Stochastic Calculus

Standard Wiener process is denoted by W_t.

1. [10 points] Consider a well-shuffled deck of 52 cards. You open the cards one by one until the second King appears.

   (a) What is the probability that the next card will be Queen of Diamonds?

   (b) What is the probability that the next card will be King of Diamonds?

   Hint: a good martingale may be very useful here :)

2. [10 points] Find $\mathbb{E}(W_{2019}^3|W_{2018})$ and $\mathbb{E}(W_{2018}^3|W_{2019})$.

3. [10 points] Suppose X_t satisfies the stochastic differential equation

$$dX_t = 2018X_t\,dt + X_t^{2019}\,dW_t$$

   Determine constants a, b and c such that $Y_t = \exp(aX_t^b + ct)$ is a martingale.

4. [10 points] Consider the framework of the Black and Scholes model. The asset X will pay you one share at fixed time T if $S_T > 2S_{T/2}$, where S_t is the price of a share.

   What is the non-arbitrage price X_0 of this asset?

5. [20 points] Consider the stochastic differential equation

$$dY_t = (2Y_t/t + 3(t^4Y_t)^{1/3})\,dt + 3(tY_t)^{2/3}\,dW_t, \quad Y_0 = 0.$$

   (a) [15 points] Solve the stochastic differential equation.

   (b) [5 points] Find $\mathbb{E}(Y_t)$ and $\mathbb{V}\text{ar}(Y_t)$.

   You are free to use or not to use the following guiding steps:

   (c) [5 points] Suppose that $Y_t = g(W_t)h(t)$. Find dY_t.

   (d) [5 points] By looking at the term before dW_t provide the equations for $g(W_t)$ and $h(t)$.

   (e) [5 points] Find $g(W_t)$ and $h(t)$ and check your solution.