1. Consider the optimal control problem

$$\int_0^\infty e^{-rt}\left(\bar{U} - \frac{1}{2}(c(t))^{-1}\right)\, dt \to \max,$$

subject to $\dot{k} = \ln(1+k) - c - \frac{1}{2}k$ and $k(0) = k_0 > 0$. It is known that $0 < r < 1/2$ and $\bar{U}$ is sufficiently large in order to keep the integrand positive.

   (a) Derive the system of differential equations in (k, c)–plane, using current value Hamiltonian, where k-axis is horizontal.

   (b) Prove that the steady-state solution exists and is unique.

   (c) Sketch the phase portrait of the system in the neighborhood of the equilibrium.

   (d) Show by pointing arrows the directions along the paths.

2. Let $Y_t = W_t^3 - 3tW_t + 1$.

   (a) Using Ito's lemma find dY_t.

   (b) Using your previous result find $\mathrm{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.

3. Find the price of the «Asset-or-nothing» call option at time $t = 0$ in the framework of Black and Scholes model. The risk-free interest rate is equal to r. The volatility of the share is equal to σ. The current share price is S_0. The «Asset-or-nothing» call option pays you at fixed time T the sum S_T if S_T is higher than the strike price K or nothing otherwise.

   Hint: the correct answer will contain the normal cumulative distribution function $F()$.