The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2 **Exercises for Week 4 (beginning 21 August)** 2017

It might be useful to attempt the Revision and Exploration Exercises before the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Volumes of Revolution, the Disc Method: The volume of revolution of the curve $y = f(x)$ about the x-axis over the interval $[a, b]$ (assuming this part of the curve is contained in the first quadrant) is

$$\int_a^b \pi\big[f(x)\big]^2\,dx\ .$$

(ii) Volumes of Revolution, the Shell Method: The volume of revolution of the curve $y = f(x)$ about the y-axis over the interval $[a, b]$ (assuming this part of the curve is contained in the first quadrant) is

$$\int_a^b 2\pi x f(x)\,dx\ .$$

(iii) Length of a Curve: The length of a paramatrised curve $\mathcal{C}$ in the plane where $x = x(t)$, $y = y(t)$ are differentiable functions, and $a \le t \le b$, is

$$\int_a^b \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}\;dt\ .$$

In the special case that $x = t$ and $y = f(x)$, the length of $\mathcal{C}$ becomes

$$\int_a^b \sqrt{1 + \big[f'(x)\big]^2}\;dx\ .$$

(iv) Surface Area of Revolution: The surface area of revolution of the curve $y = f(x)$ about the x-axis over the interval $[a, b]$ (assuming this part of the curve is contained in the first quadrant) is

$$\int_a^b 2\pi f(x)\sqrt{1 + \big[f'(x)\big]^2}\;dx\ .$$

Revision and Exploration:

1. What do we mean by a *polynomial function*? Explain briefly how you know that polynomial functions are continuous.

2. Write out the Intermediate Value Theorem.

3. Let $f(x) = x^2 - 2$, and verify that

$$f(1.4)\ <\ 0\ < f(1.5)\ .$$

Use the Intermediate Value Theorem to deduce that the square root of 2 exists and lies between 1.4 and 1.5.

4. Suppose $a > 1$ and n is an integer ≥ 2. Define a function f by the rule $f(x) = x^n - a$. Verify that

$$f(0) \; < \; 0 \; < f(a) \; .$$

Use the Intermediate Value Theorem, to deduce that the real number $\sqrt[n]{a}$ exists (and lies between 0 and a).

(See the last exercise on this sheet to prove the existence of $\sqrt[n]{a}$ using elementary properties of $\mathbb{R}$ only.)

5. By the previous exercise, the real number $\sqrt[3]{5}$ exists. Prove that $\sqrt[3]{5}$ is not rational.

Tutorial Exercises:

6. (for general discussion) Observe the following facts:

(i) The area A of a circle in terms of the radius r is $A = \pi r^2$, and $dA/dr = 2\pi r$, which is the perimeter of the circle.

(ii) The volume V of a sphere in terms of the radius r is $V = \frac{4}{3}\pi r^3$, and $dV/dr = 4\pi r^2$, which is the surface area of the sphere.

(iii) The area S of a square of side length x is $S = x^2$, and $dS/dx = 2x$, which is half of the perimeter of the square.

(iv) The volume C of a cube of side length x is $C = x^3$, and $dC/dx = 3x^2$, which is half of the surface area of the cube.

Where does the half come from in parts (iii) and (iv)?

7. Find the area of the region bounded by the curves $y = \sqrt{1 - x^2}$ and $y = \sqrt{2}\, x^2$.

8. Use both discs and shells to find a formula for the volume of a right circular cone of height h and radius r.

9. Find the length of the catenary $y = \cosh x$ for $-1 \leq x \leq 1$.

*__10.__ Find the volume of a solid torus obtained by rotating the circle of centre $(R, 0)$ and radius $r \leq R$ about the y-axis. Compare your answer with the cylinder that would result if you sliced the torus vertically and straightened it out.

**__11.__ Find the surface area of the torus of the previous question. Use the formula

$$2\pi \int_a^b x\sqrt{1 + (dx/dy)^2}\; dy$$

with appropriate choices of a, b and function $x = f(y)$. Again, compare with a cylinder, and also with the derivative of your answer to the previous exercise!

Further Exercises:

12. Find the volume generated by rotating the region bounded by the x-axis, the line $x = 2$ and the graph of $y = x$ about

(i) the x-axis. (ii) the y-axis. *(iii) the line $x = 4$. *(iv) the line $y = 2$.

13. Suppose that a bagel cut horizontally in half has the shape given by rotating about the y-axis the area bounded by the curve $y = 3x - x^2 - 2$ and the x-axis. Find the volume of the top half of the bagel.

***14.** Sketch the curve with equation $x^{2/3} + y^{2/3} = 1$ and find its length.

***15.** The curve traced out by a point on the circumference of a circle as the circle rolls along a straight line is called a *cycloid*. If the circle has radius r and rolls along the x-axis, and the cycloid passes through the origin, then the curve has parametric equations

$$x = r(\theta - \sin\theta)\,, \qquad y = r(1 - \cos\theta)\ .$$

Show that the length of one arch of the cycloid (for $0 \le \theta \le 2\pi$) is 8 times the radius.

***16.** Generalise (without proof) the formula for arc length for a curve in the plane for a curve in space, and use it to find the length of the spiral $x = a\cos t$, $y = a\sin t$, $z = bt$ for $0 \le t \le 2\pi$.

*****17.** (This difficult exercise avoids theorems about continuity, which makes one appreciate them even more.)

Suppose $a > 1$ and n is an integer ≥ 2. For any positive integer k, put

$$m_k \;=\; \max\left\{\, z \in \mathbb{Z} \;\middle|\; \left(\frac{z}{2^k}\right)^n \;\le\; a \,\right\}$$

and put $a_k = \dfrac{m_k}{2^k}$. Verify that

$$1 \le a_1 \le a_2 \le \cdots \le a_k \le \cdots \le a\ ,$$

so $\{a_i\}_{i=1}^{\infty}$ is a nondecreasing sequence bounded above, so has a least upper bound L, by completeness of $\mathbb{R}$. (In fact $L = \lim_{k\to\infty} a_k$.) Prove that $L^n = a$, so the real number $\sqrt[n]{a}$ exists.

Short Answers to Selected Exercises:

7. $\frac{\pi}{4} + \frac{1}{6}$ **8.** $\frac{\pi r^2 h}{3}$ **9.** $e - e^{-1}$

10. $2\pi^2 R r^2$ **11.** $4\pi^2 R r$

12. (i) $\frac{8\pi}{3}$ (ii) $\frac{16\pi}{3}$ (iii) $\frac{32\pi}{3}$ (iv) $\frac{16\pi}{3}$

13. $\frac{\pi}{2}$ **14.** 6 **16.** $2\pi\sqrt{a^2 + b^2}$