Chapter 23: Vectors and transformations

Key words

- Transformation
- Object
- Reflection
- Rotation
- Translation
- Enlargement
- Image
- Vector
- Magnitude
- Scalar

In this chapter you will learn how to:

- reflect, rotate, translate and enlarge plane shapes
- recognise and describe transformations
- use vectors to describe translations
- add and subtract vectors and multiply them by scalars

EXTENDED

- calculate the magnitude of a vector
- represent vectors in conventional ways
- use the sum and difference of vectors to express them in terms of coplanar vectors
- use position vectors
- recognise and use combined transformations
- precisely describe transformations using co-ordinates

This is a batik printed fabric from Ghana. The fabric designers repeat shapes by moving them and turning them in a regular ways. In Mathematics we call this a transformation.

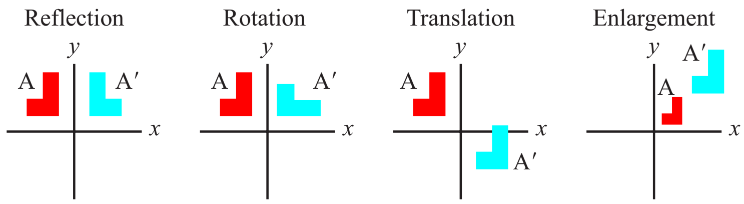
Transformation geometry deals with moving or changing shapes in set ways. You are going to revise what you know about transformations, use vectors and work with more precise mathematical descriptions of transformations.

RECAP

You should already be familiar with these ideas from work on transformations and using column vectors.

Transformations (Year 9 Mathematics)

When carrying out transformations we start by talking about an object (for example A), and after it has undergone a transformation we call it an image (A').

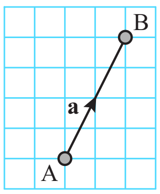


Reflection, rotation and translation move shapes but don't change them.

Enlargement changes a shape and produces a similar shape.

Column vectors (Year 9 Mathematics)

You can describe movement using a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ the top number (x) represents horizontal movement and the bottom number (y) represents vertical movement.



	Direction of movement	
	x	***y***
If number is positive	right	Up
If number is negative	left	Down

23.1 Simple plane transformations

Transformation means change. In Mathematics, a transformation is a change in the position or size of an **object** (or point). In this section you will deal with four types of transformations:

- **Reflection** (a flip or mirror image)
- **Rotation** (a turn)
- **Translation** (a slide movement)
- **Enlargement** (making the object larger or smaller).

A transformation produces an **image** of the original object in a new position or at a different size. A point, P, on the object is labelled as P′ on the image.

REWIND

You learned about congruency and similarity in chapter 11. ◀

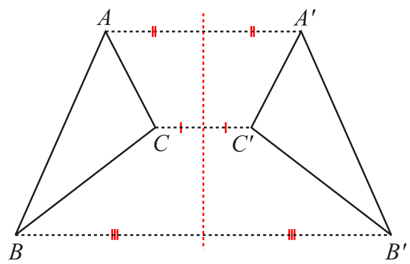
Reflections, rotations and translations change the position of an object, but they do not change its size. So, the object and its image are congruent. If you place the object and its image on top of each other, they coincide exactly.

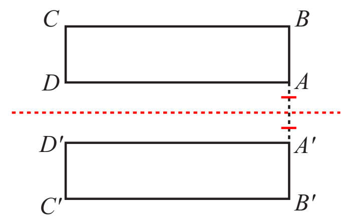
When you enlarge an object, you change its size. The object and its image are similar. In other words, the lengths of corresponding sides on the image are in the same proportion as on the object.

Reflection

A reflection is a mirror image of the shape. The line of reflection is called the mirror line. Corresponding points on the object and the image are the same distance from the mirror line. These distances are always measured perpendicular to the mirror line. (In other words, the mirror line is the perpendicular bisector of the distance between any point and its image.) You can see this on the following diagrams.

Note that the mirror line tends to be drawn as a dashed line.



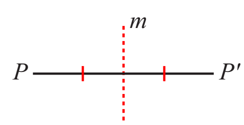


To fully define a reflection, you need to give the equation of the mirror line.

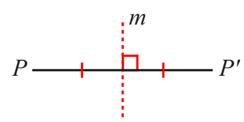
You need to be able to work with reflections in horizontal and vertical lines only.

Properties of reflection

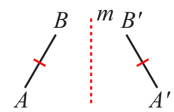
- A point and its image are equidistant from the mirror line (m) after reflection about the line m.



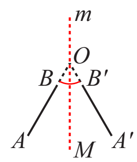
- The mirror line bisects the line joining a point and its image at right angles.



- A line segment and its image are equal in length. $AB = A'B'$.

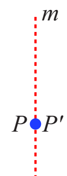


- A line and its image are equally inclined to the mirror line. $A\hat{O}M = A'\hat{O}M$.

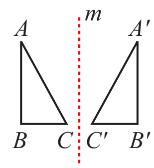


- Points on the mirror line are their own images and are invariant.

Invariant means a point, or a line, remains unchanged in its position and size.



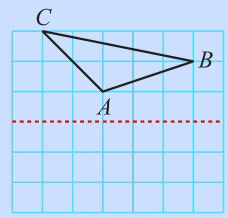
- Under reflection, a figure and its image are congruent.

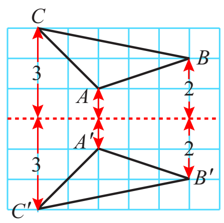


When the mirror line is one of the grid lines this makes it easy to reflect any point. You simply count the squares from the point to the mirror line and the reflection is the same distance the other side of the mirror line.

Worked example 1

Reflect ΔABC about the mirror line.





In the diagram, A is 1 unit from the mirror line, so its image A' is also 1 unit from the mirror line. Point B is 2 units from the mirror line, so its image B' is also 2 units from the mirror line. This is also true for C and its image C'.

The reflection of a straight line is a straight line. So, to obtain the reflection of ΔABC, join A' to B', B' to C' and C' to A'.

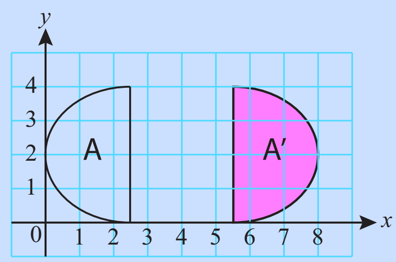
The mirror line is the perpendicular bisector of the line joining any point and its image.

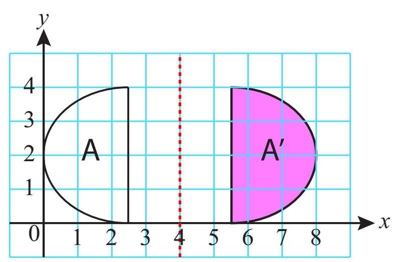
Worked example 2

A shape and its reflection are shown on the grid.

a Draw the mirror line.

b What is the equation of the mirror line?





a The mirror line must be the same distance from corresponding points on A and A′.

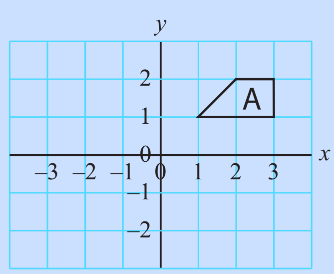
b The mirror line is parallel to the y-axis. The x values of any point on it is 4, so the equation of the line is $x = 4$.

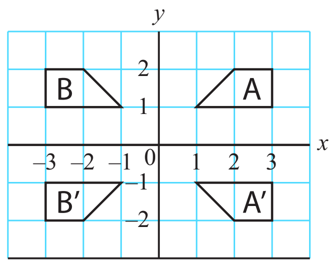
Worked example 3

Shape A is the object.

a Reflect shape A in the y-axis. Label the image B.

b Reflect shape A and shape B in the x-axis. Label the images A′ and B′ respectively.





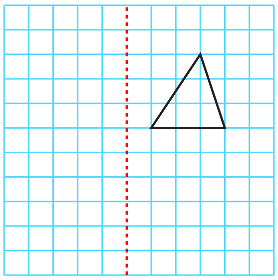
a The y-axis ($x = 0$) is the mirror line.

b The x-axis ($y = 0$) is the mirror line.

Exercise 23.1

1 Copy the shapes and the mirror lines onto squared paper.
Draw the image of each object.

a



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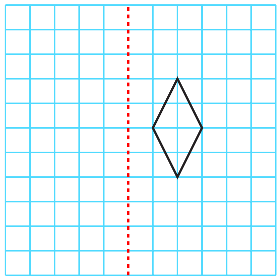

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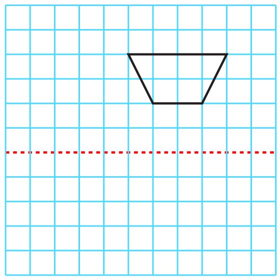


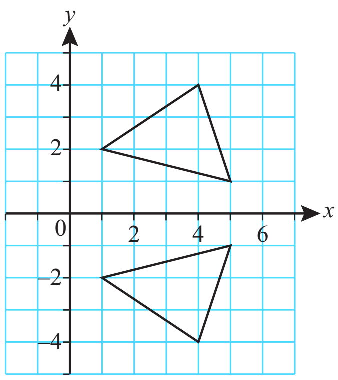
2 Copy the axes and shapes onto squared paper.

For each diagram:

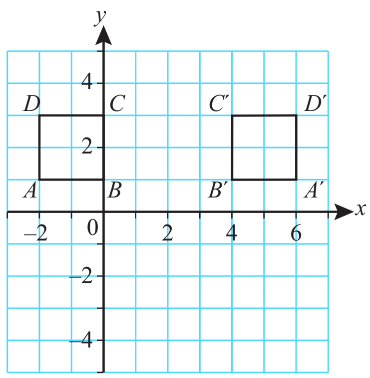
i draw in the mirror line

ii give the equation of the mirror line.

a



b



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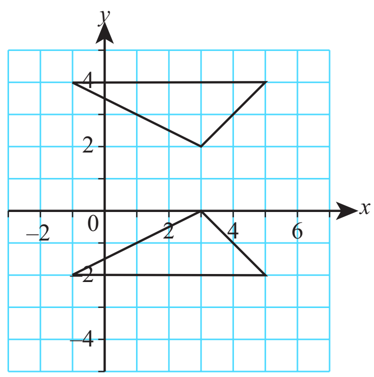
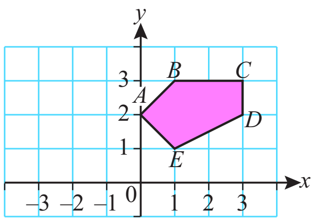


3 Copy the axes and the shape onto squared paper.



a Reflect polygon $ABCDE$ in the y-axis.

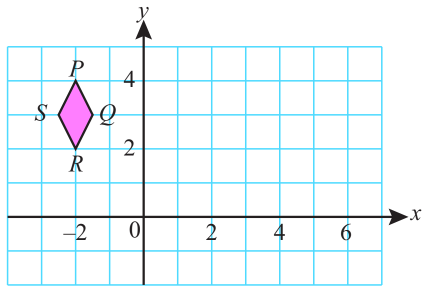
b Give the co-ordinates of point B after reflection (B').

c Which point on the shape $ABCDE$ is invariant? Why?

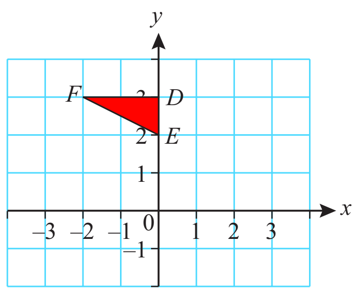
4 Copy the axes and the shape onto squared paper.

a Reflect the shape in the line $x = 1$. Label the image $P'Q'R'S'$.

b Reflect $P'Q'R'S'$ in the line $y = 2$. Label the image $P''Q''R''S''$.



5 Copy the axes and the diagram onto squared paper.



a Draw the image of ΔDEF when reflected in the y-axis. Label it $D'E'F'$.

b Give the co-ordinates of point F before and after reflection.

c Reflect ΔDEF in the line $y = 1$. Label the image $D''E''F''$.

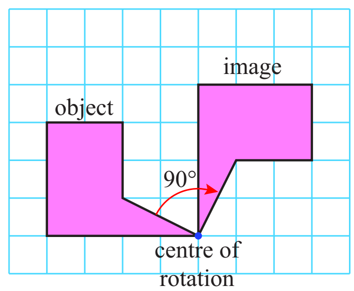
Rotation

REWIND

You dealt with rotation when you studied rotational symmetry in chapter 19. ◀

A rotation is a turn around a fixed point. Rotation occurs when an object is turned around a given point. Rotation can be clockwise or anti-clockwise. The fixed point is called the centre of rotation and the angle through which the shape is rotated is called the angle of rotation.

In this diagram, the object has been rotated 90° clockwise about the centre of rotation (a vertex of the object).

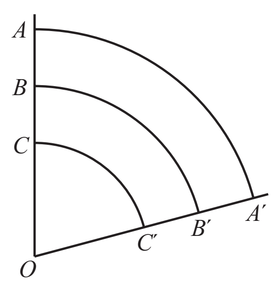


Properties of rotation

- A rotation through 180° is a half turn; a rotation through 90° is a quarter turn.
- Anti-clockwise rotation is positive and clockwise rotation is negative.
- A point and its image are equidistant from the centre of rotation.
- Each point of an object moves along the arc of a circle whose centre is the centre of rotation. All the circles are concentric:

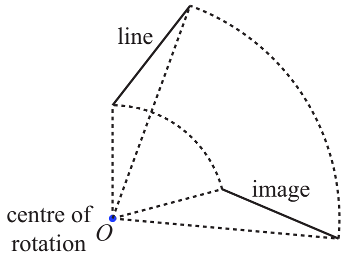
REWIND

You learned in chapter 19 that concentric circles have differ radii but the same centre. ◀

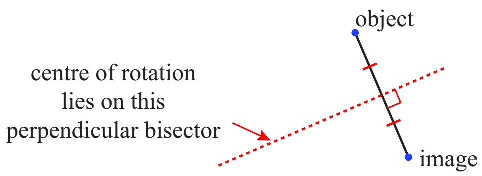


- Only the centre of rotation is invariant.

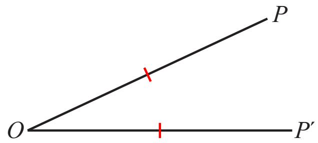
- The object and the image are congruent after rotation.



- The perpendicular bisector of a line joining a point and its image passes through the centre of rotation.



- A line segment and its image are equal in length.



Tip

The centre of rotation will normally be the origin (0, 0), a vertex of the shape or the midpoint of a side of the shape. The amount of turn will normally be a multiple of 90°.

To describe a rotation you need to give:

- the centre of rotation
- the amount of turn (90°, 180° or 270°)
- the direction of the turn (clockwise or anti-clockwise).

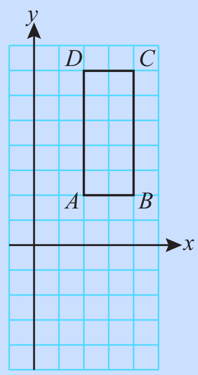
You can use thin paper or tracing paper to help you do rotations:

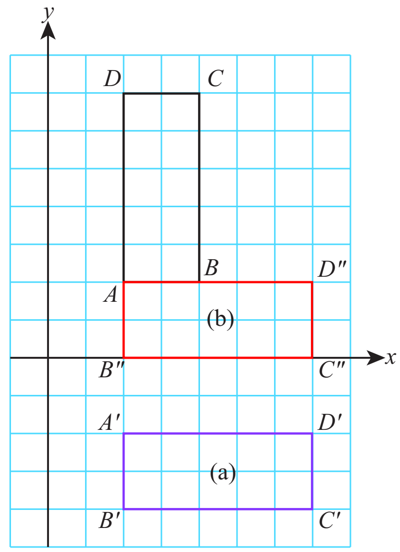
1. Trace the shape and label the vertices.
2. Place the tracing over the object.
3. Use the point of a pair of compasses or pen to hold the paper at the point of rotation.
4. Turn the paper through the given turn.
5. The new position of the shape is the image.

Worked example 4

Rotate this shape 90° clockwise about:

a the origin (label the image *A′B′C′D′*)
b point *A* (label the image *A″B″C″D″*).

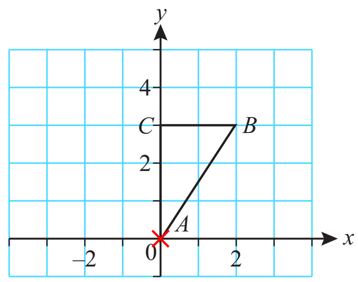




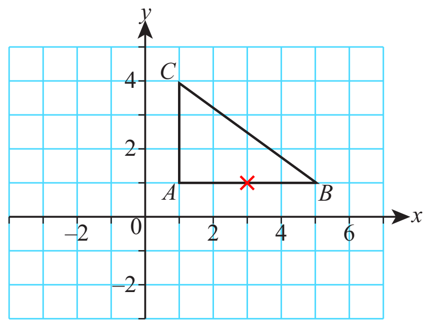
Exercise 23.2

1 Copy the diagrams in parts (**a**) to (**c**).
Draw the images of the given triangle under the rotations described.

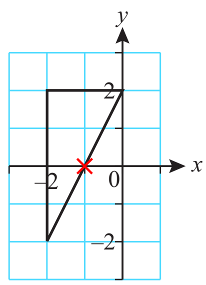
a Centre of rotation (0, 0); angle of rotation 90° anti-clockwise.



b Centre of rotation (3, 1); angle of rotation 180°.
(Note that (3, 1) is the midpoint of *AB*.)

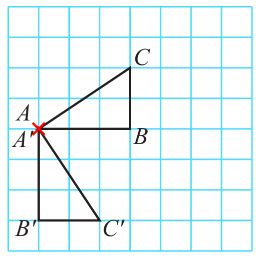


c Centre of rotation (–1, 0); angle of rotation 180°.



2 Fully describe the rotation that maps ΔABC onto $\Delta A'B'C'$ in each case.

a



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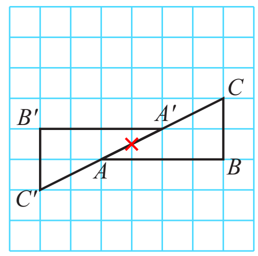

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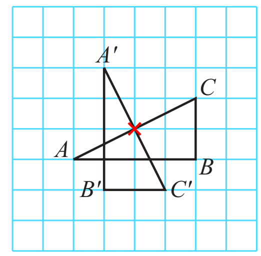
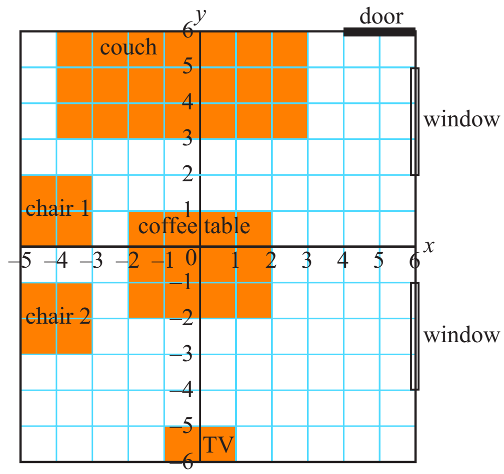


Applying your skills

3 Nick wants to rearrange the furniture in his living room. He drew this scale diagram showing the original layout of the room.

Is it possible for him to rotate all the furniture through point (0, 0) by the following amounts and still have it fit into the room?

a 90° clockwise?
b 90° anti-clockwise?
c 180° clockwise?



Translation

A translation, or slide, is the movement of an object over a specified distance along a line. The object is not twisted or turned. The movement is indicated by positive or negative signs according to the direction of movement along the axes of a plane. For example, movements to the left or down are negative and movements to the right or upwards are positive.

FAST FORWARD

You will deal with vectors in more detail later in this chapter.

A translation should be described by a **column vector**: $\begin{pmatrix} x \\ y \end{pmatrix}$. This means a movement of x units in the x-direction (left or right) and a movement of y units in the y-direction (up or down).

In other words, a translation of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ means the object moves two units to the right and three units downwards.

Be careful when writing column vectors. There is no dividing line, so they should *not* look like fractions. Write $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$ rather than $\left(\frac{3}{8}\right)$; they mean different things.

In this diagram, the triangle T is translated to five positions. Each translation is described below:

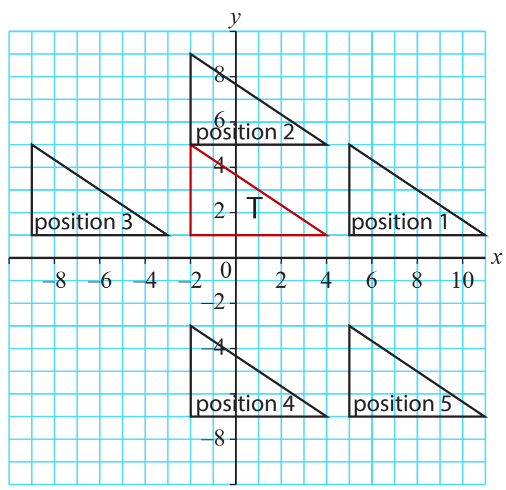
Position 1 $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$

Position 2 $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

Position 3 $\begin{pmatrix} -7 \\ 0 \end{pmatrix}$

Position 4 $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$

Position 5 $\begin{pmatrix} 7 \\ -8 \end{pmatrix}$



Properties of translation

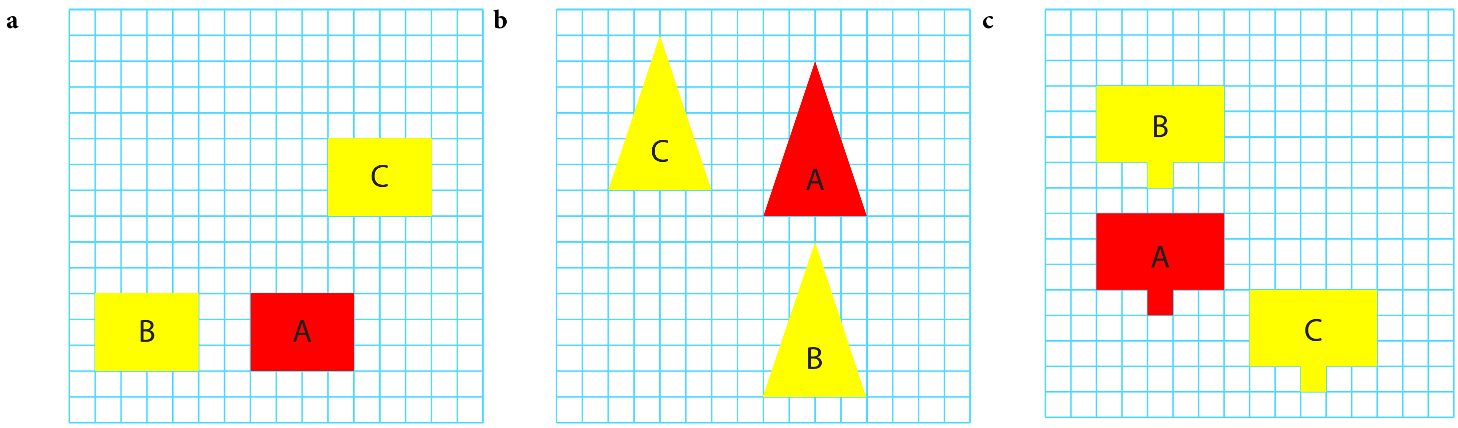
- A translation moves the entire object the same distance in the same direction.
- Every point moves through the same distance in the same direction.
- To specify the translation, both the distance and direction of translation must be given by a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.
- The translation of an entire object can be named by specifying the translation undergone by any one point.
- No part of the figure is invariant.
- The object and the image are congruent.

Exercise 23.3

1 Draw sketches to illustrate the following translations:

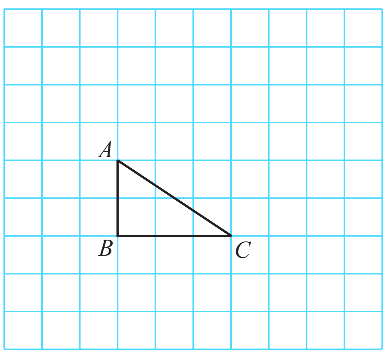
a a square is translated 6 cm to the left
b a triangle is translated 5 cm to the right.

2 Write a column vector to describe the translation from A to B and from A to C in each of the following sets of diagrams.



3 Copy the diagram onto squared paper. Translate the triangle ABC:

a three units to the right and two units down
b three units to the left and two units down
c three units upwards and one unit to the left
d three units downwards and four units right.



4 On squared paper, draw x- and y-axes and mark the points $A(3, 5)$, $B(2, 1)$ and $C(-1, 4)$.

a Draw $\Delta A'B'C'$, the image of ΔABC under the translation $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

b Draw $\Delta A''B''C''$, the image of ABC under the translation $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

5 ΔXYZ with $X(3, 1)$, $Y(2, 6)$ and $Z(-1, -5)$ is transformed onto $\Delta X'Y'Z'$ by a translation of $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$. Determine the co-ordinates of X', Y' and Z'.

6 A rectangle $MNOP$ with vertices $M(1, 6)$, $N(5, 6)$, $O(5, 3)$ and $P(1, 3)$ is transformed by the translation $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ to produce rectangle $M'N'O'P'$.

a Represent this translation accurately on a set of axes.
b Give the co-ordinates of the vertices of the image.

REWIND

'Scale factor' was introduced in chapter 11. It is the multiplier that tells you how much one shape is larger than another.

Enlargement

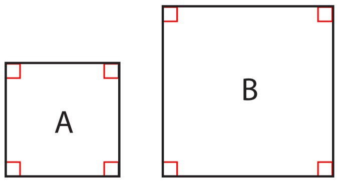
When a shape is enlarged it is made bigger. In an enlargement the lengths of sides on the object are multiplied by a scale factor (k) to form the image. The sizes of angles do not change during an enlargement, so the object and its image are similar. The scale factor can be a whole number or a fraction. To find the scale factor, you use the ratio of corresponding sides on the object and the image:

$$\text{Scale factor} = \frac{\text{image length}}{\text{original length}}$$

If the scale factor is given, you can find the lengths of corresponding sides by multiplication.

This diagram shows a square which has been enlarged by a scale factor of 1.5. This means that

$$\frac{\text{side } B}{\text{side } A} = 1.5$$



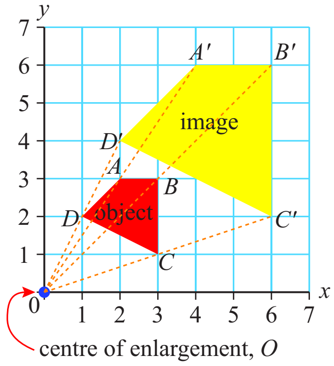
When an object is enlarged from a fixed point, it has a centre of enlargement. The centre of enlargement determines the position of the image. Lines drawn through corresponding points on the object and the image will meet at the centre of enlargement. You can see this on the following diagram.

The scale factor can be determined by comparing any two corresponding sides, for example:

$$\frac{A'B'}{AB} = \frac{2}{1} = 2,$$

or by comparing the distances of two corresponding vertices from the origin, for example:

$$\frac{OC'}{OC} = 2$$

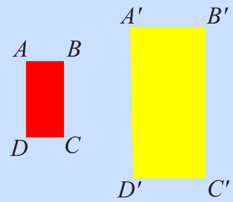


Properties of enlargement

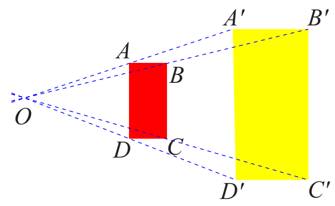
- The centre of enlargement can be anywhere (inside the object, outside the object or on a vertex or line).
- A scale factor greater than 1 enlarges the object whilst a scale factor smaller than 1 (a fraction < 1) reduces the size of the object, although this is sometimes still described as an enlargement.
- An object and its image are similar (not congruent) with sides in the ratio $1 : k$ where k is the scale factor.
- Angles and orientation of the object are invariant.

Worked example 5

The figure shows quadrilateral *ABCD* and its image *A′B′C′D′* under an enlargement. Find the centre of enlargement and the scale factor.



Join the point *A* and its image *A′*.
Extend *AA′* in both directions. Similarly, draw and extend *BB′*, *CC′* and *DD′*.
The point of intersection of these lines is the centre of enlargement, *O*.



$OA = 25\,\text{mm}$
$OA' = 50\,\text{mm}$

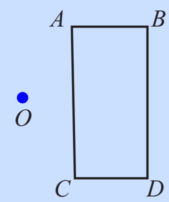
Scale factor $= \dfrac{50}{25} = 2$

Measure *OA* and *OA′*.

The ratio *OA* : *OA′* gives the scale factor.

Worked example 6

Draw the image of rectangle *ABCD* with *O* as the centre of enlargement and a scale factor of two.

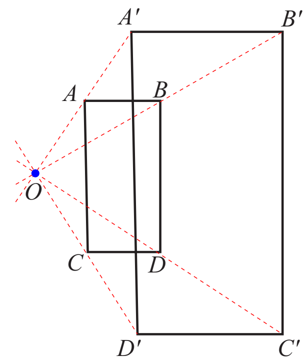


Join *OA*. Continue (produce) the line beyond *A*.
Measure *OA*.

Multiply the length of *OA* by 2.

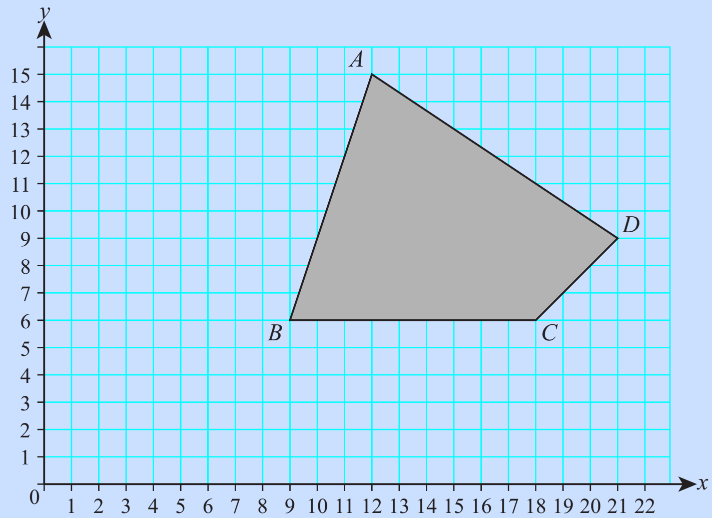
Mark the position of *A′* on the produced line so that *OA′* = 2*OA*.

Repeat for the other vertices.
Join *A′B′C′D′*.



Worked example 7

Draw A′B′C′D′, the image of ABCD under an enlargment of scale factor $\frac{1}{3}$ through the origin.

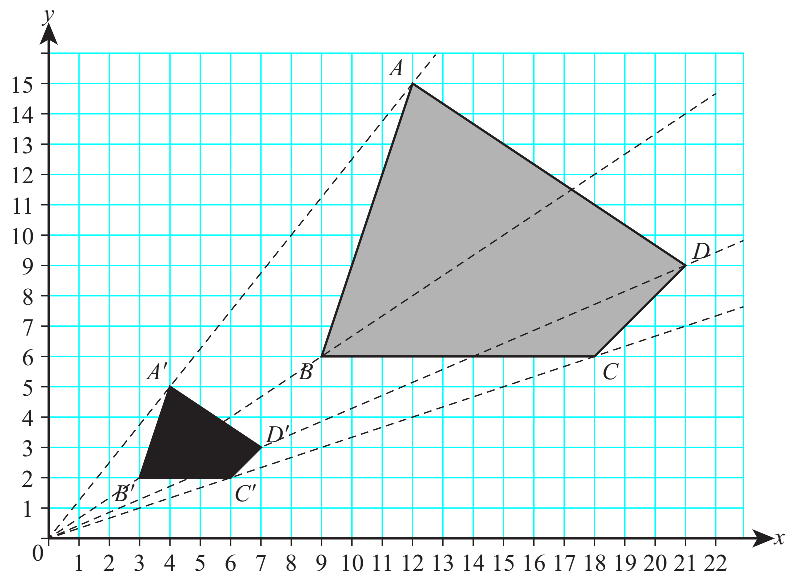


A scale factor of $\frac{1}{3}$ means the image will be smaller than the object.

Determine the coordinates of each vertex on the image. You can do this by multiplying the (x, y) coordinates of the vertices on the object by $\frac{1}{3}$.

A = (12, 15), so A′ = (4, 5)
B = (9, 6), so B′ = (3, 2)
C = (18, 6), so C′ = (6, 2)
D = (21, 9), so D′ = (7, 3)

Plot the point and draw and label the enlargement.

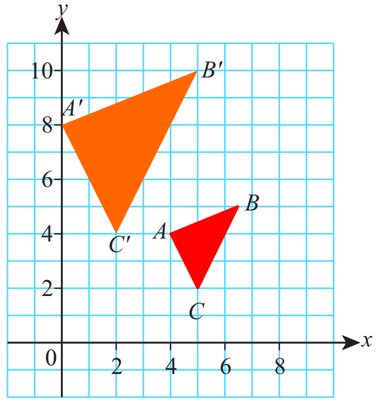


You can also measure the length of the line from the origin to each vertex on the object and divide those lengths by 3 to determine the position of each vertex on the image. This method is useful when the diagram is not on a coordinate grid.

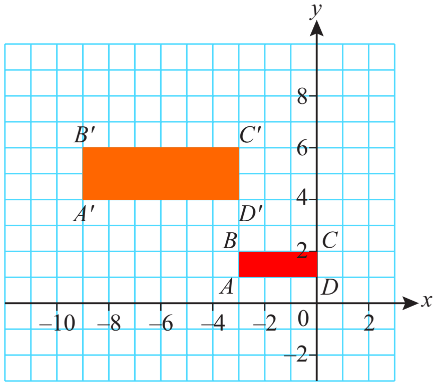
Exercise 23.4

1 For each enlargement, give the co-ordinates of the centre of enlargement and the scale factor of the enlargement.

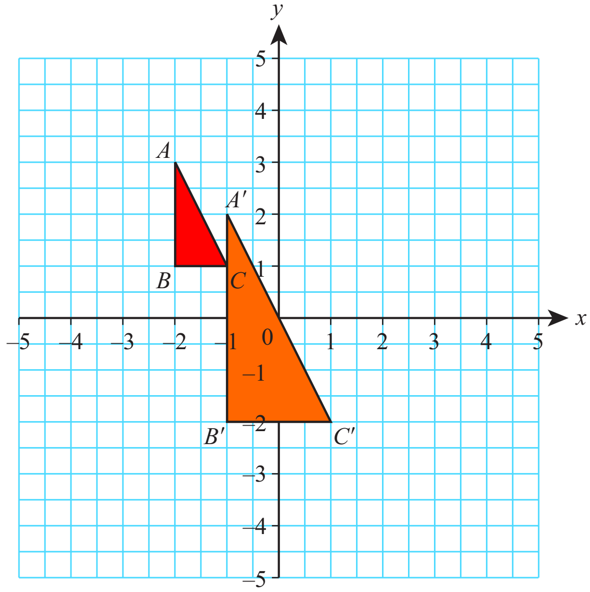
a



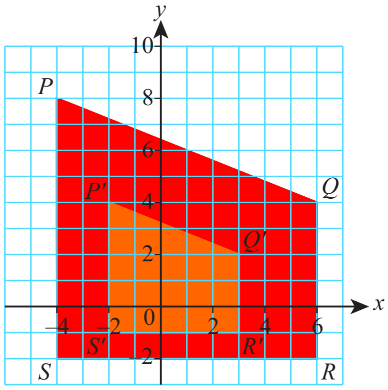
b



c

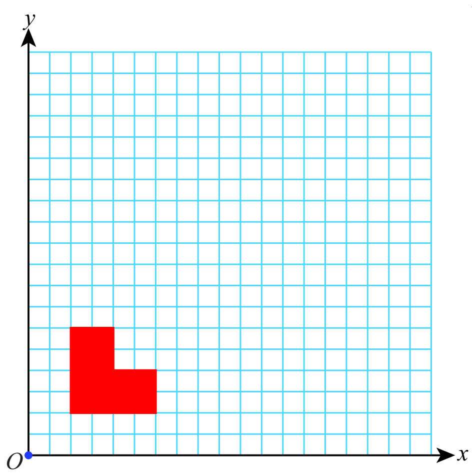


d

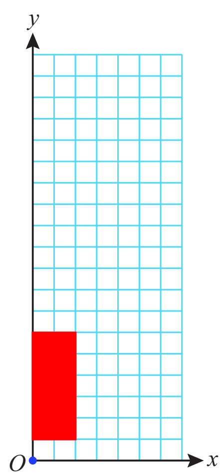


2 Copy the axes and shapes onto squared paper. Using the origin as the centre of enlargement, and a scale factor of three, draw the image of each shape under enlargement.

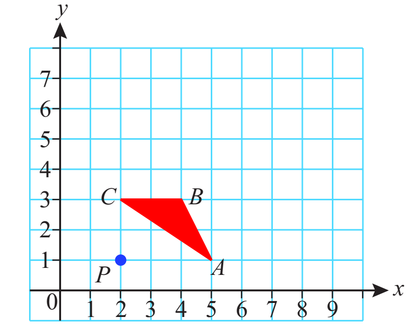
a



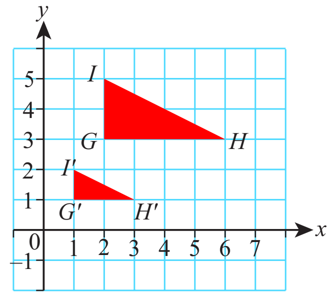
b



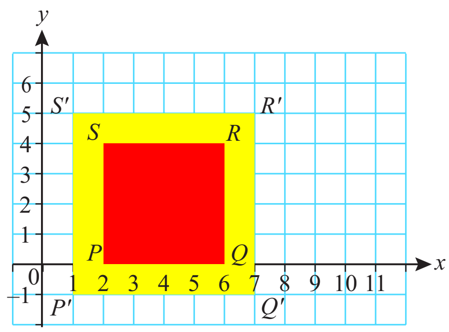
3 Copy the axes and shape onto squared paper. Draw the image of ΔABC under an enlargement of scale factor 2 and centre of enlargement $P(2, 1)$.



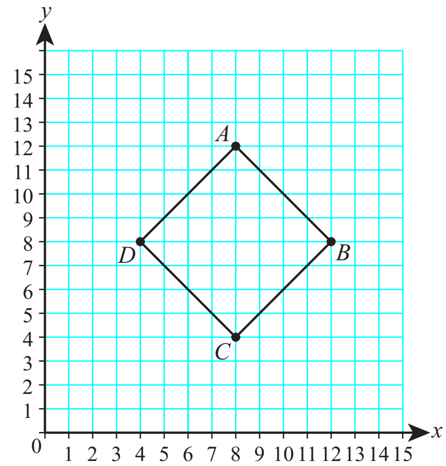
4 $\Delta G'H'I'$ is the image of ΔGHI under an enlargement. Find the scale factor of the enlargement and the co-ordinates of the centre of enlargement.



5 In the diagram, square $P'Q'R'S'$ is the image of square $PQRS$ under an enlargement. Find the scale factor of the enlargement and the co-ordinates of the centre of enlargement.



6 Draw the enlargment of object ABCD with a scale factor $\frac{1}{4}$ and centre of enlargement at (0, 0).



Applying your skills

7 Sheldon uses his computer software to enlarge and reduce pictures.

- **a** He has a picture that is 10 cm long and 6 cm wide. If he enlarges it to be 16 cm long, how wide will it be?
- **b** If Sheldon triples the width of the photograph, what will happen to its length?
- **c** Is it possible to enlarge the picture by increasing the length and leaving the width the same? Give a reason for your answer.
- **d** Sheldon needs to reduce the picture so it is a quarter of its original size. What will the new dimensions be?

8 Maria has a rectangular painting which is 240 mm by 180 mm. She wants to make a reduced colour copy of the painting to put into a small silver frame. The display area on the frame is 18 cm by 13.5 cm.

- **a** What scale factor should she use to make the photocopy?
- **b** How many times smaller is the area of the picture in the frame than the area of the original painting?

23.2 Vectors

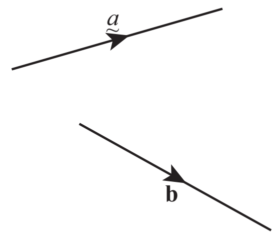
Some quantities are best described by giving both a **magnitude** (size) and a direction. For example, a wind speed of 35 km/h from the southeast or an acceleration upwards of 2 m/s². Force, velocity, displacement and acceleration are all **vector quantities**.

Other quantities such as time, temperature, speed, mass and area can be described by only giving their magnitude (they don't have a direction). In Mathematics, these quantities are called **scalars**.

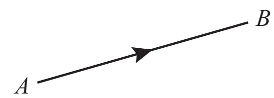
In Mathematics, a vector is an ordered pair of numbers that can be used to describe a translation. The ordered pair gives both magnitude and direction.

Vector notation

Vectors can be represented by a directed line segment as shown in the diagrams on the right. Note that the notation is either a small letter with a wavy line beneath it or a bold letter: e.g. $\underset{\sim}{a}$ or $\boldsymbol{a}$.



Vectors can also be represented by a named line such as AB. In such cases, the vector is denoted by $\mathbf{AB}$ or $\overrightarrow{AB}$. The order of letters is important because they give the direction of the line. $\overrightarrow{AB}$ is not the same as $\overrightarrow{BA}$.

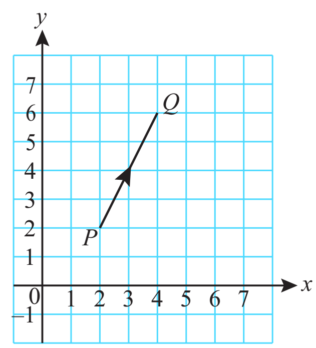


LINK

Vectors have applications in physics, for example, to model how friction affects movement down a slope or how far an object can tilt before it falls over. These applications have real world relevance - for example, making sure aircraft don't crash into each other in a flight path, landing safely in high winds and making sure double-decker buses can turn corners without falling over.

Writing vectors as number pairs

Vectors can also be written as a **column vector** using number pair notation. Look at line PQ on the diagram.



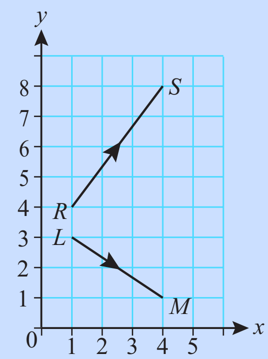
This line represents the translation of P to Q. The translation is two units in the positive x-direction and four units in the positive y-direction. This can be written as the ordered pair $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

The top number shows the horizontal movement (parallel to the x-axis) and the bottom number shows the vertical movement (parallel to the y-axis). A negative sign indicates movements down or to the left.

You can therefore write $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Worked example 8

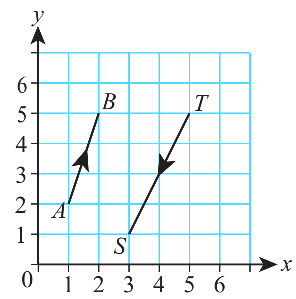
Express $\overrightarrow{RS}$ and $\overrightarrow{LM}$ as column vectors.



$\overrightarrow{RS} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$	Translation from R to S is three units right and four up.
$\overrightarrow{LM} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$	Translation from L to M is three units right and two down.

Worked example 9

Draw the column vectors $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}-2\\-4\end{pmatrix}$.



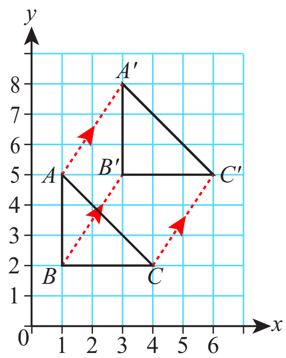
Start at any point, for example A, and move one right and three up to B. Join the points and indicate the direction using an arrow.

Start at any point, for example T, and move two left and then four down to S. Join the points and indicate the direction using an arrow.

Using vectors to describe a translation

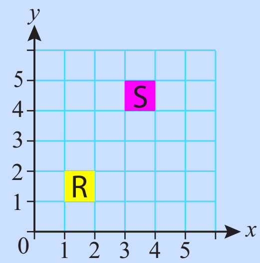
You have already seen that column vectors can be used to describe translations.

In the diagram, ΔABC is translated to $\Delta A'B'C'$. All points on the object have moved two units to the right and three units upwards, so the column vector that describes this translation is $\begin{pmatrix}2\\3\end{pmatrix}$.



Worked example 10

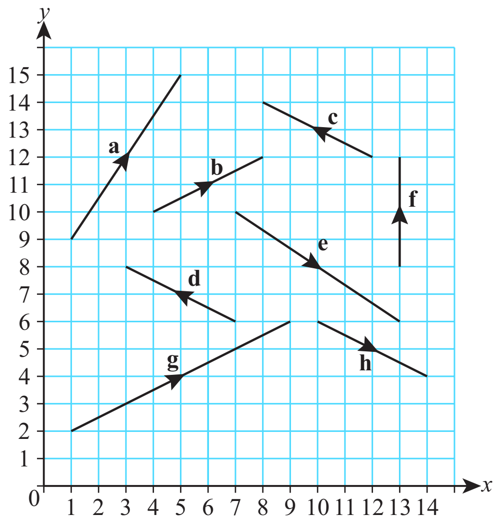
Square R is to be translated to square S.
Find the column vector for the translation.



The column vector is $\begin{pmatrix}2\\3\end{pmatrix}$.	Use one vertex of the object and the same vertex in its image to work out the translation.

Exercise 23.5

1 Write a column vector for each of the vectors shown on the diagram.



2 Represent these vectors on squared paper.

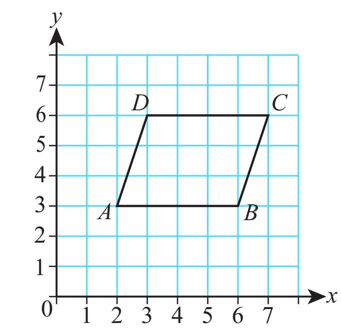
a $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ **b** $\overrightarrow{CD} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ **c** $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **d** $\overrightarrow{RS} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

e $\overrightarrow{TU} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ **f** $\overrightarrow{MN} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$ **g** $\overrightarrow{KL} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$ **h** $\overrightarrow{VW} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$

i $\overrightarrow{EF} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ **j** $\overrightarrow{JL} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ **k** $\overrightarrow{MP} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ **l** $\overrightarrow{QT} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

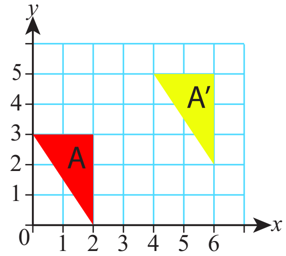
3 In the diagram, *ABCD* is a parallelogram. Write column vectors for the following:

a $\overrightarrow{AB}$ and $\overrightarrow{DC}$
b $\overrightarrow{BC}$ and $\overrightarrow{AD}$.
c What can you say about the two pairs of vectors?

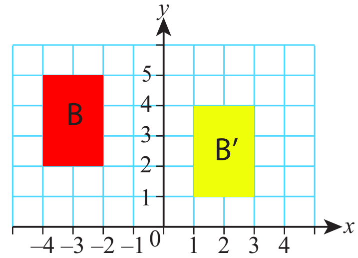


4 In the diagrams below, shapes *A*, *B*, *C*, *D*, *E* and *F* are mapped onto images *A*′, *B*′, *C*′, *D*′, *E*′ and *F*′ by translation. Find the column vector for the translation in each case.

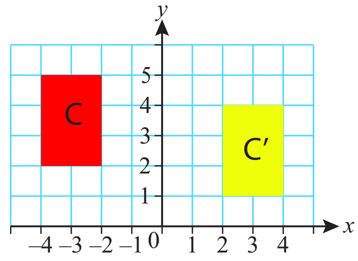
a



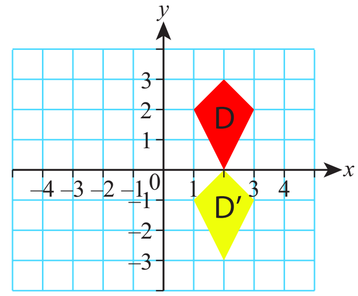
b



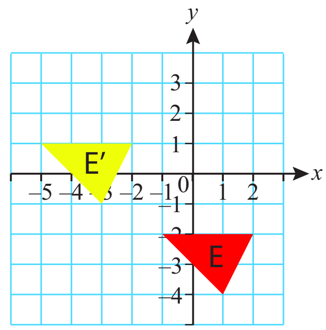
c



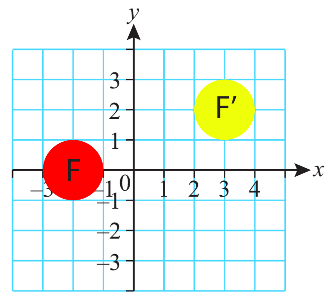
d



e



f

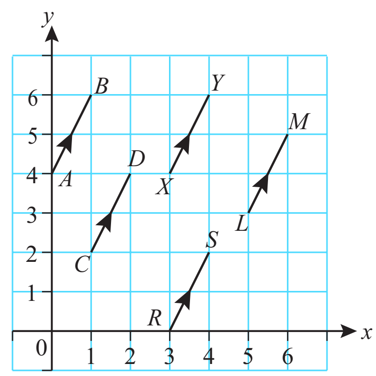


Equal vectors

Equal vectors have the same size (magnitude) and direction. As vectors are usually independent of position, they can start at any point. The same vector can be at many places in a diagram.

In the diagram, $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{XY}$, $\overrightarrow{LM}$ and $\overrightarrow{RS}$ are equal vectors.

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{XY} = \overrightarrow{LM} = \overrightarrow{RS} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



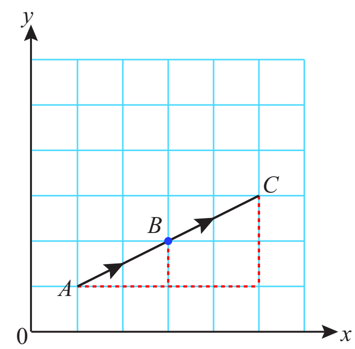
Multiplying a vector by a scalar

Remember a scalar is basically just a number. It has magnitude but no direction.

Look at this diagram. Vector $\overrightarrow{AC}$ is two times as long as vector $\overrightarrow{AB}$.

You can say:

$$\overrightarrow{AC} = 2\overrightarrow{AB} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

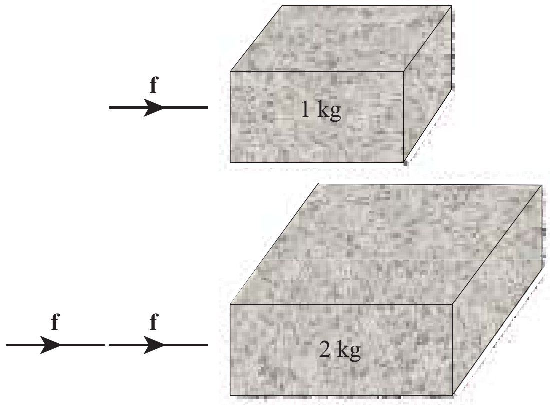


Here is another example.

A force, represented by vector **f**, is needed to move a 1 kg concrete block.

If you want to move a 2 kg concrete block, you would need to apply twice the force. In other words, you would need to apply **f** + **f** or 2**f**.

A force of 2**f** would have the same direction as **f**, but it would be twice its magnitude.



Multiplying any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by a scalar k, gives $k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$.

Vectors cannot be multiplied by each other, but they can be multiplied by a constant factor or scalar.

Vector **a** multiplied by 2 is the vector 2**a**. Vector 2**a** is twice as long as vector **a**, but they have the same direction. In other words, they are either parallel or in a straight line.

If $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ then $2\mathbf{a} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$.

Vector **a** multiplied by −1 is the vector −**a**, opposite in direction to **a**, but with the same magnitude as **a**.

*k***a** (where *k* is positive [+]) is in the same direction as **a** and *k* times as long. When *k* is negative [−], then *k***a** is opposite in direction to **a** but still *k* times as long.

Worked example 11

If $\mathbf{u} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$, find $\frac{1}{4}\mathbf{u}$.

$$\frac{1}{4}\mathbf{u} = \frac{1}{4}\begin{pmatrix} 8 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \times 8 \\ \frac{1}{4} \times -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Worked example 12

If $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find $-5\mathbf{v}$.

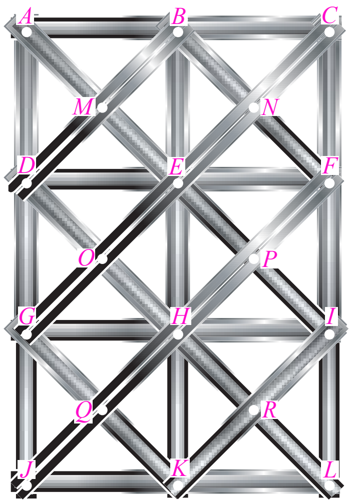
$$-5\mathbf{v} = -5\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \times -3 \\ -5 \times 2 \end{pmatrix} = \begin{pmatrix} 15 \\ -10 \end{pmatrix}$$

Exercise 23.6

1 If $\mathbf{a} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$, calculate:

a $3\mathbf{a}$ **b** $\frac{1}{2}\mathbf{a}$ **c** $-2\mathbf{a}$ **d** $-\mathbf{a}$ **e** $-\frac{3}{4}\mathbf{a}$ **f** $1.5\mathbf{a}$

Applying your skills



2 The diagram shows a rectangular metal burglar bar. Each section of the burglar bar can be represented by a vector. Sections can also be compared in terms of vectors. So, for example, $\overrightarrow{AJ} = 3\overrightarrow{AD}$.

Copy and complete these comparisons:

a $\overrightarrow{DF} = \_\_\overrightarrow{JK}$ **b** $\overrightarrow{JQ} = \_\_\overrightarrow{JF}$ **c** $\overrightarrow{HP} = \_\_\overrightarrow{HF}$
d $2\overrightarrow{GO} = \_\_\overrightarrow{GC}$ **e** $3\overrightarrow{DG} = \_\_\overrightarrow{CL}$ **f** $6\overrightarrow{BE} = \_\_\overrightarrow{CL}$

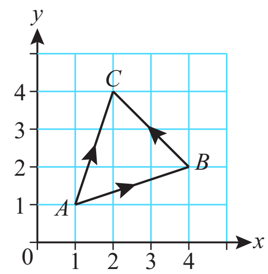
3 If $\mathbf{a} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ calculate:

a $-2\mathbf{a}$ **b** $3\mathbf{b}$ **c** $\frac{3}{2}\mathbf{b}$ **d** $-\frac{3}{4}\mathbf{a}$ **e** $-1.5\mathbf{a}$

f $-12\mathbf{b}$ **g** $-\frac{3}{2}\mathbf{a}$ **h** $-\frac{5}{9}\mathbf{b}$

Addition of vectors

In this diagram, point A is translated to point B and then translated again to end up at point C. However, if you translated the point directly from A to C, you end up at the same point. In other words, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.



You can represent each translation as a column vector:

$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

You know that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, so:

$\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ Add the corresponding x (top) and y (bottom) values.

So, to add the vectors, you add the corresponding x and y co-ordinates.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

This is called the 'nose to tail' method or the triangle law.

Subtraction of vectors

Subtracting a vector is the same as adding its negative. So $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

Think about $\overrightarrow{AC} - \overrightarrow{AB}$:

adding the negative of $\overrightarrow{AB}$ is the same as adding $\overrightarrow{BA}$.

$\therefore\ \overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{BA}$

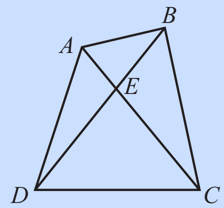
If you rearrange the vectors, you can apply the nose to tail rule and add them:

$\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$

Any section of a line joining two points is called a line segment.

Worked example 13

In the figure, the various line segments represent vectors.



Find in the figure directed line segments equal to the following:

a $\overrightarrow{AE} + \overrightarrow{EC}$ **b** $\overrightarrow{DB} + \overrightarrow{BE}$ **c** $\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BC}$

d $\overrightarrow{CB} + \overrightarrow{BE} + \overrightarrow{EA} + \overrightarrow{AD}$

a $\overrightarrow{AE} + \overrightarrow{EC} = \overrightarrow{AC}$

b $\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$ (If you travel from D to B and then from B to E, you have gone back on yourself and ended up at E, which is the same as travelling from D to E.)

c $\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BC} = \overrightarrow{AC}$ ($\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$, and $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ $\therefore$ $\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BC} = \overrightarrow{AC}$)

d $\overrightarrow{CB} + \overrightarrow{BE} + \overrightarrow{EA} + \overrightarrow{AD} = \overrightarrow{CE} + \overrightarrow{EA} + \overrightarrow{AD} = \overrightarrow{CA} + \overrightarrow{AD} = \overrightarrow{CD}$

Worked example 14

If $\mathbf{a} = \begin{pmatrix}3\\4\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\-1\end{pmatrix}$, find the column vectors equal to:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$ **c** $3\mathbf{a}$ **d** $\mathbf{a} + 4\mathbf{b}$ **e** $2\mathbf{a} - 3\mathbf{b}$

a $\mathbf{a} + \mathbf{b} = \begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3+2\\4-1\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix}$

b $\mathbf{a} - \mathbf{b} = \begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3-2\\4+1\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$ $\left(\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}); -\mathbf{b} \text{ is same asis } \begin{pmatrix}-2\\1\end{pmatrix}\right)$

c $3\mathbf{a} = 3\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}3\times 3\\3\times 4\end{pmatrix} = \begin{pmatrix}9\\12\end{pmatrix}$

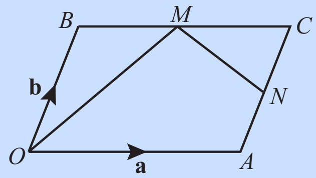
d $\mathbf{a} + 4\mathbf{b} = \begin{pmatrix}3\\4\end{pmatrix} + 4\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3+8\\4+(-4)\end{pmatrix} = \begin{pmatrix}11\\0\end{pmatrix}$

e $2\mathbf{a} - 3\mathbf{b} = 2\begin{pmatrix}3\\4\end{pmatrix} - 3\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}6\\8\end{pmatrix} - \begin{pmatrix}6\\-3\end{pmatrix}\begin{pmatrix}0\\11\end{pmatrix}$

If a question on vectors does not provide a diagram, you should draw one.

Worked example 15

OACB is a parallelogram in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
M is the midpoint of *BC* and *N* is the midpoint of *AC*.



a Find in terms of **a** and **b**:

i $\overrightarrow{OM}$ **(ii)** $\overrightarrow{MN}$

b Show that $\overrightarrow{OM} + \overrightarrow{MN} = \overrightarrow{OA} + \overrightarrow{AN}$.

a **i** $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$

$\overrightarrow{OB} = \mathbf{b}$

M is the midpoint *BC*, so $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC}$

$\therefore\ \overrightarrow{OM} = \mathbf{b} + \frac{1}{2}\mathbf{a}$

ii $\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CN} = \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA}$

$= \frac{1}{2}\mathbf{a} + -\frac{1}{2}\mathbf{b}\ (\overrightarrow{CA} = -\overrightarrow{AC} = -\overrightarrow{OB})$

$= \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} - \mathbf{b})$

b $\overrightarrow{OM} + \overrightarrow{MN} = (\mathbf{b} + \frac{1}{2}\mathbf{a}) + (\frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b})$

$= \mathbf{a} + \frac{1}{2}\mathbf{b}$

$\overrightarrow{OA} + \overrightarrow{AN} = \mathbf{a} + \frac{1}{2}\mathbf{b}$

$\therefore\ \overrightarrow{OM} + \overrightarrow{MN} = \overrightarrow{OA} + \overrightarrow{AN}$.

Exercise 23.7

1 $\mathbf{p} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

Express in column vector form:

a $3\mathbf{p}$ **b** $\mathbf{p} + \mathbf{q}$

2 Given that $\mathbf{a} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$, express $2\mathbf{a} - \mathbf{b}$ as a column vector.

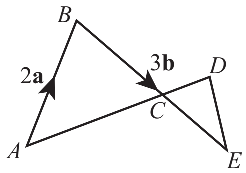
3 If $\mathbf{a} = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, calculate:

a $\mathbf{a} + \mathbf{b}$ **b** $2\mathbf{a} - 2\mathbf{b}$ **c** $\mathbf{b} - \mathbf{a}$

d $\frac{1}{2}\mathbf{b} - \mathbf{c}$ **e** $\mathbf{a} - 2(\mathbf{b} - \mathbf{c})$ **f** $2\mathbf{a} - \mathbf{c}$

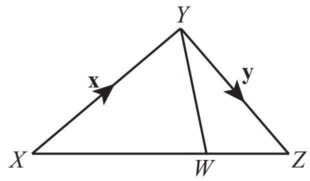
g $\frac{1}{2}(2\mathbf{a} + \mathbf{b})$ **h** $\mathbf{c} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$

4 In the diagram, BCE and ACD are straight lines. $\overrightarrow{AB} = 2\mathbf{a}$ and $\overrightarrow{BC} = 3\mathbf{b}$. The point C divides AD in the ratio 2 : 1 and divides BE in the ratio 3 : 1. Express in terms of $\mathbf{a}$ and $\mathbf{b}$, the vectors:



a $\overrightarrow{AC}$ **b** $\overrightarrow{CD}$
c $\overrightarrow{CE}$ **d** $\overrightarrow{ED}$

5 In ΔXYZ, $\overrightarrow{XY} = \mathbf{x}$ and $\overrightarrow{YZ} = \mathbf{y}$ and $WZ = \frac{1}{4}(XZ)$. Find in terms of $\mathbf{x}$ and $\mathbf{y}$:

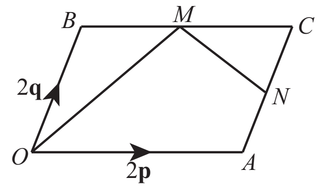


a $\overrightarrow{XZ}$ **b** $\overrightarrow{XW}$ **c** $\overrightarrow{YW}$

6 $OACB$ is a parallelogram in which $\overrightarrow{OA} = 2\mathbf{p}$ and $\overrightarrow{OB} = 2\mathbf{q}$.

M is the midpoint of BC and N is the midpoint of AC.

Find in terms of $\mathbf{p}$ and $\mathbf{q}$:



a $\overrightarrow{AB}$ **b** $\overrightarrow{ON}$ **c** $\overrightarrow{MN}$.

The magnitude of a vector is sometimes called the modulus.

The magnitude of a vector

The magnitude of a vector is its length. The notation $|\overrightarrow{AB}|$ or $|\mathbf{a}|$ is used to write the magnitude of a vector ($\overrightarrow{AB}$).

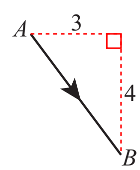
You use Pythagoras' theorem to calculate the magnitude of a vector.

In general, if $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$.

Worked example 16

Find the magnitude of the vector $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

Draw $\overrightarrow{AB}$ as the hypotenuse of a right-angled triangle.



$AB^2 = 3^2 + 4^2$ (Pythagoras' theorem)
$AB^2 = 9 + 16$
$AB^2 = 25$
$AB = 5$

$\therefore |\overrightarrow{AB}| = 5$ units.

Worked example 17

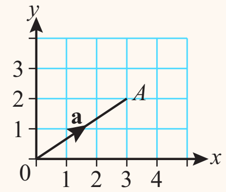
If $\mathbf{a} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$ find $|\mathbf{a}|$.

$|\mathbf{a}| = \sqrt{(-5)^2 + (12)^2}$
$= \sqrt{169}$
$= 13$

E

Position vectors

A vector that starts from the origin (O) is called a position vector. On this diagram, point A has the position vector $\overrightarrow{OA}$ or $\mathbf{a}$.



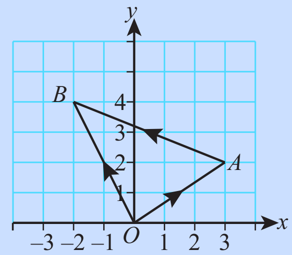
If $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ then the co-ordinates of point A will be (3, 2).

Because the co-ordinates of the point A are the same as the components of the column vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ you can use position vectors to find the magnitude of any vector.

Worked example 18

The position vector of A is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and the position vector of B is $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$.

Find the vector $2\overrightarrow{AB}$.



$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$

So, $2\overrightarrow{AB} = 2\begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ 4 \end{pmatrix}$.

You could also find the column vector for $\overrightarrow{AB}$ by counting the movements parallel to the x-axis followed by those parallel to the y-axis.

Movement parallel to x-axis = –5 units. Movement parallel to the y-axis = 2 units

$\overrightarrow{AB} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$

so, $2\overrightarrow{AB} = 2\begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ 4 \end{pmatrix}$.

Worked example 19

If A is point (–1, –2) and B is (5, 6), find $|\overrightarrow{AB}|$.

$\overrightarrow{OA} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$.

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB}$

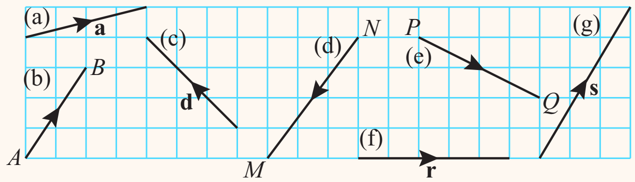
$= \begin{pmatrix} -(-1) + 5 \\ -(-2) + 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$

Exercise 23.8

E

1 Calculate the magnitude of each vector. Give your answers to 2 decimal places where necessary.



2 Find the magnitude of the following vectors. Give your answers to 2 decimal places where necessary.

a $\mathbf{a} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$ **b** $\overrightarrow{MN} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$ **c** $\mathbf{x} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ **d** $\overrightarrow{PQ} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$

3 O is the point (0, 0), P is (3, 4), Q is (−5, 12) and R is (−8, −15).
Find the values of:

a $|\overrightarrow{OP}|$ **b** $|\overrightarrow{OQ}|$ **c** $|\overrightarrow{OR}|$

4 Points A, B and C have position vectors $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

a Write down the co-ordinates of A, B and C.
b Write down the vectors $\overrightarrow{AB}$, $\overrightarrow{CB}$ and $\overrightarrow{AC}$ in column vector form.

5 $OATB$ is a parallelogram. M, N and P are midpoints of BT, AT and MN respectively. O is the origin and the position vectors of A and B are **a** and **b** respectively. Find in terms of **a** and/or **b**:

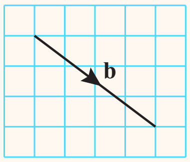
a $\overrightarrow{MT}$ **b** $\overrightarrow{TN}$ **c** $\overrightarrow{MN}$
d the position vector of P, giving your answer in simplest form.

6 Find the magnitude of:

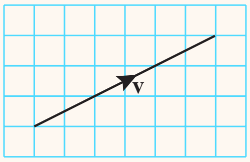
a the vector joining the points (−3, −3) and (3, 5)
b the vector joining the points (−2, 6) and (3, −1).

Applying your skills

7 Vector **b** shows the velocity (in km/h) of a car on a highway. The sides of each square on the grid represent a speed of 20 km/h. Find the speed at which the car was travelling.

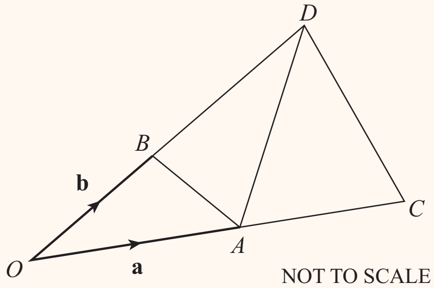


8 Vector **v** represents the velocity in km/h of a person jogging. The sides of each block on the grid represent a speed of 1 km/h. Calculate the speed at which the person was jogging.



9 In the diagram $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

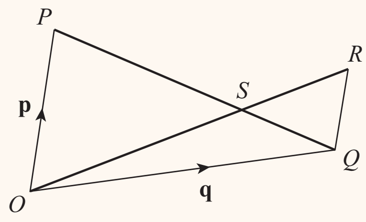
Also $\overrightarrow{AC} = \mathbf{2a}$ and $\overrightarrow{AD} = \mathbf{3b} - \mathbf{a}$.



E

a Write $\overrightarrow{AB}$ in terms of **a** and **b**.
b $\overrightarrow{OD} = n\mathbf{b}$ where n is a whole number. Find n.
c Prove that OAB and ODC are similar triangles.

10



In the diagram $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$, $\mathbf{QR} = \frac{1}{2}\mathrm{OP}$ and $\mathrm{SQ} = \frac{1}{3}\mathrm{PQ}$.

RQ is parallel to OP.

Find in terms of **p** and **q**:

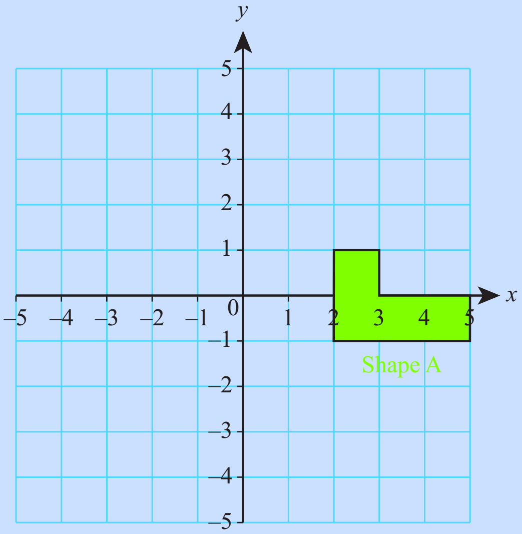
a $\overrightarrow{PQ}$ **b** $\overrightarrow{PS}$ **c** $\overrightarrow{OS}$ **d** $\overrightarrow{OR}$

23.3 Further transformations

In addition to the transformations you've already dealt with, you need to be able to reflect an object through any line, rotate an object about any point and use negative scale factors for enlargements. The properties of the object and its image under these transfromations are the same as those you already learned.

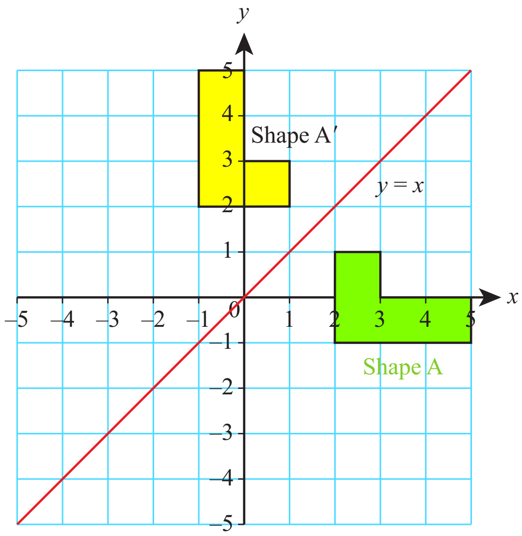
Worked example 20

Reflect shape A in the line $y = x$.



Draw the line $y = x$.

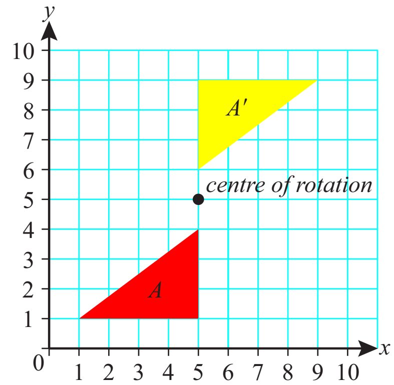
Apply the rules you know for reflecting shapes to draw the image A′.



E

Worked example 21

Sketch triangle A with vertices at (1, 1), (5, 1) and (5, 4). Rotate this shape 180° about the point (5, 5).



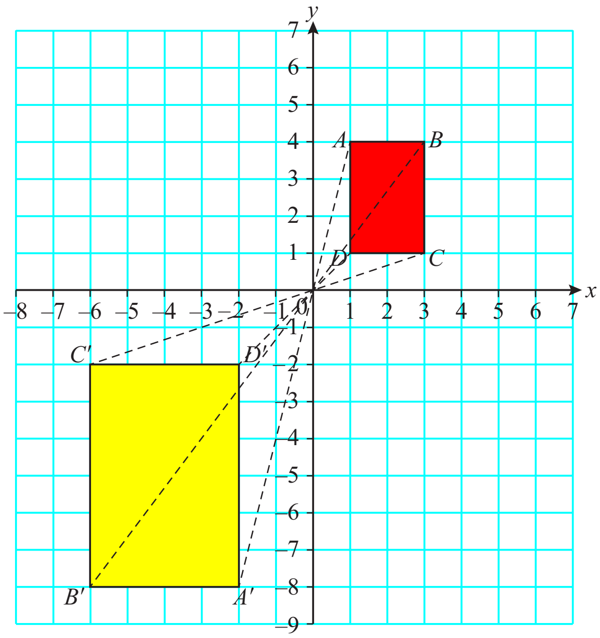
Plot the points and join them to draw triangle A.
Mark the centre of rotation.
Draw the image and label it A′.

Worked example 22

Enlarge rectangle ABCD by a scale factor of -2 with the origin as a centre of enlargement.

Multiply each set of coordinates by -2.
A(1, 4), so A′(-2, -8)
B(3, 4), so B′ (-6, -8)
C(3, 1), so C′(-6, -2)
D(1, 1), so D′(-2, -2)

Plot the points.
Make sure you label them correctly.



If a point and its image are on opposite sides of the centre of enlargement, then the scale factor is neagive

Tip

Drawing in the rays from each vertex allows you to check the points on the image are in line with the corresponding ones on the object.

E

Combining transformations

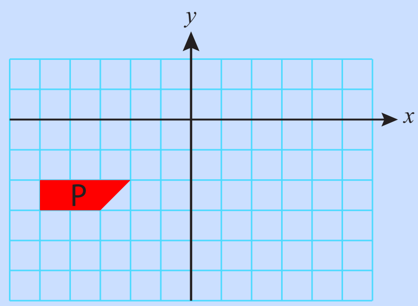
You've already seen that an object can undergo a single transformation to map it to an image. An object can also undergo two transformations in succession. For example, it could be reflected in the x-axis and then rotated through a quarter turn, or it could be rotated and then reflected in the y-axis. Sometimes a combined transformation can be described by a single, equivalent transformation.

The following capital letters are conventionally used to represent different transformations:

M Reflection (remember the M is for mirror!)
R Rotation
T Translation
E Enlargement

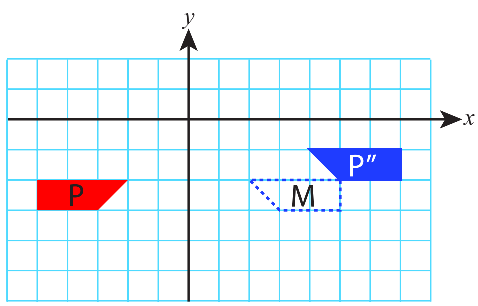
Worked example 23

For the shape P shown in the diagram, let T be the translation $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and M be the reflection in the y-axis.

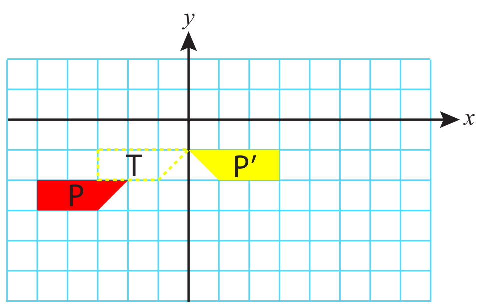


a Draw the image P' after the transformation TM(P).
b Draw the image P'' after the transformation MT(P).
c What single transformation maps P' onto P''?

a TM(P) means do M first then do T.
Use a pencil. Do the first transformation and (faintly) draw the shape. Do the second transformation, draw the image.
Label it correctly.

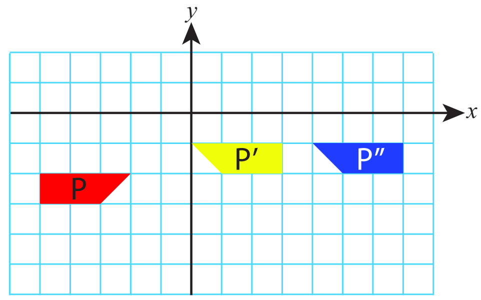


b MT(P) means do T first then do M.
Use a pencil. Do the first transformation and (faintly) draw the shape. Do the second transformation, draw the image.
Label it correctly.



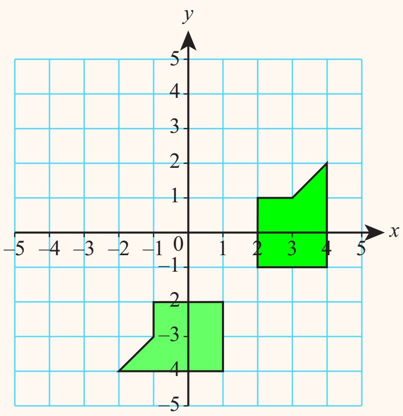
E

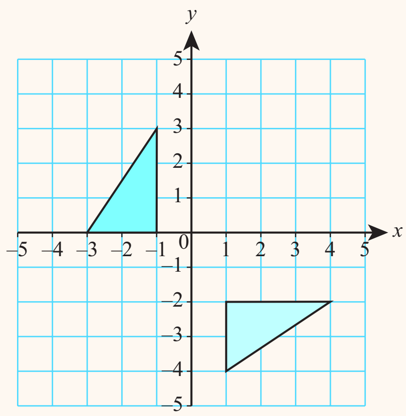
c P' can be mapped to P'' by the translation $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

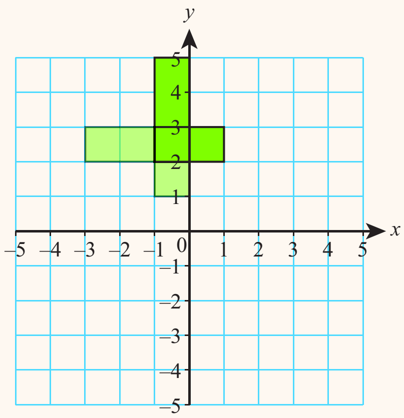


Exercise 23.9

1 For each pair of reflected shapes, give the equation of the mirror line.

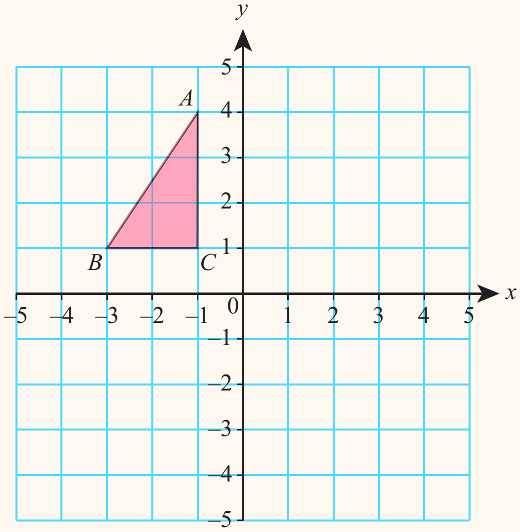




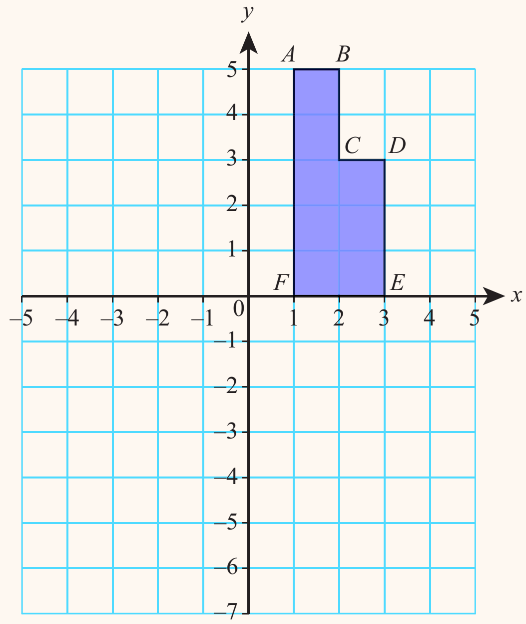


2 Draw each shape on a coordinate grid and perform the given rotation.

a Rotate the ABC 90° anticlockwise about the point (-2, 2).



b Rotate object ABCDEF 180° about the point (1, -1).

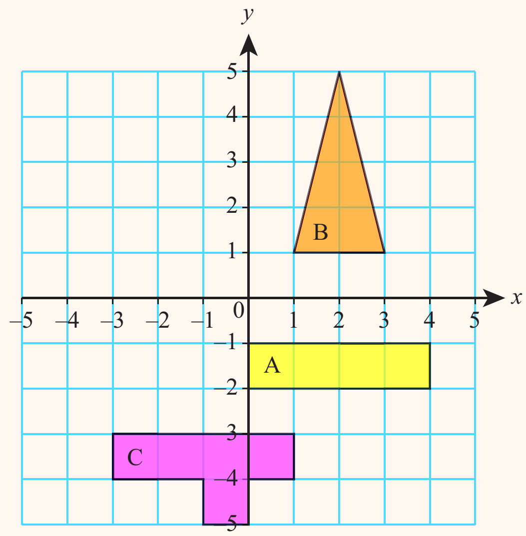


3 Draw the image of each shape under the given transformation on the same set of axes.

a Reflect shape A in the line $y = -x - 2$

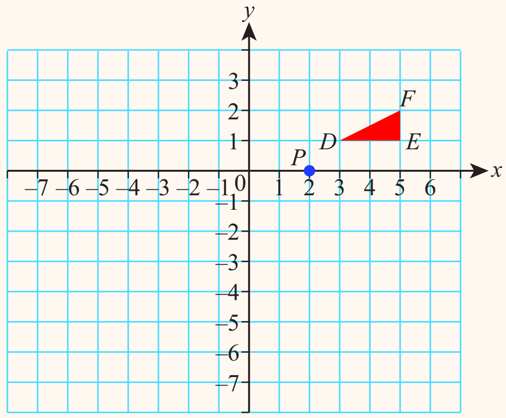
b Rotate shape B 90° anticlockwise around point (2, 4)

c Reflect shape C in the line $y = -3$ and then rotate it 270° clockwise about the point $(-1, -1)$

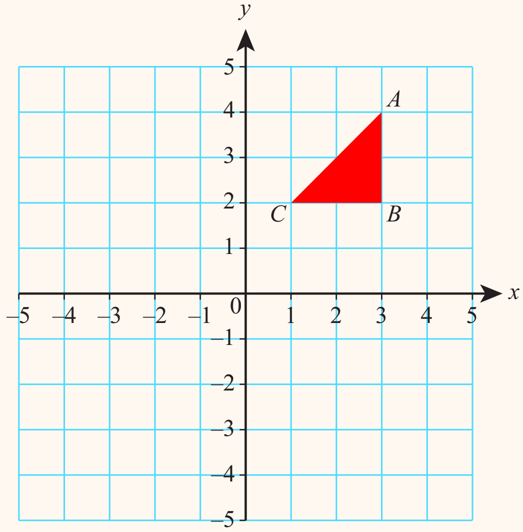


E

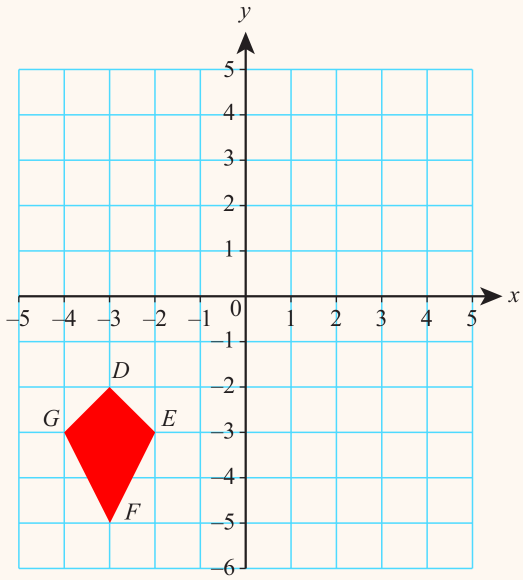
4 Copy the axes and shape onto squared paper. Draw the image of ΔDEF under an enlargement with scale factor -3 and centre of enlargement $P(2, 0)$.



5 Enlarge the given shape by a scale factor of -1. Use the origin as the centre of enlargement.

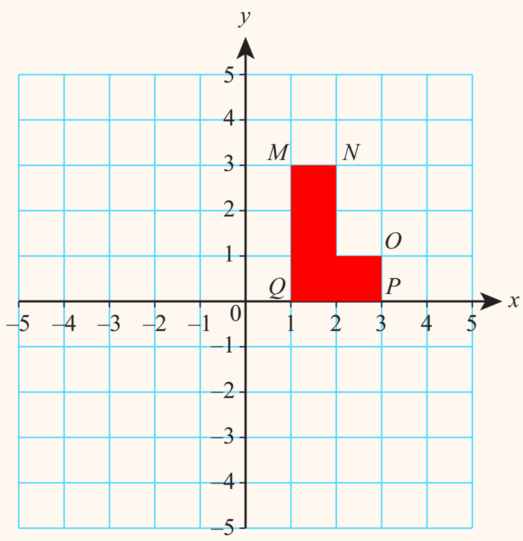


6 Enlarge shape DEFG by a scale factor of -2 with $(-1, -1)$ as the centre of enlargment.

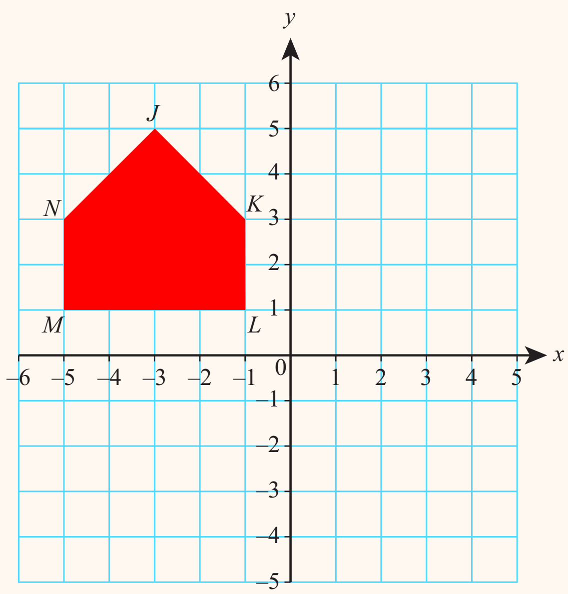


E

7 Enlarge the object shown by a scale factor of -1.5 using the point $(1, 0)$ as the centre of enlargement.

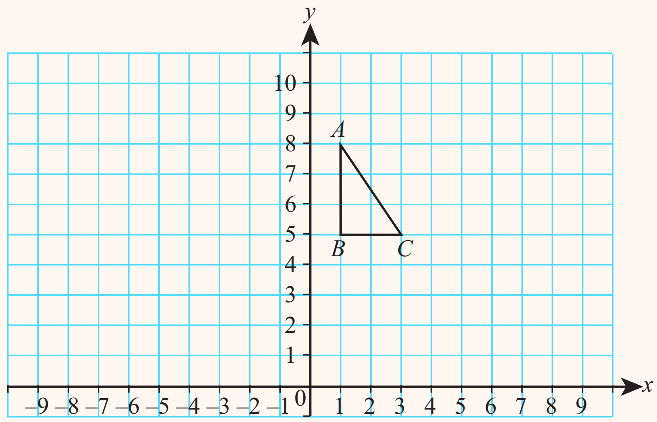


8 Enlarge the shape by a scale factor of $-\frac{1}{2}$ using the origin as the centre of enlargement.



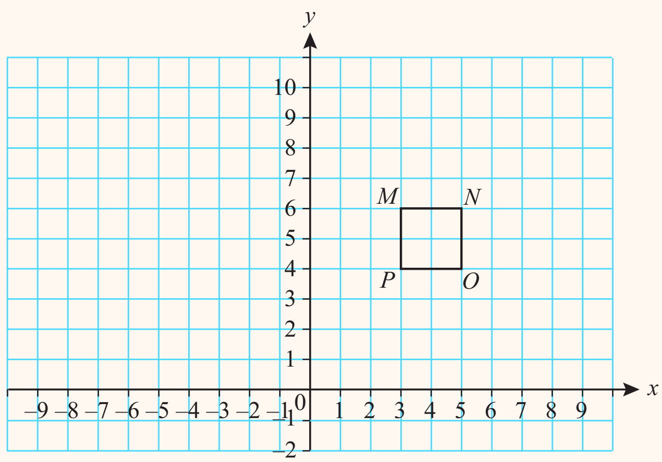
E

9 ΔABC maps onto $A'B'C'$ after an enlargement of scale factor two from the centre of enlargement (2, 5). $A'B'C'$ is then mapped onto $A''B''C''$ by reflection in the line $x = 1$.



a Draw and label image $A'B'C'$.
b Draw and label the image $A''B''C''$.

10 A square $MNOP$ maps onto $M'N'O'P'$ after an enlargement of scale factor 1.5 with the centre of enlargement (3, 4). $M'N'O'P'$ is then rotated 180° about the point (0, 6) to give the image $M''N''O''P''$. Copy the diagram and show the position of both $M'N'O'P'$ and $M''N''O''P''$.



Summary

Do you know the following?

- A transformation involves a change in the position and/or size of a shape.
- A reflection is a mirror image, a rotation is a turn, a translation is a slide and an enlargement is an increase in size.
- To fully describe a reflection you need to give the equation of the mirror line.
- To fully describe a rotation you need to give the angle and centre of rotation.
- To describe a translation you can use a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.
- To describe an enlargement you need to give the scale factor and the centre of enlargement.
- A vector has both magnitude and direction. You can add and subtract vectors but you cannot multiply or divide vectors. You can multiply a vector by a scalar.

E

- The magnitude of a vector $\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{x^2 + y^2}$. You write the magnitude as $|\overrightarrow{XY}|$ or $|\mathbf{x}|$.
- A position vector is a vector that starts at the origin.

Are you able to …?

- reflect points and plane figures about horizontal and vertical lines
- rotate plane figures about the origin, vertices of the object and midpoints of the sides
- translate shapes using a column vector
- construct enlargements of simple shapes using the scale factor and centre of enlargement
- recognise and describe single and combined translations
- describe translations using column vectors
- add and subtract vectors and multiply vectors by a scalar

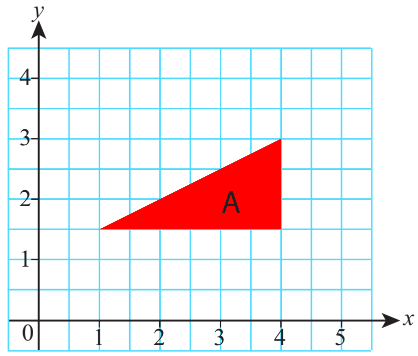
E

- calculate the magnitude of a vector
- use position vectors to find the magnitude of vectors

Examination practice

Exam-style questions

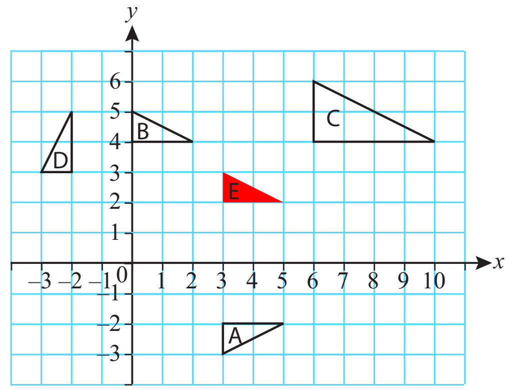
1 The diagram shows a triangle, labelled A.



a On a grid of squared paper, draw accurately the following transformations:

i the reflection of ΔA in the y-axis, labelling it ΔB
ii the rotation of ΔA through 180° about the point (4, 3), labelling it ΔC
iii the enlargement of ΔA, scale factor two, centre (4, 5), labelling it ΔD.

2 Describe fully the transformations of the shaded triangle E onto triangles A, B, C and D in the diagram.



3 $\mathbf{m} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

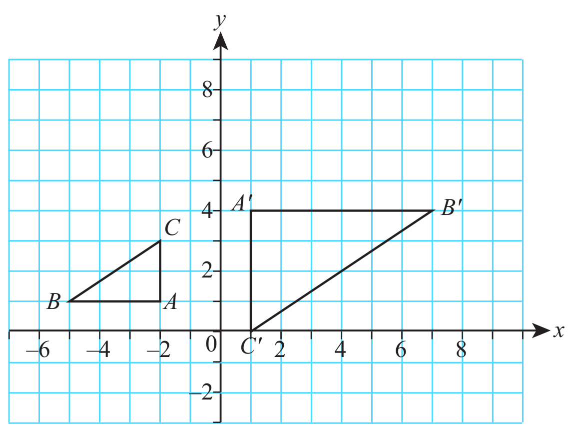
a Find:

i $\mathbf{m} + \mathbf{n}$
ii $3\mathbf{n}$

b Draw the vector **m** on a grid or on squared paper.

4 ΔABC is mapped onto $\Delta A'B'C'$ by an enlargement.

a Find the centre of the enlargement.
b What is the scale factor of the enlargement?

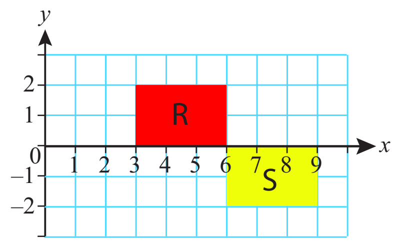


5 **a** Write down the column vector of the translation that maps rectangle R onto rectangle S.

b Describe fully another single transformation (not a translation) that would also map rectangle R onto rectangle S.

c **i** Copy the diagram onto a grid. Enlarge R with centre of enlargement $A(10, 2)$ and scale factor two.

ii Write down the ratio area of the enlarged rectangle to the area of rectangle R in its simplest terms.

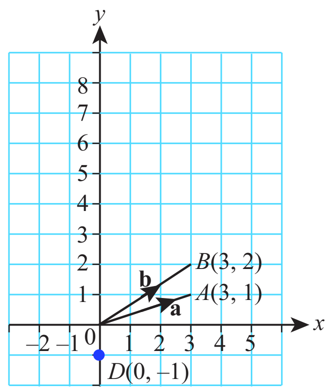


6 $OA = \mathbf{a}$ and $OB = \mathbf{b}$.

a $OC = \mathbf{a} + 2\mathbf{b}$. Make a copy of the diagram and label the point C on your diagram.

b $D = (0, -1)$. Write OD in terms of $\mathbf{a}$ and $\mathbf{b}$.

c Calculate $|\mathbf{a}|$ giving your answer to 2 decimal places.



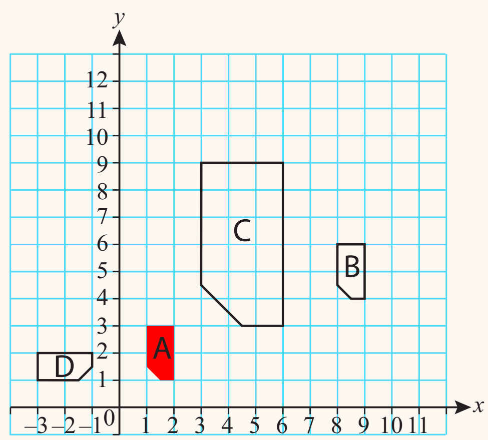
7 **a** In each case, describe fully the transformation that maps A onto:

i B

ii C

iii D

b State which shapes have an area equal to that of A.



8 Answer the whole of this question on a sheet of graph paper.

a Draw axes from –6 to +6, using a scale of 1 cm to represent 1 unit on each axis.

i Plot the points $A(5, 0)$, $B(1, 3)$ and $C(-1, 2)$ and draw ΔABC.

ii Plot the points $A'(3, 4)$, $B'(3, -1)$ and $C'(1, -2)$ and draw $\Delta A'B'C'$.

b **i** Draw and label the line l in which $\Delta A'B'C'$ is a reflection of ΔABC.

ii Write down the equation of the line l.

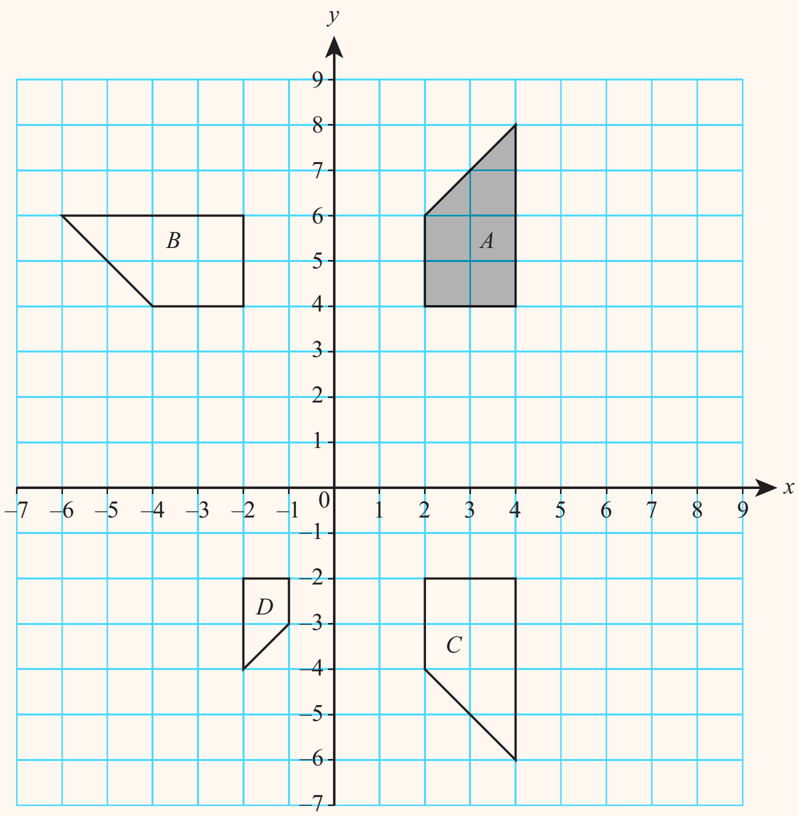
Past paper questions

1 $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

Find $|\overrightarrow{AB}|$. [2]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q4 October/November 2015]

2



a Describe fully the **single** transformation that maps

i shape A onto shape B, [3]

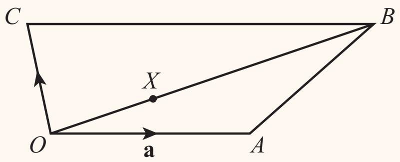
ii shape A onto shape C, [2]

iii shape A onto shape D. [3]

b On the grid, draw the image of shape A after a translation by the vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$. [2]

[Cambridge IGCSE Mathematics 0580 Paper 42 Q7 (a) & (c) October/November 2015]

3



The diagram shows a quadrilateral *OABC*.

$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{CB} = 2\mathbf{a}$.

X is a point on OB such that $OX:XB = 1:2$.

a Find, in terms of **a** and **c**, in its simplest form

i $\overrightarrow{AC}$ [1]

ii $\overrightarrow{AX}$ [3]

b Explain why the vectors $\overrightarrow{AC}$ and $\overrightarrow{AX}$ show that *C*, *X* and *A* lie on a straight line. [2]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q19 October/November 2014]