THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Week 2** 2012

**1.** (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Express the following in the form $x + iy$ (*Cartesian* or *standard* form):

(a) $(2+3i)+(4-5i)$;

(b) $(1+i)(1-i)$;

(c) $(2+3i)-(4-5i)$;

(d) $\dfrac{1+i}{1-i}$;

(e) $\dfrac{1+2i}{3-4i}$;

(f) $(1+i)^2$;

(g) $i^9$;

(h) $i^{123} - 4i^8 - 4i$.

## Questions for the Tutorial

**2.** Solve the following equations in $\mathbb{C}$:

(a) $z^2 + 3z + 2 = 0$

(b) $z^2 + z + 1 = 0$

(c) $z^2 + 2\overline{z} + 1 = 0$

(d) $z^4 = 16$

**3.** (a) Find all solutions of the equation $z^2 + 3 + 4i = 0$ by setting $z = a + bi$ for some real numbers $a$ and $b$.

(b) Solve $z^2 + z + 1 + i = 0$. (*Hint*: Use your solution to the previous part.)

**4.** For all complex numbers $z_1$ and $z_2$, prove that

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, (b) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, (c) $\overline{z_1} = z_1$ if and only if $z_1$ is real,

and (d) $\overline{\left(\dfrac{1}{z_1}\right)} = \dfrac{1}{\overline{z_1}}$ for $z_1 \neq 0$.

**5.** Sketch the following sets in the complex plane.
(*Hint*: Note that $|z-c|$ is the distance between $z$ and $c$ in the complex plane, hence $|z+c| = |z-(-c)|$ is the distance between $z$ and $-c$.)

(a) $\{z \in \mathbb{C} \mid |z+i| = 5\}$,

(b) $\{z \in \mathbb{C} \mid \frac{1}{2} \leq |z+i| < 1\}$.

(c) $\{ z \in \mathbb{C} \mid |z| \leq 3 \}$

(d) $\{ z \in \mathbb{C} \mid |z+i| > 2 \}$

(e) $\{ z \in \mathbb{C} \mid \operatorname{Re} z < -1 \}$

(f) $\{ z \in \mathbb{C} \mid \operatorname{Im} z \geq -1 \}$

(g) $\{ z \in \mathbb{C} \mid |z-i| \leq |z-1| \}$

(h) $\{ z \in \mathbb{C} \mid \left|\dfrac{z-1}{z-2}\right| \leq 3 \}$

(i) $\{ z \in \mathbb{C} \mid \operatorname{Im}(2z - \overline{z}(1+i)) = 0$ and $\operatorname{Re}(2z - \overline{z}(1+i)) < 4 \}$

(j) $\{ z \in \mathbb{C} \mid \operatorname{Im}(z^2) < \operatorname{Re} z \}$

**6.** (a) Write the following in *polar* form, $z = r(\cos\theta + i\sin\theta) = r \operatorname{cis} \theta$:

(*i*) $1+i$ (*ii*) $1+\sqrt{3}i$
(*iii*) $3\sqrt{3}+3i$

(b) Using your answers to the previous part, find the following, expressing your answers first in polar then Cartesian (standard) form.

(*i*) $(1+i)^{11}$ (*ii*) $(1+\sqrt{3}i)^7$
(*iii*) $(3\sqrt{3}+3i)^3$ (*iv*) $\dfrac{1+i}{1+\sqrt{3}i}$
(*v*) $\dfrac{3\sqrt{3}+3i}{1+i}$ (*vi*) $\dfrac{1+\sqrt{3}i}{3\sqrt{3}+3i}$

**7.** Prove the *triangle inequality* $|z_1+z_2| \leq |z_1|+|z_2|$, for all $z_1, z_2 \in \mathbb{C}$.

**8.** If the complex number $z$ is imagined as a point in the complex plane, then its conjugate $\overline{z}$ is the point obtained from $z$ by reflecting in the real axis. What are the complex numbers obtained from $z$ by the following geometric transformations?

(a) $180°$ rotation about 0. (b) Reflection in the imaginary axis.
(c) $45°$ clockwise rotation about 0. (d) Reflection in the line $y=x$.

## Extra Questions

**9.** Express $\cos 5\theta$ and $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$, respectively.
(*Hint*: Recall the binomial expansion:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Use this as well as de Moivre's Theorem to expand $(\cos\theta + i\sin\theta)^5$.)

**10.** (a) Let $r$ be a real constant greater than 2. The set $\{z \in \mathbb{C} \mid |z+1|+|z-1| = r\}$ is a curve in the plane. Describe it and then find its equation in terms of $x$ and $y$.

(b) Next, assume that $-2 < r < 2$ and $r \neq 0$.
Describe the curve $\{z \in \mathbb{C} \mid |z+1|-|z-1| = r\}$ and find its equation.

## Solution to Question 1

**1.** (a) $(2+3i)+(4-5i) = 6-2i$; (b) $(1+i)(1-i) = 1^2 - i^2 = 1+1 = 2$;

(c) $(2+3i)-(4-5i) = -2+8i$; (d) $\dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{1^2-i^2} = \dfrac{(1+i)^2}{2} = i$;

(e) $\dfrac{1+2i}{3-4i} = \dfrac{1}{5}(-1+2i)$; (f) $(1+i)^2 = 2i$;

(g) $i^9 = i^8 i = i$; (h) $i^{123} - 4i^8 - 4i = -4-5i$.