Stochastic Calculus

Standard Wiener process is denoted by W_t.

1. Consider the following stochastic integral:

$$I_t = \int_0^t 4W_u^2 + 3W_u + 6\,dW_u$$

   (a) [3 points] Find $d\,\mathbb{E}(I_t|I_s)$ for $t > s$.

   (b) [7 points] Find $\mathbb{V}\text{ar}(I_t|I_s)$ for $t > s$.

2. [10 points] Find $\mathbb{E}(W_2|W_1, W_5)$ and $\mathbb{E}(W_2^3|W_1, W_5)$.

3. [10 points] Suppose X_t satisfies the stochastic differential equation

$$dX_t = X_t\,dt + X_t^2\,dW_t$$

   Determine constants a, b and c such that $Y_t = \exp(aX_t^b + ct)$ is a martingale.

4. [10 points] Consider the framework of the Black and Scholes model. The asset X will pay you one share at fixed time T if the price of a share has increased by more than 10% during the time period $[0; T]$.

   What is the non-arbitrage price X_0 of this asset?

5. [20 points] Consider the Vasicek interest rate model, $dR_t = a(b - R_t)\,dt + s\,dW_t$, where a, b and s are positive constants.

   (a) Using the substitution $Y_t = e^{at}R_t$ find the solution of the stochastic differential equation;

   (b) Find $\mathbb{E}(R_t)$ and $\mathbb{V}\text{ar}(R_t)$.