Chapter 6: Equations and rearranging formulae

Key words

- Expansion
- Linear equation
- Solution
- Common factor
- Factorisation
- Variable
- Subject

In this chapter you will learn how to:

- expand brackets that have been multiplied by a negative number
- solve a linear equation
- factorise an algebraic expression where all terms have common factors
- rearrange a formula to change the subject.

Leonhard Euler (1707–1783) was a great Swiss mathematician. He formalised much of the algebraic terminology and notation that is used today.

Equations are a shorthand way of recording and easily manipulating many problems. Straight lines or curves take time to draw and change but their equations can quickly be written. How to calculate areas of shapes and volumes of solids can be reduced to a few, easily remembered symbols. A formula can help you work out how long it takes to cook your dinner, how well your car is performing or how efficient the insulation is in your house.

RECAP

You should already be familiar with the following algebra work:

Expanding brackets (Chapter 2)

$y(y-3) = y \times y - y \times 3$

Solving equations (Year 9 Mathematics)

Expand brackets and get the terms with the variable on one side by performing inverse operations.

$2(2x+2) = 2x-10$	
$4x+4 = 2x-10$	Remove the brackets first
$4x-2x = -10-4$	Subtract $2x$ from both sides. Subtract 4 from both sides.
$2x = -14$	Add or subtract like terms on each side
$x = -7$	Divide both sides by 2 to get x on its own.

Factorising (Year 9 Mathematics)

You can think of factorising as 'putting the brackets back into an expression'.

To remove a common factor:

- find the highest common factor (HCF) of each term. This can be a variable, it can also be a negative integer
- write the HCF in front of the brackets and write the terms divided by the HCF inside the brackets.

$2xy + 3xz = x(2y + 3z)$

$-2xy - 3xz = -x\ (2y + 3z)$

Changing the subject of a formula (Year 9 Mathematics)

You can rearrange formulae to get one letter on the left hand side of the equals sign. Use the same methods you use to solve an equation.

$A = lb \qquad b = \frac{1}{A} \qquad l = \frac{A}{b}$

6.1 Further expansions of brackets

REWIND

You dealt with expanding brackets in chapter 2. ◄

You have already seen that you can re-write algebraic expressions that contain brackets by expanding them. The process is called **expansion**. This work will now be extended to consider what happens when negative numbers appear before brackets.

The key is to remember that a '+' or a '–' is attached to the number immediately following it and should be included when you multiply out brackets.

Tip

Watch out for negative numbers in front of brackets because they always require extra care. Remember:

$+ \times + = +$

$+ \times - = -$

$- \times - = +$

LINK

Physicists often rearrange formulae. If you have a formula that enables you work out how far something has travelled in a particular time, you can rearrange the formula to tell you how long it will take to travel a particular distance, for example.

Worked example 1

Expand and simplify the following expressions.

a $-3(x + 4)$ **b** $4(y - 7) - 5(3y + 5)$ **c** $8(p + 4) - 10(9p - 6)$

a	$-3(x + 4)$ $-3(x + 4) = -3x - 12$	Remember that you must multiply the number on the outside of the bracket by everything inside *and* that the negative sign is attached to the 3. Because $-3 \times x = -3x$ and $-3 \times 4 = -12$.
b	$4(y - 7) - 5(3y + 5)$ $4(y - 7) = 4y - 28$ $-5(3y + 5) = -15y - 25$ $4(y-7) - 5(3y+5) = 4y - 28 - 15y - 25$ $= -11y - 53$	Expand each bracket first and remember that the '–5' must keep the negative sign when it is multiplied through the second bracket. Collect like terms and simplify.
c	$8(p + 4) - 10(9p - 6)$ $8(p + 4) = 8p + 32$ $-10(9p - 6) = -90p + 60$ $8(p+4) - 10(9p-6) = 8p + 32 - 90p + 60$ $= -82p + 92$	It is important to note that when you expand the second bracket '–10' will need to be multiplied by '–6', giving a positive result for that term. Collect like terms and simplify.

Exercise 6.1

1 Expand each of the following and simplify your answers as far as possible.

a $-10(3p + 6)$ **b** $-3(5x + 7)$
c $-5(4y + 0.2)$ **d** $-3(q - 12)$
e $-12(2t - 7)$ **f** $-1.5(8z - 4)$

2 Expand each of the following and simplify your answers as far as possible.

a $-3(2x + 5y)$ **b** $-6(4p + 5q)$
c $-9(3h - 6k)$ **d** $-2(5h + 5k - 8j)$
e $-4(2a - 3b - 6c + 4d)$ **f** $-6(x^2 + 6y^2 - 2y^3)$

3 Expand each of the following and simplify your answers as far as possible.

a $2 - 5(x + 2)$ **b** $2 - 5(x - 2)$
c $14(x - 3) - 4(x - 1)$ **d** $-7(f + 3) - 3(2f - 7)$
e $3g - 7(7g - 7) + 2(5g - 6)$ **f** $6(3y - 5) - 2(3y - 5)$

Try not to carry out too many steps at once. Show every term of your expansion and then simplify.

4 Expand each of the following and simplify your answers as far as possible.

a $4x(x - 4) - 10x(3x + 6)$ **b** $14x(x + 7) - 3x(5x + 7)$
c $x^2 - 5x(2x - 6)$ **d** $5q^2 - 2q(q - 12) - 3q^2$
e $18pq - 12p(5q - 7)$ **f** $12m(2n - 4) - 24n(m - 2)$

5 Expand each expression and simplify your answers as far as possible.

a $8x - 2(3 - 2x)$ **b** $11x - (6 - 2x)$
c $4x + 5 - 3(2x - 4)$ **d** $7 - 2(x - 3) + 3x$
e $15 - 4(x - 2) - 3x$ **f** $4x - 2(1 - 3x) - 6$
g $3(x + 5) - 4(5 - x)$ **h** $x(x - 3) - 2(x - 4)$
i $3x(x - 2) - (x - 2)$ **j** $2x(3 + x) - 3(x - 2)$
k $3(x - 5) - (3 + x)$ **l** $2x(3x + 1) - 2(3 - 2x)$

You will now look at solving linear equations and return to these expansions a little later in the chapter.

6.2 Solving linear equations

REWIND

It is important to remind yourself about BODMAS before working through this section. (Return to chapter 1 if you need to.)

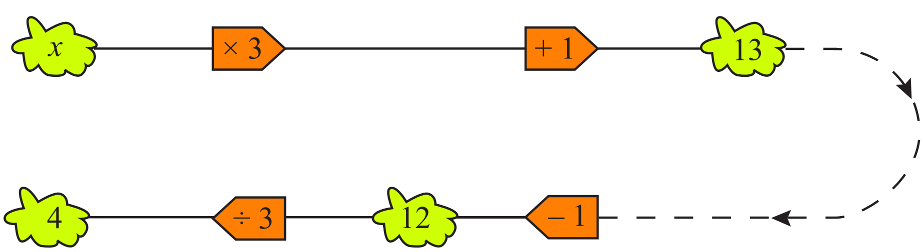
LINK

Accounting uses a great deal of mathematics. Accountants use computer spreadsheets to calculate and analyse financial data. Although the programs do the calculations, the user has to know which equations and formula to insert to tell the program what to do.

I think of a number. My number is x. If I multiply my number by three and then add one, the answer is 13. What is my number?

To solve this problem you first need to understand the stages of what is happening to x and then undo them in reverse order:

This diagram (sometimes called a function machine) shows what is happening to x, with the reverse process written underneath. Notice how the answer to the problem appears quite easily:



So $x = 4$

Even if you can see what the solution is going to be easily you *must* show working.

A more compact and efficient solution can be obtained using algebra. Follow the instructions in the question:

1	The number is x:	x
2	Multiply this number by three:	$3x$
3	Then add one:	$3x + 1$
4	The answer is 13:	$3x + 1 = 13$

This is called a **linear equation**. 'Linear' refers to the fact that there are no powers of x other than one.

The next point you must learn is that you can change this equation without changing the **solution** (the value of x for which the equation is true) *provided* you do the same to both sides at the same time.

Follow the reverse process shown in the function machine above but carry out the instruction on both sides of the equation:

$$3x + 1 = 13$$

$$3x + 1 - 1 = 13 - 1 \qquad \text{(Subtract one from both sides.)}$$

$$3x = 12$$

$$\frac{3x}{3} = \frac{12}{3} \qquad \text{(Divide both sides by three.)}$$

$$x = 4$$

Always line up your '=' signs because this makes your working much clearer.

Sometimes you will also find that linear equations contain brackets, and they can also contain unknown values (like x, though you can use any letter or symbol) on *both* sides.

The following worked example demonstrates a number of possible types of equation.

Worked example 2

An equation with *x on both sides and all x terms with the same sign:*

a Solve the equation $5x - 2 = 3x + 6$

$5x - 2 = 3x + 6$ $5x - 2 - 3x = 3x + 6 - 3x$ $2x - 2 = 6$	Look for the *smallest* number of *x's* and subtract this from both sides. So, subtract $3x$ from both sides.
$2x - 2 + 2 = 6 + 2$ $2x = 8$	Add two to both sides.
$\frac{2x}{2} = \frac{8}{2}$ $x = 4$	Divide both sides by two.

An equation with *x on both sides and x terms with different sign:*

By adding $11x$ to both sides you will see that you are left with a *positive x* term. This helps you to avoid errors with '–' signs!

b Solve the equation $5x + 12 = 20 - 11x$

$5x+12=20-11x$	This time add the negative x term to both sides.
$5x+12+11x=20-11x+11x$	Add $11x$ to both sides.
$16x+12=20$	
$16x+12-12=20-12$	Subtract 12 from both sides.
$16x=8$	
$\frac{16x}{16}=\frac{8}{16}$	Divide both sides by 16.
$x=\frac{1}{2}$	

An equation with *brackets on at least one side:*

c Solve the equation $2(y - 4) + 4(y + 2) = 30$

$2(y-4)+4(y+2)=30$	Expand the brackets and collect like terms together.
$2y-8+4y+8=30$	Expand.
$6y=30$	Collect like terms.
$\frac{6y}{6}=\frac{30}{6}$	Divide both sides by 6.
$y=5$	

An equation that contains fractions:

d Solve the equation $\frac{6}{7}p=10$

$\frac{6}{7}p\times7=10\times7$	Multiply both sides by 7.
$6p=70$	
$p=\frac{70}{6}=\frac{35}{3}$	Divide both sides by 6. Write the fraction in its simplest form.

Unless the question asks you to give your answer to a specific degree of accuracy, it is perfectly acceptable to leave it as a fraction.

Exercise 6.2

1 Solve the following equations.

a $4x+3=31$ **b** $8x+42=2$
c $6x-1=53$ **d** $7x-4=-66$
e $9y+7=52$ **f** $11n-19=102$
g $12q-7=14$ **h** $206t+3=106$
i $\frac{2x+1}{3}=8$ **j** $\frac{2x}{3}+1=8$
k $\frac{3}{5}x+11=21$ **l** $\frac{x+3}{2}=x$
m $\frac{2x-1}{3}=3x$ **n** $\frac{3x}{2}+5=2x$

2 Solve the following equations.

a $12x+1=7x+11$ **b** $6x+1=7x+11$ **c** $6y+1=3y-8$
d $11x+1=12-4x$ **e** $8-8p=9-9p$ **f** $\frac{1}{2}x-7=\frac{1}{4}x+8$

3 Solve the following equations.

a $4(x+1)=12$	**b** $2(2p+1)=14$
c $8(3t+2)=40$	**d** $5(m-2)=15$
e $-5(n-6)=-20$	**f** $2(p-1)+7(3p+2)=7(p-4)$
g $2(p-1)-7(3p-2)=7(p-4)$	**h** $3(2x+5)-(3x+2)=10$

4 Solve for x.

a $7(x+2)=4(x+5)$	**b** $4(x-2)+2(x+5)=14$
c $7x-(3x+11)=6-(5-3x)$	**d** $-2(x+2)=4x+9$
e $3(x+1)=2(x+1)+2x$	**f** $4+2(2-x)=3-2(5-x)$

Tip

Some of the numbers in each equation are powers of the same base number. Re-write these as powers and use the laws of indices from chapter 2

E

5 Solve the following equations for x

a $3^{3x}=27$	**b** $2^{3x+4}=32$
c $8.1^{4x+3}=1$	**d** $5^{2(3x+1)}=625$
e $4^{3x}=2^{x+1}$	**f** $9^{3x+4}=27^{4x+3}$

6.3 Factorising algebraic expressions

You have looked in detail at expanding brackets and how this can be used when solving some equations. It can sometimes be helpful to carry out the opposite process and put brackets back into an algebraic expression.

REWIND

If you need to remind yourself how to find HCFs, return to chapter 1.

Consider the algebraic expression $12x-4$. This expression is already simplified but notice that 12 and 4 have a **common factor**. In fact the HCF of 12 and 4 is 4.

Now, $12 = 4 \times 3$ and $4 = 4 \times 1$.

$$\begin{aligned}\text{So,}\quad 12x-4 &= 4\times 3x - 4\times 1\\ &= 4(3x-1)\end{aligned}$$

Notice that the HCF has been 'taken out' of the bracket and written at the front. The terms inside are found by considering what you need to multiply by 4 to get $12x$ and -4.

The process of writing an algebraic expression using brackets in this way is known as **factorisation**. The expression, $12x-4$, has been factorised to give $4(3x-1)$.

Some factorisations are not quite so simple. The following worked example should help to make things clearer.

Worked example 3

Factorise each of the following expressions as fully as possible.

a $15x+12y$ **b** $18mn-30m$ **c** $36p^2q-24pq^2$ **d** $15(x-2)-20(x-2)^3$

a	$15x+12y$ $15x+12y=3(5x+4y)$	The HCF of 12 and 15 is 3, but x and y have no common factors. Because $3\times 5x = 15x$ and $3\times 4y = 12y$.
b	$18mn-30m$ $18mn-30m=6m(3n-5)$	The HCF of 18 and 30 = 6 and HCF of mn and m is m. Because $6m\times 3n = 18mn$ and $6m\times 5 = 30m$.

Make sure that you have taken out *all* the common factors. If you don't, then your algebraic expression is not *fully* factorised.

c	$36p^2q - 24pq^2$	The HCF of 36 and 24 = 12 and p^2q and pq^2 have common factor pq.
	$36p^2q - 24pq^2 = 12pq(3p - 2q)$	Because $12pq \times 3p = 36p^2q$ and $12pq \times -2q = -24pq^2$.

Sometimes, the terms can have an expression in brackets that is common to both terms.

Take care to put in all the bracket symbols.

d	$15(x-2) - 20(x-2)^3$	The HCF of 15 and 20 is 5 and the HCF of $(x-2)$ and $(x-2)^3$ is $(x-2)$.
	$15(x-2) - 20(x-2)^3 = 5(x-2)[3 - 4(x-2)^2]$	Because $5(x-2) \times 3 = 15(x-2)$ and $5(x-2) \times 4(x-2)^2 = 20(x-2)^3$.

Exercise 6.3

1 Factorise.

a $3x + 6$ **b** $15y - 12$ **c** $8 - 16z$ **d** $35 + 25t$
e $2x - 4$ **f** $3x + 7$ **g** $18k - 64$ **h** $33p + 22$
i $2x + 4y$ **j** $3p - 15q$ **k** $13r - 26s$ **l** $2p + 4q + 6r$

2 Factorise as fully as possible.

a $21u - 49v + 35w$ **b** $3xy + 3x$ **c** $3x^2 + 3x$ **d** $15pq + 21p$
e $9m^2 - 33m$ **f** $90m^3 - 80m^2$ **g** $36x^3 + 24x^5$ **h** $32p^2q - 4pq^2$

Once you have taken a common factor out, you may be left with an expression that needs to be simplified further.

3 Factorise as fully as possible.

a $14m^2n^2 + 4m^3n^3$ **b** $17abc + 30ab^2c$ **c** $m^3n^2 + 6m^2n^2(8m + n)$
d $\frac{1}{2}a + \frac{3}{2}b$ **e** $\frac{3}{4}x^4 + \frac{7}{8}x$ **f** $3(x-4) + 5(x-4)$
g $5(x+1)^2 - 4(x+1)^3$ **h** $6x^3 + 2x^4 + 4x^5$ **i** $7x^3y - 14x^2y^2 + 21xy^2$
j $x(3 + y) + 2(y + 3)$

6.4 Rearrangement of a formula

FAST FORWARD

You will look again at rearranging formulae in chapter 22. ▶

Very often you will find that a formula is expressed with one **variable** written alone on one side of the '=' symbol (usually on the left but not always). The variable that is written alone is known as the **subject** of the formula.

Consider each of the following formulae:

$s = ut + \frac{1}{2}at^2$ (s is the subject)

$F = ma$ (F is the subject)

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (x is the subject)

Now that you can recognise the subject of a formula, you must look at how you *change* the subject of a formula. If you take the formula $v = u + at$ and note that v is currently the subject, you can change the subject by rearranging the formula.

Another word sometimes used for changing the subject is 'transposing'.

To make a the subject of this formula:

$v = u + at$ Write down the starting formula.

$v - u = at$ Subtract u from both sides (to isolate the term containing a).

Remember that what you do to one side of the formula must be done to the other side. This ensures that the formula you produce still represents the same relationship between the variables.

$\frac{v-u}{t}=a$ Divide both sides by t (notice that everything on the left is divided by t).

You now have a on its own and it is the new subject of the formula.

This is usually re-written so that the subject is on the left:

$a=\frac{v-u}{t}$

Notice how similar this process is to solving equations.

$\Rightarrow$ is a symbol that can be used to mean 'implies that'.

Worked example 4

Make the variable shown in brackets the subject of the formula in each case.

a $x+y=c$ (y) **b** $\sqrt{x}+y=z$ (x) **c** $\frac{a-b}{c}=d$ (b)

a	$x+y=c$	
	$\Rightarrow y=c-x$	Subtract x from both sides.
b	$\sqrt{x}+y=z$	
	$\Rightarrow \sqrt{x}=z-y$	Subtract y from both sides.
	$\Rightarrow x=(z-y)^2$	Square both sides.
c	$\frac{a-b}{c}=d$	
	$\Rightarrow a-b=cd$	Multiply both sides by c to clear the fraction.
	$\Rightarrow a=cd+b$	Make the number of b's positive by adding b to both sides.
	$\Rightarrow a-cd=b$	Subtract cd from both sides.
	So $b=a-cd$	Re-write so the subject is on the left.

Exercise 6.4

Make the variable shown in brackets the subject of the formula in each case.

1 **a** $a+b=c$ (a) **b** $p-q=r$ (r) **c** $fh=g$ (h)
d $ab+c=d$ (b) **e** $\frac{a}{b}=c$ (a) **f** $an-m=t$ (n)

2 **a** $an-m=t$ (m) **b** $a(n-m)=t$ (a) **c** $\frac{xy}{z}=t$ (x)
d $\frac{x-a}{b}=c$ (x) **e** $x(c-y)=d$ (y) **f** $a-b=c$ (b)

3 **a** $p-\frac{r}{q}=t$ (r) **b** $\frac{x-a}{b}=c$ (b) **c** $a(n-m)=t$ (m)
d $\frac{a}{b}=\frac{c}{d}$ (a) **e** $\frac{x-a}{b}=c$ (a) **f** $\frac{xy}{z}=t$ (z)

4 **a** $\sqrt{b}=c$ (b) **b** $\sqrt{ab}=c$ (b) **c** $a\sqrt{b}=c$ (b)
d $\sqrt{b+c}=c$ (b) **e** $\sqrt{x-b}=c$ (b) **f** $\frac{x}{\sqrt{y}}=c$ (y)

Applying your skills

5 A rocket scientist is trying to calculate how long a Lunar Explorer Vehicle will take to descend towards the surface of the moon. He knows that if u = initial speed and v = speed at time t seconds, then:

$v = u + at$

where a is the acceleration and t is the time that has passed.

If the scientist wants to calculate the time taken for any given values of u, v, and a, he must rearrange the formula to make a the subject. Do this for the scientist.

6 Geoff is the Headmaster of a local school, who has to report to the board of Governors on how well the school is performing. He does this by comparing the test scores of pupils across an entire school. He has worked out the mean but also wants know the spread about the mean so that the Governors can see that it is representative of the whole school. He uses a well-known formula from statistics for the upper bound b of a class mean:

$$b = a + \frac{3s}{\sqrt{n}}$$

where s = sample spread about the mean, n = the sample size, a = the school mean and b = the mean maximum value.

If Geoff wants to calculate the standard deviation (diversion about the mean) from values of b, n and a he will need to rearrange this formula to make s the subject. Rearrange the formula to make s the subject to help Geoff.

7 If the length of a pendulum is l metres, the acceleration due to gravity is g m s^{-2} and T is the period of the oscillation in seconds then:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Rearrange the formula to make l the subject.

Summary

Do you know the following?

- Expanding brackets means to multiply all the terms inside the bracket by the term outside.
- A variable is a letter or symbol used in an equation or formula that can represent many values.
- A linear equation has no variable with a power greater than one.
- Solving an equation with one variable means to find the value of the variable.
- When solving equations you must make sure that you always do the same to *both* sides.
- Factorising is the reverse of expanding brackets.
- A formula can be rearranged to make a different variable the subject.
- A recurring fraction can be written as an exact fraction. E

Are you able to . . . ?

- expand brackets, taking care when there are negative signs
- solve a linear equation
- factorise an algebraic expressions by taking out any common factors
- rearrange a formulae to change the subject by treating the formula as if it is an equation. E

Examination practice

Exam-style questions

1 Given that $T = 3p - 5$, calculate T when $p = 12$.

2 In mountaineering, in general, the higher you go, the colder it gets. This formula shows how the height and temperature are related.

$$\text{Temperature drop (°C)} = \frac{\text{height increase (m)}}{200}$$

a If the temperature at a height of 500 m is 23 °C, what will it be when you climb to 1300 m?
b How far would you need to climb to experience a temperature drop of 5 °C?

3 The formula $e = 3n$ can be used to relate the number of sides (n) in the base of a prism to the number of edges (e) that the prism has.
a Make n the subject of the formula.
b Find the value of n for a prism with 21 edges.

Past paper questions

1 Factorise $2x - 4xy$. [2]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q2 Feb/March 2016]

2 Make r the subject of this formula.

$v = \sqrt[3]{p + r}$ [2]

[Cambridge IGCSE Mathematics 0580 Paper 22 Q5 October/November 2014]

3 Expand the brackets. $y(3 - y^3)$ [2]

[Cambridge IGCSE Mathematics 0580 Paper 13 Q9 October/November 2012]

4 Factorise completely. $4xy + 12yz$ [2]

[Cambridge IGCSE Mathematics 0580 Paper 13 Q13 October/November 2012]

5 Solve the equation. $5(2y - 17) = 60$ [3] E

[Cambridge IGCSE Mathematics 0580 Paper 22 Q12 May/June 2013]

6 Solve the equation $(3x - 5) = 16$. [2]

[Cambridge IGCSE Mathematics 0580 Paper 13 Q5 May/June 2013]

7 Factorise completely. $6xy^2 + 8y$ [2]

[Cambridge IGCSE Mathematics 0580 Paper 13 Q9 May/June 2013]