The University of Sydney
MATH1901/06 Differential Calculus (Advanced)

Semester 1 **Short answers to exam questions** **2008**

1. (a) Annulus between two concentric circles, including the circles themselves, radii 1 and 2, centre i on imaginary axis, inner circle passing through the origin.

(b) Roots: $z = 1,\ -i,\ (i \pm \sqrt{3})/2$, or $z = 1,\ e^{-\pi i/2},\ e^{\pi i/6},\ e^{5\pi i/6}$.

(c) $g : \mathbf{C} \to \mathbf{C}$ is not injective because nonzero complex numbers have two square roots and infinitely many logarithms. It is enough to give one failure of the horizontal line test: $g(0) = g(2)$ or $g(z) = g(2-z)$.

(d) $x^2 - 2x$ is increasing on $[1, \infty)$, and so also is $f(x) = e^{x^2-2x}$. So f is injective. Its range is the interval $[e^{-1}, \infty)$.

2. (a) (i). Limit is 2. (Use the squeeze law on $x^2\cos(1/x)$.)

(ii). Limit is 3. (Rationalise numerator or use binomial series.)

(b) Let $f(x) = x\sinh x - \cosh x$, continuous on $[1, 2]$. According to IVT, $f(1) < 0$ and $f(2) > 0$ imply that $f(x)$ has at least one root in $(1, 2)$. But $f(x)$ is increasing for $x > 0$ (because $f'(x) = x\cosh x > 0$), and so $f(x)$ has exactly one root in $(1, 2)$.

(c) Given $\epsilon > 0$, there exists $\delta > 0$ such that $\ell - \epsilon < f(x) < \ell + \epsilon$ whenever $0 < |x-a| < \delta$. If $\ell < 0$, we may choose $\epsilon = -\ell/2$. This forces $f(x) < \ell/2 < 0$ on the intervals $(a, a+\delta)$ and $(a-\delta, a)$, contradicting the statement that $f(x) \geq 0$ for all x. This proves $\ell \geq 0$.

3. (a) (i). Limit is 0. (Apply ∞/∞ version of l'Hôpital's rule to $(\ln x)/(x^{-1})$.)

(ii). For $n > 1$, $x^n \ln x = x^{n-1} \cdot x\ln x$. Both factors tend to zero.

(b) $\sinh x/\cosh x = (x + x^3/3! + \dots)/(1 + x^2/2! + \dots) = x - x^3/3 + \dots$, so $T_3(x)$ for $\tanh x$ is $x - x^3/3$.

(c) (i). $T_n(x) = 1 + x + x^2/2! + x^3/3! + \dots + x^n/n!$.
$R_n(x) = e^x - T_n(x) = e^c\, x^{n+1}/(n+1)!$, for some c between 0 and x.

(ii). $R_n(1) = e - T_n(1) = e^c/(n+1)!$, $0 < c < 1$.
So $1/(n+1)! < e - T_n(1) < 3/(n+1)!$. $6! = 720$ implies $n = 5$.

4. (a) (i). Level curves $c = \pm 1$ rectangular hyperbolae, $c = 0$ pair of straight lines.

(ii). Tangent plane: $z = 2x - 4y + 3$.

(iii). $\nabla f = 2x\,\mathbf{i} - 2y\,\mathbf{j}$. Greatest slope $2\sqrt{5}$, direction $(\mathbf{i} - 2\mathbf{j})/\sqrt{5}$ (or $\mathbf{i} - 2\mathbf{j}$).

(b) (i). $\left|(r\cos\theta)/(r^{2p})\right| \leq r^{1-2p} \to 0$ as $r \to 0^+$ (on all paths) whenever $p < 1/2$.

(ii). Different limits on different paths (for example, the axes).