CHAPTER **NINE**

Circular Motion

9A Introduction

Exercise 9A

1. A wheel is rotating at 129 revolutions per minute about its centre. Show that the angular velocity of a point on the wheel about the centre is approximately 13·5 radians per second.

2. A wheel attached to an electric motor is rotating at 300 revolutions per minute. The radius of the wheel is 2 cm. Show that the speed of a notch on the edge of the wheel is about 0·63 m/s.

3. An object of mass 12 kg rests on a smooth horizontal surface and is attached by a string 1·2 m long to a fixed point O on the surface. If the object moves in a horizontal circle at 3·6 m/s, show that the tension in the string is 129·6 Newtons.

4. A towel of mass 2 kg is spinning in a drier of diameter 1 metre at 50 rev/s. Show that the force exerted by the towel on the bearings of the drier is about 98 696 Newtons.

5. A go-kart is being driven around a flat circular track of radius 35 m at 54 km/h.
(a) Show that the angular velocity of the go-kart is $\frac{3}{7}$ rad/s.
(b) If the combined mass of the go-kart and driver is 120 kg, show that the centripetal force is approximately 771 Newtons.

6. A boy ties a rock to one end of a piece of string, then swings it in a horizontal circle of radius 2 metres. If the mass of the rock is 200 grams and the tension in the string is 2·5 Newtons, show that:
(a) the speed of the stone is 5 m/s,
(b) the angular velocity of the stone is 2·5 rad/s.

7. A string of length 50 cm can just sustain a mass of 20 kg without breaking. A mass of 4 kg is attached to one end of the string, while the other end is fixed to a point on a smooth horizontal table. The mass is then revolved at a uniform speed on the table. Take $g = 9{\cdot}8\ \text{m/s}^2$.
(a) Show that the maximum tension the string can sustain is 196 N.
(b) Show that the maximum number of complete revolutions per minute the mass can make without the string breaking is 94.

8. A nylon cord is 60 cm long and will break if a mass exceeding 40 kg is hung from it. A mass of 2 kg is attached to one end of the cord and the other end is attached to a fixed point. The mass then undergoes circular motion in a horizontal plane. Show that the greatest speed the mass can achieve without the cord breaking is approximately 10·84 m/s.

9B The Conical Pendulum

Exercise **9B**

1. A particle is attached to one end of a string, and the other end of the string is attached to a fixed point. The particle moves in a horizontal circle of radius 60 cm with a constant angular velocity of $\frac{4\pi}{3}$ rad/s. Show that the string is inclined at approximately 47° to the vertical. (Assume that $g = 9{\cdot}8\text{m/s}^2$.)

2. A small object of mass 4 kg is attached by a string of length 35 cm to a fixed point and moves in a horizontal circle with uniform angular velocity 1 rev/s. Show that the tension in the string is approximately 55 N.

3. A 0·5 kg mass is attached to one end of a cord 1·5 m long. The cord and the mass rotate as a conical pendulum at 60 rev/min. Assume that $g = 9.8\text{m/s}^2$.

(a) Show that the tension in the cord is about 29·6 N.

(b) Show that the cord makes an angle of about $80^\circ 28'$ with the vertical.

(c) Show that the radius of the circle that the mass moves in is about 1·48 m.

4. A small object is attached by a string 1225 mm long to a fixed point, and moves with uniform speed in a horizontal circle. The tension in the string is twice the weight of the object. Show that the angular velocity is 4 rad/s.

5. The mass of the bob in a conical pendulum is 4 kg and the length of the string is 60 cm. The greatest tension that the string can sustain is 600 N. Show that the maximum number of revolutions per second of the bob without the string breaking is approximately 2·52.

6. The number of revolutions per minute of a conical pendulum is increased from 75 to 80.

(a) Show that $h = \dfrac{g}{\omega^2}$, where h is the perpendicular height of the cone.

(b) Hence show that the rise in the level of the bob is about 1·92 cm.

7. A particle is attached to one end of a string of length 80 cm, and the other end of the string is attached to a fixed point. The particle moves at a constant speed in a horizontal circle so that the string is inclined at 30° to the vertical. Show that the particle is rotating at about 36 revolutions per minute.

8. A small mass is suspended by a light rod from a pivot P. The mass moves with constant speed in a horizontal circle. The rod has length 1 metre and makes an angle of 30° with the vertical. Assume that $g = 9{\cdot}8\text{m/s}^2$.

(a) Show that the mass takes about 1·87 seconds to complete one revolution.

(b) Show that the speed of the mass is about 1·68 m/s.

(c) If the speed of the mass is doubled, show that the rod will make an angle of $54^\circ 44'$ with the vertical.

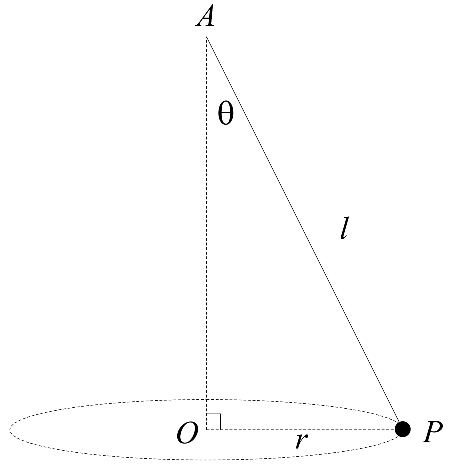
9. A particle of mass 4 kg is attached to a string 2 m long. The particle and string revolve as a conical pendulum. The constant speed of the particle is $\sqrt{g}$ m/s, where $g\,\text{m/s}^2$ is the acceleration due to gravity. Let θ be the angle of inclination of the string to the vertical, and let r m be the radius of the horizontal circle in which the particle is revolving.

(a) Show that $\tan\theta = \frac{1}{r}$.

(b) Hence show that $\theta = \cos^{-1}\frac{\sqrt{17}-1}{4}$.

(c) Show that the tension in the string is about 50·2 Newtons. (Take $g = 9{\cdot}8\,\text{m/s}^2$.)

10.



In the diagram above, A and P are the endpoints of a light string of length ℓ metres. A is a fixed point, while an object of mass m kg is attached to the string at P. The object moves in a horizontal circle of radius r metres about the point O. Let the constant angular velocity of the object be w rad/s, and the acceleration due to gravity be g m/s^2. Let θ be the angle between the string and the vertical, and T Newtons the tension in the string.

(a) Draw a diagram showing the forces acting on the object.

(b) Deduce that $\cos\theta = \dfrac{g}{\ell w^2}$.

(c) Suppose the angular velocity of the object is increased to w_1 rad/s, so that the angle θ is doubled. Show that:

$$w_1 = \sqrt{\frac{g\ell w^4}{2g^2 - \ell^2 w^4}}$$

11. A particle P of mass m is attached to one end of a light inelastic string of length ℓ, while the other end is fixed at O. The particle moves with velocity v in a horizontal circle of radius r so that the string describes a cone whose vertical axis passes through the centre C of the circle. Let T be the tension in the string as the particle moves, let $OC = h$ and let θ be the angle between the string and the vertical.

(a) Draw a diagram showing the forces acting on P.

(b) Show that $h = \dfrac{mg\ell}{T}$ and that $r^2 = \dfrac{mv^2\ell}{T}$.

(c) Hence show that $T = \dfrac{m}{2\ell}\left(v^2 + \sqrt{v^4 + 4g^2\ell^2}\right)$.

9C Banked Tracks

Exercise 9C

1. A car travels round a circular bend in a road of radius 45 metres at a speed of 48 km/h. There is no sideways frictional force between the road surface and the tyres. Show that the circular bend is banked at $21^\circ 57'$ to the horizontal. (Assume $g = 9{\cdot}8\,\text{m/s}^2$.)

2. A railway line has been constructed around a circular bend of radius 400 m. The distance between the rails is 1·5 m and the outside rail is 8 cm above the inside rail. Show that the optimum speed of a train on this bend (that is, the speed at which the wheels exert no sideways force on the rails) is about 52 km/h. (Assume $g = 9{\cdot}8\,\text{m/s}^2$.)

3. A car of mass 1·2 tonnes is rounding a circular bend of radius 150 m. The bend is banked at 10° to the horizontal. Assume that $g = 9{\cdot}8\,\text{m/s}^2$.

(a) Show that the car must travel at about 58 km/h so that there is no tendency to skid sideways.

(b) Suppose that the car travels into the bend at 72 km/h. Show that the sideways frictional force exerted by the tyres on the road is approximately 1109 Newtons.

4. A railway track around a circular curve of radius 200 m is designed for an optimum speed of 50 km/h. Assume that $g = 9{\cdot}8\,\text{m/s}^2$.

(a) If the gauge of the track is 1·52 m, show that the difference in height between the outer and inner rails is about 15 cm.

(b) Show that the sideways thrust on the rails is about 110 577 Newtons if a train of mass 120 tonnes travels around the curve at 70 km/h.

5. The sleepers of a railway line at a point on a circular bend of radius 100 metres are sloped such that a train travelling at 48 km/h exerts no lateral force on the rails. Show that a locomotive of mass 100 tonnes at rest on this bend would exert a lateral force of about $1{\cdot}75 \times 10^5$ Newtons on the rails.

6. A car travels at v m/s around a curved track of radius R metres.

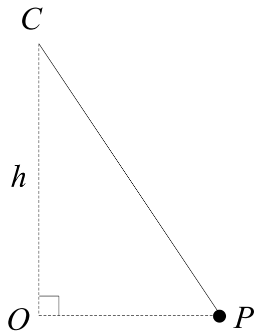
(a) Show that the inclination θ of the track to the horizontal satisfies $\tan\theta = \dfrac{v^2}{Rg}$ if there is no tendency for the car to slip sideways.

(b) A second car of mass M kg is travelling around the same curved track at V m/s. Prove that the sideways frictional force exerted by the surface of the track on the tyres of this car is $\dfrac{Mg(V^2 - v^2)}{\sqrt{v^4 + R^2 g^2}}$ Newtons.

9D Miscellaneous Problems

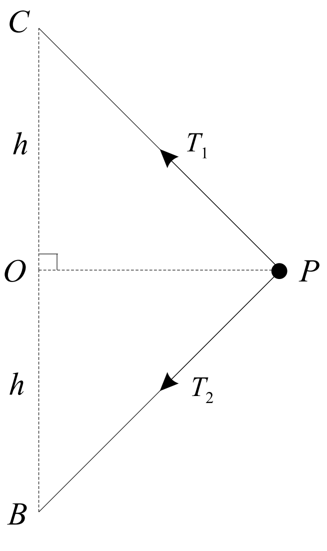
Exercise **9D**

1. (a)



The diagram above shows a particle P attached by an inelastic string to a fixed point C. The particle moves in uniform circular motion about a fixed point O that is at a distance h below C. Show that the angular velocity of P about O is $\sqrt{\frac{g}{h}}$, where g is acceleration due to gravity.

(b)

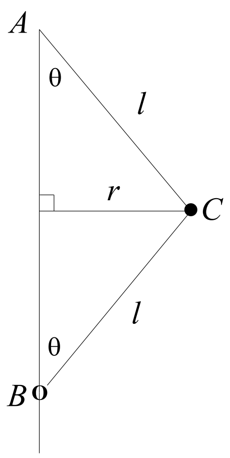


Suppose now, as shown in the diagram above, that P is attached by a second string, identical to the first, to another fixed point B which is at a distance $2h$ below C.

(i) Write down two equations of motion by resolving forces vertically and horizontally at P.

(ii) If the angular velocity of P about O is $3\sqrt{\frac{g}{h}}$, show that the ratio $T_1 : T_2$ of the tensions in the two strings is $5 : 4$.

2.



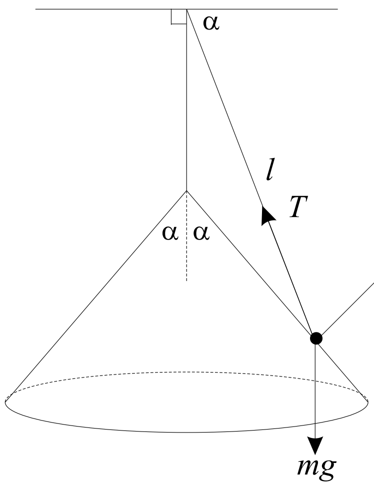
A mass m at a point C is freely jointed to two identical light rods CA and CB of length ℓ, as shown in the diagram above. The point A is fixed, and at B there is a collar, also of mass m, which is free to slide along the smooth vertical bar AB. The mass at C rotates in a horizontal circle with uniform angular velocity ω. While this happens, the inclination of the rods to the vertical is θ. Let T_1 be the tension in the rod CA, and T_2 the tension in the rod CB.

(a) By resolving forces vertically at B, show that $T_2 \cos\theta = mg$.

(b) By resolving forces vertically and horizontally at C, show that $T_1 \cos\theta = mg + T_2 \cos\theta$ and that $(T_1 + T_2)\sin\theta = mrw^2$.

(c) Deduce that $\cos\theta = \dfrac{3g}{\ell\omega^2}$.

3.



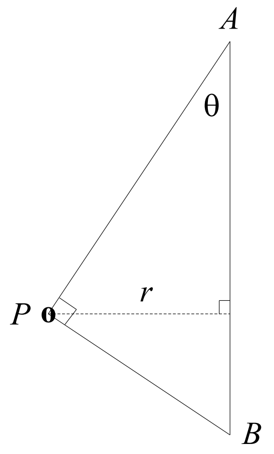
A particle of mass m is suspended by a string of length ℓ from a point directly above the vertex of a smooth cone, which has a vertical axis. The particle remains in contact with the cone and rotates as a conical pendulum with angular velocity ω. The angle of the cone at its vertex is 2α, where $\alpha > \frac{\pi}{4}$, and the string makes an angle of α with the horizontal. The forces acting on the particle are the tension in the string T, the normal reaction of the cone N and the gravitational force mg.

(a) Show, with the aid of a diagram, that the vertical component of N is $N \sin \alpha$.

(b) Show that $T + N = \dfrac{mg}{\sin \alpha}$, and find an expression for $T - N$ in terms of m, ℓ and ω.

(c) The angular velocity is increased until $N = 0$, that is, when the particle is about to lose contact with the cone. Find an expression for this value of ω in terms of α, ℓ and g.

4.



The ends of a light string are fixed to two points A and B, as shown in the diagram above. The string passes through a small ring of mass m. The ring is fastened to the string at P. When the string is taut, $\angle APB = 90^\circ$, $\angle BAP = \theta$ and the distance of P from AB is r. Suppose that the ring revolves in a horizontal circle with constant angular velocity ω and that while this happens the string is taut.

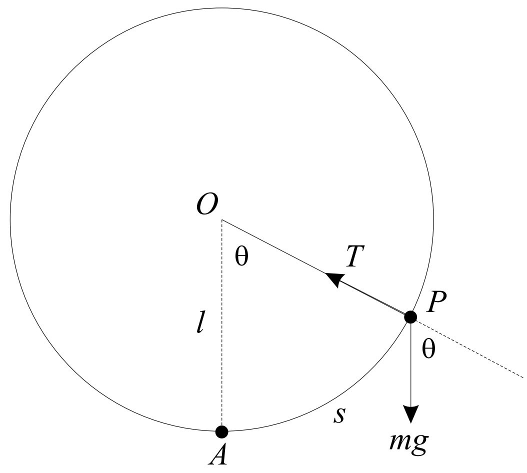
(a) Draw a diagram showing the forces acting on the ring.

(b) Show that the tensions T_1 and T_2 in the parts AP and PB respectively of the string are $T_1 = m(r\omega^2 \sin\theta + g\cos\theta)$ and $T_2 = m(r\omega^2 \cos\theta - g\sin\theta)$.

(c) Given that $AB = 13$ units and $AP = 12$ units, show that $144\omega^2 > 13g$.

(d) Suppose that the ring is now free to move on the string instead of being fastened to the string. Show that the condition for the ring to remain at the point P on the string is $420\omega^2 = 221g$.

5.



A string of length ℓ is initially vertical and has a mass P of m kg attached to it. The mass P is given an initial velocity V and begins to move along the arc of a circle in an anticlockwise direction. O is the centre of the circle and A is the initial position of P. Let s denote the arc length AP, $v = \dfrac{ds}{dt}$ and $\theta = \angle AOP$ and let the tension in the string be T.

(a) Starting with $s = \ell\theta$, show that the tangential acceleration of P is given by

$$\frac{d^2s}{dt^2} = \frac{1}{\ell}\frac{d}{d\theta}(\frac{1}{2}v^2).$$

(b) Show that $\dfrac{1}{\ell}\dfrac{d}{d\theta}(\dfrac{1}{2}v^2) = -g\sin\theta$.

(c) Deduce that $V^2 = v^2 + 2\ell g(1-\cos\theta)$.

(d) Explain why $T - mg\cos\theta = \dfrac{1}{\ell}mv^2$.

(e) Suppose that $V^2 = 3g\ell$. Find the value of θ at which $T = 0$.

(f) Consider the situation in part (v). Briefly describe the path of P after the tension T becomes zero.

Chapter Nine

