Assignment 2 solutions. (Chapter 1 of Pitman.)

Exercises 1.4.

5. (a,b) It is difficult to draw pictures in LaTeX, so I will describe the tree diagram in words. Corresponding to the choice of urn, there are two branches with probability 1/2 each. From the first branch (urn 1), there are two branches, with conditional probabilities 2/5 for black and 3/5 for white. From the second branch (urn 2), there are two branches, with conditional probabilities 4/7 for black and 3/7 for white.

(c) $(1/2)(2/5) + (1/2)(4/7) = 17/35$.

8. This could be done using Bayes's rule, but since that doesn't appear until the next section, we'll just use conditional probabilities.

If we label the three types of cards as (w_1, w_2) (30%), (b_3, w_3) (50%), and (b_1, b_2) (20%), then there are six outcomes in the outcome space corresponding to the face that is exposed, namely, $w_1, w_2, b_3, w_3, b_1, b_2$ with probabilities $0.15, 0.15, 0.25, 0.25, 0.10, 0.10$, respectively. We want to find the conditional probability of $A = \{\text{opposite face is white}\} = \{w_1, w_2, b_3\}$ given $B = \{\text{exposed face is black}\} = \{b_1, b_2, b_3\}$. Note that $AB = \{b_3\}$, so $P(A \mid B) = P(b_3)/[P(b_1)+P(b_2)+P(b_3)] = 0.25/(0.10+0.10+0.25) = 5/9$.

9. They are different because, for example, a student from the smaller school gets picked with probability 1/1000 in scheme A and 1/300 in scheme B; a student from the larger school gets picked with probability 1/1000 in scheme A and 1/1500 in scheme B. Scheme C will be equivalent to scheme A if p_i is proportional to the size of school i. This happens if $p_1 = 1/10$, $p_2 = 4/10$, and $p_3 = 5/10$.

11. Use a tree diagram. The first node has two branches, for identical (prob. p) and fraternal (prob. q). From the first branch there are two branches with conditional probabilities 1/2 for boys and 1/2 for girls. From the second brach, there are two branches for the firstborn, boy or girl, and from both of these branches there are two branches for the secondborn, boy or girl. All conditional probabilities are 1/2. This diagram yields the following answers.

(a) $p/2 + q/4 = p/2 + (1-p)/4 = (1+p)/4$.

(b) $q/4 = (1-p)/4$.

(c) $(q/4)/(p/2 + q/2) = (q/4)/(1/2) = q/2 = (1-p)/2$.

(d) $(p/2+q/4)/(p/2+q/2) = (p/2+q/4)/(1/2) = p+q/2 = p+(1-p)/2 = (1+p)/2$.

Exercises 1.5.

1. (a) $P(\text{black}) = P(\text{odd})P(\text{black} \mid \text{odd}) + P(\text{even})P(\text{black} \mid \text{even}) = (1/2)(1/4) + (1/2)(2/6) = 7/24$.

(b)

$$P(\text{even} \mid \text{white}) = \frac{P(\text{even})P(\text{white} \mid \text{even})}{P(\text{odd})P(\text{white} \mid \text{odd}) + P(\text{even})P(\text{white} \mid \text{even})}$$
$$= \frac{(1/2)(4/6)}{(1/2)(3/4) + (1/2)(4/6)} = \frac{8}{17}.$$

2. (a) The first two branches correspond to the first draw, white with probability 4/10, black with probability 6/10. From the first branch (white) there are two branches, white with conditional probability 7/13 and black with probability 6/13. From the second branch (black) there are two branches, white with conditional probability 4/13 and black with probability 9/13. So the answer is $(4/10)(7/13) + (6/10)(4/13) = 52/130 = 2/5$.

(b)

$$P(\text{first black} \mid \text{second white}) = \frac{(6/10)(4/13)}{(4/10)(7/13) + (6/10)(4/13)} = 6/13.$$

(c)

$$\frac{w}{w+b}\,\frac{w+d}{w+b+d} + \frac{b}{w+b}\,\frac{w}{w+b+d} = \frac{w(w+d)+bw}{(w+b)(w+b+d)} = \frac{w}{w+b}.$$

5. (a) $(0.01)(0.8) + (0.99)(0.05) = 0.0575$.
(b) $(0.01)(0.2) = 0.002$.
(c) $(0.99)(0.95) = 0.9405$.
(d)

$$P(\text{disease} \mid \text{positive diagnosis}) = \frac{(0.01)(0.8)}{(0.01)(0.8) + (0.99)(0.05)} = \frac{16}{115} = 0.13913.$$

(e) Yes, the experiment can be repeated (with other patients).

9. (a) The probability of a six is 1/6 with the first shape (a fair die), 1/4 with the second shape, and 1/3 with the third. $(1/3)(1/6) + (1/3)(1/4) + (1/3)(1/3) = 9/36 = 1/4$.

(b)

$$\frac{(1/3)(1/6)}{(1/3)(1/6) + (1/3)(1/4) + (1/3)(1/3)} = \frac{2}{9}.$$