30 marks.

Math 2015 Applied Multivariate and Vector Calculus: Test1

Friday September 25, 2009
10:30am to 11:20am

Name: SOLUTIONS **Student Number:**

Instructions: Complete all 5 of the following problems in the space provided. Notes and calculators are not permitted. All cell phones and pagers are to be turned off.

1. Consider the real vectors $\mathbf{u} = (3,1,1)$, $\mathbf{v} = (2,-5,-1)$. Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal and find a *unit* vector $\hat{\mathbf{w}}$ which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. 5 marks

$\vec{u} \cdot \vec{v} = 6 - 5 - 1 = 0 \Rightarrow$ orthogonal.

A vector $\vec{w}$ perpendicular to $\vec{u}$ and $\vec{v}$ is

$$\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \text{— } 1$$

$$= \hat{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 1 & 1 \\ -5 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 1 \\ 2 & -5 \end{vmatrix} \quad \text{— } 1$$

$$= 4\hat{i} + 5\hat{j} - 17\hat{k} \quad \text{— } 1$$

$$\|\vec{w}\| = \sqrt{16 + 25 + 289} = \sqrt{330} \quad \text{— } 1$$

So the UNIT vector we seek is

$$\hat{w} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{330}}(4, 5, -17) \quad \text{— } 1$$

2. Consider the real vectors $\mathbf{u} = (3, 2, 1)$, $\mathbf{v} = (2, 0, -1)$, and $\mathbf{w} = (1, 5, -3)$. Compute the vector projection

$$\text{proj}_{\mathbf{u}}(\mathbf{w} + \mathbf{v}).$$

5 marks

Let $\vec{q} = \vec{w} + \vec{v} = (1,5,-3) + (2,0,-1)$

$= (3,5,-4)$

$\text{proj}_{\vec{u}}(\vec{w}+\vec{v}) = \text{proj}_{\vec{u}}\vec{q} = \dfrac{\vec{q}\cdot\vec{u}}{\|\vec{u}\|^2}(\vec{u})$ — 2

$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u} = 3^2 + 2^2 + 1^2 = 14$ — 1

$\vec{q}\cdot\vec{u} = 3(3) + 5(2) - 4(1) = 15$ — 1

$\therefore\ \text{proj}_{\vec{u}}\vec{q} = \dfrac{15}{14}(3,2,1)$ — 1

3. Suppose $\mathbf{r}(t)$ is a differentiable vector-valued function and that

$$\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)].$$

5 marks

Show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)].$$

$$\vec{u}'(t) = \vec{r}'(t) \cdot [\vec{r}'(t) \times \vec{r}''(t)] \quad \text{— 1}$$

$$+ \vec{r}(t) \cdot \frac{d}{dt}[\vec{r}'(t) \times \vec{r}''(t)] \quad \text{— 1}$$

But $\vec{r}'(t) \cdot [\vec{r}'(t) \times \vec{r}''(t)] = 0$ — 1

since these must be orthogonal vectors.

$$\therefore \quad \vec{u}'(t) = \vec{r}(t) \cdot [\vec{r}''(t) \times \vec{r}''(t) + \vec{r}'(t) \times \vec{r}'''(t)] \quad \text{— 1}$$

Since $\vec{r}''(t) \times \vec{r}''(t) = 0$ we get — 1

$$\vec{u}'(t) = \vec{r}(t) \cdot [\vec{r}'(t) \times \vec{r}'''(t)]$$

4. Find the length of the curve defined parametrically by

$$\mathbf{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{\mathbf{k}}$$

5 marks

for $1 \leq t \leq 5$, where

$$f(t) = \mathrm{e}^t \sin t; \quad g(t) = \mathrm{e}^t \cos t; \quad h(t) = \mathrm{e}^t.$$

$$L = \int_1^5 \sqrt{\left(\frac{d}{dt} e^t \sin t\right)^2 + \left(\frac{d}{dt} e^t \cos t\right)^2 + \left(\frac{d}{dt} e^t\right)^2}\, dt \quad - 1$$

$$= \int_1^5 \left[(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2 + (e^t)^2\right]^{1/2} dt \quad - 1$$

$$= \int_1^5 \left[2e^{2t}(\cos^2 t + \sin^2 t) + e^{2t}\right]^{1/2} dt \quad - 1$$

$$= \int_1^5 \sqrt{3e^{2t}}\, dt$$

$$= \sqrt{3} \int_1^5 e^t\, dt \quad - 1$$

$$= \sqrt{3}\, e^t \Big|_1^5$$

$$= \sqrt{3}(e^5 - e) \quad - 1$$

5. Show that for any real number k, the functions

10 marks.

$$z = e^{kx}\cos(ky) \quad \text{and} \quad z = e^{kx}\sin(ky)$$

satisfy the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

(a) $\frac{\partial z}{\partial x} = ke^{kx}\cos(k\cancel{y})$ $\quad \frac{\partial^2 z}{\partial x^2} = k^2 e^{kx}\cos(ky)$ — 2

$\frac{\partial z}{\partial y} = -ke^{kx}\sin(ky)$ $\quad \frac{\partial^2 z}{\partial y^2} = -k^2 e^{kx}\cos(ky)$ — 2

$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = k^2 e^{kx}\cos(ky) - k^2 e^{kx}\cos(ky)$ — 1

$= 0$

(b) $\frac{\partial z}{\partial x} = ke^{kx}\sin(ky)$ $\quad \frac{\partial^2 z}{\partial x^2} = k^2 e^{kx}\sin(ky)$ — 2

$\frac{\partial z}{\partial y} = ke^{kx}\cos(ky)$ $\quad \frac{\partial^2 z}{\partial y^2} = -k^2 e^{kx}\sin(ky)$ — 2

$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = k^2 e^{kx}\sin(ky) - k^2 e^{kx}\sin(ky)$ — 1

$= 0$