York University
MATH 2030 3.0AF (Elementary Probability)
Midterm 2

October 30, 2006 – Salisbury

NAME:

STUDENT NUMBER:

You have 50 minutes to complete the examination. There are four questions, on three pages. No other books or notes may be used. You may use a calculator. Show all your work, and explain or justify your answers. You may leave numerical answers unsimplified.

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1. [20] A bag contains 5 dice. Four of them are standard fair dice. But the fifth has the numbers 2–3–3–4–5–6 marked on its sides (ie there is no side showing a 1, but there are two sides showing 3's). I pick a die at random from the bag, and roll it twice. Given that I get two 3's, what is the conditional probability the die is fair?

Solution: Let B be the event that the die is fair, and let A be the event that we roll two 3's. Then we know that

$$\begin{aligned} P(B) &= \frac{4}{5} \\ P(B^c) &= \frac{1}{5} \\ P(A \mid B) &= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \\ P(A \mid B^c) &= \frac{2}{6} \times \frac{2}{6} = \frac{4}{36} \end{aligned}$$

So by Bayes' rule

$$P(B \mid A) = \frac{P(B)P(A \mid B)}{P(B)P(A \mid B) + P(B^c)P(A \mid B^c)} = \frac{\frac{4}{5} \cdot \frac{1}{36}}{\frac{4}{5} \cdot \frac{1}{36} + \frac{1}{5} \cdot \frac{4}{36}} = \frac{1}{1 = 1} = \frac{1}{2}.$$

2. A random variable X has density function

$$f(x) = \begin{cases} cx, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) [10] Find c.

(b) [10] The cumulative distribution function of X has the form

$$F(x) = \begin{cases} 0, & x < 1 \\ ??, & 1 \le x < 2 \\ 1, & 2 \le x \end{cases}$$

Find what should go in place of the ??

(c) [10] Find the cumulative distribution function and density of the random variable $-X$

Solution:

(a) $1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_1^2 cx\,dx = \Big[cx^2/2\Big]_1^2 = c(4-1)/2 = c \cdot 3/2$. So $c = 2/3$.

(b) $F(x) = \int_{-\infty}^{x} f(t)\,dt$. For $1 \le x < 2$ this equals

$$\int_1^x \frac{2t}{3}\,dt = \left[\frac{t^2}{3}\right]_1^x = \frac{x^2-1}{3}$$

(c) Let $Y = -X$. Its cdf is $G(y) = P(Y \le y) = P(-X \le y) = P(X \ge -y) = 1 - P(X < -y) = 1 - F(-y)$ since X has a density. Substituting into the expression for F, we get

$$G(y) = \begin{cases} 1-0, & -y < 1 \\ 1 - \frac{(-y)^2-1}{3}, & 1 \le -y < 2 \\ 1-1, & 2 \le -y \end{cases} \quad = \begin{cases} 1, & -1 < y \\ \frac{4-y^2}{3}, & -2 < y \le -1 \\ 0, & y \le -2 \end{cases}$$

Differentiating, we get that the density of Y is

$$g(y) = G'(y) = \begin{cases} 0, & -1 < y \\ \frac{-2y}{3}, & -2 < y \le -1 \\ 0, & y \le -2 \end{cases}$$

3. X has density

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

In addition, Y satisfies $P(Y = -1) = 1/6$, $P(Y = 2) = 1/3$, and $P(Y = 5) = 1/2$.

(a) [10] Find the variance of Y.

(b) [10] Find $E[X - Y]$

Solution:

(a)

$$\begin{aligned}
E[Y] &= \sum_y yP(Y=y) = -1\cdot\frac{1}{6} + 2\cdot\frac{1}{3} + 5\cdot\frac{1}{2} \\
&= \frac{-1+2\cdot 2+5\cdot 3}{6} = 3 \\
E[Y^2] &= \sum_y y^2 P(Y=y) = -1\cdot\frac{1}{6} + 4\cdot\frac{1}{3} + 25\cdot\frac{1}{2} \\
&= \frac{1+4\cdot 2+25\cdot 3}{6} = 14 \\
\mathrm{Var}[Y] &= E[Y^2] - E[Y]^2 = 14 - 3^2 = 5
\end{aligned}$$

(b) $E[X] = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-1}^{2} x^3/3\,dx = \left[x^4\right]_{-1}^{2} = (16-1)/12 = 5/4$. We know that $E[Y] = 3$ from part (a). So $E[X-Y] = \frac{5}{4} - 3 = -7/4$.

4. [10] X has mean 5 and variance 7. Find $E[(X+2)^2]$

Solution: $\mathrm{Var}[X] = E[X^2] - E[X]^2$ so $E[X^2] = \mathrm{Var}[X] + E[X]^2 = 7 + 5^2 = 32$. Therefore $E[(X+2)^2] = E[X^2+4X+4] = E[X^2]+4E[X]+4 = 32+4\cdot 5+4 = 56$