Stochastic calculus part

Here W_t always denotes the standard Wiener process.

1. (10 points) Find $\mathrm{E}(W_6^2|W_4)$ and $\mathbb{P}(W_6 > 0|W_4)$.

   Hint: the answer may contain the normal cumulative distribution function.

2. (10 points) I wait until Wiener process will hit the level 4 or the level -3, that is up to the moment $\tau = min\{t|W_t = 4 \cup W_t = -3\}$.

   Find $\mathbb{P}(W_\tau = 4)$ and $\mathrm{E}(\tau)$.

   Hint: you may consider martingales W_t, $W_t^2 - t$

3. (10 points) The process X_t is defined as

$$X_t = \begin{cases} 1, & t \in [0;1) \\ -2, & t \in [1;2) \\ W_{1.5}, & t \in [2;\infty) \end{cases}$$

   (a) Explicitely find $Z_t = \int_0^t X_u dW_u$

   (b) Find $\mathrm{E}(Z_t)$ and $\mathrm{Var}(Z_t)$

4. (10 points) Find the price of the «Asset-or-nothing» call option at time $t = 0$ in the framework of Black and Scholes model. The risk-free interest rate is equal to r. The volatility of the share is equal to σ. The current share price is S_0. The «Asset-or-nothing» call option pays you at fixed time T the sum S_T if S_T is higher than the strike price K or nothing otherwise.

   Hint: the correct answer will contain the normal cumulative distribution function $F()$.

5. (20 points) Solve the stochastic differential equation

$$dX_t = X_t dt + 2X_t dW_t + 3dt + 4dW_t, \quad X_0 = 1.$$

   You may use or not use the following guiding steps:

   (a) Using substitution $S_t = \ln Y_t$ solve a simplier equation

$$dY_t = Y_t dt + 2Y_t dW_t$$

   (b) Represent X_t as $X_t = Y_t \cdot Z_t$ and write down the equation for dZ_t.

   (c) Solve the equation for Z_t.

   (d) Finalise the solution and find X_t.