Quantum Algorithms, Spring 2022: Lecture 14 Scribe

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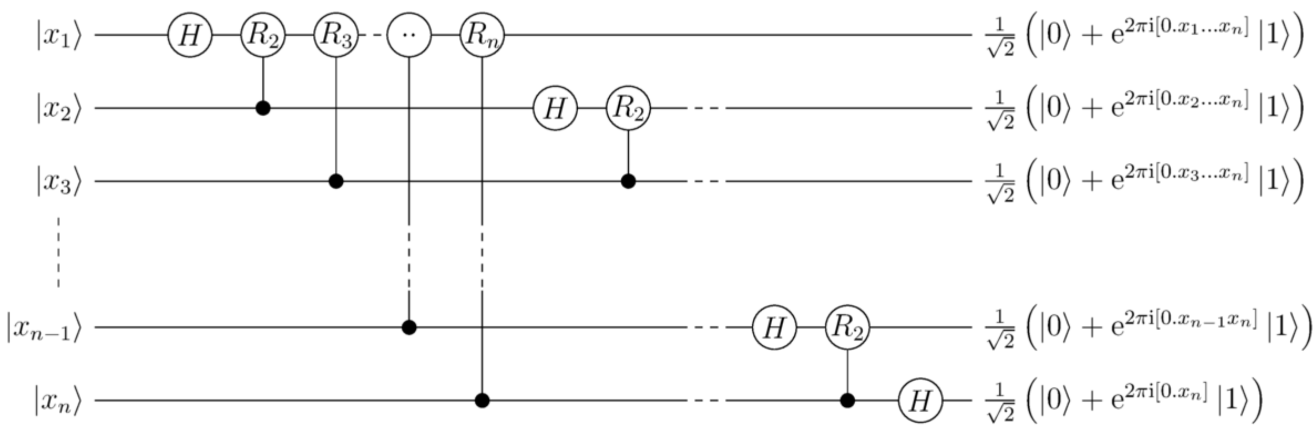
1 Recap

1.1 Circuit for Quantum Fourier Transform



$$\mathrm{QFT}(|x_1x_2\ldots x_n\rangle) = \frac{1}{\sqrt{N}}\ \left(|0\rangle + e^{2\pi i\,[0.x_n]}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i\,[0.x_{n-1}x_n]}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2\pi i\,[0.x_1x_2\ldots x_n]}|1\rangle\right)$$

Here, we have H and R_m are the Hadamard and the controlled phase gates respectively.

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \qquad R_m = \begin{bmatrix}1 & 0\\ 0 & e^{2i\pi/2^m}\end{bmatrix}$$

1.2 Gate Complexity

Implementation for Quantum Fourier Transform involves the above present circuit for QFT and $n/2$ - SWAP gates. Thus, the gate complexity of QFT is $O(n^2) = O(\log^2 N)$.

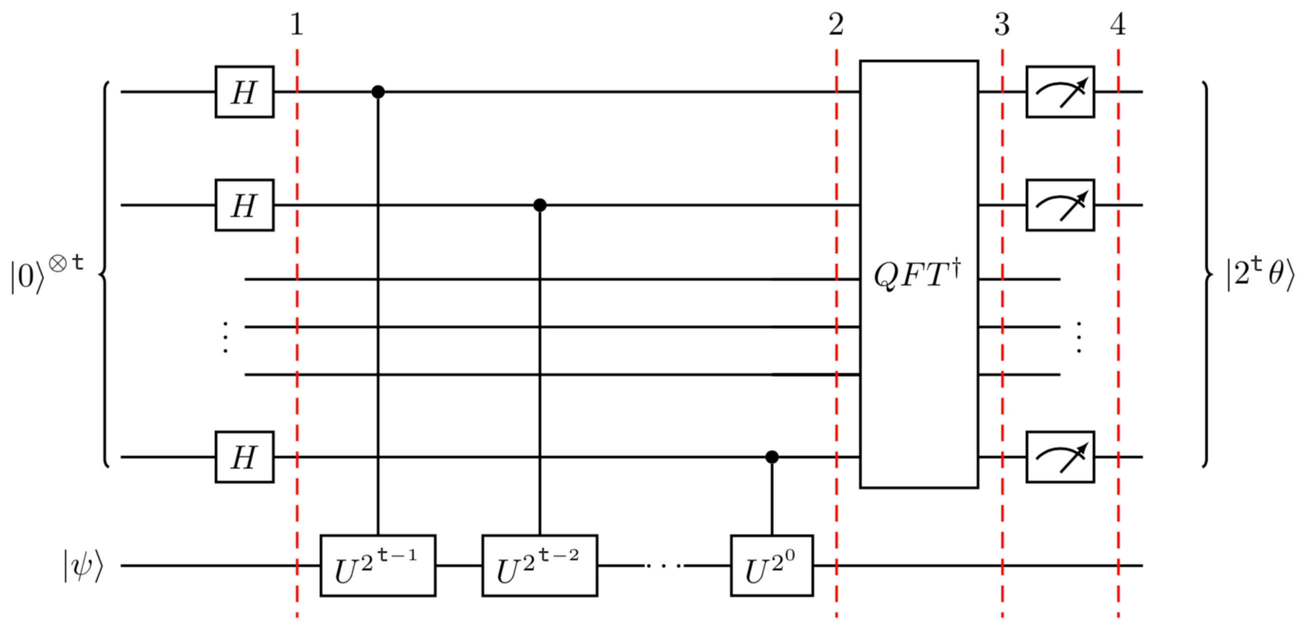
2 Quantum Phase Estimation

Suppose that U is a unitary with a eigenvector $|\phi\rangle$ and eigenvalue $e^{2i\pi\theta}$, $0 \leq \theta < 1$. If the cost of implementing U is T_u, then given $|\phi\rangle$ we can obtain $\tilde{\theta}$ such that $|\tilde{\theta} - \theta| \leq \epsilon$ with success probability at least $1-$ in time T given by the following.

$$T = O(\frac{T_u}{\delta\epsilon})$$

2.1 Circuit for QPE

The general quantum circuit for phase estimation is shown below. The top register contains t 'counting' qubits, and the bottom contains qubits.



Now, $|0\rangle^{\otimes t}|\psi\rangle \xrightarrow{H^{\otimes t}\otimes I} \frac{1}{\sqrt{2^t}}\left(|0\rangle+|1\rangle\right)^{\otimes t}|\psi\rangle$. Also, $U^{2^j}|\psi\rangle = U^{2^j-1}U|\psi\rangle = U^{2^j-1}e^{2\pi i\theta}|\psi\rangle = \cdots = e^{2\pi i 2^j\theta}|\psi\rangle$. Thus, upon applying the series of controlled unitaries we obtain the following.

$$\frac{1}{\sqrt{2^t}}\left(|0\rangle + e^{2\pi i\theta 2^{t-1}}|1\rangle\right)\otimes\cdots\otimes\left(|0\rangle + e^{2\pi i\theta 2^{1}}|1\rangle\right)\otimes\left(|0\rangle + e^{2\pi i\theta 2^{0}}|1\rangle\right)\otimes|\psi\rangle = \frac{1}{\sqrt{2^t}}\sum_{k=0}^{2^t-1} e^{2\pi i\theta k}|k\rangle\otimes|\psi\rangle$$

Furthermore, over the above expressed state QFT† is applied resulting in the following expression.

$$\frac{1}{2^{\frac{t}{2}}}\sum_{k=0}^{2^t-1} e^{2\pi i\theta k}|k\rangle\otimes|\psi\rangle \xrightarrow{QFT_t^\dagger} \frac{1}{2^t}\sum_{x=0}^{2^t-1}\sum_{k=0}^{2^t-1} e^{-\frac{2\pi i k}{2^t}(x-2^t\theta)}|x\rangle\otimes|\psi\rangle$$

Now, we can approximate the value of $\theta\in[0,1]$ by rounding $2^t\theta$ to the nearest integer. This means that $2^t\theta = a + 2^t\delta$, where a is the nearest integer to $2^t\theta$, and the difference $2^t\delta$ satisfies $0\leqslant|2^t\delta|\leqslant\frac{1}{2}$. Now, performing a measurement in the computational basis on the first register yields the result $|a\rangle$ with probability p_a.

$$\begin{aligned}
p_a &= \frac{1}{2^{2t}}\left|\frac{1-e^{2\pi i 2^t\delta}}{1-e^{2\pi i\delta}}\right|^2 && \text{for } \delta\neq 0\\
&= \frac{1}{2^{2t}}\left|\frac{2\sin\left(\pi 2^t\delta\right)}{2\sin(\pi\delta)}\right|^2 && \text{given } \left|1-e^{2ix}\right|^2 = 4\left|\sin(x)\right|^2\\
&= \frac{1}{2^{2t}}\frac{\left|\sin\left(\pi 2^t\delta\right)\right|^2}{|\sin(\pi\delta)|^2}\\
&\geqslant \frac{1}{2^{2t}}\frac{\left|\sin\left(\pi 2^t\delta\right)\right|^2}{|\pi\delta|^2} && \text{given } |\sin(\pi\delta)|\leqslant|\pi\delta| \text{ for } |\delta|\leqslant\frac{1}{2^{t+1}}
\end{aligned}$$

Thus, $p_a \geqslant \frac{1}{2^{2t}} \frac{|2 \cdot 2^t\delta|^2}{|\pi\delta|^2} \geqslant \frac{4}{\pi^2}$ given $|2 \cdot 2^t\delta| \leqslant |\sin(\pi 2^t\delta)|$ for $|\delta| \leqslant \frac{1}{2^{t+1}}$.

Now, we also know that $|\delta| < 1/2 \implies 0 < \pi|\delta| < 1/2 \implies \sin^2(\pi\delta) \geq 4\delta^2$. This is because we have the inequality $|\sin x| \geq 2|x|/\pi, x \in [-\pi/2, \pi/2]$.

$$p_a = \frac{1}{2^{2t}} \frac{\left|\sin\left(\pi 2^t \delta\right)\right|^2}{|\sin(\pi\delta)|^2} < \frac{1}{2^{2t}} \frac{1}{4\delta^2}$$

Now, we are interested in all these a's such that $|\delta| \geq \epsilon$. We will observe some $a \in [0, T-1]$ and this will be satisfied by δ given as follows. Here, $T = 2^t$.

$$\delta = \begin{cases} \epsilon, \epsilon + 1/T, \epsilon + 2/T, \ldots \\ -\epsilon, -\epsilon - 1/T, -\epsilon - 2/T, \ldots \end{cases}$$

$$\Pr[a : |\delta| \geq \epsilon] = \sum_{a:|\delta|\geq\epsilon} p_a \leq \frac{1}{4T^2} \left\{ \sum_{k=0}^{\infty} \frac{1}{\left(\epsilon + k/T\right)^2} + \sum_{k=0}^{\infty} \frac{1}{\left(-\epsilon - k/T\right)^2} \right\} \leq \frac{1}{2T^2} \left\{ \sum_{k=0}^{\infty} \frac{1}{\left(\epsilon + k/T\right)^2} \right\} \leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(k + \epsilon T)^2}$$

$$\implies \Pr[a : |\delta| \geq \epsilon] \leq \frac{1}{2} \sum_{k=0}^{\infty} 1/k^2 \leq \frac{1}{2} \int_{\epsilon T}^{\infty} \frac{1}{x^2} dx \leq \frac{1}{2\epsilon T} < \frac{1}{\epsilon T}$$

Thus, we have $\Pr[a : |\delta| \geq \epsilon] < \frac{1}{\epsilon T}$ and $\Pr[|\theta - \tilde{\theta}| < \epsilon] \geq 1 - \frac{1}{T\epsilon}$. So, we should have $\frac{1}{T\epsilon} < \delta \implies T > \frac{1}{T\epsilon} \implies t = 1 + \lceil \log_2 1/\delta \rceil + \lceil \log_2 1/\epsilon \rceil$, to have $|\theta - \tilde{\theta}| < \epsilon$ with prob $\geq 1 - \delta$.

2.2 Complexity of QPE

Now, cost of implementing $U = T_u$ whereas cost of implementing $c{-}U^{2^0}, c{-}U^{2^1}, \ldots, c{-}U^{2^{t-1}} = O\left(T_u \sum_{j=0}^{2^{t-1}} 2^j\right) = O\left(T_u 2^t\right) = O\left(T_u/(\delta\epsilon)\right)$. However, we can improve this slightly by not adding ancilla qubits.

1. Run QPE algorithm with 't' qubits in the first register $O\left(\log(1/\delta)\right)$ times.
2. Compute the median of the outcomes and output the result.
3. Thus, the new cost of implementation would be $O\left((T_u/\epsilon)\log(1/\delta)\right)$.

2.3 Remarks

In some applications, we might be interested in preparing some specific eigenstates of U and not care about the rest. In such cases QPE helps. These applications can be preparing ground states of Hamiltonian, Gibbs' state preparation, etc.

References

[1] Simon Apers, Shantanav Chakraborty, Leonardo Novo, and Jérémie Roland. Quadratic speedup for spatial search by continuous-time quantum walk, 2021. URL `https://arxiv.org/abs/2112.12746`.

[2] Yimin Ge, Jordi Tura, and J. Ignacio Cirac. Faster ground state preparation and high-precision ground energy estimation with fewer qubits, 2017. URL `https://arxiv.org/abs/1712.03193`.

[3] Lin Lin and Yu Tong. Near-optimal ground state preparation. *Quantum*, 4:372, December 2020. ISSN 2521-327X. doi: 10.22331/q-2020-12-14-372. URL `https://doi.org/10.22331/q-2020-12-14-372`.

[4] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.