CHAPTER **TWO**

Further Trigonometry

We have now established the basic calculus of the trigonometric functions and their inverse functions. Along the way, there has been much work on trigonometric equations, and on the application of trigonometry to problems in two dimensions. This chapter will give a systematic account of trigonometric identities and equations, and then extend the applications of trigonometry to problems in three dimensions. Much of this material will be used when the methods of calculus are consolidated and developed further in Chapter Three on motion and in Chapter Six on further calculus.

Although the sine and cosine waves are not so prominent here, it is important to keep in mind that they are the impulse for most of the trigonometry in this course. Remember that the tangent and cotangent functions are the ratios of the heights of the two waves, and that the secant and cosecant functions arise when these tangent and cotangent functions are differentiated.

STUDY NOTES: Trigonometric identities and equations are closely linked, because the solution of trigonometric equations so often comes down to the application of some identity. Sections 2A–2C deal systematically with identities, with particular emphasis on compound angles. Section 2D applies these identities to the solutions of trigonometric equations. Section 2E deals with the sum of sine and cosine waves in preparation for simple harmonic motion in Chapter Three. Section 2F is an extension on some 4 Unit identities called *sums to products* and *products to sums* that are best studied in the context of this chapter by those taking the 4 Unit course. Finally, Sections 2G and 2H develop the application of trigonometry to problems in three-dimensional space — they require the new ideas of the angle between a line and a plane, and the angle between two planes.

2 A Trigonometric Identities

Developing fluency in trigonometric identities is the purpose of the first three sections. Most of the identities have been established already, and are listed again here for reference. But the triple-angle formulae in this section are new (although it is not intended that they be memorised), and so are the t-formulae in the next section.

Identities Relating the Six Trigonometric Functions: Four groups of identities relating the six trigonometric functions were developed in Chapter Four of the Year 11 volume, and are listed here for reference.

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THE RECIPROCAL IDENTITIES:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

THE RATIO IDENTITIES:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

THE PYTHAGOREAN IDENTITIES:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

THE COMPLEMENTARY IDENTITIES:

$$\cos(90^\circ - \theta) = \sin\theta$$

$$\cot(90^\circ - \theta) = \tan\theta$$

$$\operatorname{cosec}(90^\circ - \theta) = \sec\theta$$

Each of these identities holds provided both LHS and RHS are well defined.

WORKED EXERCISE: Show that $\tan\theta + \cot\theta = \sec\theta\operatorname{cosec}\theta$.

SOLUTION:

$$\begin{aligned}
\text{LHS} &= \tan\theta + \cot\theta \\
&= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}, \quad \text{using the ratio identities,} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}, \quad \text{using a common denominator,} \\
&= 1 \times \operatorname{cosec}\theta\sec\theta, \quad \text{using the Pythagorean and reciprocal identities,} \\
&= \text{RHS}, \quad \text{as required.}
\end{aligned}$$

The Compound-Angle Formulae: These formulae were developed in Chapter Fourteen of the Year 11 volume.

2

THE COMPOUND-ANGLE FORMULAE:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

WORKED EXERCISE:

(a) Find $\tan 75^\circ$. (b) Use small-angle theory to approximate $\sin 61^\circ$.

SOLUTION:

(a)
$$\begin{aligned}
\tan 75^\circ &= \tan(45^\circ + 30^\circ) \\
&= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\
&= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} \\
&= \tfrac{1}{2}(4 + 2\sqrt{3}) \\
&= 2 + \sqrt{3}
\end{aligned}$$

(b)
$$\begin{aligned}
\sin 61^\circ &= \sin(60^\circ + 1^\circ) \\
&= \sin 60^\circ \cos 1^\circ + \cos 60^\circ \sin 1^\circ
\end{aligned}$$

For small angles, $\cos\theta \doteqdot 1$,
and $\sin\theta \doteqdot \theta$, where θ is in radians.
Since $1^\circ = \frac{\pi}{180}$ radians,

$$\begin{aligned}
\sin 61^\circ &\doteqdot \tfrac{1}{2}\sqrt{3} \times 1 + \tfrac{1}{2} \times \tfrac{\pi}{180} \\
&\doteqdot \tfrac{1}{360}(180\sqrt{3} + \pi).
\end{aligned}$$

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Double-Angle Formulae: These formulae are reviewed from Chapter Fourteen of the Year 11 volume — they follow immediately from the compound-angle formulae by setting α and β equal to θ. There are three forms of the $\cos 2\theta$ formula because $\sin^2\theta$ and $\cos^2\theta$ are easily related to each other by the Pythagorean identities.

3 THE DOUBLE-ANGLE FORMULAE:

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\end{aligned} \qquad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Expressing $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos 2\theta$: The second and third forms of the $\cos 2\theta$ formula above are important because they allow the squares $\sin^2\theta$ and $\cos^2\theta$ to be expressed in terms of the simple trigonometric function $\cos 2\theta$.

From $\cos 2\theta = 2\cos^2\theta - 1$,

$$\begin{aligned}2\cos^2\theta &= 1 + \cos 2\theta\\ \cos^2\theta &= \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta.\end{aligned}$$

From $\cos 2\theta = 1 - 2\sin^2\theta$,

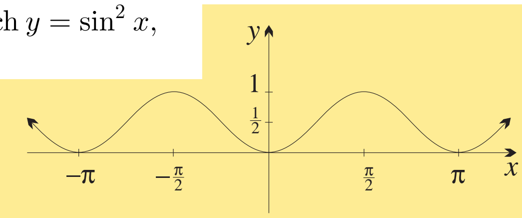
$$\begin{aligned}2\sin^2\theta &= 1 - \cos 2\theta\\ \sin^2\theta &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta.\end{aligned}$$

4 EXPRESSING $\sin^2\theta$ AND $\cos^2\theta$ IN TERMS OF $\cos 2\theta$:

$$\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta \qquad \sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta$$

Notice that $\cos^2\theta + \sin^2\theta = (\frac{1}{2} + \frac{1}{2}\cos 2\theta) + (\frac{1}{2} - \frac{1}{2}\cos 2\theta) = 1$, in accordance with the Pythagorean identities. This observation may help you to memorise them.

WORKED EXERCISE: Without using calculus, sketch $y = \sin^2 x$, and state its amplitude, period and range.



SOLUTION: Using the identities above,

$$y = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x.$$

This is the graph of $y = \cos 2x$ turned upside down, then stretched vertically by the factor $\frac{1}{2}$, then shifted up $\frac{1}{2}$. Its period is π, and its amplitude is $\frac{1}{2}$. Since it oscillates around $\frac{1}{2}$ rather than 0, its range is $0 \le y \le 1$.

The Triple-Angle Formulae: Memorisation of triple-angle formulae is not required in the course, but their proof and their application can reasonably be required. Here are the three formulae, followed by the proof of the $\sin 3\theta$ formula — the proofs of the other two are left to the following exercise.

5 THE TRIPLE-ANGLE FORMULAE:
(Memorisation is not required.)

$$\begin{aligned}\sin 3\theta &= 3\sin\theta - 4\sin^3\theta\\ \cos 3\theta &= 4\cos^3\theta - 3\cos\theta\\ \tan 3\theta &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\end{aligned}$$

PROOF OF THE FORMULA FOR $\sin 3\theta$:

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta)\\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta, \quad \text{using the formula for } \sin(\alpha+\beta),\\ &= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta, \quad \text{using the double-angle formulae,}\\ &= 2\sin\theta(1-\sin^2\theta) + (1-2\sin^2\theta)\sin\theta, \quad \text{since } \cos^2\theta = 1 - \sin^2\theta,\\ &= 3\sin\theta - 4\sin^3\theta, \quad \text{after expanding and collecting terms.}\end{aligned}$$

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Exercise 2A

1. Simplify, using the compound-angle results:

(a) $\cos 3\theta \cos\theta + \sin 3\theta \sin\theta$

(b) $\sin 50^\circ \cos 10^\circ - \cos 50^\circ \sin 10^\circ$

(c) $\dfrac{\tan 41^\circ + \tan 9^\circ}{1 - \tan 41^\circ \tan 9^\circ}$

(d) $\cos 15^\circ \cos 55^\circ - \sin 15^\circ \sin 55^\circ$

(e) $\sin 4\alpha \cos 2\alpha + \cos 4\alpha \sin 2\alpha$

(f) $\dfrac{1 + \tan 2\theta \tan\theta}{\tan 2\theta - \tan\theta}$

2. Simplify, using the double-angle results:

(a) $2\sin 2\theta \cos 2\theta$

(b) $\cos^2 \tfrac{1}{2}x - \sin^2 \tfrac{1}{2}x$

(c) $2\cos^2 3\alpha - 1$

(d) $\dfrac{2\tan 35^\circ}{1 - \tan^2 35^\circ}$

(e) $1 - 2\sin^2 25^\circ$

(f) $\dfrac{2\tan 4x}{1 - \tan^2 4x}$

3. Given that the angles A and B are acute, and that $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$, find:

(a) $\cos A$

(b) $\cos 2A$

(c) $\cos(A + B)$

(d) $\sin 2B$

(e) $\tan 2A$

(f) $\tan(B - A)$

4. (a) By writing 75° as $45^\circ + 30^\circ$, show that:

(i) $\sin 75^\circ = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$

(ii) $\cos 75^\circ = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

(b) Hence show that:

(i) $\sin 75^\circ \cos 75^\circ = \frac{1}{4}$

(ii) $\sin 75^\circ - \cos 75^\circ = \sin 45^\circ$

(iii) $\sin^2 75^\circ - \cos^2 75^\circ = \sin 60^\circ$

(iv) $\sin^2 75^\circ + \cos^2 75^\circ = 1$

5. Use the compound-angle and double-angle results to find the exact value of:

(a) $2\sin 15^\circ \cos 15^\circ$

(b) $\cos 35^\circ \cos 5^\circ + \sin 35^\circ \sin 5^\circ$

(c) $\dfrac{\tan 110^\circ + \tan 25^\circ}{1 - \tan 110^\circ \tan 25^\circ}$

(d) $1 - 2\sin^2 22\tfrac{1}{2}^\circ$

(e) $\cos \frac{\pi}{12} \sin \frac{\pi}{12}$

(f) $\sin \frac{8\pi}{9} \cos \frac{2\pi}{9} - \cos \frac{8\pi}{9} \sin \frac{2\pi}{9}$

(g) $2\cos^2 \frac{7\pi}{12} - 1$

(h) $\dfrac{\tan \frac{25\pi}{18} - \tan \frac{\pi}{18}}{1 + \tan \frac{25\pi}{18} \tan \frac{\pi}{18}}$

(i) $\dfrac{\sin 105^\circ \cos 105^\circ}{\cos^2 67\frac{1}{2}^\circ - \sin^2 67\frac{1}{2}^\circ}$

(j) $\dfrac{2\cos^2 \frac{2\pi}{5} - 1}{1 - 2\sin^2 \frac{\pi}{10}}$

6. Simplify the following using the three double-angle results $\sin A \cos A = \frac{1}{2}\sin 2A$, $1 - \cos 2A = 2\sin^2 A$ and $1 + \cos 2A = 2\cos^2 A$:

(a) $\sin \frac{\theta}{2} \cos \frac{\theta}{2}$

(b) $\frac{1}{2}(1 - \cos 2x)$

(c) $\frac{1}{2}(1 + \cos 4x)$

(d) $1 - \cos 6\theta$

(e) $\sqrt{\tfrac{1}{2}(1 + \cos 40^\circ)}$

(f) $1 + \cos\alpha$

(g) $\sqrt{\tfrac{1}{2}(1 - \cos 10x)}$

(h) $\sin^2 \alpha \cos^2 \alpha$

7. Suppose that θ is an acute angle and $\cos\theta = \frac{4}{5}$. Using the results $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, find the exact value of:

(a) $\cos \frac{1}{2}\theta$

(b) $\sin \frac{1}{2}\theta$

(c) $\tan \frac{1}{2}\theta$

8. Prove each of the following identities:

(a) $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$

(b) $\cos^4 x - \sin^4 x = \cos 2x$

(c) $\cos A - \sin 2A \sin A = \cos A \cos 2A$

(d) $\sin 2\theta(\tan\theta + \cot\theta) = 2$

(e) $\cot\alpha \sin 2\alpha - \cos 2\alpha = 1$

(f) $\dfrac{1}{1 - \tan A} - \dfrac{1}{1 + \tan A} = \tan 2A$

(g) $\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = \cos 2\theta$

(h) $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$

(i) $\dfrac{1 - \cos 2\alpha}{1 + \cos 2\alpha} = \tan^2\alpha$

(j) $\tan 2A(\cot A - \tan A) = 2$, (provided $\cot A \neq \tan A$)

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DEVELOPMENT

9. (a) If $\sin\theta = \frac{1}{3}$ and $\cos\theta < 0$, find the exact value of $\tan 2\theta$.
(b) If $\frac{3\pi}{2} < \theta < 2\pi$ and $\cos\theta = \frac{31}{50}$, find $\cos\frac{\theta}{2}$.

10. (a) By writing 3θ as $2\theta + \theta$ and using appropriate compound-angle and double-angle results, prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
(b) Hence show that $\cos 40^\circ$ is a root of the equation $8x^3 - 6x + 1 = 0$.
(c) Show also that $\cos 3\theta = \frac{5}{9}\sqrt{3}$, if $\tan\theta = \sqrt{2}$ and $\pi < \theta < \frac{3\pi}{2}$.

11. (a) Show that $\sin 3x = 3\sin x - 4\sin^3 x$.
(b) Use the identities for $\cos 3x$ (see the previous question) and $\sin 3x$ to show that

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

12. If θ is acute and $\cos\theta = \frac{2}{3}$, find the exact value of:
(a) $\sin\theta$
(b) $\cos 2\theta$
(c) $\sin 2\theta$
(d) $\sin 3\theta$
(e) $\sin 4\theta$
(f) $\cos 4\theta$
(g) $\tan 3\theta$
(h) $\tan 4\theta$
(i) $\cos\frac{1}{2}\theta$
(j) $\tan\frac{1}{2}\theta$

13. Prove each of the following identities:
(a) $\cot 2\alpha + \tan\alpha = \operatorname{cosec} 2\alpha$
(b) $\dfrac{\sin 3A}{\sin A} + \dfrac{\cos 3A}{\cos A} = 4\cos 2A$
(c) $\dfrac{2\sin^3\theta + 2\cos^3\theta}{\sin\theta + \cos\theta} = 2 - \sin 2\theta$
(d) $\tan 2x \cot x = 1 + \sec 2x$
(e) $\dfrac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta - 1} = \tan(\theta + \frac{\pi}{4})$
(f) $\dfrac{1 + \sin 2\alpha}{1 + \cos 2\alpha} = \frac{1}{2}(1 + \tan\alpha)^2$
(g) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$
(h) $8\cos^4 x = 3 + 4\cos 2x + \cos 4x$
[HINT: $\cos^4 x = \left(\frac{1}{2}(1 + \cos 2x)\right)^2$]
(i) $\operatorname{cosec} 4A + \cot 4A = \frac{1}{2}(\cot A - \tan A)$
(j) $\tan(\frac{\pi}{4} + x) = \sec 2x + \tan 2x$
(k) $\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta = \frac{1}{2}(\cos 2\alpha + \cos 2\beta)$
(l) $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4\cos^2\frac{1}{2}(A - B)$
(m) $\dfrac{\sin 2\alpha + \cos 2\alpha}{2\cos\alpha + \sin\alpha - 2(\cos^3\alpha + \sin^3\alpha)} = \operatorname{cosec}\alpha$
(n) $(\tan\theta + \tan 2\theta)(\cot\theta + \cot 3\theta) = 4$ [HINT: Use the $\tan 3x$ identity in question 11.]

14. Use the compound-angle results and small-angle theory (see the appropriate worked exercise in the notes) to show that:
(a) $\cos 46^\circ \doteqdot \frac{1}{360}\sqrt{2}(180 - \pi)$
(b) $\tan 61^\circ \doteqdot \dfrac{180\sqrt{3} + \pi}{180 - \pi\sqrt{3}}$
(c) $\sin 59^\circ \doteqdot \frac{1}{360}(180\sqrt{3} - \pi)$
(d) $\sec 29^\circ \doteqdot \dfrac{360}{180\sqrt{3} + \pi}$

15. Eliminate θ from each pair of parametric equations:
(a) $x = 2 + \cos\theta$, $y = \cos 2\theta$
(b) $x = \tan\theta + 1$, $y = \tan 2\theta$
(c) $x = 2\tan\frac{1}{2}\theta$, $y = \cos\theta$
(d) $x = 3\sin\theta$, $y = 6\sin 2\theta$

16. (a) Write down the exact value of $\cos 45^\circ$.
(b) Hence show that: (i) $\cos 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 + \sqrt{2}}$ (ii) $\cos 11\frac{1}{4}^\circ = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}$

17. (a) Show that $\sqrt{8 - 4\sqrt{3}} = \sqrt{6} - \sqrt{2}$.
(b) Show that $\tan 165^\circ = \sqrt{3} - 2$.
(c) Hence show that $\tan 82\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$.

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EXTENSION

18. (a) Eliminate θ from the equations $\cos\theta + \sin\theta = a$, $\cos 2\theta = b$.
(b) Eliminate θ and ϕ from $\sin\theta + \sin\phi = a$, $\cos\theta + \cos\phi = b$ and $\cos(\theta - \phi) = c$.

19. (a) Explain why $\sin 54° = \cos 36°$. (b) Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.
(c) Hence show that $4\sin^3 18° - 2\sin^2 18° - 3\sin 18° + 1 = 0$.
(d) Hence show that $\sin 18°$ is a root of the equation $4x^2 + 2x - 1 = 0$, and find its value.
(e) Show that: (i) $\sin 54° = \dfrac{\sqrt{5}+1}{4}$ (ii) $\cos 54° = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$
(f) Show that $\sqrt{8 + 2\sqrt{10 - 2\sqrt{5}}} = \sqrt{5 + \sqrt{5}} + \sqrt{3 - \sqrt{5}}$.
(g) Hence show that $\cos 27° = \frac{1}{4}\left(\sqrt{5 + \sqrt{5}} + \sqrt{3 - \sqrt{5}}\right)$.

20. (a) Prove by induction that $\cos\dfrac{90°}{2^n} = \frac{1}{2}\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\cdots + \sqrt{2}}}}}}_{n \text{ terms}}$.
(b) Find $\sin\dfrac{90°}{2^n}$. Hence find an expression converging to π, and investigate it as a means of approximating π.

2 B The *t*-Formulae

The t-formulae express $\sin\theta$, $\cos\theta$ and $\tan\theta$ as algebraic functions of the single trigonometric function $\tan\frac{1}{2}\theta$. In the proliferation of trigonometric identities, this can sometimes provide a systematic approach that does not rely on seeing some clever trick.

The *t*-Formulae: The first of the t-formulae is a restatement of the double-angle formula for the tangent function. The other two formulae follow quickly from it.

6 THE t-FORMULAE: Let $t = \tan\frac{1}{2}\theta$. Then:

$$\sin\theta = \frac{2t}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2} \qquad \tan\theta = \frac{2t}{1-t^2}$$

PROOF:

Let $t = \tan\frac{1}{2}\theta$. We seek to express $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of t.

First, $\tan\theta = \dfrac{2\tan\frac{1}{2}\theta}{1 - \tan^2\frac{1}{2}\theta}$, by the double-angle formula,

$$= \frac{2t}{1-t^2}. \qquad (1)$$

Secondly, $\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta$, by the double-angle formula,

$= \dfrac{\cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta}$, by the Pythagorean identity,

$= \dfrac{1 - \tan^2\frac{1}{2}\theta}{1 + \tan^2\frac{1}{2}\theta}$, dividing through by $\cos^2\frac{1}{2}\theta$,

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$$= \frac{1-t^2}{1+t^2}\,. \qquad (2)$$

Thirdly, $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$, by the double-angle formula,

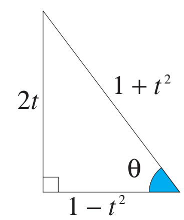
$$= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta+\sin^2\frac{1}{2}\theta}, \text{ by the Pythagorean identity,}$$

$$= \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}, \text{ dividing through by } \cos^2\tfrac{1}{2}\theta,$$

$$= \frac{2t}{1+t^2}\,. \qquad (3)$$

NOTE: The proofs given above for these identities rely heavily on the idea of expressions that are *homogeneous* of degree 2 in $\sin x$ and $\cos x$, meaning that the sum of the indices of $\sin x$ and $\cos x$ in each term is 2 — such expressions are easily converted into expressions in $\tan^2 x$ alone. Homogeneous equations will be reviewed in Section 2D.

An Algebraic Identity, and a Way to Memorise the t-formulae: On the right is a right triangle which demonstrates the relationship amongst the three formulae when θ is acute. The three sides are related by Pythagoras' theorem, and the algebra rests on the quadratic identity



$$(1-t^2)^2 + (2t)^2 = (1+t^2)^2.$$

This diagram may help to memorise the t-formulae.

WORKED EXERCISE: Use the t-formulae to prove:

(a) $\dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$ (b) $\sec 2x + \tan 2x = \tan(x+\frac{\pi}{4})$

SOLUTION:

(a) Let $t = \tan\frac{1}{2}x$.

$$\begin{aligned}\text{LHS} &= \left(1-\frac{1-t^2}{1+t^2}\right) \div \frac{2t}{1+t^2}\\ &= \frac{1+t^2-1+t^2}{1+t^2}\times\frac{1+t^2}{2t}\\ &= \frac{2t^2}{2t}\\ &= t\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{2t}{1+t^2}\times\left(1+\frac{1-t^2}{1+t^2}\right)^{-1}\\ &= \frac{2t}{1+t^2}\times\frac{1+t^2}{1+t^2+1-t^2}\\ &= \frac{2t}{2}\\ &= t\\ &= \text{LHS}\end{aligned}$$

Notice that we have proven the further identity

$$\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta} = \tan\tfrac{1}{2}\theta.$$

(b) Let $t = \tan x$.

$$\begin{aligned}\text{LHS} &= \frac{1+t^2}{1-t^2}+\frac{2t}{1-t^2}\\ &= \frac{1+2t+t^2}{(1+t)(1-t)}\\ &= \frac{(1+t)^2}{(1+t)(1-t)}\\ &= \frac{1+t}{1-t}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{\tan x+1}{1-\tan x\times 1}\\ &= \frac{1+t}{1-t}\\ &= \text{LHS}\end{aligned}$$

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Exercise 2B

1. Write in terms of t, where $t = \tan \frac{1}{2}\theta$:

(a) $\sin\theta$
(b) $\cos\theta$
(c) $\tan\theta$
(d) $\sec\theta$
(e) $1 - \cos\theta$
(f) $\dfrac{1-\cos\theta}{\sin\theta}$

2. Write in terms of t, where $t = \tan\theta$:

(a) $\cos 2\theta$
(b) $1 - \sin 2\theta$
(c) $\tan 2\theta + \sec 2\theta$

3. Use the $t = \tan\frac{1}{2}\theta$ results to simplify:

(a) $\dfrac{2\tan 10^\circ}{1-\tan^2 10^\circ}$
(b) $\dfrac{2\tan 10^\circ}{1+\tan^2 10^\circ}$
(c) $\dfrac{1-\tan^2 10^\circ}{1+\tan^2 10^\circ}$
(d) $\dfrac{2\tan 2x}{1+\tan^2 2x}$
(e) $\dfrac{2\tan 2x}{1-\tan^2 2x}$
(f) $\dfrac{1-\tan^2 2x}{1+\tan^2 2x}$

4. Use the $t = \tan\frac{1}{2}\theta$ results to find the exact value of:

(a) $\dfrac{2\tan 15^\circ}{1-\tan^2 15^\circ}$
(b) $\dfrac{2\tan 15^\circ}{1+\tan^2 15^\circ}$
(c) $\dfrac{1-\tan^2 75^\circ}{1+\tan^2 75^\circ}$
(d) $\dfrac{2\tan 112\frac{1}{2}^\circ}{1+\tan^2 112\frac{1}{2}^\circ}$
(e) $\dfrac{1-\tan^2 \frac{3\pi}{8}}{1+\tan^2 \frac{3\pi}{8}}$
(f) $\dfrac{2\tan \frac{11\pi}{12}}{1-\tan^2 \frac{11\pi}{12}}$

DEVELOPMENT

5. Prove each of the following identities using the $t = \tan\frac{1}{2}\theta$ results:

(a) $\cos\theta(\tan\theta - \tan\frac{1}{2}\theta) = \tan\frac{1}{2}\theta$
(b) $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$
(c) $\dfrac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{1}{2}\theta$
(d) $\dfrac{1+\operatorname{cosec}\theta}{\cot\theta} = \dfrac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}$
(e) $\dfrac{\tan\theta\tan\frac{1}{2}\theta}{\tan\theta - \tan\frac{1}{2}\theta} = \sin\theta$
(f) $\dfrac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1} = \tan\frac{1}{2}\theta$
(g) $\dfrac{\tan 2\alpha + \cot\alpha}{\tan 2\alpha - \tan\alpha} = \cot^2\alpha$
(h) $\tan(\frac{1}{2}x + \frac{\pi}{4}) + \tan(\frac{1}{2}x - \frac{\pi}{4}) = 2\tan x$

6. (a) Given that $t = \tan 112\frac{1}{2}^\circ$, show that $\dfrac{2t}{1-t^2} = 1$.

(b) (i) Hence show that $\tan 112\frac{1}{2}^\circ = -\sqrt{2} - 1$.
(ii) What does the other root of the equation represent?

7. Use the method of the previous question to show that:

(a) $\tan 15^\circ = 2 - \sqrt{3}$
(b) $\tan\frac{7\pi}{8} = 1 - \sqrt{2}$

8. Suppose that $\tan\alpha = -\frac{1}{3}$ and $\frac{\pi}{2} < \alpha < \pi$. Find the exact value of:

(a) $\tan 2\alpha$
(b) $\sin 2\alpha$
(c) $\cos 2\alpha$
(d) $\tan\frac{1}{2}\alpha$

9. [Alternative derivations of the t-formulae] Let $t = \tan\frac{1}{2}\theta$.

(a) (i) Express $\cos\theta$ in terms of $\cos\frac{1}{2}\theta$.
(ii) Write $\cos^2\frac{1}{2}\theta$ as $\dfrac{1}{\sec^2\frac{1}{2}\theta}$, and hence show that $\cos\theta = \dfrac{1-t^2}{1+t^2}$.

(b) (i) Write $\sin\theta$ in terms of $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$.
(ii) Write $\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ as $\dfrac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\cos^2\frac{1}{2}\theta$, and hence show that $\sin\theta = \dfrac{2t}{1+t^2}$.

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10. (a) If $x = \tan\theta + \sec\theta$, use the t-formulae to show that $\dfrac{x^2-1}{x^2+1} = \sin\theta$.

(b) If $x = \cos 2\theta$, use the t-formulae to show that $\sqrt{\dfrac{1+x}{1-x}} = |\cot\theta|$.

11. If $\cos x = \dfrac{5\cos y - 3}{5 - 3\cos y}$, prove that $\tan^2 \frac{1}{2}x = 4\tan^2 \frac{1}{2}y$.
[HINT: Let $t_1 = \tan\frac{1}{2}x$ and $t_2 = \tan\frac{1}{2}y$.]

EXTENSION

12. Consider the integral $I = \displaystyle\int \frac{1}{1+\cos x}\,dx$, and let $t = \tan\frac{1}{2}x$.

(a) Show that $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.

(b) By writing dx as $\dfrac{dx}{dt}\,dt$, show that $I = \displaystyle\int dt = \tan\frac{1}{2}x + C$.

13. Use the same approach as in the previous question to show that

$$\int \operatorname{cosec} x\,dx = \log_e(\tan\tfrac{1}{2}x) + C.$$

14. (a) Show that $\dfrac{1}{(x+2)(2x+1)} = \dfrac{1}{3}\left(\dfrac{2}{2x+1} - \dfrac{1}{x+2}\right)$.

(b) Hence show that $\displaystyle\int_0^1 \frac{1}{(x+2)(2x+1)}\,dx = \frac{1}{3}\log 2$.

(c) Using the approach of question 12, deduce that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{3}{4+5\sin x}\,dx = \log 2$.

2 C Applications of Trigonometric Identities

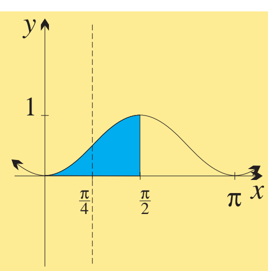
The exercise of this section contains further examples of trigonometric identities, but it also seeks to relate the trigonometric identities of the previous two sections to geometric situations and to calculus.

The Integration of $\cos^2 x$ and $\sin^2 x$: The identities expressing $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos 2\theta$ provide the standard way of finding primitives of $\sin^2 x$ and $\cos^2 x$.

WORKED EXERCISE: Find: (a) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

Explain from their graphs why these integrals are equal.

SOLUTION:



(a)
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx &= \int_0^{\frac{\pi}{2}} (\tfrac{1}{2} - \tfrac{1}{2}\cos 2x)\,dx \\ &= \Big[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\Big]_0^{\frac{\pi}{2}} \\ &= (\tfrac{\pi}{4} - \tfrac{1}{4}\sin\pi) - (0 - \tfrac{1}{4}\sin 0) \\ &= \tfrac{\pi}{4}\end{aligned}$$

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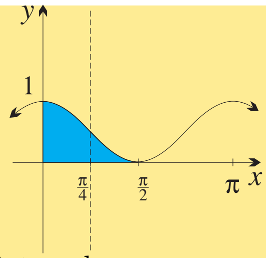
(b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,dx$

$\displaystyle = \Big[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\Big]_0^{\frac{\pi}{2}}$

$= (\tfrac{\pi}{4} + \tfrac{1}{4}\sin\pi) - (0 + \tfrac{1}{4}\sin 0)$

$= \tfrac{\pi}{4}$



Since $\cos^2 x = \sin^2(\frac{\pi}{2} - x)$, the regions represented by the two integrals are reflections of each other in this vertical line $x = \frac{\pi}{4}$, and so have the same area. Also, the answer $\frac{\pi}{4}$ can easily be seen by taking advantage of the symmetry of each graph to cut and paste the shaded region to form a rectangle.

Geometric Configurations and Trigonometric Identities: There is an endless variety of geometric configurations in which trigonometric identities play a role. The worked exercise below involves the expansion of $\sin 2\theta$ and the range of $\cos\theta$.

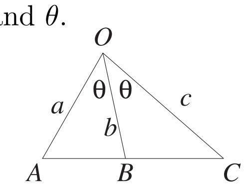
WORKED EXERCISE: Three sticks of lengths a, b and c extend from a point O so that their endpoints A, B and C respectively are collinear, and so that OB bisects $\angle AOC$. Let $\theta = \angle AOB = \angle BOC$.

(a) Find the areas of $\triangle AOB$, $\triangle BOC$ and $\triangle AOC$ in terms of a, b, c and θ.

(b) Hence show that $\cos\theta = \dfrac{b(a+c)}{2ac}$.

(c) Show that the middle stick OB cannot be the longest stick.

(d) If $a = b$ and $c = 2b$, find the area of $\triangle AOC$ in terms of b.



SOLUTION:

(a) Using the formula for the area of a triangle, area $\triangle AOB = \frac{1}{2}ab\sin\theta$,
area $\triangle BOC = \frac{1}{2}bc\sin\theta$,
area $\triangle AOC = \frac{1}{2}ac\sin 2\theta$.

(b) Since the area of $\triangle AOC$ is the sum of the areas of $\triangle AOB$ and $\triangle BOC$,

$$\tfrac{1}{2}ac\sin 2\theta = \tfrac{1}{2}ab\sin\theta + \tfrac{1}{2}bc\sin\theta$$
$$2ac\sin\theta\cos\theta = ab\sin\theta + bc\sin\theta$$
$$\cos\theta = \frac{b(a+c)}{2ac}, \text{ as required.}$$

(c) From part (b), $\cos\theta = \dfrac{b(a+c)}{2ac}$

$$= \frac{b}{2c} + \frac{b}{2a}.$$

If b were the longest stick, then both terms would be greater than $\frac{1}{2}$, and so $\cos\theta$ would be greater than 1, which is impossible.

(d) From part (b),
$$\cos\theta = \frac{b(a+c)}{2ac} = \frac{b \times 3b}{2b \times 2b} = \tfrac{3}{4}$$
so $\sin\theta = \frac{1}{4}\sqrt{7}$.

From part (a), area $\triangle AOC = ac\sin\theta\cos\theta$
$$= b \times 2b \times \tfrac{1}{4}\sqrt{7} \times \tfrac{3}{4}$$
$$= \tfrac{3}{8}b^2\sqrt{7}.$$

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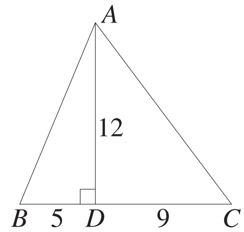
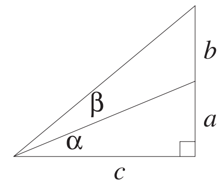
Exercise 2C



1. Find, using appropriate compound-angle results:

(a) $\sin \angle BAC$

(b) $\cos \angle BAC$



2. In the diagram opposite, suppose that $\tan \beta = \frac{1}{3}$.

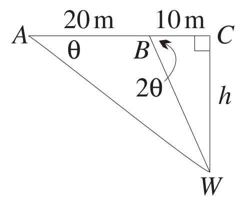
(a) Write down an expression for $\tan \alpha$.

(b) Use an appropriate compound-angle formula to show that $\tan(\alpha + \beta) = \dfrac{3a + c}{3c - a}$.

(c) Write down an alternative expression for $\tan(\alpha + \beta)$.

(d) Hence show that $b = \dfrac{a^2 + c^2}{3c - a}$.

3. Points A, B, C and W lie in the same vertical plane. A bird at A observes a worm at W at an angle of depression θ. After flying 20 metres horizontally to B, the angle of depression of the worm is 2θ. If the bird flew another 10 metres horizontally it would be directly above the worm. Let $WC = h$ metres.

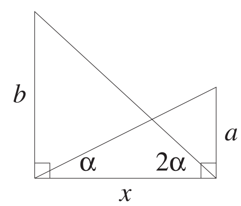


(a) Write $\tan 2\theta$ in terms of $\tan \theta$.

(b) Use the two right-angled triangles to write two equations in h and θ.

(c) Use parts (a) and (b) to show that $\dfrac{h}{10} = \dfrac{60h}{900 - h^2}$.

(d) Hence show that $h = 10\sqrt{3}$ metres.



4. (a) Using the diagram opposite, write down expressions for $\tan \alpha$ and $\tan 2\alpha$.

(b) Use the double-angle formula for $\tan 2\alpha$ to show that $\dfrac{b}{x} = \dfrac{2ax}{x^2 - a^2}$.

(c) Hence show that $x = a\sqrt{\dfrac{b}{b - 2a}}$.

(d) Why is it necessary to assume that $b > 2a$?

DEVELOPMENT

5. Use the results $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ and $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ to find:

(a) $\displaystyle\int_0^{\pi} \sin^2 x \, dx$

(b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$

(c) $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2 \tfrac{1}{2}x \, dx$

(d) $\displaystyle\int_0^{\frac{\pi}{16}} \cos^2 2x \, dx$

(e) $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 (x + \tfrac{\pi}{12}) \, dx$

(f) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 (x - \tfrac{\pi}{6}) \, dx$

6. (a) Sketch the graph of $y = \cos 2x$, for $0 \le x \le 2\pi$.

(b) Hence sketch, on the same diagram, $y = \frac{1}{2}(1 + \cos 2x)$ and $y = \frac{1}{2}(1 - \cos 2x)$.

(c) Hence show graphically that $\cos^2 x + \sin^2 x = 1$.

7. Explain why $\cos x \sin x$ cannot exceed $\frac{1}{2}$.

8. (a) If $\tan \theta = \dfrac{x \sin \phi}{1 - x \cos \phi}$, show that $x = \dfrac{\sin \theta}{\sin(\theta + \phi)}$.

(b) If also $\tan \phi = \dfrac{y \sin \theta}{1 - y \cos \theta}$, find $\dfrac{x}{y}$ in simplest form in terms of θ and ϕ.

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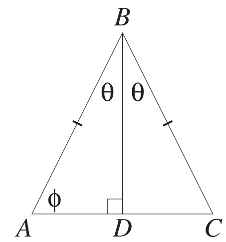
9. $\triangle ABC$ is isosceles with $AB = CB$, and D lies on AC with $BD \perp AC$. Let $\angle ABD = \angle CBD = \theta$, and $\angle BAD = \phi$.

(a) Show that $\sin\phi = \cos\theta$.

(b) Use the sine rule in $\triangle ABC$ to show that $\sin 2\theta = 2\sin\theta\cos\theta$.

(c) If $0 < \theta < \frac{\pi}{2}$, show that

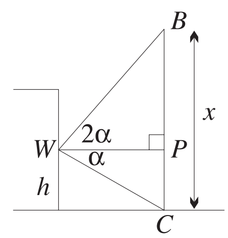
$$\sin 2\theta + \sin 2\theta\cos^2\theta + \sin 2\theta\cos^4\theta + \cdots = 2\cot\theta.$$



10. An office-worker is looking out a window W of a building standing on level ground. From W, a car C has an angle of depression α, while a balloon B directly above the car has an angle of elevation 2α. The height of the balloon above the car is x, and the height of the window above the ground is h.

(a) Show that $\dfrac{\tan\alpha}{h} = \dfrac{\tan 2\alpha}{x-h}$.

(b) Hence show that $h = \dfrac{1-\tan^2\alpha}{3-\tan^2\alpha}x$.



11. In $\triangle ABC$, $\dfrac{b}{c} = \dfrac{4}{3}$.

(a) If $B = 2C$, show that $\cos C = \frac{2}{3}$. (b) If $B = 3C$, show that $\sin C = \frac{1}{6}\sqrt{15}$.

12. (a) By writing $\sin^4 x$ as $\left(\sin^2 x\right)^2$, show that $\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$.

(b) Find a similar result for $\cos^4 x$.

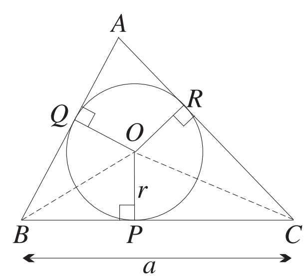
(c) Hence find: (i) $\displaystyle\int_0^{\pi} \sin^4 x\,dx$ (ii) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^4 x\,dx$

13. The diagram shows a circle with centre O and radius r inscribed in a triangle ABC.

(a) Prove that $\angle OBP = \angle OBQ$.

(b) Prove that $\dfrac{a}{r} = \cot\frac{1}{2}B + \cot\frac{1}{2}C$.

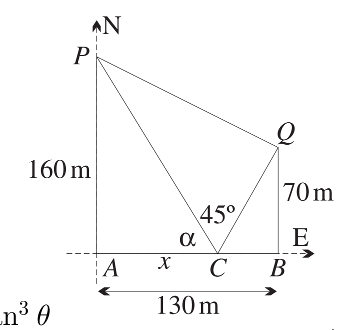
(c) Hence prove that $\dfrac{a}{r} = \dfrac{\cos\frac{1}{2}A}{\sin\frac{1}{2}B\sin\frac{1}{2}C}$.



14. In the diagram opposite, P and Q are landmarks which are 160 metres and 70 metres due north of points A and B respectively. A and B lie 130 metres apart on a west–east road. C is a point on the road between A and B and $\angle PCQ = 45°$. Let $AC = x$ and $\angle ACP = \alpha$.

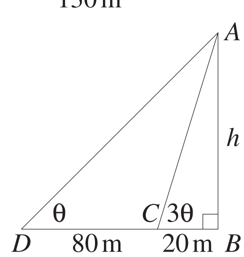
(a) Show that $\tan(135° - \alpha) = \dfrac{70}{130-x}$.

(b) Hence show that $AC = 120$ metres.



15. (a) By expressing 3θ as $2\theta + \theta$, show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

(b) A tower AB has height h metres. The angle of elevation of the top of the tower at a point C 20 metres from its base is three times the angle of elevation at a point D 80 metres further away from its base. Use the identity in part (a) to show that $h = \frac{100}{7}\sqrt{7}$ metres.



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16. Define $F(x) = \displaystyle\int_0^x \sin^2 t\,dt$, where $0 \le x \le 2\pi$.

(a) Show that $F(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x$.

(b) Explain why $F'(x) = \sin^2 x$. Hence state the values of x in the given domain for which $F(x)$ is: (i) stationary, (ii) increasing, (iii) decreasing.

(c) Explain why $F(x)$ never differs from $\frac{1}{2}x$ by more than $\frac{1}{4}$.

(d) Find any points of inflexion of $F(x)$ in the given domain.

(e) Sketch, on the same diagram, the graphs of $y = F(x)$ and $y = F'(x)$ over the given domain, and observe how they are related.

(f) (i) For what value of k is $\displaystyle\int_0^k \sin^2 x\,dx = \frac{3\pi}{2}$?

(ii) For what values of k is $\displaystyle\int_0^k \sin^2 x\,dx = \frac{n\pi}{2}$, where n is an integer?

EXTENSION

17. The lengths of the sides of a triangle form an arithmetic progression and the largest angle of the triangle exceeds the smallest by 90°. Show that the lengths of the sides of the triangle are in the ratio $\sqrt{7}-1 : \sqrt{7} : \sqrt{7}+1$. [HINT: One possible approach makes use of both the sine and cosine rules.]

18. [Harmonic conjugates] In $\triangle ABC$, the bisectors of the internal and external angles at A meet BC produced at P and Q respectively. Prove that Q divides BC externally in the same ratio as that in which P divides BC internally.

19. Suppose that $\tan^2 x = \tan(x-\alpha)\tan(x-\beta)$. Show that $\tan 2x = \dfrac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$.

2 D Trigonometric Equations

Trigonometric equations occur whenever trigonometric functions are being analysed, and careful study of them is essential. This section presents a systematic approach to their solution, and begins with the account given in Chapter Four of the Year 11 volume when the compound-angle formulae were not yet available.

Simple Trigonometric Equations: More complicated trigonometric equations eventually reduce to equations like

$$\cos x = -1, \qquad \text{or} \qquad \tan x = -\sqrt{3}, \text{ for } -2\pi \le x \le 2\pi,$$

where there may or may not be a restriction on the domain. The methods here should be familiar by now.

7 SIMPLE TRIGONOMETRIC EQUATIONS: If a trigonometric equation involves angles at the boundaries of quadrants, read the solutions off a sketch of the graph. Otherwise, draw a quadrants diagram, and read the solutions off it.

WORKED EXERCISE: Solve: (a) $\cos x = -1$ (b) $\tan x = -\sqrt{3}$, for $-2\pi \le x \le 2\pi$

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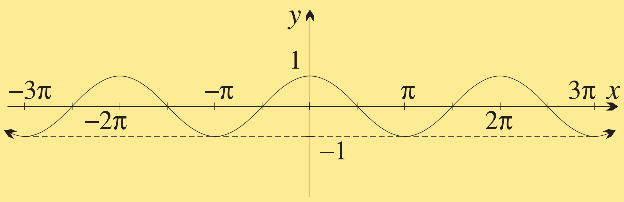
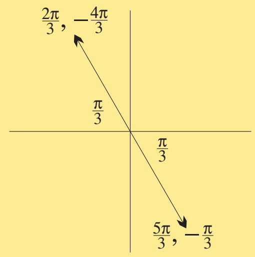
SOLUTION:





(a) Reading from the graph of $y = \cos x$,
$x = \pi,\ -\pi,\ 3\pi,\ -3\pi, \dots$
$x = (2n+1)\pi$, where n is an integer.

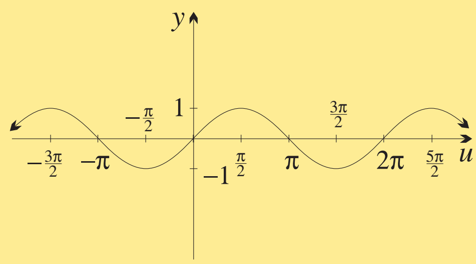
(b) The related angle is $\frac{\pi}{3}$,
and x is in quadrants 2 or 4,
so $x = \frac{2\pi}{3},\ \frac{5\pi}{3},\ -\frac{\pi}{3}$ or $-\frac{4\pi}{3}$.

Simple Trigonometric Equations with a Compound Angle: Most troubles are avoided by substitution for the compound angle. Any given restrictions on the original angle must then be carried through to restrictions on the compound angle.

8 SIMPLE EQUATIONS WITH A COMPOUND ANGLE: Let u be the compound angle. From the given restrictions on x, find the resulting restrictions on u.

WORKED EXERCISE: Solve $\sin(3x + \frac{5\pi}{4}) = 1$, for $-\pi \le x \le \pi$.

SOLUTION: Let $u = 3x + \frac{5\pi}{4}$.
Then $-3\pi \le 3x \le 3\pi$
$\boxed{+\frac{5\pi}{4}}$ $-\frac{7\pi}{4} \le 3x + \frac{5\pi}{4} \le \frac{17\pi}{4}$.
Hence $\sin u = 1$, for $-\frac{7\pi}{4} \le u \le \frac{17\pi}{4}$
$u = -\frac{3\pi}{2},\ \frac{\pi}{2}$ or $\frac{5\pi}{2}$
$3x + \frac{5\pi}{4} = -\frac{3\pi}{2},\ \frac{\pi}{2}$ or $\frac{5\pi}{2}$
$3x = -\frac{11\pi}{4},\ -\frac{3\pi}{4}$ or $\frac{5\pi}{4}$
$x = -\frac{11\pi}{12},\ -\frac{\pi}{4}$ or $\frac{5\pi}{12}$.

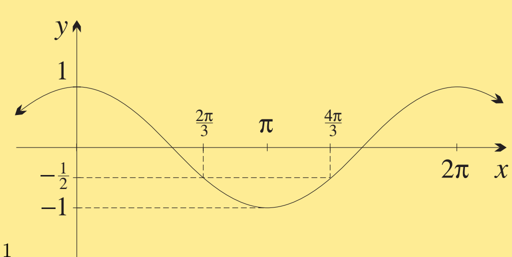


Equations Requiring Algebraic Substitutions: If there are powers or reciprocals of the one trigonometric function present, it is usually best to make a substitution for that trigonometric function.

9 ALGEBRAIC SUBSTITUTION FOR A TRIGONOMETRIC FUNCTION: Substitute u for the trigonometric function, solve the resulting algebraic equation, then solve each of the resulting trigonometric equations.

WORKED EXERCISE: Solve $2\cos x = 1 + \sec x$, for $0 \le x \le 2\pi$.

SOLUTION: Let $u = \cos x$.
Then $2u = 1 + \dfrac{1}{u}$
$2u^2 - u - 1 = 0$
$(2u+1)(u-1) = 0$
$u = 1$ or $u = -\frac{1}{2}$,
so $\cos x = 1$ or $\cos x = -\frac{1}{2}$.
Hence $x = 0,\ 2\pi,\ \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.



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Equations with More than One Trigonometric Function, but the Same Angle: This is where trigonometric identities come into play.

10 EQUATIONS WITH MORE THAN ONE TRIGONOMETRIC FUNCTION: Trigonometric identities can usually be used to produce an equation in only one trigonometric function.

WORKED EXERCISE: Solve the equation $2\tan\theta = \sec\theta$, for $0° \le \theta \le 360°$:
(a) using the ratio identities, (b) by squaring both sides.

SOLUTION:

(a)
$$2\tan\theta = \sec\theta$$
$$\frac{2\sin\theta}{\cos\theta} = \frac{1}{\cos\theta}$$
$$\sin\theta = \tfrac{1}{2}$$
$$\theta = 30° \text{ or } 150°$$

(b) Squaring,
$$4\tan^2\theta = \sec^2\theta$$
$$4\sec^2\theta - 4 = \sec^2\theta$$
$$\sec^2\theta = \tfrac{4}{3}$$
$$\cos\theta = \tfrac{1}{2}\sqrt{3} \text{ or } -\tfrac{1}{2}\sqrt{3}$$
$$\theta = 30°, 150°, 210° \text{ or } 330°.$$
Checking each solution, $\theta = 30°$ or $150°$.

The Dangers of Squaring an Equation: Squaring an equation is to be avoided if possible, because squaring may introduce extra solutions, as it did in part (b) above. If an equation does have to be squared, each solution must be checked in the original equation to see whether it is a solution or not. Here are two very simple equations, both purely algebraic, where the effect of squaring can easily be seen.

(a) Suppose that $x = 3$.
Squaring, $x^2 = 9$
so $x = 3$ or $x = -3$.
Here $x = -3$ is a spurious solution.

(b) Suppose that $\sqrt{x} = -5$.
Squaring, $x = 25$.
But $\sqrt{25} = 5$, not -5.
In fact, there are no solutions.

Equations Involving Different Angles: When different angles are involved in the same trigonometric equation, the usual approach is to use compound-angle identities to change all the trigonometric functions to functions of the one angle.

11 EQUATIONS INVOLVING DIFFERENT ANGLES: Use compound-angle identities to change all the trigonometric functions to functions of the one angle.

Frequently such an equation can be solved by more than one method.

WORKED EXERCISE: Solve $\cos 2x = 4\sin^2 x - 14\cos^2 x$, for $0 \le x \le 2\pi$:
(a) by changing all the angles to x, (b) by changing all the angles to $2x$.

SOLUTION:

(a)
$$\cos 2x = 4\sin^2 x - 14\cos^2 x$$
$$\cos^2 x - \sin^2 x = 4\sin^2 x - 14\cos^2 x$$
$$15\cos^2 x = 5\sin^2 x$$
$$\tan x = \sqrt{3} \text{ or } -\sqrt{3}$$
$$x = \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ or } \tfrac{5\pi}{3}$$

(b)
$$\cos 2x = 4\sin^2 x - 14\cos^2 x$$
$$\cos 2x = 4(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x) - 14(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)$$
$$10\cos 2x = -5$$
$$\cos 2x = -\tfrac{1}{2}$$
$$2x = \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, \tfrac{8\pi}{3} \text{ or } \tfrac{10\pi}{3}$$
$$x = \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ or } \tfrac{5\pi}{3}$$

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Homogeneous Equations: Equations homogeneous in $\sin x$ and $\cos x$ were mentioned earlier as a special case of the application of trigonometric identities.

12 HOMOGENEOUS EQUATIONS: An equation is called *homogeneous* in $\sin x$ and $\cos x$ if the sum of the indices of $\sin x$ and $\cos x$ in each term is the same.
To solve an equation homogeneous in $\sin x$ and $\cos x$, divide through by a power of $\cos x$ to produce an equation in $\tan x$ alone.

The expansions of $\sin 2x$ and $\cos 2x$ are homogeneous of degree 2 in $\sin x$ and $\cos x$. Also, $1 = \sin^2 x + \cos^2 x$ can be regarded as being homogeneous of degree 2.

WORKED EXERCISE: Solve $\sin 2x + \cos 2x = \sin^2 x + 1$, for $0 \le x \le 2\pi$.

SOLUTION: Expanding, $2\sin x\cos x + (\cos^2 x - \sin^2 x) = \sin^2 x + (\sin^2 x + \cos^2 x)$

$$3\sin^2 x - 2\sin x\cos x = 0$$

$\boxed{\div \cos^2 x}$

$$3\tan^2 x - 2\tan x = 0$$

$$\tan x(3\tan x - 2) = 0$$

$$\tan x = 0 \quad \text{or} \quad \tan x = \tfrac{2}{3}.$$

Hence $x = 0,\ \pi$ or 2π, or $x \doteqdot 0{\cdot}588$ or $3{\cdot}730$.

The Equations $\sin x = \sin\alpha$, $\cos x = \cos\alpha$ and $\tan x = \tan\alpha$: The methods associated with general solutions of trigonometric equations from the last chapter can often be very useful.

13 THE GENERAL SOLUTIONS OF $\sin x = \sin\alpha$, $\cos x = \cos\alpha$ AND $\tan x = \tan\alpha$:

- If $\tan x = \tan\alpha$, then $x = n\pi + \alpha$, where n is an integer.
- If $\cos x = \cos\alpha$, then $x = 2n\pi + \alpha$ or $2n\pi - \alpha$, where n is an integer.
- If $\sin x = \sin\alpha$, then $x = 2n\pi + \alpha$ or $(2n+1)\pi - \alpha$, where n is an integer.

WORKED EXERCISE: Solve $\tan 4x = -\tan 2x$:

(a) using the $\tan 2\theta$ formula, (b) using solutions of $\tan\alpha = \tan\beta$.

SOLUTION:

(a) $\tan 4x = -\tan 2x$

Let $t = \tan 2x$.

Then $\dfrac{2t}{1-t^2} = -t$

$$2t = -t + t^3$$

$$t^3 - 3t = 0$$

$$t(t^2 - 3) = 0.$$

Hence $\tan 2x = 0$ or $\tan 2x = \sqrt{3}$ or $\tan 2x = -\sqrt{3}$

$2x = k\pi$ or $\frac{\pi}{3} + k\pi$ or $-\frac{\pi}{3} + k\pi$, where k is an integer,

$x = \frac{1}{6}n\pi$, where n is an integer.

(b) $\tan 4x = -\tan 2x$.

Since $\tan\theta$ is an odd function,

$\tan 4x = \tan(-2x)$

$4x = -2x + n\pi$, where n is an integer

$6x = n\pi$

$x = \frac{1}{6}n\pi$.

Another Approach to Trigonometric Functions of Multiples of 18°: In Chapter Four of the Year 11 volume, we used a construction within a pentagon to generate trigonometric functions of some multiples of 18°. Here is another approach through alternative solutions of trigonometric equations.

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WORKED EXERCISE: Solve $\sin 3x = \cos 2x$, for $0° \le x \le 360°$:

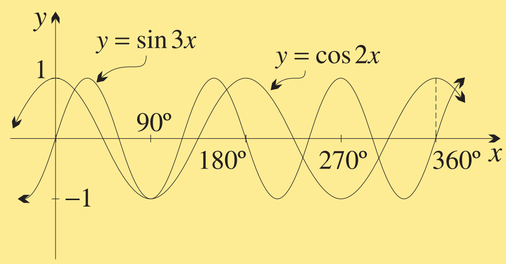
(a) graphically, (b) using solutions of $\cos\alpha = \cos\beta$.

Begin solving using compound-angle formulae, and hence find $\sin 18°$ and $\sin 54°$.

[HINT: Use the factorisation $4u^3 - 2u^2 - 3u + 1 = (u-1)(4u^2 + 2u - 1)$.]

SOLUTION:

(a) The graphs of the two functions are sketched opposite. They make it clear that there are five solutions, and from the graph, one solution is $90°$, and the other four are approximately $20°$, $160°$, $230°$ and $310°$.



(b)
$$\sin 3x = \cos 2x$$
$$\cos(90° - 3x) = \cos 2x, \quad \text{since } \sin\theta = \cos(90° - \theta),$$
$$2x = 90° - 3x + 360n° \quad \text{or} \quad 2x = -90° + 3x + 360n°$$
$$5x = 90°,\ 450°,\ 810°,\ 1170°,\ 1530° \quad \text{or} \quad x = 90°.$$

Hence $\quad x = 18°,\ 90°,\ 162°,\ 234°$ or $306°$.

Alternatively, $\quad \sin 3x = \cos 2x$
$$3\sin x - 4\sin^3 x = 1 - 2\sin^2 x, \quad \text{using compound-angle identities.}$$

Let $\quad u = \sin x$.

Then $\quad 4u^3 - 2u^2 - 3u + 1 = 0$
$$(u-1)(4u^2 + 2u - 1) = 0, \quad \text{by the given factorisation.}$$

The quadratic has discriminant 20, so the three solutions of the cubic are
$$u = 1 \quad \text{or} \quad u = \tfrac{1}{4}(-1 + \sqrt{5}\,) \quad \text{or} \quad u = \tfrac{1}{4}(-1 - \sqrt{5}\,).$$

Now $\sin x = 1$ has the one solution $x = 90°$,

and $\sin x = \frac{1}{4}(-1 + \sqrt{5}\,)$ and $\sin x = \frac{1}{4}(-1 - \sqrt{5}\,)$ each have two solutions.

From part (b), we conclude that $\sin 18° = \sin 162° = \frac{1}{4}(-1 + \sqrt{5}\,)$.

Also, $\sin 234° = \sin 306° = \frac{1}{4}(-1 - \sqrt{5}\,)$, so $\sin 54° = \frac{1}{4}(1 + \sqrt{5}\,)$.

NOTE: From these results, the values of all the trigonometric functions at $18°$, $36°$, $54°$ and $72°$ can be calculated. See the Extension to the following exercise.

Exercise 2D

1. Solve each equation for $0 \le x \le 2\pi$:

(a) $\sqrt{2}\sin x = 1$ (c) $\cot x = \sqrt{3}$ (e) $4\cos^2 x - 3 = 0$

(b) $2\cos x + 1 = 0$ (d) $\sin^2 x = 1$ (f) $\sec^2 x - 2 = 0$

2. Solve each equation for $0° \le \alpha \le 360°$:

(a) $\tan 2\alpha = \sqrt{3}$ (b) $\cos 2\alpha = 1$ (c) $\sin 3\alpha = \frac{1}{2}$ (d) $\tan 3\alpha = -1$

3. Solve, for $0 \le \theta \le 2\pi$:

(a) $\sin(\theta - \frac{\pi}{6}) = \frac{1}{2}\sqrt{3}$ (c) $\sin(2\theta - \frac{\pi}{2}) = \frac{1}{2}\sqrt{2}$

(b) $\cos(\theta + \frac{\pi}{4}) = -\frac{1}{2}\sqrt{3}$ (d) $\cos(2\theta + \frac{\pi}{6}) = -\frac{1}{2}$

4. Use the basic trigonometric identities such as $\dfrac{\sin x}{\cos x} = \tan x$ to solve, for $0 \le x \le 2\pi$:

(a) $\sin x - \sqrt{3}\cos x = 0$ (c) $4\cos 2x = 3\sec 2x$

(b) $4\sin x = \operatorname{cosec} x$ (d) $\sin^2 \frac{1}{2}x = 3\cos^2 \frac{1}{2}x$

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5. Use Pythagorean identities where necessary (such as $\sin^2 x + \cos^2 x = 1$) and factoring to solve the following for $0° \le \alpha \le 360°$. Give answers to the nearest minute where necessary.

(a) $\sin^2 \alpha = \sin \alpha$
(b) $\sec^2 \alpha = 2\sec \alpha$
(c) $\cos^2 \alpha = \sin \alpha \cos \alpha$
(d) $\tan^2 \alpha - 3\tan \alpha - 4 = 0$
(e) $2\sin^2 \alpha = \sin \alpha + 1$
(f) $2\sin^2 \alpha + 3\cos \alpha = 3$
(g) $\sec^2 \alpha - \tan \alpha - 3 = 0$
(h) $\cos^2 2\alpha - 2\sin 2\alpha + 2 = 0$
(i) $\operatorname{cosec}^3 2\alpha = 4\operatorname{cosec} 2\alpha$
(j) $\sqrt{3}\operatorname{cosec}^2 \frac{1}{2}\alpha + \cot \frac{1}{2}\alpha = \sqrt{3}$

6. Use compound-angle formulae to solve, for $0 \le \theta \le 2\pi$:

(a) $\sin(\theta + \frac{\pi}{6}) = 2\sin(\theta - \frac{\pi}{6})$
(b) $\cos(\theta - \frac{\pi}{6}) = 2\cos(\theta + \frac{\pi}{6})$
(c) $\cos 4\theta \cos \theta + \sin 4\theta \sin \theta = \frac{1}{2}$
(d) $\cos 3\theta = \cos 2\theta \cos \theta$

[HINT: In part (d), write $\cos 3\theta$ as $\cos(2\theta + \theta)$.]

7. Use double-angle formulae to solve, for $0 \le x \le 2\pi$:

(a) $\sin 2x = \sin x$
(b) $\cos 2x = \sin x$
(c) $\tan 2x + \tan x = 0$
(d) $\sin 2x = \tan x$

8. Use a sketch of the LHS in each case to help solve, for $0 \le x \le 2\pi$:

(a) $\sin x > 0$
(b) $\sin 2x > 0$
(c) $\cos x \le \frac{1}{2}$
(d) $\cos 2x \le \frac{1}{2}$
(e) $\cos(x - \frac{\pi}{4}) \le \frac{1}{2}$
(f) $\tan 2x \ge 1$

DEVELOPMENT

9. Solve, for $0° \le A \le 360°$, giving solutions correct to the nearest minute where necessary:

(a) $2\sin^2 A - 5\cos A - 4 = 0$
(b) $\tan^2 A = 3(\sec A - 1)$
(c) $3\tan A - \cot A = 2$
(d) $\sqrt{3}\operatorname{cosec}^2 A = 4\cot A$
(e) $2\cos 2A + \sec 2A + 3 = 0$
(f) $\tan^2 A + 3\cot^2 A = 4$
(g) $2(\cos A - \sec A) = \tan A$
(h) $\cot A + 3\tan A = 5\operatorname{cosec} A$
(i) $\sin^2 A - 2\sin A\cos A - 3\cos^2 A = 0$
(j) $\tan^2 A + 8\cos^2 A = 5$

10. Solve, for $0° \le \theta \le 360°$, giving solutions correct to the nearest minute where necessary:

(a) $2\sin 2\theta + \cos \theta = 0$
(b) $2\cos^2 \theta + \cos 2\theta = 0$
(c) $2\cos 2\theta + 4\cos \theta = 1$
(d) $8\sin^2 \theta \cos^2 \theta = 1$
(e) $3\cos 2\theta + \sin \theta = 1$
(f) $\cos 2\theta = 3\cos^2 \theta - 2\sin^2 \theta$
(g) $10\cos \theta + 13\cos \frac{1}{2}\theta = 5$
(h) $\tan \theta = 3\tan \frac{1}{2}\theta$
(i) $\cos^2 2\theta = \sin^2 \theta$
[HINT: Use $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$.]
(j) $\cos 2\theta + 3 = 3\sin 2\theta$
[HINT: Write 3 as $3\cos^2 \theta + 3\sin^2 \theta$.]

11. (a) Show that $\sqrt{3}\,u^2 - (1 + \sqrt{3})u + 1 = (\sqrt{3}\,u - 1)(u - 1)$.
(b) Hence solve the homogeneous equation $\sqrt{3}\sin^2 x + \cos^2 x = (1 + \sqrt{3})\sin x \cos x$, for $0 \le x \le 2\pi$. [HINT: Divide both sides by $\cos^2 x$.]
(c) Similarly, solve $\sin^2 x = (\sqrt{3} - 1)\sin x \cos x + \sqrt{3}\cos^2 x$, for $0 \le x \le 2\pi$.

12. Find general solutions of:

(a) $\cos 2x = \cos x$
(b) $2\sin 2x \cos x = \sqrt{3}\sin 2x$
(c) $\sin x + \cos 2x = 1$
(d) $\sin(x + \frac{\pi}{4}) = 2\cos(x - \frac{\pi}{4})$

13. Consider the equation $\cos 3x = \cos 2x$.
(a) Show that $x = \frac{2}{5}\pi n$, where n is an integer.
(b) Find all solutions in the domain $0 \le x \le 2\pi$.

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14. Find the x-coordinates of any stationary points on each of the following curves in the interval $0 \le x \le 2\pi$.

(a) $y = e^{\tan x - 4x}$
(b) $y = \ln \cos x + \tan x - x$
(c) $y = x + \cos(2x - \frac{\pi}{3})$
(d) $y = \sin x - \frac{1}{2}\cos 2x$
(e) $y = \sin x + \frac{1}{2}\sin 2x$
(f) $y = 2x - \sin 2x + 2\sin^2 x$

15. Consider the equation $\tan(\frac{\pi}{4} + \theta) = 3\tan(\frac{\pi}{4} - \theta)$.

(a) Show that $\tan^2\theta - 4\tan\theta + 1 = 0$.
(b) Hence use the quadratic formula to solve the equation for $0 \le \theta \le \pi$.

16. Given the equation $2\cos x - 1 = 2\cos 2x$:

(a) Show that $\cos x = \frac{1}{4}(1 + \sqrt{5})$ or $\cos x = \frac{1}{4}(1 - \sqrt{5})$.
(b) Hence solve the equation for $0 \le x \le 360°$, using the calculator.

17. (a) Show that $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.
(b) Hence solve the equation $\sin^2 3\theta - \sin^2\theta = \sin 2\theta$, for $0 \le \theta \le \pi$.

18. Use sketches to help solve, for $0 \le x \le 2\pi$:

(a) $\sin^2 x \ge \frac{1}{2}$
(b) $\tan^2 x < \tan x$
(c) $\sin^2 x \ge \cos^2 x$
(d) $2\cos^2 x + \cos x \le 1$
(e) $2\cos^2 x \ge \sin x + 2$
(f) $\sec^2 x \ge 1 + \sqrt{3}\tan x$

19. Find the values of k for which:

(a) $\displaystyle\int_0^k \sin^2 x\,dx = \int_0^k \cos^2 x\,dx$
(b) $\displaystyle\int_0^k \sin^2 x\,dx > \int_0^k \cos^2 x\,dx$

20. Sketch the curve $y = e^{\cos x}$, for $0 \le x \le 2\pi$, after finding the stationary point and the two inflexion points (approximately).

21. Sketch the curve $y = \dfrac{\sin x}{1 + \tan x}$, for $0 \le x \le 2\pi$, after finding the x-intercepts, the vertical asymptotes and the stationary points. Why are there open circles at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$?

22. (a) Show that the function $y = e^x \tan x$ is increasing for all x in its domain.
(b) Find the x-intercepts for $-\frac{\pi}{2} < x < \frac{3\pi}{2}$ and the gradient at each x-intercept.
(c) Show that the curve is concave up at each x-intercept.
(d) Sketch the curve, for $-\frac{\pi}{2} < x < \frac{3\pi}{2}$.

EXTENSION

23. It was proven in the notes that $\sin 18° = \frac{1}{4}(-1 + \sqrt{5})$ and $\sin 54° = \frac{1}{4}(1 + \sqrt{5})$. Use these results to find the sine, cosine and tangent of $18°$, $36°$, $54°$ and $72°$.

24. Consider the equation $\sin\theta + \cos\theta = \sin 2\theta$, for $0° \le \theta \le 360°$.

(a) By squaring both sides, show that $\sin^2 2\theta - \sin 2\theta - 1 = 0$.
(b) Hence solve for θ over the given domain, giving solutions to the nearest minute. [HINT: Beware of the fact that squaring can create invalid solutions.]

25. (a) Show that $\cos 3x = 4\cos^3 x - 3\cos x$.
(b) By substituting $x = 2\cos\theta$, show that the equation $x^3 - 3x - 1 = 0$ has roots $2\cos 20°$, $-2\sin 10°$ and $-2\cos 40°$.
(c) Use a similar technique to find, correct to three decimal places, the three real roots of the equation $x^3 - 12x = 8\sqrt{3}$.

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26. (a) If $t = \tan x$, show that $\tan 4x = \dfrac{4t(1-t^2)}{1-6t^2+t^4}$.

(b) If $\tan 4x \tan x = 1$, show that $5t^4 - 10t^2 + 1 = 0$.

(c) Show that $\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$ and that $\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$.

(d) Hence show that $\frac{\pi}{10}$ and $\frac{3\pi}{10}$ both satisfy $\tan 4x \tan x = 1$.

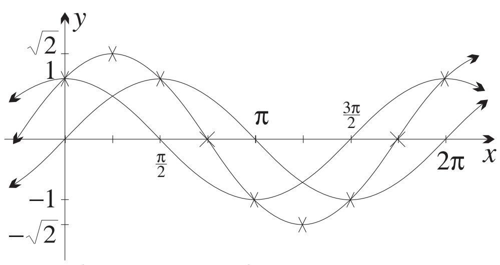
(e) Hence write down, in trigonometric form, the four real roots of the polynomial equation $5x^4 - 10x^2 + 1 = 0$.

2 E The Sum of Sine and Cosine Functions

The sine and cosine curves are the same, except that the sine wave is the cosine wave shifted right by $\frac{\pi}{2}$. This section analyses what happens when the sine and cosine curves are added, and, more generally, when multiples of the two curves are added. The surprising result is that $y = a\sin x + b\cos x$ is still a sine or cosine wave, but shifted sideways so that the zeroes no longer lie on multiples of $\frac{\pi}{2}$.

These forms for $a\sin x + b\cos x$ give a systematic method of solving any equation of the form $a\cos x + b\sin x = c$. Later in the section, an alternative method of solution using the t-formulae is developed.

Sketching $y = \sin x + \cos x$ by Graphical Methods:



The diagram to the right shows the two graphs of $y = \sin x$ and $y = \cos x$. From these two graphs, the sum function $y = \sin x + \cos x$ has been drawn on the same diagram — the crosses represent obvious points to mark on the graph of the sum.

- The new graph has the same period as $y = \sin x$ and $y = \cos x$, that is, 2π. It looks like a wave, and within $0 \le x \le 2\pi$ there are zeroes at the two values $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, where $\sin x$ and $\cos x$ take opposite values.
- The new amplitude is bigger than 1. The value at $x = \frac{\pi}{4}$ is $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \sqrt{2}$, so if the maximum occurs there, as seems likely, the amplitude is $\sqrt{2}$.

This would indicate that the resulting sum function is $y = \sqrt{2}\sin(x + \frac{\pi}{4})$, since it is the stretched sine curve shifted left $\frac{\pi}{4}$. Checking this by expansion:

$$\begin{aligned}\sqrt{2}\sin(x + \tfrac{\pi}{4}) &= \sqrt{2}\left(\sin x \cos \tfrac{\pi}{4} + \cos x \sin \tfrac{\pi}{4}\right)\\ &= \sin x + \cos x, \quad \text{as expected, since } \cos\tfrac{\pi}{4} = \sin\tfrac{\pi}{4} = \frac{1}{\sqrt{2}}.\end{aligned}$$

The General Algebraic Approach — The Auxiliary Angle: It is true in general that any function of the form $f(x) = a\sin x + b\cos x$ can be written as a single wave function. There are four possible forms in which the wave can be written, and the process is done by expanding the standard form and equating coefficients of $\sin x$ and $\cos x$.

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14 AUXILIARY-ANGLE METHOD:

- Any function of the form $f(x) = a\sin x + b\cos x$, where a and b are constants (not both zero), can be written in any one of the four forms:

$$y = R\sin(x - \alpha) \qquad y = R\cos(x - \alpha)$$
$$y = R\sin(x + \alpha) \qquad y = R\cos(x + \alpha)$$

  where $R > 0$ and $0° \le \alpha < 360°$. The constant $R = \sqrt{a^2 + b^2}$ is the same for all forms, but the *auxiliary angle* α depends on which form is chosen.
- To begin the process, expand the standard form and equate coefficients of $\sin x$ and $\cos x$. Be careful to identify the quadrant in which α lies.

The following worked exercise continues with the example given at the start of the section, and shows the systematic algorithm used to obtain the required form.

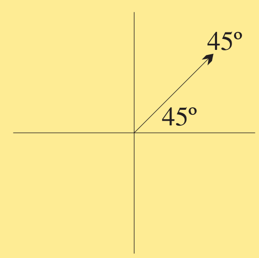
WORKED EXERCISE: Express $y = \sin x + \cos x$ in the two forms:

(a) $R\sin(x + \alpha)$, (b) $R\cos(x + \alpha)$,

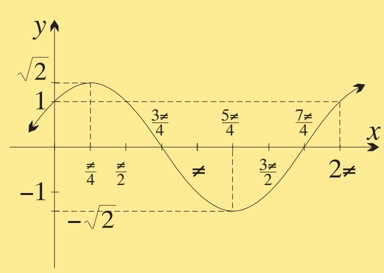
where, in each case, $R > 0$ and $0 \le \alpha < 2\pi$. Then sketch the curve, showing all intercepts and turning points in the interval $0 \le x \le 2\pi$.

SOLUTION:

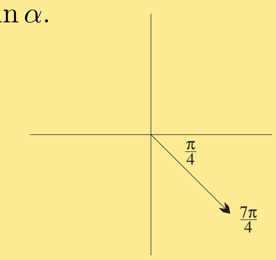
(a) Expanding, $R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$,
so for all x, $\sin x + \cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$.
Equating coefficients of $\sin x$, $R\cos\alpha = 1$, (1)
equating coefficients of $\cos x$, $R\sin\alpha = 1$. (2)
Squaring and adding, $R^2 = 2$
and since $R > 0$, $R = \sqrt{2}$.
From (1), $\cos\alpha = \dfrac{1}{\sqrt{2}}$, (1A)
and from (2), $\sin\alpha = \dfrac{1}{\sqrt{2}}$, (2A)
so α is in the 1st quadrant, with related angle $\frac{\pi}{4}$.
Hence $\sin x + \cos x = \sqrt{2}\sin(x + \frac{\pi}{4})$.



The graph is $y = \sin x$ shifted left by $\frac{\pi}{4}$
and stretched vertically by a factor of $\sqrt{2}$.
Thus the x-intercepts are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$,
there is a maximum of $\sqrt{2}$ when $x = \frac{\pi}{4}$,
and a minimum of $-\sqrt{2}$ when $x = \frac{5\pi}{4}$.



(b) Expanding, $R\cos(x + \alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$,
so for all x, $\sin x + \cos x = R\cos x\cos\alpha - R\sin x\sin\alpha$.
Equating coefficients of $\cos x$, $R\cos\alpha = 1$, (1)
equating coefficients of $\sin x$, $R\sin\alpha = -1$. (2)
Squaring and adding, $R^2 = 2$
and since $R > 0$, $R = \sqrt{2}$.
From (1), $\cos\alpha = \dfrac{1}{\sqrt{2}}$, (1A)



and from (2), $\sin\alpha = -\dfrac{1}{\sqrt{2}}$, (2A)

so α is in the 4th quadrant, with related angle $\frac{\pi}{4}$.

Hence $\sin x + \cos x = \sqrt{2}\cos(x + \frac{7\pi}{4})$.

The graph above could equally well be obtained from this.

It is $y = \cos x$ shifted left by $\frac{7\pi}{4}$ and stretched vertically by a factor of $\sqrt{2}$.

Approximating the Auxiliary Angle: Unless special angles are involved, the auxiliary angle will need to be approximated on the calculator. Degrees or radian measure may be used, but the next worked exercise uses degrees to make the working a little clearer.

WORKED EXERCISE:

(a) Express $y = 3\sin x - 4\cos x$ in the form $y = R\cos(x - \alpha)$, where $R > 0$ and $0^\circ \le \alpha < 360^\circ$, giving α correct to the nearest degree.

(b) Sketch the curve, showing, correct to the nearest degree, all intercepts and turning points in the interval $-180^\circ \le x \le 180^\circ$.

SOLUTION:

(a) Expanding, $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$,

so for all x, $3\sin x - 4\cos x = R\cos x\cos\alpha + R\sin x\sin\alpha$.

Equating coefficients of $\cos x$, $R\cos\alpha = -4$, (1)

equating coefficients of $\sin x$, $R\sin\alpha = 3$. (2)

Squaring and adding, $R^2 = 25$

and since $R > 0$, $R = 5$.

From (1), $\cos\alpha = -\frac{4}{5}$, (1A)

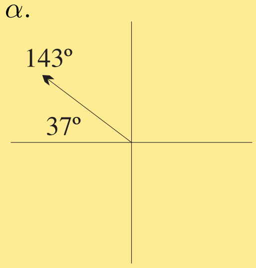
and from (2), $\sin\alpha = \frac{3}{5}$, (2A)

so α is in the 2nd quadrant,

with related angle about 37°.

Hence $3\sin x - 4\cos x = 5\cos(x - \alpha)$,

where $\alpha \doteqdot 143^\circ$.



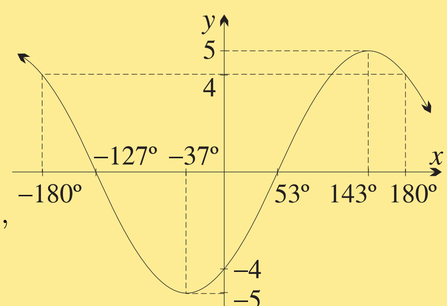
(b) The graph is $y = \cos x$ shifted right by $\alpha \doteqdot 143^\circ$

and stretched vertically by a factor of 5.

Thus the x-intercepts are $x \doteqdot 53^\circ$ and $x \doteqdot -127^\circ$,

there is a maximum of 5 when $x \doteqdot 143^\circ$,

and a minimum of -5 when $x \doteqdot -37^\circ$.



A Note on the Calculator and Approximations for the Auxiliary Angle: In the previous worked exercise, the exact value of α is $\alpha = 180^\circ - \sin^{-1}\frac{3}{5}$ (or $\alpha = 180^\circ - \cos^{-1}\frac{4}{5}$), because α is in the second quadrant. It is this value which is obtained on the calculator, and if there are subsequent calculations to do, as in the equation solved below, this value should be stored in memory and used whenever the auxiliary angle is required. Re-entry of the approximation may lead to rounding errors.

Solving Equations of the Form $a\sin x + b\cos x = c$, and Inequations: Once the LHS has been put in one of the four standard forms, the solutions can easily be obtained. It is always important to keep track of the restriction on the compound angle. The worked exercise below continues with the previous example.

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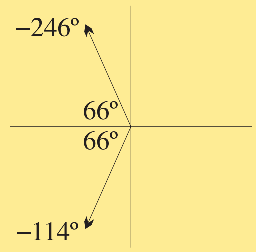
WORKED EXERCISE:

(a) Using the previous worked exercise, solve the equation $3\sin x - 4\cos x = -2$, for $-180^\circ \le x \le 180^\circ$, correct to the nearest degree.

(b) Hence use the graph to solve $3\sin x - 4\cos x \le -2$, for $-180^\circ \le x \le 180^\circ$.

SOLUTION:



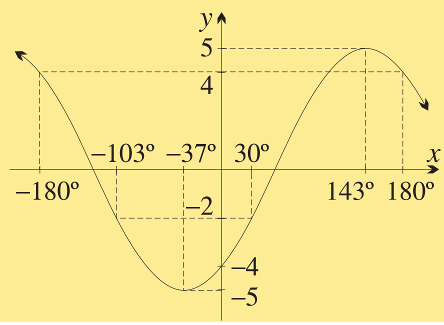
(a) Using $3\sin x - 4\cos x = 5\cos(x-\alpha)$, where $\alpha \doteqdot 143^\circ$,

$$5\cos(x-\alpha) = -2, \text{ where } -323^\circ \le x-\alpha \le 37^\circ$$
$$\cos(x-\alpha) = -\tfrac{2}{5}.$$

Hence $x-\alpha$ is in quadrant 2 or 3, with related angle about 66°,

so $$x-\alpha \doteqdot -114^\circ \text{ or } -246^\circ$$
$$x \doteqdot 30^\circ \text{ or } -103^\circ.$$

Be careful to use the calculator's memory here. Never re-enter approximations of the angles.



(b) The graph to the right shows the previously drawn graph of $y = 3\sin x - 4\cos x$ with the horizontal line $y = -2$ added. This roughly verifies the two answers obtained in part (a). It also shows that the solution to the inequality $3\sin x - 4\cos x \le -2$ is $-103^\circ \le x \le 30^\circ$.

Using the t-formulae to Solve $a\sin x + b\cos x = c$: The t-formulae provide a quite different method of solution by substituting $t = \tan\frac{1}{2}x$. The advantage of this method is that only a single approximation is involved. There are two disadvantages. First, the intuition about the LHS being a shifted wave function is lost. Secondly, if $x = 180^\circ$ happens to be a solution, it will not be found by this method, because $\tan\frac{1}{2}x$ is not defined at $x = 180^\circ$.

WORKED EXERCISE: Solve $3\sin x - 4\cos x = -2$, for $-180^\circ \le x \le 180^\circ$, correct to the nearest minute, using the substitution $t = \tan\frac{1}{2}x$.

SOLUTION: Using $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$, the equation becomes

$$\frac{6t}{1+t^2} - \frac{4-4t^2}{1+t^2} = -2, \text{ provided that } x \ne 180^\circ,$$
$$6t - 4 + 4t^2 = -2 - 2t^2$$
$$6t^2 + 6t - 2 = 0$$
$$3t^2 + 3t - 1 = 0, \text{ which has discriminant } \Delta = 21,$$
$$\tan\tfrac{1}{2}x = -\tfrac{1}{2} + \tfrac{1}{6}\sqrt{21} \text{ or } -\tfrac{1}{2} - \tfrac{1}{6}\sqrt{21}.$$

Since $-180^\circ \le x \le 180^\circ$, the restriction on $\frac{1}{2}x$ is $-90^\circ \le \frac{1}{2}x \le 90^\circ$,

so $$\tfrac{1}{2}x = 14{\cdot}775\,961\ldots^\circ \text{ or } -51{\cdot}645\,859\ldots^\circ$$
$$x \doteqdot 29^\circ 33' \text{ or } -103^\circ 18'.$$

The Problem when $x = 180^\circ$ is a Solution: The substitution $t = \tan\frac{1}{2}x$ fails when $x = 180^\circ$, because $\tan 90^\circ$ is undefined. One must always be aware, therefore, of this possibility, and be prepared to add this answer to the final solution. The situation can easily be recognised in either of the following ways:

- The terms in t^2 cancel out, leaving a linear equation in t.
- The coefficient of $\cos x$ is the opposite of the constant term.

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WORKED EXERCISE: Solve $7\sin x - 4\cos x = 4$, for $0° \le x \le 360°$, by using the substitution $t = \tan\frac{1}{2}x$.

SOLUTION: Substituting $t = \tan\frac{1}{2}x$ gives

$$\frac{14t}{1+t^2} - \frac{4-4t^2}{1+t^2} = 4, \text{ provided that } x \ne 180°,$$

$$14t - 4 + 4t^2 = 4 + 4t^2$$

$$14t = 8.$$

[WARNING: The terms in t^2 cancelled out — check $t = 180°$!]

Hence $\tan\frac{1}{2}x = \frac{4}{7}$

$$x \doteqdot 59°29'.$$

But $x = 180°$ is also a solution, since then LHS $= 7 \times 0 - 4 \times (-1) =$ RHS,
so $x = 180°$ or $x \doteqdot 59°29'$.

A Summary of Methods of Solving $a\sin x + b\cos x = c$: Here then is a summary of the two approaches to the solution.

15 SOLVING EQUATIONS OF THE FORM $a\sin x + b\cos x = c$:

- THE AUXILIARY-ANGLE METHOD: Get the LHS into one of the forms

$$R\sin(x+\alpha), \quad R\sin(x-\alpha), \quad R\cos(x+\alpha) \quad \text{or} \quad R\cos(x-\alpha),$$

  then solve the resulting equation.
- USING THE t-FORMULAE: Substitute $t = \tan\frac{1}{2}x$ and then solve the resulting quadratic in t. Be aware that $x = 180°$ will also be a solution if:
  - $*$ the terms in t^2 cancel out, leaving a linear equation in t, or equivalently,
  - $*$ the coefficient of $\cos x$ is the opposite of the constant term.

Exercise 2E

1. Find R and α exactly, if $R > 0$ and $0 \le \alpha < 2\pi$, and:
   (a) $R\sin\alpha = \sqrt{3}$ and $R\cos\alpha = 1$,
   (b) $R\sin\alpha = 3$ and $R\cos\alpha = 3$.

2. Find R (exactly) and α (correct to the nearest minute), if $R > 0$ and $0° \le \alpha < 360°$, and:
   (a) $R\sin\alpha = 5$ and $R\cos\alpha = 12$,
   (b) $R\cos\alpha = 2$ and $R\sin\alpha = 4$.

3. (a) If $\cos x - \sin x = A\cos(x+\alpha)$, show that $A\cos\alpha = 1$ and $A\sin\alpha = 1$.
   (b) Find the positive value of A by squaring and adding.
   (c) Find α, if $0 \le \alpha < 2\pi$.
   (d) State the maximum and minimum values of $\cos x - \sin x$ and the first positive values of x for which they occur.
   (e) Solve the equation $\cos x - \sin x = -1$, for $0 \le x \le 2\pi$.
   (f) Write down the amplitude and period of $\cos x - \sin x$. Hence sketch $y = \cos x - \sin x$, for $0 \le x \le 2\pi$. Indicate on your sketch the line $y = -1$ and the solutions to the equation in part (e).

4. Sketch $y = \cos x$ and $y = \sin x$ on one set of axes. Then, by taking differences of heights, sketch $y = \cos x - \sin x$. Compare your sketch with that in the previous question.

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5. (a) If $\sqrt{3}\cos x - \sin x = B\cos(x+\theta)$, show that $B\cos\theta = \sqrt{3}$ and $B\sin\theta = 1$.
(b) Find B, if $B > 0$, by squaring and adding.
(c) Find θ, if $0 \le \theta < 2\pi$.
(d) State the greatest and least possible values of $\sqrt{3}\cos x - \sin x$ and the values of x closest to $x = 0$ for which they occur.
(e) Solve the equation $\sqrt{3}\cos x - \sin x = 1$, for $0 \le x \le 2\pi$.
(f) Sketch $y = \sqrt{3}\cos x - \sin x$, for $0 \le x \le 2\pi$. On the same diagram, sketch the line $y = 1$. Indicate on your diagram the solutions to the equation in part (e).

6. Let $4\sin x - 3\cos x = A\sin(x-\alpha)$, where $A > 0$ and $0^\circ \le \alpha < 360^\circ$.
(a) Show that $A\cos\alpha = 4$ and $A\sin\alpha = 3$.
(b) Show that $A = 5$ and $\alpha = \tan^{-1}\frac{3}{4}$.
(c) Hence solve the equation $4\sin x - 3\cos x = 5$, for $0^\circ \le x \le 360^\circ$. Give the solution(s) correct to the nearest minute.

7. Consider the equation $2\cos x + \sin x = 1$.
(a) Let $2\cos x + \sin x = B\cos(x-\theta)$, where $B > 0$ and $0^\circ \le \theta < 360^\circ$. Show that $B = \sqrt{5}$ and $\theta = \tan^{-1}\frac{1}{2}$.
(b) Hence find, correct to the nearest minute where necessary, the solutions of the equation, for $0^\circ \le x \le 360^\circ$.

8. Let $\cos x - 3\sin x = D\cos(x+\phi)$, where $D > 0$ and $0^\circ \le \phi < 360^\circ$.
(a) Show that $D = \sqrt{10}$ and $\phi = \tan^{-1} 3$.
(b) Hence solve $\cos x - 3\sin x = 3$, for $0^\circ \le x \le 360^\circ$. Give the solutions correct to the nearest minute where necessary.

9. Consider the equation $\sqrt{5}\sin x + 2\cos x = -2$.
(a) Transform the LHS into the form $C\sin(x+\alpha)$, where $C > 0$ and $0^\circ \le \alpha < 360^\circ$.
(b) Find, correct to the nearest minute where necessary, the solutions of the equation, for $0^\circ \le x \le 360^\circ$.

10. Solve each equation, for $0^\circ \le x \le 360^\circ$, by transforming the LHS into a single-term sine or cosine function. Give solutions correct to the nearest minute.
(a) $3\sin x + 5\cos x = 4$
(b) $6\sin x - 5\cos x = 7$
(c) $7\cos x - 2\sin x = 5$
(d) $9\cos x + 7\sin x = 3$

11. Consider the equation $\cos x - \sin x = 1$, where $0 \le x \le 2\pi$.
(a) Using the substitutions $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$, where $t = \tan\frac{1}{2}x$, show that the equation can be written as $t^2 + t = 0$.
(b) Hence show that $\tan\frac{1}{2}x = 0$ or -1, where $0 \le \frac{1}{2}x \le \pi$.
(c) Hence solve the given equation for x.

12. Consider the equation $\sqrt{3}\sin x + \cos x = 1$.
(a) Show that the equation can be written as $t^2 = \sqrt{3}\,t$, where $t = \tan\frac{1}{2}x$.
(b) Hence solve the equation, for $0 \le x \le 2\pi$.

13. (a) Show that the equation $4\cos x + \sin x = 1$ can be written as $(5t+3)(t-1) = 0$, where $t = \tan\frac{1}{2}x$.
(b) Hence solve the equation, for $0^\circ \le x \le 360^\circ$. Give the solutions correct to the nearest minute where necessary.

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14. (a) Show that the equation $3\sin x - 2\cos x = 2$ can be written as $3t - 2 = 0$, where $t = \tan\frac{1}{2}x$.

(b) Hence solve the equation for $0° \le x \le 360°$, giving solutions correct to the nearest minute where necessary. (Remember to check $x = 180°$ as a possible solution, given that the resulting equation in t is linear.)

15. (a) Show that the equation $6\sin x - 4\cos x = 5$ can be written as $t^2 - 12t + 9 = 0$, where $t = \tan\frac{1}{2}x$.

(b) Show that $\tan\frac{1}{2}x = 6 + 3\sqrt{3}$ or $6 - 3\sqrt{3}$.

(c) Hence show that $77°35'$ and $169°48'$ are the solutions (to the nearest minute) of the given equation over the domain $0° \le x \le 360°$.

16. Solve each equation, for $0° \le x \le 360°$, by using the $t = \tan\frac{1}{2}x$ results. Give solutions correct to the nearest minute where necessary.

(a) $5\sin x + 4\cos x = 5$

(b) $7\cos x - 6\sin x = 2$

(c) $3\sin x - 2\cos x = 1$

(d) $5\cos x + 6\sin x = -5$

DEVELOPMENT

17. Find A and α exactly, if $A > 0$ and $0 \le \alpha < 2\pi$, and:

(a) $A\sin\alpha = 1$ and $A\cos\alpha = -\sqrt{3}$,

(b) $A\cos\alpha = -5$ and $A\sin\alpha = -5$.

18. Find A (exactly) and α (correct to the nearest minute), if $A > 0$ and $0° \le \alpha < 360°$, and:

(a) $A\cos\alpha = 5$ and $A\sin\alpha = -4$,

(b) $A\sin\alpha = -11$ and $A\cos\alpha = -2$.

19. (a) (i) Express $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x + \theta)$, where $A > 0$ and $0 < \theta < 2\pi$.

(ii) Hence solve $\sqrt{3}\cos x + \sin x = 1$, for $0 \le x < 2\pi$.

(b) (i) Express $\cos x - \sin x$ in the form $B\sin(x + \alpha)$, where $B > 0$ and $0 < \alpha < 2\pi$.

(ii) Hence solve $\cos x - \sin x = 1$, for $0 \le x < 2\pi$.

(c) (i) Express $\sin x - \sqrt{3}\cos x$ in the form $C\sin(x + \beta)$, where $C > 0$ and $0 < \beta < 2\pi$.

(ii) Hence solve $\sin x - \sqrt{3}\cos x = -1$, for $0 \le x < 2\pi$.

(d) (i) Express $-\cos x - \sin x$ in the form $D\cos(x - \phi)$, where $D > 0$ and $0 < \phi < 2\pi$.

(ii) Hence solve $-\cos x - \sin x = 1$, for $0 \le x < 2\pi$.

20. (a) (i) Express $2\cos x - \sin x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 360°$. (Write α to the nearest minute.)

(ii) Hence solve $2\cos x - \sin x = 1$, for $0° \le x < 360°$. Give the solutions correct to the nearest minute where necessary.

(b) (i) Express $-3\sin x - 4\cos x$ in the form $S\cos(x - \beta)$, where $S > 0$ and $0 < \beta < 2\pi$. (Write β to four decimal places.)

(ii) Hence solve $-3\sin x - 4\cos x = 2$, for $0 \le x < 2\pi$. Give the solutions correct to two decimal places.

21. (a) (i) Show that $\sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$.

(ii) Hence sketch the graph of $y = \sin x - \cos x$, for $0 \le x \le 2\pi$.

(iii) Use your sketch to determine the values of x in the domain $0 \le x \le 2\pi$ for which $\sin x - \cos x > 1$.

(b) Use a similar approach to that in part (a) to solve, for $0 \le x \le 2\pi$:

(i) $\sin x + \sqrt{3}\cos x \le 1$

(ii) $\sin x - \sqrt{3}\cos x < -1$

(iii) $\left|\sqrt{3}\sin x + \cos x\right| < 1$

(iv) $\cos x - \sin x \ge \frac{1}{2}\sqrt{2}$

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22. Solve, for $0 \le x \le 2\pi$:
(a) $\sin x - \cos x = \sqrt{1{\cdot}5}$ (b) $\sqrt{3}\sin 2x - \cos 2x = 2$ (c) $\sin 4x + \cos 4x = 1$

23. Solve, for $0° \le x \le 360°$. Give solutions correct to the nearest minute:
(a) $2\sec x - 2\tan x = 5$ (b) $2\operatorname{cosec} x + 5\cot x = 3$

24. Suppose that $a\cos x = 1 + \sin x$, where $0° < x < 90°$:
(a) Prove that $\dfrac{a-1}{a+1} = t$, where $t = \tan\frac{1}{2}x$.
(b) Hence find, to the nearest minute, the acute angle x that satisfies $2\cos x - \sin x = 1$.

25. Solve the equation $\sin\theta + \cos\theta = \cos 2\theta$, for $0 \le \theta \le 2\pi$.

26. (a) Show that $(\sqrt{3}+1)\cos 2x + (\sqrt{3}-1)\sin 2x = 2\sqrt{2}\cos(2x - \frac{\pi}{12})$.
(b) Hence find the general solution of $(\sqrt{3}+1)\cos 2x + (\sqrt{3}-1)\sin 2x = 2$.

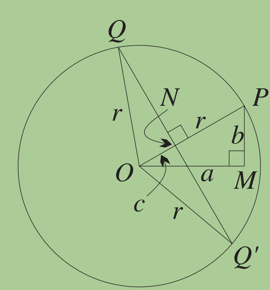
EXTENSION

27. (a) Prove that: (i) $\sin\theta = \cos(\theta - \frac{\pi}{2})$ (ii) $\cos\theta = \sin(\theta + \frac{\pi}{2})$
(b) Use the result $\sin x + \sqrt{3}\cos x = 2\sin(x + \frac{\pi}{3})$ to express $\sin x + \sqrt{3}\cos x$ in each of the other three standard forms.
(c) Repeat part (b) using the result $\cos x - \sin x = \sqrt{2}\cos(x + \frac{\pi}{4})$. Then sketch the functions $y = \sqrt{2}\cos x$ and $y = \sqrt{2}\sin x$.

28. (a) Prove that $\sin(\theta + \pi) = -\sin\theta$.
(b) Given that $\sqrt{3}\sin x + \cos x = 2\sin(x + \frac{\pi}{6})$, use appropriate reflections in the x- and y-axes, the fact that $\sin x$ is odd and $\cos x$ is even, and part (a) to prove that:
(i) $-\sqrt{3}\sin x + \cos x = 2\sin(x + \frac{5\pi}{6})$ (ii) $-\sqrt{3}\sin x - \cos x = 2\sin(x + \frac{7\pi}{6})$
(iii) $\sqrt{3}\sin x - \cos x = 2\sin(x - \frac{\pi}{6})$

29. Consider the equation $a\cos x + b\sin x = c$, where a, b and c are constants.
(a) Show that the equation can be written in the form $(a+c)t^2 - 2bt - (a-c) = 0$, where $t = \tan\frac{1}{2}x$. (b) Show that the root(s) of the equation are real if $c^2 \le a^2 + b^2$.
(c) Suppose that $\tan\frac{1}{2}\alpha$ and $\tan\frac{1}{2}\beta$ are distinct real roots of the quadratic equation in part (a). Prove that $\tan\frac{1}{2}(\alpha + \beta) = b/a$.

30. Consider the equation $a\cos x + b\sin x = c$, where a, b and c are positive constants. Let $a = r\cos\theta$ and $b = r\sin\theta$, where θ is acute.
(a) Show that $a^2 + b^2 = r^2$ and that $\tan\theta = \frac{b}{a}$.
(b) Show that the equation becomes $\cos(x - \theta) = \frac{c}{r}$, and hence write down the general solution.
(c) Show that there are no real roots if $c > \sqrt{a^2 + b^2}$.
(d) In the diagram opposite, the circle has centre O and radius $OP = r$. Suppose that $OM = a$, $MP = b$ and $MP \perp OM$. Suppose also that $ON = c$ and the chord QNQ' is drawn perpendicular to OP.



(i) State the size of $\angle MOP$.
(ii) Show that $\cos\angle NOQ = \cos\angle NOQ' = c/r$.
(iii) State which part of the general solution in part (b) contains $\angle MOQ$, and which part contains $\angle MOQ'$.
(e) What condition corresponds geometrically to the condition $c > \sqrt{a^2 + b^2}$, for which the equation has no real roots?

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2 F Extension — Products to Sums and Sums to Products

This section concerns a set of identities that convert the sum of two sine or cosine functions to the product of two sine or cosine functions, and vice versa. For example, we shall show that

$$\sin 3x + \sin 11x = 2\sin 7x \cos 4x.$$

The product form on the right is important for purposes such as finding the zeroes of the function. The sum form on the left is important, for example, when integrating the function.

These identities form part of the 4 Unit course, but are not required in the 3 Unit course. The worked exercises give only the examples mentioned above of their use, but the exercise following gives a fuller range of their applications.

Products to Sums: We begin with the four compound-angle formulae involving sine and cosine:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \text{(1A)}$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B \qquad \text{(1B)}$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad \text{(1C)}$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \qquad \text{(1D)}$$

Adding and subtracting equations (1A) and (1B), then adding and subtracting equations (1C) and (1D), gives the four products-to-sums formulae:

16 PRODUCTS TO SUMS:

$$\begin{aligned} 2\sin A\cos B &= \sin(A+B) + \sin(A-B) \\ 2\cos A\sin B &= \sin(A+B) - \sin(A-B) \\ 2\cos A\cos B &= \cos(A+B) + \cos(A-B) \\ -2\sin A\sin B &= \cos(A+B) - \cos(A-B) \end{aligned}$$

WORKED EXERCISE: Find $\displaystyle\int_0^{\frac{\pi}{3}} \sin 7x\cos 4x\,dx$.

SOLUTION:

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \sin 7x\cos 4x\,dx &= \tfrac{1}{2}\int_0^{\frac{\pi}{3}} (\sin 3x + \sin 11x)\,dx \\ &= \Big[-\tfrac{1}{6}\cos 3x - \tfrac{1}{22}\cos 11x\Big]_0^{\frac{\pi}{3}} \\ &= -\tfrac{1}{6}\cos 0 + \tfrac{1}{6}\cos\pi - \tfrac{1}{22}\cos 0 + \tfrac{1}{22}\cos\tfrac{11\pi}{3} \\ &= -\tfrac{1}{6} - \tfrac{1}{6} - \tfrac{1}{22} + \tfrac{1}{44} \\ &= -\tfrac{47}{132} \end{aligned}$$

Sums to Products: The previous formulae can be reversed to become formulae for sums to products by making a simple pair of substitutions. Let

$$S = A + B \qquad \text{and} \qquad T = A - B.$$

Then adding and subtracting these formulae gives

$$A = \tfrac{1}{2}(S+T) \qquad \text{and} \qquad B = \tfrac{1}{2}(S-T)\,.$$

Substituting these into the products-to-sums formulae, and reversing them:

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SUMS TO PRODUCTS:

17
$$\begin{aligned}\sin S + \sin T &= 2\sin\tfrac{1}{2}(S+T)\cos\tfrac{1}{2}(S-T)\\ \sin S - \sin T &= 2\cos\tfrac{1}{2}(S+T)\sin\tfrac{1}{2}(S-T)\\ \cos S + \cos T &= 2\cos\tfrac{1}{2}(S+T)\cos\tfrac{1}{2}(S-T)\\ \cos S - \cos T &= -2\sin\tfrac{1}{2}(S+T)\sin\tfrac{1}{2}(S-T)\end{aligned}$$

WORKED EXERCISE: Solve $\sin 3x + \sin 11x = 0$, for $0 \le x \le \pi$:

(a) using sums to products, (b) using solutions to $\sin\alpha = \sin\beta$.

SOLUTION:

(a) $\sin 3x + \sin 11x = 0$

$2\sin 7x \cos 4x = 0$, using sums to products,

$\sin 7x = 0$ or $\cos 4x = 0$

$7x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi$ or $4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$,

so $x = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}$ or $\frac{7\pi}{8}$.

(b) Alternatively, $\sin 11x = \sin(-3x)$, since $\sin\theta$ is odd,

so, using the solutions of $\sin\alpha = \sin\beta$,

$11x = -3x + 2n\pi$ or $11x = 3x + (2n+1)\pi$, where n is an integer,

$7x = n\pi$ or $4x = (n + \frac{1}{2})\pi$, giving the same answers as before.

Exercise 2F

1. (a) Establish the following identities by expanding the RHS:

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(b) Hence express as a sum or difference of trigonometric functions:

(i) $2\cos 35^\circ \cos 15^\circ$

(ii) $2\cos 48^\circ \sin 32^\circ$

(iii) $2\sin 3\alpha\cos\alpha$

(iv) $2\sin(x+y)\sin(x-y)$

(c) Use the products-to-sums identities to prove that

$$2\sin 3\theta\cos 2\theta + 2\cos 6\theta\sin\theta = \sin 7\theta + \sin\theta.$$

2. (a) Let $P = A + B$ and $Q = A - B$ in the identities in part (a) of the previous question to establish these identities:

(i) $\sin P + \sin Q = 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$

(ii) $\sin P - \sin Q = 2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$

(iii) $\cos P + \cos Q = 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$

(iv) $\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$

(b) Hence express as products:

(i) $\cos 16^\circ + \cos 12^\circ$

(ii) $\sin 56^\circ - \sin 20^\circ$

(iii) $\sin 6x + \sin 4x$

(iv) $\cos(2x+3y) - \cos(2x-3y)$

(c) Use the sums-to-products identities to prove that:

(i) $\sin 35^\circ + \sin 25^\circ = \sin 85^\circ$

(ii) $\dfrac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} = \tan 2\theta$

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3. (a) (i) Show that $\sin 3x + \sin x = 2\sin 2x \cos x$.

(ii) Hence solve the equation $\sin 3x + \sin x = 0$, for $0 \le x \le \pi$.

(b) Using a similar method, solve $\cos 3x + \cos x = 0$, for $0 \le x \le \pi$.

4. (a) (i) Show that $2\sin 3x \cos x = \sin 4x + \sin 2x$.

(ii) Hence find $\displaystyle\int 2\sin 3x \cos x\,dx$.

(b) Using a similar method, find $\displaystyle\int 2\cos 3x \cos x\,dx$.

DEVELOPMENT

5. Prove the following identities:

(a) $\dfrac{\cos 6\alpha - \cos 4\alpha + \cos 2\alpha}{\sin 6\alpha - \sin 4\alpha + \sin 2\alpha} = \cot 4\alpha$

(b) $4\cos 4x \cos 2x \cos x = \cos 7x + \cos 5x + \cos 3x + \cos x$

6. Evaluate: (a) $\displaystyle\int_0^{\frac{\pi}{12}} \cos 4x \sin 2x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{24}} \sin 5x \sin x\,dx$

7. [These identities are known as the *orthogonality relations*.]

(a) If m and n are positive integers, use the products-to-sums identities to prove:

(i) $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0$

(ii) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & \text{for } m \ne n, \\ \pi, & \text{for } m = n. \end{cases}$

(iii) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0, & \text{for } m \ne n, \\ \pi, & \text{for } m = n. \end{cases}$

(b) The functions $f(x)$ and $g(x)$ are defined by $f(x) = \sin x + 2\sin 2x + 3\sin 3x + 4\sin 4x$, and $g(x) = \cos x + 2\cos 2x + 3\cos 3x + 4\cos 4x$. Use parts (a)(i), (ii) and (iii) to find:

(i) $\displaystyle\int_{-\pi}^{\pi} f(x)g(x)\,dx$ (ii) $\displaystyle\int_{-\pi}^{\pi} \big(f(x)\big)^2\,dx$ (iii) $\displaystyle\int_{-\pi}^{\pi} \big(g(x)\big)^2\,dx$

8. (a) (i) Use the result $2\sin B \cos A = \sin(A+B) - \sin(A-B)$ to show that

$$2\sin x(\cos 2x + \cos 4x + \cos 6x) = \sin 7x - \sin x.$$

(ii) Deduce that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$ and hence $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$.

(b) Use the result $\sin A \sin B = \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)$ to prove that

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}.$$

9. (a) Express $\sin 3x + \sin x$ as a product.

(b) Hence solve the equation $\sin 3x + \sin 2x + \sin x = 0$, for $0 \le x < 2\pi$.

10. Solve each of the following equations, for $0 \le x \le \pi$:

(a) $\cos 5x + \cos x = 0$

(b) $\sin 4x - \sin x = 0$

(c) $\cos 3x + \cos 5x = \cos 4x$

(d) $\cos 4x + \cos 2x = \cos 3x + \cos x$

(e) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$

(f) $\sin 5x \cos 4x = \sin 3x \cos 2x$

11. (a) Solve the equation $\cos 5x = \sin x$, for $0 \le x \le \pi$. [Write $\sin x$ as $\cos(\frac{\pi}{2} - x)$.]

(b) Find general solutions of the equation $\sin 3x = \cos 2x$. [Write $\cos 2x$ as $\sin(\frac{\pi}{2} - 2x)$.]

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EXTENSION

12. [The three angles of a triangle] If A, B and C are the three angles of any triangle, prove:

(a) $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$

(b) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(c) $\sin^2 B + \sin^2 C - \sin^2 A = 2\cos A\sin B\sin C$

(d) $\dfrac{\sin 2A - \sin 2B + \sin 2C}{\sin^2 A - \sin^2 B + \sin^2 C} = 2\cot A\tan B\cot C$

13. Consider the definite integral $D = \displaystyle\int_{-\pi}^{\pi} \cos\lambda x\,\cos nx\,dx$, where n is a positive integer, and λ is any positive real number.

(a) Show that $D = \begin{cases} \pi, & \text{for } \lambda = n, \\ 0, & \text{for } \lambda \text{ a positive integer, } \lambda \neq n, \\ \dfrac{(-1)^n\, 2\lambda\sin\lambda\pi}{\lambda^2 - n^2}, & \text{for } \lambda \text{ not an integer.} \end{cases}$

(b) Show that when $0 < \lambda < n$, $|D| < \pi$.

(c) Show that when $\lambda > n + \frac{1}{2}$, $|D| < 3$.

(d) Is π the maximum value of $|D|$, for all positive integers n and all positive reals λ?

2 G Three-Dimensional Trigonometry

Trigonometry, in its application to mensuration problems, essentially deals with triangles, which are two-dimensional objects. Hence when trigonometry is applied to a three-dimensional problem, the diagram must be broken up into a collection of triangles in space, and trigonometry used for each in turn.

Three-dimensional work, however, requires two new ideas about angles — the angle between a line and a plane, and the angle between two planes — and these angles will need to be defined and discussed.

Trigonometry and Pythagoras' Theorem in Three Dimensions: As remarked above, every three-dimensional problem in trigonometry requires a careful sketch showing the triangles where trigonometry and Pythagoras' theorem are to be applied.

18 TRIGONOMETRY AND PYTHAGORAS' THEOREM IN THREE DIMENSIONS:

1. Draw a careful sketch of the situation.
2. Note carefully all the triangles in the figure.
3. Mark all right angles in these triangles.
4. Always state which triangle you are working with.

WORKED EXERCISE: The rectangular prism $ABCDEFGH$ sketched below has sides of length $AB = 5$ cm, $BC = 4$ cm and $AE = 3$ cm.

(a) Find the lengths of the three diagonals AC, AF and FC.

(b) Find the angle $\angle CAF$ between the diagonals AC and AF.

(c) Find the length of the space diagonal AG.

(d) Find the angle between the space diagonal AG and the edge AB.

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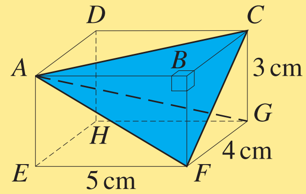
SOLUTION:



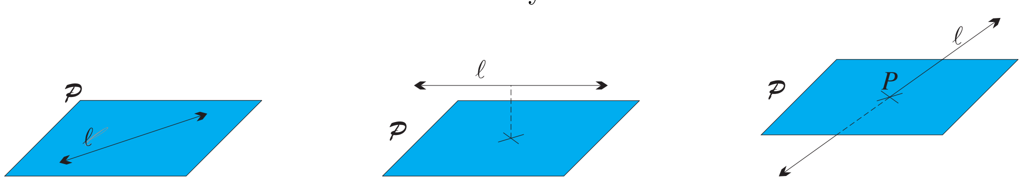
(a) In $\triangle ABC$, $AC^2 = 5^2 + 4^2$, using Pythagoras,
so $AC = \sqrt{41}$ cm.
In $\triangle ABF$, $AF^2 = 5^2 + 3^2$, using Pythagoras,
so $AF = \sqrt{34}$ cm.
In $\triangle FBC$, $FC^2 = 3^2 + 4^2$, using Pythagoras,
so $FC = 5$ cm.

(b) In $\triangle CAF$, $\cos\angle CAF = \dfrac{41 + 34 - 25}{2 \times \sqrt{41} \times \sqrt{34}}$, using the cosine rule,
$= \dfrac{50}{2 \times \sqrt{41} \times \sqrt{34}}$,
so $\angle CAF \doteqdot 47^\circ 58'$.

(c) In $\triangle ACG$, $AG^2 = AC^2 + CG^2$, since $AC \perp CG$,
$= 41 + 9$,
so $AG = 5\sqrt{2}$.

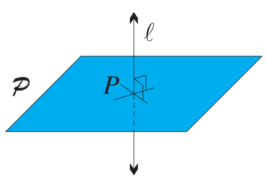
(d) In $\triangle BAG$, $\cos\angle BAG = \dfrac{5}{5\sqrt{2}}$, since $AB \perp BG$,
$= \dfrac{1}{\sqrt{2}}$,
so $\angle BAG = 45^\circ$.

The Angle Between a Line and a Plane: In three-dimensional space, a plane $\mathcal{P}$ and a line ℓ can be related in three different ways:



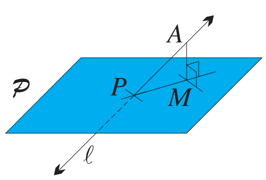
In the first diagram above, the line lies wholly within the plane. In the second diagram, the line never meets the plane, and the plane and the line are said to be *parallel*. In the third diagram, the line meets the plane in a single point P called the *intersection* of $\mathcal{P}$ and ℓ.

When the line ℓ meets the plane $\mathcal{P}$ in the single point P, it can do so in two distinct ways.



In the upper diagram to the right, the line ℓ is perpendicular to every line in the plane through P, and we say that ℓ is *perpendicular to the plane* $\mathcal{P}$.

In the lower diagram to the right, ℓ is not perpendicular to $\mathcal{P}$, and we define the angle θ between the line and the plane as follows. Choose any other point A on ℓ, and then construct the point M in the plane $\mathcal{P}$ so that $AM \perp \mathcal{P}$. Then $\angle APM$ is defined to be the angle between the plane and the line.



WORKED EXERCISE: Find the angle between a slant edge and the base in a square pyramid of height 8 metres whose base has side length 12 metres.

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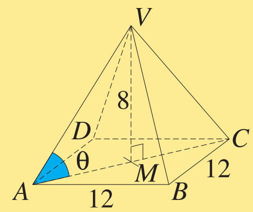
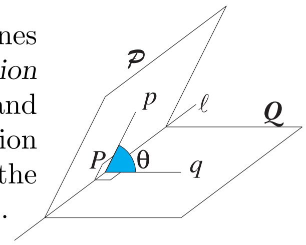
SOLUTION: Using Pythagoras' theorem in the base $ABCD$, the diagonal AC has length $12\sqrt{2}$ metres.
The perpendicular from the vertex V to the base meets the base at the midpoint M of the diagonal AC.

In $\triangle MAV$, $\tan\angle MAV = \dfrac{8}{6\sqrt{2}}$
$= \frac{2}{3}\sqrt{2}$,
so $\angle MAV \doteqdot 43°19'$,
and this is the angle between the edge and the base.



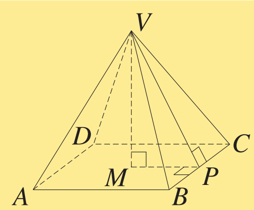
The Angle Between Two Planes: In three-dimensional space, any two planes that are not parallel meet in a line ℓ, called the *line of intersection of the two planes*. Take any point P on this line of intersection, and construct the lines p and q perpendicular to this line of intersection and lying in the planes $\mathcal{P}$ and $\mathcal{Q}$ respectively. The angle between the planes $\mathcal{P}$ and $\mathcal{Q}$ is defined to be the angle between these two lines.



19 THE ANGLE BETWEEN TWO PLANES: Suppose that the line p in the plane $\mathcal{P}$ and the line q in the plane $\mathcal{Q}$ meet at the point P on the line ℓ of intersection of the planes, and are both perpendicular to ℓ. Then the angle between the planes is the angle between the lines p and q.

WORKED EXERCISE: In the pyramid of the previous worked exercise, find the angle between an oblique face of the pyramid and the base.

SOLUTION: Let P be the midpoint of the edge BC.
Then $VP \perp BC$ and $MP \perp BC$,
so $\angle VPM$ is the angle required.
Now $\tan\angle VPM = \frac{8}{6}$
so $\angle VPM \doteqdot 53°8'$.



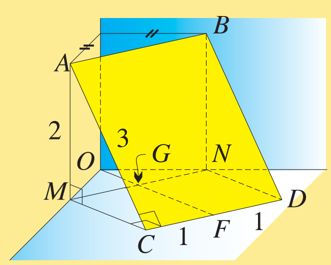
WORKED EXERCISE: [A harder question] A 2 metre × 3 metre rectangular sheet of metal leans lengthwise against a corner of a room, with its top vertices equidistant from the corner and 2 metres above the ground.

(a) What is the angle between the sheet of metal and the floor.

(b) How far is the bottom edge of the sheet from the corner of the floor?

SOLUTION:

(a) The diagram shows the piece of metal $ABCD$ and the corner O of the floor. The vertical line down the wall from A meets the floor at M. Notice that $AC \perp CD$ and $MC \perp CD$, so $\angle ACM$ is the angle between the sheet and the floor.
In $\triangle ACM$, $\sin\angle ACM = \frac{2}{3}$,
so $\angle ACM \doteqdot 41°49'$.



(b) Let the vertical line down the wall from B meet the floor at N. Let F be the midpoint of CD, and G be the midpoint of MN. Then OGF is the closest distance between the bottom edge CD of the sheet and the corner O of the floor.

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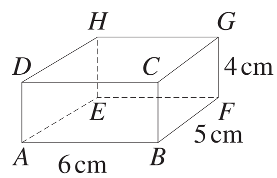
First, $\triangle OMN$ is an isosceles right triangle with hypotenuse $MN = 2$, so the altitude OG of $\triangle OMN$ has length 1.

Secondly, in $\triangle AMC$,
$$\begin{aligned} MC^2 &= 3^2 - 2^2 \\ &= 5, \\ MC &= \sqrt{5}\,. \end{aligned}$$

Since $MNDC$ is a rectangle, $\quad OF = 1 + \sqrt{5}$ metres.

Exercise 2G

1. The diagram shows a box in the shape of a rectangular prism.

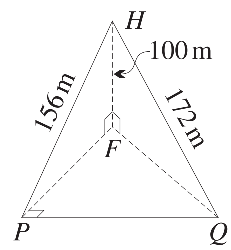


(a) Find, correct to the nearest minute, the angle that the diagonal plane $AEGC$ makes with the face $BCGF$.

(b) Find the length of the diagonal AG of the box, correct to the nearest millimetre.

(c) Find, correct to the nearest minute, the angle that the diagonal AG makes with the base $AEFB$.

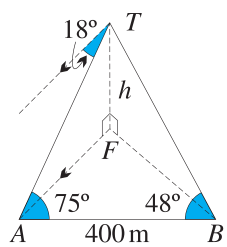
2. A helicopter H is hovering 100 metres above the level ground below. Two observers P and Q on the ground are 156 metres and 172 metres respectively from H. The helicopter is due north of P, while Q is due east of P.



(a) Find the angles of elevation of the helicopter from P and Q, correct to the nearest minute,

(b) Find the distance between the two observers P and Q, correct to the nearest metre.

3. The points A and B are 400 metres apart in a horizontal plane. The angle of depression of A from the top T of a vertical tower standing on the plane is $18°$. Also, $\angle TAB = 75°$ and $\angle TBA = 48°$.

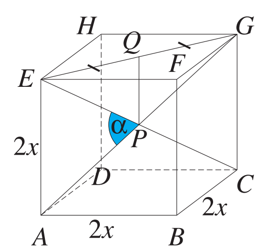


(a) Show that $TA = \dfrac{400 \sin 48°}{\sin 57°}$.

(b) Hence find the height h of the tower, correct to the nearest metre.

(c) Find, correct to the nearest degree, the angle of depression of B from T.

4. The diagram shows a cube $ABCDEFGH$. The diagonals AG and CE meet at P. Q is the midpoint of the diagonal EG of the top face. Suppose that $2x$ is the side length of the cube and α is the acute angle between the diagonals AG and CE.



(a) State the length of PQ.

(b) Show that $EQ = \sqrt{2}\,x$.

(c) Hence show that $EP = \sqrt{3}\,x$.

(d) Hence show that $\cos \angle EPQ = \frac{1}{3}\sqrt{3}$.

(e) By using an appropriate double-angle formula, deduce that $\cos \angle EPG = -\frac{1}{3}$, and hence that $\cos \alpha = \frac{1}{3}$.

(f) Confirm the fact that $\cos \alpha = \frac{1}{3}$ by using the cosine rule in $\triangle APE$.

(g) Find, correct to the nearest minute, the angle that the diagonal AG makes with the base $ABCD$ of the cube.

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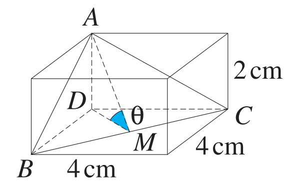
5. The prism in the diagram has a square base of side 4 cm and its height is 2 cm. ABC is a diagonal plane of the prism. Let θ be the acute angle between the diagonal plane and the base of the prism.

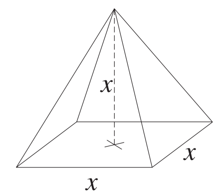


(a) Show that $MD = 2\sqrt{2}$ cm.

(b) Hence find θ, correct to the nearest minute.

DEVELOPMENT

6. The diagram shows a square pyramid whose perpendicular height is equal to the side of the base. Find, correct to the nearest minute:

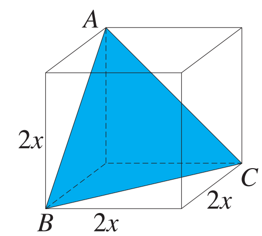


(a) the angle between an oblique face and the base,

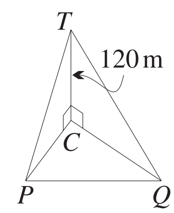
(b) the angle between a slant edge and the base,

(c) the angle between an opposite pair of oblique faces.

7. The diagram shows a cube of side $2x$ in which a diagonal plane ABC is drawn. Find, correct to the nearest minute, the angle between this diagonal plane and the base of the cube.



8. Two boats P and Q are observed from the top T of a vertical cliff CT of height 120 metres. P is on a bearing of 195° from the cliff and its angle of depression from T is 22°. Q is on a bearing of 161° from the cliff and its angle of depression from T is 27°.

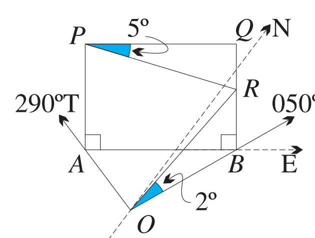


(a) Show that $\angle PCQ = 34°$.

(b) Use the cosine rule to show that the boats are approximately 166 metres apart.

9. A plane is flying along the path PR. Its constant speed is 300 km/h. It flies directly over landmarks A and B, where B is due east of A. An observer at O first sights the plane when it is over A at a bearing of 290° T, and then, ten minutes later, he sights the plane when it is over B at a bearing of 50° T and with an angle of elevation of 2°.

(a) Show that the plane has travelled 50 km in the ten minutes between observations.

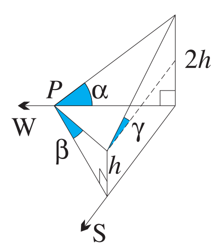


(b) Show that $\angle AOB = 120°$.

(c) Prove that the observer is 19 670 metres, correct to the nearest ten metres, from landmark B.

(d) Find the height h of the plane, correct to the nearest 10 metres, when it was directly above A.

10. Two towers of height $2h$ and h stand on a horizontal plane. The shorter tower is due south of the taller tower. From a point P due west of the taller tower, the angles of elevation of the tops of the taller and shorter towers are α and β respectively. The angle of elevation of the top of the taller tower from the top of the shorter tower is γ. Show that



$$4\cot^2\alpha = \cot^2\beta - \cot^2\gamma.$$

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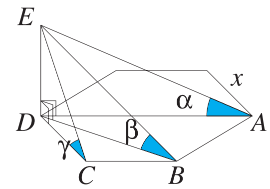
11. A, B, C and D are four of the vertices of a horizontal regular hexagon of side length x. DE is vertical and subtends angles of α, β and γ at A, B and C respectively.

(a) Show that each interior angle of a regular hexagon is $120°$.

(b) Show that $\angle BAD = 60°$ and $\angle ABD = 90°$.

(c) Show that $BD = \sqrt{3}\,x$ and $AD = 2x$.

(d) Hence show that $\cot^2\alpha = \cot^2\beta + \cot^2\gamma$.

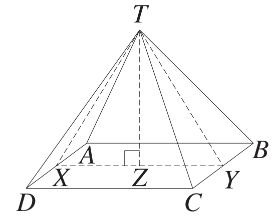


12. The diagram shows a rectangular pyramid. X and Y are the midpoints of AD and BC respectively and T is directly above Z. $TX = 15$ cm, $TY = 20$ cm, $AB = 25$ cm and $BC = 10$ cm.

(a) Show that $\angle XTY = 90°$.

(b) Hence show that T is 12 cm above the base.

(c) Hence find, correct to the nearest minute, the angle that the front face DCT makes with the base.



13. A plane is flying due east at 600 km/h at a constant altitude. From an observation point P on the ground, the plane is sighted on a bearing of $320°$. One minute later, the bearing of the plane is $75°$ and its angle of elevation is $25°$.

(a) How far has the plane travelled between the two sightings?

(b) Draw a diagram to represent the given information.

(c) Show that the altitude h metres of the plane is given by $h = \dfrac{10\,000 \sin 50° \tan 25°}{\sin 65°}$, and hence find the altitude, correct to the nearest metre.

(d) Find, correct to the nearest degree, the angle of elevation of the plane from P when it was first sighted.

14. (a) The diagonal PQ of the rectangular prism in the diagram makes angles of α, β and γ respectively with the edges PA, PB and PC.

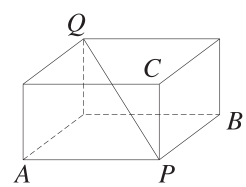
(i) Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

(ii) What is the two-dimensional version of this result?

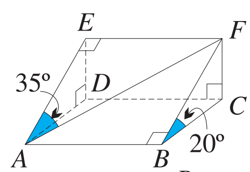
(b) Suppose that the diagonal PQ makes angles of θ, ϕ and ψ with the three faces of the prism that meet at P.

(i) Prove that $\sin^2\theta + \sin^2\phi + \sin^2\psi = 1$.

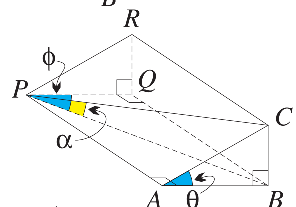
(ii) What is the two-dimensional version of this result?



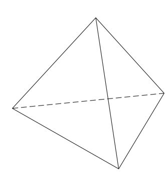
15. The diagram shows a hill inclined at $20°$ to the horizontal. A straight road AF on the hill makes an angle of $35°$ with a line of greatest slope. Find, correct to the nearest minute, the inclination of the road to the horizontal.



16. The plane surface $APRC$ is inclined at an angle θ to the horizontal plane $APQB$. Both $APRC$ and $APQB$ are rectangles. PR is a line of greatest slope on the inclined plane. $\angle BPQ = \phi$ and $\angle BPC = \alpha$. Show that $\tan\alpha = \tan\theta\cos\phi$.



17. The diagram shows a triangular pyramid, all of whose faces are equilateral triangles — such a solid is called a *regular tetrahedron*. Suppose that the slant edges are inclined at an angle θ to the base. Show that $\cos\theta = \frac{1}{3}\sqrt{3}$.



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18. A square pyramid has perpendicular height equal to the side length of its base.

(a) Show that the angle between a slant edge and a base edge it meets is $\cos^{-1}\frac{1}{6}\sqrt{6}$.

(b) Show that the angle between adjacent oblique faces is $\cos^{-1}(-\frac{1}{5})$.

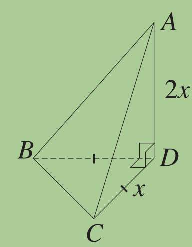
EXTENSION

19. A cube has one edge AB of its base inclined at an angle θ to the horizontal and another edge AC of its base horizontal. The diagonal AP of the cube is inclined at angle ϕ to the horizontal.

(a) Show that the height h of the point P above the horizontal plane containing the edge AC is given by $h = x\cos\theta(1+\tan\theta)$, where x is the side length of the cube.

(b) Hence show that $\cos^2\phi = \frac{2}{3}(1-\sin\theta\cos\theta)$.

20. The diagram shows a triangular pyramid $ABCD$. The horizontal base BCD is an isosceles triangle whose equal sides BD and CD are at right angles and have length x units. The edge AD has length $2x$ units and is vertical.



(a) Let α be the acute angle between the front face ABC and the base BCD. Show that $\alpha = \cos^{-1}\frac{1}{3}$.

(b) Let θ be the acute angle between the front face ABC and a side face (that is, either ABD or ACD). Show that $\theta = \cos^{-1}\frac{2}{3}$.

2 H Further Three-Dimensional Trigonometry

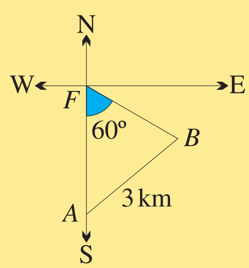
This section continues with somewhat harder three-dimensional problems. Some trigonometric problems are difficult simply because the diagram is complicated to visualise or because the necessary calculations are intricate. But other problems are difficult because no triangle in the figure can be solved — in such cases, an equation must be formed and solved in the required pronumeral.

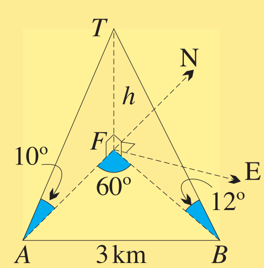
Three-dimensional Problems in which No Triangle can be Solved: In the following classic problem, there are four triangles forming a tetrahedron, but no triangle can be solved, because no more than two measurements are known in any one of these triangles. The method is to introduce a pronumeral for the height, then work around the figure until *four* measurements are known in terms of h in the base triangle — at this point an equation in h can be formed and solved.

WORKED EXERCISE: A motorist driving on level ground sees, due north of her, a tower whose angle of elevation is 10°. After driving 3 km further in a straight line, the tower is in the direction N60°W, with angle of elevation 12°.

(a) How high is the tower? (b) In what direction is she driving?

SOLUTION:





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Let the tower be TF, and let the motorist be driving from A to B.
There are four triangles, none of which can be solved.

(a) Let h be the height of the tower.
In $\triangle TAF$, $AF = h\cot 10°$.
In $\triangle TBF$, $BF = h\cot 12°$.
We now have expressions for four measurements in $\triangle ABF$,
so we can use the cosine rule to form an equation in h.

$$\text{In } \triangle ABF,\quad 3^2 = h^2\cot^2 10° + h^2\cot^2 12° - 2h^2\cot 10°\cot 12° \times \cos 60°$$
$$9 = h^2(\cot^2 10° + \cot^2 12° - \cot 10°\cot 12°)$$
$$h^2 = \frac{9}{\cot^2 10° + \cot^2 12° - \cot 10°\cot 12°},$$

so the tower is about 571 metres high.

(b) Let $\theta = \angle FAB$.

$$\text{In } \triangle AFB,\ \frac{\sin\theta}{h\cot 12°} = \frac{\sin 60°}{3}$$
$$\sin\theta = h\cot 12° \times \frac{\sqrt{3}}{6}$$
$$\theta \doteqdot 51°,$$

so her direction is about N51°E.

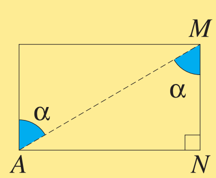
The General Method of Approach: Here is a summary of what has been said about three-dimensional problems (apart from the ideas of angles between lines and planes and between planes and planes).

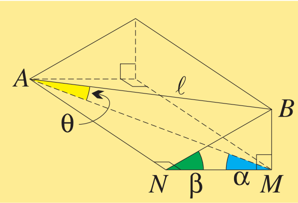
20 THREE-DIMENSIONAL TRIGONOMETRY:
1. Draw a careful diagram of the situation, marking all right angles.
2. A plan diagram, looking down, is usually a great help.
3. Identify every triangle in the diagram, to see whether it can be solved.
4. If one triangle can be solved, then work from it around the diagram until the problem is solved.
5. If no triangle can be solved, assign a pronumeral to what is to be found, then work around the diagram until an equation in that pronumeral can be formed and solved.

Problems Involving Pronumerals: When a problem involves pronumerals, there is little difference in the methods used. The solution will usually require working around the diagram, beginning with a triangle in which expressions for three measurements are known, until an equation can be formed.

WORKED EXERCISE: [A harder example] A hillside is a plane of gradient m facing due south. A map shows a straight road on the hillside going in the direction α east of north. Find the gradient of the road in terms of m and α.

SOLUTION:





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The diagrams above show a piece AB of the road of length ℓ.
Let $\theta = \angle BAM$ be the angle of inclination of the road,
and let $\beta = \angle BNM$ be the angle of inclination of the hillside.

In $\triangle ABM$, $BM = \ell \sin\theta$,
and $AM = \ell\cos\theta$.

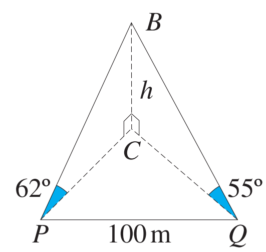
In $\triangle AMN$, $MN = AM\cos\alpha$
$= \ell\cos\theta\cos\alpha$.

In $\triangle BMN$, $BM = NM\tan\beta$
$\ell\sin\theta = \ell\cos\theta\cos\alpha\tan\beta$
$\tan\theta = \cos\alpha\tan\beta$.

But $\tan\theta$ and $\tan\beta$ are the gradients of the road and hillside respectively, so the gradient of the road is $m\cos\alpha$.

Exercise 2H

1. A balloon B is due north of an observer P and its angle of elevation is 62°. From another observer Q 100 metres from P, the balloon is due west and its angle of elevation is 55°. Let the height of the balloon be h metres and let C be the point on the level ground vertically below B.



(a) Show that $PC = h\cot 62^\circ$, and write down a similar expression for QC.

(b) Explain why $\angle PCQ = 90^\circ$.

(c) Use Pythagoras' theorem in $\triangle CPQ$ to show that

$$h^2 = \frac{100^2}{\cot^2 62^\circ + \cot^2 55^\circ}.$$

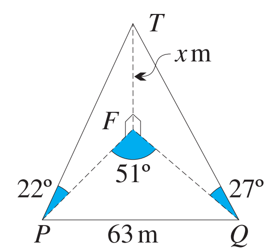
(d) Hence find h, correct to the nearest metre.

2. From a point P due south of a vertical tower, the angle of elevation of the top of the tower is 20°. From a point Q situated 40 metres from P and due east of the tower, the angle of elevation is 35°. Let h metres be the height of the tower.

(a) Draw a diagram to represent the situation.

(b) Show that $h = \dfrac{40}{\sqrt{\tan^2 70^\circ + \tan^2 55^\circ}}$, and evaluate h, correct to the nearest metre.

3. In the diagram, TF represents a vertical tower of height x metres standing on level ground. From P and Q at ground level, the angles of elevation of T are 22° and 27° respectively. $PQ = 63$ metres and $\angle PFQ = 51^\circ$.

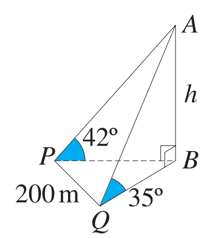


(a) Show that $PF = x\cot 22^\circ$ and write down a similar expression for QF.

(b) Use the cosine rule to show that $x^2 = \dfrac{63^2}{\cot^2 22^\circ + \cot^2 27^\circ - 2\cot 22^\circ\cot 27^\circ\cos 51^\circ}$.

(c) Use a calculator to show that $x \doteqdot 32$.

4. The points P, Q and B lie in a horizontal plane. From P, which is due west of B, the angle of elevation of the top of a tower AB of height h metres is 42°. From Q, which is on a bearing of 196° from the tower, the angle of elevation of the top of the tower is 35°. The distance PQ is 200 metres.



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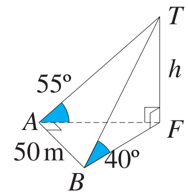
(a) Explain why $\angle PBQ = 74°$.

(b) Show that $h^2 = \dfrac{200^2}{\cot^2 42° + \cot^2 35° - 2\cot 35° \cot 42° \cos 74°}$.

(c) Hence find the height of the tower, correct to the nearest metre.

DEVELOPMENT

5. The diagram shows a tower of height h metres standing on level ground. The angles of elevation of the top T of the tower from two points A and B on the ground nearby are $55°$ and $40°$ respectively. The distance AB is 50 metres and the interval AB is perpendicular to the interval AF, where F is the foot of the tower.



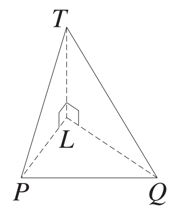
(a) Find AT and BT in terms of h.

(b) What is the size of $\angle BAT$?

(c) Use Pythagoras' theorem in $\triangle BAT$ to show that $h = \dfrac{50 \sin 55° \sin 40°}{\sqrt{\sin^2 55° - \sin^2 40°}}$.

(d) Hence find the height of the tower, correct to the nearest metre.

6. The diagram shows two observers P and Q 600 metres apart on level ground. The angles of elevation of the top T of a landmark TL from P and Q are $9°$ and $12°$ respectively. The bearings of the landmark from P and Q are $32°$ and $306°$ respectively. Let $h = TL$ be the height of the landmark.



(a) Show that $\angle PLQ = 86°$.

(b) Find expressions for PL and QL in terms of h.

(c) Hence show that $h \doteqdot 79$ metres.

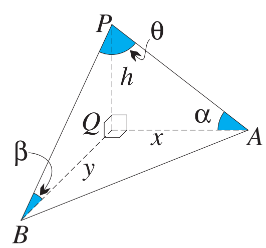
7. PQ is a straight level road. Q is x metres due east of P. A vertical tower of height h metres is situated due north of P. The angles of elevation of the top of the tower from P and Q are α and β respectively.

(a) Draw a diagram representing the situation.

(b) Show that $x^2 + h^2 \cot^2 \alpha = h^2 \cot^2 \beta$.

(c) Hence show that $h = \dfrac{x \sin\alpha \sin\beta}{\sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}}$.

8. In the diagram of a triangular pyramid, $AQ = x$, $BQ = y$, $PQ = h$, $\angle APB = \theta$, $\angle PAQ = \alpha$ and $\angle PBQ = \beta$. Also, there are three right angles at Q.



(a) Show that $x = h\cot\alpha$ and write down a similar expression for y.

(b) Use Pythagoras' theorem and the cosine rule to show that $\cos\theta = \dfrac{h^2}{\sqrt{(x^2+h^2)(y^2+h^2)}}$.

(c) Hence show that $\sin\alpha \sin\beta = \cos\theta$.

9. A man walking along a straight, flat road passes by three observation points P, Q and R at intervals of 200 metres. From these three points, the respective angles of elevation of the top of a vertical tower are $30°$, $45°$ and $45°$. Let h metres be the height of the tower.

(a) Draw a diagram representing the situation.

(b) (i) Find, in terms of h, the distances from P, Q and R to the foot F of the tower.

(ii) Let $\angle FRQ = \alpha$. Find two different expressions for $\cos\alpha$ in terms of h, and hence find the height of the tower.

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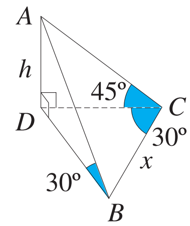
10. $ABCD$ is a triangular pyramid with base BCD and perpendicular height AD.

(a) Find BD and CD in terms of h.

(b) Use the cosine rule to show that $2h^2 = x^2 - \sqrt{3}\,hx$.

(c) Let $u = \dfrac{h}{x}$. Write the result of the previous part as a quadratic equation in u, and hence show that

$$\frac{h}{x} = \frac{\sqrt{11} - \sqrt{3}}{4}.$$

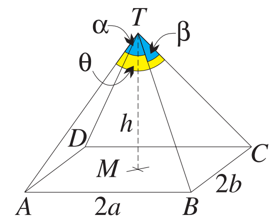


11. The diagram shows a rectangular pyramid. The base $ABCD$ has sides $2a$ and $2b$ and its diagonals meet at M. The perpendicular height TM is h. Let $\angle ATB = \alpha$, $\angle BTC = \beta$ and $\angle ATC = \theta$.

(a) Use Pythagoras' theorem to find AC, AM and AT in terms of a, b and h.

(b) Use the cosine rule to find $\cos\alpha$, $\cos\beta$ and $\cos\theta$ in terms of a, b and h.

(c) Show that $\cos\alpha + \cos\beta = 1 + \cos\theta$.

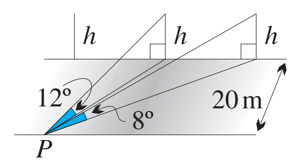


12. The diagram shows three telegraph poles of equal height h metres standing equally spaced on the same side of a straight road 20 metres wide. From an observer at P on the other side of the road directly opposite the first pole, the angles of elevation of the tops of the other two poles are $12°$ and $8°$ respectively. Let x metres be the distance between two adjacent poles.

(a) Show that $h^2 = \dfrac{x^2 + 20^2}{\cot^2 12°}$.

(b) Hence show that $x^2 = \dfrac{20^2(\cot^2 8° - \cot^2 12°)}{4\cot^2 12° - \cot^2 8°}$.

(c) Hence calculate the distance between adjacent poles, correct to the nearest metre.

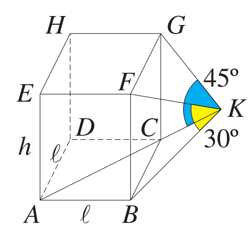


13. A building is in the shape of a square prism with base edge ℓ metres and height h metres. It stands on level ground. The diagonal AC of the base is extended to K, and from K, the respective angles of elevation of F and G are $30°$ and $45°$.

(a) Show that $BK^2 = h^2 + \ell^2 + \sqrt{2}\,h\ell$.

(b) Hence show that $2h^2 - \ell^2 = \sqrt{2}\,h\ell$.

(c) Deduce that $\dfrac{h}{\ell} = \dfrac{\sqrt{2} + \sqrt{10}}{4}$.

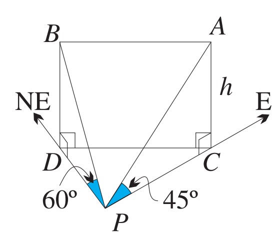


14. From a point P on level ground, a man observes the angle of elevation of the summit of a mountain due north of him to be $18°$. After walking 3 km in a direction N$50°$E to a point Q, the man finds that the angle of elevation of the summit is now $13°$.

(a) Show that $(\cot^2 13° - \cot^2 18°)h^2 + (6000\cot 18° \cos 50°)h - 3000^2 = 0$, where h metres is the height of the mountain.

(b) Hence find the height, correct to the nearest metre.

15. A plane is flying at a constant height h, and with constant speed. An observer at P sighted the plane due east at an angle of elevation of $45°$. Soon after it was sighted again in a north-easterly direction at an angle of elevation of $60°$.



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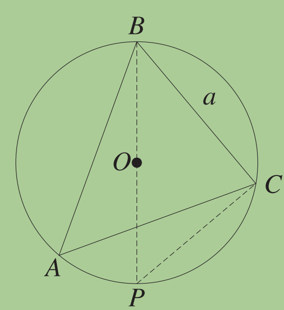
(a) Write down expressions for PC and PD in terms of h.
(b) Show that $CD^2 = \frac{1}{3}h^2(4 - \sqrt{6})$.
(c) Find, as a bearing correct to the nearest degree, the direction in which the plane is flying.

16. Three tourists T_1, T_2 and T_3 at ground level are observing a landmark L. T_1 is due north of L, T_3 is due east of L, and T_2 is on the line of sight from T_1 to T_3 and between them. The angles of elevation to the top of L from T_1, T_2 and T_3 are 25°, 32° and 36° respectively.
(a) Show that $\tan \angle LT_1T_2 = \dfrac{\cot 36°}{\cot 25°}$.
(b) Use the sine rule in $\triangle LT_1T_2$ to find, correct to the nearest minute, the bearing of T_2 from L.

EXTENSION

17. (a) Use the diagram on the right to show that the diameter BP of the circumcircle of $\triangle ABC$ is $\dfrac{a}{\sin A}$.



(b) A vertical tower stands on level ground. From three observation points P, Q and R on the ground, the top of the tower has the same angle of elevation of 30°. The distances PQ, PR and QR are 60 metres, 50 metres and 40 metres respectively.
(i) Explain why the foot of the tower is the centre of the circumcircle of $\triangle PQR$.
(ii) Use the result in part (a) to show that the height of the tower is $\frac{80}{21}\sqrt{21}$ metres.

GO Online Multiple Choice Quiz

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