038016

THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH1902

LINEAR ALGEBRA (ADVANCED)

June/July 2003

TIME ALLOWED: One and a half hours

LECTURERS: TM Gagen, DJ Ivers, N Joshi

This Examination has 3 Printed Components.

(1) AN EXTENDED ANSWER QUESTION PAPER (THIS BOOKLET, GREEN 80/16A): 4 PAGES NUMBERED 1 TO 4; 5 QUESTIONS NUMBERED 1 TO 5.

(2) A MULTIPLE CHOICE QUESTION PAPER (YELLOW 80/16B): 3 PAGES NUMBERED 1 TO 3; 15 QUESTIONS NUMBERED 1 TO 15.

(3) A MULTIPLE CHOICE ANSWER SHEET (WHITE 80/16C): 1 PAGE.

Components 2 and 3 MUST NOT be removed from the examination room.

This Examination has 2 Sections: ***Extended Answer*** *and* ***Multiple Choice****.*

The ***Extended Answer Section*** *is worth 75% of the total marks for the paper: all questions may be attempted; questions are of equal value; working must be shown.*

The ***Multiple Choice Section*** *is worth 25% of the total marks for the paper: all questions may be attempted; questions are of equal value; answers must be coded onto the* ***Multiple Choice Answer Sheet****.*

Calculators will be supplied; no other electronic calculators are permitted.

1. Let p_1 and p_2 be the two planes given by $3x + 4y - 2z = 5$ and $2x - 3y + 4z = 3$, respectively, and let ℓ be their line of intersection.

(*i*) **(3 marks)** Write down vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ which are perpendicular to p_1 and p_2 respectively, and calculate $\mathbf{n}_1 \times \mathbf{n}_2$.

(*ii*) **(1 mark)** Find the cosine of the angle θ between p_1 and p_2.

(*iii*) **(1 mark)** Verify that the point $(1, 1, 1)$ lies on ℓ.

(*iv*) **(3 marks)** Find equations of ℓ in

(a) vector parametric form

(b) scalar parametric form

(c) cartesian form.

(*v*) **(2 marks)** Find the cartesian equation of the plane through $(1, -1, 2)$ perpendicular to ℓ.

2. (*i*) **(4 marks)** Let $ABCD$ be a parallelogram and let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{BC}$.

(a) Write down $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Show that

$$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = 2\left(|\mathbf{a}|^2 + |\mathbf{b}|^2\right).$$

(*ii*) Two faces BCD and ACD of a tetrahedron $ABCD$ have centroids K, L, respectively. You are given that the centroid K of the face BCD satisfies $\mathbf{k} = \frac{1}{3}(\mathbf{b}+\mathbf{c}+\mathbf{d})$, and so on, where the vector $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, etc, and O is the origin.

(a) **(4 marks)** Find $\overrightarrow{KL}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and show that $KL \parallel AB$. and that AK and BL meet at a point P with $AP : PK = 3 : 1 = BP : PL$.

(b) **(2 marks)** Hence show that the four 'spatial medians' AK, BL, CM, DN from each vertex to the centroid of the opposite face intersect at a point P which divides each of them in the ratio $3 : 1$.

3. (*i*) Consider the system of linear equations,

$$\begin{aligned} x + 2y + z &= a \\ 2x + 5y + 3z &= b \\ 2y + 2z &= c\,. \end{aligned}$$

(a) **(4 marks)** Use Gaussian elimination without row interchanges to solve the system. What condition must a, b, c satisfy in order for the system to have a solution?

(b) **(1 mark)** If the system is written in the matrix form $\mathbf{Ax} = \mathbf{d}$, show that a solution exists if and only if $\mathbf{d}$ is of the form

$$\mathbf{d} = a\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(*ii*) Answer true or false to each of the following giving a counter-example when the statement is false.

(a) **(1 mark)** If a system of r linear equations in n unknowns has a unique solution, then $r = n$.

(b) **(1 mark)** A system of r linear equations in n unknowns has no solution if $r > n$.

(c) **(1 mark)** A system of n equations in n unknowns $A\mathbf{x} = \mathbf{b}$ in which $\det A = 0$ always has infinitely many solutions.

(d) **(2 marks)** Suppose that a system of r linear equations in n unknowns is written as a matrix equation $A\mathbf{x} = \mathbf{b}$. If B is an $n \times r$ matrix such that $BA = I_n$, then $\mathbf{x} = B\mathbf{b}$ is a solution of the system.

4. (*i*) **(3 marks)** Show that if a, b, c, d are integers such that $a + b = c + d$, then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has integer eigenvalues given by $\lambda_1 = a + b$ and $\lambda_2 = a - c$. Find the corresponding eigenvectors.

(*ii*) (a) **(3 marks)** Find all matrices $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $AX = XA$, where $A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}$ and show that every such matrix has the form $\alpha A + \beta I$, for some real numbers α, β.

(b) **(2 marks)** Calculate α, β such that $A^2 = \alpha A + \beta I$, where A is the matrix in Part(a).

(*iii*) **(2 marks)** The matrix $Y = \begin{pmatrix} 0 & 5 & 10 \\ 1 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix}$ commutes with $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$.
Show that Y is not a polynomial in B. (You do not need to show $YB = BY$.)

5. (*i*) (a) **(3 marks)** By expanding by cofactors down the last column, show that the characteristic polynomial $\det(M - \lambda I)$ of the $n \times n$ matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & 0 & \dots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -c_{n-1} \end{pmatrix}$$

is given by

$$(-1)^n(\lambda^n + c_{n-1}\lambda^{n-1} + \ldots + \lambda c_1 + c_0).$$

(b) **(3 marks)** Consider the case $n = 3$ in Part (a). Suppose c_0, c_1, c_2 are given by symmetric combinations of a, b, c, i.e. $c_0 = -abc$, $c_1 = ab + bc + ca$, $c_2 = -(a + b + c)$, where $a > b > c$. Find the eigenvalues and the eigenspace belonging to the largest eigenvalue of M.

(*ii*) (a) **(2 marks)** Suppose that $A = (a_{ij})$ is a 3×3 matrix in which $a_{11} + a_{22} + a_{33} = 0$. Show that $\det(I + A) = 1 + \det A$ if and only if

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = 0\,.$$

(b) **(2 marks)** Give an example of a 3×3 non-zero complex symmetric matrix A such that $\det(I + A) = 1 + \det A$. (A matrix A is symmetric if $A^T = A$.)