The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 10

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/
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Questions marked with * are more difficult questions.

Material covered

- ☐ linear first order equations.
- ☐ method of integrating factors for solving inhomogeneous linear first order equations.
- ☐ change of variables.

Outcomes

After completing this tutorial you should

- ☐ be able to solve linear first order differential equations using the method of integrating factors
- ☐ be able to do simple change of variables in differential equations

Summary of essential material

First order linear homogeneous equations. These are equations of the form

$$a(x)y'(x) + b(x)y(x) = 0.$$

The equation is called *linear* since the dependence on y and y' is linear and homogeneous since its right hand side is zero. The fact that it is linear means that any sum and scalar multiples of solutions are also solutions, a property often referred to as the *superposition principle*. If $a(x) \neq 0$ the equation can be written in *standard form*

$$y'(x) + p(x)y(x) = 0.$$

The equation is separable with solution

$$y(x) = Ae^{-\int p(x)\,dx} \quad A \text{ is a constant, } \int p(x)\,dx \text{ is some anti-derivative of } p, \text{ no constant required.}$$

(You can just use this formula, or derive it by separation of variables.)

First order linear inhomogeneous equations. These are equations of the form

$$a(x)y'(x) + b(x)y(x) = f(x).$$

The equation is called *linear* since the dependence on y and y' is linear and inhomogeneous since its right hand side f is non-zero. Such equations are solved by division by $a(x)$ to bring them into *standard form*

$$y'(x) + p(x)y(x) = q(x)$$

and then multiplying with an *integrating factor*. An integrating factor is an (arbitrary) non-zero solution of the homogeneous $w'(x) - p(x)w(x) = 0$ (note the changed sign!), that is,

$$w(x) = e^{\int p(x)\,dx}.$$

Then

$$(yw)' = qw \qquad \text{and} \qquad y = \frac{1}{w}\int qw\,dx,$$

where the last integral involves an integration constant. When initial conditions are given it is convenient (but not necessary) to take a definite integral:

$$y(x) = y(x_0)\frac{w(0)}{w(x)}\int_{x_0}^{x} q(\xi)w(\xi)\,d\xi.$$

The solution exists as long as $a(x) \neq 0$ and $w(x) \neq 0$.

Questions to do before the tutorial

1. Find the general solution to the following differential equations.

(a) $\dfrac{dy}{dx} - y\tan x = x$

Solution: The equation is linear in y. An integrating factor is $\cos x$, so $d\big(y\cos x\big)/dx = x\cos x$. Integration by parts gives $y\cos x = x\sin x + \cos x + C$ and hence $y = x\tan x + 1 + C\sec x$.

(b) $\dfrac{dx}{dt} + 2tx = 2t^3$

Solution: The equation is linear in x. An integrating factor is e^{t^2}, so $d(e^{t^2}x)/dt = 2t^3e^{t^2}$. Integration by parts gives $e^{t^2}x = (t^2-1)e^{t^2} + C$ and hence $x = t^2 - 1 + Ce^{-t^2}$.

Questions to complete during the tutorial

2. Which of the following differential equations are linear? Can any of the nonlinear cases be transformed into a linear differential equation by a simple change of variables?

(a) $(x-1)^3\dfrac{dy}{dx} + 4(x-1)^2y = x+1$

Solution: $\dfrac{dy}{dx} + \dfrac{4}{x-1}y = \dfrac{x+1}{(x-1)^3}$ is linear.

(b) $(x-y)\dfrac{dy}{dx} + y = e^x$

Solution: The given differential equation is not linear. The nonlinear term is almost of the form

$$z\frac{dz}{dx} = \frac{1}{2}\frac{dz^2}{dx}$$

We can artificially rewrite the nonlinear term in that form by setting

$$(x-y)\frac{dy}{dx} = -(x-y)\frac{d(x-y)-x}{dx} = -(x-y)\frac{d(x-y)}{dx} + (x-y) = -\frac{1}{2}\frac{d(x-y)^2}{dx} + (x-y)$$

This suggests that we can use the change of variable $v = (y-x)^2$. Using the given differential equation we have

$$v' = 2(y-x)(y'-1) = 2(y-x)y' - 2(y-x) = 2(y-e^x) - 2y + 2x = 2(x-e^x).$$

Integrating we get $v = x^2 - 2e^x + C$ and so the general solution is $y = x \pm \sqrt{x^2 - 2e^x + C}$. The sign is determined by the initial conditions given.

3. Solve the following differential equations.

(a) $dx - (\sec y + 2x\tan y)dy = 0$

Solution: The equation is linear in x, and its standard form is $dx/dy - (2\tan y)x = \sec y$. The integrating factor is $\cos^2 y$, so $d(x\cos^2 y)/dy = \cos y$, leading to $x = (\sin y + C)/\cos^2 y$. Optional inverse: $y = \sin^{-1}\big\{(-1 \pm \sqrt{4x^2 - 4Cx + 1})/(2x)\big\}$.

(b) $\dfrac{dy}{dx} = \dfrac{2y}{y - x - y^3}$

Solution: The equation is not linear, and not separable, but it can be written as a linear equation for the inverse function $x = x(y)$. The standard form of the equation is

$$\frac{dx}{dy} + \frac{x}{2y} = \frac{1-y^2}{2}.$$

The integrating factor is

$$w = e^{\int \frac{1}{2y}} = e^{\frac{1}{2}\ln|y|} = e^{\ln\sqrt{|y|}} = |y|^{1/2}.$$

We therefore have

$$|y|^{1/2}x = \int \frac{|y|^{1/2}(1-y)}{2}\,dy = \frac{1}{3}|y|^{1/2}y - \frac{1}{7}|y|^{5/2}y + C.$$

Hence the general solution is

$$x = \frac{1}{3}y - \frac{1}{7}y^3 + \frac{C}{\sqrt{|y|}}.$$

(c) $(1+x)\dfrac{dy}{dx} + y = 3x^2$, given $y(0) = 2$.

Solution: The equation is linear in y, and its standard form is $dy/dx + (1+x)^{-1}y = 3x^2(1+x)^{-1}$. The integrating factor is $1+x$. So $d\big((1+x)y\big)/dx = 3x^2$, leading to $y = (x^3+C)/(1+x)$. Imposing the condition $y = 2$ when $x = 0$ gives $C = 2$ and so the particular solution is $y = (x^3+2)/(1+x)$.

(d) $2dx + (2x+3y)dy = 0$, given $y(2) = 0$.

Solution: The equation is linear in x, and its standard form is $dx/dy + x = -3y/2$. The integrating factor is e^y, so $d(xe^y)/dy = -3ye^y/2$, leading to $2x = -3y + 3 + Ce^{-y}$. Imposing the condition $y = 0$ when $x = 2$ gives $C = 1$, and so the particular solution is $e^{-y} = 2x + 3y - 3$.

4. Consider a logistic equation $\dfrac{dP}{dt} = a(t)P - b(t)P^2$, where P models the size of a population and $a, b\colon \mathbb{R} \to \mathbb{R}$ are continuous functions. We assume that a and b are T-periodic, that is, $a(t+T) = a(t)$ for all $t \in \mathbb{R}$ and likewise for b. We further assume that $b(t) > 0$ for all $t \in \mathbb{R}$.

The periodicity assumption is natural if we consider seasonal changes in the environment.

(a) Show that $v = 1/P$ satisfies the linear differential equation $v' + a(t)v = b(t)$.

Solution: Differentiating v and using that P is the solution of the given differential equation we get

$$v' = -\frac{P'}{P^2} = -\frac{a(t)P - b(t)P^2}{P^2} = -a(t)\frac{1}{P} + b(t) = -a(t)v + b(t).$$

If we rearrange we get $v' + a(t)v = b(t)$ as claimed.

(b) Hence, deduce that if $P(0) = P_0$, then

$$P(t) = \frac{P_0 \exp\big(\int_0^t a(\tau)\,d\tau\big)}{1 + P_0 \displaystyle\int_0^t \exp\big(\int_0^s a(\tau)\,d\tau\big)b(s)\,ds}$$

Solution: We solve the differential equation $v' + av = b$ by using the method of integrating factors. An integrating factor is given by $\exp(\int_0^t a(\tau)\,d\tau)$. Hence

$$v(t)\exp\Big(\int_0^t a(\tau)\,d\tau\Big) - v(0) = \int_0^t \exp\Big(\int_0^s a(\tau)\,d\tau\Big)b(s)\,ds$$

if we use a definite integral. Using that $v = 1/P$ we get

$$\frac{1}{P(t)}\exp\Big(\int_0^t a(\tau)\,d\tau\Big) - \frac{1}{P_0} = \int_0^t \exp\Big(\int_0^s a(\tau)\,d\tau\Big)b(s)\,ds.$$

Hence

$$\begin{aligned}\frac{1}{P(t)}\exp\Big(\int_0^t a(\tau)\,d\tau\Big) &= \frac{1}{P_0} + \int_0^t \exp\Big(\int_0^s a(\tau)\,d\tau\Big)b(s)\,ds \\ &= \frac{1 + P_0\displaystyle\int_0^t \exp\big(\int_0^s a(\tau)\,d\tau\big)b(s)\,ds}{P_0}.\end{aligned}$$

Rearranging we get the solution required. Alternatively we can compute the general solution and then use the initial condition to determine the required answer.

(c) A solution is T-periodic if $P_0 = P(0) = P(T)$. Find the initial condition of the T-periodic solution. Show that the periodic solution is positive if and only if $\int_0^T a(\tau)\,d\tau > 0$.

Solution: We first see whether there is a T-periodic solution by choosing P_0 so that $P(T) = P_0$. Using the result from the previous part

$$P_0 = P(T) = \frac{P_0 \exp\bigl(\int_0^T a(\tau)\,d\tau\bigr)}{1 + P_0 \displaystyle\int_0^T \exp\bigl(\int_0^s a(\tau)\,d\tau\bigr) b(s)\,ds}.$$

Hence either $P_0 = 0$ which leads to the zero solution or

$$1 = \frac{\exp\bigl(\int_0^T a(\tau)\,d\tau\bigr)}{1 + P_0 \displaystyle\int_0^T \exp\bigl(\int_0^s a(\tau)\,d\tau\bigr) b(s)\,ds}.$$

Therefore

$$1 + P_0 \int_0^T \exp\bigl(\textstyle\int_0^s a(\tau)\,d\tau\bigr) b(s)\,ds = \exp\bigl(\textstyle\int_0^T a(\tau)\,d\tau\bigr)$$

and so

$$P_0 = \frac{\exp\bigl(\int_0^T a(\tau)\,d\tau\bigr) - 1}{\displaystyle\int_0^T \exp\bigl(\int_0^s a(\tau)\,d\tau\bigr) b(s)\,ds}$$

is the initial condition of the T-periodic solution. Since the exponential and b are positive the denominator is always positive. Thus the periodic solution is positive if and only if the numerator is positive, and that is the case if and only if

$$\exp\Bigl(\int_0^T a(\tau)\,d\tau\Bigr) > 1,$$

that is, $\int_0^T a(\tau)\,d\tau > 0$.

*(d) If $\int_0^T a(\tau)\,d\tau < 0$, then the differential equation has no positive periodic solution. Show that $P(t) \to 0$ for all positive initial values $P_0 > 0$. What does this mean for the population?

Solution: Since $P_0 > 0$ and $b > 0$ it follows from part (b) that

$$0 \le P(t) \le P_0 \exp\Bigl(\int_0^t a(\tau)\,d\tau\Bigr).$$

Hence it is sufficient to show that the integral on the right hand side goes to zero as $t \to \infty$. Equivalently we can show that

$$\lim_{t\to\infty} \int_0^t a(\tau)\,d\tau = -\infty.$$

By assumption $K := -\int_0^T a(\tau)\,d\tau > 0$. If $t > 0$ we can write $t = nT + r$ with $n \in \mathbb{N}$ and $r \in [0, T)$. By the T-periodicity

$$\int_0^t a(\tau)\,d\tau = \sum_{k=1}^{n} \int_{(k-1)T}^{kT} a(\tau)\,d\tau + \int_{kT}^{t} a(\tau)\,d\tau = -nK + \int_0^r a(\tau)\,d\tau.$$

As a is continous, the extreme value theorem implies that it bounded on the closed and bounded interval $[0, T]$, that is, $|a(\tau)| \leq M$ for some constant $M > 0$ for all $\tau \in [0, M]$. Hence

$$\left|\int_0^r a(\tau)\, d\tau\right| \leq \int_0^r |a(\tau)|\, d\tau M \leq \int_0^T M\, d\tau MT$$

for all $r \in [0, T)$ and therefore

$$\int_0^t a(\tau)\, d\tau \leq -nK + MT \to -\infty$$

as $n \to \infty$. Hence $\int_0^t a(\tau)\, d\tau \to -\infty$ as $t \to \infty$. It means that the population will become extinct no matter how large the initial population was. This is intuitively expected since $\int_0^T a(\tau)\, d\tau$ can be interpreted as the average growth rate over one season. If that is negative we expect extinction over a long period of time.

5. A tank initially contains 700 litres of fresh water. A pipe is opened which admits salty water at a rate of 10 litres/min. At the same time, a drain is opened to allow 8 litres/min of the mixture to leave the tank. If the inflowing salty water contains 0.01 kg of salt per litre, what is the mass of salt in the tank after 60 minutes? What is the concentration of the salt?

Solution: Let $m(t)$ be the mass of salt (in kilograms) in the tank at time t. Since the tank initially contains fresh water, $m(0) = 0$. In this problem, the volume $V(t)$ of the water in the tank is not constant since water is leaving the tank at a slower rate than the water entering. The net gain is 2 litres/min. Hence the volume of the water in the tank at time t is $V(t) = 700 + 2t$. The concentration of salt in the water leaving the tank is $m(t)/V(t)$. The differential equation governing the mass $m(t)$ is given by

$$\begin{aligned}
\frac{dm}{dt} &= \{\text{rate in}\} - \{\text{rate out}\} \\
&= 10 \times 0.01 - 8 \times m/V \\
&= \frac{1}{10} - \frac{8m}{700 + 2t}, \\
\frac{dm}{dt} + \frac{8m}{700 + 2t} &= \frac{1}{10}.
\end{aligned}$$

This is a linear differential equation of the first order. Its integrating factor is

$$w(t) = e^{\int 8/(700+2t)dt} = e^{4\ln(700+2t)} = (700 + 2t)^4.$$

Hence, the equation can be written in the exactly integrable form,

$$\frac{d}{dt}\{(700 + 2t)^4 m\} = \frac{1}{10}(700 + 2t)^4.$$

Integrating gives

$$(700 + 2t)^4 m = \frac{(700 + 2t)^5}{10 \cdot 2 \cdot 5} + C = \frac{(700 + 2t)^5}{100} + C.$$

The initial condition $m(0) = 0$ implies that $C = -(700)^5/100 = -7 \cdot (700)^4$. Hence, the required particular solution of the differential equation is

$$m(t) = \frac{700 + 2t}{100} - \frac{7 \cdot (700)^4}{(700 + 2t)^4}.$$

After 60 minutes, the mass of salt is

$$m(60) = \frac{820}{100} - \frac{7 \cdot (700)^4}{(820)^4} = 8.2 - 3.717 = 4.483\text{kg}.$$

The corresponding concentration is $m(60)/V(60) = 4.483/820 = 0.00547$ kg/L.

Extra questions for further practice

6. Some rocks contain a radioactive isotope of radium, Ra^{226}, which has a half-life of 1590 years and decays into an isotope of lead, Pb^{210}. This lead isotope is itself radioactive, and decays with a half-life of 22 years. Let $R(t)$ be the amount of radium in the rock and $L(t)$ be the amount of lead. Then the rate of change of L is the rate at which lead is produced by the decay of radium, minus the rate at which the lead decays; so $dL/dt = \lambda R - \mu L$ where λ and μ are the decay constants of radium and lead respectively. Given that $R = R_0 e^{-\lambda t}$ and that $L = 0$ at $t = 0$, solve this equation to show that

$$L(t) = \frac{\lambda R_0}{\mu - \lambda}(e^{-\lambda t} - e^{-\mu t}).$$

What are the values of λ and μ?

Solution: The differential equation,

$$\frac{dL}{dt} = \lambda R_0 e^{-\lambda t} - \mu L,$$

is linear, with $p(t) = \mu$ so that the integrating factor is $w(t) = e^{\int \mu dt} = e^{\mu t}$. Multiplying through by this factor, we can put the DE in the form,

$$\frac{d}{dt}\left(e^{\mu t} L\right) = \lambda R_0 e^{\mu t} e^{-\lambda t} = \lambda R_0 e^{(\mu - \lambda)t}.$$

Integrating gives

$$e^{\mu t} L = \frac{\lambda R_0}{\mu - \lambda} e^{(\mu-\lambda)t} + C, \quad \text{or} \quad L = \frac{\lambda R_0}{\mu - \lambda} e^{-\lambda t} + Ce^{-\mu t}.$$

Putting $L = 0$ when $t = 0$, we find that $C = -\lambda R_0/(\mu - \lambda)$ and so

$$L(t) = \frac{\lambda R_0}{\mu - \lambda}\left(e^{-\lambda t} - e^{-\mu t}\right).$$

In each case, the decay constant is given by $k = (\ln 2)/T_{1/2}$, where $T_{1/2}$ is the half-life. Thus $\lambda = 0.6931/1590 = 4.36 \times 10^{-4}$, and $\mu = 0.6931/22 = 3.15 \times 10^{-2}$.

7. Obtain first-order differential equations that govern the following one-parameter families of curves (this means $y(x)$ is a solution to some first order differential equation):

(a) $y = Cx^4$;

Solution: For the class of curves $y = Cx^4$ (quartic parabolas), eliminate the parameter C by differentiation:

$$C = \frac{y}{x^4} \quad \text{implies} \quad \frac{d}{dx}\left(\frac{y}{x^4}\right) = 0.$$

Taking the derivative and clearing the denominator, we arrive at the differential equation,

$$x\frac{dy}{dx} - 4y = 0.$$

(b) $\dfrac{x^2}{C} + \dfrac{y^2}{C-1} = 1.$

Solution: For the class of confocal ellipses and hyperbolas $x^2/C + y^2/(C-1) = 1$, rearrange as a quadratic equation for the parameter:

$$C^2 - (x^2 + y^2 + 1)C + x^2 = 0.$$

To eliminate C, one could solve for C and then differentiate, but that method would be slow. A better way is to differentiate the quadratic equation directly with respect to x, thereby eliminating C^2, and then solve for C:

$$C = \frac{x}{x + yy'}, \qquad C - 1 = -\frac{yy'}{x + yy'}.$$

Substitute these into the original equation to get the differential equation:

$$\left(xy' - y\right)\left(x + yy'\right) = y'.$$

8. Find the general solutions of the following differential equations.

(a) $\dfrac{dx}{dt} - tx = t$

Solution: The integrating factor is $e^{\int (-t)dt} = e^{-t^2/2}$. Multiplying the equation by this integrating factor gives $(d/dt)\left\{e^{-t^2/2}x\right\} = te^{-t^2/2}$, and then integration gives $e^{-t^2/2}x = \int te^{-t^2/2}dt = -e^{-t^2/2} + C$ and so $x = -1 + Ce^{t^2/2}$.

(b) $\dfrac{dy}{dx} = \dfrac{4x^3 - y}{x}$

Solution: Rewrite as $y' + x^{-1}y = 4x^2$. The integrating factor is $e^{\int x^{-1}dx} = e^{\ln|x|} = |x|$, but x will do. Multiplying the equation by this factor gives $(xy)' = 4x^3$, and then integrating gives $xy = x^4 + C$ and so $y = x^3 + Cx^{-1}$.

(c) $\dfrac{dy}{dx} + 2y = e^{-x}$

Solution: The integrating factor is $e^{\int 2dx} = e^{2x}$. Multiplying the equation by this factor gives $(d/dx)(e^{2x}y) = e^x$, and then integrating gives $e^{2x}y = e^x + C$ and so $y = e^{-x} + Ce^{-2x}$.

(d) $x^2\dfrac{dy}{dx} + (1 - 2x)y = x^2$

Solution: Rewrite as $y' + \left(x^{-2} - 2x^{-1}\right)y = 1$. The integrating factor is $e^{\int \left(x^{-2} - 2x^{-1}\right)dx} = e^{-x^{-1} - 2\ln|x|} = x^{-2}e^{-1/x}$. Multiplying the equation by this factor gives $(d/dx)(x^{-2}e^{-1/x}y) = x^{-2}e^{-1/x}$, and then integrating gives $x^{-2}e^{-1/x}y = e^{-1/x} + C$ and so $y = x^2 + Cx^2e^{1/x}$.

9. Find the particular solutions of the following differential equations under the given conditions.

(a) $\dfrac{dy}{dx} + y\tan x = \sec x, \quad y = 2$ when $x = 0$

Solution: The integrating factor is $e^{\int \tan x dx} = e^{\ln|\sec x|} = |\sec x|$, but $\sec x$ will do. Multiplying the equation by this factor gives $(d/dx)(y \sec x) = \sec^2 x$, and then integrating gives $y \sec x = \tan x + C$ and so $y = \sin x + C\cos x$. Putting $x = 0$ then gives $y(0) = C$. But we are told that $y(0) = 2$, and so $C = 2$. Hence the particular solution is $y = \sin x + 2\cos x$.

(b) $\dfrac{dy}{dx} = \dfrac{2y}{x} + x^4, \quad y = 1$ when $x = 1$

Solution: Rewrite as $y' - 2x^{-1}y = x^4$. The integrating factor is $e^{\int (-2/x)dx} = e^{-2\ln|x|} = x^{-2}$. Multiplying the equation by this factor gives $(d/dx)(x^{-2}y) = x^2$ and then integrating gives $x^{-2}y = (1/3)x^3 + C$ and so $y = (1/3)x^5 + Cx^2$. Putting $x = 1$ then gives $y(1) = 1/3 + C$. But we are told that $y(1) = 1$, and so $C = 2/3$. Hence the particular solution is $y = (1/3)(x^5 + 2x^2)$.

(c) $\dfrac{dx}{dt} + 4x = e^{-4t}\sin 2t, \quad x(0) = 1/2$

Solution: The integrating factor is $e^{\int 4dt} = e^{4t}$. Multiplying the equation by this factor gives $(d/dt)(e^{4t}x) = \sin 2t$ and then integrating gives $e^{4t}x = -(1/2)\cos 2t + C$ and so

$x = \{-(1/2)\cos 2t + C\}e^{-4t}$. Putting $t = 0$ gives $x(0) = -1/2 + C$. But we are told that $x(0) = 1/2$, and so $C = 1$. Hence the particular solution is $x = e^{-4t}\{1 - (1/2)\cos 2t\}$.

(d) $(1 + x^2)\dfrac{dy}{dx} + 2xy = 4 + 2x, \quad y(0) = 4$

Solution: Rewrite as $y' + 2x(1 + x^2)^{-1}y = (4 + 2x)(1 + x^2)^{-1}$. The integrating factor is $e^{\int 2x/(1+x^2)dx} = e^{\ln(1+x^2)} = 1 + x^2$. Multiplying the equation by this factor gives $(d/dx)\big((1 + x^2)y\big) = 4 + 2x$ and then integrating gives $(1 + x^2)y = 4x + x^2 + C$ and so $y = (x^2 + 4x + C)/(x^2 + 1)$. Putting $x = 0$ gives $y(0) = C$. But we are told that $y(0) = 4$, and so $C = 4$ and the particular solution is $y = (x + 2)^2/(x^2 + 1)$.

10. The Howard family borrows \$176 000 to buy a house, and plans to make frequent regular repayments of increasing amounts so that the rate of repayment t years after the start of the loan will be \$$R(t)$ per year, where $R(t) = R_0\big(1 + t^2/80\big)$ and R_0 is the initial repayment rate. The interest rate is fixed at 5% per annum, and interest charges are added to the loan amount at frequent regular intervals.

(a) Assuming repayments and interest charges are so frequent that they are effectively continuous, show that the loan amount L varies with time according to the differential equation,

$$\frac{dL}{dt} = \frac{L}{20} - R_0\left(1 + \frac{t^2}{80}\right).$$

Solution: If the time interval Δt between consecutive repayments is small, then the amount repaid each time will be $R(t) \times \Delta t$. Also the amount of interest charged each time will be $L \times 5/100 \times \Delta t$. Hence, at the end of each such interval, the loan amount will increase by an amount $\Delta L = (L \times 5/100 \times \Delta t) - (R(t) \times \Delta t)$. Dividing this equation by Δt gives

$$\frac{\Delta L}{\Delta t} = \frac{5L}{100} - R(t) = \frac{L}{20} - R_0\left(1 + \frac{t^2}{80}\right).$$

Since Δt is small, this equation is approximated well by the differential equation,

$$\frac{dL}{dt} = \frac{L}{20} - R_0\left(1 + \frac{t^2}{80}\right).$$

(b) Solve this equation, and hence obtain an expression for the amount still owed after t years.

Solution: Rearranging the DE as $dL/dt - L/20 = -R_0\big(1 + (t^2/80)\big)$, we see that the integrating factor is $e^{\int(-1/20)dt} = e^{-t/20}$. Multiplying the DE by this factor gives $(d/dt)\big(e^{-t/20}L\big) = -R_0\big(1 + (t^2/80)\big)e^{-t/20}$. Integrating by parts then gives

$$\begin{aligned}
e^{-t/20}L(t) &= 20R_0\int\left(1 + \frac{t^2}{80}\right)\frac{d}{dt}e^{-t/20}dt\\
&= 20R_0\left[\left(1 + \frac{t^2}{80}\right)e^{-t/20} - \int\frac{t}{40}e^{-t/20}dt\right]\\
&= 20R_0\left[\left(1 + \frac{t^2}{80}\right)e^{-t/20} + 20\int\frac{t}{40}\frac{d}{dt}e^{-t/20}dt\right]\\
&= 20R_0\left[\left(1 + \frac{t^2}{80}\right)e^{-t/20} + 20\left(\frac{t}{40}e^{-t/20} - \int\frac{1}{40}e^{-t/20}dt\right)\right]\\
&= 20R_0\left[\left(1 + \frac{t^2}{80}\right)e^{-t/20} + 20\left(\frac{t}{40}e^{-t/20} + \frac{1}{2}e^{-t/20}\right)\right] + C,
\end{aligned}$$

and hence,

$$L(t) = 20R_0\left(11 + \frac{t}{2} + \frac{t^2}{80}\right) + Ce^{t/20},$$

where C is an arbitrary constant of integration. Taking $t = 0$ gives $L(0) = 220R_0 + C$; but we are told that the initial amount of the loan is $L(0) = 176\,000$, and so we deduce that $C = 176\,000 - 220R_0$. Hence the amount (in dollars) owed after t years is

$$L(t) = 20R_0\left(11 + \frac{t}{2} + \frac{t^2}{80}\right) + (176000 - 220R_0)e^{t/20}.$$

(c) Show that the initial repayment rate R_0 must exceed \$800/year or else the debt will eventually grow out of control.

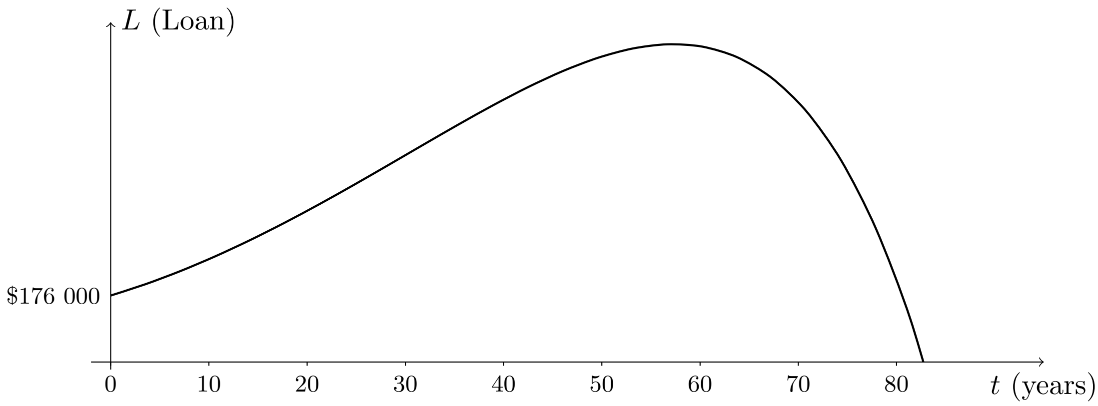
Solution: If $R_0 < 176\,000/220 = 800$, then both terms in the above expression are positive and increasing for all values of t; in fact the second term increases exponentially. If $R_0 = 800$, the second term is always zero but the first term increases forever. In either case, the debt will increase without limit. However, if $R_0 > 800$ then the second term is negative, and at some time in the future (i.e., at some positive value of t) it will cancel the first term. At this time $L(t)$ will vanish and the debt will be fully paid off.

(d) If $R_0 =$ \$1000/year, what is the remaining debt after 20 years?

Solution: We substitute $R_0 = 1000$ and $t = 20$ into the solution from the previous part. Then

$$L(20) = 20\,000(11 + 10 + 5) + (176\,000 - 220\,000)e = 400\,395.60$$

This is still more than owed at the start. Note however that the term involving $e^{t/20}$ has a negative coefficient. As an exponential grows faster than any polynomial the debt decreases eventually as shown on the graph below.



11. Which of the following differential equations are linear? Can any of the nonlinear cases be transformed into a linear differential equation by a simple change of variables? Try to solve the equation.

(a) $\dfrac{dy}{dx} + \dfrac{3y}{x} = \sin x$

Solution: The differential equation is linear. We can solve it by using an integrating factor. The integrating factor is

$$\exp\left(3\int \frac{1}{x}\,dx\right) = \exp(3\ln|x|) = \exp(\ln|x|^3) = |x|^3.$$

For a given initial condition the solution is defined either for positive or for negative x because the equation is singluar at $x = 0$. Hence on each part we can choose the sign of the integrating factor. We therefore choose x^3 which is smooth on all of $\mathbb{R}$. Multiplying the equation and integrating we get

$$x^3 y = \int x^3 \sin x\,dx.$$

Integrating by parts three times we get

$$\begin{aligned}\int x^3 \sin x &= -x^3 \cos x + 3\int x^2 \cos x\, dx = -x^3 \cos x + 3x^2 \sin x - 6\int x \sin x\, dx \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\int \cos x\, dx \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C\end{aligned}$$

Therefore the general solution is

$$y = \frac{6 - x^2}{x^2} \cos x + \frac{3(x^2 - 2)}{x^3} \sin x + \frac{C}{x^3}.$$

(b) $\dfrac{dy}{dx} = \dfrac{y^2 + 1}{2xy + 1}$

Solution: The equation is not linear. However, the reciprocal equation,

$$\frac{dx}{dy} - \frac{2y}{y^2 + 1} x = \frac{1}{y^2 + 1}$$

is linear, and can be solved as such, using the method of integrating factors. An integrating factor is given by

$$\exp\Big(-\int \frac{2y}{y^2 + 1}\, dy\Big) = \exp\big(-\ln(1 + y^2)\big) = \frac{1}{1 + y^2}.$$

Multiplying the equation and integrating yields

$$\frac{x}{1 + y^2} = \int \frac{1}{(1 + y^2)^2}\, dy$$

If we use the substitution $y = \tan t$ we get $dy = \sec^2\, dt$ and $1 + y^2 = \sec^2 t$. Hence

$$\begin{aligned}\int \frac{1}{(1 + y^2)^2}\, dy = \int \frac{1}{\sec^2 t}\, dt = \int \cos^2 t\, dt &= \frac{1}{2}\big(t + \sin t \cos t\big) + C \\ &= \frac{1}{2}\Big(t + \frac{\tan t}{1 + \tan^2 t}\Big) + C = \frac{1}{2}\Big(\tan^{-1} y + \frac{y}{1 + y^2}\Big) + C\end{aligned}$$

Hence the general solution is

$$x = \frac{1}{2}\Big(y + (y^2 + 1)(C + \tan^{-1} y)\Big)$$