Chapter 22: More equations, formulae and functions

Key words

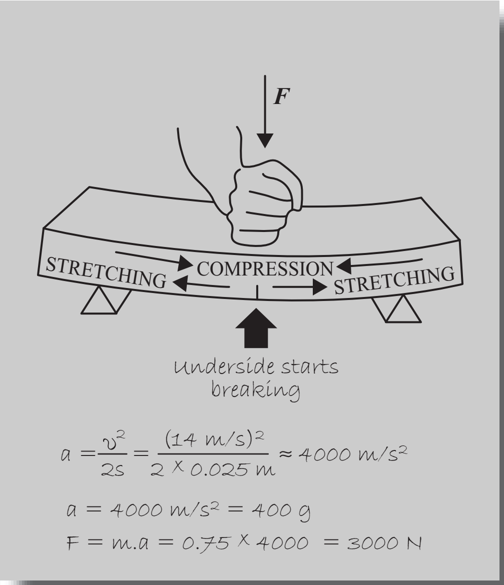
- Equation
- Subject
- Substitute
- Function
- Function notation
- Composite function
- Inverse function

In this chapter you will learn how to:

- make your own equations and use them to solve worded problems

EXTENDED

- construct and transform more complex formulae
- use function notation to describe simple functions and their inverses
- form composite functions.



Formulae can be used to describe the very simple as well as the complex. They can be used to calculate the area of a shape, decide where best to strike a block to smash it or how to launch a rocket into space.

You have already worked with algebraic expressions and learned how to solve equations. Now you are going to apply what you know to solve worded problems by setting up your own equations.

You will also work with more complicated formulae and equations. You will need to be able to rearrange formulae to solve related problems.

Lastly, you will revise what you know about functions and learn how to use a more formal mathematical notation to describe functions and their inverses. You will also work with composite functions.

RECAP

You should already be familiar with the following algebra work:

Equations (Chapters 2 and 6)

Remove brackets and/or fractions
↓
Perform inverse operations to collect terms that include the variable on the same side
↓
Add or subtract like terms to solve the equation

Remember to do the same to both sides to keep the equation balanced.

Formulae (Chapter 6)

You can change the subject of a formula using algebra and the same rules you follow for solving equations.

Functions (Year 9 Mathematics and Chapter 15)

A function is a rule for changing one number into another. For example $y = 3x + 4$ or $x \rightarrow 3x + 4$.

$\sin\theta$, $\cos\theta$ and $\tan\theta$ are examples of trigonometric functions.

22.1 Setting up equations to solve problems

REWIND

You used algebra to write expressions and simple equations in chapter 2. Read through that section again if you have forgotten how to do this. ◀

You already know that you can translate worded problems (story sums) into **equations** using variables to represent unknown quantities. You can then solve the equation to find the solution to the problem.

Working through the simple problems in exercise 22.1 will help you remember how to set up equations that represent the sum, difference, product and quotient of quantities and use these to solve problems.

Exercise 22.1

Translating information from words to diagrams or equations is a very useful problem-solving strategy.

LINK

The idea of taking an input, manipulating it and getting an output applies to computer programming.

1 For each statement, make an equation in terms of x and then and solve it.

- **a** A number multiplied by four gives 32.
- **b** If a certain number is multiplied by 12 the result is 96.
- **c** A number added to 12 gives 55.
- **d** The sum of a number and 13 is 25.
- **e** When six is subtracted from a certain number, the result is 14.
- **f** If a number is subtracted from nine the result is −5.
- **g** The result of dividing a number by seven is 2.5.
- **h** If 28 is divided by a certain number, the result is four.

2 Represent each situation using an equation in terms of y. Solve each equation to find the value of y.

- **a** A number is multiplied by three, then five is added to get 14.
- **b** When six is subtracted from five times a certain number, the result is 54.
- **c** Three times the sum of a number and four gives 150.
- **d** When eight is subtracted from half of a number, the result is 27.

3 Solve each problem by setting up an equation.

a When five is added to four times a certain number, the result is 57. What is the number?
b If six is subtracted from three times a certain number the result is 21. What is the number?
c Four more than a number is divided by three and then multiplied by two to give a result of four. What is the number?
d A number is doubled and then six is added. When this is divided by four, the result is seven. What is the number?

Solving more complex problems

The problems in exercise 22.1 are simple algebraic manipulations. You need to be able to set up equations to solve any problem. To do this, you need to read and make sense of the written problem, represent the situation as an equation and then solve it.

To solve problems by setting up equations:

- Read the problem carefully, paying attention to the words used.
- Decide what you need to find and what information is already given.
- Ask yourself if there is anything to be assumed or deduced from the given information. For example, if the problem mentions equal lengths and breadths of a room can you assume the room is a rectangle? Or, if you are working with a pack of cards, can you assume it is a standard pack with 52 cards?
- Consider whether there is a formula or mathematical relationship that you can use to connect the information in the problem. For example, if you are asked to find the distance around a round shape, you can use the formula $C = \pi d$, or if the problem involves time, distance and speed, you can use the time–distance–speed triangle to form an equation.

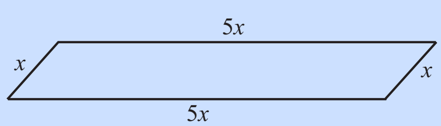
Worked example 1

My mother was 26 years old when I was born. She is presently three times as old as I am. What are our present ages?

Let my present age be x. $\therefore$ my mother's present age is $3x$.	She is 3 times as old as me.
The difference in ages is 26 years, so:	Mother will always be 26 years older.
$3x - x = 26$ $\therefore \quad 2x = 26$ $\therefore \quad x = 13$	
My present age is 13. My mother's present age is 39.	

Worked example 2

A parallelogram has its longest sides five times longer than its shorter sides. If it has a perimeter of 9.6 m, what are the lengths of the long and short sides?



Always say what the variables you are using represent.

Let the shorter side be x metres. $\therefore$ the longer side is $5x$.	The longer side is five times the shorter side.
$5x + x + 5x + x = 9.6\,\text{m}$ $\therefore \quad 12x = 9.6\,\text{m}$ $\therefore \quad x = \frac{9.6}{12} = 0.8\,\text{m}$	Perimeter is the sum of the sides.
So, the shorter side is 0.8 m and the longer side is $5 \times 0.8 = 4\,\text{m}$.	

Exercise 22.2

1 A father is three times as old as his daughter. If the father is 31 years older than his daughter, what are their ages?

2 Jess and Silvia have 420 marbles between them. If Jess has five times as many marbles as Silvia, how many do they each have?

Remember to follow the basic steps in problem-solving when you are faced with a word problem.

3 Soumik has \$5 less than his friend Kofi. If they have \$97.50 altogether, how much does each person have?

4 Two competition winners are to share a prize of \$750. If one winner receives twice as much as the other, how much will they each receive?

5 A grandfather is six times as old as his grandson. If the grandfather was 45 when his grandson was born, how old is the grandson?

6 A rectangle of perimeter 74 cm is 7 cm longer than it is wide. What is the length of each side?

7 Smitville is located between Jonesville and Cityville. Smitville is five times as far away from Cityville as it is from Jonesville. If the distance between Jonesville and Cityville is 288 km, how far is it from Jonesville to Smitville?

8 Amira is twice as old as her cousin Pam. Nine years ago, their combined age was 18. What are their present ages?

9 Jabu left town A to travel to town B at 6.00 a.m. He drove at an average speed of 80 km/h. At 8.30 a.m., Sipho left town A to travel to town B. He drove at an average speed of 100 km/h. At what time will Sipho catch up with Jabu?

10 Cecelia took 40 minutes to complete a journey. She travelled half the distance at a speed of 100 km/h and the other half at 60 km/h. How far was her journey?

E

Deriving quadratic equations

To solve some word problems you might need to derive and solve a quadratic equation. Before you can solve the equation you must translate the word problem and derive the equations. You may need to use geometry, number facts, probability or any other appropriate techniques that relate to the topic you are working with.

E

Worked example 3

The product of two consecutive integers is 42. Form and solve a quadratic equation to find both possible pairs of integers.

Use the letter n to represent the smaller of the two numbers. Then the larger of the two numbers is $n + 1$.

The product of the two number is 42, so

$$
\begin{aligned}
&n(n+1) = 42\\
&\Rightarrow n^2 + n = 42\\
&\Rightarrow n^2 + n - 42 = 0\\
&\Rightarrow (n+7)(n-6) = 0\\
&\Rightarrow n = -7 \text{ or } n = 6
\end{aligned}
$$

If $n = -7$, then $n + 1 = -6$.

Check: -7 and -6 are consecutive integers and $-6 \times -7 = 42$

If $n = 6$, then $n + 1 = 7$.

Check: 6 and 7 are consecutive integers and $6 \times 7 = 42$

So the pairs of integers are 6, 7 and -7, -6

Tip

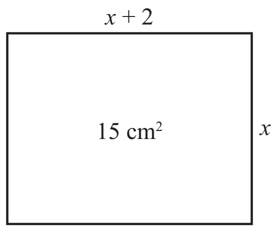
You might have spotted the solution 6, 7 without using algebra, but writing a quadratic equation means you will definitely find *all* the solutions.

Worked example 4

A rectangle has length 2 cm greater than its width. The area of the rectangle is 15 cm. Find its perimeter.

Always draw a diagram to show the information and to help you visualise the situation.

Use x to represent the width of the rectangle. This means that the length will be $x + 2$



The area is the length × width, so

$$
\begin{aligned}
&x(x+2) = 15\\
&\Rightarrow x^2 + 2x = 15\\
&\Rightarrow x^2 + 2x - 15 = 0\\
&\Rightarrow (x+5)(x-3) = 0\\
&\Rightarrow x = -5 \text{ or } x = 3
\end{aligned}
$$

x is the width of the rectangle so it can't be a negative number. This means x must be 3. The dimensions of the rectangle are 3 cm × 5 cm.

Its perimeter = 3 + 5 + 3 + 5 = 16 cm

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Tip

Where possible we make one unknown the subject of the equation, so that it can be substituted into any later equation

Worked example 5

A right-angled triangle has height h cm and base b cm. The hypotenuse of the triangle has length $\sqrt{13}$ cm. If the area of the triangle is 3cm² find the possible values for b and h.

Using the area first, we have

$$\frac{1}{2}bh = 3$$

$$\Rightarrow bh = 6$$

$$\Rightarrow h = \frac{6}{b} \qquad (1)$$

Now using Pythagoras' theorem and the fact that the hypotenuse has length $\sqrt{13}$ cm:

$$b^2 + h^2 = 13 \qquad (2)$$

Subsituting (1) into (2)

$$b^2 + \left(\frac{6}{b}\right)^2 = 13$$

$$\Rightarrow (b^2)^2 + 36 = 13b^2$$

Multiplying by b^2 on the right

$$\Rightarrow (b^2)^2 - 13(b^2) + 36 = 0$$

$$\Rightarrow (b^2 - 9)(b^2 - 4) = 0$$

$$\Rightarrow b = 3 \text{ or } 2$$

$$\Rightarrow h = 2 \text{ or } 3$$

Exercise 22.3

1 A number is 3 more than another number and the product of these two numbers is 40. Find the possible pairs of numbers.

2 A ball starts to roll down a slope. If the ball is d metres from its starting point at time t seconds and $d = t^2 + 3t$, find the time at which the ball is 10m from its starting point.

3 The n^{th} term of the sequence

1 3 6 10 15 ....

is $\frac{n(n+1)}{2}$ where n is the position of each term in the sequence.

Use algebra to find the position of the number 78.

REWIND

This is the sequence of *triangular numbers*.

4 The sum of two integers is 11 and the product of the same two integers is 28. Use algebra to show that the two integers must be 4 and 7.

5 The base of a triangle is 2cm longer than its perpendicular height. If the area of the triangle is 24 cm², find the height of the triangle.

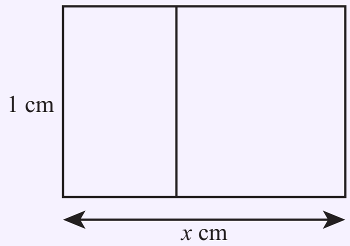
E

6 A trapezium has area 76 cm^2. The parallel sides differ in length by 3cm, and the shorter of the two is equal in length to the perpendicular height of the trapezium. Find the distance between the two parallel sides of the trapezium.

7 The number of diagonals of a convex polygon with n sides is $\frac{1}{2}n(n-3)$

a How many sides does a polygon with 54 diagonals have?
b Show that it is not possible for a polygon to have 33 diagonals.

8



The diagram shows a rectangle that has been divided into a square and a smaller rectangle. The smaller rectangle is similar to the larger one.

a Show that $x^2 - x - 1 = 0$
b Solve this equation, giving your answers to two decimal places.
c Explain why one of your solutions does not work in this case.
d Find the perimeter of the rectangle.

A rectangle that can be divided up in this way is known as a *golden rectangle* and the positive solution of the equation that you have solved is known as the *Golden Ratio*. It is a very, very special number in mathematics and is important in the arts and sciences as well as mathematics

9 The product of three consecutive integers is 30 times the smallest of the three integers. Find the two possible sets of integers that will satisfy this condition.

10 The square of a number is 14 more than 5 times the number. Find the two possible numbers for which this is true.

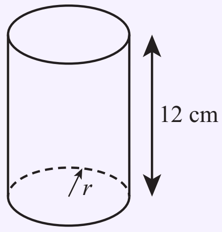
11 A rectangle has sides of length x cm and y cm. If the perimeter of the rectangle is 22 cm and the area of the rectangle is 24 cm^2, use algebra to find the dimensions of the rectangle.

12 A ball is thrown from a building and falls $3t + 5t^2$ metres in t seconds. After how many seconds has the ball fallen 6m?

13 A stone is thrown up into the air from ground level. If the height of the stone is $16t - 5t^2$, for how long will the stone be more than 8 metres above the ground?

14 The cube of a number is 152 more than the cube of another number that is 2 smaller. What are the two possible numbers for which this is true.

15



The diagram shows an open-topped cylinder with radius r cm and height 12 cm. If the outer surface area of the cylinder is 81π cm^2, find the radius of the cylinder.

16 The product of two consecutive integers is 11 more than 3 times their sum. Use algebra to find the two pairs of integers which meet these conditions.

22.2 Using and rearranging formulae

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REWIND

You have already seen basic rearrangements of formulae in chapter 6. The methods in this chapter are a little more involved.

You already know that the variable which is written alone on one side of the '=' sign (usually the left) of a formula is called the **subject** of the formula. For example, in the formula for finding the circumference of a circle, $C = \pi d$, C is the subject of the formula. This means that it is very easy to find the value of C if you know the diameter of the circle.

In many cases, however, you know the value of the subject and you have to find the value of another variable. To do this, you need to rearrange the formula to make the other variable the subject of the formula.

To change the subject of a formula:

- expand to get rid of any brackets if necessary
- use inverse operations to isolate the variable required.

Inverse operations 'undo' the 'original'.

If you are told to solve for x, or find x, it means the same as 'make x the subject of the formula'.

Worked example 6

Given that $c = ax + b$, find x.

$ax + b = c$	Reorganise the formula so the term with the x is on the left-hand side of the '=' sign.
$ax = c - b$	Subtract b from both sides.
$x = \dfrac{c-b}{a}$	Divide both sides by a.

Worked example 7

Given that $m = \frac{1}{2}(x + y)$, solve this formula for x.

$m = \dfrac{1}{2}(x + y)$	
$\therefore\ 2m = x + y$	Multiply both sides by 2 to remove the fraction.
$\therefore\ 2m - y = x$	Subtract y from both sides.
$x = 2m - y$	Rewrite the formula so x is on the left-hand side of the '=' sign.

Worked example 8

Solve for h if $A = 2\pi r(r + h)$.

$A = 2\pi r(r + h)$	
$\therefore A = 2\pi r^2 + 2\pi rh$	Expand to remove the brackets.
$\therefore A - 2\pi r^2 = 2\pi rh$	Subtract to isolate the term with h in it.
$\therefore \dfrac{(A - 2\pi r^2)}{2\pi r} = h$	Divide to get rid of $2\pi r$ on the right-hand side.
$\therefore h = \dfrac{(A - 2\pi r^2)}{2\pi r}$	Express the formula in terms of h.

Exercise 22.4

E

1 Make x the subject of each formula.

a $m = x + bp$ **b** $n = pr - x$ **c** $4x = m$ **d** $ax^2 - b = c$ **e** $d - 2b = mx + c$

f $\frac{x}{y} = 3b$ **g** $m = \frac{p}{x}$ **h** $\frac{mx}{n} = p$ **i** $m = \frac{2x}{k}$ **j** $p = \frac{20}{x}$

2 Solve for x.

a $m = 3(x + y)$ **b** $c = 4(t - x)$ **c** $y = 3(x - 5)$

d $r = 2r(3 - x)$ **e** $m = 4c(x - y)$ **f** $a = \pi r(2r - x)$

3 Express $E = mc^2$ in terms of m.

4 Express $\mathrm{I} = \frac{\mathrm{PRT}}{100}$ in terms of R.

5 Express $k = \frac{1}{2}mv^2$ in terms of m.

6 Express $A = \frac{h(a+b)}{2}$ in terms of b.

7 Express $V = \frac{Ah}{3}$ in terms of h.

8 Express $V = \frac{\pi r^2 h}{3}$ in terms of h.

In real life applications, you will often know the value of the subject of the formulae and some of the other values. In these examples, you need to change the subject of the formula and **substitute** the given values before solving it like an equation.

Applying your skills

9 If $V = LBH$, find B when $V = 600$, $L = 34$ and $H = 26$. Give your answer correct to 2 decimal places.

10 Given that $V = Ah$, find h if $V = 1.26$ and $A = 0.41$. Give your answer correct to 2 decimal places.

11 The formula for converting temperatures from Celsius to Fahrenheit is $F = 32 + \frac{9C}{5}$. Find the temperature in degrees C to the nearest degree when F is:

a 100 **b** 212 **c** 32

12 You can find the area of a circle using the formula $A = \pi r^2$. If $\pi = 3.14$, find the radius r, of circular discs of metal with the following areas (A):

a 14 **b** 120 **c** 0.5

Formulae containing squares and square roots

Some formulae have squared terms and square roots. You need to remember that a squared number has both a negative and a positive root when you solve these equations.

The inverse of $\sqrt{x}$ is $(x)^2$ but note, '$\sqrt{x}$' means the positive square root. There is only one value. So, $\sqrt{9} = 3$ and 3 only. But, if $x^2 = 9$ then $x = \pm\sqrt{9} = \pm 3$. You only use $\pm$ when undoing a square.

Worked example 9

Make x the subject of the formula $ax^2 = b$.

$ax^2 = b$	
$x^2 = \frac{b}{a}$	Divide both side by a.
$x = \pm\sqrt{\frac{b}{a}}$	Take square root of both sides to get x.

E

Worked example 10

Given $r = \sqrt{\frac{A}{\pi}}$ express the formula in terms of A.

$r = \sqrt{\frac{A}{\pi}}$	
$r^2 = \frac{A}{\pi}$	Square both sides to get rid of the square root.
$\pi r^2 = A$	Multiply each side by π.
$A = \pi r^2$	

REWIND

You learned how to factorise in chapter 6.

Formulae where the subject appears in more than one term

When the variable that is to be the subject occurs more than once, you need to gather the like terms and factorise before you can express the formula in terms of that variable.

Worked example 11

Given that $m = 6 - \frac{12}{p}$, make p the subject of the formula.

$m = 6 - \frac{12}{p}$	
$mp = 6p - 12$	Multiply both sides by p to remove the fraction.
$mp - 6p = -12$	Gather like terms.
$p(m - 6) = -12$	Factorise.
$p = \frac{-12}{(m-6)}$	

Exercise 22.5

1 Make x the subject of each formula.

a $m = ax^2$ **b** $x^2 - y = m$ **c** $m = n - x^2$ **d** $\frac{x^2}{y} = a$

e $a = \frac{bx^2}{c}$ **f** $a = x^2 - b^2$ **g** $m = \frac{n}{x^2}$ **h** $\sqrt{xy} = m$

i $a = \sqrt{5x}$ **j** $y = \sqrt{x - z}$ **k** $y = \sqrt{x} - z$ **l** $a = b + \frac{c}{\sqrt{x}}$

m $a - b\sqrt{x} = m$ **n** $\sqrt{3x - 1} = y$ **o** $a = \sqrt{y - 2x}$ **p** $y = \frac{a}{\sqrt{4x - b}}$

2 Express each of these formulae in terms of a.

a $x + a = ax + b$ **b** $L = Ba + (1 + C)a$ **c** $b = \frac{a}{a - 5}$

d $y = \frac{a + x}{a - x}$ **e** $y = \frac{a + 3}{1 + a}$ **f** $ma^2 = na^2 + 2$

3 Einstein developed the formula $E = mc^2$ when he worked on relativity. Express this formula in terms of c.

E

4 Pythagoras' theorem can expressed as $a^2 + b^2 = c^2$. Express this in terms of a.

5 Given that $y = \frac{a}{a+2}$, express this formula in terms of a.

6 Given that in a square, $A = s^2$, rearrange the formula to find the length of one side (s).

7 In each of these formulae, make y the subject.

a $\frac{x}{3} = \frac{y}{2} - 1$ **b** $x = \frac{y+c}{3}$ **c** $\frac{x+z}{3} = \frac{y+z}{4}$ **d** $a = b - \frac{3y}{2}$

Applying your skills

8 In physics, the kinetic energy (E) of a particle can be found using the formula $E = \frac{1}{2}mv^2$, where m is the mass, and v is the velocity of the particle.

a Find E when $m = 8$ and $v = 3.5$.
b Show how you could rearrange the formula to find v.

9 The volume (V) of a cylinder is found using the formula $V = \pi r^2 h$, where r is the radius and h is the height of the cylinder.

a Find the volume, correct to the nearest cm^3, of a cylinder with a radius of 0.8 m and height of 1 m.
b Rearrange the formula to make r the subject.

10 You can use the formula $A = \frac{\pi d^2}{4}$ to find the area (A) of a circle, where d is the diameter of the circle.

a Find the area of a circle of diameter 1.2 m.
b Use the formula $A = \pi r^2$ to find the area of the same circle.
c Express the formula $A = \frac{\pi d^2}{4}$ in a way that would allow you to find the diameter of the circle when the area is known.

REWIND

The word function is sometimes used interchangeably with equation (although this is not always strictly true). You met equations of this type, where a value of x led to a value of y, when you worked with straight lines and quadratics in chapter 10. It will help you to read through that chapter again before you do this work.

22.3 Functions and function notation

A **function** is a rule or set of instructions for changing one number (the input) into another (the output). If y is a function of x, then the value of y depends on the values you use for x. In a function, there is only one possible value of y for each value of x.

Function notation

Function notation is a mathematical way of writing equations (functions). Function notation is widely used in computer applications and also in technical fields.

Think about the equation $y = x + 2$.
When you write this in function notation it becomes $f(x) = x + 2$.
$f(x)$ is read as, 'the function of x' or 'f of x'.
If $f(x) = x + 2$, then $f(5)$ means the value of the function when $x = 5$.
In other words, $f(5) = 5 + 2 = 7$.
Similarly $f(-2) = -2 + 2 = 0$.

Functions can also be written as, $f : x \rightarrow 6 - 3x$.
This is read as, 'f is the function that maps x onto $6 - 3x$'.
The number $6 - 3x$ is sometimes called the image of x (under function f).

When there are two or more functions involved in a problem, you use different letters to represent them. For example, you could have:

$g(x) = x^2 - 2x + 3$ and $h(x) = 5x - 3$.

In function notation you leave out the y and replace it with the conventional notation $f(x)$. So, if 'f' is a function and 'x' is an input then, $f(x)$ is the output when f is applied to x.

Here are three ways of writing the SAME function using different letters.

$x \rightarrow 3x - 1$
$t \rightarrow 3t - 1$
$y \rightarrow 3y - 1$

E

The steps taken to work out the value of any function f(x) can be shown on a simple flow diagram. For example, the function f(x) = $2x + 5$ can be represented as:

$x \rightarrow \boxed{\times 2} \rightarrow \boxed{+5} \rightarrow 2x+5$

The function g(x) = $2(x + 5)$ can be represented as:

$x \rightarrow \boxed{+5} \rightarrow \boxed{\times 2} \rightarrow 2(x+5)$

Note that the flow charts show the same operations but, as they are done in a different order, they produce different results.

Worked example 12

Given that f(x) = $x^2 - 3x$ and g(x) = $4x - 6$, find the value of:

a f(6) **b** f(–3) **c** $g\left(\frac{1}{2}\right)$ **d** g(6).

a $f(6) = 6^2 - 3(6) = 36 - 18 = 18$

b $f(-3) = (-3)^2 - 3(-3) = 9 + 9 = 18$

c $g\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right) - 6 = 2 - 6 = -4$

d $g(6) = 4(6) - 6 = 24 - 6 = 18$

Worked example 13

Given h : $x \rightarrow 9 - x^2$,

a write down the expression for h(x).

b Find:

i h(0) **ii** h(3) **iii** h(9) **iv** h(–9)

a $h(x) = 9 - x^2$

b **i** $h(0) = 9 - (0)^2 = 9 - 0 = 9$

ii $h(3) = 9 - (3)^2 = 9 - 9 = 0$

iii $h(9) = 9 - (9)^2 = 9 - 81 = -72$

iv $h(-9) = 9 - (-9)^2 = 9 - 81 = -72$

Worked example 14

If f(x) = $3 + 2x$ and f(x) = 6, find x.

$$\begin{aligned} 3 + 2x &= 6 \\ 2x &= 6 - 3 \\ 2x &= 3 \\ x &= 1.5 \end{aligned}$$

The functions are equivalent.

E

Worked example 15

Given the functions $f(x) = x^2$ and $g(x) = x + 2$,

a solve the equation $f(x) = g(x)$
b solve the equation $4g(x) = g(x) - 3$.

a

$f(x) = g(x)$	
$\therefore x^2 = x + 2$	The functions are equivalent.
$x^2 - x - 2 = 0$	
$(x - 2)(x + 1) = 0$	Factorise.
so, $x = 2$ or $x = -1$	

b

$4g(x) = g(x) - 3$	
$\therefore 3g(x) = -3$	Subtract $g(x)$ from both sides.
$g(x) = -1$	Divide both sides by three.
$x + 2 = -1$	Replace $g(x)$ with $x + 2$.
$x = -3$	

Exercise 22.6

1 For each function, calculate:

i $f(2)$ **ii** $f(-2)$ **iii** $f(0.5)$ **iv** $f(0)$

a $f(x) = 3x + 2$ **b** $f(x) = 5x - 2$ **c** $f(x) = 2x - 1$
d $f(x) = 2x^2 + 3$ **e** $f(x) = x^2 - 2x$ **f** $f(x) = x^3 - 2$

2 $f(x) = 4x - 1$, find:

a $f(-1)$ **b** $f(0)$ **c** $f(1.5)$ **d** $f(-4)$

3 $f : x \to x^2 - 4$, find:

a $f(2)$ **b** $f(0)$ **c** $f(-3)$ **d** $f(0.25)$

4 Given the functions $f(x) = x^3 - 8$ and $g(x) = 3 - x$, find the value of:

a $f(2)$ **b** $f(-1)$ **c** $g(5)$ **d** $g(-2)$

5 Given the function $h : x \to 4x^2$, find:

a $h(2)$ **b** $h(-2)$ **c** $h\left(\frac{1}{2}\right)$

6 If $f(x) = 3x - 1$ and $f(x) = 3f(x) = 3$, find x.

7 If $h(x) = \frac{1}{x} + 1$ and $h(x) = 4$, find the value of x.

8 If $g(x) = \sqrt{4x + 1}$ and $g(x) = 5$, find the value of x.

9 Given the functions $f(x) = x^2 - x$ and $g(x) = x^2 - 3x - 12$,

a solve the equation $f(x) = 6$
b solve the equation $f(x) = g(x)$.

10 Given $f : x \to 2x$, find:

a $f(a)$ **b** $f(a + 2)$ **c** $f(4a)$ **d** $4f(a)$

E

11 $f(x)=\dfrac{4+x}{x}$ $(x \neq 0)$

a Calculate $f\left(\dfrac{1}{2}\right)$, simplifying your answer.

b Solve $f(x) = 3$.

12 $f(x) = (2x + 1)(x + 1)$, find:

a $f(2)$ **b** $f(-2)$ **c** $f(0)$

Composite functions

The order of the letters in a composite function is important.
$gf(4) \neq fg(4)$.
$gf(x)$ means do f first then g.
$fg(x)$ means do g first then f.
So, the function closest to the x is applied first.

A **composite function** is a function of a function. You get a composite function when you apply one function to a number and then apply another function to the result.

Look at these two functions: $f(x) = 2x + 1$ and $g(x) = x^2$.

$f(4) = 2(4) + 1 = 8 + 1 = 9$ (9 is the result of the first function.)

$g(9) = 9^2 = 81$ (The function g has been applied to result.)

You can write what has been done as $g[f(4)] = 81$. However, normally the square brackets are left out and you just write $gf(4) = 81$. $gf(x)$ is a composite function.

Worked example 16

Given the functions $f(x) = x^2 - 2x$ and $g(x) = 3 - x$, find the value of:

a $gf(4)$ **b** $fg(4)$ **c** $ff(-1)$ **d** $gg(100)$

a $gf(4) = g[f(4)] = g[16 - 8] = g[8] = 3 - 8 = -5$

b $fg(4) = f[g(4)] = f[3 - 4] = f[-1] = (-1)^2 - 2(-1) = 1 + 2 = 3$

c $ff(-1) = f[f(-1)] = f[1 + 2] = f[3] = 9 - 6 = 3$

d $gg(100) = g[g(100)] = g[3 - 100] = g[-97] = 3 - (-97) = 3 + 97 = 100$

Exercise 22.7

1 For each pair of functions, evaluate $fg(x)$ and $gf(x)$.

a $f(x) = x + 6$
$g(x) = x - 3$

b $f(x) = 2x^2 - 3x + 1$
$g(x) = 5x$

c $f(x) = 3x^2 - 4x + 2$
$g(x) = 3x - 2$

d $f(x) = \dfrac{4x}{3}$
$g(x) = x^2 - 9$

2 Given $f(x) = 2x$ and $g(x) = -x$, find:

a $fg(x)$ **b** $fg(2)$ **c** $ff(4)$ **d** $gf(1)$

3 $f(x) = 3x + 1$ and $h(x) = 6x^2$, find:

a $ff(x)$ **b** $fh(x)$ **c** $hh(-2)$ **d** $hf(-2)$ **e** $hf\left(\dfrac{2}{5}\right)$

4 Given the functions $g(x) = x^2 + 1$ and $h(x) = 2x + 3$, find the values of:

a $gh(1)$ **b** $hg(1)$ **c** $gg(2)$ **d** $hh(5)$

5 Find $gh(4)$ and $hg(4)$ if $g(x) = \dfrac{1}{x}$ and $h(x) = \dfrac{1}{x+1}$.

E

6 Given $f(x) = 8 - x^2$ and $g(x) = x^2 - 8$, find:

a $fg(x)$ **b** $gf(x)$ **c** $ff(x)$ **d** $gg(x)$

7 Given $f(x) = 2x - 5$ and $g(x) = \frac{1}{x}$, evaluate:

a $f(-10)$ **b** $g(\frac{2}{3})$ **c** $gf(\frac{5}{70})$ **d** $gf(4)$ **e** $ff(0)$

8 If $f(x) = x^4$ and $g(x) = \sqrt{(x^2 + 36)}$, evaluate:

a $fg(x)$ **b** $gf(x)$ **c** $ff(0)$ **d** $gg(-2)$

9 Given that $f(x) = -x$, $g(x) = x - 1$ and $h(x) = \frac{1}{x+2}$, show why it is not possible to evaluate $hgf(1)$.

10 The function $f(x)$ is given by $f(x) = \frac{x+1}{x-1}$

a Show that $ff(x) = x$

b Write down $f^{-1}(x)$

REWIND

Inverse functions were met briefly with trigonometry in chapter 15. ◀

Inverse functions

The **inverse** of any function (f) is the function that will do the opposite of f. In other words, the function that will undo the effects of f. So, if f maps 4 onto 13, then the inverse of f will map 13 onto 4.

In effect, when f is applied to a number and the inverse of f is applied to the result, you will get back to the number you started with.

In simple cases, you can find the inverse of a function by inspection. For example, the inverse of $x \rightarrow x + 5$ must be $x \rightarrow x - 5$ because subtraction is the inverse of addition; to undo add five you have to subtract five.

Similarly, the inverse of $x \rightarrow 2x$ is $x \rightarrow \frac{x}{2}$, because to undo multiply by two you have to divide by two.

The inverse of the function (f) is written as f^{-1}.

So, if $f(x) = x + 5$, then $f^{-1}(x) = x - 5$

and, if $g(x) = 2x$, then $g^{-1}(x) = \frac{x}{2}$.

Some functions do not have an inverse. Think about the function $x \rightarrow x^2$. This is a function because for every value of x, there is only one value of x^2. The inverse (in other words, the square root) is not a function because a positive number has two square roots, one negative, and one positive.

Finding the inverse of a function

There are two methods of finding the inverse:

- Method 1: using a flow diagram.

  In this method you draw a flow diagram for the function and then work out the inverse by 'reversing' the flow to undo the operations in the boxes.

- Method 2: reversing the mapping.

  In this method you use the fact that if f maps x onto y, then f^{-1} maps y onto x. To find f^{-1} you have to find a value of x that corresponds to a given value of y.

The worked examples 14–17 show you the two methods of finding the inverse of the same functions.

E

Worked example 17

Find the inverse of $f(x) = 3x - 4$.

Let x be the input to f^{-1}

$\frac{x+4}{3} \leftarrow \boxed{\div 3} \leftarrow \boxed{+4} \leftarrow x$

$\therefore f^{-1}(x) = \frac{x+4}{3}$

Using Method 1, the flow diagram, you get:

f : input $\rightarrow \boxed{\times 3} \rightarrow \boxed{-4} \rightarrow$ output

f^{-1} : output $\leftarrow \boxed{\div 3} \leftarrow \boxed{+4} \leftarrow$ input

Worked example 18

Given $g(x) = 5 - 2x$, find $g^{-1}(x)$.

Let x be the input to g^{-1}

$\frac{x-5}{-2} \leftarrow \boxed{\div(-2)} \leftarrow \boxed{-5} \leftarrow x$

$\therefore g^{-1}(x) = \frac{x-5}{-2} = \frac{5-x}{2}$

Using Method 1, the flow diagram, you get:

g : input $\rightarrow \boxed{\times(-2)} \rightarrow \boxed{+5} \rightarrow$ output

g^{-1} : output $\leftarrow \boxed{\div(-2)} \leftarrow \boxed{-5} \leftarrow$ input

Worked example 19

Find the inverse of the function $f(x) = 3x - 4$.

$y = 3x - 4$

$y + 4 = 3x$

$x = \frac{y+4}{3}$

Using Method 2, reversing the mapping.
Suppose the function maps x onto y (y is the subject).
Make x the subject of the formula, so that y maps onto x.

You know that f^{-1} maps y onto x, so $f^{-1}(y) = \frac{y+4}{3}$

This is usually written in terms of x so, $f^{-1}(x) = \frac{x+4}{3}$

Worked example 20

Given $g(x) = 5 - 2x$, find $g^{-1}(x)$.

Let $y = 5 - 2x$

$2x = 5 - y$

$x = \frac{5-y}{2}$

This means g maps x onto y.

Make x the subject of the formula, so that y maps onto x.

g^{-1} maps y onto x, so $g^{-1}(y) = \frac{5-y}{2}$

This is usually written in terms of x so, $g^{-1}(x) = \frac{5-x}{2}$

Exercise 22.8

E

1 Find the inverse of each function.

a $f(x) = 7x$ **b** $f(x) = \frac{1}{7x^3}$ **c** $f(x) = x^3$

d $f(x) = 4x + 3$ **e** $f(x) = \frac{1}{2}x + 5$ **f** $f(x) = \frac{x+2}{2}$

g $f(x) = 3(x - 2)$ **h** $f(x) = \frac{2x+9}{2}$ **i** $f(x) = \frac{2(x+1)}{4-x}$

j $f(x) = x^3 + 5$ **k** $f(x) = \sqrt{3x + 8}$ **l** $f(x) = \frac{x + 1}{x - 1}$

2 For each pair of functions, determine whether $g(x)$ is the inverse of $f(x)$.

a $f(x) = 2x - 6$
$g(x) = \frac{x}{2} + 3$

b $f(x) = 12x$
$g(x) = \frac{x}{12}$

c $f(x) = 3x + 2$
$g(x) = x + \frac{3}{2}$

d $f(x) = x^3 - 2$
$g(x) = \sqrt[3]{x+2}$

3 Given the function $g(x) = \frac{x}{3} - 44$, find $g^{-1}(x)$.

4 For each function, find:

i $f^{-1}(x)$ **ii** $ff^{-1}(x)$ **iii** $f^{-1}f(x)$

a $f(x) = 5x$ **b** $f(x) = x + 4$ **c** $f(x) = 2x - 7$

d $f(x) = x^3 + 2$ **e** $f(x) = \sqrt{2x - 1}$ **f** $f(x) = \frac{9}{x}$

g $f(x) = x^3 - 1$

5 Given the function $h(x) = 2(x - 3)$, find the value of:

a $h^{-1}(10)$ **b** $hh^{-1}(20)$ **c** $h^{-1}h^{-1}(26)$

6 $f(x) = \frac{1}{2}x + 5$ and $g(x) = 4x - \frac{2}{5}$

a Solve $f(x) = 0$
b Find $g^{-1}(x)$
c Solve $f(x) = g(x)$ giving your answer correct to 2 decimal places.
d Find the value of:

i $gf^{-1}(-2)$ **ii** $f^{-1}f(3)$ **iii** $f^{-1}g^{-1}(4)$

Summary

Do you know the following?

- Algebraic expressions and equations are useful for representing situations and solving worded problems.
- When you set up your own equations to represent problems you need to state what the variables stand for.
- A formula is an equation that links variables. The subject of the formula is the variable on the left-hand side of the formula.
- You can rearrange formulae to make any of the variables the subject. This is called changing the subject of the formula. It may also be called solving the formula for (x) or expressing the formula in terms of (x).

E

- More complex formulae can be rearranged, including:
  - formulae that contain squares and square roots
  - formulae where the subject appears in more than one term.
- A function is a rule for changing one variable into another.
- Functions are written using conventional notation of $f(x) = x + 2$ and $f : x \to 2 - 3x$.
- You can use a flow diagram to represent the steps in a function.
- A composite function is a function of a function. The order of a composite function is important fg(x) means do g first then f.
- An inverse function is function that undoes the original function. The reverse of the function.

Are you able to …?

- set up your own equations and use them to solve worded problems
- change the subject of formula

E

- set up and rearrange even more complicated formulae such as those that contain squares, square roots or where the subject appears in more than one term
- substitute values to find the given subject of a formula
- read, understand and use function notation to describe simple functions
- form composite functions such as gf(x) and ff(x)
- find the inverse of a function using a flow diagram
- find the inverse of a function by reversing the mapping.

Examination practice

Exam-style questions

1 Six litres of white paint are mixed with three litres of blue paint that costs \$2 per litre more. The total price of the mixture is \$24. Find the price of the white paint.

2 A trader has a mixture of 5c and 10c coins. He has 50 coins in all, with a total value of \$4.20. How many of each coin does he have?

E

3 If $S = \dfrac{a}{1-r}$, find a when $S = 5.2$ and $r = 0.3$.

4 f and g are the functions $f : x \rightarrow x - 5$ and $g : x \rightarrow 5 - x$. Which of the following are true and which are false?

a $f^{-1} = g$

b $g^{-1} : x \rightarrow 5 - x$

c $fg : x \rightarrow -x$

d $fg = gf$

5 $f(x) = 3x^2 - 3x - 4$ and $g(x) = 4 - 3x$.

a State the value of f(−2).

b Solve the equation $f(x) = -3$.

c Solve the equation $f(x) = 0$, giving your answer correct to 2 decimal places.

d Solve the equation $g(x) = 2g(x) - 1$.

e Find $g^{-1}(x)$.

6 $f : x \rightarrow 3 - 4x$.

a Find f(−1)

b Find $f^{-1}(x)$

c Find $ff^{-1}(4)$

7 If $f(x) = \dfrac{5}{2x-1}$ and $f(x) = -2$, find x.

Past paper questions

1 Rearrange the formula to make x the subject.

$y = x^2 + 4$ [2]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q06 October/November 2013]

E

2 Make y the subject of the formula. $A = \pi x^2 - \pi y^2$ [3]

[Cambridge IGCSE Mathematics 0580 Paper 23 Q16 October/November 2012]

3 $f(x) = 5x - 2$ $g(x) = \frac{7}{x-3}, x \neq 3$ $h(x) = 2x^2 + 7x$.

a Work out

i f(2), [1]

ii hg(17). [2]

b Solve $g(x) = x + 3$. [3]

c Find $fh(x) = 11$, showing all your working and giving your answers correct to 2 decimal places. [5]

d Find $f^{-1}(x)$. [2]

e Solve $g^{-1}(x) = -0.5$. [1]

[Cambridge IGCSE Mathematics 0580 Paper 42 Q5 October/November 2014]

4 $f(x) = 2x + 5$ $g(x) = 2^x$ $h(x) = 7 - 3x$.

a Find

i f(3), [1]

ii gg(3). [2]

b Find $f^{-1}(x)$. [2]

c Find $fh(x)$, giving your answers in its simplest form. [2]

d Find the integer values of x which satisfy this inequality.

$$1 < f(x) \leqslant 9$$ [2]

[Cambridge IGCSE Mathematics 0580 Paper 42 Q9 October/November 2015]