The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Board Tutorial for Week 10** 2017

Important Ideas and Useful Facts:

(ii) The *determinant* of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

(iv) The *determinant* of A is denoted by $\det A$ or $|A|$. If $A = [a_{ij}]$ is an $n \times n$ matrix and A_{ij} denotes the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A, then expanding along the ith row (for fixed i) gives

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} ,$$

and down the jth column (for fixed j) becomes

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} .$$

(v) Determinant method for cross products: If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and $\mathbf{w} = d\,\mathbf{i} + e\,\mathbf{j} + f\,\mathbf{k}$ then

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} .$$

(vi) Multiplicative property: $\det(AB) = (\det A)(\det B)$.

(vii) Invertibility criterion: A square matrix is invertible if and only if its determinant is nonzero.

(viii) If B is obtained from A by swapping two rows or swapping two columns then

$$\det B = -\det A .$$

(ix) If B is obtained from A by multiplying a row or column by λ then

$$\det B = \lambda \det A .$$

(x) If B is obtained from A by adding a multiple of one row [column] to another row [column] then

$$\det B = \det A .$$

(xi) If B is the *transpose* of A, that is, obtained by interchanging rows and columns, then

$$\det B = \det A .$$

(xii) If A is *triangular*, that is all entries above or below the diagonal are zero, then $\det A$ is the product of the diagonal elements.

Tutorial Exercises:

6. Justify briefly the following calculation:

$$\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 2 & -3 & -2 \\ 3 & -3 & 0 \\ 1 & -14 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ 3 & 0 & 0 \\ 1 & -13 & 0 \end{vmatrix} = -2 \begin{vmatrix} 3 & 0 \\ 1 & -13 \end{vmatrix} = 78$$

7. Use elementary row and column operations, or otherwise, to find the following:

(i) $\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix}$

8. Find $\mathbf{v} \times \mathbf{w}$ using (v) above in each of the following cases:

(i) $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{w} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ (ii) $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$, $\mathbf{w} = -\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

9. Make sense of the expression $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$ where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are geometric vectors. Explain how

$$\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad \text{follows from} \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} .$$

10. Calculate $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$ in each of the following cases:

(i) $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{w} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

(ii) $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$, $\mathbf{w} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$

11. Use the multiplicative property of the determinant to verify that if A is an invertible matrix then $\det A \neq 0$ and $\det A^{-1} = (\det A)^{-1}$.

12. Explain briefly, using properties of determinants, why a square matrix with two identical rows or two identical columns has zero determinant.

13. Decide whether the following statements are true for all 2×2 matrices A and B:

(i) $\det(AB) = (\det B)(\det A)$ (ii) $\det(A + B) = (\det A) + (\det B)$

(iii) $\det(2A) = 2 \det A$ (iv) $\det(-A) = \det A$

14. Without evaluating it, but using simple properties of determinants, explain why the following determinant is divisible by 17, given that 867, 459 amd 187 are each divisible by 17:

$$\begin{vmatrix} 8 & 6 & 7 \\ 4 & 5 & 9 \\ 1 & 8 & 7 \end{vmatrix}$$

15.* Determine the values of λ for which $\det(A - \lambda I) = 0$ in each case:

(i) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ (iii) $A = \begin{bmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{bmatrix}$