MATH2601 - Assignment 1 | Keegan Gyoery - z5197058

We have $V = \mathbb{R}^n$, and

$$V_1 = \{\mathbf{x} \in V \mid x_1 + \cdots + x_n = 0\}$$
$$V_2 = \{\mathbf{x} \in V \mid x_1 = \cdots = x_n\}.$$

Consider the zero vector, $\mathbf{0}$, for $V = \mathbb{R}^n$. The zero vector, $\mathbf{0}$, has the components $x_i = 0$, $\forall i \in \mathbb{Z}$, $1 \leq i \leq n$. Clearly, for $\mathbf{0}$, $x_1 + \cdots + x_n = 0$ and $x_1 = \cdots = x_n$. As a result, $\mathbf{0} \in V_1$, and $\mathbf{0} \in V_2$. Thus, V_1 and V_2 are non-empty. Both V_1 and V_2 define their elements as an $\mathbf{x} \in V$, with a further restriction applied. It is then clear to see that $V_1 \subseteq V$, and $V_2 \subseteq V$. Therefore, both V_1 and V_2 are non-empty subsets of V. Applying the Subspace Lemma, we will prove that $V_1 \leq V$, and $V_2 \leq V$.

Consider the vectors $\mathbf{u}, \mathbf{v} \in V_1$, and the scalar $\alpha \in \mathbb{R}$. As $\mathbf{u}, \mathbf{v} \in V_1$,

$$u_1 + \cdots + u_n = 0 \ldots (A)$$
$$v_1 + \cdots + v_n = 0 \ldots (B).$$

Consider now the components of the vector $\alpha\mathbf{u} + \mathbf{v}$, which are $\alpha u_i + v_i$, $\forall i \in \mathbb{Z}$, $1 \leq i \leq n$..

$$\begin{aligned}(\alpha u_1 + v_1) + \cdots + (\alpha u_n + v_n) &= (\alpha u_1 + \cdots + \alpha u_n) + (v_1 + \cdots + v_n) \\ &= \alpha(u_1 + \cdots + u_n) + (v_1 + \cdots + v_n) \\ &= \alpha(0) + 0 \text{ by } (A) \text{ and } (B) \\ &= 0\end{aligned}$$

Therefore, the sum of the components of $\alpha\mathbf{u} + \mathbf{v}$ is 0. Thus, $\alpha\mathbf{u} + \mathbf{v} \in V_1$, and thus $V_1 \leq V$ by the Subspace Lemma.

Consider the vectors $\mathbf{s}, \mathbf{t} \in V_2$, and the scalar $\beta \in \mathbb{R}$. As $\mathbf{s}, \mathbf{t} \in V_2$,

$$s_1 = \cdots = s_n \ldots (C)$$
$$t_1 = \cdots = t_n \ldots (D).$$

Consider now the components of the vector $\beta\mathbf{s} + \mathbf{t}$, which are $\beta s_i + t_i$, $\forall i \in \mathbb{Z}$, $1 \leq i \leq n$.

$$\begin{aligned}&\beta s_1 = \cdots = \beta s_n \text{ from } (C) \\ &\therefore \beta s_1 + t_1 = \beta s_2 + t_1 = \cdots = \beta s_n + t_1 \\ &\therefore \beta s_1 + t_1 = \beta s_2 + t_2 = \cdots = \beta s_n + t_n \text{ by } (D)\end{aligned}$$

Therefore, all of the components of $\beta\mathbf{s} + \mathbf{t}$ are equal. Thus, $\beta\mathbf{s} + \mathbf{t} \in V_2$, and thus $V_2 \leq V$ by the Subspace Lemma.

Consider $V_1 \cap V_2$, and let $\mathbf{y} \in V_1 \cap V_2$. As $\mathbf{y} \in V_1$, and $\mathbf{y} \in V_2$,

$$\begin{aligned}
&y_1 + \cdots + y_n = 0 \ldots (E)\\
&y_1 = \cdots = y_n \ldots (F).
\end{aligned}$$

Considering (E),

$$\begin{aligned}
&y_1 + \cdots + y_n = 0 \text{ from } (E)\\
&y_1 + y_1 + \cdots + y_1 = 0 \text{ by } (F)\\
&ny_1 = 0\\
\therefore\, &y_1 = 0\\
\therefore\, &y_1 = y_2 = \cdots = y_n = 0\\
\therefore\, &\mathbf{y} = \mathbf{0}\\
\therefore\, &V_1 \cap V_2 = \{\mathbf{0}\}
\end{aligned}$$

We can now justify that the sum of the vector spaces V_1 and V_2 is a direct sum, $V_1 \oplus V_2$. Furthermore, we know V with the standard dot product, is an inner product space. Let $\mathbf{q} \in V_1$, and $\mathbf{r} \in V_2$, so we have,

$$\begin{aligned}
&q_1 + \cdots + q_n = 0 \ldots (G)\\
&r_1 = \cdots = r_n \ldots (H).
\end{aligned}$$

Consider their inner product $\langle \mathbf{q}, \mathbf{r} \rangle = \mathbf{q} \cdot \mathbf{r}$, which is the standard dot product.

$$\begin{aligned}
\mathbf{q} \cdot \mathbf{v} &= q_1 r_1 + q_2 r_2 + \cdots + q_n r_n\\
&= q_1 r_1 + q_2 r_1 + \cdots + q_n r_1 \text{ by } (H)\\
&= r_1(q_1 + q_2 + \ldots q_n)\\
&= r_1(0) \text{ by } (G)\\
&= 0\\
\therefore\, \langle \mathbf{q}, \mathbf{r} \rangle &= 0
\end{aligned}$$

Therefore by Definition 4.10, as $V_1 \leq V$, $V_2 = V_1^{\perp}$, that is, V_2 is the orthogonal complement of V_1. Using the standard basis $\mathcal{S}$ for V, it is clear that $\dim(V) = n$, and thus V is finite dimensional. Futhermore, $V_1 \leq V$, and so by Theorem 4.11, $V = V_1 \oplus V_1^{\perp} = V_1 \oplus V_2$.

Consider the small finite field $\mathbb{F} = \{0, 1\}$, where $1 + 1 = 0$. Replace $\mathbb{R}$ with $\mathbb{F}$, so now $V = \mathbb{F}^n$, and consider the case when $n = 2$. By the definition of V_1, we get that $V_1 = \{(0,0), (1,1)\}$, and likewise, by the definition of V_2, we get that $V_2 = \{(0,0), (1,1)\}$. Note that $\mathbf{0} = (0,0)$ when $n = 2$. Therefore, it is clear that $V_1 \cap V_2 = \{(0,0), (1,1)\} \neq \{(0,0)\}$. Thus, $V_1 \cap V_2 \neq \mathbf{0}$, and so the sum of the vector spaces V_1, and V_2, cannot be direct.