The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1	**Exercises for Week 2**	2017

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) If $\mathbf{v}$ is a geometric vector and λ is a scalar then $|\lambda\mathbf{v}| = |\lambda||\mathbf{v}|$.

(ii) A *unit vector* is a vector of length one. There is always a unit vector pointing in any given direction.

(iii) If λ is a nonzero scalar and $\mathbf{v}$ a vector then we write $\dfrac{\mathbf{v}}{\lambda}$ for the scalar multiple $\dfrac{1}{\lambda}\mathbf{v}$.

(iv) Let $\mathbf{v}$ be a nonzero vector. Then $\widehat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ is called *the hat of* $\mathbf{v}$, or simply $\mathbf{v}$-*hat*, and is the unit vector pointing in the direction of $\mathbf{v}$. It is designed to satisfy the equation

$$\mathbf{v} \;=\; |\mathbf{v}|\,\widehat{\mathbf{v}}\,,$$

which captures precisely the idea of $\mathbf{v}$ being characterised by length and direction.

(v) Parallel vectors: Nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if and only if $\mathbf{v} = \lambda\mathbf{w}$ for some nonzero scalar λ. The zero vector $\mathbf{0}$ is parallel to all vectors $\mathbf{v}$ and $\mathbf{0} = 0\,\mathbf{v}$.

(vi) Linear independence of two vectors: Two vectors $\mathbf{v}$ and $\mathbf{w}$ are *linearly independent* if they are not parallel, which is equivalent to saying that the vector equation

$$a\,\mathbf{v} + b\,\mathbf{w} \;=\; \mathbf{0}$$

implies that the scalars a and b must both be zero.

(vii) Cartesian form: The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ point in the directions of the positive x, y and z-axes respectively. If $\mathbf{v}$ is the position vector of the point $P(a,b,c)$ then

$$\mathbf{v} \;=\; a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}\,,$$

called the *Cartesian form* of $\mathbf{v}$. In this case, a, b, c are called the *components* of $\mathbf{v}$.

(viii) Component-wise operations: To add, subtract or negate vectors, simply add, subtract or negate respective components. To multiply a vector by a scalar, simply multiply the components by the scalar.

(ix) If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ then $|\mathbf{v}| = \sqrt{a^2+b^2+c^2}\,$.

(x) If $P(a_1,b_1,c_1)$ and $Q(a_2,b_2,c_2)$ are points in space then

$$\overrightarrow{PQ} \;=\; (a_2-a_1)\mathbf{i} + (b_2-b_1)\mathbf{j} + (c_2-c_1)\mathbf{k}\,.$$

(xi) If a point R divides a line PQ in the ratio $\lambda : \mu$ and O is any point in space then

$$\overrightarrow{OR} \;=\; \frac{\mu\,\overrightarrow{OP} + \lambda\,\overrightarrow{OQ}}{\lambda+\mu}\,.$$

(xii) Linear combination of vectors: A *linear combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an expression of the form $\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n$ where $\lambda_1, \ldots, \lambda_n$ are scalars.

(xiii) Linear independence of n vectors: We say that vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are *linearly independent* if, whenever $\lambda_1, \ldots, \lambda_n$ are scalars,

$$\lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n = \mathbf{0} \qquad \text{implies} \qquad \lambda_1 = \ldots = \lambda_n = 0 .$$

(xiv) Linear dependence of n vectors: We call a sequence of vectors *linearly dependent* if the vectors are not linearly independent, which is equivalent to one of the vectors in the sequence being equal to a linear combination of the remaining vectors in the sequence.

(xv) Span of n vectors: The span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the set of all linear combinations of these vectors.

Preparatory Exercises:

1. In the xy-plane the z-coordinate is ignored and $\mathbf{i}$ and $\mathbf{j}$ are, as usual, unit vectors in the positive x and y-directions respectively. Let P be the point $(3, 1)$ and Q the point $(4, -2)$ in the xy-plane. As usual the origin $(0, 0)$ is denoted by O.

(i) Write down the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(ii) Write down the displacement vector $\overrightarrow{PQ}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(iii) Write down the coordinates of the point R such that $\overrightarrow{OR} = \overrightarrow{PQ}$.

2. Given points $A(-2, 3)$ and $B(4, -1)$ in the xy-plane, find

(i) the position vectors of A and B in terms of $\mathbf{i}$ and $\mathbf{j}$;

(ii) the displacement vector $\overrightarrow{AB}$ in terms of $\mathbf{i}$ and $\mathbf{j}$;

(iii) the unit vector pointing from A towards B;

(iv) the unit vector pointing from B towards A.

3. Given $\mathbf{a} = \sqrt{3}\,\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \dfrac{1}{\sqrt{2}}\big(\mathbf{i} - \mathbf{j}\big)$, find

(i) $|\mathbf{a}|$ (ii) $|\mathbf{b}|$ (iii) $\widehat{\mathbf{a}}$ (iv) $\widehat{\mathbf{b}}$ (v) $2\sqrt{3}\,\widehat{\mathbf{a}} + \sqrt{2}\,\widehat{\mathbf{b}}$

4. Let P be the point $(3, 1, -2)$ and Q the point $(4, -2, 5)$ in space. As usual the origin $(0, 0, 0)$ is denoted by O.

(i) Write down the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

(ii) Write down the displacement vector $\overrightarrow{PQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

(iii) Write down the coordinates of the point R such that $\overrightarrow{OR} = \overrightarrow{PQ}$.

5. Given $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} - 4\mathbf{k}$, find

(i) $\mathbf{a} + \mathbf{b}$ (ii) $\mathbf{a} + 3\mathbf{b} - 2\mathbf{c}$ (iii) $|\mathbf{a}|$ (iv) $\widehat{\mathbf{a}}$ (v) $\widehat{\mathbf{c}}$

Exercises:

15. Consider the following points in space:

$$P(1,-1,2)\ ,\quad Q(2,3,0)\ ,\quad R(-2,5,0)\ ,\quad S(0,1,\lambda)$$

where λ is some real number.

(i) Find the values of λ for which $|\overrightarrow{PQ}| = |\overrightarrow{PS}|$.

(ii) Find the value of λ for which $\overrightarrow{PR}$ is parallel to $\overrightarrow{RS}$.

16. Consider the points $A(1,2,-3)$, $B(-2,1,1)$ and $C(0,2,1)$.

(i) Find the point D such that $ABCD$ is a parallelogram.

(ii) Let P be the midpoint of AC. Find the Cartesian form of $\overrightarrow{OP}$.

(iii) Find the Cartesian forms of $\overrightarrow{BP}$ and $\overrightarrow{PD}$, and deduce that the diagonals AC and BD bisect each other.

(iv) Find the lengths of $\overrightarrow{AC}$ and $\overrightarrow{BD}$. Is the parallelogram $ABCD$ a rectangle?

17. Let $\mathbf{v} = \overrightarrow{PQ}$ where $P = (-3,2,0)$ and $Q = (4,-2,3)$. Find the Cartesian form of $\mathbf{v}$, the length of $\mathbf{v}$ and the angles $\mathbf{v}$ makes (to the nearest degree) with each of the positive x, y and z-axes. (The cosine of each angle will be the relevant component divided by the length of the vector.)

18. Find the scalars α, β and γ such that

(i) $3\mathbf{i}+\mathbf{j}$ is parallel to $\alpha\mathbf{i}-4\mathbf{j}$ (ii) $3\mathbf{i}+\beta(\mathbf{j}-\mathbf{k})$ is parallel to $\mathbf{i}-4\mathbf{j}+4\mathbf{k}$

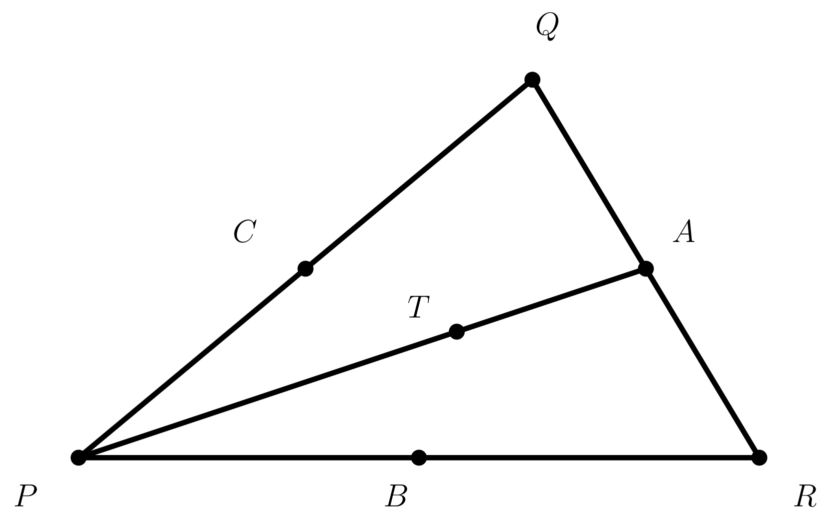
(iii) $3\mathbf{i}+\gamma(\mathbf{j}+3\mathbf{k})$ has the same length as $12\mathbf{i}-5\mathbf{k}$.

19.* The line joining the vertex of a parallelogram to the midpoint of an opposite side divides one of the diagonals into two pieces of unequal lengths. Find the ratio of the lengths of these pieces.

20.* Given points P and Q that divide the sides AB and AC of a triangle ABC in the ratios $\alpha : 1-\alpha$ and $\beta : 1-\beta$ respectively, verify that if $\overrightarrow{PQ} = \gamma\overrightarrow{BC}$ then $\alpha = \beta = \gamma$.

21.* Given a quadrilateral $ABCD$ in the plane, suppose that the diagonals AC and BD intersect at P and the lines through AB and CD intersect at Q. Suppose that Q divides AB in the ratio $3 : -1$ and CD in the ratio $-5 : 7$. Prove that P divides AC in the ratio $7 : 1$ and BD in the ratio $5 : 3$.

22.* This exercise gives a direct proof (with the benefit of knowing in advance what the ratios ought to be) that the medians of a triangle intersect. Let PQR be a triangle with medians PA, QB and RC, where A, B, C are midpoints of the sides opposite P, Q, R respectively.



Let T be the point on PA which divides it in the ratio $2:1$. Put

$$\mathbf{u} \;=\; \overrightarrow{PQ}\,, \quad \text{and} \quad \mathbf{v} \;=\; \overrightarrow{PR}\,.$$

Express $\overrightarrow{QT}$ and $\overrightarrow{QB}$ in terms of $\mathbf{u}$ and $\mathbf{v}$ and observe that they are parallel. This proves T lies on QB, and similarly on RC, so that the medians intersect.

23.* Prove carefully from the definition of linear independence, that a collection of vectors is linearly dependent if and only if one of the vectors is a linear combination of the others.

24.* Prove that any four geometric vectors in space are linearly dependent.

25.* Suppose that A, B, C are distinct points in space. Prove that these points lie on a line if and only if there exist non-zero scalars α, β, γ such that $\alpha+\beta+\gamma=0$ and

$$\alpha\,\overrightarrow{OA} + \beta\,\overrightarrow{OB} + \gamma\,\overrightarrow{OC} \;=\; \mathbf{0}\,.$$

26.* The notion of parallel lines induces an equivalence relation on the set of lines.

a) Show that the notion of "parallel" for non-zero vectors induces an equivalence relation on the set of non-zero vectors.

b) Show that the notion of linear dependence of two vectors is not transitive, and hence is not an equivalence relation.

27.** Define *addition* and *scalar multiplication* of real-valued functions of a real variable as follows. Let $f, g : \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. Define $f+g$ and λf by the rules:

$$(f+g)(x) \;=\; f(x)+g(x)\,, \qquad (\lambda f)(x) \;=\; \lambda f(x)\,.$$

It is routine to verify that all of the usual properties of vector arithmetic hold, with the zero constant function taking the role of the zero vector. The definitions of linear combinations, linear dependence and independence make sense with functions in place of vectors. (In fact, functions form a *vector space*, the abstract theory of which you will learn about next year). For an integer $n \geq 0$ define $f_n : \mathbb{R} \to \mathbb{R}$ by the rule

$$f_n(x) \;=\; x^n\,.$$

(i) What is the common name for functions that are linear combinations of $f_0, \ldots, f_n$?

(ii) Prove that $f_0, \ldots, f_n$ are linearly independent for all nonnegative integers n. (This implies that the vector space of real-valued functions of a real variable is *infinite dimensional*.)

Short Answers to Selected Exercises:

1. (i) $\overrightarrow{OP} = 3\mathbf{i} + \mathbf{j}$, $\overrightarrow{OQ} = 4\mathbf{i} - 2\mathbf{j}$ (ii) $\overrightarrow{PQ} = \mathbf{i} - 3\mathbf{j}$ (iii) $R = (1, -3)$

2. (i) $-2\mathbf{i} + 3\mathbf{j}$, $4\mathbf{i} - \mathbf{j}$ (ii) $2(3\mathbf{i} - 2\mathbf{j})$ (iii) $\frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$ (iv) $-\frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$

3. (i) 2 (ii) 1 (iii) $\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$ (iv) $\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$ (v) $4\,\mathbf{i} + (\sqrt{3} - 1)\,\mathbf{j}$

4. (i) $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ (ii) $\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$ (iii) $(1, -3, 7)$

5. (i) $3\mathbf{i} + \mathbf{k}$ (ii) $-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$ (iii) 3 (iv) $\frac{2}{3}\,\mathbf{i} - \frac{1}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}$ (v) $\frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{k}$

6. (i) $4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $6\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ (ii) $2\mathbf{i} - 7\mathbf{k}$ (iii) $\frac{1}{\sqrt{53}}(2\mathbf{i} - 7\mathbf{k})$ (iv) $-\frac{1}{\sqrt{53}}(2\mathbf{i} - 7\mathbf{k})$

7. (i) $-2\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}$ (ii) $2\mathbf{i} + 8\mathbf{j} - 13\mathbf{k}$ (iii) $4\mathbf{i} - 12\mathbf{j} + 18\mathbf{k}$ (iv) $12\mathbf{i} + 6\mathbf{j} - 12\mathbf{k}$
(v) $-8\mathbf{i} - 18\mathbf{j} + 30\mathbf{k}$ (vi) 11 (vii) 6 (viii) $\frac{1}{11}(2\mathbf{i} - 6\mathbf{j} + 9\mathbf{k})$ (ix) $\frac{1}{3}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ (x) $\sqrt{77}$

8. (i) True (ii) False (iii) False (iv) True (v) False (vi) True

9. (i) $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $-\mathbf{i} - \mathbf{j}$, $\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $-2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $-\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ (ii) The diagonals have different lengths.

10. (i) $(150\sqrt{2} - 75)\,\mathbf{i} + (75\sqrt{3} - 150\sqrt{2})\,\mathbf{j}$ (ii) 160 km , 31°

11. 1/3 , 2/3

15. (i) -2 , 6 (ii) 4/3

16. (i) $(3, 3, -3)$ (ii) $\frac{1}{2}\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ (iii) $\frac{5}{2}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (iv) $\sqrt{17}$, $3\sqrt{5}$

17. $7\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$, $\sqrt{74}$, 36° , 118° , 70°

18. -12 , -12 , ± 4

19. 1 : 2

27. (i) polynomial functions