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## **Abstract**

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No eros nemore impedit his, per at salutandi eloquentiam, ea semper euismod meliore sea. Mutat scaevola cotidieque cu mel. Eum an convenire tractatos, ei duo nulla molestie, quis hendrerit et vix. In aliquam intellegam philosophia sea. At quo bonorum adipisci. Eros labitur deleniti ius in, sonet congrue ius at, pro suas meis habeo no.

## Acknowledgements

Lorem ipsum

Andreas Salhus Bakseter  
Tuesday 21<sup>st</sup> February, 2023

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# Chapter 1

## Background

### 1.1 Formalizing Mathematical Problems

#### 1.1.1 Proofs

When solving mathematical problems, one often uses proofs to justify some claim. We can group proofs into two types; *informal* and *formal* proofs.

An informal proof is often written in a natural language, and the proof is adequate if most readers are convinced by the proof [3]. As a proof grows larger and more complex, it becomes harder to follow, which can ultimately lead to errors in the proof's reasoning. This might cause the whole proof to be incorrect [2].

A formal proof is written in a formal language, and can be compared to a computer program written in a programming language. Writing a formal proof is more difficult than writing an informal proof.

### 1.2 Type theory

Type theory groups mathematical objects with similar properties together by assigning them a "type". Similarly to data types in computer programming, we can use types to represent mathematical objects. For example, we can use the data type `nat` to represent natural numbers.

### 1.2.1 Propositions as types

The concept of propositions as types sees the proving of a mathematical proposition as the same process as constructing a value of that type. For example, to prove a proposition  $P$  which states "all integers are the sum of four squares", we must construct a value of the type  $P$  that shows that this is true for all integers. Proofs are mathematical objects; thus a proposition can be viewed as having the type of all its proofs (if any!). We can use this correspondence to model a proof as a typed computer program. The power of this concept comes from the fact that we can use a type checker to verify that our program is typed correctly, and thus that the corresponding proof is valid.

## 1.3 Proof assistants

Using a *proof assistant*, we can verify a formal proof mechanically.

### 1.3.1 Coq

*Coq* is an example of a proof assistant. *Coq* uses type theory to formulate and verify proofs, but can also be used as a functional programming language [4].

### 1.3.2 Agda

### 1.3.3 Isabelle

### 1.3.4 Lean

### 1.3.5 Higher-Order Logic

## 1.4 Extraction of programs from verified proofs

# Chapter 2

## Our case

We will use the Coq proof assistant to formalize parts of the proofs of the following paper, Bezem and Coquand [1]. This paper solves two problems that occur in dependent type systems where typings depend on universe-level constraints. Since this proof is complex enough that mistakes are possible it is a good candidate for formalization. We can also use this process to gain further insight into the algorithm that lies behind the proof. It might also be interesting to use Coq to extract programs from the final proofs.

# Chapter 3

## Approach & Design Choices

When translating an informal proof to a formal proof or specification, one often has to decide how to model certain mathematical objects and/or properties. For example, in Coq, there are several implementations of the mathematical notion of a *set*. When choosing which implementation to use, there are often tradeoffs to consider.

### 3.1 Modeling Sets in Coq

Sets in mathematics are seemingly simple structures. A set is a collection of elements, where the elements are of a similar type. The set cannot contain more than one of the same element (*no duplicates*), and the elements are not arranged in any specific order (*no order*). This is the most basic definition, ignoring more complex paradoxes and different set theories etc...

Sets are easy to work with when writing informal proofs. We do not care about how our elements or sets are represented, we only care about their properties. This does not hold for formal proofs though. In a formal proof, we need to specify exactly what happens when you take the union of two sets, or how you determine if a set contains an element.

One of the most important data structures in functional computer programming is the *list*. Unlike a set, a list *can* contain more than one of the same element, and the elements *are* arranged in a specific order. List also has a simple definition, but it is also a formal one. The definition from Coq’s standard library is as follows:

```

1 Inductive list {A : Type} : list A :=
2   | nil : list
3   | cons : A → list → list.

```

Listing 3.1: The definition of a list in Coq

Using the `cons` constructor, we can easily define any list containing any elements of the same type; we can even have lists of lists. The problem is of course that lists are not sets. We want to find a way to include the two important properties of *no duplicates* and *no order* into our definition of lists. In Coq, there are several ways to do this.

### 3.1.1 List & ListSet

As stated previously, Coq gives us a traditional definition of a list in the **List** module of the standard library. Due to the nature of its definition, it is very easy to construct proofs using induction on lists; we only need to check two cases. The list is polymorphic, meaning any type can be used to construct a list. We do not need to give Coq any more information about the properties of the type of the elements other than the type itself.

The **List** module also gives us a tool to combat the possibility of duplicates in a list, with `NoDup` and `nodup`. `NoDup` is an inductively defined proposition that gives evidence (?) on whether a list has duplicates or not. `nodup` is a function that takes in a list and returns a list without duplicates. These can be used effectively in proofs, since we still keep the underlying list type.

Having just the implementation of the set structure is rarely enough; we also want to do operations on the set, and reason about these. That is where the **ListSet** module comes in, which defines a new type called `set`. This type is just an alias for the `list` type from the **List** module, but the module also contains some useful functions. Most of these treat the input as a mathematical set, meaning that they try to preserve the properties of *no duplicates* and *no order*. Examples of some of these functions are `set_add`, `set_mem`, `set_diff`, and `set_union`. We also get useful lemmas that prove common properties about these functions. One thing to note is that all these functions use `bool` when reasoning

about if something is true or false. This makes them decidable, but it also requires the equality of the underlying type of the set to be decidable. A proof of the decidability of the underlying type must be supplied as an argument to all the functions. An example of such a proof for the `string` type would be:

```
1 Lemma string_eq_dec :
2   forall x y : string, {x = y} + {x <> y}.
3 Proof.
4 Admitted.
```

These proofs are often given for the standard types in Coq such as `nat`, `bool` and `string`. As such, they can just be passed to the functions as arguments. This convention of always passing the proof as an argument can be cumbersome and make the code hard to read, but it is a necessary evil to get the properties we want.

Many of these set functions, such as `set_union`, take in two sets as arguments and pattern match on the structure of one of them. For example, `set_union` pattern matches on the second set given as an argument. This makes proofs where we destruct or use induction on the second argument easy, such as this example:

```
1 Lemma set_union_nodup_l (dec : forall x y : A, {x = y} + {x <> y}) :
2   forall s1 s2,
3     set_union dec s1 s2 = set_union dec s1 (nodup dec s2).
4 Proof.
5 Admitted.
```

The downside is that even easy and seemingly trivial proofs that reason about the other argument are frustratingly hard, for example:

```
1 Lemma set_union_nodup_r (dec : forall x y : A, {x = y} + {x <> y}) :
2   forall s1 s2,
3     set_union dec s1 s2 = set_union dec (nodup dec s1) s2.
4 Proof.
5 Admitted.
```

With all this, we still have the problem with order of elements in the list. The **ListSet** module gives us no concrete way to combat this, but there are ways to circumvent the problem. Since we often reason about if an element is in a list, or if the list has a certain length, we do not care about the order of the elements. If we construct our proofs with

this in mind, **ListSet** is a viable implementation. There might however be cases where strict equality of two lists are needed, and that is where this implementation falls short.

Another thing to note is because of the polymorphic nature of the set type, any additional lemmas proven about a set can be used for any decidable type. This is useful if one needs sets of different types of elements.

### 3.1.2 MSetWeakList

The Coq standard library also gives us another implementation of sets, **MSetWeakList**. This implementation is a bit more complicated than the previous one, but gives us more guarantees about the properties of the set. The module is expressed as a functor, which in this case is a "function" that takes in a module as an argument, and again returns a module. The module we give to the functor must define some basic properties about the type we want to create a set of, namely equality, decidability of equality and the equivalence relation of (on?) equality. The output from the functor is a module containing functions and lemmas about set operations, with our input type being the type of the elements of the set.

This means that for every type we want to use as an element in the set, we have to go through this process. In **List** and **ListSet**, we just had to pass in the proof of the equality of the type as an argument to the set functions and lemmas. The structure of the sets in **MSetWeakList** is also a lot more complicated than the simple and intuitive definition of **List**. This makes it harder to reason about the sets in proofs.

### 3.1.3 Ensembles

Another implementation of sets is given by the **Ensembles** module, which defines the structure of a set as inductive propositions. This means it uses **Prop** instead of **bool**, making **Ensembles** useful for proofs where we do not care about decidability. The biggest downside to this implementation, is that we cannot reason about the size of the set. We can only determine if an element is in the set, not how big the set is. In our case, this makes the **Ensembles** module useless, since the theorem we are formalizing requires us to reason about the size of the set.

### 3.1.4 finset

If we venture past Coq's standard library and into the *Mathematical Components* library, we find the **finset** module.

## 3.2 Prop vs. Bool



# Chapter 4

## Implementation

### 4.1 Choice of set implementation

The simplest set (or set-like) implementation in Coq are the **List** and **ListSet** modules. These require minimal knowledge of advanced Coq syntax and behave like lists, making proofs by induction easy.

### 4.2 Atom & Clause

# Chapter 5

## Examples & Results

# Chapter 6

## Evaluation

## Chapter 7

## Conclusion

# Bibliography

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## Appendix A

### Coq examples