MATH 1907 (SEMESTER 2, 2017)
ASSIGNMENT 2

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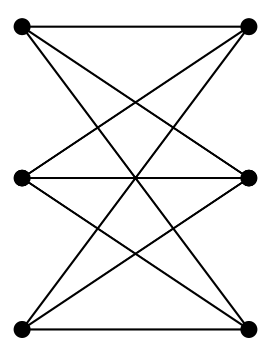
This assignment is due by **5pm Thursday 5 October 2017**, via Turnitin.

The guidelines for this assignment are the same as those from your first assignment: A PDF copy of your answers must be uploaded in the Learning Management System (Blackboard) at https://elearning.sydney.edu.au. Please submit only a PDF document (scan or convert other formats). It should include your name and SID. It is your responsibility to preview each page of your assignment after uploading to ensure each page is included in correct order and is legible (not sideways or upside down) before confirming your submission, and then to check your submission was successful. The School of Mathematics and Statistics encourages some collaboration between students when working on problems, but students must write up and submit their own version of the solutions.

This assignment is worth 3.33% of your final assessment for this course. Your answers should be well written, neat, thoughtful, mathematically concise, and a pleasure to read. Please cite any resources used and show all working. Present your arguments clearly using words of explanation and diagrams where relevant. After all, mathematics is about communicating your ideas. This is a worthwhile skill which takes time and effort to master. The assignment will be scored out of 10 points.

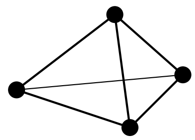
(1) (a) (1 point) Let G be the graph below. Recall that a *cycle* in a graph is a path the begins and ends at the same vertex, and traverses any edge of the graph at most once. Prove that any cycle in G has length at least four.



(b) (2 points) Use Euler's formula to show that G is not planar.

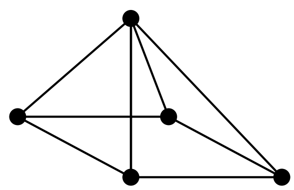
(2) (3 points) Let G be a simple plane graph with $n > 2$ vertices. (Recall that *simple* means the graph has no loops and no parallel edges.) Use Euler's formula to prove that G has at most $3n - 6$ edges.

(3) Now we are going to apply Euler's formula to *polyhedra* in three dimensional space. As a warm-up consider the tetrahedron:


$\subset \mathbb{R}^3$

To be clear, we regard this as a surface in $\mathbb{R}^3$ which consists of 4 vertices, 6 edges, and 4 faces. Here each of the faces is a *solid* triangle.

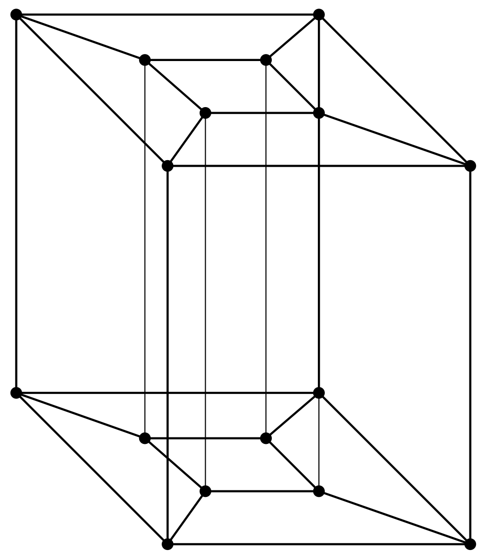
If we apply Euler's formula to the tetrahedron we see that we get $4-6+4 = 2$, which should look familiar. What if we try to do this for a slightly different polyhedron? Consider a pyramid:



The pyramid has 5 vertices, 8 edges, and 5 faces, so again Euler's formula holds! (The faces here consist of four triangles and one square.) Trying do the same calculation for a cube - you'll see again that $n - e + f = 2$.

What's happening here is that the tetrahedron, the pyramid, the cube are *toplogically* the same. We won't worry about the precise definitions here. Informally, what this means is that if we picture these as being made of a flexible material which we can smooth out and reshape (but we cannot tear or glue it), then these are the same, i.e. they are all (topoligically speaking) spheres. An amazing theorem in algebraic topology says that Euler's formula can't tell apart polyhedra that are topologically the same. Conversely if Euler's formula applied to two different polyhedra yields different answers, then those polyhedra are topologically distinct. This is an example of what's called a *topological invariant*.

For any polyhedron that's built out of n vertices, e edges, and f faces, then the number $n - e + f$ is called the *Euler characteristic* of the polyhedron. So far we've only seen polyhedra of Euler characteristic 2, because all we've looked at are spheres. Let's consider something topologically different, namely a "sphere" with one hole which is called a *torus*. As a polyhedron, we will regard the torus as a cube with a middle shaft removed:



(a) (1 point) Compute the Euler characteristic of the polyhedron above.
(b) (1 point) Draw a diagram similar to the above but with two holes, i.e. consider a cube with two shafts removed. Compute its Euler characteristic.
(c) (1 point) Conjecture a formula for the Euler characteristic of a similar polyhedron but with g holes, i.e. consider a cube with g shafts removed, and conjecture a formula for its Euler characteristic.
(d) (1 point) Prove your formula from part (c).