Model-theoretic embeddings

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  - all syntactic
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- what do we actually mean by "semantics"?
  - formal definition of "truth" relies on "models"
  - universal algebra over formal languages (with signature Σ)

Description Logic EL++

Name	Syntax	Semantics
top	$\top$	$\Delta^{\mathcal{I}}$
bottom	$\bot$	$\emptyset$
nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
existential restriction	$\exists r.C$	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} : (x, y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
generalized concept inclusion	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
role inclusion	$r_1 \circ \ldots \circ r_n \sqsubseteq r$	$r_1^{\mathcal{I}} \circ \ldots \circ r_n^{\mathcal{I}} \subseteq r^{\mathcal{I}}$

Models

- Interpretations and Σ-structures
- Model $\mathfrak{A}$ of a formula ϕ: ϕ is true in $\mathfrak{A}$ ($\mathfrak{A} \models \phi$)
- Theory T: set of formulas
- $\mathfrak{A}$ is a model of T if $\mathfrak{A}$ is a model of all formulas in T
- Ontologies are (special kinds of) theories

EL Embeddings

- given a theory/ontology T with signature $\Sigma(T)$
- aim: find $f_e : \Sigma(T) \mapsto \mathbb{R}^n$ s.t. $f_e(\Sigma(T))$ is a model of T ($f_e(\Sigma(T)) \models T$)

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- more general: find an algorithm that maps symbols (signatures) into $\mathbb{R}^n$ so that the *semantics* of the symbol (expressed through axioms and explicit in model structures) is preserved
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- any consistent $\mathcal{EL}^{++}$theory has infinite models
- any consistent $\mathcal{EL}^{++}$theory has models in $\mathbb{R}^n$ (Loewenheim-Skolem, upwards; compactness)

Key idea

- for all $r \in \Sigma(T)$ and $C \in \Sigma(T)$, define $f_e(r)$ and $f_e(C)$
- $f_e(C)$ maps to points in an open n-ball such that $f_e(C) = C^{\mathcal{I}}$: $C^{\mathcal{I}} = \{x \in \mathbb{R}^n | \, \|f_e(C) - x\| < r_e(C)\}$
  - these are the *extension* of a class in $\mathbb{R}^n$
- $f_e(r)$ maps a binary relation r to a vector such that $r^{\mathcal{I}} = \{(x, y) | x + f_e(r) = y\}$
  - that's the TransE property for *individuals*
- use the axioms in T as constraints

Algorithm

- normalize the theory:
  - every $\mathcal{EL}^{++}$theory can be expressed using four normal forms (Baader et al., 2005)
- eliminate the ABox: replace each individual symbol with a singleton class: a becomes $\{a\}$
- rewrite relation assertions $r(a, b)$ and class assertions $C(a)$ as $\{a\} \sqsubseteq \exists r.\{b\}$ and $\{a\} \sqsubseteq C$
  - something to remember for the next class-vs-instance discussion?
- normalization rules to generate:
  - $C \sqsubseteq D$
  - $C \sqcap D \sqsubseteq E$
  - $C \sqsubseteq \exists R.D$
  - $\exists R.C \sqsubseteq D$

Algorithm: loss functions

$$\begin{aligned}
&loss_{C \sqsubseteq D}(c, d) = \\
&\max(0, \|f_\eta(c) - f_\eta(d)\| + r_\eta(c) - r_\eta(d) - \gamma) \qquad (1) \\
&+ |\,\|f_\eta(c)\| - 1| + |\,\|f_\eta(d)\| - 1|
\end{aligned}$$

Algorithm: loss functions

$$
\begin{aligned}
&loss_{C \sqcap D \sqsubseteq E}(c, d, e) = \\
&\max(0, \|f_\eta(c) - f_\eta(d)\| - r_\eta(c) - r_\eta(d) - \gamma) \\
&+ \max(0, \|f_\eta(c) - f_\eta(e)\| - r_\eta(c) - \gamma) \\
&+ \max(0, \|f_\eta(d) - f_\eta(e)\| - r_\eta(c) - \gamma) \\
&+ \max(0, \min(r_\eta(c), r_\eta(d)) - r_\eta(e) - \gamma) \\
&+ |\,\|f_\eta(c)\| - 1| + |\,\|f_\eta(d)\| - 1| + |\,\|f_\eta(e)\| - 1|
\end{aligned}
\tag{2}
$$

Algorithm: loss functions

$$
\begin{aligned}
&loss_{C \sqsubseteq \exists R.D}(c, d, r) = \\
&\max(0, \|f_\eta(c) + f_\eta(r) - f_\eta(d)\| + r_\eta(c) - r_\eta(d) - \gamma) \\
&+ |\,\|f_\eta(c)\| - 1| + |\,\|f_\eta(d)\| - 1|
\end{aligned} \tag{3}
$$

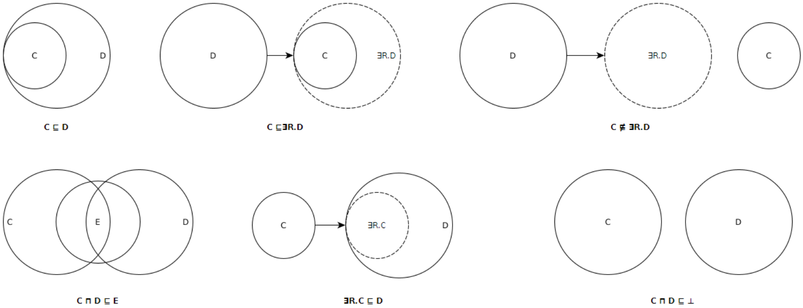
Algorithm: loss functions

$$
\begin{aligned}
&loss_{\exists R.C \sqsubseteq D}(c, d, r) = \\
&\max(0, \|f_\eta(c) - f_\eta(r) - f_\eta(d)\| - r_\eta(c) - r_\eta(d) - \gamma) \\
&+ |\,\|f_\eta(c)\| - 1| + |\,\|f_\eta(d)\| - 1|
\end{aligned} \tag{4}
$$

Algorithm: loss functions

$$
\begin{aligned}
&loss_{C \sqcap D \sqsubseteq \bot}(c, d, e) = \\
&\max(0, r_\eta(c) + r_\eta(d) - \|f_\eta(c) - f_\eta(d)\| + \gamma) \\
&+ |\, \|f_\eta(c)\| - 1| + |\, \|f_\eta(d)\| - 1|
\end{aligned} \tag{5}
$$

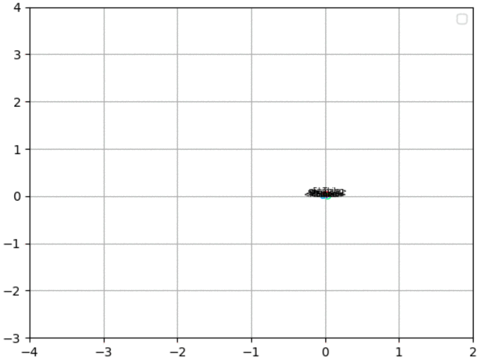
Algorithm: loss functions



EL Embeddings

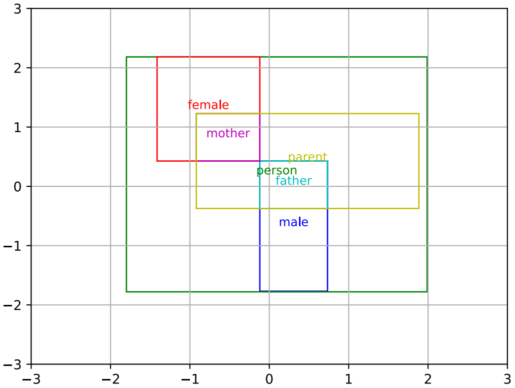
$$
\begin{align}
Male &\sqsubseteq Person \tag{6}\\
Female &\sqsubseteq Person \tag{7}\\
Father &\sqsubseteq Male \tag{8}\\
Mother &\sqsubseteq Female \tag{9}\\
Father &\sqsubseteq Parent \tag{10}\\
Mother &\sqsubseteq Parent \tag{11}\\
Female \sqcap Male &\sqsubseteq \bot \tag{12}\\
Female \sqcap Parent &\sqsubseteq Mother \tag{13}\\
Male \sqcap Parent &\sqsubseteq Father \tag{14}\\
\exists hasChild.Person &\sqsubseteq Parent \tag{15}\\
Parent &\sqsubseteq Person \tag{16}\\
Parent &\sqsubseteq \exists hasChild.\top \tag{17}
\end{align}
$$

EL Embeddings



- model with $\Delta = R^n$
- support quantifiers, negation, conjunction,...

EL BOX Embeddings



Name	Syntax	Semantics	F$\mathcal{ALC}$ON model
top	$\top$	$\Delta^{\mathcal{I}}$	1?
bottom	$\bot$	$\emptyset$	0?
instantiation	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$	$m(a, C^{\mathcal{I}}) = \sigma(f_e(a) \cdot f_e(C))$
role assertion	$R(a, b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$	$m((a, b), R^{\mathcal{I}}) = \sigma((f_e(a) + f_e(R)) \cdot f_e(b))$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$	$m(a, (C^{\mathcal{I}} \cap D^{\mathcal{I}})) = \theta(m(a, C^{\mathcal{I}}), m(a, D^{\mathcal{I}}))$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$	$m(a, (C^{\mathcal{I}} \cup D^{\mathcal{I}})) = \kappa(m(a, C^{\mathcal{I}}), m(a, D^{\mathcal{I}}))$
negation	$\neg C(a)$	$a^{\mathcal{I}} \notin C^{\mathcal{I}}$	$m(a, \neg C^{\mathcal{I}}) = \nu(m(a, C^{\mathcal{I}}))$
existential restriction	$\exists r.C$	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} : (x, y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$	$m(x, (\exists R.C)^{\mathcal{I}}) = \max_{y \in \Delta} \theta(m(y, C^{\mathcal{I}}), m((x, y), R^{\mathcal{I}}))$
universal restriction	$\forall r.C$	$\{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}} : (x, y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$	$m(x, (\exists R.C)^{\mathcal{I}}) = \min_{y \in \Delta} \theta(m(y, C^{\mathcal{I}}), m((x, y), R^{\mathcal{I}}))$
generalized concept inclusion	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ or $C^{\mathcal{I}} \cap \neg D^{\mathcal{I}} \subseteq \emptyset$	$m(a, (C^{\mathcal{I}} \cap \neg D^{\mathcal{I}})) = \theta(m(a, C^{\mathcal{I}}), \nu(m(a, D^{\mathcal{I}})))$