**Stochastic calculus part**

Here $W_t$ always denotes the standard Wiener process.

1. (10 points) You throw a standard fair die $n$ times. Let $X$ be the number of «fives» in the first $(n-1)$ throws and $Y$ — the number of «fives» in the last $(n-1)$ throws.

   For $n \geq 3$ calculate $\mathrm{E}(Y|X)$, $\mathrm{E}(X|Y)$, $\mathrm{Var}(Y|X)$.

2. (10 points) Find a constant $b$ such that $Y_t = \int_0^t W_u^3\, du + bW_t^5$ is a martingale.

3. (10 points) The process $X_t$ evolves according to the formula

$$X_t = 1 - t + (1-t)\int_0^t \frac{1}{1-u}\, dW_u$$

   (a) Is $X_t$ a martingale?

   (b) Find $\mathrm{E}(X_t)$, $\mathrm{Var}(X_t)$ and $\mathrm{Cov}(W_t, X_t)$

   (c) Draw $\mathrm{E}(X_t)$ and $\mathrm{Var}(X_t)$ as functions of $t$

   (d) Draw two possible trajectories of $X_t$

4. (10 points) Find the price of the «Asset-or-nothing» call option at time $t = 0$ in the framework of Black and Scholes model. The risk-free interest rate is equal to $r$. The volatility of the share is equal to $\sigma$. The current share price is $S_0$. The «Asset-or-nothing» call option pays you at fixed time $T$ the sum $S_T$ if $S_T$ is higher than the strike price $K$ or nothing otherwise.

   Hint: the correct answer will contain the normal cumulative distribution function $F()$.

5. (20 points) Interest rate evolves according to the stochastic differential equation

$$dr_t = -\lambda(r_t - c)\, dt + \sigma dW_t$$

   where $r_0$, $c$, $\lambda$ and $\sigma$ are some positive constants.

   Solve this stochastic differential equation.

   You may use or not use the following hints:

   (a) Obtain a stochastic differential equation for $X_t = r_t - c$.

   (b) Solve the ordinary differential equation $dY_t = -\lambda Y_t\, dt,\ Y_0 = 1$

   (c) Represent $X_t$ as $X_t = Y_t \cdot Z_t$ and find the equation for $dZ_t$.

   (d) Solve the equation for $Z_t$.

   (e) The correct answer for $r_t$ will contain an Ito's integral that cannot be simplified.

**Optimal control part**

6. (10 points) Solve the bounded control problem:

$$\int_0^T (u-x)^2 dt \to \min$$

subject to $\dot{x} = \frac{1}{2}(u-x)$, $x(0) = 1$, $|u| \leqslant 1$, $T > 0$.

7. (10 points) Find the extremals for the calculus of variations problem:

$$\int_0^1 \frac{1}{2}\dot{x}^2 + x\dot{x} + x\, dt$$

when both endpoint values $x(0)$ and $x(1)$ can be chosen freely.

8. (20 points) Costs of the farmer's production $C(x,y)$ depends on the output $y(t)$ and the fertility of soil $x(t)$ by the formula $C = y^2 + \frac{1}{\sqrt{x}}$. This industry is perfectly competitive and the price of the harvest per unit is $p$. Fertility changes over time by the dynamic law $\dot{x} = B - \alpha y$, where $p\alpha > 2B > 0$, and $x(0) > 0$. Constant $B$ is defined by the government subsidy. Profit of the firm is discounted with the rate $r > 0$.

   (a) Write down the problem of maximizing the stream of discounted profit over the infinite time horizon.

   (b) Write down the system of first order conditions using the current value Hamiltonian. Draw the phase portrait in $(x,y)$ coordinate system, in which $x$ — axis is drawn horizontally. Find the steady-state $(x_s, y_s)$ if it exists, classify it using Jacobian. With the help of the arrows provide the sketch of the dynamic paths of the system.

   (c) Find the sign of $\partial x_s/\partial p$. What justification of this sign can you provide from the economic point of view?