York University

MATH 2030 3.0AF (Elementary Probability)
Midterm I - SOLUTIONS

October 12, 2007

NAME:

STUDENT NUMBER:

You have 50 minutes to complete the examination. There are 5 questions on 4 pages. You may bring one letter-sized two-sided formula sheet to the exam. No other books or notes may be used. You may use a calculator. Show all your work, and explain or justify your solutions to the extent possible. You may leave numerical answers or binomial coefficient unsimplified, unless specifically told otherwise. Use the back of your page if you run out of room.

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1. [10] $P(A) = 0.3$ and $P(B) = 0.2$; What value of $P(A \cup B)$ ensures that
   (a) A and B are independent?
   (b) A and B are disjoint?

Solution:
   (a) If A and B are independent then $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = 0.3 + 0.2 - 0.3 \times 0.2 = 0.44$
   (b) If A and B are disjoint then $P(A \cup B) = P(A) + P(B) = 0.3 + 0.2 = 0.5$

2. [10] Suppose $P(A) = 0.4$, $P(B) = 0.2$, and $P(A \mid B) = 0.7$
   Find $P(A \cap B^c)$.

Solution:
$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(B)P(A \mid B) = 0.4 - 0.2 \times 0.7 = 0.26$

3. You are dealt 10 cards from a regular deck of 52 cards.
   (a) [10] What is the probability that of these 10 cards, exactly 5 will be hearts and exactly 4 will be aces?
   (b) [3] Briefly describe a model (Ω, P) on which your answer to (a) could be based, and find the number of elements in Ω.
   (c) [2] (Unrelated to the other parts.) Work out and simplify $\binom{11}{3}$.

Solution:

(a) To specify such a 10 card hand, we need the following information: the 4 aces, the 4 other hearts (out of 12 possibilities, remembering that the ace of hearts is already counted among the aces), and the 2 cards that are neither aces nor hearts (out of $36 = 52 - 4 - 12$ possibilities). Thus the probability of such a hand is

$$\frac{\binom{4}{4}\binom{12}{4}\binom{36}{2}}{\binom{52}{10}} = \frac{311850}{15820024220} = 0.0000197$$

(b) One possible model would let Ω consist of all possible 10-element set of cards, with P assigning equal probability to each outcome. In this case $\#\Omega = \binom{52}{10}$.

(c) $\binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = 165$

4. [15] Amit and Betsy are the proofreaders at an advertising firm. An advertisement might be checked only by Amit – the probability of this is 0.4; Or it might be checked only by Betsy – the probability of this is 0.5; Or it might be checked by both of them – the probability of this is 0.1;

Betsy on average finds 80% of the typos in the ads she checks. Amit on average finds 70% of the typos in the ads he checks.

An ad in today's paper has the firm's name spelled wrong. Given this, what is the (conditional) probability that the copy was checked by both Amit and Betsy?

Solution: Let A be the event that only Amit checked the ad, so $P(A) = 0.4$; Let B be the event that only Betsy checked the ad, so $P(B) = 0.5$; Let C be the event that both Amit and Betsy checked the ad, so $P(C) = 0.1$

Let M be the event that the typo in the firm's name is missed (and so appears in print). We are asked to find $P(C \mid M)$ and we do this using Bayes rule.

When Amit checks an ad, the probability that he finds a typo is $1 - 0.7 = 0.3$; When Betsy checks an ad, the probability that she finds a typo is $1 - 0.5 = 0.2$; Amit and Betsy are assumed to work independently of each other, so we get the following conditional probabilities:

$$P(M \mid A) = 0.3 \qquad P(M \mid B) = 0.2 \qquad P(M \mid C) = 0.2 \times 0.3 = 0.06$$

So by Bayes rule,

$$\begin{aligned} P(C \mid M) &= \frac{P(C)P(M \mid C)}{P(C)P(M \mid C) + P(B)P(M \mid B) + P(A)P(M \mid A)} \\ &= \frac{0.1 \times 0.06}{0.1 \times 0.06 + 0.5 \times 0.2 + 0.4 \times 0.3} = 0.02655 \end{aligned}$$

5. [15] In a psychology experiment about learning, a mouse is placed in a maze and is given 2 minutes to find a piece of cheese. This experiment is repeated 3 times.

The first time, the mouse has probability 0.1 of success. Each time the mouse succeeds, its probability of success on the next trial gets doubled.

What is the probability that out of these 3 trials, the mouse succeeds at least twice?

Solution: In symbols, let's denote by S_1 the event that the mouse succeeds on the first trial, and by F_1 the event that it fails on the first trial. Likewise S_2 denotes the event that it succeeds on the second trial, etc. Then

$$P(S_1 \cap S_2 \cap S_3) = P(S_1)P(S_2 \mid S_1)P(S_3 \mid S_2 \cap S_1) = 0.1 \times 0.2 \times 0.4 = 0.008$$
$$P(S_1 \cap S_2 \cap F_3) = P(S_1)P(S_2 \mid S_1)P(F_3 \mid S_2 \cap S_1) = 0.1 \times 0.2 \times 0.6 = 0.012$$
$$P(S_1 \cap F_2 \cap S_3) = P(S_1)P(F_2 \mid S_1)P(S_3 \mid F_2 \cap S_1) = 0.1 \times 0.8 \times 0.2 = 0.016$$
$$P(F_1 \cap S_2 \cap S_3) = P(F_1)P(S_2 \mid F_1)P(S_3 \mid S_2 \cap F_1) = 0.9 \times 0.1 \times 0.2 = 0.018$$

These are the four ways the mouse can succeed at least twice, and adding up the probabilities we get a total probability of 0.54

The above calculations can also be based on the following tree diagram (lines 1, 2, 3, 5):