ICEF master program. Maths for economists. Exam collection.

Kirill Bukin, Dmitri Pervouchine, Boris Demeshev

4 декабря 2017 г.

Содержание

1. 2008-2009

1.1. Exam, 12.01.2009

Final exam consists of the two parts: A and B. Part A lasts for 110 minutes. Upon completion of that part the papers will be collected and the students will have 10 minutes break. Part B lasts for 70 minutes. Students should answer eight of the following nine questions: six from Part A and two from Part B. Points will be deducted for the insufficient explanation within your answers.
Part A. Answer all SIX questions of this section. Each question is worth 10 points.

1. The joint distribution of vector (X, Y) is given by $\mathbb{P}(X = i, Y = j) = 0.1$ for $1 \leqslant i \leqslant j \leqslant 4$. Find $\mathbb{E}(Y \mid X)$.
2. The random variable X is exponentially distributed with parameter λ. The random variable Y is exponentially distributed with parameter $1/X$. Find $\mathbb{E}(Y|X)$, $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.
3. Let $Y_t = W_t^3 - 3tW_t$.
   (a) Using Ito's lemma find dY_t.
   (b) Using your previous result find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.
4. Using a current value Hamiltonian maximize the integral

$$\int_0^2 e^{-t}\left(x - \frac{5}{2}x^2 - 2y^2\right) dt$$

subject to the conditions $\dot{x} = y - x/2$, $x(0) = 0$, $x(2)$ is free. Find x, y, μ.

5. Solve the bounded control problem

$$\int_0^T e^{-rt}(1-u)x\,dt \to \max$$

, subject to $\dot{x} = xu$, $x(0) = 1$, $0 \leqslant u \leqslant 1$, where $0 < r < 1$.

6. Consider the following optimization problem

$$\sum_0^\infty u(a_t) \to \max$$

, subject to $\sum_0^\infty a_t \leqslant s$, $s > 0$, $a_t \geqslant 0$.

   (a) Show that if $u(a)$ is increasing and strictly concave this problem has no solution

   (b) What happens if the sum in the maximization problem is changed to $\sum_0^\infty \delta^t u(a_t)$, where $0 < \delta < 1$?

Part B. Answer two questions out of the three from this section. Each question is worth 20 points. Part B lasts for 70 minutes.

7. In the framework of the Black and Scholes model find the price at $t = 0$ of an asset that pays $\max\{0, \ln S_T\}$ at time T, where S_T denotes the price of one share at time T.

8. Let's consider the following system of stochastic differential equations

$$\begin{cases} dX_t = aX_t dt - Y_t dW_t \\ dY_t = aY_t dt + X_t dW_t \end{cases}$$

with initial conditions $X_0 = x_0$ and $Y_0 = 0$

   (a) Find the solution of the form $X_t = f(t)\cos W_t$ and $Y_t = g(t)\sin W_t$

   (b) Prove that for any solution $D_t = X_t^2 + Y_t^2$ is nonstochastic

9. Consider the profit-maximizing problem for a representative competitive firm

$$\int_0^\infty (p - wn(t))q(t)e^{-rt}\,dt \to \max$$

, subject to (*) $\dot{x} = x(1-x) - q$, where the state variable $x(t)$ represents a renewable stock resource (fish) that evolves according to the equation (*) and $q(t) = 2\sqrt{x(t)n(t)}$ is the extraction rate. Here $n(y)$ is a labor effort with a constant wage rate w. The price of fish is assumed to be constant and equal p. The optimization problem is to choose $n(t)$ to maximize the discounted profits, $0 < r < 1$.

   (a) Derive necessary conditions

   (b) Draw the phase diagram for this problem with the fish stock and the multiplier labeled on the axes

   (c) Show that if p/w is sufficiently large the fish stock will be driven to zero, while p/w is low there is a steady-state with a positive stock

1.2. Exam, 12.01.2009, solutions

1. Density table of the joint distribution of X and Y
   Fixing a certain value of X we find the conditional expectation of Y:

X/Y	1	2	3	4
1	0.1	0.1	0.1	0.1
2	0	0.1	0.1	0.1
3	0	0	0.1	0.1
4	0	0	0	0.1

$$\mathbb{E}(Y|X) = \begin{cases} \frac{1+2+3+4}{4} = 2.5, \text{ if } x = 1 \\ \frac{2+3+4}{3} = 3, \text{ if } x = 2 \\ \frac{3+4}{2} = 3.5, \text{ if } x = 3 \\ \frac{4}{1} = 4, \text{ if } x = 4 \end{cases}$$

Or, simply, $\mathbb{E}(Y|X) = 2 + X/2$.

2. Variable X has exponential distribution, so

$$f_X(x) = \lambda e^{-\lambda x}$$

And we know conditional distribution of Y:

$$f_{Y|X}(x, y) = \frac{1}{x} e^{-\frac{1}{x} y}$$

So,

$$\mathbb{E}(Y|X) = \int_{-\infty}^{+\infty} y f_{Y|X}(X, y) dy = \int_0^{+\infty} y \frac{1}{X} e^{-\frac{1}{X} y} dy = \left(\frac{1}{X}\right)^{-1} = X$$

Expected value:

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\, dx = \int_0^{+\infty} x \lambda e^{-\lambda x}\, dx = \lambda^{-1}$$

Variance:

$$\text{Var}(Y) = \text{Var}(\mathbb{E}(Y|X)) + \mathbb{E}(\text{Var}(Y|X)) =$$
$$= \text{Var}(X) + \mathbb{E}\left(\left(\frac{1}{X}\right)^{-2}\right) = \lambda^{-2} + \mathbb{E}(X^2)$$

We know that $\mathbb{E}(X^2) = \text{Var}(X) + (\mathbb{E}(X))^2$, so:

$$\text{Var}(Y) = \lambda^{-2} + \lambda^{-2} + \lambda^{-2} = 3\lambda^{-2}$$

3. Using Ito's lemma $dY_t = (3W_t^2 - 3t)\, dW_t + (-3W_t + \frac{1}{2} 6W_t)\, dt = (3W_t^2 - 3t)\, dW_t$. Hence, Y_t is a martingale. $Y_0 = W_0^3 - 3 \cdot 0 W_0 = 0$.
Now $\mathbb{E}(Y_t) = \mathbb{E}(Y_0) = 0$ and using Ito's isometry:

$$\text{Var}(Y_t) = \int_0^t \mathbb{E}((3W_s^2 - 3s)^2)\, ds = \int_0^t 27s^2 + 9s^2 - 18s^2\, ds = \int_0^t 18s^2\, ds = 6t^3$$

Here we have used the facts that $\mathbb{E}(W_s^2) = s$ and $\mathbb{E}(W_s^4) = 3s^2$.

4.
5.
6.
7.
8. Using Ito's lemma we find dX_t and dY_t

$$dX_t = f_t'(t)\cos W_t dt - f(t)\sin W_t\, dW_t - 0.5f(t)\cos W_t\, dt = \\ = (f_t'(t) - 0.5f(t))\cos W_t dt - f(t)\sin W_t\, dWt$$

$$dY_t = g_t'(t)\sin W_t\, dt + g(t)\cos W_t\, dW_t - 0.5g(t)\sin W_t\, dt = \\ = (g_t'(t) - 0.5g(t))\sin W_t\, dt + g(t)\cos W_t\, dWt$$

Comparing these expressions with the system we receive:

$$\begin{array}{rcl} f_t'(t) - 0.5f(t) & = & af(t) \\ f(t) & = & g(t) \\ g_t'(t) - 0.5g(t) & = & ag(t) \\ g(t) & = & f(t) \end{array}$$

The general solution has the form

$$f(t) = g(t) = Ae^{(a+0.5)t}, \quad A \in \mathbb{R}$$

From initial condition we get $A = x_0$ and

$$f(t) = g(t) = x_0 e^{(a+0.5)t}$$

Answer:

$$\begin{array}{l} X_t = x_0 + x_0 \int_0^t ae^{(a+0.5)u}\cos W_u du - x_0 \int_0^t e^{(a+0.5)t}\sin W_u\, dWu \\ Y_t = x_0 \int_0^t ae^{(a+0.5)u}\sin W_u du + x_0 \int_0^t e^{(a+0.5)t}\cos W_u\, dWu \end{array}$$

Now we use Ito's lemma once again:

$$dD_t = 2X_t\, dX_t + 2Y_t\, dY_t = 2X_t((f_t'(t) - 0.5f(t))\cos W_t\, dt - f(t)\sin W_t\, dWt) + \\ + 2Y_t((g_t'(t) - 0.5g(t))\sin W_t], dt + g(t)\cos W_t\, dWt) = \\ = 2f(t)(f_t'(t) - 0.5f(t))\cos^2 W_t\, dt - f^2(t)\cos W_t \sin W_t\, dWt + \\ + 2f(t)(f_t'(t) - 0.5f(t))\sin^2 W_t\, dt + f^2(t)\cos W_t \sin W_t\, dWt = \\ = 2f(t)(f_t'(t) - 0.5f(t)\, dt = 2af^2(t)\, dt$$

$$D_t = D_0 + \int_0^t 2ax_0 e^{(2a+1)u} du = x_0^2 + \frac{2ax_0}{2a+1}(e^{(2a+1)t} - 1)$$

1.3. Retake, ??.??.2009

Final exam consists of the two parts: A and B. Part A lasts for 110 minutes. Upon completion of that part the papers will be collected and the students will have 10 minutes break. Part B lasts for

70 minutes. Students should answer eight of the following nine questions: six from Part A and two from Part B. Points will be deducted for the insufficient explanation within your answers.

Part A. Answer all SIX questions of this section. Each question is worth 10 points.

1. In the first bag there balls numbered from 0 to 9, in the second bag there are balls numbered from 1 to 10. Two balls were selected. You know that one ball was selected from the first bag and one from the second one. You will select at random one ball from these two and you will know only its number. Let's denote its number by X and the number of the other of the two balls by Y.
   Find $\mathbb{E}(Y \mid X)$
2. The random variable X is uniformly distributed on $[0; a]$. The random variable Y is uniformly distributed on $[0; X]$.
   Find $\mathbb{E}(Y \mid X)$, $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.
3. Let $Y_t = \exp\left(-aW_t - \frac{a^2}{2}t\right)$.
   (a) Using Ito's lemma find dY_t
   (b) Using your previous result find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$
4. Find extremals for the integral $\int_0^1 \left(\frac{1}{2}\dot{x}^2 + x\dot{x} + x\right) dt$ when $x(0) = 0$ and $x(1)$ is chosen freely.
5. Solve the bounded control problem $\int_0^2 (2x - 3u)\, dt \to \max$, subject to $\dot{x} = x + u$, $x(0) = 5$, $0 \leqslant u \leqslant 2$, where $x(2)$ is free.
6. Consider the following optimization problem: maximize $\sum_0^\infty \beta^t u(c_t)$, subject to $c_t + k_{t+1} = f(k_t)$, $0 < \beta < 1$, where both the utility function $u(t)$ and the production function $f(k)$ have the standard properties of monotonicity and strict concavity. Let the state variable be k and denote the next period value of k as k'. Substitute $c = f(k) - k'$ into utility function and write down the Bellman equation.
   Using the formal differentiation of the Bellman equation with respect to k under the sign of max, and drawing FOC from Bellman equation, exclude the value function and find the equation which combines the values of u' and f' at the adjacent time periods.

Part B. Answer two questions out of the three from this section. Each question is worth 20 points. Part B lasts for 70 minutes.

7. In the framework of the Black and Scholes model find the price at $t = 0$ of an asset that pays $\min\{M, \ln S_t\}$ at time T, where S_T denotes the price of one share at time T, M — arbitrary constant, specified at the moment of the issue.
8. The price of a share in euros is driven by the equation $dS = \sigma S dW + \alpha S dt$, the dollar/euro exchange rate is driven by the equation $dU = bU dW + cU dt$. Find the current price in dollars of a European call option with maturity date T, strike price K.
9. Consider the profit-maximizing problem for a representative competitive firm $\int_0^\infty (p - c(x(t)))q(t)e^{-rt}\, dt \to \max$, subject to (*) $\dot{x} = 1 - x - q$, where the state variable $x(t) < 1$ represents a nonrenewable stock resource (oil) that depletes according to the equation (*) and $q(t)$ is the extraction rate. Here $c(x)$ is a cost function of the extraction that is defined by $c(x) = e^{-x}$. The price of oil is assumed to be constant and equal p where $1/e < p < 1$. The optimization problem is to choose $q(t)$ to maximize the discounted profits, $0 < r < 1$.
   (a) Derive necessary conditions.
   (b) Prove that the steady-state exists and is unique.

1.4. Retake, ??.??.2010-solutions

1. If $X=0$ we know that that the chosen ball was in the first basket, consequently Y was in the second one.

$$\mathbb{E}(Y|X=0)=\mathbb{E}(Y|\text{ second basket })=\frac{1+2+\cdots+10}{10}=5.5$$

Similarly

$$\mathbb{E}(Y|X=10)=\mathbb{E}(Y|\text{ first basket })=\frac{0+1+\cdots+9}{10}=4.5$$

Finaly,

$$\mathbb{E}(Y|0<X<10)=\frac{1}{2}\mathbb{E}(Y|\text{ first basket })+\frac{1}{2}\mathbb{E}(Y|\text{ second basket })=0.5(5.5+4.5)=5$$

Answer:

$$\mathbb{E}(Y|X)=\begin{cases}5.5, \text{ if } X=0\\ 5, \text{ if } 0<X<10\\ 4.5, \text{ if } X=10\end{cases}$$

2. $\mathbb{E}(Y|X)=X/2, \mathbb{E}(Y)=\mathbb{E}(X/2)=a/4$,

$$\text{Var}(Y)=\mathbb{E}(\text{Var}(Y|X))+\text{Var}(\mathbb{E}(Y|X))=\mathbb{E}(X^2/12)+\text{Var}(X/2)=\frac{7a^2}{12^2}$$

3. $dY_t=-aY_t\,dW_t$, so Y_t is a martingale, and $\mathbb{E}(Y_t)=Y_0=1$. To find variance one may use Ito's isometry.

4. This is the autoconversion from the ugly word:

$$X_0=E_{\tilde{p}}\left(exp\left(-rT\right)X_T\left|\mathcal{F}_0\right.\right)$$

$$X_T=Min\left\{M,\ lnS_T\right\}$$
$$S_T=S_0\exp\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma\tilde{W}_T\right)\ ;\ \tilde{W}_T\sim N\left(0,T\right)$$
$$\ln S_T=lnS_0+\left(r-\frac{\sigma^2}{2}\right)T+\sigma\tilde{W}_T$$
$$X_0=E_{\tilde{p}}\left(exp\left(-r_{\$}T\right)X_T\left|\mathcal{F}_0\right.\right)=M*\tilde{p}\left(\ln S_T>M\right)+\int_{-\infty}^{M}\ln S_T*\tilde{p}\left(\ln S_T\right)\text{dln}S_T$$
$$1)\ \tilde{p}\left(\ln S_T>M\right)=\tilde{p}\left(\ln S_0+\left(r-\frac{\sigma^2}{2}\right)T+\sigma\tilde{W}_T>M\right)=\left[Z=\frac{\tilde{W}_T}{\sqrt{T}}\sim N(0,1)\right]=$$
$$=\tilde{p}\left(Z<\frac{\ln S_0+\left(r-\frac{\sigma^2}{2}\right)T-M}{\sigma\sqrt{T}}\right)=\Phi\left(\frac{\ln S_0+\left(r-\frac{\sigma^2}{2}\right)T-M}{\sigma\sqrt{T}}\right)$$

$$\Phi - cummulative\ standard\ normal\ distribution$$

$$2)\ M = lnS_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}m\ \ ;\ \ m = \frac{M - lnS_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\int_{-\infty}^{M} \ln S_T * \tilde{p}\left(\ln S_T\right) \mathrm{dln}S_T = \int_{-\infty}^{m} \left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z^2}{2}\right) \mathrm{dZ} =$$

$$= \left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T\right)\Phi\left(\frac{M - lnS_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \int_{-\infty}^{m} Z \exp\left(-\frac{Z^2}{2}\right) \mathrm{dZ} =$$

$$\left[\int Z \exp\left(-\frac{Z^2}{2}\right) \mathrm{dZ} = \begin{bmatrix} Let\ s = -\frac{Z^2}{2} \\ ds = -ZdZ \\ dZ = -\frac{\mathrm{ds}}{Z} \end{bmatrix} = \int \left(-e^s\right) \mathrm{ds} = -e^s = -\exp\left(-\frac{Z^2}{2}\right)\right]$$

$$= \left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T\right)\Phi\left(\frac{M - lnS_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \exp\left(-\frac{m^2}{2}\right)$$

$$3)\ X_0 = M\Phi\left(\frac{\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T - M}{\sigma\sqrt{T}}\right) + \left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T\right)\Phi\left(\frac{M - lnS_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) -$$

$$- \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \exp\left(-\frac{m^2}{2}\right) = M + \left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T - M\right)\Phi\left(\frac{M - lnS_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \exp\left(-\frac{m^2}{2}\right) =$$

$$= M - m\sigma\sqrt{T}\Phi\left(m\right) - \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \exp\left(-\frac{m^2}{2}\right)$$

$$\textbf{Answer} : \mathbf{X_0} = \mathbf{M} - \sigma\sqrt{\mathbf{T}}\left(\mathrm{m}\Phi\left(\mathbf{m}\right) + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\mathbf{m}^2}{2}\right)\right)\ ,\ \textbf{where } \mathbf{m} = \frac{\mathbf{M} - \mathbf{lnS_0} - \left(\mathbf{r} - \frac{\sigma^2}{2}\right)\mathbf{T}}{\sigma\sqrt{\mathbf{T}}}$$

5. This is the autoconversion from the ugly word:

We need to find a price of the classic European option in dollars, while stock prices are in euros. K-strike price. Xt-option price at time t.

$$X_0 = E_{\tilde{p}}\left(exp\left(-r_\$ T\right) X_T \left|\mathcal{F}_0\right.\right) = exp\left(-r_\$ T\right) \int_0^\infty X_T \tilde{p}\left(X_T\right) dX_T$$
$$X_T = U_T S_T - K \; if \; U_T S_T > K, \; 0 \; otherwise$$
$$1)\; S_T = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma \tilde{W}_T\right) \;;\; \tilde{W}_T \sim N\left(0, T\right)$$
$$2)\; U_T = U_0 * \exp\left(\left(r_\$ - \frac{b^2}{2}\right) T + \sigma \tilde{w}_T\right);\; \tilde{w}_T \sim N\left(0, T\right)$$
under condition of no arbitrage $\exp\left(r_\$ T\right) = exp\left(r_\$ T\right) \exp\left(rT\right) \rightarrow \; r_\$ = r_\$ + r$
$$3)\; X_T = U_0 S_0 \exp\left(\left(r_\$ + r - \frac{\sigma^2}{2} - \frac{b^2}{2}\right) T + \sigma \tilde{W}_T + b \tilde{w}_T\right)$$
$\tilde{w}_T$ and $\tilde{W}_T$ *are independent (in common sense), then* $\sigma \tilde{W}_T + b \tilde{w}_T \sim \left(0, \left(\sigma^2 + b^2\right) T\right)$
let$'$*s make a substitution* $Z = \frac{\sigma \tilde{W}_T + b \tilde{w}_T}{\sqrt{\left(\sigma^2 + b^2\right) T}} \sim \left(0, 1\right)$
$$4)\; X_0 = exp\left(-r_\$ T\right) \int_a^\infty \left(U_0 S_0 \exp\left(\left(r_\$ + r - \frac{\sigma^2}{2} - \frac{b^2}{2}\right) T + Z\sqrt{\left(\sigma^2 + b^2\right) T}\right) - K\right) \tilde{p}\left(Z\right) \mathrm{dZ}$$
$$where \; a = -\frac{\ln\left(\frac{U_0 S_0}{K}\right) + \left(r_\$ + r - \frac{\sigma^2}{2} - \frac{b^2}{2}\right) T}{\sqrt{\left(\sigma^2 + b^2\right) T}}$$
$$\int_a^\infty K \tilde{p}\left(Z\right) \mathrm{dZ} = K \int_a^\infty \tilde{p}\left(Z\right) dZ = K \int_{-\infty}^{-a} \tilde{p}\left(Z\right) dZ = K * \Phi\left(\frac{\ln\left(\frac{U_0 S_0}{K}\right) + \left(r_\$ + r - \frac{1}{2}\left(\sigma^2 + b^2\right)\right) T}{\sqrt{\left(\sigma^2 + b^2\right) T}}\right)$$
$$\Phi - cummulative \; standard \; normal \; distribution$$
$$\int_a^\infty \left(U_0 S_0 \exp\left(\left(r_\$ + r - \frac{\sigma^2}{2} - \frac{b^2}{2}\right) T + Z\sqrt{\left(\sigma^2 + b^2\right) T}\right)\right) \tilde{p}\left(Z\right) dZ = dZ =$$
$$= U_0 S_0 \exp\left(\left(r_\$ + r\right) T\right) \int_a^\infty \left(\exp\left(-\frac{1}{2}\left(\sigma^2 + b^2\right) T + Z\sqrt{\left(\sigma^2 + b^2\right) T}\right)\right) \frac{1}{\sqrt{2\pi}} exp\left(-\frac{Z^2}{2}\right) dZ =$$
$$= U_0 S_0 \exp\left(\left(r_\$ + r\right) T\right) \int_a^\infty \exp\left(-\frac{1}{2}\left(Z - \sqrt{\left(\sigma^2 + b^2\right) T}\right)^2\right) dZ =$$
$$\left[\text{let}' s \; make \; a \; substitution \; V = Z - \sqrt{\left(\sigma^2 + b^2\right) T}\right]$$
$$= \left[b = -\frac{\ln\left(\frac{U_0 S_0}{K}\right) + \left(r_\$ + r + \frac{\sigma^2}{2} + \frac{b^2}{2}\right) T}{\sqrt{\left(\sigma^2 + b^2\right) T}}\right] = U_0 S_0 \exp\left(\left(r_\$ + r\right) T\right) \int_b^\infty \exp\left(-\frac{V^2}{2}\right) dV =$$

$$= U_0 S_0 \exp\left(\left(r_\$ + r\right) T\right) \int_{-\infty}^{-b} \exp\left(-\frac{V^2}{2}\right) dV =$$
$$= U_0 S_0 \exp\left(\left(r_\$ + r\right) T\right) * \Phi\left(\frac{\ln\left(\frac{U_0 S_0}{K}\right) + \left(r_\$ + r + \frac{1}{2}\left(\sigma^2 + b^2\right)\right) T}{\sqrt{\left(\sigma^2 + b^2\right) T}}\right)$$
$$X_0 = U_0 S_0 \frac{\exp\left(\left(r_\$ + r\right) T\right)}{exp\left(r_\$ T\right)} * \Phi\left(\frac{\ln\left(\frac{U_0 S_0}{K}\right) + \left(r_\$ + r + \frac{1}{2}\left(\sigma^2 + b^2\right)\right) T}{\sqrt{\left(\sigma^2 + b^2\right) T}}\right) -$$
$$-\frac{K}{\exp\left(r_\$ T\right)} * \Phi\left(\frac{\ln\left(\frac{U_0 S_0}{K}\right) + \left(r_\$ + r - \frac{1}{2}\left(\sigma^2 + b^2\right)\right) T}{\sqrt{\left(\sigma^2 + b^2\right) T}}\right) = \left[r_\$ = r_\$ + r\right] =$$
$$= U_0 S_0 \Phi\left(N_1\right) - \exp\left(-r_\$ T\right) K \Phi\left(N_1 - \sqrt{\left(\sigma^2 + b^2\right) T}\right), \; where \; N_1 = \frac{\ln\left(\frac{U_0 S_0}{K}\right) + \left(r_\$ + r + \frac{1}{2}\left(\sigma^2 + b^2\right)\right) T}{\sqrt{\left(\sigma^2 + b^2\right) T}}$$
$$\textbf{Answer}: \; \mathbf{X_0 = U_0 S_0 ⊠ \left(N_1\right) - \exp\left(-r_\$ T\right) K ⊠ \left(N_1 - \sqrt{\left(\sigma^2 + b^2\right) T}\right)}, \; \textbf{where}$$
$$\mathbf{N_1 = \frac{\ln\left(\frac{U_0 S_0}{K}\right) + \left(r_\$ + r + \frac{1}{2}\left(\sigma^2 + b^2\right)\right) T}{\sqrt{\left(\sigma^2 + b^2\right) T}}}$$

2. 2009-2010

2.1. Exam, 14.01.2010

Final exam consists of the two parts: A and B. Part A lasts for 120 minutes. Upon completion of that part the papers will be collected and the students will have 10 minutes break. Part B lasts for 60 minutes. Students should answer eight of the following eight questions: six from Part A and two from Part B. Points will be deducted for the insufficient explanation within your answers.

Part A. Answer all SIX questions of this section. Each question is worth 10 points.

1. The joint distribution of the random vector (X, Y) is given by its p.d.f
$$f(x,y) = \begin{cases} ce^{x-y}, \text{ for } 0 \leqslant x, y \leqslant 1 \\ 0, \text{ otherwise} \end{cases}$$
where c is a normalization constant. Find $\mathbb{E}(X \mid Y)$.
2. Let $Y_t = W_t^3 - tW_t^4$. Find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$. You don't have to use Ito Lemma here.
3. Let X_n be a discrete time stochastic process that converges in probability to a random number X as $n \to \infty$. Does this condition imply that X_n converges to X in mean? Almost surely? In distribution? Support at least one of your answers with a proof or counterexample. Give a definition for each type of convergence.
4. Seek to optimize the integral $\int_0^{\frac{\ln 2}{2}} (4u - u^2 - x - 3x^2)\, dt$ subject to the conditions $\dot{x} = u + x$, $x(0) = 5/8$, $x\left(\frac{\ln 2}{2}\right)$ is free. Find x, u, λ. What kind of optimum did you find?
5. Solve the bounded control problem $\int_0^1 (2 - 5t)u\, dt$, subject to $\dot{x} = 2x + 4te^{2t}u$, $x(0) = 0$, $x(1) = e^2$, $-1 \leqslant u \leqslant 1$.
6. Consider the following optimization problem: maximize $\sum_0^\infty \left(\frac{3}{4}\right)^t \ln c_t$, subject to $c_t + k_{t+1} = \sqrt{k_t}$, $k_0 > 0$. Let the state variable be k and denote the next period value of k as k'.
   (a) Write down the Bellman equation for the value function $V(k)$
   (b) Using method of undetermined coefficients find $V(k)$
   (c) Find the optimal policy function $k' = h(k)$

Part B. Answer both questions from this section. Each question is worth 20 points. Part B lasts for 60 minutes.

7. Let X_t be a stochastic process such that $dX_t = \frac{X_\infty - X_t}{\tau} dt + \sigma dW_t$, where X_∞ and τ are non-random constants, W_t is a Wiener process, and let $Y_t = X_t e^{t/\tau}$.
   (a) Use Ito Lemma to find both differential and integral expressions for Y_t and use them to express X_t (a.s.) in terms of X_∞, X_0, τ, σ and t. Here X_0 is the value of X_t at time $t = 0$.
   (b) Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$. Sketch the graph of $\mathbb{E}(X_t)$ as a function of t for $X_\infty = 1$, $\tau = 1$, and $X_0 = 0$, 1, and 2. Plot a possible trajectory of X_t in each case. Is there any name or names associated with X_t?
8. Utility $U(C, P) = U_1(C) + U_2(P)$ increases with the consumption C and decreases with the level of pollution P. For $C > 0$, $P > 0$ it is known that $U_1' > 0$, $U_1'' < 0$, $U_2' < 0$ and $U_2'' < 0$. It is known that $\lim_{C \to 0} U_1'(C) = \infty$ and $\lim_{P \to 0} U_2'(C) = 0$.
   Consumption lies within the range $0 \leqslant C \leqslant \bar{C}$. Consumption contributes to pollution, while pollution control reduces it; moreover environment absorbs pollution at a constant rate $b > 0$. Pollution dynamics is governed by equation $\dot{P} = C^2 - (C^*)^2 - bP$ in which the first two terms represent the net contribution to the pollution flow, where $0 < C^* < \bar{C}$.
   Consider the problem of maximizing the discounted ($r > 0$) utility stream $\int_0^\infty e^{-rt} U(C, P)\, dt \to$ max subject to $\dot{P} = C^2 - (C^*)^2 - bP$, $P(0) = P_0 > 0$, $P \geqslant 0$, $0 \leqslant C \leqslant \bar{C}$.
   (a) Derive necessary conditions, using the current value Hamiltonian.
   (b) Sketch the phase diagram for this problem with the pollution and the consumption labeled on the axes.
   (c) Find the condition under which the steady state solution (P_s, C_s) exists and $0 < C_s < \bar{C}$.

(d) Explore the stability of the steady state. Hint. You may find the return to the variables (P, m) easier for solving that part.

2.2. Exam-solutions

1. The probability density should integrate to one, so from $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 1$ we infer the value of c, $c = e/(e-1)^2$.
Using the formula $f(x|y) = f(x,y)/f(y)$ we obtain $f(x|y) = e^x/(e-1)$.
We obtain marginal density, $f(y) = \int_0^1 f(x,y)\,dx = e^{1-y}/(e-1)$ and hence the conditional density $f(x|y) = e^x/(e-1)$. As one may notice it does not depend on y. The reason is that the random variables X and Y are independent and $f(x,y) = f_X(x)f_Y(y)$. So we may expect that $\mathbb{E}(X|Y) = \mathbb{E}(X)$.
And *coup de grâce*
$$\mathbb{E}(X|Y) = \int_0^1 xf(x|Y)\,dx = 1/(e-1)$$
2. $\mathbb{E}(Y_t) = \mathbb{E}(W_t^3 - tW_t^4) = \mathbb{E}(W_t^3) - \mathbb{E}(tW_t^4) = \mathbb{E}(W_t^3) - t\mathbb{E}(W_t^4) = 0 - 3t^3 = -3t^3$, as
$\mathbb{E}(W_t^3) = \frac{3(3-1)}{2}\int_0^t \mathbb{E}(W_s^{3-2})ds = 0$
$\mathbb{E}(W_t^4) = \frac{4(4-1)}{2}\int_0^t \mathbb{E}(W_s^{4-2})ds = 6\int_0^t sds = 3t^2$
$\mathrm{Var}(Y_t) = \mathbb{E}(Y_t^2) - (\mathbb{E}(Y))^2$
$\mathbb{E}(Y_t^2) = \mathbb{E}(W_t^3 - tW_t^4)^2 = \mathbb{E}(W_t^6 - 2tW_t^7 + t^2W_t^8) = \mathbb{E}(W_t^6) - 2t\mathbb{E}(W_t^7) + t^2\mathbb{E}(W_t^8) = 15t^3 + 105t^6$,
as $\mathbb{E}(W_t^5) = \frac{5(5-1)}{2}\int_0^t \mathbb{E}(W_s^{5-2})ds = 0$
$\mathbb{E}(W_t^6) = \frac{6(6-1)}{2}\int_0^t \mathbb{E}(W_s^{6-2})ds = 15\int_0^t 3s^2ds = 15t^3$
$\mathbb{E}(W_t^7) = \frac{7(7-1)}{2}\int_0^t \mathbb{E}(W_s^{7-2})ds = 0$
$\mathbb{E}(W_t^8) = \frac{8(7-1)}{2}\int_0^t \mathbb{E}(W_s^{8-2})ds = 105t^4$
$\mathrm{Var}(Y_t) = \mathbb{E}(Y_t^2) - (\mathbb{E}(Y))^2 = 15t^3 + 105t^6 - (-3t^3)^2 = 15t^3 + 96t^6$
3.
4.
5.
6.
7.
$$dY_t = \frac{1}{\tau}X_te^{\frac{t}{\tau}}dt + e^{\frac{t}{\tau}}dX_t = \frac{1}{\tau}X_te^{\frac{t}{\tau}}dt + \frac{X_\infty - X_t}{\tau}e^{\frac{t}{\tau}}dt + \sigma e^{\frac{t}{\tau}}dW_t =$$
$$= \frac{X_\infty}{\tau}e^{\frac{t}{\tau}}dt + \sigma e^{\frac{t}{\tau}}dW_t$$
$$Y_t = Y_0 + \int_0^t \frac{X_\infty}{\tau}e^{\frac{u}{\tau}}du + \int_0^t \sigma e^{\frac{u}{\tau}}dW_u = X_0 + X_\infty(e^{\frac{t}{\tau}} - 1) + \int_0^t \sigma e^{\frac{u}{\tau}}dW_u$$
$$X_t = e^{-\frac{t}{\tau}}Y_t = e^{-\frac{t}{\tau}}(X_0 + X_\infty(e^{\frac{t}{\tau}} - 1) + \int_0^t \sigma e^{\frac{u}{\tau}}dW_u) = X_0e^{-\frac{t}{\tau}} + X_\infty(1 - e^{-\frac{t}{\tau}}) + \int_0^t \sigma e^{\frac{u-t}{\tau}}dW_u$$
$$\mathbb{E}(X_t) = X_0e^{-\frac{t}{\tau}} + X_\infty(1 - e^{-\frac{t}{\tau}})$$

$$\text{Var}(X_t) = \text{Var}(\int_0^t \sigma e^{\frac{u-t}{\tau}} dW_u) = \int_0^t \mathbb{E}(\sigma e^{\frac{u-t}{\tau}})^2 du = \int_0^t \sigma^2 e^{\frac{2u-2t}{\tau}} du = \frac{\tau\sigma^2}{2}(1 - e^{\frac{-2t}{\tau}})$$

3. 2010-2011

3.1. Exam, 12.01.2011

Notation: W_t is the standard Wiener process.

Part A (10 points each problem). Time allowed: 120 minutes.

1. The joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} x + y, & x \in [0; 1],\ y \in [0; 1] \\ 0, & \text{otherwise} \end{cases}$$

   Find $\mathbb{E}(Y|X)$ in terms of X, find the probability density function of $\mathbb{E}(Y|X)$
2. Consider the process $X_t = \int_0^t sW_s\, dW_s$. Find $\mathbb{E}(X_t)$, $\text{Var}(X_t)$, $\text{Cov}(X_t, W_t)$
3. The process Y_t is given by $Y_t = 2W_t + 5t$. The stopping time τ is given by $\tau = \min\{t|Y_t^2 = 100\}$. Find the distribution of the random variable Y_τ and the expected value $\mathbb{E}(\tau)$.
   Hint: you may find the martingales a^{Y_t} and $Y_t - f(t)$ useful
4. Find $\mathbb{P}(W_2 - W_1 > 2)$
5. In the framework of Black and Scholes model find the price of an asset which gives you the payoff of 1 rubble only if the final price S_t is at least two times bigger than the initial price S_0 of the asset.
6. Consider the free end problem, where $T > 0$ is not given

$$\int_0^T (\dot{x}^2 - x + 1)dt \to extr$$

   At the left end $x(0) = 0$ Find the optimal T value and the extremal. Check that you solved the optimality problem or show the opposite.

Part B (20 points each prolem). Time allowed: 60 minutes.

7. Consider the stochastic differential equation

$$dX_t = (\sqrt{1 + X_t^2} + 0.5X_t)dt + \sqrt{1 + X_t^2} dW_t, \quad X_0 = 0$$

   (a) Suppose that Y_t is another process that depens only on X_t, i.e. $Y_t = f(X_t)$. Find dY using the Ito's lemma.

   (b) Find such function f that the term before dW in dY is constant.

   (c) Find X_t

   (d) Sketch $\mathbb{P}(X_t > 0)$ as the function of t.

8. Consider the neoclassical optimal growth model

$$\int_0^\infty e^{-rt} \left(\bar{U} - \frac{1}{c(t)} \right) dt \to \max$$

   subject to $\dot{k} = A \ln(1 + k) - c - \delta k$, where $A > r > \delta > 0$, $k(0) = k_0$, $\bar{U} > 0$.

   (a) Derive necessary conditions, using the current value Hamiltonian

(b) Sketch the phase diagram for this problem with the capital intensity and consumption labeled on the horizontal and vertival axes, respectively

(c) Check that the steady state solution exists. Provide explanation.

(d) Explore the stability of the steady state, using the Jacobian

(e) Why are you sure the found growth path maximizes the discounted stream of utility?

3.2. Exam-solutions

1. Let's start, $f(x) = \int_0^1 f(x,y)\,dy = x + 1/2$ for $x \in [0;1]$. The conditional density, $f(y|x) = f(x,y)/f(x) = (x+y)/(x+1/2)$ for x and y in $[0;1]$.
The conditional expected value:
$$\mathbb{E}(Y|X) = \int_0^1 yf(y|X)\,dy = \frac{3X+2}{6X+3} = \frac{1}{2} + \frac{1}{12X+6}$$
We also need to find the density of $\hat{Y} = \frac{3X+2}{6X+3}$.
Let's first find the probability distribution function:
$$F_{\hat{Y}}(t) = \mathbb{P}(\hat{Y} \leqslant t) = \mathbb{P}\left(\frac{1}{2} + \frac{1}{12X+6} \leqslant t\right) = \mathbb{P}\left(X \geqslant \frac{2-3t}{6t-3}\right) = 1 - F_X(a),$$
where $a = \frac{2-3t}{6t-3}$ and $F_X(a) = \int_0^a f(t)\,dt$.
Take the derivative with respect to t to obtain:
$$f_{\hat{Y}}(t) = -f_X(a)\frac{da}{dt} = -\left(\frac{2-3t}{6t-3} + \frac{1}{2}\right)\frac{-3}{(6t-3)^2}$$

2. First, $\mathbb{E}(X_t) = 0$. Using Ito's isometry:
$$\text{Var}(X_t) = \int_0^t \mathbb{E}(s^2W_s^2)\,ds = \int_0^t s^3\,ds = t^4/4$$
Again using Ito's isometry:
$$\text{Cov}(X_t, W_t) = \text{Cov}\left(\int_0^t sW_s\,dW_s, \int_0^t 1\,dW_s\right) = \int_0^t \mathbb{E}(sW_s \cdot 1)\,ds = 0$$

3. First we find distribution of Y_τ:

Y_τ	-10	10
	p	$1-p$

We introduce $Z_t = a^{Y_t}$. To make Z_t martingale we should eliminate dt part in dZ_t. Hence $\ln a = -2.5$. And $Z_t = e^{-2.5Y_t}$ is a martingale.

Z_τ	e^{25}	e^{-25}
	p	$1-p$

From Doob's theorem: $\mathbb{E}(Z_\tau) = Z_0 = 1$. And we obtain p from linear equation
$$pe^{25} + (1-p)e^{-25} = 1$$
The process $R_t = Y_t - 5t$ is a martingale. By Doob's theorem $\mathbb{E}(R_\tau) = R_0 = 0$, but

$$\mathbb{E}(R_\tau) = \mathbb{E}(Y_\tau) - 5\mathbb{E}(\tau)$$

So,

$$\mathbb{E}(\tau) = \mathbb{E}(Y_{tau})/5 = -2p + 2(1-p) = 2 - 4p$$

4. We know that $W_2 - W_1 \sim \mathcal{N}(0; 1)$, therefore
$\mathbb{P}(W_2 - W_1 > 2) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228$
5.
$$X_T = \begin{cases} 1 & \text{if } \frac{S_T}{S_0} > 2, \\ 0 & \text{otherwise} \end{cases}$$

So,

$$X_0 = e^{-rT} E_{\tilde{\mathbb{P}}}(X_T|\mathcal{F}_0) = e^{-rT}\tilde{\mathbb{P}}\left(\frac{S_T}{S_0} > 2|\mathcal{F}_0\right) = e^{-rT}\tilde{\mathbb{P}}\left(\left(\exp\left(r - \frac{\sigma^2}{2}\right)T + \sigma\tilde{W_T}\right) > 2|\mathcal{F}_0\right)$$

$$\sigma\tilde{W_T} > \ln 2 - \left(r - \frac{\sigma^2}{2}\right)T \Rightarrow \tilde{W_T} > \frac{\ln 2 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma}$$

We want standard normal distribution of $\tilde{W_T}$, so we standardize by $\sqrt{T}$:

$$\frac{\tilde{W_T}}{\sqrt{T}} > \frac{\ln 2 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

So,

$$X_0 = e^{-rT}\tilde{\mathbb{P}}\left(\frac{\tilde{W_T}}{\sqrt{T}} > \frac{\ln 2 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = e^{-rT}\left(1 - \Phi\left(\frac{\ln 2 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right)$$

3.3. Retake, ??.01.2011

Answer all SIX questions of this section. Each question is worth 10 points.

1. The joint distribution of the random vector (X, Y) is given by its p.d.f

$$f(x, y) = \begin{cases} ce^{x-y}, \text{ for } 0 \leqslant x, y \leqslant 1 \\ 0, \text{ otherwise} \end{cases}$$

   where c is a normalization constant. Find $\mathbb{E}(X \mid Y)$.
2. Let $Y_t = W_t^3 - tW_t^4$. Find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$. You don't have to use Ito Lemma here.
3. Let X_n be a discrete time stochastic process that converges in probability to a random number X as $n \to \infty$. Does this condition imply that X_n converges to X in mean? Almost surely? In distribution? Support at least one of your answers with a proof or counterexample. Give a definition for each type of convergence.
4. Solve the calculus of variations problem to optimize the integral $\int_{1/2}^{1} \sqrt{1 + \dot{x}^2}/x \, dt \to \max$ subject to the conditions $x(1/2) = \sqrt{3}/2$, $x(1) = 1$. Justify your answer referring to the sufficiency conditions. What kind of extremum did you find?
5. Let $x(t)$ represent the revenue of a firm. The fraction of it, namely $xu(t)$, where $0 \leqslant u \leqslant 1$, is spent on investments allowing the revenue to grow according to the rule $\dot{x} = \alpha x u$ with

$\alpha = const > 0$.

Another fraction of the revenue βx, $\beta = const > 0$ serves to reimburse the costs. Maximize the profit of the firm over the finite time horizon

$$\int_0^T (1 - \beta - u)x\,dt \to \max$$

subject to $\dot{x} = \alpha x u$, $x(0) = x_0$, $0 \leqslant u \leqslant 1$.

6. Consider the following <<cake-eating>> problem: maximize

$$\sum_0^\infty \beta^t \ln c_t$$

subject to $W_{t+1} = W_t - c_t$, $W_t \leqslant W$, $W > 0$, $0 < \beta < 1$. Let the state variable be W and denote the next period value of W as W'.

   (a) Write down the Bellman equation for the value function $V(W)$

   (b) Using method of undetermined coefficients find $V(W)$

   (c) Find the optimal policy function $c = h(W)$

Part B. Answer both questions from this section. Each question is worth 20 points. Part B lasts for 60 minutes.

7. Let X_t be a stochastic process such that $dX_t = \frac{X_\infty - X_t}{\tau} dt + \sigma dW_t$, where X_∞ and τ are non-random constants, W_t is a Wiener process, and let $Y_t = X_t e^{t/\tau}$.

   (a) Use Ito Lemma to find both differential and integral expressions for Y_t and use them to express X_t (a.s.) in terms of X_∞, X_0, τ, σ and t. Here X_0 is the value of X_t at time $t = 0$.

   (b) Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$. Sketch the graph of $\mathbb{E}(X_t)$ as a function of t for $X_\infty = 1$, $\tau = 1$, and $X_0 = 0$, 1, and 2. Plot a possible trajectory of X_t in each case. Is there any name or names associated with X_t?

8. Consider the profit maximizing problem over the infinite time horizon

$$\int_0^\infty e^{-rt}(2\sqrt{K} - cI)\,dt \to \max$$

where K is capital, I is investment, $c = const$ is the unit cost of investment, r is discount rate. Let $K(0) = K_0 > 0$. The capital changes under the investment equation $\dot{K} = I - bK$, where $b > 0$ is the depreciation rate. Investment is bounded $0 \leqslant I \leqslant \bar{I}$. Suppose the parameters of the problem satisfy conditions

$$K_0 < \frac{1}{c^2(r+b)^2} < \frac{\bar{I}}{b}$$

   (a) Derive necessary conditions, using the current value Hamiltonian

   (b) Does the steady-state solution(s) (K_s, I_s) exist? Explain. If you answer is positive explore its stability.

3.4. Retake-solutions

1. Let's go:

$$f(x,y) = \begin{cases} ce^{(x-y)}, 0 \leqslant x, y \leqslant 1 \\ 0, else \end{cases}$$

First, we need to find constant c.
$1 = \int_0^1 \int_0^1 (ce^{x-y}) dxdy = \int_0^1 (-ce^{x-y}) \Big|_0^1 dx = \int_0^1 (-ce^{x-1} + ce^x) dx = -ce^{x-1} + +ce^x \Big|_0^1 = -c + ce + \frac{c}{e} - c = 1$
From this simple equation we get $c = \frac{e}{(e-1)^2}$
So the problem now looks like:

$$f(x,y) = \begin{cases} \frac{e}{(e-1)^2} e^{(x-y)}, 0 \leqslant x, y \leqslant 1 \\ 0, else \end{cases}$$

$E(X|Y) = \int_0^1 x f(x|y) dx$, where $f(x|y) = \frac{f(x,y)}{f(y)}$
$f(y) = \int_0^1 f(x,y) dx = \frac{e}{(e-1)^2}(e^{1-y} - e^{-y})$
$f(x|y) = \frac{e^x}{e-1}$
$E(X|Y) = \int_0^1 \frac{xe^x}{e-1} dx = \frac{1}{e-1} \int_0^1 x de^x = \frac{1}{e-1}(xe^x - e^x) \Big|_0^1 = \frac{1}{e-1}$
So the final answer will be $E(X|Y) = \frac{1}{e-1}$

2. $\mathbb{E}(Y_t) = \mathbb{E}(W_t^3 - tW_t^4) = \mathbb{E}(W_t^3) - \mathbb{E}(tW_t^4) = \mathbb{E}(W_t^3) - t\mathbb{E}(W_t^4) = 0 - 3t^3 = -3t^3$, as
$\mathbb{E}(W_t^3) = \frac{3(3-1)}{2} \int_0^t \mathbb{E}(W_s^{3-2}) ds = 0$
$\mathbb{E}(W_t^4) = \frac{4(4-1)}{2} \int_0^t \mathbb{E}(W_s^{4-2}) ds = 6 \int_0^t s ds = 3t^2$
$\text{Var}(Y_t) = \mathbb{E}(Y_t^2) - (\mathbb{E}(Y))^2$
$\mathbb{E}(Y_t^2) = \mathbb{E}(W_t^3 - tW_t^4)^2 = \mathbb{E}(W_t^6 - 2tW_t^7 + t^2 W_t^8) = \mathbb{E}(W_t^6) - 2t\mathbb{E}(W_t^7) + t^2 \mathbb{E}(W_t^8) = 15t^3 + 105t^6$,
as $\mathbb{E}(W_t^5) = \frac{5(5-1)}{2} \int_0^t \mathbb{E}(W_s^{5-2}) ds = 0$
$\mathbb{E}(W_t^6) = \frac{6(6-1)}{2} \int_0^t \mathbb{E}(W_s^{6-2}) ds = 15 \int_0^t 3s^2 ds = 15t^3$
$\mathbb{E}(W_t^7) = \frac{7(7-1)}{2} \int_0^t \mathbb{E}(W_s^{7-2}) ds = 0$
$\mathbb{E}(W_t^8) = \frac{8(7-1)}{2} \int_0^t \mathbb{E}(W_s^{8-2}) ds = 105t^4$
$\text{Var}(Y_t) = \mathbb{E}(Y_t^2) - (\mathbb{E}(Y))^2 = 15t^3 + 105t^6 - (-3t^3)^2 = 15t^3 + 96t^6$

3.

3.5. Reretake, 08.02.2011

Stochastic calculus part

1. (10 pts) The joint density of random variables X and Y is given by $f(x,y) = x+y$ for $x \in [0;1]$ and $y \in [0;1]$. Find $\mathbb{E}(Y|X)$ and $\text{Var}(Y|X)$
2. (10 pts) The random variables X_i are independent and uniformly distributed on $[0;2011]$. We define $Y_n = (X_1 \cdot X_2 \cdot \ldots \cdot X_n)^{1/n}$. In what sense does the sequence Y_n converge? What is the limit? Find the expected value and the variance of the limit.
3. (10 pts) Let $W_1(t)$ and $W_2(t)$ be two independent Wiener processes. Consider $Y_t = aW_1(t) + \sqrt{1-a^2} W_2(t)$. Is Y_t a Wiener process?

4. (20 pts) Consider the Vasicek interest rate model, $dR_t = a(b - R)\,dt + s\,dW_t$, where a, b and s are positive constants.
   (a) Using the substitution $Y_t = e^{-at}R_t$ find the solution of the stochastic differential equation
   (b) Find $\mathbb{E}(R_t)$ and $\mathrm{Var}(R_t)$

4. 2011-2012

4.1. Retake, 03.02.2012

Section A. 10 points for each problem.

1. Solve the bounded control problem with the free end value where $T > 1$ is specified in advance but $x(T)$ is not.
$$\int_0^T (x - u)\,dt \to \max,\ x(0) = 1$$
$x' = u$, $0 \leqslant u \leqslant x$. What guarantees that a maximizer would be found?
2. Two stochastic processes are defined by the system of SDE with the Brownian motions W_{1t}, W_{2t} independent of each other
$$\begin{cases} dX_t = (2 + 5t + X_t)dt + 3dW_{1t} \\ dY_t = 4Y_t dt + 8Y_t dW_{1t} + 6dW_{2t} \end{cases}$$
Calculate $d(X_tY_t)$
3. Find the value of the constant a such that the process $X_t = W_t^3 - a\int_0^t W_s\,ds$ is a martingale.
4. The process Y_t is given by $Y_t = 2W_t + 5t$. The stopping time τ is given by $\tau = \min\{t | Y_t^2 = 100\}$. Find the distribution of the random variable Y_τ and the expected value $\mathbb{E}(\tau)$.
Hint: you may find the martingales a^{Y_t} and $Y_t - f(t)$ useful
5. What is the expected value and variance of W_t^2 for $t > s$ given that $W_s = x$?
6. In the framework of Black and Scholes model find the price at the time 0 of an asset which gives you the payoff $\max\{\ln(S_t), 0\}$ at the time t. Here S_t is the price of the underlying asset.

Section B. 20 points for each problem.

7. The goal of this exercise is to solve the SDE
$$dX_t = \frac{1}{X_t}dt + X_t dW_t$$
with initial condition $X_0 = 1$.
   (a) Apply the Ito's lemma to the process $Y_t = \exp(f(t) + g(W_t))X_t^2$
   (b) Find non-constant functions f and g such that the coefficient before dW_t in the expression for dY_t is zero.
   (c) Find X_t. The final expression may contain a Riemann integral of some stochastic process.
8. A typical firm with the production function $x = f(l)$ in an economy employs 1 unit of the capital paying for it $\bar{r}$. There are n identical firms in this economy where $n(t)$ is a function of time. Let the output of a firm be $x(t)$. Then the aggregate supply equals nx. Assuming that the market is in the equilibrium we use the inverse demand function $p(nx)$, where $p'(y) < 0$. Equation defining the dynamics of labor is given by $\frac{dl}{dt} = a(w - \bar{w})$, where $a > 0$ and $\bar{w}$

is some equilibrium wage rate. The growth (or fall) in number of firms is governed by the equation $\frac{dn}{dt} = b(px - \bar{w}l - \bar{r})$, where $b > 0$. Write down the system of ODE for the unknowns $l(t)$, $n(t)$, using the first-order condition for the profit-maximizing firm, find and classify the steady-state solutions (if any exist). Production function is twice differentiable and concave everywhere.

4.2. Retake-solutions

1.
2.
$$d(X_t Y_t) = 0\,dt + X_t\,dY_t + Y_t\,dX_t + \frac{1}{2}dX_t dY_t$$
Here W_{1t} and W_{2t} are independent and $dW_{1t}\,dW_{2t} = 0$.
3. Using Ito-Doeblin lemma, $dW_t^3 = 3W_t^2\,dW_t + \frac{1}{2}6W_t\,dt$. Hence,
$$dX_t = 3W_t^2\,dW_t + (3-a)W_t\,dt$$
Ito's integral is always a martingale, so $a = 3$.
4. The distribution of Y_τ is given by the table

y	-10	10
$\mathbb{P}(Y_\tau = y)$	$1-p$	p

First we find two martingales, $Z_t = Y_t - 5t$, and $X_t = \exp(-2.5Y_t)$. Here Ito's lemma is useful. According to Doob's theorem $\mathbb{E}(X_t) = X_0 = 1$ and we have an equation for p:
$$p \cdot \exp(-25) + (1-p) \cdot \exp(25) = 1$$
We use once again Doob's theorem, $\mathbb{E}(Z_t) = Z_0 = 0$, so
$$\mathbb{E}(Y_\tau - 5\tau) = 0$$
The value of $\mathbb{E}(\tau)$ may be calculated as $\mathbb{E}(\tau) = \mathbb{E}(Y_\tau)/5 = \frac{10p-10(1-p)}{5}$.
5. Let's first consider the unconditional version of the problem.
$$E(W_t^2) = E((W_t - W_0)^2) = Var(W_t - W_0) + E(W_t - W_0)^2 = t - 0 = t$$
$$Var(W_t^2) = E(W_t^4) - E(W_t^2)^2 = E(W_t^4) - t^2$$
In order to compute $E(W_t^4)$ we use the following formula:
$$E(W_t^k) = \frac{k(k-1)}{2}\int_0^t E(W_s^{k-2})ds$$
So,
$$E(W_t^4) = 6\int_0^t E(W_s^2)ds = 6\int_0^t sds = 3t^2 \Rightarrow \text{Var}(W_t^2) = 3t^2 - t^2 = 2t^2$$
However we know that $W_s = x$. Using the properties of Wiener process we may assume that the timer is started again from 0. So we define $W'_{t'} = W_{t'+s} - x$.
So,
$$\mathbb{E}(W_t^2|W_s = x) = \mathbb{E}((W'_{t'} + x)^2) = \mathbb{E}(W'^2_{t'} + 2xW'_{t'} + x^2) = t' + x^2 = (t-s) + x^2$$

And

$$\mathrm{Var}(W_t^2|W_s = x) = \mathrm{Var}((W'_{t'} + x)^2) = \mathrm{Var}(W'^2_{t'} + 2xW'_{t'} + x^2) = \\ = \mathrm{Var}(W'^2_{t'}) + 4x^2\,\mathrm{Var}(W'_{t'}) + 0 = 2(t-s)^2 + 4x^2(t-s) \quad (1)$$

5. 2012-2013

5.1. Midterm, 13.11.2012

1. (25 points) Solve the calculus of variations problem (use necessary conditions)

$$\int_0^{\pi/2} (x^2 + \dot{x}^2 - x\dot{x} + 4x\cos^2 t)\,dt \to extr,$$

where $x(0) = -6/5$, $x(\pi/2) = -4/5$.
What kind of a sufficient condition is applicable to this problem and why?

2. (25 points) Solve the control problem with the free end value

$$\int_0^T (-u^2/2 - x)\,dt \to \max,$$

where $x(0) = x_0 > 0$, $x(T)$ is unknown, $\dot{x} = u^2 x$.
Can you apply any of the sufficiency theorems to prove that the maximizer has been found? Why we are sure in the validity of the solution?

3. (20 points) Consider an optimal growth problem $\sum_{t=0}^{\infty} \frac{1}{5^t} \ln c_t \to \max$, subject to $k_{t+1} = k_t^{7/8} - c_t$, $k(0) = k_0 > 0$.
   (a) State the Bellman equation
   (b) Use the iteration method to find the first two iterations of the value function. Set $V_0(k) = 0$.
   (c) Use the guess-and-verify method to find the value function.

4. (30 points) Consider a profit-maximizing firm over the infinite horizon planning period

$$\int_0^{\infty} (pf(K,I) - cK - gI)e^{-rt}\,dt \to \max,$$

subject to $\dot{K} = I - \delta K$, $K(0) = K_0 > 0$.
The production function $f(K,I) = (1 - \alpha I^2)\ln K$, where parameter α is so small that f takes only positive values within the reasonable values of the gross investment rate I which can take any sign (investment/disinvestment). Moreover, the production function is concave everywhere. Using the current value Hamiltonian derive the system of differential equations in $K(t)$ and $I(t)$. Prove that the steady-state solution exists. By calculating the Jacobian at the equilibrium prove that (K_s, I_s) is the saddle point. All exogeneous parameters in this model are positive.

5.2. Exam, 10.01.2013

Optimal control part.

1. [10 points] Solve the optimal control problem $\int_0^1 (x+u)\,dt \to \max$ subject to $\dot{x} = -x+u+t$, $x(0)=0$, $x(1)$ is free, and $0 \leqslant u \leqslant 1$.
2. [10 points] Consider an intertemporal utility maximization problem over the finite horizon in discrete time $t = 1, \ldots, T$. The utility is $u(c) = \ln c$, the initial wealth is w and if the remaining wealth at time t was w_t, then by the beginning of the next period it becomes $w_{t+1} = (1+r)(w_t - c_t)$, where r is the discount rate and c_t is the consumption. To find the maximum value of the utility stream one can use the finite version of the Bellman equation written in the form $V_t(w) = \max\{u(c) + V_{t+1}((1+r)(w-c))\}$ where the function within the braces is maximized over the values of $0 \leqslant c \leqslant w$. It is important to note that the value function V_t is the maximum value of the utility stream from time t onwards.
   (a) [3 points] Let $V_T = \ln w$. Find V_{T-1}.
   (b) [7 points] Try to verify the conjecture that $V_t = \gamma_t \ln(1+r) + (T-t+1)\ln\frac{w}{T-t+1}$ where γ_t is some (unknown) function of time.
3. [20 points] A spill of toxic substance should be cleaned up by a company. If a cleanup rate is $u(t)$, then the area of the spill shrinks by the law $x(t) = x_0 - \int_0^t u(s)\,ds$, where x_0 is the initial area. According to the government contract by the time T (and not earlier) the area of the spill should be reduced to x_T, where $x_T < x_0$. Clearly $u(t) \geqslant 0$, and by technology requirement $u(t) \leqslant \bar{u}$, where $\bar{u} > (x_0 - x_T)/T$. Costs of cleaning are directly proportional to $u(t)$. Let the unit costs equal $c > 0$ and the costs are discounted with the discount rate $r > 0$.
   (a) [5 points] Formulate the minimization problem with the bounded control. Hint: this is a fixed endpoints problem.
   (b) [15 points] By using the current value Hamiltonian set the system of equations and solve it.

Stochastic calculus part. Here W_t always denotes the standard Wiener process.

4. [10 points] Waves are arriving on the seashore. The sizes of the waves are independent uniform on $[0;1]$ random variables X_i. If the size of i-th wave is greater than the size of its neighboring waves, $X_i > \max\{X_{i-1}, X_{i+1}\}$, then we call it a «high» wave. Let H_i be the indicator of the «high» waves: $H_i = 1$ if $X_i > \max\{X_{i-1}, X_{i+1}\}$ and $H_i = 0$ otherwise.
   Find $\mathbb{E}(X_i \mid H_i)$ and $\mathbb{E}(H_i \mid X_i)$
5. [10 points] Find $\mathbb{E}(W_s \cdot W_t)$ and $\mathbb{E}(W_r \cdot W_s \cdot W_t)$ where $r < s < t$.
6. [10 points] Consider the process $X_t = \int_0^t s^2 W_s\,dW_s$. Find $\mathbb{E}(X_t)$, $\mathrm{Var}(X_t)$, $\mathrm{Cov}(X_t, W_t)$
7. [10 points] In the framework of Black and Scholes model find the price at the time 0 of an asset which gives you the payoff $\ln(S_T)$ at the time T. Here S_t is the price of the underlying asset at time t.
8. [20 points] The goal of this exercise is to solve the SDE

$$dX_t = -9X_t^2(1-X_t)dt + 3X_t(1-X_t)dW_t, \quad X_0 = 1/2$$

   (a) Using Ito's lemma find dY_t for the process $Y_t = f(X_t)$
   (b) Find such a function $f()$ that the term before dt in dY_t is zero
   (c) Obtain a simple differential equation for Y_t. It should not contain X_t.
   (d) Solve the stochastic differential equation for Y_t
   (e) Express X_t as a function of W_t and t.

5.3. exam 10.01.2013-hints

1. $\mathbb{E}(H_i \mid X_i) = X_i^2$, $\mathbb{E}(X_i \mid H_i = 1) = 3/4$, $\mathbb{E}(X_i \mid H_i = 0) = 3/8$, $\mathbb{E}(X_i \mid H_i) = (3 + 3H_i)/8$
2. $\mathbb{E}(W_s \cdot W_t) = s$ and
   Let's introduce deltas: $D_1 = W_s - W_r$ and $D_2 = W_t - W_s$. The bonus of this notation is that D_1, D_2 and W_r are independent random variables. And

   $$\mathbb{E}(W_r \cdot (W_r + D_1) \cdot (W_r + D_1 + D_2)) = \mathbb{E}(W_r^3 + W_r^2 D_1 + W_r^2 D_2 + W_r^2 D_1 + W_r D_1^2 + W_r D_1 D_2) = 0+0+0+0+0$$

3. $\mathbb{E}(X_t) = 0$, $\mathrm{Var}(X_t) = t^6/6$, $\mathrm{Cov}(X_t, W_t) = 0$
4. $C_0 = e^{-rT}(\ln S_0 + (r - \sigma^2/2)T$
5. Equation for f has the form $f''(x)(1-x) - 2f'(x) = 0$. Use the substituion $f'(x) = h(x)$. This ordinary differential equation has many solutions, one of them is $f(x) = 1/(1-x)$. We may take any function f that makes initial stochastic differential equation less diffucult, so we take $f(x) = 1/(1-x)$, and $Y_t = 1/(1-X_t)$.
   And
   $$dY_t = 3X_t/(1-X_t)\,dW_t$$
   We may notice that $X_t/(1-X_t) = 1/(1-X_t) - 1 = Y_t - 1$, so
   $$dY_t = 3(Y_t - 1)\,dW_t$$
   We use the substitution $Z_t = Y_t - 1$, so $dZ_t = 3Z_t\,dW_t$ and then the substitution $Q_t = \ln Z_t$.

5.4. Retake, 08.02.2013

Stochastic calculus part. Here W_t denotes standard Wiener process

1. (10 points) Random variables X and Y are jointly normal with zero expected values, unit variances and correlation ρ. Find $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(Y^2 \mid X)$
2. (10 points) Consider the processes $X_t = \int_0^t s^3 W_s\,dW_s$ and $Y_t = \int_0^t W_s\,dW_s$. Find $\mathbb{E}(X_t)$, $\mathrm{Var}(X_t)$, $\mathrm{Cov}(X_t, Y_t)$
3. (10 points) The stochastic process Y_t is given by equation $Y_t = W_t^4 - 6tW_t^2 + 3t^2$. Find dY_t and $\mathbb{E}(Y_7 \mid Y_2)$
4. (20 points) Consider the stochastic differential equation
   $$dX_t = X_t dt + X_t dW_t, \quad X_0 = 1$$
   (a) Solve this stochastic differential equation

   (b) Let $\tau = \inf_{t>0}\{t : X_t \geqslant R\}$. Find $\mathbb{E}(\tau)$ applying the optional stopping theorem to the process W_t.

Optimal control part.

5. (10 points) Solve the optimal control problem: $\int_0^1 u^2\,dt \to \min$, subject to $\dot{x} = x + u$, $x(0) = 1$.

6. (10 points) Consider a maximization problem over the finite horizon in discrete time $t = 1, \ldots, T$:

$$\sum_{t=1}^{T} \left(\frac{t^3}{3} + \frac{t^2}{2} \right) a_t^2 \to \max$$

subject to constraint $\sum_{t=1}^{T} a_t = c$, $a_t \geqslant 0$.
By introducing $w_t = \sum_{i=1}^{t} a_i$ reduce that problem to dynamic programming (3 points) and solve it (7 points).

7. (20 points) Consider the profit-maximizing problem for a representative competitive firm

$$\int_0^\infty (p - c(x(t)))q(t)e^{-rt}\, dt \to \max$$

subject to (*) $\dot{x} = 1 - x - q$, where the state variable $x(t) < 1$ represents a nonrenewable stock resource (oil) that depletes according to the equation (*) and $q(t)$ is the extraction rate. Here $c(x)$ is a cost function of the extraction that is defined by $c(x) = e^{-x}$. The price of oil is assumed to be constant and equal p where $1/e < p < 1$. The optimization problem is to choose $q(t)$ to maximize the discounted profits, $0 < r < 1$.

   (a) (10 points) Derive necessary conditions.

   (b) (10 points) Prove that the steady-state exists.

5.5. retake — short hints

1. First we decompose Y as $Y = aX + bZ$, where Z is independent from X and has unit variance. To obtain a and b we make system:

$$\begin{cases} \operatorname{Var}(Y) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Z) = 1 \\ \operatorname{Cov}(Y, X) = a \operatorname{Cov}(X, X) = \rho \end{cases}$$

   So, $a = \rho$ and $b = \sqrt{1 - \rho^2}$. The variable Y is decomposed as $Y = \rho X + \sqrt{1 - \rho^2} Z$. Then

$$\mathbb{E}(Y|X) = \mathbb{E}(\rho X + \sqrt{1 - \rho^2} Z|X) = \rho X + 0$$

   And

$$\mathbb{E}(Y^2|X) = \mathbb{E}(\rho^2 X^2 + (1 - \rho^2)Z^2 + 2\rho\sqrt{1 - \rho^2} XZ|X) = \rho^2 X^2 + (1 - \rho^2) \cdot 1 + 2 \cdot 0$$

2. $\mathbb{E}(X_t) = \mathbb{E}(Y_t) = 0$, for variance and covariance we use isometry property

$$\operatorname{Var}(X_t) = \operatorname{Var}\left(\int_0^t s^3 W_s \, dW_s \right) = \int_0^t \mathbb{E}(s^3 W_s)^2 \, ds = \int_0^t s^7 \, ds = \frac{t^8}{8}$$

$$\operatorname{Cov}(X_t, Y_t) = \operatorname{Cov}\left(\int_0^t s^3 W_s dW_s, \int_0^t W_s dW_s \right) = \int_0^t \mathbb{E}(s^3 W_s^2) ds = \int_0^t s^4 ds = \frac{t^5}{5}$$

3. Calculate dY_t. We observe that $dY_t = (4W_t^3 - 12tW_t)\, dW_t$, so Y_t is a martingale and $\mathbb{E}(Y_7 \mid Y_2) = Y_2$.
4. Use the substitution $Y_t = \ln X_t$. Then, using Ito's lemma we obtain:

$$dY_t = \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(dX_t)^2$$

Initial SDE gives us $dX_t = X_t dt + X_t dW_t$, so we can compute also $(dX_t)^2$ and plug it in the dY_t.

$$dX_t^2 = (X_t dt + X_t dW_t)^2 = X_t^2 dt$$

Recall that $dW_t \cdot dW_t = dt$, $dW_t \cdot dt = 0$ and $dt \cdot dt = 0$
Plugging in we get:

$$dY_t = \frac{X_t dt + X_t dW_t}{X_t} - \frac{X_t^2 dt}{2X_t^2} = dt + dW_t - \frac{dt}{2} = dW_t + \frac{dt}{2}$$

The full form for Y_t:

$$Y_t = Y_0 + \int_0^t dW_s + \int_0^t \frac{ds}{2} = Y_0 + W_t + \frac{t}{2} = \ln X_t$$

Now we are ready to write the solution:

$$X_t = e^{Y_0} e^{W_t} e^{\frac{t}{2}} = X_0 e^{W_t + \frac{t}{2}}$$

6. 2013-2014

6.1. Stochastic calculus hometask

1. Let X and Y be independent exponentially distributed with parameter λ. Find $\mathbb{E}(X + Y \mid X - Y)$.
2. Let S_n be symmetric random walk with $S_0 = 2013$. The stopping time τ is the first moment when $|S_n|$ reaches the value 2014 or the value 2000. Find $\mathbb{E}(\tau S_\tau)$.
   Hint: You may construct a martingale of the form $M_n = S_n^3 - f(n)S_n$ for some function f.
3. Conditional variance is defined as $\operatorname{Var}(Y \mid X) = \mathbb{E}(Y^2 \mid X) - (\mathbb{E}(Y \mid X))^2$. Find $\operatorname{Var}(W_s \mid W_t)$.
   Hint: do not forget two cases, $t > s$ and $s < t$. You may find inversion property of the Wiener process useful.
4. Let f be a real function such that f'' is continuous. Find all such functions f that $X_t = \exp(\alpha t) f(W_t)$ is a martingale.
5. Solve the stochastic differential equation

   $$dX_t = (X_t/t + t)dt + 2\sqrt{tX_t}dW_t$$

   Hint: You may suppose without a proof that the solution has the form $X_t = f(t)g(W_t)$.
6. In the framework of Black and Scholes model find the price of the asset, which pays you S_T at the fixed moment of time $T > 1$ if $S_{T-1} > 1$ and 0 otherwise.

6.2. Solution for stochastic calculus hometask

Author: Vladimir Shmarov
Problem 1.
Denote $\xi = X + Y$ and $\eta = X - Y$.

Then we are gonna find $E(\xi|\eta)$.
Since $X \sim \text{Exp}(\lambda)$, it has a nice density: $p_X(u) = \lambda e^{-\lambda u} \cdot I_{\{u \geqslant 0\}}$.
Similarly, $p_Y(v) = \lambda e^{-\lambda v} \cdot I_{\{v \geqslant 0\}}$.
Since X and Y are independent, the vector (X, Y) has a two-dimensional density, which equals to the product of densities of X and Y, i.e.

$$p_{(X,Y)}(u, v) = I_{\{u \geqslant 0, v \geqslant 0\}} \lambda^2 e^{-\lambda(u+v)}$$

We have $X = X(\xi, \eta) = \dfrac{\xi + \eta}{2}$ and $Y = Y(\xi, \eta) = \dfrac{\xi - \eta}{2}$.
Then $J = \begin{vmatrix} \dfrac{\partial X}{\partial \xi} & \dfrac{\partial X}{\partial \eta} \\ \dfrac{\partial Y}{\partial \xi} & \dfrac{\partial Y}{\partial \eta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$.
Therefore

$$p_{(\xi,\eta)}(s, t) = p_{(X,Y)}\left(\frac{s+t}{2}, \frac{s-t}{2}\right) \cdot |J| = \frac{\lambda^2}{2} I_{\{s+t \geqslant 0\}} I_{\{s-t \geqslant 0\}} e^{-\lambda s} = \frac{\lambda^2}{2} I_{\{s \geqslant |t|\}} e^{-\lambda s}$$

is a density of a vector (ξ, η).
Further,

$$p_\eta(t) = \int\limits_{\mathbb{R}} p_{(\xi,\eta)}(s, t) \mathrm{d}s = \frac{\lambda e^{-\lambda|t|}}{2}$$

Then the conditional density is

$$p_{(\xi|\eta)}(s|t) = \frac{p_{(\xi,\eta)}(s,t)}{p_\eta(t)} = \lambda e^{-\lambda(s-|t|)} \cdot I_{\{s-|t| \geqslant 0\}}$$

And finally, denoting $z = s - |t|$, we get

$$E(\xi|\eta) = \int\limits_{\mathbb{R}} s \cdot p_{(\xi|\eta)}(s|t) \mathrm{d}s \Bigg|_{t=\eta} = \int\limits_{\mathbb{R}} (z + |t|) \cdot \lambda e^{-\lambda z} \cdot I_{\{z \geqslant 0\}} \mathrm{d}z \Bigg|_{t=\eta} =$$

$$= \int\limits_{\mathbb{R}} z \cdot \lambda e^{-\lambda z} \cdot I_{\{z \geqslant 0\}} \mathrm{d}z \Bigg|_{t=\eta} + |t| \int\limits_{\mathbb{R}} \cdot \lambda e^{-\lambda z} \cdot I_{\{z \geqslant 0\}} \mathrm{d}z \Bigg|_{t=\eta} = \left(\frac{1}{\lambda} + |t|\right)\Bigg|_{t=\eta} = \frac{1}{\lambda} + |\eta|$$

(the first integral is just the expectation of exponential distribution with parameter λ,
and the second is $|t|$ times the integral of the density, which equals 1)

Answer: $E(X + Y|X - Y) = \frac{1}{\lambda} + |X - Y|$
Problem 2 — part 1 of 2.
At first, $S_n = S_0 + \sum\limits_{i=1}^{n} \varepsilon_i$, where $\{\varepsilon_i\}$ is the sequence of independent random variables with $P(\varepsilon_i = -1) = P(\varepsilon_i = 1) = \frac{1}{2}$. Also denote $\mathcal{F}_n = \sigma(\varepsilon_1, \ldots, \varepsilon_n)$.
Observe that τ is really a stopping time, because at each moment of time we precisely know, are we in one of the points $\{2000, 2014\}$ or not.

Let's prove that $E(\tau) < \infty$ (which will also imply that $P(\tau = +\infty) = 0$).

We shall obviously reach τ after any 14 consecutive steps in one direction. Then

$$P(\tau \geqslant 14n) \leqslant P(\text{At least one of } \varepsilon_1, \ldots, \varepsilon_{14} \text{ equals } -1) \times \ldots \times P(\text{At least one of } \varepsilon_{14(n-1)+1}, \ldots, \varepsilon_{14n} \text{ equals -}$$
$$= \left(1 - 2^{-14}\right)^n = q^n \quad (2)$$

where $q < 1$.

Then

$$E(\tau) = \sum_{n=1}^{+\infty} P(\tau \geqslant n) \leqslant 14 \sum_{n=0}^{+\infty} P(\tau \geqslant 14n+1) \leqslant 14 \sum_{n=0}^{\infty} q^n = 14 \cdot 2^{14} < \infty$$

as required.

$\boxed{\text{The sequence } \{S_t\} \text{ is a martingale}}$ w.r.t. filtration $\{\mathcal{F}_n\}$, because it is obviously $\{\mathcal{F}_t\}$-adapted, has a finite (zero) expectation and

$$E(S_{t+1}|\mathcal{F}_t) = E(S_t + \varepsilon_{t+1}|\mathcal{F}_t) = S_t + E(\varepsilon_{t+1}|\mathcal{F}_t) = S_t$$

Further, $S_{\min\{t,\tau\}}$ is bounded in t: it always lies between 2000 and 2014. The Doob's theorem conditions are satisfied, which means that

$$E(S_\tau) = S_0 = 2013$$

But $E(S_\tau) = 2000 \cdot P(S_\tau = 2000) + 2014 \cdot P(S_\tau = 2014)$, which after the substitution $E(S_\tau) = 2013$ gives us

$\boxed{P(S_\tau = 2000) = \frac{1}{14}}$ and $\boxed{P(S_\tau = 2014) = \frac{13}{14}}$

Further, let's prove that $\boxed{\text{the sequence } \{S_t^3 - 3tS_t\} \text{ is a } \{\mathcal{F}_t\}\text{-martingale}}$. At first, it is obviously adapted and $|E(S_t^3 - 3tS_t)| < \infty$

Then

$$E(\,S_{t+1}^3 - 3(t+1)S_{t+1} \mid \mathcal{F}_t) = E\left((S_t + \varepsilon_{t+1})^3 - 3(t+1)(S_t + \varepsilon_{t+1}) \mid \mathcal{F}_t\right) =$$
$$E\left(\,S_t^3 + 3S_t^2\underline{\varepsilon_{t+1}} + 3\underline{(\varepsilon_{t+1}^2 - 1)}S_t - 3tS_t + \underline{\varepsilon_{t+1}^3} - 3(t+1)\underline{\varepsilon_{t+1}}\middle| \mathcal{F}_t\right) = S_t^3 - 3tS_t \quad (3)$$

(the underlined terms are going out after taking conditional expectation; $(\varepsilon_{t+1}^2 - 1)$ goes out even without it)

Denote $X_t = S_t^3 - 3tS_t$.

For $t \geqslant \tau$ we have $|X_{\min\{t+1,\tau\}} - X_{\min\{t,\tau\}}| = 0$, and for $t < \tau$ we have $|X_{\min\{t+1,\tau\}} - X_{\min\{t,\tau\}}| = |X_{t+1} - X_t| = |3S_t^2\varepsilon_{t+1} - (3t+2)\varepsilon_{t+1}| \leqslant 3 \times 2014^2 + 3\tau + 2$, which expectation is bounded by the finite number $3 \times 2014^2 + 2 + 3E(\tau)$.

Then, the overall expectation $E\left(|X_{\min\{t+1,\tau\}} - X_{\min\{t,\tau\}}|\right) \leqslant 3 \times 2014^2 + 2 + 3E(\tau)$ for every t, i.e. is bounded in t.

Problem 2 — part 2 of 2.

Then again, the conditions of the Doob's theorem are satisfied, i.e.

$$E(X_\tau) = X_0 = 2013^3$$

But also $E(X_\tau) = E(S_\tau^3) - 3E(\tau S_\tau)$, i.e.

$$E(\tau S_\tau) = \frac{1}{3}\left(E(S_\tau^3) - 2013^3\right) = \frac{1}{3}\left(\frac{1}{14}2000^3 + \frac{13}{14}2014^3 - 2013^3\right) = 13 \times 2009 = 26117$$

Answer: $E(\tau S_\tau) = 26117$.

Problem 3.
Case 1. Let $s \geqslant t$.
Then

$$E\left(W_s^2 \middle| W_t\right) = E\left(W_t^2 + 2W_t(W_s - W_t) + (W_s - W_t)^2 \middle| W_t\right) =$$
$$= W_t^2 + 2W_t E(W_s - W_t) + E\left((W_s - W_t)^2\right) = W_t^2 + (s - t)$$

Further,

$$E(W_s|W_t) = W_t + E(W_s - W_t|W_t) = W_t$$

Then we have $Var(W_s|W_t) = E(W_s^2|W_t) - E(W_s|W_t)^2 = (s - t)$

Case 2. Let $s < t$.
By the inverse property, $V_t = tW_{\frac{1}{t}}$ is also a standard Wiener process.
From the case 1 we have

$$\frac{1}{s} - \frac{1}{t} = Var\left(V_{\frac{1}{s}} \middle| V_{\frac{1}{t}}\right) = Var\left(\frac{1}{s}W_s \middle| \frac{1}{t}W_t\right) = \frac{1}{s^2}Var\left(W_s \middle| \frac{1}{t}\, W_t\right) = \frac{1}{s^2}Var(W_s|W_t)$$

the last equation is because obviously $\sigma(W_t) = \sigma\left(\frac{1}{t}W_t\right)$.
Then $Var(W_s|W_t) = s - \frac{s^2}{t}$.

Summing up these two cases, we get the

Answer: $Var(W_s|W_t) = \max\left\{s - t, s - \frac{s^2}{t}\right\}$
Problem 4.
Let $g(t, W_t) = X_t = e^{\alpha t} f(W_t)$, then $g_t = \alpha g$, $g_{W_t} = e^{\alpha t} f'(W_t)$ and $g_{W_t W_t} = e^{\alpha t} f''(W_t)$.
By the Itô's lemma we have

$$\mathrm{d}X_t = \alpha X_t \delta + e^{\alpha t} f'(W_t)\mathrm{d}W_t + \frac{1}{2}e^{\alpha t} f''(W_t)\delta =$$

$$= e^{\alpha t}\left(\alpha f(W_t) + \frac{1}{2}f''(W_t)\right)\delta + e^{\alpha t} f'(W_t)\mathrm{d}W_t$$

Therefore

$$X_t = f(0) + \int\limits_0^t e^{\alpha u} \left(\alpha f(W_u) + \frac{1}{2} f''(W_u) \right) \mathrm{d}u + \int\limits_0^t e^{\alpha u} f'(W_u) \mathrm{d}W_u$$

The last term in the right-hand side is a martingale, so X_t is a martingale if and only if the coefficient near δ equals zero for all t, i.e.

$$\alpha f(W_t) + \frac{1}{2} f''(W_t) = 0 \qquad \text{a.s.}$$

which (together with continuity of f'') implies

$$\alpha f(x) + \frac{1}{2} f''(x) = 0 \qquad \forall x$$

We know how to solve the linear differential equations, so I will not go into details and I'll just write the answer.

Answer: If $\alpha < 0$, then $f(x) = C_1 e^{t\sqrt{-2\alpha}} + C_2 e^{-t\sqrt{-2\alpha}}$, where C_1, C_2 are any real constants;
if $\alpha = 0$, then $f(x) = C_1 x + C_2$;
if $\alpha > 0$, then $f(x) = C_1 \cos(t\sqrt{2\alpha}) + C_2 \sin(t\sqrt{2\alpha})$

Problem 5 — part 1 of 2.

Assume for simplicity that $X_t = f(t)g(W_t)$, where f and g are kinda nice functions (defined on $[0, +\infty)$ and continuously differentiable the required number of times on this interval; right-differentiable at zero, of course).

By the Itô's lemma we have

$$\mathrm{d}X_t = \left(f'(t)g(W_t) + \frac{1}{2} f(t)g''(W_t) \right) \delta + f(t)g'(W_t)\mathrm{d}W_t$$

Substituting this into the condition of the problem and separating the parts with δ and $\mathrm{d}W_t$ we get

$$\begin{cases} f'(t)g(W_t) + \frac{1}{2} f(t)g''(W_t) = \frac{f(t)}{t} g(W_t) + t & (1) \\ f(t)g'(W_t) = 2\sqrt{tf(t)g(W_t)} & (2) \end{cases}$$

Both equations are satisfied for every $t > 0$ almost surely.

Step 1. Let's prove that $f(t) \neq 0$ if $t > 0$. Indeed, if $f(t_0) = 0$, (1) gives us

$$f'(t_0)g(W_{t_0}) = t_0 \qquad \text{a.s.}$$

so

$$g(W_{t_0}) = \frac{t_0}{f'(t_0)} \qquad \text{a.s.}$$

But g is continuous, so this gives us

$$g(x) \equiv C = Const$$

But then (2) gives

$$X_t \equiv 0 \qquad \text{a.s.}$$

which does not satisfy the condition of the problem.
Therefore, $f(t) \neq 0$, when $t \neq 0$.

Now (2) can be transformed into

$$\frac{f(t)}{4t} g'(W_t)^2 = g(W_t) \tag{3}$$

Step 2. $g'(x)$ cannot be zero in any interval.
Suppose that $g'(x) = 0$ for all $x \in (a, b)$. From (3) we have $g(x) = 0$ almost everywhere at (a, b) (because $P(W_t \in (a, b)) \neq 0$ for $t > 0$), and also $g'(x) = 0$ at (a, b). But substituting these values to (1) provides us contradiction.

Let $A = \{x \in \mathbb{R} | g'(x) \neq 0\}$. Since g' is continuous and is not identical zero (from the step 2), the set A has a positive Lebesgue measure; therefore $P(W_t \in A) \neq 0$ for all $t > 0$.

Problem 5 — pat 2 of 2.

Now fix some $t_0 > 0$. From (3) we have that almost everywhere in the event $\{W_{t_0} \in A\}$ the equality

$$\frac{g(W_{t_0})}{g'(W_{t_0})^2} = \frac{f(t_0)}{4t_0}$$

holds. Then for almost all $x \in A$ we have $\dfrac{g(x)}{g'(x)^2} = \dfrac{f(t_0)}{4t_0}$. Now fix any other $t_1 > 0$.
We have that almost everywhere in A the equality $\dfrac{g(x)}{g'(x)^2} = \dfrac{f(t_1)}{4t_1}$ holds. Since A has a positive measure, we have

$$\frac{f(t_0)}{4t_0} = \frac{f(t_1)}{4t_1}$$

i.e. $f(t) = Ct$ for all $t > 0$ and some $C = Const$. From the continuity $\boxed{f(t) = Ct \text{ for all } t \geqslant 0}$

Substitute this to (1):

$$Cg(W_t) + \frac{1}{2} Ctg''(W_t) = Cg(W_t) + t$$

$$g''(W_t) = \frac{2}{C} \tag{4}$$

(4) holds almost everywhere, but since g'' is continuous, we have $\boxed{g''(x) \equiv \frac{2}{C}}$

Then $g'(x) = \frac{2x}{C} + D$ and $g(x) = \frac{x^2}{C} + Dx + E$ for some constants D, E.
From (3) we have

$$g(x) = \frac{f(t)}{4t} g'(x)^2 = \frac{C}{4} g'(x)^2$$

Substitute the values of $g'(x)$ and $g(x)$:

$$\frac{x^2}{C} + Dx + E = \frac{C}{4}\left(\frac{2x}{C} + D\right)^2$$

$$\frac{x^2}{C} + Dx + E = \frac{x^2}{C} + Dx + \frac{CD^2}{4}$$

which gives us $E = \frac{CD^2}{4}$ and $g(x) = (Ax+B)^2$, where $A^2 = \frac{1}{C}$ and $2AB = D$.

Answer: $X_t = t(W_t + C)^2$ for some constant C.
(in our notation this constant equals $\frac{B}{A}$)
REMARK. Precisely speaking, this is not a correct answer, because $\sqrt{tX_t}$ is always positive , so (2) does not always hold, because the left side of (2), which is $2t(W_t + C)$, can be negative with nonzero probability. In other words, we solved similar, but different equation

$$\mathrm{d}X_t = \left(\frac{X_t}{t} + t\right)\delta \;\pm\; 2\sqrt{tX_t}\mathrm{d}W_t$$

and the original problem has no solutions of the type $f(t)g(W_t)$.
Problem 6 — part 1 of 4.
The risky asset S_t satisfies the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t\delta + \sigma S_t \mathrm{d}W_t$$

which implies

$$S_t = S_0 \exp\left(t\left(\mu - \frac{\sigma^2}{2}\right) + \sigma W_t\right)$$

The risk-free asset satisfies the equation

$$\mathrm{d}B_t = rB_t\delta$$

which implies

$$B_t = B_0 e^{rt}$$

Let X_t be our self-financing portfolio at time t, which contains Δ_t shares of the risky asset.
The total amount of money in the risky asset is Δ_t, in the risk-free asset — $(X_t - \Delta_t S_t)$.
Therefore

$$\mathrm{d}X_t = \Delta_t \mathrm{d}S_t + r(X_t - \Delta_t S_t)\delta$$

Let $\tilde{W}_t = \frac{\mu - r}{\sigma}t + W_t$, and $\tilde{P}$ is the probability measure, w.r.t. which $\tilde{W}_t$ is a standard Wiener process (the existence of such measure is the statement of Girsanov's theorem).
In class we explicitly derived that

$$e^{-rt}X_t = X_0 + \int_0^t \Delta_u e^{-ru}\sigma S_u \mathrm{d}\tilde{W}_u$$

The last term is a $\tilde{P}$-martingale, which means that

$$E_{\tilde{P}}\left(e^{-rT}X_T\middle|\,\mathcal{F}_0\right)=X_0 \tag{1}$$

Now remember that

$$X_T=S_T I_{\{S_{T-1}>1\}}$$

so we need to calculate

$$E_{\tilde{P}}(S_T I_{\{S_{T-1}>1\}})=\iint\limits_{u>0,v>1} u\cdot\tilde{p}_{(S_T,S_{T-1})}(u,v)\mathrm{d}u\mathrm{d}v$$

where $\tilde{p}_{(S_T,S_{T-1})}$ is a two-dimensional density of a corresponding vector w.r.t. measure $\tilde{P}$.

Problem 6 — part 2 of 4.

But

$$\iint\limits_{u>0,v>1} u\cdot\tilde{p}_{(S_T,S_{T-1})}(u,v)\mathrm{d}u\mathrm{d}v=\int\limits_1^{+\infty}\left(\int\limits_0^{+\infty} u\cdot\tilde{p}_{(S_T,S_{T-1})}(u,v)\mathrm{d}u\right)\mathrm{d}v=\int\limits_1^{+\infty}\tilde{p}_{S_{T-1}}(v)\cdot E_{\tilde{P}}\left(S_T|S_{T-1}=v\right)\mathrm{d}v$$

Further,

$$S_{T-1}=S_0\exp\left((T-1)\left(\mu-\frac{\sigma^2}{2}\right)+\sigma W_{T-1}\right)=S_0\exp\left((T-1)\left(r-\frac{\sigma^2}{2}\right)+\sigma\tilde{W}_{T-1}\right)$$

which means that with notation $A(x)=S_0\exp\left((T-1)\left(r-\frac{\sigma^2}{2}\right)+\sigma x\right)$ we have

$$\tilde{p}_{\tilde{W}_{T-1}}(x)=\tilde{p}_{S_{T-1}}\left(A(x)\right)\cdot\sigma A(x)$$

We have $x=\sigma^{-1}\left(\ln\left(\frac{A(x)}{S_0}\right)-(T-1)\left(r-\frac{\sigma^2}{2}\right)\right)$. Therefore

$$\tilde{p}_{S_{T-1}}(v)=\frac{1}{\sigma v}\cdot\frac{1}{\sqrt{2\pi(T-1)}}\exp\left(-\frac{1}{2\sigma^2(T-1)}\left(\ln(v)-\left(\ln(S_0)+(T-1)\left(r-\frac{\sigma^2}{2}\right)\right)\right)^2\right) \tag{2}$$

Very ugly expression! Let's believe most of those terms will cancel out.

Now

$$S_T=S_0\exp\left(T\left(r-\frac{\sigma^2}{2}\right)+\sigma\tilde{W}_T\right)=S_{T-1}\cdot\exp\left(r-\frac{\sigma^2}{2}+\sigma\left(\tilde{W}_T-\tilde{W}_{T-1}\right)\right)$$

and $\left(\tilde{W}_T-\tilde{W}_{T-1}\right)$ is independent of the σ-algebra $\sigma(S_{T-1})\subseteq\mathcal{F}_{T-1}$.

Then

$$\begin{aligned}E_{\tilde{P}}\left(S_T|S_{T-1}=v\right)&=E_{\tilde{P}}\left(S_{T-1}\cdot\exp\left(r-\frac{\sigma^2}{2}+\sigma\left(\tilde{W}_T-\tilde{W}_{T-1}\right)\right)\middle|\,S_{T-1}=v\right)= \\ &=v\cdot\exp\left(r-\frac{\sigma^2}{2}\right)E_{\tilde{P}}\left(\exp\left(\sigma\left(\tilde{W}_T-\tilde{W}_{T-1}\right)\right)\right)\end{aligned} \quad (3)\quad(4)$$

Further, if $\xi \sim \mathcal{N}(0,1)$, then

$$E(e^{\sigma\xi}) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-\frac{x^2}{2}+\sigma x} \mathrm{d}x = e^{\frac{\sigma^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-\frac{(x-\sigma)^2}{2}} \mathrm{d}x = e^{\frac{\sigma^2}{2}}$$

which gives us

$$E_{\tilde{P}}\left(S_T | S_{T-1} = v\right) = v \cdot \exp\left(r - \frac{\sigma^2}{2}\right) \exp\left(\frac{\sigma^2}{2}\right) = ve^r \tag{4}$$

which, I think, is a very nice formula (especially if we compare it with (2)).

Problem 6 — part 3 of 4.

Taking (2) and (4), we have

$$\int\limits_1^{+\infty} \tilde{p}_{S_{T-1}}(v) \cdot E_{\tilde{P}}\left(S_T | S_{T-1} = v\right) \mathrm{d}v =$$
$$= e^r \int\limits_1^{+\infty} \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi(T-1)}} \exp\left(-\frac{1}{2\sigma^2(T-1)} \left(\ln(v) - \left(\ln(S_0) + (T-1)\left(r - \frac{\sigma^2}{2}\right)\right)\right)^2\right) \mathrm{d}v =$$
$$= e^r \int\limits_0^{+\infty} e^s \cdot \frac{1}{\sqrt{2\pi\sigma^2(T-1)}} \exp\left(-\frac{1}{2\sigma^2(T-1)} \left(s - \left(\ln(S_0) + (T-1)\left(r - \frac{\sigma^2}{2}\right)\right)\right)^2\right) \mathrm{d}s \tag{5}$$

We substituted $s = \ln(v)$. Of course, $\mathrm{d}v = e^s \mathrm{d}s$.

Denote also $B = \ln(S_0) + (T-1)\left(r - \frac{\sigma^2}{2}\right)$.

Then our integral equals

$$e^r \int\limits_0^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-1)}} \exp\left(-\frac{s^2 - 2Bs - 2\sigma^2(T-1)s + B^2}{2\sigma^2(T-1)}\right) \mathrm{d}s =$$
$$= \exp\left(r - \frac{B^2}{2\sigma^2(T-1)} + \frac{\left(B + \sigma^2(T-1)\right)^2}{2\sigma^2(T-1)}\right) \int\limits_0^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-1)}} \exp\left(-\frac{\left(s - B - \sigma^2(T-1)\right)^2}{2\sigma^2(T-1)}\right) \mathrm{d}s =$$
$$= \exp\left(r - \frac{B^2}{2\sigma^2(T-1)} + \frac{\left(B + \sigma^2(T-1)\right)^2}{2\sigma^2(T-1)}\right) \cdot \tilde{P}\left(\eta > -\left(B + \sigma^2(T-1)\right)\right) \tag{6}$$

where $\eta \sim \mathcal{N}(0, \sigma^2(T-1))$. The integral magically turns into the probability, because under the integral there is a well-known density.

Therefore our integral finally equals

$$\exp\left(r+\frac{\left(B+\sigma^2(T-1)\right)^2-B^2}{2\sigma^2(T-1)}\right)\cdot\tilde{P}\left(\frac{\eta}{\sigma\sqrt{T-1}}>-\frac{B+\sigma^2(T-1)}{\sigma\sqrt{T-1}}\right)=$$
$$=\exp\left(r+B+\frac{\sigma^2(T-1)}{2}\right)\Phi\left(\frac{B+\sigma^2(T-1)}{\sigma\sqrt{T-1}}\right)=e^{\ln(S_0)+rT}\cdot\Phi\left(\frac{\ln(S_0)+(T-1)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-1}}\right) \tag{7}$$

where $\Phi(x)$ is a distribution function of a standard normal random value, $\Phi(x)=\int\limits_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)\delta$.

Problem 6 — part 4 of 4.
And now I'll finish this huge problem.
We had

$$X_0=e^{-rT}E_{\tilde{P}}\left(S_TI_{\{S_{T-1}>1\}}\right)=e^{-rT}\cdot e^{\ln(S_0)+rT}\Phi\left(\frac{\ln(S_0)+(T-1)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-1}}\right)=$$
$$=S_0\Phi\left(\frac{\ln(S_0)+(T-1)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-1}}\right) \tag{8}$$

This is our answer.
Answer: The price of our asset equals

$$S_0\cdot\Phi\left(\frac{\ln(S_0)+(T-1)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-1}}\right)$$

where $\Phi(x)$ is a distribution function of a standard normal random value, $\Phi(x)=\int\limits_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)\delta$

6.3. Exam, 17.01.2014

Optimal control part

1. [10 points] A collector of wine is told by a doctor that he is going to die in T years and he develops a consumption plan that maximizes the utility of consumption of wine over remaining lifetime. At present the stock of wine equals $w(0)=w_0>0$. It will be consumed fully by time T, that is $w(T)=0$. The utility function is $U(c)=c$, where c is consumption of wine. The stock of wine follows the rule $w(t)=w_0-\int_0^t c(s)\,ds$. Consumption is bounded $0\leqslant c\leqslant\bar{c}$, where $\bar{c}>w_0/T$.

State the dynamic optimization problem, if the discount rate is $r > 0$. Introduce the current value Hamiltonian for that problem. Write down the first-order conditions. Solve the system of equations.

2. [10 points] Consider the following optimization problem: maximize $\sum_{t=0}^{\infty} 0.75^t \ln c_t$, subject to $c_t + k_{t+1} = \sqrt{k_t}$, where $k_0 > 0$. Let the state variable be k and denote the next period value of k as k'.

   (a) Write down the Bellman equation for the value function $V(k)$

   (b) Using the method of undetermined coefficients find $V(k)$

   (c) Find the optimal policy function $k' = h(k)$

3. [20 points] Consider the nonlinear system of differential equations

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2) \\ \dot{y} = x + ay(x^2 + y^2) \end{cases}$$

   where a is a parameter.

   (a) [5 points] Linearize the system and classify the steady-state solution

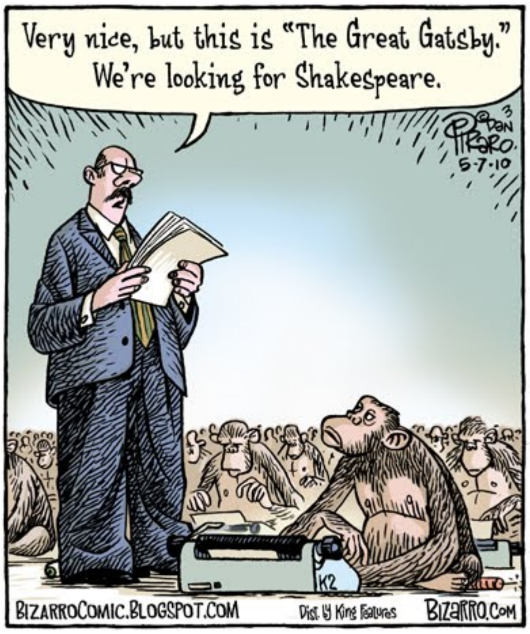
   (b) [10 points] Restore the nonlinear system and by changing the Carthesian into the polar coordinates show that the system becomes

$$\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{cases}$$

   (c) [5 points] Solve this system and draw the phase diagram for $a < 0$, $a = 0$, $a > 0$.

Stochastic calculus part

Here W_t always denotes the standard Wiener process.



1. [10 points] It is known that $\text{Var}(X) = 16$, $\text{Var}(Y) = 9$, $\text{Cov}(X, Y) = -1$, $\mathbb{E}(X) = \mathbb{E}(Y) = 0$. Let's denote $\hat{Y} = \mathbb{E}(Y|X)$ and $\hat{X} = \mathbb{E}(X|Y)$. It is also known that $\mathbb{E}(\hat{X}^2) = 4$ and $\mathbb{E}(\hat{Y}^2) = 1$. Find $\text{Var}(\hat{Y})$, $\text{Cov}(\hat{X}, Y)$, $\text{Cov}(\hat{Y}, Y)$.
2. [10 points] Let's consider a Wiener process with drift, $X_t = W_t + \mu t$.
   (a) Find a non-trivial martingale of the form $M_t = e^{\beta X_t}$
   (b) Let τ be a stopping time, the first moment when X_t hits 2 or -1. Find the probability $\mathbb{P}(X_\tau = 2)$

   You may assume without proof that some version of Doob's theorem may be applied here.
3. [10 points] Is $X_t = \cos W_t$ a martingale? If not, find any non-zero function $f(t)$ such that $Y_t = f(t)X_t$ is a martingale. Find the variance of Y_t.
   Hint: You may use the fact that $\mathbb{E}(\cos W_t) = e^{-t/2}$.
4. [10 points] In the framework of Black and Scholes model find the price at the time 0 of an asset which gives you the payoff S_T^2 at the time T. Here S_t is the price of the underlying asset at time t.
5. [20 points] Consider the following stochastic differential equation

$$dX_t = -0.5e^{-2X_t}dt + e^{-X_t}dW_t,$$

   with deterministic initial value X_0.
   (a) Is X_t a martingale?
   (b) Using the substitution $Y_t = f(X_t)$ solve this differential equation. Hint: try to find a function f such that the term before dt cancels out.
   (c) Sketch the possible path of the process X_t
   (d) Using basic facts about Wiener process find the limit $\lim_{t\to\infty} \mathbb{P}(X_t \geqslant a)$ for all values of the parameter a.

6.4. marking scheme

1. 3 pts:

$$\text{Var}(\hat{Y}) = \mathbb{E}(\hat{Y}^2) - (\mathbb{E}(\hat{Y}))^2 = 1 - 0^2 = 1$$

   3 pts:

$$\text{Cov}(\hat{X}, Y) = \mathbb{E}(\mathbb{E}(X|Y)Y) - \mathbb{E}(\hat{X})\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(YX|Y)) - \mathbb{E}(X)\mathbb{E}(Y) = \text{Cov}(X, Y)$$

   4 pts:

$$\text{Cov}(\hat{Y}, Y) = \ldots = \text{Cov}(\hat{Y}, \hat{Y})$$

2. $M_t = e^{-2\mu X_t}$ 5 pts, probability — 5 pts, $p = \frac{1-\exp(2\mu)}{\exp(-4\mu)-\exp(2\mu)}$
3. X_t is not a martingale — 1 pt,

$$dY_t = \cos W_t f'(t)dt + f(t)(-\sin W_t dW_t + \frac{1}{2}(-\cos W_t)dt)$$

   We obtain simple differential equation $f'(t) - f(t)/2 = 0$, we choose any solution, for example, $f(t) = e^{t/2}$.
   $Y_t = e^{t/2}X_t$ or proportional — 4 pts, variance — 5 pts

$$\text{Var}(Y_t) = \mathbb{E}(Y_t^2) - (\mathbb{E}(Y_t))^2$$

We note that Y_t is a martingale by construction, so $\mathbb{E}(Y_t) = Y_0 = 1$.
So, $\mathbb{E}(e^{t/2}\cos W_t) = 1$. Hence $\mathbb{E}(\cos W_t) = e^{-t/2}$.
Let's recall from middle school that $\cos^2 W_t = \frac{1+\cos 2W_t}{2}$.
We also note that $2W_t \sim \mathcal{N}(0;4t)$ and $W_{4t} \sim \mathcal{N}(0;4t)$. In any expectation we may safely replace $2W_t$ by W_{4t}.

$$\mathbb{E}(Y_t^2) = \mathbb{E}(e^t\cos^2 W_t) = \mathbb{E}\left(e^t\frac{1+\cos 2W_t}{2}\right) = \frac{e^t}{2} + \frac{e^t}{2}\mathbb{E}(\cos 2W_t)$$

Where $\mathbb{E}(\cos 2Wt) = \mathbb{E}(\cos(W_{4t})) = e^{-4t/2} = e^{-2t}$.
Finally we obtain,

$$\mathrm{Var}(Y_t) = (e^t + e^{-t} - 2)/2$$

4. Pricing formula — 2 pts, risk-neutral substitution — 2 pts, calculations — 6 pts.
Using the pricing formula $X_0 = \mathbb{E}_{\tilde{\mathbb{P}}}\left[e^{-rT}X_T\right]$:

$$X_0 = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp(-rT)X_T\right] = \exp(-rT)\mathbb{E}_{\tilde{\mathbb{P}}}\left[S_T^2\right]$$

We substitute $S_T = S_0\exp((r-\sigma^2/2)T + \sigma\tilde{W}_T)$:

$$X_0 = \exp(-rT)\mathbb{E}_{\tilde{\mathbb{P}}}\left[S_0^2\exp(2(r-\sigma^2/2)T + 2\sigma\tilde{W}_T)\right] = \\ = S_0^2\exp(-rT)\exp(2(r-\sigma^2/2)T)\mathbb{E}_{\tilde{\mathbb{P}}}\left[e^{2\sigma\tilde{W}_T}\right] \quad (9)$$

We use the fact that $\mathbb{E}[\exp(aW_t)] = \exp(a^2t/2)$, so:

$$X_0 = S_0^2e^{-rT}\exp(2(r-\sigma^2/2)T)\exp(4\sigma^2T/2) = S_0^2\exp((2r+\sigma^2)T)$$

5. (a) X_t is not a martingale — 1 pt
   (b) use of Ito's formula — 5 pts, equation $f' = f$ — 5 pts, solution $X_t = \ln(W_t + e^{X_0})$. — 5 pts
   (c) Key features: ``jiggly'' — 1 pt, blows down to minus infinity in finite time — 2 pts
   (d) The limit is zero.

6.5. Retake, 08.02.2014

Stochastic calculus part

1. [10 points] Is $X_t = \sin W_t$ a martingale? If not, find any non-zero function $f(t)$ such that $Y_t = f(t)X_t$ is a martingale. Find the expected value $\mathbb{E}(X_t)$
2. [10 points] In the framework of Black and Scholes model find the price of the asset, which pays you S_2/S_1 at the fixed moment of time $T = 2$
3. [10 points] Let τ be a stopping time, the first moment when W_t hits 2 or -1.
   (a) Find the probability $\mathbb{P}(W_\tau = 2)$
   (b) Find a martingale of the form $X_t = W_t^2 + f(t)$.
   (c) Find $\mathbb{E}(\tau)$

4. [10 points] The joint distribution of the random vector (X, Y) is given by its probability density function
$$f(x,y) = \begin{cases} ce^{x-y}, & \text{for } 0 \leqslant x, y \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$
where c is a normalization constant. Find $\mathbb{E}(X \mid Y)$.
5. [20 points] Let's consider the following system of stochastic differential equations
$$\begin{cases} dX_t = aX_t dt - Y_t dW_t \\ dY_t = aY_t dt + X_t dW_t \end{cases}$$
with initial conditions $X_0 = x_0$ and $Y_0 = 0$

   (a) Ignoring the initial conditions find solution of the form $X_t = f(t)\cos W_t$ and $Y_t = g(t)\sin W_t$

   (b) Modify your solution to take into account the initial conditions

   (c) Prove that for any solution $D_t = X_t^2 + Y_t^2$ is nonstochastic

6.6. Answers and hints

1. $f(t) = e^{t/2}$ or proportional, $\mathbb{E}(X_t) = 0$.
2. Using the pricing formula $X_0 = \mathbb{E}_{\tilde{\mathbb{P}}}\left[e^{-rT}X_T\right]$:
$$X_0 = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp(-rT)X_T\right] = \exp(-2r)\mathbb{E}_{\tilde{\mathbb{P}}}\left[S_2/S_1\right]$$
We substitute $S_T = S_0 \exp((r - \sigma^2/2)T + \sigma\tilde{W}_T)$:
$$X_0 = \exp(-2r)\mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp(r - \sigma^2/2 + \sigma\tilde{W}_2 - \sigma\tilde{W}_1)\right] = \\ = \exp(-2r)\exp(r - \sigma^2/2)\mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp(\sigma\tilde{W}_2 - \sigma\tilde{W}_1)\right] \quad (10)$$
The distribution of $\tilde{W}_2 - \tilde{W}_1$ is similar to the distribution of $\tilde{W}_1$ and is $N(0;1)$. So, we simplify:
$$X_0 = \exp(-2r)\exp(r - \sigma^2/2)\mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp(\sigma\tilde{W}_1)\right]$$
We use the fact that $\mathbb{E}[\exp(aW_t)] = \exp(a^2 t/2)$, so:
$$X_0 = \exp(-2r)\exp(r - \sigma^2/2)\exp(\sigma^2/2) = \exp(-r)$$
3. Using Doob's theorem, $\mathbb{E}(W_\tau) = 0$, so $\mathbb{P}(W_\tau = 2) = 1/3$. Using Ito's lemma or otherwise $X_t = W_t^2 - t$ is a martingale, so $\mathbb{E}(X_\tau) = 0$ and $\mathbb{E}(\tau) = 2$.
4. $\mathbb{E}(X \mid Y) = \int_0^1 x f(x \mid Y)\, dx$. Here $f(x,y)$ maybe factored, so X and Y are independent, and $\mathbb{E}(X \mid Y) = const$.
5. See the solution of problem 8 from exam of 2008-2009 on page 5

7. 2014-2015

7.1. Practice problems

These problems were not graded.

1. An enemy submarine is somewhere on the number line. The initial coordinate of the submarine is some unknown integer number. It is moving at some constant integer speed (units per minute).
   You can launch a torpedo each minute at any integer on the number line. If the the submarine is there, you hit it and it sinks. You have infinite number of torpedoes. You must sink this enemy sub.
   (a) Draw a picture of number line and the submarine. Just for fun!
   (b) Devise a strategy that is guaranteed to eventually hit the enemy submarine
2. Compare the power of two sets: $A = \mathbb{Q}$ — the set of all rational numbers, $B = \mathbb{Q}^2$ — the set of all possible pairs of rational numbers.
3. Using the fact that the Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra containing all subsets of the form $(-\infty; t]$ show that $\mathbb{N} \in \mathcal{B}$.
4. Let $\Omega = \mathbb{R}$. Find explicitely the smallest σ-algebra which contains the sets $A = [0; 1]$ and $B = [10; 100]$.
5. Let X be uniform on $[0; 1]$ and $Y = 1_{X<0.7} + 1_{X>0.1}$. Describe the σ-algebra $\sigma(Y)$. How many events $\sigma(Y)$ contains? Is the set $\sigma(X)$ countable?
6. How many different σ-algebras one may construct if Ω contains three elements? Four?
7. We throw a coin infinite number of times. Let's define the sequence of random variables X_n such that X_n is equal to 1, if the result of n-th throw is tail, and 0 otherwise. We also define a bunch of σ-algebras: $\mathcal{F}_n := \sigma(X_1, X_2, ..., X_n)$, $\mathcal{H}_n := \sigma(X_n, X_{n+1}, X_{n+2}, \ldots)$.
   Give two non-trivial (other than Ω and $\emptyset$) examples of event A, such that:
   - $A \in \mathcal{F}_{2014}$
   - $A \notin \mathcal{F}_{2014}$
   - A belongs to every $\mathcal{H}_n$

   Which σ-algebras contains the events:
   - $A = \{X_{37} > 0\}$
   - $B = \{X_{37} > X_{2014}\}$
   - $C = \{X_{37} > X_{2014} > X_{12}\}$

   Simplify where possible: $\mathcal{F}_{11} \cap \mathcal{F}_{25}$, $\mathcal{F}_{11} \cup \mathcal{F}_{25}$, $\mathcal{H}_{11} \cap \mathcal{H}_{25}$, $\mathcal{H}_{11} \cup \mathcal{H}_{25}$
8. Veniamin throws a coin until three consecutive tails appear. Let the random variable T be the number of throws. Let $\mathcal{F}_T$ be the σ-algebra of all events distinguishable by Veniamin.
   (a) Provide two non-trivial (not equal to Ω or $\emptyset$) examples of event A such that $A \in \mathcal{F}_T$ but $A \notin \sigma(T)$
   (b) Provide two non-trivial (not equal to Ω or $\emptyset$) examples of event A such that $A \in \sigma(T)$
   (c) Is it true, that $\sigma(T) \subset \mathcal{F}_T$?

7.2. Practice-solutions

1.
2. The set A and B have the same power.
3.
4. The σ-algebra contains $8 = 2^3$ events: all combinations of A, B and $\mathbb{R}/(A \cup B)$.

5. $\sigma(Y) = \{\emptyset, \Omega, \{Y = 1\}, \{Y = 2\}\}$, $\sigma(X)$ is not countable.

7.3. Hometask

1. Researcher Veniamin throws a fair dice until 6 appears. Let denote by T the total number of throws and by N the number of throws when 5 appeared. Find $\mathbb{E}(N|T)$, $\text{Var}(N|T)$, $\mathbb{E}(N)$, $\text{Var}(N)$ and $\mathbb{E}(T|N)$.
2. It is $25°C$ in Australia today. Each day the temperature goes one degree up or down with equal probability. Each day I will put the sum equal to the temperature in my piggy bank. So today I will put 25 roubles. I will stop my investment strategy when the temperature will reach $15°C$ or $30°C$ for the first time.

What will be the expected value of my account in my piggy bank?

3. Veniamin and Varvara are managers of two gladiators-vampires teams. Veniamin has 4 gladiators-vampires with initial strengths 1, 2, 3 and 4. Varvara has 3 glagiators-vampires with initial strengths 1, 3 and 5.
The competition between two teams is organised as a sequence of rounds. In each round two gladiators-vampires (one from each team) will meet and fight to the death. Varvara always selects the best gladiator from her team to fight in the next round.
When the gladiators of strengths a and b meet the first will win with probability $a/(a+b)$, the second — with probability $b/(a+b)$. The gladiators are vampires, so the strength of the winner will become $a+b$.
   (a) Let's denote by τ the number of the final round. What is the maximum value of τ?
   (b) Let's denote by X_t the total strength of the Veniamin's team. Is X_t a martingale?
   (c) What is best strategy for Veniamin? What is the probability that Veniamin's team will win?
4. Consider the processes $X_t = \int_0^t (sW_s)^3 \, dW_s$ and $Y_t = \int_0^t W_s \, dW_s$. Find $\mathbb{E}(X_t)$, $\text{Var}(X_t)$, $\text{Cov}(X_t, Y_t)$
5. The process X_t is given by
$$dX_t = tW_t e^{W_t} \, dt + \sin(tW_t) \, dW_t, \; X_0 = 1$$
Using Ito-Doeblin lemma find Z_0 and dZ_t if $Z_t = X_t^2 + t\cos(X_t)$
6. Solve the stochastic differential equation:
$$dX_t = X_t(1 - X_t) \, dt + X_t \, dW_t, \; X_0 = 1$$
Some hints: you may find the substitution $Y_t = \ln X_t$ useful.
7. In the framework of Black and Scholes model find the price at $t = 0$ of the asset, which pays you 1\$ if $S_{T-1} > 1$ and 0 otherwise. Here S_t denotes the price of a share at time t. The payment is made at the fixed moment of time $T > 1$.

7.4. Hometask-solution

1. If the total number of throws it T then the last throw has value 6. Each of the preceeding $(T-1)$ throws may have a value from 1 to 5. So if T is considered fixed, then N has binomial distribution with $n = T-1$ and $p = 1/5$. It follows that $\mathbb{E}(N|T) = \frac{T-1}{5}$ and $\mathrm{Var}(N|T) = \frac{4(T-1)}{25}$.
   Using tower property of conditional expectation
   $$\mathbb{E}(N) = \mathbb{E}(\mathbb{E}(N|T)) = \mathbb{E}\left(\frac{T-1}{5}\right) = \frac{\mathbb{E}(T)-1}{5}$$
   Here T is geometric with $p = 1/6$, so $\mathbb{E}(T) = 6$ and $\mathbb{E}(N) = 1$.
   We use the following decomposition of variance:
   $$\mathrm{Var}(N) = \mathbb{E}(\mathrm{Var}(N|T)) + \mathrm{Var}(\mathbb{E}(N|T)) = \\ = \mathbb{E}\left(\frac{4(T-1)}{25}\right) + \mathrm{Var}\left(\frac{T-1}{5}\right) = (\mathrm{Var}(T-1) + 4\mathbb{E}(T-1))/25 \quad (11)$$
   As T is geometric $\mathrm{Var}(T) = (1-p)/p^2 = 30$. And, hence, $\mathrm{Var}(N) = 2$.
   If N is fixed, then the total time T is equal to the following sum: from start to first five, from first five to second five, ..., from N-th five to six, so $\mathbb{E}(T|N) = aN + b$. From the properties of geometric distribution, $b = 5$ (no five) and $a = 5$ (no six).
2. $X_0 = 25$, $X_{t+1} = X_t + D_{t+1}$ — is a martingale. The random variable X_τ can take only the values 30 and 15. Applying the Doob's theorem we get $\mathbb{E}(X_\tau) = 25$ and, hence, $\mathbb{P}(X_\tau = 15) = 1/3$.
   Using the second martingale, $X_t^2 - t$ and again Doob's theorem, $\mathbb{E}(X_\tau^2) - \mathbb{E}(\tau) = X_0^2 = 625$, and $\mathbb{E}(\tau) = 50$.
   Using the third martingale, $M_t = \sum_{i=0}^t X_i - 25(t+1)$. Here we have $(t+1)$ as there are $(t+1)$ terms in $\sum_{i=0}^t X_i$. Using Doob's theorem, $\mathbb{E}(M_\tau) = M_0 = 0$, and, finally, $\mathbb{E}(\sum_{i=0}^\tau X_i) = 25 \cdot 51 = 1275$.
3. $\tau \in \{3, 4, 5, 6\}$, X_t is a martingale. The random variable X_τ can take only two values, 0 or $1+2+3+4+1+3+5 = 19$. According to the Doob's theorem, $\mathbb{E}(X_\tau) = X_0 = 10$. And hence $\mathbb{P}(\text{Veniamin wins}) = \mathbb{P}(X_\tau = 19) = 10/19$. The strategy does not matter.
4. $\mathbb{E}(X_t) = 0$, $\mathrm{Var}(X_t) = 3t^10/2$, and $\mathrm{Cov}(X_t, Y_t) = t^6/2$.
   The expected value
   $$\mathbb{E}(W_t^{2k}) = t^k \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)$$
   may be useful with Ito's isometry.
5.
   $$dZ_t = [\cos X_t + 2X_t t W_t e^{W_t} - t^2 W_t e^{W_t} \sin X_t + \sin^2(tW_t) - 0.5t \cos X_t \sin^2(tW_t)]\, dt + \\ [\sin(tW_t)(2X_t - t\sin(X_t))]\, dW_t \quad (12)$$
6. Using the substitution $Y_t = \ln X_t$ we obtain equation $dY_t = (0.5 - X_t)\, dt + dW_t$. And $\ln X_t + \int_0^t X_s\, ds = 0.5t + W_t$.
   We exponentiate and then take integral
   $$\int_0^t X_s\, ds = \ln\left(1 + \int_0^t \exp(0.5s + W_s)\, ds\right)$$

Differentiate to obtain

$$X_t = \frac{\exp(0.5t + W_t)}{1 + \int_0^t \exp(0.5s + W_s)\, ds}$$

7.

$$X_0 = e^{-rT} F\left(\frac{\ln S_0 - (\sigma^2/2 - r)(T-1)}{\sigma\sqrt{T-1}}\right),$$

where $F()$ is the cumulative probability function for standard normal distribution.

7.5. Exam, 27.12.2014

Optimal control part

1. (10 points) Consider the nonlinear system of DE

$$\begin{cases} \dot{x}_1 = e^{x_1+x_2} - x_2 \\ \dot{x}_2 = -x_1 + x_1 x_2 \end{cases}$$

   Determine the points of rest of the system, classify their type and stability and draw the local phase diagram at each of them with the found eigenvectors.
2. (10 points) Solve the optimal control problem

$$\int_0^T \left[pK(t) - wK(t) - cI^2(t)\right]\, dt \to \max$$

   subject to $\dot{K}(t) = I(t)$, $K(0) = K_0 > 0$, $K(T)$ not given, $p > w > 0$, $c > 0$.
   Also show that $K(t) > 0$ for $t \in [0; T]$ and $\dot{K}(T) = 0$ (4 points for that part).
3. (20 points) Consider the optimal problem for the infinite horizon

$$\int_0^\infty e^{-rt}\left[-\frac{1}{2}u^2 + xu - \frac{1}{2}x^2\right]\, dt \to \max$$

   subject to $\dot{x} = u$, $x(0) = x_0 > 0$ and $1 < r < 2$.
   (a) (5 points) Write down the system of equations based on the current value Hamiltonian
   (b) (3 points) Is the integrand a concave or a strictly concave function?
   (c) (5 points) Reduce the necessary conditions down to a pair of linear DE involving only x and λ
   (d) (3 points) Find all steady-states of the system
   (e) (4 points) Classify them

Stochastic calculus part

Here W_t always denotes the standard Wiener process.

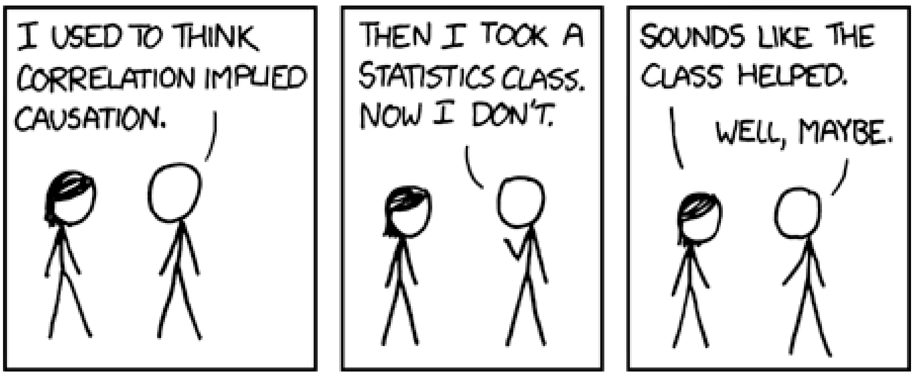
1. [10 points] Let τ denote the first moment of time when $|W_t| = 100$.
   (a) What is the distribution of W_τ?
   (b) Are W_τ and τ independent?
   (c) Assuming that some version of Doob's theorem may be applied find $\mathbb{E}(e^{-2\tau})$.

Hint: maybe $\exp(2W_t - 2t)$ will help?

2. [10 points] The random variables X_1, X_2, ..., X_n, ...are independent uniformly distributed on $[0; 1]$. I am summing them until the first X_i greater than 0.5 is added. After this term I stop. Let's denote by S the total sum and by N — the number of terms added. Find $\mathbb{E}(S|N)$, $\text{Var}(S|N)$, $E(S)$
   Hints: If U is uniform on $[a; b]$ then $\text{Var}(U) = (b-a)^2/12$. If G has geometric distribution (the number of throws to get the first success) then $\mathbb{E}(G) = 1/p$ where p is the probability of success.
3. [10 points] Find $\text{Var}\left(\int_0^t W_s\, ds\right)$.
   You may use the following guiding steps:
   (a) Find $d(tW_t)$ in short and full form
   (b) Find $\mathbb{E}\left(2tW_t \int_0^t s\, dW_s\right)$
   (c) Find $\mathbb{E}\left(\left(\int_0^t s\, dW_s\right)^2\right)$
   (d) Find $\mathbb{E}\left(\int_0^t W_s\, ds\right)$
   (e) $(a-b)^2 = a^2 - 2ab + b^2$:)
4. [10 points] The risk-free interest rate is equal to 0.1. The volatility of the share is equal to $\sigma = 1$. You have an option to receive 1\$ two years later if the percentage change of price of the share, S_t, during the first year is less than during the second year. Assume the framework of the Black and Scholes model. What is the fair price of this option?
5. [20 points] Solve the stochastic differential equation

$$dX_t = t\, dt + X_t\, dW_t,\ X_0 = 1$$

   You may use the following guiding steps:
   (a) Solve a less difficult stochastic differential equation, $dY_t = -Y_t\, dW_t,\ Y_0 = 1$
   (b) Find dA_t, where $A_t = X_t Y_t$
   (c) Find any deterministic function $f(t)$ such that $d(f(t)A_t)$ contains neither X_t nor A_t
   (d) Using full form of $d(f(t)A_t)$ find X_t. It may depend on some integrals :)



7.6. Exam, 27.12.2014 - solution

1. W_τ takes values 100 and -100 with equal probabilities. Yes, independent. $\mathbb{E}\left(e^{-2\tau}\right) = \frac{2}{e^{200}+e^{-200}}$

2. $\mathbb{E}(S|N) = 0.25N + 0.5$, $\text{Var}(S|N) = 0.25N/12$, $\mathbb{E}(N) = 1/p = 2$, $\mathbb{E}(S) = \mathbb{E}(\mathbb{E}(S|N)) = 1$. One may also find the variance, $\text{Var}(S) = 1/8 + 1/24 = 1/6$.
3. By hints:
   (a) $d(tW_t) = t\,dW_t + W_t\,dt$
   (b) Using Ito's isometry:

$$\mathbb{E}\left(2tW_t \int_0^t s\,dW_s\right) = 2t\mathbb{E}\left(\int_0^t 1\,dW_s \int_0^t s\,dW_s\right) = 2t\,\text{Cov}\left(\int_0^t 1\,dW_s, \int_0^t s\,dW_s\right) = \\ = 2t\int_0^t \mathbb{E}(1\cdot s)\,ds = t^3 \quad (13)$$

   (c) $\mathbb{E}\left(\left(\int_0^t s\,dW_s\right)^2\right) = t^3/3$ using Ito's isometry
   (d) $\mathbb{E}\left(\int_0^t W_s\,ds\right) = 0$
   (e) $\text{Var}\left(\int_0^t W_s\,ds\right) = t^3/3$
4. Event $\left\{\frac{S_1 - S_0}{S_0} < \frac{S_2 - S_1}{S_1}\right\}$ simplifies to $S_1^2 < S_0 S_2$ and further simplifies to $2\tilde{W}_1 - \tilde{W}_2 < 0$. And

$$X_0 = e^{-0.2}\tilde{P}(2\tilde{W}_1 - \tilde{W}_2 < 0) = e^{-0.2}/2$$

5. By steps:
   (a) Using the substitution $U_t = \ln Y_t$, one obtains

$$dU_t = -dW_t - \frac{1}{2}dt$$

   And the solution $Y_t = \exp(-W_t - t/2)$
   (b) As $A_t = X_t Y_t$ we have:

$$dA_t = X_t dY_t + Y_t dX_t + \frac{1}{2}2dX_t dY_t$$

   Finally

$$dA_t = Y_t(t - X_t)dt$$

   (c)

$$d(f(t)A_t) = f'(t)A_t dt + f(t)dA_t = f'(t)X_t Y_t dt + f(t)Y_t(t - X_t)dt$$

   To kill X_t in this expression we need to find a function f such that $f'(t) - f(t) = 0$. For example, $f(t) = e^t$.
   (d) From previous point we have

$$d(e^t A_t) = e^t Y_t t dt$$

   And

$$e^t A_t = e^0 A_0 + \int_0^t e^s Y_s s\,ds$$

   From initial condition $A_0 = 0$ and finally

$$X_t = \frac{1 + \int_0^t s \exp(-W_s + s/2)\, ds}{\exp(-W_t + t/2)}$$

8. 2015-2016

8.1. Stochastic calculus hometask

1. Find the expected value of $\mathbb{E}(\exp(aW_t))$, $\mathbb{E}(1_{W_t \leqslant b})$, $\mathbb{E}(\exp(aW_t) \cdot 1_{W_t \leqslant b})$ and $\mathbb{E}(W_t 1_{W_t \leqslant b})$. Naturally, you may use the standard normal cumulative distribution function F in your answer
2. It is known that $\mathbb{E}(Y|X) = 0$. Which of the following quantities must be zero: $\mathbb{E}(Y)$? $\mathbb{E}(X)$? $\mathrm{Cov}(X, Y)$? $\mathrm{Cov}(X^2, Y)$? $\mathrm{Cov}(X, Y^2)$? Prove or provide a counter-example.
3. Alisa and Bob throw a fair coin until either the sequence THTH or the sequence HTHH appears. Alisa wins if the sequence THTH appears first and Bob wins if the sequence HTHH appears first.

   (a) What is the probability that Alisa will win?

   (b) What is the expected duration of the game in tosses?

   Hint: you may introduce a martingale from lecture or solve this problem without martingales at all
4. Consider the process $Y_t = \exp(2W_t - 2t)$.

   (a) Find dY_t

   (b) Find $\int_0^t Y_u \, dW_u$

   (c) Find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$
5. Consider stochastic differential equation

   $$dX_t = (a - bX_t)dt + cdW_t,\ X_0 = x_0$$

   (a) Solve this differential equation[1].

   (b) Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$
6. In the framework of Black and Scholes model find the price at $t = 0$ of the classic European call option by calculating corresponding expected value.
   European call option with strike price K is the right to buy at time t one share at price K. So, at time t it pays you $S_t - K$ if $S_t > K$ and zero otherwise.

8.2. Stochastic calculus hometask — Solution

1. $\mathbb{E}(\exp(aW_t)) = e^{a^2 t/2}$, $\mathbb{E}(1_{W_t \leqslant b}) = F(b/\sqrt{t})$, $\mathbb{E}(\exp(aW_t) \cdot 1_{W_t \leqslant b}) = \ldots$, $\mathbb{E}(W_t 1_{W_t \leqslant b}) = \ldots$
2. Short answers:

   (a) $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = 0$.

   (b) The value $\mathbb{E}(X)$ can be non-zero, counterexample: $X \sim N(42; 42)$, $Y = 0 = const$.

[1] The answer may contain an Ito integral that cannot be simplified.

(c) $\text{Cov}(Y, f(X)) = \mathbb{E}(f(X)Y) - \mathbb{E}(Y)\mathbb{E}(f(X)) = \mathbb{E}(f(X)Y)$.

$$\mathbb{E}(f(X)Y) = \mathbb{E}(\mathbb{E}(f(X)Y|X)) = \mathbb{E}(f(X)\mathbb{E}(Y|X)) = \mathbb{E}(0) = 0$$

So, $\text{Cov}(Y, X) = \text{Cov}(Y, X^2) = 0$

(d) $\text{Cov}(X, Y^2)$ may be non-zero. Let Y take the values -1, 1, -2 and 2 with equal probability. And $X = Y^2$. In this case $\text{Cov}(X, Y^2) = \text{Var}(X) > 0$.

3. Let's denote the moment of time when the sequences A and B appear by N_A and N_B correspondingly. The game ends at moment $N = \min\{N_A, N_B\}$.
We need to find $p_A = \mathbb{P}(N = N_A)$, $p_B = \mathbb{P}(N = N_B)$ and $\mathbb{E}(N)$. We have one equation, $p_A + p_B = 1$ so we need two more equations:
We build them from nothing :)

$$\begin{cases} \mathbb{E}(N_A) = \mathbb{E}(N) + \mathbb{E}(N_A - N) \\ \mathbb{E}(N_B) = \mathbb{E}(N) + \mathbb{E}(N_B - N) \end{cases}$$

Here $\mathbb{E}(N_A) = 16 + 4 = 20$, $\mathbb{E}(N_B) = 16 + 2 = 18$.
Or

$$\begin{cases} \mathbb{E}(N_A) = \mathbb{E}(N) + p_A\mathbb{E}(N_A - N|N = N_A) + p_B\mathbb{E}(N_A - N|N = N_B) \\ \mathbb{E}(N_B) = \mathbb{E}(N) + p_A\mathbb{E}(N_B - N|N = N_A) + p_B\mathbb{E}(N_B - N|N = N_B) \end{cases}$$

$$\begin{cases} \mathbb{E}(N_A) = \mathbb{E}(N) + p_B\mathbb{E}(N_A - N|N = N_B) \\ \mathbb{E}(N_B) = \mathbb{E}(N) + p_A\mathbb{E}(N_B - N|N = N_A) \end{cases}$$

...
Based on http://projecteuclid.org/download/pdf_1/euclid.aop/1176994578
There is also a solution using Markov chains.

8.3. Exam, 12.01.2016

Stochastic calculus part

Here W_t always denotes the standard Wiener process.

1. [10 points] You throw a fair coin until «head» appears. Let's denote the result of the first toss by Y_1 (0 for tail and 1 for head) and the total number of throws by N. Find $\mathbb{E}(Y_1|N)$, $\text{Var}(Y_1|N)$ and $\mathbb{E}(N|Y_1)$
2. [10 points] Consider τ, the first moment of time when the standard Wiener process will touch the barrier $4y^2 = x + 1$, or formally, $\tau = \inf\{t \mid t \geqslant 0,\ |W_t| = 0.5\sqrt{t+1}\}$. Find $\mathbb{E}(\tau)$.
Hint: you may find the process $M_t = W_t^2 - t$ useful, you may also suppose that technical conditions of Doob's theorem are satisfied
3. [10 points] Let

$$Y_t = \exp\left(-6t^3 + \int_0^t f(s)\,dW_s\right),$$

where f is some deterministic function.

(a) Using Ito's lemma find dY_t

(b) Find at least one function f such that Y_t is a martingale

4. [10 points] The risk-free interest rate is equal to 0.1. The volatility of the share is equal to $\sigma = 1$. You have an option to receive 1$ two years later if the price growth during the second year is higher than 5%. Assume the framework of the Black and Scholes model. What is the fair price of this option?
5. [20 points] Consider the stochastic differential equation

$$dX_t = X_t^3\,dt + X_t^2\,dW_t,$$

   (a) Apply Ito's lemma to $Y_t = f(X_t)$
   (b) Find all the functions f that makes Y_t a martingale
   (c) Find all the solutions of the stochastic differential equation
   (d) Find the solution such that $X_0 = 2$.
   (e) Find the probability that the sample path of X_t will be continuous for $t \in [0; \infty)$

Optimal control part

6. (10 points) Given the system of differential equations

$$\begin{cases} \dot{x} = \sin(x+y) \\ \dot{y} = \sin(x-y) \end{cases}$$

   explore the behavior of its solutions in the neighborhood of the 2 points: $A(\pi,\pi)$ and $B(3\pi/2, 3\pi/2)$. Draw the phase portraits near A and B based on the knowledge of their eigenvalues, eigenvectors where possible.
7. (10 points) Solve the bounded control problem: maximize

$$\int_0^1 (2x - u^2/2)\,dt$$

   subject to constraints $\dot{x} = u - x + t^2$, $x(0) = 0$, $-1 \leqslant u \leqslant 0$. Verify that the maximizer has been found by applying one of the sufficient conditions.
8. (20 points) Solve so-called "limit pricing problem". A homogeneous product is produced by a dominant firm along with the "competitive fringe" consisting of the $x(t)$ identical firms (x is a continuous variable). Demand on good is given by $f(p) \in C^2$, where $p(t)$ is the price, $f'(p) < 0$ for $p > 0$. Let the dominant firm has a constant returns to scale technology with the marginal costs $c = const > 0$. Then $(p-c)f(p)$ is a strictly concave function. The problem of the firm is to maximize the discounted stream of profits

$$\int_0^{\infty} e^{-rt}(p-c)(f(p)-x)\,dt$$

   (each fringe firm produces only one unit of good) subject to constraint $\dot{x} = k(p - \bar{p})$ where k is some number, $\bar{p}$ is the equilibrium price and $r > 0$. Also $x(0) = x_0$.

   (a) Application of the current value Hamiltonian is required here. Derive the system of the first-order conditions.
   (b) By eliminating the Lagrange multiplier reduce the system to 2 equations with respect to (x, p). Check that $H_{pp} < 0$.

(c) Show that if the dominant firm operates at the price level $\bar{p} = c$ then the fringe firms supply the entire market in the equilibrium.

(d) Let the equilibrium price $\bar{p}$ be slightly above c. By evaluating the derivative $\frac{\partial x^s}{\partial \bar{p}}$ at $\bar{p} = c$ show that the dominant firm's market share becomes positive if $\bar{p}$ is slightly above c (x^s is the equilibrium number of the fringe firms).

8.4. Solution, exam 12.01.2016

1. If we know the value N then we know the value of Y_1, so $\mathrm{Var}(Y_1|N) = 0$ and

$$\mathbb{E}(Y_1|N) = \begin{cases} 0, & \text{if } N > 1; \\ 1, & \text{if } N = 1 \end{cases}$$

$$\mathbb{E}(N|Y_1) = \begin{cases} 1, & \text{if } Y_1 = 1; \\ 1+2, & \text{if } Y_1 = 0 \end{cases}$$

   Or, simply, $\mathbb{E}(N|Y_1) = 3 - 2Y_1$

2. We check that M_t is a martingale: $dM_t = -dt + 2W_t\,dW_t + \frac{1}{2}2\,dt = 2W_t\,dW_t$. And we apply Doob's theorem:

$$\mathbb{E}(W_\tau^2 - \tau) = \mathbb{E}(M_\tau) = \mathbb{E}(M_0) = 0$$

   So, $\mathbb{E}(W_\tau^2) = \mathbb{E}(\tau)$
   From the definition of τ, $W_\tau^2 = (\tau + 1)/4$. We get the equation

$$\mathbb{E}(\tau) = \mathbb{E}(\tau + 1)/4$$

   And $\mathbb{E}(\tau) = 1/3$
   Based on http://www-stat.wharton.upenn.edu/~shepp/publications/14.pdf.

3. Let's write the process Y_t as $Y_t = \exp(Z_t)$ where $Z_t = -6t^3 + \int_0^t f(s)\,dW_s$. So, $dY_t = \exp(Z_t)\,dZ_t + \frac{1}{2}\exp(Z_t)\,(dZ_t)^2$.
   We find dZ_t:

$$dZ_t = -18t^2\,dt + f(t)\,dW_t$$

   So, $(dZ_t)^2 = f^2(t)\,dt$ and

$$dY_t = \exp(Z_t)(-18t^2\,dt + f(t)\,dW_t) + \frac{1}{2}\exp(Z_t)f^2(t)\,dt$$

   To cancel the term before dt one should have

$$-18t^2 + \frac{1}{2}f^2(t) = 0$$

   Possible solutions are $f(t) = 6t$ and $f(t) = -6t$

4. We should find the discounted expected price:

$$X_0 = e^{-2r}\tilde{\mathbb{P}}(S_2/S_1 > 1.05) = e^{-2r}\tilde{\mathbb{P}}(\exp((r - \sigma^2/2) + \sigma(\tilde{W}_2 - \tilde{W}_1)) > 1.05)$$

   We remark that $\tilde{W}_2 - \tilde{W}_1 \sim N(0;1)$ under $\tilde{\mathbb{P}}$. So

$$X_0 = e^{-2r}\tilde{P}(N(0;1) > (\ln 1.05 - r + \sigma^2/2)/\sigma) = e^{-2r}F((r - \sigma^2/2 - \ln 1.05)/\sigma) = e^{-0.2}F(-0.45)$$

5. Step by step
   (a) Apply Ito's lemma:
   $$dY_t = f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,(dX_t)^2 = f' \cdot (X_t^3\,dt + X_t^2\,dW_t) + \frac{1}{2}f'' \cdot X_t^4\,dt$$
   (b) To cancel the term before dt the following condition must be satisfied
   $$f' \cdot X_t^3 + \frac{1}{2}f'' \cdot X_t^4 = 0$$
   Let's denote f' by g, so $g + g' \cdot x/2 = 0$. This equation may be solved by separation of variables:
   $$-2\frac{dx}{x} = \frac{dg}{g}$$
   The general solution is $g(x) = cx^{-2}$, so the general solution for f is
   $$f(x) = c_1 x^{-1} + c_2$$
   (c) To find all the solutions we use the simpliest possible f, $f(x) = x^{-1}$. In this case
   $$dY_t = 0\,dt + f' \cdot X_t^2\,dW_t = -dW_t$$
   So, $Y_t = Y_0 - W_t$ and
   $$X_t = \frac{1}{c - W_t}$$
   (d) If $X_0 = 2$ than $X_t = 1/(0.5 - W_t)$
   (e) The probability that W_t will forever stay below 0.5 is zero, so the probability that X_t will have continuous sample path for all t is also zero.

8.5. Retake, 15.02.2016

Stochastic calculus part

Here W_t always denotes the standard Wiener process.

1. [10 points] You throw a fair coin infinite number of times. Let's denote the result of the second toss by Y_2 (0 for tail and 1 for head) and the number of throws to get the first «head» by N. Find $\mathbb{E}(Y_2|N)$, $\mathrm{Var}(Y_2|N)$ and $\mathbb{E}(N|Y_2)$
2. [10 points] The process Y_t is given by $Y_t = 2W_t + 5t$. The stopping time τ is given by $\tau = \min\{t|Y_t^2 = 100\}$. Find the distribution of the random variable Y_τ and the expected value $\mathbb{E}(\tau)$.
   Hint: you may find the martingales a^{Y_t} and $Y_t - f(t)$ useful
3. [10 points] Let $X_0 = 2016$ and $dX_t = 2t\,dt + t^2\,dW_t$. Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$.

4. [10 points] The risk-free interest rate is equal to 0.1. The volatility of the share is equal to $\sigma = 1$. The price of a share at $t = 0$ is $S_0 = 100$. You have an option to receive 1$ **two** years later if the price of the share after **one** year is more than 105. Assume the framework of the Black and Scholes model. What is the fair price of this option?
5. [20 points] Consider the stochastic differential equation

$$dX_t = 8W_t^2 X_t\, dt + 4W_t X_t\, dW_t, \text{ where } X_0 = 1$$

   (a) Apply Ito's lemma to $Y_t = \ln X_t$

   (b) Find the solution of the initial stochastic differential equation

Optimal control part

6. Find extremals that provide the highest or lowest values of the following integral

$$J(y) = \int_1^e \left[\frac{1}{2}x(y')^2 + \frac{2yy'}{x} - \frac{y^2}{x^2}\right] dx$$

   with the boundary values $y(1) = 1$, $y(e) = 2$.

   (a) (10 points) Find the extremal(s).

   (b) (10 points) Let $\tilde{y}(x)$ be the extremal you have found. Let $h(x) \in C^1$ and $h(1) = h(e) = 0$ ($h(x)$ is not identically zero). Prove that $J(\tilde{y} + h) - J(\tilde{y}) > 0$.

7. Consider Ramsey's model. Maximize the integral $I = \int_0^\infty [u(c) - B]dt$ subject to $\dot{k} = f(k) - c - \delta k$, $k(0) = k_0$. Function $u(c)$ monotonically increases and tends to B at the infinity, moreover I converges.

   (a) (5 points) Derive Ramsey's Law $\frac{d}{dt}u'(c) = u'(c)[\delta - f'(k)]$.

   (b) (5 points) Let

$$u(c) = \begin{cases} 2c - \frac{c^2}{B}, \text{ for } c \leqslant B \\ B, \text{ otherwise} \end{cases}$$

   Production function $f(k) = \alpha k$ and $\alpha > \delta$. Find the optimal solutions c^*, k^*. Do these solutions necessarily have an economic sense?

8. (10 points) Solve the problem on the bounded optimal control $\int_0^T (1 - \beta - u)x dt \to \max$, subject to $\dot{x} = \alpha x u$, $x(0) = x_0$, $0 \leqslant u \leqslant 1$. In this problem $\alpha > 0$ and $0 < \beta < 1$.

8.6. Retake, solution

1.

$$\mathbb{E}(Y_2|N) = \begin{cases} 0.5, \text{ if } N = 1; \\ 1, \text{ if } N = 2; \\ 0, \text{ if } N > 2; \end{cases}$$

$$\operatorname{Var}(Y_2|N) = \begin{cases} 0.25, \text{ if } N = 1; \\ 0, \text{ if } N > 1; \end{cases}$$

9. 2016-2017

9.1. Hometask

1. Consider the two independent Brownian motions, (W_t) and (V_t). Which of the following processes is Brownian motion:
   (a) $Z_t = \frac{1}{2}W_t + \frac{1}{2}V_t$
   (b) $Q_t = \frac{1}{\sqrt{2}}W_t + \frac{1}{\sqrt{2}}V_t$
2. Let $Y_t = \int_0^t (W_u + u)^2 \, dW_u$. Find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.
3. James Bond plays in a casino. At every bet he wins one pound with probability 0.5, looses one pound with probability 0.4 or wins nothing with probability 0.1. His initial fortune is $X_0 = 10$ pounds. He stops playing if he goes bankrupt or if he achieves the fortune of 300 pounds (airline ticket price from London to Moscow).
   (a) Find the constant a such that $M_t = a^{X_t}$ is a martingale.
   (b) What is the probability that James Bond will win enough to buy the ticket?
4. Let R_t be the exchange rate at time t. We suppose that $dR_t = \mu R_t dt + \sigma R_t dW_t$. Consider the inverse exchange rate $I_t = 1/R_t$. Find the expression for dI_t. The expression should not contain R_t.
5. It is known that M_t is a martingale. We also know that in short-hand notation $(dM_t)^2 = dt$. What can we say about the process $Y_t = M_t^2 - t + 2017$?
6. Solve the stochastic differential equation

$$dX_t = \frac{-1}{1-t} X_t \, dt + dW_t, \; X_0 = 0 \tag{14}$$

   You may use or not use the following hints:
   (a) The correct answer will contain the integral $\int_0^t \frac{1}{1-u} dW_u$ that cannot be simplified.
   (b) Solve the ordinary differential equation

$$dY_t = \frac{-1}{1-t} Y_t \, dt, \; Y_0 = 1 \tag{15}$$

   (c) Represent X_t as $X_t = Y_t \cdot Z_t$ and find the equation for dZ_t. Find the expression for Z_t.
7. Consider the framework of the Black and Scholes model. Let $\mathbb{P}$ be the original probability measure and $\tilde{\mathbb{P}}$ be the risk-neutral probability measure. Provide an example of three events A, B and C such that $\mathbb{P}(A) > \tilde{\mathbb{P}}(A)$, $\mathbb{P}(B) < \tilde{\mathbb{P}}(B)$, $\mathbb{P}(C) = \tilde{\mathbb{P}}(C)$.
8. Consider the framework of the Black and Scholes model. The asset pays you 1 dollar at fixed time T if and only if the price of a share S_T is above the strike-price K. Find the current price X_0 of this asset.

9.2. ht, solution

Solution by Anastasia Andreeva:

1. (a)

$$Z_t = \frac{1}{2}W_t + \frac{1}{2}V_t$$

$\text{Var}(Z_t - Z_s) = \text{Var}[0.5 \cdot (W_t - W_s + V_t - V_s)] = \frac{1}{4}\text{Var}(W_t - W_s) + \frac{1}{4}\text{Var}(V_t - V_s) = \frac{1}{2}(t - s)$
Thus, $Z_t - Z_s \sim \mathcal{N}\left(0, \frac{t-s}{2}\right)$, therefore, Z_t is not a Brownian motion.

(b)

$$Q_t = \frac{1}{\sqrt{2}} W_t + \frac{1}{\sqrt{2}} V_t$$

Let's check all the conditions:

i. $Q_0 = \frac{1}{\sqrt{2}}(W_0 + V_0) = 0$

ii. $P(Q_t \text{trajectory is continuous}) = 1$ since W_t and V_t correspond to this condition

iii. $Q_t - Q_s = \frac{1}{\sqrt{2}}(W_t - W_s) + \frac{1}{\sqrt{2}}(V_t - V_s)$ is independent on $\mathcal{F}_s$ since $W_t - W_s$ and $V_t - V_s$ are independent on $\mathcal{F}_s$

iv. $\mathbb{E}(Q_t - Q_s) = \frac{1}{\sqrt{2}}\mathbb{E}(W_t - W_s) + \frac{1}{\sqrt{2}}\mathbb{E}(V_t - V_s) = 0$
$\text{Var}(Q_t - Q_s) = \text{Var}\left(\frac{1}{\sqrt{2}} \cdot (W_t - W_s + V_t - V_s)\right) = \frac{1}{2}\text{Var}(W_t - W_s) + \frac{1}{2}\text{Var}(V_t - V_s) = t - s$ Thus, $Q_t - Q_s \sim \mathcal{N}(0, t - s)$

v. Q_t is measurable relative to $\mathcal{F}_t$

Thus, Q_t is a Brownian motion.

2.

$$Y_t = \int_0^t (W_u + u)^2 dW_u$$

Y_t is a martingale because it is a stochastic integral, so $\mathbb{E}(Y_t) = \mathbb{E}(Y_0) = 0$.

$$\mathbb{E}(W_u^4) = 3u^2, \mathbb{E}(W_u^2) = u, \mathbb{E}(W_u^3) = \mathbb{E}(W_u) = 0$$

$$\text{Var}(Y_t) = \int_0^t \mathbb{E}(W_u + u)^4 du = \int_0^t \mathbb{E}(W_u^4 + 4W_u^3 u + 6W_u^2 u^2 + 4W_u u^3 + u^4) du =$$

$$= \int_0^t (3u^2 + 6u^3 + u^4) du = u^3 + \frac{3u^4}{2} + \frac{u^5}{5}\Big|_0^t = t^3 + \frac{3t^4}{2} + \frac{t^5}{5} \quad (16)$$

3. (a) $\mathbb{E}(M_{t+1}|\mathcal{F}_t) = \mathbb{E}(a^{X_{t+1}}|\mathcal{F}_t) = 0.5a^{X_t+1} + 0.4a^{X_t-1} + 0.1a^{X_t} = a^{X_t}(0.5a + 0.4a^{-1} + 0.1)$
$0.5a + 0.4a^{-1} + 0.1 = 1 \Rightarrow a^2 - 1.8a + 0.8 = 0 \Rightarrow a_1 = 1, a_2 = 0.8$
$a \neq 1$ otherwise $M_t = 1$ for all t. So, $a = 0.8$.

(b) James Bond will get 300 with probability p and lose all money with probability $(1 - p)$.
τ is the stopping moment (win or lose).
According to Doob's theorem, $\mathbb{E}(M_\tau) = \mathbb{E}(M_0) = 0.8^{10}$.
At the same time $\mathbb{E}(M_\tau) = \mathbb{E}(0.8^{X_\tau}) = p \cdot 0.8^{300} + (1 - p) \cdot 0.8^0 = 0.8^{10}$
Then $p = \frac{1-0.8^{10}}{1-0.8^{300}} \approx 0.89$
Thus, the answer is 89%.

4. I will use the Ito's formula:

$$(dR_t)^2 = \sigma^2 R_t^2 dt$$

$$dI_t = -\frac{1}{R_t^2}dR_t + \frac{1}{R_t^3}(dR_t)^2 = -\frac{1}{R_t^2}(\mu R_t dt + \sigma R_t dW_t) + \frac{\sigma^2}{R_t}dt =$$
$$= \frac{1}{R_t}((\sigma^2 - \mu)dt + \sigma dW_t) = I_t((\sigma^2 - \mu)dt + \sigma dW_t) \quad (17)$$

5.
$$dY_t = 2M_t dM_t - dt + 2dt = 2M_t dM_t + dt$$

Since the expression includes dt, Y_t is not a martingale.

6.
$$dX_t = -\frac{1}{1-t}X_t dt + dW_t, X_0 = 0$$

(a)
$$dY_t = -\frac{1}{1-t}Y_t dt, Y_0 = 1$$
$$\frac{dY_t}{Y_t} = -\frac{dt}{1-t}$$
$$\ln|Y_t| = \ln|1-t| + C_0$$
$$Y_t = C(1-t), Y_0 = C = 1 \Rightarrow Y_t = 1-t$$

(b)
$$X_t = Y_t Z_t$$
$$dX_t = -\frac{X_t}{1-t}dt + dW_t = Z_t dY_t + Y_t dZ_t = -\frac{Z_t Y_t}{1-t}dt + Y_t dZ_t = -\frac{X_t}{1-t}dt + (1-t)dZ_t$$
$$dW_t = (1-t)dZ_t \Rightarrow dZ_t = \frac{dW_t}{1-t} \Rightarrow Z_t = Z_0 + \int_0^t \frac{dW_u}{1-u}$$

(c)
$$X_t = Y_t Z_t = (1-t)\left(Z_0 + \int_0^t \frac{dW_u}{1-u}\right), X_0 = 0 = Z_0$$
$$X_t = (1-t)\int_0^t \frac{dW_u}{1-u}$$

7. W_t is a Brownian motion for P. $Y_t = W_t + \frac{\mu - r}{\sigma}t$ is a Brownian motion for $\tilde{P}$.
If $\mu > r$, then $Y_t > W_t$ and $\tilde{\mathbb{P}}(Y_t > 0) = \mathbb{P}(W_t > 0) = \mathbb{P}(Y_t > \mu t) < \mathbb{P}(Y_t > 0)$.
If $\mu < r$, then $Y_t < W_t$ and $\tilde{\mathbb{P}}(Y_t > 0) > \mathbb{P}(Y_t > 0)$.
If $\mu = r$, then $W_t = Y_t$ and $\mathbb{P}(W_t > 0) = \tilde{\mathbb{P}}(W_t > 0)$.

8.
$$X_T = \begin{cases} 1, \text{if } S_T > K \\ 0, \text{otherwise} \end{cases}$$
$$X_0 = e^{-rT}\mathbb{E}_{\tilde{P}}(X_T|\mathcal{F}_0) = e^{-rT}\tilde{\mathbb{P}}(S_T > K|\mathcal{F}_0) = e^{-rT}\tilde{\mathbb{P}}(S_0 e^{(r-0.5\sigma^2)T + \sigma\tilde{W}_T} > K|\mathcal{F}_0)$$
$$\tilde{W}_T > \frac{\ln\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma}$$

Further, we standardize $\tilde{W}_T$:

$$\frac{\tilde{W}_T}{\sqrt{T}} > \frac{\ln\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Thus,

$$X_0 = e^{-rT}\tilde{P}\left(\frac{\tilde{W}_T}{\sqrt{T}} > \frac{\ln\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = e^{-rT}\left(1 - \Phi\left(\frac{\ln\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right)$$

9.3. exam, 10.01.2017

Stochastic calculus part

Here W_t always denotes the standard Wiener process.

1. (10 points) You throw a standard fair die n times. Let X be the number of «fives» in the first $(n-1)$ throws and Y — the number of «fives» in the last $(n-1)$ throws.
   For $n \geqslant 3$ calculate $\mathbb{E}(Y|X)$, $\mathbb{E}(X|Y)$, $\mathrm{Var}(Y|X)$.
2. (10 points) Find a constant b such that $Y_t = \int_0^t W_u^3\,du + bW_t^5$ is a martingale.
3. (10 points) The process X_t evolves according to the formula

$$X_t = 1 - t + (1-t)\int_0^t \frac{1}{1-u}\,dW_u$$

   (a) Is X_t a martingale?
   (b) Find $\mathbb{E}(X_t)$, $\mathrm{Var}(X_t)$ and $\mathrm{Cov}(W_t, X_t)$
   (c) Draw $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$ as functions of t
   (d) Draw two possible trajectories of X_t
4. (10 points) Find the price of the «Asset-or-nothing» call option at time $t = 0$ in the framework of Black and Scholes model. The risk-free interest rate is equal to r. The volatility of the share is equal to σ. The current share price is S_0. The «Asset-or-nothing» call option pays you at fixed time T the sum S_T if S_T is higher than the strike price K or nothing otherwise.
   Hint: the correct answer will contain the normal cumulative distribution function $F()$.
5. (20 points) Interest rate evolves according to the stochastic differential equation

$$dr_t = -\lambda(r_t - c)\,dt + \sigma dW_t$$

   where r_0, c, λ and σ are some positive constants.
   Solve this stochastic differential equation.
   You may use or not use the following hints:
   (a) Obtain a stochastic differential equation for $X_t = r_t - c$.
   (b) Solve the ordinary differential equation $dY_t = -\lambda Y_t\,dt,\ Y_0 = 1$
   (c) Represent X_t as $X_t = Y_t \cdot Z_t$ and find the equation for dZ_t.
   (d) Solve the equation for Z_t.
   (e) The correct answer for r_t will contain an Ito's integral that cannot be simplified.

Optimal control part

6. (10 points) Solve the bounded control problem:

$$\int_0^T (u-x)^2 dt \to \min$$

subject to $\dot{x} = \frac{1}{2}(u-x)$, $x(0)=1$, $|u| \leqslant 1, T > 0$.

7. (10 points) Find the extremals for the calculus of variations problem:

$$\int_0^1 \frac{1}{2}\dot{x}^2 + x\dot{x} + x\,dt$$

when both endpoint values $x(0)$ and $x(1)$ can be chosen freely.

8. (20 points) Costs of the farmer's production $C(x,y)$ depends on the output $y(t)$ and the fertility of soil $x(t)$ by the formula $C = y^2 + \frac{1}{\sqrt{x}}$. This industry is perfectly competitive and the price of the harvest per unit is p. Fertility changes over time by the dynamic law $\dot{x} = B - \alpha y$, where $p\alpha > 2B > 0$, and $x(0) > 0$. Constant B is defined by the government subsidy. Profit of the firm is discounted with the rate $r > 0$.

    (a) Write down the problem of maximizing the stream of discounted profit over the infinite time horizon.

    (b) Write down the system of first order conditions using the current value Hamiltonian. Draw the phase portrait in (x,y) coordinate system, in which x − axis is drawn horizontally. Find the steady-state (x_s, y_s) if it exists, classify it using Jacobian. With the help of the arrows provide the sketch of the dynamic paths of the system.

    (c) Find the sign of $\partial x_s/\partial p$. What justification of this sign can you provide from the economic point of view?

9.4. exam, solution

1.

$$\mathbb{E}(Y|X) = \frac{n-2}{n-1}X + \frac{1}{6}$$
$$\mathbb{E}(X|Y) = \frac{n-2}{n-1}Y + \frac{1}{6}$$

2. Take dY_t, kill term before dt, $b = -0.1$.

$$dY_t = (W_t^3 + 10bW_t^3)\,dt + 5bW_t^4\,dW_t$$

3. The process X_t is not a martingale as $\mathbb{E}(X_t) = 1-t$.

$$\mathrm{Var}(X_t) = (1-t)^2\mathbb{E}(I^2) = (1-t)^2\int_0^t \frac{1}{(1-u)^2}\,du = t(1-t)$$

$$\mathrm{Cov}(W_t, X_t) = (1-t)\cdot\left(\int_0^t dW_u \cdot \int_0^t \frac{1}{1-u}dW_u\right) = (1-t)\mathbb{E}\left(\int_0^t \frac{1}{1-u}\,du\right) = (t-1)\ln(1-t)$$

4. $$X_0 = e^{-rT}\tilde{\mathbb{E}}(S_T \cdot 1_{S_T > K})$$
5. $$dX_t = dr_t = -\lambda X_t dt + \sigma dW_t$$
   Solving ODE we obtain $Y_t = e^{-\lambda t}$.
   $$dX_t = Y_t dZ_t + Z_t dY_t + dY_t dZ_t$$
   Equation for Z_t:
   $$dZ_t = e^{\lambda t}\sigma dW_t$$
   Just recover the full form:
   $$Z_t = Z_0 + \sigma \int_0^t e^{\lambda u} dW_u$$
   Finally,
   $$r_t = e^{-\lambda t}\left(r_0 - c + \sigma \int_0^t e^{\lambda u}\, dW_u\right) + c$$

10. Some junk

10.1. Problems

1. Let $Y_t = W_t + \mu t$. Find all constants b such that $Z_t = e^{bY_t}$ is a martingale.

10.2. Solutions

1. Using Ito's lemma
   $$dZ_t = e^{bY_t} b\left(dW_t + \mu dt + \frac{1}{2} b\, dt\right)$$
   So $b = 0$ and $b = -2\mu$ are ok.