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Question 1

According to the Cauchy Schwarz Inequality,

$$|\langle u,v\rangle|^2 \leqslant \langle u,u\rangle \cdot \langle v,v\rangle \Rightarrow |\langle u,v\rangle| \leqslant \|u\| \cdot \|v\| \quad -①$$

Meanwhile the reverse triangle inequality states that $\forall$ two vectors x and y in a Hilbert space $\|x-y\| \geqslant \big|\|x\| - \|y\|\big|$ – ②

Now, suppose we have,

$$\begin{aligned} \|x-y\|^2 &= \|x\|^2 + \|y\|^2 - 2|Re(\langle x,y\rangle)| \\ &\geqslant \|x\|^2 + \|y\|^2 - 2|\langle x,y\rangle| \\ &\geqslant \|x\|^2 + \|y\|^2 - 2\|x\|\cdot\|y\| \\ &\qquad\qquad [\text{from } ①] \\ &= (\|x\| - \|y\|)^2 \end{aligned}$$

Thus, $\|x-y\| \geqslant \big|\|x\| - \|y\|\big|$

as required in ②.

Hence, we proved the reverse triangle inequality from Cauchy Schwarz inequality.

Question 2

Since radius of the nucleus is of order $10^{-15}\,m$, the maximum uncertainty in position (Δx) shall be $10^{-15}\,m$ – ①

Now, given maximum energy coming out of the nucleus is 4 MeV which is equivalent to 6.41×10^{-13} J (let E).

Thus maximum uncertainty in momentum shall be $\Delta p \lesssim \sqrt{2mE}$
$\approx 1.08 \times 10^{-21}$ kgm/s — (2)

Here, m is the mass of an electron (9.1×10^{-31} kg)

Thus the maximum uncertainty in position and momentum for an electron residing in the nucleus is $\Delta x . \Delta p$

Now, $\Delta x . \Delta p = 1e-15 + 1.08e-21 = 1.08e-36$ in SI units

Thus, we have $\Delta x . \Delta p = 1.08e-36$ in SI
$< 5.27e-35$ in SI $= \frac{\hbar}{2}$

$\Rightarrow \Delta x . \Delta p < \frac{\hbar}{2}$ which completely violates the Heisenberg Uncertainty Principle.

Therefore, we can conclude from the above that the electron cannot reside inside the nucleus.

Question 3 a

We have $H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1| \right)$

Now let $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then we have,

$$H = a\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & a \\ a & -a \end{bmatrix}$$

Now let $|\psi\rangle$ be a eigenvector of H, then

$H|\psi\rangle = \lambda|\psi\rangle$ where λ is the corresponding eigenvalue

$$\Rightarrow (H - \lambda I)|\psi\rangle = 0 \Rightarrow \det(H - \lambda I) = 0$$

Thus, $\det\left(\begin{bmatrix} a-\lambda & a \\ a & -a-\lambda \end{bmatrix}\right) = 0 \Rightarrow -(a-\lambda)(a+\lambda) - a^2 = 0$

$$\Rightarrow -2a^2 + \lambda^2 = 0 \Rightarrow \underbrace{\boxed{\lambda = \pm\sqrt{2}\,a}}_{\text{two eigenvalues}}$$

Let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ be the eigenvectors corresponding to $\sqrt{2}a$ and $-\sqrt{2}a$ then,

$$H\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \sqrt{2}a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \Rightarrow \begin{bmatrix} a & a \\ a & -a \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}a x_1 \\ \sqrt{2}a y_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} ax_1 + ay_1 \\ ax_1 - ay_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}a x_1 \\ \sqrt{2}a y_1 \end{bmatrix} \Rightarrow ax_1 + ay_1 = \sqrt{2}a x_1$$

and $ax_1 - ay_1 = \sqrt{2}a y_1$

Thus, $x_1 + y_1 = \sqrt{2}x_1$ and $x_1 - y_1 = \sqrt{2}y_1$

and assuming $y_1 = k$, we have $x_1 = (1+\sqrt{2})k$

Now by normalization we have,

$$x_1 = \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \quad \text{and} \quad y_1 = \frac{1}{\sqrt{4+2\sqrt{2}}}$$

Similarly, $x_2 = \dfrac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}}$ and $y_2 = \dfrac{1}{\sqrt{4-2\sqrt{2}}}$

Now since $\begin{bmatrix} x \\ y \end{bmatrix} = x|1\rangle + y|2\rangle$ we have the following eigenvectors

$\dfrac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}}|1\rangle + \dfrac{1}{\sqrt{4+2\sqrt{2}}}|2\rangle$ with eigenvalue $\sqrt{2}a$ and

$\dfrac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}}|1\rangle + \dfrac{1}{\sqrt{4-2\sqrt{2}}}|2\rangle$ with eigenvalue $-\sqrt{2}a$

Question 3b

Given, $A = |1\rangle\langle 2| - |2\rangle\langle 1|$ and

now $A^{\dagger} = (|1\rangle\langle 2| - |2\rangle\langle 1|)^{\dagger} = (|2\rangle\langle 1| - |1\rangle\langle 2|)$

$= -A$

Thus the operator A is not Hermitian since $A \neq A^{\dagger}$ and hence A cannot be a Hamiltonian.

Question 3c

Given $B = a|1\rangle\langle 1| + b|2\rangle\langle 2| + c(|1\rangle\langle 2| - |2\rangle\langle 1|)$

then we have $B^{\dagger} = a^{*}|1\rangle\langle 1| + b^{*}|2\rangle\langle 2| + c^{*}(|2\rangle\langle 1| - |1\rangle\langle 2|)$

Now B can be Hamiltonian iff B is Hermitian.

Thus, $B = B^{\dagger}$

$\Rightarrow \quad a = a^{*}, \; b = b^{*}$ and $c = -c^{*}$

Thus the required condition for B to be Hamiltonian is $a = a^{*}$, $b = b^{*}$ and $c = -c^{*}$

In other words, a and b are real and c is purely imaginary.

Question 3d

Given, $B = a|1\rangle\langle 1| + b|2\rangle\langle 2| + c(|1\rangle\langle 2| - |2\rangle\langle 1|)$

where a and b are real ($a, b \in \mathbb{R}$) and $c = id$ where $i = \sqrt{-1}$ and $d \in \mathbb{R}$

Now, let $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then,

$$\mathcal{B} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + id\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - id\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Let $|\psi\rangle$ be an eigenvector of $\mathcal{B}$ then,

$$\mathcal{B}|\psi\rangle = \lambda|\psi\rangle \text{ then,}$$

$$\begin{bmatrix} a & id \\ -id & b \end{bmatrix}|\psi\rangle = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}|\psi\rangle$$

$$\Rightarrow \begin{bmatrix} a-\lambda & id \\ -id & b-\lambda \end{bmatrix}|\psi\rangle = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} a-\lambda & id \\ -id & b-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (a-\lambda)(b-\lambda) + (id)(id) = 0$$

$$\Rightarrow \lambda^2 - (a+b)\lambda + ab - d^2 = 0$$

Therefore we have,

$$\lambda = \frac{a+b \pm \sqrt{(a+b)^2 - 4(ab-d^2)}}{2}$$

$$\Rightarrow \lambda = \frac{a+b \pm \sqrt{(a-b)^2 + 4d^2}}{2}$$

Now let the corresponding eigenkets be of form $|\psi\rangle = x|1\rangle + y|2\rangle$

with its corresponding matrix form as $\begin{bmatrix} x \\ y \end{bmatrix} = |\psi$

When $\lambda = \dfrac{a+b+\sqrt{(a-b)^2+4d^2}}{2}$ we have

$$|\psi\rangle = \begin{bmatrix} \dfrac{i\left(a-b+\sqrt{(a-b)^2+4d^2}\right)}{2d} k \\ k \end{bmatrix}$$

solving for $y = k$ where $|\psi\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B|\psi\rangle = \lambda|\psi\rangle$

$$\Rightarrow |\psi\rangle = \begin{bmatrix} \dfrac{i\left(a-b+\sqrt{(a-b)^2+4d^2}\right)}{2d\sqrt{1+\frac{1}{4}\left|\frac{a-b+\sqrt{(a-b)^2+4d^2}}{d}\right|^2}} \\ \dfrac{1}{\sqrt{1+\frac{1}{4}\left|\frac{a-b+\sqrt{(a-b)^2+4d^2}}{d}\right|^2}} \end{bmatrix}$$

on normalization

When $\lambda = \dfrac{a+b-\sqrt{(a-b)^2+4d^2}}{2}$ similarly we have the following eigenket,

$$|\psi\rangle = \begin{bmatrix} \dfrac{i\left(a-b-\sqrt{(a-b)^2+4d^2}\right)}{2d} k \\ k \end{bmatrix}$$

solving for $y = k$ where $|\psi\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B|\psi\rangle = \lambda|\psi\rangle$

$$\Rightarrow |\psi\rangle = \begin{bmatrix} \dfrac{i\left(a-b-\sqrt{(a-b)^2+4d^2}\right)}{2d\sqrt{1+\frac{1}{4}\left|\frac{a-b-\sqrt{(a-b)^2+4d^2}}{d}\right|^2}} \\ \dfrac{1}{\sqrt{1+\frac{1}{4}\left|\frac{a-b-\sqrt{(a-b)^2+4d^2}}{d}\right|^2}} \end{bmatrix}$$

on normalization

Thus, the required eigenkets (normalized for B under cond^{n}s described in Question 3c) are as follows:

① $$\frac{i\left(a-b+\sqrt{(a-b)^2+4d^2}\right)}{2d\sqrt{1+\frac{1}{4}\left|\frac{a-b+\sqrt{(a-b)^2+4d^2}}{d}\right|^2}}|1\rangle + \frac{1}{\sqrt{1+\frac{1}{4}\left|\frac{a-b+\sqrt{(a-b)^2+4d^2}}{d}\right|^2}}|2\rangle$$

for the eigenvalue $\lambda = \dfrac{a+b+\sqrt{(a-b)^2+4d^2}}{2}$

② $$\frac{i\left(a-b-\sqrt{(a-b)^2+4d^2}\right)}{2d\sqrt{1+\frac{1}{4}\left|\frac{a-b-\sqrt{(a-b)^2+4d^2}}{d}\right|^2}}|1\rangle + \frac{1}{\sqrt{1+\frac{1}{4}\left|\frac{a-b-\sqrt{(a-b)^2+4d^2}}{d}\right|^2}}|2\rangle$$

for the eigenvalue $\lambda = \dfrac{a+b-\sqrt{(a-b)^2+4d^2}}{2}$

Question 4

Given matrix $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and let its eigenvalue(s) be λ then,

$Av = \lambda v$ where v is some eigenvector

$$\Rightarrow \det(A - \lambda I) = 0 \quad \Rightarrow \det \begin{pmatrix} -\lambda & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -\lambda & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -\lambda \end{pmatrix} = 0$$

$$\Rightarrow -\lambda^3 + \tfrac{1}{2}\lambda + \frac{1}{2}\lambda = 0$$

$$\Rightarrow -\lambda^3 + \lambda = 0 \quad \Rightarrow \lambda^3 - \lambda = 0 \quad \Rightarrow \lambda(\lambda^2 - 1) = 0$$

$$\Rightarrow \lambda(\lambda - 1)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } \lambda = 1 \text{ or } \lambda = -1$$

When $\lambda = 0$ we have,

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{where } k \in \mathbb{C}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$ from normalization since

$k^2(1+1) = 1$

$\Rightarrow k = 1/\sqrt{2}$

When $\lambda = 1$ we have,

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \text{ where } k \in \mathbb{C}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \text{ by normalization since}$$

$$k^2(1+2+1) = 1$$

$$\Rightarrow k = 1/2$$

When $\lambda = -1$ we have,

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \text{ where } k \in \mathbb{C}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} \text{ by normalization since}$$

$$k^2(1+2+1) = 1$$

$$\Rightarrow k = 1/2$$

Therefore the required normalized eigenket and corresponding eigenvalues are $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ for $\lambda = 0$, $\begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$ for $\lambda = 1$ and $\begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}$ for $\lambda = -1$

Question 5

According to the theorem of simultaneous diagonalization we have, if A and B are Hermitian operators then iff $[A, B] = 0$, $\exists$ an orthonormal basis such that both A and B are diagonal wrt that basis.

Now if simultaneous diagonalization is not possible then we can't have simultaneous eigenkets.

Thus, we need $[A,B] = 0 \Rightarrow AB - BA = 0$.

Now given, $\{A, B\} = 0 \Rightarrow AB + BA = 0$ [known] $-①$

Thus from ① if we were to satisfy $[A,B] = 0$ then it is necessary to have $\underline{BA = 0}$.

derivation

$$\cancel{AB} - BA = \cancel{AB} + BA$$

$$\Rightarrow 2BA = 0 \Rightarrow BA = 0$$

Thus it is possible to have simultaneous eigenkets of A and B if and only if $BA = 0$.