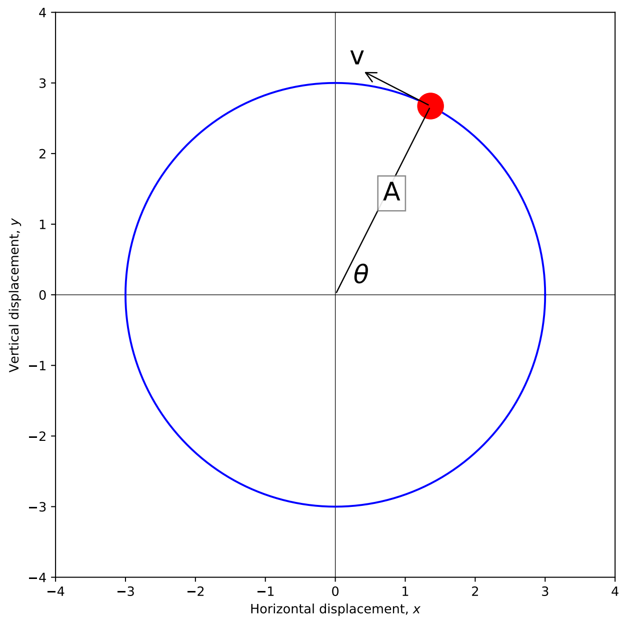

Figure 1.4: A particle moving in a circle of radius A can be assumed to have an instantaneous linear velocity v. The x and y components of the motion are found from trigonometry of the radius A and the angle θ.

The particle of mass m is moving in a circle of radius A with instantaneous linear velocity v; the radius makes an angle θ with the x-axis. We now look at how its position maps onto each of the axes:

- The angular velocity of the particle is ω; found *via* $\frac{v}{A}$
  - We can then describe θ in terms of ω:
  - $\theta = \omega t + \delta$ (δ is the angle at time $t = 0$)
- The particle's position on the x-axis is therefore found *via*:
  - $x = A\cos\theta = A\cos(\omega t + \delta)$
  - This corresponds with the expression for SHM for a particle moving in a linear fashion (Equation 1.5).

We can also consider how its position maps onto the y-axis:

- The position on the y-axis is found *via*:
  - $y = A\sin\theta = A\sin(\omega t + \delta) \equiv A\cos(\omega t + [\delta - \frac{\pi}{2}])$
  - This once again corresponds with the expression for SHM for a particle moving in a linear fashion.
  - The y-component of the motion is $\frac{\pi}{2}$ out of phase with the x-component

This illustrates that circular motion is a combination of two perpendicular SHM oscillations of the same frequency and amplitude, but a relative phase of $\frac{\pi}{2}$.

1.9 Energy in SHM

As with all isolated systems, the total energy E of the simple harmonic oscillator is constant, however the contributions from potential energy (U) and KE vary with time.

$$E = KE + U = \text{constant}$$

Let's go back to the condition for SHM; there is a restoring force proportional to the displacement:

$$F = -kx$$

Knowing that the force is the first derivative of the potential energy, we can therefore integrate this force expression (with respect to x) to get back to the energy statement:[3]

$$\begin{aligned} U &= \int F \mathrm{d}x \\ &= \tfrac{1}{2}kx^2 \quad [+C] \end{aligned} \tag{1.14}$$

[3] The constant of integration will evaluate to zero from the starting condition $U = 0$ at zero displacement.

However, we already have an expression for how the displacement, x, varies with time (Equation 1.5); let's now substitute this into the result from Equation 1.22:

$$U = \frac{1}{2}kA^2\cos^2(\omega t + \delta) \tag{1.15}$$

We can also generate an expression for the kinetic energy; remember that kinetic energy can be found from $\frac{1}{2}mv^2$; so we use the expression for v given in Equation 1.6:

$$\begin{aligned} KE &= \tfrac{1}{2}mv^2 \\ &= \tfrac{1}{2}mA^2\omega^2\sin^2(\omega t + \delta) \end{aligned} \tag{1.16}$$

We can simplify this using Equation 1.7 for a particle on a spring, where $\omega^2 = \frac{k}{m}$:

$$KE = \frac{1}{2}kA^2\sin^2(\omega t + \delta) \tag{1.17}$$

Combining the result of Equation 1.15 and Equation 1.17 we find the result in Equation 1.18:

$$\begin{aligned} E_{\text{total}} &= U + KE \\ &= \tfrac{1}{2}kA^2\cos^2(\omega t + \delta) + \tfrac{1}{2}kA^2\sin^2(\omega t + \delta) \\ &= \tfrac{1}{2}kA^2\left[\cos^2(\omega t + \delta) + \sin^2(\omega t + \delta)\right] \\ &= \tfrac{1}{2}kA^2 \end{aligned} \tag{1.18}$$

This result tells us that the total energy in a simple harmonic oscillation is **proportional to the square of the amplitude**.

Some points to bear in mind

- $U = U_{\text{max}}$ at $x = \pm x_{\text{max}}$
- $KE = KE_{\text{max}}$ at $x = 0$
- $U_{\text{average}} = KE_{\text{average}} = \frac{1}{2}E_{\text{total}}$

1.10 SHM in Real Systems

Textbook link: Tipler and Mosca: Ch 14.2 to 14.4

We will now go on to look at some applications of SHM in real-world systems.

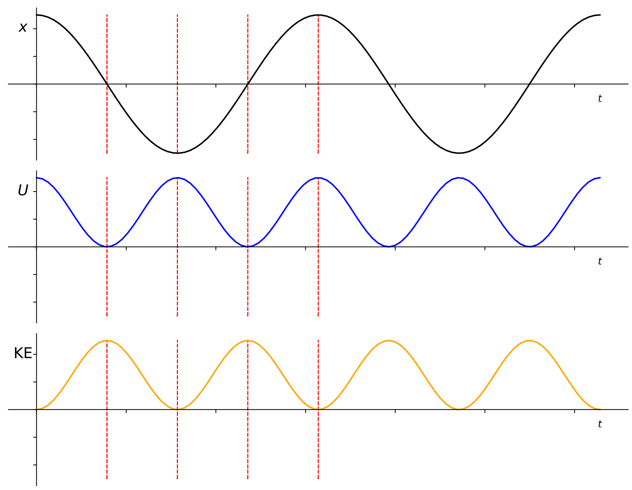

Figure 1.5: Variation of the kinetic energy (KE) and potential energy (U) of an harmonic oscillator with displacement x about the equilibrium position.

1.10.1 General motion near equilibrium

A way of thinking about SHM is that it is a point mass oscillating within a potential energy field. As with any potential energy field, the force on this particle is given by the gradient of the potential energy and is directed down the potential energy slope. Mathematically, for a potential energy field, the force may be found as follows (Equation 1.19):

$$F = -\frac{\mathrm{d}U}{\mathrm{d}r} \tag{1.19}$$

In a one-dimensional system, this is expressed as follows (Equation 1.20):

$$F_x = -\frac{\mathrm{d}U}{\mathrm{d}x} \tag{1.20}$$

As mentioned in Section 1.1, under SHM the force is proportional to the displacement from the equilibrium position and in the opposite direction; *i.e.*:

$$F_x = -kx \tag{1.21}$$

Applying the principle from Equation 1.19 we can therefore integrate this experession with respect to x to obtain the expression for our potential energy. We covered this in Section 1.9, and we found the result (Equation 1.22); remember that, due to initial conditions, the constant of integration reduces to zero).

$$U = \frac{1}{2}kx^2 \tag{1.22}$$

Simple inspection and recall of our mathematics knowledge tells us that this simple equation represents a parabola.

Some useful points on the harmonic oscillator

- A parabolic potential energy function implies SHM and *vice versa*;
- For small amplitudes of oscillation, many potential energy functions may be *approximated* by a parabola (*e.g.* a pendulum, vibrating molecules)
- A system undergoing SHM is called a **harmonic oscillator**.

The simplicity of the **simple harmonic oscillator** model is what makes it such a generally useful system to consider.

1.10.2 Example: a diatomic molecule

A diatomic molecule is a useful system to consider as an example because it can be approximated to a harmonic oscillator at small displacements about the equilibrium. The potential energy curve for a vibrating diatomic molecule (in this case the H_2 hydrogen molecule) is shown in Figure 1.6.

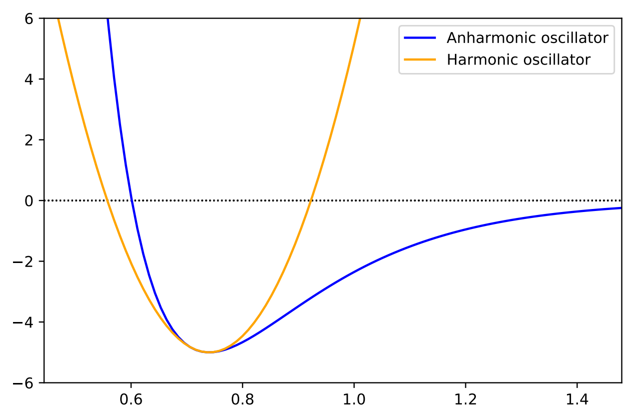

Figure 1.6: Comparing the Morse potential of the anharmonic oscillator with the harmonic approximation. Near equilibrium, the harmonic oscillator model approximates diatomic behaviour, however this rapidly deviates from reality.

The potential of a vibrating diatomic is known as the Morse potential; the form of this is outwith this discussion, however it is useful to think that, for small displacements around the equilibrium separation the potential energy curve approximates a parabola. We can therefore re-draw our potential energy curve as such, and show this in Figure Figure 1.7).[4]

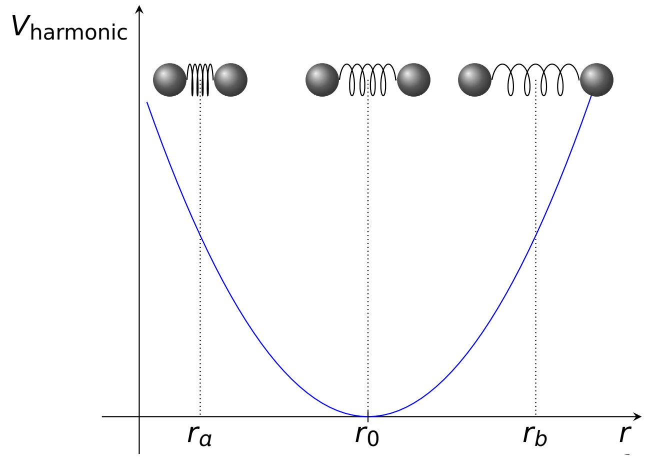

Figure 1.7: The parabolic approximation of a diatomic molecule, showing the potential varying with compression or extension from equilibrium separation, r_0.

The equation of the parabola shown in Figure 1.7 takes the following form:

$$U(r) = A + B(r - r_0)^2 \tag{1.23}$$

...where A and B are constants relating to the molecular system under consideration, and r_0 is the equilibrium bond length.

The force on the bond can then be found from the first derivative of the bond potential described in Equation 1.23:

$$\begin{aligned} F_r &= -\frac{\mathrm{d}U}{\mathrm{d}r} \\ &= -2B(r - r_0) \end{aligned} \tag{1.24}$$

Since the term $(r - r_0)$ is the displacement from the equilibrium position, we see that the force, F_r is a **restoring force** and is proportional to the displacement (and in the opposite direction!), telling us that the motion is SHM. In this example however, the parabolic approximation fails at larger amplitudes.

1.10.3 Example: Mass on a vertical spring

In Section 1.1 we considered a mass on a horizontal spring; there was only a single force acting on the mass (the force from the spring), however we are now considering

[4]Simplistically, the deviation is due to nuclear repulsion at high compression, while at large extension the bond eventually breaks - the "zero potential" point.

a vertical spring and must consider the effects of gravity (Figure 1.8).

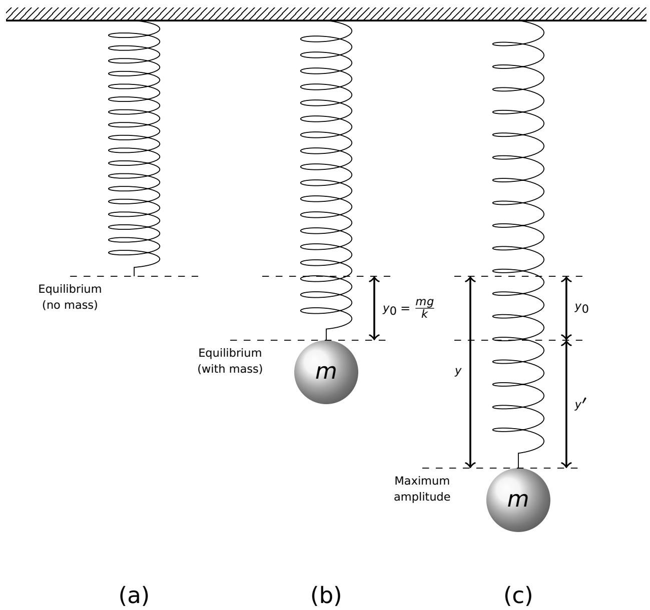

Figure 1.8: A particle oscillating on a vertical spring. There are two equilibrium positions corresponding to the equilibrium position of the unladen spring (a) and the equilibrium position of the mass loaded on spring (b), where the weight of the load is balanced by the force from the extended spring.

In this case we need to work through a slightly different process to find the equation of motion, chiefly because the equilibrium displacement of the mass (y_0) is different from the equilibrium position of the spring.

The equilibrium position of the mass y_0 is lower than the equilibrium extension of the spring (gravity on the mass causes the spring to stretch). This is found by relating the force from the spring and the force of gravity acting on the mass (Equation 1.25):

$$\begin{aligned} \text{Force due to spring extension} &= \text{Gravity acting on mass} \\ ky_0 &= mg \\ y_0 &= \frac{mg}{k} \end{aligned} \tag{1.25}$$

We now apply the Second Law of motion to obtain an expression for the acceleration on the mass due to the forces acting on it (the spring force and gravity).

$$\begin{aligned} m\ddot{y} &= \text{spring force} + \text{gravity} \\ m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= -ky + mg \end{aligned} \tag{1.26}$$

In this expression, y is the total extension of the spring (the extension to the mass equilibrium point, y_0 plus the displacement from this point, y').

We now substitute a variable; since $y = y_0 + y'$, we substitute y for y':

- $y = y_0 + y'$
- $y' = y - y_0$

Since y_0 is a constant:

$$\frac{\mathrm{d}y'}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2y'}{\mathrm{d}t^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$

Replacing y for $(y_0 + y')$ in Equation 1.26, we obtain Equation 1.27:

$$m\frac{\mathrm{d}^2y'}{\mathrm{d}t^2} = -k(y_0 + y') + mg \tag{1.27}$$

Since $ky_0 = mg$ (Equation 1.25), we can therefore eliminate these terms from Equation 1.27, and rewrite as Equation 1.28):

$$m\frac{\mathrm{d}^2y'}{\mathrm{d}t^2} = -ky' \tag{1.28}$$

This means that, in reference to Figure 1.8, we still have SHM centered on the equilibrium position of the mass. This may seem like a self-evident result, however it is useful to recognise the role of gravity; its effect is to shift the equilibrium position of the oscillation from $y = 0$ (the equilibrium position of the spring) to $y = y_0$ ($y' = 0$).

Let's now consider the energy in this system. The system already contains some elastic energy as the spring is already stretched to y_0 by the gravity acting on the mass:

$$\text{elastic potential energy} = \frac{1}{2}ky^2 - \frac{1}{2}ky_0^2$$

The gravitational potential energy (relative to the starting position y_0) is given by:

$$\text{gravitational potential energy} = mg(y - y_0)$$

The total potential energy is therefore given by Equation 1.29:

$$U = \frac{1}{2}ky^2 - \frac{1}{2}ky_0^2 - mg(y - y_0) \tag{1.29}$$

We can then show that the total potential energy expression in Equation 1.29 can be simplified to that shown in Equation Equation 1.30:

$$U = \frac{1}{2}ky'^2 \tag{1.30}$$

You should ensure you understand how this simplification is done; this is left as an exercise.[5]

Overall, the expression for the total potential energy shown in Equation 1.30 will still yield a parabola and as such the oscillation is still a simple harmonic oscillation as before.

[5]Yes; we know this sort of thing infuriates learners, however it is based in valid educational practice! Do give it a go. In this instance, remember that you will need to start with Equation 1.29, recall the relation of y and y', substitute this into the equation, expand and cancel terms.

Chapter 2

Damped oscillations

Textbook link

The oscillations we have looked at so far make the assumption that the oscillation will continue indefinitely and that no energy is gained or lost by the system. Such perpetually oscillating systems are extremely unusual, and almost every oscillation you encounter in the real world will lose energy to its surroundings, either requiring constant input of energy to maintain the oscillation or it will eventually dissipate all of its energy and the oscillation will stop. The energy is dissipated through a process known as 'damping'.

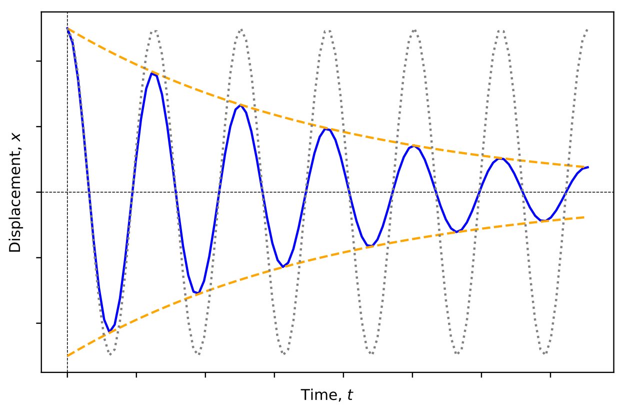

Figure 2.1: The amplitude of a damped oscillation decays exponentially with time. The observed position of the oscillation is shown in blue, while the maximum possible amplitude (related to energy stored in the system) is illustrated by the orange dotted line. The undamped oscillation is shown in grey for comparison.

The change in amplitude of a damped oscillation is illustrated in Figure 2.1. Both amplitude and energy decrease by a **constant percentage** in each cycle; this is an **exponential decrease**.

2.1 The general case of damping

For a simply damped system, the damping force is proportional to the velocity of the oscillating mass and **opposes** the direction of motion (Equation 2.1):

$$F = -bv \equiv -b\frac{\mathrm{d}x}{\mathrm{d}t} \tag{2.1}$$

This expression for the damping force can then be included with the equation for the force from the spring (Equation 1.1) to consider the overall acceleration of a mass undergoing damped harmonic motion. (Equation 2.2).

$$\begin{aligned} F = -kx - bv &= -kx - b\frac{\mathrm{d}x}{\mathrm{d}t} = ma \\ m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -kx - b\frac{\mathrm{d}x}{\mathrm{d}t} \end{aligned} \tag{2.2}$$

This can be written as a differential equation for **damped SHM** as shown in Equation 2.3:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0 \tag{2.3}$$

Let's now try to find a solution for this differential equation. Firstly, let's make a couple of assumptions:

1. The amplitude of the damped oscillation is subject to an exponential decay over a timescale of 2τ (don't worry about the factor of 2 for now), and
2. The damped oscillation has a frequency ω' which may be different from the natural frequency of the undamped oscillator, ω_0.

Our exponential decay factor then becomes $\mathrm{e}^{-\frac{t}{2\tau}}$, and the exponential form of the wave equation becomes $\mathrm{e}^{\mathrm{i}(\omega' t+\delta)}$ (combination of Equation 1.5 and Equation 12.11). When these are combined we obtain a form of the solution shown in Equation 2.4

$$x = A_0\mathrm{e}^{-\frac{t}{2\tau}}\mathrm{e}^{\mathrm{i}(\omega' t+\delta)} \tag{2.4}$$

We can now find the first and second derivatives (make sure you are able to do this; you will need the product rule for differentiation) of this expression to substitute into the differential equation (Equation 2.3):

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \left(-\frac{1}{2\tau}+\mathrm{i}\omega'\right)x \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \left(-\frac{1}{2\tau}+\mathrm{i}\omega'\right)^2 x\end{aligned} \tag{2.5}$$

We can now take our derivatives shown in Equation 2.5) and substitute into the differential equation (Equation 2.3)):

$$\begin{aligned} m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+b\frac{\mathrm{d}x}{\mathrm{d}t}+kx &= 0 \\ m\left(-\frac{1}{2\tau}+\mathrm{i}\omega'\right)^2 x+b\left(-\frac{1}{2\tau}+\mathrm{i}\omega'\right)x+kx &= 0\end{aligned} \tag{2.6}$$

Multiplying this expression out, we obtain:

$$m\left(\frac{1}{4\tau^2}-\frac{\mathrm{i}\omega'}{\tau}-\omega'^2\right)x+b\left(-\frac{1}{2\tau}+\mathrm{i}\omega'\right)x+kx=0$$

...and we can now combine the **Real** and **Imaginary** components:

- **Imaginary**:

$$-\frac{\omega' m}{\tau}+\omega' b=0$$

$$\tau=\frac{m}{b} \tag{2.7}$$

- **Real**:

$$\left(\frac{1}{4\tau^2}-\omega'^2\right)m-\frac{b}{2\tau}+k=0$$

Therefore (using result from Equation 2.7):

$$\omega'^2=\frac{k}{m}-\left(\frac{b}{2m}\right)^2$$

However, we already know that $\frac{k}{m}=\omega_0^2$ (Equation 1.9), so:

$$\omega'^2=\omega_0^2-\left(\frac{b}{2m}\right)^2 \tag{2.8}$$

Having found this result, we can now say that a general solution to damped SHM is as shown in Equation 2.4, or in trigonometric notation as shown in Equation 2.9:

$$x = A_0 e^{-\frac{t}{2\tau}} \cos(\omega' t + \delta) \tag{2.9}$$

...where

- $A_0 =$ initial (maximum) amplitude
- $\tau = \frac{m}{b}$ is the characteristic decay time or time constant
- $\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$, where ω_0 is the frequency of the undamped oscillator ($\omega_0^2 = \frac{k}{m}$ for a mass on a spring)

Having derived and defined a general expression for damped oscillations, we will now turn to look at different modes of damping.

2.2 Light Damping

In a system which is lightly damped, we can make a number of assumptions:

1. That the frequency of the damped oscillator (ω') is approximately equal to that of the undamped oscillator; $\omega' \approx \omega_0$
2. That the damping factor, $\frac{b}{2m\omega_0}$ is significantly less than one; *i.e.* $\frac{b}{2m\omega_0} \ll 1$

When we apply these assumptions to Equation 2.4 we obtain the "standard" SHM oscillation (tending towards $e^{i(\omega' t+\delta)}$ as ω' tends to ω_0) with an exponential decay on its amplitude (Equation 2.10):

$$A = A_0 e^{-\frac{t}{2\tau}} \tag{2.10}$$

The effect of this light damping is that if b (the damping coefficient on the velocity, Equation 2.1 increases, the damped frequency ω' will decrease, and the characteristic decay time, τ will also decrease.

2.3 Critical damping

In a critically damped system, the **system does not oscillate**; rather, it returns to equilibrium in the shortest possible time. This can be imagined as the damping required to **exactly stop** the vibration and no more. The damped oscillation frequency ω' then, by definition, is equal to zero ($\omega' = 0$) and, by placing this condition into Equation 2.11, we obtain the result for the damping coefficient, b (Equation 2.12):

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \tag{2.11}$$

$$b = b_c = 2m\omega_0 \tag{2.12}$$

2.4 Overdamping

An overdamped system is one which has so much damping applied that the system returns to equilibrium *even more slowly* than in the critically damped case (Section 2.3). This could be imagined as a mass on a spring which is allowed to return to equilibrium within an extremely viscous medium (honey, or treacle!) and takes a considerable time to slowly return to equilibrium. A comparison of the displacement/time curve between a critically damped system and an overdamped system is shown in Figure 2.2.

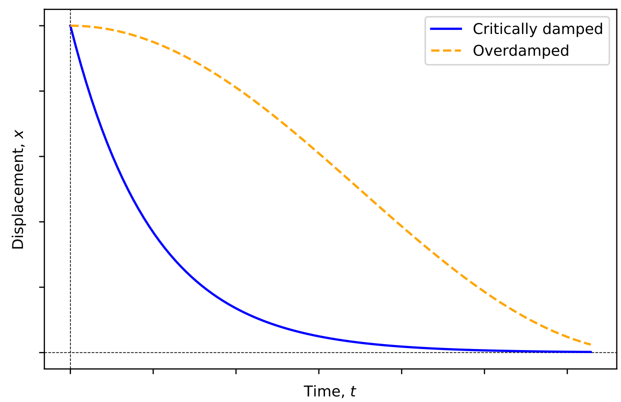

Figure 2.2: A critically damped system returns to its equlibrium position in the shortest possible time, while an overdamped system works against an overbearing damping force slowly returning it to its equilibrium position.

In the overdamped case, the damping coefficient, b is greater than the critical damping coefficient, b_c:

$$b > b_c$$

2.5 Quality factor and energy in damped SHM

Recall that we determined energies within SHM in Equation 1.17 and Equation 1.18; remembering that $\omega^2 = \frac{k}{m}$, we can write the overall energy of the system as in Equation 2.13:

$$E = \frac{1}{2} m\omega^2 A^2 \tag{2.13}$$

Within the damped regimes we have established the equations for damping, in particular how to calculate the damped amplitude at a given time, A (Equation 2.10).

By combining this with Equation 2.13 we can obtain an expression for the damped energy, E, as a proportion of the initial energy of the system, E_0 (Equation 2.14):

$$\begin{aligned} E &= \tfrac{1}{2}m\omega^2\left(A_0 \mathrm{e}^{-\frac{t}{2\tau}}\right)^2 \\ &= E_0 \mathrm{e}^{-\frac{t}{\tau}} \end{aligned} \tag{2.14}$$

The decay time τ can now be considered the time taken for the energy to decrease to $\frac{1}{\mathrm{e}}$ of its original value (see? There was a reason we considered the timescale 2τ in Section 2.1!).

A useful measure of the **persistence** of an oscillation is the **quality factor**, Q. This is defined as shown in Equation 2.15:

$$Q = \omega_0 \tau = \frac{\omega_0 m}{b} \tag{2.15}$$

Generally:

- Large Q represents lighter damping, persistent oscillation. (think **high quality oscillation**)
- Small Q represents heavier damping, oscillation stops rapidly. (think **low quality oscillation**)

Q can relate to the energy loss per cycle of oscillation; firstly define the rate of change of energy (Equation 2.16, from Equation 2.13):

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{E_0}{\tau}\mathrm{e}^{-\frac{t}{\tau}} = -\frac{E}{\tau} \tag{2.16}$$

If we are considering finite changes, we can adapt our calculus to allow $\Delta E \approx \mathrm{d}E$ and $\Delta t \approx \mathrm{d}t = T$. Applying this to Equation 2.16 and rearranging gives:

$$\frac{|\Delta E|}{E} = \frac{T}{\tau} = \frac{2\pi}{\omega_0 \tau} = \frac{2\pi}{Q}$$

{#eq-}

This gives us the result for Q shown in Equation 2.17:

$$Q = \frac{2\pi}{\left(\frac{|\Delta E|}{E}\right)_{\mathrm{cycle}}} \tag{2.17}$$

This gives us one more way to consider the quality factor: it is inversely proportional to the fractional energy loss per cycle of the oscillation. A low Q therefore corresponds to a higher fractional energy loss per cycle than a high Q factor.

Chapter 3

Forced oscillations

Textbook link: Tipler and Mosca 14.5

In Chapter 2) we explored the effect of damping on a system and we said that every system in the real world is, to a greater or lesser extent, a damped system in which energy is lost (dissipated) to the surroundings. In order to maintain the amplitude of any oscillation we must supply energy to the system at the same rate as it is lost to the surroundings; for example, pushing a child on a swing.

The equation for forced SHM is given in Equation 3.1:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0\mathrm{e}^{\mathrm{i}\omega t} \tag{3.1}$$

Compare this to Equation 2.3 for damped SHM; we have now applied an oscillating force represented as $F_0\mathrm{e}^{\mathrm{i}\omega t}$, with an amplitude of F_0 and frequency ω. Note that this driving frequency ω is different to the natural *undamped* frequency of the oscillator, $\omega_0 = \sqrt{\frac{k}{m}}$ and different again to the frequency of the damped *unforced* oscillator, $\omega' = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$.

Any forced oscillation consists of two distinct regimes:

1. An initial transient period during which the oscillations are established, and
2. A steady state period during which the oscillations have constant amplitude and a frequency equal to the driving frequency, ω.

The general solution of the equation for forced SHM shown in Equation 3.1 is a combination of these two regimes.

3.1 The Transient Solution

The solution to the transient component of the forced oscillator is identical to the solution of damped SHM (*i.e.* Equation 3.1 with the right-hand side set to zero; identical to Equation 2.3). Its solution is shown in Equation 3.2:

$$\begin{aligned} x &= A_0 \mathrm{e}^{-\left(\frac{b}{2m}\right)t} \mathrm{e}^{\mathrm{i}(\omega' t + \delta')} \\ \text{or:} \quad x &= A_0 \mathrm{e}^{-\left(\frac{b}{2m}\right)t} \cos\left(\omega' t + \delta'\right) \end{aligned} \tag{3.2}$$

This contribution to motion establishes the oscillation, but rapidly decays with time constant $\tau = \frac{m}{b}$. The term δ' is simply the phase constant for this transient oscillation.

3.2 The Steady State solution

Once in the steady state, the energy which is put into the system during each cycle is equal to the energy dissipated per cycle due to the damping in the system. If there is no damping of the system, energy keeps being added to the system and the amplitude will increase indefinitely. This is **not** a steady state, and is an **unphysical result**. The frequency then of this 'driven' oscillator in the steady state is equal to the driving frequency.

The amplitude (and hence the energy) of the system in the steady state depends on both the **amplitude** and the **frequency** of the driving force. For a steady state, the solution to Equation 3.1 is in the form shown in Equation 3.3:

$$x = A\mathrm{e}^{\mathrm{i}(\omega t - \delta)} \tag{3.3}$$

The terms A and δ are defined in Equation 3.4 and Equation 3.5 below:

$$A = \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + b^2\omega^2}} \tag{3.4}$$

$$\delta = \arctan\left(\frac{b\omega}{m\left(\omega_0^2 - \omega^2\right)}\right) \tag{3.5}$$

3.3 Steady state behaviour - Resonance

When we vary the frequency of the driving frequency we find that the response of the driven system varies. If we examine the power transferred to the system as we vary the driving frequency, we see that there is a peak around the natural frequency, ω_0, of the driven system (Figure 3.1). This is a phenomenon known as **resonance**.

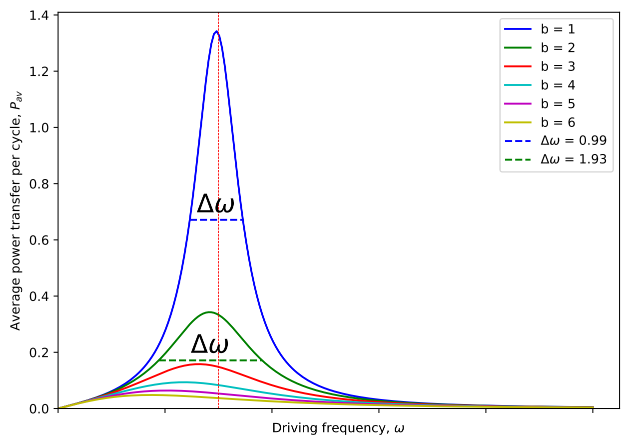

Figure 3.1: The power transferred to a system varies with the driving frequency. The power transferred to a system is proportional to ωA^2. The full-width at half-maximum value is designated $\Delta\omega$. Note that as the damping coefficient b increases, there is a shift in maximum power transmission away from the natural frequency ω_0.

If this system is damped, we also see a change in the "full-width at half-maximum" value.[1] This FWHM is designated by $\Delta\omega$ in Figure 3.1 and is related to the Q-factor of the system via Equation 3.6:

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q} \tag{3.6}$$

This allows us to determine the Q-factor of a system through measurement of resonance of the system, as a lightly-damped system (high Q) will give a sharp resonance with low $\Delta\omega$, while a more heavily damped system will lead to a more broad resonance profile.

This use of the Q-factor is important as it give us a measure of the 'sharpness' (or quality) of the resonance. It may be applied to many systems including electronic circuits.

We can also examine the phase of the oscillator's displacement x relative to that of the driving force, δ, shown in Figure 3.2.

We can see that at low driving frequencies, there is very little phase shift between the two (the force is at a minimum when the displacement is at a minimum and is in the same direction), while at very high driving frequencies we approach a maximum

[1]The "full-width at half maximum" (FWHM) is a term widely used in signals processing to describe the "spread" of a signal. A large FWHM indicates a broad signal over a range of frequencies, while a small FWHM indicates a sharp signal over a narrow band of frequencies.

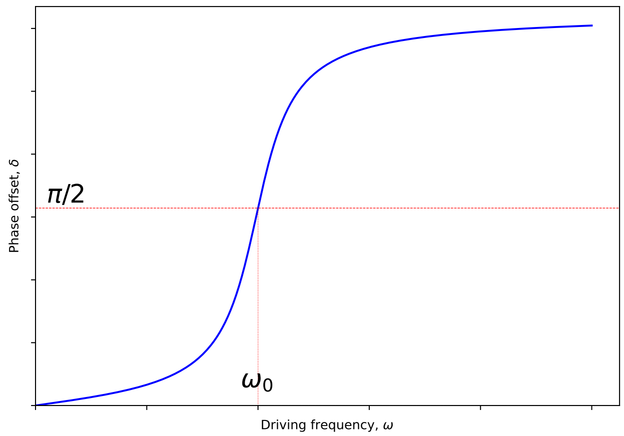

Figure 3.2: The phase shift δ of the observed oscillation varies relative to the driving frequency ω.

phase shift of $\frac{\pi}{2}$ between the two (force still at a minimum at minimum displacement, but now the force is directed in the opposite direction to the displacement; akin to SHM). The phase change is at its most rapid when the frequency of the driving force ω is similar to the natural frequency of the oscillator (ω_0). At this point, the driving force are "in quadrature" (90° out of phase), with the force at a maximum when x is changing most rapidly.

3.4 Full solution of the forced oscillator

As with the damped oscillator explored in Chapter 2, the full solution of the forced oscillator is found through the combination of the "transient state" (a free, damped system) and the "steady state" (fully forced oscillations). This is illustrated in Equation 3.7:

$$\begin{aligned} x &= \text{transient} &&+ \text{steady state} \\ \textit{i.e.} \quad x &= A_0 \mathrm{e}^{-\left(\frac{b}{2m}\right)t} \mathrm{e}^{\mathrm{i}(\omega' t + \delta')} &&+ A\mathrm{e}^{\mathrm{i}(\omega t - \delta)} \end{aligned} \tag{3.7}$$

The terms in Equation 3.7 are as follows:

- δ is given by Equation 3.5;
- A given by Equation 3.4;
- A_0 and δ' depend on the initial conditions;
- ω is the driving frequency;
- ω' is the frequency of the damped (transient) oscillations.

We can simplify our view of Equation 3.7 by applying initial conditions whereby the displacement $x = 0$ at time $t = 0$. Therefore:

$$0 = A_0 \mathrm{e}^{\mathrm{i}(\delta')} + A\mathrm{e}^{-\mathrm{i}\delta}$$

…or, to rearrange, we obtain the form shown in Equation 3.8:

$$A_0 = -A\mathrm{e}^{-\mathrm{i}(\delta+\delta')} \tag{3.8}$$

As with any complex representation, we now compare the **real** and **imaginary** parts of the solution. Remember that Equation 3.8 expands *via* De Moivre's theorem to:

$$A_0 = -A\left(\cos\left(\delta+\delta'\right) + \mathrm{i}\sin\left(\delta+\delta'\right)\right) \tag{3.9}$$

The imaginary component of Equation 3.9 reduces to zero (the solution, A_0 is fully real):

$$0 = -A\sin\left(\delta+\delta'\right)$$

We can therefore relate δ and δ' as follows:

$$\delta + \delta' = \pi \quad \textit{i.e.} \quad \delta' = \pi - \delta \tag{3.10}$$

We now examine the 'real' component of Equation 3.9, equal to A_0:

$$A_0 = -A\cos\left(\delta+\delta'\right)$$

…and doing the same analysis, now that we know that, under our initial conditions, $\delta + \delta' = \pi$:

$$A_0 = -A\cos\pi \quad \textit{i.e.} \quad A_0 = A \tag{3.11}$$

We can now revisit Equation 3.7, now that we have values for δ, δ' and A_0 under our starting conditions ($x = 0$ when $t = 0$):

$$\begin{aligned}
x &= A\left[\mathrm{e}^{\mathrm{i}(\omega t-\delta)} + \mathrm{e}^{-\left(\frac{b}{2m}\right)t}\mathrm{e}^{\mathrm{i}(\omega' t+\delta')}\right] \\
&= A\left[\mathrm{e}^{\mathrm{i}(\omega t-\delta)} + \mathrm{e}^{-\left(\frac{b}{2m}\right)t}\mathrm{e}^{\mathrm{i}(\omega' t+\pi-\delta)}\right] \\
&= A\left[\mathrm{e}^{\mathrm{i}(\omega t-\delta)} + \mathrm{e}^{-\left(\frac{b}{2m}\right)t}\mathrm{e}^{\mathrm{i}(\omega' t-\delta)}\mathrm{e}^{\mathrm{i}\pi}\right] \\
&= A\left[\mathrm{e}^{\mathrm{i}(\omega t-\delta)} - \mathrm{e}^{-\left(\frac{b}{2m}\right)t}\mathrm{e}^{\mathrm{i}(\omega' t-\delta)}\right]
\end{aligned} \tag{3.12}$$

Finally, we can find the actual displacement of the oscillator by examining the real component of Equation 3.12, summarised in Equation 3.13):

$$\mathrm{Re}(x) = A\left[\cos(\omega t - \delta) - \mathrm{e}^{-\left(\frac{b}{2m}\right)t}\cos(\omega' t - \delta)\right] \tag{3.13}$$

We can visualise the amplitude of a forced oscillator, as shown in Figure 3.3. At very low frequencies, the oscillation has the same amplitude as the driving oscillation. This rapidly increases under resonance to a peak, but then drops rapidly to a value towards zero.

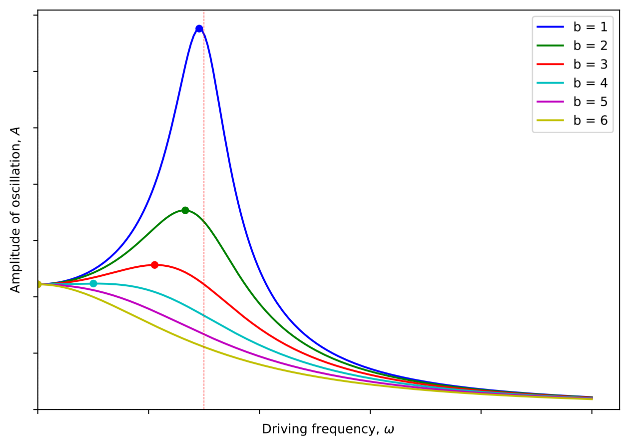

Figure 3.3: The amplitude of a forced oscillation varies with driving frequency; close to the driving amplitude at very low frequencies, close to zero at very high frequencies, with a peak occurring under resonant conditions. Note that the position of this peak varies with the damping coefficient b.

Note that the position of the resonant peak amplitude also varies with the damping coefficient b.

3.4.1 Special cases of forced oscillations

There are three special cases of forced oscillations to consider:

1. $\omega \ll \omega_0$
2. $\omega \gg \omega_0$
3. $\omega = \omega_0$

In the special case when $\omega = \omega_0$ and the damping coefficient b is small (allowing the damped frequency ω' to be approximately equal to the driving frequency ω), we can rewrite the expression of the displacement (Equation 3.13) as shown in Equation 3.14:

$$\mathrm{Re}(x) = A\cos(\omega_0 t - \delta)\left[1 - \mathrm{e}^{-\left(\frac{b}{2m}\right)t}\right] \tag{3.14}$$

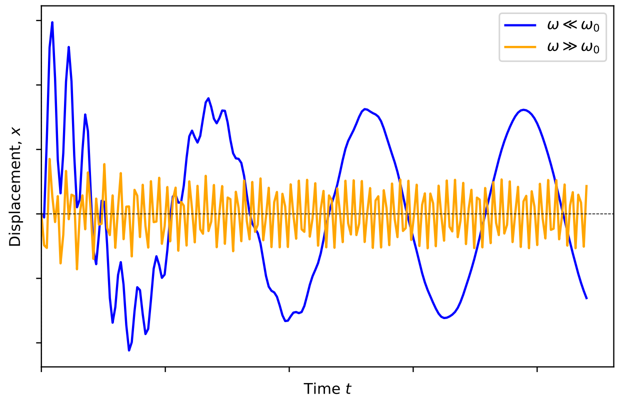

Figure 3.4: When the frequency of the driving oscillation ω is very different to the natural frequency of the system we see two regimes; firstly, where $\omega \ll \omega_0$ (blue), the natural frequency decays and we are left to observe the driving frequency at a large amplitude; when $\omega \gg \omega_0$ (orange), we see a much smaller amplitude oscillation at close to the driving frequency.

We can see that the oscillation will converge to a maximum value, at which point the energy put into the system becomes equal to the energy dissipated by the system through damping, as shown in Figure 3.5:

3.5 Energy in driven oscillators

In a driven oscillator, energy is continually added to the system. When it reaches its steady state, the rate of loss of energy in each cycle due to damping of the system is equal to the work done by the driving force.

We can demonstrate this through integration; remember that (simplistically!) work done is "force × distance", so under varying force (as we have here), we can integrate the force with respect to displacement x to find an expression for the work done.

For the sake of convenience, we will consider the "real" component of the equation for forced SHM shown in Equation 3.1 (the left-hand-side is entirely "real", so the "imaginary" component reduces to zero anyway):

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0 \cos\omega t \tag{3.15}$$

We also know that the steady state solution for the displacement x in SHM is as given in Equation 1.5. We can then find an expression for the velocity v by finding the first derivative of this expression (Equation 3.16):

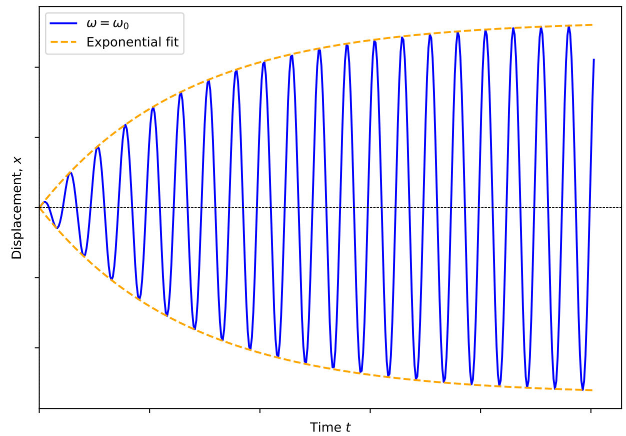

Figure 3.5: When the frequency of the driving oscillation $\omega = \omega_0$ (blue), we see the amplitude grow to a plateau at which point the energy put into the system is equal to the energy losses from the system.

$$\begin{aligned} x &= A\cos(\omega t - \delta) \\ v = \frac{\mathrm{d}x}{\mathrm{d}t} &= -\omega A \sin(\omega t - \delta) \end{aligned} \tag{3.16}$$

3.5.1 Energy input for driven oscillators

To find the energy which is put into the system over one cycle, we therefore need to find the integral of the force acting over one cycle of the oscillation, from $t = 0$ to $t = T$ (remember that T is the period of the oscillation). We do this integration by substituting our variable $\mathrm{d}x$ for $v\mathrm{d}t$ (as defined in Equation 3.16):

$$
\begin{aligned}
E_{\text{in}} = \int_0^T F \cdot \mathrm{d}x \quad &\equiv \quad \int_0^T F \cdot v\mathrm{d}t \\
&= \quad \int_0^T F_0 \cos\omega t \cdot [-\omega A \sin(\omega t + \delta)]\,\mathrm{d}t \\
&= \quad -\omega A F_0 \int_0^T \cos\omega t \cdot \sin(\omega t + \delta) \\
&= \quad -\omega A F_0 \int_0^T \cos\omega t \cdot [\sin\omega t \cos\delta + \cos\omega t \sin\delta] \\
&= \quad -\omega A F_0 \int_0^T [\cos\omega t \sin\omega t \cos\delta + \cos\omega t \cos\omega t \sin\delta] \\
&= \quad -\omega A F_0 \int_0^T \left[\tfrac{1}{2}\sin 2\omega t \cos\delta + \left(\tfrac{1}{2}\cos 2\omega t + \tfrac{1}{2}\right)\sin\delta\right] \\
&= \quad -\omega A F_0 \left[-\tfrac{1}{4\omega}\cos 2\omega t \cos\delta + \left(\tfrac{1}{4\omega}\sin 2\omega t + \tfrac{t}{2}\right)\sin\delta\right]_0^T \\
&= \quad -\omega A F_0 \left(\left[-\tfrac{1}{4\omega}\cos 2\omega T \cos\delta + \left(\tfrac{1}{4\omega}\sin 2\omega T + \tfrac{T}{2}\right)\sin\delta\right] - \left[-\tfrac{1}{4\omega}\cos 0 \cos\delta + \left(\tfrac{1}{4\omega}\sin 0 +\right.\right.\right.
\end{aligned}
$$

Evaluation of this integral gives us the end result shown in Equation 3.17:

$$E_{\text{in}} = \frac{1}{2}\omega A F_0 T \sin\delta \tag{3.17}$$

If we recall that $\delta = \frac{\pi}{2}$ at resonance, substitution of this value into Equation 3.17 tells us that the maximum energy is transferred to the oscillator when driven at a resonant frequency - in line with our expectations.

If we now use the definition of δ in Equation 3.5, we can use trigonometry and the expression for A in Equation 3.4 to give the expression $\sin\delta = \frac{Ab\omega}{F_0}$; we can also use the fact that the period of oscillation T and the angular frequency ω are related by Equation 1.12 ($T = \frac{2\pi}{\omega}$), and simplify the expression for the energy input for the forced oscillator as shown in Equation 3.18:

$$E_{\text{in}} = \pi b \omega A^2 \tag{3.18}$$

3.5.2 Energy lost in driving oscillators

To now determine the amount of energy, we follow a similar process as followed in Section 3.5.1. This time however we need to determine the work done by the oscillator on the damping force during one cycle. This integral is set up as follows:

$$E_{\text{lost}} = \int_0^T F v \mathrm{d}t$$

...but this time the force is the damping force, $F_{\text{damp}} = bv$.

$$E_{\text{lost}} = \int_0^T bv^2 \mathrm{d}t$$

As before, we have the result that $v = -\omega A \sin(\omega t - \delta)$ (Equation 3.16), and following through the integration in a similar manner as in the previous section we are led to the result in Equation 3.19 for a **driven damped oscillator** at **steady state**.

$$E_{\text{lost}} = \pi b \omega A^2 = E_{\text{in}} \tag{3.19}$$

This result should not be a surprise; at a steady state, we expect the energy lost to be equivalent to the energy put into the system, however it is nice that this is vindicated through the mathematics!

A similar result may be obtained for electrical systems in resonance.

Note: When $b = 0$ the above result appears to fail (no energy is lost, but none is put in??); in fact, in an undamped system there can be no steady state (as no energy is lost!), so the amplitude of vibration $A \to \infty$, so this result does not apply.

3.6 Impedance

As you learn about fields, you will examine electrical circuits and see that circuits containing capacitors and inductors are analogous to mechanical oscillators, with the electrical charge oscillating within the circuit. These properties of circuits have many important applications and you will find that much of the analysis we have done here can be applied to those electrical systems.

In the context of an electrical circuit it is useful to define the term **impedance** a measure of the opposition to the flow of current, and is defined as the ratio of the voltage to current ($\frac{V}{I}$) for a particular circuit component. For a resistor this is simply the resistance, however capacitors and inductors also possess impedance.

It turns out to be helpful to write this impedance as a complex number with components in both the real and imaginary plane. This works well with the complex representation of oscillations and naturally takes care of any differences in phase of the current and voltage in different components.

Bringing the analogy back to mechanical oscillations, a mechanical system can also be considered to have an impedance. We define the mechanical impedance as the **force required to produce unit velocity**, *i.e.* $Z_m = \frac{F}{v}$, or $F = vZ_m$. This is the mechanical equivalent of Ohm's law; force corresponding to the voltage, and velocity corresponding to current.

This idea of mechanical impedance will be useful in discussion of wave propagation.

Chapter 4

Coupled Oscillators

The next stage in our exploration of oscillators is to examine the principles behind coupling one oscillator to another. This is known as a 'coupled oscillator'. Two typical examples of a coupled oscillator are shown in Figure XX below; either two pendulums which are linked by a spring, or a pair of oscillating masses on springs which are linked by a further spring. In either case, the mathematics is similar.

For the purposes of discussion here, we will consider the example of two linked masses. This ostensibly keeps the mathematics simpler, however through a simple substitution it is readily adapted for the case of the linked pendulums.

4.1 The uncoupled example

Consider two masses oscillating on a smooth plane as shown in Figure 4.1.

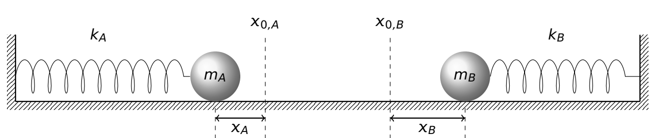

Figure 4.1: Two masses, m_A and m_B oscillating on a frictionless surface. In the absence of a coupling spring, the two masses will undergo SHM indpendently. The displacements x_A and x_B are defined relative to the respective equilibrium positions, $x_{0,A}$ and $x_{0,B}$.

This is a situation we have seen before, and we can set up the equations of motion in a straightforward manner by considering the forces on the masses ()

For A:

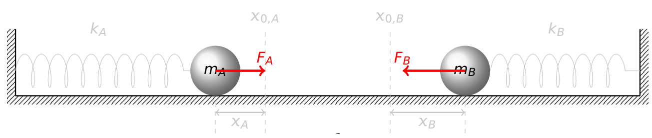

Figure 4.2: The forces on the oscillating masses due to the springs can be determined from the spring extension. For the purposes of setting up the equations, we arbitrarily set the positive direction to the right, however we will get the same result whichever way we consider this.

$$F_A = -k_A x_A = m_A \frac{\mathrm{d}^2 x_A}{\mathrm{d}t^2} \tag{4.1}$$

and for B:

$$F_B = -k_B x_B = m_B \frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} \tag{4.2}$$

4.2 Coupling the oscillators

So far, this is unremarkable. However, when we connect the two masses with a third spring, we can start to consider the coupled interactions. We illustrate this in Figure 4.3.

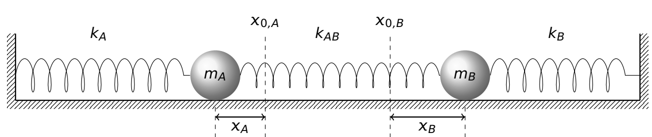

Figure 4.3: The two oscillating masses from our earlier example are now connected by a third spring, of spring constant k_{AB}. We can asssume that all other parameters retain their previous definitions.

In order to determine the equations of motion here, we need to consider the forces on each oscillating mass. Put simply; there are two forces acting on each mass to control their oscillations; the force from its original spring (F_A or F_B), and the new force from the coupling spring, F_{AB}.

From Figure 4.3 we can see that the overall extension of the coupling spring, δ will be given by:

$$\delta = x_B - x_A \tag{4.3}$$

The appearance of the negative sign in Equation 4.3 initially appears counter-intuitive, however consider the signs on each of the extensions in reference to the arbitrary 'positive' direction we defined earlier:

- If $x_B = x_A$, there will be no extension; so no force will be applied from the coupling spring
- In the diagram above, x_A is negative; this gives a double-negative in Equation 4.3, with a positive value of δ (a stretched spring)
- However, if x_A is sufficiently positive, it will result in a negative value of δ, indicating a compressed spring.

You may find it helpful to draw sketches of each of these situations to reason it through!

Ok, let's first consider the total force acting on mass A:

$$\begin{aligned} F_{\text{Total},A} &= F_A + F_{AB} \\ &= -k_A x_A + k_{AB}(\delta) \\ &= -k_A x_A + k_{AB}(x_B - x_A) \\ &= m_A \frac{\mathrm{d}^2 x_A}{\mathrm{d}t^2} \end{aligned} \tag{4.4}$$

We can reason the direction of the force $F_A B$ from a sketch; if the coupling spring is stretched (positive δ), the force will be in the positive direction. If the coupling spring is compressed (negative δ) the force will be in the negative direction.

This situation changes slightly for mass B:

$$\begin{aligned} F_{\text{Total},B} &= F_B - F_{AB} \\ &= -k_B x_B - k_{AB}(\delta) \\ &= -k_B x_B - k_{AB}(x_B - x_A) \\ &= m_B \frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} \end{aligned} \tag{4.5}$$

In this case, a stretched coupling spring (positive δ) will cause a force in the negative direction and *vice versa*.

We have therefore generated two equations of motion to describe the coupled oscillator:

$$-k_A x_A + k_{AB}(x_B - x_A) = m_A \frac{\mathrm{d}^2 x_A}{\mathrm{d}t^2} \tag{4.6}$$

and

$$-k_B x_B - k_{AB}(x_B - x_A) = m_B \frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} \tag{4.7}$$

4.3 Simplifying the expressions

We know from instinct that the motion of a component in a coupled oscillator **should** be harmonic in nature, however it is not immediately clear from the equations that this is the case; therefore some simplification is in order.

Firstly, let's add together Equation 4.6 and Equation 4.7:

$$\begin{aligned} m_A\frac{\mathrm{d}^2x_A}{\mathrm{d}t^2} + m_B\frac{\mathrm{d}^2x_B}{\mathrm{d}t^2} &= -k_Ax_A - k_Bx_B + k_{AB}(x_B - x_A) \\ &\quad -k_{AB}(x_B - x_A) \\ &= -k_Ax_A - k_Bx_B \end{aligned} \tag{4.8}$$

If the spring constants k_A and k_B are equal, and the masses m_A and m_B are equal, we can simplify further:

$$m\frac{\mathrm{d}^2(x_A + x_B)}{\mathrm{d}t^2} = -k(x_A + x_B) \tag{4.9}$$

We can gain a second expression to describe our coupled system by subtracting Equation 4.7 from Equation 4.6:

$$\begin{aligned} m_A\frac{\mathrm{d}^2x_A}{\mathrm{d}t^2} - m_B\frac{\mathrm{d}^2x_B}{\mathrm{d}t^2} &= -k_Ax_A + k_Bx_B + k_{AB}(x_B - x_A) \\ &\quad +k_{AB}(x_B - x_A) \\ &= -k_Ax_A + k_Bx_B + 2k_{AB}(x_B - x_A) \end{aligned} \tag{4.10}$$

Again, **if** the spring constants k_A and k_B are equal, and the masses m_A and m_B are equal, we can again simplify further:

$$\begin{aligned} m\frac{\mathrm{d}^2(x_A - x_B)}{\mathrm{d}t^2} &= -k(x_A - x_B) - 2k_{AB}(x_A - x_B) \\ &= -(k + 2k_{AB})(x_A - x_B) \end{aligned} \tag{4.11}$$

This does not immediately appear to simplify the situation, however if we now define two variables, y_1 and y_2 (we use numerical subscripts now to eliminate confusion with the alphabetical labels of the oscillating masses):

$$\begin{aligned} y_1 &= x_A + x_B \\ y_2 &= x_A - x_B \end{aligned} \tag{4.12}$$

we can now do a substitution into Equation 4.9 and Equation 4.11:

Equation 4.9 becomes:

$$\begin{aligned} m\frac{\mathrm{d}^2(x_A + x_B)}{\mathrm{d}t^2} &= -k(x_A + x_B) \\ m\frac{\mathrm{d}^2y_1}{\mathrm{d}t^2} &= -ky_1 \end{aligned} \tag{4.13}$$

...while Equation 4.11 becomes

$$
\begin{aligned}
m\frac{\mathrm{d}^2(x_A - x_B)}{\mathrm{d}t^2} &= -(k + 2k_{AB})(x_A - x_B) \\
m\frac{\mathrm{d}^2 y_2}{\mathrm{d}t^2} &= -(k + 2k_{AB})y_2
\end{aligned} \tag{4.14}
$$

Now we have two equations, Equation 4.13 and Equation 4.14, each of which is far simpler than the solutions in x, and each satisfies a harmonic oscillator condition in y_n. However, the y_n terms do not influence each other and are, effectively independent.

i Key observations

- y_1 represents one oscillator of mass m and a spring constant k;
- y_2 represents a second oscillator of mass m and a spring constant $(k + 2k_{AB})$

4.4 Getting back to displacement

We can therefore use the same approach as we used in Section 1.2 to give a solution for each term:

$$
\begin{cases}
y_1(t) = B_1 \cos(\omega_1 t + \phi_1) \\
y_2(t) = B_2 \cos(\omega_2 t + \phi_2)
\end{cases} \tag{4.15}
$$

Note that we are deliberately using different terms from previous examples (B_n, ϕ_n), to highlight that these are arbitrary constants. However, we can now use the same techniques as in Equation 1.2 and Equation 1.9 to obtain expressions for the frequencies ω_1 and ω_2 (Equation 4.16):

$$
\begin{aligned}
\omega_1 &= \sqrt{\frac{k}{m}} \\
\omega_2 &= \sqrt{\frac{(k + 2k_{AB})}{m}}
\end{aligned} \tag{4.16}
$$

We can now return to equations Equation 4.12 to return our expressions in y to the displacement x of each oscillating mass. Firstly, invert the equations shown in Equation 4.12 to obtain the results shown in Equation 4.17:

$$
\begin{cases}
x_A &= y_1 - x_B \\
&= y_2 + x_B \\
&= \frac{1}{2}y_1 + \frac{1}{2}y_2 \\
\\
x_B &= y_1 - x_A \\
&= y_2 + x_A \\
&= \frac{1}{2}y_1 - \frac{1}{2}y_2
\end{cases}
\tag{4.17}
$$

4.5 Solving the coupled oscillator

We can now use the general results from Equation 4.15 to give the coupled solutions Equation 4.18:

$$
\begin{cases}
x_1 &= \frac{1}{2}B_1\cos(\omega_1 t + \phi_1) + \frac{1}{2}B_2\cos(\omega_2 t + \phi_2) \\
x_2 &= \frac{1}{2}B_1\cos(\omega_1 t + \phi_1) - \frac{1}{2}B_2\cos(\omega_2 t + \phi_2)
\end{cases}
\tag{4.18}
$$

Recalling our angle formulae, we know that we can express the term $B\cos(\omega t + \phi)$ as $C\cos(\omega t) + D\sin(\omega t)$, where the new constants C and D are a combination of the original constant B and either $\cos\phi$ or $\sin\phi$ (ensure you can identify where this comes from!). This can be used to give a full general solution as:

$$
\begin{cases}
x_1 = C_1\cos(\omega_1 t) + D_1\sin(\omega_1 t) + C_2\cos(\omega_2 t) + D_2\sin(\omega_2 t) \\
x_2 = C_1\cos(\omega_1 t) + D_1\sin(\omega_1 t) - C_2\cos(\omega_2 t) - D_2\sin(\omega_2 t)
\end{cases}
\tag{4.19}
$$

This full solution is more helpful to us if we wish to consider initial conditions of the system (e.g. $x_1(0)$, $v_1(0)$ etc.).

4.6 Normal coordinates

But where are the practicalities of all this? So far, we have done a lot of mathematical manipulation to arrive at two coupled equations (Equation 4.19) which, while they ostensibly describe the position of two coupled oscillating masses, still remain frustratingly abstract! It is helpful to return to the 'simple' expressions in Equation 4.13 and Equation 4.14:

$$
\begin{cases}
m\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} = -ky_1 \\
m\frac{\mathrm{d}^2 y_2}{\mathrm{d}t^2} = -(k + 2k_{AB})y_2
\end{cases}
\tag{4.20}
$$

These underlying 'natural variables', y_1 and y_2, are much simpler to consider and work with, and we therefore call these variables "**normal coordinates**", as it is simpler to work with these directly, and then return to displacement/velocity/acceleration coordinates of the system. It also helps us to visualise what is happening within the system. To do this, we will take the two extreme situations - when $x_A = x_B$, and when $x_A = -x_B$.

It is helpful to have the following on hand:

$$\begin{aligned} y_1 &= x_A + x_B \\ y_2 &= x_A - x_B \end{aligned}$$

4.6.1 When $x_A = x_B$

In this situation, $y_1 = 2x$, and $y_2 = 0$. From equation Equation 4.18, the y_2 term reduces to zero, and we have the displacements for each oscillator as Equation 4.21:

$$\begin{cases} x_1 = \frac{1}{2}B_1 \cos(\omega_1 t + \phi_1) \\ x_2 = \frac{1}{2}B_1 \cos(\omega_1 t + \phi_1) \end{cases} \tag{4.21}$$

We can see from the mathematics that the displacement of each mass from its equilibrium point is identical, however it is often helpful to visualise this motion. In this case, the mid-point between the masses may be considered to be moving in concert with the two masses (Figure 4.4):

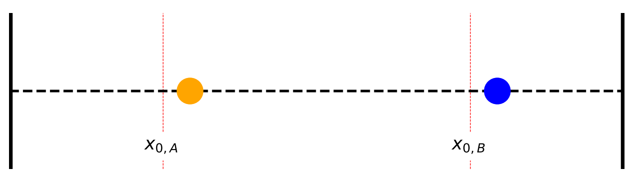

Figure 4.4: When the initial displacement of each oscillator is identical, the coupling spring is neither stretched nor compressed, so has no effect on the oscillation period of each oscillator. The oscillation of each mass is identical and perfectly in phase.

Fundamentally, from Equation 4.3 we know that if $x_A = x_B$, then the extension of the coupling spring is zero. Therefore each mass may be considered to be oscillating exactly in phase and only under the influence of its spring, k. We would expect therefore the oscillation frequency to match that of the uncoupled oscillators introduced in Section 4.1.

4.6.2 When $x_A = -x_B$

This is the exact opposite situation to that discussed above. In this situation, $y_1 = 0$, and $y_2 = 2x$. As it is the y_0 term which is now reduced to zero in Equation 4.18, this expression reduces to Equation 4.22:

$$\begin{cases} x_1 &= \frac{1}{2}B_2\cos(\omega_2 t+\phi_2) \\ x_2 &= -\frac{1}{2}B_2\cos(\omega_2 t+\phi_2) \end{cases} \tag{4.22}$$

Now we have a symmetry; at any stage of motion, mass A will be displaced by the same amount as mass B *in the opposite direction*. The consequence of this now is that the mid-point between the masses now does **not** move as the masses oscillate, and can be considered a *node* (Figure 4.5).

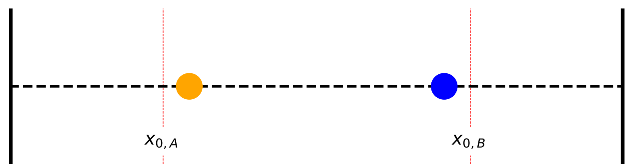

Figure 4.5: When the initial displacement of each oscillator is identical but opposite, the coupling spring is either fully stretched or fully compressed. It may now be considered to be fully active in the oscillation. This time, the oscillation of each mass, while identical, is perfectly out of phase.

4.6.3 Application of normal coordinates

These two situations describe the two vibrational "modes" of this coupled system, termed **normal modes**; any oscillation of this system is completely described by the combination of contributions from each mode (y_1 describing the uncoupled contribution, y_2 describing the fully coupled contribution)

Characteristics of normal coordinates

- Normal coordinates arise from equations of motion expressed in the form of linear differential equations, each with only a single dependent variable (y_1 and y_2 in our examples).
- A **normal mode of vibration** is a vibration of the system which involves only one dependent variable (either y_1 **or** y_2 in our examples)
- Each *normal mode of vibration* has its own characteristic frequency, its **normal frequency**
- The overall vibration of a system may be described as a series of contributions from each normal mode; each normal mode is independent of other normal modes, and energy is never exchanged between normal modes.

4.7 Particular solutions

We covered two particular solutions above, concerning the extreme cases of the normal coordinates, however there are other interesting cases. One particular case occurs when a vibration is initiated with one mass in its equilibrium position and only the other mass is disturbed. The coupling spring of course transfers energy from the moving mass to the stationary one, however due to the similarity in mass, we start to observe resonance effects (Figure 4.6).

> ⚠ A note on energy transfer
>
> When considering the individual masses in a coupled system, we of course have transfer of energy occurring between the two masses. **However** it is important to recognise that while energy is transferred between coupled oscillators, energy is **not** transferred between the normal vibrational modes describing that oscillation of the system.

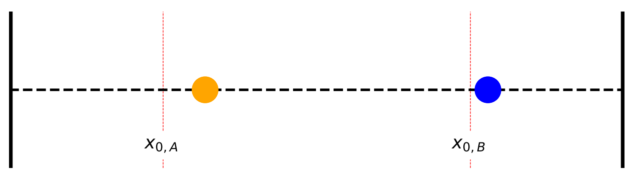

Figure 4.6: If one mass is held stationary at its equilibrium position while the other is displaced as a starting condition, then we see that energy is transferred from one oscillator to the other and back again. In this example, $k = 10k_{AB}$

This resonant energy transfer between each oscillator is particularly apparent when the coupling spring has a considerably lower spring constant than the springs connecting the oscillators to the rigid walls (the **local** springs); in Figure 4.6 the coupling spring has a constant which is one-tenth that of the local springs.

We can solve this example algebraically. Let's re-exmine to the general solution for the coupled oscillator (Equation 4.23)

$$\begin{cases} x_1 = \frac{1}{2}B_1\cos(\omega_1 t+\phi_1)+\frac{1}{2}B_2\cos(\omega_2 t+\phi_2) \\ x_2 = \frac{1}{2}B_1\cos(\omega_1 t+\phi_1)-\frac{1}{2}B_2\cos(\omega_2 t+\phi_2) \end{cases} \tag{4.23}$$

We said that at the start, our initial conditions would be that one oscillator is displaced to amplitude A while the other is held at its equilibrium. Therefore our starting conditions become:

$$\begin{array}{lllll} x_1(0) & = & A & \dot{x}_1(0) & = & 0 \\ x_2(0) & = & 0 & \dot{x}_2(0) & = & 0 \end{array}$$

The result of this is that our phase constant ϕ evaluates to zero, while the amplitude component $B_1 = B_2 = A$. We can apply these to Equation 4.23 to obtain a particular solution:

$$\begin{cases} x_1 & = & \frac{1}{2}A\cos(\omega_1 t) + \frac{1}{2}A\cos(\omega_2 t) \\ x_2 & = & \frac{1}{2}A\cos(\omega_1 t) - \frac{1}{2}A\cos(\omega_2 t) \end{cases} \tag{4.24}$$

This can then be rewritten using the appropriate trigonometric identity as:

$$\begin{cases} x_1 & = & A\cos\left(\frac{\omega_2-\omega_1}{2}t\right)\cos\left(\frac{\omega_2+\omega_1}{2}t\right) \\ x_2 & = & A\sin\left(\frac{\omega_2-\omega_1}{2}t\right)\sin\left(\frac{\omega_2+\omega_1}{2}t\right) \end{cases} \tag{4.25}$$

We said that this resonance transfer was particularly great when there is an appreciable difference between the spring constants of the coupling spring and the local springs; let's therefore consider what is going on.

When the spring constant of the coupling spring $k_A B$ is much greater than k, this will affect the frequencies ω_1 and ω_2. From Equation 4.16, when $k_A B << k$, then the values of ω_1 and ω_2 will be similar in magnitude, with ω_2 only slightly larger. This means that the value of $\omega_2 - \omega_1$ will be very small, while $\omega_2 + \omega_1$ will be considerably larger.

We therefore have two cosine terms governing x_1:

- $\cos\left(\frac{\omega_2-\omega_1}{2}t\right)$ will oscillate much slower than $\cos\left(\frac{\omega_2+\omega_1}{2}t\right)$
- We view this as an oscillator of frequency $\left(\frac{\omega_2+\omega_1}{2}\right)$, whose amplitude varies slowly with frequency $\left(\frac{\omega_2-\omega_1}{2}\right)$

A very similar situation exists for for x_2, however the terms are governed by sine functions - which are $\pi/2$ out of phase with the oscillations of the first.

A note about phases of beats

The second amplitude term described above can be called the **envelope** of the primary oscillation - in that it **contains** the amplitude of the primary oscillation within itself.
Although the oscillations of the second coupled oscillator are $\pi/2$ out of phase with the oscillations of the first, practically we observe that one is a maximum while the other is a minimum; the nature of an envelope is to contain a primary function, and its amplitude maxima of the primary function occur twice per envelope cycle giving the appearance of amplitudes being perfectly out of phase.

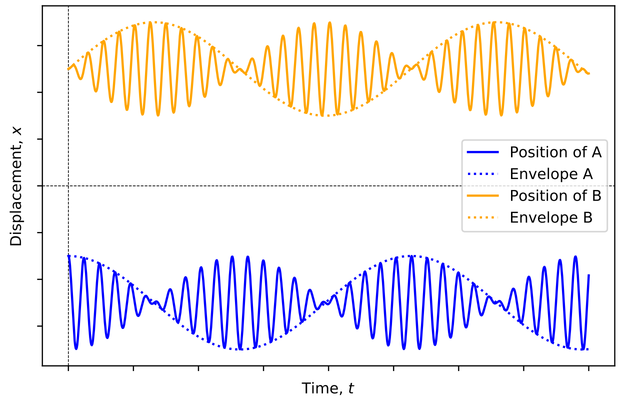

Figure 4.7: Comparing the changes of position of coupled oscillators A and B as a function of time; note that the energy transfers back and forward between A and B such that each is contained within envelopes with a $\pi/2$ phase difference, although it appears as though it is a π phase offset.

4.8 The general solution: a matrix approach

We have shown above how we can obtain a solution for a system of two coupled oscillators, together with methods for finding a general solution. However, what happens when we have three coupled oscillators? Or four? Or - given that in materials we are dealing with solid lattices of bonded molecules - an almost uncountable number of coupled oscillators? The approaches detailed above become extremely difficult with each additional oscillator. We therefore need a method which is scalable.

In Section 4.2 we set up the equations of motion for the coupled oscillator (Equation 4.6 and Equation 4.6); these are reproduced below in Equation 4.27:

$$\begin{cases} m_A \frac{\mathrm{d}^2 x_A}{\mathrm{d}t^2} = -k_A x_A + k_{AB}(x_B - x_A) \\ m_B \frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} = -k_B x_B - k_{AB}(x_B - x_A) \end{cases} \tag{4.26}$$

Let's rewrite these equations to group our x_A and x_B terms rather than the k terms:

$$\begin{cases} m_A \frac{\mathrm{d}^2 x_A}{\mathrm{d}t^2} = -(k_A + k_{AB}) x_A + k_{AB} x_B \\ m_B \frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} = -(k_B + k_{AB}) x_B + k_{AB} x_A \end{cases} \tag{4.27}$$

From your understanding of differential equations, you should recognise these as **linear differential equations**; therefore you should know that we can apply the principles of linear algebra to solve the equations. We can therefore use matrices and eigenstates to determine our solutions.

We can add together the differential equations in Equation 4.27 and then write the result in matrix form as follows (Equation 4.28)

$$\begin{pmatrix} m_A & 0 \\ 0 & m_B \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}^2 x_A}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} \end{pmatrix} = -\begin{pmatrix} (k_A + k_{AB}) & -k_{AB} \\ -k_{AB} & (k_B + k_{AB}) \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \tag{4.28}$$

For simplicity, we will now define a 'spring matrix', K, defining the springs constant, and a 'mass matrix', M (Equation 4.29)

$$K = \begin{pmatrix} (k_A + k_{AB}) & -k_{AB} \\ -k_{AB} & (k_B + k_{AB}) \end{pmatrix} \quad ; \quad M = \begin{pmatrix} m_A & 0 \\ 0 & m_B \end{pmatrix} \tag{4.29}$$

This allows us to tidily rewrite Equation 4.28 as Equation 4.30:

$$M\frac{\mathrm{d}^2 \bar{x}}{\mathrm{d}t^2} = -K\bar{x} \tag{4.30}$$

...where:

$$\bar{x} = \begin{pmatrix} x_A \\ x_B \end{pmatrix} \quad \text{and} \quad \frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \begin{pmatrix} \frac{\mathrm{d}x_A}{\mathrm{d}t} \\ \frac{\mathrm{d}x_B}{\mathrm{d}t} \end{pmatrix}$$

You will recall from your mathematics lectures that you can solve such matrix equations by multiplying both sides of the equation by the inverse of a matrix; in this case we will multiply both sides of Equation 4.30 by the inverse M^{-1}:

$$\begin{aligned} M^{-1}M\frac{\mathrm{d}^2 \bar{x}}{\mathrm{d}t^2} &= -M^{-1}K\bar{x} \\ \frac{\mathrm{d}^2 \bar{x}}{\mathrm{d}t^2} &= -M^{-1}K\bar{x} \end{aligned} \tag{4.31}$$

...where M^{-1} is:

$$M^{-1} = \begin{pmatrix} \frac{1}{m_A} & 0 \\ 0 & \frac{1}{m_B} \end{pmatrix}$$

If we express $M^{-1}K$ as D, the **dynamics equation**, we can write this matrix equation as:

$$\frac{\mathrm{d}^2 \bar{x}}{\mathrm{d}t^2} = -D\bar{x} \tag{4.32}$$

We can immediately see the connection between the matrix equation shown in Equation 4.31 and the equation of motion for a harmonic oscillator. Let's now develop this.

4.8.1 A trial solution

Let's now assume a simple case, with a single oscillation frequency. We will use a trial solution for all vibrational modes, in the form $A\cos(\omega t + \phi)$. Let's define each solution of x_n in this manner:

$$\begin{cases} x_{A,\text{trial}} = A_1\cos(\omega t + \phi) \\ x_{B,\text{trial}} = A_2\cos(\omega t + \phi) \end{cases} \tag{4.33}$$

We have been working with matrix forms; so let's represent this in a matrix form:

$$\bar{x}_{\text{trial}} = \begin{pmatrix} A_1\cos(\omega t + \phi) \\ A_2\cos(\omega t + \phi) \end{pmatrix} = \bar{A}\cos(\omega t + \phi) \quad \text{where} \quad \bar{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{4.34}$$

This matrix form is now using $\bar{A}$; the **mode amplitude vector**. We can now find the second derivative of this to be:

$$\frac{\mathrm{d}^2\bar{x}_{\text{trial}}}{\mathrm{d}t^2} = -\omega^2\bar{A}\cos(\omega t + \phi) \tag{4.35}$$

...which, when we know from Equation 4.32 that the second derivative should give us $-D\bar{A}\cos(\omega t + \phi)$, we obtain the result:

$$\begin{aligned} \omega^2\bar{A}\cos(\omega t + \phi) &= -D\bar{A}\cos(\omega t + \phi) \\ D\bar{A} &= \omega^2\bar{A} \end{aligned} \tag{4.36}$$

Equation 4.36 is the **main result** of this process; that being that the amplitude vector of a vibrational mode is an eigenvector of the dynamics matrix D with the eigenvalue being the square of the frequency of that mode.

4.8.2 The general case

We have said Equation 4.36 as an **eigenvalue equation**; this is one of the most important tools in physics, and it is essential to understand how they work. In this case, we will apply this to determine the frequencies for a coupled oscillator and check with our results above.

We already seen the end result, that $D\bar{A} = \omega^2\bar{A}$, having constructed this assuming a single oscillation frequency. But how does this scale, particularly as we said earlier that we expect *two* frequencies, one from each vibrational mode?

A refresher on eigenvalue equations

For an eigenvalue equation, an $n \times n$ matrix A will have n eigenvalues (λ_i) and n eigenvectors $\bar{x}_i$ such that:

$$A\bar{x}_i = \lambda_i \bar{x}_i$$

To find our eigenvalues, we need to rearrange and solve for λ_i:

$$\begin{aligned} A\bar{x}_i - \lambda_i \bar{x}_i &= 0 \\ A\bar{x}_i - \lambda_i \mathbf{I}\bar{x}_i &= 0 \\ (A - \lambda_i \mathbf{I})\bar{x}_i &= 0 \end{aligned}$$

Remember that we need to multiply λ_i by the identity matrix $\mathbf{I}$ because we cannot perform addition/subtraction on matrices of different dimensions.
We can now solve the eigenvalue equation; either $\bar{x}_i = 0$ (a null result), or the determinant of the matrix $|A - \lambda_i \mathbf{I}| = 0$. Evaluating this determinant will give the characteristic polynomial which, when solved for λ_i, will give the eigenvalues for the equation.

We start with our eigenvalue equation, and we can rearrange this to obtain an expression which will allow us to evaluate the eigenvalue (Equation 4.37).

$$\begin{aligned} D\bar{A} &= \omega^2 \bar{A} \\ D\bar{A} - \omega^2 \bar{A} &= 0 \\ D\bar{A} - \omega^2 \mathbf{I}\bar{A} &= 0 \\ (D - \omega^2 \mathbf{I})\bar{A} &= 0 \end{aligned} \tag{4.37}$$

From this statement, either $\bar{A}$ is equal to zero, or the determinant of the matrix $(D - \omega^2 \mathbf{I})$ is equal to zero. Clearly we use this second result to determine the value of the eigenvalue, ω^2.

Remembering from Section 4.8 we defined D as the **dynamics matrix** derived from the inverse of the 'mass matrix' and the 'spring matrix' such that:

$$\begin{aligned} D = M^{-1}K &= \begin{pmatrix} \frac{1}{m_A} & 0 \\ 0 & \frac{1}{m_B} \end{pmatrix} \begin{pmatrix} (k_A + k_{AB}) & -k_{AB} \\ -k_{AB} & (k_B + k_{AB}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{k_A + k_{AB}}{m_A} & -\frac{k_{AB}}{m_A} \\ -\frac{k_{AB}}{m_B} & \frac{k_A + k_{AB}}{m_B} \end{pmatrix} \end{aligned} \tag{4.38}$$

Now, we set up the matrix form of our eigenvalue equation from Equation 4.37; remembering that we are looking for the eigenvalues which result in a determinant of zero. Let's work it through!

$$
\begin{aligned}
(D-\omega^2\mathbf{I})\bar{A} &= 0 \\
\det\left[\begin{pmatrix} \frac{k_A+k_{AB}}{m_A} & -\frac{k_{AB}}{m_A} \\ -\frac{k_{AB}}{m_B} & \frac{k_A+k_{AB}}{m_B} \end{pmatrix} - \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] &= 0 \\
\det\begin{bmatrix} \left(\frac{k_A+k_{AB}}{m_A}\right)-\omega^2 & -\frac{k_{AB}}{m_A} \\ -\frac{k_{AB}}{m_B} & \left(\frac{k_A+k_{AB}}{m_B}\right)-\omega^2 \end{bmatrix} &= 0
\end{aligned} \tag{4.39}
$$

We will simplify the arrangement somewhat by considering the coupled oscillator we have been examining above; where the local springs k_A and k_B have the same spring constant, k, and the oscillators A and B have the same mass, m. Our determinant then becomes:

$$
\begin{aligned}
\det\begin{bmatrix} \left(\frac{k+k_{AB}}{m}\right)-\omega^2 & -\frac{k_{AB}}{m} \\ -\frac{k_{AB}}{m} & \left(\frac{k+k_{AB}}{m}\right)-\omega^2 \end{bmatrix} &= 0 \\
\left[\left(\frac{k+k_{AB}}{m}\right)-\omega^2\right]^2-\frac{k_{AB}^2}{m^2} &= 0
\end{aligned} \tag{4.40}
$$

This relation is now a quadratic in ω^2; as such, it will have two solutions - each corresponding to a different value of ω - this is what we are expecting - to find ω_1 and ω_2 above. So we solve for ω^2:

$$
\begin{aligned}
\left[\left(\frac{k+k_{AB}}{m}\right)-\omega^2\right]^2-\frac{k_{AB}^2}{m^2} &= 0 \\
\left(\frac{k+k_{AB}}{m}\right)-\omega^2 &= \pm\frac{k_{AB}}{m} \\
k+k_{AB}-m\omega^2 &= \pm k_{AB} \\
m\omega^2 &= \mp k_{AB}+(k+k_{AB})
\end{aligned} \tag{4.41}
$$

$$
\omega^2 = \frac{k}{m} \quad \text{or} \quad \omega^2 = \frac{2k_{AB}+k}{m}
$$

As we can see in Equation 4.41 we gain a result for ω^2 which gives us the two solutions for ω_1 and ω_2 which we uncovered in Section 4.3 and Section 4.4.

Caution

Remember the frequencies that we have found here correspond to the frequencies of the **normal modes of vibration of the system** (Section 4.6.3). They are used to describe the motion of individual oscillating masses, but both frequencies apply to each oscillating mass - *not* one frequency for each mass!

4.9 Why use matrices?

We have shown a few routes to look at coupled oscillators, each of which we have demonstrated to lead to the same outcome. The obvious question is **why** use so many approaches?

When we looked at the separate differential equations, it was a problem which is possible to solve here for two coupled oscillators, however it is not a method which is particularly scalable, for either different masses, different spring constants, or even a larger number of coupling oscillators.

The matrix approach however is scalable - it is simple in its form allowing us to see the nature of the oscillation and that it is - at its heart - still harmonic, however this simple form will scale readily for larger systems. We could look at a system of five linearly coupled oscillators, and we would expect a 5×5 spring (and hence dynamics) matrix, the determinant of which would be a fifth order polynomial with five solutions for the oscillation frequency, each corresponding to the frequency of a normal mode of vibration for the system.

Solving a determinant for a 3×3 matrix by hand is doable, for a 4×4 matrix is troublesome, and a 5×5 matrix is downright inconvenient. However, computers are *excellent* at resolving determinants of matrices - making solving such problems possible - provided we know what instructions to give the computer!

> **Applications of matrix methods**
>
> Matrices are universal in physics, and it is hard to understate the importance of solving eigenvalue equations. Linear algebra is central to quantum physics, and greatly simplifies the process of solving complex interlinked systems. The methods shown above are simply one application of the techniques.

Chapter 5

Pendulums

A pendulum is a mechanically simple oscillating system; its oscillation can be considered SHM for small displacements. A simple pendulum is shown in Figure 5.1, and it is worth quickly revisiting this example as we will extend this in our further examples.

At its core, a pendulum is an oscillator whose oscillation is based on a rotational, rather than a linear displacement. Consequently the dynamics governing pendulums is slightly different to a linear oscillator, however they are a good example to show that we can approximate small displacements as "simple harmonic oscillations".

We can loosely break down pendulums (pendula?) into a number of categories depending on their nature. Note this is not an exhaustive list!

- A simple pendulum. This is the simplest pendulum we can consider; a point mass attached to a light string, swinging under the influence of gravity alone. We approximate this in reality by using a thin string and a heavy mass such that the mass of the string is insignificant in the calculations.
- A physical pendulum. This is similar in setup to the simple pendulum, however we take into account the mass of the rod connecting the mass to the pivot (note a "rod" rather than a "string") as well as the physical size of the mass. This requires determination of moments of inertia.
- A torsional pendulum. This is a pendulum whose oscillations are not driven by gravity - rather by the forces induced when a horizontally rotating mass twists a fixed vertical support. This is often used in portable mechanical clocks where there is not space for a physical pendulum.
- A double (or compound) pendulum. This is a system of two (or more!) pendula connected end to end, such that the end of one pendulum forms the pivot of the next. It results in motion which is described as **chaotic**.

We will examine the case of the simple pendulum, the physical pendulum, and will take a brief look at the double pendulum.

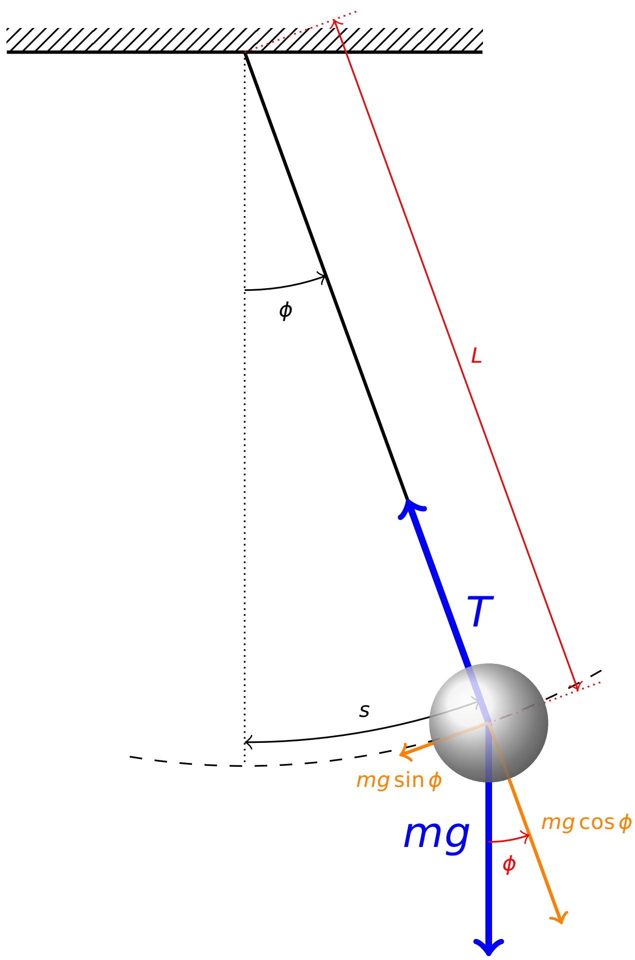

Figure 5.1: A simple pendulum, consisting of a mass m swinging on a string of length L. At angle ϕ, the weight of the mass mg can be resolved into components to determine the mechanics of the system.

5.1 The simple pendulum

The mass of the pendulum can be considered to oscillate along the arc s (Figure 5.1), where the restoring force becomes $mg\sin\phi$. This force then causes an acceleration along the arc, $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$, with the equation of motion becoming:

$$mg\sin\phi = -m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \tag{5.1}$$

This however relates the variable ϕ relative to the arc length s; it is more useful to relate this to a single variable. We can derive an expression for this oscillation with respect to angle as follows:

- The arc length s can be calculated as $s = L\phi$.
- The second derivative of s with respect to time can then be found:

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = L\frac{\mathrm{d}^2 \phi}{\mathrm{d}t^2} \tag{5.2}$$

We can then substitute this into Equation 5.1:

$$m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = mL\frac{\mathrm{d}^2 \phi}{\mathrm{d}t^2} = -mg\sin\phi \tag{5.3}$$

A quick rearrangement gives us the equation of motion for a simple pendulum.

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}t^2} = -\frac{g}{L}\sin\phi \tag{5.4}$$

For small displacements (small ϕ), the equation of motion can be considered as in Equation 5.5:

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}t^2} = -\frac{g}{L}\sin\phi \approx -\frac{g\phi}{L} \text{ for small } \phi \tag{5.5}$$

This is SHM, with angular frequency ω and period T found as in Equation 5.6:

$$\omega^2 = \frac{g}{L} \quad \textit{and} \quad T = 2\pi\sqrt{\frac{L}{g}} \tag{5.6}$$

The solution for the equation of motion of this system then becomes Equation 5.7:

$$\phi = \phi_0\cos(\omega t + \delta) \tag{5.7}$$

...where ϕ_0 is the amplitude of the system and δ is the phase constant. Notice that, for the pendulum we express the amplitude in terms of the angle of the string rather than an absolute distance (shown as s in Figure 5.1).

5.2 The Physical Pendulum

Many oscillating systems demonstrate rotational oscillations under gravity akin to the simple pendulum. In this case, the system rotates around a pivot, P, and this can then be considered as a pendulum with the centre of mass acting as the 'bob'. An example of a general system is shown in Figure 5.2.

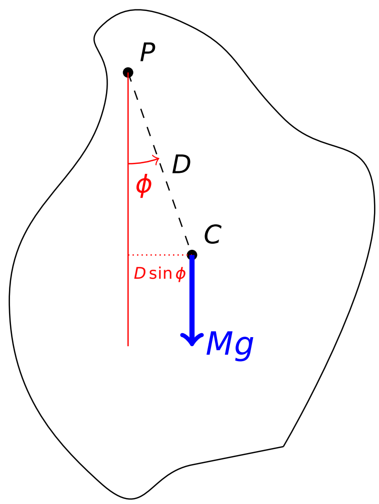

Figure 5.2: A physical pendulum, consisting of a physical object of mass M swinging on a pivot through point P of length L. At angle ϕ, the weight of the mass Mg can again be resolved into components to determine the mechanics of the system.

Now, as a restoring force, we consider the **torque** of the centre of mass around the pivot. Remember that the torque is defined as the product of the force and the **perpendicular distance of the force's line of action from the pivot**. If using vectors, this is considered as the **cross product** of the force vector $\mathbf{F}$ with the position vector of the centre of mass from the pivot, $\mathbf{D}$ (Equation 5.8):

$$\begin{aligned} \text{Torque about pivot } P &= \mathbf{D} \times \mathbf{F} \\ &= \hat{\mathbf{n}}\, |D||F| \sin\phi \\ &= \hat{\mathbf{n}}\, Dmg \sin\phi = \tau\hat{\mathbf{n}} \end{aligned} \tag{5.8}$$

In this case, the unit vector $\hat{\mathbf{n}}$ is perpendicular to the plane of rotation and is included for completeness' sake. What we are interested in is the magnitude of this torque vector, τ.

Recall from rotational motion that the angular acceleration α and the torque τ are connected via the moment of inertia I (Equation 5.9):[1]

$$\begin{aligned} I\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} &= I\alpha = \tau \\ &= -mgD\sin\phi \end{aligned} \tag{5.9}$$

We now approximate this for small ϕ:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -\frac{mgD\phi}{I} = -\omega^2\phi$$

This allows us to identify expressions again for the angular frequency, ω, and the period T (Equation 5.10):

$$\omega^2 = \frac{mgD}{I} \qquad and \qquad T = 2\pi\sqrt{\frac{I}{mgD}} \tag{5.10}$$

We can compare this result with that for the simple pendulum shown in Equation 5.6; if we remember that the moment of inertia I is defined as $I = mD^2$, we can substitute this into Equation 5.10 and see that this is a general case for any rotating body:

$$T_{\text{simple pendulum}} = 2\pi\sqrt{\frac{I_{\text{simple pendulum}}}{mgD}} = 2\pi\sqrt{\frac{mD^2}{mgD}} = 2\pi\sqrt{\frac{D}{g}}$$

...giving us our expected result (where the general term D can be replaced for the length of the simple pendulum, L).

5.3 The Double Pendulum

The double pendulum is a case study for a particular type of coupled oscillator; we take a simple pendulum (a light rod suspended from a pivot with a mass m_1 at the end), and couple to it a second simple pendulum with mass m_2; the pivot of this second pendulum being m_1. It is a deceptively complex problem, as it looks from the outset to be fairly simple. However, we rapidly find that while at small amplitudes, the oscillations are predictable, as the amplitude increases, tiny variations in the starting conditions for the pendulum can have wildly different effects, and solving the equations of motion is anything but trivial.

5.4 Chaos

[1] Note that the appearance of the negative sign indicates that the torque force is opposite to the direction of increasing ϕ

Chapter 6

From Coupled Oscillators to Wave Motion

We started our explorations of oscillations and waves by acknowledging the inherent connection between the two. We have seen that harmonic oscillations can be described by sinusoidal functions (sine ansd cosine functions), and we know that sinusoidal functions take the form of a wave. However it may not be immediately clear how we get from the oscillation of a single particle to an organised group oscillation creating a wave which is capable of transmitting energy across space. We will now examine this process.

6.1 Coupled masses on a string under tension

In this example we will consider a system consisting of a number of equally spaced masses on a light string; these behave as coupled oscillators perpendicular to the axis of the string. We consider the string to be fixed at both ends and the masses to be separated by a fixed horizontal distance a (Figure 6.1)

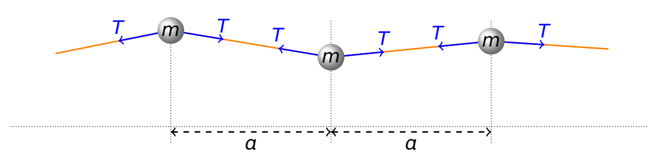

Figure 6.1: A system of equally spaced masses on a tensioned string. The masses are equally spaced, and undergo oscillations perpendicular to the axis of the string.

In order to develop the equations of motion here, we need to take a number of steps:

- Knowing the masses only oscillate vertically, we need to find the vertical component of the tension for each mass
- We use this to determine the equation of motion for a given mass on the string
- Finally, we look at what happens to each mass along a string.

Let's take each step in turn

6.1.1 Consolidating the forces

Let's consider the mass in the middle of Figure 6.1. We will label this arbitrary nth mass n. We consider the forces on it along the string towards the $(n-1)$th and $(n+1)$th mass. To do this, we identify a few lengths and the angles θ_1 and θ_2 (Figure 6.2):

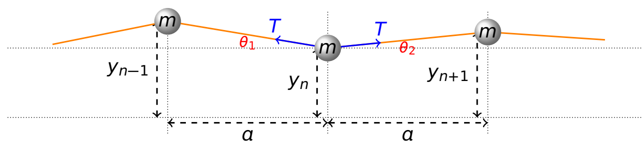

Figure 6.2: Consolidating the forces on the middle mass, mass n. We consider everythign relative to the y position of this oscillator and the effect of the masses to either side. The angles θ_1 and θ_2 indicate the angles the forces make to the horizontal.

The vertical component of the tension to the left will be given by $T\sin\theta_1$, while the vertical component of the tension to the right will be given by $T\sin\theta_2$.

Through the small-angle approximation, we can say:

$$\tan\theta_1 = \frac{(y_{n-1} - y_n)}{a} \approx \sin\theta_1 \tag{6.1}$$

$$\tan\theta_2 = \frac{(y_{n+1} - y_n)}{a} \approx \sin\theta_2 \tag{6.2}$$

The total force acting on the mass therefore becomes

$$\begin{aligned} F_{\text{total}} &= T(\sin\theta_1 + \sin\theta_2) \\ &= T\left(\frac{(y_{n-1}-y_n)}{a} + \frac{(y_{n+1}-y_n)}{a}\right) \end{aligned} \tag{6.3}$$

6.1.2 The equations of motion

Now that we have identified the vertical force acting on our mass (Equation 6.3) we can now write our equation of motion. We assume that this will be simple harmonic motion, therefore we can write our equation of motion for the nth mass as:

$$m\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = T\left(\frac{(y_{n-1} - y_n)}{a} + \frac{(y_{n+1} - y_n)}{a}\right) \tag{6.4}$$

...or...

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = \frac{T}{ma}(y_{n-1} - 2y_n + y_{n+1}) \tag{6.5}$$

> **Note**
>
> We have not placed a negative sign in Equation 6.4 in front of the force to indicate a restoring force. However, remember that we find our force using the parameters $(y_{n-1} - y_n)$ or $(y_{n+1} - y_n)$. In the example shown in Figure 6.2, $y_n < y_{n-1}$ and $y_n < y_{n+1}$, so we expect a positive value, so a positive force. If the situation is reversed, and $y_n > y_{n-1}$ and $y_n > y_{n+1}$, we would expect a negative value to arise, giving a negative value for the force.
> (While the sinusoidal functions are periodic, we are only concerned for values of θ in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$).

> **Warning**
>
> Note: remember that the a in Equation 6.23 is the separation between the masses on the tensioned string, **not** the acceleration!

Let's now use the approach used previously; insert a trial function into the equation of motion. Assuming simple harmonic motion again, we use the trial function $y_n = A_n \cos(\omega t + \delta)$, where A_n is the maximum amplitude of vibration. We can also use the trial functions $y_{n-1} = A_{n-1}\cos(\omega t + \delta)$ and $y_{n+1} = A_{n+1}\cos(\omega t + \delta)$.

We can place these into Equation 6.23 as follows:

$$\begin{aligned}
\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} &= \frac{T}{ma}(y_{n-1} - 2y_n + y_{n+1}) \\
-\omega^2 A_n \cos(\omega t + \delta) &= \frac{T}{ma}(A_{n-1} - 2A_n + A_{n+1})\cos(\omega t + \delta) \\
-\omega^2 A_n &= \frac{T}{ma}(A_{n-1} - 2A_n + A_{n+1})
\end{aligned} \tag{6.6}$$

This can then be reformulated into our key result:

$$-A_{n-1} + \left(2 - \frac{ma\omega^2}{T}\right)A_n - A_{n+1} = 0 \tag{6.7}$$

6.2 The overview

In considering the case for each of n oscillating masses connected by a tensioned string, it is all very much looking like a coupled oscillator system, akin to that introduced in Section 4.8. We would therefore expect to have a set of n coupled equations, which in turn will give n different values of ω^2. We can apply the matrix methods we introduced in Section 4.8, however let's just look at the first two cases (one mass and two masses).

6.2.1 A single mass on a tensioned string ($n = 1$)

This is a fairly straightforward analysis. Adapting Equation 6.7, A_{n-1} and A_{n+1} reduce to zero as these represent the fixed ends of the string. Our fundamental equation therefore becomes:

$$\begin{aligned} -A_0 + \left(2 - \tfrac{ma\omega^2}{T}\right) A_1 - A_2 &= 0 \\ \left(2 - \tfrac{ma\omega^2}{T}\right) A_1 &= 0 \end{aligned}$$

...with the result that

$$\omega^2 = \frac{2T}{ma} \tag{6.8}$$

We therefore have a single vibrational frequency, ω when we have a single mass oscillating on a tensioned string. This is not a great surprise, and is easily visualised.

6.2.2 Two masses on a tensioned string, $n = 2$:

If we have two masses on the string, we now need to consider the respective equations for the first ($n = 1$) and the second ($n = 2$) masses.

- For $n = 1$:
  - $A_0 = 0$
- For $n = 2$:
  - $A_3 = 0$

Equation 6.7 then becomes:

$$\begin{aligned} -A_0 + \left(2 - \tfrac{ma\omega^2}{T}\right) A_1 - A_2 &= 0 \\ -A_1 + \left(2 - \tfrac{ma\omega^2}{T}\right) A_2 - A_3 &= 0 \end{aligned}$$

which, when we apply the boundary conditions for A_0 and A_3, becomes:

$$\begin{cases} \left(2 - \tfrac{ma\omega^2}{T}\right) A_1 - A_2 = 0 \\ -A_1 + \left(2 - \tfrac{ma\omega^2}{T}\right) A_2 = 0 \end{cases} \tag{6.9}$$

This is simply a pair of simultaneous equations in A_1 and A_2; substituting through to eliminate A_2, we arrive at the result:

$$
\begin{aligned}
A_1 \left[\left(2 - \frac{ma\omega^2}{T} \right)^2 - 1 \right] &= 0 \\
\left(2 - \frac{ma\omega^2}{T} \right)^2 - 1 &= 0
\end{aligned}
$$

This now factorises:

$$
\left(2 - \frac{ma\omega^2}{T} + 1 \right) \left(2 - \frac{ma\omega^2}{T} - 1 \right) = 0
$$

This shows that we end up with two possible solutions for ω^2:

$$
\omega_1^2 = \frac{T}{ma} \quad \text{and} \quad \omega_2^2 = \frac{3T}{ma} \tag{6.10}
$$

These are the frequencies corresponding to the normal modes of vibration on the string; note that while the *values* are different to the coupled oscillator model in Chapter 4, the principle is the same - where two oscillating masses give two characteristic frequencies, each corresponding to a specific **normal mode**.

6.2.3 The general case, n masses on a tensioned string:

Let's return to our fundamental equation (Equation 6.7):

$$
-A_{n-1} + \left(2 - \frac{ma\omega^2}{T} \right) A_n - A_{n+1} = 0
$$

From Equation 6.8 for a single mass on a string we note that $\omega_1^2 = \frac{2T}{ma}$; therefore the term $\frac{T}{ma}$ is intrinsically linked to a frequency. Let's therefore define this as a fundamental frequency, ω_0. We now use this reformulate our fundamental equation to isolate the A terms

$$
\frac{A_{n-1} + A_{n+1}}{A_n} = 2 - \frac{ma\omega^2}{T} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2} \tag{6.11}
$$

In Equation 6.11, the frequency ω represents any normal mode frequency; therefore for any fixed value of this the right-hand side of the equation is a constant regardless of which oscillating mass we are considering. So; can we solve the equation to find a value for the amplitude of the oscillation of the nth mass, A_n?

From our coupled oscillator case in Chapter 4, we saw that for two oscillating masses in a coupled system, the **amplitude** varied sinusoidally (Equation 4.25, graphed in Figure 4.7). Let's therefore assume that there is a general solution for the amplitude of the jth mass, A_j.

$$A_j = B\sin(j\phi) \tag{6.12}$$

In this, B is a constant and ϕ is some constant value for a value of ω_j. Let's now place this into Equation 6.11:

$$\begin{aligned}
\frac{A_{j-1}+A_{j+1}}{A_j} &= \frac{B\left[\sin([j-1]\phi)+\sin([j+1]\phi)\right]}{B\sin(j\phi)} \\
&= \frac{\sin(j\phi)\cos\phi+\sin\phi\cos(j\phi)+\sin(j\phi)\cos\phi-\sin\phi\cos(j\phi)}{\sin(j\phi)} \\
&= 2\cos\phi
\end{aligned} \tag{6.13}$$

What does this result mean? It shows us that the ratio of the amplitudes has a constant value, independent of the number of masses oscillating on the string. Note that the value of ϕ is dependent on which oscillating element we are considering; but to subscript ϕ_j onto everything would make it even more untidy!

Now, if we can identify values for ϕ, this will allow us to determine the allowed expressions for ω, the frequencies of the normal modes in the system.

6.2.4 Finding the amplitude of the jth element

In Equation 6.13 we have presented the general solution for the amplitude ratios of the jth element of an oscillating system:

$$\frac{A_{j-1}+A_{j+1}}{A_j} = 2\cos\phi_j \tag{6.14}$$

To find an expression for ϕ_j, we can use our boundary conditions, namely that $A_0 = A_{n+1} = 0$, and use this with the trial function $A_j = B\sin(j\phi)$:

$$\begin{cases} A_0 = B\sin 0\phi = 0 \\ A_{n+1} = B\sin(n+1)\phi = 0 \end{cases} \tag{6.15}$$

The first case is not particularly useful; let's instead look at the second case. The boundary condition of the sine function being equal to zero gives us the following:

$$\begin{aligned}
B\sin(n+1)\phi &= 0 \\
\sin(n+1)\phi &= 0
\end{aligned} \tag{6.16}$$

For this to be true, then:

$$(n+1)\phi_m = 0, \pi, 2\pi, \ldots, r\pi \tag{6.17}$$

...where r is simply an arbitrary integer. This then gives us an expression for the rth value of ϕ_r in an n-oscillator system. This gives our values of ϕ_r as:

$$\phi_r = \frac{r\pi}{n+1} \tag{6.18}$$

Putting this back into the trial function we proposed ($A_j = B\sin(j\phi)$), we find that the amplitude of the jth mass at the **fixed normal mode frequency** ω_r to be:

$$A_j = B\sin\frac{jr\pi}{n+1} \tag{6.19}$$

6.2.5 Identifying the allowed frequencies

We can now use the expression for ϕ_r presented in Equation 6.18 and place this into Equation 6.13 and Equation 6.11

$$\frac{A_{j-1}+A_{j+1}}{A_j} = \frac{2\omega_0^2-\omega_r^2}{\omega_0^2} = 2\cos\phi_r = 2\cos\frac{r\pi}{n+1}$$

We can then solve this for the frequency of the rth vibrational mode (ω_r) within the system:

$$\begin{aligned} \frac{2\omega_0^2-\omega_r^2}{\omega_0^2} &= 2\cos\frac{r\pi}{n+1} \\ 2-\frac{\omega_r^2}{\omega_0^2} &= 2\cos\frac{r\pi}{n+1} \\ \omega_r^2 &= 2\omega_0^2\left[1-\cos\frac{r\pi}{n+1}\right] \end{aligned} \tag{6.20}$$

i Alphabet soup!

We've used a lot of subscripts, so it is worth taking a bit of time to remind ourselves what we have used.

- n - We have n oscillators on our tensioned string, which will give n normal vibrational modes of the system.
- j - This is the label for the jth oscillator in our n-oscillator system. *e.g.* In a system of ten oscillators ($n = 10$), we may be looking at the fourth oscillating mass ($j = 4$).
- r - This is the label for the normal mode frequency of interest. So in a ten oscillator system ($n = 10$), there are ten normal modes. Within this, we may be interested in the frequency of the sixth mode ($\omega_r, r = 6$).

6.2.6 Tying it all together - the takeaway points

We've done a lot of derivation here, but what are the take-away points?

6.2.6.1 There is a maximum frequency of oscillation

Looking at Equation 6.20, there is a maximum frequency of oscillation available. The cosine function can only vary between -1 and 1; when the cosine function is equal to -1, then:

$$\begin{aligned} \omega_{r,\mathsf{max}}^2 &= 2\omega_0^2[1-(-1)] = 4\omega_0^2 \\ \omega_{r,\mathsf{max}} &= 2\omega_0 \end{aligned} \tag{6.21}$$

This frequency is called a cutoff frequency and is a feature of many lattice vibrations.

6.2.6.2 The frequencies of normal modes of oscillation

From Equation 6.20 and the value of $\omega_0 = \frac{T}{ma}$, we can identify the frequency of any normal mode of oscillation within the system:

$$\begin{aligned} \omega_r^2 &= 2\omega_0^2\left[1-\cos\frac{r\pi}{n+1}\right] \\ &= \frac{2T}{ma}\left[1-\cos\frac{r\pi}{n+1}\right] \end{aligned} \tag{6.22}$$

Therefore, if we know the tension T in the string, the mass m of the oscillating masses (and that they are all the same mass), the number n of masses and the separation between the masses a, we can identify the frequency of a given mode.

6.3 From coupled oscillations to the wave equation

The final destination in considering such systems of coupled oscillators is to consider "what happens when the coupled oscillators on a tensioned string are so close together they can be considered continuous?". Intuition (and indeed the title of this course!) tells us that these should "of course" form a wave. But it can be helpful to work through this and validate it appropriately. After all, what *is* a wave anyway?

6.3.1 Getting started

We need to firstly identify what we know about a given system. From Equation 6.23 we identified the equations of motion of the jth mass in an n-oscillator system:

$$\frac{\mathrm{d}^2 y_j}{\mathrm{d}t^2} = \frac{T}{ma}(y_{j-1} - 2y_j + y_{j+1}) \tag{6.23}$$

Alongside this, we also know that, for a system of n oscillating masses on a tensioned string, we will have n normal modes of vibration; within this set of modes, any given mode r will have a specific frequency ω_r. This means that, in a given vibration mode, the y-displacement of each mass in the system will all have the same time dependence

Finally, we know that the displacement y_j of the jth element must depend on the value of j; if it did not, we would only see the simplest vibrations as all elements oscillate in concert, with $y_{j-1} = y_j = y_{j+1}$. Given that the value of j is intrinsically connected to the x coordinate (where $x = (j - \frac{1}{2})a$), we can say that the displacement must depend on x.

Therefore, we can say that there are two independent variables which are factored into the y-displacement; t and x.

6.3.2 Reducing the spacing between elements

We have said that x is related to the spacing between masses a and the position in the chain j - so what happens as we reduce the spacing? Let's apply our principles of calculus, and reduce the spacing such that $a \simeq \partial x$ and let $\partial x \to 0$. This makes x a continuous variable, and therefore the vertical displacement of an element becomes dependent on x and t; $y(x,t)$.

As we now have two independent variables, we will now need to enter the world of partial derivatives.

6.3.3 Modifying the equations...

We will assume that any coupled oscillation in this new continuous system will still be a sinusoidal function, but now dependent on x as well as t. We propose it takes the form:

$$y(x,t) = \sin(kx + \omega t) \tag{6.24}$$

For now, we will say that the term k is there to keep the units of kx congruent with the units of ωt; if the frequency ω is a "per second" (s^{-1}) unit to cancel the time unit of t, then k must be a "per metre" unit (m^{-1}) to cancel the length unit of x. It has deeper meaning which we will come to later.

As x is now a continuous variable, we can now return to Equation 6.23, and rewrite this as:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{ma}(y_{j-1} - 2y_j + y_{j+1}) \tag{6.25}$$

We now need to consider the meaning of y_{j-1}, y_j and y_{j+1} in the context of a continuously variable x. In Section 6.3.2 we said that the spacing a reduced to ∂x;

we can therefore say:

- y_j becomes $y(x,t)$
- y_{j-1} becomes $y(x-\partial x,t)$
- y_{j+1} becomes $y(x+\partial x,t)$

We now use a Taylor series expansion on the function $y(x \pm \partial x, t)$, and we obtain:

$$y(x \pm \partial x, t) = y(x) \pm \partial x \frac{\partial y}{\partial x} + \frac{1}{2}(\pm \partial x)^2 \frac{\partial^2 y}{\partial x^2}$$

This is when we say "so what?" Well, if we remember the *initial* form of Equation 6.23, it was actually derived from:

$$\frac{\mathrm{d}^2 y_j}{\mathrm{d}t^2} = \frac{T}{m}\left(\frac{y_{j+1}-y_j}{a} - \frac{y_r - y_{j-1}}{a}\right)$$

This immediately gives us an equation we can use our Taylor series expansion on:

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \frac{T}{m}\left(\frac{y(x+\partial x,t)-y(x,t)}{a} - \frac{y(x,t)-y(x-\partial x,t)}{a}\right) \\
&= \frac{T}{m}\left(\frac{\partial x \frac{\partial y}{\partial x} + \frac{1}{2}(\partial x)^2 \frac{\partial^2 y}{\partial x^2}}{\partial x} - \frac{\partial x \frac{\partial y}{\partial x} - \frac{1}{2}(\partial x)^2 \frac{\partial^2 y}{\partial x^2}}{\partial x}\right) \\
&= \frac{T}{m}\frac{(\partial x)^2}{\partial x}\frac{\partial^2 y}{\partial x^2} \\
&= \frac{T}{m}(\partial x)\frac{\partial^2 y}{\partial x^2}
\end{aligned} \tag{6.26}$$

You may find it helpful to ensure you can follow through the cancellation steps in this arrangement and verify your own understanding.

6.3.4 The Wave Equation

Our final step is to consider what the terms are in the final step in Equation 6.26, in particular the $\partial x/m$ term. If we invert it to obtain $m/\partial x$, we have a "mass per unit length". This is often known as a 'linear mass density', and is usually assigned the symbol ρ. When we place this into Equation 6.26, we obtain:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} \tag{6.27}$$

Equation 6.27 is the **wave equation**, and it is a tool we can use to verify the validity of a **wave function** - if, when differentiated appropriately, a function can fit this relation, then it is a valid wavefunction and can be used to describe a propagating wave.

A useful observation in the wave equation is that the T/ρ term has units m^2 s^{-2}; a speed squared. We may sometimes see this term replaced in the wave equation with a v^2 term, corresponding with the speed of propagation of the wave.

We will revisit the wave equation again in the future.

6.4 Summary of key points

We've covered a lot in this section, and it can be difficult to see the thread of the discussion, so it is helpful to revisit the key points.

- The coupled oscillator model from Chapter 4 showed us that for two oscillators coupled together, we would expect to see **two normal modes** of vibration, each with a distinct frequency.
- We extended this to many masses on a tensioned string behaving as coupled oscillators. The tension in the string is constant, however the angle of the force on each mass varies as the string moves, inducing the transverse oscillation of each mass.
- By considering the transverse component of the tension on each mass, we could create the equations of motion. Solving these for the case of one mass on a string, then two masses on a string, we verified that for n masses, we would expect to see n characteristic frequencies.
- We determined an expression for the allowed frequencies and found that, while we would obtain n frequencies for an n-component system, there is an upper limit to the frequency for any normal mode - this is the **cut-off frequency**
- Finally, we showed that, as we increase the number of oscillating masses and reduce the space between them, the movement of the system becomes closer and closer to a sinusoidal wave.

Chapter 7

Simple wave motion and the Wave Equation

Textbook link: Tipler and Mosca, Section 15.1

A wave is a means by which energy and momentum are carried through space, without transporting matter. When we consider a medium which carries a wave, the particles of that material oscillate about a mean position, but have an average displacement of zero; *i.e.* they always return to their starting position. A wave can be of any shape - there is no requirement for a wave to be sinusoidal, though this is the simplest shape which we can consider mathematically.

We also consider waves to be either **transverse**; where the displacement of the medium is perpendicular to the direction of wave propagation (*e.g.* a wave travelling along a string), or they can be **longitudinal**; where the displacement of the medium is parallel to the direction of propagation (*e.g.* a sound wave passing through air).

A note on transverse waves

In contrast with longitudinal waves, the medium carrying a transverse wave is displaced perpendicular to the direction of travel. This gives rise to the phenomenon of **polarisation**. A *plane of polarisation* is defined as a plane containing the displacement direction and the direction of propagation. For any given transverse wave, two orthogonal independent polarisations are possible. All other polarisations may be constructed from weighted combinations of these two basic polarisations. For electromagnetic waves, it is the **electric field vector** which defines the plane of polarisation in combination with the direction of propagation. We will revisit these properties of transverse waves as we go through our discussion.

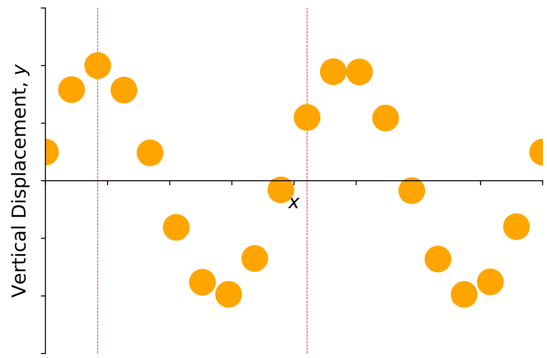

Figure 7.1: An animation to show the propagation of a wave through a material arising from an organised transverse oscillation of many particles. The two marked particles only move up and down, but the net effect of the organised motion is a propagation of energy to the right.

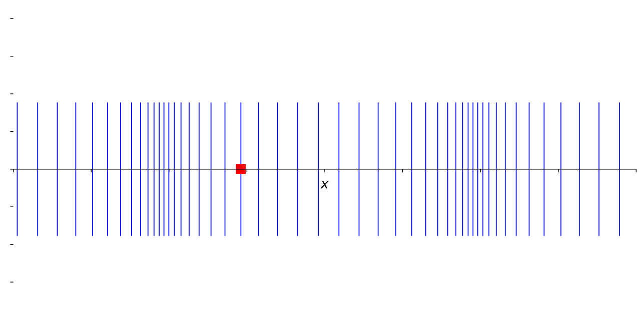

Figure 7.2: An animation to show the propagation of a wave through a material arising from an organised longitudinal oscillation of many particles. The marked particle now moves in the direction of propagation, but oscillates around a fixed point; the net effect of the organised motion is again a propagation of energy to the right.

7.1 Wave pulses

Where a wave is a sustained periodic disturbance which propagates energy through a medium, a **wave pulse** in contrast is any localised non-periodic disturbance propagating an energy pulse through the medium. A typical pulse is shown in Figure 7.3.

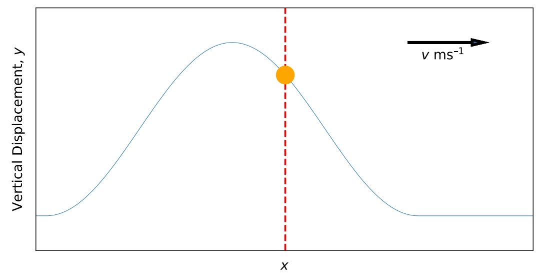

Figure 7.3: A wave pulse propagating through a system at velocity v; this causes a temporary transverse disturbance of molecules in the system from their origin.

We illustrate a pulse graphically as any function defined as $y = f(x)$, where the $+x$ direction is the direction of propagation of the pulse. If the pulse propagates without changing shape, it becomes convenient to consider a moving reference frame within which the pulse is stationary; *i.e.* rather than imagining the pulse moving to the right along fixed axes, we keep the pulse stationary in our view and move the axes to the left.

In the moving frame then, the pulse is described as $y' = f(x')$ for all times, because the pulse does not change its shape.

We can inter convert between the two frames of reference by the relation:

$$x = x' + vt \tag{7.1}$$

...where v is the velocity of the pulse. This allows us to convert the position in the moving reference frame, x', back to the position in the fixed reference frame by adding the distance vt.

If the shape of the pulse in the moving frame is defined as $y' = f(x')$ we can use Equation 7.1 to find the shape of the pulse in the static frame, y:

$$\begin{aligned} y &= f(x') \\ &= f(x - vt) \end{aligned} \tag{7.2}$$

The relation described in Equation 7.2 describes a wave moving to the **RIGHT**; for a pulse moving to the **left**, v becomes negative, and hence:

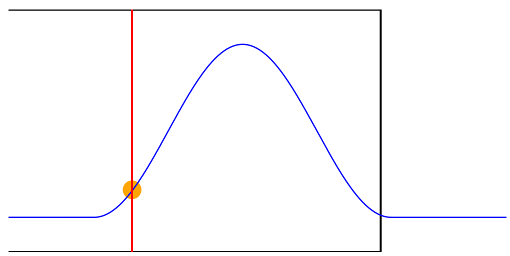

Figure 7.4: Rather than imagining a moving wave pulse, we can consider it from the reference frame of the wave pulse; in this example the wave pulse is stationary, but the space moves past it.

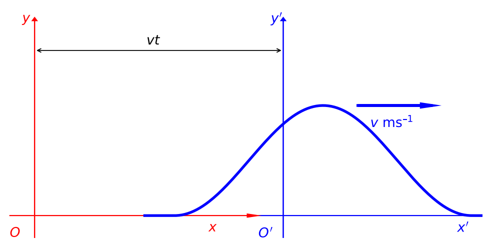

Figure 7.5: A wave pulse propagating through a system at velocity v; this causes a temporary transverse disturbance of molecules in the system from their origin.

$$y = f(x + vt)$$

The function $y = f(x \pm vt)$ is known as the **wave function**; it describes the displacement of the medium, whether the transverse displacement of a string or the longitudinal displacement of air molecules in a sound wave.[1].

The *wave function* is a solution of the **wave equation** (Equation 7.3):[2]

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{7.3}$$

Any function in the form $y = f(x \pm vt)$ is a solution of this wave equation; *i.e.* the wave equation describes the uniform propagation of **any** displacement, provided it does not change shape as it travels. There are numerous examples of such functions, including:

- $y = \exp(x - vt)^2$
- $y = \frac{\sin(x-vt)}{x-vt}$
- $y = \cos(x + vt)$

7.2 Deriving the wave equation

To better understand the wave equation, it is useful to know its derivation. To do this, we shall first consider a segment of string from a curved part of a wave pulse (Figure 7.6):

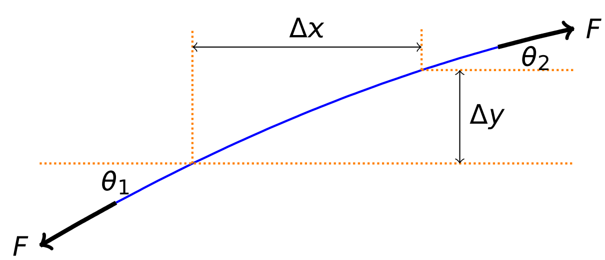

Figure 7.6: Here we consider a small portion of a string carrying a wave pulse; the tension in the string F is constant throughout, but resolving the forces via the angles θ_1 and θ_2 allows us to determine the equation of motion on this segment.

[1]It may seem counter intuitive that, for a wave moving with positive $v$in the $+x$ direction, we subtract vt; but try thinking this way: after time t, the wave will have advanced further to the right. If, after one second, the shape of the wave has its peak at x_a, after three seconds, the peak will have advanced to $x_b = x_a + vt$; **however**, the value of the wavefunction at x_a at this new time (x'_a) will have come from the value of the wavefunction three seconds earlier, to the **left** of the point; i.e. $x'_a = x_a - vt$

[2]The ∂ symbol refers to *partial differentiation*.

The length of this segment is $\approx \Delta x$ (for small angles), and the mass of this segment of string $m = \mu \Delta x$ (where μ is the mass per unit length). As we are considering a transverse wave, this segment of string will move vertically, and all forces acting on the segment arise from the tension force within the string. We can determine these forces by resolving the horizontal and vertical components of this force F using the parameters laid out in Figure 7.6.

By considering the net vertical force acting on this segment:

$$\begin{aligned} \sum F &= F \sin\theta_2 - F \sin\theta_1 \\ &= F\left(\tan\theta_2 - \tan\theta_1\right) \quad \text{for small angles} \end{aligned} \tag{7.4}$$

We can also determine an expression for the slope, S, of the segment of string at a given point:

$$S = \tan\theta = \frac{\partial y}{\partial x} \tag{7.5}$$

This means we can now express the overall force acting on the segment in terms of the slope of the segment at its start and end points (assuming small angles!) by combining Equation 7.4 and Equation 7.5 :

$$\sum F = F(S_2 - S_1) = F\Delta S \tag{7.6}$$

The quantity $F\Delta S$ in Equation 7.6 is the net force acting on the segment; so we can now apply Newton's second law:

$$F\Delta S = ma = \mu \Delta x \frac{\partial^2 y}{\partial t^2} \tag{7.7}$$

...or, to rearrange:

$$\frac{\Delta S}{\Delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{7.8}$$

Since the slope ΔS of the segment is the 'rate of change of y with respect to x' (Equation 7.5), we can redefine the term in Equation 7.7; in the limit, as $\Delta x \to 0$:

$$\frac{\Delta S}{\Delta x} \approx \frac{\partial S}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial x^2} \tag{7.9}$$

Finally, by combining this result in Equation 7.9 with the result in Equation 7.8), we obtain the wave equation for a stretched string (Equation 7.10):

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{7.10}$$

As we mentioned in Section 7.1, any function of the form $y = f(x \pm vt)$ will be a solution to the wave Equation 7.10. In the case of the string, the solution will solve the wave equation provided that $v^2 = \frac{F}{\mu}$.

7.3 The wave equation - proof by substitution

In this section, we will show the proof of the wave equation which we determined graphically in Section 7.2. Here we will use *partial differentiation*, explaining the ∂ notation we have already seen in this work.[3]

Let's work through this step by step:

1. Consider the general function $y = \cos(x - vt)$. Differentiate this first with respect to x:

$$\frac{\partial y}{\partial x} = -\sin(x - vt)$$

The second derivative therefore becomes:

$$\frac{\partial^2 y}{\partial x^2} = -\cos(x - vt) \tag{7.11}$$

2. Now we differentiate with respect to time t:

$$\frac{\partial y}{\partial t} = v\sin(x - vt)$$

... and find the second derivative:

$$\frac{\partial^2 y}{\partial t^2} = -v^2\cos(x - vt) \tag{7.12}$$

3. We can now combine Equation 7.11 and Equation 7.12 to eliminate the term $\cos(x - vt)$, we obtain the wave equation as required:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{7.13}$$

[3] Partial differentiation is needed when we have multivariate expressions. We can only differentiate one variable at a time, so *partial* differentiation indicates that we are holding all other variables constant. In our wave equations, we have position and time variables, so when differentiating x with respect to y, we are holding the time t constant.

Through unpacking of Equation 7.13 we can uncover how it might be adapted to obtain wave equations for other systems. The term x represents the direction of propagation of the wave, while y represents the disturbance of the particle carrying the wave (in the case of a transverse wave, this is perpendicular to the direction of propagation).

For longitudinal sound waves, Equation 7.13 becomes:

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{v_s^2}\frac{\partial^2 s}{\partial t^2} \tag{7.14}$$

In this example, s is the displacement of molecules of the medium parallel to the direction of the propagation, while v_s is the velocity of sound in the medium.

For electromagnetic waves, Equation 7.13 becomes either:

$$\frac{\partial^2 E_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}(\#eq - ch6 - wavederivsubs5)$$

{#eq-}

...or

$$\frac{\partial^2 B_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2} \tag{7.15}$$

...where E_z and B_y are the transverse components of the electric and magnetic field vectors (we will discuss this further later), and c is the speed of light.

Note: The speed of propagation cannot be deduced from the wave equation, instead it must be obtained from a model of the system concerned.

7.4 The Phase Velocity - the velocity of waves

We define the phase velocity as the speed of propagation of any particular point on a wave. Since the wave propagates without changing shape, it does not matter which point we pick.

The phase velocity will depend on a combination of the elastic and inertial terms.

For waves travelling on a string the phase velocity can be found from the tension in the string (F) and the linear mass density (mass per unit length) of the string (μ):

$$v = \sqrt{\frac{F}{\mu}}$$

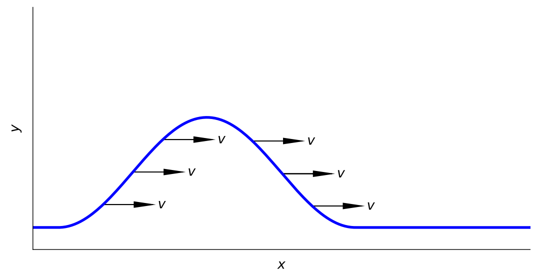

Figure 7.7: A reminder that every point in a wave pulse here moves at the same velocity v, such that the shape of the pulse is not changed.

This has parallels for sound waves in a fluid, with phase velocity found from the bulk modulus B (a factor describing the fluid's resistance to compression) and the equilibrium density ρ of the fluid:

$$v_s = \sqrt{\frac{B}{\rho}}$$

We can go further to look at the propagation of sound waves in an isotropic solid (solid of constant composition) - however we can only use this to consider a thin section, not a bulk solid (this is an acceptable model for considering earth tremors in the Earth's crust). This time the phase velocity depends on the density of the solid ρ and either the Young's modulus (Y, for longitudinal P waves) or the shear modulus (G, for transverse S waves):

$$v_s = \sqrt{\frac{Y}{\rho}} \text{or} \sqrt{\frac{G}{\rho}}$$

These equations describe how wave phase velocities can be determined in solids and liquids; in a gas, we need to use other parameters, however the core relationship is still familiar:

$$v_s = \sqrt{\frac{\gamma R T}{M}} = \sqrt{\frac{\gamma k_{\mathrm{B}} T}{m}}$$

Here, the terms R, T and M refer to the gas constant (units J K^{-1} mol^{-1}), absolute temperature (units K) and the molar mass of the gas (units kg mol^{-1}) respectively[4],

[4]Note that the molar mass is expressed in kg mol^{-1} here, rather than the more common g mol^{-1} ; the reason for this is to ensure continuity and present everything in SI units.

while k_{B} and m refer to the Boltzmann constant (units J K^{-1}) and the mass of an individual gas molecule (units kg).

Finally, we can obtain a similar expression for the phase velocity of light waves:

$$c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$$

...where μ_0 and ϵ_0 are the permeability of free space and the permittivity of free space respectively.

7.5 Simple wave summary

In this chapter, we have introduced the principle of waves and the wave equation. Waves carry energy and momentum through space by localised organised oscillations without net transport of matter. They can be transverse (oscillation perpendicular to direction of propagation) or longitudinal (oscillation parallel to direction of propagation), or can have a more complex displacement pattern. Waves do not have to be sinusoidal, and can have any shape.

Any travelling wave can be described by the function in the form of $y = f(x \pm vt)$ which satisfies the wave equation, relating the displacement of the medium (y) to the position of the wave (x) and the phase velocity (v).

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Remember that the phase velocity cannot be predicted from the wave equation, but depends on the physics of the system. It generally results from a consideration of the elastic and inertial properties of the system. For waves on a string, $v = \sqrt{\frac{F}{\mu}}$, where F is the tension in the string and μ is the mass per unit length (linear mass density).

Chapter 8

Harmonic Waves

Textbook link: Tipler and Mosca, Section 15.2

A *harmonic wave* is a general term for a wave which, at some instant of time, can be described by a sinusoidal function (*i.e.* it is a sine or a cosine function). They are the simplest of waves to consider, and we will devote this chapter to exploring their properties.

8.1 Transverse sine and cosine waves

If we consider a string which is excited by a tuning fork or other object undergoing simple harmonic motion (SHM), we can imagine the shape of the save on the string at some instant of time appearing as a sine or a cosine wave (depending on our choice of origin) (Figure 8.1):

This sinusoidal appearance is known as a **harmonic wave**. Each point on the string oscillates up and down with the same frequency as the driving frequency. During the period T, the wave moves through distance λ:

$$v = \frac{\lambda}{T} = f\lambda \tag{8.1}$$

...where v is the phase velocity of the wave and f is the frequency in hertz (Hz), and λ defined as the **wavelength** *i.e.* the spacial repeat distance of the wave.

A harmonic wave has a unique frequency and wavelength, and other waves (*e.g.* wave pulses) may be regarded as a superposition of many harmonic waves of different frequencies (we will discuss the Fourier analysis of this later).

At any instant in time, the wave can be described by the relation in Equation 8.2:

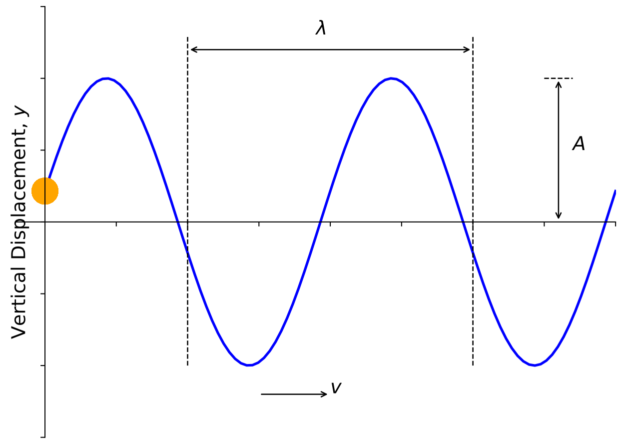

Figure 8.1: A sine wave in a string created by an oscillating particle. The wavelength λ is shown for one complete cycle, while the amplitude A is defined as the maximum deviation from the origin point.

$$y = A\sin(kx + \delta) \tag{8.2}$$

...where A is the amplitude of the wave and δ is the phase constant. Let's now choose the origin so that δ is equal to zero (*i.e.* there is no phase constant). Now we can show the periodic condition in Equation 8.3:

$$\sin kx = \sin k(x + \lambda) \tag{8.3}$$

Here we specify the periodic condition; the amplitude (and phase!) at position x is equal to the amplitude (and phase!) at position x **plus one wavelength**. Because this is a sine function, we know that, in order to achieve this condition, the value $k\lambda$ must be equal to 2π:

$$\begin{aligned} \sin\theta &= \sin(\theta + 2\pi) \\ \sin kx &= \sin(kx + k\lambda) = \sin(kx + 2\pi) \\ k\lambda &= 2\pi \end{aligned} \tag{8.4}$$

This allows us to give a value for k (Equation 8.5):

$$k = \frac{2\pi}{\lambda} \tag{8.5}$$

The parameter k is defined as the **wave number** of the wave, in units radians per metre (rad m^{-1}).

8.2 Travelling waves

The wave we showed in Section 8.1 was a snapshot in time, so the wave was, in effect, static. We instead wish to consider a travelling wave. To do this, instead of writing x as in Equation 8.2 with a phase difference, we rewrite with $x - vt$ (see Section 7.1) to illustrate the time-dependent nature of the travelling wave. Equation 8.2 then becomes Equation 8.6:

$$y = A \sin k(x - vt) = A \sin(kx - kvt) \tag{8.6}$$

As we have described, T is the period of the wave, so any point on the wave will oscillate up and down also with period T. This means that, for the wave position at time t, it will return to the same state at time $(t + T)$. We can therefore write Equation 8.6 in terms of this period:

$$\begin{aligned} y = A \sin (kx - kvt) &= A \sin (kx - kv(t + T)) \\ &= A \sin (kx - kvt - kvT) \end{aligned} \tag{8.7}$$

Applying the same reasoning as shown in Equation 8.4, we are led to the result:

$$kvT = 2\pi$$

If we revisit our discussions on SHM (Section 1.7) we defined the relationship between T and 2π (Equation 1.12) as $T = \frac{2\pi}{\omega}$ (and also $\omega = 2\pi f$). This allows us to define a number of factors as follows:

- **Phase velocity**:

$$v = \frac{2\pi}{kT} = \frac{\omega}{k} \tag{8.8}$$

- **Travelling wave to the right**:

$$y = A \sin (kx - kvt) = A \sin (kx - \omega t) \tag{8.9}$$

- **Travelling wave to the left**:

$$y = A \sin (kx + kvt) = A \sin (kx + \omega t) \tag{8.10}$$

It is worth noting for each of these factors:

1. The sign of ω:
   - negative for waves travelling to right;
   - positive for waves travelling to left
2. The wave expressions can be shown to satisfy the wave equation by substitution.

8.3 Complex representation of waves

We introduced complex numbers for descriptions of oscillations; we can use the same treatment for our wave equations. Just as $\sin(kx - \omega t)$ and $\cos(kx - \omega t)$ satisfy the wave equation, so too will $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$. It is often convenient to write the wave expression in a complex form as shown in Equation 8.11:

$$y = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{8.11}$$

In this expression, the sine wave is given by the 'imaginary' component of y, while the cosine part is given by the 'real' component.

8.4 Energy carried by waves on a string

Suppose we now have a string attached to an oscillating driver at one end. As the driver oscillates, it imparts energy to the string at $x = 0$ by:

1. Stretching the string to give it potential energy, and
2. imparting transverse speed to the string to increase its kinetic energy.

As the waves move along the string, so the energy is transported along the string.

8.4.1 Potential energy of string segment

We now consider the same string segment as we showed in Figure 7.6, but now we consider it stretched (Figure 8.2). We can picture this as the string is "relaxed" in its 'horizontal' orientation (length of segment is Δx), but when a wave passes along it, the string elongates to accommodate the curve of the wave form. This means our segment now takes on a new length Δl.

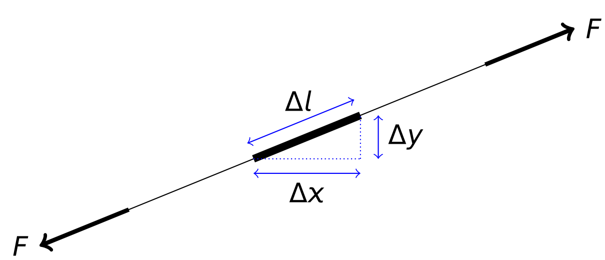

Figure 8.2: As the wave passes along a stretched string, we can consider it to be 'stretched'; considering a string element of lencth Δx, it is stretched to new length Δl as the wave passes by.

The work done (U) in stretching the segment Δx can be expressed as:

$$\Delta u = F(\Delta l - \Delta x) \tag{8.12}$$

...where F is the tension in the string and the extension is given by $(\Delta l - \Delta x)$. We can apply Pythagoras to relate Δx and Δl:

$$\begin{aligned}
\Delta l^2 &= \Delta x^2 + \Delta y^2 \\
&= \Delta x^2 \left[1 + \left(\frac{\Delta y}{\Delta x} \right)^2 \right]
\end{aligned}$$

Therefore we can isolate Δl:

$$\Delta l = \Delta x \left[1 + \left(\frac{\Delta y}{\Delta x} \right)^2 \right]^{\frac{1}{2}}$$

We can approximate this expression by using the Taylor series expansion for $\sqrt{1+n}$, where $n = \left(\frac{\Delta y}{\Delta x} \right)^2$; assuming that the fraction is significantly less than one we can write this as an approximation and disregard terms past the first two terms:[1]

$$\Delta l \approx \Delta x \left[1 + \frac{1}{2} \left(\frac{\Delta y}{\Delta x} \right)^2 + \cdots \right]$$

Rearranging this expression to obtain the expression for the extension of the string $\Delta l - \Delta x$:

$$\Delta l - \Delta x \approx + \frac{\Delta x}{2} \left(\frac{\Delta y}{\Delta x} \right)^2 + \cdots$$

This now allows us to obtain an expression for the work done in stretching the spring solely in terms of the x and y displacement (from Equation 8.12)

$$\Delta U = \frac{F \Delta x}{2} \left(\frac{\Delta y}{\Delta x} \right)^2$$

As we have an expression for y in terms of x (Equation 8.2), we can differentiate this with respect to x to approximate $\frac{\Delta y}{\Delta x}$:

$$\frac{\Delta y}{\Delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x} = kA\cos(kx - \omega t)$$

We also know that $v^2 = \frac{F}{\mu}$ (from the wave equation), and that $v = \frac{\omega}{k}$ (Equation 8.8), therefore:

[1]The Taylor expansion for this expression is $\sqrt{1+n} = 1 + \frac{n}{2} - \frac{n^2}{8} + \frac{n^3}{16} - \cdots$

$$\Delta U = \frac{1}{2}\left(\frac{\mu\omega^2}{k^2}\right)\Delta x\,(kA\cos(kx-\omega t))^2$$

Tidying up and cancelling, we obtain the expression for the potential energy stored in an element of string of length Δx (Equation 8.13):

$$\Delta U = \frac{1}{2}\mu\omega^2 A^2 \Delta x \cos^2(kx-\omega t) \tag{8.13}$$

8.4.2 Kinetic energy of string segment

We will again consider the segment of string discussed in Section 8.4.1; an element Δx of the string, of mass Δm (Figure 8.3).

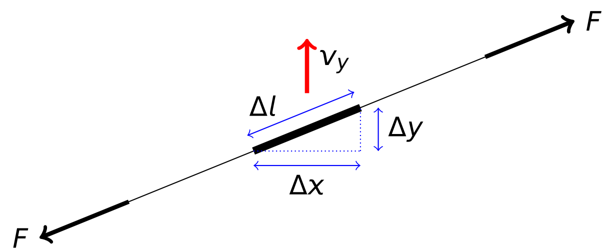

Figure 8.3: Similar to the situation shown above, the same string segment will have a kinetic energy associated with its velocity v_y.

As before, the segment is stretched to new length Δl, but the mass is still given by:

$$\Delta m = \mu\Delta x$$

...where μ is the mass per unit length. We now use the **transverse velocity** of the segment (*i.e.* **not** the wave velocity) to determine the kinetic energy of the segment:

$$\Delta KE = \frac{1}{2}\Delta m v_y^2 = \frac{1}{2}\mu\Delta x\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2$$

Again, we know the expression for the vertical displacement y in terms of x and t (Equation 8.6), so we now differentiate with respect to t:

- $y = A\sin(kx-\omega t)$
- $v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = -\omega A\cos(kx-\omega t)$

Therefore our expression for the kinetic energy becomes:

$$\Delta KE = \frac{1}{2}\mu\omega^2 A^2 \Delta x \cos^2(kx - \omega t) \quad (8.14)$$

A quick comparison of Equation 8.13 and Equation 8.14 shows that these expressions are identical; *i.e.* the KE stored in the string is the **same** as the PE stored in the spring.

8.4.3 Total energy of wave on a string

We can therefore find the total energy of the string segment carrying a harmonic wave as the total of the kinetic and potential energies:

$$\Delta E = \Delta KE + \Delta U$$

i.e.:

$$\Delta E = \mu\omega^2 A^2 \Delta x \cos^2(kx - \omega t) \quad (8.15)$$

Note that the energy of the segment varies with time with twice the frequency of the wave (since $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$).

We can also define the average energy at any point (Equation 8.16) using the time-average definition $\langle\cos^2\theta\rangle = \frac{1}{2}$:

$$\Delta E_{\mathrm{av}} = \frac{1}{2}\mu\omega^2 A^2 \Delta x \quad (8.16)$$

...and we can define the average *energy density* (per unit length) as:

$$\varepsilon = \frac{\Delta E_{\mathrm{av}}}{\Delta x} = \frac{1}{2}\mu\omega^2 A^2$$

There are several things to note from this derivation:

- KE is at a maximum when displacement is zero
- At this point the string is most stretched, so PE is at a maximum also
- PE and KE are **in phase** (unlike in a pendulum)

These points are illustrated in Figure 8.4.

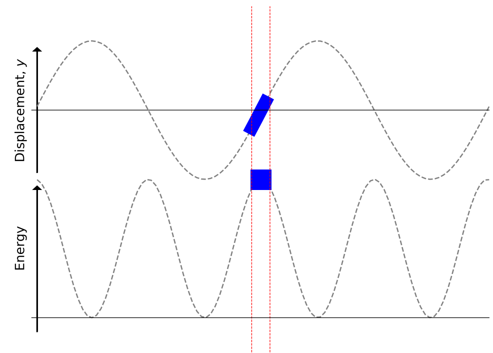

Figure 8.4: The kinetic energy and potential energy of a displaced string element have maxima and minima at the same points in the oscillation; as the element passes through the origin (top plot), it is at its most stretched (PE maximum), and it is traveling at its fastest (KE maximum).

8.4.4 Transport of energy and power

As the wave propagates along the string, energy is transported by the moving wave-front at speed v. The average energy passing a point on the string in time Δt is the average energy in the segment of length $\Delta x = v\Delta t$. This means we can rewrite Equation 8.16:

$$\Delta E_{\text{av}} = \frac{1}{2}\mu\omega^2 A^2 v\Delta t \tag{8.17}$$

Since the power transmitted is a *rate of change of energy*, *i.e.* $\frac{\Delta E}{\Delta t}$, we can obtain an expression for the average power transmitted:

$$\Delta P_{\text{av}} = \frac{\mathrm{d}E_{\text{av}}}{\mathrm{d}t} \approx \frac{\Delta E_{\text{av}}}{\Delta t} = \frac{1}{2}\mu\omega^2 A^2 v \tag{8.18}$$

From this result, we can see that both the average energy and average power transmitted are both proportional to A^2; a similar observation as in SHM (Chapter 1).

8.5 Summary

We have covered a large amount of derivations in this chapter, however the take-home points are the following:

When describing sine waves travelling through a medium, the following statements apply:

- For a wave travelling to the **right**: $y = A\sin(kx - \omega t)$
- For a wave travelling to the **left**: $y = A\sin(kx + \omega t)$
- The **phase velocity**: $v = \frac{\omega}{k}$ m s^{-1}

...where:

- $k =$ wavenumber $= \frac{2\pi}{\lambda}$
- $\lambda =$ wavelength /m
- $\omega =$ angular frequency $= 2\pi f$ /rad m^{-1}
- $f =$ frequency /Hz $= \frac{1}{T}$
- $T =$ period /s
- $A =$ amplitude /m

The average energy carried by a wave (per unit length) is given by:

$$\varepsilon = \frac{\Delta E_{\mathrm{av}}}{\Delta x} = \frac{1}{2}\mu\omega^2 A^2$$

...and the average power transmitted by the wave is given by:

$$\Delta P_{\mathrm{av}} = \frac{\Delta E_{\mathrm{av}}}{\Delta t} = \frac{1}{2}\mu\omega^2 A^2 v$$

Chapter 9

Reflection and Transmission at boundaries

We now turn our attention to what happens to waves and wave pulses when they encounter boundaries. We define a boundary as the dividing line between regions with different phase velocity. In the context of imagining our waves moving along strings, a boundary can exist either between the string and a rigid anchoring point, or at a point where the two strings join, each with a different mass density (*i.e.* a thick string joining to a thin string).

Any wave incident on such a boundary between regions with different phase velocities will be partly reflected back from the boundary and partly transmitted through the boundary.

In the case of a boundary between a thick string and a thin string, the phase velocity v is related to the tension F and the mass density μ *via* $v^2 = \frac{F}{\mu}$; *i.e.* heavier strings will have a lower phase velocity.

Considering reflection/transmission across such a boundary in a purely qualitative manner:

1. A wave propagating along a thin string towards a boundary with a thicker string will be reflected from the boundary **with inversion**, as well as a proportion of the energy transmitted as a wave into the thicker string.
2. A wave propagating along a thick string towards a boundary with a thinner string will be reflected from the boundary **without inversion**, as well as a proportion of the energy transmitted as a wave into the thinner string.

This qualitative outcome is illustrated in Figure 9.1 and Figure 9.2.

If instead we fix a uniform string to an immovable anchor rather than another, thicker, string, we will get complete reflection of the wave pulse with inversion. This is akin to saying the anchor has an infinite mass density, μ.

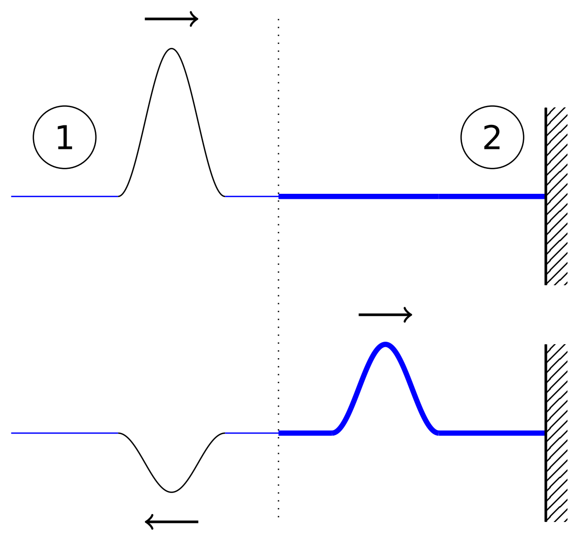

Figure 9.1: When a wavepulse originates in string 1 (where $\mu_1 \ll \mu_2$), the wave will be both reflected with inversion and partially transmitted into string 2 of greater μ.

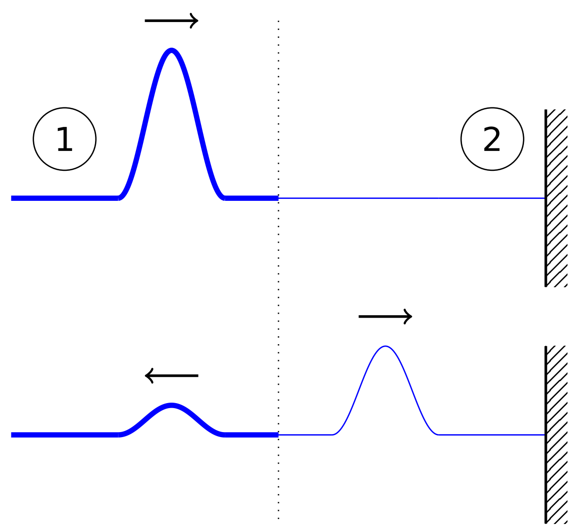

Figure 9.2: If we reverse the situation with the wavepulse originating in the string with greater μ (now $\mu_1 \gg \mu_2$), the wave is still partially transmitted into string 2, but the reflection within string 1 is no longer inverted.

To quantitatively assess these outcomes, we can obtain the reflected and transmitted amplitudes for harmonic waves by considering the power transmitted through the string as the wave propagates.

9.1 Power transmitted and reflected at a boundary

The principles of energy conversion state that, as a wave encounters a boundary, the energy in the incident wave must equal the total energy of the reflected and transmitted waves. This principle therefore also applies to the total power of the system, *i.e.*:

$$P_i = P_r + P_t \tag{9.1}$$

We can therefore use Equation 8.18 to derive an expression for the power before and the powers after the wave encounters the boundary (Equation 9.2):

$$\frac{1}{2}\mu_1\omega^2 A_i^2 v_1 = \frac{1}{2}\mu_1\omega^2 A_r^2 v_1 + \frac{1}{2}\mu_2\omega^2 A_t^2 v_2 \tag{9.2}$$

...or, since $\omega = v_n k_n$ and $F = \mu_n v_n^2$ (rearrangements of equations seen previously):

$$\frac{1}{2}Fk_1\omega A_i^2 = \frac{1}{2}Fk_1\omega A_r^2 + \frac{1}{2}Fk_2\omega A_t^2 \tag{9.3}$$

Note that:

- A_i, A_r and A_t are the amplitudes of the incident, reflected and transmitted waves respectively;
- μ_1, v_1, μ_2, v_2 refer to the mass per unit length and the phase velocities for strings 1 and 2 respectively;
- k_1 and k_2 are the wavenumbers for each of the two strings;
- ω is the **same** for all waves - this depends only on the source;
- λ will be **different** on each string; since $\lambda = \frac{v}{f} = \frac{2\pi v}{\omega}$, *i.e.* λ will be smaller on the heavier string;
- Conversely $k = \frac{2\pi}{\lambda}$ will be larger on the heavier string;
- $\mu = \frac{F}{v^2}$; the tension F will be the same in both strings.

When we compare the proportion of the incident power which is reflected, we can show that:

$$\frac{\text{Reflected power}}{\text{Incident power}} = \frac{k_1 A_r^2}{k_1 A_i^2} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \tag{9.4}$$

We can also show the proportion of the incident power which is transmitted:

$$\frac{\text{Transmitted power}}{\text{Incident power}} = \frac{k_2 A_t^2}{k_1 A_i^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2} \tag{9.5}$$

Note These results **only** hold for waves on strings where the tension, F, is the same in both strings. We will discuss a more general result in the next section.

Proof of power ratios

From our expression of the conservation of powers (Equation 9.3), we can cancel the common terms:

$$k_1 A_i^2 = k_1 A_r^2 + k_2 A_t^2 \tag{9.6}$$

However, at the interface, the wave must be continuous on both sides; therefore the amplitude on each side must be the same:

$$A_i + A_r = A_t \tag{9.7}$$

We now substitute Equation 9.7 into Equation 9.6 to eliminate A_r:

$$k_1 A_i^2 = k_1 (A_t - A_i)^2 + k_2 A_t^2 \tag{9.8}$$

We can then expand and rearrange this:

$$\begin{aligned}
k_1 A_i^2 &= k_1 (A_t - A_i)^2 + k_2 A_t^2 \\
&= k_1 (A_t^2 - 2A_t A_i + A_i^2) + k_2 A_t^2 \\
0 &= k_1 A_t^2 - 2k_1 A_t A_i + k_2 A_t^2 \\
&= k_1 \frac{A_t^2}{A_i^2} - 2k_1 \frac{A_t}{A_i} + k_2 \frac{A_t^2}{A_i^2} \\
0 &= \left(\frac{A_t}{A_i}\right)^2 (k_1 + k_2) - 2k_1 \frac{A_t}{A_i}
\end{aligned} \tag{9.9}$$

This is a quadratic equation, and we can then say the following; either:

$$\frac{A_t}{A_i} = 0$$

...which represents total reflection (zero transmission), and not what we would expect to have with two joined strings, or:

$$\left(\frac{A_t}{A_i}\right) = \frac{2k_1}{k_1 + k_2}$$

...from which we obtain:

$$\frac{k_2}{k_1}\left(\frac{A_t}{A_i}\right)^2 = \frac{4k_1k_2}{(k_1+k_2)^2}$$

...as required in Equation 9.5 for the ratio of transmitted to incident power.

We can obtain the result for Equation 9.4 for ratio of reflected to incident power in a similar manner by substitution of Equation 9.7)into Equation 9.6 to instead eliminate A_t.

9.2 Example of reflection and transmission

Consider a wave travelling from a light string to a heavy string, where $\mu_2 = 4\mu_1$.

Remember that:

$$\mu = \frac{F}{v^2} = F\frac{k^2}{\omega^2}$$

i.e.

$$k \propto \sqrt{\mu}$$

We can therefore determine:

- $k_2 = 2k_1$ via the square-root relationship
- The fraction of power reflected will be:

$$\frac{A_r^2}{A_i^2} = \left(\frac{1-2}{1+2}\right)^2 = \frac{1}{9}$$

- The fraction of power transmitted:

$$\frac{k_2A_t^2}{k_1A_i^2} = \left(\frac{4\times 2\times 1}{(1+2)^2}\right) = \frac{8}{9}$$

9.3 The impedance of a piece of string

The examples we considered in Section 9.1 were a particular result under constant tension. We can generalise the result by considering the impedances of the media on either side of the boundary.

We discussed the concept of impedance (both electrical and mechanical) in Section 3.6; however the salient points are:

- "Impedance" describes the property of a system which resists motion, either mechanical or motion of charge;

- Any material through which waves propagate presents impedance to those waves;
- In general, impedance depends on inertia and elasticity;
- For a string, we define impedance as:

$$Z = \frac{\text{transverse force}}{\text{transverse velocity}}$$

i.e. for a given force, a large Z implies a small velocity and *vice versa*; * Similar definitions can be written for longitudinal waves.

We now consider a string which is driven by an oscillating force (Figure 9.3)

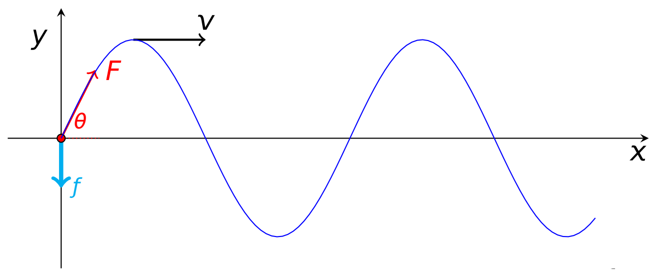

Figure 9.3: When a wave is driven by an oscillating force at its origin (f), the element of the string indicated by the dot experiences the tension within the string, F, at angle θ from the direction of propagation.

The driving force on this string is given by the relationship:

$$f = -f_0 \cos \omega t = -f_0 \mathrm{e}^{\mathrm{i}\omega t}$$

This driving force is negative because f points downwards at time $t = 0$.

As this is a wave, the vertical displacement at any point is given by the relationship:

$$y = A\mathrm{e}^{\mathrm{i}(kx - \omega t)}$$

We now consider the tension in the string, F, and resolve this in the transverse direction. At $x = 0$, we assume small angle of θ:

$$\begin{aligned} f &= -F \sin \theta \\ &\simeq -F \tan \theta \quad \text{for small angles} \\ &= -F \frac{\partial y}{\partial x} \end{aligned}$$

We can now use the definition of impedance given above ($Z = \frac{\text{force}}{\text{velocity}}$) and the expressions for the force and velocity:

$$
\begin{aligned}
Z &= \frac{f}{v_y} \\
&= -F\frac{\partial y}{\partial x} \div \frac{\partial y}{\partial t}
\end{aligned}
$$

We already have expressions for $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial t}$ by differentiating the wave equation $y = A\mathrm{e}^{\mathrm{i}(kx-\omega t)}$:

$$
\frac{\partial y}{\partial x} = ikA\mathrm{e}^{\mathrm{i}(kx-\omega t)}
$$

$$
\frac{\partial y}{\partial t} = -i\omega A\mathrm{e}^{\mathrm{i}(kx-\omega t)}
$$

Therefore:

$$
\begin{aligned}
Z &= \frac{-F \times ikA\mathrm{e}^{\mathrm{i}(kx-\omega t)}}{-i\omega A\mathrm{e}^{\mathrm{i}(kx-\omega t)}} \\
&= \frac{Fk}{\omega}
\end{aligned}
$$

Remember also that the phase velocity $v = \frac{\omega}{k}$, so we can also express the impedance as $Z = \frac{F}{v}$ and $Z = \mu v$ (because $v^2 = \frac{F}{\mu}$), where μ is the mass per unit length of the string and F is the tension within the string.

9.4 Reflection and transmission revisited

We can now express our previous result of the reflection and transmission coefficients (derived in Equation 9.9 more generally:

Reflection coefficient:

$$
\frac{A_r}{A_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{9.10}
$$

Transmission coefficient:

$$
\frac{A_t}{A_i} = \frac{2Z_1}{Z_1 + Z_2} \tag{9.11}
$$

Writing these in terms of the power (the approach used in Section 9.1), we obtain the following expressions:

$$\frac{\text{Reflected power}}{\text{Incident power}} = \frac{Z_1 A_r^2}{Z_1 A_i^2} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 \tag{9.12}$$

and:

$$\frac{\text{Transmitted power}}{\text{Incident power}} = \frac{Z_2 A_t^2}{Z_1 A_i^2} = \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2} \tag{9.13}$$

These are general expressions which apply in mechanical, electrical and optical systems. From them, we can see that:

- If $Z_2 > Z_1$, the reflected wave is inverted;
  - From Equation 9.10, the term $Z_1 - Z_2$ is negative under these conditions, leading to a negative amplitude
- If the second string is a "wall" (*i.e.* immovable, infinite μ)
  - $Z_2 \to \infty$;
  - $A_r = -A_i$ (by energy conservation);
  - $A_t = 0$;
  - Therefore the wave is fully reflected and inverted.
- If $Z_2 = Z_1$ we have:
  - Impedance matching;
  - No reflection;
  - Maximum power transfer.

9.5 Impedance - Miscellaneous cases

For **longitudinal (sound) waves**, generally we expect the impedance Z to be described by $Z = \rho_0 v_p$, where:

- ρ is the mean density of the medium
- v_p is the phase velocity of the wave

Example values for longitudinal sound waves are: * Air: ~ 400 kg m^{-2} s^{-1} * Water: 1.45×10^6 kg m^{-2} s^{-1} * Steel: 3.9×10^7 kg m^{-2} s^{-1}

For **transverse waves on a string**, the impedance Z is described by $Z = \mu v_p$, where:

- μ is the mass per unit length
- v_p is the phase velocity of the wave

For **electromagnetic waves**, the impedance depends on the medium under consideration:

- In a dielectric medium:

$$Z = \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$$

where in this case μ and ϵ are the permittivity and the permeability of the medium.

- In free space:

$$Z = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.6\Omega$$

- For a light wave in a dielectric medium:

$$Z = \frac{1}{n}$$

where n is the refractive index of the medium (see later).

Chapter 10

Sound waves and the Doppler effect

Textbook link: Tipler and Mosca, Section 15.2-4

As has been mentioned already, sound waves are a longitudinal wave propagated by the localised displacement of air molecules in the direction of propagation. This displacement of air molecules within sound waves can be described by the function shown in Equation 10.1.

$$s(x,t) = s_0 \sin(kx - \omega t) \tag{10.1}$$

In contrast to the transverse waves previously discussed, there are now only two dimensions to this function; while we considered the transverse displacement y of an element of the medium carrying the transverse wave, in this longitudinal wave the longitudinal displacement s is in the x direction, *i.e.* the same direction as the propagation of the wave.

This displacement of the molecules leads to changes in both the density (ρ) and the pressure (p) of the medium. It is worth recognising that both p and ρ are out of phase with the displacement (when the displacement is at a maximum, the pressure and density are at a minimum):

$$p = p_0 \sin(kx - \omega t - \frac{\pi}{2}) \tag{10.2}$$

... where the initial pressure $p_0 = \rho\omega v s_0$ and v is the phase velocity.

10.1 Energy of sound waves

When considering the energy of sound waves, we can examine the expression we already have for transverse waves and adapt this for our longitudinal waves.

Recall that the expression for the energy within a transverse wave is given in Equation 8.16:

$$\Delta E_{\mathrm{av}} = \frac{1}{2}\mu\omega^2 A^2 \Delta x$$

To adapt this for longitudinal sound waves, we perform the following substitutions:

- We replace the linear mass density μ (units kg m^{-1}) with the density of the medium, ρ (units kg m^{-3})
- We replace our transverse amplitude A with the longitudinal displacement s_0
- To keep units congruent, we replace the change in segment length caused by the wave Δx with the change in segment volume caused by the wave ΔV

Our result is therefore:

$$\Delta E_{\mathrm{av}} = \frac{1}{2}\rho\omega^2 s_0^2 \Delta V \tag{10.3}$$

10.2 Wave intensity

We have mentioned that with sound waves we are now considering a three-dimensional volume. This means we need to consider the effect of this on the energy of the wave at a distance r from its origin. The energy at a given distance will be spread uniformly over a spherical surface; therefore we need to consider the power per unit area of this surface. This is the **intensity** of the wave.

$$I = \frac{P_{\mathsf{av}}}{4\pi r^2} \tag{10.4}$$

We already know that the average power is defined as the rate of change of the average energy:

$$P_{\mathsf{av}} = \frac{\Delta E_{\mathsf{av}}}{\Delta t}$$

We can therefore rewrite Equation 10.4 taking this into account.

$$\begin{aligned} I &= \frac{\Delta E_{\mathsf{av}}}{4\pi r^2 \Delta r}\frac{\Delta r}{\Delta t} \\ &= \frac{\Delta E_{\mathsf{av}}}{\Delta V} v \end{aligned} \tag{10.5}$$

In Equation 10.5 we identify the term $\frac{\Delta r}{\Delta t}$; this is the speed at which the wave travels from the centre of the sphere, so is the phase velocity v. We can also say that the term $4\pi r^2 \Delta r$ is the rate of change of volume, ΔV. We now use our expression for the average energy (Equation 10.3) to simplify this expression:

$$I = \frac{1}{2}\rho\omega^2 s_0^2 v = \frac{p_0^2}{2\rho v} \tag{10.6}$$

In other words, the intensity of the sound wave travelling at constant speed v through a medium of constant density ρ at a point in space is proportional to the square of the amplitude of the wave, p_0.

10.3 Levels of intensity

The human ear perceives sounds according to the logarithm of their intensity - not the absolute value of the intensity.[1] To represent an **intensity level** we use the term **decibel** (dB). This intensity level, β is represented in Equation 10.7 as follows:

$$\beta = 10\log_{10}\left(\frac{I}{I_0}\right) \tag{10.7}$$

The term I_0 is the absolute intensity considered to be at the absolute limit of human hearing, where $I_0 = 10^{-12}$ W m^{-2}. A description of approximate intensity levels is shown in Table 10.1.

Table 10.1: A description of the approximate decibel level of particular sounds.

β /dB	Description
0	Hearing threshold
40	Library
70	Busy traffic
120	Pain threshold

10.4 The Doppler Effect (non-relativistic)

You have already met the Doppler effect for light in the context of the Special Relativity course; here we will briefly revisit it in the context of non-relativistic cases.

The general principles of the Doppler effect are unchanged, namely:

[1]When you encounter electronics you will find potentiometers labelled "audio taper"; this describes their use in audio applications in which the resistance is a logarithmic response to accommodate our logarithmic perception of sound.

- If the source and observer move relative to each other, the observed frequency is different from the emitted frequency;
- When the source and observer move towards each other, $f_{\text{obs}} > f_{\text{source}}$;
- When the source and observer move away from each other, $f_{\text{obs}} < f_{\text{source}}$;
- The frequency change, Δf, depends on whether the source or observer move relative to each other.

Consider a source moving relative to its surrounding medium at a speed of u_s. We can visualise this as a moving 'dipper' in a pool of water (Figure 10.1)

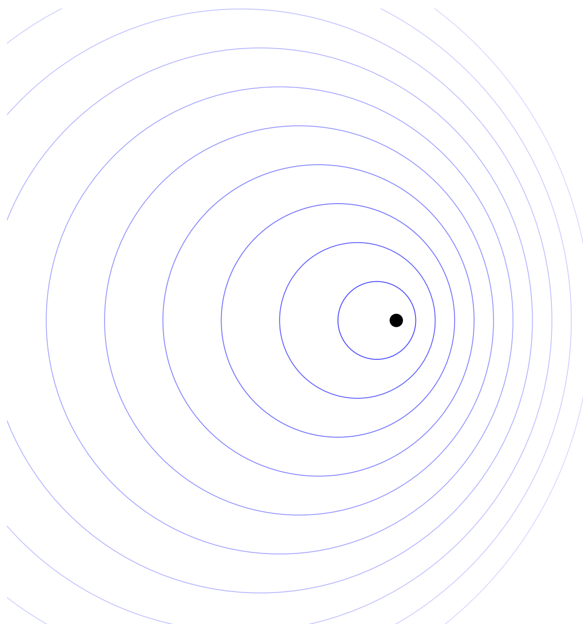

Figure 10.1: When a dipper (marked by the black dot) moves relative to the water, we see that the wavefronts 'bunch' in the direction of motion and diverge behind the motion of the dipper.

This is a visual representation of a number of key statements:

- The speed of waves v in the medium is independent of the movement of the source;
- The source produces waves at a frequency f_0;
- In a given time frame, Δt, the source will emit N_s wavefronts, where:

$$N_s = f_0 \Delta t$$

{#eq-}

From these statements, we can calculate the observed wavelength, λ', by considering the relative distance traveled by the wave in a given time-frame and the number of waves produced by the source in that time-frame (Equation 10.8):

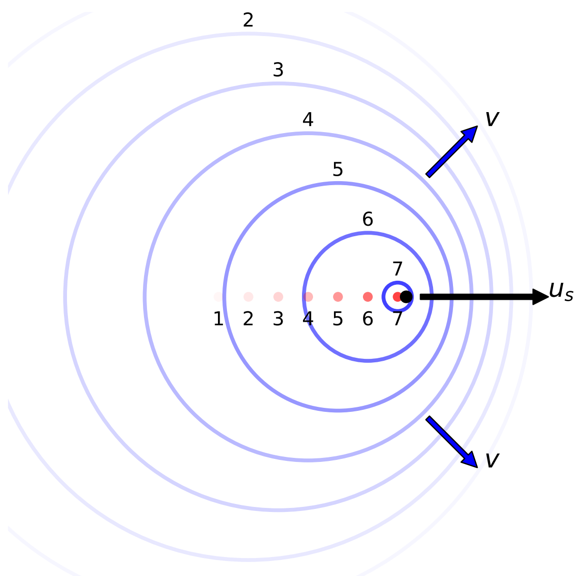

Figure 10.2: We can look at this in more detail by showing where the dipper was for each of the spreading wavefronts. Here the dipper is moving forward at speed u_r, while the wavefronts spread out from their point of origin at speed v.

$$\lambda' = \frac{\text{relative distance}}{\text{no. of waves}} = \frac{(v \pm u_s)\Delta t}{f_0 \Delta t} \tag{10.8}$$

We can now consider two extremes; the observed wavelength in front of the source (to the right in the diagram), and the observed wavelength behind the source (to the left in the diagram):

$$\lambda'_{\text{behind}} = \frac{v + u_s}{f_0} \qquad \lambda'_{\text{in front}} = \frac{v - u_s}{f_0} \tag{10.9}$$

We can now determine the frequency observed by using these expressions for the wavelength. Firstly, we determine the number of wavefronts passing the observer in the time frame Δt:

$$N_{\text{obs}} = \frac{v_{\text{obs}} \Delta t}{\lambda'} \tag{10.10}$$

...where v_{obs} is the speed of the waves relative to the observer; i.e. $v_{\text{obs}} = v \pm u_{\text{obs}}$; where u_{obs} corresponds to the observer moving to the right in the diagram (as this reduces the relative velocity between the observer and the wave). We can now rewrite the expression in Equation 10.10 in terms of the phase velocity v and the relative speed of the observer, u_{obs}:

$$N_{\text{obs}} = \frac{(v \pm u_{\text{obs}})\Delta t}{\lambda'} \tag{10.11}$$

...and finally we have the observed frequency f' of a source in motion:

$$f' = \frac{N_{\text{obs}}}{\Delta t} = \frac{v \pm u_{\text{obs}}}{\lambda'} \tag{10.12}$$

When we combine this with the equation for λ', we obtain the general result (Equation 10.13)

$$f' = \left(\frac{v \pm u_{\text{obs}}}{v \pm u_s}\right) f_0 \quad \text{or} \quad f' = \left(\frac{1 \pm \frac{u_{\text{obs}}}{v}}{1 \pm \frac{u_s}{v}}\right) f_0 \tag{10.13}$$

It is important to pay attention to the signs when using the above equations; it can be easy to confuse the positive directions. The sign convention can help with this:

- The direction from the observer towards the source is positive

You can check your results using the principles:

- The observed frequency f' **increases** when the source and observer approach each other;

- The observed frequency f' **decreases** when the source and observer move apart.

Chapter 11

Superposition and Standing Waves

Textbook link: Tipler and Mosca, Section 15.1

11.1 Superposition of harmonic waves

In Section 7.1) we showed that the wave equation (Equation 7.3) is satisfied by any function in the form $y = f(x \pm vt)$. We can go further than this, and specify general expressions which can satisfy the wave equation.

Specifically, if we have two functions, y_1 and y_2 (Equation 11.1), which satisfy the wave equation, then their sum (Equation 11.2), including scaling constants C_1 and C_2) must also satisfy the wave equation.

$$\begin{aligned} y_1 &= A_1 \sin\left[k_1(x \pm vt)\right] \\ y_2 &= A_2 \sin\left[k_2(x \pm vt)\right] \end{aligned} \tag{11.1}$$

$$y_3 = C_1 y_1 + C_2 y_2 \tag{11.2}$$

Using double angle formulae we can demonstrate that y_3 can also be written in the form $y = f(x \pm vt)$, thus satisfying the wave equation.[1] Therefore, the new wave y_3 is a linear superposition of the original waves y_1 and y_2.

It is worth remembering that the wave equation was derived in the case of small amplitude disturbances; therefore if the vibration amplitudes become too large, the

[1] When including scaling factors, this can become less than trivial, but can still be done.

principle of superposition may fail. This can lead to some very interesting effects, including **non-linear optics**[2]

11.2 Two waves with same amplitude and frequency

Let's now consider two waves with the same amplitude and frequency; but differing in phase (Equation 11.3):

$$\begin{aligned} y_1 &= y_0 \sin(kx - \omega t) \\ y_2 &= y_0 \sin(kx - \omega t + \delta) \end{aligned} \tag{11.3}$$

If the two waves are superimposed, for example if they are travelling through the same medium, the resultant wavefunction is just the sum of y_1 and y_2 (Equation 11.4):

$$\begin{aligned} y &= y_1 + y_2 \\ &= y_0 \sin(kx - \omega t) + y_0 \sin(kx - \omega t + \delta) \end{aligned} \tag{11.4}$$

In many situations it is mathematically simpler to use the complex exponential notation to treat waves and oscillations. However, in the case of a linear addition of two waves, it is simpler to just use trigonometric identities.

Here we will use the identity:

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

… we obtain:

$$y = 2y_0 \sin\left(kx - \omega t + \frac{\delta}{2}\right)\cos\left(\frac{-\delta}{2}\right) \tag{11.5}$$

As an aside, we can do the same thing with complex notation; we can write our wave superposition as follows:

$$y = y_1 + y_2 = y_0 e^{i(kx-\omega t)} + y_0 e^{i(kx-\omega t+\delta)}$$

Remembering that we are interested in the 'imaginary' component at the end since we started with y_1 and y_2 as sine waves, we can write this superposition as:

$$y = y_0 e^{i(kx-\omega t)}\left(1 + \mathrm{e}^{\mathrm{i}\delta}\right)$$

[2] Related to the non-linear response of the electromagnetic interactions with the medium, rather than light travelling in straight lines!

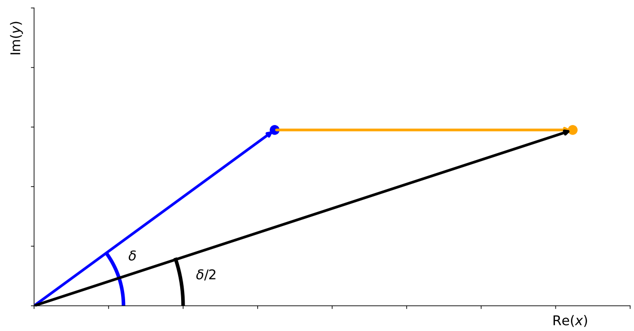

Figure 11.1: We can illustrate addition of complex numbers on an Argand diagram; here we show the sum $\left(1+\mathrm{e}^{\mathrm{i}\delta}\right)$; the result of this is the argument is halved from δ to $\frac{\delta}{2}$.

We can show using an Argand diagram (Figure 11.1) that the term $(1+\mathrm{e}^{\mathrm{i}\delta})$ is equal to $2\cos\left(\frac{\delta}{2}\right)e^{i\frac{\delta}{2}}$, and hence:

$$y = 2y_0\cos(\delta/2)e^{i(kx-\omega t+\delta/2)}$$

When we expand this expression using De Moivre's theorem, we obtain the same result as in Equation 11.5:

$$y = \underbrace{2y_0\cos\left(\frac{\delta}{2}\right)}_{\text{New amplitude}}\underbrace{\sin\left(kx-\omega t+\frac{\delta}{2}\right)}_{\text{Travelling wave}} \tag{11.6}$$

This tells us that the resulting wave has the same frequency as the component waves, but a different amplitude and phase. This new amplitude is given by the expression in Equation 11.7:

$$A_{\text{new}} = 2y_0\cos\left(\frac{\delta}{2}\right) \tag{11.7}$$

There are three special cases to be aware of:

1. If $\delta = 0$, the waves are exactly in phase and the waves add together (constructive interference), with amplitude $A = 2y_0$;
2. If $\delta = \pi$, the waves are exactly out of phase and the waves subtract (destructive interference), and the amplitude $A = 0$;

3. If $\delta = \frac{2\pi}{3}$, the resultant wave has exactly the same amplitude as the input waves.

All three of these cases can be derived by substituting the relevant value of δ into Equation 11.7).

11.3 Standing waves

A standing wave is a specific outcome which occurs when a wave is confined to space (for example on a piano string) and it reflects at the boundaries and travels back along its original path. This leads to waves travelling in both directions which combine by superposition. Only certain frequencies can exist in a standing wave, as the superposition leads to a stationary pattern called a **standing wave**.

11.4 Wave function for a standing wave

In order to fully consider a standing wave, we need to derive the form of its wave-function. To do this, we consider two waves travelling in opposite directions along a string (Equation 11.8). Because one is a reflection of the other (they each reflect from the boundaries), they will have the same frequency and phase.

$$\begin{aligned} y_1 &= y_0 \sin(kx - \omega t) \\ y_2 &= y_0 \sin(kx + \omega t) \end{aligned} \tag{11.8}$$

The resultant vertical displacement of the string is then the sum of these two waves (Equation 11.9):

$$\begin{aligned} y &= y_1 + y_2 \\ y &= y_0 \sin(kx - \omega t) + y_0 \sin(kx + \omega t) \end{aligned} \tag{11.9}$$

We can add these directly using a trigonometric identify or we can work in the complex notation:

$$\begin{aligned} y &= y_0 \sin(kx - \omega t) + y_0 \sin(kx + \omega t) \\ &= y_0 \mathrm{e}^{\mathrm{i}(kx - \omega t)} + y_0 \mathrm{e}^{\mathrm{i}(kx + \omega t)} \\ &= y_0 \mathrm{e}^{\mathrm{i}kx} \left(\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{e}^{\mathrm{i}\omega t}\right) \end{aligned} \tag{11.10}$$

We can use either an Argand diagram or De Moivre's theorem to show that the term $\left(\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{e}^{\mathrm{i}\omega t}\right) = 2\cos\omega t$, and hence:

$$y = 2y_0 \cos \omega t \mathrm{e}^{\mathrm{i}kx}$$

We can now expand the complex exponent using De Moivre's theorem again and, remembering we are interested in the 'imaginary' component (as this contains the desired sine function), and we obtain the result in Equation 11.11:

$$y = \underbrace{2y_0 \cos \omega t}_{\text{time-dependent amplitude}} \underbrace{\sin kx}_{\text{static wave}} \tag{11.11}$$

This result describes a static wave, $y = \sin kx$ whose amplitude varies in time as $A = 2y_0 \cos \omega t$. Note that it is possible to obtain this result via a trigonometric identity also.

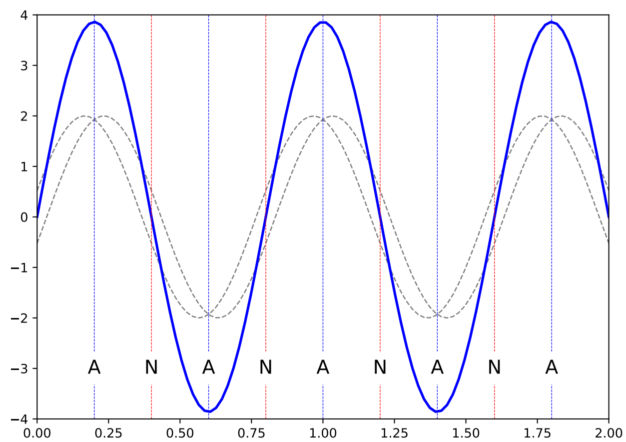

Figure 11.2: A standing wave is the result of two waves equal in frequency and amplitude moving past each other. This forms a static wave whose amplitude varies in time, and has nodes (N) and antinodes (A) present in the waveform. Here we show the fifth harmonic; 5 antinodes and a wavelength $\lambda = 2/5 = 0.8$.

This standing wave is illustrated in Figure 11.2; but we notice that there are boundary conditions enforced; namely that the ends of the string are fixed at a constant, zero displacement; *i.e.*:

- $y = 0$ at $x = 0$, and:
- $y = 0$ at $x = L$ at all times t

...where L is the length of the string. From this we deduce that $\sin kL = 0$ and therefore there are a family of solutions for k and λ, known as harmonics.

- If $\sin kL = 0$, then $kL = 0, \pi, 2\pi, ...$, or $k_n L = n\pi$
- Additionally, $\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}$ where $n = 1, 2, 3, ...$

In other words, the vibrational modes are **quantised** due to the boundary conditions.

11.5 Waves on strings fixed at both ends

We have described the mathematics of standing waves; let's now apply this to a wave travelling on a string which is fixed at both ends. Figure 11.3 illustrates the fundamental wavelength of the string, which corresponds to twice the length of the string.

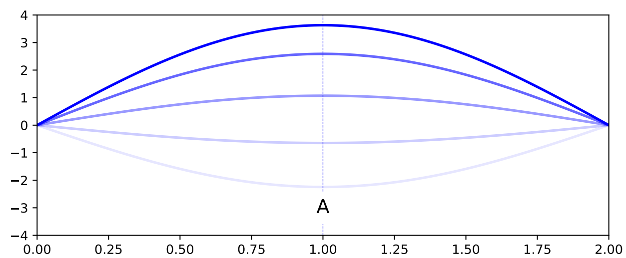

Figure 11.3: The fundamental frequency, or first harmonic. This is one half-wavelength enclosed in the boundaries and has a single antinode.

We can then visualise the harmonics within the standing wave on this fixed string in Figure 11.4:

The modes of vibration (resonances) shown in Figure 11.4 illustrate the occurrence of **nodes** (points which do not move) and **antinodes** (points with the maximum vibration amplitudes). Note also that the end-points of the string must be nodes as well, as these points are fixed.

In general, the nth harmonic will have $\frac{1}{n}\times$ wavelength and $n\times$ the frequency of the fundamental vibration shown in Figure 11.3.

Table 11.1: Showing the variation of wavelength and frequency of each harmonic with respect to the first (fundamental) wave.

Harmonic	λ	f
Fundamental, first	$2L$	f_1
Second	L	$2f_1$
Third	$\frac{2L}{3}$	$3f_1$
Fourth	$\frac{L}{2}$	$4f_1$
Fifth	$\frac{2L}{5}$	$5f_1$
nth	$\frac{2L}{n}$	nf_1

In general, for the nth harmonic:

- Wavelength is given by $\lambda_n = \frac{2L}{n}$
- Frequency is given by $f = nf_1 = \frac{nv}{\lambda_1} = \frac{nv}{2L}$

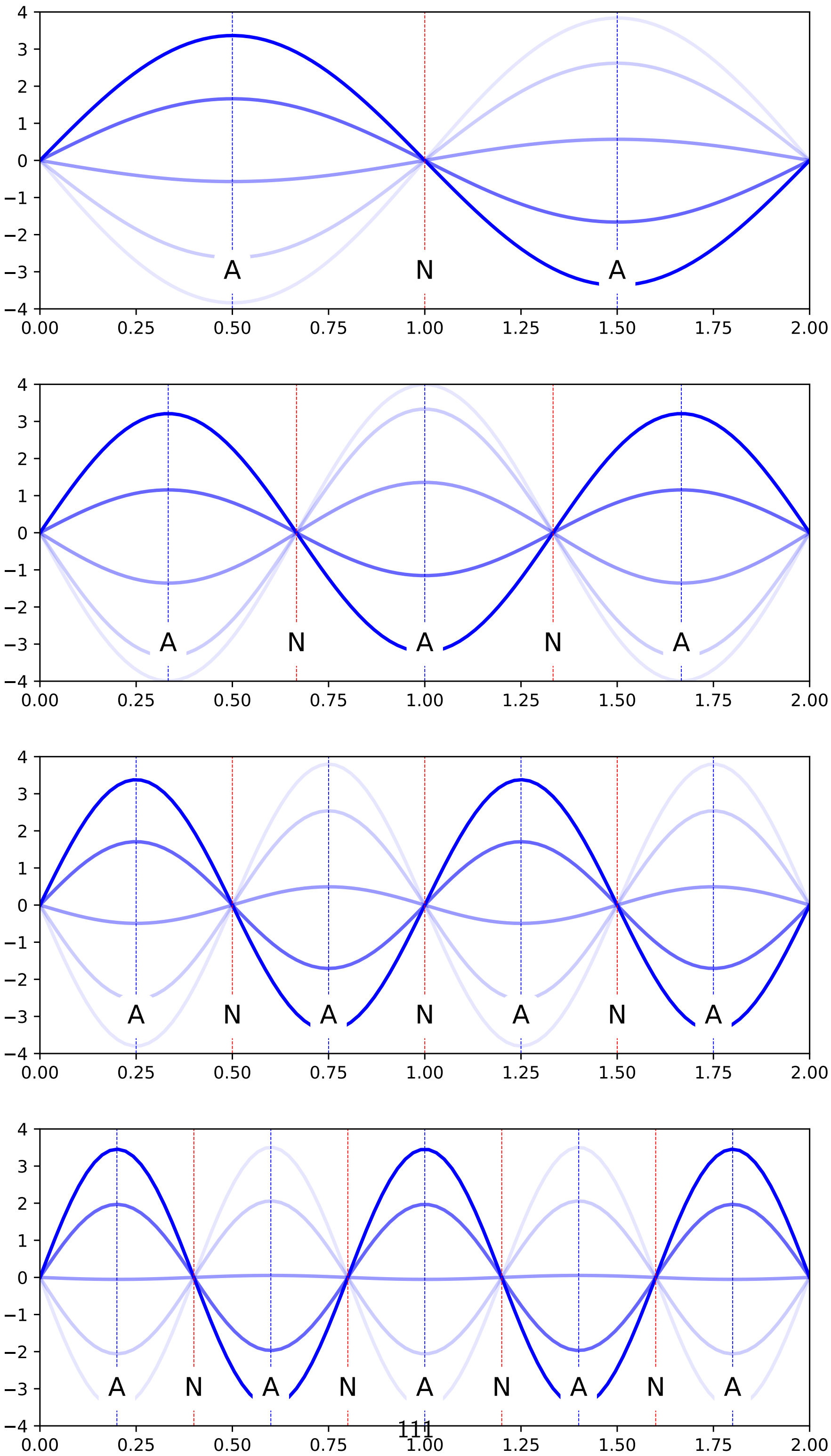

Figure 11.4: The second, third, fourth and fifth harmonics of a standing wave on a string. Notice that the nth harmonic has $\frac{n}{2}$ wavelengths contained in the space.

...where v is the phase velocity (the speed of propagation of the wave along the string).

The resonant frequencies, or harmonics, of the string are known as its natural frequencies. Any string will resonate with maximum amplitude when excited with these frequencies, and this set of harmonics are known as a harmonic series. The actual harmonics heard when the string is excited will depend on the manner of its excitation; *e.g.* a string plucked at its centre will *only* display the odd harmonics; *i.e.* those with an anti-node in the centre.

In stringed instruments (violin, piano, guitar etc.) the vibration of the string is amplified by a mechanical resonator; a soundboard in the case of the piano, or resonant cavities for a guitar or violin. These resonators must be carefully designed to resonate equally well over a wide range of frequencies.

11.6 Organ pipes and other wind instruments

In contrast to a vibrating string, wind instruments rely on a resonance within a column of air. We can model these pipes as a simple pipe, resonating at its natural frequencies when air is blown into (or across) an opening at one end. The resonant behaviour will differ depending on whether the other end of the pipe is open or closed. We will consider each of these cases in turn.

11.6.1 Pipes open at both ends

In this model, the column of air is able to vibrate at its ends, so we have a similar set of harmonics as for a string, but with displacement antinodes at its ends (the air can vibrate freely at the ends of the tube). There is a second set of nodes/antinodes corresponding to the pressure; these do not align with the displacement nodes/antinodes; rather a **pressure node** aligns with a **displacement antinode** and vice versa. (if an air molecule does not move, we have a displacement node, but it is continuously 'squashed' from both sides by the oscillating air molecules, so experiences the biggest pressure change).

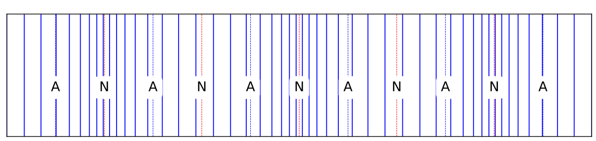

Figure 11.5: This image shows the node/antinode structure of a standing wave in a **closed tube**. Note the presence of a displacement node at either end where molecules are compressed against the end of the tube.

We observe **all** harmonics in this system; there are no concerns about 'position of plucking' that there is for the string. The displacement of the air molecules extends a little beyond the ends of the tube, so the effective length is given by $L_{\text{eff}} = L + \Delta L$, where ΔL is a small end correction. Therefore:

$$\lambda_n = \frac{2L_{\text{eff}}}{n} \quad \text{and} \quad f_n = \frac{nv}{2L_{\text{eff}}}$$

11.6.2 Pipe closed at one end

We now have a different situation with different boundary conditions:

- There must be a displacement node at the closed end
- There must be a displacement antinode at the open end

This now gives us a fundamental wavelength four times longer than the effective length of the tube (the shortest node-antinode separation is a quarter wavelength).

This means that we only observe the **odd** harmonics (the even harmonics would not allow the boundary conditions for this tube).

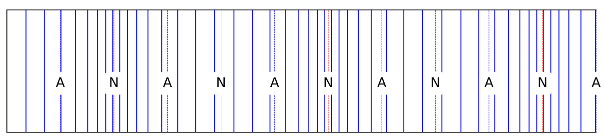

Figure 11.6: For a tube open at one end, the standing waves now have a node at the closed end, and an antinode at the open end. This changes the available harmonics within the tube.

$$\lambda_n = \frac{4L_{\text{eff}}}{n} \quad n = 1, 3, 5, ...$$

$$f_n = \frac{nv}{4L_{\text{eff}}} \quad n = 1, 3, 5, ...$$

Chapter 12

Mathematical Toolkit

Through the course of the material we have covered, there are a number of mathematical tools we have used in order to explore the physics. It is not the intention of this course to rigourously teach the mathematics, however it is helpful to see the maths that we are using and the manner in which we use it, distinct from any abstract “pure mathematical” setting.

12.1 Complex numbers

We can greatly simplify the mathematics by using **complex numbers** in our derivations. While the idea of a “complex number” sounds ... *complex*, the use of these numbers becomes straightforward as we apply our familiar mathematical techniques. In the context of mathematics, the term **complex** simply means ‘more than one part’; therefore, a **complex number** is a number with more than one part. It is this two-component nature of a complex number which makes them so useful in many aspects of Physics, and particularly when describing wave behaviour.

12.1.1 Overview of complex numbers

The general form of a complex number z is shown in Equation 12.1:

$$z = a + \mathrm{i}b \tag{12.1}$$

The symbol z is a general term for a complex number, and has two components, a “real” component a and an “imaginary” component, b. The imaginary number, i, is defined using the process shown in Equation 12.2:

$$\begin{aligned} x^2 &= -1 \\ x &= \pm \mathrm{i} \\ i^2 &= -1 \end{aligned} \tag{12.2}$$

The terms 'real' and 'imaginary' are nothing more than labels. Neither is any more or less "realistic" than the other nor is it any less valid. Some may claim that the number i is a 'pretend' number; however were this to be true, it would not be as useful as it is![1].

The next useful concept to recall is the **complex conjugate**, z^*. This is defined as in Equation 12.3:

$$\begin{aligned} z &= a + \mathrm{i}b \\ z* &= a - \mathrm{i}b \\ zz^* &= a^2 + b^2 \end{aligned} \tag{12.3}$$

In general, for any complex number of the form $z = a \pm \mathrm{i}b$, there exists its complex conjugate, $z^* = a \mp \mathrm{i}b$ such that zz^* is a wholly real number and equal to $a^2 + b^2$.

The complex conjugate is particularly useful when finding fractions of complex numbers as it is used to make the denominator of the fraction wholly "real".

Useful results with complex numbers

For a pair of complex numbers, z_1 and z_2:

$$\begin{aligned} z_1 &= a_1 + \mathrm{i}b_1 \\ z_2 &= a_2 + \mathrm{i}b_2 \end{aligned} \tag{12.4}$$

...we can establish the following principles:

- Equality:

$$\text{If} \quad a_1 = a_2 \quad \textbf{and} \quad b_1 = b_2 \quad \text{then} \quad z_1 = z_2$$

- Addition and subtraction:

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + \mathrm{i}(b_1 + b_2) \\ z_1 - z_2 &= (a_1 - a_2) + \mathrm{i}(b_1 - b_2) \end{aligned}$$

- Products:

$$\begin{aligned} z_1 \times z_2 &= (a_1 + \mathrm{i}b_1)(a_2 + \mathrm{i}b_2) \\ &= (a_1 a_2 - b_1 b_2) + \mathrm{i}(a_1 b_2 + a_2 b_1) \end{aligned}$$

- Reciprocal:

[1]Remember that negative numbers were once seen as 'pretend numbers', as you could not have negative eight apples. They have since become indispensable in many applications, not least financial transactions!

$$\frac{1}{z} = \frac{z^*}{zz^*} = \frac{a - \mathrm{i}b}{a^2 + b^2}$$

- Division:

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{(a_1 a_2 + b_1 b_2) - \mathrm{i}(a_1 b_2 - a_2 b_1)}{a_2^2 + b_2^2}$$

Applications of the complex conjugate

$$\begin{aligned}
(z_1 + z_2)^* &= z_1^* + z_2^* \\
(z_1 z_2)^* &= z_1^* z_2^* \\
\left(\frac{z_1}{z_2}\right)^* &= \frac{z_1^*}{z_2^*} \\
a = \mathrm{Re}(z) &= \frac{1}{2}(z + z^*) \\
b = \mathrm{Im}(z) &= \frac{1}{2}(z - z^*)
\end{aligned} \tag{12.5}$$

12.1.2 The Argand Diagram

Since a complex number consists of two independent components, we have another way to describe these numbers. Complex numbers can be plotted on a graph, with the '**real**' component plotted on one axis (the x-axis) and the '**imaginary**' component plotted on the other axis (the y-axis). This is the basis of the Argand diagram (Figure 12.1).

This allows us to define a complex number in terms of a **modulus** (radial distance from the origin) and an **argument** (angle from the 'real' axis). Useful properties of the modulus are listed in Equation 12.6:

$$\begin{aligned}
|z^*| &= |z| \\
zz^* &= |z^2| \\
|z_1 z_2| &= |z_1||z_2| \\
|\frac{z_1}{z_2}| &= \frac{|z_1|}{|z_2|}
\end{aligned} \tag{12.6}$$

$$|z_1 + z_2| \neq |z_1| + |z_2|$$

The Argand diagram is a representation of the **complex plane**, through which it becomes possible to visualise properties of complex numbers. One example of this is the addition of complex numbers; these can be considered to behave as vectors (Figure 12.2)

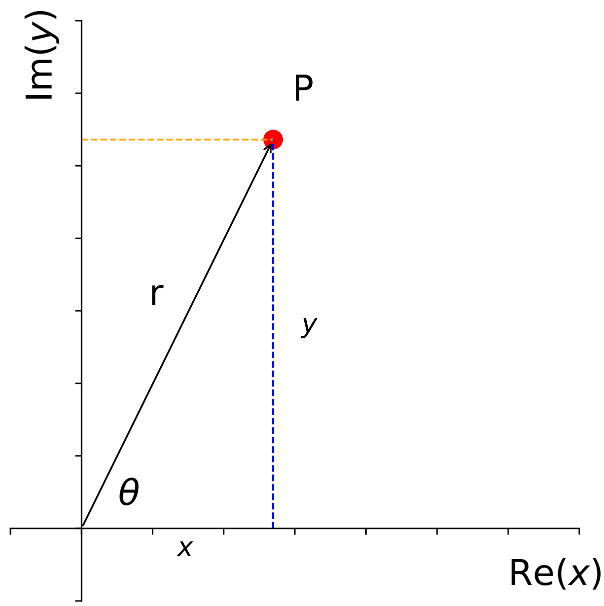

Figure 12.1: A typical Argand diagram, showing the Real (‘Re’) axis and the Imaginary (‘Im’) axis. The point P can be defined in ‘x, y’ terms (the ‘complex number’), or can be defined as polar ‘r, θ’ terms (termed ‘modulus’ and ‘argument’)

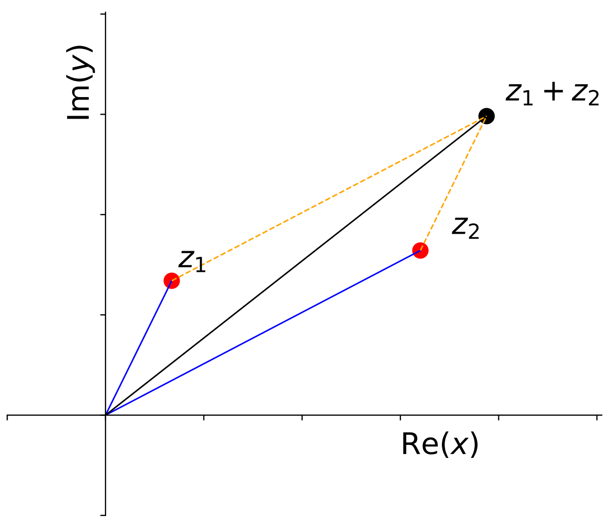

Figure 12.2: Addition of complex numbers z_1 and z_2 can be shown grapically on an Argand diagram; the separate consideration of the ‘real’ and ‘imaginary’ components is analogous to the separate consideration of vector components.

12.1.3 Polar representation of complex numbers

As well as the **Cartesian interpretation** of the Argand diagram, we can also consider a **polar representation** of a complex number; where instead of "real" and "imaginary" components acting as (x, y) coordinates, we define the position of the complex number on the complex plane as a radius and an angle, θ. We have already illustrated this in Figure 12.1

In this representation, the complex number can be expressed a different way:

$$z = a + \mathrm{i}b$$
$$r = |z| = \sqrt{a^2 + b^2}$$
$$a = r\cos\theta \qquad b = r\sin\theta \qquad \theta = \arg(z) = \arctan\left(\frac{b}{a}\right)$$
$$z = r\cos\theta + \mathrm{i}|z|\sin\theta$$

Normally, θ will lie in the range such that $-\pi < \theta \leq \pi$, meaning that our complex number representation is now shown in Equation 12.7:

$$\begin{gathered} z = a + \mathrm{i}b = r\cos\theta + \mathrm{i}r\sin\theta \\ z = r\left(\cos\theta + \mathrm{i}\sin\theta\right) \end{gathered} \tag{12.7}$$

12.1.4 Exponential representation of complex numbers

The exponential representation of a complex number takes the general form of $z = Ae^{\mathrm{i}\theta}$. This is based on series expansions of $\cos\theta$ and $\mathrm{i}\sin\theta$, which shows *De Moivre's theorem*. Key results from this are shown in Equation 12.8:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\theta} &= \cos\theta + \mathrm{i}\sin\theta \\ \left(\mathrm{e}^{\mathrm{i}\theta}\right)^n &= \left(\cos\theta + \mathrm{i}\sin\theta\right)^n = \mathrm{e}^{\mathrm{i}n\theta} \\ \left(\cos\theta + \mathrm{i}\sin\theta\right)^n &= \cos n\theta + \mathrm{i}\sin n\theta \end{aligned} \tag{12.8}$$

This means that we obtain the following representations for complex numbers:

$$\begin{gathered} z = r\left(\cos\theta + \mathrm{i}\sin\theta\right) = r\mathrm{e}^{\mathrm{i}\theta} \\ z^* = r\left(\cos\theta - \mathrm{i}\sin\theta\right) = r\mathrm{e}^{-\mathrm{i}\theta} \\ \text{where:} \quad r = |z| \quad \theta = \arg(z) \end{gathered}$$

Combining these with Equation 12.5 we also note the following useful results (Equation 12.9)

$$\begin{aligned} \cos\theta &= \tfrac{1}{2}\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right) \\ \sin\theta &= \tfrac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right) \end{aligned} \tag{12.9}$$

12.1.5 Complex representation of oscillations

Having quickly readdressed our understanding of complex numbers, we now turn our attention to the application of these in the context of oscillations and waves.

Consider the general equation of SHM (Equation 12.10), derived from Equation 1.8)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \omega^2 u = 0 \tag{12.10}$$

As has been previously discussed, sinusoidal functions can form the basis of solutions to this differential equation; so both $\cos\omega t$ and $\sin\omega t$ are solutions to this equation. Therefore, any linear combination of these solutions will also be a solution, *i.e.* the linear combination shown here:

$$u = c_1 \cos\omega t + c_2 \sin\omega t$$

... will also satisfy Equation 12.10. This can be extended using De Moivre's theorem (Equation 12.8) allowing an exponential representation of an oscillation as shown in Equation 12.11:

$$u = A(\cos\omega t + \mathrm{i}\sin\omega t) \equiv A\mathrm{e}^{\mathrm{i}\omega t} \tag{12.11}$$

Therefore the solution $u = A\mathrm{e}^{\mathrm{i}\omega t}$ represents an oscillation with amplitude A and frequency ω

12.1.6 Take-home points

- We can **always** represent an oscillation using a complex exponential function
- To obtain the actual physical displacement of the system we simply examine either the real or the imaginary part of the solution:

$$\begin{array}{lrcl} \text{Either:} & \text{displacement} & = & \mathrm{Re}(u) = A\cos\omega t \\ \text{or:} & \text{displacement} & = & \mathrm{Im}(u) = A\sin\omega t \end{array}$$

The main advantage of working with complex exponentials is that they are considerably easier to manipulate than the trigonometric functions sine and cosine. In general it is far easier to use this exponential notation when multiplying oscillations (such as you will explore in electrical circuits later). However, when adding oscillations or waves you may find it easier using a trigonometric identity.

You should be comfortable using either approach to represent an oscillation.

References