CHAPTER **TWELVE**

The Logarithmic Function

So far the calculus has been developed only for algebraic functions like

$$x^2 - 2x + 4, \qquad \sqrt{1 - x^2}\,, \qquad \text{and} \qquad x^2 + \frac{1}{x^2}\,,$$

which can be written using powers, roots and reciprocals. The first step in extending calculus to non-algebraic or *transcendental functions* is to study the logarithmic and exponential functions, which are so important in dealing with applications of calculus to the natural world.

Direct first principles differentiation of logarithmic and exponential functions, however, is blocked by some intractable limits, and by the fact that the most natural base to use for logarithmic and exponential functions is an irrational real number which is given the symbol e, and which makes its first appearance in this chapter. The most satisfying account of the theory begins not with any particular logarithmic or exponential function, but with the problem of integrating the reciprocal function $y = 1/x$. It is quite surprising that such an indirect approach makes the theory so clear.

Study Notes: The natural logarithmic function $y = \log_e x$ is the central subject of this chapter, but it is vital that logarithmic and exponential functions to more familiar bases like 2 and 10 be well understood before the arguments involving calculus begin. Particularly important are their graphs, their domains and ranges, their asymptotic behaviour, and above all the fact that they are inverses of each other. The short account in Section 12A reviews the earlier discussion of the graphs and the algebra. The theorem on the derivative of the logarithmic functions in Section 12B is the fundamental step in the theory, and it may be best on first reading to be convinced simply by the graphical argument of Steps 1–4 of the explanation, leaving until later the tricky Steps 5 and 6. As always, computer, calculator and graph-paper work with these unfamiliar functions should be used to help establish an intuitive understanding of their behaviour. In particular, integration of $y = 1/x$ by counting squares on graph paper should be used to establish approximations of both the function $\log x$ and the value of e.

12 A Review of Logarithmic and Exponential Functions

The algebra of the logarithmic and exponential functions was reviewed in Sections A and B of Chapter Six before the discussion of sequences and series. What is required for this chapter is a clear picture of the graphs, and a clear understanding that the logarithmic and exponential functions to a given base b are mutually inverse (the base b must be positive and different from 1).

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1 DEFINITION OF LOGARITHMS: The *logarithm* base b of a positive number x is the index, when the number x is expressed as a power of the base b:

$$y = \log_b x \qquad \text{means that} \qquad x = b^y.$$

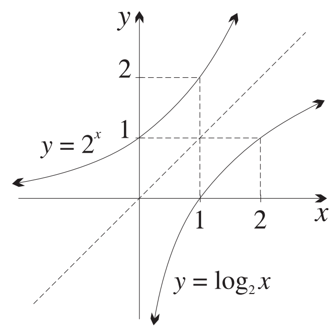
This means that the functions $y = b^x$ and $y = \log_b x$ are inverse functions. The tables of values of the two functions will be the same except with the rows reversed. Taking the particular case $b = 2$,

$y = 2^x$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$y = \log_2 x$

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	-3	-2	-1	0	1	2	3



On the right are the resulting graphs of the exponential function $y = 2^x$ and the logarithmic function $y = \log_2 x$, drawn on the one set of axes. They are reflections of each other in the diagonal line $y = x$.

Because the two function are mutually inverse, if they are applied one after the other to a number, then the number remains the same. For example, applying them in turn to 8,

$$\log_2 2^8 = \log_2 256 = 8 \qquad \text{and} \qquad 2^{\log_2 8} = 2^3 = 8.$$

The general statement of this is:

2 THE FUNCTIONS $y = b^x$ AND $y = \log_b x$ ARE MUTUALLY INVERSE:

$$\log_b b^x = x, \text{ for all real } x \qquad \text{and} \qquad b^{\log_b x} = x, \text{ for all } x > 0.$$

Here once again is a list of the index laws and the corresponding log laws:

3

THE LOG AND INDEX LAWS BASE b:	
$b^u \times b^v = b^{u+v}$	$\log_b xy = \log_b x + \log_b y$
$b^u \div b^v = b^{u-v}$	$\log_b \dfrac{x}{y} = \log_b x - \log_b y$
$(b^u)^v = b^{uv}$	$\log_b x^v = v \log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
$b^{-1} = \frac{1}{b}$	$\log_b \frac{1}{b} = -1$
$b^{-u} = \dfrac{1}{b^u}$	$\log_b \dfrac{1}{x} = -\log_b x$

Lastly there is the change of base formula. If c is some other base, then:

4 CHANGE OF BASE: $\log_c x = \dfrac{\log_b x}{\log_b c}$ (log of the number over log of the base).

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Exercise 12A

1. (a) Copy and complete these tables of values of the functions $y = 3^x$ and $y = \log_3 x$:

x	-2	-1	0	1	2
3^x					

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\log_3 x$					

(b) Sketch both curves on the one set of axes, choosing appropriate scales on the axes.

(c) Add the line $y = x$ to your graph. What transformation transforms the graph of $y = 3^x$ into the graph of $y = \log_3 x$?

(d) What are the domain and range of $y = 3^x$ and $y = \log_3 x$?

2. (a) Use your calculator to copy and complete these tables of values of $y = 10^x$ and $y = \log_{10} x$:

x	-2	-1	$-0{\cdot}75$	$-0{\cdot}5$	0	$0{\cdot}25$	$0{\cdot}5$	$0{\cdot}75$	1
10^x									

x	$0{\cdot}01$	$0{\cdot}1$	$0{\cdot}5$	$0{\cdot}8$	1	2	3	5	7	10
$\log_{10} x$										

(b) Sketch both curves on the one set of axes, choosing appropriate scales on the axes.

(c) Add the line $y = x$ to your graph. What transformation transforms the graph of $y = 10^x$ into the graph of $y = \log_{10} x$?

(d) What are the domain and range of $y = 10^x$ and $y = \log_{10} x$?

3. First rewrite each equation in index form and then solve it:

(a) $x = \log_2 16$
(b) $x = \log_3 \frac{1}{3}$
(c) $x = \log_5 25$
(d) $x = \log_{10} 0{\cdot}01$
(e) $x = \log_{\frac{1}{7}} 49$
(f) $x = \log_{\frac{1}{3}} 27$
(g) $x = \log_{\frac{2}{3}} \left(\frac{8}{27}\right)$
(h) $x = \log_{\frac{5}{2}} \left(\frac{4}{25}\right)$

4. First rewrite each equation in index form and then solve it:

(a) $\log_x 14 = 1$
(b) $\log_x 64 = 3$
(c) $\log_x 36 = 2$
(d) $\log_x 1000 = 3$
(e) $\log_x 25 = -2$
(f) $\log_x \frac{1}{8} = -3$
(g) $\log_x 27 = \frac{3}{2}$
(h) $\log_x 7 = \frac{1}{2}$

5. Use the identities $\log_b b^x = x$ and $b^{\log_b x} = x$ to simplify:

(a) $\log_2 2^3$
(b) $3^{\log_3 5}$
(c) $\log_7 7^{-2\frac{1}{2}}$
(d) $2{\cdot}71^{\log_{2\cdot71} 10}$
(e) $\theta^{\log_\theta 3\cdot5}$
(f) $\log_n n^{-2}$
(g) $t^{\log_t 2\pi}$
(h) $\log_{2c} (2c)^y$

6. Evaluate without the use of a calculator:

(a) $\log_2 32 - \log_2 128$
(b) $\log_3 \frac{1}{9} - \log_3 \frac{1}{3} + \log_3 1 - \log_3 3$
(c) $\log_{10} \sqrt{10} + \log_{10} \sqrt[3]{10}$
(d) $\log_5 125 - \log_5 \frac{1}{25} - \log_5 \sqrt{5}$

7. Simplify:

(a) $\log_6 3 + \log_6 2$
(b) $\log_5 100 - \log_5 4$
(c) $\log_2 2 - \log_2 3 + \log_2 6$,
(d) $\log_3 54 - \log_3 10 + \log_3 5$.

8. Solve each equation by converting both sides to a common base:

(a) $2^x = 2^{2x+3}$
(b) $2^{2x} = 16^{x-8}$
(c) $3^{\frac{x}{2}+1} = 1$
(d) $25^{\frac{x}{3}} = 5^{x+1}$
(e) $3 \times 2^x = 48$
(f) $\dfrac{6^x}{12} = 18$
(g) $4^{2x+3} = 8^{x+5}$
(h) $9^{4x-3} = 3^{x+7}$
(i) $5^{3x-4} = 25^{x-2}$

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9. Use the change of base formula and a calculator to evaluate to three significant figures:

(a) $\log_2 3$ (c) $\log_5 16$ (e) $\log_\pi \left(\frac{22}{7}\right)$
(b) $\log_{\sqrt{2}} 3$ (d) $\log_3 8$ (f) $\log_{\frac{2}{3}} 5$

DEVELOPMENT

10. Solve each pair of simultaneous equations by converting both sides to a common base:

(a) $2^{2x-y} = 32$, $2^{4x+y} = 128$ (b) $5^{x+y} = \frac{1}{5}$, $5^{3x+2y} = 1$ (c) $3^{x+y} = 81$, $81^{x-y} = 3$ (d) $7^{x+y} = 49$, $49^{x-y} = 7$

11. Use the formula to change the base of each logarithm to the base in the brackets, then evaluate:

(a) $\log_{36} 216$, (6) (c) $\log_8 32$, (2) (e) $\log_{\frac{1}{4}} 32$, (2) (g) $\log_{\sqrt{2}} 4$, (2)
(b) $\log_{125} 25$, (5) (d) $\log_{27} 81$, (3) (f) $\log_{\frac{1}{9}} 27$, (3) (h) $\log_{\sqrt[3]{3}} \left(3^{\frac{2}{5}}\right)$, (3)

12. Prove the first three logarithm results of Box 3 by putting $x = b^u$ and $y = b^v$.

13. Prove the last four logarithm results of Box 3 by rewriting them in index form.

14. (a) Use the change of base result to show that $\log_a b = \dfrac{1}{\log_b a}$.

(b) Hence evaluate, without the use of a calculator: (i) $\log_8 2$ (ii) $\log_{\sqrt{125}} 5$

(c) Using the change of base formula, prove:

(i) $\log_a b \times \log_b a = 1$ (ii) $\log_p q \times \log_q r \times \log_r p = 1$

15. (a) Use the change of base result to show that $\log_{a^x} b = \dfrac{\log_a b}{x}$.

(b) Hence evaluate, without the use of a calculator: (i) $\log_8 128$ (ii) $\log_{\sqrt{27}} 81$

(c) Solve for x the equation $\log_{\sqrt{a}}(x+2) - \log_{\sqrt{a}} 2 = \log_a x + \log_a 2$.

16. (a) Use the change of base result to show that $\log_{a^x} (b^x) = \log_a b$.

(b) Hence simplify: (i) $\log_{16} 81$ (ii) $\log_{\sqrt{3}} \sqrt{2}$ (iii) $\log_{\sqrt{27}} \sqrt{125}$

EXTENSION

17. (a) Show that $\log_{ab} x = \dfrac{\log_a x}{1 + \log_a b}$. (b) Hence show that $\log_2 5 = \dfrac{1 - \log_{10} 2}{\log_{10} 2}$.

(c) Use this result and a calculator to evaluate $\log_2 5$ to four significant figures.

18. Solve for x: (a) $\log_{2x} 216 = x$ (b) $\log_{5x} 3375 = x$ [HINT: Rewrite each as an equation in index form and then consider the prime factorisation of 216 and of 3375.]

12 B The Logarithmic Function and its Derivative

The principal purpose of this section is to prove that the derivative of the logarithmic function $y = \log_e x$ is the reciprocal function $y = \dfrac{1}{x}$, where the base e is an irrational number with approximate value $e \doteqdot 2{\cdot}7183$.

5 THE DERIVATIVE OF THE LOGARITHMIC FUNCTION: $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

The number e will be defined by a definite integral as follows:

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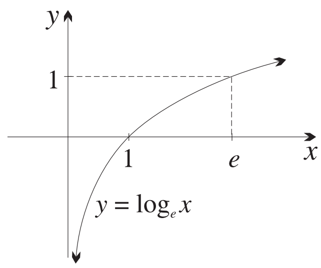
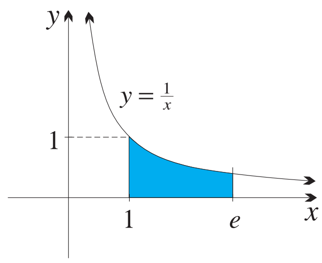
6 THE DEFINITION OF e: $\int_1^e \frac{1}{x}\,dx = 1$ (and $e \doteqdot 2{\cdot}7183$)

Graphed below on the left is the logarithmic function $y = \log_e x$. On the right is the reciprocal function $y = 1/x$, with the definite integral used to define e shaded.





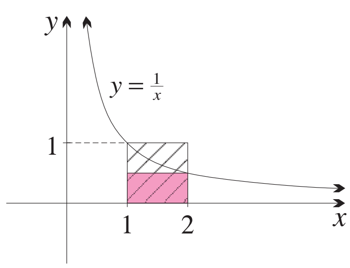
The six-step explanation that follows is rather indirect in that there is no direct attempt to differentiate any logarithmic function $y = \log x$, but instead the fundamental theorem of calculus is used to investigate the integral of the reciprocal function $y = 1/x$.

Step 1 — What is the Primitive of $1/x$: The reciprocal function $y = 1/x$ is obviously an important function which is required whenever two quantities are inversely proportional to each other. If we were to use the rule for integrating powers of x, however, we would get nonsense:

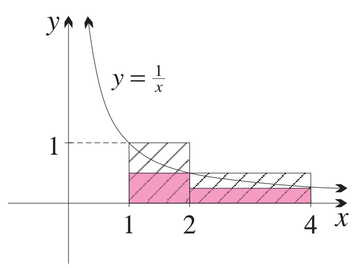
$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad \text{with } n = -1 \text{ gives} \qquad \int x^{-1}\,dx = \frac{x^0}{0}\,,$$

which is undefined because of the division by zero. Yet definite integrals involving the reciprocal function are clearly well defined, provided that the integral does not cross the discontinuity at $x = 0$.

For example, here are diagrams of $\int_1^2 \frac{1}{x}\,dx$ and $\int_1^4 \frac{1}{x}\,dx$. Some upper and lower rectangles have been drawn to establish rough bounds for these integrals.



$$\tfrac{1}{2} < \int_1^2 \frac{1}{x}\,dx < 1$$



$$1 < \int_1^4 \frac{1}{x}\,dx < 2$$

So definite integrals of $y = 1/x$ clearly exist, but we don't yet know how to find their values. A useful procedure when this sort of thing happens is to give a name to the object we want to study, and then examine some of its properties.

Step 2 — A Function Defined by a Definite Integral: Define the function $L(x)$, for all $x > 0$, by the formula

$$L(x) = \int_1^x \frac{1}{t}\,dt.$$

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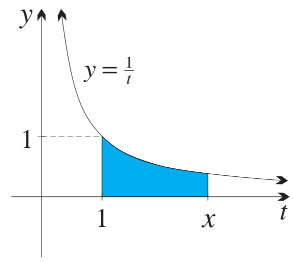


Notice that x is the variable in the function $L(x)$, and so has to be used as a bound of the definite integral. Consequently, as was done in the proof of the fundamental theorem of calculus in Chapter 11, the variable in the integrand has been changed to t (the variable is called a *dummy variable*, because it disappears before any final result). Notice also that the variable x must be positive, because it is not possible to integrate across the asymptote at $x = 0$.

Step 3 — Three Properties of the Function $L(x)$: Three properties follow quickly from the definition of $L(x)$. These will allow us to make a preliminary sketch of $L(x)$.

FIRST PROPERTY: Since the curve $y = \dfrac{1}{t}$ is always above the x-axis for $t > 0$,

$$\begin{cases} L(x) > 0, & \text{for } x > 1, \\ L(x) = 0, & \text{for } x = 1, \\ L(x) < 0, & \text{for } 0 < x < 1. \end{cases}$$

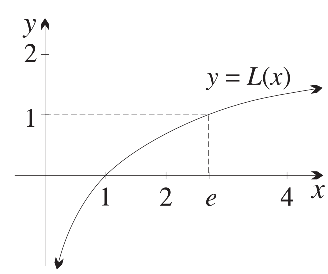
SECOND PROPERTY: The fundamental theorem of calculus says that for any function $f(x)$, $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$. Applying this to the function $L(x)$ gives the result

$$L'(x) = \frac{1}{x}\,, \quad \text{for all } x > 0.$$

THIRD PROPERTY: From the two diagrams in Step 1,

$$\tfrac{1}{2} < L(2) < 1 \qquad \text{and} \qquad 1 < L(4) < 2.$$

A PRELIMINARY SKETCH OF $L(x)$: On the right the function $y = L(x)$ is sketched using the information we have gained so far.



First, the curve is above the x-axis for $x > 0$, and below the x-axis for $0 < x < 1$.

Secondly, the gradient of $y = L(x)$ is $\dfrac{dy}{dx} = \dfrac{1}{x}$, which means that the gradient is always positive, but becomes ever flatter as $x \to \infty$, and ever steeper as $x \to 0^+$.

Thirdly, $L(x)$ reaches the value 1 between $x = 2$ and $x = 4$. Accordingly, we shall define a new number e by $L(e) = 1$.

Step 4 — The Definition of e: Already it is clear that the graph of $y = L(x)$ looks very like a logarithmic function. The next step is to establish its base, and the key to finding the base is the fact that with any logarithmic function, the log of the base is exactly 1:

$$\log_b b = 1, \quad \text{for all positive bases } b \neq 1,$$

so if $L(x)$ is going to be a log function, its base has to be the number e such that $L(e) = 1$, as is indicated on the previous graph.

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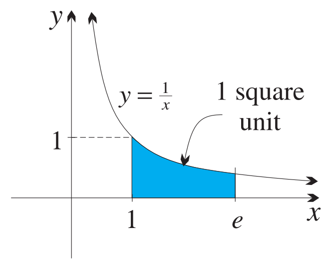

DEFINITION: Define e to be the positive real number such that

$$\int_1^e \frac{1}{x}\,dx = 1.$$

This definition works because $L(x)$ reaches the value 1 somewhere between 2 and 4. Also, $L(x)$ is always increasing, and hence it cannot take the value 1 at more than one place.

Hence e is a well-defined real number between 2 and 4. Some later, more sophisticated calculations will justify the approximation given by the calculator, $e \doteqdot 2{\cdot}718\,281\,28$. It is also possible to prove that e is an irrational number (see for example question 7(b) of the 1993 HSC 4 Unit paper).

Step 5 — A Characteristic Property of Logarithmic Functions: This step and the next are difficult, but they establish that $L(x)$ is indeed the function $y = \log_e x$. First, we show that $L(x)$ satisfies a characteristic property of logarithmic functions.

FOURTH PROPERTY: $L(x^a) = aL(x)$, for all real a and for all $x > 0$.

PROOF: In part A we prove that LHS and RHS have the same derivative and so must differ by a constant. In part B we prove that this constant is zero, so that LHS and RHS must be equal.

A. $$\frac{d}{dx}(\text{RHS}) = aL'(x)$$
$$= a \times \frac{1}{x}, \quad \text{by the second property.}$$

Using the chain rule:

$$\text{LHS} = L(x^a)$$
$$\frac{d}{dx}(\text{LHS}) = \frac{1}{x^a} \times ax^{a-1}$$
$$= a \times \frac{1}{x}.$$

Let $u = x^a$,
then $\text{LHS} = L(u)$.
So $\dfrac{du}{dx} = ax^{a-1}$
and $\dfrac{d}{du}(\text{LHS}) = \dfrac{1}{u}$.

B. Because RHS and LHS have the same derivative,

$$L(x^a) = aL(x) + C, \quad \text{for some constant } C.$$

Substituting $x = 1$, $L(1) = aL(1) + C$,

and since we already know that $L(1) = 0$, it follows that $C = 0$, as required.

Step 6 — $L(x)$ is the Logarithmic Function with Base e: We can now prove the main theorem of this section. The proof is short, but it relies on the fourth property above. It also relies on log and exponential functions being inverse functions, and in particular uses the crucial idea that every positive number x can be expressed as a power of e:

$$x = e^{\log_e x}, \quad \text{for all } x > 0.$$

THEOREM: $$\int_1^x \frac{1}{t}\,dt = \log_e x$$

PROOF: We have to prove that $L(x) = \log_e x$.

$$L(x) = L(e^{\log_e x}), \quad \text{using the remark above,}$$
$$= (\log_e x)L(e), \quad \text{using the fourth property above,}$$
$$= \log_e x, \quad \text{since } L(e) = 1 \text{ by the definition of } e.$$

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COROLLARY: Since $L'(x) = \dfrac{1}{x}$, it follows now that $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$.

NOTE: Careful readers may see difficulties with this presentation, and particularly with the final proof, in that powers have only really been properly defined for rational indices rather than for real indices, and therefore a serious question remains about whether it is possible to take logarithms of arbitrary real numbers.

A standard way around this problem in more advanced treatments is to *define* the logarithmic function to be the function $L(x)$, and then prove the various log laws with the techniques used to prove the fourth property. Powers of e can then be defined as the inverse function of the logarithmic function, and one then has to prove that for rational indices this definition of powers agrees with the earlier definition in terms of roots. In this way the existence, the continuity and the differentiability of the logarithmic function are all placed beyond question.

The Logarithmic Function: The function $y = \log_e x$ is called *the logarithmic function*, in contrast with logarithmic functions with other bases like 2, 3 or 10. As far as calculus is concerned, it is the basic log function, and is often written simply as $\log x$, so that if no base is given, base e will from now on be implied. It is also written as $\ln x$, the 'n' standing for 'natural' logarithms, or for 'Napierian' logarithms in honour of the Scottish mathematician John Napier (1550–1617), who invented tables of logarithms base e for calculations (first published in 1614). The graph of $y = \log x$ was sketched at the start of this section — it should be regarded as one of most important curves in the course.

Be careful of the different convention used on calculators, where $\log x$ stands for $\log_{10} x$.

7 NOTATION: $\log_e x$, $\log x$ and $\ln x$ all mean the same thing, except on calculators, where $\log x$ usually means $\log_{10} x$.

Differentiating Functions Involving the Logarithmic Function: The basic standard form is $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$. The following examples use this in combination with the logarithmic laws and the chain, product and quotient rules.

WORKED EXERCISE: Differentiate: (a) $3\log x$ (b) $\log 7x^2$ (c) $\log(ax+b)$

SOLUTION:

(a) Since $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$, it follows that $\dfrac{d}{dx}(3\log x) = \dfrac{3}{x}$.

(b) Let $y = \log 7x^2$.
Using the log laws, $y = \log 7 + 2\log x$,
so $\dfrac{dy}{dx} = \dfrac{2}{x}$ (notice that $\log 7$ is a constant).

(c) For $y = \log(ax+b)$,
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}, \text{ by the chain rule,}$$
$$= \frac{a}{ax+b}.$$

Let $u = ax + b$,
then $y = \log u$.
So $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \dfrac{1}{u}$.

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WORKED EXERCISE: Using the chain and product rules, differentiate:

(a) $\log \log x$ (b) $x^3 \log x$

SOLUTION:

(a) Let $y = \log \log x$.

Then $\dfrac{dy}{dx} = \dfrac{du}{dx} \times \dfrac{dy}{du}$, by the chain rule,

$= \dfrac{1}{x} \times \dfrac{1}{\log x}$

$= \dfrac{1}{x \log x}$.

Let $u = \log x$,
then $y = \log u$.
So $\dfrac{du}{dx} = \dfrac{1}{x}$
and $\dfrac{dy}{du} = \dfrac{1}{u}$.

(b) Let $y = x^3 \log x$. Then, by the product rule with $u = x^3$ and $v = \log x$,

$$\frac{dy}{dx} = 3x^2 \log x + x^3 \times \frac{1}{x}$$
$$= x^2(1 + 3\log x).$$

Standard Forms for Differentiation: It is convenient to write down two further standard forms for differentiation. The second form below was proven in part (c) of the first worked exercise above. The third is the general chain rule extension.

8

STANDARD FORMS FOR DIFFERENTIATION:

A. $\dfrac{d}{dx} \log_e x = \dfrac{1}{x}$

B. $\dfrac{d}{dx} \log_e (ax + b) = \dfrac{a}{ax + b}$

C. $\dfrac{d}{dx} \log_e u = \dfrac{1}{u} \times \dfrac{du}{dx}$ OR $\dfrac{d}{dx} \log_e f(x) = \dfrac{f'(x)}{f(x)}$

WORKED EXERCISE: Differentiate: (a) $\log(4x - 9)$ (b) $\log(4 + x^2)$

SOLUTION:

(a) $\dfrac{d}{dx} \log(4x - 9) = \dfrac{4}{4x - 9}$ (second standard form with $ax + b = 4x - 9$).

(b) $\dfrac{d}{dx} \log(4 + x^2) = \dfrac{2x}{4 + x^2}$ (third standard form with $u = 4 + x^2$).

Using the Log Laws to Make Differentiation Easier: The following example shows the use of the log laws to avoid a combination of the chain and quotient rules.

WORKED EXERCISE: Differentiate $\log \dfrac{(1 + x)^2}{(1 - x)^2}$.

SOLUTION: Let $y = \log \dfrac{(1 + x)^2}{(1 - x)^2}$.

Then
$$y = \log(1 + x)^2 - \log(1 - x)^2$$
$$= 2\log(1 + x) - 2\log(1 - x),$$

so
$$\frac{dy}{dx} = \frac{2}{1 + x} + \frac{2}{1 - x}$$
$$= \frac{4}{1 - x^2}.$$

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Logarithmic Functions to Other Bases: All other logarithmic functions can be expressed in terms of *the logarithmic function* by the change of base formula, for example,

$$\log_2 x = \frac{\log x}{\log 2},$$

and so every other logarithmic function is just a constant multiple of $\log x$. This allows any other logarithmic function to be differentiated easily.

9 DIFFERENTIATING A LOGARITHMIC FUNCTIONS WITH ANOTHER BASE:
Use the change of base formula to write it as a multiple of $\log x$.

WORKED EXERCISE: Find the derivative of $y = \log_b x$.

SOLUTION: Let $y = \log_b x$.

Then $y = \dfrac{\log_e x}{\log_e b}$

so $\dfrac{dy}{dx} = \dfrac{1}{x \log_e b}$.

A Characterisation of the Logarithmic Function: Since the derivative of $f(x) = \log x$ is $f'(x) = 1/x$, substitution of $x = 1$ shows that the tangent at the x-intercept has gradient exactly 1. This property characterises the logarithmic function amongst all other logarithmic functions.

10 THE GRADIENT AT THE x-INTERCEPT: The function $y = \log x$ is the only logarithmic function whose gradient at the x-intercept is exactly 1.

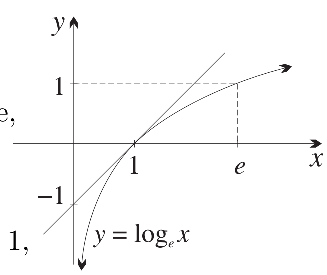
PROOF: Let $f(x) = \log_b x$ be any other logarithmic function,

then $f(1) = 0$, and so the x-intercept is at $x = 1$.

Also $f'(x) = \dfrac{1}{x \log_e b}$, by the previous worked exercise,

and so $f'(1) = \dfrac{1}{\log_e b}$.

Hence the gradient at the x-intercept is 1 if and only if $\log_e b = 1$, that is, if and only if the base b is equal to e.



Extension — The Log Laws and Implicit Differentiation: The log laws and implicit differentiation can be combined to differentiate complicated algebraic functions.

WORKED EXERCISE: Use implicit differentiation to differentiate $y = \sqrt{\dfrac{x-1}{x+1}}$.

SOLUTION: Taking logs of both sides, $\log y = \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x+1)$.

Differentiating with respect to x, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

$$= \frac{1}{(x-1)(x+1)}.$$

$\boxed{\times\, y}$ $\dfrac{dy}{dx} = \dfrac{1}{(x-1)^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}$.

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Exercise **12B**

NOTE: Remember that $\log x$ and $\ln x$ both mean $\log_e x$ (except on the calculator, where $\log x$ means $\log_{10} x$).

1. Differentiate, using the log laws:

(a) $\log 3x$ (b) $\log_e 7x$ (c) $2\log x$ (d) $x + 4\log x$ (e) $\log x + \pi$ (f) $\ln \frac{x}{2}$ (g) $3\log 5x$ (h) $4x^3 - \ln \frac{4}{3}x$

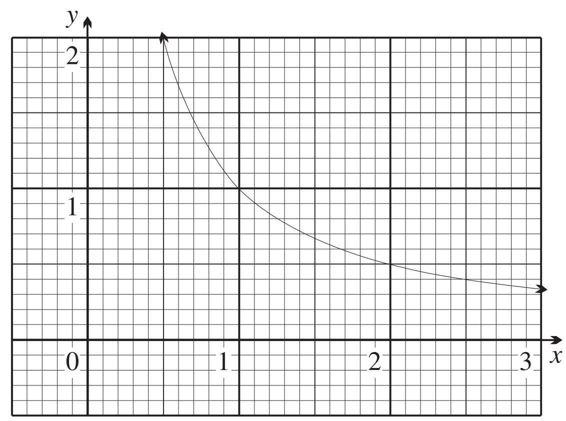
2. Differentiate, using the standard form $\dfrac{d}{dx}\log_e(ax+b) = \dfrac{a}{ax+b}$:

(a) $\log(2x+5)$ (b) $\ln(3x-7)$ (c) $\log(3+2x)$ (d) $\log(4-x)$ (e) $\log_e(4+7x)$ (f) $\log(2-5x)$ (g) $x - \ln(1-x)$ (h) $\log_e(-ex+3e)$ (i) $\log(ax-b)$ (j) $\ln(\pi x+1)$ (k) $\log(1-\frac{\pi}{2}x)$ (l) $\log_e(a-\frac{2x}{b})$

3. In Step 4 of the development of the logarithmic function, e was defined by

$$\int_1^e \frac{1}{x}\,dx = 1.$$

This question uses this definition to estimate e from a graph of $y = 1/x$. The diagram to the right shows the graph of $y = 1/x$ from $x = 0$ to $x = 3$, drawn with a scale of 10 little divisions to 1 unit, so that 100 little squares make 1 square unit. Count the squares in the column from $x = 1{\cdot}0$ to $1{\cdot}1$, then the squares in the column from $x = 1{\cdot}1$ to $1{\cdot}2$, and so on. Continue until the number of squares equals 100 — the x-value at this point will be an estimate of e.



4. The notes established the further result that $\log x = \displaystyle\int_1^x \frac{1}{t}\,dt$. By counting squares, find estimates of $\log x$ for the values of x in the table below, then sketch the graph of $y = \log x$.
[HINT: For values of x less than 1, the integral runs backwards, and so will be negative.]

x	0·5	0·6	0·8	1	1·2	1·4	1·6	1·8	2	2·2	2·4	2·6	2·8	3

5. Simplify these expresions involving logarithms to the base e:

(a) $e\log e$ (b) $\frac{1}{e}\ln\frac{1}{e}$ (c) $3\log_e e^2$ (d) $\ln\sqrt{e}$ (e) $e\log e^3 - e\log e$ (f) $\log_e e + \log_e \frac{1}{e}$ (g) $\log e^e$ (h) $\log(\log e^e)$ (i) $\log(\log(\log e^e))$

6. Solve: (a) $\ln(x^2+5x) = 2\ln(x+1)$ (b) $\log(7x-12) = 2\log x$.

7. Use the chain rule to differentiate:

(a) $\log(x^2+1)$ (b) $\log(x^2+3x+2)$ (c) $\ln(2-x^2)$ (d) $\log_e(1+2\sqrt{x})$

8. Use the logarithm laws to help differentiate:

(a) $\log 7x^2$ (b) $\log 5x^3$ (c) $\log\sqrt{x}$ (d) $\log\dfrac{3}{x}$ (e) $\log_e\sqrt{2+x}$ (f) $\ln\left(\dfrac{1+x}{1-x}\right)$ (g) $\log\sqrt[3]{x+1}$ (h) $\log(x\sqrt{x+1})$

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9. Use the change of base formula to express these to base e, then differentiate them:
(a) $\log_2 x$ (b) $\log_{10} x$ (c) $\log_2 5x$ (d) $5\log_3 7x$

10. Differentiate these functions using the product rule:
(a) $x\log x$ (b) $x\log(2x+1)$ (c) $(2x+1)\log x$ (d) $\sqrt{x}\log x$

11. Find the equation of the tangent to $y = \log x$ at the point where $x = e^2$.

12. Find the equation of the normal to $y = \log x$ at the point where $x = \dfrac{1}{e}$. What is its x-intercept?

DEVELOPMENT

13. Use the logarithm laws to simplify the following, then differentiate them:
(a) $\ln\left(\dfrac{\sqrt{x-1}}{x^2+1}\right)$ (b) $\log\left((x^2-2x)\sqrt{x}\right)$ (c) $\log_e \pi^x$

14. Differentiate the following, using the chain, product and quotient rules and the logarithm laws:
(a) $x^2\log x$ (d) $(\log x)^4$ (g) $(2\log x - 3)^4$ (j) $\log_x 3$
(b) $\dfrac{\log x}{x}$ (e) $\dfrac{1}{1+\log x}$ (h) $\dfrac{1}{\log x}$ (k) $\dfrac{x}{\log x}$
(c) $(\log x)^2$ (f) $\sqrt{\log x}$ (i) $\log(\log x)$ (l) $\log_x 3^x$

15. Find the point(s) where the tangent to each of these curves is horizontal:
(a) $y = x\log x$ (b) $y = \dfrac{1}{x} + \log x$ (c) $y = x^2\log\dfrac{1}{x}$

16. (a) Show that the gradient of $y = \log x$ at $x = 1$ is equal to 1.
(b) Find the value of the derivative of $y = \log x$ at $x = 1$ by first principles, using the formula $f'(1) = \lim\limits_{x\to 1}\dfrac{f(x)-f(1)}{x-1}$, and hence show that $\lim\limits_{x\to 1}\dfrac{\log x}{x-1} = 1$.

17. For what value of x does the tangent to $y = \log_{10} x$ have gradient 1 ?

18. Differentiate the following using any appropriate technique. Use the logarithm laws whenever possible.
(a) $\log(2x^2-3x)$ (c) $\log(1+\log x)$ (e) $(x^2+2x)\log\sqrt{x-2}$
(b) $\log\left(\dfrac{x-1-2x^2}{5}\right)$ (d) $\log(x^{\frac{1}{2}}+\log x)$ (f) $\log\left(\dfrac{(x-3)^4\sqrt{x}}{x+1}\right)$

19. (a) Show that $y = \dfrac{x}{\log x}$ is a solution of the equation $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right) - \left(\dfrac{y}{x}\right)^2$.
(b) Show that $y = \log(\log x)$ is a solution of the equation $x\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 + \dfrac{dy}{dx} = 0$.

EXTENSION

20. Use logarithms to help find the derivatives of:
(a) $y = \dfrac{(x+1)\sqrt{x-1}}{x+2}$ (c) $y = \dfrac{x^2\sqrt{x+1}}{\sqrt{x-1}}$ (e) $y = \left(\dfrac{x^2+1}{x}\right)^{\frac{1}{\pi}}$
(b) $y = \dfrac{(x-1)^3(x+2)^2}{(x-3)^4}$ (d) $y = \dfrac{\sqrt{x}(x-1)^2}{x+1}$ (f) $y = \sqrt{x}\sqrt{x+1}\sqrt{x+2}$

21. Take logarithms of both sides and use the log laws to differentiate:
(a) $y = x^x$ (b) $y = x^{\log x}$ (c) $y = x^{\frac{1}{x}}$

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22. It can be shown (with some considerable difficulty) that the continued fraction on the right approaches the value $e-1$. With the help of a calculator, use this continued fraction to find a rational approximation for e that is accurate to four significant figures.

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cdots}}}}}}}}$$

23. (a) If $y=\log x$, use differentiation by first principles to show that $y'=\lim_{h\to 0}\log\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$.

(b) Use the fact that $y'=\dfrac{1}{x}$ to show that $\lim_{h\to 0}\log\left(1+\frac{h}{x}\right)^{\frac{1}{h}}=\frac{1}{x}$.

(c) Substitute $n=\frac{1}{h}$ and $u=\frac{1}{x}$ to prove these two important limits:

(i) $\displaystyle\lim_{n\to\infty}\left(1+\frac{u}{n}\right)^n=e^u$ (ii) $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

(d) Investigate how quickly $(1+\frac{1}{n})^n$ converges to e by using your calculator with the following values of n: (i) 1 (ii) 10 (iii) 100 (iv) 1000 (v) 10 000

12 C Applications of Differentiation

Differentiation can now be used in the usual way to study tangents, turning points and inflexions of functions involving the logarithmic function. Systematic sketching of such curves, however, will need the special limits developed after the first worked example.

The Geometry of Tangents and Normals: The following example illustrates how to investigate the geometry of tangents and normals to a curve.

WORKED EXERCISE: (a) Find the point $T(a,\log a)$ on $y=\log x$ where the tangent passes through the origin. (b) Find the area of the triangle formed by the y-axis and the tangent and the normal at T.

SOLUTION:

(a) Differentiating, $\dfrac{dy}{dx}=\dfrac{1}{x}$,

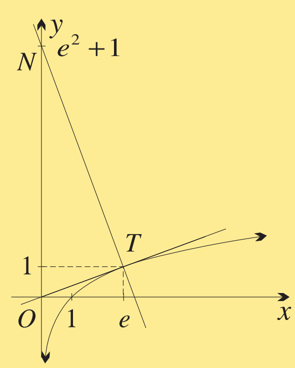
so the tangent at $T(a,\log a)$ has gradient $\dfrac{1}{a}$,

and the tangent is $y-\log a=\dfrac{1}{a}(x-a)$.

The tangent passes through $(0,0)$ if and only if

$$-\log a=\frac{1}{a}\times(-a),$$
$$a=e,$$

so T is the point $(e,1)$.



(b) The normal at T has gradient $-e$, so its equation is $y-1=-e(x-e)$, and its y-intercept N is therefore $(0,e^2+1)$. So the base ON of $\triangle ONT$ is (e^2+1), and its altitude is e. Hence the triangle has area $\frac{1}{2}e(e^2+1)$ square units.

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Two Special Limits — x Dominates $\log x$: Curve sketching involving $\log x$ requires knowledge of some special limits that arise from clashes between x and $\log x$. For example, we need to know what happens to $x \log x$ as $x \to 0^+$. On the one hand, x is approaching zero, but on the other hand, $\log x$ is diverging to $-\infty$, so what does their product do?

The answer is that x *dominates* $\log x$. To put it more colourfully, 'in a battle between x and $\log x$, x always wins'.

11 THE FUNCTION x DOMINATES THE FUNCTION $\log x$:

$$\lim_{x\to\infty} \frac{\log x}{x} = 0 \qquad \text{and} \qquad \lim_{x\to 0^+} x \log x = 0$$

PROOF: These proofs are not easy. The first limit is proven below by an interesting geometric method, then the second limit follows using a substitution.

A. Consider the definite integral $\displaystyle\int_1^{\sqrt{x}} \frac{1}{t}\,dt$ sketched below, where $x > 1$.

It is clear from the diagram that

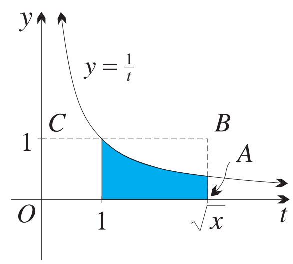
$$0 < \int_1^{\sqrt{x}} \frac{1}{t}\,dt < \text{area } ABCO,$$

$$0 < \Big[\log t\Big]_1^{\sqrt{x}} < \sqrt{x}$$

$$0 < \log\sqrt{x} - \log 1 < \sqrt{x}$$

$$0 < \tfrac{1}{2}\log x < \sqrt{x}$$

$$\boxed{\times \tfrac{2}{x}} \quad 0 < \frac{\log x}{x} < \frac{2}{\sqrt{x}}\,.$$



But $\dfrac{2}{\sqrt{x}} \to 0$ as $x \to \infty$, so by the sandwiching principle, $\displaystyle\lim_{x\to\infty} \frac{\log x}{x} = 0$.

B. Substitute $u = \dfrac{1}{x}$ into the limit proven in part (a).

Since $\log u = -\log x$, and $u \to 0^+$ as $x \to \infty$, $\displaystyle\lim_{u\to 0^+}(-u\log u) = 0$.

Then replacing u by x, and taking opposites, $\displaystyle\lim_{x\to 0^+} x\log x = 0$.

Two More General Limits: A more general version of this result is that x^k dominates $\log x$ for all $k > 0$ (proven in the Extension of the following Exercise 12C).

12 THE FUNCTION x^k DOMINATES THE FUNCTION $\log x$ FOR $k > 0$:

$$\lim_{x\to\infty} \frac{\log x}{x^k} = 0 \qquad \text{and} \qquad \lim_{x\to 0^+} x^k \log x = 0.$$

An Example of Curve Sketching: Here are the six steps of the curve sketching menu applied to $y = x\log x$. Notice in Step 5 the use of the derivative not only to find the turning point, but also to analyse the gradient of the curve near the boundary of the domain.

1. The domain is $x > 0$, because $\log x$ is undefined for $x \le 0$.
2. The domain is unsymmetric, so the function is neither even nor odd.

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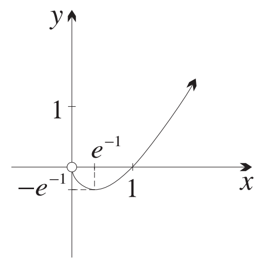
3. The only zero is at $x = 1$, and the curve is continuous for $x > 0$:

x	0	$1/e$	1	e
y	$*$	$-1/e$	0	e
sign	$*$	$-$	0	$+$



4. Since x dominates $\log x$, $y \to 0$ as $x \to 0^+$.
   Also $y \to \infty$ as $x \to \infty$.

5. Differentiating by the product rule, $f'(x) = \log x + 1$,
   $$f''(x) = \frac{1}{x},$$
   so $f'(x) = 0$ when $x = 1/e$, and $f''(1/e) = e > 0$,
   hence $(1/e, -1/e)$ is a minimum turning point.
   Also, $f'(x) \to -\infty$ as $x \to 0^+$, so the curve becomes vertical near the origin.

6. Since $f''(x)$ is always positive, there are no inflexions, and the curve is always concave up.

Exercise 12C

1. Use your knowledge of transformations to help sketch the graphs of the given functions:
(a) $y = \log(x + 1)$ (c) $y = -\log 3x$ (e) $y = \log\left(\frac{x}{2}\right)$
(b) $y = \log(-x)$ (d) $y = \log(x - 2)$ (f) $y = \log|x|$

2. (a) Write down the domain of $y = \log(1 + x^2)$.
(b) Show that $y' = \dfrac{2x}{1 + x^2}$ and $y'' = \dfrac{2(1 - x^2)}{(1 + x^2)^2}$.
(c) Hence show that $y = \log(1 + x^2)$ has one stationary point, and determine its nature.
(d) Find the coordinates of the two points of inflexion.
(e) Hence sketch the curve, and then write down its range.
(f) Hence sketch $y = \log\left(\dfrac{1 + x^2}{2}\right)$. [HINT: You will need to use the logarithm laws.]

3. (a) Find the domain of $y = (\log x)^2$.
(b) Find y' and y'', and hence show that the curve has as an inflexion at $x = e$.
(c) Classify the stationary point at $x = 1$, sketch the curve, and write down the range.

4. (a) Determine the first two derivatives of $y = x - \log x$.
(b) Deduce that the curve is concave up for all values of x in its natural domain.
(c) Find the minimum turning point. (d) Sketch the curve and write down its range.

5. (a) Write down the domain of $y = \dfrac{1}{x} + \log x$. (b) Find the first and second derivatives.
(c) Show that the curve has a minimum at $(1, 1)$ and an inflexion at $(2, \frac{1}{2} + \log 2)$.
(d) Sketch the graph and write down its range.

6. (a) Write down the domain of $y = \dfrac{\log x}{x}$, then find any horizontal or vertical asymptotes.
(b) Find y' and y''. (c) Find any stationary points and determine their nature.
(d) Find the exact coordinates of the lone point of inflexion.
(e) Sketch the curve, and write down its range.

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DEVELOPMENT

7. (a) Write down the equation of the tangent to $y = 2\log x$ at the point where $x = c$. Hence find any values of c for which the tangent passes through the origin.
(b) Repeat part (a) for these curves: (i) $y = (\log x)^2$ (ii) $y = x^2 \log x$

8. (a) Show that the equation of the tangent to $y = x\log x$ at the point $x = e$ is $y = 2x - e$.
(b) Find the distance from this tangent to the origin.

9. (a) Find the equation of the tangent to $y = (\log x)^2$ at the point where $x = t$.
(b) Find the area of the triangle cut off by this tangent and the coordinate axes when $t = e$.

10. Investigate the curve $y = -x\log x$ as follows, then sketch it and write down its range.
(a) Find its domain and any intercepts.
(b) Find and classify any stationary points.
(c) Examine the behaviour of y and y' as $x \to 0^+$.

11. (a) Find and classify the lone stationary point of $y = x^2 \log x$ in its natural domain.
(b) Show that there is an inflexion at $x = e^{-\frac{3}{2}}$.
(c) Examine the behaviour of y and y' as $x \to 0^+$.
(d) Hence sketch the graph of this function, then write down its range.

12. Carefully classify the critical points of $y = \dfrac{x}{\log x}$ and show that there is an inflexion at $(e^2, \frac{1}{2}e^2)$. Examine the behaviour of y and y' as $x \to 0^+$ and as $x \to \infty$, then sketch the curve and write down its range.

13. (a) Write down the domain of $y = \log\left(\dfrac{x^2}{x+1}\right)$.
(b) Show that $y' = \dfrac{x+2}{x(x+1)}$.
(c) Show that $y' = 0$ at $x = -2$. Explain why there is no stationary point there.
(d) How many inflexion points does this curve have? (e) Sketch the graph.

14. (a) What is the natural domain of $y = \log(\log x)$? (b) What is the x-intercept?
(c) Find y' and y'', and explain why there are no stationary points.
(d) Confirm that $y''(\frac{1}{e}) = 0$, and explain why there is no inflexion there.
(e) Sketch the curve.

15. (a) Given that $\log ax = \log x + C$, what is the value of C?
(b) Hence show that the gradient of $y = \log x$ is everywhere the same as $y = \log ax$ and explain this in terms of enlargements and translations.
(c) Do likewise for $y = \log_b ax$, where a, b and x are all positive.

16. (a) Find the gradient of the tangent to $y = \log(1 + x^2)$ at the point where $x = c$.
(b) The tangent at another point (x, y) is perpendicular to the one found in part (a). Show that $x^2 + \dfrac{4cx}{1+c^2} + 1 = 0$.
(c) Show that $\Delta = -4\left(\dfrac{1-c^2}{1+c^2}\right)^2$ for this quadratic, and hence that the only tangents to $y = \log(1 + x^2)$ which are mutually perpendicular are those at $x = -1$ and $x = 1$.

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EXTENSION

17. Use the results $\displaystyle\lim_{u\to\infty}\frac{\log u}{u}=0$ and $\displaystyle\lim_{u\to 0^+} u\log u=0$, and the substitution $u=x^k$ where $k>0$, to prove the two further limits in Box 12 of the notes above.

18. Show that $y=x^{\frac{1}{\log x}}$ is a constant function and find the value of this constant. What is the natural domain of this function? Sketch its graph.

19. (a) Differentiate $y=x^x$ by taking logs of both sides. Then examine the behaviour of $y=x^x$ near $x=0$, and show that the curve becomes vertical as $x\to 0^+$.
(b) Locate and classify any stationary points, and where the curve has gradient 1.
(c) Sketch the function.

20. (a) Find the limits of $y=x^{\frac{1}{x}}$ as $x\to 0^+$ and as $x\to\infty$.
(b) Show that there is a maximum turning point when $x=e$.
(c) Show that $y=x^x$ and $y=x^{\frac{1}{x}}$ have a common tangent at $x=1$.
(d) Sketch the graph of the function.

12 D Integration of the Reciprocal Function

Integration of the Reciprocal Function: Since $\log x$ has derivative $1/x$, it follows that $\log x$ is a primitive of $1/x$, provided that x remains positive so that $\log x$ is defined. This gives a new standard form for integration, with the following three versions (omitting constants of integration).

13 STANDARD FORMS FOR INTEGRATION:

A. $\displaystyle\int\frac{1}{x}\,dx=\log x,$ provided that $x>0$

B. $\displaystyle\int\frac{1}{ax+b}\,dx=\frac{1}{a}\log(ax+b),$ provided that $ax+b>0$

C. $\displaystyle\int\frac{1}{u}\frac{du}{dx}\,dx=\log u,$ provided that $u>0$ OR $\displaystyle\int\frac{f'(x)}{f(x)}\,dx=\log f(x)$

WORKED EXERCISE: Evaluate: (a) $\displaystyle\int_e^{e^2}\frac{1}{x}\,dx$ (b) $\displaystyle\int_0^1\frac{1}{2x+1}\,dx$ (c) $\displaystyle\int_0^{\frac{1}{2}}\frac{x}{1-x^2}\,dx$

SOLUTION:

(a) $$\begin{aligned}\int_e^{e^2}\frac{1}{x}\,dx&=\Big[\log x\Big]_e^{e^2}\\&=\log e^2-\log e\\&=1\end{aligned}$$

(b) $$\begin{aligned}\int_0^1\frac{1}{2x+1}\,dx&=\tfrac{1}{2}\Big[\log(2x+1)\Big]_0^1\\&=\tfrac{1}{2}(\log 3-\log 1)\\&=\tfrac{1}{2}\log 3\end{aligned}$$

(c) $$\begin{aligned}\int_0^{\frac{1}{2}}\frac{x}{1-x^2}\,dx&=-\tfrac{1}{2}\int_0^{\frac{1}{2}}\frac{-2x}{1-x^2}\,dx\\&=-\tfrac{1}{2}\Big[\log(1-x^2)\Big]_0^{\frac{1}{2}}\\&=-\tfrac{1}{2}(\log\tfrac{3}{4}-\log 1)\\&=\log 2-\tfrac{1}{2}\log 3\end{aligned}$$

Let $u=1-x^2$.
Then $\dfrac{du}{dx}=-2x$.
$\displaystyle\int\frac{1}{u}\frac{du}{dx}\,dx=\log u$

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WORKED EXERCISE: Find $f(x)$, if $f'(x) = \dfrac{3}{4x-1}$, and $f(1) = 0$.

SOLUTION: Integrating, $f(x) = \frac{3}{4}\log(4x-1) + C$, provided that $x > \frac{1}{4}$.

Substituting $x = 1$, $0 = \frac{3}{4}\log 3 + C$,

so $f(x) = \frac{3}{4}\log(4x-1) - \frac{3}{4}\log 3$

$f(x) = \frac{3}{4}\log\frac{1}{3}(4x-1)$.

Given a Derivative, Find an Integral: Our theory so far has not yielded a primitive of $\log x$, but the following exercise shows how the primitive of $\log x$ can be obtained.

WORKED EXERCISE: Differentiate $x\log x$, and hence find:

(a) $\displaystyle\int_1^e \log x\,dx$ (b) $\displaystyle\int_1^2 \log_2 x\,dx$

SOLUTION: First, $\dfrac{d}{dx}(x\log x) = \log x + x \times \dfrac{1}{x}$, by the product rule,

that is, $\dfrac{d}{dx}(x\log x) = 1 + \log x$.

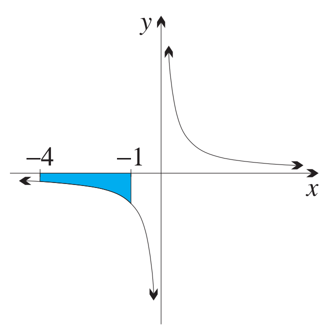
Reversing this, $\displaystyle\int (1+\log x)\,dx = x\log x + C$, for some constant C,

$$\int \log x\,dx = x\log x - x + C.$$

(a) Hence
$$\begin{aligned}\int_1^e \log x\,dx &= \Big[x\log x - x\Big]_1^e \\ &= (e\log e - e) - (1\log 1 - 1) \\ &= (e-e) - (0-1) \\ &= 1.\end{aligned}$$

(b) Also,
$$\begin{aligned}\int_1^2 \log_2 x\,dx &= \int_1^2 \frac{\log x}{\log 2}\,dx \\ &= \frac{1}{\log 2}\Big[x\log x - x\Big]_1^2 \\ &= \frac{1}{\log 2}(2\log 2 - 2 - 1\log 1 + 1) \\ &= \frac{2\log 2 - 1}{\log 2}.\end{aligned}$$

Extension — The Primitive of $y = 1/x$ on Both Sides of the Origin: The graph of the function $y = 1/x$ is a hyperbola, with two disconnected branches separated by the discontinuity at $x = 0$.



Clearly we can take definite integrals of $1/x$ provided only that the interval of integration does not cross the asymptote at $x = 0$, and there is no reason why we should not integrate over an interval like $-4 \le x \le -1$ on the negative side of the origin. If x is negative, then $\log(-x)$ is well defined, and using the chain rule:

$$\frac{d}{dx}\log(-x) = -\frac{1}{-x} = \frac{1}{x}.$$

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So $\log(-x)$ is a primitive of $1/x$ when x is negative. Combining these results, $\log|x|$ is a primitive of $1/x$ for all $x \neq 0$. We now have the following three versions of the more general standard form (omitting constants of integration).

14 FURTHER STANDARD FORMS FOR INTEGRATION (EXTENSION):

A. $$\int \frac{1}{x}\,dx = \log|x|$$

B. $$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b|$$

C. $$\int \frac{1}{u}\frac{du}{dx}\,dx = \log|u| \qquad \text{OR} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|$$

NOTE: Careful readers will notice that because $y = 1/x$ has two disconnected branches, there can be different constants of integration in the two branches. So the general primitive of $1/x$ is

$$\int \frac{1}{x}\,dx = \begin{cases} \log x + A, & \text{for } x > 0, \\ \log(-x) + B, & \text{for } x < 0, \end{cases} \quad \text{where } A \text{ and } B \text{ are constants.}$$

If a boundary condition is given in one region, this has no implication at all for the constant of integration in the other region. In any physical interpretation, however, the function would normally have meaning in only one of the two branches.

Exercise 12D

1. Determine the following indefinite integrals:

(a) $\int \frac{1}{x}\,dx$
(b) $\int \frac{2}{x}\,dx$
(c) $\int \frac{1}{3x}\,dx$
(d) $\int \frac{1}{5x+4}\,dx$
(e) $\int \frac{5}{3+2x}\,dx$
(f) $\int \frac{dx}{4x-1}$
(g) $\int \frac{dx}{2x-1}$
(h) $\int \frac{3}{2x+1}\,dx$
(i) $\int \frac{2}{2x-1}\,dx$
(j) $\int \frac{dx}{3-5x}$
(k) $\int \frac{2}{5-7x}\,dx$
(l) $\int \frac{e}{\pi x+1}\,dx$
(m) $\int \frac{dx}{2-ex}$
(n) $\int \frac{\sqrt{2}}{3x-\pi}\,dx$
(o) $\int \frac{dx}{b-ax}$
(p) $\int \frac{a}{b-cx}\,dx$

2. Evaluate the following definite integrals:

(a) $\int_1^e \frac{dx}{x}$
(b) $\int_{\sqrt{e}}^{e^2} \frac{dx}{x}$
(c) $\int_1^5 \frac{1}{x}\,dx$
(d) $\int_3^9 \frac{1}{x}\,dx$
(e) $\int_0^1 \frac{dx}{x+1}$
(f) $\int_4^{18} \frac{dx}{x-2}$
(g) $\int_1^3 \frac{dx}{3x-1}$
(h) $\int_1^2 \frac{3}{5-2x}\,dx$
(i) $\int_0^{\pi} \frac{1}{2x+\pi}\,dx$

3. Find primitives of the following by first writing them as separate fractions:

(a) $\frac{x+1}{x}$
(b) $\frac{2-x}{3x}$
(c) $\frac{3x^2-2x}{x^2}$
(d) $\frac{3x^3+4x-1}{x^2}$

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4. Use the result $\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$, or $\int \frac{1}{u}\frac{du}{dx}\,dx = \log u$, to integrate:

(a) $\frac{2x}{x^2-9}$ (c) $\frac{2x+1}{x^2+x-3}$ (e) $\frac{x+3}{x^2+6x-1}$

(b) $\frac{6x+1}{3x^2+x}$ (d) $\frac{5-6x}{2+5x-3x^2}$ (f) $\frac{3-x}{12x-3-2x^2}$

5. (a) Find y as a function of x, if $y' = \frac{1}{4x}$ and $y = 1$ when $x = e^2$. What is the x-intercept of this curve?

(b) The gradient of a curve is given by $y' = \frac{2}{x+1}$, and the curve passes through the point $(0,1)$. What is the equation of this curve?

(c) Find $y(x)$, given that $y' = \frac{2x+5}{x^2+5x+4}$ and $y = 1$ when $x = 1$.

(d) Given that the derivative of $f(x)$ is $\frac{x^2+x+1}{x}$ and $f(1) = 1\frac{1}{2}$, find $f(x)$.

(e) Write down the equation of the family of curves with the property $y' = \frac{2+x}{x}$. Hence find the curve that passes through $(1,1)$ and evaluate y at $x = 2$ for this graph.

DEVELOPMENT

6. Use the result $\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$, or $\int \frac{1}{u}\frac{du}{dx}\,dx = \log u$, to find:

(a) $\int \frac{3x^2}{x^3-5}\,dx$ (c) $\int \frac{x^3-3x}{x^4-6x^2}\,dx$ (e) $\int \frac{\sqrt{x}}{x^{\frac{3}{2}}+1}\,dx$

(b) $\int \frac{4x^3+1}{x^4+x-5}\,dx$ (d) $\int \frac{10x^3-7x}{5x^4-7x^2+8}\,dx$ (f) $\int \frac{x^2-\sqrt{x}}{x^3-2x^{\frac{3}{2}}+1}\,dx$

7. Using the methods of the previous question, evaluate:

(a) $\int_{-e}^{-2} \frac{1-3x^2}{x-x^3}\,dx$ (b) $\int_3^6 \frac{4x-5}{2x^2-5x}\,dx$ (c) $\int_1^e \frac{2x+e}{x^2+ex}\,dx$

8. (a) Differentiate $x\log x$ and hence find: (i) $\int \log x\,dx$ (ii) $\int_{\sqrt{e}}^{e} \log x\,dx$

(b) Use the change of base formula and the integral in part (a) to evaluate $\int_1^{10} \log_{10} x\,dx$.

(c) Differentiate $x^2\log x$ and hence determine $\int_{\sqrt{e}}^{e} x\log x\,dx$.

(d) Differentiate $\sqrt{x}\log x$ and hence determine the family of primitives of $\frac{\log x}{\sqrt{x}}$.

9. Find: (a) $\int_a^{a^n} \frac{1}{x}\,dx$ (b) $\int \frac{dx}{t(s+tx)}$ (c) $\int_0^1 \frac{dx}{b^2x+b}$

10. (a) Given that we may write $\frac{x}{(x+4)^2} = \frac{(x+4)-4}{(x+4)^2}$, evaluate $\int_0^1 \frac{x}{(x+4)^2}\,dx$.

(b) Show that $\frac{1}{x^2-9} = \frac{1}{6}\left(\frac{1}{x-3} - \frac{1}{x+3}\right)$. Hence find $\int \frac{dx}{x^2-9}$.

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11. Rewrite in the form $\displaystyle\int \frac{1}{u}\frac{du}{dx}\,dx = \log u$, or $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$, then evaluate:

(a) $\displaystyle\int_4^{16} \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$

(b) $\displaystyle\int_e^{e^2} \frac{dx}{x\log x}$

(c) $\displaystyle\int_e^{e^e} \frac{1}{x\log(x^2)}\,dx$

(d) $\displaystyle\int_1^{27} \frac{3}{(\sqrt[3]{x}+1)x^{\frac{2}{3}}}\,dx$

12. Given that $y = \log\left(x + \sqrt{x^2+1}\right)$, find y' and hence determine $\displaystyle\int \frac{dx}{\sqrt{x^2+1}}$ to within a constant.

EXTENSION

13. Determine a primitive of $\dfrac{1}{x+\sqrt{x}}$.

14. (a) Write down the derivatives of $\log(ax+b)$ and $\log(-ax-b)$. What do you notice? Use this to justify the result $\displaystyle\int \frac{a}{ax+b}\,dx = \log|ax+b|$, and then evaluate $\displaystyle\int_{-5}^{-1} \frac{1}{2x+1}\,dx$.

(b) A certain curve has gradient $y' = \dfrac{1}{x}$ and its two branches pass through the two points $(-1,2)$ and $(1,1)$. Find the equation of the curve.

15. [A series converging to $\log(1+x)$, and approximations to the log function]

(a) Use the formula for the partial sum of a GP to prove that for $t \neq -1$,

$$1 - t + t^2 - t^3 + \cdots + t^{2n} = \frac{1}{1+t} + \frac{t^{2n+1}}{1+t}.$$

(b) Integrate both sides of this result from $t=0$ to $t=x$ to show that for $x > -1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{t^{2n+1}}{1+t}\,dt.$$

(c) Explain why $\dfrac{t^{2n+1}}{1+t} \le t^{2n+1}$, for $0 \le t \le 1$. Hence prove that for $0 \le x \le 1$, the integral $\displaystyle\int_0^x \frac{t^{2n+1}}{1+t}\,dt$ converges to 0 as $n \to \infty$. Hence show that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad \text{for } 0 \le x \le 1.$$

(d) (i) Use this series to approximate $\log\frac{3}{2}$ to two decimal places.

(ii) Write down the series converging to $\log 2$ — called the *alternating harmonic series*.

(e) With a little more effort, it can be shown that the series in part (c) converges to the given limit for $-1 < x \le 1$ (the proof is a reasonable challenge). Use this to write down the series converging to $\log(1-x)$ for $-1 \le x < 1$, and hence approximate $\log\frac{1}{2}$ to two decimal places.

(f) Use both series to show that for $-1 < x < 1$,

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

Use this result and an appropriate value of x to find $\log 3$ to five significant figures.

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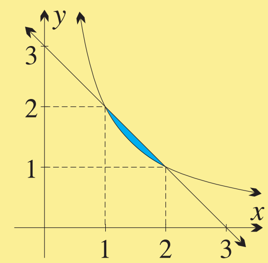
12 E Applications of Integration

The usual methods of finding areas and volumes can now be applied to the reciprocal function, whose primitive was previously unavailable.

WORKED EXERCISE: Find the area contained between the hyperbola $xy = 2$ and the line $x + y = 3$.

SOLUTION: Substitution shows that the curves meet at $A(1, 2)$ and $B(2, 1)$, so area

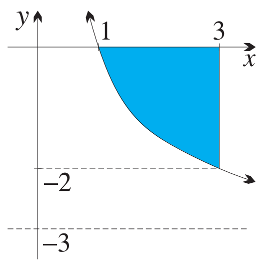
$$\begin{aligned} &= \int_1^2 \left((3 - x) - \frac{2}{x} \right) dx \\ &= \Big[3x - \tfrac{1}{2}x^2 - 2\log x \Big]_1^2 \\ &= (6 - 2 - 2\log 2) - (3 - \tfrac{1}{2} - 2\log 1) \\ &= 1\tfrac{1}{2} - 2\log 2 \text{ square units.} \end{aligned}$$



Exercise 12E

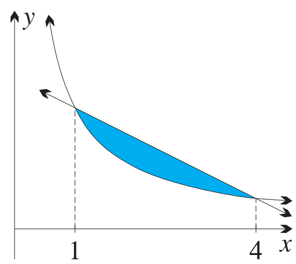
1. Find the area under the curve $y = \dfrac{1}{x}$ for: (a) $e \le x \le e^2$ (b) $2 \le x \le 8$

2. (a)



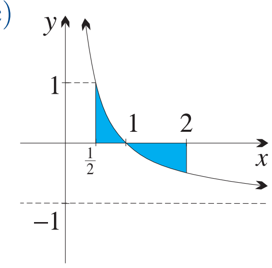
Find the area of the region bounded by $y = \dfrac{3}{x} - 3$, the x-axis and $x = 3$.

(b)



Find the area of the region between $y = \dfrac{2}{x}$ and the line $x + 2y - 5 = 0$.

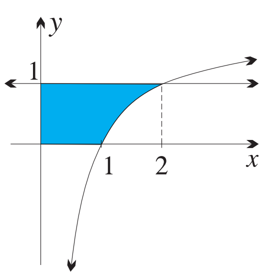
(c)



Find the area of the region bounded by $y = \dfrac{1}{x} - 1$, the x-axis, $x = \frac{1}{2}$ and $x = 2$.

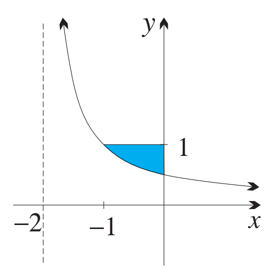
3. (a) Find the area under the graph $y = \dfrac{x}{x^2 + 1}$ between $x = 0$ and $x = 2$.

4. (a)



Find the area of the region in the first quadrant bounded by $y = 2 - \dfrac{2}{x}$ and $y = 1$.

(b)



Find the area of the region bounded by the curve $y = \dfrac{1}{x + 2}$, the y-axis and $y = 1$.

5. (a) Sketch the region bounded by the x-axis, $y = x$, $y = \dfrac{1}{x}$ and $x = e$.

(b) Hence find the area of this region by using two appropriate integrals.

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6. (a) Sketch the area bounded by the coordinate axes, $y = 1$, $x = 8$ and the curve $y = \dfrac{4}{x}$.

(b) Determine the area of this region with the aid of an appropriate integral.

DEVELOPMENT

7. (a) Find the two intersection points of the curve $y = \dfrac{1}{x}$ with $y = 4 - 3x$.

(b) Determine the area between these two curves.

8. The curve $y = \dfrac{1}{x^2}$ (called a *truncus*) is rotated about the y-axis between $y = 1$ and $y = 6$. Evaluate the resulting volume.

9. (a) Find the volume generated when the curve $y = \dfrac{1}{\sqrt{x}}$ is rotated about the x-axis between $x = 2$ and $x = 4$.

(b) A horn is created by rotating the curve $y = \dfrac{1}{\sqrt{4 - x}}$ about the x-axis between $x = 0$ and $x = 3\frac{3}{4}$. Find the volume of the horn.

(c) Another horn is generated by rotating the curve $y = 1 + \dfrac{1}{x}$ about the x-axis between $x = \frac{1}{2}$ and $x = 3$. Find its volume.

10. (a) Expand $(x + 1)(x - 1)(x - 4)$.

(b) Use part (a) to help find the intersection points of $y = \dfrac{4}{x}$ and $y = 1 + 4x - x^2$, and hence sketch the two curves.

(c) Hence find the area in the first quadrant enclosed between these two curves.

11. Carefully graph $y = \dfrac{1}{x}$ and $y = 2 - x$ on the same number plane. Find the area between these two curves, the x-axis and the line $x = 4$.

12. (a) Differentiate $y = x \log x$, and hence write down a primitive of $\log x$.

(b) Hence determine the area under $y = \log x$ between $x = e$ and $x = e^2$.

(c) Use part (a) to help determine the area in the first quadrant above $y = \log x$ and below $y = c$, for $c > 0$.

13. (a) The area between $y = \sqrt{x}$ and $y = \dfrac{1}{\sqrt{x}}$, and between $x = 1$ and $x = 4$, is rotated about the x-axis. Find the volume of the resulting solid.

(b) Compare the volume found in part (a) with the volume generated when the area below $y = \sqrt{x} - \dfrac{1}{\sqrt{x}}$, also between $x = 1$ and $x = 4$, is rotated about the x-axis.

14. The area in the first quadrant under $y = 1$ and above $y = 2 - \dfrac{2}{x}$ is rotated about the x-axis. Find the volume so formed.

15. A rod lying between $x = 1$ and $x = 3$ has density at any point x given by $\rho(x) = \dfrac{1}{x}$.

(a) Calculate its mass, given the formula $M = \displaystyle\int_1^3 \rho(x)\, dx$.

(b) Find the position $\overline{x}$ of the centre of mass, given the formula $\overline{x} = \dfrac{1}{M} \displaystyle\int_1^3 x\rho(x)\, dx$.

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16. (a) Use upper and lower rectangles to prove that $\frac{1}{2} < \int_{2^n}^{2^{n+1}} \frac{1}{x}\,dx < 1$, for $n \geq 0$.

(b) Hence prove that $\int_1^{2^n} \frac{1}{x}\,dx \to \infty$ as $n \to \infty$.

17. (a) Explain why $\log e = \int_1^3 \frac{1}{x}\,dx - \int_e^3 \frac{1}{x}\,dx$.

(b) Show that Simpson's rule with five function values estimates the first integral as $\frac{11}{10}$.

(c) Show that the trapezoidal rule with two points estimates $\int_e^3 \frac{1}{x}\,dx = \frac{1}{6e}(9 - e^2)$.

(d) Combine parts (a), (b) and (c) to show that e may be approximated with the equation $5e^2 + 3e - 45 = 0$. Solve this equation to find an approximation for e, giving your answer correct to three decimal places.

18. Sketch the region cut off in the first quadrant by $y = \dfrac{2x-3}{x^2-3x-4}$, and find its area.

19. (a) Show that $4x = 2(2x+1) - 2$.

(b) Hence evaluate the area under $y = \dfrac{4x}{2x+1}$ between $x = 0$ and $x = 1$.

20. (a) Graph the region bounded by the curve $y = 1/x$, the x-axis and the lines $x = -3$ and $x = -2$. Use the fact that $y = 1/x$ is an odd function to express the area as an integral, and evaluate the area.

(b) Find the area between the curve $y = 1/x$ and the x-axis, for:

(i) $-1 \leq x \leq -e^{-3}$ (ii) $-9 \leq x \leq -3$

21. (a) Show that the curves $y = \dfrac{6}{x}$ and $y = x^2 - 6x + 11$ intersect when $x = 1$, 2 and 3.

(b) Graph these two curves and shade the two areas enclosed by them.

(c) Find the total area enclosed by the two curves.

EXTENSION

22. Consider the area under $y = \dfrac{1}{x}$ between $x = n$ and $x = n+1$.

(a) Show that $\dfrac{1}{n+1} < \int_n^{n+1} \frac{1}{x}dx < \dfrac{1}{n}$. (b) Hence show that $\dfrac{n}{n+1} < \log(1+\frac{1}{n})^n < 1$.

(c) Take the limit of this last result as n tends to infinity to show that $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = e$.

(d) Repeat the above steps, replacing $n+1$ with $n+t$ and show that $\lim\limits_{n\to\infty}(1+\frac{t}{n})^n = e^t$.

23. What is the volume generated when the area under $y = e^{-x^2}$ between $x = 0$ and $x = 1$ is rotated about the y-axis? (A sketch will be required first.)

24. The curve $y = \dfrac{\sqrt{x}}{x+1}$ is rotated about the x-axis, generating a volume between $x = 0$ and $x = c$. Determine this volume. [HINT: $\dfrac{x}{(x+1)^2} = \dfrac{x+1}{(x+1)^2} - \dfrac{1}{(x+1)^2}$]

25. (a) Find the x coordinates of the inflexion points of $y = \dfrac{x}{x^2+1}$.

(b) Hence explain why the trapezoidal rule applied to this function between $x = 0$ and $x = \sqrt{3}$ will underestimate the area.

GO Online Multiple Choice Quiz

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