Solutions to Suggested Problems from Section 1.1

1.1.2 It's easier to read the data off this table:

word	# letters	# vowels
suppose	7	3
a	1	1
word	4	1
is	2	1
picked	6	2
at	2	1
random	6	2
from	4	1
this	4	1
sentence	8	3

(a) 7 out of 10 words have at least 4 letters, $\frac{7}{10}$.

(b) 4 out of 10 contain 2 or more vowels, $\frac{4}{10}$.

(c) The same words counted in (b) all have 4 or more letters; $\frac{4}{10}$.

1.1.3 With replacement, the the number of pairs is n^2.

(a) The probabilitly of any one ordered pair is $1/n^2$.

(b) The number of ways this could happen is $n-1$ (the first number could be any number $1, \ldots, n-1$, and the first number determines the second one), so the probability of this event is $\frac{n-1}{n^2}$

(c) There are a couple of ways to see this one. The easiest is by symmetry: there are three things that can happen – either the first number is bigger than the second, the second is bigger than the first, or they are the same. The first two situations occur the same number of times (swapping the order of the numbers takes you from one situation to the other), and the third situation happens n times (one for each number). It follows that

$$\#\text{ ways the two numbers are different} = n^2 - n,$$

so

$$\#\text{ ways the second one is bigger} = \frac{n^2 - n}{2}.$$

Another way to see the same formula is to get it as a sum over all possible first numbers. If the first number drawn is 1, we have $n-1$ choices for the second one (it just needs to be bigger than 1). If we draw a j, for $j = 1..n-1$, we have $n-j$

choices for the second one, as it should be bigger. Thus the total number of possible pairs where the second is bigger than the first is

$$\sum_{j=1}^{n-1}(n-j) = \sum_{j=1}^{n-1} n - \sum_{j=1}^{n-1} j = n(n-1) - \frac{(n-1)n}{2} = \frac{n(n-1)}{2}.$$

By either argument, the probability of this event is $\frac{n^2-n}{2n^2}$.

Without replacement, the total number of possible pairs is $n(n-1)$.

(a) By the same logic as before, the probability of any individual ordered pair is $\frac{1}{n(n-1)}$.

(b) The number of possible consecutive pairs is still $n-1$, so the probability is now $\frac{n-1}{n(n-1)} = \frac{1}{n}$.

(c) The number of pairs where the second one is bigger stays the same as before, so the probability of this event is $\frac{n^2-n}{2n(n-1)} = \frac{1}{2}$. Why is this answer so much simpler?

1.1.4

(a) If we both lose, it has to land on green, with probability $2/38$.

(b) If we don't both lose, at least one of us wins, so the probability is $1 - 2/38 = 36/38$.

(c) The ball can't land on both red and black at the same time, so the probability that we both win is zero. Hence the probability that at least one of us loses is 1.

1.1.5

(a) $52 \cdot 51$.

(b) $\frac{4}{52} = 1/13$.

(c) $\frac{1}{13}$ from the symmetry argument. Or, you can say that among all ordered pairs of cards, there are $4 \cdot 3$ where both are aces, and $48 \cdot 4$ where the second card is an ace but the first is not, for a total of 204 hands where the second card is an ace. Thus the probability that the second card is an ace is $204/(52 \cdot 51) = 1/13$.

(d) $(4 \cdot 3)/(52 \cdot 51)$.

(e) There are $48 \cdot 4$ where the second card is an ace and the first is not, $4 \cdot 48$ where the first is an ace and the second is not, and $4 \cdot 3$ where they are both aces, so 396 pairs where at least one is an ace. Hence the probability is $396/(52 \cdot 51)$. Alternatively, there are $48 \cdot 47$ pairs where neither one is an ace, and those are the only ones we need to rule out – so the probability can also be calculated as $1 - \frac{48 \cdot 47}{52 \cdot 51}$.

1.1.7 When rolling two dice, there are 36 possible rolls.

(a) If the max is 2, both numbers must be ≤ 2, so there are $2^2 = 4$ ways, that is, the probability is $4/36$.

(b) For both to be ≤ 3, the probability is $9/36$.

(c) If the max is exactly 3, it's ≤ 3 but not ≤ 2, so the probability is $9/36 - 4/36 = 5/36$.

(d) By the same logic, the probability that both dice land $\leq k$ is $k^2/36$, so the probability that the max is exactly k is $\frac{k^2-(k-1)^2}{36}$.

(e) Well, the max number has to be something between 1 and 6, so these numbers better add up to 1.