THE UNIVERSITY OF SYDNEY
FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH1902
LINEAR ALGEBRA (ADVANCED)

June 2008

LECTURER: A Molev

TIME ALLOWED: **One and a half hours**

Name: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

1. **(10 marks)**. Let π be the plane given by the equation $2x - 3y - 6z = 6$ and let M be the intersection point of the y axis and the plane π.
   (a) Let π' be the plane through M with the normal vector $\mathbf{n} = 4\,\mathbf{j} + 3\,\mathbf{k}$. Find the Cartesian equations of the intersection line ℓ of the planes π and π'.
   (b) Find the acute angle between the planes π and π'.
   (c) Find parametric scalar equations of the line m which lies on the plane π, passes through the point M, and m is parallel to the xz coordinate plane.
   (d) Find the coordinates of a point K on the line m such that the distance from K to the plane π' equals 3.

2. **(10 marks)**. A matrix A and a column-vector $\mathbf{v}$ are given by

$$A = \begin{bmatrix} 1 & 2 & -4 & 1 \\ -1 & -2 & 4 & 0 \\ -2 & -4 & 8 & -1 \\ 1 & 3 & -4 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 6 \\ -4 \\ -10 \\ 8 \end{bmatrix}.$$

   (a) Use elementary row operations to find the general solution of the system $A\,\mathbf{x} = \mathbf{v}$, where $\mathbf{x}$ is the column-vector with coordinates x_1, x_2, x_3, x_4.
   (b) By using the elementary row operations of the previous part or otherwise find a column-vector $\mathbf{u}$ such that the system $A\,\mathbf{x} = \mathbf{u}$ is inconsistent.
   (c) Is it possible to find a column-vector $\mathbf{u}$ such that the system $A\,\mathbf{x} = \mathbf{u}$ has a unique solution? Justify your answer.
   (d) Prove that if $\mathbf{u}$ and $\mathbf{w}$ are two column-vectors such that each of the systems $A\,\mathbf{x} = \mathbf{u}$ and $A\,\mathbf{x} = \mathbf{w}$ is consistent, then the system $A\,\mathbf{x} = \alpha\,\mathbf{u} + \beta\,\mathbf{w}$ is consistent for any scalars α and β.

3. (10 marks). Let A and B be square matrices of the same size.

(a) Given that $\mathbf{v}$ is an eigenvector of A with the eigenvalue λ and $\mathbf{v}$ is an eigenvector of B with the eigenvalue μ, prove that $\mathbf{v}$ is an eigenvector of each of the matrices $A + B$ and AB and find the corresponding eigenvalues.

(b) Under the assumptions of the previous part, prove that $\det(AB - BA) = 0$.

(c) Find a common eigenvector $\mathbf{v}$ of the matrices

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}.$$

(d) Show that the vector $\mathbf{v}$ you found in the previous part is an eigenvector of the matrix $A^{-3}B^2$ and calculate the corresponding eigenvalue.

4. (10 marks). Let X and Y be square matrices of the same size.

(a) Prove that the relation $(X + Y)^2 = X^2 + 2XY + Y^2$ implies $(X + Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$.

(b) Prove that if both matrices X and Y are invertible then the relation $(XY)^2 = X^2Y^2$ implies $(XY)^3 = X^3Y^3$.

(c) Would the statement in the previous part be true if only one of the matrices X or Y were invertible? Justify your answer.

End of Extended Answer Section