THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial for Week 9

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/
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Material covered

☐ Solution of first order differential equations by separation of variables
☐ Particular solutions of differential equations
☐ Applications of differential equations in various contexts

Outcomes

After completing this tutorial you should

☐ confident in solving separable first order differential equations with or without initial conditions
☐ be able to deal with differential equations arising from a variety of models and be able to interpret the solutions.

Questions to do before the tutorial

1. Find the general solutions of the following differential equations.

(a) $(1+x^2)\dfrac{dy}{dx} + xy = 0,$ (b) $x\dfrac{dy}{dx} = y^2 - 1.$

Questions to complete during the tutorial

2. Find the general solution of the following differential equations.

(a) $(x^2y^2 + x^2 + y^2 + 1)\dfrac{dy}{dx} = xy + x,$ (b) $ye^x\dfrac{dy}{dx} = y^2 + y - 2.$

3. Find the particular solutions of the following differential equations.

(a) $\dfrac{dy}{dx} = xe^{y-x^2}, \quad y(0) = 0,$ (b) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}, \quad y(0) = a,$ a a constant.

4. Consider the differential equation $y' = xy^2$.

(a) Sketch the direction field for the given differential equation for $-3 \le x, y \le 3$.

(b) Solve the differential equation with the initial condition $y(1) = -2$.

(c) The equilibrium solution $y = 0$ is stable if any solution starting near zero stays near zero for all $x > 0$. Is the zero solution $y = 0$ for the given differential equation stable? Briefly justify your answer.

5. Many special functions including the exponential function can be defined as a particular solution of a differential equation. This is in particular true for the exponential function: It is the unique function that is its own derivative and has value one at zero. More formally, we are interested in a function $u\colon \mathbb{R} \to \mathbb{R}$ satisfying the differential equation

$$u'(x) = u(x) \qquad \text{and} \qquad u(0) = 1.$$

This question shows the existence and uniqueness of such a function and its properties.

(a) Use the fundamental theorem of calculus to show that u solves the above initial value problem if and only if

$$u(x) = 1 + \int_0^x u(t)\,dt$$

for all $x \in \mathbb{R}$.

(b) We approximate solutions by an iterative process. We let

$$u_0(x) = 1 \qquad \text{and} \qquad u_{n+1}(x) = 1 + \int_0^x u_n(t)\,dt \quad \text{for } n \geq 0.$$

Using mathematical induction on n prove that

$$u_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

(This is a special case of "Picard-Lindelöf iteration" to prove the existence and uniqueness of solutions to differential equations.)

(c) Using the ratio test show that $u(x) := \lim_{n\to\infty} u_n(x)$ exists. Assuming that you can take a limit inside an integral, show that u is a solution of the differential equation under consideration (This shows the existence of a solution.)

(d) Assume that u and v are solutions of the differential equation. Show that $\dfrac{u(x)}{v(x)} = 1$ for all $x \in \mathbb{R}$ and hence that the solution to the differential equation is unique.

The next few parts establish fundamental properties of the solution to link it to the exponential function.

(e) Let u be a solution of the differential equation. Show that $u(x)u(-x) = 1$ for all $x \in \mathbb{R}$ and hence deduce that $u(x) \neq 0$.

(f) Let u be a solution of $u' = u$. Fix $y \in \mathbb{R}$ and show that $v(x) := u(x+y)/u(y)$ is a solution of the differential equation. Hence show that $u(x+y) = u(x)u(y)$ for all $x, y \in \mathbb{R}$.

(g) Set $e := u(1) = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$. Show that $u(n) = e^n$ and $u(1/n) = \sqrt[n]{e}$. This justifies the definition $e^x := u(x)$ for all $x \in \mathbb{R}$.

*(h) Let $M > 0$ be given. Show that for $|x| \leq M$ and $n + 1 \geq 2M$ we have that

$$|u_n(x) - u(x)| \leq 2\frac{M^n}{n!}.$$

Deduce that for $|x| \leq M$

$$\lim_{n\to\infty} \int_0^x u_n(t)\,dt = \int_0^x u(t)\,dt.$$

(This completes part (c), justifying the interchange of the integral and the limit.)

Extra questions for further practice

6. Einstein's Theory of Relativity predicts the existence of black holes: regions in space from which nothing can escape, due to strong gravitational forces. The theory predicts that black holes will be formed when large stars collapse.

However, Einstein's theory did not take into account quantum mechanical effects. In 1975, Stephen Hawking used quantum theory to show that a black hole should glow slightly; that is, it should radiate energy and particles in the same way that a heated object does. Assuming that nothing else falls into the black hole, this causes its mass M to decrease at the rate governed by the differential equation,

$$\frac{dM}{dt} = -\frac{\alpha}{M^2},$$

where t denotes time and α is a constant whose value is not yet known precisely.

(a) Find the general solution $M(t)$ of this differential equation.

(b) Find the particular solution which satisfies the condition that the mass is M_0 when $t = 0$.

(c) How long does it take for a black hole which initially has mass M_0 to lose half its mass? How long does it take for it to evaporate completely?

7. Find the general solutions of

(a) $\dfrac{dy}{dx} = \dfrac{x + \sin x}{3y^2}$, (b) $\dfrac{dx}{dt} = 1 + t - x - tx$, (c) $\dfrac{dy}{dx} = \dfrac{\ln x}{xy + xy^3}$.

8. Find particular solutions satisfying the given conditions.

(a) $\dfrac{dy}{dx} = \dfrac{1+x}{xy}$ $(x > 0)$, $y(1) = -4$; (b) $\dfrac{dy}{dt} = \dfrac{ty + 3t}{t^2 + 1}$, $y(2) = 2$.

9. Find a function $g(x)$ such that $g'(x) = g(x)\big(1 + g(x)\big)$ and $g(0) = 1$.

10. A molecule of substance A can combine with a molecule of substance B to form a molecule of substance X, in a reaction which is denoted $A + B \to X$. According to the Law of Mass Action, the rate of formation of X is proportional to the product of the amounts of A and B present. A test-tube initially contains amounts a and b of substances A and B, respectively, (measured in moles), but none of substance X.

(a) Let $x(t)$ denote the amount of substance X (in moles) produced within the first t seconds. Write down a differential equation for $x(t)$.

(b) Assuming that $a \neq b$, solve this equation to obtain an expression for $x(t)$.

(c) Suppose that initially there are two molecules of B for every molecule of A, and that after 10 seconds there are six molecules of B for every molecule of A. What is the ratio after 30 seconds?

(d) The experiment is repeated, but with the initial amount of substance B halved so as to equal the initial amount a of substance A. (As before, substance X is absent initially.) What fraction of A molecules remain after 30 seconds?