MATH 1901 Assignment 1

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Tutorial: Thursday 10 - 11 am
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Q1. a)

RTP $x^{n+1} - (n+1)x + n = (x-1)\left[1 + x + x^2 + \dots + x^n - (n+1)\right]$
for $n \geq 1$, $x \in \mathbb{R}$, $n \in \mathbb{N}$

$RHS = (x-1)\left[1 + x + x^2 + \dots + x^n - (n+1)\right]$

$1 + x + x^2 + \dots + x^n$ is a GP, with first term $= 1$
ratio $= x$, and $(n+1)$ terms

$\therefore 1 + x + x^2 + \dots + x^n = \dfrac{(x^{n+1} - 1)}{x-1}$

$$\therefore RHS = (x-1) \times \left[\frac{(x^{n+1}-1)}{(x-1)} - (n+1)\right]$$

$$= \frac{(x-1)(x^{n+1}-1)}{(x-1)} - (x-1)(n+1)$$

$$= x^{n+1} - 1 - (nx + x - n - 1)$$

$$= x^{n+1} - 1 - nx - x + n + 1$$

$$= x^{n+1} - (n+1)x + n$$

$$= LHS$$

$\therefore x^{n+1} - (n+1)x + n = (x-1)\left[1 + x + x^2 + \dots + x^n - (n+1)\right]$
for $n \geq 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}$

b) RTP $x^{n+1} - (n+1)x + n \geq 0$ for all $x \geq 0$

for $x \geq 1$

$\therefore (x-1) \geq (1-1)$
$= 0$

$\therefore (x-1) \geq 0$

$\left[1 + x + x^2 + \dots + x^n - (n+1)\right] \geq \left[1 + 1 + 1 + \dots + 1 - (n+1)\right]$
$= \left[n \times 1 + 1 - (n+1)\right]$
$= \left[(n+1) - (n+1)\right]$
$= 0$

$\therefore \left[1 + x + x^2 + \dots + x^n - (n+1)\right] \geq 0$

$\therefore (x-1)\left[1 + x + x^2 + \dots + x^n - (n+1)\right] \geq 0$ for $x \geq 1$

for $0 \leq x < 1$

$\therefore (x-1) < (1-1)$
$= 0$

$\therefore (x-1) < 0$

$$[1+x+x^2+\dots+x^n-(n+1)] < [1+1+1+\dots+1-(n+1)]$$
$$= [n\times 1+1-(n+1)]$$
$$= (n+1)-(n+1)$$
$$= 0$$

$\therefore [1+x+x^2+\dots+x^n-(n+1)] < 0$

$\therefore (x-1)[1+x+x^2+\dots+x^n-(n+1)] > 0$ for $0 \le x < 1$

AND $(x-1)[1+x+x^2+\dots+x^n-(n+1)] \geqslant 0$ for $x \geqslant 1$

$\therefore (x-1)[1+x+x^2+\dots+x^n-(n+1)] \geqslant 0$ for $x \geqslant 0$

NOW $(x-1)[1+x+x^2+\dots+x^n-(n+1)] = x^{n+1}-(n+1)x+n$

$\therefore x^{n+1}-(n+1)x+n \geqslant 0$ for $x \geqslant 0$ (1)

c) $a_n := \dfrac{x_1+x_2+\dots+x_n}{n}$

NOW let $x = \dfrac{a_{n+1}}{a_n}$

Substituting into (1): $x^{n+1}-(n+1)x+n \geqslant 0$

$$\therefore \frac{a_{n+1}^{n+1}}{a_n^{n+1}} - (n+1)\frac{a_{n+1}}{a_n} + n \geqslant 0$$

$$\frac{a_{n+1}^{n+1}}{a_n^{n+1}} \geqslant (n+1)\frac{a_{n+1}}{a_n} - n$$

$$a_{n+1}^{n+1} \geqslant (n+1)a_{n+1}a_n^n - na_n^{n+1}$$

$$= a_n^n\left[(n+1)a_{n+1} - na_n\right]$$

$$= a_n^n\left[(n+1)\frac{(x_1+x_2+\dots+x_n+x_{n+1})}{(n+1)} - n\frac{(x_1+x_2+\dots+x_n)}{n}\right]$$

$$= a_n^n\left[(x_1+x_2+\dots+x_n+x_{n+1})-(x_1+x_2+\dots+x_n)\right]$$

$$= a_n^n\left[x_1-x_1+x_2-x_2+\dots+x_n-x_n+x_{n+1}\right]$$

$$= a_n^n(x_{n+1})$$

$\therefore a_{n+1}^{n+1} \geqslant a_n^n x_{n+1}$ (2)

d) Let $S(n)$ be the statement that:

$$a_n^n = \left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^n \geq x_1 x_2 \dots x_n \quad \text{for } n \geq 1,\ n \in \mathbb{N}$$

Consider the case for $n = 1 \Rightarrow S(1)$:

$$LHS = \left(\frac{x_1}{1}\right)^1$$
$$= x_1$$

$$RHS = x_1$$

$\therefore$ LHS $\geq$ RHS

$\therefore$ $S(1)$ holds

Assume $S(k)$

$$\left(\frac{x_1 + x_2 + \dots + x_k}{k}\right)^k \geq x_1 x_2 \dots x_k$$

Consider $S(k+1)$

$$\left(\frac{x_1 + x_2 + \dots + x_k + x_{k+1}}{k+1}\right)^{k+1} \geq x_1 x_2 \dots x_k x_{k+1}$$

$$LHS = \left(\frac{x_1 + x_2 + \dots + x_k + x_{k+1}}{k+1}\right)^{k+1}$$

$$= a_{k+1}^{k+1}$$

$$\geq a_k^k x_{k+1} \quad \text{from } (2)$$

$$= \left(\frac{x_1 + x_2 + \dots + x_k}{k}\right)^k x_{k+1}$$

$$\geq x_1 x_2 \dots x_k x_{k+1} \quad \text{by assumption}$$

$$= RHS$$

$\therefore$ LHS $\geq$ RHS

$\therefore$ $S(k) \Rightarrow S(k+1)$

$S(1) \Rightarrow S(n) \quad n \in \mathbb{N} \setminus \{0\}$

Q2. Consider the map, $f(z) = \frac{z+i}{z-i}$, $z \neq i$

Let $z = x + iy$

$$\therefore f(z) = \frac{x+iy+i}{x+iy-i}$$
$$= \frac{x+i(y+1)}{x+i(y-1)}$$

Now, to consider the image of the real axis under $f(z)$, we set $y=0$ to restrict our map.

$$\therefore f(z) = \frac{x+i(y+1)}{x+i(y-1)}$$
$$= \frac{x+i(0+1)}{x+i(0-1)}$$
$$= \frac{x+i}{x-i}$$

Now, rationalising the denominator:

$$\therefore f(z) = \frac{x+i}{x-i}$$
$$= \frac{x+i}{x-i} \times \frac{(x+i)}{(x+i)}$$
$$= \frac{(x+i)(x+i)}{(x-i)(x+i)}$$
$$= \frac{x^2+ix+ix-1}{x^2-ix+ix+1}$$
$$= \frac{x^2+2ix-1}{x^2+1}$$
$$= \frac{x^2-1}{x^2+1} + \frac{2ix}{x^2+1} \quad (1)$$

As we can see, the equation (1), is in the form of some complex number, w_1 say, where $w_1 = u + iv$. As equation (1) is in this form, we are now able to examine the image produced through geometric properties.

$$\therefore w_1 = \frac{x^2-1}{x^2+1} + \frac{2ix}{x^2+1}$$

Now, let us consider the modulus of w_1, in other words, the radius.

$$\therefore |w_1| = \sqrt{u^2+v^2}$$

$$= \sqrt{\frac{(x^2-1)^2}{(x^2+1)^2} + \frac{(2x)^2}{(x^2+1)^2}}$$

$$= \sqrt{\frac{(x^4-2x^2+1)+4x^2}{(x^2+1)^2}}$$

$$= \sqrt{\frac{(x^4+2x^2+1)}{(x^2+1)^2}}$$

$$= \sqrt{\frac{(x^2+1)^2}{(x^2+1)^2}}$$

$$= \sqrt{1}$$

$$= 1$$

$$\therefore |w_1| = 1$$

Therefore, w_1 lies on a circle of radius one. Let us now consider the general form of an equation centered at the origin with radius equal to one:

Using a instead of x, and b instead of y to avoid confusion with the x variable in w_1, we have:

$$a^2+b^2=1 \quad \text{as the general form}$$

Now substituting in the real and imaginary parts of w_1 for a and b respectively, to prove the image of the real axis is centred at the origin, we have:

$$\frac{(x^2-1)^2}{(x^2+1)^2} + \frac{(2x)^2}{(x^2+1)^2} = \frac{x^4-2x^2+1+4x^2}{(x^2+1)^2}$$

$$= \frac{x^4+2x^2+1}{(x^2+1)^2}$$

$$= \frac{(x^2+1)^2}{(x^2+1)^2}$$

$$= 1$$

Therefore w_1 is a circle centred at the origin, with radius equal to one.

Thus, as w_1 is the image of the real axis under $f(z)$, it can be seen that this image is in fact a circle with radius equal to one, and centre at the origin.

Now, considering the cases as x moves to positive and negative infinity, we can determine any discontinuities present in the image.

$$\lim_{x\to\infty} w_1 = \lim_{x\to\infty}\left[\frac{(x^2-1)}{(x^2+1)} + \frac{2ix}{(x^2+1)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1+\frac{1}{x^2}\right)} + \frac{x^2\left(\frac{2i}{x}\right)}{x^2\left(1+\frac{1}{x^2}\right)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\left(1-\frac{1}{x^2}\right)}{\left(1+\frac{1}{x^2}\right)} + \frac{\left(\frac{2i}{x}\right)}{\left(1+\frac{1}{x^2}\right)}\right]$$

$$= \left[1+0i\right]$$

$$= 1$$

<u>AND</u>

$$\lim_{x\to-\infty} w_1 = \lim_{x\to-\infty}\left[\frac{(x^2-1)}{(x^2+1)} + \frac{2ix}{(x^2+1)}\right]$$

$$= \lim_{x\to-\infty}\left[\frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1+\frac{1}{x^2}\right)} + \frac{x^2\left(\frac{2i}{x}\right)}{x^2\left(1+\frac{1}{x^2}\right)}\right]$$

$$= \lim_{x\to-\infty}\left[\frac{\left(1-\frac{1}{x^2}\right)}{\left(1+\frac{1}{x^2}\right)} + \frac{\left(\frac{2i}{x}\right)}{\left(1+\frac{1}{x^2}\right)}\right]$$

$$= \left[1-0i\right]$$

$$= 1$$

Thus as x moves to positive and negative infinity on the real axis under $f(z)$, we get the discontinuity at $(1,0)$ on the circle of w_1.

Thus the image of the real axis under the map $f(z)$, is as follows: