NOTES ON OPTIMISATION WITH COMMUNICATION CONSTRAINTS

Consider the optimisation problem P:

$$\min_{x \in D} f(x) \tag{1}$$

$$s.t. \quad g_j(x) \le 0, \;\; j = 1, \cdots, m \tag{2}$$

where $x \in \mathcal{R}^n$, $f : \mathcal{R}^n \to \mathcal{R}$ and $g : \mathcal{R}^n \to \mathcal{R}$ are both convex. We assume that $f(\cdot)$ and $g(\cdot)$ are Lipschitz on domain D (this is quite a mild assumption for convex functions) and have bounded subgradients on D. Let D^* denote set of optima solving problem P.

Define

$$F(x, \lambda) := f(x) + \lambda \sum_{j=1}^{m} \max\{0, g_j(x)\} \tag{3}$$

and consider the iterative update

$$x(k+1) = x(k) - \alpha(k)\partial_x F(x(k), \lambda(k)) \tag{4}$$

$$\lambda(k+1) = \lambda(k) + \alpha(k) \tag{5}$$

with $\alpha(k) > 0$. For $x^* \in D^*$, following the usual Bertsekas argument we have that

(6)

$$\|x(k+1) - x^*\|^2 = \|x(k) - \alpha(k)\partial_x F(x(k), \lambda(k)) - x^*\|^2$$

$$= \|x(k) - x^*\|^2 - 2\alpha(k)\partial_x F^T(x(k), \lambda(k))(x(k) - x^*) + \alpha^2(k)\|\partial_x F(x(k), \lambda(k))\|^2 \tag{7}$$

$$\overset{(a)}{\le} \|x(k) - x^*\|^2 - 2\alpha(k)(F(x(k), \lambda(k)) - F(x^*, \lambda(k))) + \alpha^2(k)\|\partial_x F(x(k), \lambda(k))\|^2 \tag{8}$$

where (a) follows from the convexity of $F(x, \lambda)$ and the definition of the subgradient. It now follows that

(9)

$$\|x(k+1) - x^*\|^2 \le \|x(1) - x^*\|^2 - 2\sum_{i=1}^{k} \alpha(k)(F(x(i), \lambda(i)) - F(x^*, \lambda(i))) + 2\sum_{i=1}^{k} \alpha^2(i)\|\partial_x F(x(i), \lambda(i))\|^2$$

By assumption, there exists a constant $\bar{v}$ such that $\|\partial f(x)\| \le \bar{v}$ and $\|\partial g(x)\| \le \bar{v}$. Hence, $\|\partial_x F(x(i), \lambda(i))\| \le (1 + m\lambda(i))\bar{v}$ and

(10)

$$\|x(k+1) - x^*\|^2 \le \|x(1) - x^*\|^2 - 2\sum_{i=1}^{k} \alpha(k)(F(x(i), \lambda(i)) - F(x^*, \lambda(i))) + 2\bar{v}^2 \sum_{i=1}^{k} \alpha^2(i)(1 + m\lambda(i))^2$$

That is,

$$\sum_{i=1}^{k} \alpha(k)(F(x(i),\lambda(i)) - F(x^*,\lambda(i))) \leq \frac{1}{2}\|x(1)-x^*\|^2 + \bar{v}^2 \sum_{i=1}^{k} \alpha^2(i)(1+m\lambda(i))^2 \tag{11}$$

Suppose $\lim_{k\to\infty}\sum_{i=1}^{k}\alpha^2(i)(1+m\lambda(i))^2 = 0$ and $\sum_{i=1}^{k}\alpha(i)\to\infty$, $\lambda(k)\to\infty$ as $k\to\infty$. Then

$$\lim_{k\to\infty}\sum_{i=1}^{k} \alpha(k)(F(x(i),\lambda(i)) - F(x^*,\lambda(i))) \leq \frac{1}{2}\|x(1)-x^*\|^2 \tag{12}$$

and so

$$\lim_{k\to\infty}\left(\min_{i\in\{1,\cdots,k\}} F(x(i),\lambda(i)) - F(x^*,\lambda(i))\right)\sum_{i=1}^{k}\alpha(k) \leq \frac{1}{2}\|x(1)-x^*\|^2 \tag{13}$$

Since the RHS is finite and $\sum_{i=1}^{k}\alpha(i)\to\infty$ as $k\to\infty$, it follows that $\min_{i\in\{1,\cdots,k\}}(F(x(i),\lambda(i)) - F(x^*,\lambda(i))) = 0$ as $k\to\infty$ *****need to double check this argument carefully; note also that it doesn't prove convergence – can we do better ?**.

Recall that $F(x(k),\lambda(k)) - F(x^*,\lambda(k)) = f(x(k)) - f(x^*) + \lambda(k)\max\{0,g(x(k))\}$. Now $f(x(k)) - f(x^*) \geq 0$ with equality only when $x(k)\in D^*$, and $\lambda(k)\max\{0,g(x(k))\}\geq 0$ with equality only when $g(x(k)\leq 0$ (since $\lambda(k)>0$). Since $\min_{i\in\{1,\cdots,k\}}(F(x(i),\lambda(i)) - F(x^*,\lambda(i))) = 0$ as $k\to\infty$, it follows that the minimising $x(i)$ must lie in set D^*.