Optimal Control

1. Consider a profit maximization problem when a perfectly competitive firm spends $u(t)$ on advertising:

$$\int_0^\infty e^{-rt}(Px(t) - u(t))\, dt \to \max,$$

where $x(t)$ is the share of the market, P — price of sales per unit time, $r > 0$ — the discount rate. Maximization is subject to $\dot{x} = u(1-x) - x/2$, $x_0 \geq 0$ and $0 \leq u \leq 1$.

   (a) (5 points) Using current value Hamiltonian state the system of first-order conditions.

   (b) (15 points) Find the bounds on price $P_{\text{min}} < P < P_{\text{max}}$ for which the steady-state solution (x_s, u_s) with $x_s < 1$ and $u_s < 1$ exists and find it. Draw in (x, u)-plane the paths of most rapid approach to steady-state for $x_s \neq x_0$.

2. (10 points) Consider a problem in discrete time

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \to \max$$

with respect to c_t subject to $k_{t+1} = \sqrt{k_t - c_t}$, where k_0 is given and $0 < \beta < 1$. Suppose there is a constant $\delta > 0$ such that $U(c) \leq \delta c$ for all $c \geq 0$.

   (a) Write down Bellman's equation for this problem.

   (b) Prove that under assumption that Bellman's equation has a unique solution the value function of the problem satisfies the estimate $V(k) \leq \delta k + C$ for some $C > 0$ and provide lower bound for C.

3. (10 points) Solve bounded control problem $\int_0^4 (u^2 + x)\, dt \to \min$ subject to $dx/dt = u$, $x(0) = 0$ and $|u| \leq 1$.

Stochastic Calculus

Standard Wiener process is denoted by W_t.

1. [10 points] Find a function $f(t)$ such that $M_t = \exp(W_t^2/(1+2t))/f(t)$ is a martingale.
2. [10 points] Find the following limit in mean squared

$$\lim_{n\to\infty} \sum_{i=1}^{n} (W_{ti/n} - W_{t(i-1)/n})^3$$

3. [10 points] In the framework of the Black and Scholes model find the price at $t = 0$ of an asset that pays $X_T = S_T^2/S_{T/2}$ at time T. Here S_t denotes the price of one share at time t.
4. [10 points] Solve the following stochastic differential equation.

$$dX_t = \frac{X_t}{t-1} dt + dW_t, \; X_0 = 0$$

Hint: the solution has the form $X_t = a(t) \cdot \int_0^t b(u) dW_u$ for some deterministic functions $a(t)$ and $b(t)$. You only need to find $a(t)$ and $b(t)$:)

5. [20 points] Let τ be the first moment of time when the Wiener process W_t touches the line $y(t) = 10 - t$.

The goal of this exercise is to find $\mathbb{E}(\tau)$ and $\mathbb{V}\mathrm{ar}(\tau)$.

You are free to use or not to use the following hints:

(a) Check whether the process $R_t = \exp(\sigma W_t - \sigma^2 t/2)$ is a martingale.

(b) Apply the Doob's theorem and find the function $m(\lambda) = \mathbb{E}(\exp(-\lambda\tau))$. You don't need to check all the conditions of the Doob's theorem.

(c) Find $m'(0)$, find $m''(0)$.

(d) Find $\mathbb{E}(\tau)$, $\mathbb{V}\mathrm{ar}(\tau)$ using $m'(0)$ and $m''(0)$. You can do the last point without any previous point.