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# A Case Study in Dependent Type Theory: Extracting a Certified Program from the Formal Proof of its Specification

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June, 2023

## Abstract

Proofs are an important part of mathematics, but they are not without their flaws. Most proofs are written by humans, and humans make mistakes. In this thesis, we explore the use of proof assistants to construct formal versions of informal proofs, and to extract certified and correct programs from these formal proofs. We study a specific case of two problems from lattice theory, solved by Bezem and Coquand in [2]. Firstly, we ask ourselves: are the results from [2] correct? The informal proofs posed in [2], used to justify the correctness of the results, are complex enough that mistakes are possible. Using the Coq proof assistant, we formalize parts of the proofs from [2], to gain more confidence in the correctness of these informal proofs. Secondly, is the process of formalization a feasible one? Transforming an informal proof into a formal proof is not necessarily an easy or straightforward process, and there are several approaches to formalization of a proof. Lastly, is the process of formalization worth the effort? A fully formalized proof gives us almost complete confidence in the correctness of the proof, and hence the result. Moreover, if the proof is constructive, we get the added bonus of extracting a certified program from the proof. In our case, this program has some practical use cases in a real-world setting, mostly as a prototype.

**Acknowledgments**

First and foremost, I would like to extend a massive thank you to my supervisor, Marc Bezem. Without your excellent guidance and help, I would not have been able to complete this project. Thank you for all the knowledge you have shared with me. I would also like to thank all my friends at Jafu for always aiding in procrastination, and for all the fun times. Lastly, I'd like to thank my family for their support and encouragement.

`Andreas Salhus Bakseter`
Thursday 1st June, 2023

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# Chapter 1

# Background

## 1.1 Proofs, verification & formalization

When solving mathematical problems, we often use proofs to either **justify** a claim or to **explain** why the claim is true. We can distinguish between two types of proofs; *informal* and *formal* proofs.

An informal proof is often written in a natural language, and the proof is adequate if most readers are convinced by the proof [17, ch. 2]. Such proofs rely heavily on the reader's intuition and often omit logical steps to make them easier to understand for humans [10, p. 1371]. As these proofs grow larger and more complex, they become harder for humans to follow, which can ultimately lead to errors in the proofs' logic. This might cause the whole proof to be incorrect [11], and even the claim "justified" by the proof might be wrong.

A formal proof is written in a formal language, and can be compared to a computer program written in a programming language. Writing a formal proof is more difficult than writing an informal proof. Formal proofs include every logical step, and nothing is left for the reader (often a computer) to assume. This can make formal proofs extremely verbose, but the amount of logical errors is reduced [10, p. 1371]. The only possible errors in formalized proofs are false assumptions and/or flawed verification software. When we talk about *formalization*, we mean the process of translating an informal proof into a formal proof.

## 1.2 Type theory

Type theory is a foundation of mathematics [5] in which every mathematical object is viewed as being a member of a certain type. We can say that the object $a$ is a member of the type $A$, or, that $a$ has type $A$, denoted by $a : A$. For any type $A$, we must know how to construct an element $a$ of that type [15, p. 76]. A simple example would be the type of natural numbers, $\mathbb{N}$. Any object $n : \mathbb{N}$ would be a natural number, which we can easily provide by iterating a successor function on the constant $O$. Type theory gives us a framework to work with and manipulate objects and types by providing us with rules of inference for objects and their types. There are several different type theories, where each one has their own rules. As mathematical objects can be a lot more complex than just natural numbers, the types these objects belong to follow in complexity.

A related concept in computer programming is the notion of a *data type.* Akin to type theory, these are often used to represent some object or structure, e.g. the data type `bool` is usually used to represent a boolean value.

### 1.2.1 Propositions as types

The concept of proposition as types sees any proposition $P$ as representing a type, and every proof $p$ of $P$ is an element of the type $P$, or $p : P$. Thus, if we can construct an element of the type $P$, we can prove $P$, and vice versa; if we can prove $P$, we can construct an element of the type of $P$. For example, to prove a proposition $P$ which states "all natural numbers are the sum of four squares", we must construct a value of the type $P$ that shows that this is true for all natural numbers. Such a value is a function that for any input $n$ returns a proof that $n$ is the sum of four squares, that is, return four numbers $a, b, c, d$ and a proof that $n = a^2 + b^2 + c^2 + d^2$. Proofs are mathematical objects; thus a proposition can be viewed as having the type of all its proofs (if any!). We can use this correspondence to model a proof as a typed computer program. The power of this concept comes from the fact that we can use a type checker to verify that our program is typed correctly, and thus that the corresponding proof is valid. Often, the proof can be used to compute something, i.e., the numbers $a, b, c, d$ mentioned above [27] [16].

### 1.2.2 Dependent types

Dependent types allow us to define more rigorously types that depend on values. We again look at the proposition $P$ from above. It states that for all natural numbers $n$, there exists four other natural numbers $a, b, c, d$, such that $P$ holds. Or, in mathematical notation:

$$\forall n\, \exists a\, \exists b\, \exists c\, \exists d\, (n = a^2 + b^2 + c^2 + d^2) \quad \text{where } n, a, b, c, d \in \mathbb{N}$$

The proof of this proposition is, as stated previously, a function that takes an argument $n$ and returns a proof, or, evidence in the form of values for $a, b, c, d$ that satisfy $P$. Since propositions are just types, any function $f : A \to B$ will produce a proof of $B$ given a proof of $A$.

To represent the type of an universal quantified proposition, we use a $\Pi$-type, also called a dependent function type. Given a function $B$ that for any $a : A$ maps to an element of the type $B(a)$, we denote $\prod_{a:A} B(a)$. This type is the type of all functions that for any element $a : A$ maps to an element $b : B(a)$ [15, p. 78]. In the case for $P$, the type $A$ would be $\mathbb{N}$, and the element $a$ would be a natural number, corresponding to the variable $n$ in $P$. A proof of $\prod_{n:\mathbb{N}} (\exists a\, \exists b\, \exists c\, \exists d\, (n = a^2 + b^2 + c^2 + d^2)$ is a function that for any $n \in \mathbb{N}$ gives us a proof of $\exists a\, \exists b\, \exists c\, \exists d\, (n = a^2 + b^2 + c^2 + d^2)$. Thus, the type of the proposition varies with $n$, modeling universal quantification.

To represent the type of an existentially quantified proposition, we use a $\Sigma$-type, also called a dependent pair type. Given a function $B$ that for any $a : A$ maps to an element of the type $B(a)$, we denote $\sum_{a:A} B(a)$. This type is the type of all pairs $(x, y)$ where $x : A$ and $y : B(a)$ [15, p. 79]. Again, in the case for $P$, $A$ would be $\mathbb{N}$, and the element $a$ would be a natural number, corresponding to the variable $a$ in $P$. A proof of $\prod_{n:\mathbb{N}} \sum_{a:\mathbb{N}} (\exists b\, \exists c\, \exists d\, (n = a^2 + b^2 + c^2 + d^2)$ is a pair $(x, y)$, where $x$ is evidence of $P$ and $y$ is a proof of $P$. Hence, any evidence of $P$ is provided with its proof, modeling existential quantification. The other existential quantifiers are likewise replaced by $\Sigma$'s:

$$\prod_{n:\mathbb{N}} \sum_{a:\mathbb{N}} \sum_{b:\mathbb{N}} \sum_{c:\mathbb{N}} \sum_{d:\mathbb{N}} (n = a^2 + b^2 + c^2 + d^2)$$

The proposition $n = a^2 + b^2 + c^2 + d^2$ is a type dependent on the five values $n, a, b, c, d$.

As we have just shown by representing universal and existential quantification, the concept of dependent types is important for propositions as types by providing us with more expressivity. We can also look at a more practical example of dependent types, using the Coq proof assistant, which has support for dependently typed programming. We consider the type `vector`, which is a type polymorphic (the elements of the list can be of any same type) finite list with a fixed length, which is the value on which the type depends, a natural number.

```
(* brackets around 'A : Type' indicate that its type should be inferred *)
Inductive vector {A : Type} : nat → Type :=
  | Vnil : vector 0
  (* 'forall' is an instance of a dependent function type *)
  | Vcons : forall (h : A) (n : nat), vector n → vector (S n).
```

Listing 1.1: `vector` in Coq, using dependent types

This definition gives us a type with two constructors:

- `Vnil` has the type of `vector` 0, and represents the empty vector.
- `Vcons` has type of `forall (h : A) (n : nat) vector (S n)`, where `h` is the head of the vector and `n` is the length of the vector, where both are given to the constructor as arguments.

In this scenario, the length of an object of the type `vector n` is fixed by the argument `n : nat`. Any definition of a `vector` must adhere to this term, and is checked at compile time. An example of a valid and invalid definition is:

```
(* valid definition; (S 0) equal to 1 *)
Definition vec_valid : vector 2 :=
    Vcons "b" 1 (Vcons "a" 0 (Vnil)).
(* invalid definition; (S 0) not equal to 2 *)
Fail Definition vec_invalid : vector 2 :=
    Vcons "b" 2 (Vcons "a" 0 (Vnil)).
```

Listing 1.2: Examples of vectors in Coq

## 1.3 Proof assistants

Propositions as types allow us to bridge the gap between logic and computing, while dependent types allow us to define more rigorously types which depend on values, thus enhancing the expressivity of simple types. The former is a crucial aspect of *proof assistants*, while the latter gives us more expressive power when constructing proofs using a proof assistant, as we saw when modeling universal and existential quantifiers using types. An example of the expressive power of dependent types is the fact that we can define predicates that depend on the value of a term, e.g. a predicate that checks if a number is even. The purpose of a proof assistant is to get computer support to construct and verify a formal proof mechanically.

### 1.3.1 Coq

Coq is based on the higher-order type theory *Calculus of Inductive Constructions* (CIC), and functions as both a proof assistant and a dependently typed functional programming language. Coq also allow us to extract certified programs from the proofs of their specification to the programming languages OCaml and Haskell [21]. Coq implements a specification language called *Gallina*, which allows us to define logical objects within Coq. These objects are each assigned a type to ensure they are correct, and the rules used to assign these types come from CIC [19].

This is an example of the syntax of Gallina:

```
Inductive nat : Type :=
  | O
  | S : nat → nat.

Inductive le (n : nat) : nat → Prop :=
  | le_n : n ≤ n
  | le_S : forall m : nat, n ≤ m → n ≤ S m.

Definition lt_n_S_n :=
  (fun n : nat ⇒ le_n (S n)) : forall n : nat, n < S n.
```

Listing 1.3: Example of Gallina syntax

Looking at the final definition in the example, we can see the concept of propositions as types in action. `lt_n_S_n` defines a function which takes a natural number `n` as input, and returns a value of the type `forall n : nat, n < S n`, denoted by the colon before the type itself. The return value is therefore a proof of `forall n : nat, n < S n`, and since the definition has been type-checked by Coq, we know that this proof is valid! In this case, the function is `fun : nat ⇒ le_s (S n)`, where `le_n` is a constructor of the type `forall n : nat, n ≤ n`. By applying this constructor to `S n`, we get a value of the type (and a proof) of `forall n : nat, S n ≤ S n`. By Coq's definition of <, our initial theorem can be rewritten as `forall n : nat, S n ≤ S n`. This matches the type of our function, and the proof is complete.

Proving theorems like this is not really intuitive for a human prover, and that is why Coq gives us the *Ltac* meta-language for writing proofs. Ltac provides us with tactics, which are a kind of shorthand syntax for defining Gallina terms [7]. Using Ltac, we can rewrite the proof from Listing 1.3 as such:

```
Theorem lt_n_S_n : forall n : nat, n < S n.
Proof.
  intro n. destruct n.
  - apply le_n.
  - apply le_n_S. apply le_n.
Qed.
```

Listing 1.4: Example of Ltac syntax

If we check the Gallina terms generated by Ltac for this theorem using the `Print` command, we get essentially the same function as in Listing 1.3.

```
Print lt_n_S_n.

lt_n_S_n =
  fun n : nat ⇒
  match n as n0 return (n0 < S n0) with
  | 0 ⇒ le_n 1
  | S n0 ⇒ (fun n1 : nat ⇒ le_n_S (S n1) (S n1) (le_n (S n1))) n0
  end
   : forall n : nat, n < S n
```

Listing 1.5: Gallina terms generated by Ltac example

When developing proofs using Ltac, each tactic is executed or "played" one by one, much like an interpreter. The tactics are separated by punctuation marks. When the use of a tactic causes the proof to depend on the solving of multiple sub-proofs (called "goals"), we can use symbols like ”-”, ”+”, and ”*” to branch into these sub-proofs and solve their goals independently. Once a goal has been solved, we can move on to the next. When there are no more goals, the proof is complete. Coq provides us with tooling that gives us the ability to see our goals and the proof state to further simplify the process [20]. Ltac is not the only proof language, with another example being *SSReflect* [9].

### 1.3.2 Other proof assistants

Coq is a widely used proof assistant, but it is not the only one. There are also other related tools that can be used for formalization and verification.

#### Agda

Agda is a dependently typed functional programming language based on Martin-Löf’s intuitionistic type theory. Unlike Coq, Agda does not use tactics. However, by using proposition as types, Agda can also function as a proof assistant [3].

#### Lean

Lean is dependently typed functional programming language and theorem prover. The goal of Lean is to establish a connection between the automated and interactive processes of proving theorems. Thus, Lean functions both as a proof assistant and as an automated theorem prover. Like Coq, Lean is also based on the type theory of CIC [6].

### 1.3.3 Extraction of programs from verified proofs

By the notion of propositions as types, we can use a proof assistant to prove the correctness of a program [17, Preface]. However, we can also extract a program from a proof of its correctness. This type of program extraction is a common feature of proof assistants. The extracted program is guaranteed to be correct by the type system of the proof assistant, and the code for the program can be given as source code in several programming languages, such as Haskell and OCaml (as is the case for Coq) [21].

# Chapter 2

# The Case in Question

## 2.1 Overview & research questions

Recall the three questions posed in the abstract. Firstly, we want to use the Coq proof assistant to formalize parts of the proofs of the paper, Bezem and Coquand [2]. This paper solves two problems that occur in dependent type systems where typings depend on universe-level constraints, and the results can be summarized as follows:

1. The uniform word problem for an an equational theory of semilattices with one inflationary endomorphism can be solved in polynomial time.
2. Loop-checking in a type system with universe-level constraints can be done in polynomial time.

From [2], we get a definition of a uniform word problem for an equational theory $T$:

> The uniform word problem for an equational theory $T$ is to determine, given a finite set $E$ of relations between generators, whether a given relation is provable from $E$ in $T$.

In our case, the equational theory $T$ is the theory of join-semilattices with one inflationary endomorphism. This problem is shown in [2] to be solvable in polynomial time, and provability in semilattice theory is shown to be equivalent to provability in Horn clauses. We will explain in more detail in the next subsection what join-semilattices, inflationary endomorphisms, and Horn clauses are. Loop-checking, in our case, means determining if a set of constraints on some set of variables generate an infinite loop, that is, a term $t$ such that $t \geq t^+$ is provable from the constraints. This is important for algorithms such as a type-checking algorithm where type universe levels have to be determined, and the algorithm has to be decidable. We have focused on formalizing the proof of Theorem 3.2 from [2]. Since this proof is complex enough that mistakes are possible, it is a good candidate for formalization. By formalizing the proof and then verifying it using Coq, we can be sure that it is correct.

Secondly, we are also interested in finding out whether or not the process of formalizing the proof can be completed in a reasonable amount of time, and with a reasonable amount of effort. A formal proof of Theorem 3.2 is a sizable undertaking, and we want to know if it is feasible to complete it.

Finally, we want to assess if our formalization effort will be worth it. By completing a formal version of Theorem 3.2 we can be very confident in its correctness. We can also, by using an advanced feature of Coq, extract an algorithm from the formal proof, where this algorithm is certified to produce correct results. As an aside, this algorithm has direct applications to the formalization and verification of the Coq proof assistant itself, as the algorithm outlined by the proofs in [2] can be used to solve a similar problem that Coq encounters when checking the consistency of universe levels during interpretation/-compilation.

## 2.2 Relevant parts of the paper

We will paraphrase some key concepts from [2] that are needed to understand our implementation and formalization.

We will call join-semilattices with inflationary endomorphisms simply semilattices. A *join-semilattice* is a partially ordered set in which any two elements have a least upper bound, called their join. The *join* of a semilattice is a binary operation $\vee$ that is commutative, associative and idempotent. An *inflationary endomorphism*, in our case, is a function that maps an element to itself or to a greater element in the ordered set. We denote the inflationary endomorphism as $^+$, for every lattice element $x$, $x^+$ is also called the successor of $x$. The fact that $x$ is an inflationary endomorphism means that we get the following properties: $(x \vee y)^+ = x^+ \vee y^+$ and $x \vee x^+ = x^+$. The latter is also denoted as $x \leq x^+$.

A *semilattice presentation* consists of a set $V$ of *generators* (also called *variables*) and a set $C$ of *relations* (also called *constraints*). For a semilattice term $t$ and $k \in \mathbb{N}$, $t + k$ denotes the k-fold successor of $t$, thus making $t+0 = t$ and $t+1 = t^+$. A term over $V$ has the form $x_1 + k_1 \vee \ldots \vee x_m + k_m$, where $x_i \in V$ and $k_i \in \mathbb{N}$. Every semilattice expression is equal to a term over $V$.

*Horn clauses* are propositional clauses $A \to b$, with a non-empty body $A$ and conclusion $b$. The atoms are of the form $x + k$, where $x \in V, k \in \mathbb{N}$. We call this special form of Horn clauses simply *clauses*.

A constraint is an equation $s = t$, where $s$ and $t$ are terms over $V$. A constraint, like $a \vee b = c^+$ (with $a, b, c \in V$), expresses a relation between the variables $a, b, c$. For each constraint $s = t$, we generate clauses by replacing join by conjunction, denoted by ",", and $\geq$ by implication. E.g. for the constraint $a \vee b = c^+$, we first derive $a \vee b \geq c^+$ and $c^+ \geq a \vee b$. From $a \vee b \geq c^+$, we generate the clause $a, b \to c^+$. From $c^+ \geq a \vee b$, we derive $c^+ \geq a$ and $c^+ \geq b$. We then generate the clause $c^+ \to a$ from the former, and the clause $c^+ \to b$ from the latter. This generated set of clauses is denoted by $S_{s=t}$, which in our example makes it $S_{a \vee b = c^+} := \{a, b \to c^+, c^+ \to a, c^+ \to b\}$. We define $S_C$ as the union of all $S_{s=t}$, where $s = t$ is a constraint in $C$. From the axiom $x \vee x^+ = x^+$, we can derive the following clauses: $x, x^+ \to x^+$, $x^+ \to x^+$ and $x^+ \to x$. Since the first two are tautological, we are only interested in the last one, which we call a *predecessor clause*. A predecessor clauses ensures that any atom $x+k$ that is true in a model, also has $x+k-1$ true in that model.

In [2, Theorem 2.2, p. 3] it is proven that provability in the semilattice theory is equivalent to Horn clause provability, making it so that we can continue with the latter in the rest of this thesis. Downward closure of the set of atoms validates the predecessor clauses coming from $x \leq x^+$.

We define closure under shifting upwards as follows: if $A \to b$ is in the set of clauses, then so must $A + 1 \to b + 1$ be. $A + 1$ denotes the set of atoms of the form $a + 1$, where $a \in A$. Given a finite semilattice presentation $(V, C)$, and a subset $W \subseteq V$, we denote by $\overline{S_C}$ the smallest subset of $S_C$ that is closed under shifting upwards, by $\overline{S_C} \mid W$ the set of clauses in $\overline{S_C}$ mentioning only variables in $W$, and by $\overline{S_C} \downarrow W$ the set of clauses in $\overline{S_C}$ with conclusion over $W$. Closure under shifting upwards is the Horn clause equivalent of $(x \vee y)^+ = x^+ \vee y^+$.

A function $f : V \to \mathbb{N}^\infty$ specifies a downward closed set of atoms, namely $\{v + k \mid v \in V, k \in \mathbb{N}, k \leq f(v)\}$, where $\mathbb{N}^\infty$ is the set of natural numbers extended with $\infty$. This set contains all atoms $v + k$ if $f(v) = \infty$. The sets specified by such functions are models of $\overline{S_C}$.

### 2.2.1 Lemmas and theorems

**Lemma 3.1 ([2, p. 3])** *Given $f : V \to \mathbb{N}^\infty$, and a clause $A \to b$, let $P$ be the problem whether or not $A + k \to b + k$ is satisfied by $f$ for all $k \in \mathbb{N}$. Then $P$ is decidable.*

The proof of Lemma 3.1 demonstrates that the problem $P$ is decidable, meaning we can indeed write an algorithm that determines whether or not the problem holds for all $k \in N$. Lemma 3.1 is also crucial for making case distinctions in further proofs, since we know that any $S_C$ is finite, whereas $\overline{S_C}$ is not.

**Theorem 3.2 ([2, p. 3])** *Let $(V, C)$ be a finite semilattice presentation. For any $f : V \to \mathbb{N}^\infty$ we can compute the least $g \geq f$ that is a model of $\overline{S_C}$.*

The proof of Theorem 3.2 has a special case that is solved by an additional lemma:

**Lemma 3.3 ([2, p. 3])** *Let $(V, C)$ be a finite semilattice presentation. Let $W$ be a strict subset of $V$ such that for any $f : W \to \mathbb{N}^\infty$ we can compute the least $g \geq f$ that is a model of $\overline{S_C} \mid W$. Then for any $f : V \to \mathbb{N}^\infty$ with $f(V - W) \subseteq \mathbb{N}$, we can compute the least $h \geq f$ that is a model of $\overline{S_C} \downarrow W$.*

# Chapter 3

# Approach & Design Choices

When translating an informal proof or specification to a formal proof, one often has to decide how to model certain mathematical objects and their properties, and which parts of a proof that can be simplified or left out entirely. Our formalization effort has been focused mostly on Theorem 3.2 and its proof. We have left aside two concepts from [2]:

- the relation between semilattices and Horn clauses
- complexity theoretic claims about the problems

These are sections 2 and 4 from [2], respectively. We have also made some simplifications to various parts of our formalization, which we will explain in more detail in the next subsection.

## 3.1 Simplifications for the sake of time

### 3.1.1 Incomplete formal proofs for some purely logical lemmas

We have chosen to not spend too much time fully completing the formal Coq proofs of some purely logical lemmas, which are mainly used as intermediate steps in the proof of Theorem 3.2. When we say ”purely logical”, we mean that they do not contribute anything to the constructive part of the proof, and are not used in the extracted algorithm in any way (we will expand on how most logical lemmas are removed by extraction in Section 4.2.4). As long as these lemmas are true, the correctness of the extracted algorithm is not in danger, even though not formally proved. As we will see later in this section, this is mainly due to very complex or difficult-to-prove formal proofs of lemmas in our set implementation being trivial when proving the same lemma informally. The formal proofs of these lemmas are however not very interesting, which is why we have not spent too much time on them.

### 3.1.2 Leaving out the formal proof of Lemma 3.3

Lemma 3.3 is not purely logical, and has computational content. We have included a formulation of the statement of this lemma, but not a formal proof. When testing the algorithm generated by our formalization of Theorem 3.2, we have manually edited the code to use the identity function instead of crashing due to the lack of a proof of Lemma 3.3. This simplification is sufficient for a surprising large number of problems, but not all; the limitations of this simplification will be explained in more detail in Section 5.3. Thus, this clearly is a limitation to be completed in the future.

### 3.1.3 Proof of minimality

In our proof of Theorem 3.2, we have chosen to omit proving the minimality of the model generated by the algorithm. Our algorithm does however generate a minimal model, but we have not proven that it does.

To prove that the model generated by our algorithm is minimal, we would have had to include the following proposition in our definition of Theorem 3.2:

```
forall h : Frontier, sub_model Cs V V h → geq h f.
```

Listing 3.1: Proposition for minimal model

We will explain the semantics of this proposition in more detail in Chapter 4.

## 3.2 Modeling finite sets in Coq

Sets in mathematics are seemingly simple structures; a set is a collection of elements. A set cannot contain more than one of the same element (*no duplicates*), and the elements are not arranged in any specific order (*no order*). This is the naive definition of a set, not taking into account the complexity of this subtle notion, different set theories, powerful axioms, and so on.

Sets are easy to work with when writing informal proofs. We do not care about how our elements or sets are represented, we only care about their properties. This does not hold for formal proofs though. In a formal proof, we need to specify exactly what happens when you take the union of two sets, or how you determine whether or nor a set contains an element.

One of the most important data structures in functional computer programming is the *list*. Unlike a set, a list *can* contain more than one of the same element, and the elements *are* arranged in a specific order. The inductive definition of the type of polymorphic lists from Coq's standard library is as follows [22]:

```
Inductive list (A : Type) : list A :=
    | nil : list A
    | cons : A → list A → list A.
```

Listing 3.2: Inductive definition of the list type in Coq

Using the `cons` constructor, we can easily define any list containing any elements of the same type; we can even have lists of lists. The problem is of course that lists are not sets. We want to find a way to take into account the two important properties of *no duplicates* and *no order* into our definition of lists. In Coq, there are several ways to do this.

### 3.2.1 List & ListSet

As stated previously, Coq gives us a traditional definition of a list in the `List` module [1] of the standard library [24]. Due to the nature of its definition, it is very easy to construct

[1]The definition of `list` is actually located in the module `Init.Datatypes`, while everything else relating to lists is in the `List` module.

proofs using induction or case distinction on lists; we only need to check two cases. This list implementation is type polymorphic, meaning any type can be used to construct a list of that type. We do not need to give Coq any more information about the properties of the underlying type of the list other than the type itself.

The List module also gives us a tool to combat the possibility of duplicates in a list, with NoDup and nodup. NoDup is an inductively defined proposition that is true when a list has no duplicates. nodup is a function that takes in a list and returns a list with the same elements, but without duplicates. In other words, a list for which NoDup holds. These two can be used to better represent finite sets as lists, since we gain additional information about whether the list has duplicates or not. Coq does inherently not, however, understand how to compare elements when checking a list for duplicates in nodup. Hence we have to provide a proof that the equality of the underlying type of the list is decidable. An example of such a proof for the string type would be:

```
Lemma string_eq_dec :
    forall x y : string, {x = y} + {x <> y}.
Proof.
    (* ... *)
Qed.
```

Listing 3.3: Decidability proof for string equality in Coq

We will omit showing the Ltac terms (i.e. the actual proof) for proofs that are longer than a few lines in the rest of this thesis, which we will usually denote by (* ... *). Proofs as in Listing 3.3 are often given for the standard types in Coq such as nat, bool and string. As such, they can just be passed as arguments. This convention of always passing the proof as an argument can be cumbersome and make the code hard to read, but it is a necessary evil to get the properties we want.

Having only the implementation of the set structure is rarely enough; we also want to do operations on the set, and reason about these. That is where the ListSet module comes in, which defines a new type called set [25]. This type is just an alias for the list type from the List module, but the module also contains some useful functions. Most of these functions treat the input as a set in the traditional sense, meaning that they try to preserve the properties of *no duplicates* and *no order*. Examples of some of these functions are set_add, set_mem, set_diff, and set_union. We also get useful lemmas that prove common properties about these functions. As with nodup, these functions all require a proof of decidability of equality for the underlying type of the set. One thing to note is that all these functions use `bool` instead of `Prop`, and all require a decidability proof, which make the functions themselves decidable.

The module also gives us some lemmas to transform the boolean (type `bool`) set-operation functions into propositions (type `Prop`), and vice versa. An example to illustrate this is the following lemma on `set_mem`:

```
Lemma set_mem_correct1 {A : Type} (dec : forall x y : A, {x = y} + {x <> y}) :
    forall (x : A) (l : set A), set_mem dec x l = true → set_In x l.
Proof.
    (* ... *)
Qed.
```

Listing 3.4: `set_mem` lemma in `ListSet` module

`set_In` is just an alias for `In` from the `List` module, which is a proposition that is very common in many lemmas from the standard library. Lemmas such as the example above are very useful when reasoning about boolean functions such as `set_mem` in proofs, as transforming them into propositions makes them easier to work with and often enables us to use existing lemmas from the standard library.

Many of these boolean set functions, such as `set_union`, take in two sets as arguments and pattern match on the structure of one of them. For example, `set_union` pattern matches on the second set given as an argument. This makes proofs where we destruct or use induction on the second argument easy, such as this example:

```
Lemma set_union_l_nil {A : Type} (dec : forall x y : A, {x = y} + {x <> y}) :
    forall l : set A, set_union dec l [] = l.
Proof.
    destruct l; reflexivity.
Qed.
```

Listing 3.5: Easy proof of lemma in `ListSet` module

The downside is that even easy and seemingly trivial proofs that reason about the other argument are frustratingly hard (or impossible) to prove, for example:

```
Lemma set_union_nil_l {A : Type} (dec : forall x y : A, {x = y} + {x <> y}) :
    forall l : set A, set_union dec [] l = l.
Abort.
```

Listing 3.6: Wrong lemma in ListSet module

What makes this proof impossible is that the order of elements in `set_union dec [] l` is not the same as in `l` (due to how `set_union` is implemented), and since equality on lists care about order, we cannot prove this lemma. However, there are ways to circumvent this problem. Since we often reason about if an element is in a list, or if the list has a certain length, we do not care about the order of the elements. We could for example use `incl`, which is list inclusion, instead of equality. `incl l1 l2` means that all elements in `l1` are also in `l2`. We can alter the lemma in Listing 3.6 to use `incl` instead of equality:

```
Lemma incl_set_union_l_nil {A : Type} (dec : forall x y : A, {x = y} + {x <> y}) :
  forall l, incl l (set_union dec l []) ∧ incl (set_union dec [] l) l.
Proof.
    (* ... *)
Qed.
```

Listing 3.7: Tedious, but correct lemma in ListSet module using incl

Another example of a similar workaround is this definition of a strict subset relation on lists:

```
Definition strict_subset {A : Type} (s1 s2 : set A) :=
  incl s1 s2 ∧ ~(incl s2 s1).
```

Listing 3.8: Strict subset relation on lists using incl

Instead of defining a strict subset as a list that is a subset of another list and where the lists are not equal, we instead define it as a list that includes every element of another list and where some element of the other list is not in the first list. If we use these kinds of "tricks" and construct our proofs carefully, `ListSet` is a viable implementation for our purposes. There might however be cases where the order of the elements in the lists come into play (e.g. such as in Listing 3.6), and that is where this implementation falls short. Another thing to note is because of the polymorphic nature of the `set` type, any additional lemmas proven about a set can be used for any decidable type. This is useful if one needs sets with elements of different types, which is the case in our implementation; we need sets of atoms, clauses, strings, etc.

### 3.2.2 MSetWeakList

The Coq standard library also gives us another implementation of sets, `MSetWeakList` [26]. This implementation is a bit more complicated than the previous one, but gives us more guarantees about the properties of the set. The module is expressed as a functor, which in this case is a "function" that takes in a module as an argument, and again returns a module. The module we give to the functor must satisfy some basic properties about the type we want to create a set of, namely an equality relation, decidability of this relation and the fact that this relation is an equivalence relation. The "output" from the functor is a new module containing functions and lemmas about set operations, with our input type being the type of the elements of the set. An example definition for a module of a set of atoms would be as follows:

```
Module AtomEq : DecidableType.

  (* what is our base type? *)
  Definition t := Atom.

  Definition eq := (* ... *)

  Lemma eq_equiv :
    Equivalence eq.
  Proof.
    (* proof that eq is an equivalence relation *)
  Qed.

  Lemma eq_dec :
    forall x y : t, {eq x y} + {~ eq x y}.
  Proof.
    (* proof that eq is decidable *)
  Qed.

End AtomEq.

(* apply functor Make to module AtomEq *)
Module AtomSet := Make AtomEq.
```

Listing 3.9: Set of atoms in `MSetWeakList` module

The proofs of both lemmas above are trivial, but require a substantial amount of boilerplate code. For every type we want to use as an element in a set, we have to go through this entire process. In `List` and `ListSet`, we just had to pass in the proof of the decidability of equality of the type as an argument to the set functions and lemmas. The structure of the sets in `MSetWeakList` is also a lot more complicated than the simple and intuitive definitions given in `List`. This makes it harder to reason about the sets in proofs, given limited knowledge of Coq.

### 3.2.3 Ensembles

Yet another implementation of sets is given by the `Ensembles` module which defines the structure of a set as inductive propositions [23]. These inductive propositions assert some fact about if an element is in the set or not. An example that illustrates this is the definition of `Union`:

```
Inductive Union (B C:Ensemble) : Ensemble :=
  | Union_introl : forall x:U, In B x → In (Union B C) x
  | Union_intror : forall x:U, In C x → In (Union B C) x.
```

Listing 3.10: `Union` in `Ensembles` module

We see that the first constructor of `Union` asserts that if an element is in `B`, then it is in the union of `B` and `C`. `Ensembles` uses `Prop` instead of `bool`, making `Ensembles` useful for proofs where we do not care about decidability, and when the sets can be infinite. The biggest downside to this implementation is that we cannot reason about the size of the set. We can only determine if an element is in the set, not how big the set is. In our case, this makes the `Ensembles` module useless, since the theorem we are formalizing requires us to reason about the size of the set.

## 3.3 Choice of implementation of sets

The simplest set (or set-like) implementation in Coq are the `List` and `ListSet` modules. These require minimal knowledge of advanced Coq syntax and behave like lists, making proofs by induction easy. They are also polymorphic, meaning ease of use when making sets of different or self-defined types. Because of these reasons, we chose to go with `List` and `ListSet`.

# Chapter 4

# Implementation of Case-Specific Notions

We will now explain how we have formalized the concepts discussed in Chapter 2 using Coq. We will only go over the most important parts of the formalization. The full code can be found in the GitHub repository [1]. Our formal proof of Theorem 3.2 can also be found in Listing A.1.

## 4.1 Data types

As discussed in Chapter 2, we want to represent clauses as a set of atoms as premises and a single atom as a conclusion. We implement this in Coq using two types, `Atom` and `Clause`.

```
Inductive Atom : Type :=
  | atom : string → nat → Atom.

Notation "x & k" := (atom x k) (at level 80).

Inductive Clause : Type :=
  | clause : set Atom → Atom → Clause.

Notation "ps ~> c" := (clause ps c) (at level 81).
```

Listing 4.1: `Atom` and `Clause` in Coq

Note also the `Notation`-syntax, which allow us to define a custom notation, making the code easier to read. The expression on the left-hand side of the := in quotation marks is definitionally equal to the expression on the right-hand side in parentheses. The level determines which notation should take precedence, with a higher level equaling a higher precedence.

We also want to represent functions of the form $f : V \to \mathbb{N}^\infty$, which as we saw in Section 2.2 are functions that are models of a set of clauses. We implement this in Coq using two types, `Ninfty` and `Frontier`. `Ninfty` has two constructors; `infty` represents $\infty$, and `fin n` represents a natural number $n$. `Frontier` is a function from strings (variables) to `Ninfty`. In the context of our formalization, we will refer to a function of this type as a *frontier*.

```
Inductive Ninfty : Type :=
  | infty : Ninfty
  | fin   : nat → Ninfty.

Definition Frontier := string → Ninfty.
```

Listing 4.2: `Ninfty` and `Frontier` in Coq

Using these definitions of `Atom`, `Clause` and `Frontier`, we can define functions that check whether any given atom or clause is satisfied a given frontier.

```
Definition atom_true (a : Atom) (f : Frontier) : bool :=
  match a with
  | (x & k) ⇒
    match f x with
    | infty ⇒ true
    (* see explanation for ≤ ? below *)
    | fin n ⇒ k ≤ ? n
    end
  end.

Definition clause_true (c : Clause) (f : Frontier) : bool :=
  match c with
  | (conds ~> conc) ⇒
    (* every atom in conds is true *)
    if fold_right andb true (map (fun a ⇒ atom_true a f) conds)
    then (atom_true conc f)
    else true
  end.
```

Listing 4.3: `atom_true` and `clause_true` in Coq

The infix function ≤ ? is the boolean (and hence decidable) version of the Coq function ≤ , which uses `Prop` and is not inherently decidable without additional lemmas.

We can also define functions that ”shift” the number value of atoms or whole clauses by some amount `n : nat`.

```
Definition shift_atom (n : nat) (a : Atom) : Atom :=
  match a with
  | (x & k) ⇒ (x & (n + k))
end.

Definition shift_clause (n : nat) (c : Clause) : Clause :=
  match c with
  | conds ~> conc ⇒
    (map (shift_atom n) conds) ~> (shift_atom n conc)
  end.
```

Listing 4.4: `shift_atom` and `shift_clause` in Coq

Using these definitions, we can now define an important property that is possible by Lemma 3.1 [2, p. 3], since this lemma enables us to check whether or not a clause is satisfied by a frontier for any shift of `k : nat`. We will use this property later to determine if a frontier is a valid model for a set of clauses.

```
Definition all_shifts_true (c : Clause) (f : Frontier) : bool :=
  match c with
  | (conds ~> conc) ⇒
      match conc with
      | (x & k) ⇒
          match f x with
          | infty ⇒ true
          | fin n ⇒ clause_true (shift_clause (n + 1 − k) c) f
          end
      end
  end.
```

Listing 4.5: `all_shifts_true` in Coq

The idea is to shift the clause up by the smallest shift that makes the conclusion false and then to check that this shifted clause is still true (because one of the atoms in the condition also has become false).

## 4.2 Semantic functions and predicates

### 4.2.1 The function `sub_model`

Given any set of clauses and a frontier (function assigning values to the variables), we can determine if the frontier is a model of the set of clauses, i.e. whether all shifts of all clauses are satisfied by the frontier.

We translate this property to Coq as the recursive function `sub_model`. As we saw in Section 2.2, we distinguish between three different sets of clauses when talking about frontiers as models; $\overline{S_C}$, $\overline{S_C} \mid W$ and $\overline{S_C} \downarrow W$. These are represented by `sub_model Cs V V f`, `sub_model Cs W W f` and `sub_model Cs V W f`, respectively, where `V` is the set of all variables.

We have two additional arguments `V` and `W`: `V` is the the set of variables (strings) from the set of clauses `Cs`, and `W` is a subset of `V` such that for each variable there is a clause in `Cs` that can be used to generate a new atom. The function `vars_set_atom` used below simply returns all the variables used in a set of atoms, as a set of strings.

```
Fixpoint sub_model (Cs : set Clause) (V W : set string) (f : Frontier) : bool :=
  match Cs with
  | []          ⇒ true
  | (l ~> (x & k)) :: t  ⇒
    (* conclusion not in W *)
    (negb (set_mem string_dec x W) ||
    (* some premise not in V *)
     negb (
       fold_right andb true
         (map (fun x ⇒ set_mem string_dec x V) (vars_set_atom l))
     ) ||
     all_shifts_true (l ~> (x & k)) f
    ) && sub_model t V W f
  end.
```

Listing 4.6: The function `sub_model` in Coq

### 4.2.2 The functions `sub_vars_improvable` and `sub_forward`

The three conditions of the disjunction in Listing 4.6 are what determines if a clause is satisfied by a frontier. If these conditions are not met, we want to repeatedly increment the value of the variable, and check again if the clause is satisfied. We start by first determining such "improvable" variables with the function `sub_vars_improvable`:

```
Fixpoint sub_vars_improvable (Cs : set Clause) (V W : set string) (f : Frontier)
    : set string :=
  match Cs with
  | []          ⇒ []
  | (l ~> (x & k)) :: t  ⇒
      if negb (set_mem string_dec x W) ||
         negb (
            fold_right andb true
                (map (fun x ⇒ set_mem string_dec x V) (vars_set_atom l))
         ) ||
         all_shifts_true (l ~> (x & k)) f
      then sub_vars_improvable t V W f
      else set_add string_dec x (sub_vars_improvable t V W f)
  end.
```

Listing 4.7: The function `sub_vars_improvable` in Coq

We then define a new function `sub_forward` that takes these improvable variables and a frontier `f`, and defines a new frontier `f'` that increments the value that `f` assigns to each variable. The return type of `sub_forward` is a pair of a set of strings and a frontier, where the set of strings is the set of improvable variables.

```
Definition sub_forward (Cs : set Clause) (V W : set string) (f : Frontier)
    : set string * Frontier :=
  let X := sub_vars_improvable Cs V W f in
  (* Sinfty increases a Fin value by 1, and leaves Infty unchanged *)
  let f' := fun v ⇒ if set_mem string_dec v X then Sinfty (f v) else f v
  in (X, f').
```

Listing 4.8: The function `sub_forward` in Coq

### 4.2.3 The function `geq`

We want to determine whether all the values assigned to a set of variables from one frontier are greater than or equal to all the values assigned to a set of variables from another frontier. The values are of the type `Ninfty`, and the function only returns true if **all** the values from the first frontier are greater than the values from the second frontier.

```
Fixpoint geq (V : set string) (g f : Frontier) : bool :=
  match V with
  | []       ⇒ true
  | h :: t  ⇒
    match g h with
    | infty ⇒ geq t g f
    | fin n ⇒
        match f h with
        | infty ⇒ false
        | fin k ⇒ (k ≤ ? n) && geq t g f
        end
    end
  end.
```

Listing 4.9: Point-wise comparing frontiers with `geq` in Coq

### 4.2.4 The predicate `ex_lfp_geq` with `Prop` and `Set`

We can now combine `sub_model` and `geq` to construct a predicate stating that there exists a frontier `g` that is a model of the set of clauses `Cs` and is greater than or equal to another frontier `f`.

```
Definition ex_lfp_geq_P (Cs : set Clause) (V W : set string) (f : Frontier) : Prop :=
  exists g : Frontier, geq V g f = true ∧ sub_model Cs V W g = true.

Definition ex_lfp_geq_S (Cs : set Clause) (V W : set string) (f : Frontier) : Set :=
  sig (fun g : Frontier ⇒ prod (geq V g f = true) (sub_model Cs V W g = true)).
```

Listing 4.10: `ex_lfp_geq` in Coq, using both `Prop` and `Set`

One thing to note here is that we can define this predicate either with `Prop` or with `Set`. When defined with `Prop`, we define it as a logical predicate, using standard first-order logic syntax. When defined with `Set`, we define it as a type, using the `sig` type constructor in place of `exists`. `sig` denotes the Σ-type, which we explained in Section 1.2.2.

The reason for defining `ex_lfp_geq` with `Set`, is that we then can use Coq's extraction feature to generate Haskell code from the Coq definitions. As briefly mentioned in Section 1.2.1, a proof can often be used to compute something. In this case, we want to compute the actual frontier g that satisfies the above predicate. Coq distinguishes between logical objects (objects in Prop) and informative objects (objects in Set) [13, p. 1-2]. When extracting, Coq will remove as many logical objects as possible, meaning a proposition defined in Prop would simply be collapsed to "()" (the unit type) when extracted to Haskell [14, p. 8]. Logical objects are only used to ensure correctness when constructing a proof in Coq, and are not needed when actually computing something using the extracted code. Informative objects, on the other hand, are kept when extracting, and can be used to compute something. This is why we define ex_lfp_geq with Set.

An important thing we also need to keep in mind is to avoid *universe inconsistencies*. These can occur by unwise use of Type. We will not elaborate much on what type universes are, but we will give a brief explanation of how they work in Coq. In Coq, all objects of type Prop also live in the universe Set, and all objects of type Set also live in the universe Type. Moreover, Prop : Set and Set : Type. The cumulative universe hierarchy extends infinitely past Type, but we will not need to go further than that. When defining a type, we can only refer to objects in universes lower than the universe of the type we are defining. Here is an example to illustrate this:

```
(* type signature of sig *)
sig : forall A : Type, (A → Prop) → Type

(* Set ≤ Set, OK *)
Definition valid_def (A : Set) : Set :=
  sig (fun x : A ⇒ x = x).

(* Set > Prop, universe inconsistency! *)
Fail Definition invalid_def (A : Set) : Prop :=
  sig (fun x : A ⇒ x = x).
```

Listing 4.11: A universe inconsistency in Coq

## 4.3 The main proofs

We have now laid the groundwork for the formalization of Theorem 3.2. We precede the definition of Theorem 3.2 with two additional definitions, which helps us simplify its definition and the proof of the theorem itself.

### 4.3.1 The predicate `pre_thm`

Since our formal definitions of Lemma 3.3, which will be expanded on shortly, and Theorem 3.2 share some structure, we define (with `Set`) a predicate `pre_thm`:

```
Definition pre_thm (n m : nat) (Cs : set Clause) (V W : set string) (f : Frontier)
    : Set :=
  incl W V →
  Datatypes.length (nodup string_dec V) ≤ n →
  Datatypes.length
    (set_diff string_dec
      (nodup string_dec V)
      (nodup string_dec W)
    ) ≤ m ≤ n →
  ex_lfp_geq Cs (nodup string_dec W) (nodup string_dec W) f →
  ex_lfp_geq Cs (nodup string_dec V) (nodup string_dec V) f.
```

Listing 4.12: `pre_thm` in Coq

### 4.3.2 Lemma 3.3

Lemma 3.3 is used in the proof of Theorem 3.2 to solve a special case, as we saw in Section 2.2. We also need this lemma to be able to prove Theorem 3.2 in Coq. As explained in Section 3.1.2, we will only give a formal definition of the lemma in Coq, but leave out the actual proof. By using the Coq command `Admitted`, we can leave out a full proof of a lemma, but still be able to use the lemma in other Coq proofs. We define Lemma 3.3 in Coq using `pre_thm`, as follows:

```
Lemma lem_33 :
  forall Cs : set Clause,
  forall V W : set string,
  forall f : Frontier,
    (forall Cs' : set Clause,
     forall V' W' : set string,
     forall f' : Frontier,
     forall m : nat,
      pre_thm (Datatypes.length (nodup string_dec V) - 1) m Cs' V' W' f'
    ) →
    incl W V →
    ex_lfp_geq Cs (nodup string_dec W) (nodup string_dec W) f →
    ex_lfp_geq Cs (nodup string_dec V) (nodup string_dec W) f.
Admitted.
```

Listing 4.13: Lemma 3.3 in Coq

One thing to note is that when applying Lemma 3.3 in the formal proof of Theorem 3.2, the first assumption of the lemma is solved immediately by applying the primary induction hypothesis (the induction on `n`, also called `IHn` in Coq).

### 4.3.3 Theorem 3.2

We can now formulate Theorem 3.2 using `pre_thm`:

```
Theorem thm_32 :
  forall n m : nat,
  forall Cs : set Clause,
  forall V W : set string,
  forall f : Frontier,
    pre_thm n m Cs V W f.
Proof.
  (* ... *)
Qed.
```

Listing 4.14: Theorem 3.2 in Coq

The proof of Theorem 3.2 is by primary induction on `n` and secondary induction on `m`.

### Base case of `n`

The first base case is simple. We directly prove the conclusion of `pre_thm`:

(1) `ex_lfp_geq Cs (nodup string_dec V) (nodup string_dec V) f`.

Since `n` $= 0$, we get that the length of `V` is 0, and hence we get a new goal `ex_lfp_geq Cs [] [] f`. This is proven by the lemma `ex_lfp_geq_empty`, which states that `forall Cs f, ex_lfp_geq Cs [] [] f`.

### Inductive case of `n`

We start the inductive case of `n` by doing a new induction on `m`.

### Base case of `m`

The base case of `m` is similar to the base case of `n`. We again want to prove

`ex_lfp_geq Cs (nodup string_dec V) (nodup string_dec V) f`.

But now we have as assumption this same statement with `W`, and `incl V W` since `m` $= 0$. We now apply the lemma `ex_lfp_geq_incl`, which states that

`forall Cs V W f, incl V W → forall f, ex_lfp_geq Cs W W f → ex_lfp_geq Cs V V f`.

We give this lemma the arguments of `Cs`, `nodup string_dec V` and `nodup string_dec W`. This generates two new goals:

(1) `incl (nodup string_dec V) (nodup string_dec W)`

(2) `ex_lfp_geq Cs (nodup string_dec W) (nodup string_dec W) f`

The goal (1) is proven by using a hypothesis that states that

`Datatypes.length (set_diff string_dec (nodup string_dec V) (nodup string_dec W)) ≤ m ≤ n`.

Since `m` $= 0$, this means that the set difference of `V` and `W` is empty. We can now apply the lemma `set_diff_nil_incl` on this hypothesis, which states that

`forall dec V W, set_diff dec V W = [] ↔ incl V W`.

This gives us a hypothesis identical to our goal (1), and therefore proves it.

The goal (2) is proven by an existing hypothesis.

#### Inductive case of `m`

The inductive case of `m` is by far the largest part of the proof (see lines 17-199 of Listing A.1). As it is very complex and does not follow the informal proof in [2] too closely, we will go through a more "high-level" explanation of the reasoning we followed when constructing the formalized proof. We will sometimes use mathematical notation to explain the proof instead of Coq code, for the sake of being terse and clear.

Assume $W \subseteq V$, $|V| \leq n+1$, $|V-W| \leq m+1 \leq n+1$. If $|V-W| \leq m$ then we can apply the secondary induction hypothesis, and we are done. Hence, we can assume $|V-W| = m+1$, implying $|W| \leq n$. To prove `ex_lfp_geq Cs V V f`, we assert (`W, g`) = `sub_forward Cs V V f`. This gives us three cases to consider:

1. $W = \emptyset$, and we are done.
2. $W = V$, meaning all variables in $V$ map to $\infty$, and we are done.
3. $\emptyset \subset W \subset V$. Then $|W| \leq n$ and $|V-W| \leq m$. So by the primary induction hypothesis, we get `ex_lfp_geq Cs W W f` (line 31 of Listing A.1). Then by the second induction hypothesis and the application of Lemma 3.3 (line 40 of Listing A.1), we get `ex_lfp_geq Cs V V f`, and we are done.

A complete formal proof of Theorem 3.2 can be found in Listing A.1.

## 4.4 Extraction to Haskell

Using Coq's code extraction feature, we can extract Haskell code from our Coq definitions.

```
Extraction Language Haskell.

Extract Constant map ⇒ "Prelude.map".
Extract Constant fold_right ⇒ "Prelude.foldr".

Extraction "/home/user/path/to/code/ex.hs"
  thm_32
  lem_33.
```

Listing 4.15: Extraction of Coq definitions to Haskell

Coq will automatically determine definitions which depend on one another when doing extraction. In the example above, we would not have needed to specify `lem_33` to be extracted, since `thm_32` already depends on it.

By default, Coq will give its own implementation of any functions used, instead of using Haskell's native implementations. If we want, we can specify what native Haskell functions should be used when extracting a Coq function. In the example code above, we specify that when extracting, Haskell's `Prelude.map` and `Prelude.foldr` should be used for the Coq functions `map` and `fold_right`.

In the next chapter we will go more into detail about the results of the extraction, and the results of the Haskell code ran on some example input.

# Chapter 5

# Examples & Results

## 5.1 Examples using the extracted Haskell code

### 5.1.1 Defining examples for extraction in Coq

It is easiest to define as much of the example as possible in Coq and then extract it to Haskell, since Coq heavily prioritizes code correctness over readability when extracting. This makes much of the Haskell code hard to read due to unconventional syntax, e.g. using a recursion operator instead of calling a function recursively. First, we can look at the type of `thm_32` in Coq to see what arguments we need to provide it with. We do this by using the Coq command `Check`.

```
thm_32
    : forall (n m : nat) (Cs : set Clause) (V W : set string) (f : Frontier),
      pre_thm n m Cs V W f
```

Listing 5.1: Type signature of `thm_32` in Coq

We see that the type of `thm_32` is a proof of `pre_thm`, with the quantified variables `n`, `m`, `Cs`, `V`, `W`, `f`. We can now look at the type signature of `pre_thm`.

```
pre_thm
    : nat →
      nat → set Clause → set string → set string → Frontier → Set
```

Listing 5.2: Type signature of `pre_thm` in Coq

We see that `pre_thm` returns an object in `Set`. This object is the proposition [1] defined in Listing 4.12, where all the variables are the arguments given to `pre_thm`, i.e. the variables quantified in `thm_32`. This proposition states some logical assumptions, and concludes with a definition of `ex_lfp_geq`. By the notion of propositions as types, this proposition is a function that transforms a proof of the assumptions into a proof of the conclusion. Since the conclusion is a definition of `ex_lfp_geq` instantiated with some variables, we get a proof of `ex_lfp_geq` for these same variables. Looking at the definition of `ex_lfp_geq`, it defines a proposition stating that there exists a `g : Frontier` such that the proposition holds. If such a `g` exists, then the `g` itself is evidence (proof) that the proposition holds. Hence, a proof of `ex_lfp_geq` is simply a `Frontier` that satisfies the conditions in `ex_lfp_geq`.

With all this information, we can now define an example of `thm_32` in Coq.

```
(* represents the clauses: a → b + 1, b + 1 → c + 2 *)
Example Cs := [
  ["a" & 0] ~> "b" & 1;
  ["b" & 1] ~> "c" & 2
].

(* all-zero frontier *)
Example thesis_ex_1_f := frontier_fin_0.

(* extract only variables from clauses *)
Example vars' := nodup string_dec (vars Cs).

(* (partially) apply arguments to thm_32 *)
Example thesis_ex_1 :=
  thm_32
  (Datatypes.length vars')
  (Datatypes.length vars')
  Cs
  vars'
  []
  thesis_ex_1_f.
```

Listing 5.3: `thm_32` example

[1]When we use the term "proposition", we refer to the colloquial usage of a true or false statement, not necessarily the Coq notion of a proposition, i.e. an object in `Prop`. We will therefore sometimes refer to an object in `Set` as a proposition, since it is still a proposition in the colloquial sense.

Since Coq eliminates many of the logical parts of the proof when extracting, [14, p. 8], we can avoid the tedious task of proving all the assumptions of `pre_thm` by simply using the extracted Haskell function for `thesis_ex_1`. The final assumption of `pre_thm`, namely `ex_lfp_geq Cs` (`nodup string_dec W`) (`nodup string_dec W`) `f`, will however not be removed by Coq, since it is an object in `Set`. This assumption is trivially true for any `f`, since `W` is an empty list in our example. Since Coq simplifies the type of `ex_lfp_geq` to `Frontier`, we can just include the same frontier `f` as in `thesis_ex_1` to the extracted Haskell version of `thesis_ex_1`, as a replacement for a proof of the above trivially true assumption. We then receive a `Frontier` as output, which is the result of the theorem; the frontier that is a model of a set of clauses `Cs`. This frontier can then be applied to any string representing a variable to get the value of that variable in the model, which should be either a natural number or infinity.

### 5.1.2 Necessary alterations to the extracted Haskell code

Since we have not given a proof of Lemma 3.3, Coq will include a ”placeholder” definition of the lemma in the extracted code. This definition will simply call the `error` function, which will crash the program when called. We circumvent this by replacing the extracted definition of `lem_33` with the identity function for any frontier. We will look at some examples where this workaround is not sufficient in Section 5.3.

If we want to actually read the output from the extracted functions, we also need to derive a `Show` instance for `Ninfty`. What this means is that we need to define a function `show` :: `Ninfty` → `String`, which will be used by Haskell to convert a `Ninfty` to a `String`. This can be done by simply adding the line `deriving Prelude.Show` to the definition of `Ninfty`, which will make Haskell just print the constructors of `Ninfty`, which will be either `Fin n` for some natural number `n`, or `Infty` for infinity.

### 5.1.3 Example output

We can now run the example from Listing 5.3 using GHCi, which is an interactive Haskell interpreter that is included with GHC, the standard Haskell compiler.

```
ghci> (thesis_ex_1 thesis_ex_1_f) "a"
Fin 0
ghci> (thesis_ex_1 thesis_ex_1_f) "b"
Fin 1
ghci> (thesis_ex_1 thesis_ex_1_f) "c"
Fin 2
ghci> (thesis_ex_1 thesis_ex_1_f) "x"
Fin 0
```

Listing 5.4: `thm_32` example output

The expression (`thesis_ex_1 thesis_ex_1_f`) produces a function that is the minimal model of `Cs`. When given a string value (variable) from the set of clauses, the function will compute the value of that variable. When given any other variable, the function will return the value that the original frontier given as input would return for that variable, which is always `Fin` 0 in this case (since in our example the frontier is `frontier_fin_0`, which is a constant function that always returns `Fin` 0).

### 5.1.4 Example of loop-checking

In the previous example, the algorithm found a minimal model for the set of clauses `Cs`. We can also use the algorithm to check for loops in a set of clauses. As mentioned in Section 2.1, a loop is a term $t$ such that $t \geq t^+$ can be provable from the set of constraints. We give the simplest example, which is just the above definition of a loop translated to our Coq notation:

```
Example Cs := [
  ["a" & 0] ~> "a" & 1
].
Example thesis_ex_2_f := frontier_fin_0.
Example vars' := nodup string_dec (vars Cs).

Example thesis_ex_2 :=
thm_32
  (Datatypes.length vars')
  (Datatypes.length vars')
  Cs
  vars'
  []
  thesis_ex_2_f.
```

Listing 5.5: Loop-checking example

```
ghci> (thesis_ex_2 thesis_ex_2_f) "a"
Infty
ghci> (thesis_ex_2 thesis_ex_2_f) "x"
Fin 0
```

Listing 5.6: Loop-checking example output

For a model defined by a frontier `f`, any variable that maps to `Infty` part of a loop. For any other variable we get the usual result of `Fin` 0, since the given frontier is constant.

## 5.2 Real world example

As stated previously, the algorithm described Theorem 3.2 is being tested for use in checking loops and determining type universe levels in the type system of Coq. We briefly explained type universes in the case of Coq in Section 4.2.4. The error that occurred in the example of a universe inconsistency in Listing 4.11 was detected by Coq's own universe-constraint checking algorithm; we can also use our algorithm to detect these universe inconsistencies.

Using the Coq command `Print Universes` we can see the current state of the universe hierarchy for any Coq file. Doing this at the bottom of our file `Main.v` (which is where the main proofs are located), after the formal proof of Theorem 3.2, we get the following output:

```
DefaultRelation.u0 ≤ Coq.Relations.Relation_Definitions.1 ;
default_relation.u0 ≤ DefaultRelation.u0 ;

...

Coq.Structures.Equalities.1 ≤ Morphisms.proper_sym_impl_iff_2.u1 ;
```

Listing 5.7: Universe hierarchy in `Main.v`

This output has over 5000 lines, so we have truncated it to only show the first and last few lines. We can translate these constraints into a set of clauses `Cs` as such:

```
Definition Cs := [
    ["Coq.Relations.Relation_Definitions.1" & 0]
        ~> "DefaultRelation.u0" & 0;
    ["DefaultRelation.u0" & 0]
        ~> "default_relation.u0" & 0;

    ...

    ["Morphisms.proper_sym_impl_iff_2.u1" & 0]
        ~> "Coq.Structures.Equalities.1" & 0
].
```

Listing 5.8: Universe hierarchy as clauses

Not shown above are constraints of the form `x` $<$ `y` and `x` $=$ `y`, which are translated like this:

```
(* x < y *)
Example Cs1 := [
    ["y" & 0] ~> "x" & 1;
];

(* x = y *)
Example Cs2 := [
    ["y" & 0] ~> "x" & 0;
    ["x" & 0] ~> "y" & 0;
];
```

Listing 5.9: Other forms of constraints translated to clauses

We can then run our algorithm on these clauses to get a minimal model of `Cs`. This minimal model is a function that assigns to each type universe a natural number, which corresponds to its universe level. It is minimal in the sense that no other mapping from type universes to universe levels that satisfies the constraints in `Cs` has a lower universe level for any given type universe. If the algorithm had detected a loop, it would constitute a universe inconsistency. Even though we have not given a formal proof of Lemma 3.3, this "real world" example still gives a correct minimal model for the type universe constraints from Coq.

### 5.2.1 Loose comparison to Coq's universe consistency checker

To give the reader an idea of how our extracted algorithm compares to some of the work done by the Coq compiler on a similar problem, we can loosely compare the run time of our algorithm to the run time of a full Coq compilation of our Coq project, which includes all lemmas and theorems needed for Theorem 3.2, and the proof itself. It is important to note that this is **not** a one-to-one comparison, since Coq does a lot more when compiling than simply checking universe constraints. We give this comparison only to give the reader an idea of the order of magnitude of the run time of our algorithm in a real world setting.

We compile our Haskell algorithm using the Stack build tool, which is more efficient than using an interpreter such as GHCi. Using the `time` command, we can check how long any command takes. For our Haskell algorithm we run the command `stack run`, and for the Coq example we run the command `make –f CoqMakefile`.

```
stack run 18.47s user 0.94s system 100% cpu 19.369 total
make -f CoqMakefile 3.25s user 0.76s system 99% cpu 4.045 total
```

Listing 5.10: Real-world example benchmark

As we can see, our algorithm takes around 18 seconds, while a full Coq compilation takes around 3 seconds. Again, keep in mind that Coq does a lot more than just checking universe constraints when compiling. But, as we can see from these results, our extracted algorithm is not too slow to be used as a prototype.

## 5.3 Extension of Listing 5.3 that fails

There are some cases where the extracted Haskell code will crash. To show this, we can extend the set of clauses from Listing 5.3 as such:

```
Example Cs := [
  ["a" & 0] ~> "b" & 1;
  ["b" & 1] ~> "c" & 2;
  (* add new clause *)
  ["c" & 2] ~> "d" & 3
].

Example thesis_ex_3_f := frontier_fin_0.
Example vars' := nodup string_dec (vars Cs).

Example thesis_ex_3 :=
  thm_32
  (Datatypes.length vars')
  (Datatypes.length vars')
  Cs
  vars'
  []
  thesis_ex_3_f.
```

Listing 5.11: `thm_32` example extended

If we try to run the extracted Haskell code for this example, we will always encounter a runtime exception:

```
ghci> (thesis_ex_3 thesis_ex_3_f) "a"
*** Exception: absurd case
CallStack (from HasCallStack):
  error, called at ex.hs:13:3 in main:Ex
```

Listing 5.12: Output of `thm_32` example extended

The exception ”absurd case” is what happens when a logical assumption in the proof is not correct, which is the case here. To see why this happens, we can do some minor alterations to the extracted Haskell code, and print out the values of some variables during the execution of the algorithm. What we want to show is every primary and secondary induction call (`IHn` and `IHm` respectively), and the values of the variables `n`, `m`, `V` and `W` at the time of those calls. Doing so, we get the following output (truncated for brevity):

```
ghci> (thesis_ex_3 thesis_ex_3_f) "a"
IHn: n = 3, m = 4, V = ["a","b","c","d"], W = []

...

IHn: n = 0, m = 1, V = ["a","b","c","d"], W = []
IHm: n = 1, m = 0, V = [],  W = []
IHm: n = 3, m = 0, V = ["b","c","d"], W = ["d"]
*** Exception: absurd case
CallStack (from HasCallStack):
  error, called at ex.hs:13:3 in main:Ex
```

Listing 5.13: Output of `thm_32` example extended, with debug

The debug line we are most interested in is the last one, which is the last secondary induction call before the exception. If we refer to lines 133-137 of Listing A.1, we see that the proof asserts (and proves) the following claim:

```
Datatypes.length
  (set_diff string_dec V (set_union string_dec (nodup string_dec W) U)) <
Datatypes.length
  (set_diff string_dec V (nodup string_dec W)) ≤ S m.
```

Listing 5.14: False claim in `thm_32` proof

This claim is false, as we can see by looking at the values of `V`, `W` and `m` at the time of the last induction call. We replace the variables in the claim with their values, and reduce (parts of) the claim to the following:

`Datatypes.length` (`set_diff string_dec V` (`nodup string_dec W`)) $\leq$ `S m`

$\Rightarrow$ `Datatypes.length` (`set_diff string_dec` [`"b"`,`"c"`,`"d"`] [`"d"`]) $\leq$ `S` 0.

$\Rightarrow$ `Datatypes.length` [`"c"`,`"b"`] $\leq$ `S` 0.

$\Rightarrow$ $2 \leq 1$.

The final claim is obviously not true, hence why the Haskell code crashes with "absurd case". But why does this happen at all? Shouldn't the Coq proof have failed at the point where the claim was asserted? This error is due to the fact that we have not given a formal proof of Lemma 3.3. Since we do not use Lemma 3.3 to calculate a new frontier in the algorithm but instead just redefine Lemma 3.3 as the identity function, this has consequences for later steps in the algorithm. This example in particular is "constructed to fail"; as we saw in the previous section, the algorithm does work for many cases even without a formal proof of Lemma 3.3.

# Chapter 6

# Evaluation

## 6.1 Correctness of results from [2]

As we have shown in our formalization and by the results of the extraction, the results from [2] that we have formalized, appear to be correct. As mentioned in Section 3.1, we have not fully formalized nor verified the entirety of the informal proofs, and as such we cannot be 100% sure that the informal proof of Theorem 3.2 (and related results) is correct. However, we have verified the most important parts of Theorem 3.2 without encountering any problems, and we can thus be very confident that the informal proofs from [2] are correct.

## 6.2 Feasibility of formalization

The formalization of the proof of Theorem 3.2 from [2] took us about 12 months of work. This includes the time spent on learning Coq and the time spent on the formalization of the proof and related lemmas/definitions. We have shown that a formalization of the proof of Theorem 3.2 is feasible, and that it is possible to complete such a formalization in a reasonable amount of time. It is important to remember that we did not have any prior knowledge of Coq or any other proof assistants, and that we had to learn Coq from scratch. The project was also not a full-time effort, and other work and studies have been done in parallel. Given more extensive knowledge of Coq (or another other proof assistant), and possibly more effort or man-power devoted to the project, we believe that a formalization of a proof of a similar complexity would be a feasible task.

## 6.3 Value of formalization & extracted algorithm

Type theory has started the transition from research to development in the software industry, e.g. in the project CompCert [12], which aims to produce verified compilers for mission-critical computer systems. A case in point of the impact of type theory on software development is the project Deep Spec [4] that focuses on the specification and verification of full functional correctness of software and hardware. The paper [8] in this project uses Coq to certify high-level code for cryptographic arithmetic and yields the fastest-known elliptic-curve library in C. This library is deployed on an industrial scale for Chrome and Android.

All these examples show that formalization has provided great value for mission-critical systems where correctness is of utmost importance. If we imagine our correct algorithm as a starting point for further development and for eventual use in the Coq proof assistant for checking universe constraints, one could argue about whether or not our case is as mission-critical as these. However, any fault in Coq as a proof assistant would taint the correctness of anything proven using it.

As seen in Section 5.2, the extracted algorithm is useful in practice, however, the efficiency is not very good. There are also cases where the extracted algorithm crashes, as seen in Section 5.3. However, it should not be a surprise that not every case is handled by the algorithm, as we have not given a formal proof of Lemma 3.3. We can better think of this algorithm as a prototype or proof-of-concept, rather than a practical algorithm.

# Chapter 7

# Related & Future Work

## 7.1 Related work: universe consistency checking in Coq

There is an ongoing effort [18] to implement a version of the algorithm from [2] for use in universe consistency checking in Coq. This version of the algorithm is graph-based, does not include joins, and only support constraints (clauses) of the form $x + k >= y$ with $k \in \{0, 1\}$. The main focus is on speed, which explains the limited expressivity. There are plans to extend the current implementation with more expressivity later on.

## 7.2 Future work

### 7.2.1 Completing proof of remaining logical lemmas

As mentioned in Section 3.1.1, we have not fully completed some of the intermediate lemmas used directly in the proof of Theorem 3.2. We have done some work on making sure these are purely logical steps, and that they should have no effect on the final result. Solving these fully would however strengthen the argument for the correctness of the formal, and hence informal, proof of Theorem 3.2.

### 7.2.2 Formal proof of Lemma 3.3

As mentioned in Section 3.1.2, we have not given a full formal proof of Lemma 3.3. As we also saw in Section 5.3, constructing a formal proof of this lemma would fix some cases where the extracted algorithm fails with an exception. Further work on a complete formalization of Theorem 3.2 would certainly benefit from a formal proof of Lemma 3.3.

### 7.2.3 Proving minimality of model generated by Theorem 3.2

As mentioned in Section 3.1.3, our proof of Theorem 3.2 does not include a claim that the model generated is minimal. As later explained in the same subsection, we would only need to include one additional assumption in our proof of Theorem 3.2 to be able to correctly claim that the model generated is minimal. We chose not to do this as it had no effect on the results produced by the extracted algorithm, only the claim for correctness.

# Chapter 8

# Conclusion

We started this thesis by introducing relevant theoretical concepts, including propositions as types and dependent types. We then used these concepts to explore the Coq proof assistant, and compared it to other related tools.

The case presented in Chapter 2 provided an opportunity in using Coq to apply these theoretical concepts in a more practical setting. Simplifications were made to fit the time constraints of the project. Since reasoning about sets was a sizable part of the theorem we formalized, a good implementation of sets was important. Various implementations were considered, and we made a choice that best fit our needs. We then presented the "building blocks" of our formalization, in the form of various data types, functions, predicates, lemmas and theorems.

The relevance of our project was demonstrated through the extraction of our Coq proofs into a certified Haskell program. We gave an example of how the program could be run to solve a simple case, but also discussed the limitations of the algorithm. We then gave a more sizable "real-world" example, which would determine if our extracted algorithm could function as a prototype.

We then assessed how through our formalization effort, we gained more confidence in the correctness of the original informal proofs posed in [2]. Furthermore, the extracted algorithm was evaluated for its usefulness as a prototype, and we discussed what useful value is provided with a formal proof and resulting extraction.

We then explained how our work could be extended in the future, namely by completing the proofs of remaining logical lemmas, providing a formal proof of Lemma 3.3, and proving the minimality of the model generated by Theorem 3.2.

In conclusion, we have in this thesis presented a detailed exploration into formalizing the proofs from two problems in lattice theory, and extracting a certified program from their verified formal proofs using Coq. We have investigated the utility of proof assistants and dependent type theory in providing a structured approach to proofs and formalization, which ultimately led to a practical application, in the form of a certified correct program usable as a prototype.

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# Appendix A

## Full proof of Theorem 3.2 in Coq

```
Theorem thm_32 :
  forall n m : nat,
  forall Cs : set Clause,
  forall V W : set string,
  forall f : Frontier,
    pre_thm n m Cs V W f.
Proof.
  unfold pre_thm. induction n as [|n IHn].
  - intros. apply le_0_r in H0.
    apply length_zero_iff_nil in H0.
    rewrite H0. apply ex_lfp_geq_empty.
  - induction m as [|m IHm]; intros.
    + apply (ex_lfp_geq_incl Cs (nodup string_dec V) (nodup string_dec W));
      try assumption. destruct H1. apply le_0_r in H1.
      apply length_zero_iff_nil in H1.
      apply set_diff_nil_incl in H1. assumption.
    + inversion H1.
      apply le_lt_eq_dec in H3. destruct H3.
      * apply (IHm Cs V W f); try assumption. lia.
      * apply (ex_lfp_geq_nodup_iff) in H2.
        assert (Datatypes.length (nodup string_dec W) ≤ n).
        {
           eapply (set_diff_succ string_dec) in H; try apply H3.
           apply succ_le_mono. eapply le_trans.
           apply H.
           - eapply le_trans in H0. apply H0. apply le_refl.
           - apply e.
         }
    assert (ex_lfp_geq Cs (nodup string_dec W) (nodup string_dec W) f).
    {
      apply (IHn n Cs W [] f);
      try assumption; try apply incl_nil_l.
      - eapply (set_diff_succ string_dec) in H; try apply H3.
        simpl. rewrite set_diff_nil. apply conj; lia. apply e.
      - unfold ex_lfp_geq. exists f. split.
        apply geq_refl. apply sub_model_W_empty.
    }
    assert (H': incl W V) by assumption.
    apply (nodup_incl2 string_dec) in H.
    apply (lem_33 Cs V (nodup string_dec W) f) in H;
    try assumption. elim H. intros h [H8 H9].
    destruct
      (sub_forward
        Cs
        (nodup string_dec V)
        (nodup string_dec V)
        h) as [U h'] eqn:Hforward.
    assert
      (sub_forward
        Cs
        (nodup string_dec V)
        (nodup string_dec V)
        h = (U, h')) by assumption.
    assert
      (sub_forward
        Cs
        (nodup string_dec V)
        (nodup string_dec V)
        h = (U, h')) by assumption.
    rewrite nodup_rm in H9.
    apply
      (sub_forward_incl_set_diff
        Cs
        h
        h'
        (nodup string_dec V)
        (nodup string_dec W)
        U) in H9;
    try apply Hforward.
    inversion Hforward. apply sub_forward_incl in Hforward.
    destruct U as [|u U'] eqn:Hu.
    -- apply sub_forward_empty in H7.
       destruct H7. unfold ex_lfp_geq. exists h.
       split; assumption.
    -- destruct
        (incl_dec
          string_dec
          V
          (nodup
            string_dec
            (set_union string_dec (nodup string_dec W) U))).
       ++ unfold ex_lfp_geq. exists (update_infty_V V f). split.
          ** apply geq_nodup_true. apply geq_update_infty_V.
          ** rewrite <- sub_model_nodup. apply sub_model_update_infty_V.
       ++ assert (incl (nodup string_dec (set_union string_dec W U)) V).
          {
            apply nodup_incl2. eapply incl_set_union_trans.
            assumption. apply nodup_incl in Hforward.
            rewrite Hu. assumption.
          }
          assert
            (Datatypes.length
              (nodup string_dec (set_union string_dec W U)) <
            Datatypes.length
              (nodup string_dec V)).
          {
            assert (Datatypes.length
              (nodup string_dec (set_union string_dec W U)) <=
                Datatypes.length
                  (nodup string_dec V)).
            {
              eapply NoDup_incl_length. apply NoDup_nodup.
              apply nodup_incl. assumption.
            }
            apply le_lt_eq_dec in H13. destruct H13; try assumption.
            assert
              (strict_subset
                (nodup string_dec (set_union string_dec W U))
                (nodup string_dec V)).
            {
              unfold strict_subset. split.
              – apply nodup_incl. assumption.
              – unfold not. intros. apply n0.
                apply nodup_incl2 in H13.
                assert
                  (incl
                    (nodup string_dec (set_union string_dec W U))
                    (nodup string_dec V)).
                {
                  apply nodup_incl. assumption.
                }
                rewrite (incl_set_union_nodup_l string_dec). assumption.
            }
            apply (strict_subset_lt_length string_dec).
            unfold strict_subset in H13.
            destruct H13. unfold strict_subset. split.
            – apply nodup_incl in H13.
              apply nodup_incl2 in H13. assumption.
            – unfold not. intros. apply n0.
              rewrite (incl_set_union_nodup_l string_dec).
              apply nodup_incl. assumption.
          }
          assert
            (Datatypes.length
              (set_diff string_dec V (set_union string_dec (nodup string_dec W) U)) <
            Datatypes.length
              (set_diff string_dec V (nodup string_dec W)) ≤ S m).
          {
            apply conj.
            – rewrite Hu. apply set_diff_incl_lt_length;
              try assumption. apply nodup_incl2. assumption.
              rewrite ← set_diff_nodup_l in H9. assumption.
            – rewrite ← set_diff_nodup_r.
              rewrite ← set_diff_nodup_eq. rewrite e. lia.
          }
          apply (ex_lfp_geq_monotone Cs (nodup string_dec V) h' f).
          eapply (IHm Cs V (nodup string_dec (set_union string_dec W U)) h');
          try assumption.
```

```
** apply conj; try lia. inversion H14.
   apply le_lt_eq_dec in H16. destruct H16;
   rewrite nodup_rm; rewrite set_diff_nodup_eq in *;
   rewrite ← length_set_diff_set_union_nodup_l;
   lia.
** apply (IHn n Cs (nodup string_dec (set_union string_dec W U)) [] h');
   try apply incl_nil_l.
   --- assert (Datatypes.length (nodup string_dec V) ≤ Datatypes.length V).
       {
         apply NoDup_incl_length. apply NoDup_nodup.
         apply nodup_incl2. apply incl_refl.
       }
       rewrite nodup_rm. rewrite set_diff_nodup_eq in *;
       try rewrite set_union_nodup_l in *; lia.
   --- apply conj; try lia.
       assert
         (Datatypes.length
           (set_diff
              string_dec
              (nodup string_dec (set_union string_dec W U))
              [])  ≤
         Datatypes.length
           (nodup string_dec (set_union string_dec W U))).
       apply (set_diff_nil_length string_dec).
       eapply le_trans.
       rewrite set_diff_nodup_eq. apply H15.
       apply lt_le_pred in H13.
       eapply le_trans. apply H13.
       assert
        (Datatypes.length (nodup string_dec V) ≤
         Datatypes.length V).
       {
          apply NoDup_incl_length. apply NoDup_nodup.
          apply nodup_incl2. apply incl_refl.
       }
       assert
        (pred (Datatypes.length (nodup string_dec V)) ≤
         pred (Datatypes.length V)).
       {
         apply Nat.pred_le_mono. assumption.
```

```
                        }
                        apply le_n_Sm_pred_n_m in H0. assumption.
                    - - - unfold ex_lfp_geq. exists h'.
                        split. apply geq_refl.
                        apply sub_model_W_empty.
                  ** eapply geq_trans with h; try assumption.
                    rewrite ← H12.
                    apply geq_Sinfty_f2.
          - - unfold pre_thm. intros. apply (IHn m0 Cs' V' W' f');
            try assumption; try lia.
          - - rewrite nodup_rm. assumption.
Qed.
```

Listing A.1: Full proof of Theorem 3.2 in Coq