A case study in dependent type theory: extracting a certified program from the formal proof of its specification

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Abstract

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Andreas Salhus Bakseter

Saturday 13th May, 2023

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Chapter 1

Background

1.1 Formalizing mathematics

1.1.1 Proofs

When solving mathematical problems, we often use proofs to either **justify** a claim or to **explain** why the claim is true. We can distinguish between two types of proofs; *informal* and *formal* proofs.

An informal proof is often written in a natural language, and the proof is adequate if most readers are convinced by the proof [10]. Such proofs rely heavily on the reader's intuition and often omit logical steps to make them easier to understand for humans [6]. As these proofs grow larger and more complex, they become harder for humans to follow, which can ultimately lead to errors in the proofs' logic. This might cause the whole proof to be incorrect [7], and even the claim justified by it might be wrong.

A formal proof is written in a formal language, and can be compared to a computer program written in a programming language. Writing a formal proof is more difficult than writing an informal proof. Formal proofs include every logical step, and nothing is left for the reader to assume. This can make them extremely verbose, but the amount of logical errors is reduced [6]. The only possible errors in formalized proofs are false assumptions and/or flawed verification software.

1.2 Type theory

Type theory groups mathematical objects with similar properties together by assigning them a "type". Similarily to data types in computer programming, we can use types to represent mathematical objects. For example, we can use the data type `nat` to represent natural numbers, or we can create our own data types which allows us to represent e.g., clauses in logic.

1.2.1 Propositions as types

The concept of propositions as types sees proving a mathematical proposition as the same process as constructing a value of a type, in this case, of the proposition as a type. For example, to prove a proposition P which states "all natural numbers are the sum of four squares", we must construct a value of the type P that shows that this is true for all natural numbers. Such a value is a function that for any input n returns a proof that n is the sum of four squares, that is, return four numbers a, b, c, d and a proof that $n = a^2 + b^2 + c^2 + d^2$. Proofs are mathematical objects; thus a proposition can be viewed as having the type of all its proofs (if any!). We can use this correspondance to model a proof as a typed computer program. The power of this concept comes from the fact that we can use a type checker to verify that our program is typed correctly, and thus that the corresponding proof is valid. Often, the proof can be used to compute something, i.e., the numbers a, b, c, d mentioned above [20].

1.2.2 Dependent types

Dependent types allow us to define more rigorously types that depend on values. This is important for propositions as types. For example, the proposition P in Section 1.2.1 uses $n = a^2 + b^2 + c^2 + d^2$, which is a type depending on five values n, a, b, c, d. Since Coq has support for dependent types, we can give an example to illustrate what this means. We consider the type `vector`, which is a type polymorphic (the elements of the list can be of any same type) finite list with a fixed length, which is the value on which the type depends, a natural number.

```
Inductive vector (A : Type) : nat → Type :=
  | Vnil : vector A 0
  | Vcons : forall (h : A) (n : nat), vector A n → vector A (S n).
```

Listing 1.1: `vector` in Coq, using dependent types

This definition gives us a type with two constructors:

- `Vnil` has the type of `vector A` 0, and represents the empty vector.
- `Vcons` has type of `forall (h : A) (n : nat) vector A (S n)`, where `h` is the head of the vector and `n` is the length of the vector, where both are given to the constructor as arguments. This makes the type of a `vector` *depend* on its length.

In this scenario, the length of a `vector` is fixed by the argument `n : nat` and the term `vector A n → vector A (S n)`. Any definition of a `vector` must adhere to this term, and is checked at compile time. An example of a valid and invalid definition is:

```
(* valid definition; (S 0) equal to 1 *)
Definition vec_valid : vector string 2 :=
    Vcons string "b" 1 (Vcons string "a" 0 (Vnil string)).
(* invalid definition; (S 0) not equal to 2 *)
Fail Definition vec_invalid : vector string 2 :=
    Vcons string "b" 2 (Vcons string "a" 0 (Vnil string)).
```

Listing 1.2: Examples of vectors in Coq

1.3 Proof assistants

Propositions as types allow us to bridge the gap between logic and computing, while dependent types allow us to define more rigorously types which depend on values, thus enhancing the expressivity of simple types. The former is a crucial aspect of *proof assistants*, while the latter gives us more expressive power when constructing proofs using a proof assistant. An example of the expressive power of dependent types is the fact that we can define predicates that depend on the value of a term, e.g. a predicate that checks if a number is even. The purpose of a proof assistant is to get computer support to construct and verify a formal proof mechanically.

1.3.1   Coq

Coq is based on the higher-order type theory *Calculus of Inductive Constructions* (CIC), and functions as both a proof assistant and a dependently typed functional programming language. Coq also allow us to extract certified programs from the proofs of their specification to the programming langauges OCaml and Haskell [14]. Coq implements a specification language called *Gallina*, which allows us to define logical objects within Coq. These objects are typed to ensure their correctness (is quote too direct?), and the typing rules used are from CIC [12].

This is an example of the syntax of Gallina:

```
Inductive nat : Type :=
  | O
  | S : nat → nat.


Definition lt_n_S_n :=
  (fun n : nat ⇒ le_n (S n)) : forall n : nat, n < S n.
```

Listing 1.3: Example of Gallina syntax

Looking at the final definition in the example, we can see the concept of propositions as types in action. `lt_n_S_n` defines a function which takes a natural number `n` as input, and returns a value of the type `forall n : nat, n < S n`, denoted by the colon before the type itself. The return value is therefore a proof of `forall n : nat, n < S n`, and since the definition has been type-checked by Coq, we know that this proof is valid! In this case, the function is `fun : nat ⇒ le_s (S n)`, where `le_n` is a constructor of the type `forall n : nat, n ≤ n`. By applying this constructor to `S n`, we get a value of the type (and a proof) of `forall n : nat, S n ≤ S n`. By Coq's definition of <, our initial theorem can be rewritten as `forall n : nat, S n ≤ S n`. This matches the type of our function, and the proof is complete.

Proving theorems like this is not really intuitive for a human prover, and that is why Coq gives us the *Ltac* meta-language for writing proofs. Ltac provides us with tactics, which are a kind of shorthand syntax for defining Gallina terms [4]. Using Ltac, we can rewrite the proof from 1.3 as such:

```
Theorem lt_n_S_n : forall n : nat, n < S n.
Proof.
  intro n. destruct n.
  – apply le_n.
  – apply le_n_S. apply le_n.
Qed.
```

Listing 1.4: Example of Ltac syntax

When developing proofs using Ltac, each tactic is executed or "played" one by one, much like an interpreter. The tactics are seperated by punctuation marks. When the use of a tactic causes the proof to depend on the solving of multiple sub-proofs (called "goals"), we can use symbols like "-", "+", and "*" to branch into these sub-proofs and solve their goals independently. Once a goal has been solved, we can move on the next. When there are no more goals, the proof is complete. Coq provides us with tooling that gives us the ability to see our goals and the proof state to further simplify the process [13]. Ltac is not the only proof langauge, with another example being *SSReflect* [5].

1.3.2 Other proof assistants

Agda

Agda is a depdently typed functional programming language based on Martin-Löf's intuitionistic type theory. Unlike Coq, Agda does not use tactics. [2] However, by using proposition as types, Agda can also function as a proof assistant.

Isabelle

Lean

Lean is proof assistant, automated theorem prover and dependently typed functional programming language. Lean can be instantiated using either CIC or an Martin-Löf's intuitionistic type theory [3].

1.3.3 Extraction of programs from verified proofs

By the the notion of propositions as types, we can use a proof assistant to prove the correctness of a program. However, we can also extract a program from a proof of its correctness. This type of code extraction is a common feature of proof assistants. The extracted program is guaranteed to be correct by the type system of the proof assistant, and the resulting code can be extracted to a variety programming languages, such as Haskell and OCaml (as is the case for Coq) [14].

Chapter 2

The Case in Question

2.1 Overview

Recall the questions posed in the abstract. We want to use the Coq proof assistant to formalize parts of the proofs of the following paper, by Bezem and Coquand [1]. This paper solves two problems that occur in dependent type systems where typings depend on universe-level constraints:

1. an equational theory of semilattices with one inflationary endomorphism has a uniform word problem that can be solved in polynomial time.

2. loop-checking in a type system with universe-level constraints can be done in polynomial time.

first problem is maybe summarized to generally, while second is more specific for the case of type systems? ask Marc.

We focused on formalizing the proof of Theorem 3.2 from [1]. Since this proof is complex enough that mistakes are possible, it is a good candidate for formalization. By formalizing the proof and then verifying it using Coq, we can be sure that it is correct.

We are also interested in finding out whether or not the process of formalizing the proof can be completed in a reasonable amount of time, and with a reasonable amount of effort.

Is the promise (find better word?) of a complete verification of the proof worth the effort of formalizing it?

By using an advanced feature of Coq, we can also extract an algorithm from the proof, where this algorithm is certified to produce correct results. As an aside, this algorithm has direct applications to the formalization and verification of the Coq proof assistant itself, as Coq employs a similar algorithm to resolve universe-level constraints in its type system.

2.2   Relevant parts of the paper

We will paraphrase some key concepts from [1] that are needed to understand our implementation and formalization.

We will call join-semilattices with inflationary endomorphisms simply semilattices. A join-semilattice is a partially ordered set in which any two elements have a least upper bound, called their join., denoted by $\vee$. The join of a semilattice is a binary operation $\vee$ that is commutative, associative and idempotent. An inflationary endomorphism, in our case, is a function that maps an element to itself or to a greater element in the ordered set. We denote the inflationary endomorphism as $^+$, for every lattice element x, x^+ is also called the successor of x. The fact that x is an endomorphism means that we get the folloring properties $(x \vee y)^+ = x^+ \vee y^+$ and $x \vee x^+ = x^+$.

A semilattice presentation consists of a set V of generators (also called variables) and a set C of constraints (also called relations). For a semilattice t and $k \in \mathbb{N}$, $t + k$ denotes the k-fold successor of t, thus making $t + 0 = t$ and $t + 1 = t^+$. A term over V has the form $x_1 + k_1 \vee \ldots \vee x_m + k_m$, where $x_i \in V$ and $k_i \in \mathbb{N}$. Every semilattice expression is equal to a term over V.

Horn clauses are propositional clauses $A \to b$, with a non-empty body A and conclusion b. The atoms are of the form $x + k$, where $x \in V, k \in \mathbb{N}$. We call this special form of Horn clauses simply *clauses*.

A relation is an equation $s = t$, where s and t are terms over V. A constraint (relation), like $a \vee b = c^+$ (with $a, b, c \in V$), expresses a relation between the generators (variables) a, b, c. For each constraint $s = t$, we generate clauses by replacing join by conjunction, denoted by ",", and $\geq$ by implication. E.g. for the constraint $a \vee b = c^+$, we first derive

$a \vee b \geq c^+$ and $c^+ \geq a \vee b$. From $a \vee b \geq c^+$, we generate the clause $a, b \rightarrow c^+$. From $c^+ \geq a \vee b$, we derive $c^+ \geq a$ and $c^+ \geq b$. We then generate the clause $c^+ \rightarrow a$ from the former, and the clause $c^+ \rightarrow b$ from the latter. This generated set of clauses is denoted by $S_{s=t}$, which in our example makes it $S_{a \vee b = c^+} := \{a, b \rightarrow c^+, c^+ \rightarrow a, c^+ \rightarrow b\}$. We define S_C as the union of all $S_{s=t}$, where $s = t$ is a constraint in C.

In [1, Theorem 2.2, p. 3] it is proven that provability in the semilattice theory is equivalent to Horn clause provability, making it so that we can continue with the latter in the rest of this thesis.

We define closure under shifting upwards as follows: if $A \rightarrow b$ is in the set of clauses, then so must $A + 1 \rightarrow b + 1$ be. $A + 1$ denotes the set of atoms of the form $a + 1$, where $a \in A$. Given a finite semilattice presentation (V, C), and a subset $W \subseteq V$, we denote by $\overline{S_C}$ the smallest set of clauses that is closed under shifting upwards, by $\overline{S_C} \mid W$ the set of clauses in $\overline{S_C}$ mentioning only variables in W, and by $\overline{S_C} \downarrow W$ the set of clauses in $\overline{S_C}$ with conclusion over W.

A function $f : V \rightarrow \mathbb{N}^\infty$ specifies a downward closed set of atoms, namely $\{v + k \mid v \in V, k \in \mathbb{N}, k \leq f(v)\}$, where $\mathbb{N}^\infty$ is the set of natural numbers extended with ∞. This set contains all atoms $v + k$ if $f(v) = \infty$. The sets specified by such functions are models of $\overline{S_C}$.

2.2.1 Lemmas and theorems

Lemma 3.1 ([1, p. 3]) *Given $f : V \rightarrow N^\infty$, and a clause $A \rightarrow b$, let P be the problem whether or not $A + k \rightarrow b + k$ is satisfied by f for all $k \in N$. Then P is decidable.*

The proof of Lemma 3.1 demonstrates that the problem P is decidable, meaning we can indeed write an algorithm that determines whether or not the problem holds for all $k \in N$. Lemma 3.1 is also crucial for making case distinctions in further proofs, since we know that any S_C is finite.

Theorem 3.2 ([1, p. 3]) *For any finite semilattice representation (V, C) and any function $f : V \rightarrow N^\infty$, the least $g \geq f$ that is a model of $\overline{S_C}$ can be computed.*

Theorem 3.2 has a special case that is solved by an additional lemma:

Lemma 3.3 ([1, p. 3-4]) *Given a finite semilattive presentation (V, C) and a strict subset $W \subset V$, if for any function $f : W \to N^\infty$, the least $g \geq f$ that is a model of $\overline{S_C}|W$ can be computed, then for any function $f : V \to N^\infty$ with $f(V - W) \subseteq N$, the least $h \geq f$ that is a model of $\overline{S_C} \downarrow W$ can be computed.*

Chapter 3

Approach & Design Choices

When translating an informal proof or specification to a formal proof. one often has to decide how to model certain mathematical objects and their properties. For example, in Coq, there are several implementations of the mathematical notion of a *set*. When choosing which implementation to use, there are often tradeoffs to consider. Examples of such trade-offs are simplicity of the implementation, ease of use, and performance.

3.1   Simplifications for the sake of time

3.1.1   Proof of minimality

In our proof of Theorem 3.2, we have chosen to omit proving the minimality of the model generated by the algorithm. Our algorithm does however generate a minimal model, but we have not proven that it does.

To prove that the model generated by our algorithm is minimal, we would have had to include the following proposition in our definition of Theorem 3.2:

```
forall h : Frontier, sub_model Cs V V h → geq h f.
```

Listing 3.1: Proposition for minimal model

3.1.2   Omission of formal proof of Lemma 3.3

We have included a formulation of this lemma, but not a proof. When testing the algorithm generated by our formalization of Theorem 3.2, we have manually edited the code to use the identity function instead of crashing due to the lack of a proof of Lemma 3.3. This simplification is sufficient for a surprising large number of problems; the limitations of this simplification will be explained in more detail in Section 5.3.

3.1.3   Incomplete proofs for some purely logical lemmas

We have chosen to not waste too much time fully completing the proofs of some purely logical lemmas, which are mainly used as intermediate steps in the proof of Theorem 3.2. As we will se later in this section, this is mainly due to very complex or impossible-to-prove lemmas in our set implementation being trivial in informal mathematics. The proofs of these lemmas are however not very interesting, and they do not contribute to the correctnes of the implementation nor the results of extracting code from the Coq formalization.

3.2   Modeling finite sets in Coq

Sets in mathematics are seeminlgy simple structures; a set is a collection of elements. The set cannot contain more than one of the same element (*no duplicates*), and the elements are not arranged in any specific order (*no order*). This is the naive definition of a set, not taking into account the complexity of this subtle notion, different set theories, powerful axioms, and so on.

Sets are easy to work with when writing informal proofs. We do not care about how our elements or sets are represented, we only care about their properties. This does not hold for formal proofs though. In a formal proof, we need to specify exactly what happens when you take the union of two sets, or how you determine whether or nor a set contains an element.

One of the most important data structures in functional computer programming is the *list*. Unlike a set, a list *can* contain more than one of the same element, and the elements

are arranged in a specific order. The inductive definition of a list from Coq's standard library is as follows [15]:

```
Inductive list (A : Type) : list A :=
    | nil : list A
    | cons : A → list A → list A.
```

Listing 3.2: Inductive definition of list type in Coq

Using the `cons` constructor, we can easily define any list containing any elements of the same type; we can even have lists of lists. The problem is of course that lists are not sets. We want to find a way to take into account the two important properties of *no duplicates* and *no order* into our definition of lists. In Coq, there are several ways to do this.

3.2.1 List & ListSet

As stated previously, Coq gives us a traditional definition of a list in the **List** module [1] of the standard library [17]. Due to the nature of its definition, it is very easy to construct proofs using induction or case distinction on lists; we only need to check two cases. This list implementation is type polymorphic, meaning any type can be used to construct a list of that type. We do not need to give Coq any more information about the properties of the underlying type of the list other than the type itself.

The **List** module also gives us a tool to combat the possibility of duplicates in a list, with `NoDup` and `nodup`. `NoDup` is an inductively defined proposition that asserts whether a list has duplicates or not. `nodup` is a function that takes in a list and returns a list with the same elements, but without duplicates. In other words, a list for which `NoDup` holds. These two can be used to better represent finite sets as lists, since we gain additional information about whether the list has duplicates or not. Coq does not however inherently understand how to compare elements when checking a list for duplicates in `nodup`. Hence we have to provide a proof that the equality of the underlying type of the list is decidable. An example of such a proof for the `string` type would be:

---

[1]The definition of `list` is actually located in the module **Init.Datatypes**, while everything else relating to lists is in the **List** module.

```
Lemma string_eq_dec :
    forall x y : string, {x = y} + {x <> y}.
Proof.
    (* proof goes here *)
Qed.
```

Listing 3.3: Decidability proof for string equality in Coq

Proofs as in Listing 3.3 are often given for the standard types in Coq such as `nat`, `bool` and `string`. As such, they can just be passed as arguments. This convention of always passing the proof as an argument can be cumbersome and make the code hard to read, but it is a necessary evil to get the properties we want.

Having just the implementation of the set structure is rarely enough; we also want to do operations on the set, and reason about these. That is where the **ListSet** module comes in, which defines a new type called `set` [18]. This type is just an alias for the `list` type from the **List** module, but the module also contains some useful functions. Most of these functions treat the input as a set in the traditional sense, meaning that they try to preserve the properties of *no duplicates* and *no order*. Examples of some of these functions are `set_add`, `set_mem`, `set_diff`, and `set_union`. We also get useful lemmas that prove common properties about these functions. As with `nodup`, these functions all require a proof of decidability of equality for the underlying type of the set. One thing to note is that all these functions use `bool` instead of `Prop`, and all require a decidability proof, which make the functions themselves decidable.

The module also gives us some lemmas to transform the boolean (type `bool`) set-operation functions into propositions (type `Prop`), and vice versa. An example to illustrate this is the following lemma on `set_mem`:

```
Lemma set_mem_correct1 {A : Type} (dec : forall x y : A, {x = y} + {x <> y}) :
    forall (x : A) (l : set A), set_mem dec x l = true → set_In x l.
Proof.
    (* ... *)
Qed.
```

Listing 3.4: `set_mem` lemma in `ListSet` module

`set_In` is just an alias for `In` from the **List** module, which is a proposition that is very common in many lemmas from the standard library. Lemmas such as the example above

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are very useful when reasoning about boolean functions such as `set_mem` in proofs, as transforming them into propositions makes them easier to work with and often enables us to use existing lemmas from the standard library.

Many of these boolean set functions, such as `set_union`, take in two sets as arguments and pattern match on the structure of one of them. For example, `set_union` pattern matches on the second set given as an argument. This makes proofs where we destruct or use induction on the second argument easy, such as this example:

```
Lemma set_union_l_nil {A : Type} (dec : forall x y : A, {x = y} + {x <> y}) :
    forall l : set A, set_union dec l [] = l.
Proof.
    destruct l; reflexivity.
Qed.
```

Listing 3.5: Easy proof of lemma in `ListSet` module

The downside is that even easy and seamingly trivial proofs that reason about the other argument are frustratingly hard (or impossible) to prove, for example:

```
Lemma set_union_nil_l {A : Type} (dec : forall x y : A, {x = y} + {x <> y}) :
    forall l : set A, set_union dec [] l = l.
Proof.
    (* ... *)
Qed.
```

Listing 3.6: Impossible proof of lemma in `ListSet` module

What makes this proof impossible is that the order of elements in `set_union dec [] l` is not the same as in `l` (due to how `set_union` is implemented), and since equality on lists care about order, we cannot prove this lemma. There are ways to circumvent this problem. Since we often reason about if an element is in a list, or if the list has a certain length, we do not care about the order of the elements. If we construct our proofs with this in mind, **ListSet** is a viable implementation. There might however be cases where the order of the elements in the lists come into play (e.g. such as in Listing 3.6), and that is where this implementation falls short. Another thing to note is because of the polymorphic nature of the `set` type, any additional lemmas proven about a set can be used for any decidable type. This is useful if one needs sets with elements of different types, which is the case in our implementation; we need sets of atoms, clauses, strings, etc.

3.2.2 MSetWeakList

The Coq standard library also gives us another implementation of sets, **MSetWeakList**
[19]. This implementation is a bit more complicated than the previous one, but gives us
more guarantees about the properties of the set. The module is expressed as a functor,
which in this case is a "function" that takes in a module as an argument, and again
returns a module. The module we give to the functor must define some basic properties
about the type we want to create a set of, namely an equality relation, decidability of this
relation and the fact that this relation is an equivalence relation. The "output" from the
functor is a new module containing functions and lemmas about set operations, with our
input type being the type of the elements of the set. An example definition for a module
of a set of atoms would be as follows:

```
Module AtomEq : DecidableType.


  (* what is our base type? *)
  Definition t := Atom.


  Definition eq := (* ... *)


  Lemma eq_equiv :
    Equivalence eq.
  Proof.
    (* proof that eq is an equivalence relation *)
  Qed.


  Lemma eq_dec :
    forall x y : t, {eq x y} + { eq x y}.
  Proof.
    (* proof that eq is decidable *)
  Qed.


End AtomEq.


(* apply functor Make to module AtomEq *)
Module AtomSet := Make AtomEq.
```

Listing 3.7: Set of atoms in `MSetWeakList` module

The proofs of both lemmas above are trivial, but require a substantial amount of boiler-plate code. For every type we want to use as an element in a set, we have to go through this entire process. In **List** and **ListSet**, we just had to pass in the proof of the decidability of equality of the type as an argument to the set functions and lemmas. The structure of the sets in **MSetWeakList** is also a lot more complicated than the simple and intuitive definition of **List**. This makes it harder to reason about the sets in proofs, given limited knowledge of Coq.

3.2.3 Ensembles

Yet another implementation of sets is given by the **Ensembles** module which defines the structure of a set as inductive propositions [16]. These inductive propositions assert some fact about if an element is in the set or not. An example that illustrates this is the definition of `Union`:

```
Inductive Union (B C:Ensemble) : Ensemble :=
  | Union_introl : forall x:U, In B x → In (Union B C) x
  | Union_intror : forall x:U, In C x → In (Union B C) x.
```

Listing 3.8: `Union` in **Ensembles** module

We see that the first constructor of `Union` asserts that if an element is in `B`, then it is in the union of `B` and `C`. **Ensembles** uses `Prop` instead of `bool`, making **Ensembles** useful for proofs where we do not care about decidability. The biggest downside to this implementation is that we cannot reason about the size of the set. We can only determine if an element is in the set, not how big the set is. In our case, this makes the **Ensembles** module useless, since the theorem we are formalizing requires us to reason about the size of the set.

3.3 Choice of implementation of sets

The simplest set (or set-like) implementation in Coq are the **List** and **ListSet** modules. These require minimal knowledge of advanced Coq syntax and behave like lists, making proofs by induction easy. They are also polymorphic, meaning ease of use when making sets of different or self-defined types. Because of these reasons, we chose to go with **List** and **ListSet**.

Chapter 4

Implementation of Logical Notions

4.1 Data types

As discussed in Chapter 2, we want to represent clauses as a set of atoms as premises and a single atom as a conclusion. We implement this in Coq using two types, `Atom` and `Clause`.

```
Inductive Atom : Type :=
  | atom : string → nat → Atom.


Notation "x & k" := (atom x k) (at level 80).


Inductive Clause : Type :=
  | clause : set Atom → Atom → Clause.


Notation "ps ⤳ c" := (clause ps c) (at level 81).
```

Listing 4.1: `Atom` and `Clause` in Coq

Note also the `Notation`-syntax, which allow us to define a custom notation, making the code easier to read. The expression on the left-hand side of the := in quotation marks is definitionally equal to the expression on the right-hand side in parentheses. The level determines which notation should take precedence, with a higher level equaling a higher precedence.

We also want to represent functions of the form $f : V \to \mathbb{N}^\infty$, which as we saw in Section 2.2 are functions that are models of a set of clauses. We implement this in Coq using two types, `Ninfty` and `Frontier`. `Ninfty` has two constructors; `infty` represents ∞, and `fin n` represents a natural number n. `Frontier` is a function from a string (variable) to `Ninfty`.

```
Inductive Ninfty : Type :=
  | infty : Ninfty
  | fin   : nat → Ninfty.


Definition Frontier := string → Ninfty.
```

Listing 4.2: `Ninfty` and `Frontier` in Coq

Using these definitions of `Atom`, `Clause` and `Frontier`, we can define functions that check whether any given atom or clause is satisfied for any frontier.

```
Definition atom_true (a : Atom) (f : Frontier) : bool :=
  match a with
  | (x & k) ⇒
    match f x with
    | infty ⇒ true
    (* se explantation for ≤? below *)
    | fin n ⇒ k ≤? n
    end
  end.


Definition clause_true (c : Clause) (f : Frontier) : bool :=
  match c with
  | (conds ↝ conc) ⇒
    (* every atom in conds is true *)
    if fold_right andb true (map (fun a ⇒ atom_true a f) conds)
    then (atom_true conc f)
    else true
  end.
```

Listing 4.3: `atom_true` and `clause_true` in Coq

The infix function $\leq?$ is the boolean (and hence decidable) version of the Coq function $\leq$, which uses `Prop` and is not inherently decidable without additional lemmas.

We can also define functions that "shift" the number value of atoms or whole clauses by some amount `n : nat`.

```
Definition shift_atom (n : nat) (a : Atom) : Atom :=
  match a with
  | (x & k) ⇒ (x & (n + k))
end.


Definition shift_clause (n : nat) (c : Clause) : Clause :=
  match c with
  | conds ⇝ conc ⇒
    (map (shift_atom n) conds) ⇝ (shift_atom n conc)
  end.
```

Listing 4.4: `shift_atom` and `shift_clause` in Coq

Using these definitions, we can now define an important property that is possible by Lemma 3.1 [1, p. 3], since this lemma enables us to check whether or not a clause is satisfied by a frontier for any shift of `k : nat`. We will use this property later to determine if a frontier is a valid model for a set of clauses.

```
Definition all_shifts_true (c : Clause) (f : Frontier) : bool :=
  match c with
  | (conds ⇝ conc) ⇒
    match conc with
    | (x & k) ⇒
      match f x with
      | infty ⇒ true
      | fin n ⇒ clause_true (shift_clause (n + 1 − k) c) f
      end
    end
  end.
```

Listing 4.5: `all_shifts_true` in Coq

4.2 Semantic functions and predicates

4.2.1 The function `sub_model`

Given any set of clauses and a frontier (function assigning values to the variables), we can determine if the frontier is a model of the set of clauses, i.e. whether all shifts of all clauses are satisfied by the frontier.

We translate this property to Coq as the recursive function `sub_model`. We have two additional arguments `V` and `W`; these are the set of variables (strings) from the set of clauses, and all insert explanation here, respectively. The function `vars_set_atom` simply returns all the variables used in a set of atoms as a set of strings.

```
Fixpoint sub_model (Cs : set Clause) (V W : set string) (f : Frontier) : bool :=
  match Cs with
  | []          ⇒ true
  | (l ↝ (x & k)) :: t  ⇒
    (* conclusion not in W *)
    (negb (set_mem string_dec x W) ||
    (* some premise not in V *)
     negb (
       fold_right andb true
         (map (fun x ⇒ set_mem string_dec x V) (vars_set_atom l))
     ) ||
    all_shifts_true (l ↝ (x & k)) f
    ) && sub_model t V W f
  end.
```

Listing 4.6: The function `sub_model` in Coq

4.2.2 The function `geq`

We want to determine whether all the values assigned to a set of variables from one frontier are greater than or equal to all the values assigned to a set of variables from another frontier. The values are of the type `Ninfty`, and the function only returns true if **all** the values from the first frontier are greater than the values from the second frontier.

```
Fixpoint geq (V : set string) (g f : Frontier) : bool :=
  match V with
  | []        ⇒ true
  | h :: t  ⇒
    match g h with
    | infty ⇒ geq t g f
    | fin n ⇒
        match f h with
        | infty ⇒ false
        | fin k ⇒ (k ≤ ? n) && geq t g f
        end
    end
  end.
```

Listing 4.7: Pointwise comparing frontiers with `geq` in Coq

4.2.3 The predicate `ex_lfp_geq`

We can now combine `sub_model` and `geq` to construct a predicate stating that there exists
a frontier `g` that is a model of the set of clauses `Cs` and is greater than or equal to another
frontier `f`.

```
Definition ex_lfp_geq_P (Cs : set Clause) (V W : set string) (f : Frontier) : Prop :=
  exists g : Frontier, geq V g f = true ∧ sub_model Cs V W g = true.
```

```
Definition ex_lfp_geq_S (Cs : set Clause) (V W : set string) (f : Frontier) : Set :=
  sig (fun g : Frontier ⇒ prod (geq V g f = true) (sub_model Cs V W g = true)).
```

Listing 4.8: `ex_lfp_geq` in Coq, using both `Prop` and `Set`

One thing to note here is that we can define this predicate either as having the type
`Prop` or as having the type `Set`. When defined with `Prop`, we define it as a logical
predicate, using standard first-order logic syntax. When defined with `Set`, we define it as
a type, using the `sig` type constructor in place of `exists`. `sig` is an implementation of a
dependent product type. Dependent product types are also known as Σ-types, and are
often used to represent existential quantification. We also use the `prod` type constructor
to represent the conjunction of two propositions. Akin to `sig`, `prod` is an implementation

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of a dependent sum type. Dependent sum types are also known as Π-types, and are often used to represent universal quantification. find source for this, maybe `https://hottheory.files.wordpress.com/2013/03/hott-online-323-g28e4374.pdf`. would have been nice to explain this earlier or tie it in better with section about depdendent types in chap 1?

The reason for defining `ex_lfp_geq` with `Set`, is that we can then use Coq's extraction feature to generate Haskell code from the Coq definitions. As briefly mentioned in Section 1.2.1, a proof can often be used to compute something. In this case, we want to compute the actual frontier `g` that satisfies the above predicate. Coq distinguishes between logical objects (objects in `Prop`) and infomative objects (objects in `Set`) [9, p. 1-2]. When extracting, Coq will remove as many logical objects as possible, meaning a proposition defined in `Prop` would simply be collapsed to "()" (the unit type) in Haskell [8, p. 8]. Logical objects are only used to ensure correctness when constructing a proof in Coq, and are not needed when actually computing something using the extracted code.

4.3   The main proofs

We have now laid the groundwork for the formalization of Theorem 3.2. We precede the definition of Theorem 3.2 with two additional definitions, which helps us simplify its definition and the proof of the theorem itself.

4.3.1   The predicate `pre_thm`

Since (in our case) the formal definitions of Lemma 3.3, which will be expanded on shortly, and Theorem 3.2 share some structure, we define a proposition `pre_thm`:

```
Definition pre_thm (n m : nat) (Cs : set Clause) (V W : set string) (f : Frontier) :=
  incl W V →
  Datatypes.length (nodup string_dec V) ≤ n →
  Datatypes.length
    (set_diff string_dec
      (nodup string_dec V)
      (nodup string_dec W)
    ) ≤ m ≤ n →
  ex_lfp_geq Cs (nodup string_dec W) (nodup string_dec W) f →
  ex_lfp_geq Cs (nodup string_dec V) (nodup string_dec V) f.
```

Listing 4.9: `pre_thm` in Coq

4.3.2 Lemma 3.3

Lemma 3.3 from [1] is used in the proof of Theorem 3.2 to solve fill inn explanation here...

We define it using `pre_thm` as follows:

```
Lemma lem_33 :
  forall Cs : set Clause,
  forall V W : set string,
  forall f : Frontier,
    (forall Cs' : set Clause,
     forall V' W' : set string,
     forall f' : Frontier,
     forall m : nat,
       pre_thm (Datatypes.length (nodup string_dec V) − 1) m Cs' V' W' f'
    ) →
    incl W V →
    ex_lfp_geq Cs (nodup string_dec W) (nodup string_dec W) f →
    ex_lfp_geq Cs (nodup string_dec V) (nodup string_dec W) f.
Proof.
  (* ... *)
Qed.
```

Listing 4.10: Lemma 3.3 in Coq

4.3.3 Theorem 3.2

We can now formulate Theorem 3.2 using `pre_thm`:

```
Theorem thm_32 :
  forall n m : nat,
  forall Cs : set Clause,
  forall V W : set string,
  forall f : Frontier,
    pre_thm n m Cs V W f.
Proof.
  (* ... *)
Qed.
```

Listing 4.11: Theorem 3.2 in Coq

The proof of Theorem 3.2 is based on a primary induction on `n` and a secondary induction on `m`.

Base case of `n`

The first base case is simple. We want to prove

(1) `ex_lfp_geq Cs (nodup string_dec V) (nodup string_dec V) f`.

Since `n` = 0, we get that the length of `V` is 0, and hence we get a new goal `ex_lfp_geq Cs [] [] f`.
This is proven by the lemma `ex_lfp_geq_empty`, which states that `forall Cs f, ex_lfp_geq Cs [] [] f`.

Inductive case of `n`

We start the inductive case of `n` by doing a new induction on `m`.

Base case of `m`

The first base case is similar to the first base case of `n`. We again want to prove

   `ex_lfp_geq Cs (nodup string_dec V) (nodup string_dec V) f.`

We now apply the lemma `ex_lfp_geq_incl`, which states that

   `forall Cs V W f, incl V W → forall f, ex_lfp_geq Cs W W f → ex_lfp_geq Cs V V f.`

We give this lemma the arguments of `Cs`, `nodup string_dec V` and `nodup string_dec W`. This generates to new goals,

(1) `incl (nodup string_dec V) (nodup string_dec W)`

and

(2) `ex_lfp_geq Cs (nodup string_dec W) (nodup string_dec W) f.`

The goal (1) is proven by using a hypothesis that states that

   `Datatypes.length (set_diff string_dec (nodup string_dec V) (nodup string_dec W))` $\leq$ `m` $\leq$ `n.`

Since `m` $= 0$, this means that the set difference of `V` and `W` is empty. We can now apply the lemma `set_diff_nil_incl` on this hypothesis, which states that

   `forall dec V W, set_diff dec V W = []` $\leftrightarrow$ `incl V W.`

This gives us a hypothesis identical to our goal (1), and therefore proves it.

The goal (2) is proven by an existing hypothesis.

Inductive case of `m`

insert brief explanation here (from proof overview)...

4.4   Extraction to Haskell

Using Coq's code extraction feature, we can extract Haskell code from our Coq definitions.

```
Extraction Language Haskell.

Extract Constant map ⇒ "Prelude.map".
Extract Constant fold_right ⇒ "Prelude.foldr".

Extraction "/home/user/path/to/code/ex.hs"
  thm_32
  lem_33.
```

Listing 4.12: Extraction of Coq definitions to Haskell

Coq will automatically determine definitions which depend on one another when doing extraction. In the example above, we would not have needed to specify `lem_33` to be extracted, since `thm_32` already depends on it.

By default, Coq will give its own implementation of any functions used, instead of using Haskell's native implementations. If we want, we can specify what native Haskell functions should be used when extracting a Coq function. In the example code above, we specify that when extracting, `Prelude.map` and `Prelude.foldr` should be used for the Coq functions `map` and `fold_right`.

In the next chapter we will go more into detail about the results of the extraction, and the results of the Haskell code ran on some example input.

Chapter 5

Examples & Results

5.1 Examples using the extracted Haskell code

5.1.1 Defining examples for extraction in Coq

It is easiest to define as much of the example as possible in Coq and then extract it to Haskell, since Coq heavily prioritizes code correctness over readability when extracting. This makes much of the Haskell code hard to read due to unconventional syntax, e.g. using a recursion operator insted of calling a function recursively. First, we can look at the type signature of `thm_32` in Coq to see what arguments we need to give it.

```
thm_32
    : forall (n m : nat) (Cs : set Clause) (V W : set string) (f : Frontier),
      pre_thm n m Cs V W f
```

Listing 5.1: Type signature of `thm_32` in Coq

We see that the type of `thm_32` is a proof of `pre_thm`, with the quantified variables n, m, Cs, V, W, f. We can now look at the type signature of `pre_thm`.

```
pre_thm
    : nat →
      nat → set Clause → set string → set string → Frontier → Set
```

Listing 5.2: Type signature of `pre_thm` in Coq

We see that `pre_thm` returns an object in `Set`. This object is the proposition defined in Listing 4.9, where all the variables are the arguments given to `pre_thm`, i.e. the variables quantified in `thm_32`. This proposition states some logical assumptions, and concludes with a definition of `ex_lfp_geq`. By the notion of propositions as types, this proposition is a function that transforms a proof of the assumptions into a proof of the conclusion. Since the conlusion is a definition of `ex_lfp_geq` instantiated with some variables, we get a proof of `ex_lfp_geq` for these same variables. Looking at the definition of `ex_lfp_geq`, it defines a proposition stating that there exists a `g : Frontier` such that the proposition holds. If such a `g` exists, then the `g` itsself is evidence (proof) that the proposition holds. Hence, a proof of `ex_lfp_geq` is simply a `Frontier` that satisfies the conditions in `ex_lfp_geq`.

With all this information, we can now define an example of `thm_32` in Coq.

```
Example Cs := [
  ["a" & 0] ⇝ "b" & 1;
  ["b" & 1] ⇝ "c" & 2
].
Example f := frontier_fin_0.
Example vars' := nodup string_dec (vars Cs).


Example thm_32_example :=
  thm_32
    (Datatypes.length vars')
    (Datatypes.length vars')
    Cs
    vars'
    []
    f.
```

Listing 5.3: `thm_32` example

Since Coq elimiates many of the logical parts of the proof when extracting, [8, p. 8], we can avoid the tedious task of proving all the assumptions of `pre_thm` by simply using the extracted Haskell function for `thm_32_example`. The final assumption of `pre_thm`, namely `ex_lfp_geq Cs (nodup string_dec W) (nodup string_dec W) f`, will however not be removed by Coq, since it is an object in `Set`. This assumption is trivially true for any `f`, since `W` is an empty list in our example. Since Coq simplifies the type of `ex_lfp_geq` to `Frontier`, we can just include the same frontier `f` as in `thm_32_example` to the extracted Haskell version of `thm_32_example`, as a replacement for a proof of the above trivially true

assumption. We then receive a `Frontier` as output, which is the result of the theorem; the frontier that is a model of a set of clauses `Cs`. This frontier can then be applied to any string representing a variable to get the value of that variable in the model, which should be either a natural number or infinity.

5.1.2   Necessary alterations to the extracted Haskell code

Since we have not given a proof of Lemma 3.3, Coq will include a definition of the lemma in the extracted code, but will immidiately crash the program if the extracted function representing the lemma is ever called. We circumvent this by replacing the extracted definition of `lem_33` with the identity function for any frontier. We will look at some examples where this workaround is not sufficient in Section 5.3.

If we want to actually read the output from the extracted functions, we also need to derive a `Show` instance for `Ninfty`. What this means is that we need to define a function `show :: Ninfty → String`, which will be used by Haskell to convert a `Ninfty` to a `String`. This can be done by simply adding the line `deriving Prelude.Show` to the definition of `Ninfty`, which will make Haskell just print the constructors of `Ninfty`, which will be either `Fin n` for some natural number `n`, or `Infty` for infinity.

5.1.3   Example output

We can now run the example from Listing 5.3 using GHCi, which is an interactive Haskell interpreter that is included with GHC, the standard Haskell compiler.

```
ghci> (thm_32_example f) "a"
Fin 0
ghci> (thm_32_example f) "b"
Fin 1
ghci> (thm_32_example f) "c"
Fin 2
ghci> (thm_32_example f) "x"
Fin 0
```

Listing 5.4: `thm_32` example output

When given a string value (variable) from the set of clauses, the function will compute the value of that variable. When given any other variable, the function will return the value that the original frontier given as input would return for that variable, which is always `Fin 0` in this case (since in our example the frontier is `frontier_fin_0`, which is a constant function that always returns `Fin 0`).

5.2   Real world example

As stated previously, the algorithm described Theorem 3.2 is being tested for use in checking loops and determining universe levels in the type system of Coq.

explain more about type universes in Coq, see URL for possible citation `http://adam.chlipala.net/cpdt/html/Universes.html`

5.3   Limitations

Chapter 6

Evaluation

6.1 Correctness of informal proof

As we have shown in our formalization and by the results of the extraction, the informal proof of Theorem 3.2 from [1] appears to be correct. As mentioned in Section 3.1, we have not fully formalized nor verified the entirety of the informal proofs, and as such we cannot be 100% sure that the informal proof of Theorem 3.2 is correct. However, we have verified the most important parts of Theorem 3.2 without encountering any problems, and we can then be very confident that the informal proof is correct.

6.2 Worthwhileness of formalization

The formalization of the proof of Theorem 3.2 from [1] took us about X months of work. This includes the time spent on learning Coq and the time spent on the formalization of the proof.

6.3 Usefulness of extracted algorithm

As seen in Section 5.2, the extracted algorithm is useful in practice. However, the efficiency is not very good (is it? should check).

Chapter 7

Related & Future Work

7.1 Related work: Universe consistency checking in Coq

Currently no joins, only $x+k >= y$ with $k = 0, 1$. Graph-based algorithm. Ongoing effort [11] to implement algorithm from [1]. Plans to extend with joins for more expressivity for universe levels in Coq. Focus on speed with current (limited) expressivity. Later more expressivity.

7.2 Future work

7.2.1 Completing proof of remaining logical lemmas

7.2.2 Formal proof of Lemma 3.3

7.2.3 Proving minimality of model generated by Theorem 3.2

Chapter 8

Conclusion

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Appendix A

Full proof of Theorem 3.2 in Coq

```
1   Theorem thm_32 :
2     forall n m : nat,
3     forall Cs : set Clause,
4     forall V W : set string,
5     forall f : Frontier,
6       pre_thm n m Cs V W f.
7   Proof.
8     unfold pre_thm. induction n as [|n IHn].
9     - intros. apply le_0_r in H0.
10      apply length_zero_iff_nil in H0.
11      rewrite H0. apply ex_lfp_geq_empty.
12    - induction m as [|m IHm]; intros.
13      + apply (ex_lfp_geq_incl Cs (nodup string_dec V) (nodup string_dec W));
14        try assumption. destruct H1. apply le_0_r in H1.
15        apply length_zero_iff_nil in H1.
16        apply set_diff_nil_incl in H1. assumption.
17      + inversion H1.
18        apply le_lt_eq_dec in H3. destruct H3.
19        * apply (IHm Cs V W f); try assumption. lia.
20        * apply (ex_lfp_geq_nodup_iff) in H2.
21          assert (Datatypes.length (nodup string_dec W) ≤ n).
22          {
23            eapply (set_diff_succ string_dec) in H; try apply H3.
24            apply succ_le_mono. eapply le_trans.
25            apply H.
26            - eapply le_trans in H0. apply H0. apply le_refl.
27            - apply e.
28          }
```

```
29          assert (ex_lfp_geq Cs (nodup string_dec W) (nodup string_dec W) f).
30          {
31             apply (IHn n Cs W [] f);
32             try assumption; try apply incl_nil_l.
33             – eapply (set_diff_succ string_dec) in H; try apply H3.
34               simpl. rewrite set_diff_nil. apply conj; lia. apply e.
35             – unfold ex_lfp_geq. exists f. split.
36               apply geq_refl. apply sub_model_W_empty.
37          }
38          assert (H': incl W V) by assumption.
39          apply (nodup_incl2 string_dec) in H.
40          apply (lem_33 Cs V (nodup string_dec W) f) in H;
41          try assumption. elim H. intros h [H8 H9].
42          destruct
43            (sub_forward
44               Cs
45               (nodup string_dec V)
46               (nodup string_dec V)
47               h) as [U h']  eqn:Hforward.
48          assert
49            (sub_forward
50               Cs
51               (nodup string_dec V)
52               (nodup string_dec V)
53               h = (U, h'))  by assumption.
54          assert
55            (sub_forward
56               Cs
57               (nodup string_dec V)
58               (nodup string_dec V)
59               h = (U, h'))  by assumption.
60          rewrite nodup_rm in H9.
61          apply
62            (sub_forward_incl_set_diff
63               Cs
64               h
65               h'
66               (nodup string_dec V)
67               (nodup string_dec W)
68               U) in H9;
```

```
69          try apply Hforward.
70          inversion Hforward. apply sub_forward_incl in Hforward.
71          destruct U as [|u U']  eqn:Hu.
72          \−\− apply sub_forward_empty in H7.
73            destruct H7. unfold ex_lfp_geq. exists h. split; assumption.
74          \−\− destruct
75            (incl_dec
76              string_dec
77              V
78              (nodup
79                string_dec
80                (set_union string_dec (nodup string_dec W) U))).
81          \+\+ unfold ex_lfp_geq. exists (update_infty_V V f). split.
82            ** apply geq_nodup_true. apply geq_update_infty_V.
83            ** rewrite ← sub_model_nodup. apply sub_model_update_infty_V.
84          \+\+ assert (incl (nodup string_dec (set_union string_dec W U)) V).
85            {
86              apply nodup_incl2. eapply incl_set_union_trans.
87              assumption. apply nodup_incl in Hforward.
88              rewrite Hu. assumption.
89            }
90            assert
91              (Datatypes.length
92                (nodup string_dec (set_union string_dec W U)) <
93              Datatypes.length
94                (nodup string_dec V)).
95            {
96              assert (Datatypes.length
97                (nodup string_dec (set_union string_dec W U)) ≤
98                  Datatypes.length
99                    (nodup string_dec V)).
100             {
101               eapply NoDup_incl_length. apply NoDup_nodup.
102               apply nodup_incl. assumption.
103             }
104             apply le_lt_eq_dec in H13. destruct H13; try assumption.
105             assert
106               (strict_subset
107                 (nodup string_dec (set_union string_dec W U))
108                 (nodup string_dec V)).
```

```
109                        {
110                          unfold strict_subset. split.
111                          — apply nodup_incl. assumption.
112                          — unfold not. intros. apply n0.
113                            apply nodup_incl2 in H13.
114                            assert
115                              (incl
116                                (nodup string_dec (set_union string_dec W U))
117                                (nodup string_dec V)).
118                              {
119                                apply nodup_incl. assumption.
120                              }
121                            rewrite (incl_set_union_nodup_l string_dec). assumption.
122                          }
123                        apply (strict_subset_lt_length string_dec).
124                        unfold strict_subset in H13.
125                        destruct H13. unfold strict_subset. split.
126                        — apply nodup_incl in H13.
127                          apply nodup_incl2 in H13. assumption.
128                        — unfold not. intros. apply n0.
129                          rewrite (incl_set_union_nodup_l string_dec).
130                          apply nodup_incl. assumption.
131                        }
132                      assert
133                        (Datatypes.length
134                          (set_diff string_dec V (set_union string_dec (nodup string_dec W) U)) <
135                        Datatypes.length
136                          (set_diff string_dec V (nodup string_dec W)) ≤ S m).
137                      {
138                        apply conj.
139                        — rewrite Hu. apply set_diff_incl_lt_length;
140                          try assumption. apply nodup_incl2. assumption.
141                          rewrite ← set_diff_nodup_l in H9. assumption.
142                        — rewrite ← set_diff_nodup_r.
143                          rewrite ← set_diff_nodup_eq. rewrite e. lia.
144                      }
145                      apply (ex_lfp_geq_monotone Cs (nodup string_dec V) h' f).
146                      eapply (IHm Cs V (nodup string_dec (set_union string_dec W U)) h');
147                      try assumption.
148                      ** apply conj; try lia. inversion H14.
```

```
149            apply le_lt_eq_dec in H16. destruct H16;
150            rewrite nodup_rm; rewrite set_diff_nodup_eq in *;
151            rewrite ← length_set_diff_set_union_nodup_l;
152            lia.
153          ** apply (IHn n Cs (nodup string_dec (set_union string_dec W U)) [] h');
154            try apply incl_nil_l.
155            \−\−\− assert (Datatypes.length (nodup string_dec V) ≤ Datatypes.length V).
156                {
157                    apply NoDup_incl_length. apply NoDup_nodup.
158                    apply nodup_incl2. apply incl_refl.
159                }
160            rewrite nodup_rm. rewrite set_diff_nodup_eq in *;
161            try rewrite set_union_nodup_l in *; lia.
162            \−\−\− apply conj; try lia.
163            assert
164              (Datatypes.length
165                (set_diff
166                    string_dec
167                    (nodup string_dec (set_union string_dec W U))
168                    []) ≤
169              Datatypes.length
170                (nodup string_dec (set_union string_dec W U))).
171            apply (set_diff_nil_length string_dec).
172            eapply le_trans.
173            rewrite set_diff_nodup_eq. apply H15.
174            apply lt_le_pred in H13.
175            eapply le_trans. apply H13.
176            assert
177              (Datatypes.length (nodup string_dec V) ≤
178                Datatypes.length V).
179              {
180                  apply NoDup_incl_length. apply NoDup_nodup.
181                  apply nodup_incl2. apply incl_refl.
182              }
183            assert
184              (pred (Datatypes.length (nodup string_dec V)) ≤
185                pred (Datatypes.length V)).
186              {
187                  apply Nat.pred_le_mono. assumption.
188              }
```

```
189            apply le_n_Sm_pred_n_m in H0. assumption.
190        \-\-\- unfold ex_lfp_geq. exists h'.
191            split. apply geq_refl.
192            apply sub_model_W_empty.
193      ** eapply geq_trans with h; try assumption.
194          rewrite ← H12.
195          apply geq_Sinfty_f2.
196    \-\- unfold pre_thm. intros. apply (IHn m0 Cs' V' W' f');
197      try assumption; try lia.
198    \-\- rewrite nodup_rm. assumption.
199  Qed.
```

Listing A.1: Full proof of Theorem 3.2 in Coq