Description of deformation

Deformation is the change in the metric properties of a continuous body, meaning that a curve drawn in the initial body placement changes its length when displaced to a curve in the final placement. If none of the curves changes length, it is said that a rigid body displacement occurred.

It is convenient to identify a reference configuration or initial geometric state of the continuum body which all subsequent configurations are referenced from. The reference configuration need not be one the body actually will ever occupy. Often, the configuration at $t = 0$ is considered the reference configuration, $\kappa_0(\mathbf{B})$. The configuration at the current time t is the *current configuration*.

For deformation analysis, the reference configuration is identified as *undeformed configuration*, and the current configuration as *deformed configuration*. Additionally, time is not considered when analyzing deformation, thus the sequence of configurations between the undeformed and deformed configurations are of no interest.

The components X_i of the position vector $\mathbf{X}$ of a particle in the reference configuration, taken with respect to the reference coordinate system, are called the *material or reference coordinates*. On the other hand, the components x_i of the position vector $\mathbf{x}$ of a particle in the deformed configuration, taken with respect to the spatial coordinate system of reference, are called the *spatial coordinates*

There are two methods for analysing the deformation of a continuum. One description is made in terms of the material or referential coordinates, called material description or Lagrangian description. A second description is of deformation is made in terms of the spatial coordinates it is called the spatial description or Eulerian description.

There is continuity during deformation of a continuum body in the sense that:

- The material points forming a closed curve at any instant will always form a closed curve at any subsequent time.
- The material points forming a closed surface at any instant will always form a closed surface at any subsequent time and the matter within the closed surface will always remain within.

Affine deformation

A deformation is called an affine deformation if it can be described by an affine transformation. Such a transformation is composed of a linear transformation (such as rotation, shear, extension and compression) and a rigid body translation. Affine deformations are also called homogeneous deformations.

Therefore an affine deformation has the form

$$\mathbf{x}(\mathbf{X},t) = \boldsymbol{F}(t)\cdot\mathbf{X} + \mathbf{c}(t)$$

where $\mathbf{x}$ is the position of a point in the deformed configuration, $\mathbf{X}$ is the position in a reference configuration, t is a time-like parameter, $\boldsymbol{F}$ is the linear transformer and $\mathbf{c}$ is the translation. In matrix form, where the components are with respect to an orthonormal basis,

$$\begin{bmatrix} x_1(X_1,X_2,X_3,t) \\ x_2(X_1,X_2,X_3,t) \\ x_3(X_1,X_2,X_3,t) \end{bmatrix} = \begin{bmatrix} F_{11}(t) & F_{12}(t) & F_{13}(t) \\ F_{21}(t) & F_{22}(t) & F_{23}(t) \\ F_{31}(t) & F_{32}(t) & F_{33}(t) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix}$$

The above deformation becomes *non-affine* or *inhomogeneous* if $\boldsymbol{F} = \boldsymbol{F}(\mathbf{X},t)$ or $\mathbf{c} = \mathbf{c}(\mathbf{X},t)$.

Rigid body motion

A rigid body motion is a special affine deformation that does not involve any shear, extension or compression. The transformation matrix $\boldsymbol{F}$ is proper orthogonal in order to allow rotations but no reflections.

A rigid body motion can be described by

$$\mathbf{x}(\mathbf{X},t) = \boldsymbol{Q}(t)\cdot\mathbf{X} + \mathbf{c}(t)$$

where

$$\boldsymbol{Q}\cdot\boldsymbol{Q}^T = \boldsymbol{Q}^T\cdot\boldsymbol{Q} = \boldsymbol{1}$$

In matrix form,

$$\begin{bmatrix} x_1(X_1,X_2,X_3,t) \\ x_2(X_1,X_2,X_3,t) \\ x_3(X_1,X_2,X_3,t) \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & Q_{33}(t) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix}$$

Displacement

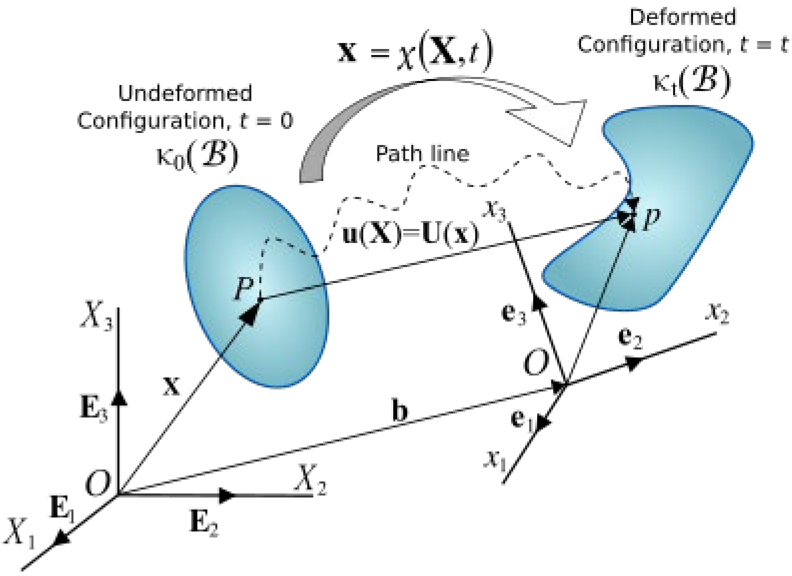

Figure 1. Motion of a continuum body.

A change in the configuration of a continuum body results in a displacement. The displacement of a body has two components: a rigid-body displacement and a deformation. A rigid-body displacement consists of a simultaneous translation and rotation of the body without changing its shape or size. Deformation implies the change in shape and/or size of the body from an initial or undeformed configuration $\kappa_0(\mathcal{B})$ to a current or deformed configuration $\kappa_t(\mathcal{B})$ (Figure 1).

If after a displacement of the continuum there is a relative displacement between particles, a deformation has occurred. On the other hand, if after displacement of the continuum the relative displacement between particles in the current configuration is zero, then there is no deformation and a rigid-body displacement is said to have occurred.

The vector joining the positions of a particle P in the undeformed configuration and deformed configuration is called the displacement vector $\mathbf{u}(\mathbf{X},t) = u_i\mathbf{e}_i$ in the Lagrangian description, or $\mathbf{U}(\mathbf{x},t) = U_J\mathbf{E}_J$ in the Eulerian description.

A *displacement field* is a vector field of all displacement vectors for all particles in the body, which relates the deformed configuration with the undeformed configuration. It is convenient to do the analysis of deformation or motion of a continuum body in terms of the displacement field. In general, the displacement field is expressed in terms of the material coordinates as

$$\mathbf{u}(\mathbf{X},t) = \mathbf{b}(\mathbf{X},t) + \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad \text{or} \qquad u_i = \alpha_{iJ}b_J + x_i - \alpha_{iJ}X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{b}(\mathbf{x},t) + \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad \text{or} \qquad U_J = b_J + \alpha_{Ji}x_i - X_J$$

where α_{Ji} are the direction cosines between the material and spatial coordinate systems with unit vectors $\mathbf{E}_J$ and $\mathbf{e}_i$, respectively. Thus

$$\mathbf{E}_J \cdot \mathbf{e}_i = \alpha_{Ji} = \alpha_{iJ}$$

and the relationship between u_i and U_J is then given by

$$u_i = \alpha_{iJ}U_J \qquad \text{or} \qquad U_J = \alpha_{Ji}u_i$$

Knowing that

$$\mathbf{e}_i = \alpha_{iJ}\mathbf{E}_J$$

then

$$\mathbf{u}(\mathbf{X},t) = u_i\mathbf{e}_i = u_i(\alpha_{iJ}\mathbf{E}_J) = U_J\mathbf{E}_J = \mathbf{U}(\mathbf{x},t)$$

It is common to superimpose the coordinate systems for the undeformed and deformed configurations, which results in **b** = 0, and the direction cosines become Kronecker deltas:

$$\mathbf{E}_J \cdot \mathbf{e}_i = \delta_{Ji} = \delta_{iJ}$$

Thus, we have

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad \text{or} \qquad u_i = x_i - \delta_{iJ}X_J = x_i - X_i$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad \text{or} \qquad U_J = \delta_{Ji}x_i - X_J = x_J - X_J$$

Displacement gradient tensor

The partial differentiation of the displacement vector with respect to the material coordinates yields the *material displacement gradient tensor* $\nabla_{\mathbf{X}}\mathbf{u}$. Thus we have:

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad u_i = x_i - \delta_{iJ}X_J = x_i - X_i$$
$$\nabla_{\mathbf{X}}\mathbf{u} = \nabla_{\mathbf{X}}\mathbf{x} - \mathbf{I}$$
$$\nabla_{\mathbf{X}}\mathbf{u} = \mathbf{F} - \mathbf{I} \qquad \text{or} \quad \frac{\partial u_i}{\partial X_K} = \frac{\partial x_i}{\partial X_K} - \delta_{iK}$$

where $\mathbf{F}$is the *deformation gradient tensor*.

Similarly, the partial differentiation of the displacement vector with respect to the spatial coordinates yields the *spatial displacement gradient tensor* $\nabla_{\mathbf{x}}\mathbf{U}$. Thus we have,

$$\mathbf{U}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad U_J = \delta_{Ji}x_i - X_J = x_J - X_J$$
$$\nabla_{\mathbf{x}}\mathbf{U} = \mathbf{I} - \nabla_{\mathbf{x}}\mathbf{X}$$
$$\nabla_{\mathbf{x}}\mathbf{U} = \mathbf{I} - \mathbf{F}^{-1} \qquad \text{or} \quad \frac{\partial U_J}{\partial x_k} = \delta_{Jk} - \frac{\partial X_J}{\partial x_k}$$