THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial for Week 4

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

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Topics covered

In lectures last week:

- ☐ Area, volume, length, and surface area.

Objectives

After completing this tutorial sheet you will be able to:

- ☐ Use integration to compute areas.
- ☐ Apply and adapt the disc and shell methods to find volumes.
- ☐ Use hyperbolic changes of variables to compute integrals.
- ☐ Compute lengths of curves.
- ☐ Compute surface areas of solids of revolution.

Preparation questions to do *before* class

1. Sketch the region bounded by the curves $y = \sqrt{1 - x^2}$ and $y = \sqrt{2}x^2$. Compute the area of this region.

2. Let S be the sphere of radius R.

(a) Calculate the volume of S using the disc method.

(b) Calculate the volume of S using the shell method.

(c) Find the surface area of S.

Questions to attempt in class

3. Let D be the region bounded by the x-axis, the y-axis, $x = 1$, and $y = \cosh x$.

(a) Find the perimeter of D.

(b) Find the area of D.

(c) Find the volume of the solid obtained by rotating D about the x-axis.

(d) Find the volume of the solid obtained by rotating D about the y-axis.

(e) Find the surface area of the solid obtained by rotating D about the x-axis.

4. Let $a > 0$ be a constant. Sketch the *hypocycloid*

$$x(t) = a\cos^3 t, \qquad y(t) = a\sin^3 t, \qquad t \in [0, 2\pi],$$

and compute its circumference.

5. Let $\lfloor x \rfloor$ denote the *integer part of* $x \in \mathbb{R}$. That is: $\lfloor x \rfloor \in \mathbb{Z}$ is the largest integer with the property that $\lfloor x \rfloor \leq x$. Let $\{x\} \in [0, 1)$ be the *fractional part of* $x \in \mathbb{R}$. That is, $\{x\} = x - \lfloor x \rfloor$. The function $f(x) = \{x\}$ has a jump discontinuity at each integer. Let $x \geq 0$. Compute the integral

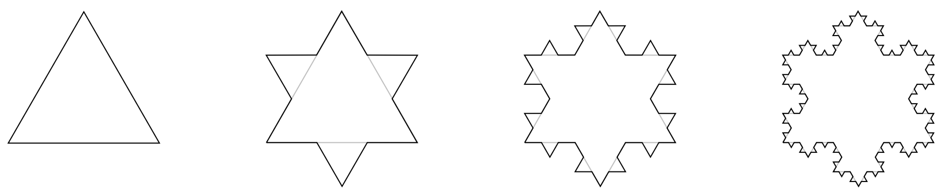
$$\int_0^x \{t\}\, dt \qquad \text{by thinking about areas.}$$

6. Compute the length of the graph $y = \ln x$ for $0 < a \leq x \leq b$.

Hint: A change of variables involving $\sinh t$ *might help, and if you get stuck trying to integrate* $\operatorname{cosech} t$*, try the change of variables* $u = \cosh t$.

Discussion question

7. The *Koch snowflake* is the curve constructed inductively according to the following picture, where the initial equilateral triangle has side length 1.



At each stage of the construction, each line segment is divided into 3 equal parts and an equilateral triangle is placed on the middle third. The snowflake curve is the "limit curve" of this procedure. What is the area enclosed by the Koch curve? What is the length of the Koch curve? Is this a continuous curve? Do you think it is differentiable?

Questions for extra practice

8. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the x-axis, the lines $x = a$ and $x = b$, and the curve $y = \sqrt{1 + x^2}$, $a \leq x \leq b$, where $a \geq 0$.

9. Let T be the solid torus obtained by rotating the circle of centre $(R, 0)$ and radius r about the y-axis (assume that $r \leq R$).

(a) Find the volume of T.

(b) Find the surface area of T. You might like to reposition the circle at $(0, R)$ and rotate about the x-axis.

10. Sketch the region bounded by the curves $y = x\sqrt{1-x^2}$ and $y = x - x^3$, and find the area of the region. *Note: The area consists of two crescent-shaped pieces.*

11. Find the interval $[a, b]$ which maximises the value of the integral $\displaystyle\int_a^b (2 + x - x^2)\,dx$.

12. Write down two integrals for the length of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (one using the length formula for a graph, and another using the formula for the length of a parametrised curve). Try to compute your integrals.

13. Find the volume of the solid obtained by:

(a) Rotating about the x-axis the region bounded by the curve $y = a\cosh(x/a)$, the x-axis, and the line $x = b$. Here $a, b > 0$.

(b) Rotating about the y-axis the region bounded by the curve $y = x\sqrt{1+x^3}$, the x-axis, and the line $x = 2$.

14. Compute the length of:

(a) The curve given by $x = a\cos t$, $y = a\sin t$, $z = bt$ with $0 \le t \le 2\pi$.

(b) The parabola $y = x^2$ between $(0, 0)$ and (a, a^2), where $a > 0$.

15. Use shells to find the volume of a right circular cone of height h and radius r.

Challenging questions

16. Find the surface area of the *spheroid* obtained by rotating the half ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $y \ge 0$ and $-a \le x \le a$ about the x-axis. Be careful with the two cases $a < b$ and $a > b$.

17. A bowl is in the shape of a hemisphere of radius r cm.

(a) If there is water in the bowl with a depth h at the centre of the bowl, what is the volume of this water?

(b) Suppose that water is poured into the bowl at a constant rate of C cubic centimeters per second. At what rate is the water level rising when $h = r/2$?

18. A *polyhedron* is a closed surface formed by joining a finite number of polygons (faces) edge-to-edge. The polygons need not be regular. Restrict attention to polyhedra that have a well-defined inside and outside. Then the inside together with the boundary forms a solid polyhedron.

(a) Suppose that a particular polyhedron has the property that every face touches a given sphere of radius R tangentially. Prove that the volume V and surface area S of such a polyhedron are related by $V = (1/3)RS$.

(b) By taking a suitable limit, prove that the sphere has the same property and deduce the surface area of the sphere from its volume.