1 Discrete time

1. Some questions about σ-algebras.

   (a) You observe the result of 10 coin tosses. How many elements the σ-algebra of your information contains?

   (b) Prove that a finite σ-algebra can contain only 2^k elements.

   (c) Is union of two σ-algebras always a σ-algebra? Prove your statement.

   (d) Is intersection of two σ-algebras always a σ-algebra? Prove your statement.

2. Prove the following statement or provide a counter-example. For any two σ-algebras $\mathcal{F}$ and $\mathcal{H}$ and a random variable Y

$$\mathrm{E}(\mathrm{E}(Y|\mathcal{F})|\mathcal{H}) = \mathrm{E}(Y|\mathcal{F} \cap \mathcal{H})$$

3. I throw a fair die until the first six appears. Let's denote the total number of throws by X and the number of odd integers thrown by Y.

   (a) Find $\mathbb{P}(Y = y|X)$, $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$;

   (b) Find $\mathrm{E}(X|Y)$.

4. I throw 100 coins. Let's denote by X the number of coins that show «heads». I throw these X coins once again, leaving other coins as they are. Let's denote by Y the number of coins that show «heads» now.

   Find $\mathbb{P}(Y = y|X)$, $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$, $\mathrm{E}(Y)$, $\mathrm{Var}(Y)$.

5. Random variables X and Y have joint normal distribution with zero means and covariance matrix

$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$

   (a) Find $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$, $\mathrm{E}(XY|X)$ and $\mathrm{Var}(XY|X)$.

   (b) Using standard normal cumulative distribution function find $\mathbb{P}(YX > 2019|X)$.

6. The random variables Z_1, Z_2, ... are independent and identically distributed with $\mathbb{P}(Z_n = 1) = p$ and $\mathbb{P}(Z_n = -1) = 1 - p$. Consider the cumulative sum process, $S_n = Z_1 + \ldots + Z_n$ with $S_0 = 0$.

   (a) For which value of p the process 1.5^{S_n} will be a martingale.

   (b) Let $p = 0.4$. If possible find the constants α and β such that $Y_n = S_n^2 + \alpha S_n + \beta n$ is a martingale?

7. Anna and Boris throw a coin infinite number of times. Anna wins if the sequence HTHH appears first, Boris wins if the sequence TTHH appears first. The coin is biased with 0.4 probability of head.

   (a) What is the expected number of throws to obtain HTHH?

   (b) What is the expected number of throws to obtain TTHH?

   (c) What is the probability that Anna will win?

   (d) What is the expected number of throws to obtain HTHH or TTHH?

8. Have a look in the past exams collection. How many pages does it contain?

2 Continuous time stochastic processes

Due to late posting date this part is not mandatory. Be happy and study stochastic calculus :)

1. The processes (X_t) and (Y_t) are independent Wiener processes with respect to filtration $(\mathcal{F}_t)$. The process $Z_t = aX_t + bY_t$ is also a Wiener process.
   (a) For which values of constants a and b is it possible?
   (b) Find covariance $\mathrm{Corr}(Z_t, X_t)$.
   (c) Find $\mathrm{E}(Z_3|X_2)$ and $\mathrm{Var}(Z_3|X_2)$.
   (d) Find $\mathrm{E}(Z_3|\mathcal{F}_2)$ and $\mathrm{Var}(Z_3|\mathcal{F}_2)$.

2. The process $C_t = W_t^3 + aW_t^2 + bW_t + c + d \cdot t \cdot W_t$ is a martingale.
   (a) For which values of constants a, b, c and d is it possible?
   (b) Find covariance $\mathrm{Cov}(C_t, \int W_u^2 dW_u)$.

3. In the framework of Black and Scholes model derive the current price of a European call option with strike price K and maturity date T.

   The European call option pays you the sum

$$X_T = \begin{cases} S_T - K, \text{ if } S_T > K; \\ 0, \text{ otherwise.} \end{cases}$$

4. Let $Y_t = W_t + 3t$. The moment τ is the first moment when Wiener process hits 10.
   (a) Let α be a constant. Find the function $f(t)$ such that $M_t = f(t)\exp(\alpha Y_t)$ is a martingale.
   (b) Using Doob's theorem find $\mathrm{E}(\exp(-s\tau))$ for arbitrary constant s.

5. Let's consider the process $X_t = \int_0^t u^2 dW_u$. Prove that this process can be represented as a time changed Wiener process. That means that there is a deterministic time-scaling $t(s)$ such that $Y_s = X_{t(s)}$ is a Wiener process with respect to some filtration.

6. Solve the stochatic differential equation

$$dY_t = -Y_t dt + dW_t, \ Y_0 = 1$$

   If you are have no clues you may try a substitution $Z_t = f(t)Y_t$. Do not forget that the final answer may contain integrals that can't be calculated explicitely. It's ok.

7. Solve the stochatic differential equation

$$dY_t = Y_t dt + (t^3 + 4Y_t)dW_t, \ Y_0 = 1$$

   If you are have no clues you may try to represent the process as $Y_t = A_t B_t$, where A_t is the solution of the equation $dA_t = A_t dt + 4A_t dW_t$.