Alapan Chaudhuri **Quantum Algorithms** (Assignment 1)

1. a) $$|\psi\rangle = \begin{bmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{bmatrix} = \sqrt{\tfrac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sqrt{\tfrac{1}{3}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \sqrt{2/3}\, |0\rangle + \sqrt{1/3}\, |1\rangle$$

1. b) $$\hat{A} = 2|0\rangle\langle 0| + (1-i)|0\rangle\langle 1| + (1+i)|1\rangle\langle 0| - |1\rangle\langle 1|$$

$$= 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (1-i)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (1+i)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1-i \\ 1+i & -1 \end{bmatrix}$$

1. c) $$A = \begin{bmatrix} 2 & 1-i \\ 1+i & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{1}{2} I + \hat{\sigma}_x + \hat{\sigma}_y + \frac{3}{2}\hat{\sigma}_z$$

1. d)

$$|\psi_1\rangle = (I \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_2\rangle = (\hat{\sigma}_x \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

$$|\psi_3\rangle = (\hat{\sigma}_y \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(i|10\rangle - i|01\rangle) = \frac{i}{\sqrt{2}}(|10\rangle - |01\rangle)$$

$$|\psi_4\rangle = (\hat{\sigma}_z \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

Now,

$$\langle\psi_1|\psi_2\rangle = \frac{1}{2}(\langle 00|10\rangle + \langle 00|01\rangle + \langle 11|10\rangle + \langle 11|01\rangle) = 0$$

$$\langle\psi_1|\psi_3\rangle = \frac{i}{2}(\langle 00|10\rangle - \langle 00|01\rangle + \langle 11|10\rangle - \langle 11|01\rangle) = 0$$

$$\langle\psi_1|\psi_4\rangle = \frac{1}{2}(\langle 00|00\rangle - \langle 00|11\rangle + \langle 11|00\rangle - \langle 11|11\rangle) = 0$$

$$\langle \Psi_2 | \Psi_3 \rangle = \frac{i}{2} \left(\langle 10 | 10 \rangle - \langle 10 | 01 \rangle + \langle 01 | 10 \rangle - \langle 01 | 01 \rangle \right) = 0$$

$$\langle \Psi_2 | \Psi_4 \rangle = \frac{1}{2} \left(\langle 10 | 00 \rangle - \langle 10 | 11 \rangle + \langle 01 | 00 \rangle - \langle 01 | 11 \rangle \right) = 0$$

$$\langle \Psi_4 | \Psi_3 \rangle = \frac{i}{2} \left(\langle 00 | 10 \rangle - \langle 00 | 01 \rangle - \langle 11 | 10 \rangle + \langle 11 | 01 \rangle \right) = 0$$

Hence all the states $\{ |\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle \}$ are orthogonal.

2. a) Given that $\{ |\lambda_1\rangle, |\lambda_2\rangle \dots |\lambda_n\rangle \}$ form the orthonormal basis for the Hilbert space $\mathcal{H}$,

we can represent $|x\rangle$ as,

$$|x\rangle = c_1 |\lambda_1\rangle + c_2 |\lambda_2\rangle + \dots + c_n |\lambda_n\rangle$$

where $\forall_i \; c_i \in \mathbb{C}$ and

$$\sum_{i=1}^{n} |c_i|^2 = 1$$

Now since all $|\lambda_i\rangle$s are orthonormal,

$$\langle \lambda_i | x \rangle = c_1 \langle \lambda_i | \lambda_1 \rangle + \dots + c_n \langle \lambda_i | \lambda_n \rangle$$

$$\Rightarrow \quad \langle \lambda_i | x \rangle = c_i \langle \lambda_i | \lambda_i \rangle = c_i$$

since $\langle \lambda_i | \lambda_j \rangle = 0$ for all $j \neq i$

Thus, $$|x\rangle = \langle \lambda_1 | x \rangle |\lambda_1\rangle + \langle \lambda_2 | x \rangle |\lambda_2\rangle + \dots + \langle \lambda_n | x \rangle |\lambda_n\rangle$$

2. b) In the orthonormal basis $\{|\lambda_1\rangle, |\lambda_2\rangle, \dots, |\lambda_n\rangle\}$

we can represent $|\lambda_i\rangle$ as $\begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}$ such that

$$\forall_j \quad a_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Thus, $$|\lambda_j\rangle\langle\lambda_k| = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

such that $$\forall_{x,y} \quad a_{pq} = \begin{cases} 1 & \text{if } p = j \text{ and } q = k \\ 0 & \text{otherwise} \end{cases}$$

3. a) According to problem we can represent

$$|\phi_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |\phi_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{A} = |\phi_1\rangle\langle\phi_1| + i|\phi_1\rangle\langle\phi_2| - i|\phi_2\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - i\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

Now let λ be an eigenvalue for some eigenvector $|x\rangle$, then, $A|x\rangle = \lambda|x\rangle$

$$\Rightarrow (A - \lambda I)|x\rangle = 0$$

$$\Rightarrow \det(A - \lambda I) = 0$$

$$\Rightarrow (1-\lambda)^2 - (i \cdot -i) = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + \cancel{1} + \cancel{i^2} = 0$$

$$\Rightarrow \lambda(\lambda - 2) = 0$$

$$\Rightarrow \lambda = \{0, 2\}$$

Thus the spectrum of $\hat{A}$ is $\{0, 2\}$ which basically represents the set of eigenvalues of $\hat{A}$ in its matrix representation.

3. b) $$\hat{A} = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

Now, $$\hat{A}^{\dagger} = \begin{bmatrix} 1^* & i^* \\ (-i)^* & 1^* \end{bmatrix}^T = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \hat{A}$$

Thus, $\hat{A} = \hat{A}^{\dagger}$ and hence we conclude that $\hat{A}$ is Hermitian.

4. a) The Taylor series expansion of e^{iAx} is given as

$$e^{iAx} = \sum_{k=0}^{\infty} \frac{(i x A)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(i(x)A)^{2n}}{(2n)!} + \frac{(i(x)A)^{2n+1}}{(2n+1)!} \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{(i)^{2n}(x)^{2n}(A)^{2n}}{(2n)!} + \frac{(i)^{2n+1}(x)^{2n+1}(A)^{2n+1}}{(2n+1)!} \right]$$

$$\Rightarrow e^{-i\theta\hat{\sigma}_x} = \sum_{n=0}^{\infty}\left[\frac{(ix)^{2n}}{(2n)!}I + \frac{(ix)^{2n+1}}{(2n+1)!}A\right]$$

$$= \sum_{n=0}^{\infty}\frac{(ix)^{2n}}{(2n)!}I + \sum_{n=0}^{\infty}\frac{(ix)^{2n+1}}{(2n+1)!}A$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)I + \left(x - \frac{x^3}{3!} + \cdots\right)A$$

$$= \cos(x)\,I + \sin(x)\,A$$

b) $$e^{-i\theta\hat{\sigma}_x} = e^{i(-\theta)\hat{\sigma}_x} = \cos(-\theta)\,I + \sin(-\theta)\,\hat{\sigma}_x$$

$$= \cos(\theta)\,I - \sin(\theta)\,\hat{\sigma}_x$$

[from the result $e^{iAx} = \cos(x)\,I + i\sin(x)\,A$]

c) Given, $\hat{n} = (n_x, n_y, n_z)$ such that $n_x^2 + n_y^2 + n_z^2 = 1$

$$\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

Thus $\hat{n}\cdot\hat{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$

$$\Rightarrow (\hat{n}\cdot\hat{\sigma})^2 = n_x^2\sigma_x^2 + n_y^2\sigma_y^2 + n_z^2\sigma_z^2$$

$$+ \; 2 n_x n_y \left(\hat{\sigma}_x \cdot \hat{\sigma}_y + \hat{\sigma}_y \cdot \hat{\sigma}_x \right)$$

$$+ \; 2 n_y n_z \left(\hat{\sigma}_y \cdot \hat{\sigma}_z + \hat{\sigma}_z \cdot \hat{\sigma}_y \right)$$

$$+ \; 2 n_z n_x \left(\hat{\sigma}_z \cdot \hat{\sigma}_x + \hat{\sigma}_x \cdot \hat{\sigma}_z \right)$$

$$= \left(n_x^2 + n_y^2 + n_z^2 \right) I = I$$

Thus we can use the result $e^{iAx} = \cos x \, I + i \sin x \, A$

resulting in $e^{-i\theta \, \hat{n} \cdot \hat{\sigma}} = \cos(\theta) I - i \sin(\theta) \, \hat{n} \cdot \hat{\sigma}$

since $(\hat{n} \cdot \hat{\sigma})^2 = I$

5. Let $\frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right) = |a\rangle \otimes |b\rangle$

$$\Rightarrow \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}$$

$\Rightarrow a_0 b_0 = 1/\sqrt{2}$ and $a_1 b_1 = 1/\sqrt{2}$

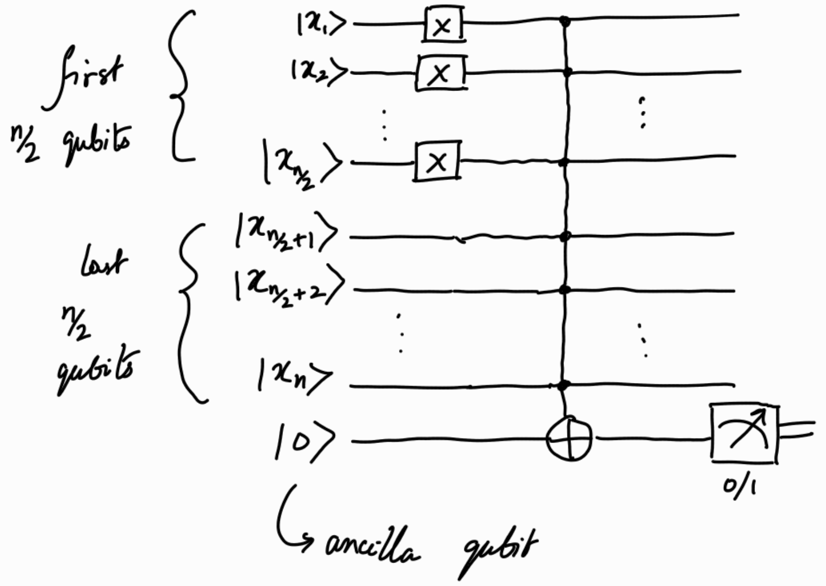
$\Rightarrow a_0, b_0, a_1, b_1 \neq 0$ thus all of the variables must be non zero

However, $a_0 b_1 = 0$ and $a_1 b_0 = 0$

$\Rightarrow (a_0 = 0 \text{ or } b_1 = 0)$ and $(a_1 = 0 \text{ or } b_0 = 0)$

thus we arrive at a contradiction.

6. Let $|x\rangle = |x_1 x_2 x_3 \dots x_n\rangle$ then we have the following



Now the ancilla qubit $|0\rangle$ becomes $|1\rangle$ (with probability 1) if and only if

$\forall i > n/2 \quad |x_i\rangle = |1\rangle$ and

$\forall i \leq n/2 \quad |x_i\rangle = |0\rangle$.

Furthermore, if the number of $|1\rangle$s in the first $n/2$ qubits plus the number of $|0\rangle$s in the second $n/2$ qubits is $n/2$ then $\exists\, i \leq n$ such that $|x_i\rangle = |1\rangle$ and $X|x_i\rangle = |0\rangle$

$\Rightarrow$ upon measurement, the ancilla qubit will always output 0 (because it won't be flipped).

Thus the above given circuit serves as a quantum algorithm that efficiently solves the given problem.

① We apply X gate on the first $n/2$ qubits

② We apply multi-controlled X gate where all n qubits are control and the ancilla $|0\rangle$ is the target.

③ We measure the ancilla.

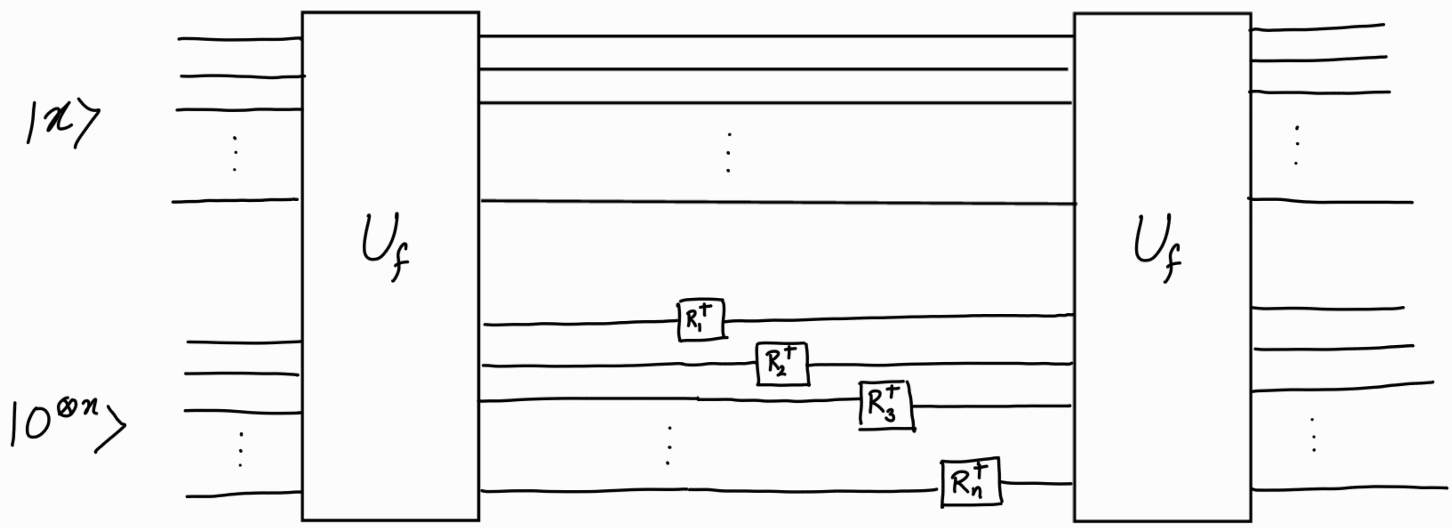
④ If we measure the classical output as 1 then, we conclude that the first $n/2$ qubits are 0 and the second $n/2$ qubits are 1.

⑤ Else if we measure 0 then, we conclude that the number of $|1\rangle$s in the first $n/2$ qubits plus the number of $|0\rangle$s in the second $n/2$ qubits is $n/2$.

7. Given assumption, $\exists$ a reversible classical circuit that uses T-Toffoli gates to implement a function

$$f: \{0,1\}^m \to \{0,1\}^n$$

such that $f: (x, y) \to (x, y \oplus f(x))$.



$$|x\rangle|0^{\otimes n}\rangle \xrightarrow{U_f} |x\rangle|f(x)\rangle \xrightarrow{R_1^\dagger} e^{-2\pi i\, 0.f_x^1}|x\rangle|f(x)\rangle$$

$$e^{-2\pi i\, 0.f_x^1} |x\rangle |f(x)\rangle \xrightarrow{R_2^\dagger} \cdots \xrightarrow{R_n^\dagger} e^{-2\pi i\, 0.f_x^1 f_x^2 \cdots f_x^n} |x\rangle |f(x)\rangle$$

$$= e^{-2\pi i f(x)/2^n} |x\rangle |f(x)\rangle$$

here, $|f(x)\rangle = |f_x^1 f_x^2 f_x^3 \cdots f_x^n\rangle$

Now, $$e^{-\frac{2\pi i f(x)}{2^n}} |x\rangle |f(x)\rangle \xrightarrow{U_f} e^{-\frac{2\pi i f(x)}{2^n}} |x\rangle |0^{\otimes n}\rangle$$

Thus considering the above circuit be U_{rev}, we have

$$U_{rev} |x\rangle |0^{\otimes n}\rangle = e^{-\frac{2\pi i f(x)}{2^n}} |x\rangle |0^{\otimes n}\rangle$$

effectively implementing the required quantum circuit for $U : |x\rangle \longrightarrow e^{-\frac{2\pi i f(x)}{2^n}} |x\rangle$