CHAPTER **THREE**

Graphs and Inequations

The interrelationship between algebra and graphs is the theme of this chapter. Graphs are used here to solve various equations and inequations, including those involving absolute value. Conversely, algebraic techniques are used to investigate unfamiliar graphs, resulting in a curve sketching menu that allows a systematic approach to the shapes of a wide variety of curves.

Study Notes: Solving inequations and equations by using graphs is the theme of Sections 3A–3E, but Sections 3B and 3C also introduce algebraic approaches to the features of general curves. Section 3E also introduces the important absolute value function and its associated graphs. Section 3F relates regions in the coordinate plane to inequations in two variables.

The final Section 3G discusses asymptotes, and brings together four distinct features of graphs into a systematic method of sketching unknown graphs. After the derivative has been introduced, Chapter Ten will extend this menu with two further steps. If it proves too demanding at this stage, Section 3G could be delayed until Chapter Ten.

This chapter is probably the most useful place for machine drawing of curves to help clarify how the features of a graph are related to the algebraic properties of its equation, and to gain familiarity with the various graphs. An alternative approach to many questions requiring sketches would be to display the shape on a machine first and then give the required algebraic explanation. Sufficient questions should, however, be left to give practice in purely algebraic analysis.

3 A Inequations and Inequalities

Statements involving the four symbols $<$ and $\le$ and $>$ and $\ge$ occur frequently. This section begins a systematic approach to them.

There is a distinction between inequations and inequalities. A statement such as $x^2 \le 16$ is an *inequation*; it has the solution $-4 \le x \le 4$, meaning that it is true for these numbers and not for any others. But a statement such as $x^2 + y^2 \ge 0$ is an *inequality*; it is true for all real numbers x and y, in the same way that an identity such as $(x-y)^2 = x^2 - 2xy + y^2$ is true for all real numbers.

The Meaning of 'Less than': There are both a geometric and an algebraic interpretation of the phrase 'less than'. Suppose that a and b are real numbers.

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1 THE GEOMETRIC INTERPRETATION OF $a < b$:

We say that $a < b$ if a is to the left of b on the number line:

THE ALGEBRAIC INTERPRETATION OF $a < b$:

We say that $a < b$ if $b - a$ is positive.

The first interpretation is geometrical, relying on the idea of a 'line' and of one point being 'on the left-hand side of' another. The second interpretation requires that the term 'positive number' be already understood. This second interpretation turns out to be very useful later in solving inequations and proving inequalities.

Solving Linear Inequations: As discussed in Chapter One, the rules for adding and subtracting from both sides, and for multiplying or dividing both sides, are exactly the same as for equations, with one qualification — the inequality symbol reverses when multiplying or dividing by a negative.

2 LINEAR INEQUATIONS: When multiplying or dividing both sides of an inequation by a negative, the inequality symbol is reversed.

WORKED EXERCISE:

(a)
$$3x - 7 \le 8x + 18$$
$$\boxed{+(-8x+7)} \quad -5x \le 25$$
$$\boxed{\div(-5)} \quad x \ge -5$$

(b)
$$20 > 2 - 3x \ge 8$$
$$\boxed{-\,2} \quad 18 > -3x \ge 6$$
$$\boxed{\div(-3)} \quad -6 < x \le -2$$

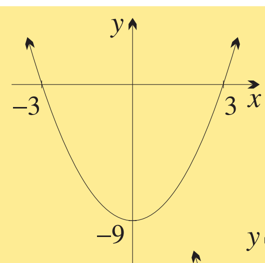
Solving Quadratic Inequations: The clearest way to solve a quadratic inequation is to sketch the graph of the associated parabola.

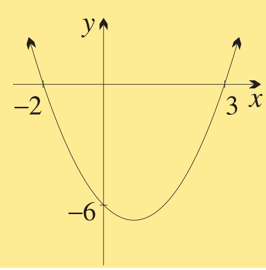
3 QUADRATIC INEQUATIONS: To solve a quadratic inequation, move everything to the LHS, sketch the graph of the LHS, showing the x-intercepts, then read the solution off the graph.

WORKED EXERCISE: Solve: (a) $x^2 > 9$ (b) $x + 6 \ge x^2$

SOLUTION:

(a) Moving everything onto the left, $x^2 - 9 > 0$
then factoring, $(x-3)(x+3) > 0$.
This is the part of the graph above the x-axis,
so from the graph opposite, $x > 3$ or $x < -3$.
[This example is easy, and could be done at sight.]





(b) Moving everything onto the left, $x^2 - x - 6 \le 0$
then factoring, $(x-3)(x+2) \le 0$.
This is the part of the graph below the x-axis,
so from the graph opposite, $-2 \le x \le 3$.

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Solving Inequations with a Variable in the Denominator: There is a problem with

$$\frac{5}{x-4} \ge 1.$$

The denominator $x-4$ is sometimes positive and sometimes negative, so if both sides were multiplied by the denominator $x-4$, the inequality symbol would reverse sometimes and not other times. The most straightforward approach is to multiply through instead by the square of the denominator, which must always be positive or zero.

4 VARIABLE IN THE DENOMINATOR: Multiply through by the *square* of the denominator, being careful to exclude the zeroes of the denominator.

Once the fractions have been cleared, there will usually be common factors on both sides. These should *not* be multiplied out, because the factoring will be easier if they are left unexpanded.

WORKED EXERCISE: Solve $\dfrac{5}{x-4} \ge 1$.

SOLUTION: The key step is to multiply both sides by $(x-4)^2$.

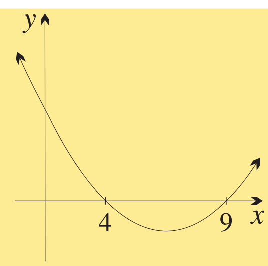
$\boxed{\times (x-4)^2}$ $\quad 5(x-4) \ge (x-4)^2$, and $x \ne 4$,

$(x-4)^2 - 5(x-4) \le 0$, and $x \ne 4$,

$(x-4)(x-4-5) \le 0$, $x \ne 4$,

$(x-4)(x-9) \le 0$, $x \ne 4$.

From the diagram, $4 < x \le 9$.



Solving Logarithmic and Exponential Inequations: When the base is greater than 1, the exponential function and its inverse the logarithmic function are both increasing functions, so:

5 LOGARITHMIC AND EXPONENTIAL INEQUATIONS: The inequality symbol is unchanged when moving between exponential and logarithmic statements, provided the base is greater than 1.

WORKED EXERCISE: Solve: (a) $\log_5 x < 3$ (b) $-5 \le \log_2 x \le 5$

SOLUTION: Note that $\log x$ is only defined for $x > 0$.

(a) Changing to exponentials,
$0 < x < 5^3$
$0 < x < 125$.

(b) Changing to exponentials,
$2^{-5} \le x \le 2^5$
$\frac{1}{32} \le x \le 32$.

Proving Inequalities — A. Standard Operations: There are many approaches to proving inequalities. The first of the three methods presented here uses only the standard operations.

6 PROVING INEQUALITIES (A): A proof of an inequality may proceed from a known result to the desired result using the standard operations.

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WORKED EXERCISE: [A standard result which may be quoted] Prove that:
(a) if $x > 1$, then $x^2 > x$, (b) if $0 < x < 1$, then $x^2 < x$.

SOLUTION:
(a) Suppose that $x > 1$.
$\boxed{\times x}$ $x^2 > x$, (since we multiplied by a positive number).
(b) Suppose that $x < 1$, where $x > 0$.
$\boxed{\times x}$ $x^2 < x$, (since we multiplied by a positive number).

Proving Inequalities — B. Everything on the Left: The second approach is based on the algebraic interpretation that $a < b$ means $b - a > 0$.

7 **PROVING INEQUALITIES (B):** To prove that LHS < RHS, prove instead that RHS − LHS > 0.

WORKED EXERCISE: [A standard result which may be quoted]
If $0 < a < b$, prove that $\dfrac{1}{b} < \dfrac{1}{a}$, by proving that RHS − LHS > 0.

SOLUTION:
$$\begin{aligned} \text{RHS} - \text{LHS} &= \frac{1}{a} - \frac{1}{b} \\ &= \frac{b-a}{ab} \\ &> 0, \text{ since } b - a \text{ is positive and } ab \text{ is positive.} \end{aligned}$$

Proving Inequalities — C. Squares Can't be Negative: The third approach uses the fact that a square can never be negative.

8 **PROVING INEQUALITIES (C):** Begin with a suitable statement that some square is positive or zero.

WORKED EXERCISE: Use the fact that $(x - y)^2 \ge 0$ to prove that $x^2 + y^2 \ge 2xy$, for all real numbers x and y.

SOLUTION: We know that $(x - y)^2 \ge 0$, for all x and y.
Expanding this, $x^2 - 2xy + y^2 \ge 0$, for all x and y,
$\boxed{+2xy}$ $x^2 + y^2 \ge 2xy$, for all x and y.

Exercise 3A

1. Solve, and graph on the number line, the solutions of:
(a) $x > 1$
(b) $x \le 2$
(c) $-2x < 4$
(d) $2x < 6$
(e) $x + 4 \ge 3$
(f) $3 - x > 1$
(g) $3x - 1 < 5$
(h) $5 - 2x \le -1$
(i) $5x - 5 \ge 10$
(j) $2 - 3x \ge 8$
(k) $\frac{1}{3}x - 1 > -\frac{1}{3}$
(l) $\frac{1}{4}x + 2 \le 1\frac{1}{2}$

2. Solve the following double inequations, and sketch the solutions on the number line:
(a) $-8 \le 4x < 12$
(b) $4 < 3x \le 15$
(c) $-2 \le 2x - 1 \le 3$
(d) $-1 \le 4x - 3 < 13$

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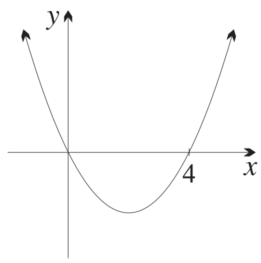
3. Solve these inequations:

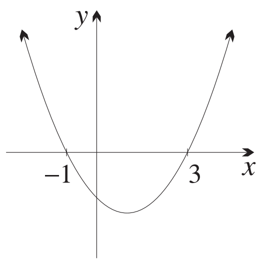
(a) $2x + 3 > x + 7$
(b) $3x - 2 \le \frac{1}{2}x + 3$
(c) $2 - x > 2x - 4$
(d) $1 - 3x \ge 2 - 2x$
(e) $2 < 3 - x \le 5$
(f) $-4 \le 1 - \frac{1}{3}x \le 3$

4. Use the given graph of the LHS to help solve each inequation:

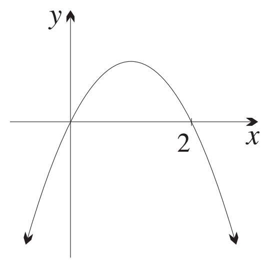
(a) $x(x-4) < 0$



(b) $(x-3)(x+1) \ge 0$



(c) $x(2-x) \le 0$



5. Draw the associated parabola and hence solve:

(a) $(x+2)(x-4) < 0$
(b) $(x-3)(x+1) > 0$
(c) $(2-x)(x-5) \ge 0$
(d) $(x+1)(x+3) \ge 0$
(e) $(2x-1)(x-5) > 0$
(f) $(3x+5)(x+4) \le 0$

6. Factor the LHS and draw an appropriate parabola in order to solve:

(a) $x^2 + 2x - 3 < 0$
(b) $x^2 - 5x + 4 \ge 0$
(c) $x^2 + 6x + 8 > 0$
(d) $x^2 - x - 6 \le 0$
(e) $2x^2 - x - 3 \le 0$
(f) $4 + 3x - x^2 > 0$

7. Collect terms on one side, factor and sketch the associated parabola, and hence solve:

(a) $x^2 \le 1$
(b) $x^2 > 3x$
(c) $x^2 \ge 144$
(d) $x^2 > 0$
(e) $x^2 + 9 \le 6x$
(f) $4x - 3 \ge x^2$

8. Multiply through by the square of the denominator and hence solve:

(a) $\dfrac{1}{x} \le 2$
(b) $\dfrac{2}{x-3} > 1$
(c) $\dfrac{3}{x+4} \ge 2$
(d) $\dfrac{5}{2x-3} < 1$
(e) $\dfrac{2}{3-x} > 1$
(f) $\dfrac{4}{5-3x} \le -1$

9. Draw a sketch of the curve $y = 2^x$ and the line $y = -1$. Hence explain why the inequation $2^x \le -1$ has no solutions.

10. State whether these are true or false, and if false, give a counterexample:

(a) $x^2 > 0$
(b) $x^2 \ge x$
(c) $2^x > 0$
(d) $x \ge \dfrac{1}{x}$
(e) $2x \ge x$
(f) $x + 2 > x$
(g) $x \ge -x$
(h) $2x - 3 > 2x - 7$

11. Given that $x - y > y - z$, prove that $y < \frac{1}{2}(x + z)$.

12. If $a > b$ and $b \ne 0$, prove: (a) $-a < -b$ (b) $ab^2 > b^3$

DEVELOPMENT

13. Multiply through by the square of the denominator and hence solve:

(a) $\dfrac{5x}{2x-1} \ge 3$
(b) $\dfrac{2x+5}{x+3} < 1$
(c) $\dfrac{x+1}{x-1} \le 2$
(d) $\dfrac{4x+7}{x-2} > -3$

14. Draw $y = 2x - 1$ and $y = 2x + 3$ on the same number plane, and hence explain why the inequation $2x - 1 \le 2x + 3$ is true for all real values of x.

15. (a) Draw $y = 1 - x$, $y = 2$ and $y = -1$ on the same number plane and find the points of intersection.

(b) Solve the inequation $-1 < 1 - x \le 2$, and relate the answer to the graph.

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16. Write down and solve a suitable inequation to find where the line $y = 5x - 4$ is below the line $y = 7 - \frac{1}{2}x$.

17. Solve the following inequations involving logarithms and exponentials:
(a) $3^x \geq 27$
(b) $1 < 5^x \leq 125$
(c) $\frac{1}{16} \leq 2^x \leq 16$
(d) $2^{-x} > 16$
(e) $\log_2 x < 3$
(f) $-2 \leq \log_5 x \leq 4$

18. State whether these are true or false, and if false, give a counterexample:
(a) If $0 < a < b$, then $\dfrac{1}{a} > \dfrac{1}{b}$.
(b) If $a < b$, then $a^2 < b^2$.
(c) If $a^2 + b^2 = 0$, then $a = b = 0$.
(d) If $a < b$ and $a, b \neq 0$, then $\dfrac{1}{a} > \dfrac{1}{b}$.
(e) If $a < b$, then $-a > -b$.
(f) If $0 < a < b$, then $\sqrt{a^2 + b^2} = a + b$.

19. If $-1 \leq t < 3$, what is the range of values of:
(a) $4t$
(b) $-t$
(c) $t + 7$
(d) $2t - 1$
(e) $\frac{1}{2}(t + 1)$
(f) $\frac{1}{2}(3t - 1)$
(g) 2^t
(h) $\sqrt{t + 1}$

20. What range of values may $x^2 + 3$ take if: (a) $2 < x < 4$ (b) $-1 < x \leq 3$

21. (a) Given that $x < y < 0$, show that $xy > y^2$.
(b) Suppose that $x > y > 0$. (i) Show that $x^2 > y^2$. (ii) For what values of n is $x^n > y^n$?

22. In the notes it was proven that $x^2 + y^2 \geq 2xy$. Use this result and appropriate substitutions to prove: (a) $a + \dfrac{1}{a} \geq 2$, for $a > 0$, (b) $\dfrac{a + b}{2} \geq \sqrt{ab}$, for a and b both positive.

EXTENSION

23. Prove that $x^2 + xy + y^2 > 0$ for any non-zero values of x and y.

24. (a) Prove that $(x + y)^2 \geq 4xy$. (b) Hence prove that $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{4}{x^2 + y^2}$.

25. (a) Expand $(a - b)^2 + (b - c)^2 + (a - c)^2$, and hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$.
(b) Expand $(a + b + c)\big((a - b)^2 + (b - c)^2 + (a - c)^2\big)$, and hence prove the identity $a^3 + b^3 + c^3 \geq 3abc$, for positive a, b and c.

3 B Intercepts and Sign

When an unknown graph is being sketched, it is important to know the x-intercepts or zeroes — usually factoring is required for this. If the zeroes can be found, a table of test points can then determine where the graph is above the x-axis and where it is below the x-axis. Most functions in this section are *polynomials*, meaning that they can be written as a sum of multiples of powers of x, like $y = 3x^3 - 2x^2 + 7x + 1$.

The x- and y-intercepts: The places where the graph meets the x-axis and the y-axis are found by putting the other variable equal to zero.

9 THE x- AND y-INTERCEPTS: To find the y-intercept, substitute $x = 0$.
To find the x-intercept, substitute $y = 0$.

The x-intercepts are also called the *zeroes* of the function. Finding them will usually involve factoring the function.

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WORKED EXERCISE: Find the x- and y-intercepts of $y = x^3 - x^2 - x + 1$.

SOLUTION: When $x = 0$, $y = 1$ (this is the y-intercept).

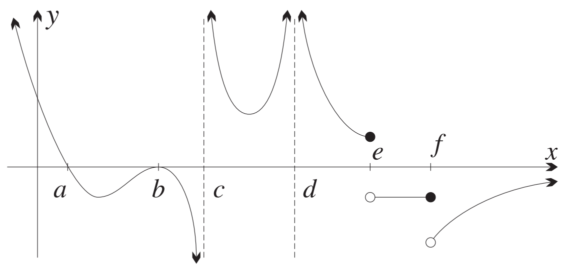
Factoring by grouping,
$$\begin{aligned} y &= x^2(x-1) - (x-1) \\ &= (x^2-1)(x-1) \\ &= (x+1)(x-1)^2, \end{aligned}$$
so $y = 0$ when $x = -1$ or 1 (these are the x-intercepts).

The complete graph is sketched in the next worked exercise.

The Sign of the Function: Given the zeroes, the sign of the function as x varies can be found by using a set of test values. This method requires a major theorem called the *intermediate value theorem*.

10 THE INTERMEDIATE VALUE THEOREM: The only places where a function may possibly change sign are zeroes and discontinuities.

The word *discontinuity* needs explanation. Informally, a function is said to be *continuous* at a point if its graph can be drawn through the point without lifting the pen off the paper — otherwise there is a *discontinuity* at the point. In the graph below, there are discontinuities at $x = c$, d, e and f.



PROOF: This theorem goes to the heart of what the real numbers are and what continuity means, but for this course an example will be sufficient. The function sketched above changes sign at the zero at $x = a$ and at the discontinuities at $x = c$ and $x = e$. Notice that the function does not change sign at the zero at $x = b$ or at the discontinuities at $x = d$ and $x = f$.

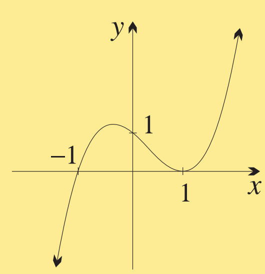
11 EXAMINING THE SIGN OF A FUNCTION: To examine the sign of a function, draw up a table of test values around any zeroes and discontinuities.

WORKED EXERCISE: Examine the sign of $y = (x+1)(x-1)^2$, and sketch the graph.

SOLUTION: There are zeroes at 1 and -1, and no discontinuities.

x	-2	-1	0	1	2
y	-9	0	1	0	3
sign	$-$	0	$+$	0	$+$

So y is positive for $-1 < x < 1$ or $x > 1$,
and y is negative for $x < -1$.



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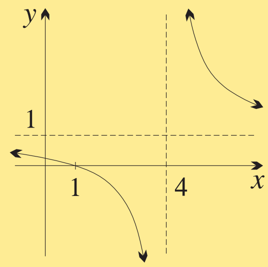
WORKED EXERCISE: [A harder example] Examine the sign of $y = \dfrac{x-1}{x-4}$.

SOLUTION: Here y has a zero at $x = 1$,
but there is also a discontinuity at $x = 4$.

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	$*$	4
sign	$+$	0	$-$	$*$	$+$

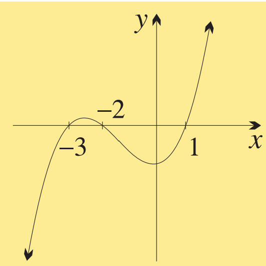


So y is positive for $x < 1$ or $x > 4$, and is negative for $1 < x < 4$.
[The graph is drawn opposite, but it won't be explained until Section 3G.]

NOTE: This procedure is unnecessary for many functions whose sign is more easily established. For example, the function $y = \dfrac{1}{1+x^2}$ must always be positive, since $x^2 + 1$ is always at least 1.

Solving Rational Inequations involving Cubics: The following rational inequation is solved in the usual way by multiplying through by $(x + 2)^2$ to give a cubic inequation. Notice how the resulting common factor $(x+2)$ is never multiplied out.

WORKED EXERCISE: [A harder example involving a cubic graph]
Solve $\dfrac{3}{x+2} \le x$.



SOLUTION: $\boxed{\times\,(x+2)^2}$ $3(x + 2) \le x(x + 2)^2$, and $x \ne -2$,

$$x(x + 2)^2 - 3(x + 2) \ge 0, \text{ and } x \ne -2,$$
$$(x + 2)(x^2 + 2x - 3) \ge 0, \text{ and } x \ne -2,$$
$$(x + 2)(x + 3)(x - 1) \ge 0, \text{ and } x \ne -2.$$

From the diagram, $x \ge 1$ or $-3 \le x < -2$.

An Alternative Approach to Rational Inequations: The previous inequation could also be solved by collecting all terms on the left and using the methods of 'intercepts and sign'.

SOLUTION: The given inequation is $\dfrac{3}{x+2} \le x$.

Collecting everything on the left, $\dfrac{3}{x+2} - x \le 0$,

using a common denominator, $\dfrac{3 - x^2 - 2x}{x+2} \le 0$,

and factoring, $\dfrac{(3 + x)(1 - x)}{x+2} \le 0$.

The LHS has zeroes at $x = -3$ and $x = 1$, and a discontinuity at $x = -2$.

x	-4	-3	$-2\frac{1}{2}$	-2	0	1	2
LHS	$2\frac{1}{2}$	0	$-3\frac{1}{2}$	$*$	$1\frac{1}{2}$	0	$-1\frac{1}{4}$
sign	$+$	0	$-$	$*$	$+$	0	$-$

So the solution is $x \ge 1$ or $-3 \le x < -2$.

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Exercise **3B**

1. Explain why the zeroes of $y = (x+1)^2(1-x)$ are $x = 1$ and $x = -1$. Then copy and complete the table of values and sketch the graph.

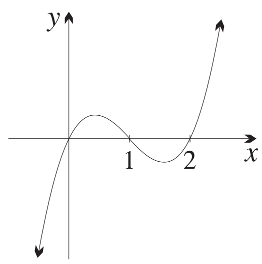
x	-2	-1	0	1	2
y					
sign					

2. Apply the methods used in the previous question to sketch the following quadratics, cubics and quartics. Mark all x- and y-intercepts.

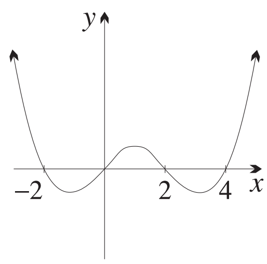
(a) $y = (x+1)(x+3)$
(b) $y = x(x-2)(x-4)$
(c) $y = (x-1)(x+2)^2$
(d) $y = x(x-2)(x+2)$
(e) $y = (x-2)x(x+2)(x+4)$
(f) $y = (x-1)^2(x-3)^2$

3. Use the given graph of the LHS to help solve each inequation:

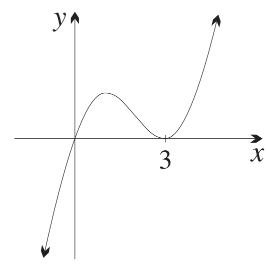
(a) $x(x-1)(x-2) \le 0$



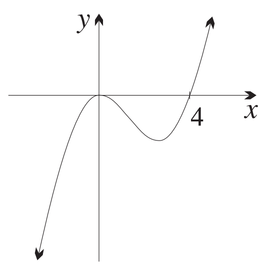
(b) $x(x+2)(x-2)(x-4) < 0$



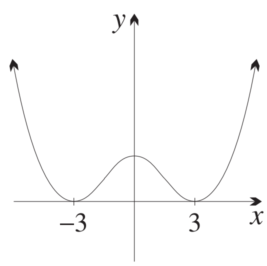
(c) $x(x-3)^2 > 0$



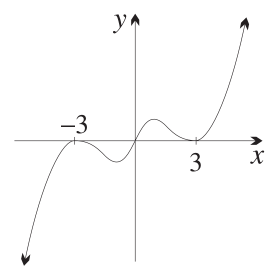
(d) $x^2(x-4) \ge 0$



(e) $(x-3)^2(x+3)^2 \le 0$



(f) $x(x-3)^2(x+3)^2 \ge 0$



4. First factor each polynomial completely, then use the methods of the first two questions to sketch its graph (take out any common factors first):

(a) $f(x) = x^3 - 4x$
(b) $f(x) = x^3 - 5x^2$
(c) $f(x) = x^3 - 4x^2 + 4x$

5. From the graphs in the previous question, or from the tables of values used to construct them, solve the following inequations. Begin by getting all terms onto the one side:

(a) $x^3 > 4x$
(b) $x^3 < 5x^2$
(c) $x^3 + 4x \le 4x^2$

DEVELOPMENT

6. If necessary, collect all terms on the LHS and factor. Then solve the inequation by finding any zeroes and discontinuities and drawing up a table of test values around them:

(a) $(x-1)(x-3)(x-5) < 0$
(b) $(x-1)^2(x-3)^2 > 0$
(c) $\dfrac{x-4}{x+2} \le 0$
(d) $x^3 > 9x$
(e) $\dfrac{x+3}{x+1} < 0$
(f) $\dfrac{x^2}{x-5} < 0$
(g) $x^4 \ge 5x^3$
(h) $\dfrac{x^2-4}{x} \ge 0$
(i) $\dfrac{x-2}{x^2+3x} \le 0$

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7. Factor each equation completely, and hence find the x-intercepts of the graph (factor parts (b) and (c) by grouping in pairs):
(a) $y = x^3 - x$ (b) $y = x^3 - 2x^2 - x + 2$ (c) $y = x^3 + 2x^2 - 4x - 8$

8. For each function in the previous question, examine the sign of the function around each zero and hence draw a graph of the function.

9. Find all zeroes of these functions, and any values of x where the function is discontinuous. Then analyse the sign of the function by taking test points around these zeroes and discontinuities:
(a) $f(x) = \dfrac{x}{x-3}$ (b) $f(x) = \dfrac{x-4}{x+2}$ (c) $f(x) = \dfrac{x+3}{x+1}$

10. Multiply through by the square of the denominator, collect all terms on one side and then factor to obtain a factored cubic. Sketch this cubic by examining the intercepts and the sign. Hence solve the original inequality:
(a) $\dfrac{4}{x+3} \ge x$ (b) $\dfrac{2}{2x+3} < x$ (c) $\dfrac{8}{2x-3} \le 2x - 1$

11. Solve these inequations by the alternative method of collecting everything on the LHS, finding a common denominator, factoring, and drawing up a table of test values. (This alternative method could also be applied to the relevant questions in Exercise 3A.)

EXTENSION

12. (a) Prove that $f(x) = 1 + x + x^2$ is positive for all x.
(b) Prove that $f(x) = 1 + x + x^2 + x^3 + x^4$ is positive for all x. You will probably need to consider separately the three cases $x \ge 0$, $x \le -1$ and $-1 < x < 0$.
(c) Similarly prove that for all positive integers n, $f(x) = 1 + x + x^2 + \cdots + x^{2n-1} + x^{2n}$ is positive for all x.
(d) Prove that $x = -1$ is the only zero of $f(x) = 1 + x + x^2 + \cdots + x^{2n-1}$.

3 C Domain and Symmetry

A systematic investigation of an unknown curve should normally begin with a consideration of three features important in most graphs — the domain, whether the function has even or odd symmetry, and the analysis of its intercepts and sign explained in Section 3B. These features will become the first three steps of a systematic curve sketching menu at the end of this chapter.

Domain: Warning — no work should be done on an unfamiliar function without first finding its domain. If no specific restriction has been made, use the natural domain, which, as mentioned in Section 2F, is the set of all values of x that can be validly substituted into the equation of the function.

WORKED EXERCISE: Find the domain of $y = \dfrac{1}{\sqrt{4 - x^2}}$.

SOLUTION: The square root can only exist when $-2 \le x \le 2$. Zero, however, has no reciprocal, so the domain of the function is $-2 < x < 2$.

Even Functions and Symmetry in the y-axis: It has been said that all mathematics is the study of symmetry. Two simple types of symmetry occur so often in the functions of this course that every function should be tested routinely for them.

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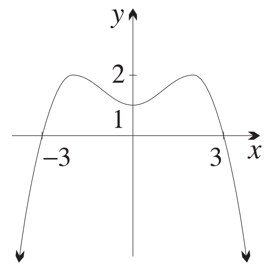
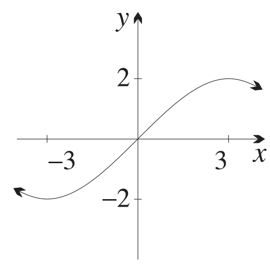
First, a graph with line symmetry in the y-axis is called *even*. This means that the graph is unchanged by reflection in the y-axis, like the graph on the right.



For any function $f(x)$, the graph of $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis. So the function is even if the graphs of $f(x)$ and $f(-x)$ coincide, that is, if the function satisfies the identity $f(-x) = f(x)$.

12 EVEN FUNCTIONS: $f(x)$ is called *even* if $f(-x) = f(x)$, for all x in its domain. A function is even if and only if its graph has line symmetry in the y-axis.

Odd Functions and Symmetry in the Origin: Secondly, a graph with point symmetry in the origin is called *odd*. This means that the graph is unchanged by a rotation of 180° about the origin, or equivalently by successive reflections in the x-axis and the y-axis.



Reflecting $y = f(x)$ in the x- and y-axis gives $y = -f(x)$ and $y = f(-x)$ respectively. So $f(x)$ is odd if the graphs of $f(-x)$ and $-f(x)$ coincide, that is, if $f(-x) = -f(x)$.

13 ODD FUNCTIONS: $f(x)$ is called *odd* if $f(-x) = -f(x)$, for all x in its domain. A function is odd if and only if its graph has point symmetry in the origin.

WORKED EXERCISE: Test these functions for evenness or oddness, then sketch them:

(a) $f(x) = x^4 - 3$ (b) $f(x) = x^3$ (c) $f(x) = x^2 - 2x$

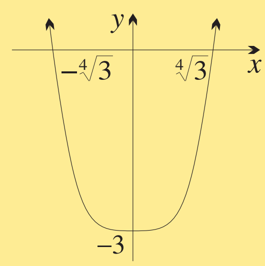
14 TESTING FOR EVENNESS AND ODDNESS: To test whether a function is even or odd or neither, work out $f(-x)$ and compare it with $f(x)$. Most functions are neither even nor odd.

SOLUTION:

(a) $f(x) = x^4 - 3$,

so $f(-x) = (-x)^4 - 3$
$= x^4 - 3$
$= f(x)$.

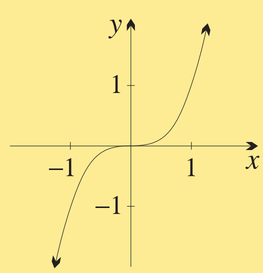
Hence $f(x)$ is an even function.



(b) $f(x) = x^3$,

so $f(-x) = (-x)^3$
$= -x^3$
$= -f(x)$.

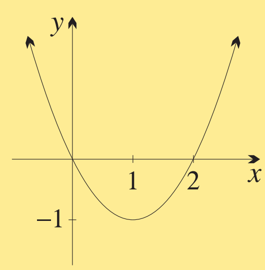
Hence $f(x)$ is an odd function.



(c) $f(x) = x^2 - 2x$,

so $f(-x) = (-x)^2 - 2(-x)$
$= x^2 + 2x$.

Since $f(-x)$ is equal neither to $f(x)$ nor to $-f(x)$, the function is neither even nor odd.



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NOTE: The last curve is a parabola, and has line symmetry about its axis of symmetry, the line $x = 1$. The test described here, however, is only a test for two particular types of symmetry, not for symmetry in general. One other type of symmetry has been touched on in Section 2H, in that a curve is symmetric in the line $y = x$ if its inverse is the same as itself, which means algebraically that the equation is unchanged when x and y are exchanged.

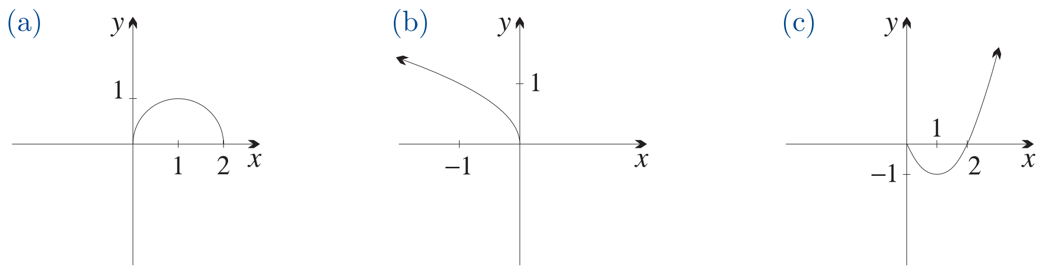
Exercise 3C

1. What are the natural domains of the following functions of x?

(a) $\dfrac{1}{x+1}$ (c) $x^3 + x + 1$ (e) $\sqrt{x}$ (g) $\sqrt{7-x}$

(b) $\dfrac{2}{2x-3}$ (d) $x^2 - x + 1$ (f) $\sqrt{x-1}$ (h) $\sqrt{x+4}$

2. For each graph below, complete the graph so that: (i) $f(x)$ is even, (ii) $f(x)$ is odd.



3. Pick up a book, reflect it in the vertical axis and then in the horizontal axis, and observe that it is now rotated 180° from its original position. Hence explain why the graph of $y = -f(-x)$ is the graph of $y = f(x)$ rotated 180°.

4. Simplify $f(-x)$ for each function, and hence determine whether it is odd, even or neither:

(a) $f(x) = x^2 - 9$ (e) $f(x) = x^3 + 5x^2$

(b) $f(x) = x^2 - 6x + 5$ (f) $f(x) = x^5 - 16x$

(c) $f(x) = x^3 - 25x$ (g) $f(x) = x^5 - 8x^3 + 16x$

(d) $f(x) = x^4 - 4x^2$ (h) $f(x) = x^4 + 3x^3 - 9x^2 - 27x$

Deduce a relationship between the powers of x and the answers for the previous parts.

5. Factor each polynomial in the previous question and write down its zeroes. Then use a table of test points to sketch its graph. Confirm that the graph exhibits the symmetry established above.

DEVELOPMENT

6. Determine the natural domains of the following functions of x:

(a) $\dfrac{x-4}{x-1}$ (c) $\dfrac{x^2-1}{x+1}$ (e) $\dfrac{1}{\sqrt{x+4}}$ (g) $3^x + 3$

(b) $\dfrac{x-1}{x-4}$ (d) $\dfrac{(x+1)(x-2)}{(x-2)}$ (f) $\dfrac{4}{\sqrt{x-1}}$ (h) $\dfrac{1}{2^x - 8}$

7. (a) Solve $x^2 - 4 \geq 0$, and hence write down the natural domain of $\sqrt{x^2-4}$.

(b) Solve $x^2 - 4 > 0$, and hence write down the natural domain of $\dfrac{5}{\sqrt{x^2-4}}$.

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8. Use the methods of the previous question to write down the domains of:

(a) $\sqrt{4-x^2}$ (c) $\sqrt{25-x^2}$ (e) $3\sqrt{x^2-4}$

(b) $\dfrac{1}{\sqrt{4-x^2}}$ (d) $\dfrac{1}{\sqrt{25-x^2}}$ (f) $\dfrac{5}{\sqrt{x^2-4}}$

9. Determine whether the following functions are odd, even or neither:

(a) $f(x)=\sqrt{3-x^2}$ (b) $f(x)=\dfrac{1}{x^2+1}$ (c) $f(x)=\dfrac{4x}{x^2+4}$ (d) $f(x)=2^x+x^2$

EXTENSION

10. (a) Given that $h(x)=f(x)\times g(x)$, determine what symmetry $h(x)$ has if: (i) both f and g are even, (ii) both f and g are odd, (iii) one is even and the other odd.

(b) Given that $h(x)=f(x)+g(x)$, determine what symmetry $h(x)$ has if: (i) both f and g are even, (ii) both f and g are odd, (iii) one is even and the other odd.

11. (a) Prove that an odd function defined at $x=0$ must pass through the origin.

(b) Show that $f(x)=\dfrac{\sqrt{x^2}}{x}$ is odd, and explain why it does not pass through the origin. [HINT: What does its graph look like?]

12. (a) Given that $f(x)$ is odd and has an inverse which is also a function, show that $f^{-1}(x)$ is also odd.

(b) Given that $f(x)$ is even, show that its inverse is not a function unless its graph is a point.

13. For any function $f(x)$, define $g(x)=\frac{1}{2}\big(f(x)+f(-x)\big)$ and $h(x)=\frac{1}{2}\big(f(x)-f(-x)\big)$.

(a) Show that $f(x)=g(x)+h(x)$, that $g(x)$ is even and that $h(x)$ is odd.

(b) Hence write each function as the sum of an even and an odd function:

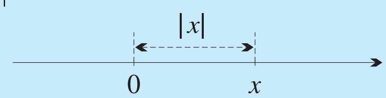
(i) $f(x)=1-2x+x^2$ (ii) $f(x)=2^x$

(c) Why is there a problem with this process if $f(x)=\dfrac{1}{x-1}$ or $f(x)=\sqrt{x}$?

3 D The Absolute Value Function

Often it is the size of a number that is significant rather than whether it is positive or negative.

15 ABSOLUTE VALUE: The absolute value $|x|$ of a number x is the distance from x to the origin on the number line.



For example, $|-5|=5$ and $|0|=0$ and $|5|=5$. Since distance is always positive or zero, so also is the absolute value. Thus absolute value is a measure of the *size* or *magnitude* of a number; in our example, the numbers -5 and $+5$ both have the same size 5, and differ only in their signs.

NOTE: Absolute value is generalised with the complex numbers of the 4 Unit course, where the *modulus* $|x|$ of a complex number x is the distance from the origin on the two-dimensional Argand diagram. Hence $|x|$ is often called the 'modulus of x' or just 'mod x', which is much easier to say.

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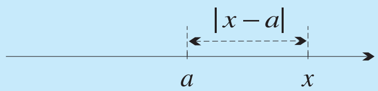
Distance Between Numbers: Replacing x by $x - a$ in the previous definition gives a measure of the distance from a on the number line.

16 DISTANCE BETWEEN NUMBERS: The distance from x to a on the number line is $|x - a|$.



Solving Equations and Inequations on the Number Line: Most equations and inequations involving absolute values in this course are simple enough to be solved using distances on the number line. More complicated equations may require the graphical methods of Section 3E.

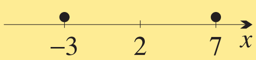
17 METHOD FOR SOLVING SIMPLE ABSOLUTE VALUE (IN)EQUATIONS:

1. Force the equation or inequation into one of the following forms:
   $|x - a| = b$, or $|x - a| < b$, or $|x - a| \ge b$ or ....
2. Find the solution using distance on a number line.

WORKED EXERCISE:

(a) $|x - 2| = 5$

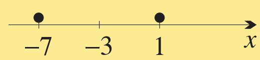
(distance from x to 2) $= 5$



so $x = -3$ or $x = 7$.

(b) $|x + 3| = 4$

(distance from x to -3) $= 4$

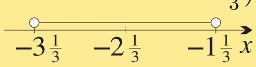


so $x = -7$ or $x = 1$.

(c) $|3x + 7| < 3$

$\boxed{\div 3}$ $\quad |x + 2\frac{1}{3}| < 1$

(distance from x to $-2\frac{1}{3}$) < 1

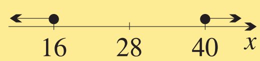


so $-3\frac{1}{3} < x < -1\frac{1}{3}$.

(d) $\left|7 - \frac{1}{4}x\right| \ge 3$

$\boxed{\times 4}$ $\quad |28 - x| \ge 12$

(distance from x to 28) ≥ 12



so $x \le 16$ or $x \ge 40$.

An Alternative Method: If $a \ge 0$ is a constant, then the statement $|f(x)| = a$ means that $f(x) = a$ or $f(x) = -a$. This gives a useful alternative method.

18 AN ALTERNATIVE METHOD: Suppose that $a \ge 0$, and that $f(x)$ is a function of x.

1. To solve $|f(x)| = a$, write '$f(x) = a$ or $f(x) = -a$'.
2. To solve $|f(x)| < a$, write '$-a < f(x) < a$'.
3. To solve $|f(x)| > a$, write '$f(x) > a$ or $f(x) < -a$'.

WORKED EXERCISE:

(a) $|x - 2| = 5$

$x - 2 = 5$ or $x - 2 = -5$

$x = -3$ or $x = 7$

(b) $|3x + 7| < 3$

$-3 < 3x + 7 < 3$

$\boxed{-7}$ $\quad -10 < 3x < -4$

$\boxed{\div 3}$ $\quad -3\frac{1}{3} < x < -1\frac{1}{3}$

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(c) $|7 - \frac{1}{4}x| \ge 3$
$7 - \frac{1}{4}x \ge 3$ or $7 - \frac{1}{4}x \le -3$
$\boxed{-7}$ $-\frac{1}{4}x \ge -4$ or $-\frac{1}{4}x \le -10$
$\boxed{\times(-4)}$ $x \le 16$ or $x \ge 40$

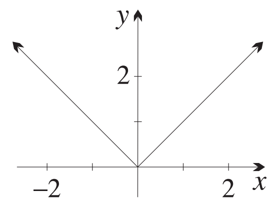
(d) $|\log_2 x| > 3$
$\log_2 x > 3$ or $\log_2 x < -3$
$x > 8$ or $0 < x < \frac{1}{8}$

An Expression for Absolute Value Involving Cases: The absolute value of a negative number is the opposite of that number. Since the opposite of x is $-x$:

19 ABSOLUTE VALUE INVOLVING CASES: $|x| = \begin{cases} x, & \text{for } x \ge 0, \\ -x, & \text{for } x < 0. \end{cases}$

This expression, with its two cases, allows us to draw the graph of $y = |x|$. Alternatively, a table of values makes clear the sharp point at the origin where the two branches meet at right angles:

x	-2	-1	0	1	2
$\lvert x\rvert$	2	1	0	1	2



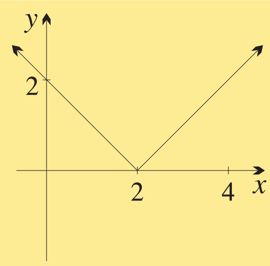
The domain is the set of all real numbers, and the range is $y \ge 0$. The function is even, the graph having line symmetry in the y-axis. The function has a zero at $x = 0$, and is positive for all other values of x.

Graphing Functions with Absolute Value: The transformations of the last chapter can now be applied to the graph of $y = |x|$ in order to sketch many functions involving absolute value. The expression involving cases, however, is needed to establish the equations of the separate branches. More complicated functions often require an approach through cases.

WORKED EXERCISE: Sketch $y = |x - 2|$.

SOLUTION: This is just $y = |x|$ shifted 2 units to the right, or it is $y = x - 2$ with the bit under the x-axis reflected back above the x-axis.
Alternatively, from the expression using cases,

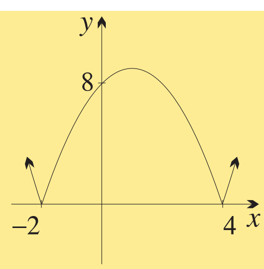
$$y = \begin{cases} x - 2, & \text{for } x \ge 2, \\ -x + 2, & \text{for } x < 2. \end{cases}$$



WORKED EXERCISE: Sketch $y = |x^2 - 2x - 8|$.

SOLUTION: Since $x^2 - 2x - 8 = (x - 4)(x + 2)$, this is $y = (x - 4)(x + 2)$, with the bit under the x-axis reflected back above the x-axis.
Alternatively, using cases,

$$y = \begin{cases} (x - 4)(x + 2), & \text{for } x \le -2 \text{ or } x \ge 4, \\ -(x - 4)(x + 2), & \text{for } -2 < x < 4. \end{cases}$$



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WORKED EXERCISE: [A harder example] Sketch $y = |x-2| - |x+4| + 1$.

SOLUTION: Considering the terms separately,

$$|x-2| = \begin{cases} x-2, & \text{for } x \geq 2, \\ -x+2, & \text{for } x < 2, \end{cases}$$

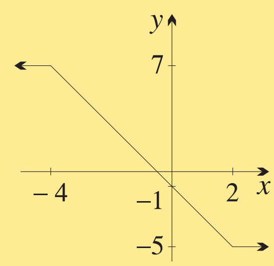
and $$|x+4| = \begin{cases} x+4, & \text{for } x \geq -4, \\ -x-4, & \text{for } x < -4. \end{cases}$$

There are therefore three possible cases:

for $x \geq 2$, $\quad y = (x-2) - (x+4) + 1$
$\quad = -5$,

for $-4 \leq x < 2$, $\quad y = (-x+2) - (x+4) + 1$
$\quad = -2x - 1$,

and for $x < -4$, $\quad y = (-x+2) - (-x-4) + 1$
$\quad = 7$.



Absolute Value as the Square Root of the Square: Taking the absolute value of a number can be thought of as separating the number into sign and size, stripping the sign, and replacing it by a positive sign. There already is an algebraic function capable of doing this job:

20 ABSOLUTE VALUE AS THE POSITIVE SQUARE ROOT OF THE SQUARE:
For all real numbers x, $\quad |x|^2 = x^2 \quad$ and $\quad |x| = \sqrt{x^2}$.

These examples with $x = -3$ should make this clear:

$$|-3|^2 = 9 = (-3)^2 \qquad \text{and} \qquad |-3| = \sqrt{9} = \sqrt{(-3)^2}\,.$$

Identities Involving Absolute Value: Here are some standard identities involving absolute value, to be proven in the following exercise.

21 IDENTITIES INVOLVING ABSOLUTE VALUE:

1. $|-x| = |x|$, for all x
2. $|x-y| = |y-x|$, for all x and y
3. $|xy| = |x||y|$, for all x and y
4. $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, for all x, and for all $y \neq 0$

Proof is best done using $|x|^2 = x^2$. For example, here is the proof of the third identity:

$$\text{LHS}^2 = (xy)^2 = x^2y^2, \qquad \text{RHS}^2 = x^2y^2 = \text{LHS}^2.$$

Since neither LHS nor RHS is negative, it follows that LHS = RHS.

Inequalities Involving Absolute Value: Here are some important inequalities.

22 INEQUALITIES INVOLVING ABSOLUTE VALUE:

1. $|x| \geq 0$, for all x
2. $-|x| \leq x \leq |x|$, for all x
3. $|x+y| \leq |x| + |y|$, for all x and y

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The first is clear. The second needs separate consideration of positive and negative values of x. The third (called the *triangle inequality*) can be proven using $|x|^2 = x^2$:

$$\text{LHS}^2 = (x+y)^2 = x^2 + y^2 + 2xy, \qquad \text{RHS}^2 = x^2 + y^2 + 2|xy|,$$

Since $xy \le |xy|$ (by 2), so $\text{LHS}^2 \le \text{RHS}^2$. But LHS and RHS are non-negative, so LHS $\le$ RHS.

Exercise 3D

1. (a) Copy and complete these tables of values of the functions $y = |x-2|$ and $y = |x| - 2$:

x	-1	0	1	2	3
$\lvert x-2\rvert$					

x	-1	0	1	2	3
$\lvert x\rvert-2$					

(b) Draw the graphs of the two functions on the same number plane and observe the differences between them. How is each graph obtained by shifting $y = |x|$?

2. Evaluate:

(a) $|5|$ (b) $|-3|$ (c) $|14-9-12|$ (d) $|7-4|$ (e) $|4-7|$ (f) $\sqrt{(4-7)^2}$ (g) $|3^2-5^2|$ (h) $|11-16|-8$

3. Solve the following equations:

(a) $|x| = 3$ (b) $|2x| = 10$ (c) $|x-3| = 7$ (d) $|x+1| = 6$ (e) $|2x-1| = 11$ (f) $|3x+2| = 8$ (g) $|5x+2| = 9$ (h) $|7x-3| = 11$

4. State whether these are true or false, and if false, give a counterexample (difficulties will usually involve negative numbers):

(a) $|x| > 0$ (b) $|-x| = |x|$ (c) $-|x| \le x \le |x|$ (d) $|x+2| = |x|+2$ (e) $|7x| = 7|x|$ (f) $|x|^2 = x^2$ (g) $|x|^3 = x^3$ (h) $|x-7| = |7-x|$

5. Use either transformations or a table of values to obtain each graph from the graph of $y = |x|$. Also write down the equations of the two branches in each case.

(a) $y = |2x|$ (b) $y = |\tfrac{1}{2}x|$ (c) $y = |x-1|$ (d) $y = |x+3|$ (e) $y = |x|-1$ (f) $y = |x|+3$ (g) $y = |2-x|$ (h) $y = 2-|x|$

6. Explain why $|x| = c$ has no solution if the constant c is negative.

7. Use the fact that $|-x| = |x|$ to decide whether these functions are odd, even or neither:

(a) $f(x) = |x|+1$ (b) $f(x) = |x|+x$ (c) $f(x) = x \times |x|$ (d) $f(x) = |x^3 - x|$

8. Solve the following inequations and graph the solutions on the number line:

(a) $|x-2| < 3$ (b) $|3x-5| \le 4$ (c) $|x-7| \ge 2$ (d) $|2x+1| < 3$ (e) $|6x-7| > 5$ (f) $|5x+4| \ge 6$

DEVELOPMENT

9. (a) (i) Sketch a graph of $y = (x-3)(x-1)$. (ii) Hence obtain the graph of $y = |x^2-4x+3|$ by reflecting in the x-axis those parts of the parabola that are below the x-axis.

(b) Similarly sketch graphs of:

(i) $y = |x^2-x-2|$ (ii) $y = |2x^2-5x-3|$ (iii) $y = |(x-1)x(x+1)|$

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10. (a) Explain why the double inequation $2 \le |x| \le 6$ is equivalent to $2 \le x \le 6$ or $-6 \le x \le -2$.

(b) Similarly solve: (i) $2 < |x+4| < 6$ (ii) $1 \le |2x-5| < 4$

11. (a) Where is $y = \dfrac{|x|}{x}$ undefined?

(b) Use a table of values from $x = -3$ to $x = 3$ to sketch the graph.

(c) Hence write down the equations of the two branches of $y = \dfrac{|x|}{x}$.

12. (a) Simplify $y = |x| + x$, for $x \ge 0$ and for $x < 0$, then sketch.

(b) Simplify $y = |x| - x$, for $x \ge 0$ and for $x < 0$, then sketch.

13. State whether these are true or false. If false, give a counterexample. If true, provide examples with: (i) $x > 0$ and $y > 0$, (ii) $x > 0$ and $y < 0$, (iii) $x < 0$ and $y > 0$, (iv) $x < 0$ and $y < 0$.

(a) $|x+y| = |x| + |y|$
(b) $|x+y| \le |x| + |y|$
(c) $|x-y| \le |x| - |y|$
(d) $|x-y| \le |x| + |y|$
(e) $|x-y| \ge \big||x| - |y|\big|$
(f) $2^{|x|} = 2^x$

14. Consider the function $y = \dfrac{1}{|x-1|}$.

(a) What is its natural domain ?

(b) Write down the equations of the two branches of the function and sketch its graph.

15. The identity $|xy| = |x||y|$ was proven in the notes above.

(a) Noting that $-x = (-1)x$, prove that $|-x| = |x|$.

(b) Prove the remaining identities from Table (21).

16. Use a similar proof to the one given in the text to prove:

(a) $|x-y| \le |x| + |y|$
(b) $|x-y| \ge \big||x| - |y|\big|$

17. Write down the equations of the two branches of the function, then sketch its graph:

(a) $y = 2(x+1) - |x+1|$
(b) $y = x^2 - |2x|$

18. Consider the inequation $\left|x + \dfrac{1}{x}\right| < 2x$.

(a) Explain why x must be positive. (b) Hence solve the inequation.

19. Carefully write down the equations of the branches of each function, then sketch its graph:

(a) $y = |x+1| - |x-3|$
(b) $y = |x-2| + |x+1| - 4$
(c) $y = 2|x+1| - |x-1| - 1$

EXTENSION

20. The function $u(x)$ is defined by $u(x) = \begin{cases} \frac{1}{2}\left(1 + \dfrac{|x|}{x}\right), & \text{for } x \ne 0, \\ 1, & \text{for } x = 0. \end{cases}$

(a) Sketch: (i) $u(x)$ (ii) $u(x-1)$

(b) Hence sketch $u(x) - u(x-1)$.

21. Sketch the relation $|y| = |x|$ by considering the possible cases.

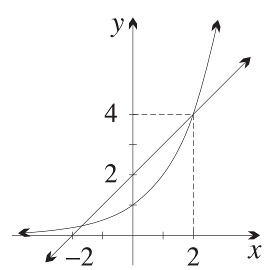
22. Consider the inequation $|x-a|+|x-b|<c$, where $a<b$.
(a) If $a \le x \le b$, show, using distances on a number line, that there can only be a solution if $b-a<c$.
(b) If $b<x$, show, using distances on a number line, that $x<\dfrac{a+b+c}{2}$.
(c) If $x<a$, show, using distances on a number line, that $x>\dfrac{a+b-c}{2}$.
(d) Hence show that either $\left|x-\dfrac{a+b}{2}\right|<\dfrac{c}{2}$ or there is no solution to the original problem.
(e) Hence find the solution to $|x+2|+|x-6|<10$.

3 E Using Graphs to Solve Equations and Inequations

In this section, graphs are used to solve equations and inequations. The advantage of this method is that once the graphs are drawn, it is usually obvious from the picture how many solutions there are, and indeed if there are any solutions at all, as well as their approximate values. Often exact solutions can then be calculated once the situation has been sorted out from the picture.

Constructing Two Functions from a Given Equation: Here is an equation that cannot be solved algebraically:

$$2^x = x+2.$$

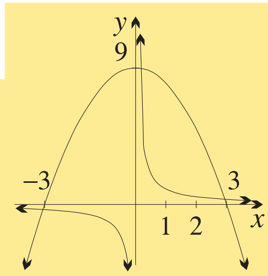


One solution of this equation is $x=2$, but this is not the only solution. If we draw the graphs of the LHS, $y=2^x$, and of the RHS, $y=x+2$, then the situation becomes clear. From the graph, the LHS and RHS are equal at $x=2$ (where they are both equal to 4), and at $x \doteqdot -1{\cdot}69$ (where they are both about $0{\cdot}31$), and these two values of x are the solutions to the original equation.

23 GRAPHICAL SOLUTION OF EQUATIONS: To solve an equation graphically, sketch the graphs of $y=\text{LHS}$ and $y=\text{RHS}$ on one pair of axes, and read off the x-coordinates of any points of intersection.
The original equation may need to be rearranged first.

WORKED EXERCISE: Graph $y=1/x$ and $y=9-x^2$ on the one set of axes. Hence find from your graph how many solutions the following equation has, and approximately where they are:

$$x^2+\frac{1}{x}=9.$$



SOLUTION: Transforming the equation to $\dfrac{1}{x}=9-x^2$, its solutions are the x-coordinates of the points of intersection of $y=1/x$ and $y=9-x^2$. From the graph there are three solutions, one between -4 and -3, one between 0 and 1, and one between 2 and 3.

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Solving an Inequation using Graphs: Now consider the inequation

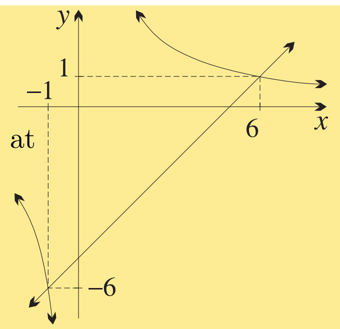
$$2^x < x + 2.$$

From the sketch above, the curve $y = 2^x$ is only below the curve $y = x + 2$ between the two points of intersection, and so the solution of the inequation is approximately $-1{\cdot}69 < x < 2$.

24 GRAPHICAL SOLUTION OF INEQUATIONS: Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes. Then examine which curve lies above the other at each value of x.

Inequations with x in the Denominator — Graphical Solution: This is a way of avoiding the problem that an inequation cannot be multiplied through by a variable when the variable may be positive or negative.

WORKED EXERCISE: Solve $\dfrac{6}{x} \ge x - 5$.



SOLUTION: The graphs of $y = 6/x$ and $y = x - 5$ meet as shown at the points A and B whose x-coordinates are the solutions of

$$x - 5 = \frac{6}{x}$$
$$x^2 - 5x - 6 = 0$$
$$x = 6 \text{ or } -1.$$

So from the graph, the solution of the inequation is $0 < x \le 6$ or $x \le -1$.

Absolute Value Equations — Graphical Solutions: If the graph can be used to sort out the situation, then the exact values can usually be found algebraically.

WORKED EXERCISE: Solve $|2x - 5| = x + 2$.

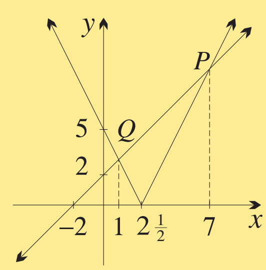
SOLUTION: The graphs $y = |2x - 5|$ and $y = x + 2$ intersect at P and at Q, and these points can be found algebraically.
Here P is the intersection of $y = x + 2$ with $y = 2x - 5$:

$$x + 2 = 2x - 5$$
$$x = 7,$$

and Q is the intersection of $y = x + 2$ with $y = -2x + 5$:

$$x + 2 = -2x + 5$$
$$x = 1.$$

So $x = 7$ or $x = 1$.



Absolute Value Inequations — Graphical Solution: The solutions of the inequation

$$|2x - 5| \ge x + 2$$

can be read off the graph sketched above. We look at where the graph of the LHS, $y = |2x - 5|$, is above the graph of the RHS, $y = x + 2$. This is to the right of P and to the left of Q, so the solution of the inequation is

$$x \le 1 \text{ or } x \ge 7.$$

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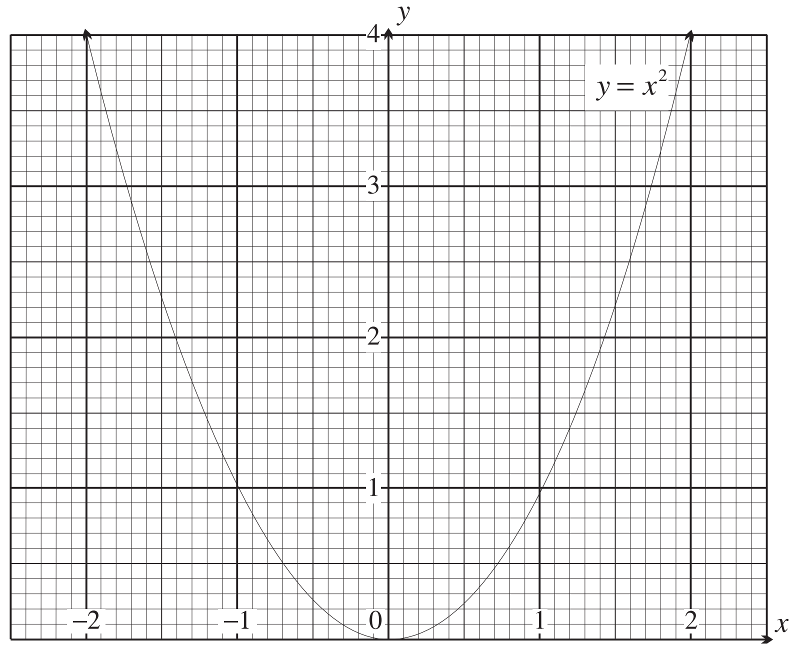
Exercise 3E

NOTE: Machine drawing of curves on a computer or a graphics calculator could be very helpful in this exercise.

1.



Photocopy the above sketch of the graph of $y = x^2$, for $-2 \le x \le 2$, in preparation for the following questions.

(a) Read $\sqrt{2}$ and $\sqrt{3}$ off the graph to one decimal place.

(b) What lines should be drawn on the graph to solve $x^2 = 2$ and $x^2 = 3$?

(c) Draw the line $y = x + 2$ on the graph, and hence read off the graph the solutions to $x^2 = x + 2$. Then check your solution by solving $x^2 = x + 2$ algebraically.

(d) From the graph, write down the solution of $x^2 > x + 2$.

(e) Draw a suitable line to solve $x^2 = 2 - x$ and $x^2 \le 2 - x$.

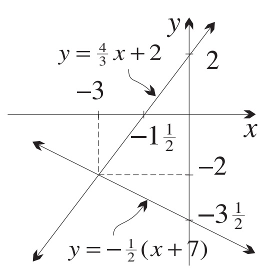
(f) Draw $y = x + 1$, and hence solve $x^2 = x + 1$ approximately. Check your result algebraically.

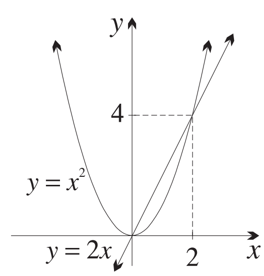
(g) Find approximate solutions for these quadratic equations by rearranging them with x^2 as subject, and drawing a suitable line on the graph:

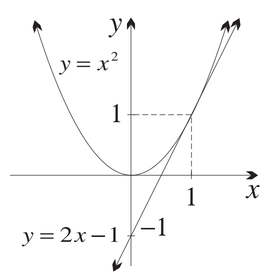
(i) $x^2 + x = 0$ (ii) $x^2 - x - \frac{1}{2} = 0$ (iii) $2x^2 - x - 1 = 0$

2. Use the given graphs to help solve each inequation:

(a) $\frac{4}{3}x + 2 \le -\frac{1}{2}(x + 7)$ (b) $x^2 \le 2x$ (c) $x^2 \le 2x - 1$







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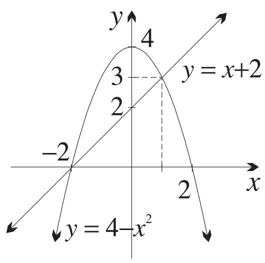
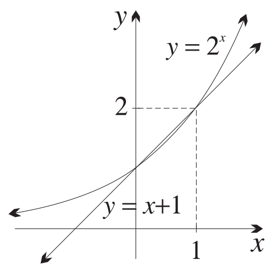
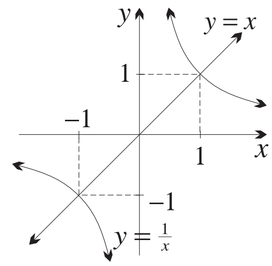
3. Solve each inequation, given the accompanying graphs of LHS and RHS:

(a) $4 - x^2 < x + 2$ (b) $2^x \le x + 1$ (c) $\dfrac{1}{x} < x$







4. For each pair below: (i) Carefully sketch each pair of equations. (ii) Hence find the solution to the simultaneous equations, given that all points of intersection have integer coordinates. (iii) Write down the equation satisfied by the x-coordinates of the points of intersection.

(a) $y = x - 2$ and $y = 3 - \frac{1}{4}x$ (c) $y = \dfrac{2}{x}$ and $y = x - 1$

(b) $y = x$ and $y = 2x - x^2$ (d) $y = x^3$ and $y = x$

5. Use your graphs in the previous question to help solve the following inequations:

(a) $x - 2 \ge 3 - \frac{1}{4}x$ (b) $x < 2x - x^2$ (c) $\dfrac{2}{x} > x - 1$ (d) $x^3 > x$

6. Draw graphs of the LHS and RHS of each equation on the same number plane in order to find the number of solutions. Do not attempt to solve them:

(a) $1 - \frac{1}{2}x = x^2 - 2x$ (c) $x^3 - x = \frac{1}{2}(x + 1)$ (e) $|x + 1| - 1 = \log_2 x$

(b) $|2x| = 2^x$ (d) $4x - x^2 = \dfrac{1}{x}$ (f) $2^{-x} - 1 = \dfrac{1}{x}$

7. (a) Sketch on the same number plane the functions $y = |x + 1|$ and $y = \frac{1}{2}x - 1$.

(b) Hence explain why all real numbers are solutions of the inequation $|x + 1| > \frac{1}{2}x - 1$.

8. Sketch each pair of equations and hence find the points of intersection:

(a) $y = |x + 1|$ and $y = 3$ (c) $y = |2x|$ and $2x - 3y + 8 = 0$

(b) $y = |x - 2|$ and $y = x$ (d) $y = |x| - 1$ and $y = 2x + 2$

9. Use your answers to the previous question to help solve:

(a) $|x + 1| \le 3$ (c) $|2x| \ge \dfrac{2x + 8}{3}$

(b) $|x - 2| > x$ (d) $|x| > 2x + 3$

DEVELOPMENT

10. (a) Sketch $y = x^2 - 6$ and $y = |x|$ on one set of axes.

(b) Find the x-coordinates of the points of intersection.

(c) Hence solve $x^2 - 6 \le |x|$.

11. (a) Draw $y = x^2 - 2$, $y = x$ and $y = -x$ on the same number plane and find all points of intersection of the three functions.

(b) Hence find the solutions of $x^2 - 2 = |x|$.

(c) Hence solve $x^2 - 2 > |x|$.

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12. (a) Sketch $y = |2x + 1|$.

(b) Draw on the same number plane $y = x + c$ for $c = -1$, $c = 0$ and $c = 1$.

(c) For what values of c does $|2x + 1| = x + c$ have two solutions?

13. (a) Use a diagram and Pythagoras' theorem to show that for $b > 0$, the perpendicular distance from the line $x + y = 2b$ to the origin is $b\sqrt{2}$.

(b) Hence find the range of values of b for which the line intersects the circle $x^2 + y^2 = 9$ twice.

14. (a) Sketch $y = |7x - 4|$ and $y = 4x + 3$ on the same number plane to find the number of solutions of $|7x - 4| = 4x + 3$.

(b) Why is it inappropriate to use the graph to find the exact solutions?

(c) Find the solutions by separately considering the two branches of the absolute value.

15. Sketch LHS and RHS on one pair of axes, then solve:

(a) $|x - 1| \le |x - 4|$

(b) $|x + 1| \ge |\frac{1}{2}x - 1|$

(c) $|2x| < |x - 2|$

(d) $|x - 3| < |2x + 1|$

16. Draw appropriate graphs on graph or grid paper, or on a machine, in order to find the solutions, or estimates to one decimal place:

(a) $x^3 = 2(x - 2)^2$

(b) $x^3 = \sqrt{4 - x^2}$

(c) $2^{-x} - (2x - x^2) = 0$

(d) $x^2 - x - \log_2(x + 1) = 0$

17. Find the values of x for which the LHS and RHS are equal. Then sketch LHS and RHS on the same number plane and hence solve each inequation:

(a) $x \ge \dfrac{2}{x - 1}$

(b) $\dfrac{x}{4} \le -\dfrac{2}{x - 2}$

(c) $\dfrac{x + 2}{2} > \dfrac{2}{x - 1}$

EXTENSION

18. (a) Carefully sketch the graph of $y = |2x + 4| + |x - 1| - 5$ and write down the equation of each branch.

(b) On the same number plane draw the lines $y = -1$ and $y = 2$. Hence solve the inequation $-1 \le |2x + 4| + |x - 1| - 5 \le 2$.

19. (a) Show that $y = mx + b$ must intersect $y = |x + 1|$ if $m > 1$ or $m < -1$.

(b) Given that $-1 \le m \le 1$, find the relationship between b and m so that the two graphs do not intersect.

(c) Generalise these results for $y = |px - q|$.

20. Sketch the LHS and RHS of $\dfrac{|2x - 3|}{4x - 6} \ge \dfrac{x}{4}$, paying attention to the branches of the LHS, and hence solve the inequality.

21. (a) Draw $y = a^x$ and $y = \log_a x$ for: (i) $a = 3$ (ii) $a = 2$ (iii) $a = \sqrt{2}$

(b) Conclude how many solutions $a^x = \log_a x$ may have.

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3 F Regions in the Number Plane

The circle $x^2 + y^2 = 25$ divides the plane into two *regions* — inside the circle and outside the circle. The graph of the inequation $x^2 + y^2 > 25$ will be one of these regions. It remains to work out which of these regions should be shaded.

Graphing Regions: To sketch the region of an inequation, use the following method.

GRAPHING THE REGION CORRESPONDING TO AN INEQUATION:

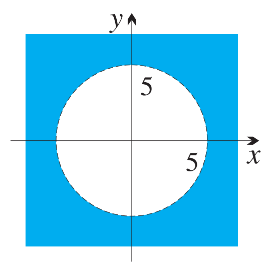
1. THE BOUNDARY: Replace the inequality symbol by an equal symbol, and graph the curve. This will be the boundary of the region, and should be drawn broken if it is excluded, and unbroken if it is included.
2. SHADING: Determine which parts are included and which are excluded, and shade the parts that are included. This can be done in two ways:

25

   (a) [Always possible] Take one or more test points not on any boundary, and substitute into the LHS and RHS of the original inequation. The origin is the easiest test point, otherwise try to choose points on the axes.
   (b) [Quicker, but not always possible] Alternatively, solve the inequation for y if possible, and shade the region above or below the curve. Or solve for x, and sketch the region to the right or left of the curve.
3. CHECKING BOUNDARIES AND CORNERS: Check that boundaries are correctly broken or unbroken. Corner points must be marked clearly with a closed circle if they are included, or an open circle if excluded.

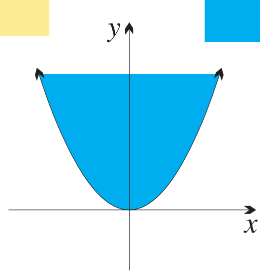
NOTE: [A nasty point] There may be points in the plane where the LHS or RHS of the inequation is undefined. For example, the RHS of $y > 1/x$ is undefined at all points on the y-axis, because $1/x$ is undefined when $x = 0$. If so, the set of all these points will usually be a boundary of the region too, and will be excluded.

WORKED EXERCISE: Sketch the region $x^2 + y^2 > 25$.



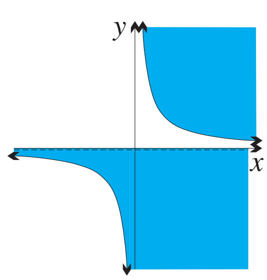
SOLUTION: The boundary is $x^2 + y^2 = 25$, and is excluded.
Take a test point $(0, 0)$. Then RHS $= 25$,
LHS $= 0$.
So $(0, 0)$ does not lie in the region.

WORKED EXERCISE: Sketch $y \geq x^2$.



SOLUTION: The boundary is $y = x^2$, and is included.
Because the inequation is $y \geq x^2$,
the region involved is the region *above* the curve.

WORKED EXERCISE: [A harder example] Sketch $x \geq \dfrac{1}{y}$.



SOLUTION: The boundary is $x = 1/y$, and is included.

Also, the x-axis $y = 0$ is a boundary, because the RHS is undefined when $y = 0$. This boundary is excluded.

Because the inequation is $x \geq 1/y$, the region to be shaded is the region to the right of the curve.

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Intersections and Unions of Regions: Some questions will ask explicitly for the intersection or union of two regions. Other questions will implicitly ask for intersections. For example,

$$|2x + 3y| < 6$$

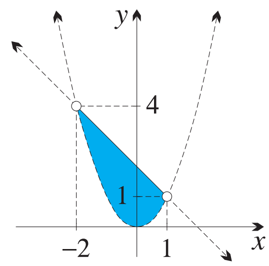
means $-6 < 2x + 3y < 6$, and so is the intersection of $2x + 3y > -6$ and $2x + 3y < 6$. Or there may be a restriction on x or on y, as in

$$x^2 + y^2 < 25, \text{ where } x \le 3 \text{ and } y > -4,$$

which means the intersection of three different regions.

26 INTERSECTIONS AND UNIONS OF REGIONS: Draw each region, then sketch the intersection or union.
Pay particular attention to whether corner points are included or excluded.

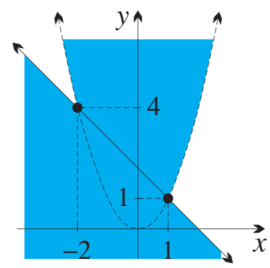
WORKED EXERCISE: Graph the intersection and union of the regions $y > x^2$ and $x + y \le 2$.



SOLUTION: The boundary of the first region is $y = x^2$, and the region lies above the curve (with the boundary excluded).

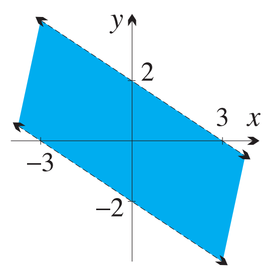
The boundary of the second region is $x + y = 2$. Solving for y gives $y \le 2 - x$, and so the region lies below the curve (with the boundary included).

By inspection, or by simultaneous equations, the parabola and the line meet at $(1, 1)$ and $(-2, 4)$. These points are excluded from the intersection because they are not in the region $y > x^2$, but included in the union because they are in the region $x + y \le 2$.



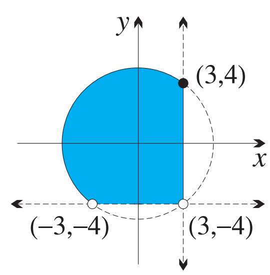
WORKED EXERCISE: Graph the region $|2x + 3y| < 6$.

SOLUTION: This is the region $-6 < 2x + 3y < 6$.
The boundaries are the parallel lines
$2x + 3y = 6$ and $2x + 3y = -6$,
both of which are excluded.
The required region is the region between these two lines.



WORKED EXERCISE: Graph the region $x^2 + y^2 \le 25$, for $x \le 3$ and $y > -4$, giving the coordinates of each corner point.

SOLUTION: The boundaries are $x^2 + y^2 = 25$ (included), and the vertical and horizontal lines
$x = 3$ (included) and $y = -4$ (excluded).
The points of intersection are $(3, 4)$ (included), $(3, -4)$ (excluded) and $(-3, -4)$ (excluded).



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Exercise 3F

1. For each inequation: (i) sketch the boundary, (ii) shade the region above or below the boundary, as required.

(a) $y < 1$
(b) $y \geq -3$
(c) $y > x - 1$
(d) $y \leq 3 - x$
(e) $y \leq 2x + 2$
(f) $y < \frac{1}{2}x - 1$

2. For each inequation: (i) sketch the boundary, (ii) shade the region to the left or right of the boundary, as required.

(a) $x < -2$
(b) $x > 1$
(c) $x \geq y + 2$
(d) $x < 2y - 1$
(e) $x > 3 - y$
(f) $x \leq \frac{1}{2}y + 2$

3. For each inequation, sketch the boundary line, then use a suitable test point to decide which side of the line to shade.

(a) $2x + 3y - 6 > 0$
(b) $x - y + 4 \geq 0$
(c) $y - 2x + 3 < 0$

4. For each inequation, sketch the boundary circle, then use a suitable test point to decide which region to shade.

(a) $x^2 + y^2 < 4$
(b) $x^2 + y^2 \geq 1$
(c) $(x - 2)^2 + y^2 \leq 4$
(d) $(x + 1)^2 + (y - 2)^2 > 9$

5. Sketch the following regions (some of the quadratics need factoring):

(a) $y \geq x^2 - 1$
(b) $y < x^2 - 2x - 3$
(c) $y \geq x^2 + 2x + 1$
(d) $y > 4 - x^2$
(e) $y \leq x^2 + 3x$
(f) $y \leq 2 + x - x^2$
(g) $y < (5 - x)(1 + x)$
(h) $y > (2x - 3)(x + 1)$
(i) $y \leq (2x + 1)(x - 3)$

6. Draw the following regions of the number plane:

(a) $y > 2^x$
(b) $y \geq |x|$
(c) $y \leq |x + 1|$
(d) $y > x^3$
(e) $y \leq \log_2 x$
(f) $y < |\frac{1}{2}x - 1|$

7. (a) Find the intersection point of the lines $x = -1$ and $y = 2x - 1$.

(b) Hence sketch the intersection of the regions $x > -1$ and $y \leq 2x - 1$, paying careful attention to the boundaries and their point of intersection.

(c) Likewise sketch the union of the two regions.

8. (a) Sketch on separate number planes the two regions $y < x$ and $y \geq -x$. Hence sketch:

(i) the union of these two regions,
(ii) the intersection of the regions.

Pay careful attention to the boundaries and their points of intersection.

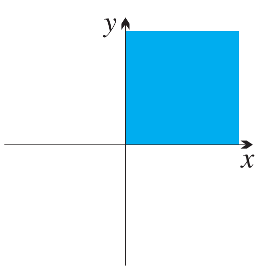
(b) Similarly, graph the union and intersection of:

(i) $y > x$ and $y \leq 2 - x$
(ii) $y > \frac{1}{2}x + 1$ and $y \leq -x - 2$

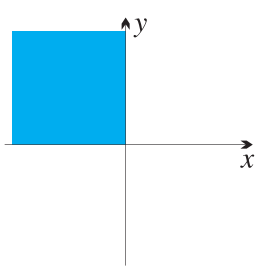
DEVELOPMENT

9. Identify the inequations that correspond to the following regions:

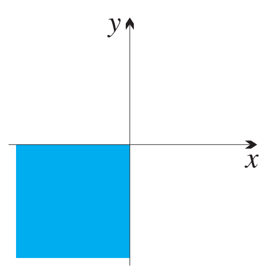
(a)



(b)



(c)



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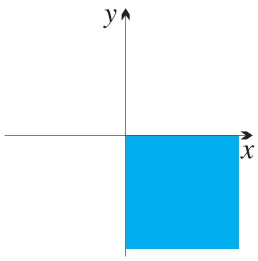
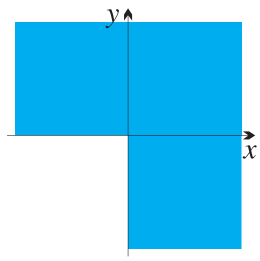
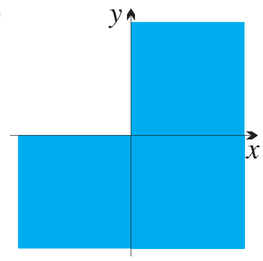
(d)



(e)

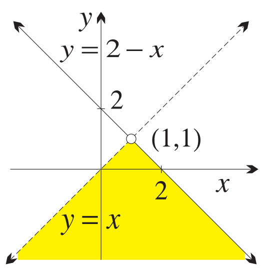


(f)

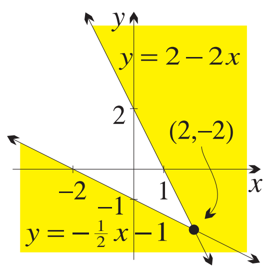


10. Write down intersections or unions that correspond to the following regions:

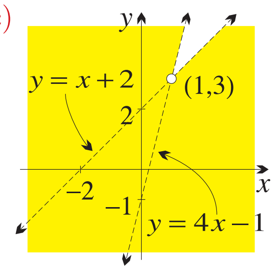
(a)



(b)



(c)



11. (a) Show that the lines $y = x + 1$, $y = -\frac{1}{2}x - 2$, and $y = 4x - 2$ intersect at $(-2, -1)$, $(0, -2)$ and $(1, 2)$. Then sketch all three on the same number plane.

(b) Hence sketch the regions indicated by:

(i) $y < x + 1$ and $y \geq -\frac{1}{2}x - 2$

(ii) $y < x + 1$ and $y \geq -\frac{1}{2}x - 2$ and $y < 4x - 2$

(iii) $y > x + 1$ or $y < -\frac{1}{2}x - 2$ or $y < 4x - 2$

12. (a) Sketch the intersection of $x^2 + y^2 > 1$ and $x^2 + y^2 \leq 9$.

(b) What is the union of these two regions?

13. (a) Sketch the union of $x^2 + y^2 \leq 1$ and $y > 2 - x$.

(b) What is the intersection of these two regions?

14. (a) Find the intersection points of the line $y = 4 - x$ and the circle $x^2 + y^2 = 16$.

(b) Hence sketch (i) the intersection and (ii) the union of $y \geq 4 - x$ and $x^2 + y^2 < 16$.

15. (a) The inequation $|x| < 2$ implies the intersection of two regions in the number plane. Write down the equations of these two regions. Hence sketch the region $|x| < 2$.

(b) The inequation $|x - y| \leq 2$ implies the intersection of two regions. Write down the equations of these two regions. Hence sketch $|x - y| \leq 2$.

16. (a) The inequation $|y| \geq 1$ implies the union of two regions in the number plane. Write down the equations of these two regions. Hence sketch the region given by $|y| \geq 1$.

(b) The inequation $|y+2x| > 1$ implies the union of two regions. Write down the equations of these two regions. Hence sketch the region $|y + 2x| > 1$.

17. Sketch the region $x^2 + y^2 \geq 5$ for the domain $x > -1$ and range $y < 2$, and give the coordinates of each corner.

18. Sketch the region $y \leq x^2 - 2x + 2$ with $y \geq 0$ and $0 \leq x \leq 2$.

19. (a) Draw the curve $y = \sqrt{x}$.

(b) Explain why the y-axis $x = 0$ is a boundary for $y < \sqrt{x}$.

(c) Hence sketch the region $y < \sqrt{x}$.

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20. (a) Explain why $x = 0$ is a boundary for $y > \dfrac{1}{x}$.

(b) Hence sketch the region $y > \dfrac{1}{x}$.

21. Carefully sketch the following regions, paying attention to implied boundaries:

(a) $y < \sqrt{4 - x^2}$ (b) $x > \sqrt{9 - y^2}$

22. Sketch the region defined by $x > |y+1|$. [HINT: $x = |y+1|$ is the inverse of what function?]

EXTENSION

23. (a) How many regions do the coordinate axes and the hyperbola $y = \dfrac{1}{x}$ divide the number plane into? (b) Carefully sketch the following regions. [HINT: It may help to take test points in each of the regions found in the previous part.]

(i) $y < \dfrac{1}{x}$ (ii) $xy < 1$ (iii) $1 > \dfrac{1}{xy}$

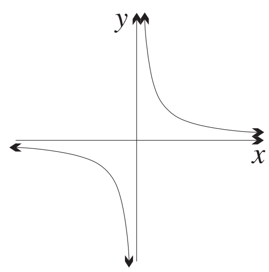
24. Graph the regions: (a) $|y| > |x|$ (b) $|xy| \geq 1$ (c) $\dfrac{1}{x} > \dfrac{1}{y}$

25. (a) Consider the region A with $x^2 + y^2 \geq 4$ and $x^2 + y^2 \leq 9$, and the region B which is the union of A with $x^2 + y^2 \leq 1$. Region A is called *connected* but region B is not connected. Discuss what might define a connected region.

(b) Consider the region A with $x^2 + y^2 \leq 2$, and the region B which is the intersection of A with $y \leq |x|$. Region A is *convex* but region B is not convex. Discuss what might define a region that is not convex.

3 G Asymptotes and a Curve Sketching Menu

The chapter concludes with a study of vertical and horizontal asymptotes, principally of rational functions. The three techniques of Sections 3B and 3C, together with asymptotes, are then combined into a systematic four-step approach to sketching an unknown graph. No such simple menu could possibly deal with the great variety of possible graphs; nevertheless, it will allow the main features of a surprising number of functions to be found. Two further steps involving calculus will be added in Chapter Ten.



Vertical Asymptotes: The rectangular hyperbola $y = 1/x$ has the y-axis as an asymptote, as discussed in Section 2G. This is because the values of y become very large in size, positive or negative, when x is near $x = 0$. In this course, discontinuities mostly arise from zeroes in the denominator. The test for a vertical asymptote is then very simple:

27 TESTING FOR VERTICAL ASYMPTOTES: If the denominator goes to zero as $x \to a$, and the numerator is *not* zero at $x = a$, then the vertical line $x = a$ is an asymptote.

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Questions still remain about the behaviour of the function on each side of the asymptote. The table of signs is the easiest way to distinguish the two cases.

28 BEHAVIOUR NEAR A VERTICAL ASYMPTOTE: The choice between $y \to \infty$ and $y \to -\infty$ can be made by looking at a table of signs.

WORKED EXERCISE: Find any vertical asymptotes of the function $y = \dfrac{x-1}{x-4}$, and use a table of values to discuss the behaviour of the curve near them. (The curve itself is sketched in the next worked exercise).

SOLUTION: The vertical line $x = 4$ is an asymptote, because at $x = 4$ the denominator vanishes but the numerator does not. From the table of values opposite, around the zero at $x = 1$ and the discontinuity at $x = 4$ (or from the sketch below):

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	$*$	4
sign	$+$	0	$-$	$*$	$+$

$$y \to \infty \text{ as } x \to 4^+\,, \quad \text{and} \quad y \to -\infty \text{ as } x \to 4^-\,.$$

NOTE: Functions like the one in the previous example, which are ratios of two polynomials, are called *rational functions*. Almost all the functions in this section are rational functions.

Behaviour as x and as x – — Horizontal Asymptotes: For most functions in this course, the following method will be sufficient.

29 BEHAVIOUR FOR LARGE x: Divide top and bottom by the highest power of x in the denominator. Then use the fact that

$$\frac{1}{x} \to 0, \text{ as } x \to \infty \text{ and as } x \to -\infty.$$

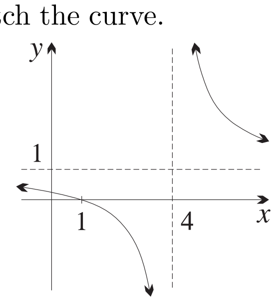
If $f(x)$ tends to a definite limit b as $x \to \infty$ or as $x \to -\infty$, then the horizontal line $y = b$ is a horizontal asymptote.

WORKED EXERCISE: Examine the behaviour of the earlier function $f(x) = \dfrac{x-1}{x-4}$ as $x \to \infty$ and as $x \to -\infty$, noting horizontal asymptotes, then sketch the curve.

SOLUTION: Dividing top and bottom by x,

$$f(x) = \frac{1 - \frac{1}{x}}{1 - \frac{4}{x}}\,,$$

and so $f(x) \to 1$ as $x \to \infty$ and as $x \to -\infty$.
Hence $y = 1$ is a horizontal asymptote.



WORKED EXERCISE: Examine the behaviour of these functions as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes:

(a) $f(x) = \dfrac{3 - 5x - 4x^2}{4 - 5x - 3x^2}$

(b) $f(x) = \dfrac{x - 1}{x^2 - 4}$

(c) $f(x) = \dfrac{x^2 - 1}{x - 4}$

(d) $f(x) = \dfrac{1}{x} + \dfrac{1}{x - 3}$

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SOLUTION:

(a) Dividing top and bottom by x^2, $f(x) = \dfrac{\frac{3}{x^2} - \frac{5}{x} - 4}{\frac{4}{x^2} - \frac{5}{x} - 3}$,

so $f(x) \to \frac{4}{3}$ as $x \to \infty$ and as $x \to -\infty$,
and $y = \frac{4}{3}$ is a horizontal asymptote.

(b) Dividing top and bottom by x^2, $f(x) = \dfrac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{4}{x^2}}$,

so $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$,
and the x-axis $y = 0$ is a horizontal asymptote.

(c) Dividing top and bottom by x, $f(x) = \dfrac{x - \frac{1}{x}}{1 - \frac{4}{x}}$,

so $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$,
and there are no horizontal asymptotes.

(d) Here $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$,
so $y = 0$ is a horizontal asymptote.

A Curve Sketching Menu: Here is a systematic approach to sketching a curve whose function is not easily analysed in terms of transformations of known curves. A 'sketch' of a graph is not an accurate plot. It is a neat diagram showing the main features of the curve, and unless there are major difficulties involved:

30 SKETCHES: A sketch should normally show any x- and y-intercepts, give some indication of scale on both axes, and have labels on both axes.

Suppose that $f(x)$ is an unfamiliar function, and that a sketch of $y = f(x)$ is required:

31 A CURVE SKETCHING MENU:

0. PREPARATION: Combine any fractions using a common denominator, then factor top and bottom as far as possible.
1. DOMAIN: Find the domain (*always* do this first).
2. SYMMETRY: Find whether the function is odd, or even, or neither.
3. A. INTERCEPTS: Find the y-intercept and the x-intercepts (zeroes).
   B. SIGN: Find where the function is positive, and where it is negative.
4. A. VERTICAL ASYMPTOTES: Examine the behaviour near any discontinuities, noting any vertical asymptotes.
   B. HORIZONTAL ASYMPTOTES: Examine the behaviour of $f(x)$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

NOTE: Finding the domain and finding the zeroes may both require factorisation, which is the reason why the preparatory Step 0 is useful. Factorisation, however, may not always be possible, even with the formula for the roots of a quadratic, and in such cases approximation methods may be useful. Questions in exercises and examinations will normally give guidance as to what is required.

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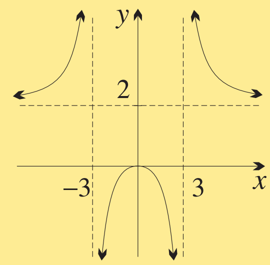
Putting it All Together — Example 1: All that remains is to give two examples of the whole process. First, here is the method applied to $f(x) = \dfrac{2x^2}{x^2 - 9}$.

SOLUTION:



0. PREPARATION: $f(x) = \dfrac{2x^2}{(x-3)(x+3)}$

1. DOMAIN: $x \neq 3$ and $x \neq -3$.

2. SYMMETRY: $$f(-x) = \frac{2(-x)^2}{(-x)^2 - 9} = \frac{2x^2}{x^2 - 9} = f(x)$$
so $f(x)$ is even, and has line symmetry in the y-axis.

3. INTERCEPTS AND SIGN: There is a zero at $x = 0$, and discontinuities at $x = 3$ and $x = -3$:

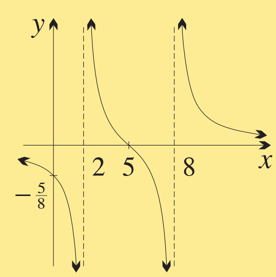
x	-4	-3	-1	0	1	3	4
$f(x)$	$\frac{32}{7}$	$*$	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$*$	$\frac{32}{7}$
sign	$+$	$*$	$-$	0	$-$	$*$	$+$

4A. VERTICAL ASYMPTOTES: At $x = 3$ and $x = -3$ the denominator vanishes but the numerator does not, so $x = 3$ and $x = -3$ are vertical asymptotes. From the table of signs,
$$f(x) \to \infty \text{ as } x \to 3^+ \quad \text{and} \quad f(x) \to -\infty \text{ as } x \to 3^-,$$
$$f(x) \to -\infty \text{ as } x \to (-3)^+ \quad \text{and} \quad f(x) \to \infty \text{ as } x \to (-3)^-.$$

4B. HORIZONTAL ASYMPTOTES: Dividing through by x^2, $f(x) = \dfrac{2}{1 - \frac{9}{x^2}}$,
so $f(x) \to 2$ as $x \to \infty$ and as $x \to -\infty$,
and $y = 2$ is a horizontal asymptote.

Putting it All Together — Example 2: The second example is much more difficult and requires more algebraic manipulation. The calculations involving sign show an alternative approach using signs rather than numbers:
$$f(x) = \frac{1}{x-2} + \frac{1}{x-8}$$

SOLUTION:



0. PREPARATION: $$f(x) = \frac{(x-8)+(x-2)}{(x-2)(x-8)} = \frac{2x-10}{(x-2)(x-8)} = \frac{2(x-5)}{(x-2)(x-8)}$$

1. DOMAIN: $x \neq 2$ and $x \neq 8$.

2. SYMMETRY: $f(x)$ is neither even nor odd.

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3. INTERCEPTS AND SIGN: There is a zero at $x = 5$, and discontinuities at $x = 2$ and $x = 8$:

x	0	2	3	5	6	8	9
$x-2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-5$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-8$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	$*$	$+$	0	$-$	$*$	$+$

4A. VERTICAL ASYMPTOTES: At $x = 2$ and $x = 8$ the denominator vanishes, but the numerator does not, so $x = 2$ and $x = 8$ are vertical asymptotes. From the table of signs,

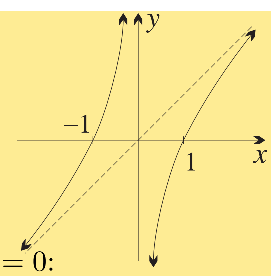
$$f(x) \to \infty \text{ as } x \to 2^+ \quad \text{and} \quad f(x) \to -\infty \text{ as } x \to 2^-,$$
$$f(x) \to \infty \text{ as } x \to 8^+ \quad \text{and} \quad f(x) \to -\infty \text{ as } x \to 8^-.$$

4B. HORIZONTAL ASYMPTOTES: From the original form of the given equation, $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$, so $y = 0$ is a horizontal asymptote.

An Example with an Oblique Asymptote: Sometimes it becomes obvious from examination of a function for large x that the curve has an oblique asymptote, and although a systematic treatment is not appropriate, the following example is quite straightforward.

WORKED EXERCISE: Sketch the graph of $y = x - \dfrac{1}{x}$.

SOLUTION:



0. $y = \dfrac{x^2-1}{x} = \dfrac{(x-1)(x+1)}{x}$

1. The domain is $x \neq 0$.

2. $f(-x) = -x + \dfrac{1}{x} = -f(x)$, so the function is odd.

3. There are zeroes at $x = 1$ and $x = -1$, and a discontinuity at $x = 0$:

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	$-1\frac{1}{2}$	0	$1\frac{1}{2}$	$*$	$-1\frac{1}{2}$	0	$1\frac{1}{2}$

4A. The y-axis is a vertical asymptote.
As $x \to 0^+$, $y \to -\infty$, and as $x \to 0^-$, $y \to \infty$.

4B. As $|x| \to \infty$, $\dfrac{1}{x} \to 0$, so $y - x \to 0$ and $y = x$ is an oblique asymptote.

Extension — Long Division and Oblique Aymptotes: Systematic examination of oblique asymptotes is not required in this course, but is described here for those who may be interested. A rational function has an oblique asymptote when its numerator has degree one more than the degree of its denominator. The equation of the oblique asymptotes is then obtained by long division. For example,

$$\frac{2x^3+9x^2+8x+1}{x^2+3x-1} = 2x+3+\frac{x+4}{x^2+3x-1},$$

and so $y = 2x+3$ is an oblique asymptote of the graph of $y = \dfrac{2x^3+9x^2+8x+1}{x^2+3x-1}$.

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Exercise 3G

NOTE: Purely algebraic approaches to sketching curves like these can be rather demanding. As an alternative, some questions could be investigated first by machine drawing, followed by algebraic explanation of the features.

1. Find the horizontal asymptotes of these functions by dividing through by the highest power of x in the denominator, and taking the limit as $x \to \infty$ and as $x \to -\infty$:

(a) $f(x) = \dfrac{1}{x+1}$ (c) $f(x) = \dfrac{2x+1}{3-x}$ (e) $\dfrac{1}{x^2+1}$

(b) $f(x) = \dfrac{x-3}{x+4}$ (d) $f(x) = \dfrac{5-x}{4-2x}$ (f) $\dfrac{x}{x^2+4}$

2. Sketch each rational function below after carrying out the following steps: (i) State the natural domain. (ii) Find the y-intercept. (iii) Explain why $y = 0$ is a horizontal asymptote. (iv) Draw up a table of values and examine the sign. (v) Identify any vertical asymptotes, and use the table of signs to write down its behaviour near any vertical asymptotes.

(a) $y = \dfrac{1}{x-1}$ (b) $y = \dfrac{2}{3-x}$ (c) $y = -\dfrac{2}{x+2}$ (d) $y = \dfrac{5}{2x+5}$

3. Sketch the curve $y = \dfrac{x}{x-2}$ after performing the following steps:

(a) Write down the natural domain. (b) Find the intercepts and examine the sign.
(c) Show that $y = 1$ is the horizontal asymptote.
(d) Investigate the behaviour near the vertical asymptote.

4. Consider $y = \dfrac{x-1}{x+3}$.

(a) Where is the function undefined?
(b) Find the intercepts and examine the sign of the function.
(c) Identify and investigate the vertical and horizontal asymptotes.
(d) Hence sketch the curve.

5. Investigate the domain, intercepts, sign and asymptotes of the function $y = \dfrac{2}{(x-1)^2}$ and hence sketch its graph.

6. (a) Show that $y = \dfrac{3x}{x^2+9}$ is an odd function.

(b) Show that it has only one intercept with the axes at the origin.
(c) Show that the x-axis is the horizontal asymptote.
(d) Hence sketch the curve.

7. (a) Investigate whether $y = \dfrac{10}{x^2+5}$ is even or odd. (b) What are its intercepts?

(c) Show that $y = 0$ is a horizontal asymptote. (d) Hence sketch the curve.

8. Factor if necessary, and find any vertical and horizontal asymptotes:

(a) $\dfrac{2(x+2)(x+3)}{(x-1)(x-3)}$ (b) $\dfrac{1-4x^2}{1-9x^2}$ (c) $\dfrac{x^2+2x+2}{x^2+5x+4}$ (d) $\dfrac{x-5}{x^2+3x-10}$

[Machine sketching of these curves would be useful to put these features in context.]

9. (a) Show that $y = \dfrac{4-x^2}{4+x^2}$ is even. (b) Find its three intercepts with the axes.

(c) Determine the equation of the horizontal asymptote. (d) Sketch the curve.

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DEVELOPMENT

10. This question looks at graphs that have holes rather than vertical asymptotes.

(a) Show that $\dfrac{x^2-4}{x-2} = x+2$, provided $x \neq 2$. Hence sketch the graph of $y = \dfrac{x^2-4}{x-2}$.

(b) Similarly sketch graphs of:

(i) $y = \dfrac{(x+1)(x-3)}{x+1}$ (ii) $y = \dfrac{x^3-1}{x-1}$ (iii) $y = \dfrac{(x+2)(x-2)}{(x-2)(x+1)}$

11. Factor the numerator of $y = \dfrac{x^2-2x-3}{x-3}$ and hence show that the curve does not have a vertical asymptote at $x = 3$. Sketch the curve.

12. Factor the numerator of $y = \dfrac{x^3-2x^2-x+2}{x-1}$ by grouping in pairs. Hence show that there is no vertical asymptote at $x = 1$. Sketch the curve.

13. (a) Show that $y = \dfrac{1}{x+1} - \dfrac{1}{x}$ can be written as $y = -\dfrac{1}{x(x+1)}$. Then identify the domain and any zeroes, examine the asymptotes and sign, and hence sketch the graph.

(b) Likewise express $y = \dfrac{1}{x+3} + \dfrac{1}{x-3}$ with a common denominator and sketch it.

14. (a) Examine the sign and asymptotes of $y = \dfrac{1}{x(x-2)}$ and hence sketch the curve.

(b) Likewise sketch $y = \dfrac{2}{x^2-4}$.

15. [Two harder sketches with oblique asymptotes] (a) Identify the oblique asymptote of the function $y = x + \dfrac{1}{x}$. Then use the appropriate steps of the curve sketching menu to sketch it. (b) Similarly sketch $y = 2^{-x} - x - 3$, using the fact that $2^{-x} \to 0$ as $x \to \infty$ to identify the oblique asymptote.

16. Use the curve sketching menu as appropriate to obtain the graphs of:

(a) $y = \dfrac{1+x^2}{1-x^2}$ (c) $y = \dfrac{x-1}{(x+1)(x-2)}$ (e) $y = \dfrac{x^2-4}{(x+2)(x-1)}$

(b) $y = \dfrac{x+1}{x(x-3)}$ (d) $y = \dfrac{x^2-2x}{x^2-2x+2}$ (f) $y = \dfrac{x^2-2}{x}$

EXTENSION

17. (a) Show that $\dfrac{(x-1)(x+2)}{x-3} = x+4+\dfrac{10}{x-3}$, and deduce that $y = \dfrac{(x-1)(x+2)}{x-3}$ has an oblique asymptote $y = x+4$. Then sketch the graph.

(b) Likewise sketch $y = \dfrac{x^2-4}{x+1} = x-1-\dfrac{3}{x+1}$, showing the oblique asymptote.

18. Consider carefully the asymptotes and intercepts of the following functions, and then sketch them:

(a) $y = \dfrac{1-2^x}{1+2^x}$ (b) $y = \dfrac{1+2^x}{1-2^x}$

19. Investigate the asymptotic behaviour of the following functions, and graph them:

(a) $y = \dfrac{x^3-1}{x}$ (b) $y = \dfrac{1}{x} + \sqrt{x}$ (c) $y = |x| + \dfrac{1}{x}$

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