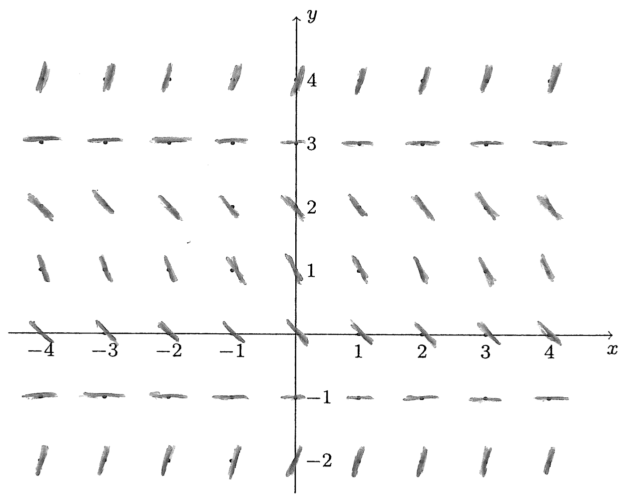
THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

# Solutions to Quiz 2a

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2015

1. Sketch the direction field of the differential equation $y' = (y+1)(y-3)$ in the region below.

Answer Q1:




2. Find the general solution of the differential equation $3y^2 \dfrac{dy}{dx} = 2x(y^3+1)$, where $y \neq -1$.

***Solution:*** $y = (Ce^{x^2} - 1)^{1/3}$, where $C \in \mathbb{R} \setminus \{0\}$ is a constant.

3. Find an integrating factor for the differential equation $\ln x + y\dfrac{\sin x}{x^2} = \dfrac{y'}{x^2}$, where $x > 0$.

***Solution:*** $e^{\cos x}$

4. Find the particular solution $x = x(t)$ of the differential equation $x\dfrac{dx}{dt} = \tan t$ in the interval $-\pi/4 < t < \pi/2$, which satisfies the condition $x(0) = 2$.

***Solution:*** $x(t) = \sqrt{4 - 2\ln(\cos t)}$.

5. Suppose that $y(x)$ is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{xy + x^2}$. We make the transformation $z = \dfrac{y}{x}$. Find a differential equation for $z = z(x)$.

***Solution:*** $x\dfrac{dz}{dx} = \dfrac{1}{z+1}$ (or equivalent formulation).

6. Find the equilibrium solution of the equation $\dfrac{dy}{dx} = -(y+3)\ln(y-1)$.

***Solution:*** $y = 2$.

7. Determine whether the equilibrium solution in the above question is stable or unstable.

***Solution:*** $y = 2$ is a stable equilibrium solution.

8. Find the particular solution of $\dfrac{dy}{dx} - 2xy = x$ with $y(0) = 1$.

***Solution:*** $y = -\dfrac{1}{2} + \dfrac{3}{2}e^{x^2}$.

9. Find the general solution $x(t)$ of the differential equation $\dfrac{dx}{dt} = \dfrac{t}{t^2 + 2t - 3}$, where $t > 1$.

***Solution:*** $x(t) = \dfrac{1}{4}\ln(t-1) + \dfrac{3}{4}\ln(t+3) + C$, where $C \in \mathbb{R}$ is a constant.

10. The amount $x(t)$ of a radioactive isotope in a sample at time $t$ decays according to the Malthusian law $\dfrac{dx}{dt} = -kx$, where $k$ is a positive constant. The half-life of a radioactive substance is the time required to decay to one-half of the initial amount of the substance. Carbon-14 is a radioactive isotope of carbon that has half-life of 5600 years. What percentage of the original amount of Carbon-14 in a sample would be present after 10,000 years?

***Solution:*** $\exp\left\{-\dfrac{25}{14}\ln 2\right\}$, which is approximately 30%.

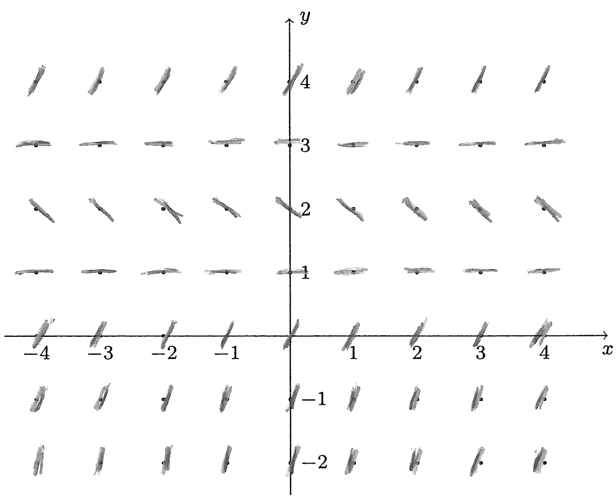
The University of Sydney
School of Mathematics and Statistics

# Solutions to Quiz 2b

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2015

**1.** Sketch the direction field of the differential equation $y' = (y-1)(y-3)$ in the region below.

Answer Q1:




**2.** Find the general solution of the differential equation $y\dfrac{dy}{dx} = x^2(y^2+9)$.

***Solution:*** $y = \pm\sqrt{Ae^{2x^3/3} - 9}$ for some constant $A > 0$.

**3.** Find an integrating factor for the differential equation $1 + y = e^x y'$.

***Solution:*** $e^{e^{-x}}$.

**4.** Find the particular solution $x = x(t)$ of the differential equation $x^2\dfrac{dx}{dt} = \tan t$ in the interval $-\pi/4 < t < \pi/2$, which satisfies the condition $x(0) = 1$.

***Solution:*** $x(t) = \sqrt[3]{1 - 3\ln(\cos t)}$.

5. Suppose that $y(x)$ is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^2 - xy + y^2}{xy + x^2}$. We make the transformation $z = \dfrac{y}{x}$. Find a differential equation for $z = z(x)$.

***Solution:*** $x\dfrac{dz}{dx} = \dfrac{1-2z}{z+1}$.

6. Find the equilibrium solution of the equation $\dfrac{dy}{dx} = (y+3)\ln(y+2)$.

***Solution:*** $y = -1$.

7. Determine whether the equilibrium solution in the above question is stable or unstable.

***Solution:*** $y = -1$ is an unstable equilibrium solution.

8. Find the particular solution of $\dfrac{dy}{dx} + 2xy = x$ with $y(0) = \dfrac{3}{2}$.

***Solution:*** $y = \dfrac{1}{2} + e^{-x^2}$.

9. Find the general solution $x(t)$ of the differential equation $\dfrac{dx}{dt} = \dfrac{t}{t^2 + t - 2}$, where $t > 1$.

***Solution:*** $x(t) = \dfrac{1}{3}\ln(t-1) + \dfrac{2}{3}\ln(t+2) + C$, where $C \in \mathbb{R}$ is a constant.

10. The amount $x(t)$ of a radioactive isotope in a sample at time $t$ decays according to the Malthusian law $\dfrac{dx}{dt} = -kx$, where $k$ is a positive constant. The half-life of a radioactive substance is the time required to decay to one-half of the initial amount of the substance. If a sample of some radioactive element is reduced from 1.0 gram to 0.91 gram after 37 days, what is the half-life of the element to nearest day?

***Solution:*** $\dfrac{-37\ln 2}{\ln 0.91}$, which is approximately 272 days (cobalt-57).

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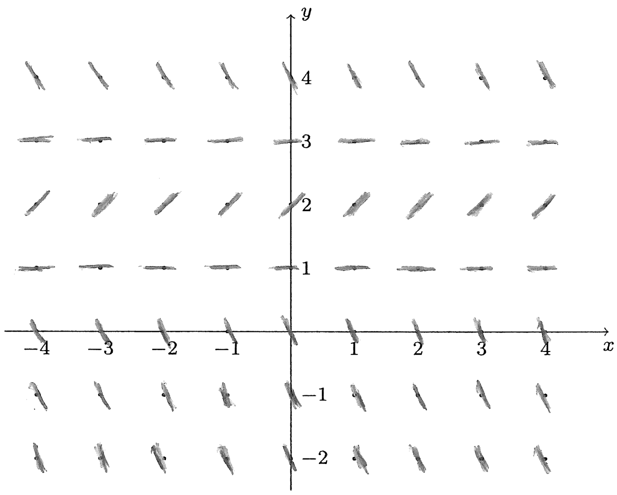
# Solutions to Quiz 2c

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2015

1. Sketch the direction field of the differential equation $y' = (1-y)(y-3)$ in the region below.

Answer Q1:




2. Find the general solution of the differential equation $2y\dfrac{dy}{dx} = -x^3(y^2+4)$.

***Solution:*** $y = \pm\sqrt{Ae^{-\frac{x^4}{4}} - 4}$ for some positive constant $A$.

3. Find an integrating factor for the differential equation $(x^2 + ye^{2x})\cos x = e^{2x}y'$.

***Solution:*** $e^{-\sin x}$.

4. Find the particular solution $x = x(t)$ of the differential equation $x^3\dfrac{dx}{dt} = \tan t$ in the interval $-\pi/4 < t < \pi/2$, which satisfies the condition $x(0) = 2$.

***Solution:*** $x = 2\left[1 - \frac{1}{4}\ln(\cos t)\right]^{1/4}$.

5. Suppose that $y(x)$ is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^3 + x^2y + y^3}{x(x^2 + y^2)}$. We make the transformation $z = \dfrac{y}{x}$. Find a differential equation for $z = z(x)$.

***Solution:*** $x\dfrac{dz}{dx} = \dfrac{1}{1 + z^2}$.

6. Find the equilibrium solution of the equation $\dfrac{dy}{dx} = -y\ln(y^2 + 1)$.

***Solution:*** $y = 0$.

7. Determine whether the equilibrium solution in the above question is stable or unstable.

***Solution:*** $y = 0$ is a stable equilibrium solution.

8. Find the particular solution of $\dfrac{dy}{dx} - \dfrac{y}{x^2} = \dfrac{1}{x^2}$ with $y(1) = 2$.

***Solution:*** $y = 3e^{1-\frac{1}{x}} - 1$.

9. Find the general solution $x(t)$ of the differential equation $\dfrac{dx}{dt} = \dfrac{t}{t^2 - 2t - 3}$, where $t > 3$.

***Solution:*** $x(t) = \dfrac{3}{4}\ln(t-3) + \dfrac{1}{4}\ln(t+1) + C$, where $C$ is a real constant.

10. The amount $x(t)$ of a radioactive isotope in a sample at time $t$ decays according to the Malthusian law $\dfrac{dx}{dt} = -kx$, where $k$ is a positive constant. The half-life of a radioactive substance is the time required to decay to one-half of the initial amount of the substance. The initial mass of an Iodine isotope was 200g. Determine the Iodine mass to nearest gram after 30 days if the half-life of the isotope is 8 days.

***Solution:*** $200e^{-\frac{15}{4}\ln 2}$, approximately 15g.

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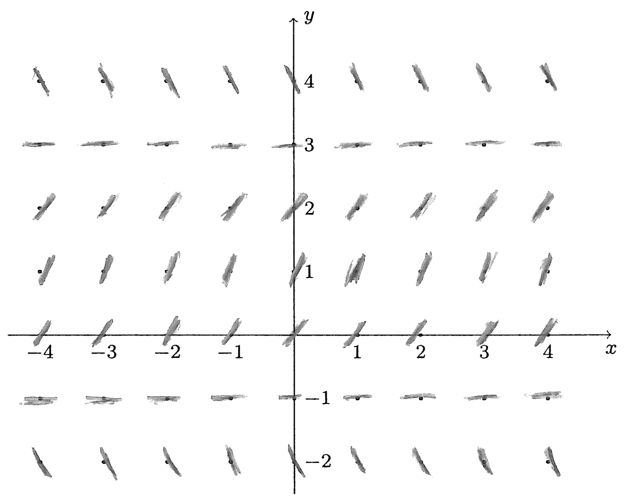
# Solutions to Quiz 2d

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2015

**1.** Sketch the direction field of the differential equation $y' = (y+1)(3-y)$ in the region below.

Answer Q1:




**2.** Find the general solution of the differential equation $y^2\dfrac{dy}{dx} = x^2(y^3+8)$, where $y \neq -2$.

***Solution:*** $y = \left(Ae^{x^3} - 8\right)^{\frac{1}{3}}$, where $A$ is a positive constant.

**3.** Find an integrating factor for the differential equation $(x^3 + ye^x)x = e^x y'$.

***Solution:*** $e^{-\frac{x^2}{2}}$.

**4.** Find the particular solution $x = x(t)$ of the differential equation $x^4\dfrac{dx}{dt} = \tan t$ in the interval $-\pi/4 < t < \pi/2$, which satisfies the condition $x(0) = 1$.

***Solution:*** $x(t) = [1 - 5\ln(\cos t)]^{\frac{1}{5}}$.

5. Suppose that $y(x)$ is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{xy - x^2}$. We make the transformation $z = \dfrac{y}{x}$. Find a differential equation for $z = z(x)$.

***Solution:*** $x\dfrac{dz}{dx} = \dfrac{2z+1}{z-1}$.

6. Find the equilibrium solution of the equation $\dfrac{dy}{dx} = \ln\left(\dfrac{2y}{y^2+1}\right)$.

***Solution:*** $y = 1$.

7. Determine whether the equilibrium solution in the above question is stable or unstable.

***Solution:*** $y = 1$ is unstable.

8. Find the particular solution of $\dfrac{dy}{dx} + \dfrac{y}{x^2} = \dfrac{1}{x^2}$ with $y(1) = 3$.

***Solution:*** $y = 1 + 2e^{-1+\frac{1}{x}}$.

9. Find the general solution $x(t)$ of the differential equation $\dfrac{dx}{dt} = \dfrac{t}{t^2 - t - 2}$, where $t > 2$.

***Solution:*** $x(t) = \dfrac{2}{3}\ln(t-2) + \dfrac{1}{3}\ln(t+1) + C$, where $C \in \mathbb{R}$ is a constant.

10. The amount $x(t)$ of a radioactive isotope in a sample at time $t$ decays according to the Malthusian law $\dfrac{dx}{dt} = -kx$, where $k$ is a positive constant. The half-life of a radioactive substance is the time required to decay to one-half of the initial amount of the substance. The radioactive isotope Indium-111 has half-life of 2.8 days. What was the initial mass to nearest gram of the isotope before decay, if the mass in 14 days was 5g?

***Solution:*** $5\,e^{\frac{14}{2.8}\ln 2} = 160$g.

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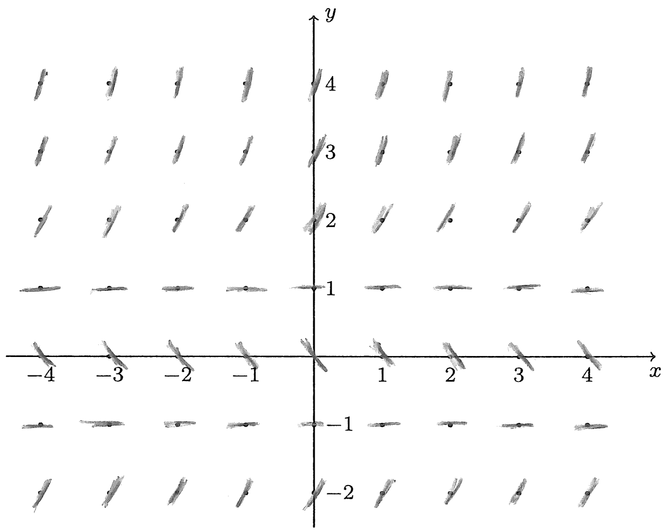
# Solutions to Quiz 2e

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2015

**1.** Sketch the direction field of the differential equation $y' = (y+1)(y-1)$ in the region below.

Answer Q1:




**2.** Find the general solution of the differential equation $\dfrac{dy}{dx} = x(y+1)^2$, where $y \neq -1$.

***Solution:*** $y = -1 - \dfrac{2}{x^2 + A}$ for some real constant $A$.

**3.** Find an integrating factor for the differential equation $e^x - y = e^{2x}y'$.

***Solution:*** $e^{-\frac{1}{2}e^{-2x}}$

**4.** Find the particular solution $x = x(t)$ of the differential equation $x\dfrac{dx}{dt} = 1 + \tan t$ in the interval $-\pi/4 < t < \pi/2$, which satisfies the condition $x(0) = 2$.

***Solution:*** $x(t) = \sqrt{4 + 2[t - \ln(\cos t)]}$.

**5.** Suppose that $y(x)$ is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^3 + xy^2 + y^3}{x(xy + y^2)}$. We make the transformation $z = \dfrac{y}{x}$. Find a differential equation for $z = z(x)$.

***Solution:*** $x\dfrac{dz}{dx} = \dfrac{1}{z(1+z)}$.

**6.** Find the equilibrium solution of the equation $\dfrac{dy}{dx} = -(y-1)\ln\left(\dfrac{y^2+1}{2y}\right)$.

***Solution:*** $y = 1$.

**7.** Determine whether the equilibrium solution in the above question is stable or unstable.

***Solution:*** $y = 1$ is a stable equilibrium solution.

**8.** Find the particular solution of $\dfrac{dy}{dx} + \dfrac{y}{x} = e^x$ with $y(1) = 2$ (here, $x > 0$).

***Solution:*** $y = e^x\left(1 - \dfrac{1}{x}\right) + \dfrac{2}{x}$.

**9.** Find the general solution $x(t)$ of the differential equation $\dfrac{dx}{dt} = \dfrac{t}{t^2 + 5t + 6}$, where $t > -2$.

***Solution:*** $x(t) = -2\ln(t+2) + 3\ln(t+3) + C$, where $C$ is a real constant.

**10.** The amount $x(t)$ of a radioactive isotope in a sample at time $t$ decays according to the Malthusian law $\dfrac{dx}{dt} = -kx$, where $k$ is a positive constant. The half-life of a radioactive substance is the time required to decay to one-half of the initial amount of the substance. Find the mass of a radioactive isotope to nearest gram if 2.5 half lives occurred. The initial mass of the material was 80g.

***Solution:*** $10\sqrt{2}$g, approximately 14g.