York University

MATH 2030 3.0AF (Elementary Probability)
Midterm 2 - Corrected Solutions

November 7, 2007

NAME:

STUDENT NUMBER:

You have 50 minutes to complete the examination. There are 5 pages, containing 5 questions and a Normal table. You may bring one letter-sized two-sided formula sheet to the exam. No other books or notes may be used. You may use a calculator. Show all your work, and explain or justify your solutions to the extent possible. You may leave numerical answers unsimplified, and you do not have to interpolate when using a normal table. Use the back of your page if you run out of room.

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1. A discrete random variable X has the following distribution:

x	-1	0	2	4	5
$P(X = x)$	0.2	0.1	0.3	c	0.2

All other values have probability 0 of occurring.

(a) [5] Find c, explaining what properties you use to reach your conclusion.
(b) [5] Find $P(X \le 3)$.
(c) [5] Find $E[X]$.

Solution:

(a) $P(\Omega) = 1$, so $1 = 0.2 + 0.1 + 0.3 + c + 0.2 = 0.8 + c$ and hence $c = 0.2$
(b) $P(X \le 3) = P(X = -1) + P(X = 0) + P(X = 2) = 0.2 + 0.1 + 0.3 = 0.6$
(c) $E[X] = \sum x \cdot P(X = x) = (-1)\times 0.2 + 0\times 0.1 + 2\times 0.3 + 4\times 0.2 + 5\times 0.2 = 2.2$

2. [10] Let X have a binomial distribution with $n = 8$ and $p = 0.3$, and let Y be uniformly distributed on $[0, 2]$. Set $V = 1 + Y - 2X$. Find $E[V]$.

Solution:
We know that $E[X] = np = 8 \times 0.3 = 2.4$ and
$E[Y] = (a + b)/2 = (0 + 2)/2 = 1$. So
$E[V] = E[1 + Y - 2X] = E[1] + E[Y] - 2E[X] = 1 + 1 - 2 \times 2.4 = -2.8$

Note: If you didn't remember the formula for $E[Y]$, it was easy to work it out, as

$$\int_0^2 y \cdot \frac{1}{2}\, dy = \left[\frac{y^2}{4}\right]_0^2 = 1.$$

3. The random variable X has a density. Its cumulative distribution function is

$$F(x) = \begin{cases} 0, & x < 1 \\ c(x^2 - 1), & 1 \le x < 3 \\ 1, & x \ge 3. \end{cases}$$

(a) [5] Find c, explaining what properties you use to reach your conclusion.
(b) [5] Find $E[X]$.
(c) [5] Find $P(X > 2)$.

Solution:

(a) Because X has a density, the cdf $F(x)$ will be continuous. So the values on either side of $x = 3$ must be equal.
In other words, $1 = c(3^2 - 1) = 8c$, so $c = 1/8$

(b) The density is $f(x) = F'(x) = 2cx = x/4$ for $1 < x < 3$ and $f(x) = 0$ elsewhere. So

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_1^3 \frac{x^2}{4}\, dx = \left[\frac{x^3}{12}\right]_1^3 = \frac{27 - 1}{12} = \frac{13}{6}.$$

(c) $P(X > 2) = 1 - P(X \le 2) = 1 - F(2) = 1 - (2^2 - 1)/8 = 5/8$

Note: You could also have answered part (a) using the density, though this would have been more work. In particular, $f(x) = F'(x) = 2cx$ for $1 < x < 3$. So

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_1^3 2cx\, dx = \left[cx^2\right]_1^3 = c(9 - 1),$$

and again $c = 1/8$.

4. To play a certain midway game at the CNE, you toss four coins at a milk jug. Each coin has a probability of 0.35 of landing in the jug, and you win the game if at least two of the coins go in.

(a) [5] What is the probability that you win the game?

(b) [15] 90 people each play the game. What is a good approximation to the probability that at least 45 of them win?

Solution:

(a) Let X be the number of coins which go in. Then $X \sim \mathrm{Bin}(4, 0.35)$ so

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{4}{0}(0.35)^0(0.65)^4 - \binom{4}{1}(0.35)^1(0.65)^3 \\ &= 1 - (0.65)^4 - 4(0.35)(0.65)^3 = 0.4370 \end{aligned}$$

(b) Let Y be the number of people who win. Then $Y \sim \mathrm{Bin}(90, p)$ where we have calculated $p = 0.4370$ in part (a). We approximate Y by a normal random variable $\mu + \sigma Z$ where $\mu = np = 90 \times 0.4370 = 39.33$ and $\sigma = \sqrt{np(1-p)} = \sqrt{90 \times 0.5630 \times 0.4370} = 4.706$;
Using the continuity correction, and the nearest value in our normal table,

$$\begin{aligned} P(Y \geq 45) = P(Y \geq 44.5) &= P\Big(\frac{Y - 39.33}{4.706} \geq \frac{44.5 - 39.33}{4.706}\Big) \\ &\approx P(Z \geq 1.098) = 1 - P(Z \leq 1.098) \\ &\approx 1 - P(Z \leq 1.10) = 1 - 0.8643 = 0.1357 \end{aligned}$$

This is an acceptable answer, as is 0.1360 (which would be the answer if you interpolated between table values).

5. [10] A game is played as follows: You pay \$1.50 to play, and then roll 8 dice. You get back \$1 for each number that doesn't come up.
(So for example, if your 8 dice yield numbers 2, 1, 2, 3, 1, 5, 5, 2 then the two numbers 4 and 6 fail to come up, so you get back \$2)
If you play this game repeatedly, then over the long run do you expect to win money or lose money? On average, how much will you win or lose per game?

Solution:

Let X be the amount you win or lose in one play of the game. Over the long run, the amount you will win or lose per play is $E[X]$.

We have that $X = -1.5 + Y$ where Y counts the numbers which don't come up. In other words,

$$Y = \sum_{i=1}^{6} 1_{A_i}$$

where A_i is the event that the number i does not come up. Then $P(A_i) = (5/6)^8$ because each die has a probability $5/6$ of not giving the number i, and the 8 rolls are independent. Thus

$$\begin{aligned} E[X] &= -1.5 + E[Y] = -1.5 + \sum_{i=1}^{6} E[1_{A_i}] = -1.5 + \sum_{i=1}^{6} P(A_i) \\ &= -1.5 + 6 \times (5/6)^8 = -0.1046 \end{aligned}$$

Because this is negative, we expect to lose money in the long run, at a rate of about 10 cents per game.