CHAPTER **ONE**

The Inverse Trigonometric Functions

A proper understanding of how to solve trigonometric equations requires a theory of inverse trigonometric functions. This theory is complicated by the fact that the trigonometric functions are periodic functions — they therefore fail the horizontal line test quite seriously, in that some horizontal lines cross their graphs infinitely many times. Understanding inverse trigonometric functions therefore requires further discussion of the procedures for restricting the domain of a function so that the inverse relation is also a function. Once the functions are established, the usual methods of differential and integral calculus can be applied to them.

This theory gives rise to primitives of two purely algebraic functions

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x \quad (\text{or } -\cos^{-1} x) \qquad \text{and} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x,$$

which are similar to the earlier primitive $\displaystyle\int \frac{1}{x}\,dx = \log x$ in that in all three cases, a purely algebraic function has a primitive which is non-algebraic.

Study Notes: Inverse relations and functions were first introduced in Section 2H of the Year 11 volume. That material is summarised in Section 1A in preparation for more detail about restricted functions, but some further revision may be necessary. Sections 1B–1E then develop the standard theory of inverse trigonometric functions and their graphs, and the associated derivatives and integrals. In Section 1F these functions are used to establish some formulae for the general solutions of trigonometric equations.

1 A Restricting the Domain

Section 2H of the Year 11 volume discussed how the inverse relation of a function may or may not be a function, and briefly mentioned that if the inverse is not a function, then the domain can be restricted so that the inverse of this restricted function is a function. This section revisits those ideas and develops a more systematic approach to restricting the domain.

Inverse Relations and Inverse Functions: First, here is a summary of the basic theory of inverse functions and relations. The examples given later will illustrate the various points. Suppose that $f(x)$ is a function whose inverse relation is being considered.

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1 INVERSE FUNCTIONS AND RELATIONS:

- The graph of the inverse relation is obtained by reflecting the original graph in the diagonal line $y = x$.
- The inverse relation of a given relation is a function if and only if no horizontal line crosses the original graph more than once.
- The domain and range of the inverse relation are the range and domain respectively of the original function.
- To find the equations and conditions of the inverse relation, write x for y and y for x every time each variable occurs.
- If the inverse relation is also a function, the inverse function is written as $f^{-1}(x)$. Then the composition of the function and its inverse, in either order, leaves every number unchanged:

$$f^{-1}\big(f(x)\big) = x \qquad \text{and} \qquad f\big(f^{-1}(x)\big) = x.$$

- If the inverse is not a function, then the domain of the original function can be restricted so that the inverse of the restricted function is a function.

The following worked exercise illustrates the fourth and fifth points above.

WORKED EXERCISE: Find the inverse function of $f(x) = \dfrac{x-2}{x+2}$. Then show directly that $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(x)\big) = x$.

SOLUTION: Let $\quad y = \dfrac{x-2}{x+2}$.

Then the inverse relation is $\quad x = \dfrac{y-2}{y+2}$ (writing y for x and x for y)

$$\begin{aligned} xy + 2x &= y - 2 \\ y(x-1) &= -2x - 2 \\ y &= \frac{2+2x}{1-x}. \end{aligned}$$

Since there is only one solution for y, the inverse relation is a function,

and $\quad f^{-1}(x) = \dfrac{2+2x}{1-x}$.

Then
$$\begin{aligned} f\big(f^{-1}(x)\big) &= f\left(\frac{2+2x}{1-x}\right) \\ &= \frac{\frac{2+2x}{1-x} - 2}{\frac{2+2x}{1-x} + 2} \times \frac{1-x}{1-x} \\ &= \frac{(2+2x) - 2(1-x)}{(2+2x) + 2(1-x)} \\ &= \frac{4x}{4} \\ &= x, \text{ as required.} \end{aligned}$$

and
$$\begin{aligned} f^{-1}\big(f(x)\big) &= f^{-1}\left(\frac{x-2}{x+2}\right) \\ &= \frac{2 + \frac{2(x-2)}{x+2}}{1 - \frac{x-2}{x+2}} \times \frac{x+2}{x+2} \\ &= \frac{2(x+2) + 2(x-2)}{(x+2) - (x-2)} \\ &= \frac{4x}{4} \\ &= x \text{ as required.} \end{aligned}$$

Increasing and Decreasing Functions: Increasing means getting bigger, and we say that a function $f(x)$ is an *increasing function* if $f(x)$ increases as x increases:

$$f(a) < f(b), \quad \text{whenever } a < b.$$

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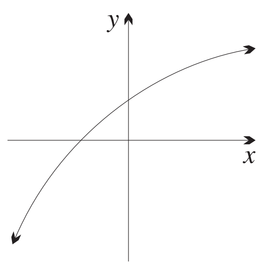
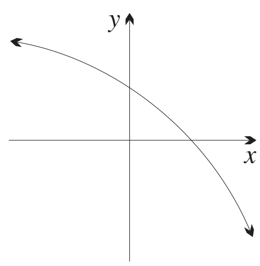
For example, if $f(x)$ is an increasing function, then provided $f(x)$ is defined there, $f(2) < f(3)$, and $f(5) < f(10)$. In the language of coordinate geometry, this means that *every chord slopes upwards*, because the ratio $\dfrac{f(b) - f(a)}{b - a}$ must be positive, for all pairs of distinct numbers a and b. *Decreasing functions* are defined similarly.

2 INCREASING AND DECREASING FUNCTIONS: Suppose that $f(x)$ is a function.

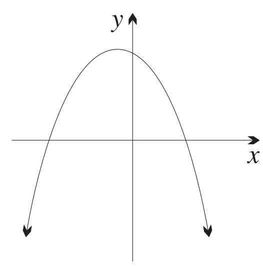
- $f(x)$ is called an *increasing function* if every chord slopes upwards, that is,
  $$f(a) < f(b), \quad \text{whenever } a < b.$$
- $f(x)$ is called a *decreasing function* if every chord slopes downwards, that is,
  $$f(a) > f(b), \quad \text{whenever } a < b.$$



An increasing function



A decreasing function



Neither of these

NOTE: These are *global* definitions, looking at the graph of the function as a whole. They should be contrasted with the *pointwise* definitions introduced in Chapter Ten of the Year 11 volume, where a function $f(x)$ was called *increasing at* $x = a$ if $f'(a) > 0$, that is, if the tangent slopes upwards at the point.

Throughout our course, a tangent describes the behaviour of a function at a particular point, whereas a chord relates the values of the function at two different points.

The exact relationship between the global and pointwise definitions of *increasing* are surprisingly difficult to state, as the examples in the following paragraphs demonstrate, but in this course it will be sufficient to rely on the graph and common sense.

The Inverse Relation of an Increasing or Decreasing Function: When a horizontal line crosses a graph twice, it generates a horizontal chord. But every chord of an increasing function slopes upwards, and so an increasing function cannot possibly fail the horizontal line test. This means that the inverse relation of every increasing function is a function. The same argument applies to decreasing functions.

3 INCREASING OR DECREASING FUNCTIONS AND THE INVERSE RELATION:

- The inverse of an increasing or decreasing function is a function.
- The inverse of an increasing function is increasing, and the inverse of a decreasing function is decreasing.

To justify the second remark, notice that reflection in $y = x$ maps lines sloping upwards to lines sloping upwards, and maps lines sloping downwards to lines sloping downwards.

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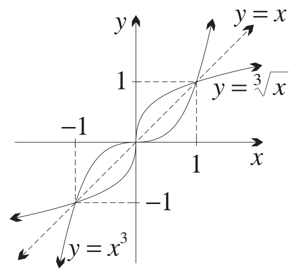
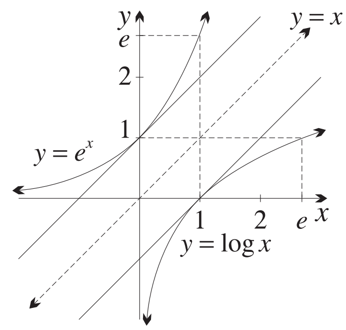
Example — The Cube and Cube Root Functions: The function $f(x) = x^3$ and its inverse function $f^{-1}(x) = \sqrt[3]{x}$ are graphed to the right.



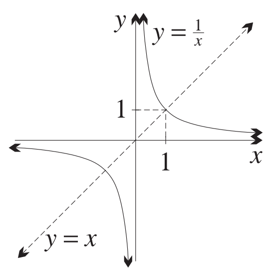
- $f(x) = x^3$ is an increasing function, because every chord slopes upwards. Hence it passes the horizontal line test, and its inverse is a function, which is also increasing.
- $f(x)$ is not, however, increasing at every point, because the tangent at the origin is horizontal. Correspondingly, the tangent to $y = \sqrt[3]{x}$ at the origin is vertical.
- For all x, $\sqrt[3]{x^3} = x$ and $(\sqrt[3]{x}\,)^3 = x$.

Example — The Logarithmic and Exponential Functions: The two functions $f(x) = e^x$ and $f^{-1}(x) = \log x$ provide a particularly clear example of a function and its inverse.



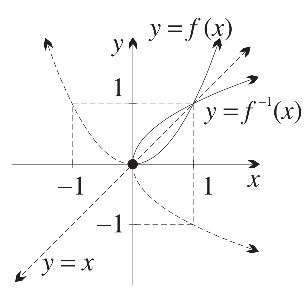
- $f(x) = e^x$ is an increasing function, because every chord slopes upwards. Hence it passes the horizontal line test, and its inverse is a function, which is also increasing.
- $f(x) = e^x$ is also increasing at every point, because its derivative is $f'(x) = e^x$ which is always positive.
- For all x, $\log e^x = x$, and for $x > 0$, $e^{\log x} = x$.

Example — The Reciprocal Function: The function $f(x) = 1/x$ is its own inverse, because the reciprocal of the reciprocal of any nonzero number is always the original number. Correspondingly, its graph is symmetric in $y = x$.



- $f(x) = 1/x$ is neither increasing nor decreasing, because chords joining points on the same branch slope downwards, and chords joining points on different branches slope upwards. Nevertheless, it passes the horizontal line test, and its inverse (which is itself) is a function.
- $f(x) = 1/x$ is decreasing at every point, because its derivative is $f'(x) = -1/x^2$, which is always negative.

Restricting the Domain — The Square and Square Root Functions: The two functions $y = x^2$ and $y = \sqrt{x}$ give our first example of restricting the domain so that the inverse of the restricted function is a function.



- $y = x^2$ is neither increasing nor decreasing, because some of its chords slope upwards, some slope downwards, and some are horizontal. Its inverse $x = y^2$ is not a function — for example, the number 1 has two square roots, 1 and -1.
- Define the restricted function $f(x)$ by $f(x) = x^2$, where $x \geq 0$. This is the part of $y = x^2$ shown undotted in the diagram on the right. Then $f(x)$ is an increasing function, and so has an inverse which is written as $f^{-1}(x) = \sqrt{x}$, and which is also increasing.
- For all $x > 0$, $\sqrt{x^2} = x$ and $(\sqrt{x}\,)^2 = x$.

Further Examples of Restricting the Domain: These two worked exercises show the process of restricting the domain applied to more general functions. Since $y = x$ is the mirror exchanging the graphs of a function and its inverse, and since points on a mirror are reflected to themselves, it follows that if the graph of the function intersects the line $y = x$, then it intersects the inverse there too.

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WORKED EXERCISE: Explain why the inverse relation of $f(x) = (x-1)^2+2$ is not a function. Define $g(x)$ to be the restriction of $f(x)$ to the largest possible domain containing $x = 0$ so that $g(x)$ has an inverse function. Write down the equation of $g^{-1}(x)$, then sketch $g(x)$ and $g^{-1}(x)$ on one set of axes.

SOLUTION: The graph of $y = f(x)$ is a parabola with vertex $(1, 2)$.
This fails the horizontal line test, so the inverse is not a function.
(Alternatively, $f(0) = f(2) = 3$, so $y = 3$ meets the curve twice.)
Restricting $f(x)$ to the domain $x \le 1$ gives the function

$$g(x) = (x-1)^2 + 2, \quad \text{where } x \le 1,$$

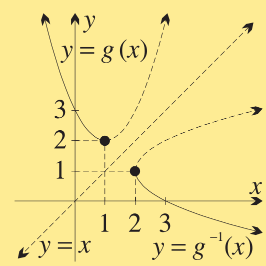
which is sketched opposite, and includes the value at $x = 0$.
Since $g(x)$ is a decreasing function, it has an inverse with equation

$$x = (y-1)^2 + 2, \quad \text{where } y \le 1.$$

Solving for y, $(y-1)^2 = x - 2$, where $y \le 1$,

$$y = 1 + \sqrt{x-2} \ \text{ or } \ 1 - \sqrt{x-2}, \quad \text{where } y \le 1.$$

Hence $g(x) = 1 - \sqrt{x-2}$, since $y \le 1$.



WORKED EXERCISE: Use calculus to find the turning points and points of inflexion of $y = (x-2)^2(x+1)$, then sketch the curve. Explain why the restriction $f(x)$ of this function to the part of the curve between the two turning points has an inverse function. Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on one set of axes, and write down an equation satisfied by the x-coordinate of the point M where the function and its inverse intersect.

SOLUTION: For

$$y = (x-2)^2(x+1) = x^3 - 3x^2 + 4,$$
$$y' = 3x^2 - 6x = 3x(x-2),$$

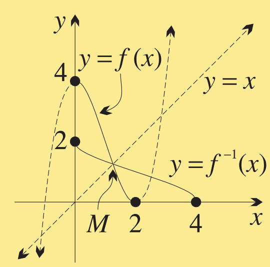
and

$$y'' = 6x - 6 = 6(x-1).$$

So there are zeroes at $x = 2$ and $x = -1$, and (after testing) turning points at $(0, 4)$ (a maximum) and $(2, 0)$ (a minimum), and a point of inflexion at $(1, 2)$.
The part of the curve between the turning points is decreasing, so the function $f(x) = (x-2)^2(x+1)$, where $0 \le x \le 2$, has an inverse function $f^{-1}(x)$, which is also decreasing.
The curves $y = f(x)$ and $y = f^{-1}(x)$ intersect on $y = x$, and substituting $y = x$ into the function,

$$x = x^3 - 3x^2 + 4,$$

so the x-coordinate of M satisfies the cubic $x^3 - 3x^2 - x + 4 = 0$.



Exercise 1A

1. Consider the functions $f = \{(0,2), (1,3), (2,4)\}$ and $g = \{(0,2), (1,2), (2,2)\}$.
   (a) Write down the inverse relation of each function.
   (b) Graph each function and its inverse relation on a number plane, using separate diagrams for f and g.
   (c) State whether or not each inverse relation is a function.

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2. The function $f(x) = x + 3$ is defined over the domain $0 \le x \le 2$.
(a) State the range of $f(x)$.
(b) State the domain and range of $f^{-1}(x)$.
(c) Write down the rule for $f^{-1}(x)$.

3. The function F is defined by $F(x) = \sqrt{x}$ over the domain $0 \le x \le 4$.
(a) State the range of $F(x)$.
(b) State the domain and range of $F^{-1}(x)$.
(c) Write down the rule for $F^{-1}(x)$.
(d) Graph F and F^{-1}.

4. Sketch the graph of each function. Then use reflection in the line $y = x$ to sketch the inverse relation. State whether or not the inverse is a function, and find its equation if it is. Also, state whether $f(x)$ and $f^{-1}(x)$ (if it exists) are increasing, decreasing or neither.
(a) $f(x) = 2x$
(b) $f(x) = x^3 + 1$
(c) $f(x) = \sqrt{1 - x^2}$
(d) $f(x) = x^2 - 4$
(e) $f(x) = 2^x$
(f) $f(x) = \sqrt{x - 3}$

5. Consider the functions $f(x) = 3x + 2$ and $g(x) = \frac{1}{3}(x - 2)$.
(a) Find $f\,(g(x))$ and $g\,(f(x))$.
(b) What is the relationship between $f(x)$ and $g(x)$?

6. Each function $g(x)$ is defined over a restricted domain so that $g^{-1}(x)$ exists. Find $g^{-1}(x)$ and write down its domain and range. (Sketches of g and g^{-1} will prove helpful.)
(a) $g(x) = x^2,\ x \ge 0$
(b) $g(x) = x^2 + 2,\ x \le 0$
(c) $g(x) = -\sqrt{4 - x^2}\,,\ 0 \le x \le 2$

7. (a) Write down $\dfrac{dy}{dx}$ for the function $y = x^3 - 1$.
(b) Make x the subject and hence find $\dfrac{dx}{dy}$.
(c) Hence show that $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$.

8. Repeat the previous question for $y = \sqrt{x}$.

DEVELOPMENT

9. The function $F(x) = x^2 + 2x + 4$ is defined over the domain $x \ge -1$.
(a) Sketch the graphs of $F(x)$ and $F^{-1}(x)$ on the same diagram.
(b) Find the equation of $F^{-1}(x)$ and state its domain and range.

10. (a) Solve the equation $1 - \ln x = 0$.
(b) Sketch the graph of $f(x) = 1 - \ln x$ by suitably transforming the graph of $y = \ln x$.
(c) Hence sketch the graph of $f^{-1}(x)$ on the same diagram.
(d) Find the equation of $f^{-1}(x)$ and state its domain and range.
(e) Classify $f(x)$ and $f^{-1}(x)$ as increasing, decreasing or neither.

11. (a) Carefully sketch the function defined by $g(x) = \dfrac{x+2}{x+1}$, for $x > -1$.
(b) Find $g^{-1}(x)$ and sketch it on the same diagram. Is $g^{-1}(x)$ increasing or decreasing?
(c) Find any values of x for which $g(x) = g^{-1}(x)$. [HINT: The easiest way is to solve $g(x) = x$. Why does this work?]

12. The previous question seems to imply that the graphs of a function and its inverse can only intersect on the line $y = x$. This is not always the case.
(a) Find the equation of the inverse of $y = -x^3$.
(b) At what points do the graphs of the function and its inverse meet?
(c) Sketch the situation.

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13. (a) Explain how the graph of $f(x) = x^2$ must be transformed to obtain the graph of $g(x) = (x+2)^2 - 4$.
(b) Hence sketch the graph of $g(x)$, showing the x and y intercepts and the vertex.
(c) What is the largest domain containing $x = 0$ for which $g(x)$ has an inverse function?
(d) Let $g^{-1}(x)$ be the inverse function corresponding to the domain of $g(x)$ in part (c). What is the domain of $g^{-1}(x)$? Is $g^{-1}(x)$ increasing or decreasing?
(e) Find the equation of $g^{-1}(x)$, and sketch it on your diagram in part (b).
(f) Classify $g(x)$ and $g^{-1}(x)$ as either increasing, decreasing or neither.

14. (a) Show that $F(x) = x^3 - 3x$ is an odd function.
(b) Sketch the graph of $F(x)$, showing the x-intercepts and the coordinates of the two stationary points. Is $F(x)$ increasing or decreasing?
(c) What is the largest domain containing $x = 0$ for which $F(x)$ has an inverse function?
(d) State the domain of $F^{-1}(x)$, and sketch it on the same diagram as part (b).

15. (a) State the domain of $f(x) = \dfrac{e^x}{1+e^x}$. (b) Show that $f'(x) = \dfrac{e^x}{(1+e^x)^2}$.
(c) Hence explain why $f(x)$ is increasing for all x.
(d) Explain why $f(x)$ has an inverse function, and find its equation.

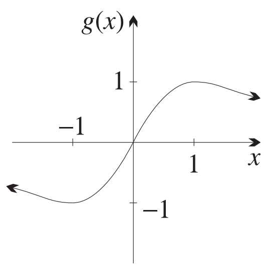
16. (a) Sketch $y = 1 + x^2$ and hence sketch $f(x) = \dfrac{1}{1+x^2}$. Is $f(x)$ increasing or decreasing?
(b) What is the largest domain containing $x = -1$ for which $f(x)$ has an inverse function?
(c) State the domain of $f^{-1}(x)$, and sketch it on the same diagram as part (a).
(d) Find the rule for $f^{-1}(x)$.
(e) Is $f^{-1}(x)$ increasing or decreasing?

17. (a) Show that any linear function $f(x) = mx + b$ has an inverse function if $m \neq 0$.
(b) Does the constant function $F(x) = b$ have an inverse function?

18. The function $f(x)$ is defined by $f(x) = x - \frac{1}{x}$, for $x > 0$.
(a) By considering the graphs of $y = x$ and $y = \frac{1}{x}$ for $x > 0$, sketch $y = f(x)$.
(b) Sketch $y = f^{-1}(x)$ on the same diagram.
(c) By completing the square or using the quadratic formula, show that

$$f^{-1}(x) = \tfrac{1}{2}\left(x + \sqrt{4 + x^2}\right).$$

19. The diagram shows the function $g(x) = \dfrac{2x}{1+x^2}$, whose domain is all real x.



(a) Show that $g(\frac{1}{a}) = g(a)$, for all $a \neq 0$.
(b) Hence explain why the inverse of $g(x)$ is not a function.
(c) (i) What is the largest domain of $g(x)$ containing $x = 0$ for which $g^{-1}(x)$ exists?
(ii) Sketch $g^{-1}(x)$ for this domain of $g(x)$.
(iii) Find the equation of $g^{-1}(x)$ for this domain of $g(x)$.
(d) Repeat part (c) for the largest domain of $g(x)$ that does not contain $x = 0$.
(e) Show that the two expressions for $g^{-1}(x)$ in parts (c) and (d) are reciprocals of each other. Why could we have anticipated this?

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20. Consider the function $f(x) = \frac{1}{6}(x^2 - 4x + 24)$.

(a) Sketch the parabola $y = f(x)$, showing the vertex and any x- or y-intercepts.

(b) State the largest domain containing only positive numbers for which $f(x)$ has an inverse function $f^{-1}(x)$.

(c) Sketch $f^{-1}(x)$ on your diagram from part (a), and state its domain.

(d) Find any points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

(e) Let N be a negative real number. Find $f^{-1}\left(f(N)\right)$.

21. (a) Prove, both geometrically and algebraically, that if an odd function has an inverse function, then that inverse function is also odd.

(b) What sort of even functions have inverse functions?

22. [The hyperbolic sine function] The function $\sinh x$ is defined by $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

(a) State the domain of $\sinh x$.

(b) Find the value of $\sinh 0$.

(c) Show that $y = \sinh x$ is an odd function.

(d) Find $\dfrac{d}{dx}(\sinh x)$ and hence show that $\sinh x$ is increasing for all x.

(e) To which curve is $y = \sinh x$ asymptotic for large values of x?

(f) Sketch $y = \sinh x$, and explain why the function has an inverse function $\sinh^{-1} x$.

(g) Sketch the graph of $\sinh^{-1} x$ on the same diagram as part (f).

(h) Show that $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$, by treating the equation $x = \frac{1}{2}(e^y - e^{-y})$ as a quadratic equation in e^y.

(i) Find $\dfrac{d}{dx}(\sinh^{-1} x)$, and hence find $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$.

EXTENSION

23. Suppose that f is a one-to-one function with domain D and range R. Then the function g with domain R and range D is the inverse of f if

$$f\left(g(x)\right) = x \text{ for every } x \text{ in } R \quad \text{and} \quad g\left(f(x)\right) = x \text{ for every } x \text{ in } D.$$

Use this characterisation to prove that the functions

$$f(x) = -\tfrac{2}{3}\sqrt{9 - x^2}\,, \text{ where } 0 \le x \le 3, \quad \text{and} \quad g(x) = \tfrac{3}{2}\sqrt{4 - x^2}\,, \text{ where } -2 \le x \le 0,$$

are inverse functions.

24. THEOREM: If f is a differentiable function for all real x and has an inverse function g, then $g'(x) = \dfrac{1}{f'\left(g(x)\right)}$, provided that $f'\left(g(x)\right) \ne 0$.

(a) It is known that $\dfrac{d}{dx}(\ln x) = \frac{1}{x}$ and that $y = e^x$ is the inverse function of $y = \ln x$.

Use this information and the above theorem to prove that $\dfrac{d}{dx}(e^x) = e^x$.

(b) (i) Show that the function $f(x) = x^3 + 3x$ is increasing for all real x, and hence that it has an inverse function, $f^{-1}(x)$. (ii) Use the theorem to find the gradient of the tangent to the curve $y = f^{-1}(x)$ at the point $(4, 1)$.

(c) Prove the theorem in general.

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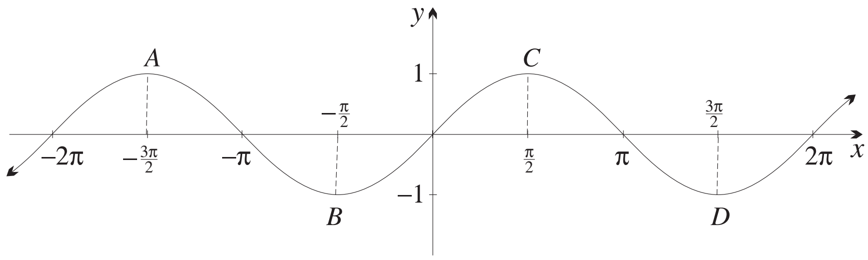
1 B Defining the Inverse Trigonometric Functions

Each of the six trigonometric functions fails the horizontal line test completely, in that there are horizontal lines which cross each of their graphs infinitely many times. For example, $y = \sin x$ is graphed below, and clearly every horizontal line between $y = 1$ and $y = -1$ crosses it infinitely many times.

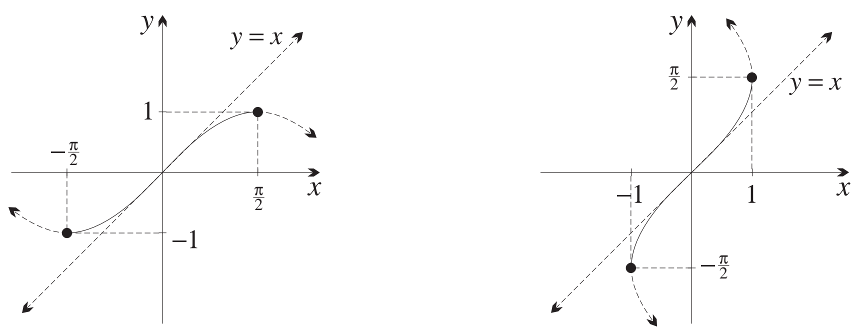


To create an inverse function from $y = \sin x$, we need to restrict the domain to a piece of the curve between two turning points. For example, the pieces AB, BC and CD all satisfy the horizontal line test. Since acute angles should be included, the obvious choice is the arc BC from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

The Definition of $\sin^{-1} x$: The function $y = \sin^{-1} x$ (which is read as 'inverse sine ex') is accordingly defined to be the inverse function of the restricted function

$$y = \sin x, \quad \text{where } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}.$$

The two curves are sketched below. Notice, when sketching the graphs, that $y = x$ is a tangent to $y = \sin x$ at the origin. Thus when the graph is reflected in $y = x$, the line $y = x$ does not move, and so it is also the tangent to $y = \sin^{-1} x$ at the origin. Notice also that $y = \sin x$ is horizontal at its turning points, and hence $y = \sin^{-1} x$ is vertical at its endpoints.



$y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $\qquad$ $y = \sin^{-1} x$

4 **THE DEFINITION OF $y = \sin^{-1} x$:**
- $y = \sin^{-1} x$ is not the inverse relation of $y = \sin x$, it is the inverse function of the restriction of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- $y = \sin^{-1} x$ has domain $-1 \le x \le 1$ and range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
- $y = \sin^{-1} x$ is an increasing function.
- $y = \sin^{-1} x$ has tangent $y = x$ at the origin, and is vertical at its endpoints.

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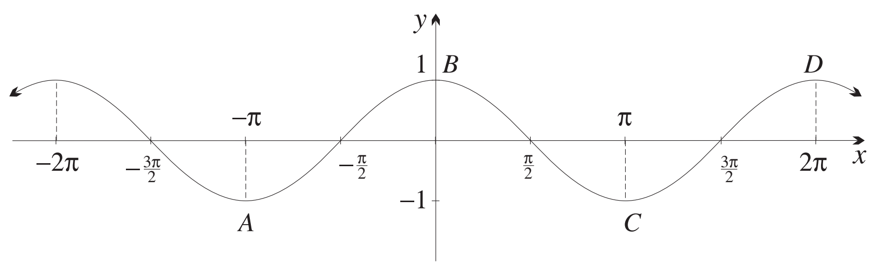
NOTE: In this course, radian measure is used exclusively when dealing with the inverse trigonometric functions. Calculations using degrees should be avoided, or at least not included in the formal working of problems.

5 RADIAN MEASURE: Use radians when dealing with inverse trigonometric functions.

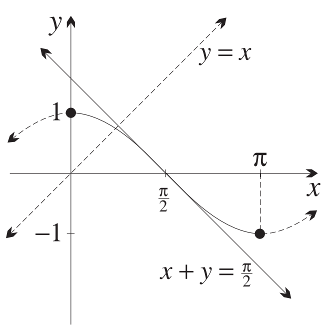
The Definition of $\cos^{-1} x$: The function $y = \cos x$ is graphed below. To create a satisfactory inverse function from $y = \cos x$, we need to restrict the domain to a piece of the curve between two turning points. Since acute angles should be included, the obvious choice is the arc BC from $x = 0$ to $x = \pi$.



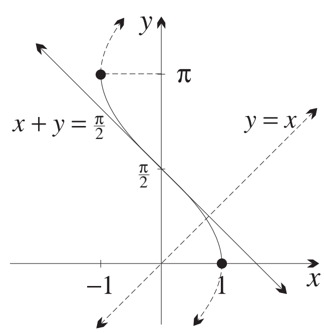
Thus the function $y = \cos^{-1} x$ (read as 'inverse cos ex') is defined to be the inverse function of the restricted function

$$y = \cos x,\quad \text{where } 0 \le x \le \pi,$$

and the two curves are sketched below. Notice that the tangent to $y = \cos x$ at its x-intercept $(\frac{\pi}{2}, 0)$ is the line t: $x + y = \frac{\pi}{2}$ with gradient -1. Reflection in $y = x$ reflects this line onto itself, so t is also the tangent to $y = \cos^{-1} x$ at its y-intercept $(0, \frac{\pi}{2})$. Like $y = \sin^{-1} x$, the graph is vertical at its endpoints.



$y = \cos x$, $0 \le x \le \pi$



$y = \cos^{-1} x$

6 THE DEFINITION OF $y = \cos^{-1} x$:
- $y = \cos^{-1} x$ is not the inverse relation of $y = \cos x$, it is the inverse function of the restriction of $y = \cos x$ to $0 \le x \le \pi$.
- $y = \cos^{-1} x$ has domain $-1 \le x \le 1$ and range $0 \le y \le \pi$.
- $y = \cos^{-1} x$ is a decreasing function.
- $y = \cos^{-1} x$ has gradient -1 at its y-intercept, and is vertical at its endpoints.

The Definition of $\tan^{-1} x$: The graph of $y = \tan x$ on the next page consists of a collection of disconnected branches. The most satisfactory inverse function is formed by choosing the branch in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

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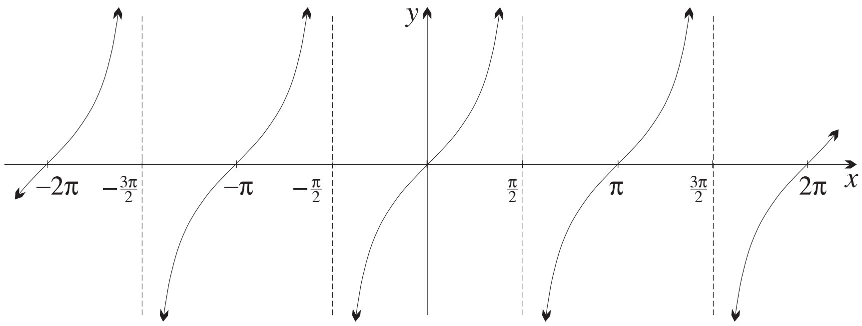
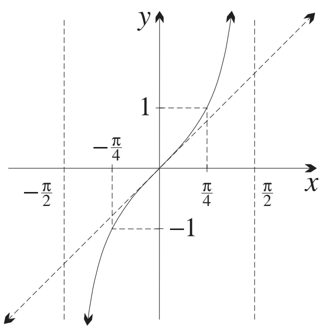


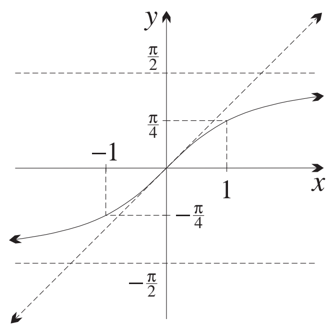
Thus the function $y = \tan^{-1} x$ is defined to be the inverse function of

$$y = \tan x, \quad \text{where} \quad -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}.$$

The line of reflection $y = x$ is the tangent to both curves at the origin. Notice also that the vertical asymptotes $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are reflected into the horizontal asymptotes $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.



$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



$y = \tan^{-1} x$

7 THE DEFINITION OF $y = \tan^{-1} x$:

- $y = \tan^{-1} x$ is not the inverse relation of $y = \tan x$, it is the inverse function of the restriction of $y = \tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
- $y = \tan^{-1} x$ has domain the real line and range $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
- $y = \tan^{-1} x$ is an increasing function.
- $y = \tan^{-1} x$ has gradient 1 at its y-intercept.
- The lines $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes.

Inverse Functions of cosec x, sec x and cot x: It is not convenient in this course to define the functions $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ because of difficulties associated with discontinuities. Extension questions in Exercises 1C and 1D investigate these situations.

Calculations with the Inverse Trigonometric Functions: The key to calculations is to include the restriction every time an expression involving the inverse trigonometric functions is rewritten using trigonometric functions.

8 INTERPRETING THE RESTRICTIONS:

- $y = \sin^{-1} x$ means $x = \sin y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
- $y = \cos^{-1} x$ means $x = \cos y$ where $0 \le y \le \pi$.
- $y = \tan^{-1} x$ means $x = \tan y$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

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WORKED EXERCISE: Find: (a) $\cos^{-1}(-\frac{1}{2})$ (b) $\tan^{-1}(-1)$

SOLUTION:

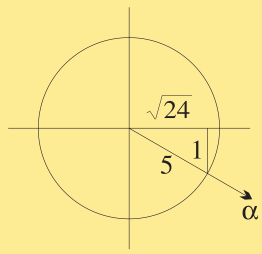
(a) Let $\alpha = \cos^{-1}(-\frac{1}{2})$.
Then $\cos\alpha = -\frac{1}{2}$, where $0 \le \alpha \le \pi$.
Hence α is in the second quadrant,
and the related angle is $\frac{\pi}{3}$,
so $\alpha = \frac{2\pi}{3}$.

(b) Let $\alpha = \tan^{-1}(-1)$.
Then $\tan\alpha = -1$, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.
Hence α is in the fourth quadrant,
and the related angle is $\frac{\pi}{4}$,
so $\alpha = -\frac{\pi}{4}$.

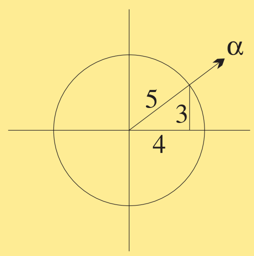
WORKED EXERCISE: Find: (a) $\tan\sin^{-1}(-\frac{1}{5})$ (b) $\sin(2\cos^{-1}\frac{4}{5})$

SOLUTION:

(a) Let $\alpha = \sin^{-1}(-\frac{1}{5})$.
Then $\sin\alpha = -\frac{1}{5}$, where $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.
Hence α is in the fourth quadrant,
and
$$\tan\alpha = \frac{-1}{\sqrt{24}} = -\tfrac{1}{12}\sqrt{6}\,.$$



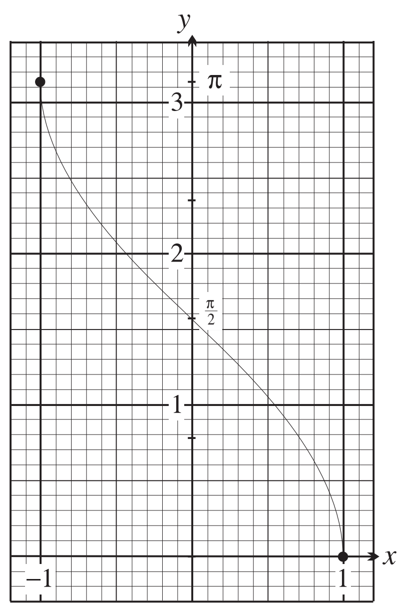
(b) Let $\alpha = \cos^{-1}\frac{4}{5}$.
Then $\cos\alpha = \frac{4}{5}$, where $0 \le \alpha \le \pi$.
Hence α is in the first quadrant,
and
$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \times \tfrac{3}{5} \times \tfrac{4}{5} = \tfrac{24}{25}.$$



Exercise 1B

1. Read off the graph the values of the following correct to two decimal places:

(a) $\cos^{-1} 0{\cdot}4$
(b) $\cos^{-1} 0{\cdot}8$
(c) $\cos^{-1} 0{\cdot}25$
(d) $\cos^{-1}(-0{\cdot}1)$
(e) $\cos^{-1}(-0{\cdot}4)$
(f) $\cos^{-1}(-0{\cdot}75)$



2. Find the exact value of each of the following:

(a) $\sin^{-1} 0$ (e) $\sin^{-1}(-1)$ (i) $\sin^{-1}(-\frac{\sqrt{3}}{2})$
(b) $\sin^{-1}\frac{1}{2}$ (f) $\cos^{-1} 0$ (j) $\cos^{-1}(-\frac{1}{\sqrt{2}})$
(c) $\cos^{-1} 1$ (g) $\tan^{-1} 0$ (k) $\tan^{-1}(-\frac{1}{\sqrt{3}})$
(d) $\tan^{-1} 1$ (h) $\tan^{-1}(-1)$ (l) $\cos^{-1}(-1)$

3. Use your calculator to find, correct to three decimal places, the value of:

(a) $\cos^{-1} 0{\cdot}123$ (c) $\sin^{-1}\frac{2}{3}$ (e) $\tan^{-1} 5$
(b) $\cos^{-1}(-0{\cdot}123)$ (d) $\sin^{-1}(-\frac{2}{3})$ (f) $\tan^{-1}(-5)$

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4. Find the exact value of:
(a) $\sin^{-1}(-\frac{1}{2}) + \cos^{-1}(-\frac{1}{2})$
(b) $\sin(\cos^{-1} 0)$
(c) $\tan(\tan^{-1} 1)$
(d) $\cos^{-1}(\sin \frac{\pi}{3})$
(e) $\sin(\cos^{-1} \frac{1}{2}\sqrt{3})$
(f) $\cos^{-1}(\cos \frac{3\pi}{4})$
(g) $\tan^{-1}(-\tan \frac{\pi}{6})$
(h) $\cos\left(2\tan^{-1}(-1)\right)$
(i) $\tan^{-1}(\sqrt{6}\sin \frac{\pi}{4})$

DEVELOPMENT

5. Find the exact value of:
(a) $\sin^{-1}(\sin \frac{4\pi}{3})$
(b) $\cos^{-1}\left(\cos(-\frac{\pi}{4})\right)$
(c) $\tan^{-1}(\tan \frac{5\pi}{6})$
(d) $\cos^{-1}(\cos \frac{5\pi}{4})$
(e) $\sin^{-1}\left(2\sin(-\frac{\pi}{6})\right)$
(f) $\tan^{-1}(3\tan \frac{7\pi}{6})$

6. (a) In each part use a right-angled triangle within a quadrants diagram to help find the exact value of:
(i) $\sin(\cos^{-1} \frac{3}{5})$
(ii) $\tan(\sin^{-1} \frac{5}{13})$
(iii) $\cos(\sin^{-1} \frac{2}{3})$
(iv) $\sin\left(\cos^{-1}(-\frac{15}{17})\right)$
(v) $\cos\left(\tan^{-1}(-\frac{1}{3})\right)$
(vi) $\tan\left(\cos^{-1}(-\frac{3}{4})\right)$

(b) Use a right-angled triangle in each part to show that:
(i) $\sin(\cos^{-1} x) = \sqrt{1-x^2}$
(ii) $\sin^{-1} x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

7. Use an appropriate compound-angle formula and the techniques of the previous question where necessary to find the exact value of:
(a) $\sin(\sin^{-1} \frac{4}{5} + \sin^{-1} \frac{12}{13})$
(b) $\cos(\tan^{-1} \frac{1}{2} + \sin^{-1} \frac{1}{4})$
(c) $\tan(\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{3}{5})$
(d) $\tan(\sin^{-1} \frac{3}{5} + \cos^{-1} \frac{12}{13})$

8. Use an appropriate double-angle formula to find the exact value of:
(a) $\cos(2\cos^{-1} \frac{1}{3})$
(b) $\sin(2\cos^{-1} \frac{6}{7})$
(c) $\tan\left(2\tan^{-1}(-2)\right)$

9. (a) If $\alpha = \tan^{-1} \frac{1}{2}$ and $\beta = \tan^{-1} \frac{1}{3}$, show that $\tan(\alpha + \beta) = 1$.
(b) Hence find the exact value of $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$.

10. Use a technique similar to that in the previous question to show that:
(a) $\sin^{-1} \frac{1}{\sqrt{5}} + \sin^{-1} \frac{1}{\sqrt{10}} = \frac{\pi}{4}$
(b) $\tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{2}{9}$
(c) $\cos^{-1} \frac{3}{11} - \sin^{-1} \frac{3}{4} = \sin^{-1} \frac{19}{44}$
(d) $\sin^{-1} \frac{1}{3} + \cos^{-1} \frac{1}{3} = \frac{\pi}{2}$

11. (a) If $\theta = \sin^{-1} \frac{3}{5}$, show that $\cos 2\theta = \frac{7}{25}$.
(b) Hence show that $\cos^{-1} \frac{7}{25} = 2\sin^{-1} \frac{3}{5}$.

12. Use techniques similar to that in the previous question to prove that:
(a) $\tan^{-1} \frac{3}{4} = 2\tan^{-1} \frac{1}{3}$
(b) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$, for $0 \le x \le 1$
(c) $2\tan^{-1} 2 = \pi - \cos^{-1} \frac{3}{5}$ [HINT: Use the fact that $\tan(\pi - x) = -\tan x$.]

13. (a) Explain why $\sin^{-1}(\sin 2) \ne 2$. (b) Sketch the curve $y = \sin x$ for $0 \le x \le \pi$, and use symmetry to explain why $\sin 2 = \sin(\pi - 2)$.
(c) What is the exact value of $\sin^{-1}(\sin 2)$?

14. Let x be a positive number and let $\theta = \tan^{-1} x$.
(a) Simplify $\tan(\frac{\pi}{2} - \theta)$.
(b) Show that $\tan^{-1} \frac{1}{x} = \frac{\pi}{2} - \theta$.
(c) Hence show that $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$, for $x > 0$.
(d) Use the fact that $\tan^{-1} x$ is odd to find $\tan^{-1} x + \tan^{-1} \frac{1}{x}$, for $x < 0$.

15. (a) If $\alpha = \tan^{-1} x$ and $\beta = \tan^{-1} 2x$, write down an expression for $\tan(\alpha+\beta)$ in terms of x.
(b) Hence solve the equation $\tan^{-1} x + \tan^{-1} 2x = \tan^{-1} 3$.

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16. Using an approach similar to that in the previous question, solve for x:
(a) $\tan^{-1} x + \tan^{-1} 2 = \tan^{-1} 7$
(b) $\tan^{-1} 3x - \tan^{-1} x = \tan^{-1} \frac{1}{2}$

17. (a) If $\alpha = \sin^{-1} x$, $\beta = \tan^{-1} x$ and $\alpha + \beta = \frac{\pi}{2}$, show that $\cos(\alpha+\beta) = \dfrac{\sqrt{1-x^2} - x^2}{\sqrt{1+x^2}}$.
(b) Hence show that $x^2 = \dfrac{\sqrt{5}-1}{2}$.

18. (a) If $u = \tan^{-1} \frac{1}{3}$ and $v = \tan^{-1} \frac{1}{5}$, show that $\tan(u+v) = \frac{4}{7}$.
(b) Show that $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$.

19. Show that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{2}{5} + \tan^{-1} \frac{8}{9} = \frac{\pi}{2}$.

20. Solve $\tan^{-1} \dfrac{x}{x+1} + \tan^{-1} \dfrac{x}{1-x} = \tan^{-1} \frac{6}{7}$.

EXTENSION

21. Prove by mathematical induction that for all positive integer values of n,
$$\tan^{-1} \frac{1}{2 \times 1^2} + \tan^{-1} \frac{1}{2 \times 2^2} + \cdots + \tan^{-1} \frac{1}{2n^2} = \frac{\pi}{4} - \tan^{-1} \frac{1}{2n+1}.$$

22. Given that $a^2 + b^2 = 1$, prove that the expression $\tan^{-1}\left(\dfrac{ax}{1-bx}\right) - \tan^{-1}\left(\dfrac{x-b}{a}\right)$ is independent of x.

23. (a) Show that $\dfrac{x^2}{x^4+x^2+1} \le \dfrac{1}{3}$, for all real x.
(b) Determine the range of $y = \tan^{-1}\left(\dfrac{1}{1+x^2}\right)$ and the range of $y = \tan^{-1}\left(\dfrac{x^2}{1+x^2}\right)$.
(c) Show that $\tan^{-1}\left(\dfrac{1}{1+x^2}\right) + \tan^{-1}\left(\dfrac{x^2}{1+x^2}\right) = \tan^{-1}\left(1 + \dfrac{x^2}{1+x^2+x^4}\right)$.
(d) Hence determine the range of $y = \tan^{-1}\left(\dfrac{1}{1+x^2}\right) + \tan^{-1}\left(\dfrac{x^2}{1+x^2}\right)$.

1 C Graphs Involving Inverse Trigonometric Functions

This section deals mostly with graphs that can be obtained using transformations of the graphs of the three inverse trigonometric functions. Graphs requiring calculus will be covered in the next section.

Graphs Involving Shifting, Reflecting and Stretching: The usual transformation processes can be applied, but substitution of key values should be used to confirm the graph. In the case of $\tan^{-1} x$, it is wise to take limits so as to confirm the horizontal asymptotes.

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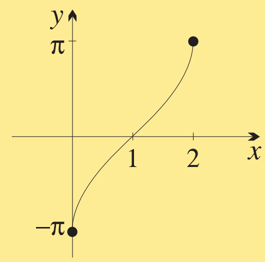
WORKED EXERCISE: Sketch, stating the domain and range:

(a) $y = 2\sin^{-1}(x-1)$ (b) $y = \pi - \tan^{-1} 3x$

SOLUTION:

(a) $y = 2\sin^{-1}(x-1)$ is $y = \sin^{-1} x$ shifted right 1 unit, then stretched vertically by a factor of 2. This should be confirmed by making the following three substitutions:

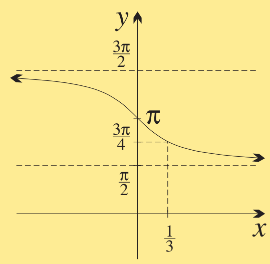
x	0	1	2
y	$-\pi$	0	π



The domain is $0 \le x \le 2$, and the range is $-\pi \le y \le \pi$.

(b) $y = \pi - \tan^{-1} 3x$ is $y = \tan^{-1} x$ stretched horizontally by a factor of $\frac{1}{3}$, then reflected in the y-axis, then shifted upwards by π. This should be confirmed by the following table of values and limits:

x	$\to -\infty$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\to \infty$
y	$\to \frac{3\pi}{2}$	$\frac{5\pi}{4}$	π	$\frac{3\pi}{4}$	$\to \frac{\pi}{2}$



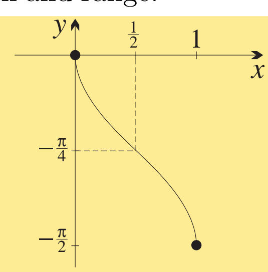
The domain is all real numbers, and the range is $\frac{\pi}{2} < y < \frac{3\pi}{2}$.

More Complicated Transformations: A curve like $y = -\frac{1}{2}\cos^{-1}(1-2x)$ could be obtained by transformations. But the situation is so complicated that the best approach is to construct an appropriate table of values, combined with knowledge of the general shape of the curve.

WORKED EXERCISE: Sketch $y = -\frac{1}{2}\cos^{-1}(1-2x)$, and state its domain and range.

SOLUTION: Using a table of values:

x	0	$\frac{1}{2}$	1
y	0	$-\frac{\pi}{4}$	$-\frac{\pi}{2}$



The domain is $0 \le x \le 1$ and the range is $-\frac{\pi}{2} \le y \le 0$.

Symmetries of the Inverse Trigonometric Functions: The two functions $y = \sin^{-1} x$ and $y = \tan^{-1} x$ are both odd, but $y = \cos^{-1} x$ has odd symmetry about its y-intercept $(0, \frac{\pi}{2})$.

9 SYMMETRIES OF THE INVERSE TRIGONOMETRIC FUNCTIONS:

- $y = \sin^{-1} x$ is odd, that is, $\sin^{-1}(-x) = -\sin^{-1} x$.
- $y = \tan^{-1} x$ is odd, that is, $\tan^{-1}(-x) = -\tan^{-1} x$.
- $y = \cos^{-1} x$ has odd symmetry about its y-intercept $(0, \frac{\pi}{2})$, that is,

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

Only the last identity needs proof.

PROOF: Let $\alpha = \cos^{-1}(-x)$.

Then $-x = \cos\alpha$, where $0 \le \alpha \le \pi$,

so $\cos(\pi - \alpha) = x$, since $\cos(\pi - \alpha) = -\cos\alpha$,

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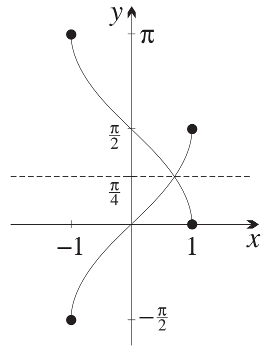
$$\pi - \alpha = \cos^{-1} x, \text{ since } 0 \le \pi - \alpha \le \pi,$$
$$\alpha = \pi - \cos^{-1} x, \text{ as required.}$$

The Identity $\sin^{-1} x + \cos^{-1} x = \pi/2$: The graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ are reflections of each other in the horizontal line $y = \frac{\pi}{4}$. Hence adding the graphs pointwise, it should be clear that



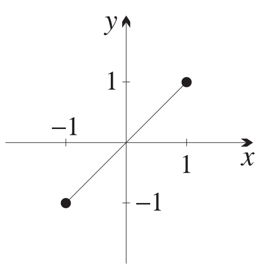
10 COMPLEMENTARY ANGLES: $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

This is really only another form of the complementary angle identity $\cos(\frac{\pi}{2} - \theta) = \sin\theta$ — here is an algebraic proof which makes this relationship clear.

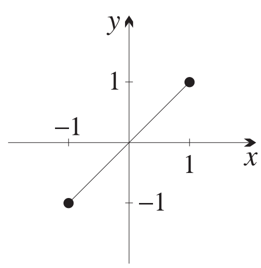
PROOF: Let $\alpha = \cos^{-1} x$.
Then $x = \cos\alpha$, where $0 \le \alpha \le \pi$,

$$\sin(\tfrac{\pi}{2} - \alpha) = x, \text{ since } \sin(\tfrac{\pi}{2} - \alpha) = \cos\alpha,$$
$$\sin^{-1} x = \tfrac{\pi}{2} - \alpha, \text{ since } -\tfrac{\pi}{2} \le \tfrac{\pi}{2} - \alpha \le \tfrac{\pi}{2},$$
$$\sin^{-1} x + \alpha = \tfrac{\pi}{2}, \text{ as required.}$$

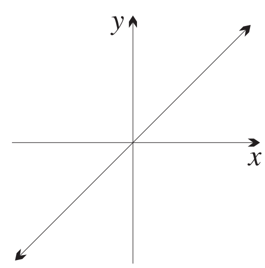
The Graphs of sin $\sin^{-1} x$, cos $\cos^{-1} x$ and tan $\tan^{-1} x$: The composite function defined by $y = \sin\sin^{-1} x$ has the same domain as $\sin^{-1} x$, that is, $-1 \le x \le 1$. Since it is the function $y = \sin^{-1} x$ followed by the function $y = \sin x$, the composite is therefore the identity function $y = x$ restricted to $-1 \le x \le 1$.



$y = \sin\sin^{-1} x$



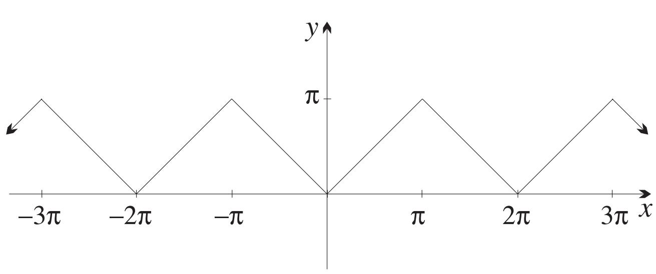
$y = \cos\cos^{-1} x$



$y = \tan\tan^{-1} x$

The same remarks apply to $y = \cos\cos^{-1} x$ and $y = \tan\tan^{-1} x$, except that the domain of $y = \tan\tan^{-1} x$ is the whole real number line.

The Graph of $\cos^{-1}$ cos x: The domain of this function is the whole real number line, and the graph is far more complicated. Constructing a simple table of values is probably the surest approach, but under the graph is an argument based on symmetries.



A. For $0 \le x \le \pi$, $\cos^{-1}\cos x = x$, and the graph follows $y = x$.

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B. Since $\cos x$ is an even function, the graph in the interval $-\pi \le x \le 0$ is the reflection of the graph in the interval $0 \le x \le \pi$.

C. We now have the shape of the graph in the interval $-\pi \le x \le \pi$. Since the graph has period 2π, the rest of the graph is just a repetition of this section.

The exercises deal with the other confusing functions $\sin^{-1} \sin x$ and $\tan^{-1} \tan x$, and also with functions like $y = \sin^{-1} \cos x$.

Exercise 1C

1. Sketch each function, stating the domain and range and whether it is even, odd or neither:
(a) $y = \tan^{-1} x$ (b) $y = \cos^{-1} x$ (c) $y = \sin^{-1} x$

2. Sketch each function, using appropriate translations of $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$. State the domain and range, and whether it is even, odd or neither.
(a) $y = \sin^{-1}(x - 1)$ (b) $y = \cos^{-1}(x + 1)$ (c) $y - \frac{\pi}{2} = \tan^{-1} x$

3. Sketch each function by stretching $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$ horizontally or vertically as appropriate. State the domain and range, and whether it is even, odd or neither.
(a) $y = 2\sin^{-1} x$ (b) $y = \cos^{-1} 2x$ (c) $y = \frac{1}{2}\tan^{-1} x$

4. Sketch each function by reflecting in the x- or y-axis as appropriate. State the domain and range, and whether it is even, odd or neither.
(a) $y = -\cos^{-1} x$ (b) $y = \tan^{-1}(-x)$ (c) $y = -\sin^{-1}(-x)$

5. Sketch each function , stating the domain and range, and whether it is even, odd or neither:
(a) $y = 3\sin^{-1} 2x$
(b) $y = \frac{1}{2}\cos^{-1} 3x$
(c) $y = \tan^{-1}(x - 1) - \frac{\pi}{2}$
(d) $3y = 2\sin^{-1} \frac{x}{2}$
(e) $\frac{1}{2}y = 2\cos^{-1}(x - 2)$
(f) $y = \frac{1}{4}\cos^{-1}(-x)$

6. (a) Consider the function $y = 4\sin^{-1}(2x + 1)$.
(i) Solve $-1 \le 2x + 1 \le 1$ to find the domain.
(ii) Solve $-\frac{\pi}{2} \le \frac{y}{4} \le \frac{\pi}{2}$ to find the range.
(iii) Hence sketch the graph of the function.
(b) Use similar steps to find the domain and range of each function, and hence sketch it:
(i) $y = 3\cos^{-1}(2x - 1)$ (ii) $y = \frac{1}{2}\sin^{-1}(3x + 2)$ (iii) $y = 2\tan^{-1}(4x - 1)$

DEVELOPMENT

7. (a) (i) Sketch the graphs of $y = \cos^{-1} x$ and $y = \sin^{-1} x - \frac{\pi}{2}$ on the same set of axes.
(ii) Hence show graphically that $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$.
(b) Use a graphical approach to show that:
(i) $\tan^{-1}(-x) = -\tan^{-1} x$ (ii) $\cos^{-1} x + \cos^{-1}(-x) = \pi$

8. (a) Determine the domain and range of $y = \sin^{-1}(1 - x)$.
(b) Complete the table to the right, and hence sketch the graph of the function.
(c) About which line are the graphs of $y = \sin^{-1}(1-x)$ and $y = \sin^{-1} x$ symmetrical?

x	0	1	2
y			

9. Find the domain and range, draw up a table of values if necessary, and then sketch:
(a) $y = 2\cos^{-1}(1 - x)$ (b) $y = \tan^{-1}(\sqrt{3} - x)$ (c) $y = \frac{1}{3}\sin^{-1}(2 - 3x)$

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10. Sketch the graph of each function using the methods of this section:
(a) $-y = \sin^{-1}(x+1)$ (b) $y = -\tan^{-1}(1-x)$ (c) $y + \frac{\pi}{2} = \frac{1}{2}\cos^{-1}(-x)$

11. Sketch these graphs, stating whether each function is even, odd or neither:
(a) $y = \sin(\sin^{-1} 2x)$ (b) $y = \cos(\cos^{-1} \frac{x}{2})$ (c) $y = \tan\left(\tan^{-1}(x-1)\right)$

12. Consider the function $f(x) = \sin 2x$.
(a) Sketch the graph of $f(x)$, for $-\pi \le x \le \pi$.
(b) What is the largest domain containing $x = 0$ for which $f(x)$ has an inverse function?
(c) Sketch the graph of $f^{-1}(x)$ by reflecting in the line $y = x$.
(d) Find the equation of $f^{-1}(x)$, and state its symmetry.

13. (a) What is the domain of $y = \sin\cos^{-1} x$? Is it even, odd or neither?
(b) By considering the range of $\cos^{-1} x$, explain why $\sin\cos^{-1} x \ge 0$, for all x in its domain.
(c) By squaring both sides of $y = \sin\cos^{-1} x$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, show that $y = \sqrt{1-x^2}$.
(d) Hence sketch $y = \sin\cos^{-1} x$.
(e) Use similar methods to sketch the graph of $y = \cos\sin^{-1} x$.

14. Consider the function $y = \tan^{-1}\tan x$.
(a) State its domain and range, and whether it is even, odd or neither.
(b) Simplify $\tan^{-1}\tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(c) What is the period of the function?
(d) Use the above information and a table of values if necessary to sketch the function.

15. In a worked exercise, $y = \cos^{-1}\cos x$ is sketched. Use $\sin^{-1} t = \frac{\pi}{2} - \cos^{-1} t$ and simple transformations to sketch $y = \sin^{-1}\cos x$. State its symmetry.

EXTENSION

16. Consider the function $y = \sin^{-1}\sin x$.
(a) State its domain, range and period, and whether it is even, odd or neither.
(b) Simplify $\sin^{-1}\sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, and sketch the function in this region.
(c) Use the symmetry of $\sin x$ in $x = \frac{\pi}{2}$ to continue the sketch for $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.
(d) Use the above information and a table of values if necessary to sketch the function.
(e) Hence sketch $y = \cos^{-1}\sin x$ by making use of the fact that $\cos^{-1} t = \frac{\pi}{2} - \sin^{-1} t$.

17. (a) Sketch $f(x) = \cos x$, for $0 \le x \le 2\pi$.
(b) What is the largest domain containing $x = \frac{3\pi}{2}$ for which $f(x)$ has an inverse function?
(c) Sketch the graph of $f^{-1}(x)$ by reflection in $y = x$.
(d) Show that $\cos(2\pi - x) = \cos x$, and that if $\pi \le x \le 2\pi$, then $0 \le 2\pi - x \le \pi$.
(e) Hence find the equation of $f^{-1}(x)$.

18. One way (and a rather bizarre way!) to define the function $y = \sec^{-1} x$ is as the inverse of the restriction of $y = \sec x$ to the domain $0 \le x < \frac{\pi}{2}$ or $\pi \le x < \frac{3\pi}{2}$.
(a) Sketch the graph of the function $y = \sec^{-1} x$ as defined above.
(b) Find the value of: (i) $\sec^{-1} 2$ (ii) $\sec^{-1}(-2)$
(c) Show that $\tan(\sec^{-1} x) = \sqrt{x^2 - 1}$.

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1 D Differentiation

Having formed the three inverse trigonometric functions, we can now apply the normal processes of calculus to them. This section is concerned with their derivatives and its usual applications to curve-sketching and maximisation.

Differentiating $\sin^{-1} x$ and $\cos^{-1} x$: The functions $y = \sin^{-1} x$ and $y = \cos^{-1} x$ can be differentiated by changing to the inverse function and using the known derivatives of the sine and cosine functions — this same procedure was used in Section 13B of the Year 11 volume when the derivative of $y = e^x$ was found by changing to the inverse function $x = \log y$. In this case, however, we need to keep track of the restrictions to the domain, which are needed later in the working to make a significant choice between positive and negative square roots.

A. Let $y = \sin^{-1} x$.
Then $x = \sin y$, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$,
so $\dfrac{dx}{dy} = \cos y$.
Since y is in the first or fourth quadrant, $\cos y$ is positive,
so $\cos y = +\sqrt{1 - \sin^2 y}$
$= \sqrt{1 - x^2}$.
Thus $\dfrac{dx}{dy} = \sqrt{1 - x^2}$
and $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$.
Hence $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1 - x^2}}$.

B. Let $y = \cos^{-1} x$.
Then $x = \cos y$, where $0 \le y \le \pi$,
so $\dfrac{dx}{dy} = -\sin y$.
Since y is in the first or second quadrant, $\sin y$ is positive,
so $\sin y = +\sqrt{1 - \cos^2 y}$
$= \sqrt{1 - x^2}$.
Thus $\dfrac{dx}{dy} = -\sqrt{1 - x^2}$
and $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1 - x^2}}$.
Hence $\dfrac{d}{dx} \cos^{-1} x = -\dfrac{1}{\sqrt{1 - x^2}}$.

Differentiating $\tan^{-1} x$: The problem of which square root to choose does not arise when differentiating $y = \tan^{-1} x$.

Let $y = \tan^{-1} x$.
Then $x = \tan y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$,
so $\dfrac{dx}{dy} = \sec^2 y$
$= 1 + \tan^2 y$.
Hence $\dfrac{dx}{dy} = 1 + x^2$
and $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$, giving the standard form $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1 + x^2}$.

11 STANDARD FORMS FOR DIFFERENTIATION:

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$$

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WORKED EXERCISE: Differentiate the functions:

(a) $y = x\tan^{-1}x$ (b) $y = \sin^{-1}(ax+b)$

SOLUTION:

(a) $y = x\tan^{-1}x$

$y' = vu' + uv'$

$= \tan^{-1}x \times 1 + x \times \dfrac{1}{1+x^2}$

$= \tan^{-1}x + \dfrac{x}{1+x^2}$

Let $u = x$
and $v = \tan^{-1}x$.
Then $u' = 1$
and $v' = \dfrac{1}{1+x^2}$.

(b) $y = \sin^{-1}(ax+b)$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= \dfrac{1}{\sqrt{1-(ax+b)^2}} \times a$

$= \dfrac{a}{\sqrt{1-(ax+b)^2}}$

Let $u = ax+b$,
then $y = \sin^{-1}u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \dfrac{1}{\sqrt{1-u^2}}$.

Linear Extensions: The method used in part (b) above can be applied to all three inverse trigonometric functions, giving a further set of standard forms.

FURTHER STANDARD FORMS FOR DIFFERENTIATION:

12

$$\frac{d}{dx}\sin^{-1}(ax+b) = \frac{a}{\sqrt{1-(ax+b)^2}}$$

$$\frac{d}{dx}\cos^{-1}(ax+b) = -\frac{a}{\sqrt{1-(ax+b)^2}}$$

$$\frac{d}{dx}\tan^{-1}(ax+b) = \frac{a}{1+(ax+b)^2}$$

WORKED EXERCISE:

(a) Find the points A and B on the curve $y = \cos^{-1}(x-1)$ where the tangent has gradient -2.

(b) Sketch the curve, showing these points.

SOLUTION:

(a) Differentiating, $y' = -\dfrac{1}{\sqrt{1-(x-1)^2}}$.

Put $y' = -2$.

Then $-\dfrac{1}{\sqrt{1-(x-1)^2}} = -2$

$1-(x-1)^2 = \frac{1}{4}$

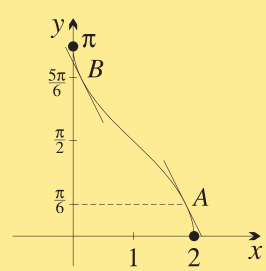
$(x-1)^2 = \frac{3}{4}$

$x - 1 = \frac{1}{2}\sqrt{3}$ or $-\frac{1}{2}\sqrt{3}$

$x = 1 + \frac{1}{2}\sqrt{3}$ or $1 - \frac{1}{2}\sqrt{3}$,

so the points are $A(1+\frac{1}{2}\sqrt{3}, \frac{\pi}{6})$ and $B(1-\frac{1}{2}\sqrt{3}, \frac{5\pi}{6})$.

(b)



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Functions whose Derivatives are Zero are Constants: Several identities involving inverse trigonometric functions can be obtained by showing that some derivative is zero, and hence that the original function must be a constant. The following identity is the clearest example — it has been proven already in Section 1C using symmetry arguments.

WORKED EXERCISE:

(a) Differentiate $\sin^{-1} x + \cos^{-1} x$.

(b) Hence prove the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

SOLUTION:

(a) $$\begin{aligned}\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) &= \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}\\ &= 0\end{aligned}$$

(b) Hence $\sin^{-1} x + \cos^{-1} x = C$, for some constant C.
Substitute $x = 0$, then $0 + \frac{\pi}{2} = C$,
so $C = \frac{\pi}{2}$, and $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, as required.

Curve Sketching Using Calculus: The usual methods of curve sketching can now be extended to curves whose equations involve the inverse trigonometric functions. The following worked example applies calculus to sketching the curve $y = \cos^{-1} \cos x$, which was sketched without calculus in the previous section.

WORKED EXERCISE: Use calculus to sketch $y = \cos^{-1} \cos x$.

SOLUTION: The function is periodic with the same period as $\cos x$, that is, 2π.
A simple table of values gives some key points:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	...
y	0	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	...

The shape of the curve joining these points can be obtained by calculus.

Differentiating using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin x}{\sqrt{1-\cos^2 x}}\\ &= \frac{\sin x}{\sqrt{\sin^2 x}}\,.\end{aligned}$$

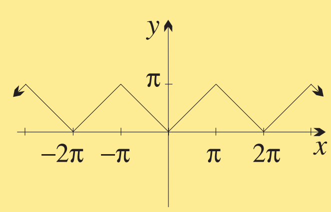
Let $u = \cos x$,
then $y = \cos^{-1} u$.
Hence $\dfrac{du}{dx} = -\sin x$
and $\dfrac{dy}{du} = -\dfrac{1}{\sqrt{1-u^2}}$.

When $\sin x$ is positive, $\sqrt{\sin^2 x} = \sin x$,
so $\dfrac{dy}{dx} = 1$.

When $\sin x$ is negative, $\sqrt{\sin^2 x} = -\sin x$,
so $\dfrac{dy}{dx} = -1$.

Hence $\dfrac{dy}{dx} = \begin{cases} 1, & \text{for } x \text{ in quadrants 1 and 2,}\\ -1, & \text{for } x \text{ in quadrants 3 and 4.}\end{cases}$

This means that the graph consists of a series of intervals, each with gradient 1 or -1.



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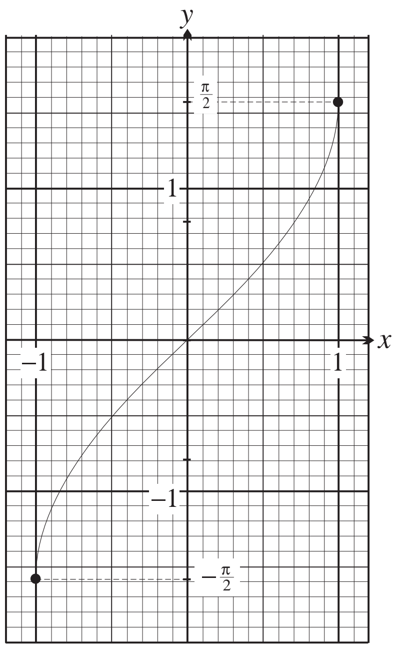
Exercise 1D

1. (a) Photocopy the graph of $y = \sin^{-1} x$ shown to the right. Then carefully draw a tangent at each x value in the table. Then, by measurement and calculation of rise/run, find the gradient of each tangent to two decimal places and fill in the second row of the table.

x	-1	$-0{\cdot}7$	$-0{\cdot}5$	$-0{\cdot}2$	0	$0{\cdot}3$	$0{\cdot}6$	$0{\cdot}8$	1
$\dfrac{dy}{dx}$									



(b) Check your gradients using $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$.

2. Differentiate with respect to x:

(a) $\cos^{-1} x$
(b) $\tan^{-1} x$
(c) $\sin^{-1} 2x$
(d) $\tan^{-1} 3x$
(e) $\cos^{-1} 5x$
(f) $\sin^{-1}(-x)$
(g) $\sin^{-1} x^2$
(h) $\tan^{-1} x^3$
(i) $\tan^{-1}(x+2)$
(j) $\cos^{-1}(1-x)$
(k) $x \sin^{-1} x$
(l) $(1+x^2)\tan^{-1} x$
(m) $\sin^{-1} \frac{1}{5}x$
(n) $\tan^{-1} \frac{1}{4}x$
(o) $\cos^{-1} \sqrt{x}$
(p) $\tan^{-1} \sqrt{x}$
(q) $\tan^{-1} \dfrac{1}{x}$

3. Find the gradient of the tangent to each curve at the point indicated:

(a) $y = 2\tan^{-1} x$, at $x = 0$
(b) $y = \sqrt{3}\sin^{-1} x$, at $x = \frac{1}{2}$
(c) $y = \tan^{-1} 2x$, at $x = -\frac{1}{2}$
(d) $y = \cos^{-1} \frac{x}{2}$, at $x = \sqrt{3}$

4. Find, in the form $y = mx + b$, the equation of the tangent and the normal to each curve at the point indicated:

(a) $y = 2\cos^{-1} 3x$, at $x = 0$
(b) $y = \sin^{-1} \frac{x}{2}$, at $x = \sqrt{2}$

5. (a) Show that $\dfrac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$.

(b) Hence explain why $\sin^{-1} x + \cos^{-1} x$ must be a constant function, and use any convenient value of x in its domain to find the value of the constant.

6. Use the method of the previous question to show that each of these functions is a constant function, and find the value of the constant.

(a) $\cos^{-1} x + \cos^{-1}(-x)$
(b) $2\sin^{-1} \sqrt{x} - \sin^{-1}(2x-1)$

DEVELOPMENT

7. (a) If $f(x) = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)$, show that $f''(x) = \dfrac{1}{1+x^2}$.

(b) Is the graph of $y = f(x)$ concave up or concave down at $x = -1$?

8. Show that the gradient of the curve $y = \dfrac{\sin^{-1} x}{x}$ at the point where $x = \frac{1}{2}$ is $\frac{2}{3}(2\sqrt{3} - \pi)$.

9. Find the derivative of each function in simplest form:

(a) $x\cos^{-1} x - \sqrt{1-x^2}$
(b) $\sin^{-1} e^{3x}$
(c) $\sin^{-1} \frac{1}{4}(2x-3)$
(d) $\tan^{-1} \frac{1}{1-x}$
(e) $\sin^{-1} e^{x}$
(f) $\log \sqrt{\sin^{-1} x}$
(g) $\sin^{-1} \sqrt{\log x}$
(h) $\sqrt{x}\sin^{-1} \sqrt{1-x}$
(i) $\tan^{-1} \frac{x+2}{1-2x}$

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10. (a) (i) If $y = (\sin^{-1} x)^2$, show that $y'' = \dfrac{2 + \frac{2x\sin^{-1} x}{\sqrt{1-x^2}}}{1 - x^2}$.

(ii) Hence show that $(1 - x^2)y'' - xy' - 2 = 0$.

(b) Show that $y = e^{\sin^{-1} x}$ satisfies the differential equation $(1 - x^2)y'' - xy' - y = 0$.

11. Consider the function $f(x) = \cos^{-1} x^2$.

(a) What is the domain of $f(x)$?

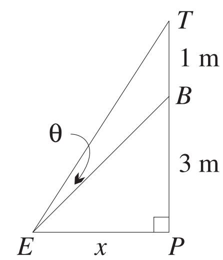
(b) About which line is the graph of $y = f(x)$ symmetrical?

(c) Find $f'(x)$. (d) Show that $y = f(x)$ has a maximum turning point at $x = 0$.

(e) Show that $f'(x)$ is undefined at the endpoints of the domain. What is the geometrical significance of this?

(f) Sketch the graph of $y = f(x)$.

12. A picture 1 metre tall is hung on a wall with its bottom edge 3 metres above the eye E of a viewer. Let the distance EP be x metres, and let θ be the angle that the picture subtends at E.

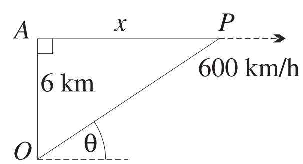


(a) Show that $\theta = \tan^{-1} \frac{4}{x} - \tan^{-1} \frac{3}{x}$.

(b) Show that θ is maximised when the viewer is $2\sqrt{3}$ metres from the wall.

(c) Show that the maximum angle subtended by the picture at E is $\tan^{-1} \frac{\sqrt{3}}{12}$.

13. A plane P at an altitude of 6 km and at a constant speed of 600 km/h is flying directly away from an observer at O on the ground. The point A on the path of the plane lies directly above O. Let the distance AP be x km, and let the angle of elevation of the plane from the observer be θ.



(a) Show that $\theta = \tan^{-1} \frac{6}{x}$.

(b) Show that $\dfrac{d\theta}{dt} = \dfrac{-3600}{x^2 + 36}$ radians per hour.

(c) Hence find, in radians per second, the rate at which θ is decreasing at the instant when the distance AP is 3 km.

14. (a) State the domain of $f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x}$, and its symmetry.

(b) Show that $f'(x) = 0$ for all values of x in the domain.

(c) Show that $f(x) = \begin{cases} \frac{\pi}{2}, & \text{for } x > 0, \\ -\frac{\pi}{2}, & \text{for } x < 0, \end{cases}$ and hence sketch the graph of $f(x)$.

15. Find $\dfrac{dy}{dx}$ in terms of t, given that:

(a) $x = \sin^{-1} \sqrt{t}$ and $y = \sqrt{1 - t}$ (b) $x = \ln(1 + t^2)$ and $y = t - \tan^{-1} t$

16. Consider the function $f(x) = \cos^{-1} \frac{1}{x}$.

(a) State the domain of $f(x)$. [HINT: Think about it rather than relying on algebra.]

(b) Recalling that $\sqrt{x^2} = |x|$, show that $f'(x) = \dfrac{1}{|x|\sqrt{x^2 - 1}}$.

(c) Comment on $f'(1)$ and $f'(-1)$.

(d) Use the expression for $f'(x)$ in part (b) to write down separate expressions for $f'(x)$ when $x > 1$ and when $x < -1$.

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(e) Explain why $f(x)$ is increasing for $x > 1$ and for $x < -1$.

(f) Find: (i) $\lim_{x\to\infty} f(x)$ (ii) $\lim_{x\to-\infty} f(x)$ (g) Sketch the graph of $y = f(x)$.

17. The function $f(x)$ is defined by the rule $f(x) = \sin^{-1} \sin x$.

(a) State the domain and range of $f(x)$, and whether it is even, odd or neither.

(b) Show that $f'(x) = \dfrac{\cos x}{|\cos x|}$. (c) Is $f'(x)$ defined whenever $\cos x = 0$?

(d) What are the only two values that $f'(x)$ takes if $\cos x \neq 0$, and when does each of these values occur?

(e) Sketch the graph of $f(x)$ using the above information and a table of values if necessary. [NOTE: This function was sketched in the previous exercise using a different approach. Look back and compare.]

EXTENSION

18. (a) What is the domain of $g(x) = \sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$?

(b) Show that $g'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{x}{|x|\sqrt{1-x^2}}$.

(c) Hence determine the interval over which $g(x)$ is constant, and find this constant.

19. In question 9(i), you proved that $\dfrac{d}{dx} \tan^{-1} \dfrac{x+2}{1-2x}$ was $\dfrac{1}{1+x^2}$, which is also the derivative of $\tan^{-1} x$. What is going on?

20. Find $\dfrac{dy}{dx}$ by differentiating implicitly:

(a) $\sin^{-1}(x + y) = 1$ (b) $\cos^{-1} xy = x^2$ (c) $\tan^{-1} \frac{y}{x} = \log \sqrt{x^2 + y^2}$

21. [The inverse cosecant function] The most straightforward way to define $\operatorname{cosec}^{-1} x$ is as the inverse function of the restriction of $y = \operatorname{cosec} x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, excluding $x = 0$.

(a) Graph $y = \operatorname{cosec}^{-1} x$, and state its domain, range and symmetry.

(b) Show that $\dfrac{d}{dx} \operatorname{cosec}^{-1} x = \dfrac{-1}{x\sqrt{x^2-1}}$ (except at endpoints).

(c) Show that $\operatorname{cosec}^{-1} x = \sin \frac{1}{x}$, for $x \ge 1$ or $x \le -1$.

22. [The inverse cotangent function] The function $y = \cot^{-1} x$ can be defined as the inverse function of the restriction of $y = \cot x$:

(i) to $0 < x < \pi$, or (ii) to $-\frac{\pi}{2} < x \le \frac{\pi}{2}$, excluding $x = 0$.

(a) Graph both functions, and state their domains, ranges and symmetries.

(b) Show that in both cases, $\dfrac{d}{dx} \cot^{-1} x = \dfrac{-1}{1+x^2}$ (except at endpoints).

(c) Is it true that in both cases $\cot^{-1} x = \tan^{-1} \frac{1}{x}$, for $x \neq 0$?

(d) What are the advantages of each definition?

23. [The inverse secant function] In the previous exercise, the function $y = \sec x$ was restricted to the domain $0 \le x < \frac{\pi}{2}$ or $\pi \le x < \frac{3\pi}{2}$, to produce an inverse function, $y = \sec^{-1} x$.

(a) Starting with $\sec y = x$, show that $\dfrac{dy}{dx} = \dfrac{1}{\sec y \tan y}$.

(b) Hence show that $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{x\sqrt{x^2-1}}$. (c) Find $\dfrac{d}{dx}\left(\sec^{-1}\sqrt{x^2-1}\right)$.

(d) The more straightforward definition of $\sec^{-1} x$ restricts $\sec x$ to $0 \le x \le \pi$, excluding $x = \frac{\pi}{2}$. Graph this version of $\sec^{-1} x$, and state its domain, range and symmetry.

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1 E Integration

This section deals with the integrals associated with the inverse trigonometric functions, and with the standard applications of those integrals to areas, volumes and the calculation of functions whose derivatives are known.

The Basic Standard Forms: Differentiation of the two inverse trigonometric functions $\sin^{-1} x$ and $\tan^{-1} x$ yields the purely algebraic functions $\dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{1}{1+x^2}$. This is a remarkable result, and is a sure sign that trigonometric functions are very closely related to algebraic functions associated with squares and square roots — a fact that was already clear when the trigonometric functions were defined using the circle, whose equation $x^2 + y^2 = r^2$ is purely algebraic. This section concerns integration, and we begin by reversing the previous standard forms for differentiation:

13 STANDARD FORMS:
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \quad \text{or} \quad -\cos^{-1} x + C$$
$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

Thus the inverse trigonometric functions are required for the integration of purely algebraic functions. These standard forms should be compared with the standard form $\displaystyle\int \frac{1}{x}\,dx = \log x$, where the logarithmic function was required for the integration of the algebraically defined function $y = 1/x$.

The Functions $y = \dfrac{1}{\sqrt{1-x^2}}$ and $y = \dfrac{1}{1+x^2}$: The primitives of both these functions have now been obtained, and they should therefore be regarded as reasonably standard functions whose graphs should be known. The sketch of each function and some important definite integrals associated with them are developed in questions 18 and 19 in the following exercise.

WORKED EXERCISE: Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ using both standard forms.

SOLUTION:

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx &= \Big[\sin^{-1} x\Big]_0^{\frac{1}{2}} \\ &= \sin^{-1}\tfrac{1}{2} - \sin^{-1} 0 \\ &= \tfrac{\pi}{6} - 0 \\ &= \tfrac{\pi}{6}\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx &= \Big[-\cos^{-1} x\Big]_0^{\frac{1}{2}} \\ &= -\cos^{-1}\tfrac{1}{2} + \cos^{-1} 0 \\ &= -\tfrac{\pi}{3} + \tfrac{\pi}{2} \\ &= \tfrac{\pi}{6}\end{aligned}$$

WORKED EXERCISE: Evaluate exactly or correct to four significant figures:

(a) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$ (b) $\displaystyle\int_0^4 \frac{1}{1+x^2}\,dx$

SOLUTION:

(a) $$\begin{aligned}\int_0^1 \frac{1}{1+x^2}\,dx &= \Big[\tan^{-1} x\Big]_0^1 \\ &= \tan^{-1} 1 - \tan^{-1} 0 \\ &= \tfrac{\pi}{4}\end{aligned}$$

(b) $$\begin{aligned}\int_0^4 \frac{1}{1+x^2}\,dx &= \Big[\tan^{-1} x\Big]_0^4 \\ &= \tan^{-1} 4 - \tan^{-1} 0 \\ &\doteqdot 1{\cdot}329\end{aligned}$$

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More General Standard Forms: When constants are involved, the calculation of the primitive becomes fiddly. The standard integrals given in the HSC papers are:

14 STANDARD FORMS WITH ONE CONSTANT: For some constant C,

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \quad \text{or} \quad -\cos^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

PROOF:

A.
$$\begin{aligned}\int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \int \frac{1}{a\sqrt{1-(\frac{x}{a})^2}}\,dx\\ &= \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}} \times \frac{1}{a}\,dx\\ &= \sin^{-1}\frac{x}{a} + C\end{aligned}$$

Let $u = \dfrac{x}{a}$.

Then $\dfrac{du}{dx} = \dfrac{1}{a}$.

$$\int \frac{1}{\sqrt{1-u^2}}\,\frac{du}{dx}\,dx = \sin^{-1} u$$

B.
$$\begin{aligned}\int \frac{1}{a^2+x^2}\,dx &= \int \frac{1}{a^2(1+(\frac{x}{a})^2)}\,dx\\ &= \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2} \times \frac{1}{a}\,dx\\ &= \frac{1}{a}\tan^{-1}\frac{x}{a} + C\end{aligned}$$

Let $u = \dfrac{x}{a}$.

Then $\dfrac{du}{dx} = \dfrac{1}{a}$.

$$\int \frac{1}{1+u^2}\,\frac{du}{dx}\,dx = \tan^{-1} u$$

WORKED EXERCISE: Here are four indefinite integrals. In parts (a) and (b), the formulae can be applied immediately, but in parts (c) and (d), the coefficients of x^2 need to be taken out first.

(a) $$\int \frac{2}{\sqrt{9-x^2}}\,dx = 2\sin^{-1}\frac{x}{3} + C$$

(b) $$\int \frac{1}{8+x^2}\,dx = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x}{2\sqrt{2}} + C$$

(c) $$\begin{aligned}&\int \frac{6}{49+25x^2}\,dx\\ &= \frac{6}{25}\int \frac{1}{\frac{49}{25}+x^2}\,dx\\ &= \frac{6}{25} \times \frac{5}{7}\tan^{-1}\frac{x}{7/5} + C\\ &= \frac{6}{35}\tan^{-1}\frac{5x}{7} + C\end{aligned}$$

(d) $$\begin{aligned}&\int \frac{1}{\sqrt{5-3x^2}}\,dx\\ &= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\frac{5}{3}-x^2}}\,dx\\ &= \tfrac{1}{3}\sqrt{3}\,\sin^{-1} x\sqrt{\tfrac{3}{5}} + C\end{aligned}$$

Because manipulating the constants in parts (c) and (d) is still difficult, some prefer to remember these fuller versions of the standard forms:

15 STANDARD FORMS WITH TWO CONSTANTS:

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx = \frac{1}{b}\sin^{-1}\frac{bx}{a} + C \quad \text{or} \quad -\frac{1}{b}\cos^{-1}\frac{bx}{a} + C$$

$$\int \frac{1}{a^2+b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\frac{bx}{a} + C$$

These forms can be proven in the same manner as the forms with a single constant, or they can be developed from those forms in the same way as was done in parts (c) and (d) above (and they are proven by differentiation in the following exercise). With these more general forms, parts (c) and (d) can be written down without any intermediate working.

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Reverse Chain Rule: In the usual way, the standard forms can be extended to give forms appropriate for the reverse chain rule.

16 THE REVERSE CHAIN RULE:

$$\int \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\,dx = \sin^{-1}u + C \quad \text{or} \quad -\cos^{-1}u + C$$

$$\int \frac{1}{1+u^2}\frac{du}{dx}\,dx = \tan^{-1}u + C$$

WORKED EXERCISE: Find a primitive of $\dfrac{x}{1+x^4}$.

SOLUTION:

$$\int \frac{x}{1+x^4}\,dx$$
$$= \tfrac{1}{2}\int \frac{2x}{1+x^4}\,dx$$
$$= \tfrac{1}{2}\tan^{-1}x^2 + C, \text{ for some constant } C.$$

Let $u = x^2$.
Then $u' = 2x$.
$$\int \frac{1}{1+u^2}\frac{du}{dx}\,dx = \tan^{-1}u$$

Given a Derivative, Find an Integral: As always, the result of a product-rule differentiation can be used to obtain an integral. In particular, this allows the primitives of the inverse trigonometric functions to be obtained.

WORKED EXERCISE:

(a) Differentiate $x\sin^{-1}x$, and hence find a primitive of $\sin^{-1}x$.

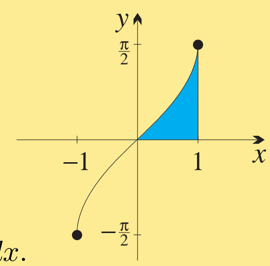
(b) Find the shaded area under the curve $y = \sin^{-1}x$ from $x = 0$ to $x = 1$.

SOLUTION:

(a) Let $y = x\sin^{-1}x$.

Using the product rule with $u = x$ and $v = \sin^{-1}x$,

$$\frac{dy}{dx} = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}.$$



Hence $\displaystyle\int \sin^{-1}x\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x$

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx.$$

Using the reverse chain rule,

$$-\int \frac{x}{\sqrt{1-x^2}}\,dx = \tfrac{1}{2}\int (1-x^2)^{-\frac{1}{2}}(-2x)\,dx$$
$$= \tfrac{1}{2}\times(1-x^2)^{\frac{1}{2}}\times\tfrac{2}{1}$$
$$= \sqrt{1-x^2},$$

Let $u = 1 - x^2$.
Then $\dfrac{du}{dx} = -2x$.
$$\int u^{-\frac{1}{2}}\frac{du}{dx}\,dx = u^{\frac{1}{2}}\times 2.$$

so $\displaystyle\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C,$

and $\displaystyle\int_0^1 \sin^{-1}x\,dx = \Big[x\sin^{-1}x + \sqrt{1-x^2}\Big]_0^1$

$$= (1\times\tfrac{\pi}{2} + 0) - (0 + 1)$$
$$= \tfrac{\pi}{2} - 1 \text{ square units.}$$

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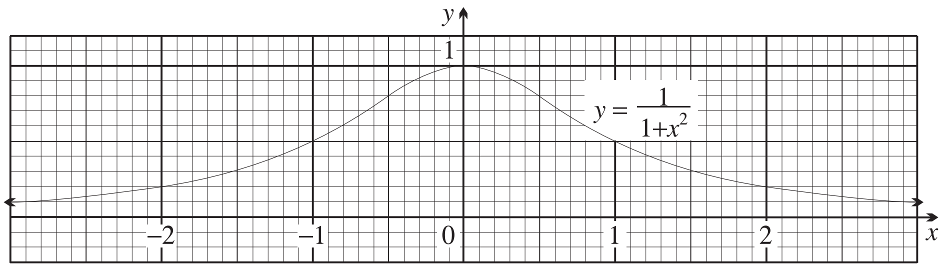
NOTE: We have already established in Section 14I of the Year 11 volume that the area under $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ is exactly 1 square unit. This means that the area between $y = \sin^{-1} x$ and the y-axis is 1, and subtracting this area from the rectangle of area $\frac{\pi}{2}$ in the diagram above gives the same value $\frac{\pi}{2} - 1$ for the shaded area.

Exercise 1E

1. (a)



Find each of the following to two decimal places from the graph by counting the number of little squares in the region under the curve:

(i) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

(ii) $\displaystyle\int_0^2 \frac{1}{1+x^2}\,dx$

(iii) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1+x^2}\,dx$

(iv) $\displaystyle\int_{-3}^{-1} \frac{1}{1+x^2}\,dx$

(b) Check your answers to (a) by using the fact that $\tan^{-1} x$ is a primitive of $\dfrac{1}{1+x^2}$.

2. Find:

(a) $\displaystyle\int \frac{-1}{\sqrt{1-x^2}}\,dx$

(b) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

(c) $\displaystyle\int \frac{1}{9+x^2}\,dx$

(d) $\displaystyle\int \frac{1}{\sqrt{\frac{4}{9}-x^2}}\,dx$

(e) $\displaystyle\int \frac{1}{2+x^2}\,dx$

(f) $\displaystyle\int \frac{-1}{\sqrt{5-x^2}}\,dx$

3. Find the exact value of:

(a) $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

(b) $\displaystyle\int_0^2 \frac{1}{4+x^2}\,dx$

(c) $\displaystyle\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx$

(d) $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{\frac{1}{2}}{\frac{1}{4}+x^2}\,dx$

(e) $\displaystyle\int_{\frac{1}{6}\sqrt{3}}^{\frac{1}{6}} \frac{-1}{\sqrt{\frac{1}{9}-x^2}}\,dx$

(f) $\displaystyle\int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$

4. Find the equation of the curve, given that:

(a) $y' = (1-x^2)^{-\frac{1}{2}}$ and the curve passes through the point $(0, \pi)$.

(b) $y' = 4(16+x^2)^{-1}$ and the curve passes through the point $(-4, 0)$.

5. (a) If $y' = \dfrac{1}{\sqrt{36-x^2}}$ and $y = \frac{\pi}{6}$ when $x = 3$, find the value of y when $x = 3\sqrt{3}$.

(b) Given that $y' = \dfrac{2}{4+x^2}$ and that $y = \frac{\pi}{3}$ when $x = 2$, find y when $x = \frac{2}{\sqrt{3}}$.

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DEVELOPMENT

6. Find:

(a) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$ (c) $\displaystyle\int \frac{-1}{\sqrt{1-2x^2}}\,dx$ (e) $\displaystyle\int \frac{1}{25+9x^2}\,dx$

(b) $\displaystyle\int \frac{1}{1+16x^2}\,dx$ (d) $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,dx$ (f) $\displaystyle\int \frac{-1}{\sqrt{3-4x^2}}\,dx$

7. Find the exact value of:

(a) $\displaystyle\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx$ (c) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-3x^2}}\,dx$ (e) $\displaystyle\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{3+4x^2}\,dx$

(b) $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1+4x^2}\,dx$ (d) $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9-4x^2}}\,dx$ (f) $\displaystyle\int_{\frac{1}{2}\sqrt{10}}^{\frac{1}{2}\sqrt{30}} \frac{1}{5+2x^2}\,dx$

8. By differentiating each RHS, prove the extended standard forms with two constants given in Box 15 of the text:

(a) $\displaystyle\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx = \frac{1}{b}\sin^{-1}\frac{bx}{a} + C$ (b) $\displaystyle\int \frac{1}{a^2+b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\frac{bx}{a} + C$

9. (a) Shade the region bounded by $y = \sin^{-1} x$, the x-axis and the vertical line $x = \frac{1}{2}$.

(b) Show that $\dfrac{d}{dx}(x\sin^{-1} x + \sqrt{1-x^2}) = \sin^{-1} x$.

(c) Hence find the exact area of the region.

10. (a) Shade the region bounded by the curve $y = \sin^{-1} x$, the y-axis and the line $y = \frac{\pi}{6}$.

(b) Find the exact area of this region.

(c) Hence use an alternative approach to confirm the area in the previous question.

11. (a) Show that $\dfrac{d}{dx}\left(\cos^{-1}(2-x)\right) = \dfrac{1}{\sqrt{4x-x^2-3}}$. (b) Hence find $\displaystyle\int_1^2 \frac{1}{\sqrt{4x-x^2-3}}\,dx$.

12. (a) Differentiate $\tan^{-1}\frac{1}{2}x^3$. (b) Hence find $\displaystyle\int \frac{x^2}{4+x^6}\,dx$.

13. (a) The portion of the curve $y = \dfrac{1}{\sqrt{7+x^2}}$ from $x = 0$ to $x = \sqrt{7}$ is rotated about the x-axis through a complete revolution. Find exactly the volume generated.

(b) Find the volume of the solid formed when the region between $y = (1-16x^2)^{-\frac{1}{4}}$ and the x-axis from $x = -\frac{1}{8}$ to $x = \frac{1}{8}\sqrt{3}$ is rotated about the x-axis.

14. (a) Show that $x^2 + 6x + 10 = (x+3)^2 + 1$. (b) Hence find $\displaystyle\int \frac{1}{x^2+6x+10}\,dx$.

15. (a) Differentiate $x\tan^{-1} x$. (b) Hence find $\displaystyle\int_0^1 \tan^{-1} x\,dx$.

16. Without finding any primitives, use symmetry arguments to evaluate:

(a) $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} \sin^{-1} x\,dx$ (c) $\displaystyle\int_{-\frac{3}{4}}^{\frac{3}{4}} \cos^{-1} x\,dx$ (e) $\displaystyle\int_{-3}^{3} \frac{x}{1+x^2}\,dx$

(b) $\displaystyle\int_{-5}^{5} \tan^{-1} x\,dx$ (d) $\displaystyle\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{x}{\sqrt{1-x^2}}\,dx$ (f) $\displaystyle\int_{-6}^{6} \sqrt{36-x^2}\,dx$

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17. (a) Given that $f(x) = \dfrac{x}{1+x^2} - \tan^{-1} x$: (i) find $f(0)$, (ii) show that $f'(x) = \dfrac{-2x^2}{(1+x^2)^2}$.

(b) Hence: (i) explain why $f(x) < 0$ for all $x > 0$, (ii) find $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$.

18. Consider the function $f(x) = \dfrac{1}{\sqrt{4-x^2}}$.

(a) Sketch the graph of $y = \sqrt{4-x^2}$. (b) Hence sketch the graph of $y = f(x)$.

(c) Write down the domain and range of $f(x)$, and describe its symmetry.

(d) Find the area between the curve and the x-axis from $x = -1$ to $x = 1$.

(e) Find the total area between the curve and the x-axis. [NOTE: This is an example of an unbounded region having a finite area.]

19. Consider the function $f(x) = \dfrac{4}{x^2+4}$.

(a) What is the axis of symmetry of $y = f(x)$? (b) What are the domain and range?

(c) Show that the graph of $f(x)$ has a maximum turning point at $(0, 1)$.

(d) Find $\displaystyle\lim_{x\to\infty} f(x)$, and hence sketch $y = f(x)$. On the same axis, sketch $y = \frac{1}{4}(x^2+4)$.

(e) Calculate the area bounded by the curve and the x-axis from $x = -2\sqrt{3}$ to $x = \frac{2}{3}\sqrt{3}$.

(f) Find the exact area between the curve and the x-axis from $x = -a$ to $x = a$, where a is a positive constant.

(g) By letting a tend to infinity, find the total area between the curve and the x-axis. [NOTE: This is another example of an unbounded region having a finite area.]

20. Show that $\displaystyle\int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$.

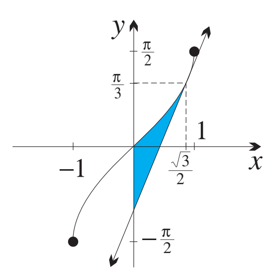
21. (a) Show that $\dfrac{d}{dx}\left(\tan^{-1}\left(\tfrac{3}{2}\tan x\right)\right) = \dfrac{6}{5\sin^2 x + 4}$.

(b) Hence find, correct to three significant figures, the area bounded by the curve $y = \dfrac{1}{5\sin^2 x + 4}$ and the x-axis from $x = 0$ to $x = 7$.

22. (a) Use Simpson's rule with five points to approximate $I = \displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$, expressing your answer in simplest fraction form.

(b) Find the exact value of I, and hence show that $\pi \doteqdot \frac{8011}{2550}$. To how many decimal places is this approximation accurate?

23. The diagram shows the region bounded by $y = \sin^{-1} x$, the y-axis and the tangent to the curve at the point $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$.



(a) Show that the area of the region is $\frac{1}{4}$ unit^2.

(b) Show that the volume of the solid formed when the region is rotated about the y-axis is $\frac{\pi}{24}(9\sqrt{3} - 4\pi)$ unit^3.

24. Find, using the reverse chain rule:

(a) $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx$ (b) $\displaystyle\int_0^1 \frac{1}{e^{-x}+e^{x}}\,dx$

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EXTENSION

25. [The power series for $\tan^{-1} x$] Suppose that x is a positive real number.

(a) Find the sum of the geometric series $1 - t^2 + t^4 - t^6 + \cdots + t^{4n}$, and hence show that for $0 < t < x$,
$$\frac{1}{1+t^2} < 1 - t^2 + t^4 - t^6 + \cdots + t^{4n}.$$

(b) Find $1 - t^2 + t^4 - t^6 + \cdots + t^{4n} - t^{4n+2}$, and hence show that for $0 < t < x$,
$$1 - t^2 + t^4 - t^6 + \cdots + t^{4n} < \frac{1}{1+t^2} + t^{4n+2}.$$

(c) By integrating the inequalities of parts (a) and (b) from $t = 0$ to $t = x$, show that
$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{x^{4n+1}}{4n+1} < \tan^{-1} x + \frac{x^{4n+3}}{4n+3}.$$

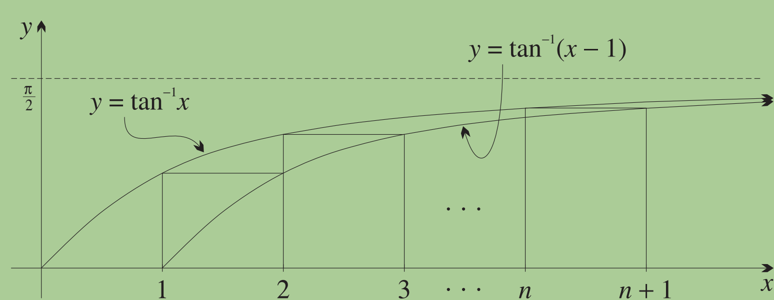
(d) By taking limits as $n \to \infty$, show that for $0 \le x \le 1$,
$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

(e) Use the fact that $\tan^{-1} x$ is an odd function to prove this identity for $-1 \le x < 0$.

(f) [Gregory's series] Use a suitable substitution to prove that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

(g) By combining the terms in pairs, show that $\dfrac{\pi}{8} = \dfrac{1}{1\times 3} + \dfrac{1}{5\times 7} + \dfrac{1}{9\times 11} + \cdots$, and use the calculator to find how close an approximation to π can be obtained by taking 10 terms.

26. [A sandwiching argument]



In the diagram, n rectangles are constructed between the two curves $y = \tan^{-1} x$ and $y = \tan^{-1}(x-1)$ in the interval $1 \le x \le n+1$.

(a) Write down an expression for S_n, the sum of the areas of the n rectangles.

(b) Differentiate $x \tan^{-1} x$ and hence find a primitive of $\tan^{-1} x$.

(c) Show that for all $n \ge 1$,
$$n \tan^{-1} n - \tfrac{1}{2}\ln(n^2+1) < S_n < (n+1)\tan^{-1}(n+1) - \tfrac{1}{2}\ln(\tfrac{n^2}{2} + n + 1) - \tfrac{\pi}{4}$$

(d) Deduce that $1562 < \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 + \cdots + \tan^{-1} 1000 < 1565$.

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1 F General Solutions of Trigonometric Equations

Using the inverse trigonometric functions, we can write down general solutions to trigonometric equations of the type $\sin x = a$ and $\sin x = \sin \alpha$.

Solving Trigonometric Equations Without Restrictions: Because each of the trigonometric functions is periodic, any unrestricted trigonometric equation that has one solution must have infinitely many. This section will later develop formulae for those general solutions, but they can always be found using the methods already established, as is demonstrated in the following worked exercise. The key to general solutions is provided by the periods of the trigonometric functions:

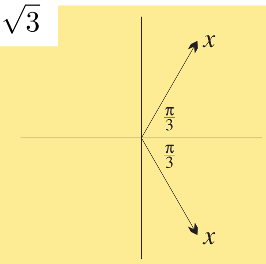
17 PERIODS OF THE TRIGONMETRIC FUNCTIONS:

$\sin x$ and $\cos x$ have period 2π, $\tan x$ has period π.

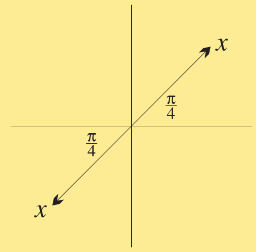
WORKED EXERCISE: Find the general solution, in radians, of:

(a) $\cos x = \frac{1}{2}$ (b) $\tan x = 1$ (c) $\sin x = \frac{1}{2}\sqrt{3}$

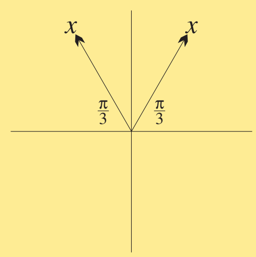
SOLUTION:



(a) Since $\cos x$ is positive, x must be in the 1st or 4th quadrants.
Also, the related acute angle is $\frac{\pi}{3}$.
Hence $x = \frac{\pi}{3}$ and $x = -\frac{\pi}{3}$ are the solutions within a revolution.
Since $\cos x$ has period 2π, the general solution is
$$x = \frac{\pi}{3} + 2n\pi \text{ or } -\frac{\pi}{3} + 2n\pi, \text{ where } n \text{ is an integer.}$$



(b) Since $\tan x$ is positive, x must be in the 1st or 3rd quadrants.
Also, the related angle is $\frac{\pi}{4}$.
Hence $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ are the solutions within a revolution.
Since $\tan x$ has period π, the general solution is
$$x = \frac{\pi}{4} + n\pi, \text{ where } n \text{ is an integer.}$$
(Notice that this includes the other solution $x = \frac{5\pi}{4}$, which is obtained by putting $n = 1$.)



(c) Since $\sin x$ is positive, x must be in the 1st or 2nd quadrants.
Also, the related acute angle is $\frac{\pi}{3}$.
Hence $x = \frac{\pi}{3}$ and $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ are both solutions.
Since $\sin x$ has period 2π, the general solution is
$$x = \frac{\pi}{3} + 2n\pi \text{ or } \frac{2\pi}{3} + 2n\pi, \text{ where } n \text{ is an integer.}$$

The Equation $\cos x = a$: More generally, suppose that $\cos x = a$, where $-1 \le a \le 1$.
First, $x = \cos^{-1} a$ is a solution.
Secondly, $x = -\cos^{-1} a$ is a solution, because $\cos x$ is an even function.
This gives two solutions within a revolution, so the general solution is
$x = \cos^{-1} a + 2n\pi$ or $x = -\cos^{-1} a + 2n\pi$, where n is an integer.

18 THE GENERAL SOLUTION OF $\cos x = a$: The general solution of $\cos x = a$ is

$$x = \cos^{-1} a + 2n\pi \quad \text{or} \quad x = -\cos^{-1} a + 2n\pi, \text{ where } n \text{ is an integer.}$$

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The Equation $\tan x = a$: Suppose that $\tan x = a$, where a is a constant.

One solution is $x = \tan^{-1} a$.

But $\tan x$ has period π, and only one solution within each period,

so the general solution is $x = \tan^{-1} a + n\pi$, where n is an integer.

19 THE GENERAL SOLUTION OF $\tan x = a$: The general solution of $\tan x = a$ is

$$x = \tan^{-1} a + n\pi, \quad \text{where } n \text{ is an integer.}$$

The Equation $\sin x = a$: Suppose that $\sin x = a$, where $-1 \le a \le 1$.

First, $x = \sin^{-1} a$ is a solution.

Also, if $\sin\theta = a$, then $\sin(\pi - \theta) = a$, so $x = \pi - \sin^{-1} a$ is a solution.

This gives two solutions within each revolution, so the general solution is

$x = \sin^{-1} a + 2n\pi$ or $x = (\pi - \sin^{-1} a) + 2n\pi$, where n is an integer.

[Alternatively, we can write $x = 2n\pi + \sin^{-1} a$ or $x = (2n+1)\pi - \sin^{-1} a$.

The first can be written as $x = m\pi + \sin^{-1} a$, where m is even,

and the second can be written as $x = m\pi - \sin^{-1} a$, where m is odd.

Using the switch $(-1)^m$, which changes sign according as m is even or odd,

we can write both families together as $x = (-1)^m \sin^{-1} a + m\pi$, where m is an integer.]

20 THE GENERAL SOLUTION OF $\sin x = a$: The general solution of $\sin x = a$ is

$$x = \sin^{-1} a + 2n\pi \quad \text{or} \quad x = (\pi - \sin^{-1} a) + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

[Alternatively, we can write these two families together using the switch $(-1)^m$:

$$x = (-1)^m \sin^{-1} a + m\pi, \quad \text{where } m \text{ is an integer.]}$$

NOTE: The alternative notation for solving $\sin^{-1} x = a$ is very elegant, and is very quick if properly applied, but it is not at all easy to use or to remember. In this text, we will enclose it in square brackets when it is used.

WORKED EXERCISE: Use these formulae to find the general solution of:

(a) $\cos x = -\frac{1}{2}$ (b) $\sin x = \frac{1}{2}\sqrt{2}$ (c) $\tan x = -2$

SOLUTION:

(a) $x = \cos^{-1}(-\frac{1}{2}) + 2n\pi$ or $-\cos^{-1}(-\frac{1}{2}) + 2n\pi$, where n is an integer,

$\quad = \frac{2\pi}{3} + 2n\pi$ or $-\frac{2\pi}{3} + 2n\pi$.

(b) $x = \sin^{-1} \frac{1}{2}\sqrt{2} + 2n\pi$ or $(\pi - \sin^{-1} \frac{1}{2}\sqrt{2}) + 2n\pi$, where n is an integer,

$\quad = \frac{\pi}{4} + 2n\pi$ or $\frac{3\pi}{4} + 2n\pi$.

[Alternatively, $x = (-1)^m \frac{\pi}{4} + m\pi$, where m is an integer.]

(c) $x = \tan^{-1}(-2) + n\pi$, where n is an integer,

$\quad = -\tan^{-1} 2 + n\pi$, which can be approximated if required.

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The Equations $\sin x = \sin\alpha$, $\cos x = \cos\alpha$ and $\tan x = \tan\alpha$: Using similar methods, the general solutions of these three equations can be written down.

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GENERAL SOLUTIONS OF $\sin x = \sin\alpha$, $\cos x = \cos\alpha$ and $\tan x = \tan\alpha$:

The general solution of $\cos x = \cos\alpha$ is

$$x = \alpha + 2n\pi \quad \text{or} \quad x = -\alpha + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

The general solution of $\tan x = \tan\alpha$ is

$$x = \alpha + n\pi, \quad \text{where } n \text{ is an integer.}$$

The general solution of $\sin x = \sin\alpha$ is

$$x = \alpha + 2n\pi \quad \text{or} \quad x = (\pi - \alpha) + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

[Alternatively, $x = m\pi + (-1)^m\alpha$, where m is an integer.]

PROOF:

A. One solution of $\cos x = \cos\alpha$ is $x = \alpha$.
Also, $\cos\alpha = \cos(-\alpha)$, since cosine is even, so $x = -\alpha$ is also a solution.
This gives the required two solutions within a single period of 2π,
so the general solution is $x = \alpha + 2n\pi$ or $x = -\alpha + 2n\pi$, where n is an integer.

B. One solution of $\tan x = \tan\alpha$ is $x = \alpha$.
This gives the required one solution within a single period of π,
so the general solution is $x = \alpha + n\pi$, where n is an integer.

C. One solution of $\sin x = \sin\alpha$ is $x = \alpha$.
Also, $\sin\alpha = \sin(\pi - \alpha)$, so $x = \pi - \alpha$ is also a solution.
This gives the required two solutions within a single period of 2π,
so the general solution is $x = \alpha + 2n\pi$ or $x = (\pi - \alpha) + 2n\pi$, where n is an integer.

WORKED EXERCISE: Use these formulae to find the general solution of $\sin x = \sin\frac{\pi}{5}$.

SOLUTION:

$$x = \tfrac{\pi}{5} + 2n\pi \quad \text{or} \quad x = (\pi - \tfrac{\pi}{5}) + 2n\pi, \quad \text{where } n \text{ is an integer,}$$
$$x = \tfrac{\pi}{5} + 2n\pi \quad \text{or} \quad x = \tfrac{4\pi}{5} + 2n\pi.$$

[Alternatively, $x = (-1)^m\frac{\pi}{5} + m\pi$, where n is an integer.]

WORKED EXERCISE: Solve: (a) $\cos 4x = \cos x$ (b) $\sin 4x = \cos x$

SOLUTION:

(a) Using the general solution of $\cos x = \cos\alpha$ from Box 21,

$$\begin{aligned} 4x &= x + 2n\pi &&\text{or}\quad 4x = -x + 2n\pi, \quad \text{where } n \in \mathbf{Z},\\ 3x &= 2n\pi &&\text{or}\quad 5x = 2n\pi, \quad \text{where } n \in \mathbf{Z},\\ x &= \tfrac{2}{3}n\pi &&\text{or}\quad x = \tfrac{2}{5}n\pi, \quad \text{where } n \in \mathbf{Z}. \end{aligned}$$

(b) First, $\sin 4x = \sin(\frac{\pi}{2} - x)$, using the identity $\cos x = \sin(\frac{\pi}{2} - x)$.
Hence, using the general solution of $\sin x = \sin\alpha$ from Box 21,

$$\begin{aligned} 4x &= \tfrac{\pi}{2} - x + 2n\pi &&\text{or}\quad 4x = \pi - (\tfrac{\pi}{2} - x) + 2n\pi, \quad \text{where } n \in \mathbf{Z},\\ 5x &= (2n + \tfrac{1}{2})\pi &&\text{or}\quad 4x = x + \tfrac{\pi}{2} + 2n\pi, \quad \text{where } n \in \mathbf{Z},\\ x &= (4n+1)\tfrac{\pi}{10} &&\text{or}\quad 3x = (2n + \tfrac{1}{2})\pi, \quad \text{where } n \in \mathbf{Z},\\ & &&\phantom{\text{or}}\quad\ x = (4n+1)\tfrac{\pi}{6}, \quad \text{where } n \in \mathbf{Z}. \end{aligned}$$

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Exercise 1F

1. Consider the equation $\tan x = 1$.
 (a) Draw a diagram showing x in its two possible quadrants, and show the related angle.
 (b) Write down the first six positive solutions.
 (c) Write down the first six negative solutions.
 (d) Carefully observe that each of these twelve solutions can be written as an integer multiple of π plus $\frac{\pi}{4}$, and hence write down a general solution of $\tan x = 1$.
 (e) Sketch the graphs of $y = \tan x$ (for $-2\pi \le x \le 2\pi$) and $y = 1$ on the same diagram and show as many of the above solutions as possible.

2. Consider the equation $\cos x = \frac{1}{2}$.
 (a) Draw a diagram showing x in its two possible quadrants, and show the related angle.
 (b) Write down the first six positive solutions.
 (c) Write down the first six negative solutions.
 (d) Carefully observe that each of these twelve solutions can be written either as an integer multiple of 2π plus $\frac{\pi}{3}$ or as an integer multiple of 2π minus $\frac{\pi}{3}$, and hence write down a general solution of $\cos x = \frac{1}{2}$.
 (e) Sketch the graphs of $y = \cos x$ (for $-2\pi \le x \le 2\pi$) and $y = \frac{1}{2}$ on the same diagram and show as many of the above solutions as possible.

3. Consider the equation $\sin x = \frac{1}{2}$.
 (a) Draw a diagram showing x in its two possible quadrants, and show the related angle.
 (b) Write down the first six positive solutions.
 (c) Write down the first six negative solutions.
 (d) Carefully observe that each of these twelve solutions can be written either as a multiple of 2π plus $\frac{\pi}{6}$ or as a multiple of 2π plus $\frac{5\pi}{6}$, and hence write down a general solution of $\sin x = \frac{1}{2}$.
 (e) Sketch the graphs of $y = \sin x$ (for $-2\pi \le x \le 2\pi$) and $y = \frac{1}{2}$ on the same diagram and show as many of the above solutions as possible.

4. Write down a general solution of:
 (a) $\tan x = \sqrt{3}$
 (b) $\cos x = \frac{1}{2}\sqrt{2}$
 (c) $\sin x = \frac{1}{2}\sqrt{3}$
 (d) $\tan x = -1$
 (e) $\cos x = -\frac{1}{2}$
 (f) $\sin x = -\frac{1}{2}$

5. Write down a general solution of:
 (a) $\cos\theta = \cos\frac{\pi}{6}$
 (b) $\tan\theta = \tan\frac{\pi}{4}$
 (c) $\sin\theta = \sin\frac{\pi}{5}$
 (d) $\sin\theta = \sin\frac{4\pi}{3}$
 (e) $\tan\theta = \tan(-\frac{\pi}{3})$
 (f) $\cos\theta = \cos\frac{5\pi}{6}$

6. Write down a general solution for each of the following by referring to the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$.
 (a) $\sin x = 0$
 (b) $\cos x = 1$
 (c) $\tan x = 0$
 (d) $\cos x = 0$
 (e) $\sin x = 1$
 (f) $\sin x = -1$

DEVELOPMENT

7. In each case: (i) find a general solution, (ii) write down all solutions in $-\pi \le x \le \pi$.
 (a) $\cos 2x = 1$
 (b) $\sin\frac{1}{2}x = \frac{1}{2}\sqrt{2}$
 (c) $\tan 3x = \frac{1}{3}\sqrt{3}$
 (d) $\sin(x - \frac{\pi}{4}) = 0$
 (e) $\cos(x + \frac{\pi}{6}) = -\frac{1}{2}\sqrt{2}$
 (f) $\tan(2x - \frac{\pi}{6}) = -\sqrt{3}$
 (g) $\cos 2x = \cos\frac{\pi}{5}$
 (h) $\sin 3x = \sin\frac{\pi}{2}$
 (i) $\tan 4x = \tan\frac{\pi}{3}$
 (j) $\tan(x + \frac{\pi}{4}) = \tan\frac{5\pi}{8}$
 (k) $\cos(x - \frac{\pi}{7}) = \cos\frac{4\pi}{7}$
 (l) $\sin(2x + \frac{3\pi}{10}) = \sin(-\frac{\pi}{10})$

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8. In each case: (i) find a general solution, (ii) write down all solutions in $-\pi \le \theta \le \pi$.

(a) $\sin^2\theta + \sin\theta = 0$
(b) $\sin 2\theta = \cos\theta$
(c) $\cot^2(\theta - \frac{\pi}{6}) = 3$
(d) $2\sin^2\theta = 3 + 3\cos\theta$
(e) $\sin 2\theta + \sqrt{3}\cos 2\theta = 0$
(f) $\sec^2 2\theta = 1 + \tan 2\theta$

9. Consider the equation $\tan 4x = \tan x$.

(a) Show that $4x = n\pi + x$.
(b) Hence show that $x = \frac{n\pi}{3}$, where $n \in \mathbf{Z}$.
(c) Hence write down all solutions in the domain $0 \le x \le 2\pi$.

10. Consider the equation $\sin 3x = \sin x$.

(a) Show that $3x = x + 2n\pi$ or $3x = (\pi - x) + 2n\pi$. Hence show that $x = n\pi$ or $x = (2n+1)\frac{\pi}{4}$. [Alternatively, show that $3x = n\pi + (-1)^n x$, and hence show that if n is even, $x = \frac{n\pi}{2}$, and if n is odd, $x = \frac{n\pi}{4}$.]
(b) Hence write down all solutions in the domain $0 \le x \le 2\pi$.

11. Consider the equation $\cos 3x = \sin x$.

(a) Show that $3x = 2n\pi + (\frac{\pi}{2} - x)$ or $2n\pi - (\frac{\pi}{2} - x)$.
(b) Hence show that $x = n\pi - \frac{\pi}{4}$ or $\frac{n\pi}{2} + \frac{\pi}{8}$, where $n \in \mathbf{Z}$.
(c) Hence write down all solutions in the domain $0 \le x \le 2\pi$.

12. Using methods similar to those in the previous two questions, solve for $0 \le x \le \pi$:

(a) $\sin 5x = \sin x$
(b) $\cos 5x = \cos x$
(c) $\sin 5x = \cos x$
(d) $\cos 5x = \sin x$

EXTENSION

13. Sketch the graphs of the following relations:

(a) $\cos y = \cos x$
(b) $\sin y = \sin x$
(c) $\cos y = \sin x$
(d) $\tan y = \tan x$
(e) $\cot y = \tan x$
(f) $\sec y = \sec x$

Which graphs are symmetric in the x-axis, which in the y-axis, and which in $y = x$?

GO Online Multiple Choice Quiz

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