Certified Programming

with Isabelle/HOL

Peter Lammich

Virginia Tech / Technische Universität München

2017-10-2

Chapter 1

Introduction

1. General Information

2. About this Course

Cellphones

Put your cellphones into airplane mode!
May interfere with audio

Use the microphones when you ask questions, such that the remote site can also hear you.

About the Instructor

Peter Lammich

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Made my phd in Münster, Germany

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Now in Logic and Verification group in Munich

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Now in Logic and Verification group in Munich
(Where Oktoberfest was invented)

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Made my phd in Münster, Germany
Now in Logic and Verification group in Munich
(Close to some really cool mountains)

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Office hours Mon, 4:00PM – 5:00PM, Durham 352
Email `lpeter1@vt.edu`

General Information

Course Reference Numbers (CRNs)

- **Physical presence on VT Blacksburg campus:**
  - ECE 4984 - Certified Programming: **89530**
  - ECE 5984 - Advanced Certified Programming: **89528**
- **Off-campus online through WebEx:**
  - ECE 5984 - Advanced Certified Programming: **89536**

Website via Canvas `https://canvas.vt.edu/`

Meeting time Mon Wed, 5:30PM-6:45PM, Torg 1050

General Information

Prerequisites

- *4984*: ECE 2574 Intro to DS and Algos
- *5984*: Graduate standing
- *Both levels*: Experience with imperative PL

Laptop Bring in a Laptop with Isabelle2016-1 installed
`http://isabelle.in.tum.de`

Texts

Nipkow, T. and Klein, G. (2014). *Concrete Semantics*. Springer.
`http://www.concrete-semantics.org/`

Nipkow, T. *Programming and Proving in Isabelle/HOL* (`http://isabelle.in.tum.de/dist/Isabelle2016-1/doc/prog-prove.pdf`)

Homeworks, Projects, Exam

Approx. 10 homeworks (all equal weight)
homework submission: Canvas

Graduate Section

- Project (3 or 4 weeks)
- Take-home exam (Dec 15, 7PM, 24h to solve)
- 60% homework, 25% project, 15% exam

Undergraduate Section

- Project (2 weeks)
- In-class exam, Dec 15, 7PM–9PM, Torg 1050. Bring two **handwritten** sheets (legal or smaller).
- 70% homework, 15% project, 15% exam

Bonus points and Grading

Bonus points

- Count on your side, but not for max. points
- awarded for bonus questions

Grading

- Compute final score in range 0–100
- from homework, project, exam (see weights)
- capped at 100 (bonus points)
- Mapping of final score to letter grade:
  Not fixed in advance

Policies

- Submissions after due date are not accepted (except if extraortdinary circumstances exist **and** arrangement with instructor has been made **prior** to due date.)
- You are expected to adhere to VT's honor code `www.honorsystem.vt.edu`.
    - Homework, project: Please discuss your approaches with your fellow students, but do not copy solutions!
    - Exam (also take-home): Solve it completely on your own!
- Special needs (disability, religious, medical/personal/family emergencies) Feel free to contact instructor.

  I will not discuss such things in front of class!

1. General Information

2. About this Course

Certified Programming

With Isabelle/HOL

Content of this Course:

Certified Programming

With Isabelle/HOL

Content of this Course:
Semantics of programming languages

Certified Programming

With Isabelle/HOL

Content of this Course:
Semantics of programming languages
with theorem prover Isabelle/HOL

Why Semantics?

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we do not really know what our programs mean.

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We merely have a good intuition and a warm feeling.

Like the state of mathematics in the 19th century
— before set theory and logic entered the scene.

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- To understand the average accounting program, intuition suffices.
- To write a bug-free accounting program may require more than intuition!
- I assume you have the necessary intuition.
- This course is about "beyond intuition".

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Example:
What does the correctness of a type checker even mean?

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Example:
What does the correctness of a type checker even mean? How is it proved?

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- They provide the worst possible semantics.

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- A compiler gives each individual program a semantics.
- It does not help with reasoning about the PL or individual programs.
- Because compilers are far too complicated.
- They provide the worst possible semantics.
- Moreover: compilers may differ!

The sad facts of life

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- Most compilers have bugs.
- Few languages have a (separate, abstract) semantics.
- If they do, it will be informal (English).

Bugs

- Google “compiler bug”

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- Google "hostile applet"
  Early versions of Java had various security holes.

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  Early versions of Java had various security holes.
  Some of them had to do with an incorrect
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  GI Dissertation Award 2003:
  Gerwin Klein: *Verified Java Bytecode Verification*

Standard ML (SML)

First real language with a mathematical semantics:

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The Definition of Standard ML. 1990.

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Main achievements: LCF (theorem proving)
SML (functional programming)
CCS, pi (concurrency)

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SML semantics hardly used:

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- too difficult to read to answer simple questions quickly

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SML semantics hardly used:

- too difficult to read to answer simple questions quickly
- too much detail to allow reliable informal proof
- not processable beyond LaTeX, not even executable

More sad facts of life

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- Real programming languages *are* complex.
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- Complex designs are error-prone.
- Informal mathematical proofs of complex designs are also error-prone.

A solution

Machine-checked language semantics and proofs

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- Semantics at least type-correct

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The tool:

Proof Assistant (PA)
or
Interactive Theorem Prover (ITP)

Proof Assistants

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- Can prove hard and huge theorems

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- You give the structure of the proof
- The PA checks the correctness of each step
- Can prove hard and huge theorems
- May be time consuming

Terminology

This lecture course:

Formal = machine-checked
Verification = formal correctness proof

Terminology

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Formal = machine-checked
Verification = formal correctness proof

Traditionally:

Formal = mathematical

Two landmark verifications

C compiler

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Competitive with `gcc -O1`

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Xavier Leroy
INRIA Paris
using Coq

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Operating system
microkernel (L4)

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INRIA Paris
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Operating system
microkernel (L4)

Gerwin Klein (& Co)
NICTA Sydney
using Isabelle

A happy fact of life

Programming language researchers
are increasingly using PAs

Why verification pays off

Short term: *The software works!*

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Long term:

Tracking effects of changes by rerunning proofs

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Short term: *The software works!*

Long term:

Tracking effects of changes by rerunning proofs

Incremental changes of the software
typically require only incremental changes of the proofs

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- Comparison of PLs or PL paradigms

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- Hot or trendy PLs
- Comparison of PLs or PL paradigms
- Compilers (although they will be one application)

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- Techniques for the description and analysis of
  - PLs
  - PL tools
  - Programs

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Both informally and formally (PA!)

Our PA: Isabelle/HOL

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Learning to use Isabelle/HOL
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All homeworks require the use of Isabelle/HOL

Overview of course

- Introduction to Isabelle/HOL

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- IMP (assignment and while loops) and its semantics

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- Introduction to Isabelle/HOL
- IMP (assignment and while loops) and its semantics
- A compiler for IMP
- Hoare logic for IMP

The semantics part of the course is mostly traditional

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The use of a PA is leading edge

What you learn in this course goes far beyond PLs

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It has applications in compilers, security, software engineering etc.

How this course works

I will give lectures and hands-on tutorials on Isabelle

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Do not hesitate to aks questions during lectures/tutorials.

There will be homework regularly: Solving (small) problems with Isabelle.

Solving homework is essential for learning Isabelle and surviving this course!

Part I

Isabelle

Chapter 2

Programming and Proving

Quick Introduction to Functional Programming

Isabelle/HOL is based on higher-order logic

Quick Introduction to Functional Programming

Isabelle/HOL is based on higher-order logic

HOL $=$ Logic $+$ Functional Programming

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HOL = Logic + Functional Programming

I assume you are not familiar with functional programming

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Isabelle/HOL is based on higher-order logic

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I'll try to give a very basic introduction of what is needed for Isabelle/HOL

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Lists of elements of any type. $'a$ may be instantiated to any type.

datatype $bintree = Leaf \mid Node\ bintree\ bintree$

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Binary trees (without data)

Datatype Examples

$S\ (S\ (S\ Z))$

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$S\ (S\ (S\ Z))$ The number 3

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$Cons\ a\ (Cons\ b\ (Cons\ c\ Nil))$

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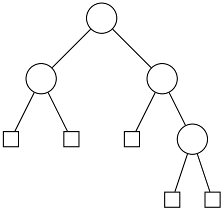

FunProg_Demo.thy

Functions

Recursive functions. No side effects!

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$$\begin{array}{l} \textbf{fun}\ add\ \textbf{where} \\ \quad add\ Z\ m = m \\ \mid\ add\ (S\ n)\ m = S\ (add\ n\ m) \end{array}$$

Functions

Recursive functions. No side effects!

fun add **where**

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fun $appnd$ **where**

$appnd\ Nil\ l = l$

$\mid\ appnd\ (Cons\ x\ l)\ ll = Cons\ x\ (appnd\ l\ ll)$

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Types

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$Z :: nat$

Types

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$S :: nat \Rightarrow nat$

Types

Every term must be typeable

$Z :: nat$

$S :: nat \Rightarrow nat$ — function taking nat and returning nat

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What is the type of $Cons\ (S\ Z)\ Nil$? $nat\ list$

More Types

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So has $(Cons{::}nat \Rightarrow \_ \Rightarrow \_)\ a\ Nil$.

Type Annotations to Functions

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$$\textbf{fun } add :: nat \Rightarrow nat \textbf{ where}$$
$$add\ Z\ m = m$$
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May also restrict inferred type:

$$\textbf{fun } appnd :: nat\ list \Rightarrow nat\ list \Rightarrow nat\ list \textbf{ where}$$
$$appnd\ Nil\ l = l$$
$$|\ appnd\ (Cons\ x\ l)\ ll = Cons\ x\ (appnd\ l\ ll)$$

FunProg_Demo.thy

Standard Library

Standard library with basic types

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nat, int, bool,

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nat, int, bool, $'a \times 'b$, $'a \ list$,

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$42{::}nat$, $-41{::}int$, $(3{,}4)$, $[1{,}2{,}3]$, $1\#2\#3\#Nil$

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and many standard functions (also with syntax)

$5 + 3$, $3{*}3$, $l_1 @ l_2$, ...

List Functions

$$map :: ('a \Rightarrow 'b) \Rightarrow 'a\ list \Rightarrow 'b\ list$$

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$filter :: ('a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow 'a\ list$ — Filter elements of list

$filter\ (\lambda x.\ x < 5)\ [7{,}3{,}4{,}9{,}5 :: int] = [3{,}4]$

FunProg_Demo.thy

Functional Quicksort

Recall: Choose pivot element, partition, sort partitions recursively

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$$\textbf{fun}\ qsort :: int\ list \Rightarrow int\ list\ \textbf{where}$$
$$qsort\ [] = []$$
$$|\ qsort\ (p\#xs) =$$
$$qsort\ (filter\ (\lambda x.\ x{\le}p)\ xs)$$
$$@\ p\ \#\ qsort\ (filter\ (\lambda x.\ x{>}p)\ xs)$$

Functional Quicksort

Recall: Choose pivot element, partition, sort partitions recursively

$$
\begin{aligned}
&\textbf{fun } \mathit{qsort} :: \mathit{int\ list} \Rightarrow \mathit{int\ list} \textbf{ where}\\
&\quad \mathit{qsort}\ [] = []\\
&|\ \mathit{qsort}\ (p\#xs) =\\
&\qquad\quad \mathit{qsort}\ (\mathit{filter}\ (\lambda x.\ x{\le}p)\ xs)\\
&\qquad @\ p\ \#\ \mathit{qsort}\ (\mathit{filter}\ (\lambda x.\ x{>}p)\ xs)
\end{aligned}
$$

$\mathit{qsort}\ [7,3,4,9,5] = [3,4,5,7,9]$

Sorting Algorithm Spec

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$mset\ xs$ — (multiset of) elements in xs

FunProg_Demo.thy

Quiz

Which of the following formulas have the same meaning?

1. $A \implies (B \implies C)$
2. $(A \implies B) \implies C$
3. $(A \land B) \implies C$

Notation

Implication associates to the right:

$$A \Longrightarrow B \Longrightarrow C \quad \text{means} \quad A \Longrightarrow (B \Longrightarrow C)$$

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$$\frac{A_1 \quad \dots \quad A_n}{B} \quad \text{means} \quad A_1 \Longrightarrow \cdots \Longrightarrow A_n \Longrightarrow B$$

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HOL Formulas:

- For the moment: only $term = term$,
  e.g. $1 + 2 = 4$
- Later: $\wedge$, $\vee$, $\longrightarrow$, $\forall$, $\ldots$

4 Overview of Isabelle/HOL

Types and terms

By example: types $bool$, nat and $list$

Summary

Types

Basic syntax:

$$\tau \quad ::=$$

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\begin{array}{rcll}
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This language of terms is known as the λ-*calculus*.

The computation rule of the λ-calculus is the replacement of formal by actual parameters:

$$(\lambda x.\ t)\ u \ = \ t[u/x]$$

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- Isabelle performs β-reduction automatically.

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$$\frac{t :: \tau_1 \Rightarrow \tau_2 \qquad u :: \tau_1}{t\ u :: \tau_2}$$

Type inference

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In the presence of *overloaded* functions (functions with multiple types) this is not always possible.

User can help with *type annotations* inside the term.
Example: $f\ (x{::}nat)$

Currying

Functions in Isabelle usually Curried

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Haskell Brooks Curry (1900–1982)

Currying

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Advantage:

Currying allows *partial application*

$f\ a_1$ where $a_1 :: \tau_1$

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Prefix binds more strongly than infix:

! $f\,x + y \;\equiv\; (f\,x) + y \;\not\equiv\; f\,(x + y)$ **!**

Enclose *if*, *let*, and *case* in parentheses:

! $(\mathit{if}\ \_\ \mathit{then}\ \_\ \mathit{else}\ \_)$ **!**

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Syntax:

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theory MyTh
imports T_1 ... T_n
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Usually: **imports** `Main`

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Types, terms and formulas need to be inclosed in "

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4 Overview of Isabelle/HOL

Types and terms

By example: types $bool$, nat and $list$

Summary

Type $bool$

Again

datatype $bool = True \mid False$

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Predefined functions:

$\wedge, \vee, \longrightarrow, \ldots :: bool \Rightarrow bool \Rightarrow bool$

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A *formula* is a term of type $bool$

E.g. $Suc\ (a + Suc\ b) = a + b + 2$

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if-and-only-if: $\longleftrightarrow$

E.g. $(a \wedge (b \vee c)) = (a \wedge b \vee a \wedge c)$

Type nat

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datatype $nat = 0 \mid Suc\ nat$

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Values of type nat: $0, \;\; Suc\ 0, \;\; Suc(Suc\ 0), \ldots$

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! Numbers and arithmetic operations are overloaded:
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You need type annotations: $1\ ::\ nat,\ x + (y{::}nat)$

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! Numbers and arithmetic operations are overloaded:
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You need type annotations: $1 :: nat$, $x + (y{::}nat)$
unless the context is unambiguous: $Suc\ z$

Nat_Demo.thy

An informal proof

Lemma $add\ m\ 0 = m$

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Proof by induction on m.

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Type $'a\ list$

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Syntactic sugar:

- $[] = Nil$: empty list

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- $[] = Nil$: empty list
- $x\ \#\ xs = Cons\ x\ xs$:
  list with first element x (*"head"*) and rest xs (*"tail"*)

Type $'a\ list$

Again

Lists of elements of type $'a$

datatype $'a\ list\ =\ Nil\ \mid\ Cons\ 'a\ \ ('a\ list)$

Some lists: Nil, $Cons\ 1\ Nil$, $Cons\ 1\ (Cons\ 2\ Nil)$, ...

Syntactic sugar:

- $[] = Nil$: empty list
- $x\ \#\ xs = Cons\ x\ xs$:
  list with first element x (*"head"*) and rest xs (*"tail"*)
- $[x_1, \ldots, x_n] = x_1\ \#\ \ldots\ x_n\ \#\ []$

Structural Induction for lists

Given formula $P :: 'a\ list \Rightarrow bool$ over lists.
To prove that $P(xs)$ for all lists xs, prove

- $P([])$ and
- for arbitrary but fixed x and xs,
  $P(xs)$ implies $P(x\#xs)$.

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- for arbitrary but fixed x and xs,
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$$\frac{P([]) \qquad \bigwedge x\ xs.\ P(xs) \Longrightarrow P(x\#xs)}{P(xs)}$$

List_Demo.thy

An informal proof

Lemma $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$

Proof by induction on xs.

- Case Nil: $app\ (app\ Nil\ ys)\ zs = app\ ys\ zs = app\ Nil\ (app\ ys\ zs)$ holds by definition of app.
- Case $Cons\ x\ xs$: We assume $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$ (IH), and we need to show $app\ (app\ (Cons\ x\ xs)\ ys)\ zs = app\ (Cons\ x\ xs)\ (app\ ys\ zs)$.

  The proof is as follows:

  $$\begin{array}{lll} & app\ (app\ (Cons\ x\ xs)\ ys)\ zs & \\ = & Cons\ x\ (app\ (app\ xs\ ys)\ zs) & \text{by definition of } app \\ = & Cons\ x\ (app\ xs\ (app\ ys\ zs)) & \text{by IH} \\ = & app\ (Cons\ x\ xs)\ (app\ ys\ zs) & \text{by definition of } app \end{array}$$

Large library: `HOL/List.thy`

Included in `Main`.

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Predefined: $xs\ @\ ys$ (append), $length$, and map

$$map\ f\ [x_1,\ \ldots,\ x_n] = [f\ x_1,\ \ldots,\ f\ x_n]$$

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Predefined: $xs\ @\ ys$ (append), $length$, and map

$$map\ f\ [x_1, \ldots, x_n] = [f\ x_1, \ldots, f\ x_n]$$

fun $map :: ('a \Rightarrow 'b) \Rightarrow 'a\ list \Rightarrow 'b\ list$ **where**
$map\ f\ [] = []\ |$
$map\ f\ (x\ \#\ xs) = f\ x\ \#\ map\ f\ xs$

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Note: map takes *function* as argument.

- **datatype** defines (possibly) recursive data types.

- **fun** defines (possibly) recursive functions by pattern-matching over datatype constructors.

Proof methods

- $induction$ performs structural induction on some variable (if the type of the variable is a datatype).

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- *auto* solves as many subgoals as it can, mainly by simplification (symbolic evaluation):

  "$=$" is used only from left to right!

Proofs

General schema:

lemma $name$: "..."
apply (...)
apply (...)
$\vdots$
done

Proofs

General schema:

lemma *name*: `"..."`
apply `(...)`
apply `(...)`
⋮
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If the lemma is suitable as a simplification rule:

lemma *name*`[simp]:  "..."`

Top down proofs

Command

sorry

"completes" any proof.

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Allows top down development:

Assume lemma first, prove it later.

The proof state

1. $\bigwedge x_1 \ \ldots \ x_p. \ \ A \Longrightarrow B$

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B	actual (sub)goal

Multiple assumptions

$$[\![A_1; \dots ; A_n]\!] \Longrightarrow B$$

abbreviates

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$;$ $\approx$ "and"

5 Type and function definitions

Type definitions

Function definitions

Type synonyms

type_synonym $name = \tau$

Introduces a *synonym* $name$ for type τ

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Introduces a *synonym* $name$ for type τ

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Type synonyms are expanded after parsing
and are not present in internal representation and output

datatype — the general case

$$
\begin{array}{lcl}
\textbf{datatype}\ (\alpha_1, \ldots, \alpha_n)t & = & C_1\ \tau_{1,1} \ldots \tau_{1,n_1} \\
& | & \ldots \\
& | & C_k\ \tau_{k,1} \ldots \tau_{k,n_k}
\end{array}
$$

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- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)t$

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Distinctness and injectivity are applied automatically
Induction must be applied explicitly

Case expressions

Datatype values can be taken apart with $case$:

$$(case\ xs\ of\ \ [] \Rightarrow \ldots\ \ \mid\ \ y\#ys \Rightarrow \ \ldots\ y\ \ldots\ ys\ \ldots)$$

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Complicated patterns mean complicated proofs!

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Need ( ) in context

Tree_Demo.thy

The *option* type

datatype $'a\ option = None \mid Some\ 'a$

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If $'a$ has values a_1, a_2, ...
then $'a\ option$ has values $None$, $Some\ a_1$, $Some\ a_2$, ...

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Typical application:

fun $lookup :: ('a \times\ 'b)\ list \Rightarrow\ 'a \Rightarrow\ 'b\ option$ **where**

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fun $lookup :: ('a \times\ 'b)\ list \Rightarrow\ 'a \Rightarrow\ 'b\ option$ **where**
$lookup\ []\ x = None \mid$
$lookup\ ((a,b)\ \#\ ps)\ x =$
$\quad (if\ a = x\ then\ Some\ b\ else\ lookup\ ps\ x)$

5 Type and function definitions

Type definitions

Function definitions

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n{*}n$

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n \ = \ n{*}n$

No pattern matching, just $f\ x_1\ \ldots\ x_n \ = \ \ldots$

The danger of nontermination

How about $f\,x = f\,x + 1$?

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Subtract $f\,x$ on both sides.

$$\Longrightarrow 0 = 1$$

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! All functions in HOL must be total **!**

Key features of **fun**

- Pattern-matching over datatype constructors

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- Order of equations matters
- Termination must be provable automatically by size measures
- Proves customized induction schema

Example: separation

fun $sep :: {'a} \Rightarrow {'a}\ list \Rightarrow {'a}\ list$ **where**
$sep\ \ a\ (x\#y\#zs) = x\ \#\ \ a\ \#\ \ sep\ \ a\ (y\#zs)\ \ \ |$
$sep\ \ a\ xs = xs$

Example: Ackermann

$$
\begin{array}{l}
\textbf{fun } ack :: nat \Rightarrow nat \Rightarrow nat \textbf{ where} \\
ack\ 0 \qquad\qquad\quad n \qquad\qquad = Suc\ n \ \mid \\
ack\ (Suc\ m)\ 0 \qquad\quad\ \ = ack\ m\ (Suc\ 0) \ \mid \\
ack\ (Suc\ m)\ (Suc\ n) = ack\ m\ (ack\ (Suc\ m)\ n)
\end{array}
$$

Example: Ackermann

fun $ack :: nat \Rightarrow nat \Rightarrow nat$ **where**
$ack\ 0\ n = Suc\ n\ \ |$
$ack\ (Suc\ m)\ 0 = ack\ m\ (Suc\ 0)\ \ |$
$ack\ (Suc\ m)\ (Suc\ n) = ack\ m\ (ack\ (Suc\ m)\ n)$

Terminates because the arguments decrease *lexicographically* with each recursive call:

- $(Suc\ m,\ 0) > (m,\ Suc\ 0)$
- $(Suc\ m,\ Suc\ n) > (Suc\ m,\ n)$
- $(Suc\ m,\ Suc\ n) > (m,\ \_)$

Basic induction heuristics

Theorems about recursive functions
are proved by induction

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Induction on argument number i of f
if f is defined by recursion on argument number i

A tail recursive reverse

Our initial reverse:

fun $rev :: {}'a\ list \Rightarrow {}'a\ list$ **where**
$rev\ [] \quad = [] \quad |$
$rev\ (x \# xs) \quad = rev\ xs\ @\ [x]$

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lemma $itrev\ xs\ [] = rev\ xs$

Induction_Demo.thy

Generalisation

Generalisation

- Replace constants by variables

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- Replace constants by variables
- Generalize free variables
  - by $arbitrary$ in induction proof
  - (or by universal quantifier in formula)

So far, all proofs were by structural induction

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In each induction step, 1 constructor is added.
In each recursive call, 1 constructor is removed.

Now: induction for complex recursion patterns.

Computation Induction

Example

fun $div2 :: nat \Rightarrow nat$ **where**
$div2\ 0 = 0 \ \ |$
$div2\ (Suc\ 0) = 0 \ \ |$
$div2\ (Suc(Suc\ n)) = Suc(div2\ n)$

Computation Induction

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$\leadsto$ induction rule `div2.induct`:

$$\frac{P(0) \quad P(Suc\ 0) \qquad\qquad P(n) \Longrightarrow P(Suc(Suc\ n))}{P(m)}$$

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prove $P(e)$ *assuming* $P(r_1)$, $\ldots$, $P(r_k)$.

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Computation Induction

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prove $P(e)$ *assuming* $P(r_1)$, $\ldots$, $P(r_k)$.

Induction follows course of (terminating!) computation
Motto: properties of f are best proved by rule $f.induct$

How to apply $f.induct$

If $f :: \tau_1 \Rightarrow \cdots \Rightarrow \tau_n \Rightarrow \tau'$:

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How to apply $f.induct$

If $f :: \tau_1 \Rightarrow \cdots \Rightarrow \tau_n \Rightarrow \tau'$:

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Heuristic:

- there should be a call $f\ a_1\ \ldots\ a_n$ in your goal

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If $f :: \tau_1 \Rightarrow \cdots \Rightarrow \tau_n \Rightarrow \tau'$:

$$(\textit{induction } a_1 \ \ldots \ a_n \textit{ rule: f.induct})$$

Heuristic:

- there should be a call $f\, a_1 \ \ldots \ a_n$ in your goal
- ideally the a_i should be variables.

Induction_Demo.thy

Computation Induction

Auxiliary Lemmas

Sometimes one gets stuck in induction proof

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But obviously, the goal should be provable

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But obviously, the goal should be provable

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Identifying such situations and coming up with good auxiliary lemma requires some practice!

Induction_Demo.thy

Generalisation

Simplification means . . .

Using equations $l = r$ from left to right

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As long as possible

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Using equations $l = r$ from left to right

As long as possible

Terminology: equation $\rightsquigarrow$ *simplification rule*

Simplification means . . .

Using equations $l = r$ from left to right

As long as possible

Terminology: equation $\leadsto$ *simplification rule*

Simplification = (Term) Rewriting

An example

Equations:

$$
\begin{aligned}
0 + n &= n & (1)\\
(Suc\ m) + n &= Suc\ (m + n) & (2)\\
(Suc\ m \leq Suc\ n) &= (m \leq n) & (3)\\
(0 \leq m) &= True & (4)
\end{aligned}
$$

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$$0 + Suc\ 0 \ \leq\ Suc\ 0 + x$$

Rewriting:

An example

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$$
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0 + n &= n & (1)\\
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(Suc\ m \leq Suc\ n) &= (m \leq n) & (3)\\
(0 \leq m) &= True & (4)
\end{aligned}
$$

Rewriting:

$$
\begin{aligned}
0 + Suc\ 0 &\leq Suc\ 0 + x & \stackrel{(1)}{=}\\
Suc\ 0 &\leq Suc\ 0 + x & \stackrel{(2)}{=}\\
Suc\ 0 &\leq Suc\ (0 + x) & \stackrel{(3)}{=}\\
0 &\leq 0 + x &
\end{aligned}
$$

An example

Equations:

$$\begin{aligned} 0+n &= n & (1)\\ (Suc\ m)+n &= Suc\ (m+n) & (2)\\ (Suc\ m \le Suc\ n) &= (m \le n) & (3)\\ (0 \le m) &= True & (4) \end{aligned}$$

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Conditional rewriting

Simplification rules can be conditional:

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Example

$$\begin{array}{rcl} p(0) & = & True \\ p(x) \Longrightarrow \; f(x) & = & g(x) \end{array}$$

We can simplify $f(0)$ to $g(0)$ but we cannot simplify $f(1)$ because $p(1)$ is not provable.

Termination

Simplification may not terminate.
Isabelle uses $simp$-rules (almost) blindly from left to right.

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$$n < m \Longrightarrow (n < Suc\ m) = True \quad \text{YES}$$
$$Suc\ n < m \Longrightarrow (n < m) = True \quad \text{NO}$$

Proof method *simp*

Goal: 1. $\llbracket\ P_1; \ldots; P_m\ \rrbracket \Longrightarrow C$

apply$(simp\ add\colon\ eq_1\ \ldots\ \ eq_n)$

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- rules from **fun** and **datatype**

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Variations:

- $(simp \ \ldots \ del\colon \ \ldots)$ removes $simp$-lemmas
- add and del are optional

auto versus *simp*

- *auto* acts on all subgoals
- *simp* acts only on subgoal 1

$auto$ versus $simp$

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- $auto$ can also be modified:
  $(auto\ simp\ add\colon \ldots\ simp\ del\colon \ldots)$

Rewriting with definitions

Definitions (**definition**) must be used explicitly:

$$(simp\ add:\ f\_def\ \ldots)$$

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f is the function whose definition is to be unfolded.

Case splitting with *simp/auto*

Automatic:

$$P(\mathit{if}\ A\ \mathit{then}\ s\ \mathit{else}\ t)$$
$$=$$
$$(A \longrightarrow P(s)) \wedge (\neg A \longrightarrow P(t))$$

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By hand:

$$P(\mathit{case}\ e\ \mathit{of}\ 0 \Rightarrow a \mid \mathit{Suc}\ n \Rightarrow b)$$
$$=$$
$$(e = 0 \longrightarrow P(a)) \wedge (\forall\, n.\ e = \mathit{Suc}\ n \longrightarrow P(b))$$

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Proof method: $(\mathit{simp}\ \mathit{split}\!: \mathit{nat.split})$

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Case splitting with *simp/auto*

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Proof method: *(simp split: nat.split)*
Or *auto*. Similar for any datatype t: *t.split*

Simp_Demo.thy

This section introduces

arithmetic and boolean expressions

of our imperative language IMP.

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of our imperative language IMP.

IMP *commands* are introduced later.

8 Case Study: IMP Expressions

- Arithmetic Expressions
- Boolean Expressions
- Stack Machine and Compilation

Concrete and abstract syntax

Concrete syntax: strings, eg `"a+5*b"`

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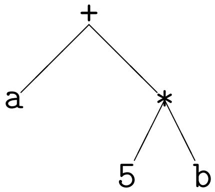
Abstract syntax: trees, eg



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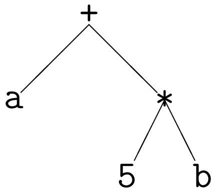

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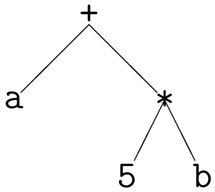

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Linear view of trees: terms, eg $Plus\ a\ (Times\ 5\ b)$

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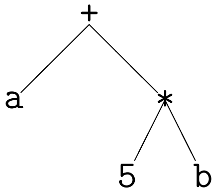

Parser: function from strings to trees

Linear view of trees: terms, eg $Plus\ a\ (Times\ 5\ b)$

Abstract syntax trees/terms are datatype values!

Concrete syntax is defined by a context-free grammar, eg

$$a ::= n \mid x \mid (a) \mid a + a \mid a * a \mid \ldots$$

where n can be any natural number and x any variable.

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We focus on *abstract* syntax
which we introduce via datatypes.

Datatype $aexp$

Variable names are strings, values are integers:

type_synonym $vname = string$
datatype $aexp = N\ int \mid V\ vname \mid Plus\ aexp\ aexp$

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Concrete	Abstract
5	$N\ 5$
`x`	$V\ ''x''$
`x+y`	$Plus\ (V\ ''x'')\ (V\ ''y'')$
`2+(z+3)`	$Plus\ (N\ 2)\ (Plus\ (V\ ''z'')\ (N\ 3))$

Warning

This is syntax, not (yet) semantics!

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$$N\,0 \;\neq\; Plus\;(N\,0)\;(N\,0)$$

The (program) state

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- The value of an expression
  depends on the value of its variables.
- The value of all variables is recorded in the *state*.
- The state is a function from variable names to values:

  type_synonym $val = int$
  type_synonym $state = vname \Rightarrow val$

Function update notation

If $f :: \tau_1 \Rightarrow \tau_2$ and $a :: \tau_1$ and $b :: \tau_2$ then

$$f(a := b)$$

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is the function that behaves like f
except that it returns b for argument a.

$$f(a := b) = (\lambda x.\ \mathit{if}\ x = a\ \mathit{then}\ b\ \mathit{else}\ f\ x)$$

How to write down a state

Some states:

- $\lambda x.\ 0$

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Nicer notation:

$$<''a'' := 5,\ ''x'' := 3,\ ''y'' := 7>$$

How to write down a state

Some states:

- $\lambda x.\ 0$
- $(\lambda x.\ 0)(''a'' := 3)$
- $((\lambda x.\ 0)(''a'' := 5))(''x'' := 3)$

Nicer notation:

$$< ''a'' := 5,\ ''x'' := 3,\ ''y'' := 7 >$$

Maps everything to 0, but $''a''$ to 5, $''x''$ to 3, etc.

AExp.thy

8 Case Study: IMP Expressions

BExp.thy

8 Case Study: IMP Expressions

- Arithmetic Expressions
- Boolean Expressions
- Stack Machine and Compilation

ASM.thy

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Because evaluation of expressions always terminates.
But execution of programs may *not* terminate.
Hence we cannot define it by a total recursive function.

We need more logical machinery
to define program execution and reason about it.

Chapter 3

Logic and Proof Beyond Equality

Syntax (in decreasing precedence):

$$
\begin{array}{lclclcl}
\mathit{form} & ::= & (\mathit{form}) & \mid & \mathit{term} = \mathit{term} & \mid & \neg \mathit{form} \\
& \mid & \mathit{form} \wedge \mathit{form} & \mid & \mathit{form} \vee \mathit{form} & \mid & \mathit{form} \longrightarrow \mathit{form} \\
& \mid & \forall x.\ \mathit{form} & \mid & \exists x.\ \mathit{form} & &
\end{array}
$$

Syntax (in decreasing precedence):

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Examples:

$$\neg\ A \wedge B \vee C \quad \equiv \quad ((\neg\ A) \wedge B) \vee C$$

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Examples:

$$
\begin{array}{rcl}
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s =\ t \wedge\ C & \equiv & (s =\ t) \wedge\ C
\end{array}
$$

Syntax (in decreasing precedence):

$$\begin{array}{rcl|l|l} form & ::= & (form) & term = term & \neg form \\ & | & form \wedge form & form \vee form & form \longrightarrow form \\ & | & \forall x.\ form & \exists x.\ form & \end{array}$$

Examples:

$$\begin{array}{rcl} \neg\ A\ \wedge\ B \vee\ C & \equiv & ((\neg\ A)\ \wedge\ B)\ \vee\ C \\ s = t \wedge\ C & \equiv & (s = t)\ \wedge\ C \\ A\ \wedge\ B = B\ \wedge\ A & \equiv & A\ \wedge\ (B = B)\ \wedge\ A \end{array}$$

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Examples:

$$
\begin{array}{rcl}
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s = t \wedge C & \equiv & (s = t) \wedge C \\
A \wedge B = B \wedge A & \equiv & \color{red}{A \wedge (B = B) \wedge A} \\
\forall\, x.\ P\ x \wedge Q\ x & \equiv & \forall\, x.\ (P\ x \wedge Q\ x)
\end{array}
$$

Syntax (in decreasing precedence):

$$\begin{array}{rclclcl} \mathit{form} & ::= & (\mathit{form}) & \mid & \mathit{term} = \mathit{term} & \mid & \neg \mathit{form} \\ & \mid & \mathit{form} \wedge \mathit{form} & \mid & \mathit{form} \vee \mathit{form} & \mid & \mathit{form} \longrightarrow \mathit{form} \\ & \mid & \forall x.\ \mathit{form} & \mid & \exists x.\ \mathit{form} & & \end{array}$$

Examples:

$$\begin{array}{rcl} \neg\ A \wedge B \vee C & \equiv & ((\neg\ A) \wedge B) \vee C \\ s = t \wedge C & \equiv & (s = t) \wedge C \\ A \wedge B = B \wedge A & \equiv & \color{red}{A \wedge (B = B) \wedge A} \\ \forall\, x.\ P\ x \wedge Q\ x & \equiv & \forall\, x.\ (P\ x \wedge Q\ x) \end{array}$$

Input syntax: $\longleftrightarrow$ (same precedence as $\longrightarrow$)

Variable binding convention:

$$\forall\, x\ y.\ P\ x\ y\ \equiv\ \forall\, x.\ \forall\, y.\ P\ x\ y$$

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Similarly for $\exists$ and λ.

Warning

Quantifiers have low precedence
and need to be parenthesized (if in some context)

$$! \quad P \wedge \forall x.\ Q\ x \ \rightsquigarrow\ P \wedge (\forall x.\ Q\ x) \quad !$$

Mathematical symbols

... and their shortcuts:

<table>
<tr><td>$\forall$</td><td>\<forall></td><td>ALL</td><td>!</td></tr>
<tr><td>$\exists$</td><td>\<exists></td><td>EX</td><td>?</td></tr>
<tr><td>λ</td><td>\<lambda></td><td>%</td><td></td></tr>
<tr><td>$\longrightarrow$</td><td>\<longrightarrow></td><td>--></td><td></td></tr>
<tr><td>$\longleftrightarrow$</td><td>\<longleftrightarrow></td><td><--></td><td></td></tr>
<tr><td>$\wedge$</td><td>\<and></td><td>/\</td><td>&</td></tr>
<tr><td>$\vee$</td><td>\<or></td><td>\/</td><td>|</td></tr>
<tr><td>$\neg$</td><td>\<not></td><td>~</td><td></td></tr>
<tr><td>$\neq$</td><td>\<noteq></td><td>~=</td><td></td></tr>
</table>

Sets over type $'a$

$'a\ set$

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- $\{\}, \quad \{e_1, \ldots, e_n\}$

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- $e \in A$, $A \subseteq B$

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- $A \cup B$, $A \cap B$, $A - B$, $-\ A$

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- $\{\}$, $\{e_1, \ldots, e_n\}$
- $e \in A$, $A \subseteq B$
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$\in$	`\<in>`	`:`
$\subseteq$	`\<subseteq>`	`(=`
$\cup$	`\<union>`	`Un`
$\cap$	`\<inter>`	`Int`

Set comprehension

- $\{x.\ P\}$ where x is a variable

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- Instead: $\{t \mid x\ y\ z.\ P\}$

Set comprehension

- $\{x.\ P\}$ where x is a variable
- But not $\{t.\ P\}$ where t is a proper term
- Instead: $\{t\ | x\ y\ z.\ P\}$
  is short for $\{v.\ \exists\, x\ y\ z.\ v = t \wedge P\}$
  where x, y, z are the free variables in t

$simp$ and $auto$

$simp$: rewriting and a bit of arithmetic

$auto$: rewriting and a bit of arithmetic, logic and sets

simp and *auto*

simp: rewriting and a bit of arithmetic

auto: rewriting and a bit of arithmetic, logic and sets

- Show you where they got stuck

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- highly incomplete

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- Extensible with new $simp$-rules

simp and *auto*

simp: rewriting and a bit of arithmetic

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- Show you where they got stuck
- highly incomplete
- Extensible with new *simp*-rules

Exception: *auto* acts on all subgoals

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.

fastforce

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Automating arithmetic

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- complete for first-order theory of nat and int (Presburger arithmetic)

Sledgehammer

Architecture:

Isabelle

external
ATPs[1]

[1] Automatic Theorem Provers

Architecture:

$$\begin{array}{rc} & \textbf{Isabelle} \\ \text{Goal} & \\ \text{\& filtered library} & \downarrow \\ & \text{external} \\ & \textbf{ATPs}^{1} \end{array}$$

---

[1] Automatic Theorem Provers

Architecture:

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Characteristics:

- Sometimes it works,

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Do you feel lucky?

[1] Automatic Theorem Provers

by($proof\text{-}method$)

$$\approx$$

apply($proof\text{-}method$)
done

Auto_Proof_Demo.thy

Step-by-step proofs can be necessary if automation fails and you have to explore where and why it failed by taking the goal apart.

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  unifying $?P \wedge ?Q$ with $a{=}b \wedge False$
  sets $?P$ to $a{=}b$ and $?Q$ to $False$.

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Example: rule: $[\![?P;\ ?Q]\!] \Longrightarrow ?P \wedge ?Q$

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"Backchaining"

Typical backwards rules

$$\frac{?P \quad ?Q}{?P \wedge\ ?Q}\ \texttt{conjI}$$

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They are known as introduction rules
because they *introduce* a particular connective.

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Can greatly increase the search space!

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$$\leadsto$$

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From now on: $?$ mostly suppressed on slides

Single_Step_Demo.thy

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$\Longrightarrow$ is part of the Isabelle framework. It structures theorems and proof states: $[\![\ A_1;\ \ldots;\ A_n\ ]\!] \Longrightarrow A$

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Phrase theorems like this	$[\![\ A_1;\ \ldots;\ A_n\ ]\!] \Longrightarrow A$
not like this	$A_1 \wedge \ldots \wedge A_n \longrightarrow A$

9 Logical Formulas

10 Proof Automation

11 Single Step Proofs

12 Inductive Definitions

Example: even numbers

Informally:

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inductive $ev :: nat \Rightarrow bool$
where
$ev\ 0 \quad |$
$ev\ n \Longrightarrow ev\ (n + 2)$

An easy proof: $ev\ 4$

$$ev\ 0 \Longrightarrow ev\ 2 \Longrightarrow ev\ 4$$

Consider

fun $evn :: nat \Rightarrow bool$ **where**
$evn\ 0 = True \mid$
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  $\Longrightarrow evn\ m = evn\ (n{+}2) = evn\ n = True$

Rule induction for ev

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by *rule induction* on $ev\ n$ we must prove

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Rule `ev.induct`:

$$\frac{ev\ n \quad P\ 0 \quad \bigwedge n.\ [\![\ ev\ n;\ P\ n\ ]\!] \Longrightarrow P(n{+}2)}{P\ n}$$

Format of inductive definitions

inductive $I :: \tau \Rightarrow bool$ **where**

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- I may have multiple arguments.

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- Each rule may also contain *side conditions* not involving I.

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! Rule induction is absolutely central to (operational) semantics and the rest of this lecture course **!**

Inductive_Demo.thy

Inductively defined sets

inductive_set $I :: \tau\ set$ **where**

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Difference to **inductive**:

- arguments of I are tupled, not curried

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Difference to **inductive**:

- arguments of I are tupled, not curried
- I can later be used with set theoretic operators, eg $I \cup \ldots$

Chapter 4

Isar: A Language for Structured Proofs

13 Isar by example

14 Proof patterns

15 Streamlining Proofs

16 Proof by Cases and Induction

Apply scripts

- unreadable

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- unreadable
- hard to maintain

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No structure!

Apply scripts versus Isar proofs

Apply script = assembly language program

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with assertions

Apply scripts versus Isar proofs

Apply script = assembly language program

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But: **apply** still useful for proof exploration

A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** $simp$
  $\vdots$
  have $formula_n$ **by** $blast$
  show $formula_{n+1}$ **by** $\ldots$
qed

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qed

proves $formula_0 \Longrightarrow formula_{n+1}$

Isar core syntax

$$\begin{array}{lcl} \text{proof} & = & \textbf{proof}\ [\text{method}]\ \ \text{step}^*\ \ \textbf{qed} \\ & | & \textbf{by}\ \text{method} \end{array}$$

Isar core syntax

proof = **proof** [method] step* **qed**
| **by** method

method = $(simp \ldots) \mid (blast \ldots) \mid (induction \ldots) \mid \ldots$

Isar core syntax

$$
\begin{array}{lcl}
\text{proof} & = & \textbf{proof}\ [\text{method}]\ \ \text{step}^*\ \ \textbf{qed} \\
 & \mid & \textbf{by}\ \text{method}
\end{array}
$$

$$
\text{method} \ = \ (\mathit{simp}\ \ldots) \mid (\mathit{blast}\ \ldots) \mid (\mathit{induction}\ \ldots) \mid \ldots
$$

$$
\begin{array}{lcll}
\text{step} & = & \textbf{fix}\ \text{variables} & (\bigwedge) \\
 & \mid & \textbf{assume}\ \text{prop} & (\Longrightarrow) \\
 & \mid & [\textbf{from}\ \text{fact}^+]\ \ (\textbf{have} \mid \textbf{show})\ \text{prop}\ \ \text{proof} &
\end{array}
$$

Isar core syntax

proof = **proof** [method] step* **qed**
| **by** method

method = $(simp\ \ldots) \mid (blast\ \ldots) \mid (induction\ \ldots) \mid \ldots$

step = **fix** variables ($\bigwedge$)
| **assume** prop ($\Longrightarrow$)
| [**from** fact$^+$] (**have** | **show**) prop proof

prop = [name:] "formula"

Isar core syntax

$$\begin{array}{lcl}
\text{proof} & = & \textbf{proof}\ [\text{method}]\ \ \text{step}^*\ \ \textbf{qed} \\
 & | & \textbf{by}\ \text{method}
\end{array}$$

$$\text{method} \ = \ (\mathit{simp}\ \ldots) \mid (\mathit{blast}\ \ldots) \mid (\mathit{induction}\ \ldots) \mid \ldots$$

$$\begin{array}{lcll}
\text{step} & = & \textbf{fix}\ \text{variables} & (\bigwedge) \\
 & | & \textbf{assume}\ \text{prop} & (\Longrightarrow) \\
 & | & [\textbf{from}\ \text{fact}^+]\ \ (\textbf{have} \mid \textbf{show})\ \text{prop}\ \ \text{proof} &
\end{array}$$

$$\text{prop} \ = \ [\text{name:}]\ \text{"formula"}$$

$$\text{fact} \ = \ \text{name} \mid \ldots$$

Example: Cantor's theorem

lemma $\neg\ surj(f :: {'a} \Rightarrow {'a}\ set)$

Example: Cantor's theorem

lemma $\neg\ surj(f :: \ 'a \Rightarrow \ 'a\ set)$
proof

Example: Cantor's theorem

lemma $\neg\ surj(f :: {}'a \Rightarrow {}'a\ set)$
proof default proof: assume *surj*, show *False*

Example: Cantor's theorem

lemma $\neg\ surj(f :: {'a} \Rightarrow {'a}\ set)$
proof default proof: assume *surj*, show *False*
 assume a: $surj\ f$

Example: Cantor's theorem

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
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assume a: $surj\ f$
from a **have** b: $\forall\ A.\ \exists\ a.\ A = f\ a$

Example: Cantor's theorem

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proof default proof: assume *surj*, show *False*
  assume a: $surj\ f$
  from a **have** b: $\forall\ A.\ \exists\ a.\ A = f\ a$
    by$(simp\ add\!: surj\_def)$

Example: Cantor's theorem

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof default proof: assume *surj*, show *False*
  assume a: $surj\ f$
  from a **have** b: $\forall\ A.\ \exists\ a.\ A = f\ a$
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  from b **have** c: $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$

Example: Cantor's theorem

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof default proof: assume *surj*, show *False*
  assume *a*: $surj\ f$
  from *a* **have** *b*: $\forall\ A.\ \exists\ a.\ A = f\ a$
    by$(simp\ add{:}\ surj\_def)$
  from *b* **have** *c*: $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$
    by *blast*

Example: Cantor's theorem

lemma $\neg$ $surj(f :: {'a} \Rightarrow {'a}\ set)$
proof default proof: assume *surj*, show *False*
  assume a: $surj\ f$
  from a **have** b: $\forall\ A.\ \exists\ a.\ A = f\ a$
    by$(simp\ add{:}\ surj\_def)$
  from b **have** c: $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$
    by $blast$
  from c **show** $False$

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Example: Cantor's theorem

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof default proof: assume *surj*, show *False*
  assume a: $surj\ f$
  from a **have** b: $\forall\ A.\ \exists\ a.\ A = f\ a$
    by$(simp\ add$: $surj\_def)$
  from b **have** c: $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$
    by $blast$
  from c **show** $False$
    by $blast$
qed

Isar_Demo.thy

Cantor and abbreviations

Abbreviations

$this$	$=$	the previous proposition proved or assumed
then	$=$	**from** $this$
thus	$=$	**then show**
hence	$=$	**then have**

using and **with**

(**have**|**show**) prop **using** facts

using and **with**

(**have**|**show**) prop **using** facts

=

from facts (**have**|**show**) prop

using and **with**

(**have**|**show**) prop **using** facts

=

from facts (**have**|**show**) prop

with facts

=

from facts *this*

Structured lemma statement

lemma
  fixes $f :: 'a \Rightarrow 'a\ set$
  assumes s: $surj\ f$
  shows $False$

Structured lemma statement

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  fixes $f :: 'a \Rightarrow 'a\ set$
  assumes s: $surj\ f$
  shows $False$
proof $-$

Structured lemma statement

lemma
  fixes $f :: {}'a \Rightarrow {}'a\ set$
  assumes s: $surj\ f$
  shows $False$
proof $-$ no automatic proof step

Structured lemma statement

lemma
  fixes $f :: 'a \Rightarrow 'a\ set$
  assumes s: $surj\ f$
  shows $False$
proof $-$ no automatic proof step
  have $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$ **using** s
    by$(auto\ simp: surj\_def)$

Structured lemma statement

lemma
fixes $f :: 'a \Rightarrow 'a\ set$
assumes s: $surj\ f$
shows $False$
proof $-$ no automatic proof step
have $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$ **using** s
by$(auto\ simp:\ surj\_def)$
thus $False$ **by** $blast$
qed

Structured lemma statement

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  thus $False$ **by** $blast$
qed

Proves $surj\ f \Longrightarrow False$

Structured lemma statement

lemma
  fixes $f :: 'a \Rightarrow 'a\ set$
  assumes s: $surj\ f$
  shows $False$
proof $-$ no automatic proof step
  have $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$ **using** s
    by$(auto\ simp:\ surj\_def)$
  thus $False$ **by** $blast$
qed

Proves $surj\ f \Longrightarrow False$
but $surj\ f$ *becomes local fact* s *in proof.*

The essence of structured proofs

Assumptions and intermediate facts
can be named and referred to explicitly and selectively

Structured lemma statements

fixes $x :: \tau_1$ **and** $y :: \tau_2 \ldots$
assumes *a:* P **and** *b:* $Q \ldots$
shows R

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- **fixes** and **assumes** sections optional

Structured lemma statements

fixes $x :: \tau_1$ **and** $y :: \tau_2 \ \ldots$
assumes *a:* P **and** *b:* $Q \ \ldots$
shows R

- **fixes** and **assumes** sections optional
- **shows** optional if no **fixes** and **assumes**

Case distinction

show R
proof $cases$
  assume P
  $\vdots$
  show R $\langle proof \rangle$
next
  assume $\neg\ P$
  $\vdots$
  show R $\langle proof \rangle$
qed

Case distinction

show R
proof $cases$
  assume P
  $\vdots$
  show R $\langle proof \rangle$
next
  assume $\neg P$
  $\vdots$
  show R $\langle proof \rangle$
qed

have $P \vee Q$ $\langle proof \rangle$
then show R
proof
  assume P
  $\vdots$
  show R $\langle proof \rangle$
next
  assume Q
  $\vdots$
  show R $\langle proof \rangle$
qed

Contradiction

show $\neg\ P$
proof
  assume P
  $\vdots$
  show $False\ \langle proof \rangle$
qed

Contradiction

show $\neg\ P$
proof
  assume P
  $\vdots$
  show $False\ \langle proof \rangle$
qed

show P
proof $(rule\ ccontr)$
  assume $\neg P$
  $\vdots$
  show $False\ \langle proof \rangle$
qed

$\longleftrightarrow$

show $P \longleftrightarrow Q$
proof
  assume P
  $\vdots$
  show Q $\langle proof \rangle$
next
  assume Q
  $\vdots$
  show P $\langle proof \rangle$
qed

$\forall$ and $\exists$ introduction

show $\forall\, x.\ P(x)$
proof
  fix x local fixed variable
  show $P(x)$ $\langle proof \rangle$
qed

$\forall$ and $\exists$ introduction

show $\forall x.\ P(x)$
proof
  fix x local fixed variable
  show $P(x)$ $\langle proof \rangle$
qed

show $\exists x.\ P(x)$
proof
  $\vdots$
  show $P(witness)$ $\langle proof \rangle$
qed

$\exists$ elimination: **obtain**

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have $\exists\, x.\ P(x)$
then obtain x **where** *p:* $P(x)$ **by** *blast*
$\vdots$ $\quad x$ fixed local variable

$\exists$ elimination: **obtain**

have $\exists\, x.\ P(x)$
then obtain x **where** *p:* $P(x)$ **by** *blast*
$\vdots$ $\quad x$ fixed local variable

Works for one or more x

obtain example

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof
assume $surj\ f$
hence $\exists\, a.\ \{x.\ x \notin f\ x\} = f\ a$ **by**$(auto\ simp\text{:}\ surj\_def)$

obtain example

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof
  assume $surj\ f$
  hence $\exists\, a.\ \{x.\ x \notin f\ x\} = f\ a$ **by**$(auto\ simp\!: surj\_def)$
  then obtain a **where** $\{x.\ x \notin f\ x\} = f\ a$ **by** $blast$

obtain example

lemma $\neg\ surj(f :: {}'a \Rightarrow {}'a\ set)$
proof
assume $surj\ f$
hence $\exists\, a.\ \{x.\ x \notin f\ x\} = f\ a$ **by**$(auto\ simp\colon surj\_def)$
then obtain a **where** $\{x.\ x \notin f\ x\} = f\ a$ **by** $blast$
hence $a \notin f\ a \longleftrightarrow a \in f\ a$ **by** $blast$

obtain example

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof
  assume $surj\ f$
  hence $\exists\, a.\ \{x.\ x \notin f\ x\} = f\ a$ **by**$(auto\ simp\!: surj\_def)$
  then obtain a **where** $\{x.\ x \notin f\ x\} = f\ a$ **by** $blast$
  hence $a \notin f\ a \longleftrightarrow a \in f\ a$ **by** $blast$
  thus $False$ **by** $blast$
qed

Set equality and subset

show $A = B$
proof
  show $A \subseteq B$ $\langle proof \rangle$
next
  show $B \subseteq A$ $\langle proof \rangle$
qed

Set equality and subset

show $A = B$
proof
  show $A \subseteq B$ $\langle proof \rangle$
next
  show $B \subseteq A$ $\langle proof \rangle$
qed

show $A \subseteq B$
proof
  fix x
  assume $x \in A$
  $\vdots$
  show $x \in B$ $\langle proof \rangle$
qed

Isar_Demo.thy

Exercise

15 Streamlining Proofs

Pattern Matching and Quotations

Top down proof development

moreover

Local lemmas

Raw proof blocks

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ (**is** $?L \longleftrightarrow ?R$)

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ (**is** $?L \longleftrightarrow ?R$)
proof
  assume $?L$
  $\vdots$
  show $?R$ $\langle proof \rangle$
next
  assume $?R$
  $\vdots$
  show $?L$ $\langle proof \rangle$
qed

$?thesis$

show $formula$
proof -
  $\vdots$
  show $?thesis$ $\langle proof \rangle$
qed

$?thesis$

show $formula$ (is $?thesis$)
proof -
$\vdots$
show $?thesis$ $\langle proof \rangle$
qed

$?thesis$

show $formula$ (is $?thesis$)
proof -
$\vdots$
show $?thesis$ $\langle proof \rangle$
qed

Every show implicitly defines $?thesis$

let

Introducing local abbreviations in proofs:

let $?t$ = *"some-big-term"*
$\vdots$
have *"$\ldots$ $?t$ $\ldots$ "*

Quoting facts by value

By name:

> **have** *x0:* $''x > 0''$ ...
> $\vdots$
> **from** *x0* ...

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By value:

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> $\vdots$
> **from** `` `x>0` `` ...

Quoting facts by value

By name:

> **have** *x0:* $''x > 0''$ ...
> $\vdots$
> **from** *x0* ...

By value:

> **have** $''x > 0''$ ...
> $\vdots$
> **from** \`$x{>}0$\` ...
> $\uparrow$ $\uparrow$
> *back quotes*

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Pattern matching and quotations

15 Streamlining Proofs

Example

lemma

$$(\exists\, ys\ zs.\ xs = ys\ @\ zs \wedge length\ ys = length\ zs) \vee$$
$$(\exists\, ys\ zs.\ xs = ys\ @\ zs \wedge length\ ys = length\ zs + 1)$$

Example

lemma

$$(\exists\, ys\ zs.\ xs = ys\ @\ zs \wedge length\ ys = length\ zs) \vee$$
$$(\exists\, ys\ zs.\ xs = ys\ @\ zs \wedge length\ ys = length\ zs + 1)$$

proof ???

`Isar_Demo.thy`

Top down proof development

When automation fails

Split proof up into smaller steps.

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... **using** ...

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have . . . **using** . . .
apply - to make incoming facts part of proof state

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apply *auto* or whatever

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Split proof up into smaller steps.

Or explore by **apply**:

have . . . **using** . . .
apply - to make incoming facts part of proof state
apply *auto* or whatever
apply . . .

At the end:

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... **using** ...	
apply -	to make incoming facts part of proof state
apply *auto*	or whatever
apply ...	

At the end:

- **done**

When automation fails

Split proof up into smaller steps.

Or explore by **apply**:

have ... **using** ...
apply - to make incoming facts part of proof state
apply *auto* or whatever
apply ...

At the end:

- **done**
- Better: convert to structured proof

15 Streamlining Proofs

moreover—ultimately

have P_1 $\ldots$
moreover
have P_2 $\ldots$
moreover
$\vdots$
moreover
have P_n $\ldots$
ultimately
have P $\ldots$

moreover—ultimately

have P_1 $\dots$
moreover
have P_2 $\dots$
moreover
$\vdots$
moreover
have P_n $\dots$
ultimately
have P $\dots$

$\approx$

have lab_1: P_1 $\dots$
have lab_2: P_2 $\dots$
$\vdots$
have lab_n: P_n $\dots$
from lab_1 lab_2 $\dots$
have P $\dots$

With names

15 Streamlining Proofs

Local lemmas

have B **if** *name:* $A_1 \ \ldots \ A_m$ **for** $x_1 \ \ldots \ x_n$

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proves $[\![\ A_1; \ \dots \ ; A_m \ ]\!] \Longrightarrow B$

Local lemmas

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proves $\llbracket\ A_1; \ \ldots\ ; A_m\ \rrbracket \Longrightarrow B$
where all x_i have been replaced by $?x_i$.

15 Streamlining Proofs

- Pattern Matching and Quotations
- Top down proof development
- **moreover**
- Local lemmas
- Raw proof blocks

Raw proof blocks

$\{$ **fix** $x_1 \ \dots \ x_n$
assume $A_1 \ \dots \ A_m$
$\vdots$
have B
$\}$

Raw proof blocks

$$
\begin{array}{l}
\{\ \textbf{fix}\ x_1\ \ldots\ x_n \\
\quad \textbf{assume}\ A_1\ \ldots\ A_m \\
\quad \vdots \\
\quad \textbf{have}\ B \\
\}
\end{array}
$$

proves $[\![\ A_1;\ \ldots\ ;\ A_m\ ]\!] \Longrightarrow B$

Raw proof blocks

$$
\begin{array}{l}
\{\ \textbf{fix}\ x_1\ \dots\ x_n \\
\quad \textbf{assume}\ A_1\ \dots\ A_m \\
\quad \vdots \\
\quad \textbf{have}\ B \\
\}
\end{array}
$$

proves $[\![\ A_1;\ \dots\ ;\ A_m\ ]\!] \Longrightarrow B$
where all x_i have been replaced by $?x_i$.

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moreover and $\{\quad\}$

Proof state and Isar text

Proof state and Isar text

In general: **proof** *method*

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ \dots \ x_n. \ \llbracket \ A_1; \ \dots \ ; \ A_m \ \rrbracket \Longrightarrow B$$

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ \ldots \ x_n.\ [\![\ A_1; \ \ldots \ ; \ A_m \ ]\!] \Longrightarrow B$$

How to prove each subgoal:

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ \dots \ x_n.\ \llbracket\ A_1; \ \dots\ ; A_m\ \rrbracket \Longrightarrow B$$

How to prove each subgoal:

fix $x_1 \ \dots \ x_n$
assume $A_1 \ \dots \ A_m$
$\vdots$
show B

Proof state and Isar text

In general: **proof** *method*

Applies *method* and generates subgoal(s):

$$\bigwedge x_1 \ \ldots \ x_n.\ [\![\ A_1; \ \ldots \ ; A_m\ ]\!] \Longrightarrow B$$

How to prove each subgoal:

fix $x_1 \ \ldots \ x_n$
assume $A_1 \ \ldots \ A_m$
$\vdots$
show B

Separated by **next**

Isar_Induction_Demo.thy

Proof by cases

Datatype case analysis

$\textbf{datatype}\ t \ = \ C_1\ \vec{\tau} \ \mid \ \ldots$

Datatype case analysis

datatype $t \;=\; C_1 \; \vec{\tau} \;\mid\; \ldots$

proof $(cases$ "$term$"$)$
  case $(C_1 \; x_1 \; \ldots \; x_k)$
  $\ldots \; x_j \; \ldots$
next
$\vdots$
qed

Datatype case analysis

datatype $t \;=\; C_1\ \vec{\tau} \ \mid\ \ldots$

proof $(cases$ "$term$"$)$
  case $(C_1\ x_1\ \ldots\ x_k)$
  $\ldots\ x_j\ \ldots$
next
$\vdots$
qed

where **case** $(C_i\ x_1\ \ldots\ x_k)$ $\equiv$

fix $x_1\ \ldots\ x_k$
assume $\underbrace{C_i:}_{\text{label}}\ \underbrace{term = (C_i\ x_1\ \ldots\ x_k)}_{\text{formula}}$

Isar_Induction_Demo.thy

Structural induction for nat

Structural induction for nat

show $P(n)$
proof $(induction\ n)$
  case 0
  $\vdots$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$
  $\vdots$
  show $?case$
qed

Structural induction for nat

show $P(n)$
proof $(induction\ n)$
  case 0 $\qquad\qquad \equiv \quad$ **let** $?case = P(0)$
  $\vdots$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$
  $\vdots$
  show $?case$
qed

Structural induction for nat

show $P(n)$
proof $(induction\ n)$
  case 0 $\equiv$ **let** $?case = P(0)$
  ⋮
  show $?case$
next
  case $(Suc\ n)$ $\equiv$ **fix** n **assume** $Suc{:}\ P(n)$
  ⋮ **let** $?case = P(Suc\ n)$
  ⋮
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof $(induction\ n)$
  case 0
  $\vdots$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$

  $\vdots$
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof $(induction\ n)$
  case 0 $\equiv$ **assume** 0: $A(0)$
  let $?case = P(0)$
  $\vdots$
  show $?case$
next
  case $(Suc\ n)$
  $\vdots$

  $\vdots$
  show $?case$
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$		
proof $(induction\ n)$		
case 0	$\equiv$	**assume** $0\colon\ A(0)$
$\vdots$		**let** $?case = P(0)$
show $?case$		
next		
case $(Suc\ n)$	$\equiv$	**fix** n
$\vdots$		**assume** $Suc\colon\ A(n) \Longrightarrow P(n)$
		$A(Suc\ n)$
$\vdots$		**let** $?case = P(Suc\ n)$
show $?case$		
qed		

Named assumptions

In a proof of

$$A_1 \implies \ldots \implies A_n \implies B$$

by structural induction:

Named assumptions

In a proof of

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:
In the context of

case C

Named assumptions

In a proof of

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:
In the context of

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we have

$C.IH$ the induction hypotheses

Named assumptions

In a proof of

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:
In the context of

case C

we have

$C.IH$	the induction hypotheses
$C.prems$	the premises A_i

Named assumptions

In a proof of

$$A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

by structural induction:
In the context of

case C

we have

$C.IH$	the induction hypotheses
$C.prems$	the premises A_i
C	$C.IH + C.prems$

A remark on style

- **case** $(Suc\ n)$ ... **show** $?case$
  is easy to write and maintain

A remark on style

- **case** $(Suc\ n)$ ... **show** $?case$
  is easy to write and maintain
- **fix** n **assume** $formula$ ... **show** $formula'$
  is easier to read:
  - all information is shown locally
  - no contextual references (e.g. $?case$)

16 Proof by Cases and Induction

Rule Induction

Rule Inversion

Isar_Induction_Demo.thy

Rule induction

Rule induction

inductive $I :: \tau \Rightarrow \sigma \Rightarrow bool$
where
$rule_1$: $\ldots$
$\vdots$
$rule_n$: $\ldots$

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Rule induction

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where
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$\vdots$
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show $I\ x\ y \Longrightarrow P\ x\ y$
proof $(induction\ rule\colon\ I.induct)$

Rule induction

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where
$rule_1$: . . .
⋮
$rule_n$: . . .

show $I\ x\ y \Longrightarrow P\ x\ y$
proof (*induction rule: I.induct*)
  case $rule_1$
  . . .
  show *?case*
next
⋮
next
  case $rule_n$
  . . .
  show *?case*
qed

Fixing your own variable names

case $(rule_i\ x_1\ \ldots\ x_k)$

Renames the first k variables in $rule_i$ (from left to right) to $x_1\ \ldots\ x_k$.

Named assumptions

In a proof of

$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \ldots$:

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by rule induction on $I \ldots$:
In the context of

case R

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we have

$R.IH$ the induction hypotheses

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by rule induction on $I \ldots$:
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we have

$R.IH$	the induction hypotheses
$R.hyps$	the assumptions of rule R

Named assumptions

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$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

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$R.hyps$	the assumptions of rule R
$R.prems$	the premises A_i

Named assumptions

In a proof of

$$I \ldots \Longrightarrow A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

by rule induction on $I \ldots$:
In the context of

case R

we have

$R.IH$	the induction hypotheses
$R.hyps$	the assumptions of rule R
$R.prems$	the premises A_i
R	$R.IH + R.hyps + R.prems$

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Rule Induction

Rule Inversion

Rule inversion

inductive $ev :: nat \Rightarrow bool$ **where**
$ev0$: $ev\ 0$ |
$evSS$: $ev\ n \Longrightarrow ev(Suc(Suc\ n))$

What can we deduce from $ev\ n$?

Rule inversion

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$$ev\ n \Longrightarrow n = 0 \vee (\exists\, k.\ n = Suc\ (Suc\ k) \wedge ev\ k)$$

Rule inversion

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$$ev\ n \Longrightarrow n = 0 \lor (\exists\, k.\ n = Suc\ (Suc\ k) \land ev\ k)$$

Rule inversion = case distinction over rules

Isar_Induction_Demo.thy

Rule inversion

Rule inversion template

from $`ev\ n`$ **have** P
proof $cases$
  case $ev0$ $\qquad\qquad\qquad\qquad n = 0$
  $\vdots$
  show $?thesis$ $\ldots$
next
  case $(evSS\ k)$ $\qquad\qquad\qquad n = Suc\ (Suc\ k),\ ev\ k$
  $\vdots$
  show $?thesis$ $\ldots$
qed

Rule inversion template

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proof $cases$
  case $ev0$ $\qquad\qquad\qquad\qquad n = 0$
  $\vdots$
  show $?thesis$ $\ldots$
next
  case $(evSS\ k)$ $\qquad\qquad\qquad\qquad n = Suc\ (Suc\ k),\ ev\ k$
  $\vdots$
  show $?thesis$ $\ldots$
qed

Impossible cases disappear automatically