CHAPTER **SEVEN**

De Moivre's Theorem

7A De Moivre's Theorem

Recall that in the section on multiplication of complex numebrs we derived the result for the argument of a product, viz

$$\arg(wz) = \arg(w) + \arg(z)\,.$$

Put $w = z = \operatorname{cis}\theta$, that is both are equal with modulus 1, to get

$$\begin{aligned} z^2 &= z \times z \\ &= \operatorname{cis}(\theta + \theta) \\ &= \operatorname{cis} 2\theta\,. \end{aligned}$$

Next put $w = z^2$, so that

$$\begin{aligned} z^3 &= z^2 \times z \\ &= \operatorname{cis}(2\theta + \theta) \\ &= \operatorname{cis} 3\theta\,. \end{aligned}$$

These simple calculations should make it obvious that $z^n = \operatorname{cis} n\theta$, at least for positive integers n. In fact the result is true for all integers, which we now prove.

de Moivre's Theorem: Let $z = \cos\theta + i\sin\theta$. It can be proven that

$$z^n = \cos n\theta + i\sin n\theta$$

for all integers n. The proof comes in two parts, beginning with a proof by induction for $n \ge 0$. Conjugates are then used to extend the proof to negative integers.

PROOF: As always with proof by induction, we first prove the result true for the starting value.

A. When $n = 0$

$$\begin{aligned} \text{LHS} &= z^0 \\ &= 1\,, \end{aligned} \qquad\qquad \begin{aligned} \text{RHS} &= \cos 0 + i\sin 0 \\ &= 1 + 0i \\ &= \text{LHS}\,. \end{aligned}$$

Hence the statement is true for $n = 0$.

B. Suppose that the result is true for some integer $k \ge 0$, that is

$$z^k = \cos k\theta + i\sin k\theta\,. \qquad (**)$$

We now prove the statement for $n = k + 1$, that is, we prove that

$$\begin{aligned} z^{k+1} &= \cos(k+1)\theta + i\sin(k+1)\theta\,. \\ \text{LHS} &= z^k \times z \\ &= (\cos k\theta + i\sin k\theta) \times (\cos\theta + i\sin\theta) \quad \text{by (**)} \\ &= \cos(k+1)\theta + i\sin(k+1)\theta \quad \text{by the sum of arguments} \\ &= \text{RHS}\,. \end{aligned}$$

Hence the result is true for $n = k + 1$.

C. It follows from parts A and B by mathematical induction that the statement is true for all integers $n \geq 0$.

D. Recall from Chapter 1 that if $|w| = 1$ then $w^{-1} = \overline{w}$. Now consider z^{-n} for some positive integer n.

$$\begin{aligned} z^{-n} &= (z^n)^{-1} \\ &= (\cos n\theta + i\sin n\theta)^{-1} \qquad \text{(by part C.)} \\ &= \overline{(\cos n\theta + i\sin n\theta)} \qquad (\text{since } |\operatorname{cis} n\theta| = 1) \\ &= \cos(-n\theta) + i\sin(-n\theta)\,, \end{aligned}$$

and the proof is now complete.

1 DE MOIVRE'S THEOREM: Let $z = \cos\theta + i\sin\theta$ be a complex number with modulus 1, then for all integers n,

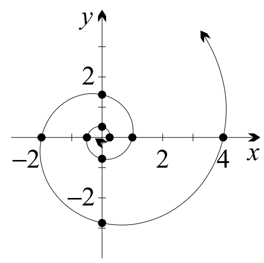
$$z^n = \cos n\theta + i\sin n\theta\,.$$

One immediate consequence of the above theorem is that if $z = r\operatorname{cis}\theta$ then $z^n = r^n\operatorname{cis} n\theta$. Thus if $r > 1$ and $\theta > 0$ then as n increases so too does the modulus and argument of z^n. That is, the points representing z^n lie on an anticlockwise spiral.

WORKED EXERCISE: Let $z = i\sqrt{2}$. Plot the points corresponding to z^n for values of n in the domain $-4 \leq n \leq 4$, and draw the spiral that these points lie on.

SOLUTION: Here is the table for z^n.

n	-4	-3	-2	-1	0	1	2	3	4
z^n	$\frac{1}{4}$	$i\frac{1}{2\sqrt{2}}$	$-\frac{1}{2}$	$-i\frac{1}{\sqrt{2}}$	1	$i\sqrt{2}$	-2	$-i2\sqrt{2}$	4



Notice that in the graph the spiral does not cut the axes at right angles.

A more practical application is to quickly simplify integer powers of complex numbers, as in the following example.

WORKED EXERCISE: (a) Write $z = -\sqrt{3} + i$ in modulus-argument form.
(b) Hence express z^7 in factored real-imaginary form.

SOLUTION: (a) It should be clear that $z = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$.
(b) Using de Moivre's theorem,

$$\begin{aligned} z^7 &= 2^7(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6})^7 \\ &= 128(\cos\tfrac{35\pi}{6} + i\sin\tfrac{35\pi}{6}) \\ &= 128(\cos\tfrac{-\pi}{6} + i\sin\tfrac{-\pi}{6}) \\ &= 64(\sqrt{3} - i)\,. \end{aligned}$$

Worked Exercise: For which values of k is $(1+i)^k$ imaginary?

Solution: Now $(1+i) = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$
so $(1+i)^k = \sqrt{2}^k(\cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4})$ (by de Moivre)
which is imaginary when $\frac{k\pi}{4}$ is an odd multiple of $\frac{\pi}{2}$.
Thus $\frac{k\pi}{4} = \frac{(2n+1)\pi}{2}$ where n is an integer,
that is $k = 4n+2$,
hence $k = \ldots, -6, -2, 2, 6, 10, \ldots$

Exercise **7A**

1. Write each expression in the form $\operatorname{cis} n\theta$:
(a) $(\cos\theta + i\sin\theta)^5$
(b) $(\cos\theta + i\sin\theta)^{-3}$
(c) $(\cos 2\theta + i\sin 2\theta)^4$
(d) $\cos\theta - i\sin\theta$
(e) $(\cos\theta - i\sin\theta)^{-7}$
(f) $(\cos 3\theta - i\sin 3\theta)^2$

2. Simplify as fully as possible:
(a) $\dfrac{(\cos\theta + i\sin\theta)^6(\cos\theta + i\sin\theta)^{-3}}{(\cos\theta - i\sin\theta)^4}$
(b) $\dfrac{(\cos 3\theta + i\sin 3\theta)^5(\cos 2\theta - i\sin 2\theta)^{-4}}{(\cos 4\theta - i\sin 4\theta)^{-7}}$

3. Write each expression in the form $a+ib$, where a and b are real:
(a) $(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^4$
(b) $(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^3$
(c) $(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^5$
(d) $(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})^{-2}$
(e) $(\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8})^{-6}$
(f) $(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12})^4$

4. (a) Write $1+i$ in the form $r(\cos\theta + i\sin\theta)$.
(b) Hence, or otherwise, find $(1+i)^{17}$ in the form $a+ib$, where a and b are integers.

5. Let $z = 1 + i\sqrt{3}$.
(a) Express z in mod-arg form.
(b) Express z^{11} in the form $a+ib$, where a and b are real.

6. Let $z = -\sqrt{3} + i$.
(a) Find the values of $|z|$ and $\arg z$.
(b) Hence, or otherwise, show that $z^7 + 64z = 0$.

7. (a) Express $\sqrt{3} - i$ in mod-arg form.
(b) Express $(\sqrt{3} - i)^7$ in mod-arg form.
(c) Hence express $(\sqrt{3} - i)^7$ in the form $x+iy$, where x and y are real.

8. (a) Express $-1 - i\sqrt{3}$ in mod-arg form.
(b) Express $(-1 - i\sqrt{3})^5$ in mod-arg form.
(c) Hence express $(-1 - i\sqrt{3})^5$ in the form $x+iy$, where x and y are real.

9. (a) Express $z = \sqrt{2} - i\sqrt{2}$ in mod-arg form.
(b) Hence write z^{22} in the form $a+ib$, where a and b are real.

10. Show that:
(a) $(1+i)^{10}$ is purely imaginary,
(b) $(1 - i\sqrt{3})^9$ is real,
(c) $-1+i$ is a fourth root of -4,
(d) $-\sqrt{3} - i$ is a sixth root of -64.

11. If k is a multiple of 4, prove that $(-1+i)^k$ is real.

12. (a) Find the minimum value of the positive integer m for which $(\sqrt{3}+i)^m$ is:
(i) real,
(ii) purely imaginary.
(b) Evaluate $(\sqrt{3}+i)^m$ for each of the above values of m.

13. (a) Prove that $(1+i)^n + (1-i)^n$ is real for all positive integer values of n.
(b) Determine the values of n for which $(1+i)^n + (1-i)^n = 0$.

14. Use de Moivre's theorem to prove that:

$$(-\sqrt{3}+i)^n - (-\sqrt{3}-i)^n = 2^{n+1}\sin\tfrac{5\pi n}{6}\, i$$

15. (a) Show that the expression $(1+\sqrt{3}\,i)^{2n} + (1-\sqrt{3}\,i)^{2n}$ simplifies to 2^{2n+1} if n is divisible by 3.
(b) Simplify the expression if n is not divisible by 3.

16. Show that $\left(\dfrac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}\right)^n = \operatorname{cis} 2n\theta$.

17. Prove that $(1+\cos\alpha + i\sin\alpha)^k + (1+\cos\alpha - i\sin\alpha)^k = 2^{k+1}\cos\tfrac{1}{2}k\alpha\cos^k\tfrac{1}{2}\alpha$.

18. Let $z = \operatorname{cis}\frac{\pi}{n}$, where n is a positive integer.
Show that:
(a) $1 + z + z^2 + \cdots + z^{2n-1} = 0$
(b) $1 + z + z^2 + \cdots + z^{n-1} = 1 + i\cot\frac{\pi}{2n}$

7B Trigonometric Identities

De Moivre's theorem is particularly useful when combined with the binomial theorem to obtain various trigonometric identities.

WORKED EXERCISE: (a) Express $\cos 3\theta$ in terms of powers of $\cos\theta$.
(b) Hence show that $x = \cos\frac{\pi}{9}$ is a solution of $8x^3 - 6x - 1 = 0$.
(c) Find the value of $\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$.

SOLUTION: (a) Let $z = \cos\theta + i\sin\theta$, then

$$z^3 = (\cos\theta + i\sin\theta)^3$$

so by de Moivre's theorem we have

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\,.$$

Take the real part to get

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)\\ &= 4\cos^3\theta - 3\cos\theta\,.\end{aligned}$$

(b) Put $\theta = \frac{\pi}{9}$ in this result to get

$$\cos\tfrac{\pi}{3} = 4\cos^3\tfrac{\pi}{9} - 3\cos\tfrac{\pi}{9}$$

or $\quad \frac{1}{2} = 4x^3 - 3x \qquad$ where $x = \cos\frac{\pi}{9}$,

thus $\quad 8x^3 - 6x - 1 = 0\,.$

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(c) Since $\cos 3\theta = \frac{1}{2}$ for $\theta = \frac{\pi}{9}, \frac{5\pi}{9},$ and $\frac{7\pi}{9}$, it follows that the given expression is the product of the roots of the equation in part (b). Hence

$$\cos\tfrac{\pi}{9}\cos\tfrac{5\pi}{9}\cos\tfrac{7\pi}{9} = \tfrac{1}{8}\,.$$

Worked Exercise: (a) Let $z = \cos\theta + i\sin\theta$. Show that $z^n - z^{-n} = 2i\sin n\theta$.

(b) Expand $(z - z^{-1})^5$.

(c) Use parts (a) and (b) to show that $16\sin^4\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(d) Hence find $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^5\theta\, d\theta$.

Solution: (a)
$$\begin{aligned} z^n - z^{-n} &= z^n - \overline{z^n} \qquad \text{(since } |z| = 1) \\ &= 2i\,\mathrm{Im}(z^n) \\ &= 2i\sin n\theta\,. \end{aligned}$$

(b) $(z - z^{-1})^5 = z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$.

(c) Rearranging part (b),

$$(z - z^{-1})^5 = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$$

so $\quad (2i\sin\theta)^5 = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \qquad$ by part (a)

thus $\quad 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(d) Dividing by 16 and integrating yields

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^5\theta\, d\theta &= \frac{1}{16}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 5\theta - 5\sin 3\theta + 10\sin\theta\, d\theta \\ &= \frac{1}{16}\left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= 0 - \frac{1}{16}\left(\frac{1}{5\sqrt{2}} - \frac{5}{3\sqrt{2}} - \frac{10}{\sqrt{2}}\right) \\ &= \frac{43\sqrt{2}}{120} \qquad \text{(you should check this.)} \end{aligned}$$

Exercise **7B**

1. (a) Use the identity $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$ to show that:

(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(b) Show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

2. Use similar methods to the previous question to show that:

(a) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(b) $\sin 4\theta = 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

(c) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

3. Let $z = \cos\theta + i\sin\theta$.

(a) Use de Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$.

(b) Show that $(z + z^{-1})^4 = (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$.

(c) Hence show that $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

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4. Repeat the methods of the previous question to show that:

$$\sin^4\theta = \tfrac{1}{8}\cos 4\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}$$

(Start by showing that $z^n - z^{-n} = 2i\sin n\theta$.)

5. (a) Use the methods of questions 1 and 2 to show that:

$$\cos 6\alpha = 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1$$

(b) Hence show that the polynomial equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$ has roots of the form $x = \cos\frac{n\pi}{12}$, where $n = 1, 3, 5, 7, 9, 11$.

(c) Use the product of these six roots to deduce that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$.

6. (a) Use the methods of question 3 to show that:

$$\cos^5\theta = \tfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

(b) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$.

7. (a) Use de Moivre's theorem to show that:

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

(b) Hence show that the equation $16x^5 - 20x^3 + 5x - 1 = 0$ has roots $x = 1$, $\sin\frac{\pi}{10}$, $\sin\frac{9\pi}{10}$, $\sin\frac{13\pi}{10}$, $\sin\frac{17\pi}{10}$.

(c) By equating coefficients, or otherwise, find the values of b and c for which $16x^4 + 16x^3 - 4x^2 - 4x + 1 = (4x^2 + bx + c)^2$, and hence explain why the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ has two double roots.

(d) Use part (b) to show that the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ has roots $x = \sin\frac{\pi}{10}$, $\sin\frac{9\pi}{10}$, $\sin\frac{13\pi}{10}$, $\sin\frac{17\pi}{10}$. Does this contradict part (c) which asserts that the equation has two double roots?

(e) Hence find exact values for $\sin\frac{\pi}{10}$ and $\sin\frac{3\pi}{10}$.

8. (a) Show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

(b) Hence solve the equation $16\sin^5\theta = \sin 5\theta$ for $0 \le \theta < 2\pi$.

9. (a) Use de Moivre's theorem to show that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

(b) Hence show that the equation $x^4 - 10x^2 + 5 = 0$ has roots $x = \pm\tan\frac{\pi}{5}$, $\pm\tan\frac{2\pi}{5}$.

(c) Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$ and that $\tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} = 10$.

10. Let $z = \cos\theta + i\sin\theta$.

(a) Show that $2\cos n\theta = z^n + \dfrac{1}{z^n}$ and that $2i\sin n\theta = z^n - \dfrac{1}{z^n}$.

(b) Hence show that:

$$128\cos^3\theta\sin^4\theta = \left(z^7 + \frac{1}{z^7}\right) - \left(z^5 + \frac{1}{z^5}\right) - 3\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

(c) Deduce that $\cos^3\theta\sin^4\theta = \frac{1}{64}(\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta)$.

11. Consider the polynomial equation $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$, which has four complex roots with modulus one.
Let $z = \operatorname{cis}\theta$.

(a) Show that $5\cos 2\theta - 11\cos\theta + 8 = 0$.

(b) Hence determine the four roots of the equation in the form $a + ib$, where a and b are real.

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12. (a) Use de Moivre's theorem to express $\dfrac{\sin 8\theta}{\sin\theta\cos\theta}$ as a polynomial in s, where $s = \sin\theta$.

(b) Hence solve the equation $x^6 - 6x^4 + 10x^2 - 4 = 0$, leaving the roots in trigonometric form.

13. Let n be a positive integer.

(a) Use de Moivre's theorem to show that:

$$\sin(2n+1)\theta = {}^{2n+1}\mathrm{C}_1 \cos^{2n}\theta \sin\theta - {}^{2n+1}\mathrm{C}_3 \cos^{2n-2}\theta \sin^3\theta + \cdots + (-1)^n \sin^{2n+1}\theta$$

(b) Hence show that the polynomial $P(x) = {}^{2n+1}\mathrm{C}_1\, x^n - {}^{2n+1}\mathrm{C}_3\, x^{n-1} + \cdots + (-1)^n$ has roots of the form $\cot^2\left(\dfrac{k\pi}{2n+1}\right)$ where $k = 1, 2, 3, \ldots, n$.

(c) Deduce that $\cot^2\left(\dfrac{\pi}{2n+1}\right) + \cot^2\left(\dfrac{2\pi}{2n+1}\right) + \cdots + \cot^2\left(\dfrac{n\pi}{2n+1}\right) = \dfrac{n(2n-1)}{3}$.

(d) Use the fact that $\cot\theta < \frac{1}{\theta}$ for $0 < \theta < \frac{\pi}{2}$ to show that:

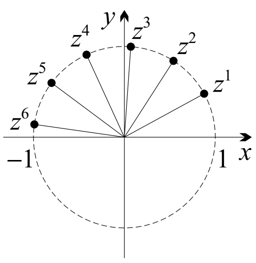
$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right)\frac{(2n+1)^2}{2n(2n-1)} > \frac{\pi^2}{6}$$

7C Roots of Unity

Recall from a previous worked exercise that the points in the Argand diagrm which represent z^n, where n is an integer, lie on a spiral whenever $|z| \neq 1$. When $|z| = 1$, it should be clear that the points lie on the unit circle. Further, if $z = \cos\theta + i\sin\theta$ then the angle at the origin subtended by successive points is

$$\begin{aligned}\arg(z^n) - \arg(z^{n-1}) &= \arg\left(\frac{z^n}{z^{n-1}}\right)\\ &= \arg(z)\\ &= \theta\,.\end{aligned}$$

That is, the angle is constant. Thus successive points are regularly spaced about the unit circle.



For example, the sketch on the right shows the points z^n for $n = 1, 2, 3, 4, 5, 6$, where $z = \cos\frac{1}{2} + i\sin\frac{1}{2}$. Note that

$$\arg(z) = \tfrac{1}{2} \doteqdot 28°39'\,,$$

which is the angle subtended at the origin by any pair of successive points. It should be clear that $2\pi \div \frac{1}{2} = 4\pi$ is irrational, and hence none of the points coincide, even for larger values of n. In that sense, this is not a very interesting example.

Significant configurations of points arise when we solve equations of the form $z^n = w$, where $|w| = 1$. There are always n solutions and the points are equally spaced about the unit circle. Further, if $z = 1$ is a solution and if α is another solution, then α^k will always coincide with one of the points, regardless of the integer value of k.

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WORKED EXERCISE: (a) Solve $z^6 = 1$.

(b) Plot the solutions on the unit circle in the complex plane.
  (i) What is the angle subtended at the origin by successive roots?
  (ii) What regular polygon has these points as vertices?

(c) Let $\alpha = \operatorname{cis}(-\frac{\pi}{3})$. Show that the list 1, α, α^2, α^3, α^4 and α^5 includes all six roots of $z^6 = 1$.

(d) Let $\beta = \operatorname{cis}\frac{2\pi}{3}$. Which roots of $z^6 = 1$ can be written in the form β^k, where k is an integer?

SOLUTION:

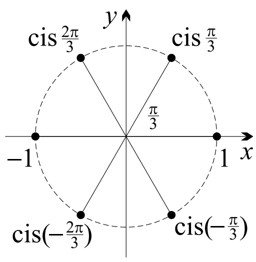
(a) Let $z = \operatorname{cis}\theta$ and note that $1 = \operatorname{cis} 2n\pi$, where n is an integer. Thus

$$\operatorname{cis} 6\theta = \operatorname{cis} 2n\pi \qquad \text{(by de Moivre)}$$
$$\text{so} \quad 6\theta = 2n\pi$$
$$\text{hence} \quad \theta = \frac{n\pi}{3}.$$

Apply the restriction $-\pi < \theta \le \pi$ to obtain all the distinct solutions. Thus

$$-\pi < \frac{n\pi}{3} \le \pi$$
$$\text{so} \quad -3 < n \le 3.$$

Hence the six roots of $z^6 = 1$ are $\operatorname{cis}(-\frac{2\pi}{3})$, $\operatorname{cis}(-\frac{\pi}{3})$, 1, $\operatorname{cis}\frac{\pi}{3}$, $\operatorname{cis}\frac{2\pi}{3}$ and -1.



(b) The graph on the right shows these six roots.
  (i) Clearly the angle at the centre is $\frac{\pi}{3}$.
  (ii) These are the vertices of a regular hexagon.

(c) Using de Moivre's theorem, the given list is:

$$1, \operatorname{cis}(-\tfrac{\pi}{3}), \operatorname{cis}(-\tfrac{2\pi}{3}), \operatorname{cis}(-\tfrac{3\pi}{3}) = -1,$$
$$\operatorname{cis}(-\tfrac{4\pi}{3}) = \operatorname{cis}\tfrac{2\pi}{3} \text{ and } \operatorname{cis}(-\tfrac{5\pi}{3}) = \operatorname{cis}\tfrac{\pi}{3}.$$

This is the same list as given in the answer to part (a), but simply in a different order.

(d) Now $\beta^k = \operatorname{cis}\frac{2k\pi}{3}$ by de Moivre's theorem. Hence $\arg(\beta^k)$ is a multiple of $\frac{2\pi}{3}$. Thus the possible values that β^k may take are:

$$\operatorname{cis}(-\tfrac{2\pi}{3}), 1 \text{ and } \operatorname{cis}\tfrac{2\pi}{3}.$$

That is, only these three roots can be written as a power of β.

WORKED EXERCISE: Consider the equation $z^5 + 1 = 0$.

(a) Find the roots of this equation and show them on the Argand diagram.

(b) Factorise $z^5 + 1$:
  (i) as a product of linear factors,
  (ii) as a product of linear and quadratic factors with real coefficients.

(c) Evauate $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$.

(d) Let α be a complex root of $z^5 + 1 = 0$, that is $\alpha \ne 1$.
  (i) Show that $1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 = 0$.
  (ii) Find a quadratic equation with roots $(\alpha^4 - \alpha)$ and $(\alpha^2 - \alpha^3)$.

(e) Put $\alpha = \operatorname{cis}\frac{\pi}{5}$ in part (d), and hence evaluate $\cos\frac{\pi}{5}$.

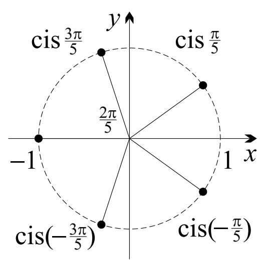
Solution:

(a) Let $z = \operatorname{cis}\theta$ and note that $-1 = \operatorname{cis}(2n+1)\pi$, where n is an integer. Thus

$$\begin{aligned} \operatorname{cis} 5\theta &= \operatorname{cis}(2n+1)\pi \qquad \text{(by de Moivre)} \\ \text{so} \quad 5\theta &= (2n+1)\pi \\ \text{hence} \quad \theta &= \tfrac{(2n+1)\pi}{5}\,. \end{aligned}$$

Apply the restriction $-\pi < \theta \le \pi$ to obtain all the distinct solutions. Thus

$$\begin{aligned} &-\pi < \tfrac{(2n+1)\pi}{5} \le \pi \\ \text{so} \quad &-5 < (2n+1) \le 5 \\ \text{or} \quad &-3 < n \le 2\,. \end{aligned}$$



Hence the five roots are:

$\operatorname{cis}(-\frac{3\pi}{5})$, $\operatorname{cis}(-\frac{\pi}{5})$, $\operatorname{cis}\frac{\pi}{5}$, $\operatorname{cis}\frac{3\pi}{5}$ and -1,

or in conjugate pairs,

$\operatorname{cis}\frac{\pi}{5}$, $\overline{\operatorname{cis}\frac{\pi}{5}}$, $\operatorname{cis}\frac{3\pi}{5}$, $\overline{\operatorname{cis}\frac{3\pi}{5}}$ and -1.

(b) Using the roots of the given equation,

$$\begin{aligned} z^5+1 &= (z+1)(z-\operatorname{cis}\tfrac{\pi}{5})(z-\overline{\operatorname{cis}\tfrac{\pi}{5}})(z-\operatorname{cis}\tfrac{3\pi}{5})(z-\overline{\operatorname{cis}\tfrac{3\pi}{5}}) \\ &= (z+1)(z^2-2z\cos\tfrac{\pi}{5}+1)(z^2-2z\cos\tfrac{3\pi}{5}+1)\,. \end{aligned}$$

(c) By the sum of the roots

$$\begin{aligned} \operatorname{cis}\tfrac{\pi}{5}+\overline{\operatorname{cis}\tfrac{\pi}{5}}+\operatorname{cis}\tfrac{3\pi}{5}+\overline{\operatorname{cis}\tfrac{3\pi}{5}}-1 &= 0 \\ \text{whence} \qquad 2\cos\tfrac{\pi}{5}+2\cos\tfrac{3\pi}{5} &= 1\,, \\ \text{that is} \qquad \cos\tfrac{\pi}{5}+\cos\tfrac{3\pi}{5} &= \tfrac{1}{2}\,. \end{aligned}$$

(d) (i) Since α is a complex root,

$$\begin{aligned} \alpha^5+1 &= 0 \\ \text{so} \quad (\alpha+1)(1-\alpha+\alpha^2-\alpha^3+\alpha^4) &= 0 \\ \text{thus} \quad 1-\alpha+\alpha^2-\alpha^3+\alpha^4 &= 0 \qquad (\text{since } \alpha \ne -1) \end{aligned}$$

(ii) The sum of the roots is $-\alpha+\alpha^2-\alpha^3+\alpha^4 = -1$ from part (i). The product of the roots is

$$\begin{aligned} (\alpha^4-\alpha)(\alpha^2-\alpha^3) &= \alpha^6-\alpha^7-\alpha^3+\alpha^4 \\ &= -\alpha+\alpha^2-\alpha^3+\alpha^4 \qquad (\text{since } \alpha^5=-1) \\ &= -1\,. \end{aligned}$$

Hence the required quadratic is $z^2+z-1=0$.

(e) With $\alpha = \operatorname{cis}\frac{\pi}{5}$ the roots of the equation in part (d) are

$$\begin{aligned} \alpha^4-\alpha &= -\alpha^{-1}-\alpha \qquad (\text{since } \alpha^5=-1) \\ &= -(\overline{\alpha}+\alpha) \qquad (\text{since } |\alpha|=1) \\ &= -2\cos\tfrac{\pi}{5}\,, \\ \text{and} \quad \alpha^2-\alpha^3 &= \alpha^2+\alpha^{-2} \qquad (\text{since } \alpha^5=-1) \\ &= \alpha^2+\overline{\alpha^2} \qquad (\text{since } |\alpha|=1) \\ &= 2\cos\tfrac{2\pi}{5}\,. \end{aligned}$$

Also, by direct calculation we have

$$z = \tfrac{-1-\sqrt{5}}{2} \text{ or } \tfrac{-1+\sqrt{5}}{2}\,,$$

hence $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$ and $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$.

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Exercise **7C**

1. (a) Find the three cube roots of unity, expressing the complex roots in both $r \operatorname{cis} \theta$ and $x + iy$ form.

(b) Show that the points in the complex plane representing these three roots form an equilateral triangle.

(c) If ω is one of the complex roots, show that the other complex root is ω^2.

(d) Write down the values of:

(i) ω^3 (ii) $1 + \omega + \omega^2$

(e) Show that:

(i) $(1 + \omega^2)^3 = -1$

(ii) $(1 - \omega - \omega^2)(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 8$

(iii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9$

2. (a) Solve the equation $z^6 = 1$, expressing the complex roots in the form $a + ib$, where a and b are real.

(b) Plot these roots on an Argand diagram, and show that they form a regular hexagon.

(c) If α is the complex root with smallest positive principal argument, show that the other three complex roots are α^2, α^{-1} and α^{-2}.

(d) Show that $z^6 - 1 = (z^2 - 1)(z^4 + z^2 + 1)$.

(e) Hence write $z^4 + z^2 + 1$ as a product of quadratic factors with real coefficients.

3. (a) Find, in the form $a + ib$, the four fourth roots of -1.

(b) Hence write $z^4 + 1$ as a product of two quadratic factors with real coefficients.

4. (a) Find, in the form $a + ib$, the six roots of the equation $z^6 + 1 = 0$.

(b) Hence show that $z^6 + 1 = (z^2 + 1)(z^2 - \sqrt{3}\, z + 1)(z^2 + \sqrt{3}\, z + 1)$.

(c) Divide both sides of this identity by z^3, and then let $z = \operatorname{cis} \theta$ to show that:

$$\cos 3\theta = 4 \cos \theta (\cos \theta - \cos \tfrac{\pi}{6})(\cos \theta - \cos \tfrac{5\pi}{6})$$

5. (a) Find, in mod-arg form, the five fifth roots of i.

(b) Find, in mod-arg form, the four fourth roots of $-i$.

(c) Find, in the form $a + ib$, the four fourth roots of $-8 - 8\sqrt{3}\, i$.

(d) Find, in mod-arg form, the five fifth roots of $16\sqrt{2} - 16\sqrt{2}\, i$.

DEVELOPMENT

6. (a) Find the five fifth roots of -1, writing the complex roots in mod-arg form.

(b) If α is the complex root with least positive principal argument, show that α^3, α^7 and α^9 are the other three complex roots.

(c) Show that $\alpha^7 = -\alpha^2$ and that $\alpha^9 = -\alpha^4$.

(d) Use the sum of the roots to show that $\alpha + \alpha^3 = 1 + \alpha^2 + \alpha^4$.

7. (a) Find the seven seventh roots of unity.

(b) By considering the sum of the real parts of these seven roots, show that:

$$\cos \tfrac{2\pi}{7} + \cos \tfrac{4\pi}{7} + \cos \tfrac{6\pi}{7} = -\tfrac{1}{2}$$

(c) Write $z^7 - 1$ as a product of one linear and three quadratic factors, all with real coefficients.

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(d) If α is the complex seventh root of unity with the least positive principal argument, show that α^2, α^3, α^4, α^5 and α^6 are the other five complex roots.

(e) By considering the relationships between the roots and the coefficients, show that the cubic equation $x^3 + x^2 - 2x - 1 = 0$ has roots $\alpha + \alpha^6$, $\alpha^2 + \alpha^5$ and $\alpha^3 + \alpha^4$.

8. (a) (i) Find the five fifth roots of unity, writing the complex roots in mod-arg form.

(ii) Show that the points in the complex plane representing these roots form a regular pentagon.

(iii) By considering the sum of these five roots, show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

(b) (i) Show that $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$.

(ii) Hence show that $z^4 + z^3 + z^2 + z + 1 = (z^2 - 2\cos\frac{2\pi}{5}\, z + 1)(z^2 - 2\cos\frac{4\pi}{5}\, z + 1)$.

(iii) By equating the coefficients of z in this identity, show that $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$.

(c) (i) Use the substitution $x = u + \dfrac{1}{u}$ to show that the equation $x^2 + x - 1 = 0$ has roots $2\cos\frac{2\pi}{5}$ and $2\cos\frac{4\pi}{5}$.

(ii) Deduce that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$.

9. (a) Find the ninth roots of unity.

(b) Hence show that:

$$z^6 + z^3 + 1 = (z^2 - 2\cos\tfrac{2\pi}{9}\, z + 1)(z^2 - 2\cos\tfrac{4\pi}{9}\, z + 1)(z^2 - 2\cos\tfrac{8\pi}{9}\, z + 1)$$

(c) Deduce that:

$$2\cos 3\theta + 1 = 8(\cos\theta - \cos\tfrac{2\pi}{9})(\cos\theta - \cos\tfrac{4\pi}{9})(\cos\theta - \cos\tfrac{8\pi}{9})$$

10. Let $\omega = \operatorname{cis}\frac{2\pi}{9}$.

(a) Show that ω^k, where k is an integer, is a solution of the equation $z^9 = 1$.

(b) Show that $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1$.

(c) Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$.

(d) Deduce that $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$.

11. Let $\rho = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. The complex number $\alpha = \rho + \rho^2 + \rho^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real.

(a) Prove that $1 + \rho + \rho^2 + \ldots + \rho^6 = 0$.

(b) The second root of the quadratic equation is β. Express β in terms of positive powers of ρ. Justify your answer.

(c) Find the values of the coefficients a and b.

(d) Deduce that $-\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2}$.

EXTENSION

12. (a) Show that the equation $(z+1)^8 - z^8 = 0$ has roots $z = -\frac{1}{2}$, $-\frac{1}{2}\left(1 \pm i\cot\frac{k\pi}{8}\right)$, where $k = 1, 2, 3$.

(b) Hence show that:

$$(z+1)^8 - z^8 = \tfrac{1}{8}(2z+1)(2z^2+2z+1)(4z^2+4z+\operatorname{cosec}^2\tfrac{\pi}{8})(4z^2+4z+\operatorname{cosec}^2\tfrac{3\pi}{8})$$

(c) By making a suitable substitution into this identity, deduce that:

$$\cos^{16}\theta - \sin^{16}\theta = \tfrac{1}{16}\cos 2\theta(\cos^2 2\theta + 1)(\cos^2 2\theta + \cot^2\tfrac{\pi}{8})(\cos^2 2\theta + \cot^2\tfrac{3\pi}{8})$$

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13. Suppose that $\omega^3 = 1$, and $\omega \neq 1$.
Let k be a positive integer.
(a) What are the two possible values of $1 + \omega^k + \omega^{2k}$?
(b) Use the binomial theorem to expand $(1+\omega)^n$ and $(1+\omega^2)^n$, where n is a positive integer.
(c) Let ℓ be the largest integer for which $3\ell \leq n$.
Show that:
$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{3\ell} = \tfrac{1}{3}\left(2^n + (1+\omega)^n + (1+\omega^2)^n\right)$$
(d) If n is a multiple of 6, show that:
$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{n} = \tfrac{1}{3}\left(2^n + 2\right)$$

14. Consider the equation $(z+1)^{2n} + (z-1)^{2n} = 0$, where n is a positive integer.
(a) Show that every root of the equation is purely imaginary.
(b) Let the roots be represented by the points $P_1, P_2, \ldots, P_{2n}$ in the Argand diagram, and let O be the origin.
Show that:
$$OP_1{}^2 + OP_2{}^2 + \cdots + OP_{2n}{}^2 = 2n(2n-1)$$

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Chapter Seven

Exercise **7A** (Page 46)

1(a) $\operatorname{cis} 5\theta$ **(b)** $\operatorname{cis}(-3\theta)$ **(c)** $\operatorname{cis} 8\theta$ **(d)** $\operatorname{cis}(-\theta)$
(e) $\operatorname{cis} 7\theta$ **(f)** $\operatorname{cis}(-6\theta)$
2(a) $\operatorname{cis} 7\theta$ **(b)** $\operatorname{cis}(-5\theta)$
3(a) -1 **(b)** $-i$ **(c)** $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ **(d)** $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
(e) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ **(f)** $\frac{1}{2} + \frac{\sqrt{3}}{2}i$
4(a) $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ **(b)** $256 + 256i$
5(a) $2\operatorname{cis}\frac{\pi}{3}$ **(b)** $1024 - 1024\sqrt{3}\,i$
6(a) $2,\ \frac{5\pi}{6}$
7(a) $2\operatorname{cis}\left(-\frac{\pi}{6}\right)$ **(b)** $128\operatorname{cis}\frac{5\pi}{6}$ **(c)** $-64\sqrt{3} + 64i$
8(a) $2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$ **(b)** $32\operatorname{cis}\frac{2\pi}{3}$ **(c)** $-16 + 16\sqrt{3}\,i$
9(a) $2\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **(b)** $2^{22}i$
12(a)(i) 6 **(ii)** 3 **(b)** $-64,\ 8i$
13(b) $n = 2,\ 6,\ 10,\ \ldots$
15(b) -2^{2n}

Exercise **7B** (Page 48)

6(b) $\frac{8}{15}$
7(c) $b = 2,\ c = -1$
(d) No, since $\sin\frac{\pi}{10} = \sin\frac{9\pi}{10}$ and $\sin\frac{13\pi}{10} = \sin\frac{17\pi}{10}$
(e) $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$, $\sin\frac{3\pi}{10} = \frac{\sqrt{5}+1}{4}$
8(b) $\theta = 0,\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \pi,\ \frac{7\pi}{6},\ \frac{11\pi}{6}$
11(b) $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ or $\frac{3}{5} \pm \frac{4}{5}i$
12(a) $8(1 - 10s^2 + 24s^4 - 16s^6)$
(b) $x = 2\sin\frac{n\pi}{8}$ for $n = 1,\ 2,\ 3,\ 5,\ 6,\ 7$

Exercise **7C** (Page 53)

1(a) $1,\ \operatorname{cis}\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i,\ \operatorname{cis}\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
(d)(i) 1 **(ii)** 0
2(a) $z = \pm 1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}\,i,\ \frac{1}{2} - \frac{\sqrt{3}}{2}\,i,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i,$
$-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$ **(e)** $(z^2 - z + 1)(z^2 + z + 1)$
3(a) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i,\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i,\ -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i,\ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i$
(b) $(z^2 - \sqrt{2}\,z + 1)(z^2 + \sqrt{2}\,z + 1)$
4(a) $i,\ -i,\ \frac{\sqrt{3}}{2} + \frac{1}{2}\,i,\ \frac{\sqrt{3}}{2} - \frac{1}{2}\,i,\ -\frac{\sqrt{3}}{2} + \frac{1}{2}\,i,\ -\frac{\sqrt{3}}{2} - \frac{1}{2}\,i$
5(a) $\operatorname{cis}\left(-\frac{7\pi}{10}\right),\ \operatorname{cis}\left(-\frac{3\pi}{10}\right),\ \operatorname{cis}\frac{\pi}{10},\ \operatorname{cis}\frac{\pi}{2} = i,\ \operatorname{cis}\frac{9\pi}{10}$
(b) $\operatorname{cis}\left(-\frac{5\pi}{8}\right),\ \operatorname{cis}\left(-\frac{\pi}{8}\right),\ \operatorname{cis}\frac{3\pi}{8},\ \operatorname{cis}\frac{7\pi}{8}$
(c) $1 + \sqrt{3}\,i,\ -1 - \sqrt{3}\,i,\ \sqrt{3} - i,\ -\sqrt{3} + i$
(d) $2\operatorname{cis}\left(-\frac{17\pi}{20}\right),\ 2\operatorname{cis}\left(-\frac{9\pi}{20}\right),\ 2\operatorname{cis}\left(-\frac{\pi}{20}\right),\ 2\operatorname{cis}\frac{7\pi}{20},$
$2\operatorname{cis}\frac{3\pi}{4}$
6(a) $-1,\ \operatorname{cis}\frac{\pi}{5},\ \operatorname{cis}\left(-\frac{\pi}{5}\right),\ \operatorname{cis}\frac{3\pi}{5},\ \operatorname{cis}\left(-\frac{3\pi}{5}\right)$
7(a) $1,\ \operatorname{cis}\left(\pm\frac{2\pi}{7}\right),\ \operatorname{cis}\left(\pm\frac{4\pi}{7}\right),\ \operatorname{cis}\left(\pm\frac{6\pi}{7}\right)$
(c) $(z - 1) \times (z^2 - 2\cos\frac{2\pi}{7}\,z + 1) \times$
$(z^2 - 2\cos\frac{4\pi}{7}\,z + 1) \times (z^2 - 2\cos\frac{6\pi}{7}\,z + 1)$
8(a)(i) $1,\ \operatorname{cis}\frac{2\pi}{5},\ \operatorname{cis}\left(-\frac{2\pi}{5}\right),\ \operatorname{cis}\frac{4\pi}{5},\ \operatorname{cis}\left(-\frac{4\pi}{5}\right)$
9(a) $\operatorname{cis}\frac{2k\pi}{9}$ for $k = -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4$
13(a) 3, when k is a multiple of 3, 0 otherwise.
(b) $(1+\omega)^n = \sum_{r=0}^{n} \binom{n}{r} \omega^r$ and
$(1+\omega^2)^n = \sum_{r=0}^{n} \binom{n}{r} \omega^{2r}$
14(a) The roots are $-i\cot\frac{(2k-1)\pi}{4n}$
for $k = 1,\ 2,\ 3,\ \ldots,\ 2n$.

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