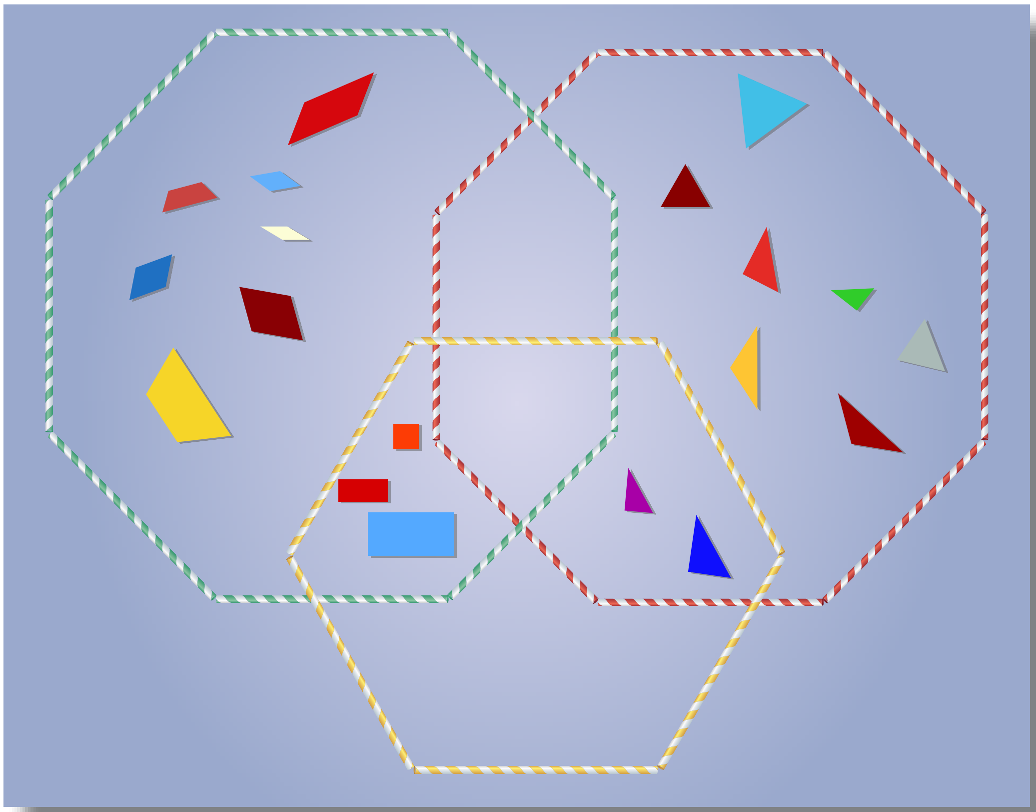
Chapter 9: Sequences and sets

Key words

- Sequence
- Term
- Term-to-term rule
- n^{th} term
- Rational number
- Terminating decimals
- Recurring decimals
- Set
- Element
- Empty set
- Universal set
- Complement
- Union
- Intersection
- Subset
- Venn diagram
- Set builder notation

In this chapter you will learn how to

- describe the rule for continuing a sequence
- find the n^{th} term of some sequences
- use the n^{th} term to find terms from later in a sequence
- generate and describe sequences from patterns of shapes
- list the elements of a set that have been described by a rule
- find unions and intersections of sets
- find complements of sets (EXTENDED)
- represent sets and solve problems using Venn diagrams
- express recurring decimals as fractions (EXTENDED)



Collecting shapes with the same properties into groups can help to show links between groups. Here, three-sided and four-sided shapes are grouped as well as those shapes that have a right angle.

How many students at your school study History and how many take French? If an event was organised that was of interest to those students who took either subject, how many would that be? If you chose a student at random, what is the probability that they would be studying both subjects? Being able to put people into appropriate sets can help to answer these types of questions!

RECAP

You should already be familiar with the following number sequences and patterns work:

Sequences (Year 9 Mathematics)

A sequence is a list of numbers in a particular order.

Each number is called a term.

The position of a term in the sequence is written using a small number like this: T_5

T_1 means the first term and T_n means any term.

Term to term rule (Year 9 Mathematics)

The term to term rule describes how to move from one term to the next in words.

For the sequence on the right the term to term rule is 'subtract 4 from the previous term to find the next one'.

$$27 \xrightarrow{-4} 23 \xrightarrow{-4} 19 \xrightarrow{-4} 15 \ldots$$

Position to term rule (Year 9 Mathematics; Chapter 2)

When there is a clear rule connecting the terms you can use algebra to write a function (equation) for finding any term.

For example, the sequence above has the rule $T_n = 27 - 4(n - 1)$

$T_1 = 27 - 4 \times (1 - 1) = 27$

$T_2 = 27 - 4 \times (2 - 1) = 23$

and so on.

9.1 Sequences

REWIND

In chapter 1 you learned that a set is a list of numbers or other items. ◄

A **sequence** can be thought of as a set whose elements (items in the list) have been listed in a particular order, with some connection between the elements. Sets are written using curly brackets { }, whereas sequences are generally written without the brackets and there is usually a rule that will tell you which number, letter, word or object comes next. Each number, letter or object in the sequence is called a **term**. Any two terms that are next to each other are called consecutive terms.

When trying to spot the pattern followed by a sequence, keep things simple to start with. You will often find that the simplest answer is the correct one.

LINK

Chemists will often need to understand how quantities change over time. Sometimes an understanding of sequences can help chemists to understand how a reaction works and how results can be predicted.

The term-to-term rule

Here are some sequences with the rule that tells you how to keep the sequence going:

2, 8, 14, 20, 26, 32, . . . (get the next term by adding six to the previous term).

The pattern can be shown by drawing it in this way:

$$2 \xrightarrow{+6} 8 \xrightarrow{+6} 14 \xrightarrow{+6} 20 \xrightarrow{+6} 26 \xrightarrow{+6} 32 \quad \ldots$$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ (divide each term by two to get the next term).

Again, a diagram can be drawn to show how the sequence progresses:

$$1 \xrightarrow{\div 2} \frac{1}{2} \xrightarrow{\div 2} \frac{1}{4} \xrightarrow{\div 2} \frac{1}{8} \xrightarrow{\div 2} \frac{1}{16} \quad \ldots$$

1, 2, 4, 8, 16, 32, . . . (get the next term by multiplying the previous term by two).

In diagram form:

$$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} 32 \quad \ldots$$

The rule that tells you how to generate the next term in a sequence is called the **term-to-term rule**.

Sequences can contain terms that are not numbers. For example, the following sequence is very well known:

a, b, c, d, e, f, g, h, i, . . .

In this last example, the sequence stops at the 26^{th} element and is, therefore, a *finite* sequence. The previous three sequences do not necessarily stop, so they may be *infinite* sequences (unless you choose to stop them at a certain point).

Exercise 9.1

1 Draw a diagram to show how each of the following sequences continues and find the next three terms in each case.

a 5, 7, 9, 11, 13, . . .
b 3, 8, 13, 18, 23, . . .
c 3, 9, 27, 81, 243, . . .
d 0.5, 2, 3.5, 5, 6.5, . . .
e 8, 5, 2, –1, –4, . . .
f 13, 11, 9, 7, 5, . . .
g 6, 4.8, 3.6, 2.4, 1.2, . . .
h 2.3, 1.1, –0.1, –1.3, . . .

2 Find the next three terms in each of the following sequences and explain the rule that you have used in each case.

a 1, –3, 9, –27, . . .
b Mo, Tu, We, Th, . . .
c a, c, f, j, o, . . .
d 1, 2, 2, 4, 3, 6, 4, 8, . . .

Relating a term to its position in the sequence

Think about the following sequence:

1, 4, 9, 16, 25, . . .

You should have recognised these as the first five square numbers, so:

first term $= 1 \times 1 = 1^2 = 1$

second term $= 2 \times 2 = 2^2 = 4$

third term $= 3 \times 3 = 3^3 = 9$

and so on.

You could write the sequence in a table that also shows the position number of each term:

Term number (n)	1	2	3	4	5	6	7	8	9
Term value (n^2)	1	4	9	16	25	36	49	64	81

Notice that the term number has been given the name 'n'. This means, for example, that $n = 3$ for the third term and $n = 100$ for the hundredth term. The rule that gives each term from its position is:

term in position $n = n^2$

An expression for the term in position n is called the **n^{th} term**. So for this sequence:

n^{th} term $= n^2$

FAST FORWARD

You will carry out similar calculations when you study equations of straight lines in chapter 10.

Now think about a sequence with n^{th} term $= 3n + 2$

For term 1, $n = 1$ so the first term is $3 \times 1 + 2 = 5$

For term 2, $n = 2$ so the second term is $3 \times 2 + 2 = 8$

For term 3, $n = 3$ so the third term is $3 \times 3 + 2 = 11$

Continuing this sequence in a table you will get:

n	1	2	3	4	5	6	7	8	9
Term	5	8	11	14	17	20	23	26	29

You should always try to include a diagram like this. It will remind you what to do and will help anyone reading your work to understand your method.

If you draw a diagram to show the sequence's progression, you get:

5 8 11 14 17 20 …

+ 3 + 3 + 3 + 3 + 3

Notice that the number *added* to each term in the diagram appears in the n^{th} term formula (it is the value that is multiplying n, or the *coefficient* of n).

This happens with any sequence for which you move from one term to the next by adding (or subtracting) a fixed number. This fixed number is known as the *common difference*.

For example, if you draw a sequence table for the sequence with n^{th} term $= 4n - 1$, you get:

n	1	2	3	4	5	6	7	8	9
Term	3	7	11	15	19	23	27	31	35

Here you can see that 4 is added to get from one term to the next and this is the coefficient of n that appears in the n^{th} term formula.

The following worked example shows you how you can find the n^{th} term for a sequence in which a common difference is added to one term to get the next.

Worked example 1

a Draw a diagram to show the rule that tells you how the following sequence progresses and find the n^{th} term.

2, 6, 10, 14, 18, 22, 26, . . .

b Find the 40th term of the sequence.
c Explain how you know that the number 50 is in the sequence and work out which position it is in.
d Explain how you know that the number 125 is *not* in the sequence.

a

2 6 10 14 18 22 …

+ 4 + 4 + 4 + 4 + 4

Notice that 4 is added on each time, this is the common difference. This means that the coefficient of n in the n^{th} term will be 4. This means that '$4n$' will form part of your n^{th} term rule.

If $n = 3$

Then

$4n = 4 \times 3 = 12$

Now think about any term in the sequence, for example the third (remember that the value of n gives the position in the sequence). Try $4n$ to see what you get when $n = 3$. You get an answer of 12 but you need the third term to be 10, so you must subtract 2.

It appears that the n^{th} term rule should be $4n - 2$.

You should check this.

Try for $n = 5$

$4n - 2 = 4 \times 5 - 2 = 18$

So the n^{th} term $= 4n - 2$

Test it using any term, say the 5th term. Substitute $n = 5$ into the rule. Notice that the 5th term is indeed 18.

b	40^{th} term $\therefore n = 40$ $4 \times 40 - 2 = 158$	To find the 40^{th} term in the sequence you simply need to let $n = 40$ and substitute this into the n^{th} term formula.
c	$4n - 2 = 50$ $4n - 2 = 50$ $4n = 52$ $n = \frac{52}{4} = 13$ Since this has given a whole number, 50 must be the 13^{th} term in the sequence.	If the number 50 is in the sequence there must be a value of n for which $4n - 2 = 50$. Rearrange the rule to make n the subject: Add 2 to both sides Divide both sides by 4
d	$4n - 2 = 125$ $4n = 127$ $n = \frac{127}{4} = 31.75$ Since n is the position in the sequence it must be a whole number and it is not in this case. This means that 125 cannot be a number in the sequence.	If the number 125 is in the sequence then there must be a value of n for which $4n - 2 = 125$. Rearrange to make n the subject. Add 2 to both sides Divide both sides by 4

Exercise 9.2

1 Find the **(i)** 15^{th} and **(ii)** n^{th} term for each of the following sequences.

a 5, 7, 9, 11, 13, . . .
b 3, 8, 13, 18, 23, . . .
c 3, 9, 27, 81, 243, . . .
d 0.5, 2, 3.5, 5, 6.5, . . .
e 8, 5, 2, −1, −4, . . .
f 13, 11, 9, 7, 5, . . .
g 6, 4.8, 3.6, 2.4, 1.2, . . .
h 2, 8, 18, 32, 50, …

2 Consider the sequence:

4, 12, 20, 28, 36, 44, 52, . . .

Remember that 'n' is always going to be a positive integer in n^{th} term questions.

a Find the n^{th} term of the sequence.
b Find the 500^{th} term.
c Which term of this sequence has the value 236? Show full working.
d Show that 154 is not a term in the sequence.

Not all sequences progress in the same way. You will need to use your imagination to find the n^{th} terms for each of these.

Questions 3 to 6 involve much more difficult n^{th} terms.

3 **a** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$
b $\frac{3}{8}, \frac{7}{11}, \frac{11}{14}, \frac{15}{17}, \ldots$
c $\frac{9}{64}, \frac{49}{121}, \frac{121}{196}, \frac{225}{289}, \ldots$
d $-\frac{2}{3}, -\frac{1}{6}, \frac{1}{3}, \frac{5}{6}, \frac{4}{3}, \ldots$

4 List the first three terms and find the 20^{th} term of the number patterns given by the following rules, where T = term and n = the position of the term.

a $T_n = 4 - 3n$
b $T_n = 2 - n$
c $T_n = \frac{1}{2}n^2$
d $T_n = n(n + 1)(n - 1)$
e $T_n = \frac{3}{1+n}$
f $T_n = 2n^3$

5 If $x + 1$ and $-x + 17$ are the second and sixth terms of a sequence with a common difference of 5, find the value of x.

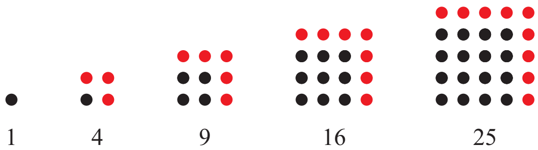
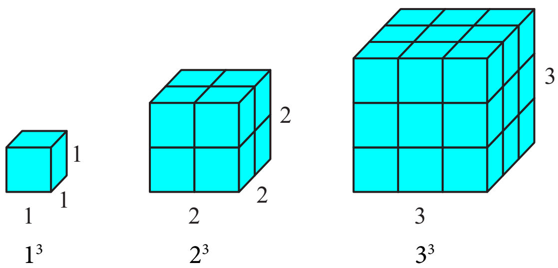
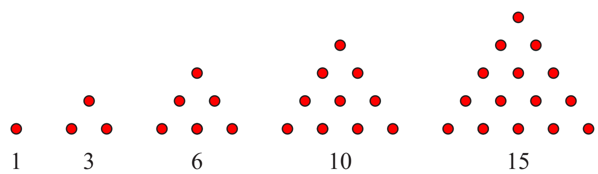
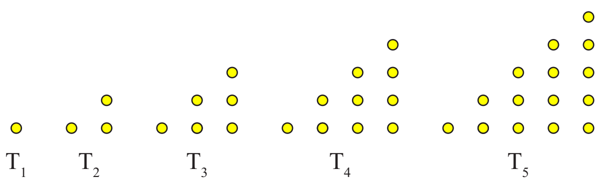
6 If $x + 4$ and $x - 4$ are the third and seventh terms of a sequence with a common difference of -2, find the value of x.

7 Write down the next three terms in each of the following sequences.

a	3	7	11	15	19	…
b	4	9	16	25	36	…
c	23	19	13	5	−5	…

Some special sequences

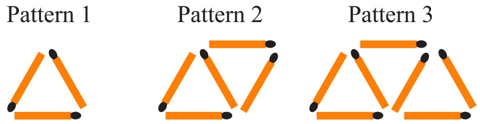
You should be able to recognise the following patterns and sequences.

Sequence	Description
Square numbers $T_n = n^2$	A square number is the product of multiplying a whole number by itself. Square numbers can be represented using dots arranged to make squares.  Square numbers form the (infinite) sequence: 1, 4, 9, 16, 25, 36, … Square numbers may be used in other sequences: $\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$ 2, 8, 18, 32, 50, … (each term is double a square number)
Cube numbers $T_n = n^3$	A cube number is the product of multiplying a whole number by itself and then by itself again.  Cube numbers form the (infinite) sequence: 1, 8, 27, 64, 125, …
Triangular numbers $T_n = \frac{1}{2}n(n+1)$	Triangular numbers are made by arranging dots to form either equilateral or right-angled isosceles triangles. Both arrangements give the same number sequence.   Triangular numbers form the (infinite) sequence: 1, 3, 6, 10, 15, …

Sequence	Description
Fibonacci numbers	Leonardo Fibonacci was an Italian mathematician who noticed that many natural patterns produced the sequence: 1, 1, 2, 3, 5, 8, 13, 21, … These numbers are now called Fibonacci numbers. They have the term-to-term rule 'add the two previous numbers to get the next term'.

Generating sequences from patterns

The diagram shows a pattern using matchsticks.



The table shows the number of matchsticks for the first five patterns.

Pattern number (n)	1	2	3	4	5
Number of matches	3	5	7	9	11

Notice that the pattern number can be used as the position number, n, and that the numbers of matches form a sequence, just like those considered in the previous section.

The number added on each time is two but you could also see that this was true from the original diagrams. This means that the number of matches for pattern n is the same as the value of the n^{th} term of the sequence.

The n^{th} term will therefore be: $2n \pm$ something.

Use the ideas from the previous section to find the value of the 'something'.

Taking any term in the sequence from the table, for example the first:

$n = 1$, so $2n = 2 \times 1 = 2$. But the first term is 3, so you need to add 1.

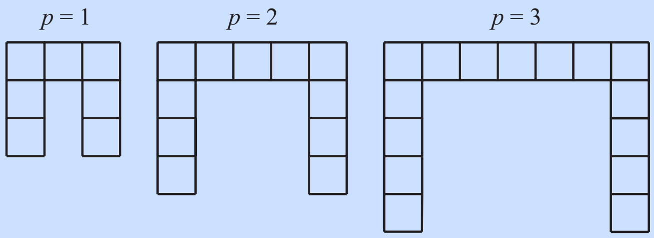
So, n^{th} term $= 2n + 1$

Which means that, if you let 'p' be the number of matches in pattern n then,

$p = 2n + 1$.

Worked example 2

The diagram shows a pattern made with squares.



a Construct a sequence table showing the first six patterns and the number of squares used.

b Find a formula for the number of squares, s, in terms of the pattern number 'p'.

c How many squares will there be in pattern 100?

Notice that 'p' has been used for the pattern number rather than 'n' here. You can use any letters that you like – it doesn't have to be n every time.

a

Pattern number (p)	1	2	3	4	5	6
Number of squares (s)	7	11	15	19	23	27

b

$4p$ is in the formula	Notice that the number of squares increases by 4 from shape to shape. This means that there will be a term '$4p$' in the formula.
If $p = 1$ then $4p = 4$ $4 + 3 = 7$	Now, if $p = 1$ then $4p = 4$. The first term is seven, so you need to add three.
so, $s = 4p + 3$	This means that $s = 4p + 3$.
If $p = 5$ then $4p + 3 = 20 + 3 = 23$, the rule is correct.	Check: if $p = 5$ then there should be 23 squares, which is correct.

c For pattern 100, $p = 100$ and $s = 4 \times 100 + 3 = 403$.

Exercise 9.3

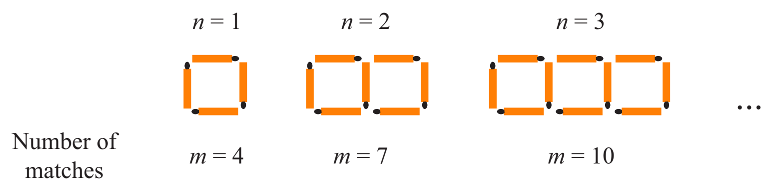
For each of the following shape sequences:

i draw a sequence table for the first six patterns, taking care to use the correct letter for the pattern number and the correct letter for the number of shapes

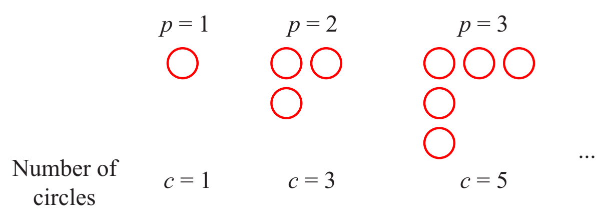
ii find a formula for the number of shapes used in terms of the pattern number

iii use your formula to find the number of shapes used in the 300th pattern.

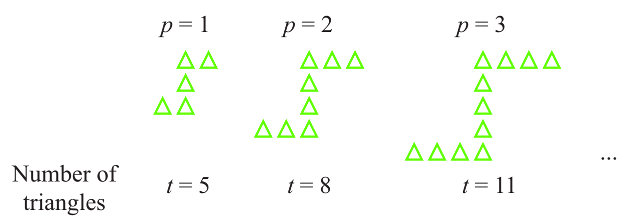
a



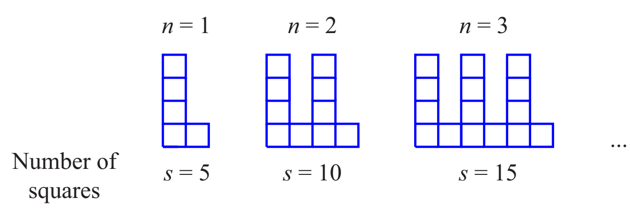
b



c



d



E

In any sequence ***n*** must be a positive integer. There are no negative 'positions' for terms. For example, ***n*** can be 7 because it is possible to have a 7th term, but ***n*** cannot be –7 as it is not possible to have a –7th term.

Subscript notation

The n^{th} term of a sequence can be written as u_n. This is called subscript notation and u represents a sequence. You read this as 'u sub n'. Terms in a specific position (for example, the first, second and hundredth term) are written as u_1, u_2, u_{100} and so on.

Term-to-term rules and position-to-term rules may be given using subscript notation. You can work out the value of any term or the position of a term by substituting known values into the rules.

Worked example 3

The position to term rule for a sequence is given as $u_n = 3n - 1$.

What are the first three terms of the sequence?

Substitute $n = 1$, $n = 2$ and $n = 3$ into the rule.

$u_1 = 3(1) - 1 = 2$ $u_2 = 3(2) - 1 = 5$ $u_3 = 3(3) - 1 = 8$	For the first term, $n = 1$

The first three terms are 2, 5 and 8.

Worked example 4

The number 149 is a term in the sequence defined as $u_n = n^2 + 5$.

Which term in the sequence is 149?

$149 = n^2 + 5$ $149 - 5 = n^2$ $144 = n^2$ $12 = n$	Find the value of n, when $u_n = 149$ $\sqrt{144} = 12$ and -12, but n must be positive as there is no -12th term.

149 is the 12th term in the sequence.

Exercise 9.4

1 Find the first three terms and the 25th term of each sequence.

a $u_n = 4n + 1$ **b** $u_n = 3n - 5$

c $u_n = 5n - \frac{1}{2}$ **d** $u_n = -2n + 1$

e $u_n = \frac{n}{2} + 1$ **f** $u_n = 2n^2 - 1$

g $u_n = n^2$ **h** $u_n = 2^n$

2 The numbers 30 and 110 are found in the sequence $u_n = n(n - 1)$. In which position is each number found?

3 Which term in the sequence $u_n = 2n^2 + 5$ has a value of 167?

4 For the sequence $u_n = 2n^2 - 5n + 3$, determine:

a the value of the tenth term.
b the value of n for which $u_n = 45$

5 The term-to-term rule for a sequence is given as $u_{n+1} = u_n + 2$.

a Explain in words what this means.
b Given that $u_3 = -4$, list the first five terms of the sequence.

9.2 Rational and irrational numbers

Rational numbers

You already know about decimals and how they are used to write down numbers that are not whole. Some of these numbers can be expressed as fractions, for example:

$0.5 = \frac{1}{2}$ $\quad$ $2.5 = \frac{5}{2}$ $\quad$ $0.125 = \frac{1}{8}$ $\quad$ $0.33333333\ldots = \frac{1}{3}$

. . . and so on.

Any number that can be expressed as a fraction in its lowest terms is known as a **rational number**.

Notice that there are two types of rational number: **terminating decimals** (i.e. those with a decimal part that doesn't continue forever) and **recurring decimals** (the decimal part continues forever but repeats itself at regular intervals).

Recurring decimals can be expressed by using a dot above the repeating digit(s):

$0.333333333\ldots = 0.\dot{3}$ $\quad$ $0.302302302302\ldots = 0.\dot{3}0\dot{2}$

$0.454545454\ldots = 0.\dot{4}\dot{5}$

E

Converting recurring decimals to fractions

Remember that the dot above one digit means that you have a *recurring* decimal. If more than one digit repeats we place a dot above the first and last repeating digit. For example $0.\dot{4}1\dot{8}$ is the same as $0.418418418418418\ldots$ and $0.34\dot{2} = 0.3422222222\ldots$.

What can we do with a decimal that continues forever but *does* repeat? Is this kind of number rational or irrational?

As an example we will look at the number $0.\dot{4}$.

We can use algebra to find another way of writing this recurring decimal:

Let

$$x = 0.\dot{4} = 0.444444\ldots$$

Then

$$10x = 4.444444\ldots$$

We can then subtract x from $10x$ like this:

$$\begin{aligned} 10x &= 4.444444\ldots \\ x &= 0.444444\ldots \\ \hline 9x &= 4 \\ \Rightarrow x &= \frac{4}{9} \end{aligned}$$

Every recurring decimal is a rational number. It is always possible to write a recurring decimal as a fraction.

Notice that this shows how it is possible to write the recurring decimal $0.\dot{4}$ as a fraction. This means that $0.\dot{4}$ is a rational number. Indeed all recurring decimals can be written as fractions, and so are rational.

E

Worked example 5

Use algebra to write each of the following as fractions. Simplify your fractions as far as possible.

a $0.\dot{3}$ **b** $0.\dot{2}\dot{4}$ **c** $0.\dot{9}3\dot{4}$ **d** $0.52\dot{4}$

a

$x = 0.33333...$ $10x = 3.33333...$	Write your recurring decimal in algebra. It is easier to see how the algebra works if you write the number out to a handful of decimal places. Multiply by 10, so that the recurring digits still line up
Subtract $10x = 3.33333...$ $x = 0.33333...$ $9x = 3$	Subtract
$\Rightarrow x = \frac{3}{9} = \frac{1}{3}$	Divide by 9

b

Let $x = 0.242424...$	(1)
then, $100x = 24.242424....$	(2) Multiply by 100
$99x = 24.24 - 0.24$	Subtract (2) – (1)
$99x = 24$	
so, $x = \frac{24}{99} = \frac{8}{33}$	Divide both sides by 99

Notice that you start by multiplying by 100 to make sure that the '2's and '4's started in the correct place after the decimal point.

c

$x = 0.934934...$ $1000x = 934.934934...$	This time we have three recurring digits. To make sure that these line up we multiply by 1000, so that all digits move three places.
$1000x = 934.934934...$ $x = 0.934934...$ $999x = 934$ $\Rightarrow x = \frac{934}{999}$	Notice that the digits immediately after the decimal point for both x and 1000x are 9, 3 and 4 in the same order.

d

$x = 0.52444444...$ $100x = 52.4444444...$	Multiply by 100 so that the recurring digits begin immediately after the decimal point.
$1000x = 524.444444...$	Then proceed as in the first example, multiplying by a further 10 to move the digits one place.
$1000x = 524.444444...$ $100x = 52.4444444...$ $900x = 472$ $\Rightarrow x = \frac{472}{900} = \frac{118}{225}$	Subtract and simplify.

The key point is that you need to subtract two different numbers, but in such a way that the recurring part disappears. This means that sometimes you have to multiply by 10, sometimes by 100, sometimes by 1000, depending on how many digits repeat.

Tip

Once you have managed to get the recurring decimals to start immediately after the decimal point you will need to multiply again, by another power of 10. The power that you choose should be the same as the number of digits that recur. In the second example the digits 9, 3 and 4 recur, so we multiply by $10^3 = 1000$.

Exercise 9.5

E

1 Copy and complete each of the following by filling in the boxes with the correct number or symbol.

a Let $x = 0.\dot{6}$

Then $10x = \boxed{}$

Subtracting:

$10x = \boxed{}$

$-\ x = 0.\dot{6}$

$\boxed{}\ x = \boxed{}$

So $\boxed{}\ x = \boxed{}$

Simplify:

$x = \boxed{}$

b Let $x = 0.\dot{1}\dot{7}$

Then $100x = \boxed{}$

Subtracting:

$100x = \boxed{}$

$-\ x = 0.\dot{1}\dot{7}$

$\boxed{}\ x = \boxed{}$

So $\boxed{}\ x = \boxed{}$

Simplify:

$x = \boxed{}$

2 Write each of the following recurring decimals as a fraction in its lowest terms.

a $0.\dot{5}$ **b** $0.\dot{1}$ **c** $0.\dot{8}$ **d** $0.\dot{2}\dot{4}$

e $0.\dot{6}\dot{1}$ **f** $0.\dot{3}\dot{2}$ **g** $0.\dot{6}1\dot{8}$ **h** $0.\dot{2}3\dot{3}$

i $0.\dot{2}0\dot{8}$ **j** $0.0\dot{2}$ **k** $0.1\dot{8}$ **l** $0.0\dot{3}\dot{1}$

m $2.\dot{4}\dot{5}$ **n** $3.\dot{1}0\dot{5}$ **o** $2.\dot{5}\dot{0}$ **p** $5.\dot{4} + 4.\dot{5}$

q $2.\dot{3}\dot{6} + 3.\dot{6}\dot{3}$ **r** $0.\dot{1}\dot{7} + 0.\dot{7}\dot{1}$ **s** $0.\dot{9}$

3 **a** Write down the numerical value of each of the following

i $1 - 0.9$ **ii** $1 - 0.99$ **iii** $1 - 0.999$ **iv** $1 - 0.999999999$

b Comment on your answers to (a). What is happening to the answer as the number of digits in the subtracted number increases? What is the answer getting closer to? Will it ever get there?

c Use algebra to express $0.\dot{6}$ and $0.\dot{2}$ as fractions in their simplest form.

d Express $0.\dot{6} + 0.\dot{2}$ as a recurring decimal.

E

e Use your answer to (c) to express $0.\dot{6} + 0.\dot{2}$ as a fraction in its lowest terms.

f Now repeat parts c, d and e using the recurring decimals $0.\dot{4}$ and $0.\dot{5}$.

g Explain how your findings for part f relate to your answers in parts a and b.

4 Jessica's teacher asks a class to find the largest number that is smaller than 4.5. Jessica's friend Jeevan gives the answer 4.4.

a Why is Jeevan not correct?

Jessica's friend Ryan now suggests that the answer is 4.49999.

b Why is Ryan not correct?

Jessica now suggests the answer $4.4\dot{9}$

c Is Jessica correct? Give full reasons for your answer, including any algebra that helps you to explain. Do you think that there is a better answer than Jessica's?

Exercise 9.6

1 Say whether each number is rational or irrational.

a $\frac{1}{4}$	**b** 4	**c** -7	**d** 3.147
e π	**f** $\sqrt{3}$	**g** $\sqrt{25}$	**h** 0
i 0.45	**j** -0.67	**k** -232	**l** $\frac{3}{8}$
m $9.\dot{4}\dot{5}$	**n** $\sqrt{123}$	**o** 2π	**p** $3\sqrt{2}$

2 Show that the following numbers are rational.

a 6 **b** $2\frac{3}{8}$ **c** $1.\dot{1}\dot{2}$ **d** $0.\dot{8}$ **e** 0.427 **f** $3.\dot{1}\dot{4}$

3 Find a number in the interval $-1 < x < 3$ so that:

a x is rational **b** x is a real number but not rational

c x is an integer **d** x is a natural number

4 Which set do you think has more members: rational numbers or irrational numbers? Why?

5 Mathematicians also talk about imaginary numbers. Find out what these are and give one example.

9.3 Sets

A **set** is a list or collection of objects that share a characteristic. The objects in a set can be anything from numbers, letters and shapes to names, places or paintings, but there is usually something that they have in common.

The list of members or **elements** of a set is placed inside a pair of curly brackets { }.

Some examples of sets are:

When writing sets, never forget to use the curly brackets on either side.

{2, 4, 6, 8, 10} – the set of all even integers greater than zero but less than 11

{a, e, i, o, u} – the set of vowels

{Red, Green, Blue} – the set containing the colours red, green and blue.

Capital letters are usually used as names for sets:

Notice, for set B, that elements of a set are not repeated.

If A is the set of prime numbers less than 10, then: $A = \{2, 3, 5, 7\}$

If B is the set of letters in the word 'HAPPY', then: $B = \{H, A, P, Y\}$.

Two sets are equal if they contain exactly the same elements, even if the order is different, so:

$\{1, 2, 3, 4\} = \{4, 3, 2, 1\} = \{2, 4, 1, 3\}$ and so on.

The number of elements in a set is written as $n(A)$, where A is the name of the set. For example in the set $A = \{1, 3, 5, 7, 9\}$ there are five elements so $n(A) = 5$.

E

A set that contains no elements is known as the **empty set**. The symbol $\varnothing$ is used to represent the empty set.

For example:

{odd numbers that are multiples of two} $= \varnothing$ because no odd number is a multiple of two.

Now, if x is a member (an element) of the set A then it is written: $x \in A$.

If x is *not* a member of the set A, then it is written: $x \notin A$.

For example, if H = {Spades, Clubs, Diamonds, Hearts}, then:

Spades $\in H$ but Turtles $\notin H$.

Some sets have a number of elements that can be counted. These are known as *finite* sets. If there is no limit to the number of members of a set then the set is *infinite*.

If A = {letters of the alphabet}, then A has 26 members and is finite.

If B = {positive integers}, then $B = \{1, 2, 3, 4, 5, 6, \ldots\}$ and is infinite.

So, to summarise:

- sets are listed inside curly brackets { }
- $\varnothing$ means it is an empty set
- $a \in B$ means a is an element of the set B
- $a \notin B$ means a is not an element of the set B
- $n(A)$ is the number of elements in set A

The following exercise requires you to think about things that are outside of mathematics. In each case you might like to see if you can find out ALL possible members of each set.

Exercise 9.7 *Applying your skills*

1 List all of the elements of each set.

a {days of the week}
b {months of the year}
c {factors of 36}
d {colours of the rainbow}
e {multiples of seven less than 50}
f {primes less than 30}
g {ways of arranging the letters in the word 'TOY'}

2 Find two more members of each set.

a {rabbit, cat, dog, . . .}
b {carrot, potato, cabbage, . . .}
c {London, Paris, Stockholm, . . .}
d {Nile, Amazon, Loire, . . .}
e {elm, pine, oak, . . .}
f {tennis, cricket, football, . . .}
g {France, Germany, Belgium, . . .}
h {Bush, Obama, Truman, . . .}
i {Beethoven, Mozart, Sibelius, . . .}
j {rose, hyacinth, poppy, . . .}
k {3, 6, 9, . . .}
l {Husky, Great Dane, Boxer, . . .}
m {Mercury, Venus, Saturn, . . .}
n {happy, sad, angry, . . .}
o {German, Czech, Australian, . . .}
p {hexagon, heptagon, triangle, . . .}

3 Describe each set fully in words.

a {1, 4, 9, 16, 25, . . .}
b {Asia, Europe, Africa, . . .}
c {2, 4, 6, 8}
d {2, 4, 6, 8, . . .}
e {1, 2, 3, 4, 6, 12}

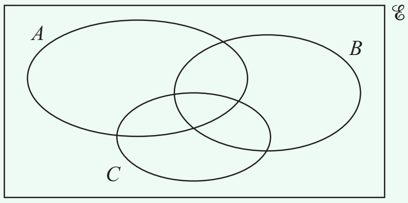
E

4 True or false?

a If $A = \{1, 2, 3, 4, 5\}$ then $3 \notin A$
b If $B = \{\text{primes less than } 10\}$, then $n(B) = 4$
c If $C = \{\text{regular quadrilaterals}\}$, then square $\in C$
d If $D = \{\text{paint primary colours}\}$, then yellow $\notin D$
e If $E = \{\text{square numbers less than } 100\}$, then $64 \in E$

5 Make 7 copies of this Venn diagram and shade the following sets:

a $A \cup B$
b $A \cup B \cup C$
c $A \cup B'$
d $A \cap (B \cup C)$
e $(A \cup B) \cap C$
f $A \cup (B \cup C)'$
g $(A \cap C) \cup (A \cap B)$



6 In a class of 30 students, 22 like classical music and 12 like Jazz. 5 like neither. Using a Venn diagram find out how many students like both classical and jazz music.

7 Students in their last year at a school must study at least one of the three main sciences: Biology, Chemistry and Physics. There are 180 students in the last year, of whom 84 study Biology and Chemistry only, 72 study Chemistry and Physics only and 81 study Biology and Physics only. 22 pupils study only Biology, 21 study only Chemistry and 20 study only Physics. Use a Venn diagram to work out how many students study all three sciences.

Universal sets

The following sets all have a number of things in common:

$M = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$N = \{1, 5, 9\}$

$O = \{4, 8, 21\}$

All three are contained within the set of whole numbers. They are also all contained in the set of integers less than 22.

When dealing with sets there is usually a 'largest' set which contains all of the sets that you are studying. This set can change according to the nature of the problem you are trying to solve.

Here, the set of integers contains all elements from M, N or O. But then so does the set of all positive integers less than 22.

Both these sets (and many more) can be used as a **universal set**. A universal set contains all possible elements that you would consider for a set in a particular problem. The symbol $\mathscr{E}$ is used to mean the universal set.

E

Complements

The **complement** of the set A is the set of all things that are in $\mathscr{E}$ but NOT in the set A. The symbol A' is used to denote the complement of set A.

For example, if: $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

and $F = \{2, 4, 6\}$

then the complement of F would be $F' = \{1, 3, 5, 7, 8, 9, 10\}$.

So, in summary:
- $\mathscr{E}$ represents a universal set
- A' represents the complement of set A.

Unions and intersections

The **union** of two sets, A and B, is the set of all elements that are members of A or members of B or members of both. The symbol $\cup$ is used to indicate union so, the union of sets A and B is written:

$A \cup B$

The **intersection** of two sets, A and B, is the set of all elements that are members of *both* A and B. The symbol $\cap$ is used to indicate intersection so, the intersection of sets A and B is written:

$A \cap B$.

For example, if $C = \{4, 6, 8, 10\}$ and $D = \{6, 10, 12, 14\}$, then:

$C \cap D$ = the set of all elements common to both = $\{6, 10\}$

$C \cup D$ = the set of all elements that are in C or D or both = $\{4, 6, 8, 10, 12, 14\}$.

Tip

Note that taking the union of two sets is rather like adding the sets together. You must remember, however, that you do not repeat elements within the set.

E

Subsets

Let the set A be the set of all quadrilaterals and let the set B be the set of all rectangles. A rectangle is a type of quadrilateral. This means that every element of B is also a member of A and, therefore, B is *completely* contained within A. When this happens B is called a **subset** of A, and is written:

$B \subseteq A$. The $\subseteq$ symbol can be reversed but this does not change its meaning. $B \subseteq A$ means B is a subset of A, but so does $A \supseteq B$. If B is not a subset of A, we write $B \nsubseteq A$. If B is not equal to A, then B is known as a *proper subset*. If it is possible for B to be equal to A, then B is not a proper subset and you write: $B \subset A$. If A is not a proper subset of B, we write $A \not\subset B$.

Note that the symbol, $\subset$, has a open end and a closed end. The subset goes at the closed end.

So in summary:

- $\cup$ is the symbol for union
- $\cap$ is the symbol for intersection
- $B \subset A$ indicates that B is a proper subset of A
- $B \subseteq A$ indicates that B is a subset of A but also equal to A i.e. it is *not* a proper subset of A.
- $B \not\subset A$ indicates that B is not a proper subset of A.
- $B \nsubseteq A$ indicates that B is not a subset of A.

Worked example 6

If $W = \{4, 8, 12, 16, 20, 24\}$ and $T = \{5, 8, 20, 24, 28\}$.

i List the sets:

a $W \cup T$ **b** $W \cap T$

ii Is it true that $T \subset W$?

i **a** $W \cup T$ = set of all members of W or of T or of both = $\{4, 5, 8, 12, 16, 20, 24, 28\}$.

b $W \cap T$ = set of all elements that appear in both W and T = $\{8, 20, 24\}$.

ii Notice that $5 \in T$ but $5 \notin W$. So it is not true that every member of T is also a member of W. So T is not a subset of W.

Exercise 9.8

Unions and intersections can be reversed without changing their elements, for example $A \cup B = B \cup A$ and $C \cap D = D \cap C$.

1 $A = \{2, 4, 6, 8, 10\}$ and $B = \{1, 3, 5, 6, 8, 10\}$.

a List the elements of:

i $A \cap B$ **ii** $A \cup B$

b Find:

i $n(A \cap B)$ **ii** $n(A \cup B)$

2 $C = \{a, b, g, h, u, w, z\}$ and $D = \{a, g, u, v, w, x, y, z\}$.

a List the elements of:

i $C \cap D$ **ii** $C \cup D$

b Is it true that u is an element of $C \cap D$? Explain your answer.

c Is it true that g is not an element of $C \cup D$? Explain your answer.

E

3 $F = \{\text{equilateral triangles}\}$ and $G = \{\text{isosceles triangles}\}$.

a Explain why $F \subset G$.

b What is $F \cap G$? Can you simplify $F \cap G$ in any way?

4 $T = \{1, 2, 3, 6, 7\}$ and $W = \{1, 3, 9, 10\}$.

a List the members of the set:

(i) $T \cup W$ **(ii)** $T \cap W$

b Is it true that $5 \notin T$? Explain your answer fully.

5 If $\mathscr{E} = \{\text{rabbit, cat, dog, emu, turtle, mouse, aardvark}\}$ and $H = \{\text{rabbit, emu, mouse}\}$ and $J = \{\text{cat, dog}\}$:

a list the members of H'

b list the members of J'

c list the members of $H' \cup J'$

d what is $H \cap J$?

e find $(H')'$

f what is $H \cup H'$?

FAST FORWARD

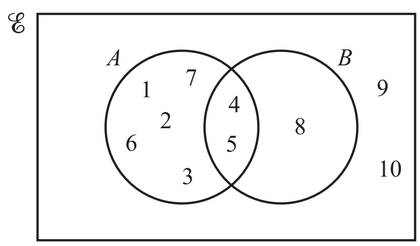
You need to understand Venn diagrams well as you will need to use them to determine probabilities in Chapter 24. ▶

Venn diagrams

In 1880, mathematician John Venn began using overlapping circles to illustrate connections between sets. These diagrams are now referred to as **Venn diagrams**.

For example, if $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{4, 5, 8\}$

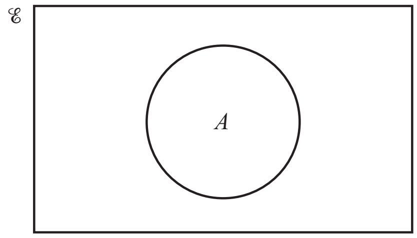
then the Venn diagram looks like this:



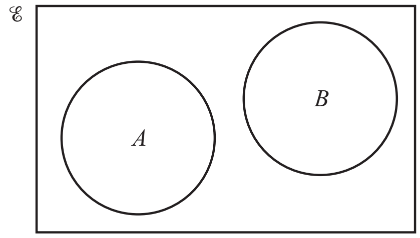
Always remember to draw the box around the outside and mark it, ℰ, to indicate that it represents the universal set.

Notice that the universal set is shown by drawing a rectangle and then each set within the universal set is shown as a circle. The intersection of the sets A and B is contained within the overlap of the circles. The union is shown by the region enclosed by at least one circle. Here are some examples of Venn diagrams and shaded regions to represent particular sets:

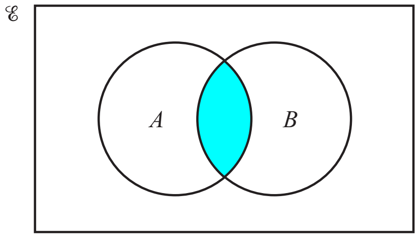
The rectangle represents ℰ.



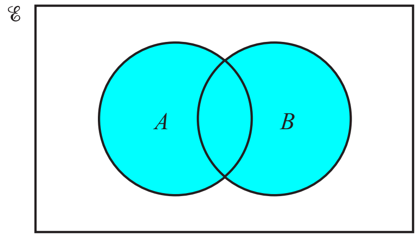
The circle represents set A.



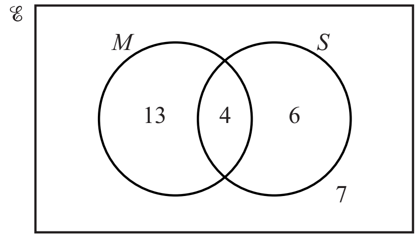
Set A and set B are disjoint, they have no common elements.



$A \cap B$ is the shaded portion.

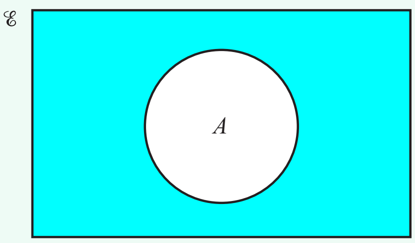


$A \cup B$ is the shaded portion.

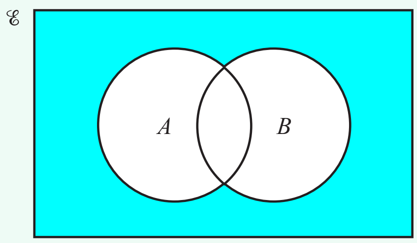


Venn diagrams can also be used to show the number of elements n(A) in a set.
In this case:
M = {students doing Maths},
S = {students doing Science}.

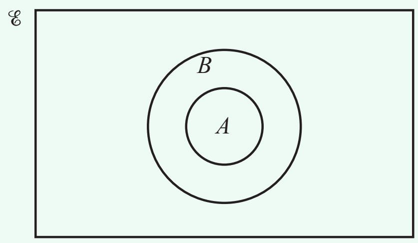
E



A' is the shaded portion.



$(A \cup B)'$ is the shaded portion.



$A \subset B$

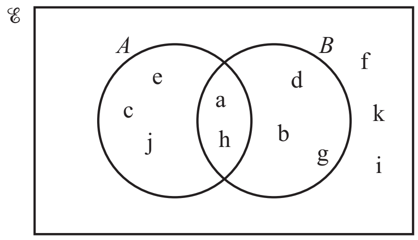
Worked example 7

For the following sets:

ℰ = {a, b, c, d, e, f, g, h, i, j, k}
A = {a, c, e, h, j}
B = {a, b, d, g, h}

a illustrate these sets in a Venn diagram
b list the elements of the set $A \cap B$
c find n($A \cap B$)
d list the elements of the set $A \cup B$
e find n($A \cup B$)
f list the set $A \cap B'$.

a



b Look in the region that is contained within the overlap of both circles. This region contains the set {a, h}. So $A \cap B$ = {a, h}.

c n($A \cap B$) = 2 as there are two elements in the set $A \cap B$.

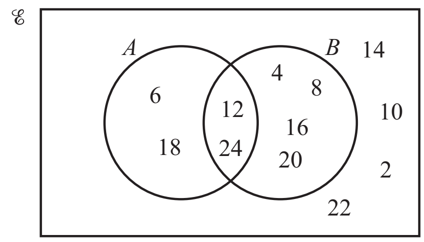
d $A \cup B$ = set of elements of A or B or both = {a, b, c, d, e, g, h, j}.

e n($A \cup B$) = 8

f $A \cap B'$ = set of all elements that are both in set A and not in set B = {c, e, j}

Exercise 9.9

1 Use the given Venn diagram to answer the following questions.

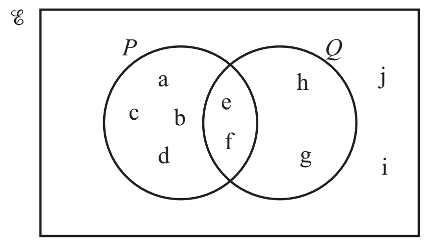


a List the elements of A and B

b List the elements of $A \cap B$.

c List the elements of $A \cup B$.

2 Use the given Venn diagram to answer the following questions.



a List the elements that belong to:

i P **ii** Q

b List the elements that belong to both P and Q.

c List the elements that belong to:

i neither P nor Q

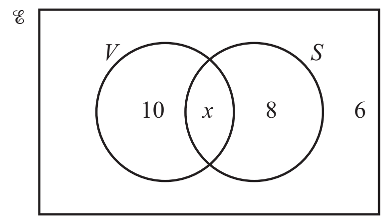
ii P but not Q.

3 Draw a Venn diagram to show the following sets and write each element in its correct space.

a The universal set is {a ,b, c, d, e, f, g, h}.
A = {b, c, f, g} and B = {a, b, c, d, f}.

b $\mathscr{E}$ = {whole numbers from 20 to 36 inclusive}.
A = {multiples of four} and B = {numbers greater than 29}.

4 The universal set is: {students in a class}.
V = {students who like volleyball}.
S = {students who play soccer}.
There are 30 students in the class.
The Venn diagram shows numbers of students.



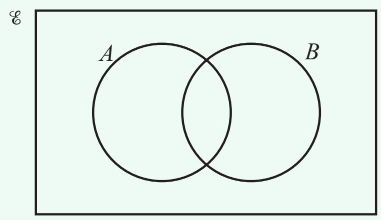
a Find the value of x.

b How many students like volleyball?

c How many students in the class do not play soccer?

E

5 Copy the Venn diagram and shade the region which represents the subset $A \cap B'$.



Set builder notation

So far the contents of a set have either been given as a list of the elements or described by a rule (in words) that defines whether or not something is a member of the set. We can also describe sets using **set builder notation**. Set builder notation is a way of describing the elements of a set using the properties that each of the elements must have.

For example:

$A = \{x : x \text{ is a natural number}\}$

This means:

A = { x : x is a natural number }

Set A is the set of | all values (x) | such that | each value of x is a natural number

In other words, this is the set: $A = \{1, 2, 3, 4, \ldots\}$

Sometimes the set builder notation contains restrictions.

For example, $B = \{\{x : x \text{ is a letter of the alphabet}, x \text{ is a vowel}\}$

In this case, set $B = \{\text{a, e, i, o, u}\}$

Here is another example:

$A = \{\text{integers greater than zero but less than 20}\}$.

In set builder notation this is:

$A = \{x : x \text{ is an integer}, 0 < x < 20\}$

This is read as: 'A is the set of all x such that x is an integer and x is greater than zero but less than 20'.

The following examples should help you to get used to the way in which this notation is used.

Worked example 8

List the members of the set C if: $C = \{x : x \in \text{primes}, 10 < x < 20\}$.

Read the set as: 'C is the set of all x such that x is a member of the set of primes and x is greater than 10 but less than 20'.

The prime numbers greater than 10 but less than 20 are 11, 13, 17 and 19.

So, $C = \{11, 13, 17, 19\}$

Worked example 9

Express the following set in set builder notation:

D = {right-angled triangles}.

So, $D = \{x : x$ is a triangle, x has a right-angle$\}$	If D is the set of all right-angled triangles then D is the set of all x such that x is a triangle and x is right-angled.

As you can see from this last example, set builder notation can sometimes force you to write more, but this isn't always the case, as you will see in the following exercise.

Exercise 9.10

1 Describe each of these sets using set builder notation.

- **a** square numbers less than 101
- **b** days of the week
- **c** integers less than 0
- **d** whole numbers between 2 and 10
- **e** months of the year containing 30 days

2 Express each of the following in set builder notation.

- **a** {2, 3, 4, 5, 6, 7, 8}
- **b** {a, e, i, o, u}
- **c** {n, i, c, h, o, l, a, s}
- **d** {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}
- **e** {1, 2, 3, 4, 6, 9, 12, 18, 36}

3 List the members of each of the following sets.

- **a** $\{x : x$ is an integer, $40 < x < 50\}$
- **b** $\{x : x$ is a regular polygon and x has no more than six sides$\}$
- **c** $\{x : x$ is a multiple of 3, $16 < x < 32\}$

Set builder notation is very useful when it isn't possible to list all the members of set because the set is infinite. For example, all the numbers less than –3 or all whole numbers greater than 1000.

E

4 Describe each set in words and say why it's not possible to list all the members of each set.

- **a** $A = \{x, y : y = 2x + 4\}$
- **b** $B = \{x : x^3$ is negative$\}$

5 If $A = \{x : x$ is a multiple of three$\}$ and $B = \{y : y$ is a multiple of five$\}$, express $A \cap B$ in set builder notation.

6 $\mathscr{E} = \{y : y$ is positive, y is an integer less than 18$\}$.
$A = \{w : w > 5\}$ and $B = \{x : x \leqslant 5\}$.

- **a** List the members of the set:

  i $A \cap B$ **ii** A' **iii** $A' \cap B$ **iv** $A \cap B'$ **v** $(A \cap B')'$

- **b** What is $A \cup B$?
- **c** List the members of the set in part (b).

Summary

Do you know the following?

- A sequence is the elements of a set arranged in a particular order, connected by a rule.
- A term is a value (element) of a sequence.
- If the position of a term in a sequence is given the letter n then a rule can be found to work out the value of the n^{th} term.
- A rational number is a number that can be written as a fraction.
- An irrational number has a decimal part that continues forever without repeating.
- A set is a list or collection of objects that share a characteristic.
- An element is a member of a set.
- A set that contains no elements is called the empty set ($\varnothing$). E
- A universal set ($\mathscr{E}$) contains all the possible elements appropriate to a particular problem.
- The complement of a set is the elements that are not in the set ($'$). E
- The elements of two sets can be combined (without repeats) to form the union of the two sets ($\cup$).
- The elements that two sets have in common is called the intersection of the two sets ($\cap$).
- The elements of a subset that are all contained within a larger set are a proper subset ($\subseteq$). E
- If it is possible for a subset to be equal to the larger set, then it is not a proper subset ($\subset$).
- A Venn diagram is a pictorial method of showing sets.
- A shorthand way of describing the elements of a set is called set builder notation.

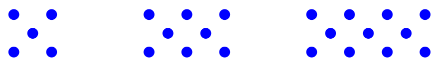
Are you able to …?

- continue sequences
- describe a rule for continuing a sequence
- find the n^{th} term of a sequence
- use the n^{th} term to find later terms
- find out whether or not a specific number is in a sequence
- generate sequences from shape patterns
- find a formula for the number of shapes used in a pattern
- write a recurring decimal as a fraction in its lowest terms E
- describe a set in words
- find the complement of a set E
- represent the members of set using a Venn diagram
- solve problems using a Venn diagram
- describe a set using set builder notation.

Examination practice

Exam-style questions

1 Pattern 1 Pattern 2 Pattern 3



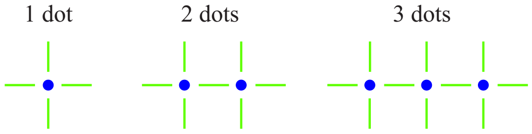
The first three patterns in a sequence are shown above.

a Copy and complete the table.

Pattern number (n)	1	2	3	4
Number of dots (d)	5			

b Find a formula for the number of dots, d, in the n^{th} pattern.
c Find the number of dots in the 60^{th} pattern.
d Find the number of the pattern that has 89 dots.

2 The diagram below shows a sequence of patterns made from dots and lines.



a Draw the next pattern in the sequence.
b Copy and complete the table for the numbers of dots and lines.

Dots	1	2	3	4	5	6
Lines	4	7	10			

c How many lines are in the pattern with 99 dots?
d How many lines are in the pattern with n dots?
e Complete the following statement:

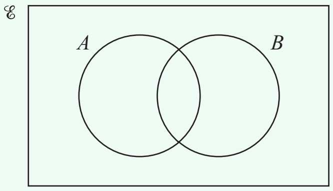
There are 85 lines in the pattern with . . . dots.

Past paper questions

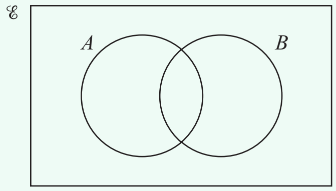
1 **a** Here are the first four terms of a sequence:
27 23 19 15
i Write down the next term in the sequence. [1]
ii Explain how you worked out your answer to **part (a)(i)**. [1]
b The nth term of a different sequence is $4n - 2$.
Write down the first three terms of this sequence. [1]
c Here are the first four terms of another sequence:
-1 2 5 8
Write down the nth term of this sequence. [2]

[Cambridge IGCSE Mathematics 0580 Paper 11 Q23 October/November 2013]

2 Shade the required region on each Venn diagram. [2]



$A' \cup B$



$A' \cap B'$

[Cambridge IGCSE Mathematics 0580 Paper 22 Q1 May/June 2013]

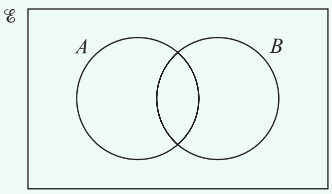
3 The first five terms of a sequence are shown below.

13 9 5 1 –3

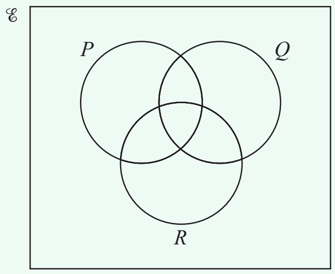
Find the *n*th term of this sequence. [2]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q3 May/June 2013]

4 Shade the required region in each of the Venn diagrams. [2]



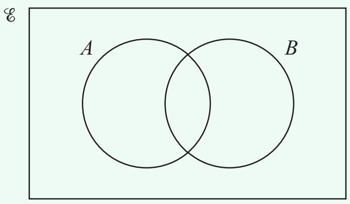
A'



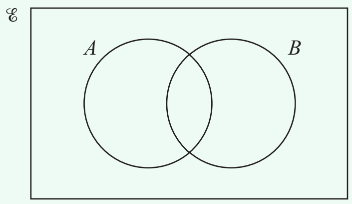
$(P \cap R) \cup Q$

[Cambridge IGCSE Mathematics 0580 Paper 23 Q9 October/November 2012]

5 Shade the region required in each Venn diagrams. [2]



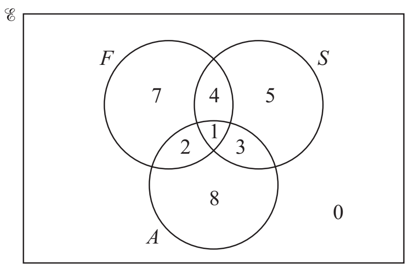
$(A \cup B)'$



$A' \cap B$

[Cambridge IGCSE Mathematics 0580 Paper 22 Q4 October/November 2014]

6 The Venn diagram shows the number of students who study French (F), Spanish (S) and Arabic (A).

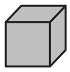


a Find $n(A \cup (F \cap S))$. [1]

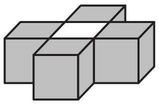
b On the Venn diagram, shade the region $F' \cap S$. [1]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q6 October/November 2015]

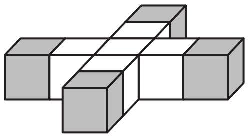
7 Layer 1



Layer 2



Layer 3



The diagrams show layers of white and grey cubes.

Khadega places these layers on top of each other to make a tower.

a Complete the table for towers with 5 and 6 layers.

Number of layers	1	2	3	4	5	6
Total number of **white** cubes	0	1	6	15		
Total number of **grey** cubes	1	5	9	13		
Total number of cubes	1	6	15	28		

[4]

b **i** Find, in terms of n, the **total** number of **grey** cubes in a tower with n layers. [2]

ii Find the total number of grey cubes in a tower with 60 layers. [1]

iii Khadega has plenty of white cubes but only 200 grey cubes.
How many layers are there in the highest tower that she can build? [2]

[Cambridge IGCSE Mathematics 0580 Paper 42 Q9 (a) & (b) October/November 2014]

8 Write the recurring decimal $0.3\dot{6}$ as a fraction.

Give your answer in its simplest form.

[$0.3\dot{6}$ means 0.3666…] [3]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q12 May/June 2016]

E