Math 1090 A **Summer 2009** **Answers to Homework – 5**

Exercise 3 from Section 4.3 (p. 149)

- Let f be a unary function, g a binary function, 7 a constant, and $<$ a binary predicate (which we write between its arguments, instead of before them).
- For each of the following, if the substitution is undefined or does not change the formula, explain why. Otherwise, just give the result.

(a) $g\big(f(x), f(y)\big)[x := 7]$

Answer

$g\big(f(7), f(y)\big)$

(b) $\big(f(x) < 7\big)[x := 7]$

Answer

$f(7) < 7$

$\big($Note that the formula $f(x) < 7$, which is a predicate and its arguments, does not have outer brackets. The brackets around it in (b) are there only to indicate that the substitution applies to the entire formula. I've removed the brackets which the the text incorrectly placed around $<$ and its arguments in parts (c)–(g).$\big)$

(c) $\big((\forall x)f(x) < 7\big)[x := 7]$

Answer

The substitution does not change the formula, because there are no free occurrences of x in the formula.

(d) $\big((\forall y)f(x) < 7\big)[x := 7]$

Answer

$(\forall y)f(7) < 7$

$\big($I've abbreviated the answer here and in part (g) by omitting the outer brackets.$\big)$

(e) $\big((\forall x)(\forall y)f(7) < g(x, y)\big)[z := g(y, 7)]$

Answer

The substitution does not change the formula, because there are no occurrences of z in the formula.

(f) $\big((\forall x)(\forall y)f(z) < g(x, y)\big)[z := g(y, 7)]$

Answer

Undefined, because replacing the free occurrence of z in the formula by $g(y, 7)$ would cause the y in $g(y, 7)$ to become bound by $(\forall y)$.

(g) $\big((\forall x)(\forall y)(\forall z)f(z) < g(x, y)\big)[z := g(y, 7)]$

Answer

The substitution does not change the formula, because there are no free occurrences of z in the formula.

Exercises from Section 6.6 (pp. 187–190)

3. Prove $(\forall \mathbf{x})(A \vee B \rightarrow C) \rightarrow (\forall \mathbf{x})(A \rightarrow C)$.

   (Hint: This can be done with a 6-line Hilbert proof, using 2.4.24.)

   Proof

   We use the Deduction Theorem and show instead that

   $$(\forall \mathbf{x})(A \vee B \rightarrow C) \vdash (\forall \mathbf{x})(A \rightarrow C).$$

(1) $(\forall \mathbf{x})(A \vee B \rightarrow C)$	$\langle$Hypothesis$\rangle$
(2) $A \vee B \rightarrow C$	$\langle$(1) + Spec$\rangle$
(3) $A \vee B \rightarrow C \equiv (A \rightarrow C) \wedge (B \rightarrow C)$	$\langle$2.4.24$\rangle$
(4) $(A \rightarrow C) \wedge (B \rightarrow C)$	$\langle$(2),(3) + Eqn$\rangle$
(5) $A \rightarrow C$	$\langle$(4) + 2.5.1(4) (Weakening 2)$\rangle$
(6) $(\forall \mathbf{x})(A \rightarrow C)$	$\langle$(5) + Gen; $\mathbf{x}$ dnof in (1)$\rangle$

4. Prove $(\forall \mathbf{x})(A \rightarrow B \wedge C) \rightarrow (\forall \mathbf{x})(A \rightarrow B)$.

   (Hint: This can be done with a 5-line Hilbert proof, using a theorem that follows from 2.5.1 by the Deduction Theorem.)

   Proof

   We use the Deduction Theorem and show instead that

   $$(\forall \mathbf{x})(A \rightarrow B \wedge C) \vdash (\forall \mathbf{x})(A \rightarrow B).$$

(1) $(\forall \mathbf{x})(A \rightarrow B \wedge C)$	$\langle$Hypothesis$\rangle$
(2) $A \rightarrow B \wedge C$	$\langle$(1) + Spec$\rangle$
(3) $B \wedge C \rightarrow B$	$\langle$Theorem $\big($2.5.1(4) (Weakening 2) + Deduction Theorem$\big)\rangle$
(4) $A \rightarrow B$	$\langle$(2),(3) + 2.5.9 (Transitivity of $\rightarrow$)$\rangle$
(5) $(\forall \mathbf{x})(A \rightarrow B)$	$\langle$(4) + Gen; $\mathbf{x}$ dnof in (1)$\rangle$

12. Prove the *One-Point Rule for* $\exists$: $(\exists\mathbf{x})(\mathbf{x} = t \wedge A) \equiv A[\mathbf{x} := t]$, if $\mathbf{x}$ dno in t. (The text has "if $\mathbf{x}$ is not free in t", but every occurrence of a variable in a term is free.)

    (Hint: This can be done with a 6-step Equational proof, using the One-Point Rule for $\forall$ (2. on p. 174).)

    Proof

    Suppose that $\mathbf{x}$ dno in t.

    $$
    \begin{array}{ll}
    & (\exists\mathbf{x})(\mathbf{x} = t \wedge A) \\
    \Leftrightarrow & \langle \text{Definition of } \exists \rangle \\
    & \neg(\forall\mathbf{x})\neg(\mathbf{x} = t \wedge A) \\
    \Leftrightarrow & \langle 2.4.17' \text{ (De Morgan 1)} + \text{WL} \rangle \\
    & \neg(\forall\mathbf{x})(\neg\, \mathbf{x} = t \vee \neg A) \\
    \Leftrightarrow & \langle 2.4.11 \text{ } (\rightarrow \text{ as } \vee) + \text{WL} \rangle \\
    & \neg(\forall\mathbf{x})(\mathbf{x} = t \rightarrow \neg A) \\
    \Leftrightarrow & \langle \text{One-Point Rule for } \forall\text{; } \mathbf{x} \text{ dno in } t + \text{WL} \rangle \\
    & \neg(\neg A)[\mathbf{x} := t] \\
    \Leftrightarrow & \langle \text{Sub} + \text{WL (See Note below)} \rangle \\
    & \neg\neg A[\mathbf{x} := t] \\
    \Leftrightarrow & \langle 2.4.4 \text{ (Double Negation)} \rangle \\
    & A[\mathbf{x} := t]
    \end{array}
    $$

    Note: "Sub" in the annotation above is an abbreviation for "the special case $B \equiv C$ of 2.1.12 (Reflexivity of $\equiv$), where B is C by definition of Substitution".

    In this proof, in particular, we use the fact that $(\neg A)[\mathbf{x} := t]$ is $\neg A[\mathbf{x} := t]$ by definition of Substitution.

13. Prove $(\exists\mathbf{x})\big(A \wedge (\exists\mathbf{y})(B \wedge C)\big) \equiv (\exists\mathbf{y})\big(B \wedge (\exists\mathbf{x})(A \wedge C)\big)$, if $\mathbf{y}$ dnof in A and $\mathbf{x}$ dnof in B.

    (Hint: This can be done with a 3-step Equational proof, using 6.4.3 and the result of Exercise 5.)

    Proof

    Suppose that $\mathbf{y}$ dnof in A and $\mathbf{x}$ dnof in B.

    $$
    \begin{array}{ll}
    & (\exists\mathbf{x})\big(A \wedge (\exists\mathbf{y})(B \wedge C)\big) \\
    \Leftrightarrow & \langle 6.4.3\text{; } \mathbf{y} \text{ dnof in } A + \text{WL} \rangle \\
    & (\exists\mathbf{x})(\exists\mathbf{y})(A \wedge B \wedge C) \\
    \Leftrightarrow & \langle \text{Theorem from Exercise 5} \rangle \\
    & (\exists\mathbf{y})(\exists\mathbf{x})(A \wedge B \wedge C) \\
    \Leftrightarrow & \langle 6.4.3\text{; } \mathbf{x} \text{ dnof in } B + \text{WL} \rangle \\
    & (\exists\mathbf{y})\big(B \wedge (\exists\mathbf{x})(A \wedge C)\big)
    \end{array}
    $$

16. Prove 6.4.5 (Dummy Renaming for $\exists$): $(\exists \mathbf{x})A \equiv (\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}]$, if $\mathbf{z}$ dno in A.

    (Hint: This can be done with a 4-step Equational proof, using 6.4.4 (Dummy Renaming for $\forall$).)

    Proof

    Suppose that $\mathbf{z}$ dno in A.

    $$\begin{array}{ll} & (\exists \mathbf{z})A[\mathbf{x} := \mathbf{z}] \\ \Leftrightarrow & \langle \text{Definition of } \exists \rangle \\ & \neg(\forall \mathbf{z})\neg A[\mathbf{x} := \mathbf{z}] \\ \Leftrightarrow & \langle \text{Sub + WL} \rangle \\ & \neg(\forall \mathbf{z})(\neg A)[\mathbf{x} := \mathbf{z}] \\ \Leftrightarrow & \langle \text{6.4.4 (Dummy Renaming for } \forall\text{); } \mathbf{z} \text{ dno in } \neg A + \text{WL} \rangle \\ & \neg(\forall \mathbf{x})\neg A \\ \Leftrightarrow & \langle \text{Definition of } \exists \rangle \\ & (\exists \mathbf{x})A \end{array}$$

22. Prove $(\exists \mathbf{x})\big(A \to (\forall \mathbf{x})A\big)$.

    (Hint: This can be done with a 5-step Equational proof, using Distributivity of $\exists$ over $\vee$ and a theorem that follows from the Dual of Spec by the Deduction Theorem.)

    Proof

    $$\begin{array}{ll} & (\exists \mathbf{x})\big(A \to (\forall \mathbf{x})A\big) \\ \Leftrightarrow & \langle \text{2.4.11 } (\to \text{ as } \vee) + \text{WL} \rangle \\ & (\exists \mathbf{x})\big(\neg A \vee (\forall \mathbf{x})A\big) \\ \Leftrightarrow & \langle \text{Distributivity of } \exists \text{ over } \vee \rangle \\ & (\exists \mathbf{x})\neg A \vee (\exists \mathbf{x})(\forall \mathbf{x})A \\ \Leftrightarrow & \langle \text{Definition of } \exists + \text{WL} \rangle \\ & \neg(\forall \mathbf{x})\neg\neg A \vee (\exists \mathbf{x})(\forall \mathbf{x})A \\ \Leftrightarrow & \langle \text{2.4.4 (Double Negation)} + \text{WL} \rangle \\ & \neg(\forall \mathbf{x})A \vee (\exists \mathbf{x})(\forall \mathbf{x})A \\ \Leftrightarrow & \langle \text{2.4.11 } (\to \text{ as } \vee) \rangle \\ & (\forall \mathbf{x})A \to (\exists \mathbf{x})(\forall \mathbf{x})A \quad \big(\text{Theorem (Dual of Spec + Deduction Theorem)}\big) \end{array}$$

    (The special case 6.5.3 of Dual of Spec was used above.)

23. Prove $\mathbf{x} = \mathbf{y} \wedge \mathbf{y} = \mathbf{z} \rightarrow \mathbf{x} = \mathbf{z}$.

(Hint: This can be done with a 6-line Hilbert proof, using Ax6 (Leibniz for =).)

Proof

$\vdash \mathbf{x} = \mathbf{y} \wedge \mathbf{y} = \mathbf{z} \rightarrow \mathbf{x} = \mathbf{z}$

if, by the Deduction Theorem,

$\mathbf{x} = \mathbf{y} \wedge \mathbf{y} = \mathbf{z} \vdash \mathbf{x} = \mathbf{z}$

if, by 2.5.2 (Hypothesis Merging/Splitting),

$\mathbf{x} = \mathbf{y},\ \mathbf{y} = \mathbf{z} \vdash \mathbf{x} = \mathbf{z}$.

(1) $\mathbf{x} = \mathbf{y}$	$\langle$Hypothesis$\rangle$
(2) $\mathbf{y} = \mathbf{z}$	$\langle$Hypothesis$\rangle$
(3) $\mathbf{y} = \mathbf{z} \rightarrow \big((\mathbf{x} = \mathbf{u})[\mathbf{u} := \mathbf{y}] \equiv (\mathbf{x} = \mathbf{u})[\mathbf{u} := \mathbf{z}]\big)$	$\langle$Ax6 (Leibniz for =); $\mathbf{u}$ is fresh$\rangle$
(4) $\mathbf{y} = \mathbf{z} \rightarrow (\mathbf{x} = \mathbf{y} \equiv \mathbf{x} = \mathbf{z})$	$\langle$(3) + Sub$\rangle$
(5) $\mathbf{x} = \mathbf{y} \equiv \mathbf{x} = \mathbf{z}$	$\langle$(2),(4) + 2.5.3 (Modus Ponens)$\rangle$
(6) $\mathbf{x} = \mathbf{z}$	$\langle$(1),(5) + Eqn$\rangle$

28. Which step is wrong in the following "proof", and why is it wrong?

"We know (generalization and specialization) that

$$\vdash A \text{ iff } \vdash (\forall \mathbf{x})A. \qquad (*)$$

By the metatheorem that says 'for any two absolute theorems B and C, we have $\vdash B \equiv C$', it follows from $(*)$ that $\vdash A \equiv (\forall \mathbf{x})A$."

Answer

The last step is wrong. The "proof" does not show that A and $(\forall \mathbf{x})A$ are absolute theorems, but only that if one is an absolute theorem, then so is the other.

29. Let ϕ be a binary predicate.

- Explain why $(\forall x)(\forall y)\phi(x, y) \rightarrow (\forall y)\phi(y, y)$ is *not* an instance of Ax2.

  Answer

  The given formula is $(\forall x)(\forall y)\phi(x, y) \rightarrow (\forall y)\phi(x, y)[x := y]$.

  To be an instance of Ax2, this formula would have to be the same as

  $$(\forall x)(\forall y)\phi(x, y) \rightarrow \big((\forall y)\phi(x, y)\big)[x := y]$$

  (note the extra pair of brackets), but the last substitution is undefined, since the y to be substituted would be bound by $(\forall y)$.

- Nevertheless, prove $(\forall x)(\forall y)\phi(x,y) \to (\forall y)\phi(y,y)$.
  (Hint: This can done with a 7-line Hilbert proof, using 6.1.8; or an 8-line Hilbert proof, using 6.4.4 (Dummy Renaming for $\forall$).)
  Answer
  We use the Deduction Theorem and show instead that
  $$(\forall x)(\forall y)\phi(x,y) \vdash (\forall y)\phi(y,y).$$
  Proof 1

(1) $(\forall x)(\forall y)\phi(x,y)$	$\langle$Hypothesis$\rangle$
(2) $(\forall x)(\forall y)\phi(x,y) \equiv (\forall y)(\forall x)\phi(x,y)$	$\langle$6.1.8$\rangle$
(3) $(\forall y)(\forall x)\phi(x,y)$	$\langle$(1),(2) + Eqn$\rangle$
(4) $(\forall x)\phi(x,y)$	$\langle$(3) + Spec$\rangle$
(5) $\phi(x,y)[x := y]$	$\langle$(4) + Spec$\rangle$
(6) $\phi(y,y)$	$\langle$(5) + Sub$\rangle$
(7) $(\forall y)\phi(y,y)$	$\langle$(6) + Gen; y dnof in (1)$\rangle$

  Proof 2

(1) $(\forall x)(\forall y)\phi(x,y)$	$\langle$Hypothesis$\rangle$
(2) $(\forall y)\phi(x,y)$	$\langle$(1) + Spec$\rangle$
(3) $\phi(x,y)[y := x]$	$\langle$(2) + Spec$\rangle$
(4) $\phi(x,x)$	$\langle$(3) + Sub$\rangle$
(5) $(\forall x)\phi(x,x)$	$\langle$(4) + Gen; x dnof in (1)$\rangle$
(6) $(\forall x)\phi(x,x) \equiv (\forall y)\phi(x,x)[x := y]$	$\langle$6.4.4 (Dummy Renaming for $\forall$); y dno in $\phi(x,x)\rangle$
(7) $(\forall x)\phi(x,x) \equiv (\forall y)\phi(y,y)$	$\langle$(6) + Sub$\rangle$
(8) $(\forall y)\phi(y,y)$	$\langle$(5),(7) and Eqn$\rangle$

31. Let ϕ and ψ be unary predicates, and c a constant. Show that
$$(\forall x)\big(\phi(x) \to \psi(x)\big),\ (\forall z)\phi(z) \vdash \psi(c).$$
(Hint: This can done with a 7-line Hilbert proof.)
Proof

(1) $(\forall x)\big(\phi(x) \to \psi(x)\big)$	$\langle$Hypothesis$\rangle$
(2) $(\forall z)\phi(z)$	$\langle$Hypothesis$\rangle$
(3) $\phi(z)[z := c]$	$\langle$(2) + Spec$\rangle$
(4) $\phi(c)$	$\langle$(3) + Sub$\rangle$
(5) $\big(\phi(x) \to \psi(x)\big)[x := c]$	$\langle$(1) + Spec$\rangle$
(6) $\phi(c) \to \psi(c)$	$\langle$(5) + Sub$\rangle$
(7) $\psi(c)$	$\langle$(4),(6) + 2.5.3 (Modus Ponens)$\rangle$