Chapter 14: Further solving of equations and inequalities

Key words

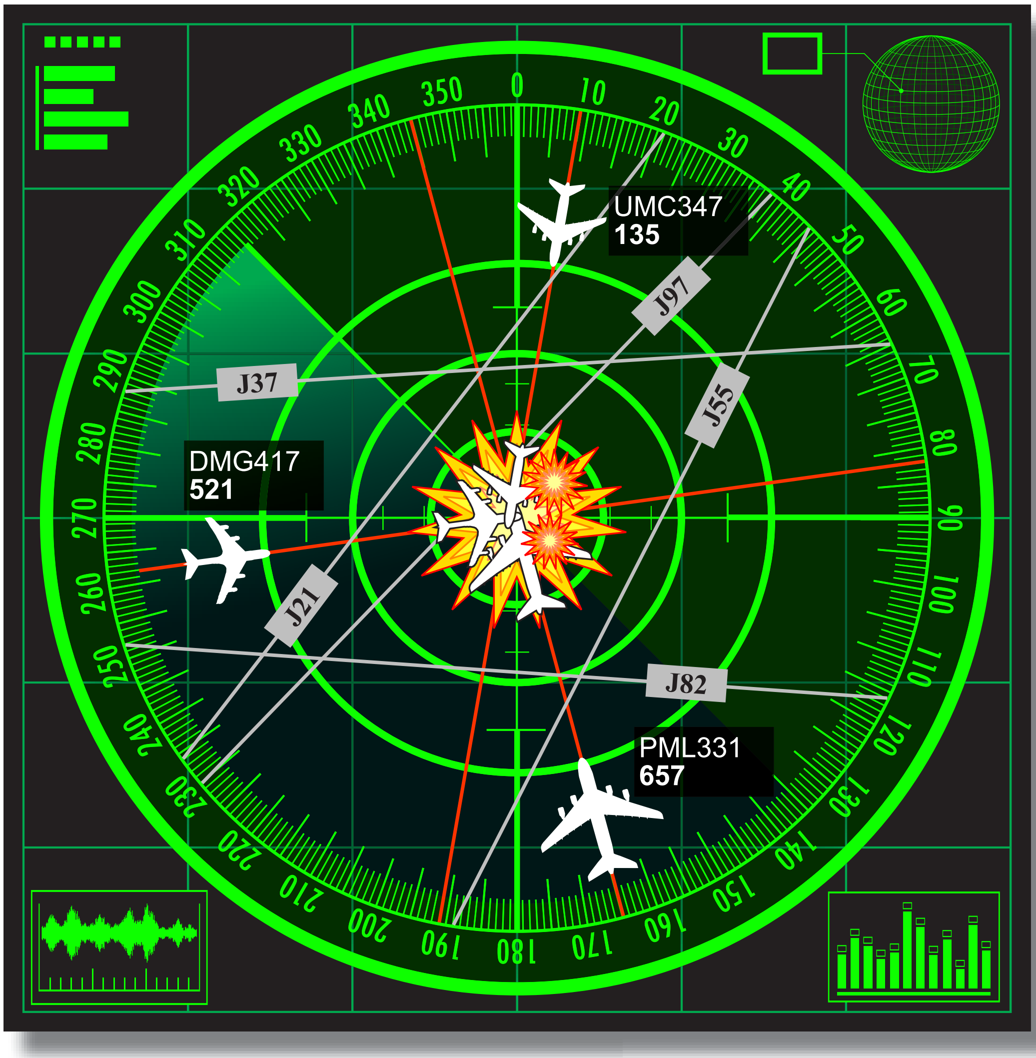
- Intersection
- Simultaneous
- Linear inequalities
- Region
- Linear programming
- Quadratic

In this chapter you will learn how to:

- derive and solve simultaneous linear equations graphically and algebraically

EXTENDED

- solve linear inequalities algebraically
- derive linear inequalities and find regions in a plane
- solve quadratic equations by completing the square
- solve quadratic equations by using the quadratic formula
- factorise quadratics where the coefficient of x^2 is not 1
- simplify algebraic fractions.



Any two airliners must be kept apart by air traffic controllers. An understanding of how to find meeting points of straight paths can help controllers to avoid disaster!

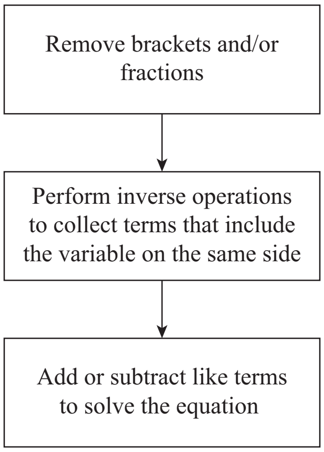
Businesses have constraints on the materials they can afford, how many people they can employ and how long it takes to make a product. They wish to keep their cost low and their profits high. Being able to plot their constraints on graphs can help to make their businesses more cost effective.

RECAP

You should be familiar with the following work on equations and inequalities:

Equations (Chapter 6)

To solve equations:



Remember to do the same things to both sides of the equation to keep it balanced.

Drawing straight line graphs (Chapter 10)

When you have two simultaneous equations each with two unknowns x and y you use a pair of equations to find the values.

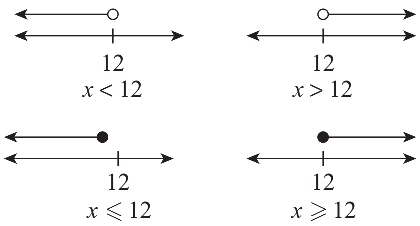
You can also draw two straight line graphs and the coordinates of the point where they meet give the solution of the equations.

Inequalities (Year 9 Mathematics)

An inequality shows the relationship between two unequal expressions.

The symbols $<$, $>$, $\leqslant$, $\geqslant$ and $\neq$ all show inequalities.

Inequalities can be shown on a number line using these conventions:



14.1 Simultaneous linear equations

Graphical solution of simultaneous linear equations

LINK

Simultaneous equations are used to solve many problems involving the momentum of particles in physics.

A little girl looks out of her window and notices that she can see some goats and some geese. From the window she can see some heads and then, when she looks out of the cat flap, she can see some feet. She knows that each animal has one head, goats have four feet and geese have two feet. Suppose that the girl counts eight heads and 26 feet. How many goats are there? How many geese are there?

If you let x = the number of goats and y = the number of geese, then the number of heads must be the same as the total number of goats and geese.

So, $x + y = 8$

Each goat has four feet and each goose has two feet. So the total number of feet must be $4x + 2y$ and this must be equal to 26.

So you have,

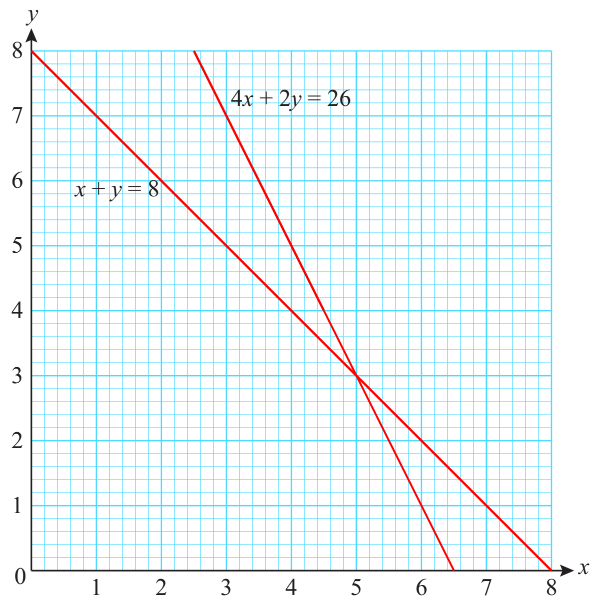
$$x + y = 8$$
$$4x + 2y = 26$$

REWIND

You plotted and drew straight line (linear) graphs in chapter 10.

The information has two unknown values and two different equations can be formed. Each of these equations is a linear equation and can be plotted on the same pair of axes. There is only one point where the values of x and y are the same for both equations – this is where the lines cross (the **intersection**). This is the **simultaneous** solution.

Simultaneous means, 'at the same time.' With simultaneous linear equations you are trying to find the point where two lines cross. i.e. where the values of ***x*** and ***y*** are the same for both equations.



Tip

It is essential that you remember to work out *both* unknowns. Every pair of simultaneous linear equations will have a pair of solutions.

Notice that the point with co-ordinates (5, 3) lies on *both* lines so, $x = 5$ and $y = 3$ satisfy *both* equations. You can check this by substituting the values into the equations:

$$x + y = 5 + 3 = 8$$

and

$$4x + 2y = 4(5) + 2(3) = 20 + 6 = 26$$

This means that the girl saw five goats and three geese.

REWIND

You learned how to plot lines from equations in chapter 10. ◀

REWIND

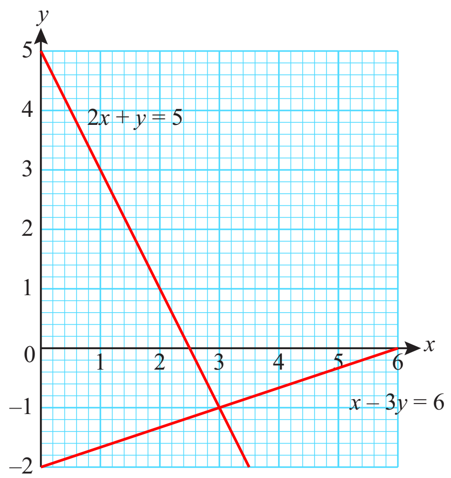
Throughout this chapter you will need to solve basic linear equations as part of the method. Remind yourself of how this was done in chapter 6. ◀

Worked example 1

By drawing the graphs of each of the following equations on the same pair of axes, find the simultaneous solutions to the equations.

$x - 3y = 6$

$2x + y = 5$



For the first equation:

if $x = 0$, $-3y = 6 \Rightarrow y = -2$
and, if $y = 0$, $x = 6$

So this line passes through the points (0, –2) and (6, 0).

For the second equation:

if $x = 0$, $y = 5$

and, if $y = 0$, $2x = 5 \Rightarrow x = \frac{5}{2}$

So this line passes through the points

(0, 5) and $\left(\frac{5}{2}, 0\right)$.

Plot the pairs of points and draw lines through them.
Notice that the two lines meet at the point with co-ordinates (3, –1)
So, the solution to the pair of equations is $x = 3$ and $y = -1$

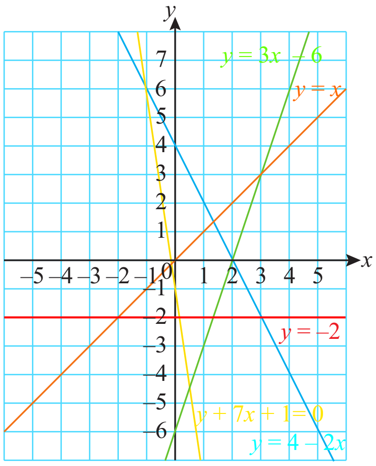
Exercise 14.1

1 Draw the lines for each pair of equations and then use the point of intersection to find the simultaneous solution. The axes that you should use are given in each case.

a $x + 2y = 11$ (x from 0 to 11 and y from 0 to 10)
$2x + y = 10$

b $x - y = -1$ (x from –2 to 3 and y from 0 to 4)
$2x + y = 4$

c $5x - 4y = -1$ (x from –1 to 5 and y from 0 to 10)
$2x + y = 10$



2 Use the graphs supplied to find the solutions to the following pairs of simultaneous equations.

a $y = x$
$y = -2$

b $y = x$
$y = 3x - 6$

c $y = 4 - 2x$
$y = -2$

d $y = 4 - 2x$
$y + 7x + 1 = 0$

e $y = -2$
$y + 7x + 1 = 0$

f $y = x$
$y = 4 - 2x$

3 For each pair of equations, find three points on each line and draw the graphs on paper. Use your graphs to estimate the solution of each pair of simultaneous equations.

a $3y = -4x + 3$
$x = 2y + 1$

b $2 - x = -y$
$8x + 4y = 7$

c $4x = 1 + 6y$
$4x - 4 = 3y$

d $3x + 2y = 7$
$4x = 2 + 3y$

4 **a** Explain why the graphical method does not always give an accurate and correct answer.
b How can you check whether a solution you obtained graphically is correct or not?

Algebraic solution of simultaneous linear equations

The graphical method is suitable for whole number solutions but it can be slow and, for non-integer solutions, may not be as accurate as you need. You have already learned how to solve linear equations with one unknown using algebraic methods. You now need to look at how to solve a pair of equations in which there are two unknowns.

You are going to learn two methods of solving simultaneous equations:

- solving by substitution
- solving by elimination.

Solving by substitution

You can solve the equations by substitution when one of the equations can be solved for one of the variables (i.e. solved for x or solved for y). The solution is then substituted into the other equation so it can be solved.

The equations have been numbered so that you can identify each equation efficiently. You should always do this.

Worked example 2

Solve simultaneously by substitution.

$3x - 2y = 29 \quad (1)$
$4x + y = 24 \quad (2)$

$4x + y = 24$ $y = 24 - 4x \quad (3)$	Solve equation (2) for y. Label the new equation (3).
$3x - 2y = 29 \quad (1)$ $3x - 2(24 - 4x) = 29$	Substitute (3) into (1) by replacing y with $24 - 4x$.
$3x - 48 + 8x = 29$	Remove brackets.
$3x + 8x = 29 + 48$	Subtract $8x$ and add 48 to both sides.
$11x = 77$	Add like terms.
$x = 7$	Divide both sides by 11.
So, $x = 7$	
$y = 24 - 4(7)$ $y = 24 - 28$ $y = -4$	Now, substitute the value of x into any of the equations to find y. Equation (3) will be easiest, so use this one.
$x = 7$ and $y = -4$	Write out the solutions.

Solving by elimination

You can also solve the equations by eliminating (getting rid of) one of the variables by adding the two equations together.

Worked example 3

Solve the following pair of equations using elimination:

$x - y = 4 \quad (1)$

$x + y = 6 \quad (2)$

$x - y = 4 \quad (1)$ $\underline{x + y = 6} \quad (2)$ $2x \quad = 10$	You can add the two equations together by adding the left-hand sides and adding the right-hand sides.
$2x = 10$ $\Rightarrow x = \frac{10}{2} = 5$	Notice that the equation that comes from this addition no longer contains a '*y*' term, and that it is now possible to complete the calculation by solving for *x*.
$x = 5$ $x + y = 6$ $\Rightarrow 5 + y = 6$ $y = 1$	As you saw in the previous section you will need a *y* value to go with this. Substitute *x* into equation (2).
$x - y = 5 - 1 = 4$	Check that these values for *x* and *y* work in equation (1).

Both equations are satisfied by the pair of values $x = 5$ and $y = 1$.

The following worked examples look at different cases where you may need to subtract, instead of add, the equations or where you may need to multiply one, or both, equations before you consider addition or subtraction.

Always 'line up' '*x*'s with '*x*'s, '*y*'s with '*y*'s and '=' with '='. It will make your method clearer.

REWIND

Remind yourself about dealing with directed numbers from chapter 1. ◀

Worked example 4

Solve the following pairs of simultaneous equations:

$2x - 3y = -8 \quad (1)$

$5x + 3y = 1 \quad (2)$

$2x - 3y = -8$ $\underline{5x + 3y = 1}$ $7x \quad = -7$ $\Rightarrow x = -1$	(1) + (2)	Notice that these equations have the same coefficient of *y* in both equations, though the signs are different. If you add these equations together, you make use of the fact that $-3y + 3y = 0$
$2x - 3y = -8$ $\Rightarrow 2(-1) - 3y = -8$ $6 - 3y = 0$ $3y = 6$ $y = 2$	Substitute in (1)	Now that you have the value of *x*, you can substitute this into either equation and then solve for *y*.
$5x + 3y = 5(-1) + 3(2)$ $= -5 + 6 = 1$		Now you should check these values in equation (2) to be sure.

The second equation is also satisfied by these values so $x = -1$ and $y = 2$.

Worked example 5

Solve simultaneously:

$4x + y = -1$ (1)

$7x + y = -4$ (2)

$7x + y = -4$ (2) − (1) $4x + y = -1$ $3x = -3$ $\Rightarrow x = -1$	Notice this time that you have the same coefficient of y again, but this time the 'y' terms have the same sign. You now make use of the fact that $y - y = 0$ and so *subtract* one equation from the other. There are more 'x's in (2) so, consider (2) − (1).
$4x + y = -1$ $\Rightarrow 4(-1) + y = -1$ $y = 3$	Substitute in (1)
$7x + y = 7(-1) + = -7 + 3 = -4$	Now check that the values $x = -1$ and $y = 3$ work in equation (2).

Equation (2) is also satisfied by these values, so $x = -1$ and $y = 3$.

Always make it clear which equation you have chosen to subtract from which.

Here, you have used the fact that $-4 - (-1) = -3$.

Manipulating equations before solving them

Sometimes you need to manipulate or rearrange one or both of the equations before you can solve them simultaneously by elimination. Worked examples 6 to 8 show you how this is done.

Worked example 6

Solve simultaneously:

$2x - 5y = 24$ (1)

$4x + 3y = -4$ (2)

$2 \times (1)$ $4x - 10y = 48$ (3) $4x + 3y = -4$ (2) $4x - 10y = 48$ (3)	With this pair of simultaneous equations notice that neither the coefficient of x nor the coefficient of y match. But, if you multiply equation (1) by 2, you can make the coefficient of x the same in each. This equation, now named (3), has the same coefficient of x as equation (2) so write both of these equations together and solve as before.
$4x + 3y = -4$ $4x - 10y = 48$ $13y = -52$ $\Rightarrow y = -4$	(2) − (3)
$2x - 5y = 24$ $\Rightarrow 2x - 5(-4) = 24$ $2x + 20 = 24$ $x = 2$	Substitute in (1)
$4x + 3y = 4(2) + 3(-4) = 8 - 12 = -4$	Check using equation (2).

So the pair of values $x = 2$ and $y = -4$ satisfy the pair of simultaneous equations.

Worked example 7

Solve simultaneously:

$2x - 21 = 5y$

$3 + 4y = -3x$

$2x - 5y = 21 \quad (1)$ $3x + 4y = -3 \quad (2)$	Before you can work with these equations you need to rearrange them so they are in the same form. In this pair, not only is the coefficient of x different but so is the coefficient of y. It is not possible to multiply through just one equation to solve this problem.
$4 \times (1) \Rightarrow 8x - 20y = 84 \quad (3)$ $5 \times (2) \Rightarrow 15x + 20y = -15 \quad (4)$	Here, you need to multiply each equation by a different value so that the coefficient of x *or* the coefficient of y match. It is best to choose to do this for the 'y' terms here because they have different signs and it is simpler to add equations rather than subtract!
$8x - 20y = 84$ $\underline{15x + 20y = -15}$ $23x = 69$	(3) + (4)
$x = 3$ $2x - 5y = 21$ $\Rightarrow 2(3) - 5y = 21$ $5y = -15$ $y = -3$	Substitute for x in (1).
$3x + 4y = 3(3) + 4(-3) = 9 - 12 = -3$	Check using equation (2).

So $x = 3$ and $y = -3$ satisfy the pair of simultaneous equations.

Worked example 8

Solve simultaneously:

$\frac{3x - 4y}{2} = 10 \quad (1)$

$\frac{3x + 2y}{4} = 2 \quad (2)$

$3x - 4y = 20 \quad (3)$	In this pair of equations it makes sense to remove the fractions before you work with them. Multiply both sides of equation (1) by 2.
$3x + 2y = 8 \quad (4)$	Multiply both sides of equation (2) by 4.
$3x - 4y = 20 \quad (3)$ $\underline{3x + 2y = 8} \quad (4)$ $-6y = 12$ $y = -2$	Subtract equation (4) from equation (3).
$3x - 4(-2) = 20$ $3x + 8 = 20$ $3x = 12$ $x = 4$	Substitute the value for y into equation (3).
$3(4) + 2(-2) = 12 - 4 = 8$ So $x = 4$ and $y = -2$	Check using equation (4).

Exercise 14.2

REWIND

Remember from chapter 1 that adding a negative is the same as subtracting a positive. ◀

Remember that you need either the same coefficient of x or the same coefficient of y. If both have the *same* sign, you should then subtract one equation from the other. If they have a *different* sign, then you should add.

1 Solve for x and y by substitution. Check each solution.

a $y+x=7$, $y=x+3$
b $y=1-x$, $x-5=y$
c $2x+y=-14$, $y=6$
d $x-8=2y$, $x+y=-2$
e $3x-2=-2y$, $2x-y=-8$
f $3x+y=6$, $9x+2y=1$
g $4x-1=2y$, $x+1=3y$
h $3x-4y=1$, $2x=4-3y$

2 Solve for x and y by elimination. Check each solution.

a $2x-y=4$, $5x+y=24$
b $-3x+2y=6$, $3x+5y=36$
c $2x+5y=12$, $2x+3y=8$
d $5x-2y=27$, $3x+2y=13$
e $x+2y=11$, $x+3y=15$
f $-2x+5y=13$, $2x+3y=11$
g $4x+y=27$, $3x-y=15$
h $4x-y=16$, $6x-y=26$
i $6x-5y=9$, $2x+5y=23$
j $6x-y=18$, $4x-y=10$
k $x+y=12$, $5x-y=24$
l $4x+3y=22$, $4x+y=18$

3 Solve simultaneously. Use the method you find easiest. Check all solutions.

a $5x+3y=22$, $10x-y=16$
b $4x+3y=25$, $x+9y=31$
c $-3x+y=5$, $-6x+5y=-20$
d $x+y=10$, $3x+5y=40$
e $5x+y=11$, $-2x+2y=1$
f $4x-3y=11$, $5x-9y=-2$
g $6x+2y=9$, $7x+4y=12$
h $12x-13y=34$, $3x-26y=19$
i $5x-17y=-3$, $25x-19y=-45$
j $3x-3y=13$, $4x-12y=-6$
k $10x=2-2y$, $2y=-7x-1$
l $-2y=1-7x$, $4x=4+2y$
m $x=12+y$, $2x=3-y$
n $3x+4y=-1$, $3x+10=2y$
o $2x+y=7$, $11+x=2y$

4 Solve simultaneously.

a $3x+7y=37$, $5x+6y=39$
b $2x-5y=-16$, $3x-5y=-14$
c $-7x+4y=41$, $-5x+6y=45$
d $7x+4y=54$, $2x+3y=21$
e $2x-y=1$, $3x+5y=34$
f $3x-4y=25$, $x-3y=15$
g $7x-4y=23$, $4x+5y=35$
h $3x-y=2$, $3x+5y=26$
i $2x+7y=25$, $x+y=5$
j $x+3=y$, $4x+y=-7$
k $3x+11=-y$, $-2x+y=4$
l $y=6x-1$, $4x-3y=-4$
m $2x+3y-8=0$, $4x+5=y$
n $y=\frac{2}{3}x+6$, $2y-4x=20$
o $8x-5y=0$, $13x=8y+1$

If an equation contains fractions, you can make everything much easier by multiplying each term by a suitable number (a common denominator). 'Clear' the fractions first.

5 Solve each pair of equations simultaneously.

a $\frac{1}{2}x+\frac{2}{3}y=6\frac{1}{5}$
$\frac{3}{4}x-\frac{1}{7}y=13\frac{3}{5}$

b $\frac{3}{7}x-\frac{5}{8}y=33\frac{1}{3}$
$64x-17y=12\frac{1}{2}$

c $456\frac{2}{17}x+987\frac{3}{4}y=1$
$233\frac{13}{22}x-94\frac{2}{3}y=4$

d $3x+\frac{2y}{3}=0$
$2x-\frac{y}{4}=14$

e $4y+x+5=0$
$y=x-5$

f $3y+\frac{6}{2}=-3$
$y-\frac{x}{2}=2$

g $2x+\frac{y}{2}=3$
$6x=-2y$

h $y=3x-6$
$2x+\frac{3y}{7}=-5$

i $\frac{3x}{7}-\frac{2y}{13}=5$
$x+\frac{1}{3}y=\frac{3}{5}$

Think carefully about these problems and consider how you can recognise problems involving simultaneous equations if you are not told to use this method to solve them.

6 Form a pair of simultaneous equations for each situation below, and use them to solve the problem. Let the unknown numbers be x and y.

a The sum of two numbers is 120 and one of the numbers is 3 times the other. Find the value of the numbers.
b The sum of two numbers is −34 and their difference is 5. Find the numbers.
c A pair of numbers has a sum of 52 and a difference of 11. Find the numbers.
d The combined ages of two people is 34. If one person is 6 years younger than the other, find their ages.

7 A computer store sold 4 hard drives and 10 pen drives for \$200. and 6 hard drives and 14 pen drives for \$290. Find the cost of a hard drive and the cost of a pen drive.

8 A large sports stadium has 21 000 seats. The seats are organised into blocks of either 400 or 450 seats. There are three times as many blocks of 450 seats as there are blocks of 400 seats. How many blocks are there?

14.2 Linear inequalities

E

The work earlier in the book on linear equations led to a single solution for a single variable. Sometimes however, there are situations where there are a range of possible solutions. This section extends the previous work on linear equations to look at **linear inequalities**.

Number lines

Suppose you are told that $x < 4$. You will remember from chapter 1 that this means each possible value of x must be less than 4. Therefore, x can be 3, 2, 1, 0, −1, −2 . . . but that is not all. 3.2 is also less than 4, as is 3.999, 2.43, −3.4, −100 . . .

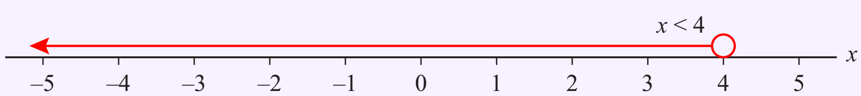
REWIND

Remind yourself how inequality symbols were used for grouped data in chapter 12. ◀

FAST FORWARD

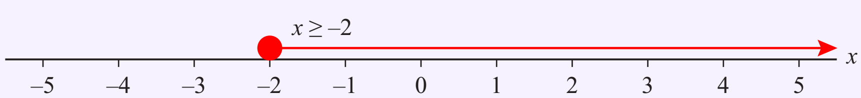
You will find it useful to review inequalities before you tackle histograms in chapter 20. ▶

If you draw a number line, you can use an arrow to represent the set of numbers:



This allows you to show the possible values of x neatly without writing them all down (there is an infinite number of values, so you can't write them all down!). Notice that the 'open circle' above the four is not filled in. This symbol is used because it is not possible for x to be *equal* to four.

Now suppose that $x \geqslant -2$. This now tells you that x can be greater than, *or equal to* −2. You can show that that x can be equal to −2 by 'filling in' the circle above −2 on the number line:



The following worked examples show that more than one inequality symbol can appear in a question.

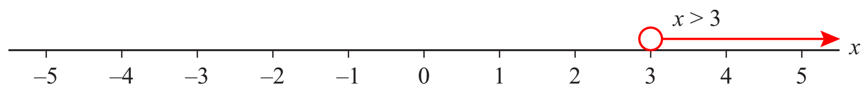
Worked example 9

Show the set of values that satisfy each of the following in equalities on a number line.

a $x > 3$ **b** $4 < y < 8$ **c** $-1.4 < x \leqslant 2.8$

d List all integers that satisfy the inequality $4.2 < x \leqslant 10.4$

a The values of x have to be larger than 3. x cannot be equal to 3, so do not fill in the circle. 'Greater than' means 'to the right' on the number line.

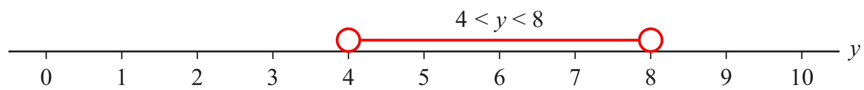


b Notice that y is now being used as the variable and this should be clearly labelled on your number line. Also, two inequality symbols have been used. In fact there are two inequalities, and *both* must be satisfied.

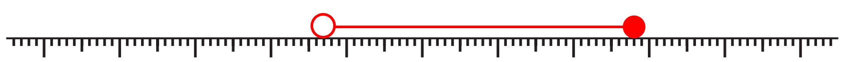
$4 < y$ tells you that y is greater than (but not equal) to 4.

$y < 8$ tells you that y is also less than (but not equal to) 8.

So y lies between 4 and 8 (not inclusive):



c This example has two inequalities that must both be satisfied. x is greater than (but not equal to) -1.4, and x is less than or equal to 2.8:



d Here x must be greater than, but not equal to 4.2. So the smallest possible value of x is 5. x must also be less than or equal to 10.4. The largest that x can be is therefore 10.

So the possible values of x are 5, 6, 7, 8, 9 or 10.

Exercise 14.3

1 Draw a number line to represent the possible values of the variable in each case.

a $x < 5$	**b** $x > 2$	**c** $p \leqslant 6$
d $y > -8$	**e** $q \geqslant -5$	**f** $x < -4$
g $1.2 < x < 3.5$	**h** $-3.2 < x \leqslant 2.9$	**i** $-4.5 \leqslant k \leqslant -3.1$

2 Write down all *integers* that satisfy each of the following inequalities.

a $3 < b < 33$	**b** $7 < h \leqslant 19$	**c** $18 \leqslant e \leqslant 27$
d $-3 \leqslant f < 0$	**e** $-3 \geqslant f \leqslant 0$	**f** $2.5 < m < 11.3$
g $-7 < g \leqslant -4$	**h** $\pi < r < 2\pi$	**i** $\sqrt{5} < w < \sqrt{18}$

E

Solving inequalities algebraically

Consider the inequality $3x > 6$.

Now, suppose that $x = 2$, then $3x = 6$ but this doesn't quite satisfy the inequality! *Any* value of x that is *larger* than 2 *will* work however. For example:

If $x = 2.1$, then $3x = 6.3$, which *is* greater than 6.

In the same way that you could divide both sides of an equation by 3, both sides of the inequality can be divided by 3 to get the solution:

$$3x > 6$$
$$\frac{3x}{3} > \frac{6}{3}$$
$$x > 2$$

Notice that this solution is a range of values of x rather than a single value. Any value of x that is greater than 2 works!

In fact you can solve any linear inequality in much the same way as you would solve a linear equation, though there are important exceptions, and this is shown in the 'warning' section on page 283. Most importantly, you should simply remember that what you do to one side of the inequality you must do to the other.

Worked example 10

Find the set of values of x for which each of the following inequalities holds.

a $3x - 4 < 14$ **b** $4(x - 7) \geqslant 16$ **c** $5x - 3 \leqslant 2x + 18$ **d** $4 - 7x \leqslant 53$

a	$3x - 4 < 14$	
	$3x < 18$	Add 4 to both sides.
	$\frac{3x}{3} < \frac{18}{3}$	Divide both sides by 3.
	So, $x < 6$	
b	$4(x - 7) \geqslant 16$	
	$4x - 28 \geqslant 16$	Expand the brackets.
	$4x \geqslant 44$	Add 28 to both sides.
	$\frac{4x}{4} \geqslant \frac{44}{4}$	Divide both sides by 4.
	So, $x \geqslant 11$	
	$4(x - 7) \geqslant 16$	Notice that you can also solve this inequality by dividing both sides by 4 at the beginning:
	$x - 7 \geqslant 4$	Divide both sides by 4.
	$x \geqslant 11$	Add 7 to both sides to get the same answer as before.
c	$5x - 3 \leqslant 2x + 18$	
	$5x - 3 - 2x \leqslant 2x + 18 - 2x$	Subtract the smaller number of 'x's from both sides ($2x$).
	$3x - 3 \leqslant 18$	Simplify.
	$3x \leqslant 21$	Add 3 to both sides.
	$x \leqslant 7$	Divide both sides by 3.

E

d	$4 - 7x \leqslant 53$	
	$4 \leqslant 53 + 7x$	Add $7x$ to both sides
	$-49 \leqslant 7x$	Subtract 53 from both sides.
	$-7 \leqslant x$	Divide both sides by 7. Notice that the x is on the right-hand side of the inequality in this answer. This is perfectly acceptable. You can reverse the entire inequality to place the x on the left without changing its meaning, but you must remember to reverse the actual inequality symbol!
	And, $x \geqslant -7$.	

A warning

Before working through the next exercise you should be aware that there is one further rule to remember. Consider this inequality:

$$3 - 5x > 18$$
$$\Rightarrow -5x > 15$$

If you divide both sides of this by -5 it *appears* that the solution will be,

$$x > -3$$

If you can avoid negatives, by adding or subtracting terms, then try to do so.

This is satisfied by any value of x that is greater than -3, for example $-2, -1, 2.4, 3.5, 10 \ldots$

If you calculate the value of $3 - 5x$ for each of these values you get $13, 8, -9, -14.5, -47 \ldots$ and not one of these works in the original inequality as they are all smaller than 18.

But here is an alternate solution:

$$3 - 5x > 18$$
$$\Rightarrow \quad 3 > 18 + 5x$$
$$-15 > 5x$$
$$-3 > x$$
or, $\quad x < -3$

This is a correct solution, and the final answer is very similar to the 'wrong' one above. The only difference is that the inequality symbol has been reversed. You should remember the following:

> If you multiply or divide both sides of an inequality by a *negative* number then you must *reverse* the direction of the inequality.

Exercise 14.4

Solve each of the following inequalities. Some of the answers will involve fractions. Leave your answers as fractions in their simplest form where appropriate.

1 **a** $18x < 36$ **b** $13x > 39$ **c** $15y \leqslant 14$ **d** $7y > -14$
e $4 + 8c \geqslant 20$ **f** $2x + 1 < 9$ **g** $\frac{x}{3} < 2$ **h** $5p - 3 > 12$
i $\frac{x}{3} + 7 > 2$ **j** $12g - 14 \geqslant 34$ **k** $22(w - 4) < 88$ **l** $10 - 10k > 3$

2 **a** $\frac{y+6}{4} > 9$ **b** $10q - 12 < 48 + 5q$ **c** $3g - 7 \geqslant 5g - 18$ **d** $3(h - 4) > 5(h - 10)$
e $\frac{y+6}{4} \leqslant 9$ **f** $\frac{1}{2}(x+5) \leqslant 2$ **g** $3 - 7h \leqslant 6 - 5h$ **h** $2(y-7) + 6 \leqslant 5(y+3) + 21$
i $6(n - 4) - 2(n + 1) < 3(n + 7) + 1$ **j** $5(2v - 3) - 2(4v - 5) \geqslant 8(v + 1)$

REWIND
Upper and Lower bounds are covered in chapter 13.

Tip
You will need to think about how the fraction could be re-written

k $\frac{z-2}{3}-7>13$ **l** $\frac{3k-1}{7}-7>7$ **m** $\frac{2e+1}{9}>7-6e$

3 **a** $2t-\frac{2t+1}{3}>12$ **b** $\frac{2}{3}t-\frac{2t+1}{9}>12$ **c** $\frac{2}{7}t-\frac{2t+1}{9}>12$

d $\frac{r}{2}+\frac{1}{3}<2$ **e** $\frac{3}{8}(2d-\frac{1}{3})-\frac{2}{9}(7-3d)\geqslant\frac{1}{4}d+\frac{2(d-8)}{7}$

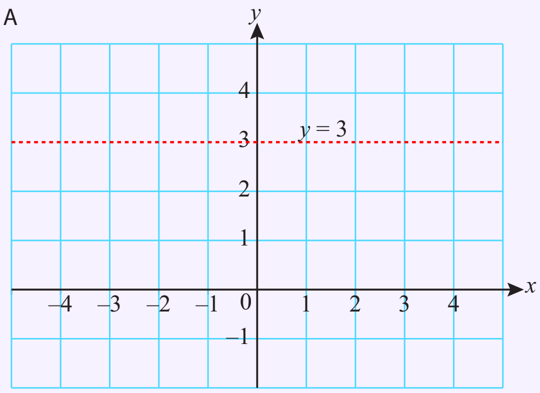
4 $a = 6.2$, correct to 1 decimal place. $b = 3.86$, correct to 2 decimal places. Find the upper and lower bounds of the fraction $\frac{a+b}{ab}$. Give your answers correct to 2 decimal places.

14.3 Regions in a plane

So far, you have only considered one variable and inequalities along a number line. You can, however, have two variables connected with an inequality, in which case you end up with a **region** on the Cartesian plane.

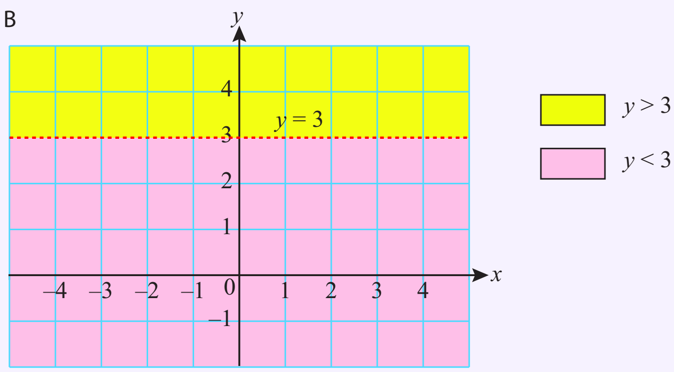
Diagram A shows a broken line that is parallel to the x-axis. Every point on the line has a y co-ordinate of 3. This means that the equation of the line is $y = 3$.

A



All of the points above the line $y = 3$ have y co-ordinates that are greater than 3. The region above the line thus represents the inequality $y > 3$. Similarly, the region below the line represents the inequality $y < 3$. These regions are shown on diagram B.

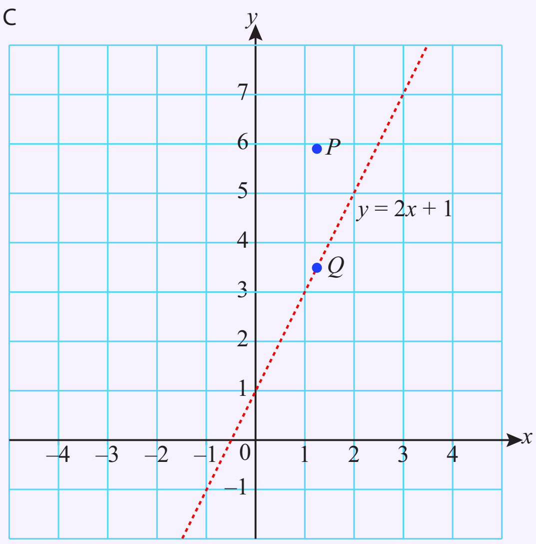
B



In diagram C, the graph of $y = 2x + 1$ is shown as a broken line. Every point on the line has co-ordinates (x, y) which satisfy $y = 2x + 1$.

Q is a point on the line. Point P has a y co-ordinate that is greater than the y co-ordinate of Q. P and Q have the same x co-ordinate. This means that for any point P in the region above the line, $y > 2x + 1$.

C



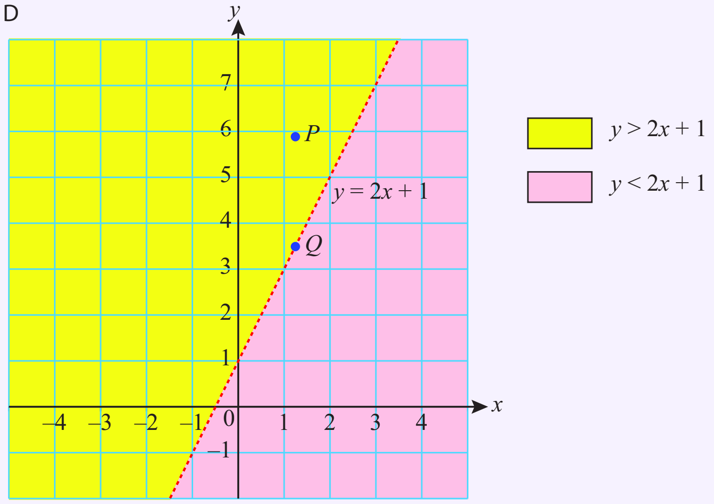
E

The region above the line represents the inequality $y > 2x + 1$.

Similarly the region below the line represents the inequality $y < 2x + 1$.

You can see this on diagram D.

D



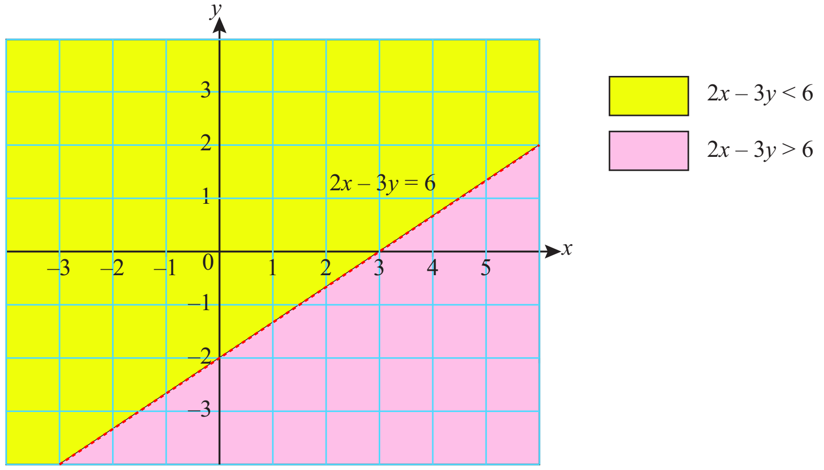
If the equation of the line is in the form $y = mx + c$, then:

- the inequality $y > mx + c$ is above the line
- the inequality $y < mx + c$ is below the line.

If the equation is not in the form $y = mx + c$, you have to find a way to check which region represents which inequality.

Worked example 11

In a diagram, show the regions that represent the inequalities $2x - 3y < 6$ and $2x - 3y > 6$.



The boundary between the two required regions is the line $2x - 3y = 6$.

This line crosses the x-axis at (3, 0) and the y-axis at (0, –2). It is shown as a broken line in this diagram.

Consider any point in the region above the line. The easiest point to use is the origin (0, 0). When $x = 0$ and $y = 0$, $2x - 3y = 0$. Since 0 is less than 6, the region above the line represents the inequality $2x - 3y < 6$.

E

Rules about boundaries and shading of regions

You have already seen inequalities are not always $<$ or $>$. They may also be $\leqslant$ or $\geqslant$. Graphical representations have to show the difference between these variations.

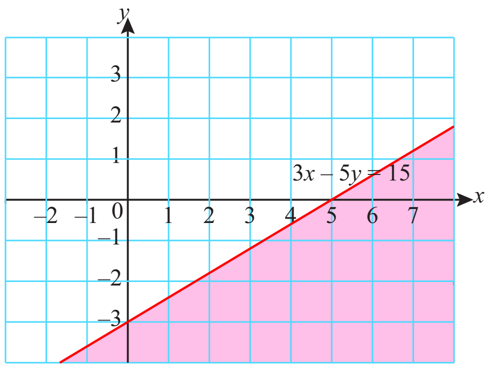
When the inequality includes equal to ($\leqslant$ or $\geqslant$), the boundary line must be included in the graphical representation. It is therefore shown as a solid line.

When the inequality does not include equal to ($<$ or $>$), the boundary line is not included in the graphical representation, so it is shown as a broken line.

Worked example 12

Sometimes it is better to shade out the *unwanted* region.

By shading the *unwanted* region, show the region that represents the inequality $3x - 5y \leqslant 15$.



The boundary line is $3x - 5y = 15$ and it is included in the region (because the inequality includes *equal to*).

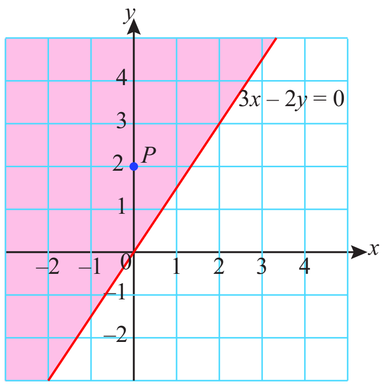
This line crosses the x-axis at (5, 0) and crosses the y-axis at (0, –3). It is shown as a solid line in this diagram.

When $x = 0$ and $y = 0$, $3x - 5y = 0$. Since 0 is less than 15, the origin (0, 0) is in the required region. (Alternatively, rearrange $3x - 5y \leqslant 15$ to get $y \geq \frac{3}{5}x - 3$ and deduce that the required region is above the line.)

The unshaded region in this diagram represents the inequality, $3x - 5y \leqslant 15$.

Worked example 13

By shading the *unwanted* region, show the region that represents the inequality $3x - 2y \geqslant 0$.



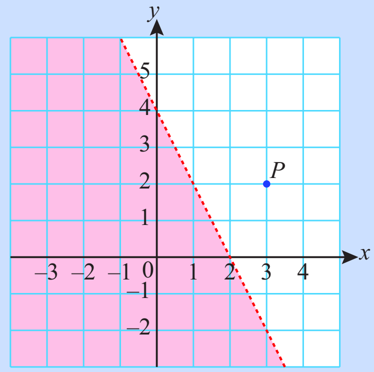
You cannot take the origin as the check-point because it lies on the boundary line. Instead take the point P (0, 2) which is *above* the line. When $x = 0$ and $y = 2$, $3x - 2y = -4$, which is less than 0. Hence P is *not* in the required region.

The boundary line is $3x - 2y = 0$ and it is included in the region. It is shown as a solid line in this diagram.

E

Worked example 14

Find the inequality that is represented by the *unshaded* region in this diagram.



First find the equation of the boundary. Its gradient $= \frac{-4}{2} = -2$ and its intercept on the y-axis is $y = 4$. Hence the boundary line is $y = -2x + 4$ or this can be re-written as $y + 2x = 4$. Take P (3, 2) in the unshaded region as the check-point: $2 + 6 = 8$. Note that 8 is greater than 4, hence, the *unshaded* region represents $y + 2x > 4$. As the boundary is a broken line, it is not included, and thus the sign is not $\geqslant$.

Exercise 14.5

For questions 1 to 3, show your answers on a grid with x- and y-axes running from −3 to +4.

1 By shading the unwanted region, show the region that represents the inequality $2y - 3x \geqslant 6$.

2 By shading the unwanted region, show the region that represents the inequality $x + 2y < 4$.

3 By shading the unwanted region, show the region that represents the inequality $x - y \geqslant 0$.

4 Shade the region that represents each inequality.

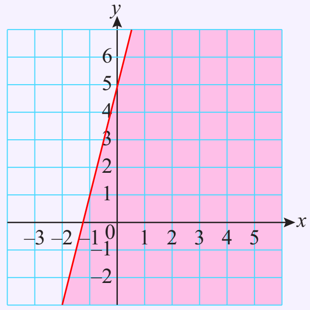
a $y > 3 - 3x$ **b** $3x - 2y \geqslant 6$ **c** $x \leqslant 5$ **d** $y > 3$
e $x + 3y \leqslant 10$ **f** $-3 < x < 5$ **g** $0 \leqslant x \leqslant 2$

5 Copy and complete these statements by choosing the correct option:

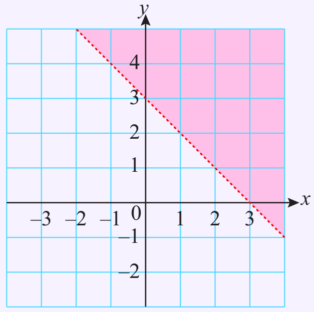
a If $y < mx + c$, shade the unwanted region **above/below** the graph of $y = mx + c$.
b If $y > mx + c$, shade the unwanted region **above/below** the graph of $y = mx + c$.
c For $y < m_1x + c_1$ and $y > m_2x + c_2$, shade the unwanted region **above/below** the graph of $y = m_1x + c_1$ **and/or above/below** the graph of $y = m_2x + c_2$.

6 For each of the following diagrams, find the inequality that is represented by the *unshaded* region.

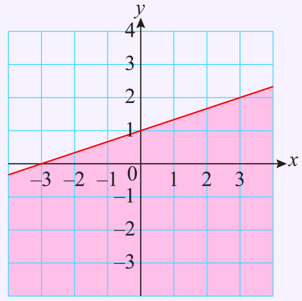
a



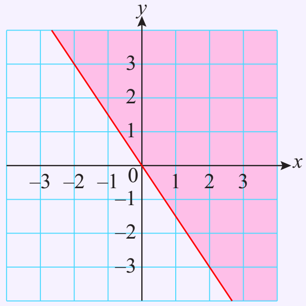
b



c



d



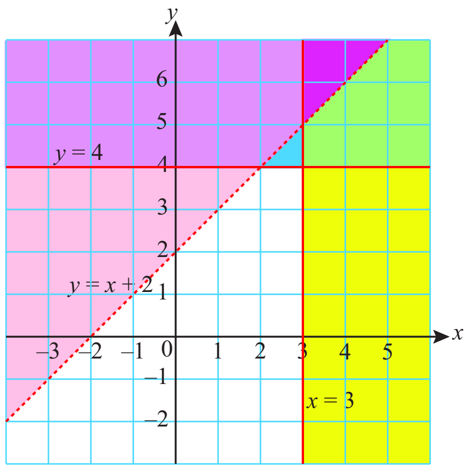
E

Representing simultaneous inequalities

When two or more inequalities have to be satisfied at the same time, they are called simultaneous inequalities. These can also be represented graphically. On the diagram in worked example 15 the inequalities are represented by regions on the same diagram. The unwanted regions are shaded or crossed out. The unshaded region will contain all the co-ordinates (x, y) that satisfy all the inequalities simultaneously.

Worked example 15

By shading the unwanted regions, show the region defined by the set of inequalities $y < x + 2$, $y \leqslant 4$ and $x \leqslant 3$.



The boundaries of the required region are $y = x + 2$ (broken line), $y = 4$ (solid line) and $x = 3$ (solid line).

The unshaded region in the diagram represents the set of inequalities $y < x + 2$, $y \leqslant 4$ and $x \leqslant 3$.

Notice that this region does not have a finite area – it is not 'closed'.

Exercise 14.6

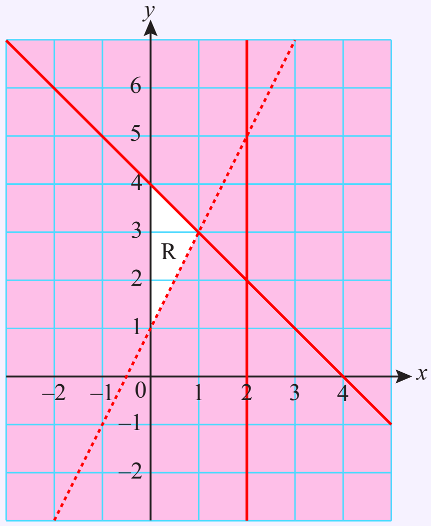
1 By shading the unwanted regions, show the region defined by the set of inequalities $x + 2y \geqslant 6$, $y \leqslant x$ and $x < 4$.

2 By shading the unwanted regions, show the region defined by the set of inequalities $x + y \geqslant 5$, $y \leqslant 2$ and $y \geqslant 0$.

3 **a** On a grid, draw the lines $x = 4$, $y = 3$ and $x + y = 5$.

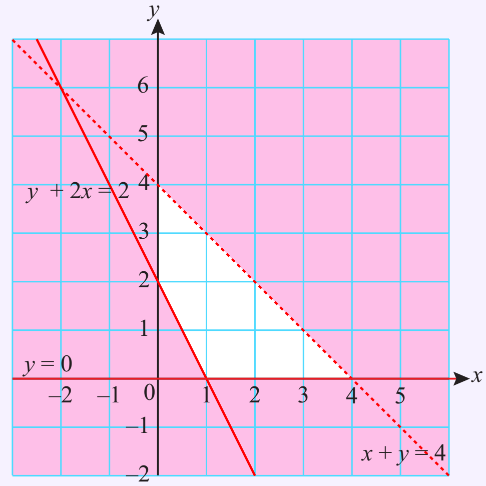
b By shading the unwanted regions, show the region that satisfies all the inequalities $x \leqslant 4$, $y \leqslant 3$ and $x + y \geqslant 5$. Label the region R.

4 Write down the three inequalities that define the unshaded triangular region R.



E

5 The unshaded region in diagram represents the set of inequalities $y \geqslant 0$, $y + 2x \geqslant 2$ and $x + y < 4$. Write down the pairs of integers (x, y) that satisfy all the inequalities.



6 Draw graphs to show the solution sets of these inequalities: $y \leqslant 4$, $y \geqslant x + 2$ and $3x + y \geqslant 4$. Write down the integer co-ordinates (x, y) that satisfy all the inequalities in this case.

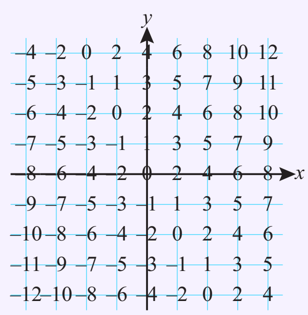
14.4 Linear programming

Many of the applications of mathematics in business and industry are concerned with obtaining the greatest profits or incurring the least cost subject to constraints (restrictions) such as the number of workers, machines available or capital available.

When these constraints are expressed mathematically, they take the form of inequalities. When the inequalities are linear (such as $3x + 2y < 6$), the branch of mathematics you would use is called **linear programming**.

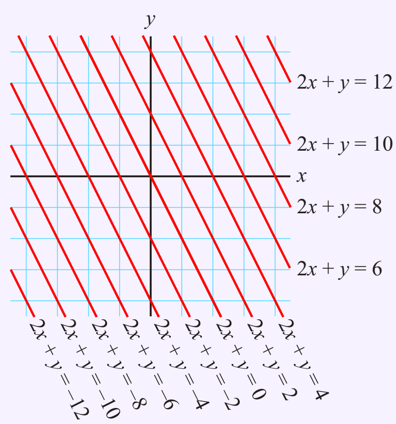
Greatest and least values.

The expression $2x + y$ has a value for every point (x, y) in the Cartesian plane. Values of $2x + y$ at some grid points are shown in the diagram.



If points that give the same value of $2x + y$ are joined, they result in a set of contour lines. These contour lines are straight lines; their equations are in the form $2x + y = k$ (k is the constant).

You can see that as k increases, the line $2x + y$ moves parallel to itself towards the top right-hand side of the diagram. As k decreases, the line moves parallel to itself towards the bottom left-hand side of the diagram. (Only the even numbered contours are shown here.)

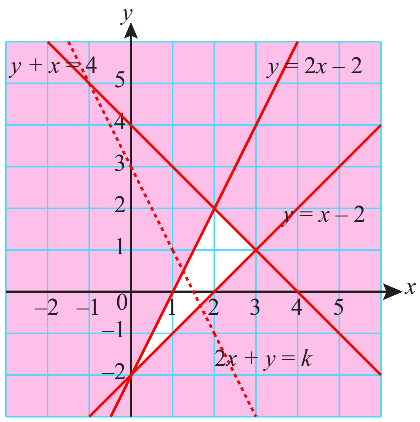


E

The expression $2x + y$ has no greatest or least value if there are no restrictions on the values of x and y. When there are restrictions on the values, there is normally a greatest and/or a least value for the expression.

Worked example 16

The numbers x and y satisfy all the inequalities $x + y \leqslant 4$, $y \leqslant 2x - 2$ and $y \geqslant x - 2$. Find the greatest and least possible values of the expression $2x + y$.



You only have to consider the values of $2x + y$ for points in the unshaded region. If $2x + y = k$, then $y = -2x + k$. Draw a line with gradient equal to -2. (The dashed line in this example has $k = 3$.)

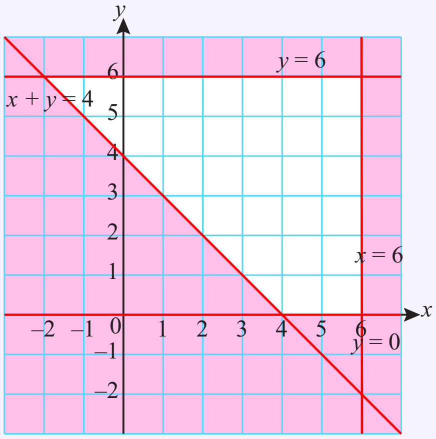
Using a set square and a ruler, place one edge of your set square on the dashed line you have drawn and your ruler against one of the other sides. If you slide your set square along the ruler, the original side will remain parallel to your dashed line.

Moving to the right, when your set square is just about to leave the unshaded region (at the point (3, 1)), $2x + y$ will have its greatest value. Substituting $x = 3$ and $y = 1$ into $2x + y$ gives a greatest value of 7.

Similarly, moving to the left will give a least value of -2 (at co-ordinates $(0, -2)$).

Exercise 14.7

1 In the diagram, the unshaded region represents the set of inequalities $x \leqslant 6$, $0 \leqslant y \leqslant 6$ and $x + y \geqslant 4$. Find the greatest and least possible values of $3x + 2y$ subject to these inequalities.



2 **a** On a grid, shade the unwanted regions to indicate the region satisfying all the inequalities $y \leqslant x$, $x + y \leqslant 6$ and $y \geqslant 0$.

b What is the greatest possible value of $2x + y$ if x and y satisfy all these inequalities?

3 The whole numbers x and y satisfy all the inequalities $y \geqslant 1$, $y \leqslant x + 3$ and $3x + y \leqslant 6$. Find the greatest and least possible values of the expression, $x + y$.

4 An IGCSE class is making school flags and T-shirts to sell to raise funds for the school. Due to time constraints, the class is able to make at most 150 flags and 120 T-shirts. The fabric is donated and they have enough to make 200 items in total. A flag sells for \$2 and a T-shirt sells for \$5. How many of each item should they make to maximise their income from sales?

E

5 A school principal wants to buy some book cases for the school library. She can choose between two types of book case. Type A costs \$10 and it requires 0.6 m^2 of floor space and holds 0.8 m^3 of books. Type B costs \$20 and it requires 0.8 m^2 of floor space and holds 1.2 m^3 of books. The maximum floor space available is 7.2 m^2 and the budget is \$140 (but the school would prefer to spend less). What number and type of book cases should the principal buy to get the largest possible storage space for books?

14.5 Completing the square

REWIND

Quadratic expressions contain an x^2 term as the highest power. You learned how to solve quadratic equations by factorising in chapter 10. ◄

It can be helpful to re-write **quadratic** expressions in a slightly different form. Although this first method will be used to solve quadratic equations, it can be used to find the co-ordinates of maximums or minimums in a quadratic. An application of this method to the general form of
a quadratic will produce the quadratic formula that is used in the next section.

Remember that when you expand $(x+a)^2$ you get:

$$(x+a)(x+a)=x^2+2ax+a^2$$

Most importantly you will see that the value of 'a' is doubled and this gives the coefficient of x in the final expansion. This is the key to the method.

Now consider x^2+6x+1 and compare with $(x+3)^2=(x+3)(x+3)=x^2+6x+9$.

The '3' has been chosen as it is half of the number of 'x's in the original expression.

This latter expression is similar, but there is constant term of 9 rather than 1. So, to make the new expression equal to the original, you must subtract 8.

$$x^2+6x+1=(x+3)^2-8$$

This method of re-writing the quadratic is called *completing the square*.

The completing the square method can only be used when the coefficient of x^2 = 1. Use this method when the trinomial (expression with three terms) cannot be factorised.

Worked example 17

Rewrite the expression $x^2-4x+11$ in the form $(x+a)^2+b$.

The number of 'x's is −4. Half of this is −2.

$(x-2)^2=x^2-4x+4$

The constant term is too small by 7, so $x^2-4x+11=(x-2)^2+7$

In chapter 10 you solved quadratic equations like $x^2-7x+12=0$ by factorisation. But some quadratic equations cannot be factorised. In such cases, you can solve the equation by completing the square.

A $\sqrt{}$ can have both a positive and negative value, which leads to the two solutions for a quadratic.

Worked example 18

Solve $x^2+4x-6=0$, giving your answer to two decimal places.

$x^2+4x-6=0$ $x^2+4x=6$	This equation cannot be factorised. Add 6 to both sides.

$(x+2)^2 - 4 = 6$	Complete the square by writing $x^2 + 4x$ in the form $(x + a)^2 + b$. Half of 4 is 2 so try $(x + 2)^2 = x^2 + 4x + 4$. The constant of +4 is too big, so it becomes $(x + 2)^2 - 4$.
$(x+2)^2 = 10$	Add 4 to both sides.
$x+2 = \pm\sqrt{10}$	Take the square root of both sides.
$x = -2 \pm \sqrt{10}$	Subtract 2 from each side.
$x = 1.1622\ldots$ or $-5.1622\ldots$	Solve for both options.
$x = 1.16$ or -5.16 (2 d.p.)	Round your solutions.

Exercise 14.8

1 Write each of the following expressions in the form $(x+a)^2 + b$.

a	$x^2 + 6x + 14$	**b**	$x^2 + 8x + 1$	**c**	$x^2 + 12x + 20$
d	$x^2 + 6x + 5$	**e**	$x^2 - 4x + 12$	**f**	$x^2 - 2x - 17$
g	$x^2 + 5x + 1$	**h**	$x^2 + 7x - 2$	**i**	$x^2 - 3x - 3$
j	$x^2 + 7x - 8$	**k**	$x^2 - 13x + 1$	**l**	$x^2 - 20x + 400$

2 Solve the following quadratic equations by the method of completing the square, giving your final answer to 2 decimal places.

a	$x^2 + 6x - 5 = 0$	**b**	$x^2 + 8x + 4 = 0$	**c**	$x^2 - 4x + 2 = 0$
d	$x^2 + 5x - 7 = 0$	**e**	$x^2 - 3x + 2 = 0$	**f**	$x^2 - 12x + 1 = 0$

3 Solve each equation by completing the square.

a	$x^2 - x - 10 = 0$	**b**	$x^2 + 3x - 6 = 0$	**c**	$x(6 + x) = 1$
d	$2x^2 + x = 8$	**e**	$5x = 10 - \dfrac{1}{x}$	**f**	$x - 5 = \dfrac{2}{x}$
g	$(x - 1)(x + 2) - 1 = 0$	**h**	$(x - 4)(x + 2) = -5$	**i**	$x^2 = x + 1$

14.6 Quadratic formula

REWIND

You saw in chapter 2 that the coefficient of a variable is the number that multiplies it. This is still true for quadratic equations: a is the coefficient of x^2 and b is the coefficient of x. c is the constant term. ◄

In the previous section the coefficient of x^2 was always 1. Applying the completing the square method when the coefficient of x^2 is not 1 is more complex but if you do apply it to the general form of a quadratic equation ($ax^2 + bx + c = 0$), the following result is produced:

$$\text{If } ax^2 + bx + c = 0 \quad \text{then} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is known as the quadratic formula.

Notice the '±' symbol. This tells you that you should calculate two values: one with a '+' and one with a '–' in the position occupied by the '±'. The quadratic formula can be used for all quadratic equations that have real solutions even if the quadratic expression cannot be factorised.

The advantage of the quadratic formula over completing the square is that you don't have to worry when the coefficient of x^2 is not 1.

E

Worked example 19

Solve the following quadratic equations, giving your answers to 3 significant figures.

a $x^2+4x+3=0$ **b** $x^2-7x+11=0$ **c** $3x^2-2x-1=0$

a Compare the quadratic equation $x^2+4x+3=0$ with $ax^2+bx+c=0$. From this you should see that $a=1$, $b=4$ and $c=3$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{4^2-4\times1\times3}}{2\times1}$$
$$=\frac{-4\pm\sqrt{16-12}}{2}$$
$$=\frac{-4\pm\sqrt{4}}{2}$$
$$=\frac{-4\pm2}{2}$$

So, $x=\frac{-4+2}{2}=\frac{-2}{2}=-1$ or $x=\frac{-4-2}{2}=\frac{-6}{2}=-3$

Notice that the original quadratic equation can be factorised to give $(x+1)(x+3)=0$ and the same solutions. If you can factorise the quadratic then you should because the method is much simpler.

Tip

Here you need to take particular care. BODMAS always applies and you should check the order of your working, and your solution, carefully.

b $x^2-7x+11=0$, $a=1$, $b=-7$ and $c=11$.

$$x=\frac{-(-7)\pm\sqrt{(-7)^2-4\times1\times11}}{2\times1}=\frac{7\pm\sqrt{49-44}}{2}$$
$$=\frac{7\pm\sqrt{5}}{2}$$

So, $x=\frac{7+\sqrt{5}}{2}=4.6180\ldots$ or $x=\frac{7-\sqrt{5}}{2}=2.3819\ldots$

$x\approx4.62$ or 2.38 (3sf)

Notice that there are brackets around the –7. If you miss these the calculation becomes $-7^2=-49$ rather than +49.

If b is negative ALWAYS use brackets to make sure that you square it correctly.

Most modern calculators will allow you to input these fractions exactly as they appear here.

c For this example you should note that a is not 1!
$3x^2-2x-1=0$, $a=3$, $b=-2$ and $c=-1$.

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4\times3\times(-1)}}{2\times3}=\frac{2\pm\sqrt{4+12}}{6}$$
$$=\frac{2\pm\sqrt{16}}{6}$$
$$=\frac{2\pm4}{6}$$

So, $x=\frac{2+4}{6}=\frac{6}{6}=1$ or $x=\frac{2-4}{6}=\frac{-2}{6}=\frac{-1}{3}$

E

Exercise 14.9

1 Each of the following quadratics will factorise. Solve each of them by factorisation and then use the quadratic formula to show that you get the same answers in both cases.

a $x^2+7x+12=0$ **b** $x^2+8x+12=0$ **c** $x^2+11x+28=0$
d $x^2+4x-5=0$ **e** $x^2+6x-16=0$ **f** $x^2+12x-160=0$
g $x^2-6x+8=0$ **h** $x^2-3x-28=0$ **i** $x^2-5x-24=0$
j $x^2-12x+32=0$ **k** $x^2-2x-99=0$ **l** $x^2-9x-36=0$
m $x^2-10x+24=0$ **n** $x^2-12x+35=0$ **o** $x^2+9x-36=0$

2 Solve each of the following equations by using the quadratic formula. Round your answers to 3 significant figures where necessary. These quadratic expressions do not factorise.

a $x^2+6x-1=0$ **b** $x^2+5x+5=0$ **c** $x^2+7x+11=0$
d $x^2+4x+2=0$ **e** $x^2-3x-1=0$ **f** $x^2-4x+2=0$
g $x^2-8x+6=0$ **h** $x^2-2x-2=0$ **i** $x^2-6x-4=0$
j $x^2-8x-2=0$ **k** $x^2-9x+7=0$ **l** $x^2+11x+7=0$

3 Solve each of the following equations by using the quadratic formula. Round your answers to 3 significant figures where necessary. Take particular note of the coefficient of x^2.

a $2x^2-4x+1=0$ **b** $3x^2-3x-1=0$ **c** $4x^2+2x-5=0$
d $-2x^2+3x+4=0$ **e** $-2x^2-2x+1=0$ **f** $5x^2+x-3=0$

4 Solve each of the following equations by using the quadratic formula. Round your answers to 3 significant figures where necessary. You must make sure that your equation takes the form of a quadratic expression *equal to zero*. If it does not, then you will need to collect all terms on to one side so that a zero appears on the other side!

a $2x^2-x+6=4x+5$ **b** $7x^2-3x-6=3x-7$ **c** $x(6x-3)-2=0$
d $0.5x^2+0.8x-2=0$ **e** $(x+7)(x+5)=9$ **f** $\frac{1}{x}+x=7$

5 A rectangle has area 12 cm². If the length of the rectangle is $(x+1)$ cm and the width of the rectangle is $(x+3)$ cm, find the possible value(s) of x.

6 A biologist claims that the average height, h metres, of trees of a certain species after t months is given by $h=\frac{1}{5}t^{\frac{2}{3}}+\frac{1}{3}t^{\frac{1}{3}}$

For this model

a Find the average height of trees of this species after 64 months.
b Find, to 3 significant figures, the number of months that the trees have been growing when the model would predict an average height of 10 metres.

Let $x=t^{\frac{1}{3}}$.
Form and solve a quadratic in x

14.7 Factorising quadratics where the coefficient of x^2 is not 1

The quadratic equation in worked example 19 (c) gave two solutions that could have been obtained by factorisation. It turns out that $(x-1)(3x+1)=3x^2-2x-1$ (you can check this by expanding the brackets).

In general, if the coefficient of x^2 in a quadratic is a number other than 1 it is harder to factorise, but there are some tips to help you.

Worked example 20

Factorise each of the following expressions:

a $2x^2+3x+1$ **b** $3x^2-14x+8$ **c** $10x^2+11x-8$

a $2x^2+3x+1$

$2x^2+3x+1=(2x\quad)(x\quad)$

The only way to produce the term $2x^2$ is to multiply $2x$ and x. These two terms are placed at the front of each bracket. There are blank spaces in the brackets because you don't yet know what else needs to be included. The clue lies in the constant term at the end, which is obtained by multiplying these two unknown values together.

E

The constant term is 1, so the only possible values are +1 or −1. Since the constant term is positive, the unknown values must be either both −1 or both +1.

Try each of these combinations systematically:

$(2x-1)(x-1)=2x^2-3x+1$ The coefficient of x is wrong.

$(2x+1)(x+1)=2x^2+3x+1$ This is correct.

So, $2x^2+3x+1=(2x+1)(x+1)$

b $3x^2-14x+8$

Start by writing $3x^2-14x+8=(3x\quad)(x\quad)$.

The two unknown terms must multiply to give 8. Since the constant term is positive, the unknowns must have the same sign. The possible pairs are:

8 and 1 2 and 4 −8 and −1 −2 and −4

Try each pair in turn, remembering that you can reverse the order:

$(3x+8)(x+1)=3x^2+3x+8x+8=3x^2+11x+8$ Incorrect

$(3x+1)(x+8)=3x^2+24x+x+8=3x^2+25x+8$ Incorrect

$(3x+2)(x+4)=3x^2+12x+2x+8=3x^2+14x+8$ Incorrect

This last one is very close. You just need to change the sign of the 'x' term. This can be done by jumping to the last pair: −2 and −4.

$(3x-2)(x-4)=3x^2-12x-2x+8=3x^2-14x+8$ Correct

So, $3x^2-14x+8=(3x-2)(x-4)$

c $10x^2+11x-8$

This question is rather more difficult because there is more than one way to multiply two expressions to get $10x^2$:

$2x$ and $5x$ or $10x$ and x.
Each possibility needs to be tried.

Start with $10x^2+11x-8=(10x\quad)(x\quad)$.

Factor pairs that multiply to give −8 are:

−8 and 1 8 and −1 2 and −4 −2 and 4

Remember that you will need to try each pair with the two values in either order.
For this particular quadratic you will find that none of the eight possible combinations works!

Instead, you must now try: $10x^2+11x-8=(5x\quad)(2x\quad)$.
Trying the above set of pairs once again you will eventually find that:

$10x^2+11x-8=(5x+8)(2x-1)$

This process does appear to be long but with practice you will find ways of making the process faster. After the following exercise is another worked example. This shows an alternative, systematic method of solving these more complex quadratics, but you should try to get a feel for the problems before you use it.

E

Exercise 14.10

1 Factorise each of the following expressions.

a $3x^2+14x+8$	**b** $2x^2+x-3$	**c** $6x^2+x-2$
d $3x^2+14x+16$	**e** $2x^2-x-10$	**f** $16x^2+32x-9$
g $3x^2+16x+5$	**h** $8x^2+2x-1$	**i** $2x^2-x-6$
j $2x^2+9x+9$	**k** $3x^2+2x-16$	**l** $10x^2-x-3$
m $5x^2+6x+1$	**n** $2x^2-19x+9$	**o** $12x^2+8x-15$

Here is another method for factorising a quadratic like those in the previous exercise.

Worked example 21

Factorise $10x^2+11x-8$.

$10\times-8=-80$	Multiply the coefficient of x^2 by the constant term.
$-1, 80$ (no) $-2, 40$ (no) $-4, 20$ (no) $-5, 16$ (yes)	List the factor pairs of -80 until you obtain a pair that totals the coefficient of x (11) (note as 11 is positive and -80 is negative, the larger number of the pair must be positive and the other negative).
$10x^2-5x+16x-8$	Re-write the x term using this factor pair.
$5x(2x-1)+8(2x-1)$	Factorise pairs of terms. (Be careful with signs here so that the second bracket is the same as the first bracket.)
$(5x+8)(2x-1)$	Factorise, using the bracket as the common term.

Exercise 14.11

1 Now go back to Exercise 14.10 and try to factorise the expressions using this new method.

2 Factorise completely. You may need to remove a common factor before factorising the trinomials.

A trinomial is an algebraic expression that contains three terms: an x^2 term, an x term and a constant term.

a $6x^2-5x-21$	**b** $-2x^2-13x-15$	**c** $4x^2+12xy+9y^2$
d $6x^2-19xy-7y^2$	**e** x^4-13x^2+36	**f** $6x^2-38xy+40y^2$
g $6x^2+7x+2$	**h** $3x^2-13x+12$	**i** $3x^2-39x+120$
j $(x+1)^2-5(x+1)+6$	**k** $(2x+1)^2-8(2x+1)+15$	**l** $3(2x+5)^2-17(2x+5)+10$

14.8 Algebraic fractions

You will now use several of the techniques covered so far in this chapter to simplify complex algebraic fractions.

You already know that you can simplify fractions by dividing the numerator and denominator by a common factor. This can also be done with algebraic fractions.

E

Worked example 22

Simplify each of the following fractions as far as possible:

a $\frac{3x}{6}$ **b** $\frac{y^2}{y^5}$ **c** $\frac{12p^3}{16p^7}$ **d** $\frac{x^2-4x+3}{x^2-7x+12}$

a	$\frac{3x}{6}$ $\frac{3x}{6} = \frac{3x \div 3}{6 \div 3} = \frac{x}{2}$	The highest common factor of 3 and 6 is 3.
b	$\frac{y^2}{y^5}$ $\frac{y^2}{y^5} = \frac{y^2 \div y^2}{y^5 \div y^2} = \frac{1}{y^3}$	The highest common factor of y^2 and y^5 is y^2.
c	$\frac{12p^3}{16p^7}$ $\frac{12p^3}{16p^7} = \frac{3p^3}{4p^7}$ $\frac{12p^3}{16p^7} = \frac{3p^3}{4p^7} = \frac{3}{4p^4}$	Consider the constants first. The HCF of 12 and 16 is 4, so you can divide both 12 and 16 by 4. Now note that the HCF of p^3 and p^7 is p^3. You can divide both the numerator and the denominator by this HCF.
d	$\frac{x^2-4x+3}{x^2-7x+12}$ $\frac{x^2-4x+3}{x^2-7x+12} = \frac{(x-3)(x-1)}{(x-3)(x-4)}$ $= \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x-4)}$ $= \frac{(x-1)}{(x-4)}$	Notice that you can factorise both the numerator and the denominator. You can see that $(x-3)$ is a factor of both the numerator and the denominator, so you can cancel this common factor.

REWIND

You might need to recap the laws of indices that you learned in chapter 2. ◀

Exercise 14.12

Simplify each of the following fractions by dividing both the numerator and the denominator by their HCF.

1 **a** $\frac{2x}{4}$ **b** $\frac{3y}{12}$ **c** $\frac{5x}{x}$ **d** $\frac{10y}{y}$ **e** $\frac{6t}{36}$

f $\frac{9u}{27}$ **g** $\frac{5t}{50}$ **h** $\frac{4y}{8}$ **i** $\frac{15z}{20}$ **j** $\frac{16t}{12}$

2 **a** $\frac{5xy}{15}$ **b** $\frac{3x}{12y}$ **c** $\frac{17ab}{34ab}$ **d** $\frac{9xy}{18x}$ **e** $\frac{25x^2}{5x}$

f $\frac{21b^2}{7b}$ **g** $\frac{14x^2}{21xy}$ **h** $\frac{12ab^2}{4ab}$ **i** $\frac{20de}{30d^2e^2}$ **j** $\frac{5a}{20ab^2}$

E

3 a $\frac{7a^2b^2}{35ab^3}$ b $\frac{(ab)^2}{ab}$ c $\frac{18abc}{36ac}$ d $\frac{13a^2bc}{52ab}$ e $\frac{12a^2b^2c^2}{24abc}$

f $\frac{36(ab)^2c}{16a^2bc^2}$ g $\frac{(abc)^3}{abc}$ h $\frac{9x^2y^3}{12x^3y^2}$ i $\frac{20x^3y^2z^2}{15xy^3z}$ j $\frac{(3y)^3}{3y^3}$

4 a $\frac{18(xy)^2z^3}{17(xyz^3)^2}$ b $\frac{334x^4y^7z^3}{668xy^8z^2}$ c $\frac{249u(vw)^3}{581u^3v^2w^7}$ d $\frac{x^2+3x}{x^2+4x}$ e $\frac{x^2+3x}{x^2+7x+12}$

f $\frac{y^3+y^4}{y^2+2y+1}$ g $\frac{x^2-8x+12}{x^2-6x+8}$ h $\frac{x^2+9x+20}{x^2+x-12}$ i $\frac{24x^2+8x}{3x^2+x}$ j $\frac{3x^2-10x-8}{3x^2-14x+8}$

k $\frac{x^2-9}{x^2+5x-24}$ l $\frac{2x^2-x-3}{x^2+2x+1}$ m $\frac{7x^2-29x+4}{x^2-8x+16}$ n $\frac{10y^2-3y-4}{2y^2-13y-7}$ o $\frac{6x^2-11x-7}{10x^2-3x-4}$

5 a $\frac{6x^2-35x+36}{14x^2-61x-9}$ b $\frac{(x^2)^2-(y^2)^2}{(x-y)(x+y)}$ c $\frac{\sqrt{x}}{(\sqrt{x})^3}$

d $\frac{x^4+2x^2+1}{x^2+1}$ e $\frac{(x^2+7x+12)(x^2+8x+12)}{(x^2+9x+18)(x^2+6x+8)}$ f $\frac{\left(\sqrt{x^3+y^3}\right)^3}{x^3+y^3}$

Multiplying and dividing algebraic fractions

You can use the ideas explored in the previous section when multiplying or dividing algebraic fractions. Consider the following multiplication: $\frac{x}{y^2}\times\frac{y^4}{x^3}$

You already know that the numerators and denominators can be multiplied in the usual way: $\frac{x}{y^2}\times\frac{y^4}{x^3}=\frac{xy^4}{y^2x^3}$

Now you can see that the HCF of the numerator and denominator will be xy^2. If you divide through by xy^2 you get: $\frac{x}{y^2}\times\frac{y^4}{x^3}=\frac{y^2}{x^2}$

The following worked examples will help you to understand the process for slightly more complicated multiplications and divisions.

Worked example 23

Simplify each of the following.

a $\frac{4}{3x^2}\times\frac{14x^3}{16y^2}$ **b** $\frac{3(x+y)^3}{16z^2}\times\frac{12z}{9(x+y)^7}$ **c** $\frac{14x^4y^3}{9}\div\frac{7x^2y}{18}$

a

$$\frac{4}{3x^2}\times\frac{14x^3}{16y^2}$$

$$\frac{4}{3x^2}\times\frac{14x^3}{16y^2}=\frac{4\times14x^3}{3x^2\times16y^2}=\frac{56x^3}{48x^2y^2}=\frac{7x}{6y^2}$$

You can simply multiply numerators and denominators and then simplify using the methods in the previous section.

b

$$\frac{3(x+y)^3}{16z^2}\times\frac{12z}{9(x+y)^7}$$

$$\frac{3(x+y)^3}{16z^2}\times\frac{12z}{9(x+y)^7}=\frac{36(x+y)^3z}{144(x+y)^7z^2}=\frac{1}{4(x+y)^4z}$$

E

c $\dfrac{14x^4y^3}{9} \div \dfrac{7x^2y}{18}$

$$\frac{14x^4y^3}{9} \div \frac{7x^2y}{18} = \frac{14x^4y^3}{9} \times \frac{18}{7x^2y} = \frac{14x^4y^3 \times 18}{9 \times 7x^2y} = 4x^2y^2$$

Exercise 14.13

Write each of the following as a single fraction in its lowest terms.

1 **a** $\dfrac{2x}{3} \times \dfrac{3x}{8}$ **b** $\dfrac{3y}{4} \times \dfrac{2y}{7}$ **c** $\dfrac{2z}{7} \times \dfrac{3z}{4}$ **d** $\dfrac{5t}{9} \times \dfrac{9t}{15}$

e $\dfrac{2x^2}{5} \times \dfrac{5}{2x^2}$ **f** $\dfrac{7x^2}{12} \times \dfrac{4}{14x^2}$ **g** $\dfrac{12e^2}{11f} \times \dfrac{33f^2}{24e^3}$ **h** $\dfrac{18g^4}{16h^2} \times \dfrac{h^4}{36g^3}$

i $\dfrac{3y}{4} \div \dfrac{3y}{8}$ **j** $\dfrac{3y}{8} \div \dfrac{3y^3}{4}$ **k** $\dfrac{4cd}{7} \div \dfrac{16c^2}{8}$ **l** $\dfrac{8pq}{r} \div \dfrac{16p^2q^2}{r^2}$

2 **a** $\dfrac{24zt^3}{x^2} \div \dfrac{8xt}{z}$ **b** $\dfrac{8}{12} \times \dfrac{x^3}{t^2} \times \dfrac{t^3}{x^2}$ **c** $\dfrac{9}{27} \times \dfrac{3x^2}{12y^3} \times \dfrac{81}{27} \times \dfrac{9y^2}{3x^3}$

d $\left(\dfrac{3}{8} \times \dfrac{64t^3y^2}{27t}\right) \div \left(\dfrac{3}{8} \times \dfrac{y^2}{t^3} \times \dfrac{t}{y^4}\right)$ **e** $\dfrac{(x+y)^2}{(x-y)^3} \times \dfrac{33(x-y)^2}{44(x+y)^7}$

f $\dfrac{3(a+b)(a-b)}{(a+b)^2} \div \dfrac{12(a-b)^2}{(a+b)}$ **g** $\dfrac{3\sqrt{x^2+y^2}}{24\sqrt{z^2+t^2}} \times \dfrac{(z^2+t^2)^2}{18\left(\sqrt{x^2+y^2}\right)^3}$

h $\dfrac{3(x+y)^{10}}{18(z-t)^{19}} \times \dfrac{10(x+y)(z-t)^4}{12(x+y)^3(z-y)} \times \dfrac{108(x+y)^2(z-t)^{20}}{15(z-t)^4(x+y)^{10}}$

Adding and subtracting algebraic fractions

You can use *common denominators* when adding together algebraic fractions, just as you do with ordinary fractions.

Worked example 24

Write as a single fraction in its lowest terms, $\dfrac{1}{x} + \dfrac{1}{y}$.

$\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{y}{xy} + \dfrac{x}{xy} = \dfrac{y+x}{xy}$	The lowest common multiple of x and y is xy. This will be the common denominator.

E

Worked example 25

Write as a single fraction in its lowest terms, $\frac{1}{x+1}+\frac{1}{x+2}$.

The lowest common multiple of $(x + 1)$ and $(x + 2)$ is $(x + 1)(x + 2)$

$$\begin{aligned}\frac{1}{x+1}+\frac{1}{x+2} &= \frac{x+2}{(x+1)(x+2)}+\frac{x+1}{(x+1)(x+2)}\\ &= \frac{x+2+x+1}{(x+1)(x+2)}\\ &= \frac{2x+3}{(x+1)(x+2)}\end{aligned}$$

Worked example 26

Write as a single fraction in its lowest terms, $\frac{3x+4}{x^2+x-6}-\frac{1}{x+3}$.

First you should factorise the quadratic expression:

$$\frac{3x+4}{x^2+x-6}-\frac{1}{x+3}=\frac{3x+4}{(x+3)(x-2)}-\frac{1}{(x+3)}$$

The two denominators have a common factor of $(x + 3)$, and the lowest common multiple of these two denominators is $(x + 3)(x - 2)$:

$$\begin{aligned}\frac{3x+4}{x^2+x-6}-\frac{1}{x+3} &= \frac{3x+4}{(x+3)(x-2)}-\frac{1}{(x+3)}\\ &= \frac{3x+4}{(x+3)(x-2)}-\frac{(x-2)}{(x+3)(x-2)}\\ &= \frac{3x+4-(x-2)}{(x+3)(x-2)}\\ &= \frac{3x+4-x+2}{(x+3)(x-2)}\\ &= \frac{2x+6}{(x+3)(x-2)}\end{aligned}$$

This may appear to be the final answer but if you factorise the numerator you will find that more can be done!

$$\begin{aligned}\frac{3x+4}{x^2+x-6}-\frac{1}{x+3} &= \frac{2x+6}{(x+3)(x-2)}\\ &= \frac{2(x+3)}{(x+3)(x-2)}\\ &= \frac{2}{(x-2)}\end{aligned}$$

Always check to see if your final numerator factorises. If it does, then there may be more stages to go.

Exercise 14.14

E

Write each of the following as a single fraction in its lowest terms.

1 **a** $\frac{y}{2}+\frac{y}{4}$ **b** $\frac{t}{3}+\frac{t}{5}$ **c** $\frac{u}{7}+\frac{u}{5}$ **d** $\frac{z}{7}-\frac{z}{14}$ **e** $\frac{(x+y)}{3}+\frac{(x+y)}{12}$

f $\frac{2x}{3}+\frac{5x}{6}$ **g** $\frac{3y}{4}+\frac{5y}{8}$ **h** $\frac{2a}{5}-\frac{3a}{8}$ **i** $\frac{2a}{7}+\frac{3a}{14}$ **j** $\frac{x}{9}+\frac{2y}{7}$

2 **a** $\frac{5(x+1)^2}{7}-\frac{3(x+1)^2}{8}$ **b** $\frac{10pqr}{17}-\frac{3pqr}{8}$ **c** $\frac{3p}{5}+\frac{3p}{7}+\frac{3p}{10}$

d $\frac{2x}{3}+\frac{3x}{7}-\frac{x}{4}$ **e** $\frac{8x^2}{9}+\frac{3x^2}{7}-\frac{x^2}{3}$ **f** $\frac{5-x}{2}-\frac{3-x}{3}+\frac{3-x}{9}$

3 **a** $\frac{x}{a}+\frac{3}{a}$ **b** $\frac{2}{3a}+\frac{5}{4a}$ **c** $\frac{3x}{2y}+\frac{5x}{3y}$

d $\frac{3}{a}+\frac{2}{a^2}$ **e** $\frac{3}{2x}+\frac{4}{3x}$ **f** $\frac{5}{4e}+\frac{3}{20e}$

4 **a** $\frac{1}{x+1}+\frac{1}{x+4}$ **b** $\frac{3}{x-2}+\frac{2}{x-1}$ **c** $\frac{5}{x+2}+\frac{2}{x+7}$

d $\frac{3}{x}-\frac{1}{2x}$ **e** $\frac{5}{2xy}-\frac{4}{3xy}$ **f** $\frac{2}{x}+x$

g $\frac{x+1}{2}+\frac{2}{x+1}$ **h** $\frac{3(x^2-1)}{7y}-\frac{2(x^2-1)}{9y^2}$ **i** $\frac{1}{x^2}-\frac{x}{2y}$

j $\frac{x+1}{3z^2}-\frac{y+z}{12xy}$ **k** $\frac{1}{(x+2)}-\frac{1}{(x+3)(x+2)}$ **l** $\frac{2}{x+1}-\frac{2}{x^2+3x+2}$

Summary

Do you know the following?

- Simultaneous means at the same time.
- The intersection of two straight lines is the simultaneous solution of their equations.
- Simultaneous linear equations can be solved graphically or algebraically.

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- Inequalities represent a range of solutions.
- Inequalities in one variable can be represented on a number line and in two variables as a region on a plane.
- A quadratic expression, x^2+bx+c can be written in the completed square form, $\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c$.
- Quadratic equations that do not factorise can be solved by the method of completing the square or by use of the quadratic formula.
- Complex algebraic fractions can be simplified by factorising and cancelling like terms.

Are you able to …?

- solve simultaneous linear equations graphically
- solve simultaneous linear equations algebraically

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- show an inequality in one variable on a number line
- show an inequality in two variables as a region in the Cartesian plane
- show a region in the Cartesian plane that satisfies more than one inequality
- use linear programming to find the great and least values to an expression in a region
- rewrite a quadratic in completed square form
- solve a quadratic using the completed square or the quadratic formula
- simplify complex algebraic fractions.

Examination practice

Exam-style questions

1 The quadratic equation $x^2 - 5x - 3 = 0$ has solutions a and b. Find the value of:

i $a - b$
ii $a + b$
Leave your answers in exact form.

2 **a** By shading the unwanted regions on a diagram, show the region that satisfies all the inequalities $y \geqslant \frac{1}{2}x + 1$, $5x + 6y \leqslant 30$ and $y \leqslant x$.
b Given that x and y satisfy these three inequalities, find the greatest possible value of $x + 2y$.

Past paper questions

1 Solve the inequality $\frac{x}{3} + 5 > 2$. [2]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q8 May/June 2016]

2 Find the co-ordinates of the point of intersection of the two lines.
$2x - 7y = 2$
$4x + 5y = 42$ [3]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q15 October/November 2013]

3 x is a positive integer and $15x - 43 < 5x + 2$.
Work out the possible values of x. [3]

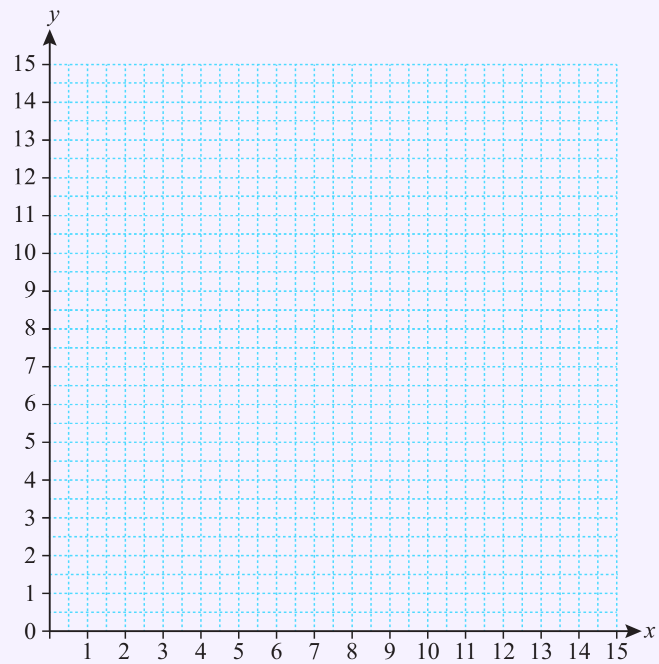
[Cambridge IGCSE Mathematics 0580 Paper 22 Q6 May/June 2012]

4 Write the following as a single fraction in its simplest form. [3]

$$\frac{x+2}{3} - \frac{2x-1}{4} + 1$$

[Cambridge IGCSE Mathematics 0580 Paper 23 Q13 October/November 2012]

5 Jay makes wooden boxes in two sizes. He makes x small boxes and y large boxes.
He makes at least 5 **small** boxes.
The greatest number of **large** boxes he can make is 8.
The greatest total number of boxes is 14.
The number of **large** boxes is at least half the number of **small** boxes.
a **i** Write down four inequalities in x and y to show this information. [4]
ii Draw four lines on the grid and write the letter R in the region which represents these inequalities.



[5]

b The price of the small box is $20 and the price of the large box is $45.

i What is the greatest amount of money he receives when he sells all the boxes he has made? [2]

ii For this amount of money, how many boxes of each size did he make? [1]

[Cambridge IGCSE Mathematics 0580 Paper 42 Q7 October/November 2012]

6 Simplify the following. $\dfrac{h^2-h-20}{h^2-25}$ [4]

[Cambridge IGCSE Mathematics 0580 Paper 23 Q21 October/November 2012]

7 Simplify.

$$\frac{x^2+6x-7}{3x+21}$$ [4]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q19 May/June 2014]

8 **a** Write as a single fraction in its simplest form.

$$\frac{3}{2x-1}-\frac{1}{x+2}$$ [3]

b Simplify.

$$\frac{4x^2-16x}{2x^2+6x-56}$$ [4]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q21 October/Nvormber 2014]

9 Solve the equation $5x^2 - 6x - 3 = 0$.

Show all your working and give your answers correct to 2 decimal places. [4]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q19 October/November 2015]

10 $y = x^2 + 7x - 5$ can be written in the form $y = (x + a)^2 + b$.

Find the value of a and the value of b. [3]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q15 May/June 2016]

11 Solve the simmultaneous equations.

Show all your working.

$$3x + 4y = 14$$
$$5x + 2y = 21$$

[3]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q16 May/June 2016]