Here W_t denotes the standard Wiener process.

1. Let $Y = 1_{W_3 \leq 2}$. Find $\mathrm{E}(W_5 \cdot Y)$, $\mathrm{Var}(W_5 \cdot Y)$, $\mathrm{Cov}(W_5, Y)$, $\mathrm{Cov}(W_5, W_5 \cdot Y)$.

2. At time $n = 0$ I have one apples and one banana. At each moment of time I choose one of all my fruits randomly with equal probabilities. I eat it and by two more of the same kind. For example, if I choose banana I will eat it and buy two bananas more. I repeat the process of choosing, eating and buying fruits indefinitely. Let A_n and B_n be the number of apples and bananas after n rounds.

   (a) Find non-trivial function f such that $M_n = f(n) \cdot A_n$ is a martingale.

   (b) Let τ be the moment when the first banana is chosen. Assuming that Doob's theorem is applicable find $E(1/(\tau + 2))$.

3. Find all constants a such that $Y_t = 42 + W_t^7 + a \int_0^t W_u^5 \, du + \int_0^t u^2 \cos W_u \, dW_u$ is a martingale. Find $\mathrm{E}(Y_t)$.

4. Consider the framework of the Black and Scholes model. The asset pays you 1 dollar at fixed time T if and only if

$$\frac{S_T}{S_{T/2}} > \frac{S_{T/2}}{S_0}.$$

   Find the current price X_0 of this asset.

5. Solve the stochastic differential equation and find $\mathrm{E}(X_t)$ and $\mathrm{Var}(X_t)$

$$dX_t + aX_t dt = bdt + cdW_t;$$

   You are free to use or not to use the following guiding steps:

   (a) Find dY_t for $Y_t = \exp(\gamma t) X_t$;

   (b) Find the value of γ such that in dY_t the term before dt is non-random.

   (c) Solve the equation for Y_t and then find X_t;

   (d) Do not forget about $\mathrm{E}(X_t)$ and $\mathrm{Var}(X_t)$.