CHAPTER **SIX**

Volumes

CHAPTER OVERVIEW: The work in this chapter demonstrates a particular application of integration to the real world, namely to find the volumes of certain solids. The solids encountered include those with rotational symmetry and those with known cross-sections. The volumes of solids with rotational symmetry is an extension of the work done in Mathematics Extension 1. The volumes of solids with known cross-sections will be new to most students.

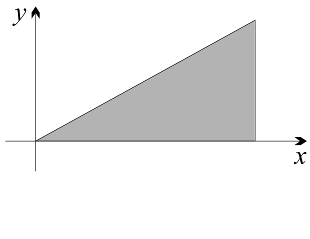
In the solution of a given problem, a full exposition would include the derivation of an expression for the tiny bits, called volume elements, that go to make up the solid. A limiting sum of these volume elements then gives the volume of the solid as an integral. Some of the Worked Exercises and some of the questions in the Exercises follow this approach, and more capable students are encouraged to attempt these harder questions. However the trend in recent HSC examinations is to obtain a formula for an area which is then integrated to yield the volume of the solid. The majority of the text and Exercise questions follows this format.

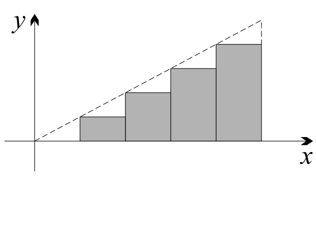
Each solid encountered in this topic has what is called a generating shape. For example, a cone is traced out when a right-angled triangle is rotated about its altitude. Thus we say that the right-angled triangle generates the cone. Sketches of the generating shape and the solid are included in the Worked Exercises and students are encouraged to draw similar sketches when tackling questions.

There are three sections in the chapter. Simple volumes of revolution are dealt with in 6A, including volumes of revolution about lines other than the coordinate axes. In 6B the method of cylindrical shells is introduced, which is used to simplify certain harder types of volumes of revolution. The final section deals with solids with known cross-sections, where similarity is an important component.

6A Volumes of Revolution

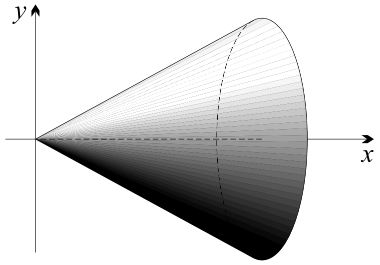
We begin this section by reviewing the work done in Mathematics Extension 1 on volumes of revolution. In the Integration topic we observed that an area can be approximated by a series of rectangles, as in the diagrams below.

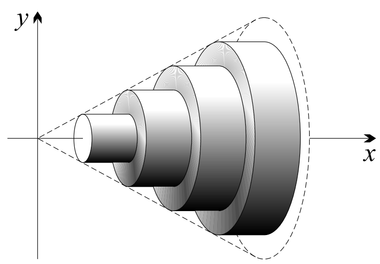




As the number of rectangles is increased, the approximation is improved. The actual area is obtained by letting the number of rectangles tend to infinity, though we only proved the result in a few simple cases. This yields an integral.

In a similar way, a volume of revolution can be approximated by a series of short cylinders, or disks, as shown in the diagrams below. The volume of each disk is easily calculated using $V = \pi r^2 h$.





It should be clear that as the number of disks is increased, the approximation is improved. The actual volume is obtained (though we did not prove the result) by letting the number of disks tend to infinity, which yields an integral.

Volumes of Revolution About the Axes: If the axis of of rotation is the x-axis, the radius of each disk is the y-coordinate and the height is dx. Thus if the solid exists between $x = a$ and $x = b$, we get the volume integral

$$V = \int_a^b \pi y^2 \, dx \, .$$

When the axis of rotation is the y-axis, the radius of each disk is the x-coordinate and the height is dy. Thus if the solid exists between $y = c$ and $y = d$, we get the volume integral

$$V = \int_c^d \pi x^2 \, dy \, .$$

Observe that in each case the radius is squared so that it does not matter if either x or y is negative. Both formulae should be familiar from the Mathematics Extension 1 course.

A Generalised Approach: Notice that in both cases above we integrated the area of a circle $A = \pi r^2$ across the height of the solid. Thus it is possible to replace both formulae with the single simple result

$$V = \int_a^b \pi r^2 \, dh \, ,$$

or more generally

$$V = \int_a^b A \, dh \, .$$

Reading this last line more carefully, it says that if the area A of a cross-section is integrated over the height of the solid, the result is the volume V of the solid. Although we will not prove this formula, it works for any shaped solid and we will use it in most problems for the remainder of this chapter.

1 **A GENERAL FORMULA FOR VOLUME:** If $A(h)$ is the area of a cross-section of a solid at height h then the volume V is given by

$$V = \int_a^b A\,dh\,,$$

where $h = a$ is the lowest point of the solid and $h = b$ is the highest point.

Volumes of Revolution About Other Axes: The previous results for volumes of revolution are now extended to problems where the axis of revolution is some other horizontal or vertical line. The approach is to carefully determine the radius of each circular cross-section, or slice, and hence find the area of that circle. Then, using the result of Box 1, integrate over the height of the solid to obtain its volume.

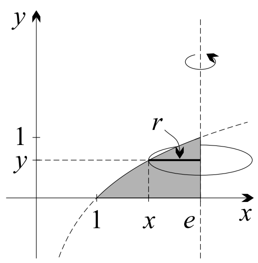
WORKED EXERCISE: The region bounded by $y = \log x$, the line $x = e$ and the x-axis is rotated about the line about $x = e$ to generate a solid.

(a) Show that the area of a slice at height y is given by

$$A = \pi(e - e^y)^2\,.$$

(b) Hence find the volume of the solid, correct to one decimal place.

SOLUTION: The situation is shown on the right.

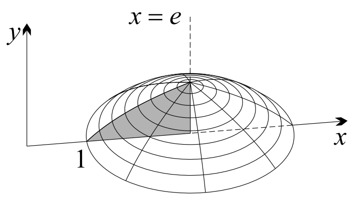


(a) First note that since $y = \log x$ we have $x = e^y$.
Consider the circular slice at height y. The radius is

$$\begin{aligned} r &= e - x \\ &= e - e^y \end{aligned}$$

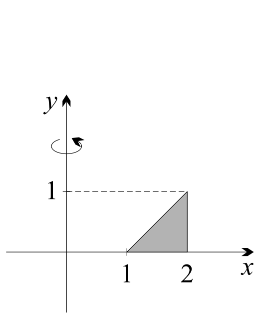
hence $\quad A = \pi(e - e^y)^2\,.$

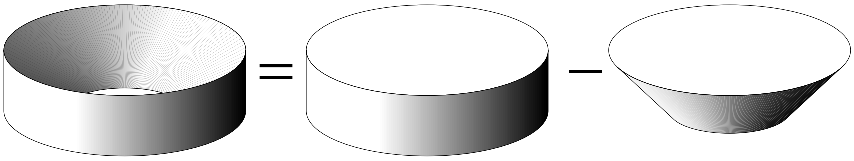
(b) Integrating part (a) from $y = 0$ to $y = 1$:



$$\begin{aligned} V &= \pi \int_0^1 (e - e^y)^2\; dy \\ &= \pi \int_0^1 e^2 - 2e^{y+1} + e^{2y}\; dy \\ &= \pi \left[e^2 y - 2e^{y+1} + \tfrac{1}{2}e^{2y}\right]_0^1 \\ &= \pi\Big((e^2 - 2e^2 + \tfrac{1}{2}e^2) - (0 - 2e + \tfrac{1}{2})\Big) \\ &= \pi\Big(2e - \tfrac{1}{2}(e^2 + 1)\Big) \\ &\doteqdot 3{\cdot}9 \text{ cubic units} \end{aligned}$$

Volumes by Subtraction: Some volume problems are best solved by taking the difference between the volumes of two simpler solids. This is covered in the Mathematics Extension 1 course. As an example, consider the solid generated when the triangle on the right is rotated about the y-axis. The volume is easily found by subtracting the solid formed by rotating the region to the left of the hypotenuse from the cylinder with radius 2 and height 1. The situation is shown in the diagram on the next page .





Thus the volume required is the difference between the bigger outer volume and the smaller inner volume. That is:

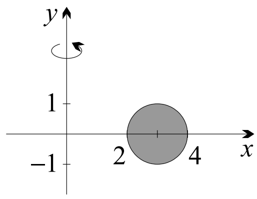
$$\begin{aligned} V &= V_{\text{outer}} - V_{\text{inner}} \\ &= \int_a^b \pi r_{\text{outer}}^2 \, dh - \int_a^b \pi r_{\text{inner}}^2 \, dh \, . \end{aligned}$$

In most problems it is easy to evaluate each integral and subtract. However in some instances it is algebraically advantageous to combine these two terms into a single integral, thus:

$$V = \int_a^b \pi (r_{\text{outer}}^2 - r_{\text{inner}}^2) \, dh \, .$$

The term $A = \pi(r_{\text{outer}}^2 - r_{\text{inner}}^2)$ represents the area of the region between two concentric circles, properly called an *annulus*, though sometimes called a washer. An annulus will result whenever the generating region does not contact the axis of revolution all the way from the lowest point to the highest point of the solid.

WORKED EXERCISE: A toroid (ring) is formed by rotating the circle $(x-3)^2 + y^2 = 1$ about the y-axis.



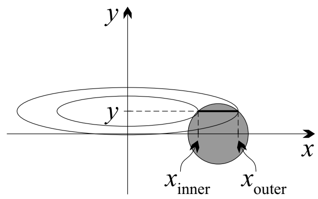
(a) Sketch a typical cross-section at height y and describe its shape.

(b) Show that the area of this cross-section is

$$A = 12\pi\sqrt{1-y^2}$$

(c) Hence find the volume of the toroid.

SOLUTION: (a) The sketch is on the right.
The cross-section is an annulus.

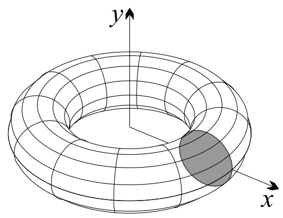


(b) First solve the equation of the circle for x.

$$\begin{aligned} (x-3)^2 &= 1 - y^2 \\ \text{so} \quad x - 3 &= \sqrt{1-y^2} \ \text{ or } \ -\sqrt{1-y^2} \\ \text{thus} \quad x_{\text{outer}} &= 3 + \sqrt{1-y^2} \\ \text{and} \quad x_{\text{inner}} &= 3 - \sqrt{1-y^2} \, . \end{aligned}$$

$$\begin{aligned} \text{Hence} \quad A &= \pi(x_{\text{outer}}^2 - x_{\text{inner}}^2) \\ &= \pi(x_{\text{outer}} + x_{\text{inner}})(x_{\text{outer}} - x_{\text{inner}}) \quad \text{(difference of two squares)} \\ &= \pi \times 6 \times 2\sqrt{1-y^2} \\ &= 12\pi\sqrt{1-y^2} \, . \end{aligned}$$

(c) Integrating A from $y = -1$ to $y = 1$



$$\begin{aligned} V &= \int_{-1}^{1} 12\pi\sqrt{1-y^2}\ dy \\ &= 12\pi \int_{-1}^{1} \sqrt{1-y^2}\ dy \\ &= 12\pi \times \tfrac{1}{2}\pi \quad \text{(area of a semi-circle, radius 1)} \\ &= 6\pi^2\,. \end{aligned}$$

2 VOLUMES BY SUBTRACTION: Whenever the cross-section of a volume of revolution is an annulus, use the formula

$$V = \int_a^b \pi(r_{\text{outer}}^2 - r_{\text{inner}}^2)\ dh\,.$$

where $h = a$ is the lowest point of the solid and $h = b$ is the highest point.

Shifts and Reflections: In some problems a judicious choice of a shift or reflection can simplify the situation. The aim is to move the axis of rotation to coincide with one of the coordinate axes.

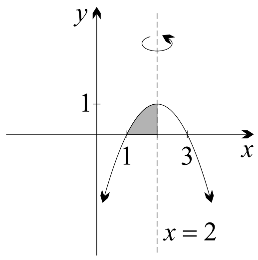
For example if a region is rotated about the line $x = 3$ then shifting the problem left by 3 units will make the y-axis the axis of rotation. In the following problem, reflection in the line $x = a$ is used, as encountered in the Mathematics Extension 2 topics Integration and Graphs. Recall that this means replacing x with $(2a - x)$.

WORKED EXERCISE: The region under the parabola $y = (x-1)(3-x)$ to the left of $x = 2$ and above the x-axis is rotated about the line $x = 2$ to generate a solid.

(a) Draw the situation.
(b) What reflection will make the y-axis become the axis of rotation?
(c) What is the equation of the reflected parabola?
(d) Draw the new configuration of the problem.
(e) Hence find the volume of the solid.

SOLUTION:

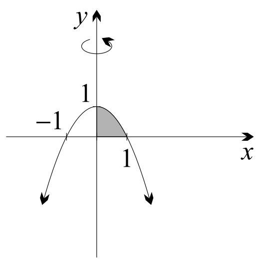
(a) The situation is shown on the right.



(b) Reflect in $x = 1$, so replace x with $(2-x)$.

(c)
$$\begin{aligned} y &= \Big((2-x)-1\Big)\Big(3-(2-x)\Big) \\ &= (1-x)(1+x) \\ &= 1-x^2\,. \end{aligned}$$

(d) The new configuration is shown on the right.



(e) This is now a 3 Unit problem, so

$$\begin{aligned} V &= \pi \int_0^1 x^2\ dy \\ &= \pi \int_0^1 1-y\ dy \\ &= \pi \Big[y - \tfrac{1}{2}y^2\Big]_0^1 \\ &= \tfrac{\pi}{2}\,. \end{aligned}$$

3

SHIFTS AND REFLECTIONS: If a region is rotated about the line $x = 2a$, do one of the following to make the y-axis become the axis of rotation.
- Shift left by replacing x with $(x + 2a)$.
- Reflect in the line $x = a$ by replacing x with $(2a - x)$.

Similar vertical shifts and reflections can be applied when the axis of rotation is the horizontal line $y = 2a$, to make the x-axis become the axis of rotation.

Exercise **6A**

1. (a) Draw the region bounded by $y = x^3 + 1$ and the coordinate axes.
(b) Find the volume of the solid formed when this region is rotated about:
(i) the x-axis, (ii) the y-axis.

2. The region bounded by $y = x^2$, $y = 2 - x$ and the x-axis is rotated about the x-axis to generate a solid. (a) Draw the region. (b) Find the volume of the solid.

3. (a) Sketch the region $\mathcal{R}$ bounded by the curve $y = x^2$, the line $x = 2$ and the x-axis.
(b) By slicing perpendicular to the axis of rotation, find the volume of the solid formed when the region $\mathcal{R}$ is rotated through 360° about: (i) the x-axis, (ii) the y-axis.

4. (a) Graph the region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$.
(b) Us the result of Box 2 to find the volume of the solid generated when this region is rotated about the y-axis.

5. The region $\mathcal{R}$ is bounded by the curve $y = x^3$, and the lines $x = 0$ and $y = 1$. Use the result of Box 2 to find the volume of the solid formed when $\mathcal{R}$ is rotated about $y = 0$.

6. In each case find the volume of the solid formed when the region with the given boundaries is rotated about the x-axis:
(a) the parabola $y^2 = 4x$, the y-axis and the line $y = 2$,
(b) the parabola $y = 3 + x^2$ and the line $y = 4$,
(c) the parabola $y = 3x - x^2$ and the line $y = 2$.

DEVELOPMENT

7. The region bounded by $y = x - 1$, $y = 3 - x$ and the x-axis is rotated about the y-axis. Find the volume of the resulting solid.

8. A solid is formed by rotating the region bounded by $y = x^2 + 1$, and $y = 3 - x^2$ about the x-axis. What is its volume?

9. By taking slices perpendicular to the axis of rotation, find the volume of the solid generated when the region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is rotated about the line $x = 4$.

10. The region $\mathcal{R}$ is bounded by the curve $y = x^2$, the line $x = 2$ and the x-axis. By slicing perpendicular to the axis of rotation, find the volume of the solid formed when the region $\mathcal{R}$ is rotated through 360° about:
(a) the line $x = 2$ (b) the line $y = 4$ (c) the line $x = 3$

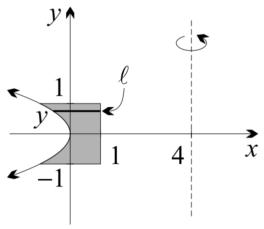
11. The region $\mathcal{R}$ is bounded by the curve $y = x^3$, and the lines $x = 0$ and $y = 1$. By slicing perpendicular to the axis of rotation, find the volume of the solid formed when $\mathcal{R}$ is rotated about: (a) the line $y = 1$, (b) the line $y = 2$.

12. The region $\mathcal{A}$ is bounded by the parabola $y = 4 - x^2$, and the lines $x = 2$ and $y = 4$. Find, by taking slices perpendicular to the axis of rotation, the volume of the solid generated when $\mathcal{A}$ is rotated about: (a) the line $x = 2$, (b) the line $x = 3$.

13. The shaded region is bounded by the lines $x = 1$, $y = 1$ and $y = -1$ and by the curve $x + y^2 = 0$. The region is rotated through $360°$ about the line $x = 4$ to form a solid. When the region is rotated, the line segment ℓ at height y sweeps out an annulus.

(a) Show that the area of the annulus is $\pi(y^4 + 8y^2 + 7)$.

(b) Hence find the volume of the solid.



14. The region $\mathcal{P}$ is bounded by the parabola $y^2 = 4ax$ and its latus rectum $x = a$. Find the volume of the solid generated when $\mathcal{P}$ is rotated about each of the following vertical lines:

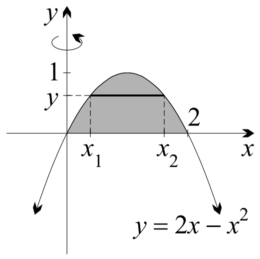
(a) the latus rectum, (b) $x = 2a$, (c) the directrix.

15. The diagram on the right shows the region $\mathcal{R}$ bounded by the parabola $y = 2x - x^2$ and the x-axis.

(a) When the interval AB at height y is rotated about the y-axis, an annulus is generated. Show that its area is given by

$$4\pi\sqrt{1-y}\,.$$

(b) Hence find the volume of the solid formed when $\mathcal{R}$ is rotated about the y-axis.



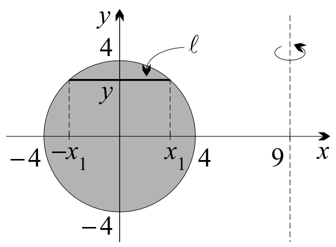
16. (a) Sketch the curve $y = 2x^2 - x^4$, clearly showing the x-intercepts and the coordinates of the stationary points.

(b) Let $\mathcal{A}$ be the region in the first quadrant bounded by the curve $y = 2x^2 - x^4$ and the x-axis. Using the methods of the previous question, find the volume of the solid formed by rotating $\mathcal{A}$ about the y-axis.

17. The circle $x^2 + y^2 = 16$ is rotated about the line $x = 9$ to form a ring. When the circle is rotated, the line segment ℓ at height y sweeps out an annulus. The endpoints of ℓ have x-values x_1 and $-x_1$.

(a) Show that the area of the annulus is $36\pi\sqrt{16 - y^2}$.

(b) Hence find the volume of the ring.



18. The circle $(x - c)^2 + y^2 = a^2$, where $c > a$, is rotated about the y-axis to form a torus. By slicing perpendicular to the y-axis, show that the torus has volume $2\pi^2 a^2 c$ cubic units.

19. Find the volume of the solid formed when the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is rotated about each of the following horizontal lines:

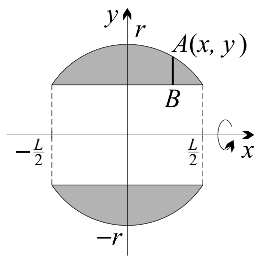
(a) $y = 0$, (b) $y = b$, (c) $y = c$. where $c > b$

20. (a) In $\triangle ABC$, $AB = 3\,\text{cm}$, $BC = 4\,\text{cm}$ and $AC = 5\,\text{cm}$. The point D is on AC such that $BD \perp AC$. Use similar triangles to find AD and CD.

(b) A spherical cap of height h is cut off a sphere of radius r by a horizontal plane. Show that the cap has volume $\frac{1}{3}\pi h^2(3r - h)$.

(c) The centres of two intersecting spheres of radii 3 cm and 4 cm are 5 cm apart. Use the results in (a) and (b) to find the volume common to the two spheres.

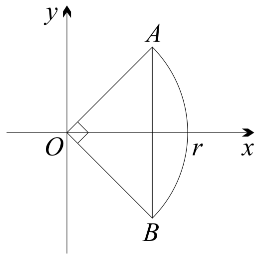
21. The diagram shows the cross-section of a sphere through which a cylindrical hole of length L has been drilled.



(a) Show that the annulus formed by rotating the interval AB about the x-axis has area $\frac{\pi}{4}(L^2 - 4x^2)$ square units, and thus is independent of r.

(b) Hence show that the volume of the sphere remaining is the same as the volume of a sphere of diameter L.

22. In the diagram, AB is an arc of a quadrant of a circle centred at O and of radius r. The chord AB is parallel to the y-axis.

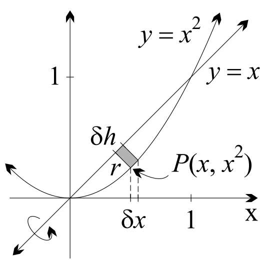


(a) The quadrant AOB is rotated about the y-axis. Show that the solid formed has volume $\frac{2\sqrt{2}}{3}\pi r^3$.

(b) The region bounded by arc AB and the chord AB is rotated about the chord AB. Show that the solid formed has volume $\frac{10-3\pi}{6\sqrt{2}}\pi r^3$ cubic units.

EXTENSION

23. [A First Principles Approach] The region bounded by $y = x^2$ and $y = x$ is rotated about the line $y = x$ to generate a solid. In this question you will find the volume of this solid by first approximating a small portion with a disk.



Let $P(x, x^2)$ be a typical point on the parabola. From x to $x + \delta x$ we will approximate the area between the curve and the line with a rectangle. One corner of this rectangle is at P and a side is on the line $y = x$. Let the dimensions of this rectangle be $r \times \delta h$ as shown in the diagram.

(a) Show that $\delta h = \sqrt{2}\,\delta x$ and find r as a function of x.

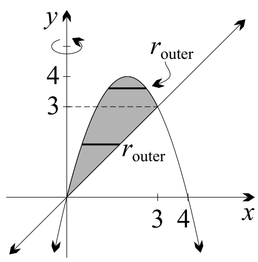
(b) When the rectangle is rotated about the line $y = x$ it generates a cylindrical prism with volume δV. Find δV.

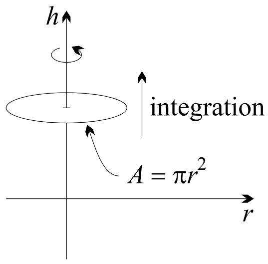
(c) Divide δV by δx. What is the limit of this ratio as $\delta x \to 0$?

(d) Hence show that the volume of the solid is $\frac{\pi}{30\sqrt{2}}$.

6B The Method of Cylindrical Shells

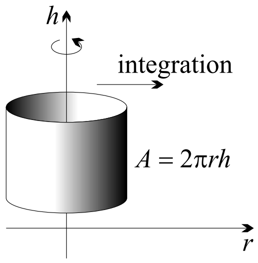
Inevitably when tackling volume problems there will be certain questions which result in awkward integrals. As an example, consider the solid generated when the region between $y = 4x - x^2$ and $y = x$ is rotated about the y-axis, as shown on the right. The volume can be found by the methods of Section 6A since each cross-section is an annulus. However the situation is complicated by the fact that the formula for the outer radius changes. Below $y = 3$ the radius is $r_{\text{outer}} = y$ whilst above $y = 3$ it is $r_{\text{outer}} = 2 + \sqrt{4 - y}$. Hence the integral must be split into two parts. (Try this as an exercise.)





In this case and in many other instances these difficulties can be overcome by using the method of cylindrical shells. By way of contrast, let us first examine more closely how volumes of revolution were found in Section 6A. At any height a slice perpendicular to the axis of revolution is a circle or annulus. The area of the slice is found. The result is then integrated in a direction perpendicular to the slice, that is along the h-axis. The situation for the circle is shown on the right. Thus the volume is:

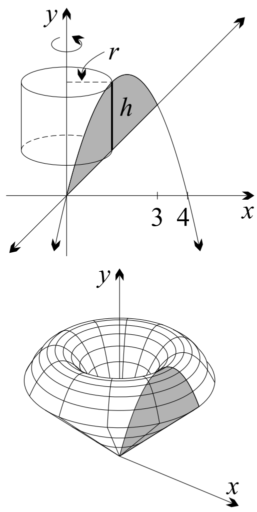
$$V = \int_a^b \pi r^2 \, dh \, .$$



In the method of cylindrical shells a slice is taken parallel with the axis of revolution. At any given radius the shape of the slice is a cylindrical shell. The surface area of the cylinder is $A = 2\pi rh$. This result is now integrated in a direction perpendicular to the surface, that is along the r-axis. Again, the situation is shown on the right. Thus, by the method of cylindrical shells:

$$V = \int_a^b 2\pi rh \; dr$$

WORKED EXERCISE: The region between $y = 4x - x^2$ and $y = x$ is rotated about the y-axis to form a solid. Find its volume by the method of cylindrical shells.



SOLUTION: Since the axis of revolution is the y-axis, the radius of a cylindrical shell is the x-coordinate, and the height is the difference between the y-coordinates on the two curves. Thus:

$$r = x$$

and $$h = (4x - x^2) - x$$
$$= 3x - x^2 \, .$$

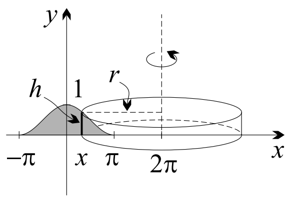
Hence
$$\begin{aligned} V &= \int_0^3 2\pi x(3x - x^2)\, dx \\ &= 2\pi \int_0^3 3x^2 - x^3 \, dx \\ &= 2\pi \Big[x^3 - \tfrac{1}{4}x^4 \Big]_0^3 \\ &= \tfrac{27\pi}{2} \, . \end{aligned}$$

4 THE METHOD OF CYLINDRICAL SHELLS: In a volume of revolution, a slice taken parallel with the axis of rotation is a cylindrical shell. The surface-area of the cylinder is $A = 2\pi rh$ and the total volume of the solid is:

$$V = \int_a^b 2\pi rh \; dr \, .$$

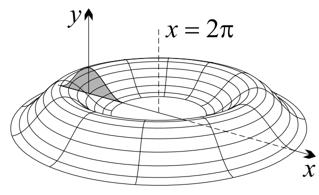
where $r = a$ is the innermost radius and $r = b$ is the outermost radius.

Rotation about another axis: Care needs to be taken that the expressions for r and h are positive. Otherwise the method is the same as before. That is, find the surface-area of a cylindrical shell then integrate to obtain the volume.



Worked Exercise: The region between $y = 1 + \cos x$ and the x-axis is rotated about the line $x = 2\pi$ to form a solid. Find the volume of this solid by the method of cylindrical shells.

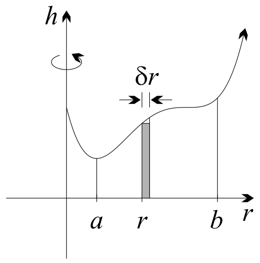
Solution: The situation is shown on the right. The radius of the cylinder is $r = 2\pi - x$, so that $dr = -dx$, and the height is $h = 1 + \cos x$. Thus:



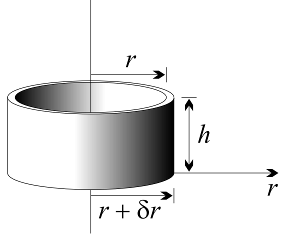
$$\begin{aligned} V &= \int_{\pi}^{-\pi} 2\pi(2\pi - x)(1 + \cos x) \times -dx \\ &= 2\pi \int_{-\pi}^{\pi} 2\pi(1 + \cos x) - x(1 + \cos x)\, dx \\ &= 2\pi \int_{-\pi}^{\pi} 2\pi(1 + \cos x)\, dx \qquad \text{(by odd symmetry)} \\ &= 4\pi^2 \Big[x + \sin x \Big]_{-\pi}^{\pi} \\ &= 8\pi^3 . \end{aligned}$$

Note that this problem could also be solved by first reflecting in the line $x = \pi$, that is, replacing x with $2\pi - x$. Try this as an exercise and compare the results.

A First Principles Approach: In harder problems you may be required to find the volume of a solid using a first principles approach. Here it is used to derive the formula in Box 4 for cylindrical shells. It should be noted that although this example is a more formal approach to the method of cylindrical shells, it is not a proper proof.



Suppose that the region below the curve $h = f(r)$ and between $r = a$ and $r = b$ is rotated about the vertical axis to generate a solid. For simplicity we will assume that $f(r)$, a and b are all positive. Now suppose that the region is approximated by a series of thin rectangular strips of width δr. One such strip is shown in the diagram on the right. The volume element generated when this thin strip is rotated about the axis is a pipe or cylindrical shell, as in the second diagram on the right. Let δV be the volume of this shell then:



$$\begin{aligned} \delta V &= \pi(r + \delta r)^2 h - \pi r^2 h \\ &= \pi h \Big((r + \delta r)^2 - r^2 \Big) \\ &= \pi h(2r + \delta r)\delta r \qquad \text{(difference of two squares)} \end{aligned}$$

The volume of the original solid will be approximately equal to the sum of all such shells between $x = a$ and $x = b$, that is:

$$V \doteqdot \sum_{r=a}^{r=b} \pi h(2r + \delta r)\delta r$$

Although we will not prove the result, it should be intuitively clear that these two volumes will be equal in the limit as $\delta r \to 0$. Thus:

$$V = \lim_{\delta r \to 0} \left(\sum_{r=a}^{r=b} \pi h(2r + \delta r)\delta r \right)$$

We know from our previous work on integration that such a limiting sum yields an integral, and hence:

$$V = \int_a^b \pi h(2r + 0)\, dr\,,$$

or more simply

$$V = \int_a^b 2\pi r h\, dr\,,$$

exactly as before.

A Derivative Approach: Here is an alternative to the first principles approach. The initial steps are identical and so the volume of a cylindrical shell is:

$$\delta V = \pi h(2r + \delta r)\delta r$$

or $$\frac{\delta V}{\delta r} = 2\pi(r + \delta r)h\,.$$

Now take the limit as $\delta r \to 0$ to get

$$\frac{dV}{dr} = 2\pi r h\,.$$

Finally integrate to get the volume, viz:

$$V = \int_a^b 2\pi r h\, dr$$

again, exactly as before.

Exercise **6B**

1. The region bounded by the curve $y = x^2$, the line $x = 2$ and the x-axis is rotated about the y-axis to form a solid.

(a) Sketch the region.

(b) Use the method of cylindrical shells to find the volume of the solid.

2. The region $\mathcal{A}$ lies in the first quadrant and is bounded by the curve $y = x^2$, the line $y = 4$ and the y-axis.

(a) Graph this region.

(b) Use the method of cylindrical shells to find the volume of the solid formed when region $\mathcal{A}$ is rotated through 360° about the x-axis.

3. Find, using cylindrical shells, the volume of the solid generated when the region with the given boundaries is rotated about the y-axis:

(a) $y = 4x - x^2$ and the x-axis, (b) $y^2 = 4x$ and $x = 9$, (c) $y = x^2 - x^3$ and the x-axis.

4. The region $\mathcal{C}$ is bounded by the curve $y = 1 - x^3$, and the coordinate axes. By using cylindrical shells, find the volume of the solid formed when $\mathcal{C}$ is rotated about:

(a) the y-axis, (b) the x-axis (let $1 - y = u^3$).

DEVELOPMENT

5. The region $\mathcal{R}$ is bounded by the curve $y = x^2$, the line $x = 2$ and the x-axis. By using the method of cylindrical shells, find the volume of the solid formed when the region $\mathcal{R}$ is rotated through 360° about: (a) the line $x = 2$, (b) the line $x = 3$.

6. The region $\mathcal{A}$ lies in the first quadrant and is bounded by the curve $y = x^2$, the line $y = 4$ and the y-axis. Use the method of cylindrical shells to find the volume of the solid formed when the region $\mathcal{A}$ is rotated through 360° about: (a) the line $y = 4$, (b) the line $y = 6$.

7. The region $\mathcal{A}$ is bounded by the parabola $y = 4 - x^2$, and the lines $x = 2$ and $y = 4$. Find, by the method of cylindrical shells, the volume of the solid generated when $\mathcal{A}$ is rotated about: (a) the line $x = 2$, (b) the line $x = 3$.

8. The region bounded by the curves $y = x^3 + 8$ and $y = x^2 + 1$, and the lines $x = 0$ and $x = 2$, is rotated about the y-axis. Use cylindrical shells to show that the solid formed has volume $\frac{164\pi}{5}$ cubic units.

9. (a) Sketch the parabola $y = x^2$ and the line $y = 4x - 3$, showing their points of intersection.
(b) The region in part (a) bounded by the parabola and the line is rotated about the y-axis to generate a solid. Find the volume of this solid using cylindrical shells.

10. (a) Sketch the region enclosed by $y = x + 1$ and $y = (x - 1)^2$.
(b) The region is rotated about the y-axis. Find the volume of the solid formed.

11. The region bounded by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = 1$ is rotated about the line $x = 0$. Use cylindrical shells to find the volume of the solid formed.

12. The region $\mathcal{R}$ is bounded by the parabolas $y = 3 - x^2$ and $y = x + x^2$ and the line $x = -1$, and lies to the right of the line $x = -1$. Use the method of cylindrical shells to find the volume of the solid generated when $\mathcal{R}$ is rotated about the line $x = -1$.

13. The region $\mathcal{R}$ is bounded by the parabola $y^2 = 9x$, the line $x = 1$ and the x-axis. A solid is formed by rotating $\mathcal{R}$ about the line $x = 2$. Find the volume of the solid by using:
(a) slices perpendicular to the axis of rotation, (b) cylindrical shells.

14. A solid is formed by rotating the region bounded by the parabola $y^2 = 16(1 - x)$ and the y-axis about the line $x = 2$.
(a) By slicing perpendicular to the axis of rotation, find the volume of the solid.
(b) (i) Use cylindrical shells to show that the volume is $V = 16\pi \int_0^1 (2 - x)\sqrt{1 - x}\, dx$.
(ii) Apply the substitution $u = 1 - x$ to evaluate this integral.

15. Find, using cylindrical shells, the volume of the solid formed when the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is rotated about each of the following horizontal lines:
(a) $y = 0$ (b) $y = b$ (c) $y = c$, where $c > b$

16. The region $\mathcal{P}$ is bounded by the parabola $y^2 = 4ax$ and its latus rectum $x = a$. Use the method of cylindrical shells to find the volume of the solid generated when $\mathcal{P}$ is rotated about each of the following vertical lines:
(a) the latus rectum, (b) $x = 2a$, (c) the directrix.

17. The circle $(x - c)^2 + y^2 = a^2$, where $c > a$, is rotated about the y-axis to form a torus. Use the method of cylindrical shells to show that the torus has volume $2\pi^2 a^2 c$ cubic units.

18. A hole of diameter R is drilled through the centre of a solid sphere of diameter $2R$. Show that the remaining solid has volume $\frac{\sqrt{3}}{2}\pi R^3$ by using:

(a) the method of cylindrical shells, (b) slices perpendicular to the axis of rotation.

19. Consider the solid formed when the region under $y = f(x)$ between $x = a$ and $x = b$ is rotated about the line $x = c$. For simplicity assume that $f(x)$, a, b and c are positive, with $a < b < c$. In a typical cylindrical shell the radius is $r = c - x$ and the height is $h = f(x)$. As in the second Worked Exercise, it seems that the volume is:

$$V = \int 2\pi rh\, dr = \int_a^b 2\pi(c - x)f(x)\, dx\,.$$

The careful reader will have observed a problem with this, since if $r = c-x$ then $dr = -dx$. Nevertheless the above result is correct. Prove it.

EXTENSION

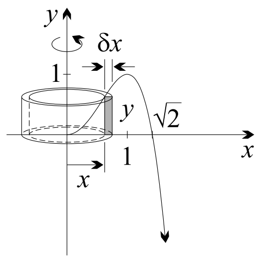
20. Let $y = f(x)$ be an even function. The region between this curve and the x-axis and between $x = -a$ and $x = a$ is rotated about the line $x = c$, where $0 < a < c$. Consider one cylindrical shell at $x = t$ and a second cylindrical shell at $x = -t$, where $0 \le t \le a$.

(a) Show that the total surface-area of the two cylinders is $A_T = 4\pi cf(t)$.

(b) Hence show that the volume of this solid is $V = 4\pi c \displaystyle\int_0^a f(t)\, dt$.

(c) Repeat the second Worked Exercise using this formula.

21. [A First Principals Approach] The region in the first quadrant below $y = 2x^2 - x^4$ is rotated about the y-axis to form a solid. Now suppose that the region is approximated by a series of thin rectangles of width δx. One such strip is shown in the diagram.



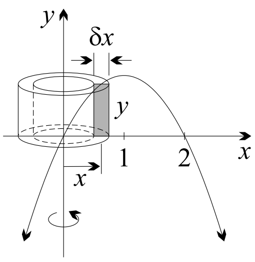
(a) Show that the volume of the cylindrical shell generated when this rectangle is rotated about the y-axis is:

$$\delta V = \pi(2x^2 - x^4)(2x + \delta x)\, \delta x\,.$$

(b) Write down a limiting sum for the volume of the solid.

(c) Rewrite the limiting sum as an integral and hence find the volume of the solid.

22. [A First Principals Approach] The region in the first quadrant below $y = 2x - x^2$ is rotated about the y-axis to form a solid. Consider the thin rectangle of width δx which generates a shell with volume δV. Notice that in this case x is the distance from the y-axis to the *midpoint* of the side of the rectangle.



(a) What are the inner and outer radii of the cylindrical shell?

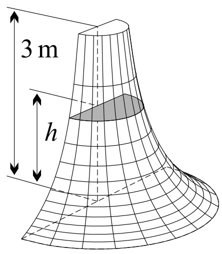
(b) Show that $\delta V = 2\pi x(2x - x^2)\, \delta x$.

(c) Write down a limiting sum for the volume of the solid.

(d) Rewrite the limiting sum as an integral and hence find the volume of the solid.

6C Solids with Known Cross-Sections

Whenever an expression is known or can be determined for the area $A(h)$ of a cross-section of a solid at height h then it is simply a matter of using the result in Box 1 to determine the volume.



WORKED EXERCISE: An artist creates a sculpture which is 3 metres high and has semi-circular cross-sections. The radius of the semi-circle at height h above its base is known to be $r = \frac{2}{h+1}$. Find the volume of the sculpture.

SOLUTION: The area of the semi-circle at height h is

$$\begin{aligned} A(h) &= \frac{1}{2} \times \pi \left(\frac{2}{h+1}\right)^2 \\ &= \frac{2\pi}{(h+1)^2} \\ \text{hence} \qquad V &= \int_0^3 \frac{2\pi}{(h+1)^2}\, dh \\ &= \left[\frac{-2\pi}{h+1}\right]_0^3 \\ &= \frac{3\pi}{2} \text{ cubic metres.} \end{aligned}$$

Note that this problem could have been done as a volume of revolution by rotating $y = \frac{2}{x+1}$ about the x-axis and then halving the answer. Try this as an exercise.

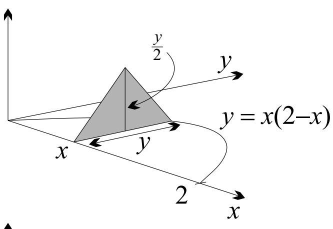
Simple Geometry: In many instances a knowledge of simple geometry will help to determine the expression for the area of the cross-section. The next Worked Exercise is an example of such a problem and the solution demonstrates two tricks that should be learnt to help make sketches look three dimensional.

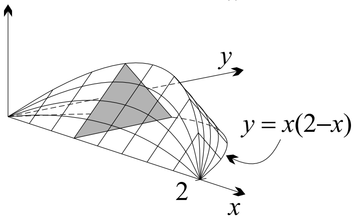
WORKED EXERCISE: A certain solid has a base which is the region between the parabola $y = x(2-x)$ and the x-axis. Each cross-section perpendicular to the base and parallel with the y-axis is an isosceles right angled triangle with the hypotenuse lying along the base of the solid.

(a) Draw a diagram showing this information.

(b) Show that the area of the cross-section at x is $A = \frac{1}{4}x^2(2-x)^2$.

(c) Hence find the volume of the solid.





SOLUTION: (a) The diagram is on the right. Notice that in order to help portray the perspective of the situation, the x- and y-axes have been skewed. The extra vertical axis indicates height above the xy-plane. These are two useful tricks to learn to help make your pictures look three dimensional. The solid is shown in the second diagram.

(b) In an isosceles right angled triangle, the distance from the hypotenuse to the oposite vertex is half the length of the hypotenuse. In this case the length of the hypotenuse is the y-coordinate hence:

$$\begin{aligned} A &= \tfrac{1}{2} \times y \times \tfrac{1}{2}y \\ &= \tfrac{1}{4}y^2 \\ &= \tfrac{1}{4}x^2(2-x)^2 . \end{aligned}$$

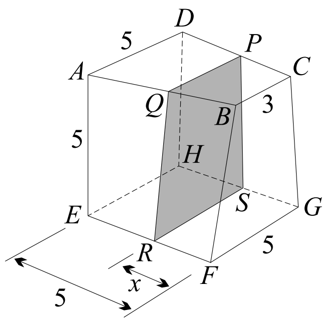
(c) Hence the volume is:

$$V = \int_0^2 \tfrac{1}{4}x^2(2-x)^2\, dx$$

$$= \tfrac{1}{4}\int_0^2 4x^2 - 4x^3 + x^4 \; dx$$
$$= \frac{1}{4}\left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right]_0^2$$
$$= \frac{4}{15}\,.$$

Similarity: In some harder problems similarity is used to find $A(h)$. It is often helpful to draw a separate diagram of the relevant part of the solid before applying the similarity argument.

WORKED EXERCISE: Two adjacent corners of a cube with edge length 5 cm are sliced off. The resulting solid is shown on the right. The faces $ABCD$ and $GCBF$ are congruent trapezia, with base 5 cm, top 3 cm and height 5 cm. Opposite these, $EFGH$ and $HDAE$ are square. The vertical slice $PQRS$ is taken x cm from $GCBF$. You may assume that this cross-section is a trapezium.



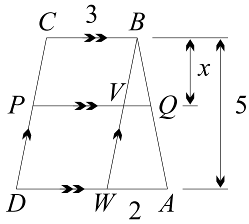
(a) Use similarity in trapezium $ABCD$ to show that

$$PQ = \tfrac{2}{5}x + 3\,.$$

(b) Find the area of $PQRS$ in terms of x.

(c) Hence find the volume of the solid.

SOLUTION:



(a) The diagram on the right has been provided for those unfamiliar with the geometry of the intercepts on transversals. Since $PQ||DA$, it should be clear that $\triangle BVQ \;|||\; \triangle BWA$. Hence:

$$\frac{PQ-3}{2} = \frac{x}{5} \quad \text{(ratio of matching sides and altitudes)}$$

so $\qquad PQ = \tfrac{2}{5}x + 3\,.$

(b) $\qquad$ Area $PQRS = \tfrac{1}{2}(PQ + RS) \times 5$
$$= \tfrac{1}{2}(\tfrac{2}{5}x + 8) \times 5$$
$$= (x + 20)\,\text{cm}^2.$$

(c)
$$V = \int_0^5 x + 20\,dx$$
$$= \left[\tfrac{1}{2}x^2 + 20x\right]_0^5$$
$$= 112\tfrac{1}{2}\,\text{cm}^3.$$

Exercise **6C**

1. A solid is 6 metres high. A horizontal cross-section at height h metres has area $(30 + h - h^2)$ square metres. Find the volume of the solid.

2. A monument has height 12 metres. A horizontal cross-section y metres from the top of the monument is an equilateral triangle of side length $\frac{y}{6}$ metres.

(a) Find the area of the cross-section. (b) Show that the monument has volume $4\sqrt{3}\,\text{m}^3$.

3. A solid is 20 cm high. A cross-section parallel to the base at height h cm is a square of side $\left(10 - e^{\frac{1}{10}h}\right)$ cm. Show that the solid has volume $990\,\text{cm}^3$, correct to the nearest cm^3.

4. The horizontal base of a solid is the circle $x^2 + y^2 = 36$. A typical vertical cross-section of the solid perpendicular to the x-axis is a square with one side in the base.

(a) Draw a diagram showing this situation.

(b) Find the area of the cross-section at position x.

(c) Hence determine the volume of the solid.

5. The base of a circus "big-top" is the ellipse $4x^2+9y^2 = 360$, with a scale of 1 unit : 1 metre. The roof is a dome where each vertical cross-section of the "big-top" perpendicular to the x-axis is a semicircle with its diameter in the base.

(a) Draw a diagram showing this situation.

(b) Show that a typical such cross-section has area $\frac{2}{9}\pi(90 - x^2)$.

(c) Hence show that the "big-top" has a capacity approximately equal to $795\,\text{m}^3$.

6. The horizontal base of a solid is the circle $x^2 + y^2 = 36$. A typical vertical cross-section of the solid perpendicular to the x-axis is a right-angled isosceles triangle with its hypotenuse in the base.

(a) Draw a diagram showing this situation.

(b) Determine the area of the cross-section at position x.

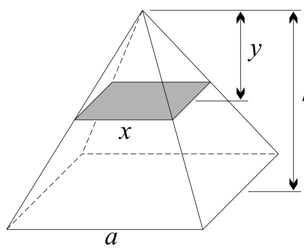
(c) Hence find the volume of the solid.

DEVELOPMENT

7. The horizontal base of a solid is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Vertical cross-sections of the solid parallel to the y-axis are equilateral triangles with one side in the base. Show that the solid has volume $\frac{4ab^2}{\sqrt{3}}$ cubic units.

8. The horizontal base of a solid is the region in the first quadrant bounded by the curves $y = x^2$ and $y = x^{\frac{1}{2}}$. Each vertical cross-section of the solid parallel to the x-axis is a right angled isosceles triangles with its hypotenuse lying in the base. Find the volume of the solid.

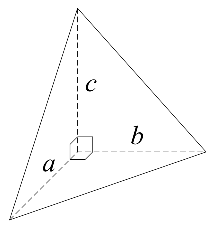
9. The diagram on the right shows a square pyramid of height h units and base base length a units. A typical square cross-section parallel to the base is shown. It is y units from the top of the pyramid, and it has side length x units.



(a) Show that $x = \frac{ay}{h}$.

(b) Hence prove that the pyramid has volume $\frac{1}{3}a^2h$ units^3.

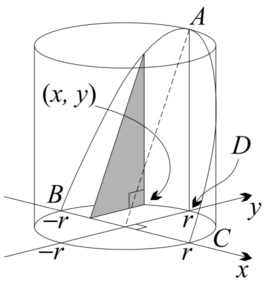
10. The diagram on the right shows a triangular pyramid. By slicing parallel to the base, prove that the pyramid has volume $\frac{1}{6}abc$.



11. (a) Show that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has area πab square units.

(b) Hence, by slicing parallel to the base, find the volume of a cone with elliptical base $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and perpendicular height h units.

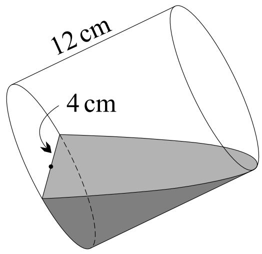
12. In the diagram on the right is a cylindrical wedge $ABCD$. The height of the cylinder is equal to the diameter of its base. Let the radius of the base be r units.



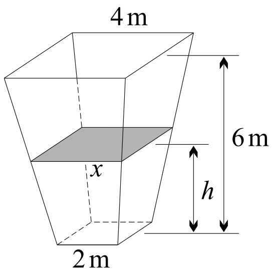
(a) Show that the typical triangular cross-section shaded has area $(r^2 - x^2)$ square units.

(b) Hence show that the wedge has volume $\frac{4}{3}r^3$ cubic units.

13. The diagram on the right shows a cylindrical drinking glass of interior radius 4 cm and perpendicular height 12 cm. The glass is filled with water which is then drunk slowly until half of the bottom of the glass is exposed. Use the methods of the previous question to find the volume of water remaining.



14. The diagram to the right shows a large tank of depth 6 metres. Its base and its top are squares with sides of 2 metres and 4 metres respectively. A typical square cross-section of side length x metres is shown h metres above the base.



(a) Show that $x = \frac{h}{3} + 2$.

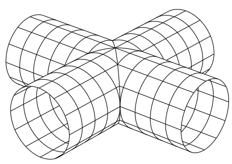
(b) Hence find the capacity of the tank.

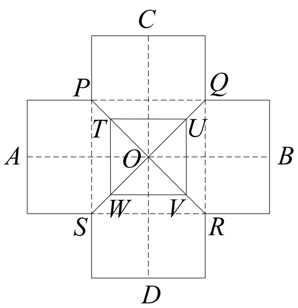
15. A rubbish skip on a building site has a rectangular base 6 metres by 3 metres, and its perpendicular height is 2 metres. Its sides are trapezia that slope outwards from bottom to top. The open top is a rectangle 7 metres by 4 metres.

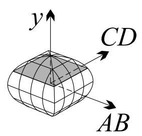
(a) Show that a rectangular cross-section h metres above the base has area $\left(\frac{h}{2} + 6\right)\left(\frac{h}{2} + 3\right)$ square metres.

(b) Hence find the capacity of the skip.

16.





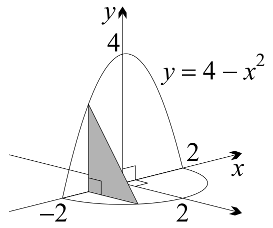


The first diagram shows two identical intersecting cylinders of radius r. The second diagram is the view from above. Their axes AB and CD intersect at 90° at the point O. The third diagram shows the solid which is common to both cylinders, bounded at its widest point by the horizontal square $PQRS$.

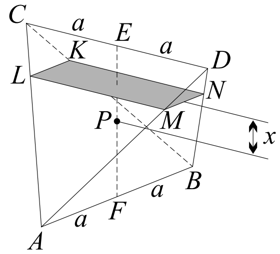
(a) The typical square cross-section $TUVW$ shown is parallel to the square $PQRS$ and y units above it. Find an expression for the area of this typical cross-section of the solid in terms of y.

(b) Hence find the volume that is common to the two intersecting cylinders.

17. The solid shown on the right has a semicircular base of radius 2 cm. Vertical cross-sections of the solid perpendicular to the diameter of the semicircle are right-angled triangles, the heights of which are bounded by the parabola $y = 4 - x^2$. Show that the solid has volume $3\pi\,\text{cm}^3$.



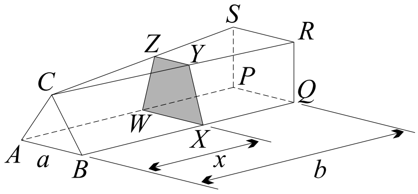
18. The diagram on the right shows tetrahedron $ABCD$. The lines AB and CD have length $2a$ and lie in horizontal planes at a distance $2a$ apart. The midpoint E of CD is vertically above the midpoint F of AB, and AB runs from South to North, whilst CD runs from West to East. Rectangle $KLMN$ is the horizontal cross-section of the tetrahedron $ABCD$ at distance x from the midpoint P of EF (so $PE = PF = a$).



(a) By considering the triangle ABE, deduce that $KL = a - x$, and find the area of the rectangle $KLMN$.

(b) Find the volume of the tetrahedron $ABCD$.

19. The diagram shows a sandstone solid with rectangular base $ABQP$ of length b metres and width a metres. The end $PQRS$ is a square, and the other end ABC is an equilateral triangle. Both ends are perpendicular to the base.



Consider cross-section of the solid $WXYZ$ which is parallel with the ends. Let $BX = x$ metres.

(a) Find the height of the equilateral triangle ABC.

(b) Given that the triangles CRS and CYZ are similar, find YZ in terms of a, b and x.

(c) Let the perpendicular height of the trapezium $WXYZ$ be h metres. Show that

$$h = \frac{a}{2}\left(\sqrt{3} + \left(2 - \sqrt{3}\right)\frac{x}{b}\right).$$

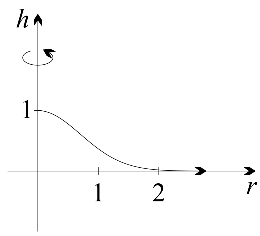
(d) Hence show that the cross-sectional area of $WXYZ$ is given by

$$\frac{a^2}{4b^2}\left((2 - \sqrt{3})x + b\sqrt{3}\right)(b + x).$$

(e) Find the volume of the solid.

EXTENSION

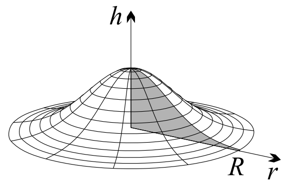
20. The graph of $h = e^{-r^2}$ is shown on the right. The region below the curve which lies in the first quadrant is rotated about the vertical axis to generate a solid which extends horizontally to infinity in all directions. A portion of this solid is shown in the second diagram.



(a) Consider the part of the solid which lies above the circle with radius $r = R$.

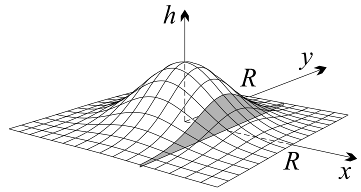
(i) Show that the volume of this is given by

$$\int_0^R 2\pi r e^{-r^2}\,dr.$$



(ii) Evaluate this integral.

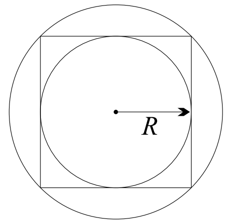
(b) Now in the Cartesian plane, $r^2 = x^2 + y^2$ so we may divide the base of our solid into a rectangular grid and write $h = e^{-(x^2+y^2)}$. Consider the part of the solid which lies above the square $-R \le x \le R$ and $-r \le y \le R$, as shown on the right.



(i) Let $I = \displaystyle\int_0^R e^{-t^2}\,dt$. Do not attempt to evaluate this integral. Show that the area of the slice parallel with the y-axis at any given value of x is equal to $2e^{-x^2} I$.

(ii) Hence show that the volume of this portion of the solid is $4I^2$.

(c) Looking from above, the base of the solid in part (a) is a circle with radius R, and the base of part (b) is a square with side $2R$. Now consider the portion of our solid above a circular base which passes through the corners of the square in (b).



(i) What is the radius of the base?

(ii) Find the volume of this portion of the solid.

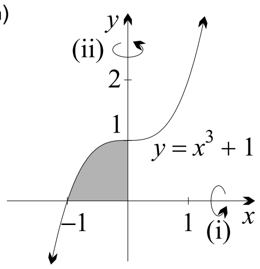
(d) Write down an inequality involving the volumes found in parts (a), (b) and (c).

(e) Hence evaluate $\displaystyle\int_0^\infty e^{-t^2}\,dt$ by taking the limit as $R \to \infty$.

Chapter Six

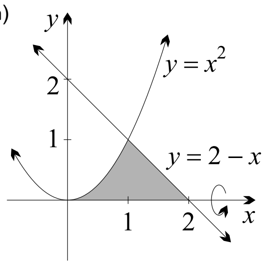
Exercise **6A** (Page 28)

1(a)



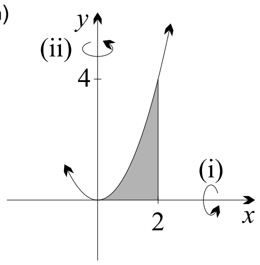
(b)(i) $\frac{9\pi}{14}$ **(ii)** $\frac{3\pi}{5}$

2(a)



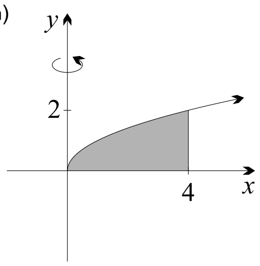
(b) $\frac{8\pi}{15}$

3(a)



(b)(i) $\frac{32\pi}{5}$ **(ii)** 8π

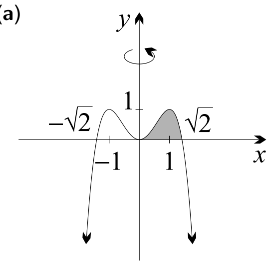
4(a)



(b) $\frac{128\pi}{5}$

5 $\frac{6\pi}{7}$
6(a) 2π **(b)** $\frac{48\pi}{5}$ **(c)** $\frac{7\pi}{10}$
7 4π
8 $\frac{32\pi}{3}$
9 $\frac{256\pi}{15}$
10(a) $\frac{8\pi}{3}$ **(b)** $\frac{224\pi}{15}$ **(c)** 8π
11(a) $\frac{9\pi}{14}$ **(b)** $\frac{15\pi}{7}$
12(a) $\frac{8\pi}{3}$ **(b)** 8π
13(b) $\frac{296\pi}{15}$
14(a) $\frac{32\pi a^3}{15}$ **(b)** $\frac{112\pi a^3}{15}$ **(c)** $\frac{128\pi a^3}{15}$
15(b) $\frac{8\pi}{3}$

16(a)

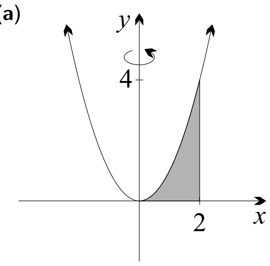


(b) $\frac{4\pi}{3}$

17(b) $288\pi^2$
19(a) $\frac{4}{3}\pi ab^2$ **(b)** $2\pi^2 ab^2$ **(c)** $2\pi^2 abc$
20(a) $AD = \frac{9}{5}$, $CD = \frac{16}{5}$ **(c)** $\frac{92\pi}{15}\text{cm}^3$
23(a) $\frac{1}{\sqrt{2}}(x - x^2)$ **(b)** $\delta V = \frac{\pi}{\sqrt{2}}(x - x^2)^2\,\delta x$
(c) $V' = \frac{\pi}{\sqrt{2}}(x - x^2)^2$

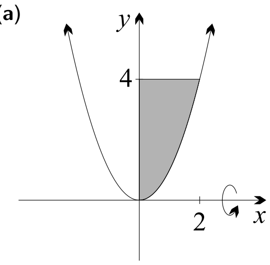
Exercise **6B** (Page 33)

1(a)



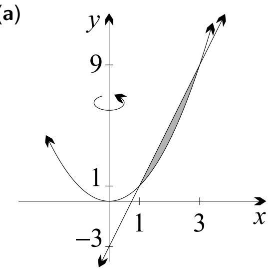
(b) 8π

2(a)



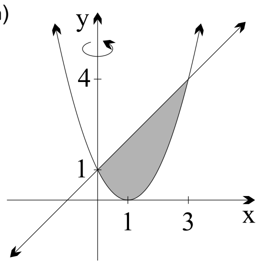
(b) $\frac{128\pi}{5}$

3(a) $\frac{128\pi}{3}$ **(b)** $\frac{3888\pi}{5}$ **(c)** $\frac{\pi}{10}$
4(a) $\frac{3\pi}{5}$ **(b)** $\frac{9\pi}{14}$
5(a) $\frac{8\pi}{3}$ **(b)** 8π
6(a) $\frac{256\pi}{15}$ **(b)** $\frac{192\pi}{5}$
7(a) $\frac{8\pi}{3}$ **(b)** 8π
9(a)



(b) $\frac{16\pi}{3}$

10(a)



(b) $\frac{27\pi}{2}$

11 2π

12 8π

13 $\frac{28\pi}{5}$

14 $\frac{256\pi}{15}$

15(a) $\frac{4}{3}\pi ab^2$ **(b)** $2\pi^2 ab^2$ **(c)** $2\pi^2 abc$

16(a) $\frac{32\pi a^3}{15}$ **(b)** $\frac{112\pi a^3}{15}$ **(c)** $\frac{128\pi a^3}{15}$

21(b) $V = \lim_{\delta x \to 0} \sum_{x=0}^{x=\sqrt{2}} \pi(2x^2 - x^4)(2x + \delta x)\,\delta x$

(c) $\frac{4\pi}{3}$

22(a) $(x - \frac{1}{2}\delta x)$, $(x + \frac{1}{2}\delta x)$

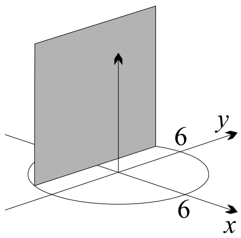
(c) $V = \lim_{\delta x \to 0} \sum_{x=0}^{x=2} 2\pi x(2x - x^2)\,\delta x$ **(d)** $\frac{8\pi}{3}$

Exercise **6C** (Page 37)

1 $126\,\mathrm{m}^3$

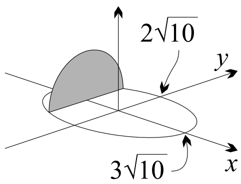
2(a) $\frac{y^2\sqrt{3}}{144}\,\mathrm{m}^2$

4(a)

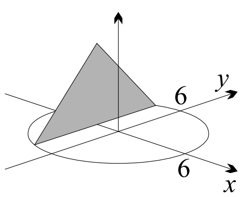


(b) $4(36 - x^2)$ **(c)** 1152

5(a)



6(a)



(b) $36 - x^2$ **(c)** 288

8 $\frac{9}{280}$

11(b) $\frac{1}{3}\pi abh$

13 $128\,\mathrm{ml}$

14(b) $56\,\mathrm{m}^3$

15(b) $45\frac{2}{3}\,\mathrm{m}^3$

16(a) $4(r^2 - y^2)$ **(b)** $\frac{16}{3}r^3$

18(a) $a^2 - x^2$ **(b)** $\frac{4}{3}a^3$

19(a) $\frac{\sqrt{3}}{2}a$ **(b)** $\frac{ax}{b}$ **(e)** $\frac{1}{12}a^2 b(5 + 2\sqrt{3})\,\mathrm{m}^3$

20(a)(ii) $\pi(1 - e^{-R^2})$ **(c)(i)** $R\sqrt{2}$ **(ii)** $\pi(1 - e^{-2R^2})$

(d) $\pi(1 - e^{-R^2}) \le 4I^2 \le \pi(1 - e^{-2R^2})$ **(e)** $\frac{\sqrt{\pi}}{2}$