York University

MATH 2030 3.0AF (Elementary Probability)
Midterm Exam I – SOLUTIONS

October 17, 2008

NAME:

STUDENT NUMBER:

You have 50 minutes to complete the examination. There are 4 pages, containing 5 questions. You may bring one letter-sized two-sided formula sheet to the exam. No other books or notes may be used. You may use a calculator. Show all your work, and explain or justify your answers. You may leave numerical answers unsimplified, including binomial coefficients. Use the back of your page if you run out of room.

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1. [20] An urn has 15 red balls and 20 yellow balls. You choose 4 balls at random, without replacement. Find

$$P(\text{at most 3 red balls are chosen} \mid \text{at least 3 red balls are chosen})$$

Solution:

$$\begin{aligned}
P(\text{at most 3 red} \mid \text{at least 3 red}) &= \frac{P(\text{at most 3 red and at least 3 red})}{P(\text{at least 3 red})} \\
&= \frac{P(\text{exactly 3 red})}{P(\text{at least 3 red})} = \frac{P(\text{exactly 3 red})}{P(\text{exactly 3 red}) + P(\text{exactly 4 red})} \\
&= \frac{\frac{\binom{15}{3}\binom{20}{1}}{\binom{35}{4}}}{\frac{\binom{15}{3}\binom{20}{1}}{\binom{35}{4}} + \frac{\binom{15}{4}\binom{20}{0}}{\binom{35}{4}}} = \frac{\binom{15}{3}\binom{20}{1}}{\binom{15}{3}\binom{20}{1} + \binom{15}{4}\binom{20}{0}} \\
&= \frac{9100}{10465} = \frac{20}{23} \approx 0.8696
\end{aligned}$$

2. [20] A, B, C are independent events, and $P(A) = 0.6$, $P(B) = 0.7$, $P(C) = 0.8$; Let D be the event that at least 2 of the events A, B, or C occur. Find $P(D)$.

Solution: Enumerating the mutually exclusive ways this can occur,

$$\begin{aligned}
P(\text{at least 2}) &= P(A \cap B \cap C) + P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C) \\
&= 0.6 \times 0.7 \times 0.8 + 0.6 \times 0.7 \times 0.2 + 0.6 \times 0.3 \times 0.8 + 0.4 \times 0.7 \times 0.8 \\
&= 0.336 + 0.084 + 0.144 + 0.244 = 0.788
\end{aligned}$$

Alternatively, you can deduce from the Venn diagram that

$$P(\text{at least 2 occur}) = P(A \cap B) + P(B \cap C) + P(A \cap C) - 2P(A \cap B \cap C)$$

and then work these out as above. Another way of obtaining the above expression is to use inclusion exclusion with $E_1 = A \cap B$, $E_2 = B \cap C$, and $E_3 = A \cap C$. Then
$P(\text{at least 2 occur}) = P(E_1 \cup E_2 \cup E_3)$
$= P(E_1)+P(E_2)+P(E_3)-P(E_1 \cap E_2)-P(E_1 \cap E_3)-P(E_2 \cap E_3)+P(E_1 \cap E_2 \cap E_3)$.
Since each of the last 4 terms is $P(A \cap B \cap C)$, the last 4 terms together give $-2P(A \cap B \cap C)$ as in the above formula.

3. [20] You conduct a phone survey. On average, 10% of calls made on weekdays (Monday – Friday) produce a response. On Saturdays this figure rises to 20%, and on Sundays it is 30%. You pick a day from a certain week (Monday – Sunday) uniformly at random to conduct your survey, and find that your first call produces a response. Given this, what is the conditional probability that you chose a weekday to conduct your survey?

Solution: Let
B_1 be the event that you choose a weekday (so $P(B_1) = 5/7$);
B_2 be the event that you choose a Saturday (so $P(B_2) = 1/7$);
B_3 be the event that you choose a Sunday (so $P(B_3) = 1/7$);
A be the event that the first person called responds.
Then $P(A \mid B_1) = 0.1$, $P(A \mid B_2) = 0.2$, and $P(A \mid B_3) = 0.3$ so by Bayes' rule

$$\begin{aligned} P(B_1 \mid A) &= \frac{P(A \mid B_1)P(B_1)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)} \\ &= \frac{0.1 \times 5/7}{0.1 \times 5/7 + 0.2 \times 1/7 + 0.3 \times 1/7} = 0.5 \end{aligned}$$

4. [15] Let X have density

$$f(x) = \begin{cases} \frac{x}{8}, & 0 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

Define a new random variable Y to be

$$Y = \begin{cases} 0, & \text{if } X \le 0 \\ 3, & \text{if } 0 < X \le 3 \\ 5, & \text{otherwise.} \end{cases}$$

Fill in the distribution table

y	0	3	5
$P(Y = y)$			

Solution:

y	0	3	5
$P(Y=y)$	0	9/16	7/16

To see this, note that

$$P(Y=0) = P(X \le 0) = \int_{-\infty}^{0} 0\,dx = 0$$
$$P(Y=3) = P(0 < X \le 3) = \int_0^3 \frac{x}{8}\,dx = \left[_0^3 \frac{x^2}{16}\right] = \frac{9}{16}$$
$$P(Y=5) = 1 - P(Y=0) - P(Y=3) = \frac{7}{16}.$$

(Or $P(Y=5) = P(X > 3) = \int_3^4 \frac{x}{8}\,dx + \int_4^\infty 0\,dx = \left[_3^4 \frac{x^2}{16}\right] = \frac{7}{16}$)

5. X has density

$$f(x) = \begin{cases} x + c, & -1 < x < 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) [5] Find c. What properties must you check to know that f is a density?
(b) [10] Find the cumulative distribution function of X.

Solution:

(a) The properties needed are that $f \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$. So

$$1 = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} (x+c)\,dx + \int_0^\infty 0\,dx$$
$$= \left[_{-1}^{0} \frac{x^2}{2} + cx\right] = 0 - \left(\frac{1}{2} - c\right) = -\frac{1}{2} + c.$$

Thus $c = 3/2$.

(b) For $-1 \le x < 0$,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{x} \frac{t+3}{2}\,dt = \left[_{-1}^{x} \frac{t^2+3t}{2}\right]$$
$$= \frac{x^2+3x}{2} - \frac{1-3}{2} = \frac{x^2+3x+2}{2}.$$

For $x < -1$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$. And for $x \ge 0$,
$F(x) = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{0} (t^2+3)/2\,dt + \int_0^x 0\,dt = 1$. Thus

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2+3x+2}{2}, & -1 \le x < 0 \\ 1, & x \ge 0 \end{cases}$$