1 Dynamic Optimization

1. [10 points] Consider a system of differential equations

$$\begin{cases} \dot{x} = -2y + x(x^2 + y^2 - 1); \\ \dot{y} = 2x + y(x^2 + y^2 - 1). \end{cases}$$

   (a) [5 points] Check that $(0, 0)$ is the only point of rest of this system. Change the variables from the Cartesian to the polar (r, ϕ). Rewrite the system in the polar coordinates.

   (b) [5 points] Draw two time paths of the system for $r_0 > 1$, where r_0 is the radius of the initial point and for $0 < r_0 < 1$. Justify your sketch. What if $r_0 = 1$?

2. [10 points] Solve the calculus of variations problem

$$\int_1^2 \left(\frac{2y^2}{x^2} + (y')^2 \right) dx \to \text{max / min}$$

   with the fixed endpoints $y(1) = 0$, $y(2) = \frac{7}{2}$.

   Hint: for solving Euler's equation try to find solutions in the form of x^a. Check all available second-order conditions.

3. [20 points] Road construction costs minimization.

   Let the terrain profile be represented by the function

$$y(t) = \begin{cases} 3 - 3|t|, \text{ if } |t| \le 1 \\ 0, \text{ otherwise.} \end{cases}$$

   The contractor minimizes the excavation costs given by the formula

$$\int_{-c}^{c} (x(t) - y(t))^2 \, dt,$$

   where $x(t)$ is the road profile we need to find, $c > 2$ and $[-c, c]$ — road section where the excavation takes place (c is not set). Allowable grade of the road satisfies $|\dot{x}| \le 1$.

   (a) [5 points] Form Hamiltonian and derive first-order conditions taking into account that multiplier λ satisfies transversality conditions $\lambda(-c) = \lambda(c) = 0$.

   (b) [15 points] Find $x(t)$ on the section $[-c, c]$.

2 Stochastic Calculus

Standard Wiener process is denoted by W_t.

1. [10 points] Let Y be equal to 1 if $W_2 > 0$ and 0 otherwise.
   Find $\mathbb{E}(Y|W_1)$, $\mathbb{V}\text{ar}(Y|W_1)$, $\mathbb{E}(Y|W_1^2)$.

2. [10 points] James Bond flips a biased coin until the sequence Head-Tail-Head-Tail appears. The probability of «Head» is equal to p. Using Doog's theorem and an appropriate martingale find the expected number of coin flips.

3. [10 points] The process X_t is given by

$$X_t = 2017 + t^2 W_t^2 + \int_0^t u\, dW_t$$

   (a) Find dX_t;

   (b) Is X_t a martingale?

   (c) Find $\mathbb{E}(X_t)$.

4. [10 points] Consider the framework of the Black and Scholes model. You agreed with Warren Buffett that at fixed time T he will pay you the strange sum

$$X_T = \ln S_T \cdot \ln S_{T/2},$$

   where S_t is the price of a share.

   What is the non-arbitrage price X_0 of this agreement?

5. [20 points] Solve the stochastic differential equation

$$dY_t = (Y_t^3 - Y_t)\, dt + Y_t^2\, dW_t$$

   You are free to use or not to use the following guiding steps:

   (a) [5 points] Consider $Z_t = Y_t^n$ and find dZ_t;

   (b) [3 points] Find a constant n such that the term before dW_t in dZ_t is non-random.

   (c) [2 points] Write down the equation for dZ_t in terms of t and Z_t only: you should get rid of Y_t.

   (d) [8 points] Solve the equation for dZ_t.
   Hint: It's just the particular case of the equation in your hometask :) Do you remember? Multiply Y_t by some exponent :)

   (e) [2 points] Finally, find Y_t.