Chapter 8: Introduction to probability

Key words

- Event
- Probability
- Probability scale
- Trial
- Experimental probability
- Outcome
- Theoretical probability
- Favourable outcomes
- Bias
- Possibility diagram
- Independent
- Mutually exclusive

In this chapter you will learn how to:

- express probabilities mathematically
- calculate probabilities associated with simple experiments
- use possibility diagrams to help you calculate probability of combined events
- identify when events are independent
- identify when events are mutually exclusive

Blaise Pascal was a French mathematician and inventor. In 1654, a friend of his posed a problem of how the stakes of a game may be divided between the players even though the game had not yet ended. Pascal's response was the beginning of the mathematics of probability.

What is the chance that it will rain tomorrow? If you take a holiday in June, how many days of sunshine can you expect? When you flip a coin to decide which team will start a match, how likely is it that you will get a head?

Questions of chance come into our everyday life from what is the weather going to be like tomorrow to who is going to wash the dishes tonight. Words like 'certain', 'even' or 'unlikely' are often used to roughly describe the chance of an event happening but probability refines this to numbers to help make more accurate predictions.

RECAP

You should already be familiar with the following probability work:

Calculating probability

Probability always has a value between 0 and 1.

The sum of probabilities of mutually exclusive events is 1.

$$\text{Probability} = \frac{\text{number of successful outcome}}{\text{total number of outcomes}}$$

Relative frequency

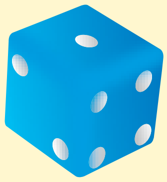
The number of times an event occurs in a number of trials is its relative frequency.

Relative frequency is also called experimental probability.

$$\text{Relative frequency} = \frac{\text{number of times an outcome occurred}}{\text{number of trials}}$$

8.1 Basic probability

A die is the singular of dice.



When you roll a die, you may be interested in throwing a prime number. When you draw a name out of a hat, you may want to draw a boy's name. Throwing a prime number or drawing a boy's name are examples of **events**.

Probability is a measure of how likely an event is to happen. Something that is impossible has a value of zero and something that is certain has a value of one. The range of values from zero to one is called a **probability scale**. A probability cannot be negative or be greater than one.

The smaller the probability, the closer it is to zero and the less likely the associated event is to happen. Similarly, the higher the probability, the more likely the event.

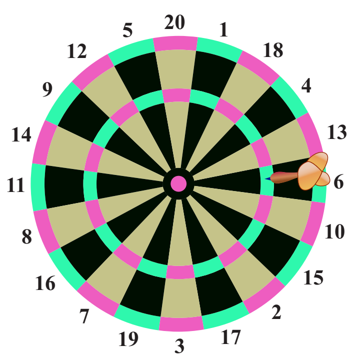
Performing an experiment, such as rolling a die, is called a **trial**. If you repeat an experiment, by carrying out a number of trials, then you can find an **experimental probability** of an event happening: this fraction is often called the **relative frequency**.

P(A) means the probability of event A happening.

$$P(A) = \frac{\text{number of times desired event happens}}{\text{number of trials}}$$

or, sometimes:

$$P(A) = \frac{\text{number of successes}}{\text{number of trials}}$$



Worked example 1

Suppose that a blindfolded man is asked to throw a dart at a dartboard.

If he hits the number six 15 times out of 125 throws, what is the probability of him hitting a six on his next throw?

$$\begin{aligned} P(\text{six}) &= \frac{\text{number times a six obtained}}{\text{number of trials}} \\ &= \frac{15}{125} \\ &= 0.12 \end{aligned}$$

Relative frequency and expected occurrences

You can use relative frequency to make predictions about what might happen in the future or how often an event might occur in a larger sample. For example, if you know that the relative frequency of rolling a 4 on particular die is 18%, you can work out how many times you'd expect to get 4 when you roll the dice 80 or 200 times.

18% of 80 = 14.4 and 18% of 200 = 36, so if you rolled the same die 80 times you could expect to get a 4 about 14 times and if you rolled it 200 times, you could expect to get a 4 thirty-six times.

Remember though, that even if you expected to get a 4 thirty six times, this is not a given and your actual results may be very different.

8.2 Theoretical probability

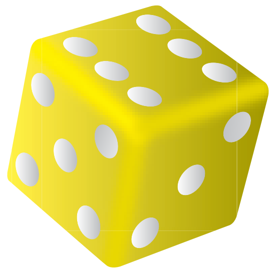
In some countries, theoretical probability is referred to as 'expected probability'. This is a casual reference and does *not* mean the same thing as mathematical 'expectation'.

Never assume that a die or any other object is unbiased unless you are told that this is so.

When you flip a coin you may be interested in the event 'obtaining a head' but this is only one possibility. When you flip a coin there are two possible **outcomes**: 'obtaining a head' or 'obtaining a tail.'

You can calculate the **theoretical** (or expected) probability easily if all of the possible outcomes are *equally likely*, by counting the number of **favourable outcomes** and dividing by the number of possible outcomes. Favourable outcomes are any outcomes that mean your event has happened.

For example, if you throw an unbiased die and need the probability of an even number, then the favourable outcomes are two, four or six. There are three of them.



Under these circumstances the event A (obtaining an even number) has the probability:

$$P(A) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} = \frac{3}{6} = \frac{1}{2}$$

Of course a die may be weighted in some way, or imperfectly made, and indeed this may be true of any object discussed in a probability question. Under these circumstances a die, coin or other object is said to be **biased.** The outcomes will no longer be equally likely and you may need to use experimental probability.

LINK

Biology students will sometimes consider how genes are passed from a parent to a child. There is never a certain outcome, which is why we are all different. Probability plays an important part in determining how likely or unlikely a particular genetic outcome might be.

Worked example 2

An unbiased die is thrown and the number on the upward face is recorded. Find the probability of obtaining:

a a three **b** an even number **c** a prime number.

a	$P(3) = \frac{1}{6}$	There is only one way of throwing a three, but six possible outcomes (you could roll a 1, 2, 3, 4, 5, 6).
b	$P(\text{even number}) = \frac{3}{6} = \frac{1}{2}$	There are three even numbers on a die, giving three favourable outcomes.
c	$P(\text{prime number}) = \frac{3}{6} = \frac{1}{2}$	The prime numbers on a die are 2, 3 and 5, giving three favourable outcomes.

Worked example 3

A card is drawn from an ordinary 52 card pack. What is the probability that the card will be a king?

$P(\text{King}) = \frac{4}{52} = \frac{1}{13}$	Number of possible outcomes is 52. Number of favourable outcomes is four, because there are four kings per pack.

Worked example 4

Jason has 20 socks in a drawer.

8 socks are red, 10 socks are blue and 2 socks are green. If a sock is drawn at random, what is the probability that it is green?

$P(\text{green}) = \frac{2}{20} = \frac{1}{10}$	Number of possible outcomes is 20. Number of favourable outcomes is two.

Worked example 5

The picture shows the famous Hollywood sign in Los Angeles, USA.

Nine painters are assigned a letter from the word HOLLYWOOD for painting at random. Find the probability that a painter is assigned:

a the letter 'Y' **b** the letter 'O' **c** the letter 'H' or the letter 'L' **d** the letter 'Z'.

For each of these the number of possible outcomes is 9.

a	$P(Y) = \frac{1}{9}$	Number of favourable outcomes is one (there is only one 'Y').
b	$P(O) = \frac{3}{9} = \frac{1}{3}$	Number of favourable outcomes is three.
c	$P(H \text{ or } L) = \frac{3}{9} = \frac{1}{3}$	Number of favourable outcomes = number of letters that are *either* H or L = 3, since there is one H and two L's in Hollywood.
d	$P(Z) = \frac{0}{9} = 0$	Number of favourable outcomes is zero (there are no 'Z's)

8.3 The probability that an event does not happen

$\overline{A}$ is usually just pronounced as 'not A'.

Something may happen or it may not happen. The probability of an event happening may be different from the probability of the event not happening but the two combined probabilities will always sum up to one.

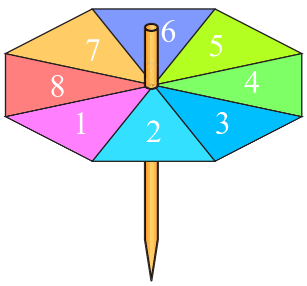
If A is an event, then $\overline{A}$ is the event that A does *not* happen and $P(\overline{A}) = 1 - P(A)$

Worked example 6

The probability that Jasmine passes her driving test is $\frac{2}{3}$. What is the probability that Jasmine fails?

$P(\text{failure}) = 1 - \frac{2}{3} = \frac{1}{3}$	P(failure) = P(not passing) = 1 – P(passing)

Exercise 8.1



1 A simple die is thrown 100 times and the number five appears 14 times. Find the experimental probability of throwing a five, giving your answer as a fraction in its lowest terms.

2 The diagram shows a spinner that is divided into exactly eight equal sectors.

Ryan spins the spinner 260 times and records the results in a table:

Number	1	2	3	4	5	6	7	8
Frequency	33	38	26	35	39	21	33	35

Calculate the experimental probability of spinning:

a the number three **b** the number five **c** an odd number **d** a factor of eight.

3 A consumer organisation commissioned a series of tests to work out the average lifetime of a new brand of solar lamps. The results of the tests as summarised in the table.

Lifetime of lamp, L hours	$0 \leqslant L < 1\,000$	$1\,000 \leqslant L < 2\,000$	$2\,000 \leqslant L < 3\,000$	$3\,000 \leqslant L$
Frequency	30	75	160	35

a Calculate the relative frequency of a lamp lasting for less than 3 000 hours, but more than 1 000 hours.

b If a hardware chain ordered 2 000 of these lamps, how many would you expect to last for more than 3 000 hours?

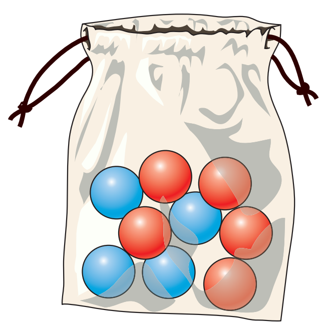
4 Research shows that the probability of a person being right-handed is 0.77. How many left-handed people would you expect in a population of 25 000?

5 A flower enthusiast collected 385 examples of a *Polynomialus mathematicus* flower in deepest Peru. Just five of the flowers were blue.

One flower is chosen at random. Find the probability that:

a it is blue **b** it is not blue.

6 A bag contains nine equal sized balls. Four of the balls are coloured blue and the remaining five balls are coloured red.



What is the probability that, when a ball is drawn from the bag:

a it is blue? **b** it is red?
c it neither blue nor red? **d** it is either blue or red?

7 A bag contains 36 balls. The probability that a ball drawn at random is blue is $\frac{1}{4}$. How many blue balls are there in the bag?

8 Oliver shuffles an ordinary pack of 52 playing cards. If he then draws a single card at random, find the probability that the card is:

a a king **b** a spade **c** a black card **d** a prime-numbered card.

8.4 Possibility diagrams

In some countries, these might be called 'probability space diagrams'.

The probability space is the set of all possible outcomes. It can sometimes simplify our work if you draw a **possibility diagram** to show all outcomes clearly.

See how drawing a possibility diagram helps solve problems in the following worked example.

Worked example 7

Two dice, one red and one blue, are thrown at the same time and the numbers showing on the dice are added together. Find the probability that:

a the sum is 7

b the sum is less than 5

c the sum is greater than or equal to 8

d the sum is less than 8.

Red

Blue

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

In the diagram above there are 36 possible sums, so there are 36 equally likely outcomes in total.

a	$P(7) = \frac{6}{36} = \frac{1}{6}$	There are six 7s in the grid, so six favourable outcomes.
b	$P(\text{less than } 5) = \frac{6}{36} = \frac{1}{6}$	The outcomes that are less than 5 are 2, 3 and 4. These numbers account for six favourable outcomes.
c	$P(\text{greater than or equal to } 8) = \frac{15}{36} = \frac{5}{12}$	The outcomes greater than or equal to 8 (which includes 8) are 8, 9, 10, 11 or 12, accounting for 15 outcomes.
d	$P(\text{less than } 8) = 1 - \frac{5}{12} = \frac{7}{12}$	P(less than 8) = P(not greater than or equal to 8) = 1 – P(greater than or equal to 8)

Exercise 8.2

1 An unbiased coin is thrown twice and the outcome for each is recorded as H (head) or T (tail). A possibility diagram could be drawn as shown.

First throw

Second throw

	H	T
H		T H
T		

a Copy and complete the diagram.

b Find the probability that:

i the coins show the same face
ii the coins both show heads
iii there is at least one head
iv there are no heads.

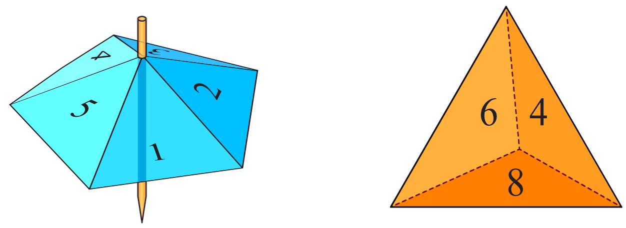
2 Two dice are thrown and the product of the two numbers is recorded.

a Draw a suitable possibility diagram to show all possible outcomes.

b Find the probability that:

i the product is 1
ii the product is 7
iii the product is less than or equal to 4
iv the product is greater than 4
v the product is a prime number
vi the product is a square number.

3



The diagram shows a spinner with five equal sectors numbered 1, 2, 3, 4 and 5, and an unbiased tetrahedral die with faces numbered 2, 4, 6 and 8. The spinner is spun and the die is thrown and the *higher* of the two numbers is recorded. If both show the same number then that number is recorded.

a Draw a possibility diagram to show the possible outcomes.

b Calculate the probability that:

i the higher number is even
ii the higher number is odd
iii the higher number is a multiple of 3
iv the higher number is prime
v the higher number is more than twice the smaller number.

REWIND

You learnt about HCF in chapter 1.

4 An unbiased cubical die has six faces numbered 4, 6, 10, 12, 15 and 24. The die is thrown twice and the highest common factor (HCF) of both results is recorded.

a Draw a possibility diagram to show the possible outcomes.

b Calculate the probability that:

i the HCF is 2
ii the HCF is greater than 2
iii the HCF is not 7
iv the HCF is not 5
v the HCF is 3 or 5
vi the HCF is equal to one of the numbers thrown.

LINK

Computer programming and software development uses probability to build applications (apps) such as voice activated dialing on a mobile phone. When you say a name to the phone, the app chooses the most likely contact from your contact list.

E

5 Two dice are thrown and the result is obtained by adding the two numbers shown. Two sets of dice are available.

Set A: one dice has four faces numbered 1 to 4 and the other eight faces numbered 1 to 8.
Set B: each dice has six faces numbered 1 to 6.

a Copy and complete the possibility diagrams below for each set.

Set A

+	1	2	3	4	5	6	7	8
1								
2								
3								
4								

Set B

+	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

b In an experiment with one of the sets of dice, the following results were obtained

Dice score	Frequency
2	15
3	25
4	44
5	54
6	68
7	87
8	66
9	54
10	43
11	30
12	14

By comparing the probabilities and relative frequencies, decide which set of dice was used.

8.5 Combining independent and mutually exclusive events

If you flip a coin once the probability of it showing a head is 0.5. If you flip the coin a second time the probability of it showing a head is still 0.5, regardless of what happened on the first flip. Events like this, where the first outcome has no influence on the next outcome, are called **independent** events.

Sometimes there can be more than one stage in a problem and you may be interested in what combinations of outcomes there are. If A and B are independent events then:

Note that this formula is only true if A and B are independent.

P(A happens and then B happens) = P(A) × P(B)

or

P(A and B) = P(A) × P(B)

There are situations where it is impossible for events A and B to happen at the same time. For example, if you throw a normal die and let:

A = the event that you get an even number

and

B = the event that you get an odd number

then A and B cannot happen together because no number is both even and odd at the same time. Under these circumstances you say that A and B are **mutually exclusive** events and

Note, this formula only works if A and B are mutually exclusive.

P(A or B) = P(A) + P(B).

The following worked examples demonstrate how these simple formulae can be used.

Worked example 8

James and Sarah are both taking a music examination independently. The probability that James passes is $\frac{3}{4}$ and the probability that Sarah passes is $\frac{5}{6}$.

What is the probability that:

a both pass **b** neither passes **c** at least one passes

d either James or Sarah passes (not both)?

Use the formula for combined events in each case.

Sarah's success or failure in the exam is independent of James' outcome and vice versa.

a P(both pass) = P(James passes and Sarah passes) $= \frac{3}{4} \times \frac{5}{6} = \frac{15}{24} = \frac{5}{8}$

b P(neither passes) = P(James fails and Sarah fails)

$$= (1 - \frac{3}{4}) \times (1 - \frac{5}{6})$$

$$= \frac{1}{4} \times \frac{1}{6}$$

$$= \frac{1}{24}$$

c P(at least one passes) = 1 – P(neither passes) $= 1 - \frac{1}{24} = \frac{23}{24}$

d P(either Sarah or James passes)

= P(James passes and Sarah fails or James fails and Sarah passes)

$$= \frac{3}{4} \times \frac{1}{6} + \frac{1}{4} \times \frac{5}{6}$$

$$= \frac{3}{24} + \frac{5}{24}$$

$$= \frac{8}{24}$$

$$= \frac{1}{3}$$

The events, 'James passes and Sarah fails' and, 'James fails and Sarah passes,' are mutually exclusive because no-one can both pass and fail at the same time. This is why you can add the two probabilities here.

Worked example 9

Simone and Jon are playing darts. The probability that Simone hits a bull's-eye is 0.1. The probability that Jon throws a bull's-eye is 0.2. Simone and Jon throw one dart each. Find the probability that:

a both hit a bull's-eye **b** Simone hits a bull's-eye but Jon does not

c exactly one bull's-eye is hit.

Simone's success or failure at hitting the bull's-eye is independent of Jon's and vice versa.

a P(both throw a bull's-eye) = $0.1 \times 0.2 = 0.02$

b P(Simone throws a bull's-eye but Jon does not) = $0.1 \times (1 - 0.2) = 0.1 \times 0.8 = 0.08$

c P(exactly one bull's-eye is thrown)

= P(Simone throws a bull's-eye and Jon does not or Simone does not throw a bull's-eye and Jon does)

$= 0.1 \times 0.8 + 0.9 \times 0.2$

$= 0.08 + 0.18$

$= 0.26$

Exercise 8.3

Usually 'AND' in probability means you will need to multiply probabilities. 'OR' usually means you will need to add them.

1 A standard cubical die is thrown twice. Calculate the probability that:

a two sixes are thrown
b two even numbers are thrown
c the same number is thrown twice
d the two numbers thrown are different.

2 A bag contains 12 coloured balls. Five of the balls are red and the rest are blue. A ball is drawn at random from the bag. It is then replaced and a second ball is drawn. The colour of each ball is recorded.

a List the possible outcomes of the experiment.
b Calculate the probability that:
i the first ball is blue
ii the second ball is red
iii the first ball is blue and the second ball is red
iv the two balls are the same colour
v the two balls are a different colour
vi neither ball is red
vii at least one ball is red.

FAST FORWARD

You will learn how to calculate probabilities for situations where objects are *not* replaced in chapter 24.

3 Devin and Tej are playing cards. Devin draws a card, replaces it and then shuffles the pack. Tej then draws a card. Find the probability that:

a both draw an ace
b both draw the king of Hearts
c Devin draws a spade and Tej draws a queen
d exactly one of the cards drawn is a heart
e both cards are red or both cards are black
f the cards are different colours.

4 Kirti and Justin are both preparing to take a driving test. They each learned to drive separately, so the results of the tests are independent. The probability that Kirti passes is 0.6 and the probability that Justin passes is 0.4. Calculate the probability that:

a both pass the test
b neither passes the test
c Kirti passes the test, but Justin doesn't pass
d at least one of Kirti and Justin passes
e exactly one of Kirti and Justin passes.

Summary

Do you know the following?

- Probability measures how likely something is to happen.
- An outcome is the single result of an experiment.
- An event is a collection of favourable outcomes.
- Experimental probability can be calculated by dividing the number of favourable outcomes by the number of trials.
- Favourable outcomes are any outcomes that mean your event has happened.
- If outcomes are equally likely then theoretical probability can be calculated by dividing the number of favourable outcomes by the number of possible outcomes.
- The probability of an event happening and the probability of that event ***not*** happening will always sum up to one. If $\overline{A}$ is an event, then A is the event that A does ***not*** happen and P(A) = 1 – P(A)
- Independent events do not affect one another.
- Mutually exclusive events cannot happen together.

Are you able to . . . ?

- find an experimental probability given the results of several trials
- find a theoretical probability
- find the probability that an event will not happen if you know the probability that it will happen
- draw a possibility diagram
- recognise independent and mutually exclusive events
- do calculations involving combined probabilities.

Examination practice

Exam-style questions

1 Rooms in a hotel are numbered from 1 to 19. Rooms are allocated at random as guests arrive.

a What is the probability that the first guest to arrive is given a room which is a prime number? (Remember: 1 is not a prime number.)

b The first guest to arrive is given a room which is a prime number. What is the probability that the second guest to arrive is given a room which is a prime number?

2 A bowl of fruit contains three apples, four bananas, two pears and one orange. Aminata chooses one piece of fruit at random. What is the probability that she chooses:

a a banana?

b a mango?

3 The probability that it will rain in Switzerland on 1 September is $\frac{5}{12}$. State the probability that it will *not* rain in Switzerland on 1 September.

4 Sian has three cards, two of them black and one red. She places them side by side, in random order, on a table. One possible arrangement is red, black, black.

a Write down all the possible arrangements.

b Find the probability that the two black cards are next to one another. Give your answer as a fraction.

5 A die has the shape of a tetrahedron. The four faces are numbered 1, 2, 3 and 4. The die is thrown on the table. The probabilities of each of the four faces finishing flat on the table are as shown.

Face	1	2	3	4
Probability	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{5}{18}$	$\frac{1}{6}$

a Copy the table and fill in the four empty boxes with the probabilities changed to fractions with a common denominator.

b Which face is most likely to finish flat on the table?

c Find the sum of the four probabilities.

d What is the probability that face 3 does not finish flat on the table?

6 Josh and Soumik each take a coin at random out of their pockets and add the totals together to get an amount. Josh has two \$1 coins, a 50c coin, a \$5 coin and three 20c coins in his pocket. Soumik has three \$5 coins, a \$2 coin and three 50c pieces.

a Draw up a possibility diagram to show all the possible outcomes for the sum of the two coins.

b What is the probability that the coins will add up to \$6?

c What is the probability that the coins add up to less than \$2?

d What is the probability that the coins will add up to \$5 or more?

Past paper questions

1 A letter is chosen at random from the following word.

S T A T I S T I C S

Write down the probability that the letter is

a **A** or **I**, [1]

b **E**. [1]

[Cambridge IGCSE Mathematics 0580 Paper 12 Q3 May/June 2011]

2 Omar rolls two fair dice, each numbered from 1 to 6, and adds the numbers shown.
He repeats the experiment 70 times and records the results in a frequency table.
The first 60 results are shown in the tally column of the table.
The last 10 results are 6, 8, 9, 2, 6, 4, 7, 9, 6, 10.

Total	Tally	Frequency
2	𝍷	
3	𝍸	
4	𝍸 𝍷𝍷𝍷𝍷	
5	𝍷𝍷𝍷	
6	𝍸 𝍸 𝍷𝍷𝍷𝍷	
7	𝍸 𝍸	
8	𝍸 𝍷𝍷𝍷	
9	𝍸 𝍷	
10	𝍷𝍷	
11	𝍷	
12	𝍷	

a **i** Complete the frequency table to show all his results. [2]

ii Write down the relative frequency of a total of 5. [3]

[Cambridge IGCSE Mathematics 0580 Paper 33 Q6 a May/June 2013]

3 $\boxed{\textbf{S}}$ $\boxed{\textbf{P}}$ $\boxed{\textbf{A}}$ $\boxed{\textbf{C}}$ $\boxed{\textbf{E}}$ $\boxed{\textbf{S}}$

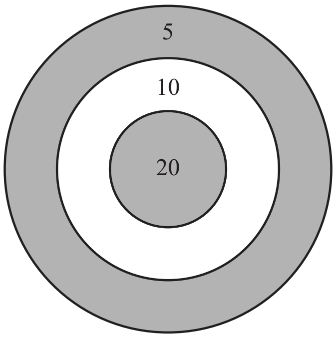
One of the 6 letters is taken at random.

a Write down the probability that the letter is S. [1]

b The letter is replaced and again a letter is taken at random. This is repeated 600 times.
How many times would you expect the letter to be S? [1]

[Cambridge IGCSE Mathematics 0580 Paper 11 Q14 October/November 2013]

4 Kiah plays a game.
The game involves throwing a coin onto a circular board.
Points are scored for where the coin lands on the board.



If the coin lands on part of a line or misses the board then 0 points are scored.
The table shows the probabilities of Kiah scoring points on the board with one throw.

Points scored	20	10	5	0
Probability	x	0.2	0.3	0.45

a Find the value of x. [2]

b Kiah throws a coin fifty times.
Work out the expected number of times she scores 5 points. [1]

c Kiah throws a coin two times.
Calculate the probability that

i she scores either 5 or 0 with her first throw, [2]

ii she scores 0 with her first throw and 5 with her scond throw, [2]

iii she scores a total of 15 points with her two throws. [3]

d Kiah throws a coin three times.
Calculate the probability that she scores a total of 10 points with her three throws. [5]

[Cambridge IGCSE Mathematics 0580 Paper 42 Q5 May/June 2016]

5 Dan either walks or cycles to school.
The probability that he cycles to school is $\frac{1}{3}$.

a Write down the probability that Dan walks to school. [1]

b There are 198 days in a school year.
Work out the expected number of days that Dan cycles to school in a school year. [1]

[Cambridge IGCSE Mathematics 0580 Paper 12 Q9 February/March 2016]