The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Board tutorial for Week 12** 2017

Important Ideas and Useful Facts:

(i) Let M be a square matrix, $\mathbf{x}$ a nonzero column vector and λ a scalar such that

$$M\mathbf{x} = \lambda\mathbf{x}\,.$$

Then λ is called an *eigenvalue* of M and $\mathbf{x}$ is called an *eigenvector* of M associated with the eigenvalue λ.

(ii) The *eigenspace* of M associated with an eigenvalue λ is the collection

$$\left\{\, \mathbf{v} \;\middle|\; M\mathbf{v} = \lambda\mathbf{v} \,\right\} \;=\; \left\{\, \mathbf{v} \;\middle|\; (M-\lambda I)\mathbf{v} = \mathbf{0} \,\right\}$$

comprising all the eigenvectors of M associated with λ and the zero vector

(iii) A scalar λ is an eigenvalue of a square matrix M if and only if $\det(M - \lambda I) = 0$.

(iv) The expression $\det(M-\lambda I)$ is a polynomial in λ and is called the *characteristic polynomial* of M. Thus the eigenvalues of a matrix are the roots of its characteristic polynomial.

(v) Finding the eigenspace corresponding to the eigenvalue λ of a matrix M is equivalent to solving the homogeneous system with coefficient matrix $M-\lambda I$. After the eigenspace has been found, substituting particular values of the parameters yields particular eigenvectors.

(vi) The eigenvalues of a triangular matrix are simply the diagonal entries.

(viii) Let M be a square $n\times n$ matrix with eigenvalues $\lambda_1,\dots,\lambda_n$ and corresponding eigenvectors $\mathbf{v}_1,\dots,\mathbf{v}_n$. Then

$$MP = PD$$

where D is the diagonal matrix with eigenvalues down the diagonal and P the matrix with corresponding eigenvectors as columns. If P is invertible then

$$M = PDP^{-1} \qquad \text{and} \qquad D = P^{-1}MP\,.$$

In this case we say that M is *diagonalisable.*

(ix) In the preceding discussion, if the eigenvalues are all different then P is invertible and M is diagonalisable.

(x) If M is diagonalisable then powers of M can be found easily by the formula

$$M^n = PD^nP^{-1}\,.$$

(xi) The Fundamental Theorem of Algebra: Every nonzero polynomial with complex number coefficients has a root in the complex numbers.

(xii) The Cayley-Hamilton Theorem: Every square matrix is a root of its own characteristic polynomial.

Tutorial Exercises:

5. Find the eigenvalues and corresponding eigenvectors for $M = \begin{bmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{bmatrix}$.

6. The matrix $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalues 2 and 4 with corresponding eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

(i) Write down an invertible matrix P and a diagonal matrix D such that

$$B = PDP^{-1}.$$

(ii) Find a formula for B^n, and use it to find B^3 and B^4.

7. The matrix $C = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has eigenvalues 0, 1 and 3 with corresponding eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ respectively.

(i) Write down an invertible matrix P and diagonal matrix D such that

$$C = PDP^{-1}.$$

(ii) Find a formula for C^n, and use it to find C^4.

8. Verify that if A is invertible and λ is an eigenvalue of A, then $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A^{-1}. What can be said about eigenvalues of A^k where k is any integer?

9. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for a matrix M corresponding to different eigenvalues λ_1 and λ_2. Explain why $\mathbf{v}_1$ cannot be a scalar multiple of $\mathbf{v}_2$.

10. Use the multiplicative property of the determinant to verify that if A and B are square matrices of the same size, and B is invertible, then A and $B^{-1}AB$ have the same eigenvalues.

11.* Suppose that $0 \leq \theta \leq \pi$. Verify that $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has real eigenvalues if and only if $\theta = 0$ or π. Interpret this result geometrically.

12.* Let A be a square matrix with eigenvalue λ. Prove the following implications:

(i) $A^2 = 0 \implies \lambda = 0$

(ii) $A^2 = A \implies \lambda = 0$ or $\lambda = 1$

(iii) $A^2 = I \implies \lambda = 1$ or $\lambda = -1$

13.* Three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are said to be *linearly independent* if

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = \mathbf{0} \qquad \implies \qquad \alpha = \beta = \gamma = 0,$$

where α, β, γ are scalars. Explain why three eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ corresponding to three different eigenvalues λ_1, λ_2, λ_3 of a matrix M must be linearly independent.