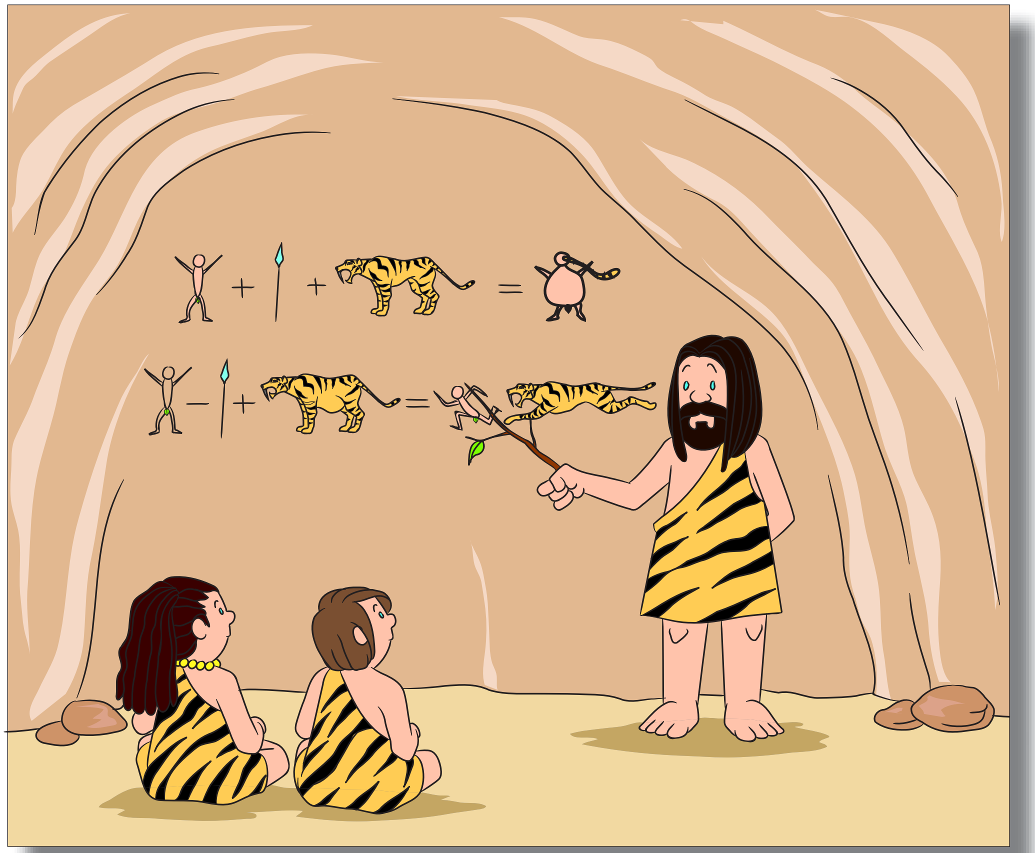
Chapter 2: Making sense of algebra

Key words

- Algebra
- Variable
- Equation
- Formula
- Substitution
- Expression
- Term
- Power
- Index
- Coefficient
- Exponent
- Base
- Reciprocal

In this chapter you will learn how to:

- use letters to represent numbers
- write expressions to represent mathematical information
- substitute letters with numbers to find the value of an expression
- add and subtract like terms to simplify expressions
- multiply and divide to simplify expressions
- expand expressions by removing grouping symbols
- use index notation in algebra
- learn and apply the laws of indices to simplify expressions.
- work with fractional indices.



Once you know the basic rules, algebra is very easy and very useful.

You can think of **algebra** as the language of mathematics. Algebra uses letters and other symbols to write mathematical information in shorter ways.

When you learn a language, you have to learn the rules and structures of the language. The language of algebra also has rules and structures. Once you know these, you can 'speak' the language of algebra and mathematics students all over the world will understand you.

At school, and in the real world, you will use algebra in many ways. For example, you will use it to make sense of formulae and spreadsheets and you may use algebra to solve problems to do with money, building, science, agriculture, engineering, construction, economics and more.

RECAP

You should already be familiar with the following algebra work:

Basic conventions in algebra

We use letters in place of unknown values in algebra.

An expression can contain numbers, variables and operation symbols, including brackets. Expressions don't have equals signs.

These are all algebraic expressions:

$x + 4 \qquad 3(x + y) \qquad \frac{3m}{n} \qquad (4 + a)(2 - a)$

Substitution of values for letters

If you are given the value of the letters, you can substitute these and work out the value of the expression.

Given that $x = 2$ and $y = 5$:

$x + y$ becomes $2 + 5$

$\frac{x}{y}$ becomes $2 \div 5$

xy becomes 2×5

$4x$ becomes 4×2 and $3y$ becomes 3×5

Index notation and the laws of indices for multiplication and division

$2 \times 2 \times 2 \times 2 = 2^4$ 2 is the base and 4 is the index.

$a \times a \times a = a^3$ a is the base and 3 is the index.

2.1 Using letters to represent unknown values

In primary school you used empty shapes to represent unknown numbers. For example, 2 + ■ = 8 and ■ + ♦ = 10. If 2 + ■ = 8, the ■ can only represent 6. But if ■ + ♦ = 10, then the ■ and the ♦ can represent many different values.

In algebra the letters can represent many different values so they are called **variables**.

In algebra, you use letters to represent unknown numbers. So you could write the number sentences above as: $2 + x = 8$ and $a + b = 10$. Number sentences like these are called **equations**. You can solve an equation by finding the values that make the equation true.

If a problem introduces algebra, you must not change the 'case' of the letters used. For example, 'n' and 'N' can represent *different* numbers in the *same* formula!

When you worked with area of rectangles and triangles in the past, you used algebra to make a general rule, or **formula**, for working out the area, A:

Area of a rectangle = length × breadth, so $A = lb$

Area of a triangle = $\frac{1}{2}$ base × height, so $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$

Notice that when two letters are multiplied together, we write them next to each other e.g. lb, rather than $l \times b$.

To use a formula you have to replace some or all of the letters with numbers. This is called **substitution**.

LINK

Algebra appears across all science subjects, in particular. Most situations in physics require motion or other physical changes to be described as an algebraic formula. An example is $F = ma$, which describes the connection between the force, mass and acceleration of an object.

Writing algebraic expressions

An algebraic **expression** is a group of letter and numbers linked by operation signs. Each part of the expression is called a **term**.

Suppose the average height (in centimetres) of students in your class is an unknown number, h. A student who is 10 cm taller than the average would have a height of $h + 10$. A student who is 3 cm shorter than the average would have a height of $h - 3$.

$h + 10$ and $h - 3$ are algebraic expressions. Because the unknown value is represented by h, we say these are expressions in terms of h.

Worked example 1

Use algebra to write an expression in terms of h for:

a a height 12 cm shorter than average
b a height $2x$ taller than average
c a height twice the average height
d a height half the average height.

a	$h - 12$	Shorter than means less than, so you subtract.
b	$h + 2x$	Taller than means more than, so you add. $2x$ is unknown, but it can still be used in the expression.
c	$2 \times h$	Twice means two times, so you multiply by two.
d	$h \div 2$	Half means divided by two.

Applying the rules

Mathematicians write the product of a number and a variable with the number first to avoid confusion with powers. For example, $x \times 5$ is written as $5x$ rather than $x5$, which may be confused with x^5.

Algebraic expressions should be written in the shortest, simplest possible way.

- $2 \times h$ is written as $2h$ and $x \times y$ is written as xy
- h means $1 \times h$, but you do not write the 1
- $h \div 2$ is written as $\frac{h}{2}$ and $x \div y$ is written as $\frac{x}{y}$
- when you have the product of a number and a variable, the number is written first, so $2h$ and not $h2$. Also, variables are normally written in alphabetical order, so xy and $2ab$ rather than yx and $2ba$
- $h \times h$ is written as h^2 (h squared) and $h \times h \times h$ is written as h^3 (h cubed). The ^{2} and the ^{3} are examples of a **power** or **index**.
- The power only applies to the number or variable directly before it, so $5a^2$ means $5 \times a \times a$
- When a power is outside a bracket, it applies to everything inside the bracket. So, $(xy)^3$ means $xy \times xy \times xy$

Worked example 2

Write expressions in terms of x to represent:

a a number times four
b the sum of the number and five
c six times the number minus two
d half the number.

a	x times 4 $= 4 \times x$ $= 4x$	Let x represent 'the number'. Replace 'four times' with $4 \times$. Leave out the $\times$ sign, write the number before the variable.
b	Sum of x and five $= x + 5$	Let x represent 'the number'. Sum of means $+$, replace five with 5.
c	Six times x minus two $= 6 \times x - 2$ $= 6x - 2$	Let x represent the number. Times means $\times$ and minus means $-$, insert numerals. Leave out the $\times$ sign.
d	Half x $= x \div 2$ $= \frac{x}{2}$	Let x represent 'the number'. Half means $\times \frac{1}{2}$ or $\div 2$. Write the division as a fraction.

Exercise 2.1

REWIND

Remember BODMAS in Chapter 1. Work out the bit in brackets first.

REWIND

Remember from Chapter 1 that a 'sum' is the result of an addition.

Also remember that the 'difference' between two numbers is the result of a subtraction. The order of the subtraction matters.

1 Rewrite each expression in its simplest form.

a $6 \times x \times y$	**b** $7 \times a \times b$	**c** $x \times y \times z$
d $2 \times y \times y$	**e** $a \times 4 \times b$	**f** $x \times y \times 12$
g $5 \times b \times a$	**h** $y \times z \times z$	**i** $6 \div x$
j $4x \div 2y$	**k** $(x + 3) \div 4$	**l** $m \times m \times m \div m \times m$
m $4 \times x + 5 \times y$	**n** $a \times 7 - 2 \times b$	**o** $2 \times x \times (x - 4)$
p $3 \times (x + 1) \div 2 \times x$	**q** $2 \times (x + 4) \div 3$	**r** $(4 \times x) \div (2 \times x + 4 \times x)$

2 Let the unknown number be m. Write expressions for:

a the sum of the unknown number and 13
b a number that will exceed the unknown number by five
c the difference between 25 and the unknown number
d the unknown number cubed
e a third of the unknown number plus three
f four times the unknown number less twice the number.

3 Let the unknown number be x. Write expressions for:

a three more than x
b six less than x
c ten times x
d the sum of -8 and x
e the sum of the unknown number and its square
f a number which is twice x more than x
g the fraction obtained when double the unknown number is divided by the sum of the unknown number and four.

Algebra allows you to translate information given in words to a clear and short mathematical form. This is a useful strategy for solving many types of problems.

4 A CD and a DVD cost x dollars.

a If the CD costs \$10 what does the DVD cost?
b If the DVD costs three times the CD, what does the CD cost?
c If the CD costs \$$(x - 15)$, what does the DVD cost?

5 A woman is m years old.

a How old will she be in ten years' time?
b How old was she ten years ago?
c Her son is half her age. How old is the son?

6 Three people win a prize of \$$p$.

a If they share the prize equally, how much will each receive?
b If one of the people wins three times as much money as the other two, how much will each receive?

2.2 Substitution

When you substitute values you need to write in the operation signs. $5h$ means $5 \times h$, so if $h = 1$, or $h = 6$, you cannot write this in numbers as 51 or 56.

Expressions have different values depending on what numbers you substitute for the variables. For example, let's say casual waiters get paid \$5 per hour. You can write an expression to represent everyone's wages like this: $5h$, where h is the number of hours worked. If you work 1 hour, then you get paid $5 \times 1 = \$5$. So the expression $5h$ has a value of \$5 in this case. If you work 6 hours, you get paid $5 \times 6 = \$30$. The expression $5h$ has a value of \$30 in this case.

'Evaluate' means to find the value of.

REWIND

You will need to keep reminding yourself about the order (BODMAS) of operations from chapter 1. ◀

Worked example 3

Given that $a = 2$ and $b = 8$, evaluate:

a ab **b** $3b - 2a$ **c** $2a^3$ **d** $2a(a + b)$

a	$ab = a \times b$ $= 2 \times 8$ $= 16$	Put back the multiplication sign. Substitute the values for a and b. Calculate the answer.
b	$3b - 2a = 3 \times b - 2 \times a$ $= 3 \times 8 - 2 \times 2$ $= 24 - 4$ $= 20$	Put back the multiplication signs. Substitute the values for a and b. Use the order of operations rules (× before –). Calculate the answer.
c	$2a^3 = 2 \times a^3$ $= 2 \times 2^3$ $= 2 \times 8$ $= 16$	Put back the multiplication signs. Substitute the value for a. Work out 2^3 first (grouping symbols first). Calculate the answer.
d	$2a(a + b) = 2 \times a \times (a + b)$ $= 2 \times 2 \times (2 + 8)$ $= 4 \times 10$ $= 40$	Put back the multiplication signs. Substitute the values for a and b. In this case you can carry out two steps at the same time: multiplication outside the bracket, and the addition inside. Calculate the answer.

LINK

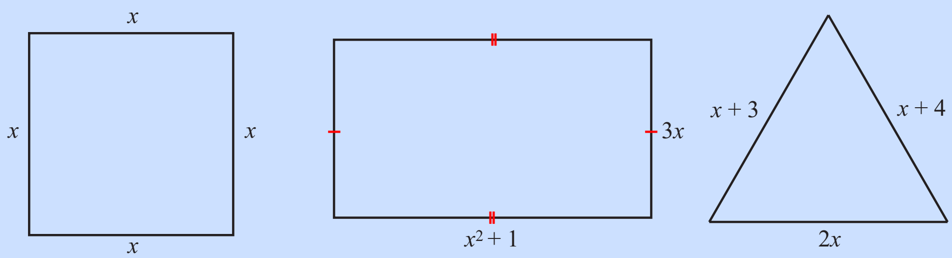
You probably don't think about algebra when you watch animated cartoons, insert emojis in messages or play games on your phone or computer but animators use complex algebra to programme all these items and to make objects move on screen.

Worked example 4

For each of the shapes in the diagram below:

i Write an expression for the perimeter of each shape.
ii Find the perimeter in cm if $x = 4$.

a **b** **c**



a	**i**	$x + x + x + x = 4x$	Add the four lengths together.
	ii	$4 \times x = 4 \times 4$ $= 16\,\text{cm}$	Substitute 4 into the expression.

b	**i**	$3x + (x^2 + 1) + 3x + (x^2 + 1) = 2(3x) + 2(x^2 + 1)$	Add the four lengths together and write in its simplest form.
	ii	$2 \times (3 \times x) + 2 \times (x^2 + 1) = 2 \times (3 \times 4) + 2 \times (4^2 + 1)$ $= 2 \times 12 + 2 \times (16 + 1)$ $= 24 + 2 \times 17$ $= 24 + 34$ $= 58\text{ cm}$	Substitute 4 into the expression.
c	**i**	$x + 3 + x + 4 + 2x$	Add the three lengths together.
	ii	$x + 3 + x + 4 + 2 \times x = 4 + 3 + 4 + 4 + 2 \times 4$ $= 4 + 3 + 4 + 4 + 8$ $= 23\text{ cm}$	Substitute 4 into the expression.

Worked example 5

Complete this table of values for the formula $b = 3a - 3$

a	0	2	4	6
b				

a	0	2	4	6
b	–3	3	9	15

Substitute in the values of a to work out b.

$3 \times 0 - 3 = 0 - 3 = -3$

$3 \times 2 - 3 = 6 - 3 = 3$

$3 \times 4 - 3 = 12 - 3 = 9$

$3 \times 6 - 3 = 18 - 3 = 15$

Exercise 2.2

FAST FORWARD

You will learn more about algebraic fractions in chapter 14.

1 Evaluate the following expressions for $x = 3$.

a $3x$ **b** $10x$ **c** $4x - 2$
d x^3 **e** $2x^2$ **f** $10 - x$
g $x^2 + 7$ **h** $x^3 + x^2$
i $2(x - 1)$ **j** $\frac{4x}{2}$ **k** $\frac{6x}{3}$
l $\frac{90}{x}$ **m** $\frac{10x}{6}$ **n** $\frac{(4x + 2)}{7}$

2 What is the value of each expression when $a = 3$ and $b = 5$ and $c = 2$?

a abc **b** a^2b **c** $4a + 2c$
d $3b - 2(a + c)$ **e** $a^2 + c^2$ **f** $4b - 2a + c$
g $ab + bc + ac$ **h** $2(ab)^2$ **i** $3(a + b)$
j $(b - c) + (a + c)$ **k** $(a + b)(b - c)$ **l** $\frac{3bc}{ac}$
m $\frac{4b}{a} + c$ **n** $\frac{4b^2}{bc}$ **o** $\frac{2(a + b)}{c^2}$

Always show your substitution clearly. Write the formula or expression in its algebraic form but with the letters replaced by the appropriate numbers. This makes it clear to your teacher, or an examiner, that you have put the correct numbers in the right places.

p $\frac{3abc}{10a}$ **q** $\frac{6b^2}{(a+c)^2}$ **r** $(\frac{1}{2}abc)^2$

s $\frac{8a}{\sqrt[3]{a+b}}$ **t** $\frac{6ab}{a^2} - 2bc$

You may need to discuss part (f)(i) with your teacher.

3 Work out the value of y in each formula when:

i $x = 0$ **i** $x = 3$ **iii** $x = 4$ **iv** $x = 10$ **v** $x = 50$

a $y = 4x$ **b** $y = 3x + 1$ **c** $y = 100 - x$

d $y = \frac{x}{2}$ **e** $y = x^2$ **f** $y = \frac{100}{x}$

g $y = 2(x + 2)$ **h** $y = 2(x + 2) - 10$ **i** $y = 3x^3$

4 A sandwich costs \$3 and a drink costs \$2.

a Write an expression to show the total cost of buying x sandwiches and y drinks.
b Find the total cost of:

i four sandwiches and three drinks
ii 20 sandwiches and 20 drinks
iii 100 sandwiches and 25 drinks.

5 The formula for finding the perimeter of a rectangle is $P = 2(l + b)$, where l represents the length and b represents the breadth of the rectangle.

Find the perimeter of a rectangle if:

a the length is 12 cm and the breadth is 9 cm
b the length is 2.5 m and the breadth is 1.5 m
c the length is 20 cm and the breadth is half as long
d the breadth is 2 cm and the length is the cube of the breadth.

REWIND

Think back to chapter 1 and the different types of number that you have already studied

6 **a** Find the value of the expression $n^2 + n + 41$ when:

i $n = 1$ **ii** $n = 3$ **iii** $n = 5$ **iv** $n = 10$

b What do you notice about all of your answers?
c Why is this different when $n = 41$?

E

In fact the outcome is the same for $n = 1$ to 39, but then breaks down for the first time at $n = 40$

2.3 Simplifying expressions

Remember, terms are not separated by × or ÷ signs. A fraction line means divide, so the parts of a fraction are all counted as one term, even if there is a + or – sign in the numerator or denominator.

So, $\frac{a+b}{c}$ is one term.

Remember, the number in a term is called a **coefficient**. In the term $2a$, the coefficient is 2; in the term $-3ab$, the coefficient is -3. A term with only numbers is called a constant. So in $2a + 4$, the constant is 4.

The parts of an algebraic expression are called *terms*. Terms are separated from each other by + or – signs. So $a + b$ is an expression with two terms, but ab is an expression with only one term and $2 + \frac{3a}{b} - \frac{ab}{c}$ is an expression with three terms.

Adding and subtracting like terms

Terms with exactly the same variables are called *like terms*. $2a$ and $4a$ are like terms; $3xy^2$ and $-xy^2$ are like terms.

The variables and any indices attached to them have to be identical for terms to be like terms. Don't forget that variables in a different order mean the same thing, so xy and yx are like terms ($x \times y = y \times x$).

Like terms can be added or subtracted to simplify algebraic expressions.

Note that a '+' or a '–' that appears within an algebraic expression, is attached to the term that sits to its right. For example: $3x - 4y$ contains two terms, $3x$ and $-4y$. If a term has no symbol written before it then it is taken to mean that it is '+'.

Worked example 6

Simplify:

a $4a + 2a + 3a$ **b** $4a + 6b + 3a$ **c** $5x + 2y - 7x$
d $2p + 5q + 3q - 7p$ **e** $2ab + 3a^2b - ab + 3ab^2$

a	$4a + 2a + 3a$ $= 9a$	Terms are all like. Add the coefficients, write the term.
b	$4a + 6b + 3a$ $= 7a + 6b$	Identify the like terms ($4a$ and $3a$). Add the coefficients of like terms. Write terms in alphabetical order.
c	$5x + 2y - 7x$ $= -2x + 2y$	Identify the like terms ($5x$ and $-7x$). Subtract the coefficients, remember the rules. Write the terms. (This could also be written as $2y - 2x$.)
d	$2p + 5q + 3q - 7p$ $= -5p + 8q$	Identify the like terms ($2p$ and $-7p$; $5q$ and $3q$). Add and subtract the coefficients. Write the terms.
e	$2ab + 3a^2b - ab + 3ab^2$ $= ab + 3a^2b + 3ab^2$	Identify like terms; pay attention to terms that are squared because a and a^2 are not like terms. Remember that ab means $1ab$.

Notice that you can rearrange the terms provided that you remember to take the '–' and '+' signs with the terms to their right. For example:

$3x - 2y + 5z$
$= 3x + 5z - 2y$
$= 5z + 3x - 2y$
$= -2y + 3x + 5z$

Exercise 2.3

1 Identify the like terms in each set.

a $6x, -2y, 4x, x$ **b** $x, -3y, \frac{3}{4}y, -5y$ **c** $ab, 4b, -4ba, 6a$
d $2, -2x, 3xy, 3x, -2y$ **e** $5a, 5ab, ab, 6a, 5$ **f** $-1xy, -yx, -2y, 3, 3x$

2 Simplify by adding or subtracting like terms.

a $2y + 6y$ **b** $9x - 2x$ **c** $10x + 3x$
d $21x + x$ **e** $7x - 2x$ **f** $4y - 4y$
g $9x - 10x$ **h** $y - 4y$ **i** $5x - x$
j $9xy - 2xy$ **k** $6pq - 2qp$ **l** $14xyz - xyz$
m $4x^2 - 2x^2$ **n** $9y^2 - 4y^2$ **o** $y^2 - 2y^2$
p $14ab^2 - 2ab^2$ **q** $9x^2y - 4x^2y$ **r** $10xy^2 - 8xy^2$

3 Simplify:

a $2x + y + 3x$ **b** $4y - 2y + 4x$ **c** $6x - 4x + 5x$
d $10 + 4x - 6$ **e** $4xy - 2y + 2xy$ **f** $5x^2 - 6x^2 + 2x$
g $5x + 4y - 6x$ **h** $3y + 4x - x$ **i** $4x + 6y + 4x$
j $9x - 2y - x$ **k** $12x^2 - 4x + 2x^2$ **l** $12x^2 - 4x^2 + 2x^2$
m $5xy - 2x + 7xy$ **n** $xy - 2xz + 7xy$ **o** $3x^2 - 2y^2 - 4x^2$
p $5x^2y + 3x^2y - 2xy$ **q** $4xy - x + 2yx$ **r** $5xy - 2 + xy$

FAST FORWARD

You will need to be very comfortable with the simplification of algebraic expressions when solving equations, inequalities and simplifying expansions throughout the course. ▶

4 Simplify as far as possible:

a $8y - 4 - 6y - 4$ **b** $x^2 - 4x + 3x^2 - x$ **c** $5x + y + 2x + 3y$
d $y^2 + 2y + 3y - 7$ **e** $x^2 - 4x - x + 3$ **f** $x^2 + 3x - 7 + 2x$
g $4xyz - 3xy + 2xz - xyz$ **h** $5xy - 4 + 3yx - 6$ **i** $8x - 4 - 2x - 3x^2$

5 Write an expression for the perimeter (P) of each of the following shapes and then simplify it to give P in the simplest possible terms.

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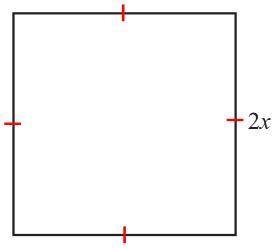

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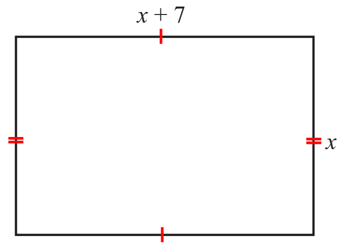

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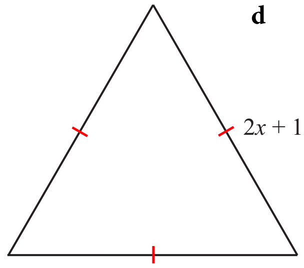

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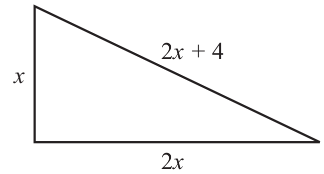

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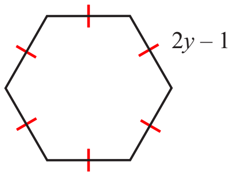

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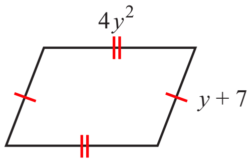

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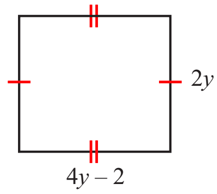

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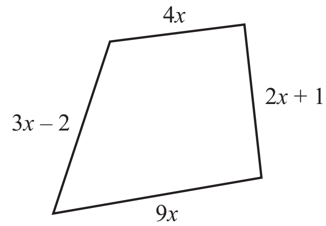


Multiplying and dividing in expressions

Although terms are not separated by × or ÷ they still need to be written in the simplest possible way to make them easier to work with.

In section 2.1 you learned how to write expressions in simpler terms when multiplying and dividing them. Make sure you understand and remember these important rules:

- $3x$ means $3 \times x$ and $3xy$ means $3 \times x \times y$
- xy means $x \times y$
- x^2 means $x \times x$ and x^2y means $x \times x \times y$ (only the x is squared)
- $\frac{2a}{4}$ means $2a \div 4$

Worked example 7

Simplify:

a $4 \times 3x$ **b** $4x \times 3y$ **c** $4ab \times 2bc$ **d** $7x \times 4yz \times 3$

a	$4 \times 3x = 4 \times 3 \times x$ $= 12 \times x$ $= 12x$	Insert the missing × signs. Multiply the numbers first. Write in simplest form.
b	$4x \times 3y = 4 \times x \times 3 \times y$ $= 12 \times x \times y$ $= 12xy$	Insert the missing × signs. Multiply the numbers. Write in simplest form.
c	$4ab \times 2bc = 4 \times a \times b \times 2 \times b \times c$ $= 8 \times a \times b \times b \times c$ $= 8ab^2c$	Insert the missing × signs. Multiply the numbers, then the variables. Write in simplest form.
d	$7x \times 4yz \times 3 = 7 \times x \times 4 \times y \times z \times 3$ $= 84 \times x \times y \times z$ $= 84xyz$	Insert the missing × signs. Multiply the numbers. Write in simplest form.

You can multiply numbers first and variables second because the order of any multiplication can be reversed without changing the answer.

FAST FORWARD

You will learn more about cancelling and equivalent fractions in chapter 5. ▶

Worked example 8

Simplify:

a $\frac{12x}{3}$ **b** $\frac{12xy}{3x}$ **c** $\frac{7xy}{70y}$ **d** $\frac{2x}{3}\times\frac{4x}{2}$

a	$\frac{12x}{3} = \frac{\overset{4}{\cancel{12}}x}{\underset{1}{\cancel{3}}} = \frac{4x}{1} = 4x$	Divide both top and bottom by 3 (making the numerator and denominator smaller so that the fraction is in its simplest form is called *cancelling*).
b	$\frac{12xy}{3x} = \frac{\overset{4}{\cancel{12}}\cancel{x}y}{\underset{1}{\cancel{3}}\cancel{x}} = \frac{4\times y}{1} = 4y$	Cancel and then multiply.
c	$\frac{7xy}{70y} = \frac{\overset{1}{\cancel{7}}x\cancel{y}}{\underset{10}{\cancel{70}}\cancel{y}} = \frac{x}{10}$	Cancel.
d	$\frac{2x}{3}\times\frac{4x}{2} = \frac{2\times x\times 4\times x}{3\times 2}$	Insert signs and multiply.
	$= \frac{\overset{4}{\cancel{8}}x^2}{\cancel{6}_3}$	Cancel.
	$= \frac{4x^2}{3}$	
	or	
	$\frac{\overset{1}{\cancel{2}}x}{3}\times\frac{4x}{\underset{1}{\cancel{2}}} = \frac{1x}{3}\times\frac{4x}{1} = \frac{4x^2}{3}$	Cancel first, then multiply.

Exercise 2.4

1 Multiply:

a $2\times 6x$	**b** $4y\times 2$	**c** $3m\times 4$
d $2x\times 3y$	**e** $4x\times 2y$	**f** $9x\times 3y$
g $8y\times 3z$	**h** $2x\times 3y\times 2$	**i** $4xy\times 2xy$
j $4xy\times 2x$	**k** $9y\times 3xy$	**l** $4y\times 2x\times 3y$
m $2a\times 4ab$	**n** $3ab\times 4bc$	**o** $6abc\times 2a$
p $8abc\times 2ab$	**q** $4\times 2ab\times 3c$	**r** $12x^2\times 2\times 3y^2$

2 Simplify:

a $3\times 2x\times 4$	**b** $5x\times 2x\times 3y$	**c** $2x\times 3y\times 2xy$
d $xy\times xz\times x$	**e** $2\times 2\times 3x\times 4$	**f** $4\times 2x\times 3x^2y$
g $x\times y^2\times 4x$	**h** $2a\times 3ab\times 2c$	**i** $10x\times 2y\times 3$
j $4\times x\times 2\times y$	**k** $9\times x^2\times xy$	**l** $4xy^2\times 2x^2y$
m $7xy\times 2xz\times 3yz$	**n** $4xy\times 2x^2y\times 7$	**o** $9\times xyz\times 4xy$
p $3x^2y\times 2xy^2\times 3xy$	**q** $9x\times 2xy\times 3x^2$	**r** $2x\times xy^2\times 3xy$

3 Simplify:

a $\frac{15x}{3}$	**b** $\frac{40x}{10}$	**c** $\frac{21x}{7}$	**d** $\frac{12xy}{2x}$
e $\frac{14xy}{2y}$	**f** $\frac{18x^2y}{9x^2}$	**g** $\frac{10xy}{40x}$	**h** $\frac{15x}{60xy}$
i $\frac{7xyz}{14xy}$	**j** $\frac{6xy}{x}$	**k** $\frac{x}{4x}$	**l** $\frac{x}{9x}$

4 Simplify:

a	$8x \div 2$	**b**	$12xy \div 2x$	**c**	$16x^2 \div 4xy$	**d**	$24xy \div 3xy$
e	$14x^2 \div 2y^2$	**f**	$24xy \div 8y$	**g**	$8xy \div 24y$	**h**	$9x \div 36xy$
i	$\dfrac{77xyz}{11xz}$	**j**	$\dfrac{45xy}{20x}$	**k**	$\dfrac{60x^2y^2}{15xy}$	**l**	$\dfrac{100xy}{25x^2}$

5 Simplify these as far as possible.

a	$\dfrac{x}{2}\times\dfrac{y}{3}$	**b**	$\dfrac{x}{3}\times\dfrac{x}{4}$	**c**	$\dfrac{xy}{2}\times\dfrac{5x}{3}$	**d**	$\dfrac{2x}{3}\times\dfrac{5}{y}$
e	$\dfrac{2x}{4}\times\dfrac{3y}{4}$	**f**	$\dfrac{5x}{2}\times\dfrac{5x}{2}$	**g**	$\dfrac{x}{y}\times\dfrac{2y}{x}$	**h**	$\dfrac{xy}{3}\times\dfrac{x}{y}$
i	$5y\times\dfrac{2x}{5}$	**j**	$4\times\dfrac{2x}{3}$	**k**	$\dfrac{x}{6}\times\dfrac{3}{2x}$	**l**	$\dfrac{5x}{2}\times\dfrac{4x}{10}$

2.4 Working with brackets

FAST FORWARD

In this section you will focus on simple examples. You will learn more about removing brackets and working with negative terms in chapters 6 and 10. You will also learn a little more about why this method works. ▶

Removing brackets is really just multiplying, so the same rules you used for multiplication apply in these examples.

When an expression has brackets, you normally have to remove the brackets before you can simplify the expression. Removing the brackets is called expanding the expression.

To remove brackets you multiply each term inside the bracket by the number (and/or variables) outside the bracket. When you do this you need to pay attention to the positive and negative signs in front of the terms:

$x(y+z) = xy + xz$

$x(y-z) = xy - xz$

Worked example 9

Remove the brackets to simplify the following expressions.

a $2(2x+6)$ **b** $4(7-2x)$ **c** $2x(x+3y)$ **d** $xy(2-3x)$

a
$$2(\overset{i}{2x}+\underset{ii}{6}) = \overset{i}{2\times 2x}+\overset{ii}{2\times 6}$$
$$= 4x + 12$$

b
$$4(\overset{i}{7}-\underset{ii}{2x}) = \overset{i}{4\times 7}-\overset{ii}{4\times 2x}$$
$$= 28 - 8x$$

c
$$2x(\overset{i}{x}+\underset{ii}{3y}) = \overset{i}{2x\times x}+\overset{ii}{2x\times 3y}$$
$$= 2x^2 + 6xy$$

d
$$xy(\overset{i}{2}-\underset{ii}{3x}) = \overset{i}{xy\times 2}-\overset{ii}{xy\times 3x}$$
$$= 2xy - 3x^2y$$

For parts (a) to (d) write the expression out, or do the multiplication mentally.

Follow these steps when multiplying by a term outside a bracket:

- Multiply the term on the left-hand inside of the bracket first - shown by the red arrow labelled i.
- Then multiply the term on the right-hand side – shown by the blue arrow labelled ii.
- Then add the answers together.

Exercise 2.5

1 Expand:

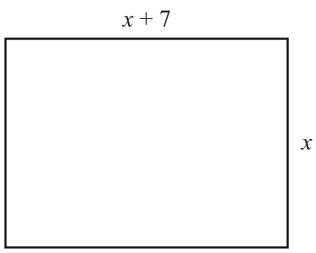
a $2(x+6)$	**b** $3(x+2)$	**c** $4(2x+3)$
d $10(x-6)$	**e** $4(x-2)$	**f** $3(2x-3)$
g $5(y+4)$	**h** $6(4+y)$	**i** $9(y+2)$
j $7(2x-2y)$	**k** $2(3x-2y)$	**l** $4(x+4y)$
m $5(2x-2y)$	**n** $6(3x-2y)$	**o** $3(4y-2x)$
p $4(y-4x^2)$	**q** $9(x^2-y)$	**r** $7(4x+x^2)$

2 Remove the brackets to expand these expressions.

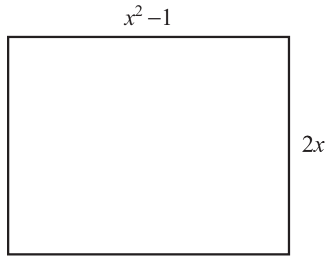
a $2x(x+y)$	**b** $3y(x-y)$	**c** $2x(x+2y)$
d $4x(3x-2y)$	**e** $xy(x-y)$	**f** $3y(4x+2)$
g $2xy(9-4y)$	**h** $2x^2(3-2y)$	**i** $3x^2(4-4x)$
j $4x(9-2y)$	**k** $5y(2-x)$	**l** $3x(4-y)$
m $2x^2y(y-2x)$	**n** $4xy^2(3-2x)$	**o** $3xy^2(x+y)$
p $x^2y(2x+y)$	**q** $9x^2(9-2x)$	**r** $4xy^2(3-x)$

3 Given the formula for area, $A = \text{length} \times \text{breadth}$, write an expression for A in terms of x for each of the following rectangles. Expand the expression to give A in simplest terms.

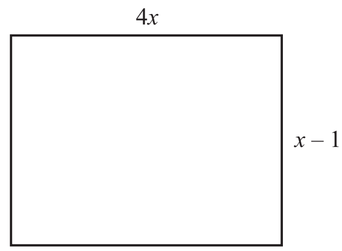
a



b



c



Expanding and collecting like terms

When you remove brackets and expand an expression you may end up with some like terms. When this happens, you collect the like terms together and add or subtract them to write the expression in its simplest terms.

Worked example 10

Expand and simplify where possible.

a $6(x+3)+4$ **b** $2(6x+1)-2x+4$ **c** $2x(x+3)+x(x-4)$

a	$6(x+3)+4 = 6x+18+4$ $= 6x+22$	Remove the brackets. Add like terms.
b	$2(6x+1)-2x+4 = 12x+2-2x+4$ $= 10x+6$	Remove the brackets. Add or subtract like terms.
c	$2x(x+3)+x(x-4) = 2x^2+6x+x^2-4x$ $= 3x^2+2x$	Remove the brackets. Add or subtract like terms.

Exercise 2.6

1 Expand and simplify:

a $2(5+x)+3x$	**b** $3(y-2)+4y$	**c** $2x+2(x-4)$
d $4x+2(x-3)$	**e** $2x(4+x)-5$	**f** $4(x+2)-7$

g $6+3(x-2)$	**h** $4x+2(2x+3)$	**i** $2x+3+2(2x+3)$
j $3(2x+2)-3x-4$	**k** $6x+2(x+3)$	**l** $7y+y(x-4)-4$
m $2x(x+4)-4$	**n** $2y(2x-2y+4)$	**o** $2y(5-4y)-4y^2$
p $3x(2x+4)-9$	**q** $3y(y+2)-4y^2$	**r** $2(x-1)+4x-4$

2 Simplify these expressions by removing brackets and collecting like terms.

a $4(x+40)+2(x-3)$	**b** $2(x-2)+2(x+3)$	**c** $3(x+2)+4(x+5)$
d $8(x+10)+4(3-2x)$	**e** $4(x^2+2)+2(4-x^2)$	**f** $4x(x+1)+2x(x+3)$
g $3x(4y-4)+4(3xy+4x)$	**h** $2x(5y-4)+2(6x-4xy)$	**i** $3x(4-8y)+3(2xy-5x)$
j $3(6x-4y)+x(3-2y)$	**k** $3x^2(4-x)+2(5x^2-2x^3)$	**l** $x(x-y)+3(2x-y)$
m $4(x-2)+3x(4-y)$	**n** $x(x+y)+x(x-y)$	**o** $2x(x+y)+2(x^2+3xy)$
p $x(2x+3)+3(5-2x)$	**q** $4(2x-3)+(x-5)$	**r** $3(4xy-2x)+5(3x-xy)$

2.5 Indices

Revisiting index notation

The plural of 'index' is 'indices'.

You already know how to write powers of two and three using indices:

$2\times2=2^2$ and $y\times y=y^2$

$2\times2\times2=2^3$ and $y\times y\times y=y^3$

Exponent is another word sometimes used to mean 'index' or 'power'. These words can be used interchangeably but 'index' is more commonly used for IGCSE.

When you write a number using indices (powers) you have written it in index notation. Any number can be used as an index including 0, negative integers and fractions. The index tells you how many times the **base** has been multiplied by itself. So:

$3\times3\times3\times3=3^4$ 3 is the base, 4 is the index

$a\times a\times a\times a\times a=a^5$ a is the base, 5 is the index

When you write a power out in full as a multiplication you are writing it in expanded form.

Worked example 11

Write each expression using index notation.

a $2\times2\times2\times2\times2\times2$ **b** $x\times x\times x\times x$ **c** $x\times x\times x\times y\times y\times y\times y$

a	$2\times2\times2\times2\times2\times2=2^6$	Count how many times 2 is multiplied by itself to give you the index.
b	$x\times x\times x\times x=x^4$	Count how many times x is multiplied by itself to give you the index.
c	$x\times x\times x\times y\times y\times y\times y=x^3y^4$	Count how many times x is multiplied by itself to get the index of x; then work out the index of y in the same way.

When you *evaluate* a number raised to a power, you are carrying out the multiplication to obtain a single value.

Worked example 12

Use your calculator to evaluate:

a 2^5 **b** 2^8 **c** 10^6 **d** 7^4

a	$2^5=32$	Enter 2 $x^{\square}$ 5 =
b	$2^8=256$	Enter 2 $x^{\square}$ 8 =
c	$10^6=1\,000\,000$	Enter 1 0 $x^{\square}$ 6 =
d	$7^4=2401$	Enter 7 $x^{\square}$ 4 =

REWIND

Quickly remind yourself, from chapter 1, how a composite number can be written as a product of primes. ◀

Index notation and products of prime factors

Index notation is very useful when you have to express a number as a product of its prime factors because it allows you to write the factors in a short form.

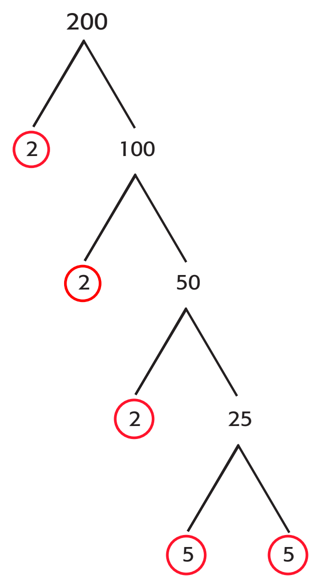
Worked example 13

Express these numbers as products of their prime factors in index form.

a 200 **b** 19 683

The diagrams below are a reminder of the factor tree and division methods for finding the prime factors.

a



$= 2 \times 2 \times 2 \times 5 \times 5$

b

3	19 683
3	6561
3	2187
3	729
3	243
3	81
3	27
3	9
3	3
	1

$= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

a $200 = 2^3 \times 5^2$ **b** $19\,683 = 3^9$

Exercise 2.7

1 Write each expression using index notation.

a $2 \times 2 \times 2 \times 2 \times 2$ **b** $3 \times 3 \times 3 \times 3$ **c** 7×7
d $11 \times 11 \times 11$ **e** $10 \times 10 \times 10 \times 10 \times 10$ **f** $8 \times 8 \times 8 \times 8 \times 8$
g $a \times a \times a \times a$ **h** $x \times x \times x \times x \times x$ **i** $y \times y \times y \times y \times y \times y$
j $a \times a \times a \times b \times b$ **k** $x \times x \times y \times y \times y \times y$ **l** $p \times p \times p \times q \times q$
m $x \times x \times x \times x \times y \times y \times y$ **n** $x \times y \times x \times y \times y \times x \times y$ **o** $a \times b \times a \times b \times a \times b \times c$

2 Evaluate:

a 10^4 **b** 7^3 **c** 6^7 **d** 4^9 **e** 10^5
f 1^{12} **g** 2^{10} **h** 9^4 **i** 2^6 **j** $2^3 \times 3^4$
k $5^2 \times 3^8$ **l** $4^5 \times 2^6$ **m** $2^6 \times 3^4$ **n** $2^8 \times 3^2$ **o** $5^3 \times 3^5$

3 Express the following as products of prime factors, in index notation.

a 64 **b** 243 **c** 400 **d** 1600 **e** 16 384
f 20 736 **g** 59 049 **h** 390 625

4 Write several square numbers as products of prime factors, using index notation. What is true about the index needed for each prime?

The laws of indices

The laws of indices are very important in algebra because they give you quick ways of simplifying expressions. You will use these laws over and over again as you learn more and more algebra, so it is important that you understand them and that you can apply them in different situations.

Multiplying the same base number with different indices

Look at these two multiplications:

$3^2 \times 3^4 \qquad x^3 \times x^4$

In the first multiplication, 3 is the 'base' number and in the second, x is the 'base' number.

You already know you can simplify these by expanding them like this:

$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 \qquad x \times x \times x \times x \times x \times x \times x = x^7$

In other words:

$3^2 \times 3^4 = 3^{2+4}$ and $x^3 \times x^4 = x^{3+4}$

This gives you the law of indices for multiplication:

When you multiply index expressions with the same base you can add the indices: $x^m \times x^n = x^{m+n}$

Remember every letter or number has a power of 1 (usually unwritten). So x means x^1 and y means y^1.

Worked example 14

Simplify:

a $4^3 \times 4^6$ **b** $x^2 \times x^3$ **c** $2x^2y \times 3xy^4$

a	$4^3 \times 4^6 = 4^{3+6} = 4^9$	Add the indices.
b	$x^2 \times x^3 = x^{2+3} = x^5$	Add the indices.
c	$2x^2y \times 3xy^4 = 2 \times 3 \times x^{2+1} \times y^{1+4} = 6x^3y^5$	Multiply the numbers first, then add the indices of like variables.

FAST FORWARD

The multiplication and division rules will be used more when you study standard form in chapter 5. ▶

Dividing the same base number with different indices

Look at these two divisions:

$3^4 \div 3^2$ and $x^6 \div x^2$

You already know you can simplify these by writing them in expanded form and cancelling like this:

$$\frac{3 \times 3 \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3}} = 3 \times 3 = 3^2 \qquad \frac{x \times x \times x \times x \times \cancel{x} \times \cancel{x}}{\cancel{x} \times \cancel{x}} = x \times x \times x \times x = x^4$$

In other words:

$3^4 \div 3^2 = 3^{4-2}$ and $x^6 \div x^2 = x^{6-2}$

This gives you the law of indices for division:

When you divide index expressions with the same base you can subtract the indices: $x^m \div x^n = x^{m-n}$

Worked example 15

Simplify:

a $\frac{x^6}{x^2}$ **b** $\frac{6x^5}{3x^2}$ **c** $\frac{10x^3y^2}{5xy}$

a	$\frac{x^6}{x^2} = x^{6-2} = x^4$	Subtract the indices.
b	$\frac{6x^5}{3x^2} = \frac{6}{3} \times \frac{x^5}{x^2} = \frac{2}{1} \times x^{5-2} = 2x^3$	Divide (cancel) the coefficients. Subtract the indices.
c	$\frac{10x^3y^2}{5xy} = \frac{10}{5} \times \frac{x^3}{x} \times \frac{y^2}{y}$ $= \frac{2}{1} \times x^{3-1} \times y^{2-1}$ $= 2x^2y$	Divide the coefficients. Subtract the indices.

Remember 'coefficient' is the *number* in the term.

The power 0

You should remember that any value divided by itself gives 1.

So, $3 \div 3 = 1$ and $x \div x = 1$ and $\frac{x^4}{x^4} = 1$.

If we use the law of indices for division we can see that:

$$\frac{x^4}{x^4} = x^{4-4} = x^0$$

This gives us the law of indices for the power 0.

Any value to the power 0 is equal to 1. So $x^0 = 1$.

Technically, there is an awkward exception to this rule when $x = 0$. 0^0 is usually defined to be 1!

Raising a power

Look at these two examples:

$$(x^3)^2 = x^3 \times x^3 = x^{3+3} = x^6 \qquad (2x^3)^4 = 2x^3 \times 2x^3 \times 2x^3 \times 2x^3 = 2^4 \times x^{3+3+3+3} = 16x^{12}$$

If we write the examples in expanded form we can see that $(x^3)^2 = x^6$ and $(2x^3)^4 = 16x^{12}$

This gives us the law of indices for raising a power to another power:

When you have to raise a power to another power you multiply the indices: $(x^m)^n = x^{mn}$

Worked example 16

Simplify:

a $(x^3)^6$ **b** $(3x^4y^3)^2$ **c** $(x^3)^4 \div (x^6)^2$

a	$(x^3)^6 = x^{3\times6}$ $= x^{18}$	Multiply the indices.

A common error is to forget to take powers of the numerical terms. For example in part (b), the '3' needs to be squared to give '9'.

b	$(3x^4y^3)^2$ $= 3^2 \times x^{4\times 2} \times y^{3\times 2}$ $= 9x^8y^6$	Square each of the terms to remove the brackets and multiply the indices.
c	$(x^3)^4 \div (x^6)^2$ $= x^{3\times 4} \div x^{6\times 2}$ $= x^{12} \div x^{12}$ $= x^{12-12}$ $= x^0$ $= 1$	Expand the brackets first by multiplying the indices. Divide by subtracting the indices.

Exercise 2.8

1 Simplify:

a $3^2 \times 3^6$ **b** $4^2 \times 4^9$ **c** $8^2 \times 8^0$ **d** $x^9 \times x^4$
e $y^2 \times y^7$ **f** $y^3 \times y^4$ **g** $y \times y^5$ **h** $x \times x^4$
i $3x^4 \times 2x^3$ **j** $3y^2 \times 3y^4$ **k** $2x \times x^3$ **l** $3x^3 \times 2x^4$
m $5x^3 \times 3$ **n** $8x^4 \times x^3$ **o** $4x^6 \times 2x$ **p** $x^3 \times 4x^5$

2 Simplify:

a $x^6 \div x^4$ **b** $x^{12} \div x^3$ **c** $y^4 \div y^3$ **d** $x^3 \div x$ **e** $\frac{x^5}{x}$
f $\frac{x^6}{x^4}$ **g** $\frac{6x^5}{2x^3}$ **h** $\frac{9x^7}{3x^4}$ **i** $\frac{12y^2}{3y}$ **j** $\frac{3x^4}{6x^3}$
k $\frac{15x^3}{5x^3}$ **l** $\frac{9x^4}{3x^3}$ **m** $\frac{3x^3}{9x^4}$ **n** $\frac{16x^2y^2}{4xy}$ **o** $\frac{12xy^2}{12xy^2}$

3 Simplify:

a $(x^2)^2$ **b** $(x^2)^3$ **c** $(x^2)^6$ **d** $(y^3)^2$ **e** $(2x^2)^5$
f $(3x^2y^2)^2$ **g** $(x^4)^0$ **h** $(5x^2)^3$ **i** $(x^2y^2)^3$ **j** $(x^2y^4)^5$
k $(xy^4)^3$ **l** $(4xy^2)^2$ **m** $(3x^2)^4$ **n** $(xy^6)^4$ **o** $\left(\frac{x^2}{y}\right)^0$

4 Use the appropriate laws of indices to simplify these expressions.

When there is a mixture of numbers and letters, deal with the numbers first and then apply the laws of indices to the letters in alphabetical order.

a $2x^2 \times 3x^3 \times 2x$ **b** $4 \times 2x \times 3x^2y$ **c** $4x \times x \times x^2$
d $(x^2)^2 \div 4x^2$ **e** $11x^3 \times 4(a^2b)^2$ **f** $4x(x^2 + 7)$
g $x^2(4x - x^3)$ **h** $x^8 \div (x^3)^2$ **i** $7x^2y^2 \div (x^3y)^2$
j $\frac{(4x^2 \times 3x^4)}{6x^4}$ **k** $\left(\frac{x^4}{y^2}\right)^3$ **l** $\frac{x^8 \times (xy^2)^4}{(2x^2)^4}$
m $(8x^2)^0$ **n** $4x^2 \times 2x^3 \div (2x)^0$ **o** $\frac{(4x^2y^3)^2}{(2xy)^3}$

Negative indices

At the beginning of this unit you read that negative numbers can also be used as indices. But what does it mean if an index is negative?

Look at the two methods of working out $x^3 \div x^5$ below.

Using expanded notation:

$$x^3 \div x^5 = \frac{x \times x \times x}{x \times x \times x \times x \times x} = \frac{1}{x \times x} = \frac{1}{x^2}$$

Using the law of indices for division:

$$x^3 \div x^5 = x^{3-5} = x^{-2}$$

In plain language you can say that when a number is written with a negative power, it is equal to 1 over the number to the same positive power. Another way of saying '1 over' is **reciprocal**, so a^{-2} can be written as the reciprocal of a^2, i.e. $\frac{1}{a^2}$.

This shows that $\frac{1}{x^2} = x^{-2}$. And this gives you a rule for working with negative indices:

$$\boxed{x^{-m} = \frac{1}{x^m} \text{ (when } x \neq 0\text{)}}$$

When an expression contains negative indices you apply the same laws as for other indices to simplify it.

FAST FORWARD

These are simple examples. Once you have learned more about working with directed numbers in algebra in chapter 6, you will apply what you have learned to simplify more complicated expressions. ▶

Worked example 17

1 Find the value of:

a 4^{-2} **b** 5^{-1}

a $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

b $5^{-1} = \frac{1}{5^1} = \frac{1}{5}$

2 Write these with a positive index.

a x^{-4} **b** y^{-3}

a $x^{-4} = \frac{1}{x^4}$

b $y^{-3} = \frac{1}{y^3}$

3 Simplify. Give your answers with positive indices.

a $\frac{4x^2}{2x^4}$ **b** $2x^{-2} \times 3x^{-4}$ **c** $(3y^2)^{-3}$

a
$$\frac{4x^2}{2x^4} = \frac{4}{2} \times x^{2-4} = 2x^{-2} = \frac{2}{x^2}$$

b
$$2x^{-2} \times 3x^{-4} = \frac{2}{x^2} \times \frac{3}{x^4} = \frac{6}{x^{2+4}} = \frac{6}{x^6}$$

c
$$(3y^2)^{-3} = \frac{1}{(3y^2)^3} = \frac{1}{3^3 \times y^{2\times 3}} = \frac{1}{27y^6}$$

Exercise 2.9

1 Evaluate:

a 4^{-1} **b** 3^{-1} **c** 8^{-1} **d** 5^{-3} **e** 6^{-4} **f** 2^{-5}

2 State whether the following are true or false.

a $4^{-2} = \frac{1}{16}$ **b** $8^{-2} = \frac{1}{16}$ **c** $x^{-3} = \frac{1}{3x}$ **d** $2x^{-2} = \frac{1}{x}$

3 Write each expression so it has only positive indices.

a x^{-2} **b** y^{-3} **c** $(xy)^{-2}$ **d** $2x^{-2}$
e $12x^{-3}$ **f** $7y^{-3}$ **g** $8xy^{-3}$ **h** $12x^{-3}y^{-4}$

4 Simplify. Write your answer using only positive indices.

a $x^{-3} \times x^4$ **b** $2x^{-3} \times 3x^{-3}$ **c** $4x^3 \div 12x^7$ **d** $\dfrac{x^{-7}}{x^4}$

e $(2x^2)^{-3}$ **f** $(x^{-2})^3$ **g** $\dfrac{x^{-3}}{x^{-4}}$ **h** $\dfrac{x^{-2}}{x^3}$

Summary of index laws

$x^m \times x^n = x^{m+n}$ When multiplying terms, add the indices.

$x^m \div x^n = x^{m-n}$ When dividing, subtract the indices.

$(x^m)^n = x^{mn}$ When finding the power of a power, multiply the indices.

$x^0 = 1$ Any value to the power 0 is equal to 1

$x^{-m} = \dfrac{1}{x^m}$ (when $x \neq 0$).

Fractional indices

The laws of indices also apply when the index is a fraction. Look at these examples carefully to see what fractional indices mean in algebra:

- $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$

  $= x^{\frac{1}{2}+\frac{1}{2}}$ Use the law of indices and add the powers.

  $= x^1$

  $= x$

  In order to understand what $x^{\frac{1}{2}}$ means, ask yourself: what number multiplied by itself will give x?

  $\sqrt{x} \times \sqrt{x} = x$

  So, $x^{\frac{1}{2}} = \sqrt{x}$

- $y^{\frac{1}{3}} \times y^{\frac{1}{3}} \times y^{\frac{1}{3}}$

  $= y^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$ Use the law of indices and add the powers.

  $= y^1$

  $= y$

  What number multiplied by itself and then by itself again will give y?

  $\sqrt[3]{y} \times \sqrt[3]{y} \times \sqrt[3]{y} = y$

  So $y^{\frac{1}{3}} = \sqrt[3]{y}$

This shows that any root of a number can be written using fractional indices. So, $x^{\frac{1}{m}} = \sqrt[m]{x}$.

Worked example 18

1 Rewrite using root signs.

a $y^{\frac{1}{2}}$ **b** $x^{\frac{1}{5}}$ **c** $x^{\frac{1}{y}}$

a $y^{\frac{1}{2}} = \sqrt{y}$ **b** $x^{\frac{1}{5}} = \sqrt[5]{x}$ **c** $x^{\frac{1}{y}} = \sqrt[y]{x}$

2 Write in index notation.

a $\sqrt{90}$ **b** $\sqrt[3]{64}$ **c** $\sqrt[4]{x}$ **d** $\sqrt[5]{(x-2)}$

a $\sqrt{90} = 90^{\frac{1}{2}}$ **b** $\sqrt[3]{64} = 64^{\frac{1}{3}}$ **c** $\sqrt[4]{x} = x^{\frac{1}{4}}$ **d** $\sqrt[5]{(x-2)} = (x-2)^{\frac{1}{5}}$

A non-unit fraction has a numerator (the number on top) that is not 1. For example, $\frac{2}{3}$ and $\frac{5}{7}$ are non-unit fractions.

Dealing with non-unit fractions

Sometimes you may have to work with indices that are non-unit fractions. For example $x^{\frac{2}{3}}$ or $y^{\frac{3}{4}}$. To find the rule for working with these, you have to think back to the law of indices for raising a power to another power. Look at these examples carefully to see how this works:

$x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 \qquad \frac{1}{3}\times 2 \text{ is } \frac{2}{3}$

$y^{\frac{3}{4}} = (y^{\frac{1}{4}})^3 \qquad \frac{1}{4}\times 3 = \frac{3}{4}$

You already know that a unit-fraction gives a root. So we can rewrite these expressions using root signs like this:

$(x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2$ and $(y^{\frac{1}{4}})^3 = (\sqrt[4]{y})^3$

So, $(x^{\frac{2}{3}}) = (\sqrt[3]{x})^2$ and $(y^{\frac{3}{4}}) = (\sqrt[4]{y})^3$.

It is possible that you would want to reverse the order of calculations here and the result will be the same. $x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$, but the former tends to work best.

$\boxed{\text{In general terms: } x^{\frac{m}{n}} = x^{m\times\frac{1}{n}} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m}$

Worked example 19

Work out the value of:

a $27^{\frac{2}{3}}$ **b** $25^{1.5}$

a	$27^{\frac{2}{3}} = (\sqrt[3]{27})^2$ $= (3)^2$ $= 9$	$\frac{2}{3} = 2\times\frac{1}{3}$ so you square the cube root of 27.
b	$25^{1.5} = 25^{\frac{3}{2}}$ $= (\sqrt{25})^3$ $= (5)^3$ $= 125$	Change the decimal to a vulgar fraction. $\frac{3}{2} = 3\times\frac{1}{2}$, so you need to cube the square root of 25.

REWIND

You saw in chapter 1 that a 'vulgar' fraction is in the form $\frac{a}{b}$. ◀

E

Sometimes you are asked to find the value of the power that produces a given result. You have already learned that another word for power is exponent. An equation that requires you to find the exponent is called an *exponential* equation.

Worked example 20

If $2^x = 128$ find the value of x.

$2^x = 128$ $2^7 = 128$ $\therefore x = 7$	Remember this means $2 = \sqrt[x]{128}$. Find the value of x by trial and improvement.

Exercise 2.10

Remembers, simplify means to write in its simplest form. So if you were to simplify $x^{\frac{1}{5}} \times x^{-\frac{1}{2}}$ you would write:

$$= x^{\frac{1}{5}-\frac{1}{2}}$$

$$= x^{\frac{2}{10}-\frac{5}{10}}$$

$$= x^{-\frac{3}{10}}$$

$$= \frac{1}{x^{\frac{3}{10}}}$$

1 Evaluate:

a $8^{\frac{1}{3}}$ **b** $32^{\frac{1}{5}}$ **c** $8^{\frac{4}{3}}$ **d** $216^{\frac{2}{3}}$ **e** $256^{0.75}$

2 Simplify:

a $x^{\frac{1}{3}} \times x^{\frac{1}{3}}$ **b** $x^{\frac{1}{2}} \times x^{\frac{2}{3}}$ **c** $\left(\frac{x^4}{x^{10}}\right)^{\frac{1}{2}}$ **d** $\left(\frac{x^6}{y^2}\right)^{\frac{1}{2}}$

e $\frac{x^{\frac{6}{7}}}{x^{\frac{2}{7}}}$ **f** $\frac{7}{8}x^{\frac{1}{2}} \div \frac{1}{2}x^{-\frac{3}{2}}$ **g** $\frac{2x^{\frac{2}{3}}}{x^{\frac{8}{3}}}$ **h** $\frac{9x^{\frac{1}{3}}}{12x^{\frac{4}{3}}}$

i $\frac{1}{2}x^{\frac{1}{2}} \div 2x^2$ **j** $-\frac{1}{2}x^{\frac{3}{4}} \div -2x^{-\frac{1}{4}}$ **k** $\frac{3}{4}x^{\frac{1}{2}} \div \frac{1}{2}x^{-\frac{1}{4}}$ **l** $-\frac{1}{4}x^{\frac{3}{4}} \div -2x^{-\frac{1}{4}}$

3 Find the value of x in each of these equations.

E

a $2^x = 64$ **b** $196^x = 14$ **c** $x^{\frac{1}{5}} = 7$

d $(x-1)^{\frac{3}{4}} = 64$ **e** $3^x = 81$ **f** $4^x = 256$

g $2^{-x} = \frac{1}{64}$ **h** $3^{x-1} = 81$ **i** $9^{-x} = \frac{1}{81}$

j $3^{-x} = 81$ **k** $64^x = 2$ **l** $16^x = 8$ **m** $4^{-x} = \frac{1}{64}$

Summary

Do you know the following?

- Algebra has special conventions (rules) that allow us to write mathematical information is short ways.
- Letters in algebra are called variables, the number before a letter is called a coefficient and numbers on their own are called constants.
- A group of numbers and variables is called a term. Terms are separated by + and – signs, but not by × or ÷ signs.
- Like terms have exactly the same combination of variables and powers. You can add and subtract like terms. You can multiply and divide like and unlike terms.
- The order of operations rules for numbers (BODMAS) apply in algebra as well.
- Removing brackets (multiplying out) is called expanding the expression. Collecting like terms is called simplifying the expression.
- Powers are also called indices. The index tells you how many times a number or variable is multiplied by itself. Indices only apply to the number or variable immediately before them.
- The laws of indices are a set of rules for simplifying expressions with indices. These laws apply to positive, negative, zero and fractional indices.

Are you able to . . . ?

- use letters to represent numbers
- write expressions to represent mathematical information
- substitute letters with numbers to find the value of an expression
- add and subtract like terms to simplify expressions
- multiply and divide to simplify expressions
- expand expressions by removing brackets and getting rid of other grouping symbols
- use and make sense of positive, negative and zero indices
- apply the laws of indices to simplify expressions
- work with fractional indices
- solve exponential equations using fractional indices.

E

Examination practice

Exam-style questions

1 Write an expression in terms of n for:

   a the sum of a number and 12
   b twice a number minus four
   c a number multiplied by x and then squared
   d the square of a number cubed.

2 Simplify:

   a $9xy + 3x + 6xy - 2x$ b $6xy - xy + 3y$

3 Simplify:

   a $\dfrac{a^3b^4}{ab^3}$ b $2(x^3)^2$ c $3x \times 2x^3y^2$

   d $(4ax^2)^0$ e $4x^2y \times x^3y^2$

4 What is the value of x, when:

   a $2^x = 32$ b $3^x = \dfrac{1}{27}$

5 Expand each expression and simplify if possible.

   a $5(x-2)+3(x+2)$ b $5x(x+7y)-2x(2x-y)$

6 Find the value of $(x+5)-(x-5)$ when:

   a $x = 1$ b $x = 0$ c $x = 5$

7 Simplify and write the answers with positive indices only.

   a $x^5 \times x^{-2}$ b $\dfrac{8x^2}{2x^4}$ c $(2x-2)^{-3}$

8 If $x \neq 0$ and $y \neq 0$, simplify:

   a $3x^{\frac{1}{2}} \times 5x^{\frac{1}{2}}$ b $(81y^6)^{\frac{1}{2}}$ c $(64x^3)^{\frac{1}{3}}$

Past paper questions

1 Simplify.

$$\left(\frac{1}{2}x^{\frac{2}{3}}\right)^3$$ [2]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q6 May/June 2016]

2 a Simplify $\left(3125t^{125}\right)^{\frac{1}{5}}$. [2]

   b Find the value of p when $3^p = \dfrac{1}{9}$. [1]

   c Find the value of w when $x^{72} + x^w = x^8$. [1]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q17 May/June 2014]