Chapter 18: Curved graphs

Key words

- Quadratic
- Parabola
- Axis of symmetry
- Turning point
- Minimum
- Maximum
- Reciprocal
- Hyperbola
- Asymptote
- Intersection
- Exponential
- Gradient
- Tangent
- Derived function
- Differentiate

In this chapter you will learn how to:

- construct a table of values to draw graphs called parabolas
- sketch and interpret parabolas
- construct a table of values to draw graphs called hyperbolas
- interpret curved graphs
- use graphs to find the approximate solutions to quadratic equations

EXTENDED

- construct tables of values to draw graphs in the form of ax^n and $\frac{a}{x}$
- recognise, sketch and interpret graphs of functions
- estimate the gradients of curves by drawing tangents
- use graphs to find the approximate solutions to associated equations
- differentiate functions to find gradients and turning points.

The water arcs from this fountain form a curved shape which is called a parabola in mathematics.

In chapter 10 you saw that many problems could be represented by linear equations and straight line graphs. Real life problems, such as those involving area; the path of a moving object; the shape of a bridge or other structure; the growth of bacteria; and variation in speed, can only be solved using non-linear equations. Graphs of non-linear equations are curves.

In this chapter you are going to use tables of values to plot a range of curved graphs. Once you understand the properties of the different graphs, you will use these to sketch the graphs (rather than plotting them). You will also learn how to interpret curved graphs and how to find the approximate solution of equations from graphs.

RECAP

You should already be familiar with the following concepts from your work on straight line graphs:

Plot graphs from a table of values (Chapter 10)

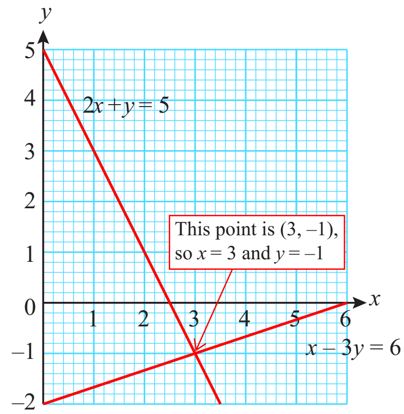
- A table of values gives you a set of ordered pairs (x, y) that you can use to plot a graph.

Gradient (Chapter 10)

- Gradient $= \dfrac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}}$
- Gradient can be positive or negative.

Graphical solution to simultaneous equations (Chapter 14)

- The point of intersection (x, y) of two straight line graphs is the simultaneous solution to the two equations (of the graphs).

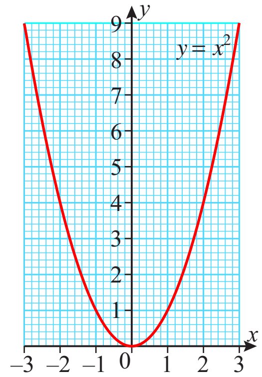


18.1 Drawing quadratic graphs (the parabola)

In chapter 10 you learned that **quadratic** equations have an x^2 term as their highest power. The simplest quadratic equation for a quadratic graph is $y = x^2$.

Here is a table showing the values for $y = x^2$ from $-3 \leqslant x \leqslant 3$.

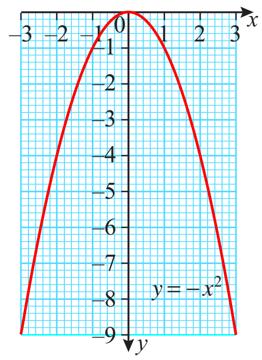
x	–3	–2	–1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9



You can use these points to plot and draw a graph just as you did with linear equations. The graph of a quadratic relationship is called a **parabola**.

Here is the table of values for $y = -x^2$ from $-3 \leqslant x \leqslant 3$.

x	–3	–2	–1	0	1	2	3
$y = -x^2$	–9	–4	–1	0	–1	–4	–9



When you plot these points and draw the parabola you can see that the negative sign in front of the x^2 has the effect of turning the graph so that it faces downwards.

If the coefficient of x^2 in the equation is positive, the parabola is a 'valley' shaped curve.

If the coefficient of x^2 in the equation is negative, the parabola is a 'hill' shaped curve.

LINK

We often draw curved graphs to help us understand how two variables might be related in geography. For example, you may find an interesting diagram arises if we take each National Park in the UK and plot the cost of maintaining visitor facilities against the number of tourists visiting each year.

For most graphs, a turning point is a local **minimum** or **maximum** value of y. For a parabola, if the x^2 term is positive the turning point will be a minimum. If the x^2 term is negative, the turning point will be a maximum.

The axis of symmetry and the turning point

The **axis of symmetry** is the line which divides the parabola into two symmetrical halves. In the two graphs above, the y-axis ($x = 0$) is the axis of symmetry.

The **turning point** or vertex of the graph is the point at which it changes direction. For both of the graphs above, the turning point is at the origin (0, 0).

Exercise 18.1

Remember that if you square a negative number the result will be positive. If using your calculator, place brackets round any negatives.

1 Complete the following tables of values and plot the graphs on the *same* set of axes. Use values of −8 to 12 on the y-axis.

a

x	−3	−2	−1	0	1	2	3
$y = x^2 + 1$							

b

x	−3	−2	−1	0	1	2	3
$y = x^2 + 3$							

c

x	−3	−2	−1	0	1	2	3
$y = x^2 - 2$							

d

x	−3	−2	−1	0	1	2	3
$y = -x^2 + 1$							

e

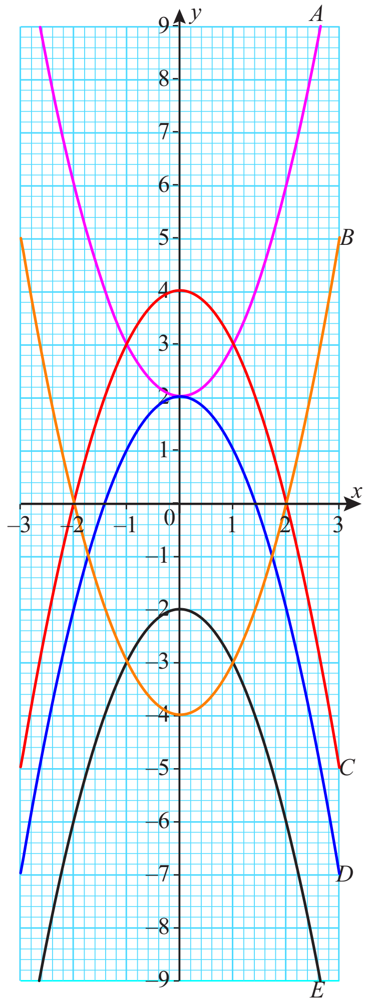
x	−3	−2	−1	0	1	2	3
$y = 3 - x^2$							

f What happens to the graph when the value of the constant term changes?

These equations are all in the form $y = -x^2 + c$, where c is the constant term. The constant term is the y-intercept of the graph in each case.

2 Match each of the five parabolas shown here to its equation.

a $y = 4 - x^2$
b $y = x^2 - 4$
c $y = x^2 + 2$
d $y = 2 - x^2$
e $y = -x^2 - 2$



Equations in the form of $y = x^2 + ax + b$

You have seen how to construct a table of values and then plot and draw a parabola from simple quadratic equations. Now you are going to see how to draw up a table of values for more usual quadratic equations with an x^2 term, an x term and a constant term. In these cases, it is easiest if you work out each term on a separate row of the table and then add them to find the value of y. Read through the two worked examples carefully to make sure you understand this.

Worked example 1

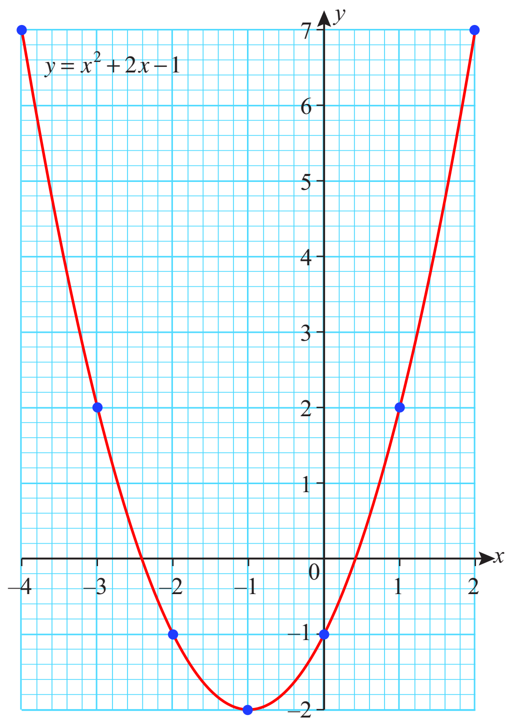
Construct a table of values for $y = x^2 + 2x - 1$ for values $-4 \leqslant x \leqslant 2$.
Plot the points to draw the graph.

x	−4	−3	−2	−1	0	1	2
x^2	16	9	4	1	0	1	4
$2x$	−8	−6	−4	−2	0	2	4
−1	−1	−1	−1	−1	−1	−1	−1
$y = x^2 + 2x - 1$	7	2	−1	−2	−1	2	7

In this table, you work out each term separately.
Add the terms of the equation in each column to get the totals for the last row (the y-values of each point).

To draw the graph:

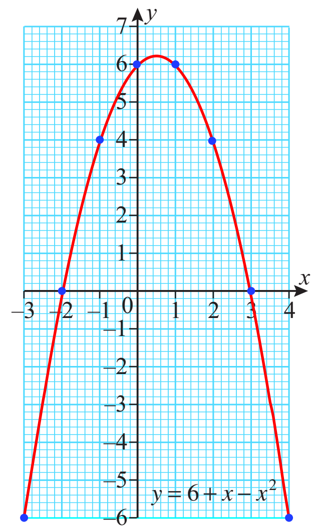
- plot the points and join them to make a smooth curve
- label the graph with its equation.



Worked example 2

Draw the graph of $y = 6 + x - x^2$ for values of x from -3 to 4.

x	-3	-2	-1	0	1	2	3	4
6	6	6	6	6	6	6	6	6
$+x$	-3	-2	-1	0	1	2	3	4
$-x^2$	-9	-4	-1	0	-1	-4	-9	-16
$y = 6 + x - x^2$	-6	0	4	6	6	4	0	-6



Some calculators have an in-built function to create tables of values. These can help you avoid errors provided you use them correctly. However, make sure that you can still do the calculations without the table function.

To plot the graph of a quadratic relationship:

- complete a table of values (often some of the values will be given)
- rule the axes and label them
- plot the (x, y) values from the table of values
- join the points with a smooth curve.

Exercise 18.2

1 Construct a table of values of $y = x^2 - 2x^2 + 2$ for $-1 \leqslant x \leqslant 3$ and use the (x, y) points from the table to plot and draw the graph.

2 Copy and complete this table of values and then draw the graph of $y = x^2 - 5x - 4$.

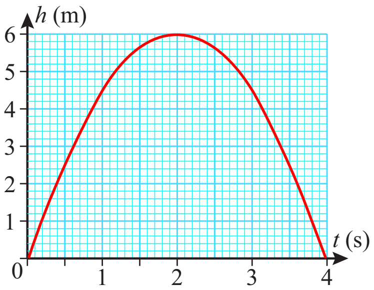
x	-2	-1	0	1	2	3	4	5	6
x^2	4								
$-5x$	10								
-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
y									

3 Construct a table of values of $y = x^2 + 2x - 3$ from $-3 \leqslant x \leqslant 2$. Plot the points and join them to draw the graph.

4 Using values of x from 0 to 4, construct a table of values and use it to draw the graph of $y = -x^2 - 4x$.

5 Using values of x from -6 to 0, construct a table of values and use it to draw the graph of $y = -x^2 - 6x - 5$.

Applying your skills



6 People who design water displays (often set to music) need to know how high water will rise from a jet and how long it will take to return to the pool. This graph shows the height of a water arc from a fountain (in metres) over a number of seconds.

- **a** What was the greatest height reached by the water arc?
- **b** How long did it take the water to reach the greatest height?
- **c** For how long was the water arc above a height of 2.5 m?
- **d** How far did the water rise in the first second?
- **e** Why do you think this graph shows only positive values of height?

Sketching quadratic functions

You can use the characteristics of the parabola to sketch a graph.

When the equation is in the standard form $y = x^2 + bx + c$ follow these steps to sketch the graph:

Step 1: Identify the shape of the graph.

If the x^2 term is positive the graph is $\cup$ shaped; if the x^2 term is negative, the graph is $\cap$ shaped.

Step 2: Find the y-intercept.

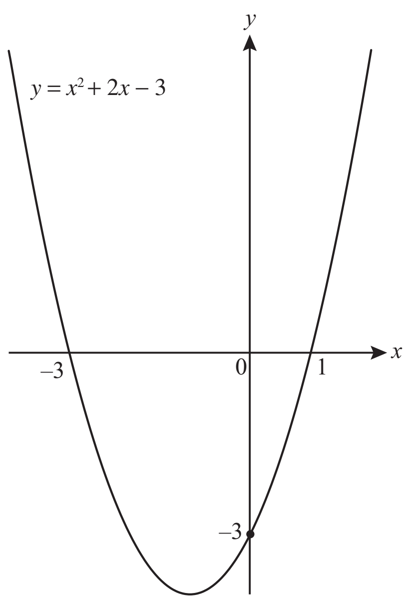
You do this by making $x = 0$ in the equation. The coordinates of the y-intercept are (0, c).

Step 3: Mark the y-intercept and x-intercept(s) and use what you know about the shape of the graph and its symmetry to draw a smooth curve. Label the graph.

Worked example 3

Sketch the graph of $y = x^2 + 2x - 3$

x^2 is positive, so the graph is $\cup$ shaped
y-intercept = (0, –3) Remember there is only ever one y-intercept.



E

x-intercepts

You can find the x-intercept(s) by making $y = 0$ in the equation and solving for x.

To find the x-intercepts of the graph in Example 3, make $y = 0$, so

$x^2 + 2x - 3 = 0$

$(x + 3)(x - 1) = 0$

$x = -3$ or $x = 1$

So, $(-3, 0)$ and $(1, 0)$ are the x-intercepts

If there is only one intercept then the graph just touches the x axis.

Turning points

To find the coordinates of the turning point of a parabola, you need to find the axis of symmetry.

When the equation is in standard form $y = ax^2 + bx + c$, the axis of symmetry can be found using $x = -\frac{b}{2a}$. This gives the x-coordinate of the turning point.

You can then find the y-coordinate of the turning point by substituting the value of x into the original equation. This y-value is the minimum or maximum value of the graph.

The turning point of a parabola is the minimum or maximum point of the graph. For the graph $y = ax^2 + bx + c$, the turning point is a maximum if a is negative and a minimum if a is positive.

Worked example 4

Sketch the graph $y = -2x^2 - 4x + 6$

$a = -2$, so the graph is $\cap$ shaped.

The y-intercept $= (0, 6)$

Find the x-intercepts:

$-2x^2 - 4x + 6 = 0$

$x^2 + 2x - 3 = 0$ — Divide both sides by common factor -2.

$(x - 1)(x + 3) = 0$ — Factorise the trinomial.

$x = 1$ or $x = -3$ — Solve for x.

$(1, 0)$ and $(-3, 0)$ are the x-intercepts.

Find the axis of symmetry using $x = -\frac{b}{2a}$

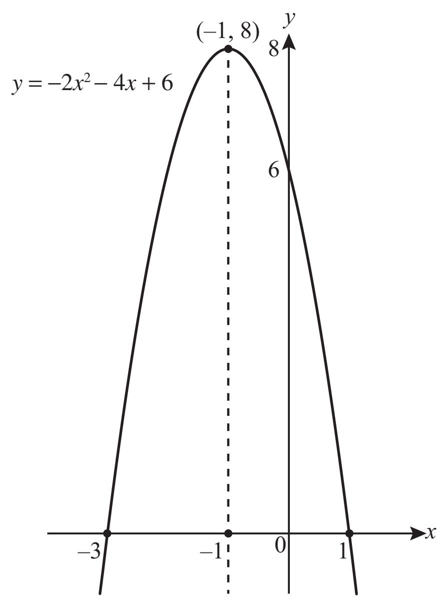
$x = \frac{4}{2(-2)} = -1$ — Remember this is the x-coordinate of the turning point.

Substitute $x = -1$ into the equation to find the y-coordinate of the turning point.

$y = -2(-1)^2 - 4(-1) + 6 = 8$

The turning point is at $(-1, 8)$ and is a maximum because a is negative.

Sketch the graph and label all the important features.



REWIND

You learned how to solve quadratic equations by completing the square in Chapter 14. Revise that section now if you've forgotten how to do this.

Find the turning point by completing the square

You can find the coordinates of the turning point of a parabola algebraically by completing the square. This involves changing the quadratic equation from the standard form $ax^2 + bx + c = 0$ to the form $a(x + p)^2 + q$. In this form, the turning point of a parabola has the coordinates $(-p, q)$.

E

Consider the equation $y = x^2 + 4x - 5$

This can be rewritten as $y = (x + 2)^2 - 9$ by completing the square.

Squaring any value results in an answer that is either positive or 0. This means that for any value of x, the smallest value of $(x + 2)^2$ is 0.

This means that the minimum value of $(x + 2)^2 - 9$ is -9 and that this occurs when $x = -2$

The turning point of the graph $y = (x + 2)^2 - 9$ has the coordinates $(-2, -9)$

Worked example 5

a Determine the equation of the axis of symmetry and turning point of $y = x^2 - 8x + 13$ by completing the square.

b Sketch the graph.

a

$y = x^2 - 8x + 13$	First complete the square.
$y = (x - 4)^2 - 16 + 13$	Half of 8 is four, but $(x - 4)^2 = x^2 - 8x + 16$ so you have to subtract 16 to keep the equation balanced.
$y = (x - 4)^2 - 3$	Simplify your solution.
Turning point: $(4, -3)$	
Axis of symmetry: $x = 4$	

b

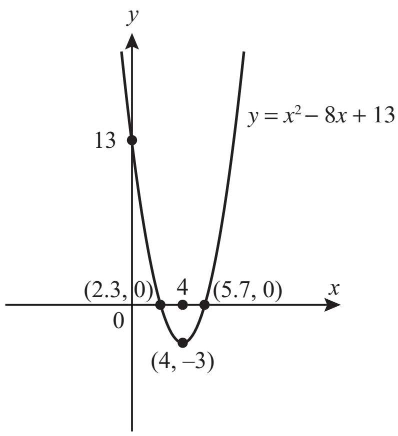
To sketch the graph, you must find the intercepts.

y-intercept = (0,13)	You can read this from the original equation.

To find the x-intercept(s), let $y = 0$ and solve.

$0 = (x - 4)^2 - 3$	
$3 = (x - 4)^2$	
$x - 4 = \pm\sqrt{3}$	Remember there is a negative and a positive root.
$x = \pm\sqrt{3} + 4$	
$x = 5.7$ or 2.3	

Sketch the graph and label it.



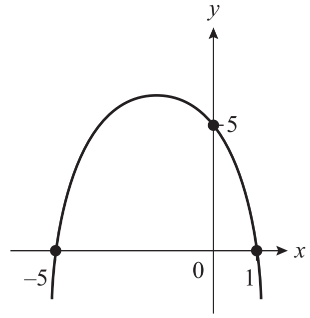
Exercise 18.3

1 Sketch the following graphs.

a $y = x^2 - 3x - 4$
b $y = x^2 - 2x - 7$
c $y = x^2 + 4x + 4$
d $y = x^2 + 4x - 5$
e $y = x^2 + 6x + 8$
f $y = x^2 - 3x - 4$
g $y = x^2 + 7x + 12$

2 Nadia sketched the following graphs and forgot to label them. Use the information on the sketch to determine the equation of each graph.

a



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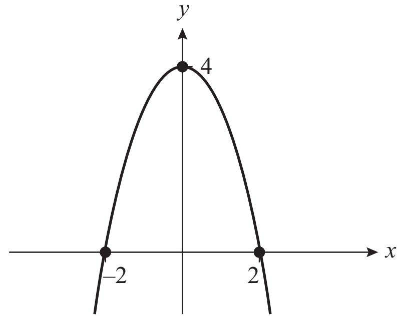
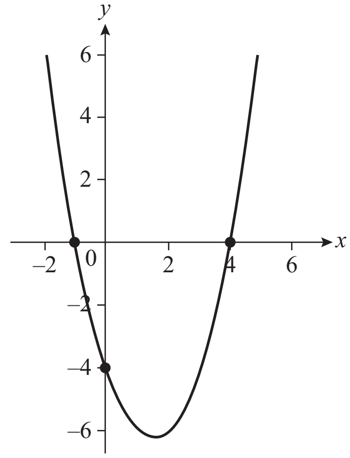
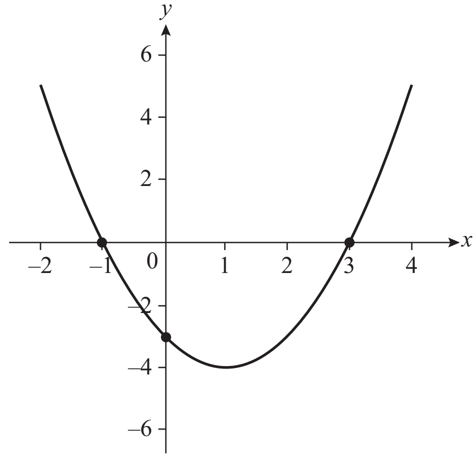


c



d



3 Sketch the following graphs. Indicate the axis of symmetry and the coordinates of the turning point on each graph.

a $y = x^2 + 6x - 5$ **b** $2x^2 + 4x = y$ **c** $y = 3 - (x + 1)^2$

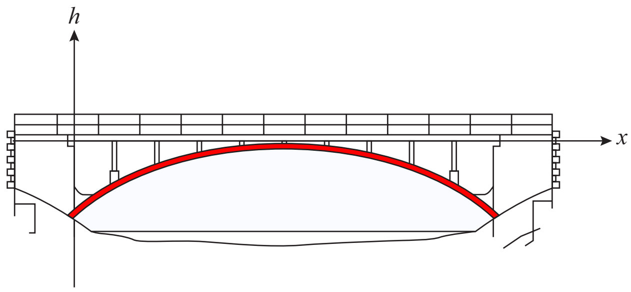
d $y = 4 - 2(x + 3)^2$ **e** $y = 17 + 6x - x^2$ **f** $y = 5 - 8x + 2x^2$

g $y = 1 + 2x - 2x^2$ **h** $y = -(x + 2)^2 - 1$

E

Applying your skills

4 The equation for the curved supporting arch of a bridge (shown in red on the diagram) is given by $h = -\frac{1}{40}(x - 20)^2$ where h m is the distance below the base of the bridge and x m is the distance from the left side.



a Determine the turning point of the graph of the relationship.
b What are the possible values for x?
c Determine the range of values of h.
d Sketch a graph of the equation within the possible values.
e What is the width of the supporting arch?
f What is the maximum height of the supporting arch?

18.2 Drawing reciprocal graphs (the hyperbola)

Reciprocal equations have a constant product. If $y = \frac{6}{x}$ then $xy = 6$. There is no value of y that corresponds with $x = 0$ because division by 0 is meaningless. Similarly, if x was 0, then xy would also be 0 for all values of y and not 6, as it should be in this example. This is what causes the two parts of the curve to be disconnected.

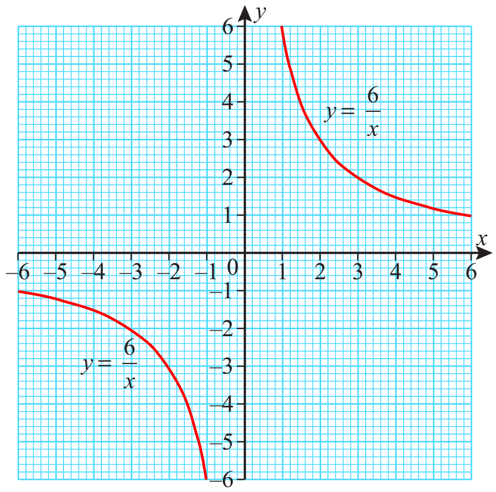
Equations in the form of $y = \frac{a}{x}$ (where a is a whole number) are called **reciprocal** equations. Graphs of reciprocal equations are called **hyperbolas**. These graphs have a very characteristic shape. Although it is one graph, it consists of two non-connected curves that are mirror images of each other, drawn in opposite quadrants.

Here is a table of values for $y = \frac{6}{x}$.

x	−6	−5	−4	−3	−2	−1	1	2	3	4	5	6
$y = \frac{6}{x}$	−1	−1.2	−1.5	−2	−3	−6	6	3	2	1.5	1.2	1

Include at least five negative and five positive values in the table of values to draw a hyperbola because it has two separate curves.

When you plot these points, you get this graph.



Notice the following about the graph:

- it has two parts which are the same shape and size, but in opposite quadrants
- the curve is symmetrical
- the curve approaches the axes, but it will never touch them
- there is no value of y for $x = 0$ and no value of x for $y = 0$.

An **asymptote** is a line that a graph approaches but never intersects. When the equation is in the form $y = \frac{a}{x}$, the curve approaches both axes and gets closer and closer to them without ever touching them.

For other reciprocal equations, the asymptotes may not be the axes, in these cases, they are normally shown on the graph as dotted lines.

Worked example 6

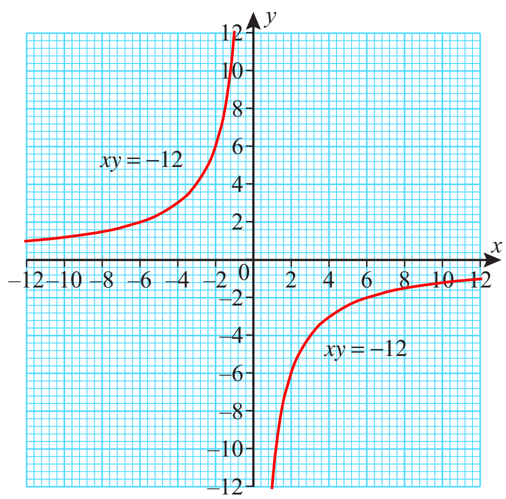
Construct a table of values and then draw a graph of $xy = -12$ ($x \neq 0$) for $-12 \leqslant x \leqslant 12$.

$xy = -12$ is the same as $y = \frac{-12}{x}$.

In this case, you can work out every second value as you will not need all 24 points to draw the graph.

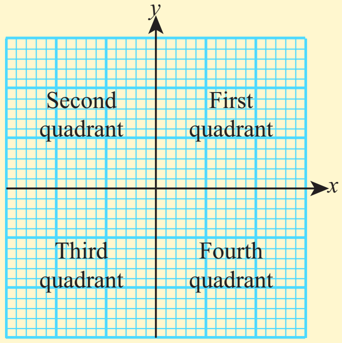
x	−12	−10	−8	−6	−4	−2	2	4	6	8	10	12
$y = \frac{-12}{x}$	1	1.2	1.5	2	3	6	−6	−3	−2	−1.5	−1.2	−1

Plot the points to draw the graph.



Notice that the graph of $xy = -12$ is in the top left and bottom right quadrants. This is because the value of the constant term (a in the equation $y = \frac{a}{x}$) is negative. When a is a positive value, the hyperbola will be in the top right and bottom left quadrants.

The quadrants are labelled in an anti-clockwise direction. The co-ordinates of any point in the first quadrant will always be positive.



To plot the graph of a reciprocal relationship:

- complete a table of values (often some of the values will be given)
- rule the axes and label them
- plot the (x, y) values from the table of values
- join the points with a smooth curve
- write the equation on both parts of the graph.

Sketching graphs of reciprocal functions

As with the parabola, you can use the features of the hyperbola (reciprocal function) to sketch the graph.

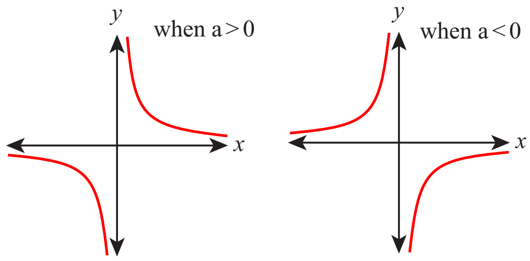
When the equation is in standard form $y = \frac{a}{x} + q$ $(x \neq 0, y \neq 0)$ follow these steps to sketch the graph.

Step 1: Identify the shape of the graph.

The value of a determines where the curves will be on the graph.

If a > 0, the y values are positive for positive x values and negative for negative x.

If a < 0, the y values are negative for positive x values and positive for negative x.



Step 2: Work out whether the graph intercepts the x-axis using q. If $q \neq 0$, the graph will have one x-intercept. Make $y = 0$ to find the value of the x-intercept.

$0 = \frac{a}{x} + q$

So, $-q = \frac{a}{x}$

$-qx = a$

$x = -\frac{a}{q}$

The graph doesn't intercept the y-axis.

If $q = 0$, the x-axis is the other asymptote.

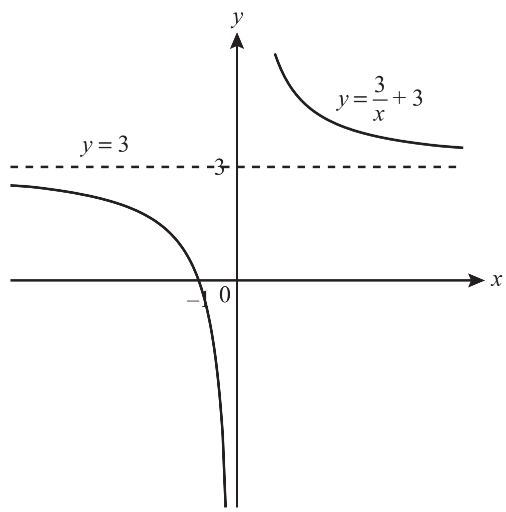
Step 3: Determine the asymptotes. One asymptote is the y-axis (the line $x = 0$). The other is the line $y = q$.

Step 4: Using the asymptotes and the x-intercept, sketch and label the graph.

Worked example 7

Sketch and label the graph of $y = \frac{3}{x} + 3$

Position of the curves:
$a = 3$, so $a > 0$ and the right hand curve is higher.
Asymptotes:
$x = 0$
$y = -3$
x-intercept:
$0 = \frac{3}{x} + 3$
$-3 = \frac{3}{x}$
$x = -1$
x-intercept $(-1, 0)$



Exercise 18.4

1 Copy and complete the following tables giving values of y correct to 1 decimal place. Use the points to plot each graph on a separate set of axes.

a

x	−6	−4	−3	−2	−1	1	2	3	4	6
$y=\frac{2}{x}$										

b

x	−5	−4	−3	−2	−1	1	2	3	4	5
$y=\frac{-1}{x}$										

c

x	−6	−4	−3	−2	−1	1	2	3	4	6
$y=\frac{-6}{x}$										

d

x	−6	−4	−3	−2	−1	1	2	3	4	6
$y=\frac{4}{x}$										

2 Sketch and label the following graphs on separate sets of axes.

a $y=\frac{3}{x}$ **b** $xy=-4$ **c** $y=\frac{1}{x}+3$

d $2y=\frac{4}{x}+7$ **e** $y=\frac{4}{x}+2$ **f** $y=-\frac{9}{x}-3$

Rewrite the equation in standard form before you sketch the graph.

Applying your skills

3 A person makes a journey of 240 km. The average speed is x km/h and the time the journey takes is y hours.

a Complete this table of corresponding values for x and y:

x	20	40	60	80	100	120
y	12		4			2

b On a set of axes, draw a graph to represent the relationship between x and y.
c Write down the relation between x and y in its algebraic form.

4 Investigate what happens when the equation of a graph is $y=\frac{1}{x^2}$.

a Copy and complete the table of values for x-values between −4 and 4.

x	−4	−3	−2	−1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y										

b Plot the points to draw the graph.
c How does your graph differ from the hyperbola?
d Why do we not use $x=0$ in the table of values?
e What are the asymptotes of the graph you have drawn?
f As with the hyperbola, the standard form $y=\frac{1}{x^2}+q$ can be used to work out the asymptotes. Given $y=\frac{1}{x^2}+3$, what would the asymptotes be?
g Use what you have learned in your investigation to sketch the graphs of:
i $y=-\frac{1}{x^2}$ **(ii)** $y=x^{-2}+2$

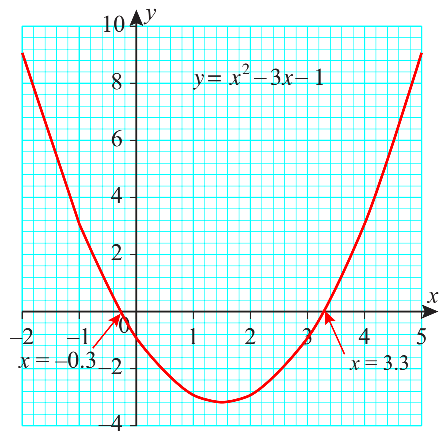
18.3 Using graphs to solve quadratic equations

Suppose you were asked to solve the equation $x^2 - 3x - 1 = 0$.

To do this, you would need to find the value or values of x that make $x^2 - 3x - 1$ equal to 0.

You can try to do this by trial and error, but you will find that the value of x you need is not a whole number (in fact, it lies between the values of 3 and 4).

It is much quicker and easier to draw the graph of the equation $y = x^2 - 3x - 1$ and to use that to find a solution to the equation. Here is the graph:



Use a sharp pencil. You will be able to correct your work more easily and it will be more accurate when looking at intersections.

The solution to the equation is the point (or points) where $y = 0$, in other words you are looking for the value of x where the graph crosses the x-axis.

If you look at the graph you can see that it crosses the x-axis in two places. The value of x at these points is 3.3 and −0.3.

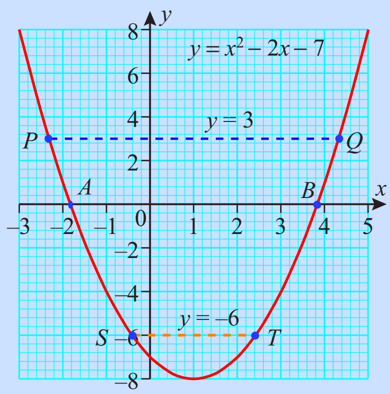
These two values are sometimes referred to as the roots of the equation $x^2 - 3x - 1 = 0$.

You can use the graph to find the solution of the equation for different values of x. Work through the worked example carefully to see how to do this.

Worked example 8

This is the graph of $y = x^2 - 2x - 7$. Use the graph to solve the equations:

a $x^2 - 2x - 7 = 0$ **b** $x^2 - 2x - 7 = 3$ **c** $x^2 - 2x = 1$



a Since this is the graph of $y = x^2 - 2x - 7$, simply find the points on the curve that have a y co-ordinate of 0 (i.e. where the curve cuts the x-axis).
There are two such points, marked A and B on the graph.
The x co-ordinates of these points are −1.8 and 3.8, so the solutions of the equation $x^2 - 2x - 7 = 0$ are $x = -1.8$ and $x = 3.8$

b Find the points on the curve that have a y co-ordinate of 3. (Draw in the horizontal line $y = 3$ to help with this.)
There are two such points, marked P and Q, on the graph.
The x co-ordinates of these points are −2.3 and 4.3, so the solutions of the equation $x^2 - 2x - 7 = 3$ are $x = -2.3$ and $x = 4.3$

c Rearrange the equation $x^2 - 2x = 1$ so that the left-hand side matches the equation whose graph you are using.
Subtracting 7 from both sides, you get $x^2 - 2x - 7 = 1 - 7$, that is $x^2 - 2x - 7 = -6$.
You can now proceed as you did in parts a and b.
Find the points on the curve that have a y co-ordinate of −6; they are marked S and T on the graph.
The x co-ordinates of S and T are −0.4 and 2.4
The solutions of the equation $x^2 - 2x = 1$ are $x = -0.4$ and $x = 2.4$

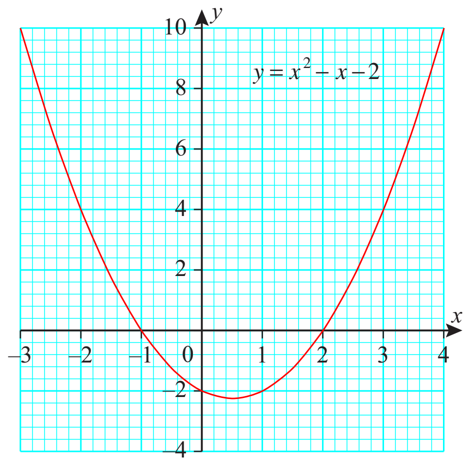
In summary, to solve a quadratic equation graphically:

- read off the x co-ordinates of any points of intersection for the given y-values
- you may need to rearrange the original equation to do this.

Exercise 18.5

1 Use this graph of the relationship $y = x^2 - x - 2$ to solve the following equations:

a $x^2 - x - 2 = 0$
b $x^2 - x - 2 = 6$
c $x^2 - x = 6$



2 **a** Construct a table of values for $y = -x^2 - x + 1$ for values $-3 \leqslant x \leqslant 2$.
b Plot the points on a grid and join them with a smooth curve.
c Use your graph to solve the equation $-x^2 - x + 1 = 0$. Give your answer correct to 1 decimal place.

3 Solve the following equations by drawing suitable graphs over the given intervals.

a $x^2 - x - 3 = 0 \ (-3 \leqslant x \leqslant 4)$
b $x^2 + 3x + 1 = 0 \ (-4 \leqslant x \leqslant 1)$

4 **a** Use an interval of $-2 \leqslant x \leqslant 4$ to draw the graph $y = 4 - x^2 + 2x$.
b Use the graph to solve the following equations:

i $0 = 4 - x^2 + 2x$
ii $0 = -x^2 + 2x$

5 **a** Draw the graph of $y = x^2 - 2x - 4$ for values of x from -3 to 5.
b Use your graph to find approximate solutions to the equations:

i $x^2 - 2x - 4 = 0$
ii $x^2 - 2x - 4 = 3$
iii $x^2 - 2x - 4 = -1$

18.4 Using graphs to solve simultaneous linear and non-linear equations

E

As you did with linear equations, you can use graphs to solve a linear and a non-linear equation, or two non-linear equations simultaneously.

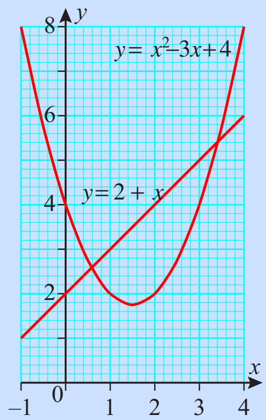
REWIND

In chapter 14 you learned how to use the point of **intersection** of two straight lines to find the solutions to simultaneous linear equations. Revise that section now if you cannot remember how to do this.

E

Worked example 9

The graphs of $y = 2 + x$ and $y = x^2 - 3x + 4$ have been drawn on the same set of axes. Use the graphs to find the x-values of the points of intersection of the line and the curve.



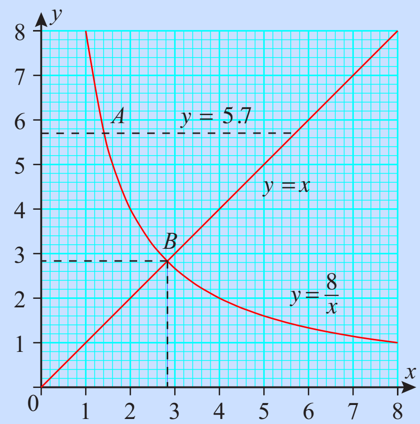
The co-ordinates of the two points of intersection are approximately (0.6, 2.6) and (3.4, 5.4), so the x-values of the points of intersection are $x = 0.6$ and $x = 3.4$

Tip

You might also be asked for the y-values, so it is important to pair up the correct x-value with the correct y-value. When $x = 0.6$, $y = 2.6$ and when $x = 3.4$, $y = 5.4$.

Worked example 10

The diagram shows the graphs of $y = \frac{8}{x}$ and $y = x$ for positive values of x.



a Use the graph of $y = \frac{8}{x}$ to solve the equation $\frac{8}{x} = 5.7$

b Find a value of x such that $\frac{8}{x} = x$.

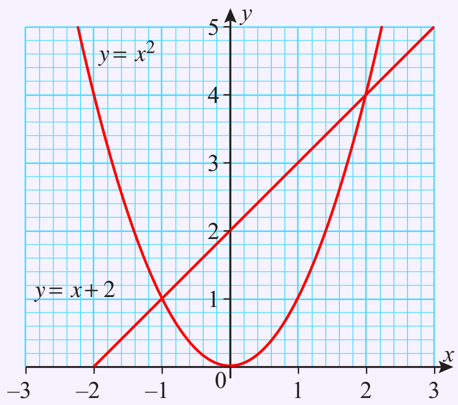
a You have to find a point on the curve that has a y co-ordinate of 5.7. Draw the line $y = 5.7$ to help find this it will be where the line cuts the curve.
The point is marked A on the diagram. Its x co-ordinate is 1.4, so the solution of the equation $\frac{8}{x} = 5.7$ is $x = 1.4$

b The straight line $y = x$ crosses the curve $y = \frac{8}{x}$ at the point B, with x co-ordinate is 2.8. Hence, a value of x such that $\frac{8}{x} = x$ is 2.8

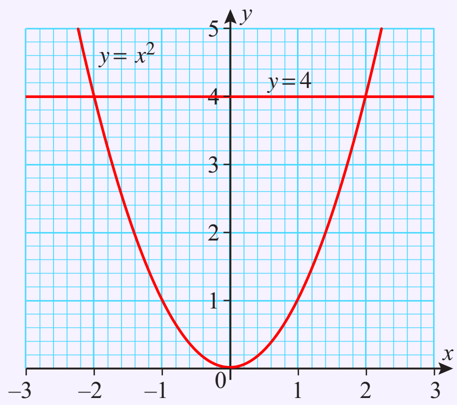
Exercise 18.6

1 Find the points of intersection of the graphs and thus give the solution to the simultaneous equations.

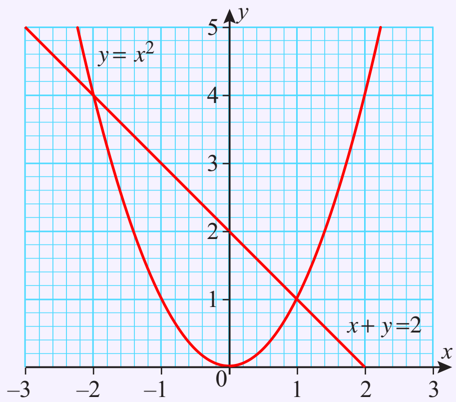
a



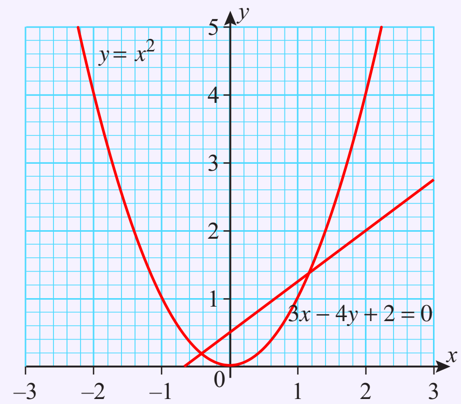
b



c



d



2 Find the points of intersection of the following graphs by drawing the graphs.

a $y = x^2$ and $y = 3x$

b $y = x$ and $y = \dfrac{2}{x}$

c $y = 2 - x$ and $y = x^2 - 5x + 6$

3 Use a graphical method to solve each pair of simultaneous equations:

a $y = x^2 - 8x + 9$ and $y = 2x + 1$
b $y = x^2 - x - 6$ and $y = 2 + x$
c $y = 4x + 4$ and $y = 2x - 3 + x^2$

4 Show graphically that there is no value of x which satisfies the pair of equations $y = -4$ and $y = x^2 + 2x + 3$ simultaneously.

18.5 Other non-linear graphs

So far you have learned how to construct a table of values and draw three different kinds of graphs:

- linear graphs (straight lines of equations in form of $y = mx + c$)
- quadratic graphs (parabolas of equations in the form of $y = x^2 + ax + b$)
- reciprocal graphs (hyperbolas of equations in the form of $y = \dfrac{a}{x}$)

In this section you are going to apply what you already know to plot and draw graphs formed by higher order equations (cubic equations) and those formed by equations that have combinations of linear, quadratic, reciprocal and cubic terms.

> **Tip**
> You are expected to deal with equations that have terms with indices that can be any whole number from –2 to 3. When you have to work with higher order equations, they will not contain more than three terms.

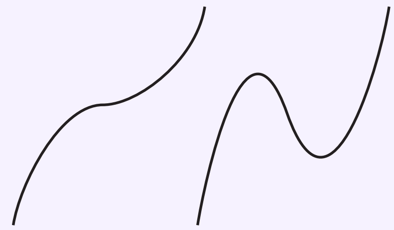
If x is positive, then x^3 is positive and $-x^3$ is negative.
If x is negative, then x^3 is negative and $-x^3$ is positive.

Plotting cubic graphs

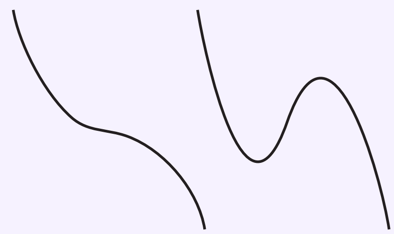
A cubic equation has a term with an index of three as the highest power of x. In other words, one of the terms is ax^3. For example, $y = 2x^3$, $y = -x^3 + 2x^2 + 3$ and $y = 2x^3 - 4x$ are all cubic equations. The simplest cubic equation is $y = x^3$.

Cubic equations produce graphs called cubic curves. The graphs you will draw will have two main shapes:

If the coefficient of the x^3 term is positive, the graph will take one of these shapes.



If the coefficient of the x^3 term is negative, the graph will be take one of these shapes.



Sketching cubic functions

You've seen that you can sketch a parabola if you know certain features of the graph. You can also sketch cubic functions if you know the following features:

- The basic shape of the graph. This is determined by the highest power of the graph.
- The orientation of the graph. This is determined by the sign of the coefficient of the term with the highest power.
- The y-intercept. This is determined by substituting $x = 0$ into the equation.
- The x-intercepts. When the cubic equation is given in factor form (for example $y = (x + a)(x + b)(x + c)$, you can let $y = 0$ and solve for x. A cubic graph may have three, two or one x-intercepts.
- The **turning point/s** of the graph. To find the turning points of a cubic function you need to use the differentiation techniques you will learn later in this chapter. For now, you need to remember that the graph of $y = ax^3 + bx^2 + cx + d$ has two basic shapes depending on whether $a > 0$ or $a < 0$.

You will learn more about how to work out the x-intercepts and turning points of a cubic function and use these to sketch cubic graphs later in this chapter when you deal with differentiation.
In this section, you are going to use a table of values and plot points to draw some cubic graphs.

> **LINK**
> Geophysicists use equations and graphs to process measurements (such as the rise or pressure of magma in a volcano) and use these to generalise patterns and make predictions.

Worked example 11

Complete the table of values and plot the points to draw the graphs on the same set of axes.

a

x	–2	–1	0	1	2
$y = x^3$					

b

x	–2	–1	0	1	2
$y = -x^3$					

E

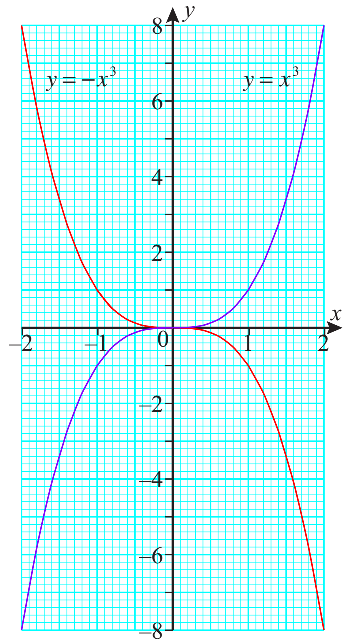
As the value of x increases, the values of x^3 increase rapidly and it becomes difficult to fit them onto the graph. If you have to construct your own table of values, stick to low numbers and, possibly, include the half points (0.5, 1.5, etc.) to find more values that will fit onto the graph.

a

x	−2	−1	0	1	2
$y = x^3$	−8	−1	0	1	8

b

x	−2	−1	0	1	2
$y = -x^3$	8	1	0	−1	−8



Worked example 12

Draw the graph of the equation $y = x^3 - 6x$ for $-3 \leqslant x \leqslant 3$.

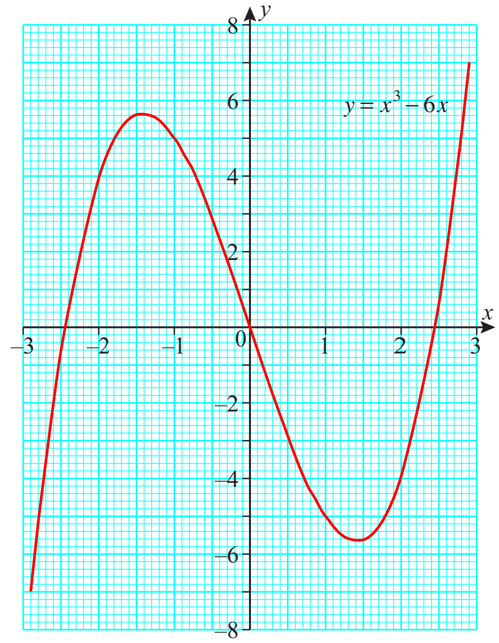
Construct a table of values for whole number values of x first.
Put each term in a separate row.
Add the columns to find $y = x^3 - 6x$. (Remember not to add the top row when calculating y.)

x	−3	−2	−1	0	1	2	3
x^3	−27	−8	−1	0	1	8	27
$-6x$	18	12	6	0	−6	−12	−18
$y = x^3 - 6x$	−9	4	5	0	−5	−4	9

Construct a separate table for 'half values' of x.

x	−2.5	−1.5	−0.5	0.5	1.5	2.5
x^3	−15.625	−3.375	−0.125	0.125	3.375	15.625
$-6x$	15	9	3	−3	−9	−15
$y = x^3 - 6x$	−0.625	5.625	2.875	−2.875	−5.625	0.625

Plot the points against the axes and join them with a smooth curve.



Using graphs to solve higher order equations

You can use cubic graphs to find approximate solutions to equations. The following worked example shows how to do this.

Worked example 13

a Draw the graph of the equation $y = x^3 - 2x^2 - 1$ for $-1 \leqslant x \leqslant 3$.

b Use the graph to solve the equations:

i $x^3 - 2x^2 - 1 = 0$

ii $x^3 - 2x^2 = -1$

iii $x^3 - 2x^2 - 5 = 0$.

a Construct a table of values of y for whole and half values x.

x	–1	–0.5	0	0.5	1	1.5	2	2.5	3
x^3	–1	–0.125	0	0.125	1	3.375	8	15.625	27
$-2x^2$	–2	–0.5	0	–0.5	–2	–4.5	–8	–12.5	–18
-1	–1	–1	–1	–1	–1	–1	–1	–1	–1
$y = x^3 - 2x^2 - 1$	–4	–1.625	–1	–1.375	–2	–2.125	–1	2.125	8

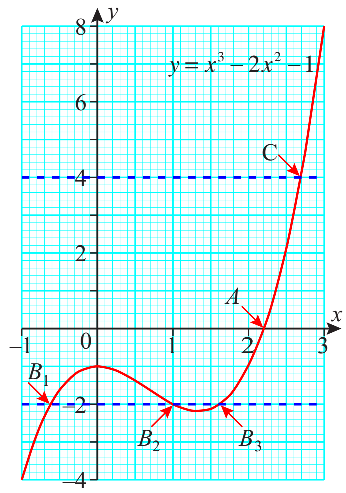
E

b Plot the points on the axes to draw the curve.

i To solve $x^3 - 2x^2 - 1 = 0$, find the point(s) on the curve that have a y co-ordinate of 0 (i.e. where the curve cuts the x-axis).
There is only one point (A on the graph).
The x co-ordinate of A is 2.2, so the solution of $x^3 - 2x^2 - 1 = 0$ is $x = 2.2$

ii To solve $x^3 - 2x^2 = -1$, rearrange the equation so that the left-hand side is the same as the equation you have just drawn the graph for.
Subtracting 1 from both sides gives $x^3 - 2x^2 - 1 = -2$.
Now find the point(s) on the curve that have a y co-ordinate of -2 (draw the line $y = -2$ to help with this).
There are three points (B_1, B_2 and B_3 on the graph).
The x co-ordinates of these points are the solutions of the equation.
So the solutions of $x^3 - 2x^2 = -1$ are $x = -0.6$, $x = 1$ and $x = 1.6$

iii Rearrange the equation $x^3 - 2x^2 - 5 = 0$ so you can use the graph of $y = x^3 - 2x^2 - 1$ to solve it.
Adding 4 to both sides of the equation, you get $x^3 - 2x^2 - 1 = 4$.
Find the point(s) on the curve that have a y co-ordinate of 4 (draw the line $y = 4$ to help with this).
There is only one point (C on the graph).
At C the x co-ordinate is 2.7.
The approximate solution is, therefore, $x = 2.7$



Exercise 18.7

Before drawing the axes, check the range of y-values required from your table.

1 Construct a table of values from $-3 \leqslant x \leqslant 3$ and plot the points to draw graphs of the following equations.

a $y = 2x^3$ **b** $y = -3x^3$ **c** $y = x^3 - 2$ **d** $y = 3 + 2x^3$
e $y = x^3 - 2x^2$ **f** $y = 2x^3 - 4x + 1$ **g** $y = -x^3 + x^2 - 9$ **h** $y = x^3 - 2x^2 + 1$

2 **a** Copy and complete the table of values for the equation $y = x^3 - 6x^2 + 8x$. (You may want to add more rows to the table as in the worked examples.)

x	–1	–0.5	0	0.5	1.5	1	2.5	3	3.5	4	4.5	5
$y = x^3 - 6x^2 + 8x$	–15	–5.6										

b On a set of axes, draw the graph of the equation $y = x^3 - 6x^2 + 8x$ for $-1 \leqslant x \leqslant 5$.
c Use the graph to solve the equations:

i $x^3 - 6x^2 + 8x = 0$
ii $x^3 - 6x^2 + 8x = 3$

3 **a** Draw the graphs of $y = \frac{x^3}{10}$ and $y = 6x - x^2$ for $-4 \leqslant x \leqslant 6$.
b Use the graphs to solve the equation $\frac{x^3}{10} + x^2 - 6x = 0$

Graphs of equations with combinations of terms

When you have to plot graphs of equations with a combination of linear, quadratic, cubic, reciprocal or constant terms you need to draw up a table of values with at least eight values of x to get a good indication of the shape of the graph.

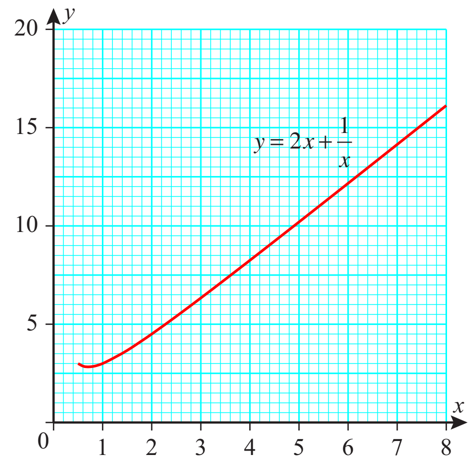
E

Worked example 14

Complete this table of values for the equation $y = 2x + \frac{1}{x}$ for $0.5 \leqslant x \leqslant 7$ and draw the graph.

x	0.5	1	2	3	4	5	6	7
$2x$	1	2	4	6	8	10	12	14
$\frac{1}{x}$								
$y = 2x + \frac{1}{x}$								

x	0.5	1	2	3	4	5	6	7
$2x$	1	2	4	6	8	10	12	14
$\frac{1}{x}$	2	1	0.5	0.33	0.25	0.2	0.17	0.14
$y = 2x + \frac{1}{x}$	3	3	4.5	6.33	8.25	10.2	12.17	14.14



Worked example 15

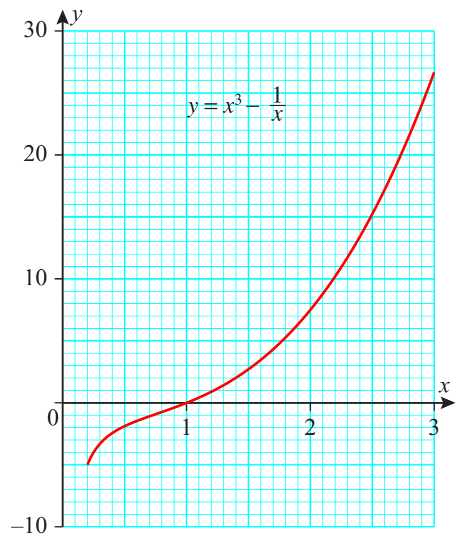
Complete this table of values and plot the graph of $y = x^3 - \frac{1}{x}$ for $0.2 \leqslant x \leqslant 3$.

x	0.2	0.5	1	1.5	2	2.5	3
x^3	0.008		1		8		27
$-\frac{1}{x}$							
$y = x^3 - \frac{1}{x}$	−5.0						

E

x	0.2	0.5	1	1.5	2	2.5	3
x^3	0.008	0.125	1	3.375	8	15.625	27
$-\frac{1}{x}$	–5	–2	–1	–0.667	–0.5	–0.4	–0.33
$y = x^3 - \frac{1}{x}$	–5.0	–1.9	0	2.7	7.5	15.2	26.7

Round the *y*-values in the last row to 1 decimal place or it will be difficult to plot them.



Exercise 18.8

REWIND

You learned about exponential growth and decay and applied a formula to calculate growth in chapter 17. ◀

1 Construct a table of values for $-3 \leqslant x \leqslant 3$ (including negative and positive values of 0.5 and 0.2) for each equation and draw the graph.

a $y = 3 + x^2 - \frac{2}{x}$

b $y = 3x - \frac{1}{x}$

c $y = -x + x^2 + \frac{2}{x}$

d $y = -x^3 - 2x + 1$ (omit the fractional values in this case)

Exponential graphs

An exponential graph in the form of $y = a^x$ will always intersect the y-axis at the point (0, 1) because $a^0 = 1$ for all values of a.

(You should remember this from the laws of indices.)

Euler's number, e = 2.71. . . , is so special that $y = e^x$ is known as *the* exponential function rather than *an* exponential function.

Exponential growth is found in many real life situations where a quantity increases by a constant percentage in a particular time: population growth and compound interest are both examples of exponential growth.

Equations in the general form of $y = a^x$ (where a is a positive integer) are called exponential equations.

The shape of $y = a^x$ is a curve which rapidly rises as it moves from left to right; this is exponential growth. As x becomes more negative, the curve gets closer and closer to the x-axis but never crosses it. The x-axis is an asymptote.

The shape of $y = a^{-x}$ is a curve which falls as it moves from left to right; this is exponential decay.

E

Worked example 16

a Complete the table of values for $y = 2^x$ for $-2 \leqslant x \leqslant 4$ and draw the graph.

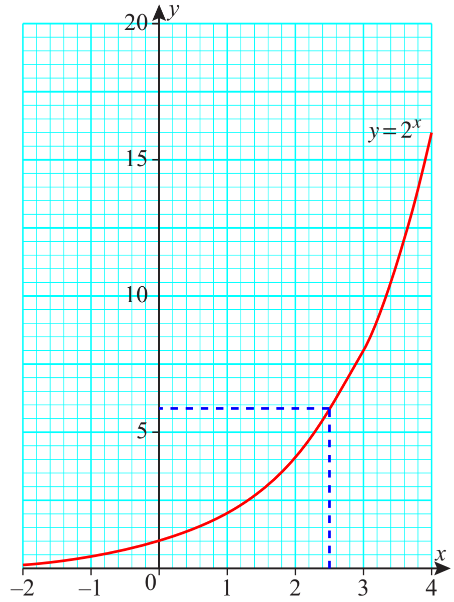
x	–2	–1.5	–1	–0.5	0	1	2	3	4
$y = 2^x$									

b Use the graph to find the value of $2^{2.5}$ and check your result using the fact that $2^{2.5} = 2^{\frac{5}{2}} = \sqrt{2^5}$.

a

x	–2	–1.5	–1	–0.5	0	1	2	3	4
$y = 2^x$	0.25	0.35	0.5	0.71	1	2	4	8	16

Plot the points to draw the graph.



b From the graph you can see that when $x = 2.5$ the value of y is 5.7, so, $2^{2.5} \approx 5.7$

Check: $2^{2.5} = 2^{\frac{5}{2}} = \sqrt{2^5} = \sqrt{32} = 5.656\ldots$

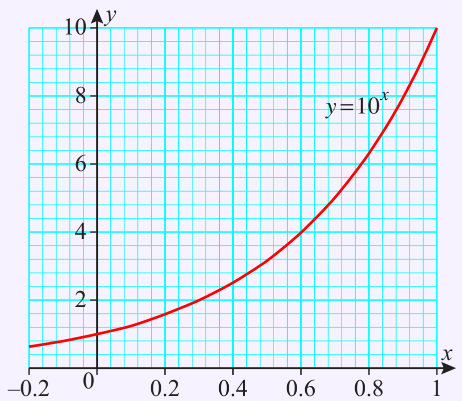
Exercise 18.9

1 **a** Draw the graph of $y = 3^x$ for x-values between –2 and 3. Give the values to 2 decimal places where necessary.

b On the same set of axes draw the graph of $y = 3^{-x}$ for x-values between –3 and 2. Give the values to 2 decimal places where necessary.

c What is the relationship between the graph of $y = 3^x$ and $y = 3^{-x}$?

2 The graph of $y = 10^x$ for $-0.2 \leqslant x \leqslant 1.0$ is shown here.

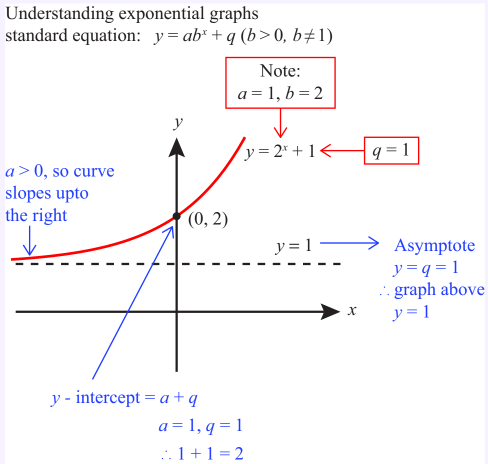


Use the graph to find the value of:

a $10^{0.3}$ **b** $10^{-0.1}$

c Copy the diagram using tracing paper and draw a straight line graph that will allow you solve the equation $10^x = 8 - 5x$.

3 Mae finds the following explanation on the internet.



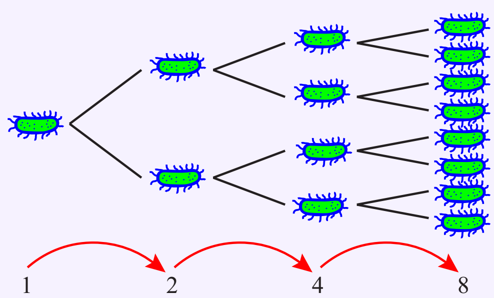
a Read the information carefully and write a step-by-step set of instructions for sketching an exponential graph.

b Sketch and label the following graphs.

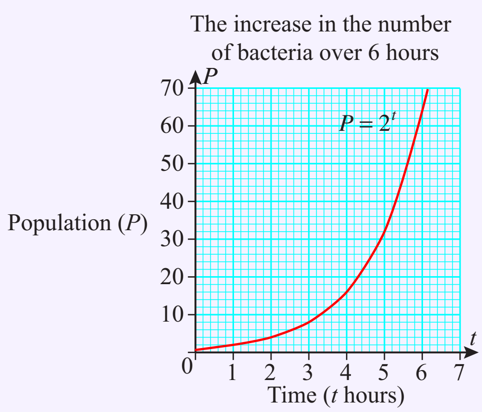
i $y = -3^x$ **ii** $y = 3^x - 4$ **iii** $y = -2^x + 1$

Applying your skills

4 Bacteria multiply rapidly because a cell divides into two cells and then those two cells divide to each produce two more cells and so on. The growth rate is exponential and we can express the population of bacteria over time using the formula $P = 2^t$ (t is the period of time).



The graph shows the increase in bacteria numbers in a six-hour period.



E

a How many bacteria are there after one hour?
b How long does it take for the number of bacteria to exceed 40 cells?
c How many cells will there be after six hours?
d When would you expect the population to exceed one million bacteria if it continued to grow at this rate?

These figures look different to the ones you are used to because the equation describing the temperature in terms of time is $T = 5 \times 3^t$; where T is the temperature and t is the time in minutes.

5 The temperature of metal in a smelting furnace increases exponentially as indicated in the table. Draw a graph to show this data.

Time (min)	0	1	2	3	4
Temp (°C)	5	15	45	135	405

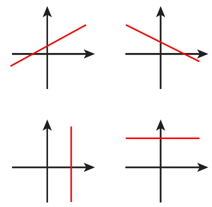
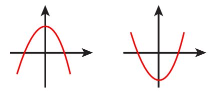
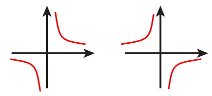
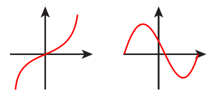
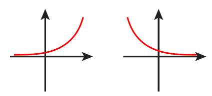
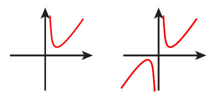
6 The population of bedbugs in New York City is found to be increasing exponentially. The changes in population are given below.

Time (months)	0	1	2	3	4
Population	1000	2000	4000	8000	16 000

a Plot a graph to show the population increase over time.
b When did the bedbug population reach 10 000?
c What will the bedbug population be after six months if it continues to increase at this rate?

Recognising which graph to draw

You need to be able to look at equations and identify which type of graph they represent. This table summarises what you have learned so far.

Type of graph	**General equation**	**Shape of graph**
Straight line (linear)	$y = mx + c$ Highest power of x is 1. When $x = a$ the line is parallel to the y-axis and when $y = b$ the line is parallel to the x-axis.	
Parabola (quadratic)	$y = x^2$ $y = ax^2 + bx + c$ Highest power of x is 2.	
Hyperbola (reciprocal)	$y = \frac{a}{x}$ or $xy = a$ Can also be $y = \frac{a}{x^2} + q$	
Cubic curve	$y = x^3$ $y = ax^3 + bx^2 + cx + d$ Highest power of x is 3.	
Exponential curve	$y = a^x$ or $y = a^{-x}$ Can also be $y = ab^x + q$	
Combined curve (linear, quadratic, cubic and/or reciprocal)	Up to three terms of: $y = ax^3 + bx^2 + cx + \frac{d}{x} + e$	

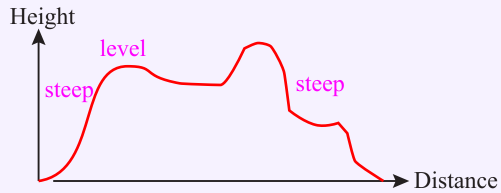
The first 3 types of graph are suitable for Core learners.

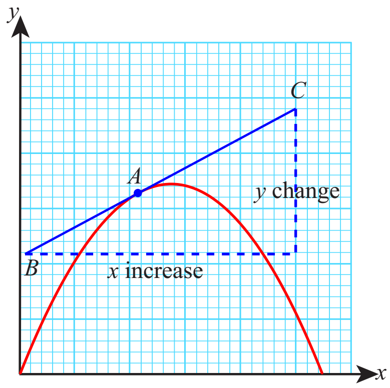
18.6 Finding the gradient of a curve

REWIND

Look again at calculating gradients from chapter 10. Make sure you understand how to do this before moving onto this section. ◀

This simple graph of height against distance shows the route followed by a mountain biker on a trail.





Some parts of the trail have a steep positive **gradient**, some have a gradual positive gradient, some parts are level and other parts have a negative gradient. It should be clear from this graph, that a curved graph never has a single gradient like a straight line has.

You cannot find the gradient of a whole curve but you can find the gradient of a point on the curve by drawing a **tangent** to it.

The gradient of a curve at a point is the gradient of the tangent to the curve at that point. Once you have drawn the tangent to a curve, you can work out the gradient of the tangent just as you would for a straight line $\left(\text{gradient} = \dfrac{y\text{-change}}{x\text{-increase}}\right)$.

Look at the graph in the margin to see how this works. BC is the tangent to the curve.

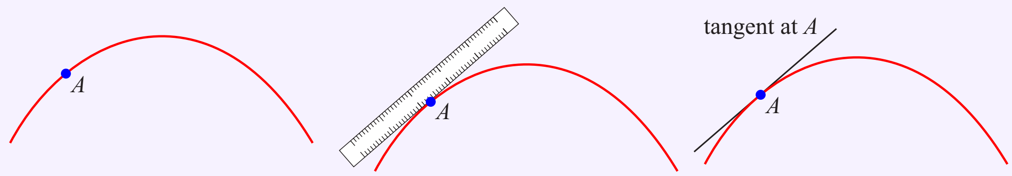
If you extend your tangent, it may touch your curve again at a completely different point but this is not a problem.

If the tangent is rising from left to right, its gradient is positive. If the tangent is falling from left to right, its gradient is negative.

Upwards = positive

Downwards = negative

How to draw the tangent



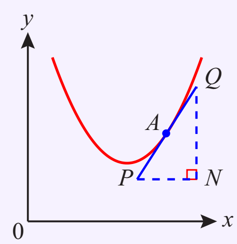
Mark a point on the curve (A).

Place your ruler against the curve so that it touches it only at point A.

Position the ruler so that the angle on either side of the point is more or less equal. Use a pencil to draw the tangent.

Calculating the gradient to a tangent

Mark two points, P and Q, on the tangent. Try to make the horizontal distance between P and Q a whole number of units (measured on the x-axis scale).
Draw a horizontal line through P and a vertical line through Q to form a right-angled triangle PNQ.



$$\text{Gradient of the curve at } A = \text{Gradient of the tangent } PAQ$$
$$= \frac{\text{distance } NQ \text{ (measured on the } y\text{-axis scale)}}{\text{distance } PN \text{ (measured on } x\text{-axis scale)}}$$

Tip

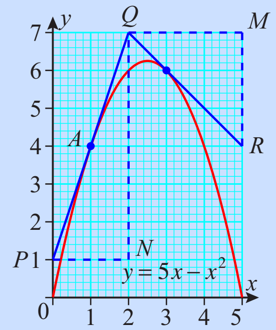
You must measure NQ and PN according to the scales on the y-axis and x-axis respectively. One of the most common mistakes is not doing this!

E

Worked example 17

The graph of the equation $y = 5x - x^2$ is shown in the diagram. Find the gradient of the graph:

a at the point (1, 4)
b at the point (3, 6).



a At the point $A(1, 4)$, gradient $= \frac{NQ}{PN} = \frac{6}{2} = 3$

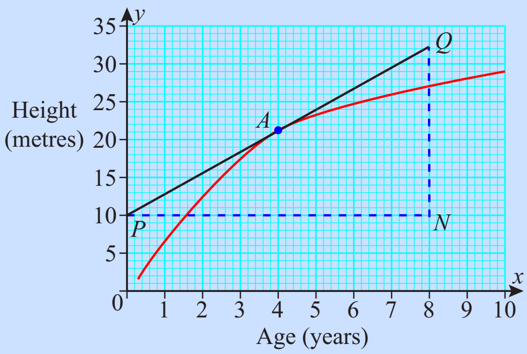
b At the point $B(3, 6)$, gradient $= \frac{MR}{QM} = \frac{-3}{3} = -1$

Tip

When estimating the gradient of a curve at a given point, it is sensible to use as long a tangent as possible on your diagram. The longer the tangent the more accurate the result.

Worked example 18

The graph shows the height of a tree (y metres) plotted against the age of the tree (x years). Estimate the rate at which the tree was growing when it was four years old.



The rate at which the tree was growing when it was four years old is equal to the gradient of the curve at the point where $x = 4$.

Draw the tangent at this point (A).

Gradient at $A = \frac{NQ}{PN} = \frac{22.5}{8} = 2.8$

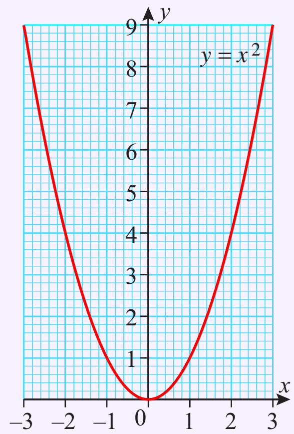
The tree was growing at a rate of 2.8 metres per year.

Remember, the gradient can be used to determine rates of change

Exercise 18.10

E

1 The graph of $y = x^2$ is shown in the diagram.

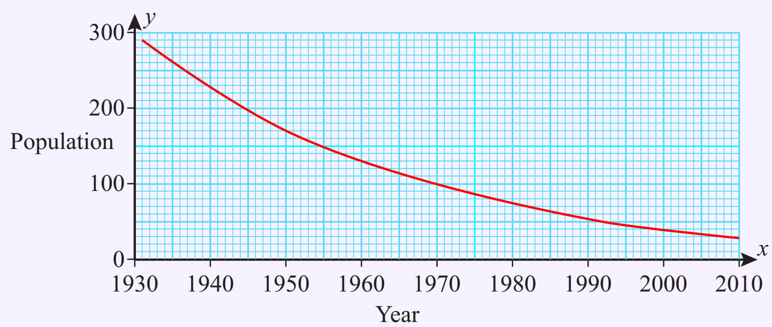


a Copy the graph using tracing paper and find the gradient of the graph at the point:

i (2, 4) **ii** (–1, 1).

b The gradient of the graph at the point (1.5, 2.25) is 3. Write down the co-ordinates of the point at which the gradient is –3.

2 The graph shows how the population of a village has changed since 1930.



a Copy the graph using tracing paper and find the gradient of the graph at the point (1950, 170).

b What does this gradient represent?

3 **a** Draw the graph of the curve $y = x^3 + 1$ for values $-2 \leqslant x \leqslant 2$.

b Find the gradient of the curve at point $A(1, 2)$

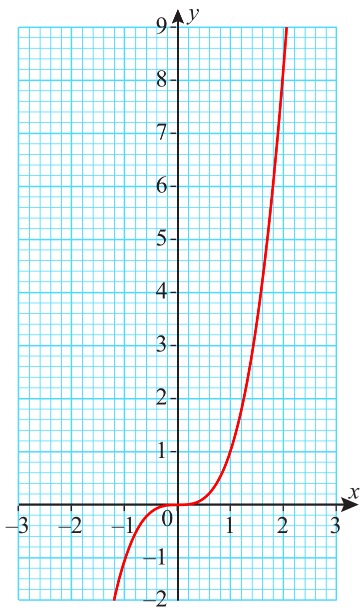
18.7 Derived functions

As well as drawing a tangent, you can calculate the gradient at any point by using the equation of the curve.

Investigation

1 In exercise 18.10 you considered the gradient of the curve with equation $y = x^2$ at the points with x-coordinates 2, –1 and, –1.5. You should have noticed that the gradient is *twice the value of the x-coordinate* at any given point.

Now use the diagram or a larger copy, to draw tangents at the points (1, 1), (–2, 4), (3, 9) and (–3, 9). Compare the gradient of each tangent to the x-coordinate of the point you have used. You should notice that the gradients are twice the x-coordinate in each case.



E

2 Here is the curve with equation $y = x^3$

Copy the graph using tracing paper and draw tangents to find the gradient of the graph at the points (1,1), (2, 8), and (−1, −1).

Copy and fill in the table below and compare the gradient to the value of x^2 in each case.

x coordinate	x^2	Gradient of tangent
1	1	
2	4	
−1	1	

What do you notice?

Try some other points on the curve, where the *x* coordinates are not integers. Does your new rule still hold?

Differentiation

You should have found that the gradient at any point on the curve $y = x^2$ is $2x$. It is helpful to write this as $2x^1$ for reasons that you will see shortly.

You should also have found that the gradient at any point on the curve $y = x^3$ is $3x^2$.

$2x$ and $3x^2$ are known as **derived functions** and the process used to find them is called **differentiation**.

Using the notation $\frac{dy}{dx}$ to stand for the derived function:

$$\text{if } y = x^2 \text{ then } \frac{dy}{dx} = 2x^1$$

$$\text{if } y = x^3 \text{ then } \frac{dy}{dx} = 3x^2$$

In both cases, to find the derived function you multiply the original equation by the power and reduce the power by 1. It was useful to write the power of 1 in the case of $y = x^2$ to make it easy to see that the power of 2 had been reduced by 1 as the rule suggests.

This rule always works for any power of *x*, even if it is negative or a fraction.

If $y = x^n$ then, differentiating, $\frac{dy}{dx} = nx^{n-1}$

Sometimes there is already a number in front of the power of *x*. When this happens you still follow the rule and multiply everything by the power before reducing the power by 1.

If $y = ax^n$ then, differentiating, $\frac{dy}{dx} = n \times ax^{n-1} = anx^{n-1}$

We will prove this result for $y = x^2$ at the end of the section.

E

Worked example 19

For the curves with the following equations, find $\frac{dy}{dx}$.

a $y = x^7$ **b** $y = x^8$ **c** $y = 4x^5$ **d** $y = -6x^3$

a Remember to multiply the expression by the power. Then you must reduce the power by 1. So:

$$\frac{dy}{dx} = 7x^6$$

b Similarly multiply by 8 and then subtract 1 from the 8:

$$\frac{dy}{dx} = 8x^7$$

c For this one you need to multiply the whole expression by 5. Then you subtract 1 from the power as before:

$$\frac{dy}{dx} = 4 \times (5 \times x^4)$$
$$= 20x^4$$

d Don't let the negative put you off! The rule is just the same:

$$\frac{dy}{dx} = 3 \times (-6x^2)$$
$$= -18x^2$$

Worked example 20

Find the gradient of the curve with equation $y = 4x^3$ at the point (2, 32).

For this question you need to remember that the derived function tells you the gradient at the point where the x co-ordinate is x. Find the derived function and then substitute the correct x co-ordinate to find the gradient at that particular point.

So $\frac{dy}{dx} = 12x^2$

You have been asked to find the gradient at the point where the x coordinate is 2.

So $\frac{dy}{dx}$ at the point where $x = 2$ is

$12 \times 2^2 = 48$

The gradient of the curve at the point (2, 32) is 48.

Note that the 12 comes from multiplying the 4 by the 3 that is pulled down from the power.

Exercise 18.11

1 For each of the following, find $\frac{dy}{dx}$.

a	$y = x^4$	**b**	$y = x^6$	**c**	$y = x^9$
d	$y = 4x^3$	**e**	$y = 12x^2$	**f**	$y = 7x^7$
g	$y = -4x^4$	**h**	$y = 7x^{12}$	**i**	$y = -16x^5$

2 Find the gradient of each of the following curves at the point indicated.

a	$y = x^2$ at the point (3, 9)	**b**	$y = x^3$ at the point (1, 1)
c	$y = x^4$ at the point (2, 16)	**d**	$y = 4x^2$ at the point (−1, 4)
e	$y = -3x^3$ at the point (2, −24)	**f**	$y = -5x^6$ at the point (−2, −320)

Remember that you will need to use the x co-ordinate in each case. Look back at worked example 20 if you need help.

E

3 Find the co-ordinates of the point at which the gradient of the curve with equation $y = 3x^2$ has gradient 18.

Differentiating Sums and Differences

The equation of a curve may involve more than one term and you may be asked to differentiate expressions like

$$y = x^2 + x^4 \quad \text{or} \quad y = 3x^2 - 4x$$

In fact you can differentiate each term independently. You can see this by differentiating $y = 4x^3$ and then rewriting the answer as a sum of separate terms.

$$y = 4x^3$$
$$\Rightarrow \frac{dy}{dx} = 12x^2$$

You can rewrite $12x^2$ as the sum $= 3x^2 + 3x^2 + 3x^2 + 3x^2$

If you now compare this to the original equation, also written as a sum of separate terms

$$y = x^3 + x^3 + x^3 + x^3$$

You can see that each term has been differentiated independently.

So we have the rule:

If $y = ax^m + bx^n$ then $\quad \dfrac{dy}{dx} = amx^{m-1} + bnx^{n-1}$

This rule can be extended to adding any number of terms together and it also works for subtraction.

Worked example 21

Find $\dfrac{dy}{dx}$ for each of the following:

a $y = 4x^7 + 3x^6$ **b** $y = \dfrac{8}{3}x^2 - 4x^5$

a Differentiate each term separately and then add together:

$$\frac{dy}{dx} = 28x^6 + 18x^5$$

b Differentiate each term separately and then subtract:

$$\frac{dy}{dx} = \frac{16}{3}x - 20x^4$$

Exercise 18.12

1 Find $\dfrac{dy}{dx}$ for each of the following.

a $y = x^4 + x^5$ **b** $y = 3x^3 - 5x^4$ **c** $y = 7x^6 + 9x^2$

d $y = \dfrac{1}{3}x^3 - 4x^7$ **e** $y = 6x^5 - \dfrac{8}{11}x^4$ **f** $y = -7x^2 + 3x^6$

g $y = 12x^3 + \dfrac{2}{3}x^8$ **h** $y = -10x^{12} - 8x^{10}$ **i** $y = 4x^2 - 12x^3 + 5x^4$

j $y = -\dfrac{8}{11}x^4 + \dfrac{2}{7}x^3 - \dfrac{3}{4}x^2$

E

2 Find the gradient of each curve at the point with the given x co-ordinate

a $y = 3x^3 + 2x^2$ $\quad x = 3$
b $y = -2x^4 + 3x^2$ $\quad x = -2$
c $y = 3x^4 + 6x^3 - 3x^2$ $\quad x = -1$

3 Find the co-ordinates of the point at which the curve with equation $y = 2x^3 + 3x^2$ has gradient 12.

4 Find the coordinates of the points at which the curve with equation $y = \frac{1}{4}x^4 - \frac{3}{2}x^2$ has gradient zero.

REWIND

You will need to remind yourself how you solved quadratic equations in Chapter 14 ◀

Special cases

Sometimes the equation of a curve will contain a multiple of x or a constant term.

In the case of a multiple of x the rule still applies, for example:

$$\begin{aligned} y &= 5x \\ &= 5x^1 \\ \Rightarrow \frac{dy}{dx} &= 5x^0 \\ &= 5 \end{aligned}$$

If $y = kx$ then $\frac{dy}{dx} = k$.

REWIND

You will need to remember that $x^1 = x$ and $x^0 = 1$

If $y = kx$ then $\frac{dy}{dx} = k$. ◀

This should not be a surprise, because the 'curve' with equation $y = kx$ is in fact a straight line with gradient k all the way along its length.

If your equation involve a constant term, you need to think about what the graph associated with $y = constant$ looks like. You will know from Chapter 10 that this is a horizontal line and has gradient zero.

So, if $y =$ constant then $\frac{dy}{dx} = 0$

You can see both of these ideas being used in the next worked example.

Worked example 22

Find $\frac{dy}{dx}$ if $y = (2x + 3)^2$.

Begin by expanding the brackets.

$$\begin{aligned} y = (2x+3)^2 &= (2x+3)(2x+3) \\ &= 4x^2 + 6x + 6x + 9 \\ &= 4x^2 + 12x + 9 \end{aligned}$$

Now you can differentiate:

$$\begin{aligned} \frac{dy}{dx} &= 8x + 12 + 0 \\ &= 8x + 12 \end{aligned}$$

Because
$4x^2$ is differentiated to give $8x$
$12x$ is differentiated to give 12
and
9 is differentiated to give 0

It is possible to differentiate an expression like this without expanding the brackets, but this is beyond the scope of IGCSE Mathematics

REWIND

You learnt how to expand a pair of brackets in Chapter 10. ◀

E

Exercise 18.13

1 Differentiate each of the following.

a $y = 5x$ **b** $y = -4x$ **c** $y = 4$
d $y = 7x - 6$ **e** $y = -3x + 6$ **f** $y = 4x^2 - 4x + 1$
g $y = 7x^3 + 2x - 4$ **h** $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}$
i $y = mx + c$, where m and c are constants

2 Differentiate each of the following.

a $y = x(x + 2)$ **b** $y = x^3(x^2 + 2x)$ **c** $y = (x + 2)(x - 3)$
d $y = (x - 4)(x - 5)$ **e** $y = 4x^3(x + 2)$ **f** $y = -5x(x - 4)$
g $y = (2x + 1)(x + 2)$ **h** $y = (3x + 2)(x - 3)$ **i** $y = (4x + 1)(3x + 5)$
j $y = (2x - 7)(3x + 4)$ **k** $y = (3x - 5)(7x - 3)$ **l** $y = (x + 3)^2$
m $y = (2x + 1)^2$ **n** $y = (3x - 2)^2$ **o** $y = \frac{1}{5}x^2(x+3)$
p $y = \frac{2}{3}x^6\left(x + \frac{1}{4}\right)$ **q** $y = 5(x + 3)(x - 7)$ **r** $y = (x + 3)(x - 3)$

3 Find the gradient of the curve with equation $y = (3x - 2)(4x + 1)$ at the point with co-ordinates (3, 91).

4 Find the point at which the gradient of the curve with equation $y = 3x^2 - 4x + 1$ is zero.

5 Find the coordinates of the point on the curve with equation $y = 3x^2 - 4x - 2$ where the curve is parallel to the line with equation $y = -2x + 1$.

What is the gradient of the *x*-axis?

6 Find the coordinates of both points on the curve with equation $y = 2x^3 - 9x^2 + 12x$ where the curve with is parallel to the *x*-axis.

7 The curve with equation $y = x^3 + 3$ has two tangents parallel to the line with equation $y = 12x - 1$. Find the co-ordinates of the two points.

8 The curve with equation $y = ax^3 - 4x + 1$ is parallel to the line $y = 50x - 1$ at the point with *x* co-ordinate 3.

a Find the gradient of the curve at the point with *x*-coordinate 4.

b Show that the tangent at the point with *x*-coordinate –3 is also parallel to the same line.

Equations of Tangents

In Chapter 10 you learned how to find the equation of a line when you know both its gradient and the co-ordinates of a point on the line. You can use the same method to find the equation of a tangent to a curve at a specific point.

You can find the gradient of the tangent by differentiating the equation and using the *x* co-ordinate of the point where the tangent is drawn. You can find a point on the line by using the given *x* co-ordinate and substituting it into the equation of the curve to find the *y* co-ordinate.

Worked example 23

Find the equation of the tangent to the curve with equation $y = 3x^2 - 4x + 1$ at the point with x co-ordinate 2.

First note that if $x = 2$ then:

$$y = 3(2)^2 - 4(2) + 1$$
$$= 12 - 8 + 1$$
$$= 5$$

E

So the tangent to the curve will pass through the point (2, 5).
Next find the gradient of the tangent by differentiation:
$\frac{dy}{dx} = 6x - 4$
Find the gradient at the point $x = 2$ by substituting:
gradient $= 6(2) - 4 = 8$
So you need to find the equation of a line with gradient 8, passing through the point with co-ordinates (2, 5).
The equation will be of the form $y = 8x + c$
But the line passes through the point with co-ordinates (2, 5), which means that $y = 5$ when $x = 2$.
This means that
$5 = 16 + c$
$\Rightarrow c = -11$
So the equation of the tangent is $y = 8x - 11$

REWIND

In Chapter 10 you leant that a line with equation $y = mx + c$ has gradient m ◀

Exercise 18.14

1 Find the equation of the tangent of each curve at the given point.

a	$y = x^2$	at the point with co-ordinates (3, 9)
b	$y = x^2$	at the point with x co-ordinate -2
c	$y = x^3 + x^2$	at the point with x co-ordinate 4
d	$y = 3x^3 - 2x + 1$	at the point with x co-ordinate 1.5
e	$y = \frac{1}{4}x^2 + \frac{1}{5}x$	at the point with x co-ordinate $\frac{1}{2}$

2 A curve has equation $y = 2x^2 + 3x - 2$. The tangent to the curve at $x = 4$ meets the x-axis at the point A. Find the coordinates of the point A.

3 A curve has equation $y = -3x^3 + x - 4$. The tangent to this curve at the point where $x = 1$ meets the x-axis at the point A and the y-axis at the point B. Find the area of triangle OAB.

4 The tangents to the curve with equation $y = x^3 - 3x$ at the points A and B with x co-ordinates -1 and 4 respectively meet at the point C. Find the co-ordinates of the point C.

Turning Points

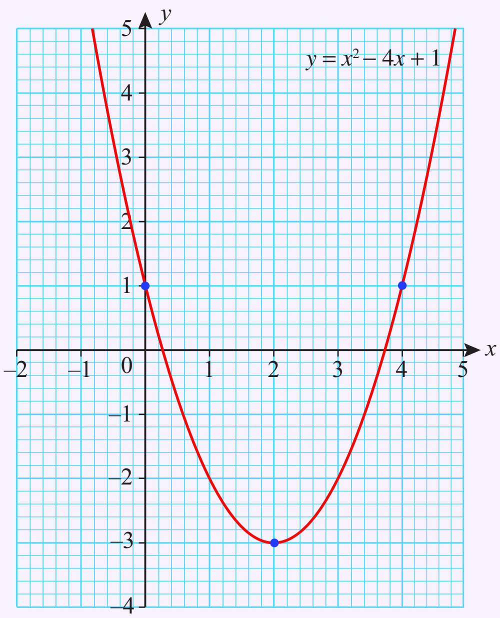
Earlier in this chapter you learnt that we can use the method of completing the square to find the points at which a quadratic curve is highest or lowest.

We will now use differentiation to do the same thing, and then compare the answer that we get from completing the square.

Think about the curve with equation $y = x^2 - 4x + 1$. The diagram below shows the curve with a tangent drawn at the lowest point.

Note that the gradient of the tangent is zero. Any point where the gradient of a quadratic curve is zero is known as a turning point.

If the gradient of the tangent is zero at this point, then so is $\frac{dy}{dx}$.



So

$$\frac{dy}{dx} = 2x - 4$$
$$= 0$$
$$\Rightarrow x = 2$$

Substitute this x-value in the original equation to find the co-ordinates of the turning point:

$$y = (2)^2 - 4(2) + 1$$
$$= -3$$

You can see that the point $(2, -3)$ is the lowest point of the graph. The value -3 is known at the minimum value of y.

If we complete the square to check the answer, we can see that

$$y = x^2 - 4x + 1$$
$$= (x-2)^2 - 4 + 1$$
$$= (x-2)^2 - 3$$

REWIND

You learnt how to complete the square in Chapter 14.

This confirms that the minimum value of y is -3 and that this occurs when $x = 2$, exactly as we found by differentiation.

Maximum and Minimum Points

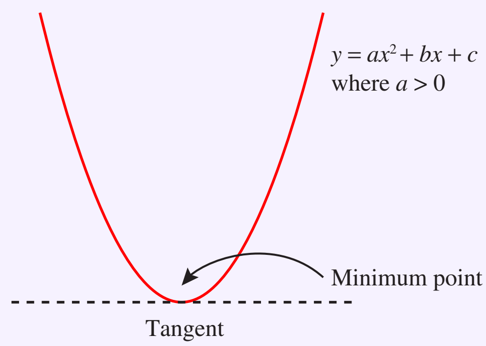
In the previous example the turning point was the minimum point on the curve. However, sometimes the turning point is a maximum point. You need to be able to tell which is which.

Tip

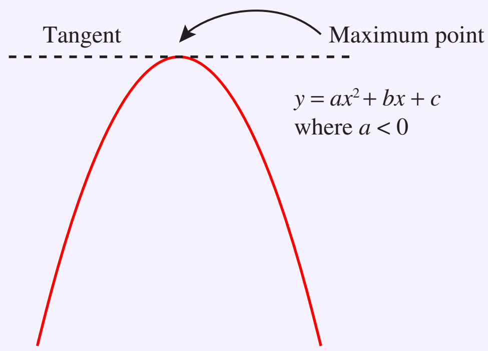
Remember that the coefficient of a power of x is the number that appears in front of it.

In the case of $y = x^2 - 4x + 1$ the coefficient of x^2 is positive, which means that the curve will be this way up:

You can see that the turning point must be a minimum as it is lower than the rest of the curve.



If the coefficient of x^2 is negative then the curve is the other way up, and the turning point is now a maximum



It is easy to see which points are maxima (the plural of *maximum*) and minima if you use the shape of the curve to help you.

The following worked example shows you how this is done.

E

Worked example 24

Find the co-ordinates of the two turning points on the curve with equation $y = 2x^3 - 3x^2 + 1$

Explain which point is a maximum and which point is a minimum.

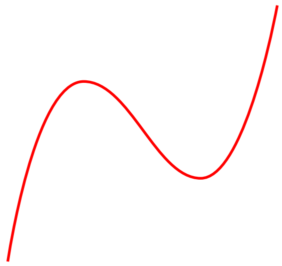
First find the turning points by differentiation:

$$\frac{dy}{dx} = 6x^2 - 6x$$

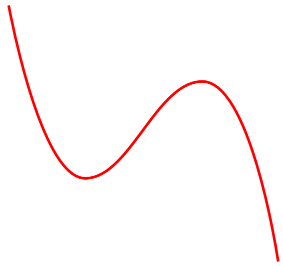
At a turning point:

$$6x^2 - 6x = 0$$
$$\Rightarrow x^2 - x = 0$$
$$\Rightarrow x(x-1) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 1$$

When $x = 0$, $y = 0 - 0 + 1 = 1$, so there is a turning point at (0,1)
When $x = 1$, $y = 2(1)^3 - 3(1)^2 + 1 = 0$, so there is a turning point at (1,0)
Now remind yourself what the graph of a cubic equation looks like. There are two possibilities:



Coefficient of x^3 is Positive



Coefficient of x^3 is Negative

Given that the coefficient of x^3 is positive the left graph must be the correct one. Notice that the maximum point lies on the left, with a lower x co-ordinate, and the minimum point lies on the right, with the higher x co-ordinate. So (0,1) is a maximum and (1,0) is a minimum.

Exercise 18.15

1 Find the turning point or points on each of the following curves and explain whether each point is a maximum or a minimum.

a $y = x^2 - 4x + 1$	**b** $y = x^2 + 6x - 4$	**c** $y = -x^2 + 8x - 2$
d $y = 3x^2 - 12x + 4$	**e** $y = -2x^2 + 4x - 3$	**f** $y = x^2 + 3x - 1$
g $y = -5x^2 + 3x + 4$	**h** $y = x^3 - 12x - 1$	**i** $y = -x^3 + 6x^2 + 3$
j $y = x(x - 4)$	**k** $y = (x - 5)(x + 5)$	**l** $y = x(2x - 3)$
m $y = x(2x^2 - 21x + 72)$	**n** $y = x^2(3 - x)$	

REWIND

You learned about the features of graphs and how to use them to sketch curves earlier in this chapter. ◀

2 Use the turning points you worked out in Question 1 and what you already know about curved graphs to sketch the graphs in parts **m** $y = x(2x^2 - 21x + 72)$ and **n** $y = x^2(3 - x)$.

Applying your skills

3 The height, h metres, of a ball above the ground is given by the formula $h = 7t - 5t^2$ at time t seconds after the ball is thrown upwards.

a Find $\frac{dh}{dt}$

b Find the greatest height of the ball above the ground.

Notice that h has taken the place of y and t has taken the place of x. The method is the same; only the letters have changed.

4 The population of bacteria in a pond is p thousand, d days after the pond is filled. It is found that $p = d^3 - 12d^2 + 45d$. Find the highest population of bacteria in the pond in the first 4 days after it is filled.

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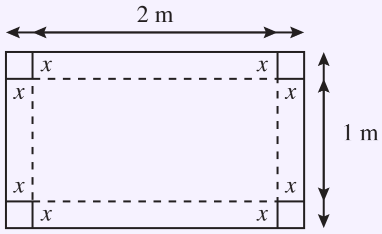
5 A manufacturer makes open-topped boxes by taking a 2m × 1m sheet of metal, cutting $xm \times xm$ squares out of the corners and folding along the dotted line shown in the diagrams.



a Show that the volume of the box produced is given by $V = x(2-2x)(1-2x)$

b Explain why x must be less than 0.5m

c Find the value of x that will give the maximum possible volume of the box.

REWIND

You will need to use the quadratic formula or complete the square to solve a quadratic equation in this question. You learnt how to do this in Chapter 14. ◀

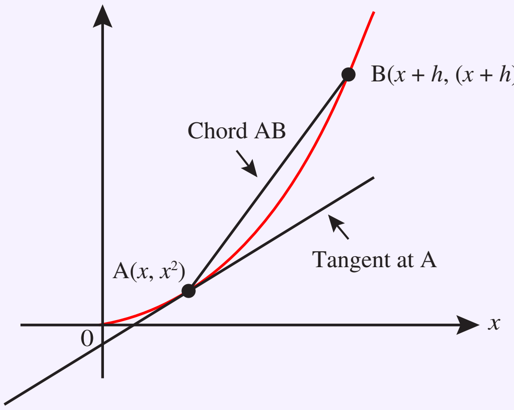
Proof that if $y = x^2$ then $\frac{dy}{dx} = 2x$

Tip

This proof is included so you can see where the result comes from, but it does not form part of the syllabus.

Point A lies , with (x, x^2) on the curve with equation $y = x^2$ and has coordinates (x, x^2). The point B with coordinates $(x+h, (x+h)^2)$, where h is a very small number, also lies on the curve.

We can draw a chord joining A and B as shown in the diagram.



REWIND

You will need to remind yourself how to find the gradient of a line that joins two points. This is covered in Chapter 10 ◀

Notice that, if h is very small, the gradient of the chord is a good approximation to the gradient of the tangent at A. By sliding the B along the curve towards A, we make h smaller and smaller–ultimately to approach zero –and the gradient of the chord tends towards the gradient of the tangent.

Work out the gradient of the chord algebraically.

$$\text{Gradient of chord} = \frac{(x+h)^2 - x^2}{(x+h)-x}$$
$$= \frac{x^2 + 2hx + h^2 - x^2}{h}$$
$$= \frac{2hx + h^2}{h}$$
$$= 2x + h$$

Notice that, as h tends towards zero this becomes $2x$, which is the result that you would expect.

You will need to work out how to expand $(x + h)^3$. You can do this by considering $(x + h)(x + h)^2$ and using the expansion of the second bracket.

Investigation

Now try this yourself for the curve with equation $y = x^3$.

Summary

Do you know the following?

- A quadratic equation is one in which the highest power of x is 2 (x^2).
- The graph of a quadratic equation is a recognisable curve called a parabola.
- A reciprocal equation is one in the form of $y = \frac{a}{x}$ or $xy = a$.
- The graph of a reciprocal equation is a two-part curve called a hyperbola.
- You can use graphs to solve equations by finding the value of x or y at different points on the graph. You can find the solution to simultaneous equations using the points of intersection of two graphs.

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- A cubic equation is one in which the highest power of x is 3 (x^3).
- The graph of a cubic equation is a curved shape.
- Linear, quadratic, cubic and reciprocal terms can occur in the same equation. It is possible to graph these curves by constructing a table of values and then plotting the points.
- An exponential equation has the form $y = a^x$. These equations produce steep curved graphs.
- You can draw a tangent to a curve and use it to find the gradient of the curve at the point where the tangent touches it.
- You can differentiate functions to find gradients and turning points.

Are you able to …?

- construct a table of values for quadratic and reciprocal equations
- plot the graph of a parabola from a table of values
- sketch the graph of a parabola using its characteristics
- find turning points by completing the square E
- plot the graph of a hyperbola from a table of values
- interpret straight line and quadratic graphs and use them to solve related equations

E

- construct tables of values and draw graphs for cubic equations and simple sums of linear and non-linear terms
- construct a table of values and draw the graph of an exponential equation
- sketch graphs of cubic, reciprocal and exponential functions using their characteristics
- interpret graphs of higher order equations and use them to solve related equations
- estimate the gradient of a curve by drawing a tangent to the curve
- differentiate functions of the form ax^n
- find the equation of the tangent to a curve
- find turning points using differentiation and work out whether they are maximum or minimum points.

Examination practice

Exam-style questions

1 **a** Write the equation for each of the graphs A, B, C and D.

b Write down the co-ordinates of the intersection of:

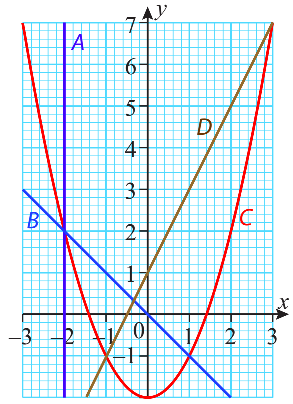
i A and B

ii C and D

c What co-ordinates satisfy the equations of B and D at the same time?

d Which graph has an x-intercept of $-\frac{1}{2}$?

e Which graph is symmetrical about the y-axis?



2 The graph of $y = x^2$ is drawn on the grid.

a The table shows some corresponding values of $y = x^2 + 3$. Copy and complete the table by filling in the missing values.

x	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2
y		5.25	4	3.25	3		4	5.25	7

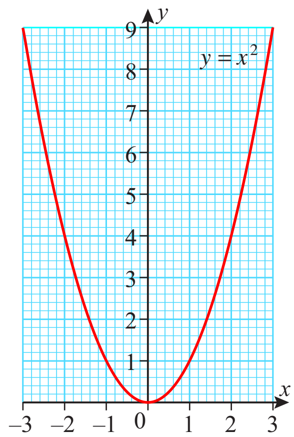
b Plot the graph of $y = x^2$ and the graph of $y = x^2 + 3$ for $-2 \leqslant x \leqslant 2$ on a grid.

c Will the two curves ever meet? Explain your answer.

d By drawing a suitable straight line on the same grid, solve the equations:

i $x^2 = 6$

ii $x^2 + 3 = 6$



E

3 Answer the whole of this question on graph paper.

x	0.6	1	1.5	2	2.5	3	3.5	4	4.5	5
y	p	–5.9	–3.7	–2.3	–1.1	0.3	1.9	3.8	q	r

Some of the values of $y = \dfrac{x^3}{12} - \dfrac{6}{x}$ are shown in the table above. Values of y are given correct to 1 decimal place.

a Find the values of p, q and r.

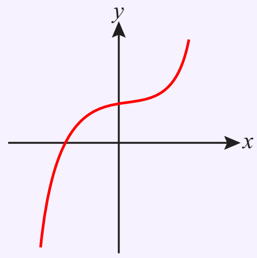
b Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = \dfrac{x^3}{12} - \dfrac{6}{x}$ for $-0.6 \leqslant x \leqslant 5$.

c From the graph, find the value of x (correct to 1 decimal place) for which $\dfrac{x^3}{12} - \dfrac{6}{x} = 0$.

d Draw the tangent to the curve at the point where $x = 1$, and estimate the gradient of the curve at that point.

4 Six sketch graphs are shown here.

i



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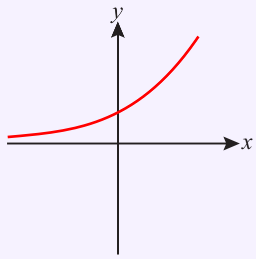

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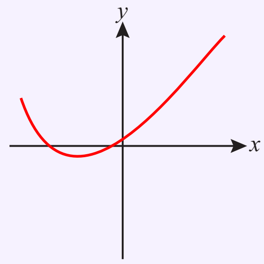

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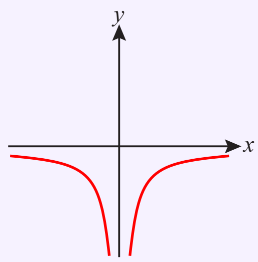

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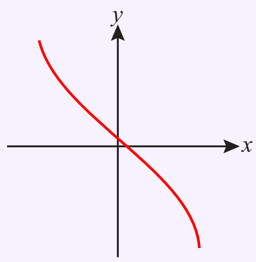

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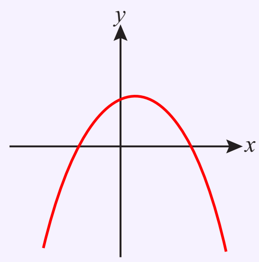


Match the graphs to the following equations.

a $y = 1 + x - 2x^2$

b $y = 3^x$

c $y = x^3 + x^2 + 1$

d $y = -\dfrac{16}{x^2}$

5 **a** In a chemical reaction, the mass, M grams, of a chemical is given by the formula $M = \dfrac{160}{2^t}$ where t is the time, in minutes, after the start.

A table of values for t and M is given below.

t (min)	0	1	2	3	4	5	6	7
M (g)	p	80	40	20	q	5	r	1.25

i Find the values of p, q and r.

ii Draw the graph of M against t for $0 \leqslant t \leqslant 7$. Use a scale of 2 cm to represent one minute on the horizontal t-axis and 1 cm to represent 10 grams on the vertical M-axis.

iii Draw a suitable tangent to your graph and use it to estimate the rate of change of mass when $t = 2$.

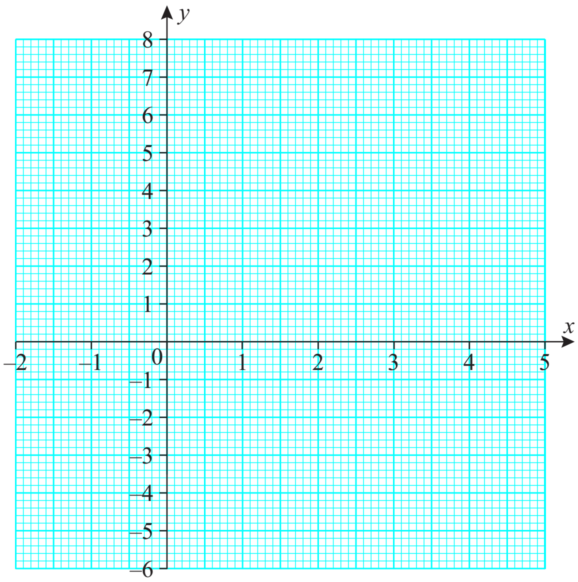
b The other chemical in the same reaction has mass m grams, which is given by $m = 160 - M$. For what value of t do the two chemicals have equal mass?

Past paper questions

1 **a** **i** Complete the table for $y = 5 + 3x - x^2$. [3]

x	−2	−1	0	1	2	3	4	5
y	−5		5	7		5		−5

ii On the grid, draw the graph of $y = 5 + 3x - x^2$ for $-2 \leqslant x \leqslant 5$. [4]



b Use your graph to solve the equation $5 + 3x - x^2 = 0$. [2]

c **i** On the grid, draw the line of symmetry of $y = 5 + 3x - x^2$. [1]

ii Write down the equation of this line of symmetry. [1]

d **i** On the grid, draw a straight line from (−1, 1) to (3, 5). [1]

ii Work out the gradient of this line. [2]

iii Write down the equation of this line in the form $y = mx + c$. [1]

[Cambridge IGCSE Mathematics 0580 Paper 33 Q05 October/November 2013]

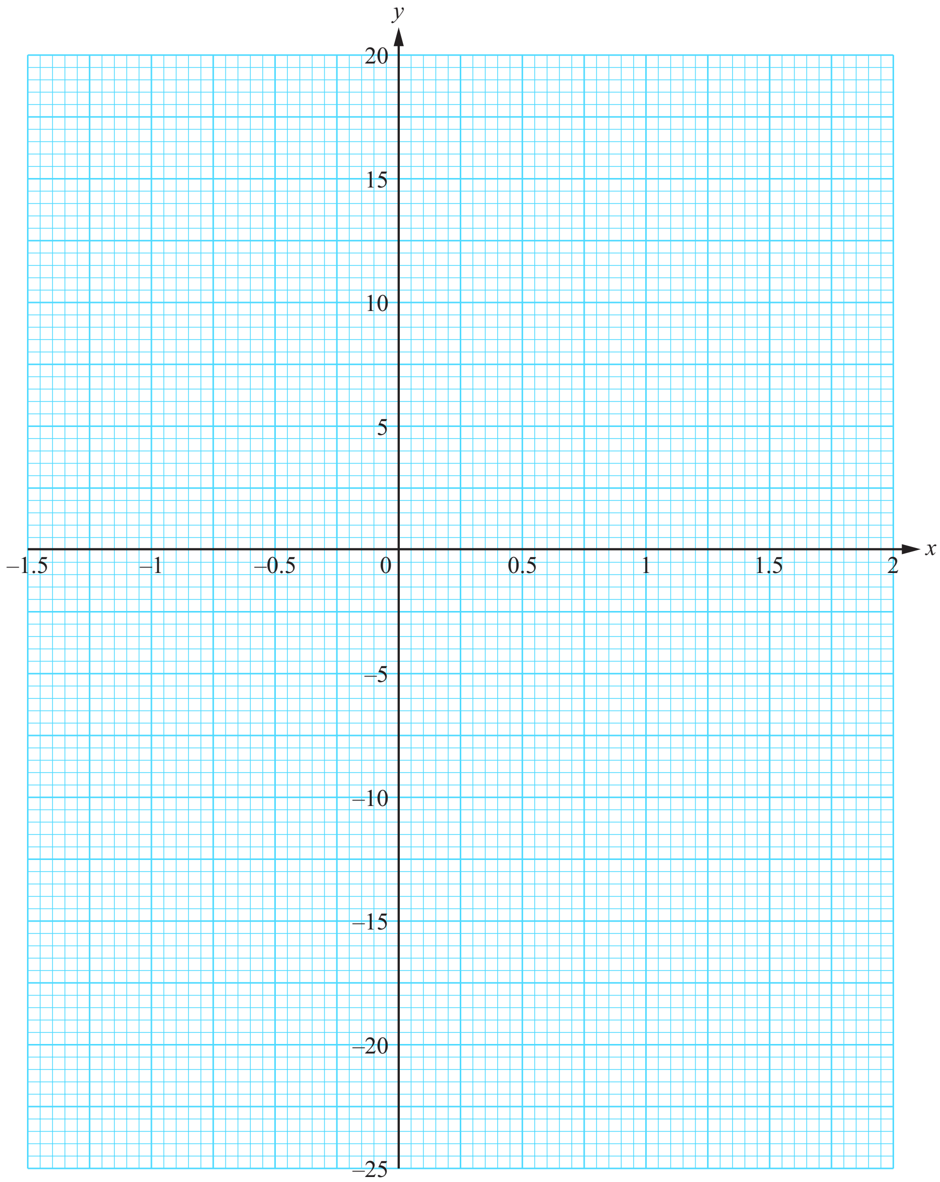
2 $f(x) = 5x^3 - 8x^2 + 10$

a Complete the table of values.

x	−1.5	−1	−0.5	0	0.5	0.75	1	1.5	2
$f(x)$	−24.9			10	8.6	7.6	7		18

[3]

b Draw the graph of $y = f(x)$ for $-1.5 \leqslant x \leqslant 2$.



[4]

c Use your graph to find an **integer** value of k so that $f(x) = k$ has

i exactly one solution, [1]

ii three solutions. [1]

d By drawing a suitable straight line on the graph, solve the equation $f(x) = 15x + 2$ for $-1.5 \leqslant x \leqslant 2$. [4]

e Draw a tangent to the graph of $y = f(x)$ at the point where $x = 1.5$.

Use your tangent to estimate the gradient of $y = f(x)$ when $x = 1.5$. [3]

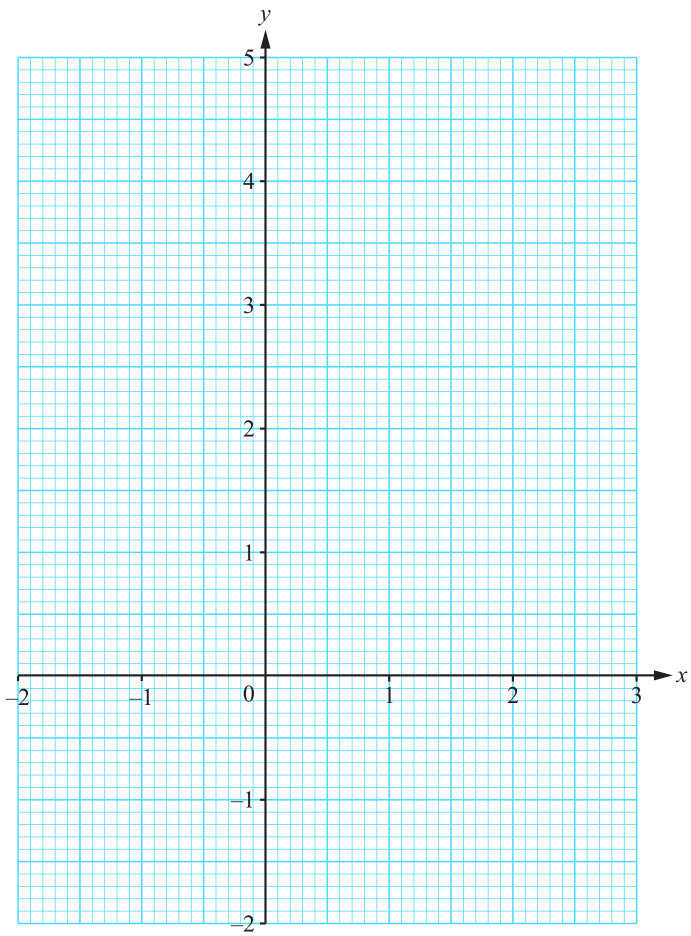
[Cambridge IGCSE Mathematics 0580, Paper 42, Q6, November 2014]

3 The table shows some values of $y = x + \frac{1}{x^2}$, $x \neq 0$

x	-2	-1.5	-1	-0.75	-0.5		0.5	0.75	1	1.5	2	3
y	-1.75	-1.06	0	1.03			4.50	2.53	2		2.25	

a Complete the table of values. [3]

b On the grid, draw the graph of $y = x + \frac{1}{x^2}$ for $-2 \leqslant x \leqslant -0.5$ and $0.5 \leqslant x \leqslant 3$.



[5]

c Use your graph to solve the equation $x + \frac{1}{x^2} = 1.5$. [1]

d The line $y = ax + b$ can be drawn on the grid to solve the equation $\frac{1}{x^2} = 2.5 - 2x$.

i Find the value of a and the value of b. [2]

ii Draw the line $y = ax + b$ to solve the equation $\frac{1}{x^2} = 2.5 - 2x$. [3]

e By drawing a suitable tangent, find an estimate of the gradient of the curve at the point where $x = 2$. [3]

[Cambridge IGCSE Mathematics 0580, Paper 42, Q7, March 2016]