Math 1090 A Summer 2009 Answers to Homework – 4

Exercises from Section 3.6 (pp. 108–109)

8. Use Resolution to prove $A \vee (B \wedge C) \to A \vee B$.

Proof

$\vdash A \vee (B \wedge C) \to A \vee B$

if, by the Deduction Theorem,

$A \vee (B \wedge C) \vdash A \vee B$

if, by 2.6.7 (Proof by Contradiction),

$A \vee (B \wedge C), \neg(A \vee B) \vdash \perp$

if, by 2.4.23 (1) (Distributivity of $\vee$ over $\wedge$),

$(A \vee B) \wedge (A \vee C), \neg(A \vee B) \vdash \perp$

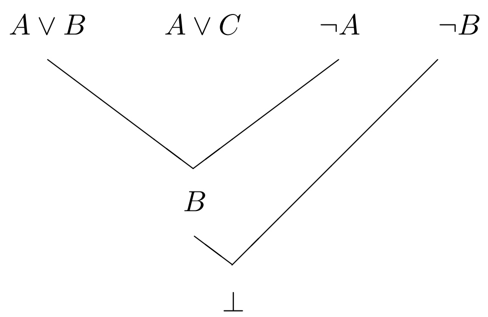
if, by 2.4.18′ (De Morgan 2),

$(A \vee B) \wedge (A \vee C), \neg A \wedge \neg B \vdash \perp$

if, by 2.5.2 (Hypothesis Merging/Splitting), twice,

$A \vee B$, $A \vee C$, $\neg A$, $\neg B \vdash \perp$.

We prove the last metatheorem by Resolution:



(Note that we didn't need to use the hypothesis $A \vee C$ in the proof by Resolution.)

9. Use Resolution to prove $(A \to B) \to (A \to C) \to A \to B \wedge C$.

Proof

$\vdash (A \to B) \to (A \to C) \to A \to B \wedge C$

if, by the Deduction Theorem, thrice,

$A \to B,\ A \to C,\ A \vdash B \wedge C$

if, by 2.6.7 (Proof by Contradiction),

$A \to B,\ A \to C,\ A,\ \neg(B \wedge C) \vdash \perp$

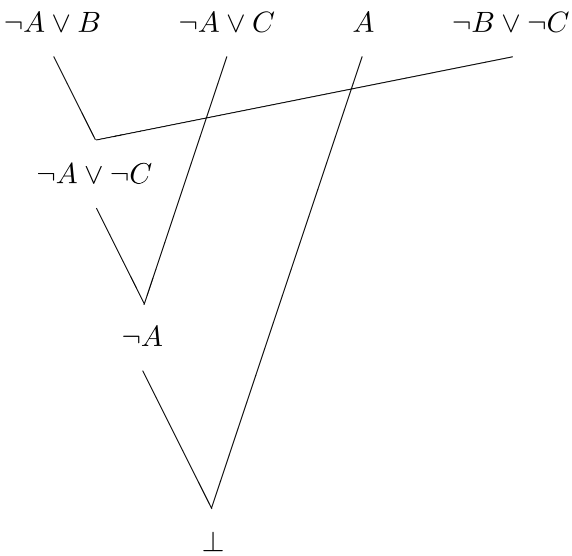
if, by 2.4.11 ($\to$ as $\vee$), twice,

$\neg A \vee B,\ \neg A \vee C,\ A,\ \neg(B \wedge C) \vdash \perp$

if, by 2.4.17′ (De Morgan 1),

$\neg A \vee B,\ \neg A \vee C,\ A,\ \neg B \vee \neg C \vdash \perp$

We prove the last metatheorem by Resolution:



10. Use Resolution to prove

$$(p \vee q \vee r) \wedge (p \to p') \wedge (q \to p') \wedge (r \to p') \;\to\; p'.$$

Proof

$\vdash\; (p \vee q \vee r) \wedge (p \to p') \wedge (q \to p') \wedge (r \to p') \;\to\; p'$

if, by the Deduction Theorem,

$(p \vee q \vee r) \wedge (p \to p') \wedge (q \to p') \wedge (r \to p') \;\vdash\; p'$

if, by 2.6.7 (Proof by Contradiction),

$(p \vee q \vee r) \wedge (p \to p') \wedge (q \to p') \wedge (r \to p'),\ \neg p' \;\vdash\; \bot$

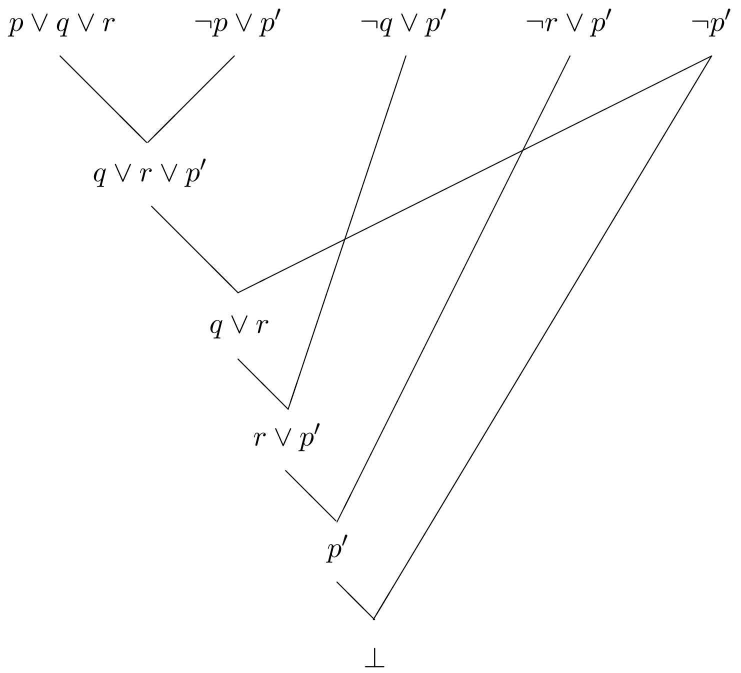
if, by 2.5.2 (Hypothesis Merging/Splitting), thrice,

$p \vee q \vee r,\ p \to p',\ q \to p',\ r \to p',\ \neg p' \;\vdash\; \bot$

if, by 2.4.11 ($\to$ as $\vee$), thrice,

$p \vee q \vee r,\ \neg p \vee p',\ \neg q \vee p',\ \neg r \vee p',\ \neg p' \;\vdash\; \bot.$

We prove the last metatheorem by Resolution:



11. Use Resolution to prove $(p \to q \to r) \to (q \to p \to r)$.

Proof

$\vdash (p \to q \to r) \to (q \to p \to r)$

if, by the Deduction Theorem, thrice,

$p \to q \to r,\ q,\ p \vdash r$

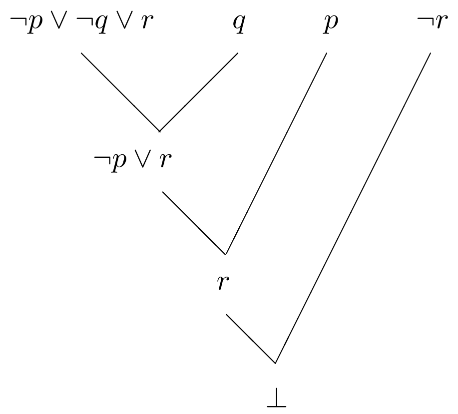
if, by 2.6.7 (Proof by Contradiction),

$p \to q \to r,\ q,\ p,\ \neg r \vdash \bot$

if, by 2.4.11 ($\to$ as $\vee$), twice,

$\neg p \vee \neg q \vee r,\ q,\ p,\ \neg r \vdash \bot$.

We prove the last metatheorem by Resolution:



12. Is it true that, for every set Γ of formulas and all formulas A and B, if $\Gamma \vdash A \vee B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$?

- If yes, give a proof.
- If no, give a counterexample, using 3.1.3 (Soundness of Boolean Logic).

Answer

No, as the following counterexample shows.

Let Γ, A, B be $\emptyset, p, \neg p$, respectively.

Also, let v_1 and v_2 be any states such that $v_1(p) = \mathbf{f}$ and $v_2(p) = \mathbf{t}$.

Then $\vdash p \vee \neg p$, by Axiom (9) (Excluded Middle).

But $\nvDash_{\text{taut}} p$, since $v_1(p) = \mathbf{f}$, and $\nvDash_{\text{taut}} \neg p$, since $v_2(\neg p) = \mathbf{f}$.

Thus $\nvdash p$ and $\nvdash \neg p$, by Soundness.

13. Use a ping-pong argument to prove *Left Distributivity of* $\rightarrow$ *over* $\rightarrow$:

$$A \rightarrow B \rightarrow C \equiv (A \rightarrow B) \rightarrow (A \rightarrow C).$$

(Hint: To prove $A \equiv B$ by a ping-pong argument means to separately prove $A \rightarrow B$ and $B \rightarrow A$. The desired theorem then follows by 2.5.1 (1) (Conjunction), 2.4.26 (Mutual $\rightarrow$) and Equanimity. You can prove the $\rightarrow$ direction using the Deduction Theorem and a 6-line Hilbert proof that uses 2.5.3 (Modus Ponens), and the $\leftarrow$ direction using the Deduction Theorem and an 8-line Hilbert proof that uses 2.5.1 and 2.5.3.)

Answer

Proof of the $\rightarrow$ *direction*

$\vdash (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$

if, by the Deduction Theorem, thrice,

$A \rightarrow B \rightarrow C,\ A \rightarrow B,\ A \vdash C.$

(1) A	$\langle$Hypothesis$\rangle$
(2) $A \rightarrow B$	$\langle$Hypothesis$\rangle$
(3) B	$\langle$(1),(2) + 2.5.3 (Modus Ponens)$\rangle$
(4) $A \rightarrow B \rightarrow C$	$\langle$Hypothesis$\rangle$
(5) $B \rightarrow C$	$\langle$(1),(4) + Modus Ponens$\rangle$
(6) C	$\langle$(3),(5) + Modus Ponens$\rangle$

Proof of the $\leftarrow$ *direction*

$\vdash \big((A \rightarrow B) \rightarrow (A \rightarrow C)\big) \rightarrow (A \rightarrow B \rightarrow C)$

if, by the Deduction Theorem, thrice,

$(A \rightarrow B) \rightarrow (A \rightarrow C),\ A,\ B \vdash C.$

(1) B	$\langle$Hypothesis$\rangle$
(2) $\neg A \vee B$	$\langle$(1) + 2.5.1 (3) (Weakening 1)$\rangle$
(3) $\neg A \vee B \equiv A \rightarrow B$	$\langle$2.4.11 ($\rightarrow$ as $\vee$)$\rangle$
(4) $A \rightarrow B$	$\langle$(2),(3) + Eqn$\rangle$
(5) $(A \rightarrow B) \rightarrow (A \rightarrow C)$	$\langle$Hypothesis$\rangle$
(6) $A \rightarrow C$	$\langle$(4),(5) + 2.5.3 (Modus Ponens)$\rangle$
(7) A	$\langle$Hypothesis$\rangle$
(8) C	$\langle$(7),(6) + Modus Ponens$\rangle$

19. Use Resolution to show that

$$p \to q \to r \to p_1,\ A \to B \to r \vdash A \to p \to B \to q \to p_1.$$

Proof

$p \to q \to r \to p_1,\ A \to B \to r \vdash A \to p \to B \to q \to p_1$

if, by the Deduction Theorem, four times,

$p \to q \to r \to p_1,\ A \to B \to r,\ A,\ p,\ B,\ q \vdash p_1$

if, by 2.6.7 (Proof by Contradiction),

$p \to q \to r \to p_1,\ A \to B \to r,\ A,\ p,\ B,\ q,\ \neg p_1 \vdash \bot$

if, by 2.4.11 ($\to$ as $\vee$), five times,

$\neg p \vee \neg q \vee \neg r \vee p_1,\ \neg A \vee \neg B \vee r,\ A,\ p,\ B,\ q,\ \neg p_1 \vdash \bot$

We prove the last metatheorem by Resolution:

$\neg p \vee \neg q \vee \neg r \vee p_1 \qquad \neg A \vee \neg B \vee r \qquad A \qquad p \qquad B \qquad q \qquad \neg p_1$

$\neg p \vee \neg q \vee p_1 \vee \neg A \vee \neg B$

$\neg p \vee \neg q \vee p_1 \vee \neg B$

$\neg q \vee p_1 \vee \neg B$

$\neg q \vee p_1$

p_1

$\bot$

20. Use Resolution to prove $(\neg B \to \neg A) \to (\neg B \to A) \to B$.

Proof

$\vdash (\neg B \to \neg A) \to (\neg B \to A) \to B$

if, by the Deduction Theorem, twice,

$\neg B \to \neg A, \neg B \to A \vdash B$

if, by 2.6.7 (Proof by Contradiction),

$\neg B \to \neg A, \neg B \to A, \neg B \vdash \perp$

if, by 2.4.11 ($\to$ as $\vee$), twice,

$\neg\neg B \vee \neg A, \neg\neg B \vee A, \neg B \vdash \perp$

We prove the last metatheorem by Resolution:

$\neg\neg B \vee \neg A \qquad \neg\neg B \vee A \qquad \neg B$

$\neg\neg B$

$\perp$

22. Use Resolution to prove $\big((A \to B) \to A\big) \to A$.

Proof

$\vdash \big((A \to B) \to A\big) \to A$

if, by the Deduction Theorem,

$(A \to B) \to A \vdash A$

if, by 2.6.7 (Proof by Contradiction),

$(A \to B) \to A, \neg A \vdash \bot$.

I'll work on the first hypothesis separately:

$$
\begin{array}{ll}
 & (A \to B) \to A \\
\Leftrightarrow & \langle 2.4.11\ (\to \text{ as } \vee) \rangle \\
 & \neg(A \to B) \vee A \\
\Leftrightarrow & \langle 2.4.11\ (\to \text{ as } \vee) + \text{Leib} \rangle \\
 & \neg(\neg A \vee B) \vee A \\
\Leftrightarrow & \langle 2.4.18'\ (\text{De Morgan } 2) + \text{Leib} \rangle \\
 & (\neg\neg A \wedge \neg B) \vee A \\
\Leftrightarrow & \langle 2.4.4\ (\text{Double Negation}) + \text{Leib} \rangle \\
 & (A \wedge \neg B) \vee A \\
\Leftrightarrow & \langle 2.4.23\ (1)\ (\text{Distributivity of } \vee \text{ over } \wedge) \rangle \\
 & (A \vee A) \wedge (\neg B \vee A) \\
\Leftrightarrow & \langle (7)\ (\text{Idempotency of } \vee) + \text{Leib} \rangle \\
 & A \wedge (\neg B \vee A)
\end{array}
$$

Thus $(A \to B) \to A, \neg A \vdash \bot$

if, by the above proof,

$A \wedge (\neg B \vee A), \neg A \vdash \bot$

if, by 2.5.2 (Hypothesis Merging/Splitting),

$A, \neg B \vee A, \neg A \vdash \bot$

We prove the last metatheorem by Resolution:

$A \qquad \neg B \vee A \qquad \neg A$

$\bot$

Additional Exercise

Is it true that, for every set Γ of formulas and all formulas A and B, if $\Gamma \vdash A \to B$, then $\Gamma \vdash A$ implies $\Gamma \vdash B$?

- If yes, give a proof.
- If no, give a counterexample.

Answer

Yes.

Assume that $\Gamma \vdash A \to B$ and $\Gamma \vdash A$.

Then the following proof shows that $\Gamma \vdash B$.

(1) A	$\langle\Gamma$-theorem$\rangle$
(2) $A \to B$	$\langle\Gamma$-theorem$\rangle$
(3) B	$\langle$(1),(2) + 2.5.3 (Modus Ponens)$\rangle$