Section 3.3

1. a)

x	28	30	31
$P(X=x)$	1/12	1/3	7/12

$E(X) = 30.42$, $SD(X) = 0.86$.

b)

x	28	30	31
$P(X=x)$	28/365	$4 \times 30/365$	$7 \times 31/365$

$E(X) = 30.44$, $SD(X) = 0.88$

2. $E(Y^2) = 3$, $Var(Y^2) = 15/2$

3. a) $E(2X+3Y) = 2E(X) + 3E(Y) = 5$.
   b) $Var(2X+3Y) = 4Var(X) + 9Var(Y) = 26$.
   c) $E(XYZ) = [E(X)][E(Y)][E(Z)] = [E(X)]^3 = 1$.
   d)

$$\begin{aligned} Var(XYZ) &= E[(XYZ)^2] - [E(XYZ)]^2 = E[X^2Y^2Z^2] - [E(XYZ)]^2 \\ &= E[X^2]E[Y^2]E[Z^2] - [E(X)E(Y)E(Z)]^2 \\ &= [E(X^2)]^3 - [E(X)]^6 = [Var(X) + [E(X)]^2]^3 - [E(X)]^6 = 26. \end{aligned}$$

4. $Var(X_1X_2) = E[(X_1X_2)^2] - [E(X_1X_2)]^2$
   $= E(X_1^2) \cdot E(X_2^2) - [E(X_1X_2)]^2$
   $= (\mu_1^2 + \sigma_1^2) \cdot (\mu_2^2 + \sigma_2^2) - (\mu_1\mu_2)^2$
   $= \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2$

5. By the computational formula for variance,

$$Var(X-a) = E\left[(X-a)^2\right] - [E(X-a)]^2 = E\left[(X-a)^2\right] - (\mu - a)^2.$$

But $Var(X-a) = Var(X) = \sigma^2$, so

$$E\left[(X-a)^2\right] = \sigma^2 + (\mu - a)^2.$$

Thus the mean square distance between X and the constant a is σ^2 + something non-negative. The mean square distance is minimized if this 'something non-negative' is actually 0, i.e., $a = \mu$. And in this case, $E\left[(X-\mu)^2\right] = \sigma^2 = Var(X)$.

6. Intuitively, 1 and 6 are the two most extreme values, so increasing the probability of these values should increase the variance.

$$\begin{aligned} Var(X_p) &= \sum_{x=1}^{6} x^2 P(X_p = x) - [E(X_p)]^2 \\ &= 1^2\frac{p}{2} + 2^2\frac{1-p}{4} + 3^2\frac{1-p}{4} + 4^2\frac{1-p}{4} + 5^2\frac{1-p}{4} + 6^2\frac{p}{2} - (3.5)^2 \\ &= \frac{2p - 4p - 9p - 16p - 25p + 72p}{4} + \frac{4+9+16+25}{4} - (3.5)^2 \\ &= 5p + \frac{5}{4} \end{aligned}$$

7. $X = X_1 + X_2 + X_3$, where X_i has binomial (n_i, p_i) distribution.
   (a) No! It is binomial only when the p_i's are all the same.
   (b) $E(X) = \sum_{i=1}^{3} n_ip_i$, $Var(X) = \sum_{i=1}^{3} n_ip_iq_i$.

8. a) $N = X_1 + X_2 + X_3$ b) $E(N) = \frac{1}{5} + \frac{1}{4} + \frac{1}{3} = \frac{47}{60}$

c) Note that N is the indicator of the event $A_1 \cup A_2 \cup A_3$, since A_i's are disjoint. So $Var(N) = (\frac{1}{5} + \frac{1}{4} + \frac{1}{3})(1 - \frac{1}{5} - \frac{1}{4} - \frac{1}{3}) = \frac{611}{3600}$

d) $Var(N) = Var(X_1) + Var(X_2) + Var(X_3) = \frac{1}{5} \cdot \frac{4}{5} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{2}{3} = \frac{2051}{3600}$

e) $Var(N) = \left[\frac{1}{5} \cdot 3^2 + (\frac{1}{4} - \frac{1}{5}) \cdot 2^2 + (\frac{1}{3} - \frac{1}{4}) \cdot 1^2\right] - (\frac{1}{5} + \frac{1}{4} + \frac{1}{3})^2 = \frac{5291}{3600}$

9. The number N_1 of individuals who vote Republican in both elections has binomial $(r, 1 - p_1)$ distribution, and the number N_2 of individuals who vote Democratic in the first election and Republican in the second has binomial $(n - r, p_2)$ distribution. The number of Republican votes in the second election is then $N_1 + N_2$. So

$$E(N_1 + N_2) = E(N_1) + E(N_2) = r(1 - p_1) + (n - r)p_2$$

Since N_1 and N_2 are independent, we have

$$Var(N_1 + N_2) = Var(N_1) + Var(N_2) = r(1 - p_1)p_1 + (n - r)p_2(1 - p_2)$$

10. a) $E(X^k) = \sum_{x=1}^n x^k P(X = x) = \sum_{x=1}^n x^k \cdot \frac{1}{n} = \frac{1}{n}(1^k + 2^k + \cdots + n^k) = s(k, n)/n$.

$E\left[(X+1)^k\right] = \sum_{x=1}^n (x+1)^k P(X = x) = \frac{1}{n}\left[2^k + 3^k + \cdots + (n+1)^k\right] = \frac{1}{n}\left[s(k, n+1) - 1\right]$.

b) By the binomial expansion, $kX^{k-1} + \binom{k}{2}X^{k-2} + \cdots + 1 = (X+1)^k - X^k$, so

$$E\left[kX^{k-1} + \binom{k}{2}X^{k-2} + \cdots + 1\right] = E\left[(X+1)^k\right] - E(X^k)$$

$$= \frac{2^k + 3^k + \cdots + (n+1)^k}{n} - \frac{1 + 2^k + \cdots + n^k}{n} = \frac{(n+1)^k - 1}{n}.$$

c) Put $k = 2$ in b):

$E(2X + 1) = \frac{(n+1)^2 - 1}{n} = n + 2 \Longrightarrow E(X) = \frac{n+1}{2}$

By a), $s(1, n)/n = E(X)$. So $s(1, n) = \frac{n(n+1)}{2}$.

d) Put $k = 3$ in b):

$E(3X^2 + 3X + 1) = \frac{(n+1)^3 - 1}{n} = n^2 + 3n + 3$.

$\Longrightarrow E(X^2) = \frac{1}{3}\left[n^2 + 3n + 3 - 3E(X) - 1\right] = \frac{1}{3}\left[n^2 + 3n + 3 - \frac{3(n+1)}{2} - 1\right] = \frac{1}{6}(n+1)(2n+1)$.

By a), $\frac{s(2,n)}{n} = E(X^2)$. So $s(2, n) = \frac{1}{6}n(n+1)(2n+1)$.

e) $Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{6}(n+1)(2n+1) - [(n+1)/2]^2 = \frac{n^2 - 1}{12}$.

f) For $n = 6$, $E(X) = 7/2$ agrees, and $Var(X) = 35/12$ agrees.

g) Put $k = 4$ in b):

$$E(4X^3 + 6X^2 + 4X + 1) = \frac{(n+1)^4 - 1}{n} = n^3 + 4n^2 + 6n + 4$$

Use part a) to conclude

$$s(3, n) = nE(X^3) = \frac{n}{4}\left(n^3 + 4n^2 + 6n + 4 - 6E(X^2) - 4E(X) - 1\right)$$

$$\frac{n}{4}\left[n^3 + 4n^2 + 6n + 4 - (n+1)(2n+1) - 2(n+1) - 1\right] = \frac{n^2(n+1)^2}{4} = [s(1, n)]^2.$$

11. $Y = (a - b) + bX$. So

$$E(Y) = a - b + bE(X) = a - b + b \cdot \frac{n+1}{2} = a + b\left(\frac{n-1}{2}\right)$$

$$Var(Y) = b^2 Var(X) = b^2 \cdot \left(\frac{n^2 - 1}{12}\right)$$

12. a) According to the Chebychev inequality,

$$P(X \geq 20) \leq P(|X-10| \geq 10) \leq \frac{5^2}{10^2} = 1/4.$$

b) If X has binomial (n,p) distribution, then $E(X) = np$ and $SD(X) = \sqrt{np(1-p)}$. Try $np = 10$, and $np(1-p) = 25$, which implies that $1-p = 25/10 = 2.5$. This is impossible!

13. Let n denote the number of scores (among the million individuals) exceeding 130, and let X denote the score of one of the million individuals picked at random. Then

$$P(X > 130) = \frac{n}{10^5} \iff n = 10^6 P(X > 130).$$

a) We have, by Chebychev's inequality,
$P(X > 130) = P(X - 100 > 30) \leq P(|X-100| > 30) \leq \frac{Var(X)}{30^2} = \frac{1}{9}$.
So $n \leq 10^6 \times \frac{1}{9} < 111112$.

b) Since the distribution of scores is symmetric about 100,
$P(X - 100 > 30) = P(100 - X > 30)$. Therefore
$P(X > 130) = P(X - 100 > 30) = \frac{1}{2}P(|X-100| > 30) \leq \frac{1}{18}$
and $n \leq 10^6 \times \frac{1}{18} \leq 55556$.

c) $P(X > 130) = P(\frac{X-100}{10} > 3) \approx 1 - \Phi(3) = 0.0013$, hence $n = 10^6 \times 0.0013 = 1300$.

Note: We can get a sharper bound in a) using Cantelli's inequality, which says: if X is a random variable with mean m and variance σ^2, then for $a > 0$

$$P(X - m > a) \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

Proof of Cantelli's inequality: Without loss of generality, we may assume $m = 0$, $\sigma = 1$. Note that if $X \geq a$ then $(1 + aX)^2 \geq (1 + a^2)^2$. So

$1_{(X>a)} \leq \frac{(1+aX)^2}{(1+a^2)^2}$,

where $1_{(X>a)}$ is the indicator of the event $(X > a)$. Take expectations, and get the result.

Application to part a): $P(X > 130) = P(X - 100 > 30) \leq \frac{10^2}{10^2+30^2} = \frac{1}{10}$.

14. a) Let X be the income in thousands of dollars of a family chosen at random. Then $E(X) = 10$ and, by Markov's inequality,

$$P(X \geq 50) \leq \frac{E(X)}{50} = \frac{1}{5}.$$

b) If $SD(X) = 8$ then by Chebychev's inequality

$$P(X \geq 50) \leq P(|X - 10| > 5 \cdot 8) \leq \frac{1}{5^2} = \frac{1}{25}.$$

15. b) $10\sqrt{8}$

16. a)

x	-2	-1	0	3
$P(X=x)$	.25	.25	.25	.25

Thus $E(X) = \frac{-2-1+0+3}{4} = 0$ and $Var(X) = \frac{(-2)^2+(-1)^2+0^2+3^2}{4} - 0^2 = \frac{7}{2} = 3.5$

b) Let $S_n = \sum_{i=1}^n X_i$ where each X_i is one play of this game. Then we wish to know $P(S_{100} \geq 25)$. $E(S_{100}) = 100E(X) = 0$ and $Var(S_{100}) = 100Var(X) = 350$. Furthermore, the sum of 100 draws should look pretty close to normal, so

$$P(S_{100} \geq 25) \approx 1 - \Phi\left(\frac{25 - .5 - 0}{\sqrt{350}}\right) = 1 - \Phi\left(\frac{24.5}{18.71}\right) = .0952$$

17. $E(X) = 1/4$, $SD(X) = \frac{1}{4}\sqrt{11}$; $E(S) = 6.25$, $SD(S) = \frac{5}{4}\sqrt{11}$.

a) $P(S < 0) \approx \Phi\left(\frac{-0.5-6.25}{(5/4)\sqrt{11}}\right) \approx .05.$

b) $P(S = 0) \approx \Phi\left(\frac{0.5-6.25}{(5/4)\sqrt{11}}\right) - \Phi\left(\frac{-0.5-6.25}{(5/4)\sqrt{11}}\right) \approx .03.$

c) $P(S > 0) \approx 1 - \Phi\left(\frac{0.5-6.25}{(5/4)\sqrt{11}}\right) \approx .92.$

18. Let X_i be the amount won on the ith bet. Thus $X_i = 6$ if you win on the ith bet, and $X_i = -1$ if you lose. Let S be the sum of 300 dollar bets, and we wish to find $P(S > 0)$.

$$\begin{aligned} E(X_i) &= 6 \times \frac{5}{38} + -1 \times \frac{33}{38} = -0.07895 \\ SD(X_i) &= \sqrt{36 \times \frac{5}{38} + 1 \times \frac{33}{38} - (-0.07895)^2} = 2.366 \\ E(S) &= 300 \times -0.07895 = -23.68 \\ SD(S) &= \sqrt{300} \times 2.366 = 40.98 \end{aligned}$$

By the normal approximation,

$$P(S > 0) \approx 1 - \Phi\left(\frac{0 - (-23.68)}{40.98}\right) = 0.2817$$

19. Let X_i be the weight of the ith guest in the sample, and let $S = X_1 + X_2 + \cdots + X_{30}$. Want $P(S > 5000)$. Use the normal approximation, with $E(S) = 4500$, and

$$SD(S) = \sqrt{30} \times 55 = 301.25$$

Need the area to the right of $(5000 - 4500)/301.25 = 1.66$ under the standard normal curve. This is $1 - \Phi(1.66) = 1 - 0.9515 = 0.0485$.

20. a) The probability that your profit is \$8,000 or more is the probability that your stock gives you a profit of either \$200 per \$1000 invested or \$100 per \$1000 invested; this probability is .5.

b) Let S_{100} be the sum of the profits for the 100 stocks invested in, and let X be the random variable representing one of the profits.
$E(X) = \frac{200+100+0-100}{4} = 50$ and $Var(X) = \frac{200^2+100^2+0^2+(-100)^2}{4} - 50^2 = 12,500.$
$E(S_{100}) = 100E(X) = 5000$, and since the investments are independent, $Var(S_{100}) = 100Var(X) = 1,250,000$. Using normal approximation,

$$P(S_{100} \geq 8000) = 1 - \Phi\left(\frac{8000 - 50 - 5000}{\sqrt{1,250,000}}\right) = 1 - \Phi\left(\frac{2950}{1118}\right) = .0042$$

In the numerator of the normal approximation term, note that the $8000 - 50$ is just the familiar continuity correction since S_{100} can only take on values which are multiples of \$100.

21. a) Let X_i be the error caused by the ith transaction; thus X_i is uniformly distributed between -49 and 50. Let S_{100} be the total accumulated error for 100 transactions, and then we wish to know $P(S_{100} > 500$ or $S_{100} < -500)$. We have

$$E(X_i) = \sum_{j=-49}^{50} \frac{j}{100} = .5$$

$$Var(X_i) = \sum_{j=-49}^{50} \frac{j^2}{100} - .5^2 = 833.25$$

So $E(S_{100}) = 50$ and $Var(S_{100}) = 83,325$. Using normal approximation,

$$P(|S_{100}| > 500) \approx 1 - \left(\Phi\left(\frac{500 + .5 - 50}{\sqrt{83,325}}\right) - \Phi\left(\frac{-500 - .5 - 50}{\sqrt{83,325}}\right)\right) = 0.0876$$

b) Using the same notation as in a), we now have that

$$E(X_i) = \sum_{j=-49}^{50} \frac{j \times .75}{99} = .3788$$

except that the coefficient when $j = 0$ is wrong, but this does not affect the answer. Similarly,

$$Var(X_i) = \sum_{j=-49}^{50} \frac{j^2 \times .75}{99} - .3788^2 = 631.30$$

Thus $E(S_{100}) = 37.88$ and $Var(S_{100}) = 63,130$. Using normal approximation,

$$P(|S_{100}| > 500) \approx 1 - \left(\Phi\left(\frac{500 + .5 - 37.88}{\sqrt{63,130}}\right) - \Phi\left(\frac{-500 - .5 - 37.88}{\sqrt{63,130}}\right)\right) = 0.0489$$

22. a) Let X_i be the face showing on the ith roll; then $E(X_i) = 7/2$, and $SD(X_i) = \sqrt{35/12}$. Let $\bar{X}_n$ be the average of the first n rolls. Then
$E(\bar{X}_n) = E(X_1) = 7/2$, and
$SD(\bar{X}_n) = SD(X_1)/\sqrt{n} = \sqrt{35/12n}$, and

$$P\left(3\frac{5}{12} \le \bar{X}_n \le 3\frac{7}{12}\right) = P\left(|Z_n| \le \sqrt{\frac{n}{12 \times 35}}\right)$$

where

$$Z_n = \frac{\bar{X}_n - E(\bar{X}_n)}{SD(\bar{X}_n)} = \frac{\bar{X}_n - (7/2)}{\sqrt{35/12n}}$$

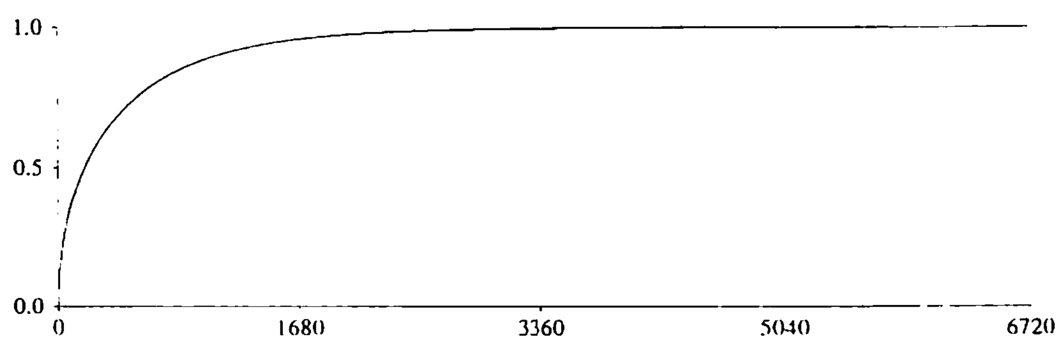
is $\bar{X}_n$ standardized, which will be approximately standard normal for large n. When $n = 105$,

$$P\left(|Z_n| \le \sqrt{\frac{n}{12 \times 35}}\right) \approx P(|Z| \le 0.5) = 0.383$$

where Z is standard normal.
The same argument gives the numbers in the following table:

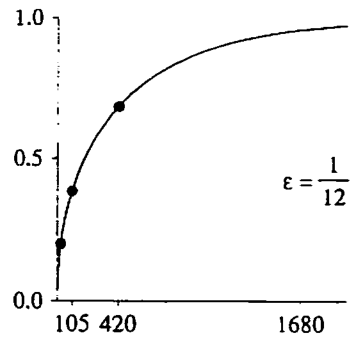
n	$P\left(3\frac{5}{12} \le \bar{X}_n \le 3\frac{7}{12}\right)$
105	$P(\lvert Z\rvert \le .5) = .383$
420	$P(\lvert Z\rvert \le 1) = .6826$
1680	$P(\lvert Z\rvert \le 2) = .9544$
6720	$P(\lvert Z\rvert \le 4) \approx 1$

b)

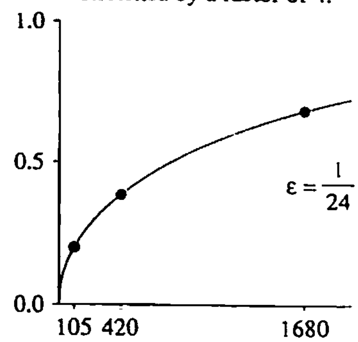


c) $P\left(|\bar{X}_n - 7/2| \le 1/24\right) \approx P\left(|Z_n| \le \sqrt{\frac{n}{4\times 12\times 35}}\right)$.

See that if ϵ is replaced by $\epsilon/2$, then probabilities that were previously obtained at n are now obtained at $4n$. So the scale on the n-axis is stretched by a factor of 4. In general, dividing ϵ by a factor of f results in a graph with the scale on the n-axis stretched by a factor of f^2.



Same curve with horizontal axis stretched by a factor of 4.



23. Let X_i be the lifetime in weeks of the ith battery. Assume $S = X_1 + \cdots + X_{27}$ is approximately normally distributed. Using $E(S) = 27 \times 4 = 108$ and $SD(S) = \sqrt{27} \times 1 = \sqrt{27}$, we have

$$\begin{aligned}
&P(\text{more than 26 replacements in a 2-year period})\\
&= P(X_1 + \cdots + X_{27} < 2 \times 52)\\
&= P(S < 104)\\
&= P\left(\frac{S-108}{\sqrt{27}} < \frac{-4}{\sqrt{27}}\right)\\
&\approx \Phi(-.77)\\
&\approx .22
\end{aligned}$$

24. a)

x	0	1	2	3	4
$P(S_2 = x)$	.0625	.25	.375	.25	.0625

b) $E(S_{50}) = 50$ and $Var(S_{50}) = 50(.5) = 25$. Thus by normal approximation,

$$P(S_{50} = 50) \approx \Phi\left(\frac{50.5-50}{5}\right) - \Phi\left(\frac{49.5-50}{5}\right) = 0.0797$$

c) $S_n = X_1 + \cdots + X_n$ where the X_i are independent, each with binomial $(2, 1/2)$ distribution. It follows (Exercise 3.1.11) that S_n has binomial $(2n, 1/2)$ distribution:

$$P(S_n = k) = \binom{2n}{k}\left(\frac{1}{2}\right)^{2n}$$

25. a) $k = 1$:

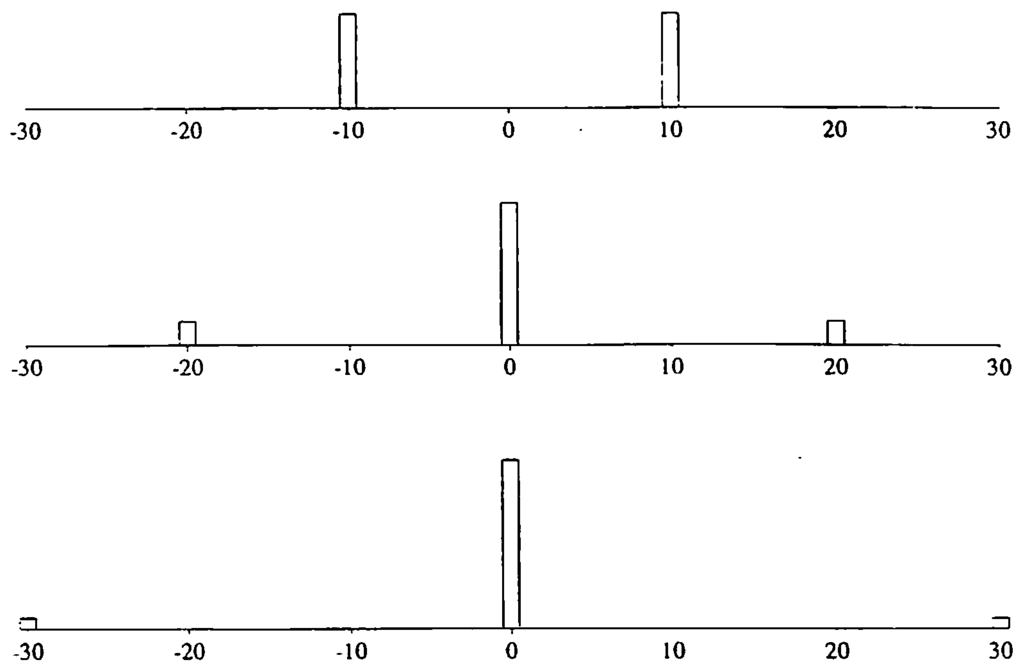
x	-10	10
$P(X = x)$	1/2	1/2

$k = 2$:

x	-20	0	20
$P(X = x)$	1/8	3/4	1/8

$k = 3$:

x	-30	0	30
$P(X = x)$	$1/18$	$8/9$	$1/18$



b) $E(X) = \mu$ by symmetry of the distribution about μ.
$Var(X) = E[(X - \mu)^2] = 2 \cdot (k\sigma)^2 \cdot \frac{1}{2k^2} + 0^2(1 - \frac{1}{k^2}) = \sigma^2$
$P(|X - \mu| \geq k\sigma) = P(X = \mu + k\sigma) + P(X = \mu - k\sigma) = 2 \cdot \frac{1}{2k^2} = \frac{1}{k^2}$,
which is the Chebychev bound on the above probability.

c) $P(|Y - \mu| < \sigma) = 0 \Longrightarrow P(|Y - \mu| \geq \sigma) = 1$. So absolute deviations from the mean are $\geq \sigma$ with probability 1. On the other hand, the average of squared deviations (variance) $= \sigma^2$. So the deviations must actually equal $\pm\sigma$. Suppose $P(Y - \mu = \sigma) = p$, so $P(Y - \mu = -\sigma) = 1 - p$. Then
$0 = E(Y - \mu) = p\sigma + (1 - p)(-\sigma)$,
so $p = 1/2$. That is, $P(Y = \mu + \sigma) = P(Y = \mu - \sigma) = \frac{1}{2}$, which is the above distribution for $k = 1$.

26. a) 1.5 b) Note that
$0 \leq Var(|X - \mu|) = E(|X - \mu|^2) - [E(|X - \mu|)]^2$. From this, the result follows immediately. That the equality holds if and only if $|X - \mu|$ is a constant follows from the first inequality.

27. a) Use the fact that expectation preserves inequalities:
(i) $0 \leq X \leq 1 \Longrightarrow 0 \leq E(X) \leq 1$.
(ii) Note that $X^2 \leq X$, so $E(X^2) \leq E(X)$ and
$0 \leq Var(X) = E(X^2) - [E(X)]^2 \leq E(X) - [E(X)]^2 = \mu - \mu^2$.
The inequality $\mu - \mu^2 \leq 1/4$ follows from calculus, or from completing the square:
$\mu - \mu^2 = \frac{1}{4} - \left(\mu - \frac{1}{2}\right)^2 \leq \frac{1}{4}$.

b) The results are trivial if $a = b$. So assume $a < b$. Let $Y = \frac{X-a}{b-a}$. Then $0 \le Y \le 1$.

(i) Use part i) of a) to conclude $0 \le E(Y) \le 1$. Substitute $E(Y) = \frac{\mu - a}{b-a}$ and rearrange.

(ii) Use part ii) of a) to conclude

$$0 \le Var(Y) \le E(Y)\,(1 - E(Y)) \le 1/4.$$

Substitute $E(Y) = \frac{\mu-a}{b-a}$ and $Var(Y) = \frac{Var(X)}{(b-a)^2}$ and rearrange.

(iii) Immediate from ii).

c) Consider a random variable X taking values $0, \ldots, 9$ with chances equal to the proportions of each digit among the list. We have $0 \le X \le 9$ and $Var(X) = \frac{1}{4}(9)^2$. By part b), we must have

$$E(X^2) - (E(X))^2 = E(X)\,(9 - E(X)) = \frac{1}{4}(9)^2.$$

The second equality yields $E(X) = 9/2$ and the first yields $9E(X) = E(X^2)$. But $9X - X^2 \ge 0$, so $9X - X^2 = 0$ with probability 1 (see below). Hence X takes values only 0 or 9. Conclude from $E(X) = 9/2$ that $P(X = 0) = P(X = 9) = 1/2$, that is, that half the list are 0's and the rest are 9's.

Claim: If $Y \ge 0$ and $E(Y) = 0$ then $Y = 0$ with probability 1.

Reason: Suppose not. Then $P(Y > a) > 0$ for some $a > 0$. But by Markov's inequality, we have $P(Y > a) \le E(Y)/a = 0$, contradiction!

28. $Var(S) = \sum_i p_i(1 - p_i) = np(1-p) - \sum_i (p_i - p)^2$.

29. $\bar{D}_n = (D_1 + \cdots + D_n)/n$, where each D_i is uniformly distributed on $\{0, \ldots, 9\}$: $P(D_i = k) = 1/10$ for each k.
So $E(D_i) = \frac{1}{10}(1 + 2 + \cdots + 9) = 9/2$.
To get the variance of D_i, note that $D_i + 1$ is uniform on $\{1, 2, \ldots, 10\}$. So by a previous problem (Moments of the Uniform Distribution), $Var(D_i) = Var(D_i + 1) = \frac{10^2 - 1^2}{12} = 33/4$ and $SD(D_i) = \sqrt{33}/2$.

a) So guess $int\ (9/2) = 4$.

b) If $n = 1$, then the chance of correct guessing is $P(D_1 = 4) = 1/10$. The chance of being correct would be the same no matter what we guessed.
If $n = 2$, the chance of correct guessing is

$$P\left(4 \le (D_1 + D_2)/2 < 5\right) = P(D_1 + D_2 = 8 \text{ or } 9) = P(D_1 + D_2 = 8) + P(D_1 + D_2 = 9) = \frac{9}{100} + \frac{10}{100} = \frac{1}{1\!\!\!}$$

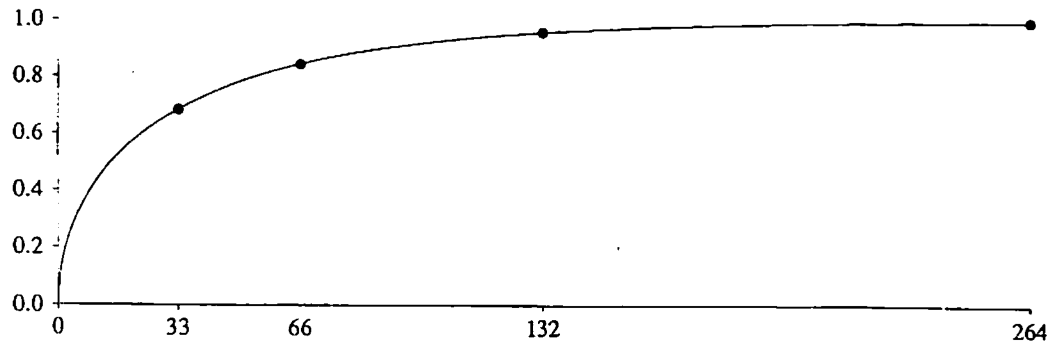
Note that the distribution of $D_1 + D_2$ is triangular with peak at $(D_1 + D_2 = 9)$, so guessing $\bar{D}_2 = 4$ indeed maximizes the chance of guessing correctly.

For large n, the standardized variable $Z_n = \frac{2\sqrt{n}(\bar{D}_n - 9/2)}{\sqrt{33}}$ has approximately standard normal distribution. So the chance of correct guessing is

$$P\left(4 \le \bar{D}_n < 5\right) \approx P\left(\left|\frac{2\sqrt{n}(\bar{D}_n - 9/2)}{\sqrt{33}}\right| \le \sqrt{\frac{n}{33}}\right) = P\left(|Z_n| \le \sqrt{\frac{n}{33}}\right).$$

The normal approximation now gives:
$n = 33$: $P(4 \le \bar{D}_n < 5) \approx P(|Z| \le 1) = 0.6826$
$n = 66$: $P(4 \le \bar{D}_n < 5) \approx P(|Z| \le \sqrt{2}) = 0.8414$

$n = 132$: $P(4 \le \bar{D}_n < 5) \approx P(|Z| \le 2) = 0.9544$



c) $P(4 \le \bar{D}_n \le 5) \ge 0.99 \iff P(|Z_n| \le \sqrt{n/33}) \ge 0.99 \iff \sqrt{n/33} \ge 2.58 \iff n \ge 220.$

30. D_i^2 takes the values 0, 1, 4, 9, 16, 25, 36, 49, 64, 81 with equal probability, so the X_i are independent with common distribution

x	0	1	4	5	6	9
$P(X_i = x)$	.1	.2	.2	.1	.2	.2

So $E(X_i) = 9/2$ and $Var(X_i) = 9.05 \Longrightarrow SD(X_i) = 3.008$.

a) By the law of averages, you expect $\bar{X}_n$ to be close to $E(\bar{X}_n) = 4.5$ for large n. So predict 4.5.

b) $P(|\bar{X}_n - 4.5| > \epsilon)$

$= P\left(\frac{\sqrt{n}|\bar{X}_n - 4.5|}{3.008} > \frac{\sqrt{n}\epsilon}{3.008}\right) \approx P\left(|Z| > \frac{\sqrt{n}\epsilon}{3.008}\right) = 2P\left(Z > \frac{\sqrt{n}\epsilon}{3.008}\right),$

where Z has standard normal distribution. For $n = 10000$, we need ϵ such that

$P\left[Z > \frac{\sqrt{n}\epsilon}{3.008}\right] = \frac{1}{400} = 0.0025 \Longrightarrow \frac{\sqrt{n}\epsilon}{3.008} = 2.81,$

therefore $\epsilon = 2.81 \times 3.008/100 = 0.085$.

c) Need n such that $P(|\bar{X}_n - 4.5| \le 0.01) \ge 0.99$ i.e.,

$P\left[|Z| \le \frac{\sqrt{n} \times 0.01}{3.008}\right] \ge 0.99 \Longrightarrow \frac{\sqrt{n}}{300.8} \ge 2.58$

therefore $n \ge 602276$.

d) We have calculated $E(X_i) = 9/2$ and $Var(X_i) = 9.05$. From the previous problem, we have $E(D_i) = 9/2$ and $Var(D_i) = 33/4 = 8.25$. Since D_i has smaller variance than does X_i, the value of $\bar{D}_n$ can be predicted more accurately.

e) Since $E(\bar{X}_{100}) = 4.5$, you should predict the first digit of $\bar{X}_{100}$ to be 4. The chance of being correct is

$P(4 \le \bar{X}_{100} < 5) \approx P\left(\left|\frac{\sqrt{100}(\bar{X}_{100} - 4.5)}{3.008}\right| \le \frac{\sqrt{100}}{2 \times 3.008}\right) \approx P(|Z| \le 1.66) = 0.903.$

31. a) $9/2, \sqrt{33}/2$

d)

$$P(|S_n - 4\tfrac{1}{2}n| \le b\sqrt{n}) = P\left(\frac{|S_n - 4\frac{1}{2}n|}{\frac{\sqrt{33}}{2}\sqrt{n}} \le \frac{2b}{\sqrt{33}}\right)$$
$$\approx 2\Phi(2b/\sqrt{33}) - 1$$

32. No Solution

33. No Solution

Section 3.4

1. a) Binomial probability: $\binom{9}{5}p^5(1-p)^4$

   b) $P(\text{first 6 tosses are tails, seventh is head}) = (1-p)^6 \cdot p$

   c) $P(\text{exactly 4 heads among first 11 tosses and 12th toss is heads}) = \binom{11}{4}p^4(1-p)^7 \cdot p$

   d) $\sum_{k=0}^{5} P(k \text{ heads among first 8 tosses and } k \text{ among next 5})$
   $= \sum_{k=0}^{5} \binom{8}{k}p^k(1-p)^{8-k} \cdot \binom{5}{k}p^k(1-p)^{5-k}$

2. a) D is distributed as 1+ a geometric random variable with parameter $\frac{1}{2}$. Whatever the first ball is, we then wait until we draw a ball of the other color.

   b) Since the expected value of a geometric random variable is $\frac{1}{p}$, $E(D) = 1 + \frac{1}{p} = 3$.

   c) The SD of a geometric random variable is $\frac{\sqrt{q}}{p}$ so in this case we have $SD(D) = \sqrt{2}$.

3. Assuming X has geometric distribution on $\{1, 2, ...\}$ with $p = 1/12$, which must be approximately correct, $E(X) = 12$.

4. a) The probability of some person being the "odd one out" is 1− the probability of having the three coins be HHH or TTT. Thus the probability is $1 - \left((\frac{1}{2})^3 + (\frac{1}{2})^3\right) = \frac{3}{4}$.

   b) Let the length of play be the random variable X, then for $r = 1, 2, \ldots$

$$P(X = r) = \left(\frac{1}{4}\right)^{r-1}\frac{3}{4}$$

   c) Since X has geometric(3/4) distribution, and the geometric (p) distribution has mean $1/p$, $E(X) = 4/3$.

5. a) Let $q_i = 1 - p_i$. The probability that Mary takes more than n tosses to get a head is the same as the probability that Mary gets tails the first n times, or ${q_2}^n$.

   b) The probability that the first person to get a head tosses more than n times is the same as the probability that everyone gets tails for the first n times, or $(q_1q_2q_3)^n$

   c) In order for the first person to get a head to toss exactly n times, there must be no heads in the first $n-1$ tosses and then at least one head on the nth toss. Thus the probability is $(q_1q_2q_3)^{n-1}(1 - q_1q_2q_3) = (q_1q_2q_3)^{n-1} - (q_1q_2q_3)^n$.

   d) Condition on the first head occuring at time n. Now in order for neither Bill nor Tom to get a head before Mary, Mary must get a head at time n; it does not matter what Bill and Tom get. Thus we want the probability that Mary gets a head given at least one head. Let M be the event that Mary gets a head and let H be the event that at least one head occurs. Then

$$P(M|H) = \frac{P(M \text{ and } H)}{P(H)} = \frac{p_2}{(1 - q_1q_2q_3)}$$

6. a) $P(W = k) = P(T = k + 1) = q^{(k+1)-1}p = q^k p$.

   b) $P(W > k) = P(T > k + 1) = q^{k+1}$.

   c) $E(W) = E(T - 1) = E(T) - 1 = \frac{1}{p} - 1 = q/p$.

   d) $Var(W) = Var(T - 1) = Var(T) = q/p^2$.

7. a) Use the craps principle. Imagine the following game: A tosses the biased coin once, then B tosses it once. If A's toss lands heads, then say that A wins the game; if A's toss does not land heads but B's does, then say that B wins the game; otherwise the game ends in a draw. The chance that A wins this game is p; the chance that B wins this game is qp; and the chance of a draw is q^2.

You can see that A and B are really repeating this game independently, over and over, until either A wins or B wins, and that the desired probability is $P(\text{A wins before B does})$. By the craps principle, this is $\frac{p}{p+qp} = \frac{1}{1+q}$.

Another way: In terms of T, the number of rolls required to produce the first head, the desired probability is

$P(T \text{ is odd}) = P(T=1) + P(T=3) + P(T=5) + \ldots = p + q^2p + q^4p + \ldots = \frac{p}{1-q^2}$.

b) Apply the craps principle to the situation in a): the desired probability is $P(\text{B wins before A does}) = \frac{qp}{p+qp} = \frac{q}{1+q}$. Or compute $P(T \text{ is even}) = \frac{qp}{1-q^2}$. Or subtract the answer in a) from 1, because one of the players must see a head sometime.

c) Apply the craps principle, this time with the game consisting of A tossing the coin once, then B tossing it twice. The chance that A wins this game is p, while the chance that B wins is $q - q^3$. So the chance that A gets the first head is $p/(1-q^3)$, and the chance that B gets the first head is $(q-q^3)/(1-q^3)$. Note that $1 - q^3 = (1-q)(1+q+q^2)$, so $P(\text{A gets the first head})$ simplifies to $1/(1+q+q^2)$. Alternatively, in terms of the random variable T, the chance that A gets the first head is the probability that T is of the form $1+3k$ for some $k = 0, 1, 2, \ldots$, similarly for B.

d) Solve $\frac{1}{1+q+q^2} = .5$ to get $q = \frac{\sqrt{5}-1}{2}$ and $p = \frac{3-\sqrt{5}}{2} = .381966$.

e) If p is very small, then the fact that A gets to toss first doesn't confer much of an advantage to A. However, since B tosses twice as often as A does, you would expect that the chance that B gets the first head is close to 2/3. Indeed, as q tends to 1:

$P(\text{A gets the first head}) = \frac{1}{1+q+q^2} \to \frac{1}{3}$,

$P(\text{B gets the first head}) \to \frac{2}{3}$.

8. a,b) Suppose that the player's point X_0 is $x = 4, 5, 6, 8, 9$, or 10. Then $P(\text{win}|X_0 = x)$ is the probability that in repeated throws of a pair of dice, the sum x appears before the sum 7 does. In the notation of the Example 2, imagine that A and B are repeating, independently, a competition consisting of the throw of a pair of dice. On each throw, if x appears, then A wins; if 7 appears, then B wins; otherwise a draw results. The desired probability is the chance that A wins before B does, which, by the craps principle, is $\frac{P(x)}{P(x)+P(7)}$.

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
$P(\text{win} \mid X_0 = x)$	0	0	3/9	4/10	5/11	1	5/11	4/10	3/9	1	0

c) $P(\text{win}) = \sum_{x=2}^{12} P(\text{win}|X_0 = x)P(X_0 = x)$

$= 0 \cdot \frac{4}{36} + \frac{3}{9} \cdot \frac{3}{36} \cdot 2 + \frac{4}{10} \cdot \frac{4}{36} \cdot 2 + \frac{5}{11} \cdot \frac{5}{36} \cdot 2 + 1 \cdot \frac{6}{36} + 1 \cdot \frac{2}{36}$

$= \frac{0+220+352+500+660+220}{36 \times 10 \times 11} = \frac{1952}{36 \times 10 \times 11}$

9. Let X be the payoff. Then

$P(X = n^2) = (\frac{1}{2})^n$ for $n = 1, 2, \ldots,$.

That is, $X = W^2$, where W has geometric (p) distribution on $\{1, 2, \ldots\}$ for $p = 1/2$. Since W has mean $1/p = 2$ and variance $q/p^2 = 2$,

$E(X) = E(W^2) = Var(W) + [E(W)]^2 = 6$.

Your net gain is given by $X - 10$, so $E(\text{net gain}) = -4$. That is, you are going to lose \$4 per game in the long run.

10. a) Condition on the first outcome and use the rule of average conditional probabilities: Let S = (first trial results in success), F = (first trial results in failure). Then

$P(X = n) = P(X = n|S)P(S) + P(X = n|F)P(F)$

Note that

$P(X = n|S) = P(W_F = n-1) = p^{n-2}q,\ n \geq 2,$

where W_F denotes the number of tosses until the first failure, which has a geometric distribution on $\{1, 2, ...\}$ with parameter q. Similarly,

$P(X = n|F) = P(W_S = n - 1) = q^{n-2}p,\ n \geq 2,$

Therefore

$P(X = n) = p^{n-1}q + q^{n-1}p$ for $n = 2, 3, ...$

b) Duplicate the method for finding the moments of the geometric distribution: $E(X) = \sum_{n=2}^{\infty} np^{n-1}q + \sum_{n=2}^{\infty} nq^{n-1}p$

$= q\left(\sum(1,p) - 1\right) + p\left(\sum(1,q) - 1\right)$

$= q\left(\frac{1}{(1-p)^2} - 1\right) + p\left(\frac{1}{(1-q)^2} - 1\right)$

$= \frac{1}{q} + \frac{1}{p} - 1$, where $\sum(1,p) = \sum_{n=1}^{\infty} np^{n-1}$.

c) Similarly $E(X^2) = \sum_{n=2}^{\infty} n^2p^{n-1}q + \sum_{n=2}^{\infty} n^2q^{n-1}p$

$= q\left(\sum(2,p) - 1\right) + p\left(\sum(2,q) - 1\right)$

$= q\left(\frac{1+p}{(1-p)^3} - 1\right) + p\left(\frac{1+q}{(1-q)^3} - 1\right)$

$= \frac{1+p}{q^2} + \frac{1+q}{p^2} - 1$.

Finally, use $Var(X) = E(X^2) - [E(X)]^2$.

11. Write $q_A = 1 - p_A,\ q_B = 1 - p_B$.

a) $P(\text{A wins})$

$= P(\text{A tosses H, B tosses T}) + P(\text{A tosses TH, B tosses TT}) + P(\text{A tosses TTH, B tosses TTT}) + \dots$

$= p_Aq_B + (q_Aq_B)p_Aq_B + (q_Aq_B)^2p_Aq_B + \dots$

$= \frac{p_Aq_B}{1-q_Aq_B}$.

b) $P(\text{B wins}) = \frac{q_Ap_B}{1-q_Aq_B}$. (Interchange A and B.)

c) $P(\text{draw}) = 1 - P(\text{A wins}) - P(\text{B wins}) = \frac{p_Ap_B}{1-q_Aq_B}$.

d) Let N be the number of trials (in which A and B both toss) required until at least one H is seen (by either A or B). N has range $\{1, 2, 3, ...\}$. For $k = 1, 2, 3, ...$

$$\begin{aligned} P(N = k) &= P(\text{first } k - 1 \text{ trials see TT, } k\text{th trial does not see TT}) \\ &= (q_Aq_B)^{k-1}(1 - q_Aq_B). \end{aligned}$$

Or: Each trial is a Bernoulli trial with success corresponding to the event (at least one H is seen among the tosses of A and B). The probability of success on each trial is then $1 - P(\text{no H is seen}) = 1 - q_Aq_B$. Therefore N, the waiting time until the first success, is geometric with parameter $(1 - q_Aq_B)$.

12. Write $q_1 = 1 - p_1, q_2 = 1 - p_2$.

a) $P(W_1 = W_2) = \sum_{k=1}^{\infty} P(W_1 = k, W_2 = k)$

$$= \sum_{k=1}^{\infty} P(W_1 = k)P(W_2 = k) = \sum_{k=1}^{\infty} q_1^{k-1}p_1q_2^{k-1}p_2 = \frac{p_1p_2}{1 - q_1q_2} = \frac{p_1p_2}{p_1 + p_2 - p_1p_2}$$

b) $P(W_1 < W_2) = \sum_{k=1}^{\infty} P(W_1 = k, W_2 < W_2)$

$= \sum_{k=1}^{\infty} P(W_1 = k, W_2 > k) = \sum_{k=1}^{\infty} q_1^{k-1}p_1q_2^k = \frac{p_1q_2}{1-q_1q_2}$

c) By symmetry, it's $\frac{p_2q_1}{1-q_2q_1}$. Check: (a) + (b) + (c) = 1.

d) Put $X = \min(W_1, W_2)$. For $k = 0, 1, 2, ...$ we have

$$P(X > k) = P(W_1 > k \text{ and } W_2 > k) = P(W_1 > k)P(W_2 > k) = q_1^kq_2^k = (q_1q_2)^k.$$

So X is geometric with parameter $1 - q_1q_2 = p_1 + p_2 - p_1p_2$.

c) Put $Y = \max(W_1, W_2)$. Y has range $\{1, 2, 3, ...\}$. For $n = 0, 1, 2, ...$ we have

$$P(Y \le n) = P(W_1 \le n \text{ and } W_2 \le n) = P(W_1 \le n)P(W_2 \le n)$$
$$= [1 - P(W_1 > n)][1 - P(W_2 > n)] = (1 - q_1^n)(1 - q_2^n).$$

For $n = 1, 2, 3, ...$ we then have

$$P(Y = n) = P(Y \le n) - P(Y \le n-1) = (1 - q_1^n)(1 - q_2^n) - (1 - q_1^{n-1})(1 - q_2^{n-1}).$$

13. a) Let $q = 1 - p$. Condition on the first two trials, and use recursion, as follows: Let B = (first draw is black), and WB = (first draw is white, second is black). Then

$P(\text{Black wins}) = P(\text{Black wins}|B) \times P(B) + P(\text{Black wins}|WB) \times P(WB)$
$= 1 \times p + P(\text{Black wins}) \times qp$

Therefore

$P(\text{Black wins}) = \frac{p}{1-qp}$

$P(\text{White wins}) = 1 - P(\text{Black wins}) = \frac{q^2}{1-qp}$.

Or use the craps principle: You can imagine that Black and White are repeating over and over, independently, a competition which consists of each player drawing *once* at random with replacement from the box. This competition results in a "win" for Black with probability p, and a "win" for White with probability q^2. The competition is repeated until one player "wins"; the chance that Black "wins" before White does is $p/(p + q^2)$, which is equal to the previously calculated chance.

b) Set $P(\text{Black wins}) = 1/2$. Solving the quadratic equation gives $p = (3 - \sqrt{5})/2 = 0.381966$. (The other root is greater than 1.)

c) No, $\frac{3-\sqrt{5}}{2}$ is irrational.

d) No player has more than a 51% chance of winning if and only if

$$.49 \le P(\text{Black wins}) \le .51$$

This happens if and only if

$$.375139206... \le p \le .388805725...$$

Reason: Observe that $f(p) = p/[1 - (1-p)p]$ is an increasing function of $p \in [0, 1]$. If $0 < c < 1$, then the equation $f(p^*) = c$ has one solution, namely $p^* = \left(c + 1 - \sqrt{(c+1)^2 - 4c^2}\right)/2c$.

Calculate all possible values of p for each value of $b + w$. The smallest value of $b + w$ for which p lies in the above range is $b + w = 13$ (with $b = 5$, $p = 5/13 = .384615$).

14. Let $n \ge 1$. V_n is a random variable having range $\{n, \dots, 2n-1\}$. For $k = n, \dots, 2n-1$ we have

$(V_n = k) = (V_n = k, k\text{th trial is success}) \cup (V_n = k\ , k\text{th trial is failure})$

The two events on the right are mutually exclusive. The first event is

$(V_n = k, k\text{th trial is success}) = (\text{exactly } n-1 \text{ successes in first } k-1 \text{ trials, } k\text{th trial is success})$

and has probability

$P(\text{exactly } n-1 \text{ successes in first } k-1 \text{ rials, } k\text{th trial is success})$

$= P(\text{exactly } n-1 \text{ successes in first } k-1 \text{ trials})P(k\text{th trial is success})$

$= \binom{k-1}{n-1}p^{n-1}q^{k-n} \cdot p = \binom{k-1}{n-1}p^n q^{k-n}$. Similarly the second event has probability $\binom{k-1}{n-1}q^n p^{k-n}$. Hence

$P(V_n = k) = \binom{k-1}{n-1}(p^n q^{k-n} + q^n p^{k-n})\ ,\ k = n, \dots, 2n-1.$

15. a) Let $k \ge 0$ and $m \ge 0$. Use problem 1:

$P(F - k = m|F \ge k) = \frac{P(F=m+k)}{P(F\ge k)} = \frac{q^{m+k}p}{q^k} = q^m p = P(F = m)$

b) Let F assume the values $0, 1, 2, \ldots$ with probabilities $p_0, p_1, p_2, \ldots$. We claim that $p_k = (1-p_0)^k p_0$.

To prove this claim, introduce the tail probabilities

$t_k = P(F \geq k) = p_k + p_{k+1} + p_{k+2} + \ldots$.

Put $m = 0$ in the Property to see

$P(F = k | F \geq k) = P(F = 0)$ for all $k \geq 0$,

which is equivalent to

$\frac{p_k}{t_k} = p_0$ for all $k \geq 0$.

Use $p_k = t_k - t_{k+1}$ and $1 - p_0 = t_1$ to get

$\frac{t_{k+1}}{t_k} = t_1$ for all $k \geq 0$.

This leads to the solution

$t_k = (t_1)^k$ for all $k \geq 0$

and

$p_k = (t_1)^k p_0 = (1-p_0)^k p_0$, for all $k \geq 0$,

as claimed.

16. a) If $k \geq 1$ then

$\frac{P(k)}{P(k-1)} = \frac{\binom{k+r-1}{r-1} p^r (1-p)^k}{\binom{k+r-2}{r-1} p^r (1-p)^{k-1}} = \frac{k+r-1}{kq}$.

b,c) If m is a mode, then $m \geq 0$ and

$\frac{P(m+1)}{P(m)} \leq 1$ and $\frac{P(m)}{P(m-1)} \geq 1$.

It follows that

$(r-1)\frac{q}{p} - 1 \leq m \leq (r-1)\frac{q}{p}$.

So if $(r-1)\frac{q}{p}$ is not an integer, then there is a unique mode, namely $int\left((r-1)\frac{q}{p}\right)$. If $(r-1)\frac{q}{p}$ is zero, then there is a unique mode, namely zero. If $(r-1)\frac{q}{p}$ is an integer greater than 0, then there are two modes, namely $(r-1)\frac{q}{p}$ and $(r-1)\frac{q}{p} - 1$.

17. Let X be the number of boys in a family and Y the number of children in a family. Observe that given $Y = n$, X has binomial$(n, p = 1/2)$ distribution; therefore

$$P(X = k) = \sum_{n=k}^{\infty} P(X = k | Y = n) P(Y = n) = \sum_{n=k}^{\infty} \binom{n}{k} (1/2)^n p^n (1-p).$$

Set $j = n - k$:

$$P(X = k) = \frac{(1-p)p^k}{2^k} \sum_{j=0}^{\infty} \binom{k+j}{k} (p/2)^j = \frac{(1-p)p^k}{2^k} \sum_{j=0}^{\infty} \binom{k+j}{j} (p/2)^j$$

Set $s = k + 1$ and multiply and divide by $\left(1 - \frac{p}{2}\right)^s$:

$$P(X = k) = \frac{(1-p)p^k}{2^k} \left(1 - \frac{p}{2}\right)^{-k-1} \sum_{j=0}^{\infty} \binom{s-1+j}{j} (p/2)^j \left(1 - \frac{p}{2}\right)^s = \frac{2(1-p)p^k}{(2-p)^{k+1}} \sum_{j=0}^{\infty} P(T_s = j)$$

where T_s denotes the number of failures before the sth success in Bernoulli$(p/2)$ trials. Thus the last sum equals 1, and

$$P(X = k) = \frac{2(1-p)p^k}{(2-p)^{k+1}}, \quad k \geq 0.$$

18. a) $P(G = n) = \begin{cases} (p^2 + q^2)(2pq)^{\frac{n}{2}-1} & n = 2, 4, \ldots \\ 0 & \text{otherwise} \end{cases}$

b) $G = 2X$ where X has geometric (p^2+q^2) distribution on $\{1, 2, ...\}$. So $E(G) = 2E(X) = \frac{2}{p^2+q^2}$.

c) $Var(G) = 4Var(X) = \frac{8pq}{(p^2+q^2)^2}$.

19. a) r plus expectation of negative binomial.

b) Let S_n = number of heads in n tosses.

$$\begin{aligned} P(T_r < 2r) &= P(T_r \leq 2r-1) \\ &= P(S_{2r-1} \geq r) = 1/2 \end{aligned}$$

by symmetry about $r - 1/2$ of the binomial $(2r-1, 1/2)$ distribution.

c) From b)

$$\frac{1}{2} = P(T_r < 2r) = P(T_r - r < r) = \sum_{i=0}^{r-1} \binom{i+r-1}{r-1} (1/2)^r (1/2)^i,$$

which gives c) with $n = r - 1$.

20. a)

$$\sum_{n=1}^{\infty} P(X \geq n) = \sum_{n=1}^{\infty}\sum_{x=n}^{\infty} P(X=x) = \sum_{x=1}^{\infty}\sum_{n=1}^{x} P(X=x) = \sum_{x=1}^{\infty} xP(X=x) = E(X)$$

b)

$$\sum_{n=1}^{\infty} nP(X \geq n) = \sum_{n=1}^{\infty} n \sum_{x=n}^{\infty} P(X=x) = \sum_{x=1}^{\infty} P(X=x)\sum_{n=1}^{x} n = \sum_{x=1}^{\infty} P(X=x)\frac{x(x+1)}{2} = E\left[\frac{X(X+1)}{2}\right]$$

c) $Var(X) = E(X^2) - [E(X)]^2 = 2\Sigma_2 - \Sigma_1 - \Sigma_1^2$

21. Let F_r be negative binomial (r, p) and N be Poisson(μ).

a) $E(F_r) = \frac{rq}{p} \to rq = \mu$ in the limit.

b) $Var(F_r) = \frac{rq}{p^2} \to rq = \mu$ in the limit.

c)

$$\begin{aligned} P(F_r = k) &= \binom{k+r-1}{r-1} p^r q^k \\ &= \frac{(k+r-1) \times \cdots \times r}{k!} p^r q^k \\ &\approx \frac{p^r}{k!}(rq)^k \\ &= (1-q)^{\frac{\mu}{q}} \frac{\mu^k}{k!} \\ &= \left((1-q)^{\frac{1}{q}}\right)^{\mu} \frac{\mu^k}{k!} \\ &\to e^{-\mu}\frac{\mu^k}{k!} \\ &= P(N = k) \end{aligned}$$

22. No Solution

23. No Solution

24. a) Let $N_1 = 1$, and for $i \geq 1$ let N_{i+1} be the additional number of trials (boxes) required t obtain an animal different from all previous. Then $T_n = N_1 + ... + N_n$, and N_i is a geometri ndom variable on { 1, 2, ... } with parameter $p_i = (n-i+1)/n$ ($i = 1, ..., n$). The variables N .. , N_n are independent, so

$$Var(T_n) = Var(\sum_{i=1}^{n} N_i) = \sum_{i=1}^{n} Var(N_i) = \sum_{i=1}^{n} \frac{q_i}{p_i^2}$$

$$= \sum_{i=1}^{n} \left(\frac{1}{p_i^2} - \frac{1}{p_i}\right) = \sum_{i=1}^{n} \left(\frac{n^2}{(n-i+1)^2} - \frac{n}{n-i+1}\right)$$

$$= \sum_{j=1}^{n} \frac{n^2}{j^2} - \sum_{j=1}^{n} \frac{n}{j} \qquad (j = n - i + 1)$$

Hence $\sigma_n = SD(T_n) = \left(n^2 \sum_{j=1}^{n} \frac{1}{j^2} - n \sum_{j=1}^{n} \frac{1}{j}\right)^{1/2}$.

b) Note that $\sigma_n^2 \leq n^2 \sum_{j=1}^{n} \frac{1}{j^2}$ for all n. The series $\sum_{j=1}^{\infty} \frac{1}{j^2}$ converges, so there exists a constant c ($0 < c < \infty$) such that $\sum_{i=1}^{n} \frac{1}{j^2} < c^2$ for all n. Hence $\sigma_n \leq nc$ for all n.

c) Chebychev's inequality states that T_n is unlikely to be more than a few standard deviations away from its expected value. Here $E(T_n) \approx n \log n$, and $SD(T_n) \leq cn$.

d) As $n \to \infty$,

$$P\left(\frac{T_n - n \log n}{n} \leq x\right) \to e^{-e^{-x}}.$$

See solution of 3.Rev.41 c) for the proof.

Section 3.5

1. The distribution is approximately binomial (200, .01), which is approximately Poisson (2). So the probability is approximately

$1 - e^{-2}\left[1 + 2 + \frac{2^2}{2} + \frac{2^3}{6}\right] = .1428.$

2. Suppose cookies contain λ raisins on average. $P(\text{cookie contains at least one raisin}) = 1 - e^{-\lambda}$, assuming a Poisson distribution for the number of raisins. We want λ so that $1 - e^{-\lambda} \geq .99$, or $\lambda \geq -\log(.01) \approx 4.6$. So an average of 5 raisins per cookie will do.

3. a) Use Poisson model. Let X be the number of raisins in a cookie. Then X has Poisson (4) distribution. So $P(X = 0) = e^{-4}$. Let N be the number of raisin-less cookies per bag. Then N has binomial $(12, e^{-4})$ distribution, which is approximately Poisson $(12 \times e^{-4})$. The long run proportion of complaint bags is

   $P(N > 0) \approx 1 - e^{-12e^{-4}} \approx 12e^{-4} \approx 0.222.$

   b) Proceed as in a). Let a be the average required. Want

   $12e^{-\frac{a \times 2}{16}} = 0.05.$

   Take logs to get $a \approx 44$.

4. Assume the number of misprints on each page has the Poisson (1) distribution. Then

$P(\text{more than 5 misprints per page}) = 1 - \sum_{x=0}^{4} \frac{e^{-1}}{x!} = .0037.$

Assume the number of misprints on any one page is independent of the number on the other pages. Then in a book of 300 pages, the number of pages having more than 5 misprints has the binomial (300, .0037) distribution, which can be approximated by the Poisson (300 x .0037) distribution. Therefore

$P(\text{at least one page contains more than 5 misprints}) = 1 - e^{-.0037 \times 300} = 1 - .33 = .67.$

5. Assume that N, the number of microbes in the viewing field, is Poisson with parameter λ. Since the average density in an area of 10^{-4} square inches is $5000 \times 10^{-4} = 0.5$ microbes, we have $\lambda = 0.5$ and

$P(N \geq 1) = 1 - P(N = 0) = 1 - e^{-\lambda} = 1 - e^{-0.5} = 0.39.$

6. Assume the number of drops falling in a given square inch in 1 minute is a Poisson random variable with mean 30. Then the number of drops falling in a given square inch in 10 seconds will be a Poisson random variable with mean 5. So

$$P(0) = e^{-5} = 0.00674$$

7. a) The number of fresh raisins per muffin has Poisson (3) distribution. The number of rotten raisins per muffin has Poisson (2) distribution. The total number of raisins per muffin has Poisson (5) distribution, assuming the number of fresh raisins per muffin is independent of the number of rotten ones.

   b) The number of raisins in 20% of a muffin has Poisson (1) distribution, so

$$P(\text{no raisins}) = e^{-1}\frac{1^0}{0!} = e^{-1} \approx .3679.$$

8. Once again assume that the number of pulses received by the Geiger counter in a given one minute period is a Poisson random variable with mean 10. Then the number received in a given half minute period will be a Poisson random variable with mean 5.

$$P(3) = \frac{e^{-5}5^3}{3!} = 0.1404$$

9. a) $P(X = 1, Y = 2) = P(X = 1)P(Y = 2) = e^{-1}\frac{e^{-2}2^2}{2!} = .09959.$

b) Note that $X+Y$ has Poisson (3) distribution. Therefore
$P((X+Y)/2 \geq 1) = P(X+Y > 2) = 1 - P(X+Y \leq 1) = 1 - (e^{-3} + e^{-3}3) = .8008$

c) $P(X=1|X+Y=4) = \binom{4}{1}\frac{1}{3}(\frac{2}{3})^3 = .3951$.

10. a) $E(3X+5) = 3 \times \lambda + 5$.

b) $Var(3X+5) = 9 \times \lambda$.

c) $E\left[\frac{1}{1+X}\right] = \sum_{x=0}^{\infty} \frac{1}{1+x} \frac{e^{-\lambda}\lambda^x}{x!}$
$= \frac{1}{\lambda}\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} = \frac{1}{\lambda}\left[-e^{-\lambda} + \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}\right] = \frac{1}{\lambda}(1-e^{-\lambda})$.

11. a) $X+Y$ is Poisson with mean 2, so $P(X+Y=4) = e^{-2}2^4/4!$

b) Again, $X+Y$ is Poisson with mean 2, so

$$E\left[(X+Y)^2\right] = Var(X+Y) + (E(X+Y))^2 = \mu + \mu^2 = 6$$

c) $X+Y+Z$ is Poisson with mean 3, so $P(X+Y+Z=4) = e^{-3}3^4/4!$

12. The total number of particles reaching the counter is the sum of two independent Poisson random variables, one with parameter 3.87, the other with parameter 5.41 (these numbers come from a famous experiment by Rutherford, Chadwick, and Ellis, in the 1920's). So the total number of particles reaching the counter follows the Poisson (9.28) distribution. So the required probability is

$e^{-9.28}\left[1 + 9.28 + \frac{(9.28)^2}{2} + \frac{(9.28)^3}{6} + \frac{(9.28)^4}{24}\right] = 0.04622$.

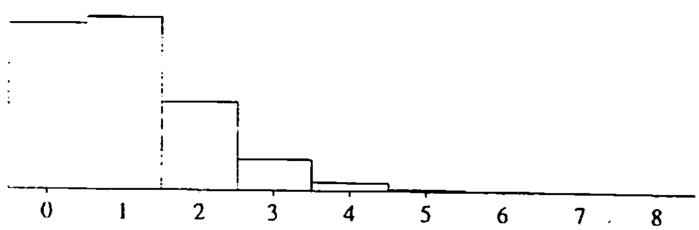
13. a) $\mu(x) = \frac{6.023 \times 10^{23}}{22.4 \times 10^3} \times x^3 \approx 2.69 \times 10^{19} x^3$

$\sigma(x) = \sqrt{\mu(x)} \approx 5.19 \times 10^9 \cdot x^{3/2}$

b) $\sigma(x) = \frac{\mu(x)}{100} \Longrightarrow x = 7.19 \times 10^{-6}$ cm.

14. a) Using the random scatter theorem, the number of tumors a single person gets in a week has approximately Poisson ($\lambda = 10^{-5}$) distribution. Since different people may be regarded as independent, and different weeks on the same person are also independent, the total number of tumors observed in the population over a year is a sum of 52×2000 independent Poisson(10^{-5}) random variables, which is a Poisson(1.04) random variable.

b) The number of tumors observed on a given person in a year will be a Poisson random variable with rate parameter $\lambda = 52 \times 10^{-5}$. Thus
$P(\text{at least one tumor}) = 1 - P(\text{none}) = 1 - e^{-0.00052} = 0.00051986$
Thus the number of people getting 1 or more tumors has distribution
binomial $(2000, 0.00051986) \approx$ Poisson $(2000 \times 0.00051986) \approx$ Poisson (1.04) .



The histograms are essentially the same: Poisson with parameter 1.04. The means and standard deviations are 1.04 and $\sqrt{1.04} \approx 1.02$ respectively.

Why are the answers to a) and b) nearly the same when they are counting different things? The reason is that the chance of a person getting 2 or more tumors is insignificant, even when compared with the chance of getting 1 tumor.

15. (a) The chance that a given pages has no mistakes is $\frac{e^{-.01}(.01)^0}{0!} = e^{-.01} = .99005$ and thus the expected number of pages with no mistakes is $200 \times .99005 = 198.01$. The number of pages with mistakes is distributed as a binomial(200, .99005), so the variance is $200 \times .99005 \times .00995 = 1.97$.

(b) The mistakes found on a given page is distributed as a Poisson(.009), so the chance that at least one mistake will be found on a given page is $1 - \frac{e^{-.009}(.009)^0}{0!} = e^{-.009} = .00896$. The expected number of pages on which at least one mistake is found is then $200 \times .00896 = 1.79$.

(c) The number of pages with mistakes can be well approximated as a Poisson(1.99) since 1.99 is the expected number of pages with mistakes. Let X = number of pages with mistakes and use the Poisson approximation to get

$$P(X \geq 2) = 1 - (P(X=0) + P(X=1)) = 1 - (e^{-1.99} + 1.99e^{-1.99}) = 0.59$$

16. a) Assume that chocolate chips are distributed in cookies according to a Poisson scatter. Let X be the number of chocolate chips in a three cubic inch cookie. Then X has Poisson (6) distribution, so

$P(X \leq 4) = \sum_{i=0}^{4} \frac{e^{-6}6^i}{i!} = .2851$.

b) Let Z_1, Z_2, and Z_3 denote the total number of goodies (either chocolate chips or marshmallows) in cookies 1, 2, and 3 respectively. Assume that marshmallows are distributed in cookies according to a Poisson scatter. By the independence assumption between marshmallows and chocolate chips, Z_1 has Poisson (6) distribution, Z_2 and Z_3 each have Poisson (9) distribution, and the Z_i's are independent. We have

$P(Z_1 = 0) = e^{-6}$, $P(Z_2 = 0) = P(Z_3 = 0) = e^{-9}$.

The complement of the desired event has probability

$P(Z_1 = Z_2 = Z_3 = 0) + P(Z_1 > 0, Z_2 = Z_3 = 0) + P(Z_2 > 0, Z_1 = Z_3 = 0) + P(Z_3 > 0, Z_1 = Z_2 = 0)$

$= e^{-6}(e^{-9})^2 + (1 - e^{-6})(e^{-9})^2 + 2(1 - e^{-9})e^{-6}e^{-9} = 6.27 \times 10^{-7}$

and the desired event has probability virtually 1.

17. Assuming the distribution of raindrops over a particular square inch during a given ten second period is a Poisson random scatter, then the number of drops hitting this square inch during the ten second period has Poisson($\lambda = 5$) distribution.

a) So $P(\text{no hit}) = e^{-5} = 0.006738$.

b) Argue that if N_1 denotes the number of big drops and N_2 the number of small, then N_1 and N_2 are independent and N_1 has Poisson($\frac{2}{3} \times 5$) distribution, N_2 has Poisson($\frac{1}{3} \times 5$) distribution. Hence

$$P(N_1 = 4, N_2 = 5) = P(N_1 = 4)P(N_2 = 5) = e^{-10/3}\frac{(10/3)^4}{4!}e^{-5/3}\frac{(5/3)^5}{5!} = 0.003714.$$

18. a) Let S_n denote the number of survivors between time n and $n+1$, and let I_n denote the number of immigrants between time n and $n+1$. Then $X_{n+1} = S_n + I_n$ where I_n has Poisson (μ) distribution, independent of S_n, and

$$P(S_n = k | X_n = x) = \binom{x}{k}(1-p)^k p^{x-k}, 0 \leq k \leq x.$$

Claim: X_n has Poisson $\left(\mu \sum_{k=0}^{n}(1-p)^k\right)$ distribution, $n = 0, 1, 2 \ldots$.
Proof. Induction. True for $n = 0$. If the claim holds for $n = m$, where $m \geq 0$, then, putting $\lambda = \mu \sum_{k=0}^{m}(1-p)^k$, we have for all $k \geq 0$

$$P(S_m = k) = \sum_{x=0}^{\infty} P(S_m = k | X_m = x)P(X_m = x)$$

$$= \sum_{x=k}^{\infty} \binom{x}{k}(1-p)^k p^{x-k} e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= \frac{e^{-\lambda}[\lambda(1-p)]^k}{k!} \sum_{x=k}^{\infty} \frac{(\lambda p)^{x-k}}{(x-k)!} = e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^k}{k!}$$

so S_m has Poisson distribution with parameter $\lambda(1-p) = \mu \sum_0^m (1-p)^{k+1}$. Since I_m is independent of S_m and has Poisson (μ) distribution, it follows that $X_{m+1} = S_m + I_m$ has Poisson distribution with parameter $\mu + \mu \sum_{k=0}^{m} (1-p)^{k+1} = \mu \sum_{k=0}^{m+1} (1-p)^k$. So claim holds for $n = m + 1$.

b) As $n \to \infty$, $\mu \sum_0^n (1-p)^k \to \frac{\mu}{p}$, so the distribution tends to the Poisson ($\frac{\mu}{p}$) distribution.

19. a) $G(z) = \sum_{i=0}^{\infty} p_i z^i = \sum_{i=0}^{\infty} e^{-\mu} \frac{\mu^i}{i!} z^i = e^{-\mu} e^{\mu z} = e^{-\mu(1-z)}$

b) $G'(z) = \mu e^{-\mu+\mu z}$, $\quad G''(z) = \mu^2 e^{-\mu+\mu z}$, $\quad G'''(z) = \mu^3 e^{-\mu+\mu z}$.
So $E(X) = \mu$, $\quad E(X(X-1)) = \mu^2$, $\quad E(X(X-1)(X-2)) = \mu^3$

c) $E(X^2) = \mu^2 + \mu$, $\quad E(X^3) = \mu^3 + 3(\mu^2 + \mu) - 2\mu = \mu^3 + 3\mu^2 + \mu$

d) $E[(X-\mu)^3] = E(X^3) - 3E(X^2)\mu + 3E(X)\mu^2 - \mu^3 = \mu$.
Skewness $(X) = \frac{\mu}{\mu^{3/2}} = \frac{1}{\sqrt{\mu}}$

20. No Solution

21. c) 0.58304 $\quad$ d) 0.5628 $\quad$ e) 0.58306

Section 3.6

1. a) 1/13

   b) 4/50

   c) $4 \times \frac{\binom{13}{5}}{\binom{52}{5}}$

   d) $1 - \frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}} - \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$

2. a) 1/13 b) 1/4 c) $(13 \times 12 \times 11 \times 10 \times 9)/(52 \times 51 \times 50 \times 49 \times 48)$ d) $(48 \times 47 \times 46 \times 45 \times 4)/(52 \times 51 \times 50 \times 49 \times 48)$

3. a) 8/47 b) $(12 \times 11 \times 10 \times 9 \times 8)/(51 \times 50 \times 49 \times 48 \times 47)$ c) 1/4 d) 1/13 e) 1/13 f) 1/4

4. a)

n	1	2	3	4
$P(T_1 = n)$	0.4	0.3	0.2	0.1

   b)

n	1	2	3	4
$P(6 - T_2 = n)$	0.4	0.3	0.2	0.1

   c)

n	2	3	4	5
$P(T_2 = n)$	0.1	0.2	0.3	0.4

   d) T_1

T_2 \ T_1	1	2	3	4
2	0.1	0	0	0
3	0.1	0.1	0	0
4	0.1	0.1	0.1	0
5	0.1	0.1	0.1	0.1

   e) $T_1 = 1+$ number of non-defectives before first defective, $T_2 - T_1 = 1+$ number of non-defectives between first and second defective, and $6 - T_2 = 1+$ number of non-defectives after second defective. $P(T_1 = n_1, T_2 - T_1 = n_2, 6 - T_2 = n_3) = .1$ where it is not zero, which is a symmetric function, so the random variables are exchangeable.

   f) By e), $T_2 - T_1$ has the same distribution as T_1, so see a).

5. $P(\text{one fixed box empty}) = \left(\frac{b-1}{b}\right)^n$ so $EX = b\left(\frac{b-1}{b}\right)^n$

$$EX^2 = b\left(\frac{b-1}{b}\right)^n + b(b-1)\left(\frac{b-2}{b}\right)^n$$

So

$$\begin{aligned} Var(X) &= EX^2 - (EX)^2 \\ &= b\left(\frac{b-1}{b}\right)^n + b(b-1)\left(\frac{b-2}{b}\right)^n - b^2\left(\frac{b-1}{b}\right)^{2n} \end{aligned}$$

6. a) Consider $M = \sum_{i=1}^n I_i$ where I_i is the indicator of the ith ball being in the ith box. Thus $E(M) = nE(I_i) = 1$.

   b)

$$\begin{aligned} E(M^2) &= E\left(\left(\sum_{i=1}^n I_i\right)^2\right) \\ &= E\left(\sum_{i=1}^n I_i^2\right) + 2E\left(\sum_{i<j} I_i I_j\right) \end{aligned}$$

$$= 1 + 2\binom{n}{2}\frac{1}{n}\frac{1}{n-1}$$
$$= 2$$

Thus $SD(M) = \sqrt{2-1} = 1$

c) For large n, the distribution of M is approximately Poisson(1). Intuitively, the distribution is very much like a binomial$(n, \frac{1}{n})$ except for the dependence between the draws, but as the number of draws gets large the dependence between draws becomes small, and the Poisson(1) becomes a good approximation.

7. a) $n \cdot \frac{26}{52}$

b) $\left(\frac{52-n}{52-1}\right) \cdot n \cdot \frac{26}{52} \cdot \frac{26}{52}$

8. a) $P(X = k) = \frac{\binom{26}{k}\binom{26}{26-k}}{\binom{52}{26}}, \qquad k = 0, 1, 2, \ldots, 26.$

b) $E(X) = 26 \cdot \frac{1}{2} = 13.$

c) $SD(X) = \sqrt{\frac{52-26}{52-1}}\sqrt{26 \times \frac{1}{2} \times \frac{1}{2}} = 1.82.$

d) $P(X \geq 15) = P(X \geq 14.5) \approx 1 - \Phi(.824) = .2061.$

The normal approximation gives a fairly good answer. You can check that

$$P(X = 13) = .21812, P(X = 14) = .18807;$$

therefore the exact answer is, by symmetry,

$$\frac{1 - (.21812 + 2 \times .18807)}{2} = .2029.$$

9. Let $b_1, b_2, \ldots, b_B$ denote the B bad elements in the population, and define

$$I_i = \begin{cases} 1 & \text{if bad element } b_i \text{ appears before the first good element} \\ 0 & \text{otherwise.} \end{cases}$$

Then $X - 1 = \#$ of bad elements before the first good element $= I_1 + I_2 + \cdots + I_B$.

a) $E(X-1) = E(I_1) + E(I_2) + \cdots + E(I_B)$. By symmetry, all the expectations on the right hand side are equal, and equal to $P(b_1$ appears before first good element$)$. This probability equals $1/(G+1)$:

Let $g_1, g_2, \ldots, g_G$ denote the G good elements. Consider the $G+1$ elements g_1, g_2, ..., g_G, b_1. We are interested in the position of b_1 relative to the g's. We can choose this position in $G+1$ ways, all equally likely. Exactly one of these choices puts b_1 before all the g's. So $P(b_1$ appears before first good element$) = 1/(G+1)$.

Conclude:

$$E(X) = E(X-1) + 1 = \frac{B}{G+1} + 1 = \frac{B+G+1}{G+1} = \frac{N+1}{G+1}.$$

b) We have $Var(X) = Var(X-1) = E\left[(X-1)^2\right] - [E(X-1)]^2$. Now

$$E\left[(X-1)^2\right] = E\left[(I_1 + I_2 + \cdots + I_B)^2\right]$$
$$= \sum_{i=1}^{B} E(I_i^2) + \sum\sum\nolimits_{i \neq j} E(I_i I_j)$$
$$= B \cdot \frac{1}{G+1} + B(B-1)E(I_1 I_2)$$

by symmetry and the fact that $I_i{}^2 = I_i$; and

$$E(I_1 I_2) = P(b_1 \text{ and } b_2 \text{ both appear before the first red card}) = \frac{1}{\binom{G+2}{2}}.$$

(Reasoning is similar to above: Consider the positions of b_1 and b_2 relative to the g's. There are $G+2$ positions to fill, and we can choose the pair of positions for b_1 and b_2 in $\binom{G+2}{2}$ ways. Only the pair $(1,2)$ gives the event we want.)

So

$$\begin{aligned}
Var(X) &= \frac{B}{G+1} + B(B-1)\frac{1}{\binom{G+2}{2}} - \left(\frac{B}{G+1}\right)^2 \\
&= \frac{B}{G+1}\left[1 + \frac{2(B-1)}{G+2} - \frac{B}{G+1}\right] \\
&= \frac{B}{G+1}\left[\frac{G^2+3G+2+2BG+2B-2G-2-BG-2B}{(G+2)(G+1)}\right] \\
&= \frac{B}{G+1}\left(\frac{G^2+G+BG}{(G+2)(G+1)}\right) = \frac{BG(N+1)}{(G+1)^2(G+2)}
\end{aligned}$$

and

$$SD(X) = \sqrt{\frac{BG(N+1)}{(G+1)^2(G+2)}}.$$

10. No Solution

11. a) $P(x_1, \ldots, x_n) = 1/\binom{n}{g}$ if $x_1 + \cdots + x_n = g$ and 0 otherwise
 b) no c) yes

12. a) There are $\binom{N}{n}$ possible ways to draw n numbers from the set $\{1 \cdots N\}$. Thus the process of taking a simple unordered random sample from this set gives the chance of any given subset of n numbers as $\frac{n!(N-n)!}{N!}$. If we consider the process of the exhaustive sample, where the X_i 's are the draws in order, the chance of getting any particular ordering is, for example,

$$P(X_1 = B, X_2 = G, X_3 = B, \cdots, X_N = B) = \frac{N-n}{N}\frac{n}{N-1}\frac{N-n-1}{N-2}\cdots\frac{1}{1}$$

where the ordering of the numerators may be different but the product must be equal to $\frac{(N-n)!n!}{N!}$ which is the same as it was for the sample of size n from $\{1 \cdots N\}$.

 b) This is essentially done in the previous problem;

$$\begin{aligned}
P(T_1 = t_1, \ldots, T_n = t_n) &= \frac{(N-n)(N-n-1)\cdots(N-n-t_1+2)(n)\cdots}{N!} \\
&= \frac{n!(N-n)!}{N!}
\end{aligned}$$

 c) The object here is to count the number of samples have $i-1$ elements less than t, t, and $n-i$ elements bigger than t. This number is $\binom{t-1}{i-1}\binom{N-t}{n-i}$ and the total number of samples is $\binom{N}{n}$, so the chance is $\frac{\binom{t-1}{i-1}\binom{N-t}{n-i}}{\binom{N}{n}}$.

 d) Given D, writing out the probability of any given sequence of $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ gives the same formula as that in part b). $P(D) = \frac{N-1}{N}\frac{N-2}{N}\cdots\frac{N-n+1}{N}$ so $P(D) > \left(\frac{N-n}{N}\right)^n \to 1$ as $N \to \infty$ for n fixed.

13. a) uniform over ordered $(n+1)$−tuples of non-negative integers that sum to $N-n$.

 c) $\frac{\binom{N-n}{w}}{\binom{N}{w}} \cdot \frac{n}{N-w}$

 d) $E(W_i) = (N-n)\frac{1}{n+1}$, $E(T_i) = i((N-n)\frac{1}{n+1} + 1)$. For $N = 52$ and $n = 4$, $E(W_i) = 9.6$, $E(T_1) = 10.6$, $E(T_2) = 21.2$, $E(T_3) = 31.8$, $E(T_4) = 42.4$

 e) $P(W_1 + W_2 = t) = P(T_2 = t+2) = \frac{\binom{t}{1}\binom{N-n}{t}}{\binom{N}{t+1}} \cdot \frac{n-1}{N-t-1}$, $0 \le t < N$ (for $t = N$, prob is 0 except in the trivial case $n = 0$ when prob. is 1).

f) Because the W_i are exchangable,
D_n has the same distribution as $N - (W_1 + W_2) - 2$.
So $P(D_n = d) = P(W_1 + W_{n+1} = N - 2 - d)$. Now use e).
$E(D_n) = E(T_n) - E(T_1) - 1 = (n-1)(\frac{N-n}{n+1} + 1) - 1$

14. No Solution

15. a) W_2 through W_n are spacing between two aces (W_1 and W_{n+1} are not). To get two consecutive aces, at least one of $W_1, W_2, \ldots, W_n$ must be 0. The expression in the problem $= 1 - P$(not all $w_i = 0$, $i = 2, \ldots, n$).

b) Put down N places in a row, and color t of them blue, the rest red. $\binom{N-t}{n}$ is the number of ways to place the good elements avoiding all the blue places. There is a 1-1 correspondence between such choices and ways to make ($W_i \geq t_i$ for each i).

c) $t_1 = 0 = t_{n+1}$, $t_2 = t_3 = \ldots = t_n = 1$. So $t = n - 1$.

16. No Solution

Chapter 3: Review

1. a) $1-(5/6)^{10}$ b) $10/6$ c) 35 d) $\frac{\binom{4}{2}\binom{6}{3}}{\binom{10}{5}} = \frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}}$

   e) $\frac{1}{2}(1 - P(\text{same number of sixes in first five rolls as in second five rolls}))$
   $= \frac{1}{2}\left\{1 - \sum_{k=0}^{5}\left[\binom{5}{k}(1/6)^k(5/6)^{5-k}\right]^2\right\}$

2. a)

$$P(\text{first 6 before tenth roll}) = P(\text{at least one 6 in first 9 rolls}) = 1-(5/6)^9 = .806194$$

   b)

$$P(\text{third 6 on tenth roll}) = P(\text{two sixes on first nine rolls, 6 on tenth roll})$$
$$= \binom{9}{2}(1/6)^2(5/6)^7(1/6) = .046507$$

   c)

$$P(\text{three 6's in first ten rolls} \mid \text{six 6's in first twenty rolls})$$
$$= P(\text{three 6's in first ten rolls, three sixes in last ten rolls})\big/P(\text{six 6's in first twenty rolls})$$
$$= \frac{\binom{10}{3}(1/6)^3(5/6)^7\binom{10}{3}(1/6)^3(5/6)^7}{\binom{20}{6}(1/6)^6(5/6)^{14}} = \frac{\binom{10}{3}^2}{\binom{10}{6}} = .371517.$$

   d) Want the expectation of the sum of six geometric $(1/6)$ random variables, each of which has expectation 6. So answer: 36.

   e) Coupon collector's problem: the required number of rolls is 1 plus a geometric $(5/6)$ plus a geometric $(4/6)$ plus etc. up to a geometric $(1/6)$, so the expectation is

$$1 + (6/5) + (6/4) + (6/3) + (6/2) + (6/1) = 14.7.$$

3. $X : \max(D_1, D_2)$ $Y = \min(D_1, D_2)$

$$P(X=x) = P(X \le x) - P(X \le x-1) = \left(\frac{x}{6}\right)^2 - \left(\frac{x-1}{6}\right)^2 = \frac{2x-1}{36} \quad (x = 1, \ldots, 6)$$

$$P(Y=y, X=3) = \begin{cases} \frac{2}{36} & \text{for } y = 1,2 \\ \frac{1}{36} & \text{for } y = 3 \\ 0 & \text{else} \end{cases}$$

$$\text{So } P(Y=y \mid X=3) = \begin{cases} \frac{2}{5} & \text{for } y = 1,2 \\ \frac{1}{5} & \text{for } y = 3 \\ 0 & \text{else} \end{cases}$$

$$P(X=x \mid Y=y) = \begin{cases} \frac{2}{36} & \text{for } 1 \le y < x \le 6 \\ \frac{1}{36} & \text{for } 1 \le y \le x \le 6 \\ 0 & \text{else} \end{cases}$$

$E(X+Y) = E(D_1 + D_2) = 7$

4. Use the fact that S has the same distribution as $200 - S$.

a) Use the convolution formula:

$$P(S=n) = P(X+Y=n) = \sum_{all\,x} P(X=x)P(Y=n-x).$$

If $0 \le n \le 100$, then

$$P(S=n) = \sum_{x=0}^{n} P(X=x)P(Y=n-x) = \sum_{x=0}^{n} \frac{1}{101} \times \frac{1}{101} = \frac{n+}{1\ldots}$$

If $100 < n \le 200$, then $0 \le 200-n \le 100$ so by the previous case

$$P(S=n) = P(200-S=200-n) = \frac{201-n}{101^2}.$$

b) $P(S \le n) = \sum_{k=0}^{n} P(S=k)$. If $0 \le n \le 100$ then

$$P(S \le n) = \sum_{k=0}^{n} \frac{k+1}{101^2} = \sum_{j=1}^{n+1} \frac{j}{101^2} = \frac{(n+1)(n+2)}{2 \cdot 101^2}.$$

The above formula works for $-2 \le n \le 100$. If $100 \le n \le 200$ then $-1 \le 199-n \le 100$ so by the previous case

$$\begin{aligned} P(S \le n) = 1 - P(S \ge n+1) &= 1 - P(200-S \le 199-n) \\ &= 1 - P(S \le 199-n) \\ &= 1 - \frac{(199-n+1)(199-n+2)}{2 \cdot 101^2} \\ &= 1 - \frac{(200-n)(201-n)}{2 \cdot 101^2}. \end{aligned}$$

5. a) Let G = gain on one spin

$$EG = \sum_{i=1}^{6} i\frac{18}{38} \cdot \frac{1}{6} - \sum_{i=1}^{6} i\frac{20}{38} \cdot \frac{1}{6}$$

$$= -\frac{1}{38 \cdot 6} \sum_{i=1}^{6} 2i = -\frac{6 \times 7}{38 \times 6} = -0.1842$$

b) $\frac{38}{18} = 2.111.$

c) $\frac{1}{\frac{1}{6} \cdot \frac{18}{38}} = 12.666$

6. a) Let Y be the number of times the gambler wins in 50 plays. Then Y has binomial (50, 9/19) distribution, and

$P(\text{ahead after 50 plays}) = P(Y > 25) = \sum_{i=26}^{50} \binom{50}{i}(9/19)^i(10/19)^{50-i}$.

Note that the gambler's capital, in dollars, after 50 plays is $100 + 10Y + (-10)(50-Y) = 20Y - 400$. So

$P(\text{not in debt after 50 plays}) = P(20Y - 400 > 0) = P(Y > 20) = \sum_{i=21}^{50} \binom{50}{i}(9/19)^i(10/19)^{50-i}$.

b) $E(\text{capital}) = E(20Y-400) = 20E(Y) - 400 = 20(50)(9/19) - 400 = 1400/19$;

$Var(\text{capital}) = Var(20Y-400) = 400Var(Y) = 400(50)(9/19)(10/19) = 1800000/19$

c) Use the normal approximation to the binomial: $E(Y) = 23.7$, and $SD(Y) = 3.53$, so

$P(Y > 25) \approx 1 - \Phi\left(\frac{25.5-23.7}{3.53}\right) = 1 - \Phi(.51) = .305$.

$P(Y > 20) \approx 1 - \Phi\left(\frac{20.5-23.7}{3.53}\right) = 1 - \Phi(-.91) = .8186$.

7. a) $P(\ge 4 \text{ heads in 5 tosses}) = \frac{6}{32} = 0.1875$

b) $P(0, 1, \text{ or } 2 \text{ heads in 5 tosses}) = \frac{16}{32} = 0.5$

c)

poss. vals	0	1	2	
probs	$\frac{26}{32}$	$\frac{5}{32}$	$\frac{1}{32}$	$EX = 0.21875$

8. a) Let $X = D_1 + \cdots + D_b$, where D_1 is the number of balls until the first black ball, D_2 is the number of balls drawn after the first black until the second black ball, and so on. When drawing with replacement, the D_i are independent geometric($\frac{b}{w+b}$) random variables, so X has the negative binomial(b, $\frac{b}{w+b}$) distribution on $b, b+1, \ldots$.

$$P(X = k) = \binom{k-1}{b-1}\left(\frac{b}{b+w}\right)^b\left(\frac{w}{b+w}\right)^{k-b} \qquad (k \geq b)$$

b) When drawing without replacement, it is possible to draw the bth black ball on any draw from b to $b + w$.

$$\begin{aligned} P(X = k) &= P(\text{b-1 blacks in first k-1 draws}) \\ &\quad \times P(k\text{th draw is black} \mid \text{b-1 blacks in first k-1 draws}) \\ &= \frac{\binom{b}{b-1}\binom{w}{k-b}}{\binom{b+w}{k-1}} \times \frac{1}{b+w-k+1} \\ &= \frac{bw!(k-1)!}{(k-b)!(b+w)!} \qquad (b \leq k \leq b+w) \end{aligned}$$

9. Let X, Y be the numbers rolled from the two doubling cubes, and let U, V be the numbers rolled from two ordinary dice. Then $(\log_2 X, \log_2 Y)$ has the same distribution as (U, V) .

a) $P(XY < 100) = P(\log_2 XY < \log_2 100) = P(U + V < 6.64) = \frac{1}{2}[1 - P(U + V = 7)] = 5/12$.

b) $P(XY < 200) = P(U + V < 7.64) = 5/12 + 1/6 = 7/12$.

c) $E(X) = 21$, so by independence $E(XY) = E(X)E(Y) = 441$.

d) $E(X^2) = 910$, so $Var(XY) = E[(XY)^2] - [E(XY)]^2 = 633619$ and $SD(XY) \approx 796$.

10. a) $\frac{1}{n} \times \left(1 - \frac{1}{n}\right)$ if $i \neq j$.

b) number of matches $= \sum_{j=1}^{n} I(\text{match occurs at place } j)$ $E(\text{number of matches}) = n \times \frac{1}{n} = 1$.

11. Assume X has Poisson(2) distribution.

$P(X < 2) = .406, P(X = 2) = .271, P(X > 2) = .323$.

12. Following the hint,

$$P_1 = p^{s-1} + (1 + p + \ldots + p^{s-2})qP_0 = p^{s-1} + (1 - p^{s-1})P_0$$
$$P_0 = (1 + q + \ldots + q^{f-2})pP_1 = (1 - q^{f-1})P_1,$$

hence

$$P_1 = p^{s-1} + (1 - p^{s-1})(1 - q^{f-1})P_1 = \frac{p^{s-1}}{1 - (1 - p^{s-1})(1 - q^{f-1})} = \frac{p^{s-1}}{p^{s-1} + q^{f-1} - p^{s-1}q^{f-1}}$$

and

$$P_0 = \frac{p^{s-1}(1 - q^{f-1})}{p^{s-1} + q^{f-1} - p^{s-1}q^{f-1}}$$

and finally

$$P(A) = pP_1 + qP_0 = \frac{p^{s-1}(1 - q^f)}{p^{s-1} + q^{f-1} - p^{s-1}q^{f-1}}$$

13. Note that $E(X^2) = VarX + [E(X)]^2 = \sigma^2 + \mu^2$, same for $E(Y)^2$. By independence we have

$$E[(XY)^2] = E(X^2)E(Y^2) = (\sigma^2 + \mu^2)^2$$
$$E(XY) = [E(X)][E(Y)] = \mu^2$$

and so

$$Var(XY) = E[(XY)^2] - [E(XY)]^2 = (\sigma^2 + \mu^2)^2 - (\mu^2)^2 = \sigma^2(\sigma^2 + 2\mu^2).$$

14. a) This is 1 minus the chance that current flows along none of the lines. The chance that current does not flow on any particular line is the chance that at least one of the switches on that line doesn't work. So answer:

$$1-(1-p_1)(1-p_2^2)(1-p_3^3)(1-p_4^4).$$

b) Let X be the number of working switches. Then X is the sum of 10 indicators, corresponding to whether each switch works or doesn't work. All the indicators are independent.

$$E(X)=p_1+2p_2+3p_3+4p_4,$$

$$Var(X)=p_1q_1+2p_2q_2+3p_3q_3+4p_4q_4,$$

and $SD(X)$ is the square root of this.

Alternatively, the number of switches working in line i has the binomial distribution with parameters i and p_i, independently of all other lines. X is the sum of these ten binomials. Formulae for $E(X)$ and $Var(X)$ can be read off the binomial formulae.

15. a) Binomial (100, 1/38). b) Poisson (100/38)

c) Negative binomial (3, 1/38) shifted to $\{3,4,\dots\}$. d) 3×38

16.

d	0	1	2	3	4	5	6	7	8	9
$P(D=d)$	.27	.04	.12	.04	.12	.09	.12	.04	.12	.04

The distribution of D on $\{1,2,\dots,9\}$ is symmetric about 5, so $E(D|D>0)=5$. Therefore

$$E(D)=E(D|D=0)P(D=0)+E(D|D>0)P(D>0)=0+5\times(73/100)=3.65.$$

17. $P(\text{one six} \mid \text{all different})=\frac{P(\text{one six and all different})}{P(\text{all different})}$.

a) Numerator $=\binom{5}{N-1}\frac{N!}{6^N}$; denominator $=\binom{6}{N}\frac{N!}{6^N}$, so the required probability is $N/6$.

b) The numerator is

$$\sum_{n=1}^{6}P(\text{one six and all different}\mid N=n)P(N=n)=\sum_{n=1}^{6}\binom{5}{n-1}\frac{n!}{6^n}\frac{1}{6}=1/6;$$

the denominator is

$$\sum_{n=1}^{6}P(\text{all different}\mid N=n)P(N=n)=\sum_{n=1}^{6}\binom{6}{n}\frac{n!}{6^n}\frac{1}{6}=.46245,$$

so the required probability is .3604.

18. a) The return (in cents) from the game is

$$X=I_1+I_2+\cdots+I_{100},$$

where

$$I_i=\begin{cases}1 & \text{if the number on deal } i \text{ is greater than those of all previous cards dealt}\\ 0 & \text{otherwise}\end{cases}$$

That is, I_i is the indicator of the event that a *record value* occurs at deal i (a record is considered to have occured at the first deal), and X simply counts the number of records seen. A counting argument shows that $P(I_i=1)=\frac{1}{i}$, so

$$E(X)=\sum_{i=1}^{100}\frac{1}{i}\approx\log 100+\gamma+\frac{1}{2\times 100}=5.19,$$

where $\gamma=.57721$ (Euler's constant). [This approximation was used in the Collector's Problem of Example 3.4.5]. So 5.19 cents is the fair price to pay in advance.

b) The gain (in cents) from 25 plays is

$$S = X_1 + X_2 + \cdots + X_{25},$$

where the X_i's are independent copies of X. The I_i's are independent so that

$$Var(X) = \sum_{i=1}^{100} Var(I_i) = \sum_{i=1}^{100} \frac{1}{i}\left(1 - \frac{1}{i}\right) \approx 5.19 - \frac{\pi^2}{6} = 3.54 \quad \Longrightarrow \quad SD(X) = 1.88.$$

Using the normal approximation, obtain $P(S > 25 \times 10) \approx 1 - \Phi(12.8) \approx 0$.

19.

Y : number of failures before first success, $P(Y \geq y) = q^y$
X : Poisson (μ) independent of Y

$$P(Y \geq X) = \sum_{k=0}^{\infty} q^k e^{-\mu} \frac{\mu^k}{k!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(q\mu)^k}{k!}$$

$$e^{-\mu} e^{q\mu} = e^{-\mu(1-q)} = 0.6065 \text{ if } p = 1/2,\ \mu = 1$$

20. a) Let $p = 1/2 + x$ (where x could be negative). So $1 - p = 1/2 - x$, and

$$p(1-p) = (1/2 + x)(1/2 - x) = 1/4 - x^2 \leq 1/4$$

since $x^2 \geq 0$.

b) The margin of error in the estimate is $\sqrt{p(1-p)/n}$, where n is the sample size, and p is the proportion of part time employed students. This assumes sampling without replacement. For sampling with replacement the margin of error would be smaller, due to the correction factor. No matter what p is,

$$\sqrt{\frac{p(1-p)}{n}} \leq \sqrt{\frac{1}{4n}}$$

by part a), so we should take the smallest n so that $\sqrt{1/4n} \leq .05$, i.e., $n = 100$.

21. X has negative binomial $(1, p)$ distribution, Y has negative binomial $(2, p)$ distribution, and X and Y are independent. Now suppose a coin having probability p of landing heads is tossed repeatedly and independently. Let X' denote the number of tails observed until the first head is observed; let Y' denote the additional number of failures until the third head. Then (X, Y) has the same distribution as does (X', Y'). So Z has the same distribution as $X' + Y'$ = the number of tails seen until the third head: negative binomial distribution on $\{0, 1, \ldots\}$ with parameters $r = 3$ and p.

22. Suppose the daily demand X has Poisson distribution with parameter $\lambda > 0$. (Here $\lambda = 100$.) Suppose the newsboy buys n (constant) papers per day, $n \geq 1$. Then $min(X, n)$ papers are sold in a day; the daily profit π_n in dollars is

$$\pi_n = \frac{1}{4}\min(X, n) - \frac{1}{10}n;$$

and the long run average profit per day is

$$E[\pi_n] = \frac{1}{4}E[\min(X, n)] - \frac{1}{10}n.$$

Claim: For each $n = 1, 2, 3, \ldots$:

$$(1) E[\min(X, n)] = \lambda P(X \leq n - 1) + nP(X \geq n + 1)$$

$$(2) E[\min(X, n)] = \sum_{k=1}^{n} P(X \geq k).$$

Part (2) holds no matter what distribution X has.

Proof: (1)

$$\begin{aligned} E[\min(X,n)] &= \sum_{k=0}^{\infty} \min(k,n)P(X=k) \\ &= \sum_{k=0}^{n} kP(X=k) + \sum_{k=n+1}^{\infty} nP(X=k) \\ &= \sum_{k=1}^{n} ke^{-\lambda}\frac{\lambda^k}{k!} + nP(X \geq n+1) \\ &= \lambda \sum_{k=1}^{n} e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} + nP(X \geq n+1) \\ &= \lambda P(X \leq n-1) + nP(X \geq n+1). \end{aligned}$$

(2) Use the tail sum formula for expectation:

$$E[\min(X,n)] = \sum_{k=1}^{n} P(\min(X,n) \geq k) = \sum_{k=1}^{n} P(X \geq k).$$

a) Say $\lambda = 100$ and the newsboy buys $n = 100$ papers per day. Then his long run average daily profit (in dollars) is

$$\begin{aligned} E[\pi_{100}] &= \frac{1}{4}E[\min(X,100)] - \frac{1}{10}\cdot 100 \\ &= \frac{1}{4}[100P(X \leq 99) + 100P(X \geq 101)] - 10 \text{ by Claim (1)} \\ &= 25[1 - P(X = 100)] - 10 \\ &= 15 - 25P(X = 100) \approx 14 \end{aligned}$$

since

$$P(X = 100) = e^{-100}\frac{(100)^{100}}{100!} \approx \frac{1}{\sqrt{2\pi \cdot 100}} \approx 0.04 \text{ by Stirling's approximation.}$$

b) If $\lambda = 100$ and the newsboy buys n papers per day, then his long run average profit per day is

$$\begin{aligned} E[\pi_n] &= \frac{1}{4}E[\min(X,n)] - \frac{1}{10}n \\ &= \frac{1}{4}\sum_{k=1}^{n} P(X \geq k) - \frac{1}{10}n \text{ by Claim (2)} \\ &= \frac{1}{4}\sum_{k=1}^{n}\left(P(X \geq k) - \frac{4}{10}\right). \end{aligned}$$

Since the function $k \to P(X \geq k) - \frac{4}{10}$ is decreasing, is positive at $k = 1$, and is negative as $k \to \infty$, it follows that $E[\pi_n]$ is maximized at $n^* =$ the largest k such that $P(X \geq k) - \frac{4}{10} \geq 0$, i.e., $n^* = 102$. Thus the newsboy should buy 102 papers per day in order to maximize his daily profit - assuming that demand has Poisson(100) distribution.

23. a) The drawing process can be described in another way as follows: think of the box as initially containing $2n$ half toothpicks in n pairs. Then half toothpicks are simply being drawn at random without replacement. The problem is to find the distribution of H, the number of halves remaining after the last pair is broken. By the symmetry of sampling without replacement, H has the same distribution as H', where H' is the number of draws preceding (i.e., not including) the first time that the remaining half of some toothpick is drawn. This may be clearer by an analogy. Think of $2n$ cards in a deck, 2 of each of n colors, and imagine dealing cards one by one off the top of the deck. Then H corresponds to the number of cards remaining in the deck after each color has been seen at least once. If the cards had been dealt from the bottom of the deck, this number would have been the number (corresponding to H') of cards dealt preceding

the first card having a color already seen. Since the order in which we deal the cards makes no difference, it follows that H and H' have the same distribution. Thus for $1 \le k \le n$

$$P(H = k) = P(H' = k)$$

$$= P(\text{first } k \text{ colors are different and } (k+1)\text{st color has been seen before})$$

$$= \frac{2n}{2n} \cdot \frac{2n-2}{2n-1} \cdots \frac{2n-2(k-1)}{2n-(k-1)} \cdot \frac{k}{2n-k}$$

$$= \frac{2^k (n)_k k}{(2n)_{k+1}}.$$

Another way to obtain $P(H = k)$ is through $P(H \ge k)$: For each $k = 1, 2, ..., n+1$ we have

$$P(H \ge k) = P(H' \ge k) = P(\text{first } k \text{ colors are all different})$$

$$= \frac{2n}{2n} \cdot \frac{2n-2}{2n-1} \cdots \frac{2n-2(k-1)}{2n-(k-1)} = \frac{2^k (n)_k}{(2n)_k}.$$

b) As $n \to \infty$,

$$P(H \ge k) = 1 \cdot \frac{1-\frac{1}{n}}{1-\frac{1}{2n}} \cdot \frac{1-\frac{2}{n}}{1-\frac{2}{2n}} \cdot \ldots \cdot \frac{1-\frac{k-1}{n}}{1-\frac{k-1}{2n}}$$

$$\approx \exp\left[\left(-\frac{1}{n} + \frac{1}{2n}\right) + \left(-\frac{2}{n} + \frac{2}{2n}\right) + \ldots + \left(-\frac{(k-1)}{n} + \frac{(k-1)}{2n}\right)\right],$$

$$= \exp\left(-\frac{1+2+\ldots+(k-1)}{2n}\right) \approx \exp\left(-\frac{k^2}{4n}\right).$$

Thus, putting $k = r\sqrt{2n}$,

$$P(\frac{H}{\sqrt{2n}} \ge r) = P(H \ge r\sqrt{2n}) \approx \exp(-r^2/2).$$

This limiting distribution of $H/\sqrt{2n}$ is called the Rayleigh distribution (see Section 5.3).

c) This approximation suggests

$$E(H) = \sum_{k=1}^{\infty} P(H \ge k) \sim \sum_{k=1}^{\infty} \exp\left(-\frac{k^2}{4n}\right) = \sqrt{2n} \sum_{k=1}^{\infty} \frac{1}{\sqrt{2n}} e^{-\frac{1}{2}(k/\sqrt{2n})^2}$$

$$\sim \sqrt{2n} \int_0^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2n}\sqrt{\pi/2}$$

That is to say

$$E(H) \sim \sqrt{\pi n} \text{ as } n \to \infty.$$

d) Expect about $\sqrt{100 \cdot \pi} \approx \sqrt{314} \approx 17$ or so.

24. a) If $P(X = 3, Y = 2, Z = 1) = 1/3$,
$P(X = 2, Y = 1, Z = 3) = 1/3$,
$P(X = 1, Y = 3, Z = 2) = 1/3$,
then $P(X > Y) = P(Y > Z) = P(Z > X) = 2/3$.

b) Let $p = \min\{P(X > Y), P(Y > Z), P(Z > X)\}$. Then
$p \le \frac{1}{3}[P(X > Y) + P(Y > Z) + P(Z > X)]$
$= \frac{1}{3}E[I(X > Y) + I(Y > Z) + I(Z > X)]$
$\le 2/3$, since $I(X > Y) + I(Y > Z) + I(Z > X) \le 2$.

c) Say each voter assigns each candidate a numerical score 1, 2, or 3, 3 for the most preferred candidate, 1 for the least preferred. Pick a voter at random, and let X be the score assigned to A, Y the score of B, Z the score of C for this voter. Then $P(X > Y) =$"Proportion of voters who prefer A to B" , etc. So a population of 3 voters with preferences as in a) above would make each of these proportions 2/3.

d) Let
$P(X_1 = n, X_2 = n-1, X_3 = n-2, \dots, X_n = 1) = 1/n,$
$P(X_1 = n-1, X_2 = n-2, X_3 = n-3, \dots, X_n = n) = 1/n,$
$\dots$
$P(X_1 = 1, X_2 = n, X_3 = n-1, \dots, X_n = 2) = 1/n.$
Then all the probabilities are $(n-1)/n$, and the minimum probability p is bounded above by $(n-1)/n$.

e) We have $P(X > Y) = 1 - (1-p_1)p_2$
$P(Y > Z) = p_2$
$P(Z > X) = 1 - p_1.$
These will all equal q if $1 - p_1 = q$, $p_2 = q$, and $1 - (1-p_1)p_2 = q$, that is, if

$$1 - q^2 = q,$$

the equation of the golden mean. (L. Dubins)

25. a)

z	0	1	2	3	4
$P(Y_1 + Y_2 = z)$	9/36	12/36	10/36	4/36	1/36

b) 10/3

c) One possibility: $f(x) = \begin{cases} 0 & \text{if } x = 1,2,3 \\ 1 & \text{if } x = 4,5 \\ 2 & \text{if } x = 6 \end{cases}$

26. a) We wish to find n such that $(.99)^n = .5$, so $n = \dfrac{\log .5}{\log .99} \approx 69.$

b) This is essentially a negative binomial with the roles of success and failure reversed. Let F_4 be the number of failures (successful honks) before the $4th$ success (failure to honk). Then $E(F_4) = \frac{4(1-.01)}{.01} = 396$, but for our problem we must also add 3 non-honks to get an answer of 399.

27.

$$E(X) = \sum_{x=1}^{5} 100xq^{x-1}p + \sum_{x=1}^{\infty}(500 + 40x)q^{5+x-1}p$$

$$= 100(p + 2qp + 3q^2p + 4q^3p + 5q^4p) + 500\sum_{x=1}^{\infty} q^{5+x-1}p + 40\sum_{x=1}^{\infty} xq^{5+x-1}p$$

$$= 100(p + 2qp + 3q^2p + 4q^3p + 5q^4p) + (500q^5) + (40q^5\frac{1}{p})$$

$$= 186.43 + 156.62$$
$$= 343.047$$

28. a) If $w \geq 1$ and $y \in A$ then

$$\begin{aligned} P(W_1 = w, Y_1 = y) &= P(X_1 \notin A, \dots, X_{w-1} \notin A, X_w = y) \\ &= P(X_1 \notin A) \cdots P(X_{w-1} \notin A)P(X_w = y) \\ &= (1 - P_1(A))^{w-1} P_1(y) \end{aligned}$$

Sum over w to get $P(Y_1 = y) = P_1(y)/P_1(A)$ and sum over y to get $P(W_1 = w) = (1 - P_1(A))^{w-1} P_1(A)$
Continue in this way to show that for all $k \geq 1$

$P(W_1 = w_1, Y_1 = y_1, \dots, W_k = w_k, Y_k = y_k) = P(W_1 = w_1)P(Y_1 = y_1) \cdots P(W_k = w_k)P(Y_k = y_k).$

(Idea: Express the event on the left in terms of the independent variables $X_1, X_2, \dots$)

b) By the weak law of large numbers: As $n \to \infty$,

$$\frac{\sum_{i=1}^{n} I(X_i \in AB)}{\sum_{i=1}^{n} I(X_i \in A)} = \frac{\sum_{i=1}^{n} I(X_i \in AB)/n}{\sum_{i=1}^{n} I(X_i \in A)/n} \to \frac{P_1(AB)}{P_1(A)} = P_1(B|A).$$

29. c) uniform on $\{0, 1, \ldots, n\}$ d) no, yes e) $\frac{b}{b+w}$ f) $\frac{b+d}{b+w+d}$

30. No Solution

31. No Solution

32. No Solution

33. a) If $n \geq k \geq 1$:

$$P(\text{all } n \text{ red in bag}|\text{pick all } k \text{ red}) = \frac{P(\text{pick all } k \text{ red}|\text{all } n \text{ red in bag})P(\text{all } n \text{ red in bag})}{P(\text{pick all } k \text{ red})}$$

$$= \frac{1 \cdot (1/2)^n}{\sum_{j=0}^{n} P(\text{pick all } k \text{ red}|j \text{ red in bag})P(j \text{ red in bag})}$$

$$= \frac{(1/2)^n}{\sum_{j=0}^{n} \frac{j^k}{n^k}\binom{n}{j}(1/2)^n}$$

$$= \frac{n^k}{2^n \sum_{j=0}^{n} j^k \binom{n}{j}(1/2)^n}$$

$$= \frac{n^k}{2^n E(X^k)},$$

where X has binomial $(n, 1/2)$ distribution.

b) If $k = 1$: then $E(X) = n/2$, so

$$P(\text{all } n \text{ red in bag}|\text{pick red}) = \frac{1}{2^{n-1}}.$$

If $k = 2$: then $E(X^2) = Var(X) + [E(X)]^2 = \frac{n}{4} + n^{\frac{2}{4}}$, so

$$P(\text{all } n \text{ red in bag}|\text{pick both red}) = \frac{1}{2^{n-2}\left(1 + \frac{1}{n}\right)}.$$

c) If the k balls are drawn without replacement from the bag, follow the argument in a), but replace $\frac{j^k}{n^k}$ with $\frac{(j)_k}{(n)_k}$:

$$P(\text{all } n \text{ red in bag}|\text{pick all } k \text{ red}) = \frac{(1/2)^n}{\sum_{j=0}^{n} \frac{(j)_k}{(n)_k}\binom{n}{j}(1/2)^n} = \frac{(n)_k}{2^n E(X)_k}$$

where again X has binomial $(n, 1/2)$ distribution. On the other hand, if the k balls are drawn without replacement from the bag, then the number of red balls seen among the k has binomial $(k, 1/2)$ distribution: it's as if we drew the k balls directly from the very large collection. So

$$P(\text{all } n \text{ red in bag}|\text{pick all } k \text{ red}) = \frac{P(\text{pick all } k \text{ red}|\text{all } n \text{ red in bag})P(\text{all } n \text{ red in bag})}{P(\text{pick all } k \text{ red})} = \frac{1 \cdot (1/2)^n}{(1/2)^k}.$$

d) If $k = 3$: Since

$$(X)_3 = X(X-1)(X-2) = X^3 - 3X^2 + 2X,$$

compute

$$E(X^3) = E(X)_3 + 3E(X^2) - 2E(X) = \frac{(n)_3}{8} + 3\left(\frac{n}{4} + \frac{n^2}{4}\right) - 2\left(\frac{n}{2}\right) = \frac{n^2(n+3)}{8}$$

and the result in a) simplifies to

$$P(\text{all } n \text{ red in bag}|\text{pick all } 3 \text{ red}) = \frac{1}{2^{n-3}\left(1 + \frac{3}{n}\right)}.$$

e) Let p be the proportion of red balls in the original collection of balls. Repeat the argument in c) to get

$$P(\text{all } n \text{ red in bag}|\text{pick all } k \text{ red}) = \frac{p^n}{\sum_{j=0}^{n}\frac{(j)_k}{(n)_k}\binom{n}{j}p^j(1-p)^{n-j}} = \frac{(n)_k p^n}{E(X)_k}$$

(where X has binomial (n,p) distribution) and

$$P(\text{all } n \text{ red in bag}|\text{pick all } k \text{ red}) = \frac{p^n}{p^k}.$$

Therefore $E(X)_k = (n)_k p^k$.
Check: This formula gives $E(X) = E(X)_1 = (n)_1 p = np$ and $E[X(X-1)] = E(X)_2 = (n)_2 p^2 = n(n-1)p^2$ from which it follows $Var(X) = E[X(X-1)] + E(X) - [E(X)]^2 = np(1-p)$.

34. No Solution

35. No Solution

36. a) The kth binomial moment b_k is just given by

$$\sum_{i_1<\cdots<i_k} p^k = \binom{n}{k}p^k$$

b)

$$\mu = b_1 = \binom{n}{1}p^1 = np$$

$$\sigma^2 = 2b_2 + b_1 - b_1^2 = 2\binom{n}{2}p^2 + np - (np)^2 = n(n-1)p^2 + np - (np)^2 = np - np^2 = np(1-p)$$

37. a) $\binom{n}{k}\frac{(G)_k}{(N)_k}$

38. a) Using the kth binomial moment results, we observe that the kth factorial moment is just $k!$ times the kth binomial moment. Let A_i be the event that the ith letter gets the correct address. Then $M_n = \sum_{i=1}^{n} A_i$ and the kth factorial moment is just

$$\begin{aligned}
f_k(M_n) &= k! \sum_{i_1<i_2<\cdots<i_k} P(A_{i_1}A_{i_2}\cdots A_{i_k}) \\
&= k! \sum_{i_1<i_2<\cdots\leq i_k} \frac{1}{n}\times\frac{1}{n-1}\times\cdots\times\frac{1}{n-k+1} \\
&= k!\binom{n}{k}\frac{1}{n}\times\frac{1}{n-1}\times\cdots\times\frac{1}{n-k+1} \\
&= 1
\end{aligned}$$

b) Let X be Poisson(1), then the kth binomial moment is

$$\begin{aligned}
f_k(X) &= E((X)(X-1)\cdots(X-k+1)) \\
&= \sum_{i=0}^{\infty}(i)\times(i-1)\times\cdots\times(i-k+1)P(X=i) \\
&= \sum_{i=k}^{\infty}(i)\times(i-1)\times\cdots\times(i-k+1)\frac{e^{-1}}{i!} \\
&= e^{-1}\sum_{i=k}^{\infty}\frac{1}{(i-k)!} \\
&= 1
\end{aligned}$$

c) Use the expression of Exercise 3.4.22 for ordinary moments in terms of factorial moments.
As $n \to \infty$, the kth moment of M_n are equals the kth moment of X for all $k \leq n$, so the convergence is immediate.

d) Using the Sieve formula,

$$\begin{aligned}
P(M_n = k) &= \sum_{j=k}^{n} \binom{j}{k} (-1)^{k-j} b_j(M_n) \\
&= \sum_{j=k}^{n} \binom{j}{k} (-1)^{k-j} \left(\frac{1}{j!}\right) \\
&= \sum_{j=k}^{n} (-1)^{k-j} \frac{1}{(k!)(j-k)!} \\
&= \frac{1}{k!} \sum_{j=0}^{n-k} (-1)^j \frac{1}{j!}
\end{aligned}$$

and as $n \to \infty$, this goes to $\frac{e^{-1}}{k!}$ as desired.

39. No Solution

40. The following identities hold:

$$\begin{aligned}
p_1 + p_2 + \cdots + p_n &= 1 \\
x_1 p_1 + x_2 p_2 + \cdots + x_n p_n &= \mu_1 \\
x_1^2 p_1 + x_2^2 p_2 + \cdots + x_n^2 p_n &= \mu_2 \\
&\cdots \\
&\cdots \\
x_1^{n-1} p_1 + x_2^{n-1} p_2 + \cdots + x_n^{n-1} p_n &= \mu_{n-1}
\end{aligned}$$

This is the same as $\mu = pM$, where M is the $n \times n$ matrix

$$M = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

To see that M must have an inverse, it suffices to show that M has rank n; equivalently, that the columns of M are linearly independent.

Proof: Let $c_0, c_1, \ldots, c_{n-1}$ be real constants such that

$$\begin{aligned}
0 &= c_0 + c_1 x_1 + c_2 x_1^2 + \cdots + c_{n-1} x_1^{n-1} \\
0 &= c_0 + c_1 x_2 + c_2 x_2^2 + \cdots + c_{n-1} x_2^{n-1} \\
&\cdots \\
&\cdots \\
&\cdots \\
0 &= c_0 + c_1 x_n + c_2 x_n^2 + \cdots + c_{n-1} x_n^{n-1}
\end{aligned}$$

Then the polynomial f defined by $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$ has n distinct roots, namely $x_1, x_2, \ldots, x_n$. But f has degree at most $n-1$. Therefore f must be identically zero. Conclude: all the c's must be zero.

41. Rephrase this problem in terms of the collector's problem: Suppose there are M objects in a complete collection, and suppose a trial consists of selecting an object at random with replacement from the set of M objects. So on each trial, the object selected is equally likely to be any one of the M possible, independently of what the other trials yielded. Now find n so that $P(T \le n) = 0.9$, where T is the number of trials required to see

a) an object specified in advance

b) at least one of the possible objects twice

c) every possible object at least once

d) at least once every object in a specified set comprising half of all possible objects

e) half of all objects.

In the present context, $M = 1024$, since each repetition of the ten toss experiment results in one of 2^{10} possible sequences.

a) T has geometric distribution on $\{1, 2, \ldots\}$ with parameter $1/M$, so

$$P(T > n) = \left(1 - \frac{1}{M}\right)^n \quad n = 1, 2, 3, \ldots.$$

Therefore

$$P(T \le n) = 0.9 \iff \left(1 - \frac{1}{M}\right)^n = 0.1 \iff n = \frac{\log 0.1}{\log\left(1 - \frac{1}{M}\right)}.$$

For example, if $M = 1024$, then require $n \approx 2350$.

b) Let n be a positive integer. Observe that

$$(T > n) = (\text{the first } n \text{ trials yielded all different objects}).$$

By comparison with the birthday problem (see index) we obtain

$$P(T > n) = \left(1 - \frac{1}{M}\right)\left(1 - \frac{2}{M}\right)\cdots\left(1 - \frac{n-1}{M}\right).$$

If M is large, the right-hand side is approximately $e^{-\frac{n(n-1)}{2M}}$, and this approximation is good over all values of n. Thus if M is large,

$$P(T > n) \approx e^{-\frac{n(n-1)}{2M}} \quad n = 1, 2, 3, \ldots.$$

and

$$\begin{aligned} P(T \le n) = 0.9 &\iff \exp -\frac{n(n-1)}{2M} \approx 0.1 \\ &\iff n^2 - n \approx 2M \log 10 \\ &\iff n \approx \frac{1 + \sqrt{1 + 8M \log 10}}{2}. \end{aligned}$$

For example, if $M = 1024$, then require $n \approx 70$.

c) We claim that for each real x,

$$\lim_{M \to \infty} P(T \le M(\log M + x)) = e^{-e^{-x}}.$$

First, a heuristic justification: We have $T = max(T_1, T_2, \ldots, T_M)$ where T_i is the number of trials (from the beginning) required to see object i. Then each T_i has geometric distribution on $\{1, 2, \ldots\}$ with parameter $1/M$, so

$$\begin{aligned} P(T_i \le M(\log M + x)) &= 1 - P(T_i > M(\log M + x)) \\ &\approx 1 - \left(1 - \frac{1}{M}\right)^{M(\log M + x)} \\ &\approx 1 - (e^{-1})^{\log M + x} \quad \text{if } M \text{ large} \end{aligned}$$

$$= 1 - \frac{e^{-x}}{M}.$$

Now if M is large, the random variables $T_1, T_2, ..., T_M$ are almost independent; hence

$$P(T \leq M(\log M + x)) \approx \Pi_{i=1}^{M} P(T_i \leq M(\log M + x))$$
$$= \left(1 - \frac{e^{-x}}{M}\right)^M \approx e^{-e^{-x}}.$$

Now, a more rigorous justification: Let k be a positive integer. We have

$$(T > k) = (\text{one of the } M \text{ objects has not been seen in the first} k \text{trials})$$
$$= \cup_{i=1}^{M}(\text{object } i \text{ not seen in } k \text{ trials}).$$

By inclusion-exclusion,

$$P(T > k) = P\left\{\cup_{i=1}^{M}(\text{object } i \text{ not seen in } k \text{ trials})\right\}$$

$$= \sum_{i=1}^{M} P(\text{object } i \text{ not seen in } k \text{ trials}) - \sum_{i<j} P(\text{objects } i,\, j \text{ not seen in } k \text{ trials})$$
$$+ ... + (-1)^{M-1} P(\text{objects } 1, \ldots, M \text{ not seen in } k \text{ trials})$$
$$= \binom{M}{1}\left(\frac{M-1}{M}\right)^k - \binom{M}{2}\left(\frac{M-2}{M}\right)^k + ... + (-1)^{M-1}\binom{M}{M}\left(\frac{M-M}{M}\right)^k$$
$$= \sum_{j=1}^{M}(-1)^{j-1}\binom{M}{j}\left(1 - \frac{j}{M}\right)^k.$$

Hence

$$P(T \leq k) = \sum_{j=0}^{M}(-1)^j\binom{M}{j}\left(1 - \frac{j}{M}\right)^k.$$

Let x (real) be fixed. Replace k by $M(\log M + x)$ and investigate the behavior of the probability as $M \to \infty$:

$$P(T \leq M(\log M + x)) = \sum_{j=0}^{M}(-1)^j\binom{M}{j}\left(1 - \frac{j}{M}\right)^{\lfloor M(\log M + x)\rfloor} = \sum_{j=0}^{\infty} a_{M,j},$$

where

$$a_{M,j} = \begin{cases} (-1)^j\binom{M}{j}\left(1 - \frac{j}{M}\right)^{\lfloor M(\log M + x)\rfloor} & 0 \leq j \leq M \\ 0 & j > M \end{cases}.$$

Note that for each $j = 0, 1, 2, ...$

$$\lim_{M\to\infty} a_{M,j} = \frac{(-1)^j e^{-jx}}{j!} = \frac{(-e^{-x})^j}{j!}.$$

We evaluate the limit as $M \to \infty$ of $P(T \leq M(\log M + x))$ as follows:

$$\lim_{M\to\infty} P(T \leq M(\log M + x)) = \lim_{M\to\infty}\sum_{j=0}^{\infty} a_{M,j} = \sum_{j=0}^{\infty}\lim_{M\to\infty} a_{M,j} = \sum_{j=0}^{\infty}\frac{(-e^{-x})^j}{j!} = e^{-e^{-x}};$$

the movement of the limit inside the summation is justified e.g. by the Weierstrass M-test (no relation to the present M).

Set $e^{-e^{-x}} = 0.9 \iff x \approx 2.25$. Then for M large, we have approximately

$$P(T \leq n) = 0.9 \iff n \approx M(\log M + 2.25).$$

For example, if $M = 1024$, then require $n \approx 9400$.

d) Without loss of generality, the desired subcollection consists of objects $1, 2, 3, ..., M/2$. By a similar analysis to (c), we obtain

$$P(T > k) = P\left\{\cup_{i=1}^{M/2}(\text{object } i \text{ has not been seen in } k \text{ trials})\right\}$$

$$= \sum_{j=1}^{M/2}(-1)^j \binom{M/2}{j}\left(1-\frac{j}{M}\right)^k;$$

$$P(T \le k) = \sum_{j=0}^{M/2}(-1)^j \binom{M/2}{j}\left(1-\frac{j}{M}\right)^k;$$

$$P(T \le M(\log\frac{M}{2}+x)) = \sum_{j=0}^{M}(-1)^j \binom{M/2}{j}\left(1-\frac{j}{M}\right)^{\lfloor M(\log\frac{M}{2}+x)\rfloor} \to e^{-e^{-x}} \text{ as } M \to \infty.$$

(Heuristically, we may view the collection of M tickets as consisting of half desirable and half undesirable. T is the time required to obtain all the desirable tickets. Since roughly half of our trials result in undesirable tickets, it would take twice as long to obtain all the desirable tickets as it normally would had there been no undesirable tickets at all. In other words, if T' denotes the number of trials needed to obtain a complete collection of $M/2$ when drawing from the set of $M/2$ objects, then T is distributed approximately like $2T'$; hence if M is large, then

$$P(T \le M(\log\frac{M}{2}+x)) \approx P(2T' \le M(\log\frac{M}{2}+x))$$

$$= P(T' \le \frac{M}{2}(\log\frac{M}{2}+x))$$

$$\approx e^{-e^{-x}}.)$$

Thus for M large,

$$P(T \le n) = 0.9 \iff n \approx M(\log\frac{M}{2}+2.25).$$

For example, if $M = 1024$, then require $n \approx 8700$.

e) $T = X_1 + X_2 + ... + X_{M/2}$ where $X_1 = 1$, and for $i = 2, 3, ..., M$, X_i is the additional number of trials [after obtaining the $(i-1)$st different object] required to obtain a new object. Then X_i has geometric distribution on $\{1, 2, ...\}$ with parameter $p_i = (M-i+1)/M$, $i = 1, ..., M$, and the $\{X_i\}$ are mutually independent. Hence

$$E(T) = \sum_{i=1}^{M/2} E(X_i) = \sum_{i=1}^{M/2}\frac{1}{p_i} = \sum_{i=1}^{M/2}\frac{M}{M-i+1}$$

$$= M\left(\frac{1}{M}+\frac{1}{M-1}+...+\frac{1}{\frac{M}{2}+1}\right) \sim M\log 2 \text{ as } M \to \infty,$$

and

$$Var(T) = \sum_{i=1}^{M/2} Var(X_i) = \sum_{i=1}^{M/2}\left(\frac{i}{p_i^2}-\frac{1}{p_i}\right)$$

$$= \sum_{i=1}^{M/2}\left(\frac{M}{M-i+1}\right)^2 - \sum_{i=1}^{M/2}\left(\frac{M}{M-i+1}\right)$$

$$= M^2\left(\frac{1}{M^2}+\frac{1}{(M-1)^2}+...+\frac{1}{(\frac{M}{2}1)^2}\right) - M\left(\frac{1}{M}\frac{1}{M-1}+...+\frac{1}{\frac{M}{2}1}\right)$$

$$\sim M^2\left(\frac{1}{M}) - M(\log 2)\right) = M(1-\log 2) \text{ as } M \to \infty.$$

Observe that the variables $X_1, X_2, ..., X_{M/2}$ are roughly identically distributed, or at least roughly of the same size (they have geometric distribution, with parameters between $1/2$ and 1;

contrast this with the full set of variables $X_1, X_2, ..., X_M$, where the latter variables have very high expectation). Hence

$$\frac{T - M\log 2}{\sqrt{M(1-\log 2)}} \approx \frac{T - E(T)}{SD(T)}$$

has approximately normal (0,1) distribution. In fact, it can be shown that for all x:

$$\lim_{M\to\infty} P\left(T \leq M\log 2 + x\sqrt{M(1-\log 2)}\right) = \lim_{M\to\infty} P\left(\frac{T - M\log 2}{\sqrt{M(1-\log 2)}} \leq x\right) = \Phi(x).$$

Set $\Phi(x) = 0.9 \iff x \approx 1.282$. Then for M large, we have

$$P(T \leq n) = 0.9 \iff n \approx M\log 2 + 1.282\sqrt{M(1-\log 2)}.$$

For example, if $M = 1024$, then require $n \approx 730$.

Section 4.1

1. a) The desired probability is the area under the standard normal density $y = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ between $z = 0$ and $z = 0.001$. The density is very nearly constant over this interval, so the desired probability is approximately

$$\text{(width of rectangle)(approx height of rectangle)} = 0.001 \times \frac{1}{\sqrt{2\pi}} = \frac{1}{1000\sqrt{2\pi}} = .000399.$$

b) Similarly the desired probability is approximately

$$0.001 \times \frac{1}{\sqrt{2\pi}} \times e^{-1/2} = \frac{1}{1000\sqrt{2\pi e}} = .000242.$$

2. a)

$$\int_1^\infty \frac{c}{x^4}dx = \left.\frac{-c}{3x^3}\right|_1^\infty = \frac{c}{3}$$

and since $f(x)$ is a density function, it must integrate to 1, so $c = 3$.

b)

$$E(X) = \int_1^\infty x\frac{3}{x^4}dx = \left.\frac{-3}{2x^2}\right|_1^\infty = \frac{3}{2}$$

c)

$$E(X^2) = \int_1^\infty x^2\frac{3}{x^4}dx = \left.\frac{-3}{x}\right|_1^\infty = 3$$

Thus $Var(X) = E(X^2) - (E(X))^2 = 3 - \frac{9}{4} = \frac{3}{4}$.

3. a) Since f is a probability density, $\int_{-\infty}^{\infty} f(x)dx = 1$, so

$1 = c\int_0^1 x(1-x)dx = c\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{c}{6} \Longrightarrow c = 6.$

b) $P(X \le \frac{1}{2}) = \int_0^{\frac{1}{2}} 6x(1-x)dx = 3x^2 - 2x^3\Big|_0^{\frac{1}{2}} = \frac{1}{2}.$

Remark. Once you note that $f(x)$ is symmetric about $\frac{1}{2}$ (draw a picture), the answer is clear without calculation.

c) $P(X \le \frac{1}{3}) = \int_0^{\frac{1}{3}} 6x(1-x)dx = 3x^2 - 2x^3\Big|_0^{\frac{1}{3}} = \frac{7}{27}.$

d) By the difference rule of probabilities,

$P(\frac{1}{3} < X \le \frac{1}{2}) = P(X \le \frac{1}{2}) - P(X \le \frac{1}{3}) = \frac{1}{2} - \frac{7}{27} = \frac{13}{54}.$

Remark. Of course you can obtain the same result by integration.

e) $E(X) = \int f(x)dx = \int_0^1 6x^2(1-x)dx = 2x^3 - \frac{3}{2}x^4\Big|_0^1 = \frac{1}{2}.$

Again, this is obvious by symmetry. But you must compute an integral for the variance:

$E(X^2) = \int_0^1 x^2 f(x)dx = \int_0^1 6x^3(1-x)dx = \frac{3}{2}x^4 - \frac{6}{5}x^5\Big|_0^1 = \frac{3}{10}$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$

4. a) We know that $\int_0^1 cx^2(1-x)^2dx = 1$, and

$$\int_0^1 cx^2(1-x)^2dx = c\int_0^1 (x^2 - 2x^3 + x^4)dx = c\left(\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right)\Big|_0^1 = \frac{c}{30}$$

so $c = 30$.

b)

$$E(X) = \int_0^1 30x^3(1-x)^2dx = 30\int_0^1 (x^3 - 2x^4 + x^5)dx = 30\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) = \frac{1}{2}$$

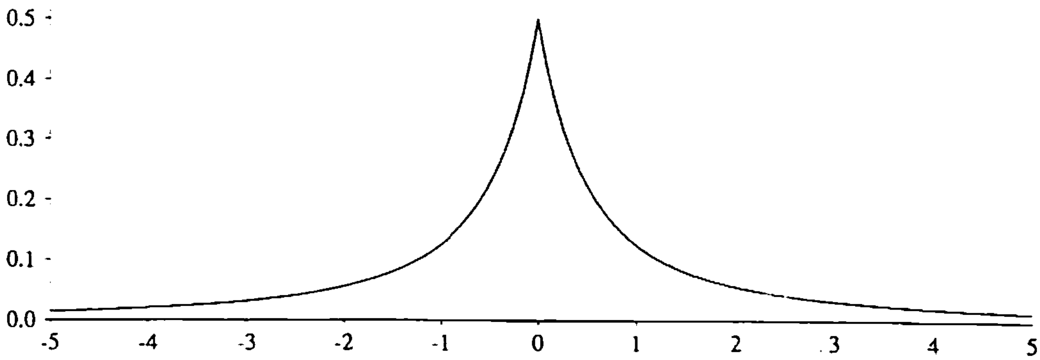
c)

$$E(X^2) = \int_0^1 30x^4(1-x)^2dx = 30\int_0^1 (x^4 - 2x^5 + x^6)dx = 30\left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right) = \frac{2}{7}$$

and so

$$Var(X) = \frac{2}{7} - \frac{1}{4} = \frac{1}{28}$$

5. a) Graph:



b)

$$P(-1 < X < 2) = \frac{1}{2}\left(\int_{-1}^{0} \frac{1}{(1-x)^2}dx + \int_0^2 \frac{1}{(1+x)^2}dx\right) = \frac{1}{2}\left(\left[\frac{1}{1-x}\right]\Big|_{-1}^{0} + \left[-\frac{1}{1+x}\right]\Big|_0^2\right) = \frac{7}{12}$$

c) $P(|X| > 1) = 2 \times \frac{1}{2}\int_1^\infty \frac{1}{(1+x)^2}dx = -\frac{1}{1+x}\Big|_1^\infty = \frac{1}{2}$

d) No, because $E(|X|) = \int_{-\infty}^{\infty} \frac{|x|}{2(1+|x|)^2}dx = \int_0^\infty \frac{x}{(1+x)^2}dx = \infty$ (see Example 3).

6. a) $\frac{1}{3} = P(X \le 0) = P\left(\frac{X-\mu}{\sigma} < -\frac{\mu}{\sigma}\right) = \Phi\left(-\frac{\mu}{\sigma}\right) \iff -\frac{\mu}{\sigma} = -.4303$;
$\frac{2}{3} = P(X \le 1) = P\left(\frac{X-\mu}{\sigma} < \frac{1-\mu}{\sigma}\right) = \Phi\left(\frac{1-\mu}{\sigma}\right) \iff \frac{1-\mu}{\sigma} = .4303$.
Subtract: $\frac{1}{\sigma} = .8606 \iff \sigma = 1.162$. Hence $\mu = .4303\sigma = 0.5$. Or you may easily see that, by symmetry, μ must be located halfway between 0 and 1.

b) If $P(X \le 1) = \frac{3}{4}$ then $\Phi\left(\frac{1-\mu}{\sigma}\right) = \frac{3}{4} \iff \frac{1-\mu}{\sigma} = .6742$. This implies that $\frac{1}{\sigma} = .6742 + .4303 = 1.1045$, and $\sigma = .9054$, and $\mu = .3896$.

7. Let X be the height of an individual picked at random from this population. We know that the distribution of X is approximately normal (μ, σ^2), with $\mu = 70$ (inches), and $P(X > 72) = .1$. That is,

$.9 = P(X \le 72) = P\left(\frac{X-70}{\sigma} \le \frac{72-70}{\sigma}\right) \approx P\left(Z \le \frac{2}{\sigma}\right) = \Phi\left(\frac{2}{\sigma}\right)$

(where Z has standard normal distribution). Hence $2/\sigma = 1.28$ from the normal table, and the chance that the height of an individual picked at random exceeds 74 inches is

$P(X > 74) = P\left(\frac{X-70}{\sigma} > \frac{74-70}{\sigma}\right) \approx P\left(Z > \frac{4}{\sigma}\right) = P(Z > 2.56) = 1 - \Phi(2.56) = .0052.$

In a group of 100, the number of individuals who are over 74 inches tall therefore has binomial(100, .0052) distribution. By the Poisson approximation, the chance that there are 2 or more such individuals is approximately (with $\mu = 100 \times .0052 = .52$)

$1 - (e^{-\mu} + e^{-\mu}\mu) = 1 - e^{-.52}(1 + .52) = .096.$

8. a)
$$\Phi\left(\frac{12.2-12}{1.1}\right)-\Phi\left(\frac{11.8-12}{1.1}\right)=.1443$$

b)
$$\Phi\left(\frac{12.2-12}{.11}\right)-\Phi\left(\frac{11.8-12}{.11}\right)=.9307$$

Since the Central Limit Theorem says that the average of a large number of measurements will be normal, it is not necessary for the measurements themselves to be normal, although if the measurements are extremely skewed then 100 may not be a large enough number.

9. The distribution of S_4 is approximately normal with mean 2 and variance $4\times\frac{1}{12}=\frac{1}{3}$.

$P(S_4\geq 3)\approx 1-\Phi\left(\frac{3-2}{1/\sqrt{3}}\right)=1-\Phi(1.73)=1-.9582=.0418$

10. a) $\Phi(\frac{9.800-9.7800}{0.0031})-\Phi(\frac{9.7840-9.7800}{0.0031})=\Phi(6.45)-\Phi(1.29)=0.0985$

b) $\Phi(\frac{9.7794-9.7800}{0.0031})=\Phi(-0.19)=1-\Phi(0.19)=0.4246$

c) $\Phi(1.28)\approx 0.90$, so the weight $=9.7800+(1.28\times 0.0031)=9.7840$ gm.

11. a) $\Phi(0.43)\approx\frac{2}{3}$, so $\frac{1.1-1}{\sigma}\approx 0.43$ and $\sigma\approx 0.2325$.

b) $\Phi\left(\frac{0.20}{0.2325}\right)-\Phi\left(-\frac{0.20}{0.2325}\right)=2\Phi(0.86)-1=0.6102$

c) $\Phi(0.675)\approx 0.75$, so the diameter is 0.675 multiples of σ below the mean. $1-(0.675\times 0.2325)\approx$ 0.84

12. a) Range of X: $[-2,2]$

If $-2\leq x\leq 2$, then
$f(x)dx=P(X\in dx)=\frac{2\times(2-|x|)dx}{4\times(\frac{1}{2}\times 2\times 2)}=\frac{1}{4}(2-|x|)dx,$

so $f(x)=(2-|x|)/4$. Elsewhere $f(x)=0$.

b) Range of X: $[-2,1]$.

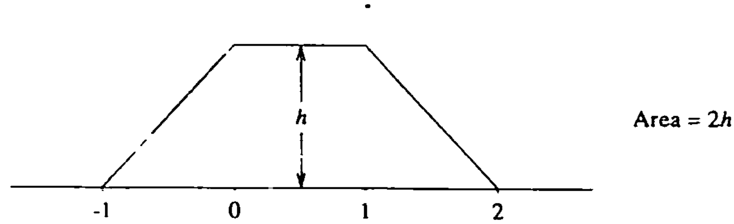
If $-2\leq x<0$, then
$f(x)dx=P(X\in dx)=\frac{(2+x)dx}{\frac{1}{2}\times 3\times 2}=\frac{1}{3}(2+x)dx$.

If $0\leq x\leq 1$, then $f(x)dx=\frac{2(1-x)dx}{3}$.

Elsewhere $f(x)=0$.

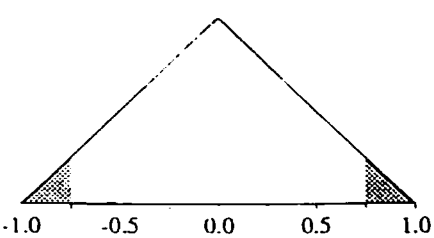
c) Range of $X:[-1,2]$.

The density will be linear on $[-1,0]$, constant on $[0,1]$, linear on $[1,2]$:



To make area $=1$, h must satisfy $2h=1$, or $h=\frac{1}{2}$.

13. Let X denote the length of a rod produced. Note that the probabilities of interest remain the same under a linear change of scale. So, without loss of generality, assume X has a triangular density from -1 to 1, as in the figure.



a) $P(|X|>.75)=1/16$ from the figure.

b) $P(|X| \le .5 \,|\, |X| \le .75) = \frac{1-(1/2)^2}{1-(1/4)^2} = .8$.

If the customer buys n rods, the number N of rods which meet his specifications (a··uming independence of rod lengths) has binomial (n, .8) distribution. Need n such that

$P(N \ge 100) \ge .95 \iff P(N < 100) \le .05$.

Now by the normal approximation to the binomial,

$P(N < 100) \approx \Phi\left(\frac{99.5-(n)(.8)}{\sqrt{(n)(.8)(.2)}}\right)$.

So solve

$\Phi\left(\frac{99.5-(n)(.8)}{\sqrt{(n)(.8)(.2)}}\right) \le .05 \iff \frac{99.5-(.8)n}{(.4)\sqrt{n}} \le -1.645 \iff n \ge 134.$

14. (continued from Exercise 13)
Again, the probabilities remain the same under a linear change of scale. Let Y be the length of a rod produced by the current manufacturing process. To get the mean and standard deviation of Y, note that the density of X is given by

$$f_X(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Therefore $E(Y) = E(X) = 0$ by symmetry; $Var(Y) = Var(X) = E(X^2) = \int_{-1}^{1} x^2(1-|x|)dx = 2\int_0^1 (x^2 - x^3)dx = \frac{1}{6}$.

So Y has normal (0, 1/6) distribution.

a) $P(|Y| > .75) = 2 \times \left[1 - \Phi(\sqrt{6} \times .75)\right] = .066193$

b) $P(|Y| \le .5) = 2 \times \Phi(\sqrt{6} \times .5) - 1 = .779328$;
therefore $P(|Y| \le .5 \,|\, |Y| \le .75) = \frac{.779328}{1-.066193} = .834571$.
So you should choose the manufacturer of this exercise, since each rod that you buy would have a greater chance of meeting your specifications (although not by much).

15. a) $(0, 1/2)$ b) $\text{erf}(x) = 2\Phi(\sqrt{2}x) - 1$ c) $\Phi(z) = (\text{erf}(z/\sqrt{2}) + 1)/2$

Section 4.2

1. Let X denote the lifetime of an atom. Then X has exponential distribution with rate $\lambda = \log 2$.

   a) $P(X > 5) = e^{-5\lambda} = (1/2)^5 = 1/32$.

   b) Find t (years) such that
   $$P(X > t) = .1 \iff e^{-t\lambda} = .1 \iff t = \frac{\log 10}{\lambda} = 3.32$$

   c) Assuming that the lifetimes of atoms are independent, the number N_t of atoms remaining after t years has binomial $(1024,\ e^{-\lambda t})$ distribution. So find t such that
   $$E(N_t) = 1 \iff 1024e^{-\lambda t} = 1 \iff t = \frac{\log 1024}{\lambda} = 10.$$

   d) N_{10} has binomial $(1024,\ 1/1024)$ distribution, which is approximately Poisson (1). So by the Poisson approximation,
   $$P(N_{10} = 0) \approx e^{-1} = .3679.$$

2. a) Let X denote the lifetime of an atom, then X is exponentially distributed with rate $\lambda = \frac{\log 2}{.5} = \log 4$ per century. Now we have 10^{20} atoms, and we wish to find the time such that we expect 1 out of 10^{18} atoms to survive, which we can do by solving $P(X > t) = \frac{1}{10^{18}}$ for t. We know that $P(X > t) = e^{-\lambda t}$ and so $-(\log 4)(t) = -18 \log 10$ and finally $t = 18\frac{\log 10}{\log 4} = 29.9 \approx 30$ centuries.

   b) This is equivalent to saying that there is about a 50% chance that no atoms are left. Let N_t be the number of atoms left at time t, and we wish to find t such that $P(N_t = 0) = .5$. Note that N_t will be a binomial (n,p) where $n = 10^{20}$. Note further that this will be approximately a Poisson with $\mu = np$. Thus we observe that
   $$P(N_t = 0) = \frac{e^{-np}(np)^0}{0!} = .5$$
   and so $np = \log 2$ and finally $p = \frac{\log 2}{n}$. We also know that $p = .5^t$ since p is the probability that a given atom will survive for t centuries, and so
   $$.5^t = \frac{\log 2}{n}$$
   $$t = \frac{\log\left(\frac{\log 2}{n}\right)}{\log .5} = 66.97 \approx 67$$
   So after 67 centuries there is about a 50% chance that no atoms are left.

3. Let T be the time until the next earthquake, then we have in general that $P(T < t) = 1 - e^{-\lambda t}$.

   a) The probability of an earthquake in the next year is $1 - e^{-1} = 0.6321$.

   b) Similarly, $P(T < .5) = 1 - e^{-.5} = 0.3935$.

   c) $P(T < 2) = 1 - e^{-2} = 0.8647$

   d) $P(T < 10) = 1 - e^{-10} = 0.99995$

4. Let W be the lifetime of a component. Then W has exponential distribution with rate $\lambda = 1/10$.

   a) $P(W > 20) = e^{-20\lambda} = e^{-2} \approx 0.135$.

   b) The median lifetime m satisfies
   $$1/2 = P(W > m) = e^{-m\lambda} \iff m = \frac{(\log 2)}{\lambda} = 6.93.$$

c) $SD(W) = 1/\lambda = 10$.

d) If X denotes the average lifetime of 100 independent components, then $E(X) = 10$ and $SD(X) = 10/\sqrt{100} = 1$ so by the normal approximation

$$P(X > 11) = 1 - P(X \le 11) \approx 1 - \Phi(1) = .1586$$

e) Let W_1 and W_2 denote the lifetimes of the first and second components respectively. Then

$$P(W_1 + W_2 > 22) = P(N < 2) = e^{-2.2} + 2.2e^{-2.2} = .35457$$

where N has Poisson $(22\lambda = 2.2)$ distribution.

5. a) $P(W_4 \le 2) = 1 - P(W_4 > 2) = 1 - e^{-2} \approx .86$.

b) $P(T_4 \le 5) = P(N(0,5] \ge 4) = 1 - P(N(0,5] \le 3) = 1 - e^{-5}(1 + 5 + 25/2 + 125/6) \approx .73$

c) $E(T_4) = E(W_1 + W_2 + W_3 + W_4) = E(W_1) + E(W_2) + E(W_3) + E(W_4) = 1 + 1 + 1 + 1 = 4$

Note. T_4 has gamma (4, 1) distribution.

6. Let N_2 be the number of hits during the first 2 minutes, and N_4 be the number of hits during the first 4 minutes. Then N_2 has Poisson (2) distribution and N_4 has Poisson (4) distribution, and

$$P(2 \le T_3 \le 4) = P(T_3 > 2) - P(T_3 > 4)$$
$$= P(N_2 \le 2) - P(N_4 \le 2)$$
$$= \left(e^{-2} + 2e^{-2} + \frac{e^{-2}2^2}{2!}\right) - \left(e^{-4} + 4e^{-4} + \frac{e^{-4}4^2}{2!}\right) = 5e^{-2} - 13e^{-4} = 0.43857.$$

7. $1 - e^{-t_p\lambda} = p \iff t_p = -\frac{1}{\lambda}\log(1-p)$

8. To compute the density f of X, argue infinitesimally:

$f(t)dt = P(X \in (t, t+dt))$

$= P(X \in (t, t+dt)|\lambda = \frac{1}{100})P(\lambda = \frac{1}{100}) + P(X \in (t, t+dt)|\lambda = \frac{1}{200})P(\lambda = \frac{1}{200})$

$= f_{Y_1}(t)dt \cdot \frac{1}{3} + f_{Y_2}(t)dt \cdot \frac{2}{3}$,

where f_{Y_1} is the density of an exponential random variable Y_1 having rate 1/100, and f_{Y_2} is the density of an exponential random variable Y_2 having rate 1/200. Hence

$f(t) = \frac{1}{3}f_{Y_1}(t) + \frac{2}{3}f_{Y_2}(t)(t > 0)$.

a) $P(X \ge 200) = \frac{1}{3}P(Y_1 \ge 200) + \frac{2}{3}P(Y_2 \ge 200) = \frac{1}{3}e^{-200/100} + \frac{2}{3}e^{-200/200} \approx .29$

b) $E(X) = \frac{1}{3}E(Y_1) + \frac{2}{3}E(Y_2) = \frac{1}{3} \times 100 + \frac{2}{3} \times 200 = \frac{500}{3}$

c) $E(X^2) = \frac{1}{3}E(Y_1^2) + \frac{2}{3}E(Y_2^2) = \frac{1}{3} \times 2 \times 100^2 + \frac{2}{3} \times 2 \times 200^2 = 6 \times 100^2$. Therefore $Var(X) = E(X^2) - [E(X)]^2 = \frac{290{,}000}{9}$.

9. a) $\Gamma(r+1) = \int_0^\infty x^r e^{-x}dx = -x^r e^{-x}\Big|_0^\infty + r\int_0^\infty x^{r-1}e^{-x}dx = r\Gamma(r)$

b) Note that $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$.

So $\Gamma(r+1) = r\Gamma(r) = r(r-1)\Gamma(r-1) = \cdots = r!\Gamma(1) = r!$.

c) $E(T^n) = \int_0^\infty t^n e^{-t}dt = \Gamma(n+1) = n!$.

$Var(T) = E(T^2) - [E(T)]^2 = 2 - 1 = 1 \Longrightarrow SD(T) = 1$.

d) $P(\lambda T > u) = P(T > u/\lambda) = e^{-\lambda u/\lambda} = e^{-u}$, so λT has exponential (1) distribution, and $E[(\lambda T)^n] = n! \Longrightarrow E(T^n) = n!/\lambda^n$.

$SD(\lambda T) = 1 \Longrightarrow SD(T) = 1/\lambda$.

10. a) Let $X = int\ (T)$. X takes values in $\{0, 1, 2, \ldots\}$ and

$P(X = k) = P(k \le T < k + 1)$
$= P(T < k + 1) - P(T < k)$
$= (1 - e^{-\lambda(k+1)}) - (1 - e^{-\lambda k})$
$= e^{-\lambda k}(1 - e^{-\lambda})$
$= q^k p$, where $p = 1 - e^{-\lambda}, q = 1 - p$.

b) If T has exponential distribution on $(0, \infty)$ with rate λ, then mT has exponential distribution with rate λ/m (why?) so by a) $mT_m = int\ (mT)$ has geometric distribution on $\{0, 1, \ldots\}$ with success parameter $p_m = 1 - e^{-\lambda/m}$.

Conversely, if for each $m = 1, 2, \ldots$ we have that $int\ (mT)$ has geometric distribution on $\{0, 1, \ldots\}$ with success parameter p_m, then: argue that there exists $\lambda > 0$ such that $p_m = 1 - e^{-\lambda/m}$ (set $q_1 = e^{-\lambda}$ and consider $P(T \ge 1)$); next argue that for each rational $t > 0$ we have $P(T \ge t) = e^{-\lambda t}$; finally argue that this last must hold for all real t by using the fact that $P(T > t)$ is a nonincreasing function of t.

c) $E(T_m) = \frac{1}{m} E(int\ (mT)) = \frac{1}{m} \cdot \frac{q_m}{p_m} = \frac{1}{m} \cdot \frac{e^{-\lambda/m}}{(1 - e^{-\lambda/m})} \to \frac{1}{\lambda}$ as $m \to \infty$
(Use l'Hôpital's rule, or series expansions).

Since $T_m \le T \le T_m + \frac{1}{m}$, we have $E(T_m) \le E(T) \le E(T_m) + \frac{1}{m}$. Let $m \to \infty$ to see $E(T) = \frac{1}{\lambda}$.
Similarly argue that $E(T_m^2) \to \frac{2}{\lambda^2}$ as $m \to \infty$, and therefore that $E(T^2) = \frac{2}{\lambda^2}$.

11. a) Want to show $P(T \ge t) \approx e^{-\lambda t}$. Write $\Delta = 10^{-6}$ seconds, and consider $t = n\Delta$ for $n = 0, 1, 2, \ldots$. By assumption,

$$P(T \le (n+1)\Delta \mid T > n\Delta) = \lambda\Delta \quad \text{for all } n = 0, 1, 2, \ldots; \quad \text{equivalently}$$

$$P(T > (n+1)\Delta | T > n\Delta) = 1 - \lambda\Delta \quad \text{for all } n = 0, 1, 2, \ldots.$$

Therefore

$$P(T \ge 0) = 1,$$

$$P(T > \Delta) = P(T > 0)P(T > \Delta | T > 0) = 1 \times (1 - \lambda\Delta) = 1 - \lambda\Delta,$$

$$P(T > 2\Delta) = P(T > \Delta)P(T > 2\Delta | T > \Delta) = (1 - \lambda\Delta)(1 - \lambda\Delta) = (1 - \lambda\Delta)^2.$$

In general, $P(T > n\Delta) = (1 - \lambda\Delta)^n$. Use the approximation $1 - \lambda\Delta \approx e^{-\lambda\Delta}$ to conclude

$$P(T > n\Delta) \approx (e^{-\lambda\Delta})^n = e^{-n\lambda\Delta}.$$

Put $t = n\Delta : P(T > t) \approx e^{-\lambda t}$.

b) $P(1 < T \le 2) \approx \int_1^2 \lambda e^{-\lambda t} dt = e^{-\lambda} - e^{-2\lambda}$.

12. a) Differentiate the gamma (r, λ) density:

$$\frac{d}{dt} f_{r,\lambda}(t) = \frac{\lambda^r}{\Gamma(r)} \left[(r-1)t^{r-2}e^{-\lambda t} - \lambda e^{-\lambda t} t^{r-1}\right] = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda t} t^{r-2}(r - 1 - \lambda t).$$

If $r \le 1$, then the derivative is negative for all $t > 0$, so $f_{r,\lambda}$ is maximized at 0.
If $r > 1$, then the derivative is zero at $t^* = (r-1)/\lambda$, is positive to the left of t^*, and is negative to its right. So t^* yields a local maximum for the density. But the density is zero at $t = 0$ and tends to zero as $t \to \infty$; hence the density achieves its overall maximum at t^*.
If $r < 1$, then the density blows up to ∞ as t approaches zero.

b) For $k = 0, 1, 2, \ldots$ we have :

$$E(T^k) = \int_0^\infty t^k \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t} dt = \frac{\lambda^r}{\Gamma(r)} \int_0^\infty t^{(r+k)-1} e^{-\lambda t} dt = \frac{\lambda^r}{\Gamma(r)} \frac{\Gamma(r+k)}{\lambda^{r+k}} = \frac{1}{\lambda^k} \frac{\Gamma(r+k)}{\Gamma(r)}.$$

Hence

$E(T) = \frac{1}{\lambda} \frac{\Gamma(r+1)}{\Gamma(r)} = \frac{r}{\lambda}$

$E(T^2) = \frac{1}{\lambda^2}\frac{\Gamma(r+2)}{\Gamma(r)} = \frac{(r+1)r}{\lambda^2}$
$Var(T) = \frac{r^2+r}{\lambda^2} - (\frac{r}{\lambda})^2 = \frac{r}{\lambda^2}$
$SD(T) = \frac{\sqrt{r}}{\lambda}$.

13. a) Estimate $\lambda = 1/20 = 5\%$ per day.

b) N_d has binomial $(10,000, e^{-d/20})$ distribution. Therefore

$$E(N_d) = 10,000e^{-d/20}, SD(N_d) = 100\sqrt{e^{-d/20}(1-e^{-d/20})}.$$

From this calculate
$E(N_{10}) = 6065$; $SD(N_{10}) = 49$;
$E(N_{20}) = 3679$; $SD(N_{20}) = 48$;
$E(N_{30}) = 2231$; $SD(N_{30}) = 42$.

14. Option b) is correct: The probability that a component fails in its first day of use is $1 - e^{-1/20}$, which is approximately $1/20 = 5\%$, because $1 - e^{-x} \approx x$ as $x \to 0$; and is less than 5% because $e^{-x} \geq 1 - x$ for all x (See Appendix III). Exact value is $4.877\ldots\%$.

15. a) 80 days

b) 40 days

c) $P(T_{total} \geq 60) = P(\text{at most 3 failures in 60 days}) = e^{-3}(1 + 3 + \frac{3^2}{2} + \frac{3^3}{6}) = 13e^{-3} \approx .64723$ since the number of failures in 60 days has Poisson $(.05 \times 60)$ distribution.

16. Say a total of k components will do. Since $P(T_{total} \geq 60) = P(N_{60} \leq k - 1)$, we require

$$P(N_{60} \leq k - 1) \geq 0.9$$

By trial and error, we find $P(N_{60} \leq 5) \approx 0.91608$, so a total of six components (five spares) will do.

17. Redoing the satellite problem:

a) 80 days

b) $20\sqrt{2}$ days

c) Guess the answer to c) should be larger, because now 60 days is more standard deviations below the mean of 80 days. In fact, T_{total} has the same distribution as the sum of 8 independent exponential (.1) random variables, so $P(T_{total} \geq 60) = P(N_{60} < 8) \approx .744$ where now N_{60} has Poisson (6) distribution.

Redoing the preceding problem: Since $P(N_{60} \leq 10) \approx .9161$, it follows that four spare components will do.

Section 4.3

1. a) $P(T \leq b) = 1 - P(T > b) = 1 - G(b)$.
   b) $P(a \leq T \leq b) = P(T \geq a) - P(T > b) = G(a) - G(b)$. (Since T is continuous, $P(T \geq a)$ equals $P(T > a)$ equals $G(a)$.)

2. Suppose T has constant hazard rate: Say $\lambda(t) = c$ for all $t > 0$. Use (7) to get

$$G(t) = e^{-\lambda t}, t > 0.$$

Then the density of T is, by (5),

$$f(t) = -\frac{dG(t)}{dt} = ce^{-ct}, t > 0,$$

so T has exponential distribution with rate c.

Conversely, if T has exponential distribution with rate λ, then for each $t > 0$:

$$f(t) = \lambda e^{-\lambda t}; G(t) = P(T > t) = e^{-\lambda t}; \lambda(t) = \frac{f(t)}{G(t)} = \lambda.$$

3. a) $\left(\frac{b}{b+t}\right)^a, t > 0$.
   b) $\left(\frac{a}{b+t}\right)\left(\frac{b}{b+t}\right)^a = \frac{ab^a}{(b+t)^{a+1}}, t > 0$.

4. (i) $\Longrightarrow$ (ii):

$$\exp\left(-\int_0^t \lambda(u)du\right) = \exp\left(-\int_0^t \lambda\alpha u^{\alpha-1}du\right) = \exp(-\lambda t^\alpha) = G(t).$$

(ii) $\Longrightarrow$ (iii): differentiate G with respect to t:

$$-\frac{d}{dt}G(t) = -\frac{d}{dt}e^{-\lambda t^\alpha} = \lambda\alpha t^{\alpha-1}e^{-\lambda t^\alpha} = f(t).$$

(iii) $\Longrightarrow$ (ii):

$$\int_t^\infty f(u)du = \int_t^\infty \lambda\alpha u^{\alpha-1}e^{-\lambda u^\alpha}du = -e^{-\lambda u^\alpha}\Big|_{u=t}^\infty = e^{-\lambda t^\alpha} = G(t).$$

(iii) & (ii) $\Longrightarrow$ (i):

$$\frac{f(t)}{G(t)} = \frac{\lambda\alpha t^{\alpha-1}e^{-\lambda t^\alpha}}{e^{-\lambda t^\alpha}} = \lambda\alpha t^{\alpha-1} = \lambda(t).$$

5. Let $k \geq 0$.
   a) $E(T^k) = \int_0^\infty t^k f(t)dt = \int_0^\infty t^k \lambda\alpha t^{\alpha-1}e^{-\lambda t^\alpha}dt = \int_0^\infty \left(\frac{x}{\lambda}\right)^{k/\alpha} e^{-x}dx = \lambda^{-k/\alpha}\int_0^\infty x^{k/\alpha}e^{-x}dx = \lambda^{-k/\alpha}\Gamma\left(\frac{k}{\alpha}+1\right)$.
   (Substitute $x = \lambda t^\alpha$.)
   b) By (a),

$$E(T) = \lambda^{-1/\alpha}\Gamma\left(\frac{1}{\alpha}+1\right);$$

$$E(T^2) = \lambda^{-2/\alpha}\Gamma\left(\frac{2}{\alpha}+1\right);$$

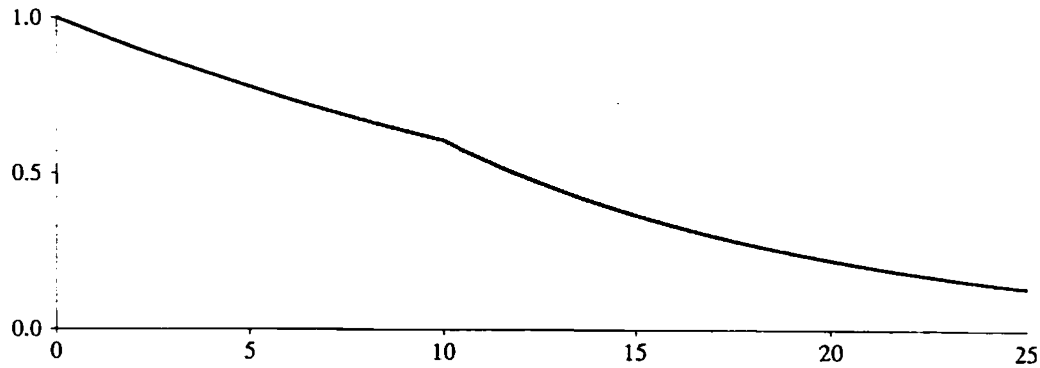
hence $$Var(T) = \lambda^{-2/\alpha}\left\{\Gamma\left(\frac{2}{\alpha}+1\right) - \left[\Gamma\left(\frac{1}{\alpha}+1\right)\right]^2\right\}.$$

6. We have $\lambda(t) = 1/20$ if $0 \leq t \leq 10$, and $\lambda(t) = 1/10$ if $t > 10$.

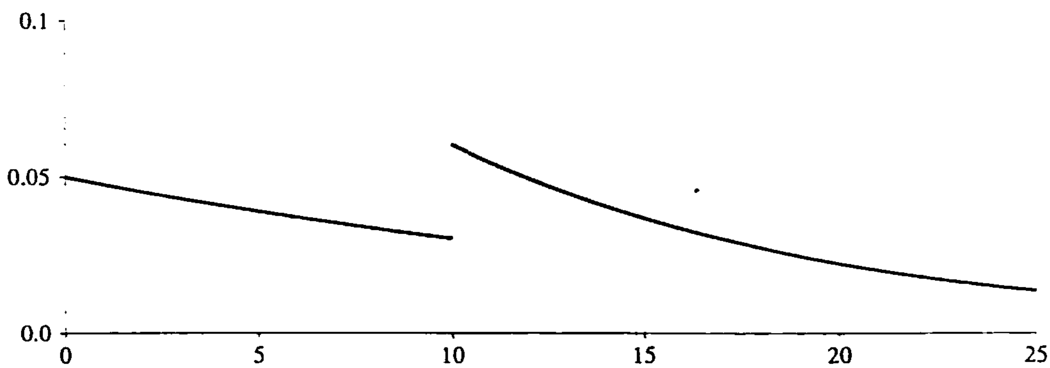
a) $P(T > 15) = G(15) = \exp\left(-\int_0^{15} \lambda(u)du\right) = \exp\left\{-[\int_0^{10}(1/20)du + \int_{10}^{15}(1/10)du\right. \quad = e^{-1} \approx$ .3679.

b) If $0 \le t \le 10$ then $G(t) = \exp\left(-\int_0^t (1/20)du\right) = e^{-t/20}$;

if $t > 10$ then $G(t) = \exp\left\{-[\int_0^{10}(1/20)du + \int_{10}^{t}(1/10)du]\right\} = e^{-(\frac{1}{2}+\frac{t-10}{10})}$.



c) $f(t) = -\frac{dG(t)}{dt} = \begin{cases} \frac{1}{20}e^{-t/20} & 0 < t < 10 \\ \frac{1}{10}e^{-(t/10-1/2)} & t > 10 \end{cases}$



d) $E(T) = \int_0^\infty G(t)dt = \int_0^{10} e^{-t/20}dt + \int_{10}^\infty e^{-(\frac{1}{2}+\frac{t-10}{10})}dt = 20\int_0^{1/2} e^{-u}du + 10e \quad \int_0^\infty e^{-v}dv =$ 13.93.

(Substitute $u = t/20$, $v = (t-10)/10$.)

7. a) Integrate by parts the relation $E(T^2) = \int_0^\infty t^2 f(t)dt$.

b) $E(T^2)$ is 400. So $SD(T)$ is $\sqrt{400 - 100\pi} \approx 9.265$.

c) If $\bar{T}$ denotes the average lifetime of 100 components, then $E(\bar{T}) = E(T_1) = 17.7245$ a… $D(\bar{T}) =$ $SD(T_1)/\sqrt{100} = 0.9265$ so by the normal approximati…

$P(\bar{T} > 20) \approx 1 - \Phi(2.456) = 0.007$.

8. a) Only for $a \ge 0$, $b \ge 0$, and either $a > 0$ or $b > 0$.

b) $G(t) = \exp -\left(\frac{at^2}{2} + bt\right)$.