THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet 1 (Week 2)

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Set notation, and number systems $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$, interval notation.
- ☐ Polynomial equations; solving quadratic equations over $\mathbb{C}$.
- ☐ Plotting regions in the complex plane.
- ☐ Polar and Cartesian forms of a complex number.
- ☐ Modulus, argument, and principal argument of a complex number.
- ☐ Arithmetic in polar form.

Outcomes

After completing this tutorial you should

- ☐ understand set notation;
- ☐ understand number systems;
- ☐ solve simple examples of polynomial equations over the complex numbers;
- ☐ construct proofs of basic properties of complex numbers;
- ☐ be able to plot regions in the complex plane;
- ☐ understand the geometric interpretation of complex numbers;
- ☐ efficiently convert between polar and Cartesian forms;
- ☐ perform arithmetic in polar form.

Summary of essential material

Rational and irrational numbers: A real number $r \in \mathbb{R}$ is called *rational* if there are integers $p, q \in \mathbb{Z}$ with $q \neq 0$ such that $r = p/q$. If it is not rational, it is called *irrational*. Interval notation if $a \leq b$:

$$[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}, \quad (a,b) := \{x \in \mathbb{R} \mid a < x < b\}, \quad (a,\infty) := \{x \in \mathbb{R} \mid a < x\}, \quad \text{etc...}$$

Intersections and unions: If A, B are subsets of a larger set X we define

- the *union* of A and B: $A \cup B := \{x \in X \mid x \in A \text{ or } x \in B\}$;
- the *intersection* of A and B: $A \cap B := \{x \in X \mid x \in A \text{ and } x \in B\}$;
- the *complement* of A: $A^c := \{x \in X \mid x \notin A\}$;
- the *complement of B in A*: $A \setminus B := A \cap B^c = \{x \in A \mid x \notin B\}$.

The Fundamental Theorem of Algebra: The complex polynomial

$$f(z) = c_0 + c_1 z + \cdots + c_n z^n \quad \text{with } c_i \in \mathbb{C} \text{ and } c_n \neq 0$$

factorises into precisely n linear factors (possibly with repetition) over the complex numbers.

Theorem: Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with $a_0, \dots, a_n \in \mathbb{R}$. If $\alpha \in \mathbb{C}$ is a root of $p(z)$, then the conjugate $\overline{\alpha}$ is also a root of $p(z)$.

Cartesian and Modulus–argument form (polar form): Every complex number $z = x + iy$ represents a point on the plane with coordinates (x, y). With that identification we obtain the *complex plane* or *Argand diagram*. We call $x + iy$ the *Cartesian form* of z. We can represent each point (x, y) in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ where r is the distance from the origin and θ is the angle from the positive x-axis measured anti-clockwise, usually in radians. The *modulus–argument form* (or *polar form*) of z is

$$z = r(\cos\theta + i\sin\theta)$$

We call $r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$ the *modulus* and θ an *argument* of z, written $\arg z$. The argument is determined up to a multiple of 2π. The unique argument in the interval $(-\pi, \pi]$ is called the *principal argument*, denoted $\operatorname{Arg} z$.

The complex exponential function: For any complex number $z = x + iy$, $x, y \in \mathbb{R}$ we let

$$e^z := e^x(\cos y + i \sin y)$$

We also sometimes write $\exp(z)$ (for larger expressions). It follows the usual index laws just as the real exponential function, that is, $e^{z+w} = e^z e^w$ and $e^{-z} = 1/e^z$ for all $z, w \in \mathbb{C}$. It coincides with the real exponential function on $\mathbb{R}$ and is 2π-periodic on $i\mathbb{R}$, that is, $e^{i(2\pi+\theta)} = e^{i\theta}$ for all $\theta \in \mathbb{R}$. Moreover, $|e^{i\theta}| = 1$ for all $\theta \in \mathbb{R}$.

De Moivre's Theorem: Let $n \in \mathbb{Z}$. If $r > 0$ and $\theta \in \mathbb{R}$ then

$$\big(r(\cos\theta + i\sin\theta)\big)^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \qquad \text{or} \qquad \big(re^{i\theta}\big)^n = r^n e^{in\theta}.$$

Questions to complete during the tutorial

1. Express the following subsets of $\mathbb{R}$ as a union of intervals.

(a) $\{x \in \mathbb{R} \mid -1 \leq x < 5\}$ (b) $(-\infty, 3] \setminus (-6, 10]$. (c) $\{x \in \mathbb{R} \mid x^2 + x > 2\}$

2. Prove that is $\log_2 3$ irrational. (*Hint:* Assume that $\log_2 3 = p/q$ is rational and derive a contradiction.)

3. What are the complex numbers obtained from z by the following geometric transformations?

(a) $180°$ rotation about 0.

(b) Reflection at the imaginary axis.

(c) $45°$ clockwise rotation about 0.

(d) Reflection in the line $y = x$.

4. Write the following complex numbers in Cartesian form:

(a) $e^{i\pi/4}e^{i2\pi/5}e^{i\pi/3}e^{i\pi/2}e^{i11\pi/60}$ (b) $(1 + \sqrt{3}\, i)^{107}$ (c) $(1 - i)^{-76}$

5. Solve the following equations for $z \in \mathbb{C}$:

(a) $z^2 + z + 1 = 0$ (b) $z^2 + 2\bar{z} + 1 = 0$

6. You are told that $\alpha = 2 + i$ is a root of the polynomial

$$p(z) = z^6 - 4z^5 + 8z^4 - 12z^3 + 5z^2 + 40z - 50.$$

You will need to use polynomial long division and the fact that $\bar{\alpha} = 2 - i$ is a root as well.

(a) Factorise the polynomial $p(z)$ into linear factors over $\mathbb{C}$.

(b) What can you say about factorisations of $p(z)$ over $\mathbb{R}$ and $\mathbb{Q}$?

7. Given any pair of complex numbers z_1 and z_2, prove the following facts. Focus on the geometric meaning, then confirm by calculation.

(a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (b) $\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$ (c) $|z_1 + z_2| \leq |z_1| + |z_2|$

8. Sketch the following sets in the complex plane. Recall that $|z - \alpha|$ is the distance between z and α in the Argand diagram.

(a) $\{z \in \mathbb{C} \mid |z + i| = 5\}$

(b) $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq -1\}$

(c) $\{z \in \mathbb{C} \mid |z - i| \leq |z - 1|\}$

(d) $\left\{z \in \mathbb{C} \,\middle|\, \left|\frac{z-1}{z-2}\right| \leq 3\right\}$

Extra questions for further practice

9. Solve the following equations.

(a) $z^4 - 16 = 0$

(b) $z^2 + 3z + 2 = 0$

(c) $z^2 + z + 1 + i = 0.$

(d) $z^2 + (2 + 3i)z - 1 + 3i = 0.$

10. Solve $z^5 - 2z^4 + 2z^3 - z^2 + 2z - 2 = 0$, given that $z = 1 + i$ is a solution.

11. For all complex numbers z, prove that

(a) $|z|^2 = z\overline{z}$.

(b) $\overline{z} = z$ if and only if z is real.

(c) $\mathrm{Re}(z) \leq |z|$ and $\mathrm{Im}(z) \leq |z|$.

(d) $\overline{1/z} = 1/\overline{z}$ for $z \neq 0$.

12. Sketch the following sets in the complex plane.

(a) $\{z \in \mathbb{C} \mid \mathrm{Re}\, z < -1\}$

(b) $\{z \in \mathbb{C} \mid \frac{1}{2} \leq |z + i| < 1\}$

(c) $\{ z \in \mathbb{C} \mid |z + i| > 2 \}$

(d) $\{ z \in \mathbb{C} \mid \mathrm{Im}(2z - \overline{z}(1 + i)) = 0 \text{ and } \mathrm{Re}(2z - \overline{z}(1 + i)) < 4 \}$

(e) $\{z \in \mathbb{C} \mid \mathrm{Im}(z^2) < \mathrm{Re}\, z\}$

13. (a) Let $r > 2$. The set $\{z \in \mathbb{C} \mid |z + 1| + |z - 1| = r\}$ is a curve in the plane. Describe it and then find its equation in terms of x and y, where x, y are real and $z = x + iy$.

(b) Now assume that $-2 < r < 2$. Describe the curve $\{z \in \mathbb{C} \mid |z + 1| - |z - 1| = r\}$ and find its equation.

14. Complex numbers can be used in proofs of things that seem to have nothing to do with complex numbers. For example, consider the following number theoretic statement: If $m = a^2 + b^2$ is a sum of two squares (with $a, b \in \mathbb{Z}$) and $n = c^2 + d^2$ is a sum of two squares (with $c, d \in \mathbb{Z}$), then $mn = u^2 + v^2$ is also a sum of two squares (with $u, v \in \mathbb{Z}$). Prove this statement by first noting that $m = |a + ib|^2$ and $n = |c + id|^2$.

Revision questions on complex numbers

The questions below are particularly relevant for those students who have not seen complex numbers at high school level.

15. Express the following complex numbers in Cartesian form:

(a) $(1 + i)(1 - i)$

(b) $(2 + 3i) - (4 - 5i)$

(c) $\dfrac{1 + 2i}{3 - 4i}$

(d) $(1 + i)^2$

(e) $(3 - 2i)\left(\dfrac{5}{2} - 7i\right)$

(f) $\dfrac{3i - 5}{i + 7}$

(g) i^{-1}

(h) i^9

(i) $i^{123} - 4i^8 - 4i.$

16. Find the principal argument of the following complex numbers.

(a) $-1 + i$

(b) $-3i$

(c) $-5e^{i7\pi/2}$

(d) $6 - 5i.$

17. Write the following complex numbers in polar form.

(a) $1+i$
(b) $1+\sqrt{3}i$
(c) $3\sqrt{3}+3i$
(d) $1+i$
(e) $-1+\sqrt{3}i$
(f) -5
(g) i
(h) $5-7i$

18. Find the following, expressing your final answers first in polar form, and then in Cartesian form.

(a) $(1+i)^{11}$
(b) $(1+\sqrt{3}i)^7$
(c) $(3\sqrt{3}+3i)^3$
(d) $\dfrac{1+i}{1+\sqrt{3}i}$
(e) $\dfrac{3\sqrt{3}+3i}{1+i}$
(f) $\dfrac{1+\sqrt{3}i}{3\sqrt{3}+3i}$

Challenge questions (optional)

19. Use the binomial expansion $(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$ and de Moivre's Theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$, respectively. Hence show that

$$\cos(\pi/5) = \frac{1+\sqrt{5}}{4} \quad \text{and} \quad \sin(\pi/5) = \frac{\sqrt{2(5-\sqrt{5})}}{4}.$$

20. Prove the following property of the real numbers: Given any two numbers $a < b$ there is a rational number r and an irrational number s such that $a < r < b$ and $a < s < b$. You may use that $\sqrt{2}$ is irrational.