The University of Sydney
School of Mathematics and Statistics

# Tutorial for Week 5

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

Lecturers: Daniel Daners and James Parkinson

## Topics covered

In lectures last week:

- ☐ Review of integration techniques.
- ☐ Integrating functions with discontinuities: Improper integrals.
- ☐ Integrating over an unbounded domain: Improper integrals.
- ☐ The $p$-integrals $\int_0^1 \frac{1}{x^p}\,dx$ and $\int_1^\infty \frac{1}{x^p}\,dx$.
- ☐ The Comparison Test for integrals.

## Objectives

After completing this tutorial sheet you will be able to:

- ☐ Determine if an improper integral exists by taking a limit of proper integrals.
- ☐ Obtain an intuition for which integrals exist/do not exist.
- ☐ Apply the comparison test to show that an integral exists of does not exist.
- ☐ Work with inequalities to find suitable bounds.
- ☐ Understand that integration by parts can be used to study convergence.
- ☐ Use integration by parts and limit laws to prove the elegant *Wallis product*.

## Preparation questions to do *before* class

**1.** Determine whether the following improper integrals exist by evaluating an appropriate limit of a proper integral. If the integral exists, compute its value.

(a) $\displaystyle\int_0^1 \frac{1}{1-x^2}\,dx$  (b) $\displaystyle\int_0^1 \frac{\ln x}{x^{1/3}}\,dx$

**2.** Use the Comparison Test to determine if the following improper integrals exist.

(a) $\displaystyle\int_0^1 \frac{e^{-x}}{x}\,dx$  (b) $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$

## Questions to attempt in class

**3.** Decide if the following improper integrals exist or not (either use the Comparison Test, or make a direct limit calculation). If they exist, try to compute their value (this is not always possible!).

(a) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

(b) $\displaystyle\int_1^\infty \sin(\pi x)\,dx$

(c) $\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$

(d) $\displaystyle\int_0^\infty \frac{\cosh x}{x^2+1}\,dx$

(e) $\displaystyle\int_{\pi/4}^{\pi/2} \sec^2 x\,dx$

(f) $\displaystyle\int_{-\infty}^{0} e^x \cos x\,dx$

(g) $\displaystyle\int_0^1 \sin\left(\frac{1}{x}\right)\,dx$

(h) $\displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx$

**4.** Does the improper integral

$$\int_1^\infty \frac{\sin x}{x}\,dx$$

exist? *Hint: Integration by parts might help.*

**5.** (a) Find a reduction formula for the integral $\displaystyle\int (\ln x)^n\,dx$ $(n \geq 0)$.

(b) Hence evaluate the improper integral $\displaystyle\int_0^1 (\ln x)^n\,dx$ $(n \geq 1)$.

## Discussion question

**6.** *Gabriel's horn* is the solid given by rotating $y = \frac{1}{x}$ $(x \geq 1)$ about the $x$-axis.

(a) Show that the volume of Gabriel's horn is finite.

(b) Show that the surface area of Gabriel's horn is infinite.

(c) Interesting: Part (a) seems to say that Gabriel's horn can be filled with a finite volume of paint (thereby painting the 'inside' of the horn), but part (b) seems to say that it takes an infinite amount of paint to paint the outside of Gabriel's horn. What's going on here?

## Questions for extra practice

**7.** Decide if the following improper integrals exist or not. If they exist, try to compute their value (this is not always possible!).

(a) $\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_0^\infty x^3 e^{-x}\,dx$

(c) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

(d) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

(e) $\displaystyle\int_1^\infty \frac{e^{-x^2}}{\sqrt{x-1}}\,dx$

(f) $\displaystyle\int_0^1 \sin\left(\frac{1}{x^2}\right)\,dx$

(g) $\displaystyle\int_0^\infty \operatorname{erf}(x)\,dx$

(h) $\displaystyle\int_0^\infty \cosh(3x)e^{-4x}\,dx$

(i) $\displaystyle\int_1^2 \frac{1}{\ln x}\,dx$

(j) $\displaystyle\int_2^\infty \frac{\operatorname{Li}(x)}{x^2}\,dx$

## Some questions involving reduction formulae

**8.** Let $x \in \mathbb{R}$, and let $n \geq 0$ be an integer.

(a) Use a reduction formula to prove that

$$\int_0^x (x-t)^n e^t\,dt = n!\left(e^x - \sum_{k=0}^n \frac{x^k}{k!}\right).$$

(b) Set $x = 1$ in (a) and deduce that $e$ is irrational.

*Hint: If $e$ is rational then the integral is an integer for sufficiently large $n$.*

(c) Use (a) to show that

$$\lim_{n\to\infty} \sum_{k=0}^n \frac{x^k}{k!} = e^x \qquad \text{for all } x \in \mathbb{R}.$$

**9.** For $n \geq 0$ let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta.$

(a) Derive a reduction formula for $I_n$, and use it to deduce that

$$I_{2n} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2} \qquad \text{and} \qquad I_{2n+1} = \frac{(2n)!!}{(2n+1)!!},$$

where $(2n)!! = 2\cdot 4\cdots(2n)$ and $(2n+1)!! = 1\cdot 3\cdots(2n+1)$.

(b) Show that $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$, and deduce that that

$$\frac{2n}{2n+1} \leq \frac{1\cdot 3\cdot 3\cdots(2n-1)(2n-1)}{2\cdot 2\cdot 4\cdots(2n-2)(2n)}\frac{\pi}{2} \leq 1\,.$$

Hence prove the *Wallis Product Formula* for $\pi$:

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}\cdots = \lim_{n\to\infty}\frac{2^{4n}n!^4}{2n(2n)!^2}\,.$$