CHAPTER **FOUR**

Polynomial Functions

The primitive of a linear function is a quadratic, the primitive of a quadratic function is a cubic, and so on, so ultimately the study even of linear functions must involve the study of polynomial functions of arbitrary degree. In this course, linear and quadratic functions are studied in great detail, but this chapter begins the systematic study of polynomials of higher degree. The intention is first to study the interrelationships between their factorisation, their graphs, their zeroes and their coefficients, and secondly, to reinterpret all these ideas geometrically by examining curves defined by algebraic equations.

Study Notes: After the terminology of polynomials has been introduced in Section 4A, graphs are drawn in Section 4B — as always, machine drawing of some of these examples may illuminate the wide variety of possible curves generated by polynomials. Sections 4C–4E concern the division of polynomials, the remainder and factor theorems, and their consequences. The work in these sections may have been introduced at a more elementary level in earlier years. Section 4F, however, which deals with the relationship between the zeroes and the coefficients, is probably quite new except in the context of quadratics. The problem of factoring a given polynomial is common to Sections 4B–4F, and a variety of alternative approaches are developed through these sections. The final Section 4G applies the methods of the chapter to geometrical problems about polynomial curves, circles and rectangular hyperbolas.

4 A The Language of Polynomials

Polynomials are expressions like the quadratic $x^2 - 5x + 6$ or the quartic $3x^4 - \frac{2}{3}x^3 + 4x + 7$. They have occurred routinely throughout the course so far, but in order to speak about polynomials in general, our language and notation needs to be a little more systematic.

1 Polynomials: A *polynomial function* is a function that can be written as a sum:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants, and n is a cardinal number.

The term a_0 is called the *constant term*. This is the value of the polynomial at $x = 0$, and so is the y-intercept of the graph. The constant term can also be written as $a_0 x^0$, so that every term is then a multiple $a_k x^k$ of a power of x in which the index k is a cardinal number. This allows sigma notation to be used, and we can write

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$$P(x) = \sum_{k=0}^{n} a_k x^k.$$

Such notation is very elegant, but it can also be confusing, and questions involving sigma notation are usually best converted into the longer notation before proceeding. In the next chapter, however, we will need such notation.

NOTE: Careful readers may notice that a_0x^0 is undefined at $x = 0$. This means that rewriting the quadratic $x^2 + 3x + 2$ as $x^2 + 3x^1 + 2x^0$ causes a problem at $x = 0$. To overcome this, the convention is made that the term a_0x^0 is interpreted as a_0 before any substitution is performed.

Leading Term and Degree: The term of highest index with nonzero coefficient is called the *leading term*. Its coefficient is called the *leading coefficient* and its index is called the *degree*. For example, the polynomial

$$P(x) = -5x^6 - 3x^4 = 2x^3 + x^2 - x + 9$$

has leading term $-5x^6$, leading coefficient -5 and degree 6, which is written as '$\deg F(x) = 6$'.

A *monic polynomial* is a polynomial whose leading coefficient is 1; for example, $P(x) = x^3 - 2x^2 - 3x + 4$ is monic. Notice that every polynomial is a multiple of a monic polynomial:

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = a_n\left(x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \cdots + \frac{a_1}{a_n}x + \frac{a_0}{a_n}\right).$$

Some Names of Polynomials: Polynomials of low degree have standard names.

- The *zero polynomial* $Z(x) = 0$ is a special case. It has a constant term 0. But it has no term with a nonzero coefficient, and therefore has no leading term, no leading coefficient, and most importantly, no degree. It is also quite exceptional in that its graph is the x-axis, so that every real number is a zero of the zero polynomial.
- A *constant polynomial* is a polynomial whose only term is the constant term, for example,

  $$P(x) = 4, \qquad Q(x) = -\tfrac{3}{5}, \qquad R(x) = \pi, \qquad Z(x) = 0.$$

  Apart from the zero polynomial, all constant polynomials have degree 0, and are equal to their leading term and to their leading coefficient.
- A *linear polynomial* is a polynomial whose graph is a straight line:

  $$P(x) = 4x - 3, \qquad Q(x) = -\tfrac{1}{2}x, \qquad R(x) = 2, \qquad Z(x) = 0.$$

  Linear polynomials have degree 1 when the coefficient of x is nonzero, and are constant polynomials when the coefficient of x is zero.
- A polynomial of degree 2 is called a *quadratic polynomial*:

  $$P(x) = 3x^2 + 4x - 1, \qquad Q(x) = -\tfrac{1}{2}x - x^2, \qquad R(x) = 9 - x^2.$$

  Notice that the coefficient of x^2 must be nonzero for the degree to be 2.
- Polynomials of higher degree are called *cubics* (degree 3), *quartics* (degree 4), *quintics* (degree 5), and so on.

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Addition and Subtraction: Any two polynomials can be added or subtracted, and the results are again polynomials:

$$(5x^3 - 4x + 3) + (3x^2 - 3x - 2) = 5x^3 + 3x^2 - 7x + 1$$
$$(5x^3 - 4x + 3) - (3x^2 - 3x - 2) = 5x^3 - 3x^2 - x + 5$$

The zero polynomial $Z(x) = 0$ is the neutral element for addition, in the sense that $P(x) + 0 = P(x)$, for all polynomials $P(x)$. The *opposite polynomial* $-P(x)$ of any polynomial $P(x)$ is obtained by taking the opposite of every coefficient. Then the sum of $P(x)$ and $-P(x)$ is the zero polynomial; for example,

$$(4x^4 - 2x^2 + 3x - 7) + (-4x^4 + 2x^2 - 3x + 7) = 0.$$

The degree of the sum or difference of two polynomials is normally the maximum of the degrees of the two polynomials, as in the first example above, where the two polynomials had degrees 2 and 3 and their sum had degree 3. If, however, the two polynomials have the same degree, then the leading terms may cancel out and disappear, for example,

$$(x^2 - 3x + 2) + (9 + 4x - x^2) = x + 11, \text{ which has degree 1,}$$

or the two polynomials may be opposites so that their sum is zero.

2 DEGREE OF THE SUM AND DIFFERENCE: Suppose that $P(x)$ and $Q(x)$ are nonzero polynomials of degree n and m respectively.
- If $n \neq m$, then $\deg\big(P(x) + Q(x)\big) = $ maximum of m and n.
- If $n = m$, then $\deg\big(P(x) + Q(x)\big) \le n$ or $P(x) + Q(x) = 0$.

Multiplication: Any two polynomials can be multiplied, giving another polynomial:

$$\begin{aligned}(3x^3 + 2x + 1) \times (x^2 - 1) &= (3x^5 + 2x^3 + x^2) - (3x^3 + 2x + 1)\\ &= 3x^5 - x^2 + x^2 - 2x - 1\end{aligned}$$

The constant polynomial $I(x) = 1$ is the neutral element for multiplication, in the sense that $P(x) \times 1 = P(x)$, for all polynomials $P(x)$. Multiplication by the zero polynomial on the other hand always gives the zero polynomial.

If two polynomials are nonzero, then the degree of their product is the sum of their degrees, because the leading term of the product is always the product of the two leading terms.

3 DEGREE OF THE PRODUCT: If $P(x)$ and $Q(x)$ are nonzero polynomials, then

$$\deg\big(P(x) \times Q(x)\big) = \deg P(x) + \deg Q(x).$$

Factorisation of Polynomials: The most important problem of this chapter is the factorisation of a given polynomial. For example,

$$x(x+2)^2(x-2)^2(x^2 + x + 1) = x^7 + x^6 - 7x^5 - 8x^4 + 8x^3 + 16x^2 + 16x$$

is a reasonably routine expansion of a factored polynomial, but it is not clear how to move from the expanded form back to the factored form.

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Identically Equal Polynomials: We need to be quite clear what is meant by saying that two polynomials are the same.

4

IDENTICALLY EQUAL POLYNOMIALS: Two polynomials $P(x)$ and $Q(x)$ are called *identically equal*, written as $P(x) \equiv Q(x)$, if they are equal for all values of x:

$$P(x) \equiv Q(x) \qquad \text{means} \qquad P(x) = Q(x), \text{ for all } x.$$

For two polynomials to be equal, the corresponding coefficients in the two polynomials must all be equal.

WORKED EXERCISE: Find a, b, c, d and e if $ax^4 + bx^3 + cx^2 + dx + e \equiv (x^2 - 3)^2$.

SOLUTION: Expanding, $(x^2 - 3)^2 = x^4 - 6x^2 + 9$.
Now comparing coefficients, $a = 1$, $b = 0$, $c = -6$, $d = 0$ and $e = 9$.

Polynomial Equations: If $P(x)$ is a polynomial, then the equation formed by setting $P(x) = 0$ is a *polynomial equation*. For example, using the polynomial in the previous paragraph, we can form the equation

$$x^7 + x^6 - 7x^5 - 8x^4 + 8x^3 + 16x^2 + 16x = 0.$$

Solving polynomial equations and factoring polynomial functions are very closely related. For example, using the factoring of the previous paragraph,

$$x(x+2)^2(x-2)^2(x^2 + x + 1) = 0,$$

so the solutions are $x = 0$, $x = 2$ and $x = -2$. Notice that the quadratic factor $x^2 + x + 1$ has no zeroes, because its discriminant is $\Delta = -3$.

The solutions of a polynomial equation are called *roots*, whereas the *zeroes* of a polynomial function are the values of x where the value of the polynomial is zero. The distinction between the words is not always strictly observed.

Exercise 4A

1. State whether or not the following are polynomials.

(a) $3x^2 - 7x$
(b) $\dfrac{1}{x^2} + x$
(c) $\sqrt{x} - 2$
(d) $3x^{\frac{2}{3}} - 5x + 11$
(e) $\sqrt{3}x^2 + \sqrt{5}x$
(f) $2^x - 1$
(g) $(x+1)^3$
(h) $\dfrac{7x^{13} + 3x}{4}$
(i) $\log_e x$
(j) $\frac{4}{3}x^3 - ex^2 + \pi x$
(k) 5
(l) $\dfrac{x-2}{x+1}$

2. For each polynomial, state: (i) the degree, (ii) the leading coefficient, (iii) the leading term, (iv) the constant term, (v) whether or not the polynomial is monic. Expand the polynomial first where necessary.

(a) $4x^3 + 7x^2 - 11$
(b) $10 - 4x - 6x^3$
(c) 2
(d) x^{12}
(e) $x^2(x-2)$
(f) $(x^2 - 3x)(1 - x^3)$
(g) 0
(h) $x(x^3 - 5x + 1) - x^2(x^2 - 2)$
(i) $6x^7 - 4x^6 - (2x^5 + 1)(5 + 3x^2)$

3. If $P(x) = 5x + 2$ and $Q(x) = x^2 - 3x + 1$, find:

(a) $P(x) + Q(x)$
(b) $Q(x) + P(x)$
(c) $P(x) - Q(x)$
(d) $Q(x) - P(x)$
(e) $P(x)Q(x)$
(f) $Q(x)P(x)$

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4. If $P(x) = 5x + 2$, $Q(x) = x^2 - 3x + 1$ and $R(x) = 2x^2 - 3$, show, by expanding separately the LHS and RHS, that:
(a) $P(x)\,(Q(x) + R(x)) = P(x)Q(x) + P(x)R(x)$
(b) $(P(x)Q(x))\,R(x) = P(x)\,(Q(x)R(x))$
(c) $(P(x) + Q(x)) + R(x) = P(x) + (Q(x) + R(x))$

5. Express each of the following polynomials as a multiple of a monic polynomial:
(a) $2x^2 - 3x + 4$
(b) $3x^3 - 6x^2 - 5x + 1$
(c) $-2x^5 + 7x^4 - 4x + 11$
(d) $\frac{2}{3}x^3 - 4x + 16$

DEVELOPMENT

6. Factor the following polynomials completely, and state all the zeroes.
(a) $x^3 - 8x^2 - 20x$
(b) $2x^4 - x^3 - x^2$
(c) $x^4 - 5x^2 - 36$
(d) $x^3 - 8$
(e) $x^4 - 81$
(f) $x^6 - 1$

7. (a) The polynomials $P(x)$ and $Q(x)$ have degrees p and q respectively, and $p \neq q$. What is the degree of: (i) $P(x)Q(x)$, (ii) $P(x) + Q(x)$?
(b) What differences would it make if $P(x)$and $Q(x)$ both had the same degree p?
(c) Give an example of two polynomials, both of degree 2, which have a sum of degree 0.

8. Write down the monic polynomial whose degree, leading coefficient, and constant term are all equal.

9. Find the values of a, b and c if:
(a) $ax^2 + bx + c \equiv 3x^2 - 4x + 1$
(b) $(a-b)x^2 + (2a+b)x \equiv 7x - x^2$
(c) $a(x-1)^2 + b(x-1) + c \equiv x^2$
(d) $a(x+2)^2 + b(x+3)^2 + c(x+4)^2 \equiv 2x^2 + 8x + 6$

10. For the polynomial $(a-4)x^7 + (2-3b)x^3 + (5c-1)$, find the values of a, b and c if it is:
(a) of degree 3, (b) of degree 0, (c) of degree 7 and monic, (d) the zero polynomial.

11. Suppose that $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ and $P(3x) \equiv P(x)$.
(a) Show that $81ax^4 + 27bx^3 + 9cx^2 + 3dx + e \equiv ax^4 + bx^3 + cx^2 + dx + e$.
(b) Hence show that $P(x)$ is a constant polynomial.

12. (a) Show that if $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, then $b = d = 0$.
(b) Show that if $Q(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is odd, then $b = d = f = 0$.
(c) Give a general statement of the situation in parts (a) and (b).

13. $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ are polynomials. Indicate whether the following statements are true or false, giving reasons for your answers.
(a) If $P(x)$ is even, then $P'(x)$ is odd.
(b) If $Q'(x)$ is even, then $Q(x)$ is odd.
(c) If $R(x)$ is odd, then $R'(x)$ is even.
(d) If $S'(x)$ is odd, then $S(x)$ is even.

EXTENSION

14. Real numbers a and b are said to be *multiplicative inverses* if $ab = ba = 1$.
(a) What can be said about two polynomials if they are multiplicative inverses.
(b) Explain why a polynomial of degree ≥ 1 cannot have a multiplicative inverse.

15. We have assumed in the notes above that if two polynomials $P(x)$ and $Q(x)$ are equal for all values of x (that is, if their graphs are the same), then their degrees are equal and their corresponding coefficients are equal. Here is a proof using calculus.
(a) Explain why substituting $x = 0$ proves that the constant terms are equal.
(b) Explain why differentiating k times and substituting $x = 0$ proves that the coefficients of x^k are equal.

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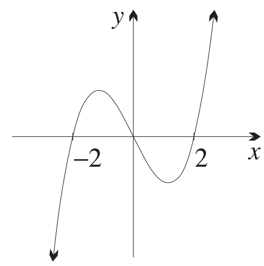
4 B Graphs of Polynomial Functions

A lot of work has already been done on sketching polynomial functions. We know already that the graph of any polynomial function will be a continuous and differentiable curve, whose domain is all real numbers, and which possibly intersects the x-axis at one or more points. This section will concentrate on two main concerns. First, how does the graph behave for large positive and negative values of x? Secondly, given the full factorisation of the polynomial, how does the graph behave near its various x-intercepts? We will not be concerned here with further questions about turning points and inflexions which are not zeroes.

The Graphs of Polynomial Functions: It should be intuitively obvious that for large positive and negative values of x, the behaviour of the curve is governed entirely by the sign of its leading term. For example, the cubic graph sketched on the right below is

$$P(x) = x^3 - 4x = x(x-2)(x+2).$$



For large positive values of x, the degree 1 term $-4x$ is negative, but is completely swamped by the positive values of the degree 3 term x^3. Hence $P(x) \to \infty$ as $x \to \infty$. On the other hand, for large negative values of x, the term $-4x$ is positive, but is negligible compared with the far bigger negative values of the term x^3. Hence $P(x) \to -\infty$ as $x \to \infty$.

In the same way, every polynomial of odd degree has a graph that disappears off diagonally opposite corners. Being continuous, it must therefore be zero somewhere. Our example actually has three zeroes, but however much it were raised or lowered or twisted, only two zeroes could ever be removed. Here is the general situation.

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BEHAVIOUR OF POLYNOMIALS FOR LARGE x: Suppose that $P(x)$ is a polynomial of degree at least 1 with leading term $a_n x^n$.
- As $x \to \infty$, $P(x) \to \infty$ if a_n is positive, and $P(x) \to -\infty$ if a_n is negative.
- As $x \to -\infty$, $P(x)$ behaves the same as when $x \to \infty$ if the degree is even, but $P(x)$ behaves in the opposite way if the degree is odd.
- It follows that every polynomial of odd degree has at least one zero.

PROOF: Clearly the leading term dominates proceedings as $x \to \infty$ and as $x \to -\infty$, but here is a more formal proof, should it be required.

A. Let
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$
then
$$\frac{P(x)}{x^n} = a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}.$$
As $x \to \infty$ or $x \to -\infty$, $\dfrac{P(x)}{x^n} \to a_n$.
Hence for large positive x, $P(x)$ has the same sign as a_n. For large negative x, $P(x)$ has the same sign as a_n when n is even, and the opposite sign to a_n when n is odd.

B. If $P(x)$ is a polynomial of odd degree, then $P(x) \to \infty$ on either the left or right side, and $P(x) \to -\infty$ on the other side. Hence, being a continuous function, $P(x)$ must cross the x-axis somewhere.

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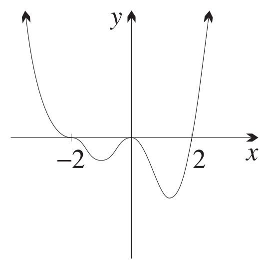
Zeroes and Sign: If the polynomial can be completely factored, then its zeroes can be read off very quickly, and our earlier methods would then have called for a table of test values to decide its sign. Here, for example, is the table of test values and the sketch of

$P(x) = (x+2)^3 x^2 (x-2)$.

x	-3	-2	-1	0	1	2	3
y	45	0	-9	0	-27	0	1125



The function changes sign around $x = -2$ and $x = 2$, where the associated factors $(x+2)^3$ and $(x-2)$ have odd degrees, but not around $x = 0$ where the factor x^2 has even degree.

The curve has a horizontal inflexion on the x-axis at $x = -2$ corresponding to the factor $(x+2)^3$ of odd degree, and a turning point on the x-axis at $x = 0$ corresponding to the factor x^2 of even degree — proving this will require calculus, although the result is fairly obvious by comparison with the known graphs of $y = x^2$, $y = x^3$ and $y = x^4$.

Multiple Zeroes: Some machinery is needed to describe the situation. The zero $x = -2$ of the polynomial $P(x) = (x+2)^3 x^2 (x-2)$ is called a *triple zero*, the zero $x = 0$ is called a *double zero*, and the zero $x = 2$ is called a *simple zero*.

6 MULTIPLE ZEROES: Suppose that $x - \alpha$ is a factor of a polynomial $P(x)$, and

$$P(x) = (x-\alpha)^m Q(x), \quad \text{where } Q(x) \text{ is not divisible by } x-\alpha.$$

Then $x = \alpha$ is called a *zero of multiplicity m*.

A zero of multiplicity 1 is called a *simple zero*, and a zero of multiplicity greater than 1 is called a *multiple zero*.

Behaviour at Simple and Multiple Zeroes: In general:

7 MULTIPLE ZEROES AND THE SHAPE OF THE CURVE: Suppose that $x = \alpha$ is a zero of a polynomial $P(x)$.

- If $x = \alpha$ has even multiplicity, the curve is tangent to the x-axis at $x = \alpha$, and does not cross the x-axis there.
- If $x = \alpha$ has odd multiplicity at least 3, the curve has a point of inflexion on the x-axis at $x = \alpha$.
- If $x = \alpha$ is a simple zero, then the curve crosses the x-axis at $x = \alpha$ and is not tangent to the x-axis there.

Because the proof relies on the factor theorem, it cannot be presented until Section 4E (where it is proven as Consequence G of the factor theorem). Sketching the curves at the outset seems more appropriate than maintaining logical order.

WORKED EXERCISE: Sketch, showing the behaviour near any x-intercepts:

(a) $P(x) = (x-1)^2(x-2)$

(b) $Q(x) = x^3(x+2)^4(x^2+x+1)$

(c) $R(x) = -2(x-2)^2(x+1)^5(x-1)$

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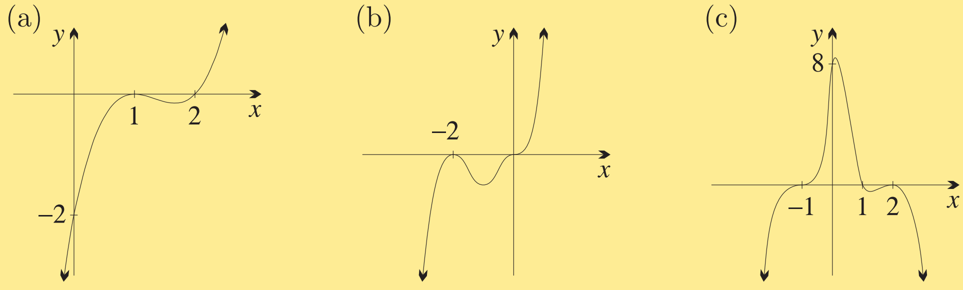
SOLUTION: In part (b), $x^2 + x + 1$ is irreducible, because $\Delta = 1 - 4 < 0$.



Exercise 4B

1. Without the aid of calculus, sketch graphs of the following linear polynomials, clearly indicating all intercepts with the axes:

(a) $P(x) = 2$ (b) $P(x) = x$ (c) $P(x) = x - 4$ (d) $P(x) = 3 - 2x$

2. Without the aid of calculus, sketch graphs of the following quadratic polynomials, clearly indicating all intercepts with the axes:

(a) $P(x) = x^2$
(b) $P(x) = (x - 1)(x + 3)$
(c) $P(x) = (x - 2)^2$
(d) $P(x) = 9 - x^2$
(e) $P(x) = 2x^2 + 5x - 3$
(f) $P(x) = 4 + 3x - x^2$

3. Without the aid of calculus, sketch graphs of the following cubic polynomials, clearly indicating all intercepts with the axes:

(a) $y = x^3$
(b) $y = x^3 + 2$
(c) $y = (x - 4)^3$
(d) $y = (x - 1)(x + 2)(x - 3)$
(e) $y = x(2x + 1)(x - 5)$
(f) $y = (1 - x)(1 + x)(2 + x)$
(g) $y = (2x + 1)^2(x - 4)$
(h) $y = x^2(1 - x)$
(i) $y = (2 - x)^2(5 - x)$

4. Without the aid of calculus, sketch graphs of the following quartic polynomials, clearly indicating all intercepts with the axes:

(a) $F(x) = x^4$
(b) $F(x) = (x + 2)^4$
(c) $F(x) = x(3x + 2)(x - 3)(x + 2)$
(d) $F(x) = (1 - x)(x + 5)(x - 7)(x + 3)$
(e) $F(x) = x^2(x + 4)(x - 3)$
(f) $F(x) = (x + 2)^3(x - 5)$
(g) $F(x) = (2x - 3)^2(x + 1)^2$
(h) $F(x) = (1 - x)^3(x - 3)$
(i) $F(x) = (2 - x)^2(1 - x^2)$

DEVELOPMENT

5. These polynomials are not factored, but the positions of their zeroes can be found by trial and error. Copy and complete each tables of values, and sketch a graph, stating how many zeroes there are, and between which integers they lie.

(a) $y = x^2 - 3x + 1$

x	-1	0	1	2	3	4
y						

(b) $y = 1 + 3x - x^3$

x	-2	-1	0	1	2	3
y						

6. Without the aid of calculus, sketch graphs of the following polynomial functions, clearly indicating all intercepts with the axes.

(a) $P(x) = x(x - 2)^3(x + 1)^2$
(b) $P(x) = (x + 2)^2(3 - x)^3$
(c) $P(x) = x(2x + 3)^3(1 - x)^4$
(d) $P(x) = (x + 1)(4 - x^2)(x^2 - 3x - 10)$

7. Use the graphs drawn in the previous question to solve the following inequalities.

(a) $x(x - 2)^3(x + 1)^2 > 0$
(b) $(x + 2)^2(3 - x)^3 \geq 0$
(c) $x(2x + 3)^3(1 - x)^4 \geq 0$
(d) $(x + 1)(4 - x^2)(x^2 - 3x - 10) < 0$

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8. (a) Without using calculus, sketch a graph of the function $P(x) = x(x-2)^2(x+1)$.

(b) Hence by translating or reflecting this graph, sketch the following functions:

(i) $R(x) = -x(x-2)^2(x+1)$ (iii) $U(x) = (x-2)(x-4)^2(x-1)$

(ii) $Q(x) = -x(-x-2)^2(-x+1)$ (iv) $V(x) = (x+3)(x+1)^2(x+4)$

9. (a) Find the monic quadratic polynomial that crosses the y-axis at $(0,-6)$ and the x-axis at $(3,0)$.

(b) Find the quadratic polynomial that has a minimum value of -3 when $x=-2$, and passes through the point $(1,6)$.

(c) Find the cubic polynomial that has zeroes 0, 1 and 2, and in which the coefficient of x^3 is 2.

10. Consider the polynomial $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$.

(a) What condition on the coefficients is satisfied if $P(x)$ is: (i) even, (ii) odd?

(b) Find the monic, even quartic that has y-intercept 9 and a zero at $x = 3$.

(c) Find the odd quintic with zeroes at $x = 1$ and $x = 2$ and leading coefficient -3.

11. (a) Prove that every odd polynomial function is zero at $x = 0$.

(b) Prove that every odd polynomial $P(x)$ is divisible by x.

(c) Find the polynomial $P(x)$ that is known to be monic, of degree 3, and an odd function, and has one zero at $x = 2$.

12. By making a suitable substitution, factor the following polynomials. Without using calculus, sketch graphs showing all intercepts with the axes.

(a) $P(x) = x^4 - 13x^2 + 36$ (c) $P(x) = (x^2-5x)^2 - 2(x^2-5x) - 24$

(b) $P(x) = 4x^4 - 13x^2 + 9$ (d) $P(x) = (x^2-3x+1)^2 - 4(x^2-3x+1) - 5$

13. (a) Sketch graphs of the following polynomials, clearly labelling all intercepts with the axes. Do not use calculus to find further turning points.

(i) $F(x) = x(x-4)(x+1)$ (iii) $F(x) = x(x+3)^2(5-x)$

(ii) $F(x) = (x-1)^2(x+3)$ (iv) $F(x) = x^2(x-3)^3(x-7)$

(b) Without the aid of calculus, draw graphs of the derivatives of each of the polynomials in part (a). You will not be able to find the x-intercepts or y-intercepts accurately.

(c) Suppose that $G(x)$ is a primitive of $F(x)$. For each of the polynomials in part (a), state for what values of x the function $G(x)$ is increasing and decreasing.

14. Sketch, over $-\pi < x < \pi$: (a) $y = \cos x$ (b) $y = \cos^2 x$ (c) $y = \cos^3 x$

15. [Every cubic has odd symmetry in its point of inflexion.]

(a) Suppose that the origin is the point of inflexion of $f(x) = ax^3 + bx^2 + cx + d$.

(i) Prove that $b = d = 0$, and hence that $f(x)$ is an odd function.

(ii) Hence prove that if ℓ is a line through the origin crossing the curve again at A and B, then O is the midpoint of the interval AB.

(b) Use part (a), and arguments based on translations, to prove that if ℓ is a line through the point of inflexion I crossing the curve again at A and B, then I is the midpoint of AB.

(c) Prove that if a cubic has turning points, then the midpoint of the interval joining them is the point of inflexion.

EXTENSION

16. At what points do the graphs of the polynomials $f(x) = (x+1)^n$ and $g(x) = (x+1)^m$ intersect? [HINT: Consider the cases where m and n are odd and even.]

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17. [The motivation for this question is the power series $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$.]

For each integer $n > 0$, let $E_n(x) = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$.

(a) Show that: (i) $E_n(x) = E_{n-1}(x) + \dfrac{x^n}{n!}$ (ii) $E_n{}'(x) = E_{n-1}(x)$

(b) Show that if $x = \alpha$ is a zero of $E_n{}'(x)$, then $E_n(\alpha) = \dfrac{\alpha^n}{n!}$.

(c) Suppose that n is even.
   (i) Show that every stationary point of $E_n(x)$ lies above the x-axis,
   (ii) Show that $E_n(x)$ is positive, for all x, and concave up, for all x.
   (iii) Show that $E_n(x)$ has one stationary point, which is a minimum turning point.

(d) Suppose that n is odd.
   (i) Show that $E_n(x)$ is increasing for all x, and has exactly one zero.
   (ii) Show that $E_n(x)$ has exactly one point of inflexion.
   (iii) By factoring in pairs, show that $E_n(-n) < 0$.
   (iv) Show that the inflexion is above the x-axis.

4 C Division of Polynomials

The previous exercise had examples of adding, subtracting and multiplying polynomials, operations which are quite straightforward. The division of one polynomial by another, however, requires some explanation.

Division of Polynomials: It can happen that the quotient of two polynomials is again a polynomial; for example,

$$\frac{6x^3 + 4x^2 - 9x}{3x} = 2x^2 + \tfrac{4}{3}x - 3 \qquad \text{and} \qquad \frac{x^2 + 4x - 5}{x + 5} = x - 1.$$

But usually, division results in rational functions, not polynomials:

$$\frac{x^4 + 4x^2 - 9}{x^2} = x^2 + 4 - \frac{9}{x^2} \qquad \text{and} \qquad \frac{x + 4}{x + 3} = 1 + \frac{1}{x + 3}.$$

In this respect, there is a very close analogy between the set $\mathbf{Z}$ of all integers and the set of all polynomials. In both cases, everything works nicely for addition, subtraction and multiplication, but the results of division do not usually lie within the set. For example, although $20 \div 5 = 4$ is an integer, the division of two integers usually results in a fraction rather than an integer, as in $23 \div 5 = 4\frac{3}{5}$.

The Division Algorithm for Integers: On the right is an example of the well-known long division algorithm for integers, applied here to $197 \div 12$. The number 12 is called the *divisor*, 197 is called the *dividend*, 16 is called the *quotient*, and 5 is called the *remainder*.

$$\begin{array}{r|l} & \;\;16 \text{ remainder } 5 \\ \hline 12 & 197 \\ & \underline{12} \\ & \;\;77 \\ & \;\;\underline{72} \\ & \;\;\;\;5 \end{array}$$

The result of the division can be written as $\frac{197}{12} = 16\frac{5}{12}$, but we can avoid fractions completely by writing the result as:

$$197 = 12 \times 16 + 5.$$

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The remainder 5 had to be less than 12, otherwise the division process could have been continued. Thus the general result for division of integers can be expressed as follows:

8 DIVISION OF INTEGERS: Suppose that p (the dividend) and d (the divisor) are integers, with $d > 0$. Then there are unique integers q (the quotient) and r (the remainder) such that

$$p = dq + r \qquad \text{and} \qquad 0 \le r < d.$$

When the remainder r is zero, then d is a *divisor of p, and the integer p factors into the product $p = d \times q$.*

The Division Algorithm for Polynomials: The method of dividing one polynomial by another is similar to the method of dividing integers.

9 THE METHOD OF LONG DIVISION OF POLYNOMIALS:
- At each step, divide the leading term of the remainder by the leading term of the divisor. Continue the process for as long as possible.
- Unless otherwise specified, express the answer in the form

$$\text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder}.$$

WORKED EXERCISE: Divide $3x^4 - 4x^3 + 4x - 8$ by: (a) $x - 2$ (b) $x^2 - 2$
Give results first in the standard manner, then using rational functions.

SOLUTION: The steps have been annotated to explain the method.

(a)
$$
\begin{array}{rrl}
 & 3x^3 + 2x^2 + 4x + 12 & \text{(leave a gap for the missing term in } x^2\text{)} \\
x-2 \,\big) & \overline{3x^4 - 4x^3 \qquad + 4x - 8} & \text{(divide } x \text{ into } 3x^4\text{, giving the } 3x^3 \text{ above)} \\
 & 3x^4 - 6x^3 \qquad\qquad\qquad & \text{(multiply } x-2 \text{ by } 3x^3 \text{ and then subtract)} \\
 & \overline{2x^3 \qquad + 4x - 8} & \text{(divide } x \text{ into } 2x^3\text{, giving the } 2x^2 \text{ above)} \\
 & 2x^3 - 4x^2 \qquad\qquad & \text{(multiply } x-2 \text{ by } 2x^2 \text{ and then subtract)} \\
 & \overline{4x^2 + 4x - 8} & \text{(divide } x \text{ into } 4x^2\text{, giving the } 4x \text{ above)} \\
 & 4x^2 - 8x \qquad & \text{(multiply } x-2 \text{ by } 4x \text{ and then subtract)} \\
 & \overline{12x - 8} & \text{(divide } x \text{ into } 12x\text{, giving the 12 above)} \\
 & 12x - 24 & \text{(multiply } x-2 \text{ by 12 and then subtract)} \\
 & \overline{16} & \text{(this is the final remainder)}
\end{array}
$$

Hence $3x^4 - 4x^3 + 4x - 8 = (x-2)(3x^3 + 2x^2 + 4x + 12) + 16$,
or, writing the result using rational functions,

$$\frac{3x^4 - 4x^3 + 4x - 8}{x - 2} = 3x^3 + 2x^2 + 4x + 12 + \frac{16}{x-2}.$$

(b)
$$
\begin{array}{rrl}
 & 3x^2 - 4x + 6 & \\
x^2-2 \,\big) & \overline{3x^4 - 4x^3 \qquad + 4x - 8} & \text{(divide } x^2 \text{ into } 3x^4\text{, giving the } 3x^2 \text{ above)} \\
 & 3x^4 \qquad - 6x^2 \qquad\qquad & \text{(multiply } x^2-2 \text{ by } 3x^2 \text{ and then subtract)} \\
 & \overline{-4x^3 + 6x^2 + 4x - 8} & \text{(divide } x^2 \text{ into } -4x^3\text{, giving the } -4x \text{ above)} \\
 & -4x^3 \qquad + 8x \qquad & \text{(multiply } x^2-2 \text{ by } -4x \text{ and then subtract)} \\
 & \overline{6x^2 - 4x - 8} & \text{(divide } x^2 \text{ into } 6x^2\text{, giving the 6 above)} \\
 & 6x^2 \qquad - 12 & \text{(multiply } x^2-2 \text{ by 6 and then subtract)} \\
 & \overline{-4x + 4} & \text{(this is the final remainder)}
\end{array}
$$

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Hence $3x^4 - 4x^3 + 4x - 8 = (x^2 - 2)(3x^2 - 4x + 6) + (-4x + 4)$,

or $\dfrac{3x^4 - 4x^3 + 4x - 8}{x^2 - 2} = 3x^2 - 4x + 6 + \dfrac{-4x + 4}{x^2 - 2}$.

The Division Theorem:

The division process illustrated above can be continued until the remainder is zero or has degree less than the degree of the divisor. Thus the general result for polynomial division is:

10 DIVISION OF POLYNOMIALS: Suppose that $P(x)$ (the dividend) and $D(x)$ (the divisor) are polynomials with $D(x) \neq 0$. Then there are unique polynomials $Q(x)$ (the quotient) and $R(x)$ (the remainder) such that

1. $P(x) = D(x)Q(x) + R(x)$,
2. either $\deg R(x) < \deg D(x)$, or $R(x) = 0$.

When the remainder $R(x)$ is zero, then $D(x)$ is called a *divisor of* $P(x)$, *and the polynomial* $P(x)$ *factors into the product* $P(x) = D(x) \times Q(x)$.

For example, in the two worked exercises above:

- the remainder after division by the degree 1 polynomial $x-2$ was the constant polynomial 16,
- the remainder after division by the degree 2 polynomial $x^2 - 2$ was the linear polynomial $-4x + 4$.

Exercise 4C

1. Perform each of the following integer divisions, and write the result in the form $p = dq + r$, where $0 \le r < d$. For example, $30 = 4 \times 7 + 2$.

(a) $63 \div 5$ (b) $125 \div 8$ (c) $324 \div 11$ (d) $1857 \div 23$

2. Use long division to perform each of the following divisions. Express each result in the form $P(x) = D(x)Q(x) + R(x)$.

(a) $(x^2 - 4x + 1) \div (x + 1)$
(b) $(x^2 - 6x + 5) \div (x - 5)$
(c) $(x^3 - x^2 - 17x + 24) \div (x - 4)$
(d) $(2x^3 - 10x^2 + 15x - 14) \div (x - 3)$
(e) $(4x^3 - 4x^2 + 7x + 14) \div (2x + 1)$
(f) $(x^4 + x^3 - x^2 - 5x - 3) \div (x - 1)$
(g) $(6x^4 - 5x^3 + 9x^2 - 8x + 2) \div (2x - 1)$
(h) $(10x^4 - x^3 + 3x^2 - 3x - 2) \div (5x + 2)$

3. Express the answers to parts (a)–(d) of the previous question in rational form, that is, as $\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$, and hence find the primitive of the quotient $\dfrac{P(x)}{D(x)}$.

4. Use long division to perform each of the following divisions. Express each result in the form $P(x) = D(x)Q(x) + R(x)$.

(a) $(x^3 + x^2 - 7x + 6) \div (x^2 + 3x - 1)$
(b) $(x^3 - 4x^2 - 2x + 3) \div (x^2 - 5x + 3)$
(c) $(x^4 - 3x^3 + x^2 - 7x + 3) \div (x^2 - 4x + 2)$
(d) $(2x^5 - 5x^4 + 12x^3 - 10x^2 + 7x + 9) \div (x^2 - x + 2)$

5. (a) If the divisor of a polynomial has degree 3, what are the possible degrees of the remainder?

(b) On division by $D(x)$, a polynomial has remainder $R(x)$ of degree 2. What are the possible degrees of $D(x)$?

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DEVELOPMENT

6. Use long division to perform each of the following divisions. Take care to ensure that the columns line up correctly. Express each result in the form $P(x) = D(x)Q(x) + R(x)$.

(a) $(x^3 - 5x + 3) \div (x - 2)$
(b) $(2x^3 + x^2 - 11) \div (x + 1)$
(c) $(x^3 - 3x^2 + 5x - 4) \div (x^2 + 2)$
(d) $(2x^4 - 5x^2 + x - 2) \div (x^2 + 3x - 1)$
(e) $(2x^3 - 3) \div (2x - 4)$
(f) $(x^5 + 3x^4 - 2x^2 - 3) \div (x^2 + 1)$

Write the answers to parts (c) and (f) above in rational form, that is, in the form $\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$, and hence find the primitive of the quotient $\dfrac{P(x)}{D(x)}$.

7. Find the quotient and remainder in each of the following divisions. Fractions will be needed throughout the calculations.

(a) $(x^2 + 4x + 7) \div (2x + 1)$
(b) $(6x^3 - x^2 + 4x - 2) \div (3x - 1)$
(c) $(x^3 - x^2 + x + 1) \div (2x - 3)$

8. (a) Use long division to show that $P(x) = x^3 + 2x^2 - 11x - 12$ is divisible by $x - 3$, and hence express $P(x)$ as the product of three linear factors.
(b) Find the values of x for which $P(x) > 0$.

9. (a) Use long division to show that $F(x) = 2x^4 + 3x^3 - 12x^2 - 7x + 6$ is divisible by $x^2 - x - 2$, and hence express $F(x)$ as the product of four linear factors.
(b) Find the values of x for which $F(x) \le 0$.

10. (a) Write down the division identity statement when $30 \div 4$ and $30 \div 7$.
(b) Division of the polynomial $P(x)$ by $D(x)$ results in the quotient $Q(x)$ and remainder $R(x)$. Show that if $P(x)$ is divided by $Q(x)$, the remainder will still be $R(x)$. What is the quotient?

11. (a) Find the quotient and remainder when $x^4 - 2x^3 + x^2 - 5x + 7$ is divided by $x^2 + x - 1$.
(b) Find a and b if $x^4 - 2x^3 + x^2 + ax + b$ is exactly divisible by $x^2 + x - 1$.
(c) Hence factor $x^4 - 2x^3 + x^2 + 8x - 5$.

12. (a) Use long division to divide the polynomial $f(x) = x^4 - x^3 + x^2 - x + 1$ by the polynomial $d(x) = x^2 + 4$. Express your answer in the form $f(x) = d(x)q(x) + r(x)$.
(b) Hence find the values of a and b such that $x^4 - x^3 + x^2 + ax + b$ is divisible by $x^2 + 4$.
(c) Hence factor $x^4 - x^3 + x^2 - 4x - 12$.

13. If $x^4 - 2x^3 - 20x^2 + ax + b$ is exactly divisible by $x^2 - 5x + 2$, find a and b.

EXTENSION

14. Two integers are said to be *relatively prime* if their highest common factor is 1. If a and b are relatively prime it is possible to find integers x and y such that $ax + by = 1$. For example 51 and 44 are relatively prime.

Repeated use of the division identity leads to:

$$\begin{aligned} 51 &= 44 \times 1 + 7 \\ 44 &= 7 \times 6 + 2 \\ 7 &= 3 \times 2 + 1 \end{aligned}$$

Reversing these steps leads to:

$$\begin{aligned} 1 &= 7 - 3 \times 2 \\ &= 7 - 3(44 - 7 \times 6) \\ &= 19 \times 7 - 3 \times 44 \\ &= 19(51 - 44 \times 1) - 3 \times 44 \\ &= 19 \times 51 - 22 \times 44 \end{aligned}$$

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(a) Use this method to find integers a and b such that $87a + 19b = 1$

(b) Find polynomials $A(x)$ and $B(x)$ such that $1 = A(x)(x^2 - x) + B(x)(x^4 + 4x^2 - 4x + 4)$.

15. [The uniqueness of integer division and polynomial division]

(a) Suppose that $p = dq + r$ and $p = dq' + r'$, where p, d, q, q', r and r' are integers with $d \neq 0$, and where $0 \le r < d$ and $0 \le r' < d$. Prove that $q = q'$ and $r = r'$.

(b) Suppose that $P(x) = D(x)Q(x) + R(x)$ and $P(x) = D(x)Q'(x) + R'(x)$, where $P(x)$, $D(x)$, $Q(x)$, $Q'(x)$, $R(x)$ and $R'(x)$ are polynomials with $D(x) \neq 0$, and where $R(x)$ and $R'(x)$ each has degree less than $D(x)$ or is the zero polynomial. Prove that $Q(x) = Q'(x)$ and $R(x) = R'(x)$.

4 D The Remainder and Factor Theorems

Long division of polynomials is a cumbersome process. It is therefore very useful to have the remainder and factor theorems, which provide information about the results of that division without the division actually being carried out. In particular, the factor theorem gives a simple test as to whether a particular linear function is a factor or not.

The Remainder Theorem: The remainder theorem is a remarkable result which, in the case of linear divisors, allows the remainder to be calculated without the long division being performed.

11 THE REMAINDER THEOREM: Suppose that $P(x)$ is a polynomial and α is a constant. Then the remainder after division of $P(x)$ by $x - \alpha$ is $P(\alpha)$.

PROOF: Since $x - \alpha$ is a polynomial of degree 1, the division theorem tells us that there are unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = (x - \alpha)Q(x) + R(x),$$

and $\qquad$ either $R(x) = 0$ or $\deg R(x) = 0$.

Hence $R(x)$ is a zero or nonzero constant, which we can simply write as r,

and so $\qquad P(x) = (x - \alpha)Q(x) + r$.

Substituting $x = \alpha$ gives $P(\alpha) = (\alpha - \alpha)Q(\alpha) + r$

and rearranging, $\qquad r = P(\alpha)$, as required.

WORKED EXERCISE: Find the remainder when $3x^4 - 4x^3 + 4x - 8$ is divided by $x - 2$: (a) by long division, (b) by the remainder theorem.

SOLUTION: In the previous worked exercise, performing the division showed that

$$3x^4 - 4x^3 + 4x - 8 = (x - 2)(3x^3 + 2x^2 + 4x + 12) + 16,$$

that is, that the remainder is 16. Alternatively, substituting $x = 2$ into $P(x)$,

$$\begin{aligned} \text{remainder} &= P(2) \quad \text{(this is the remainder theorem)} \\ &= 48 - 32 + 8 - 8 \\ &= 16, \quad \text{as expected.} \end{aligned}$$

WORKED EXERCISE: The polynomial $P(x) = x^4 - 2x^3 + ax + b$ has remainder 3 after division by $x - 1$, and has remainder -5 after division by $x + 1$. Find a and b.

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SOLUTION: Applying the remainder theorem for each divisor,

$$P(1) = 3$$
$$1 - 2 + a + b = 3$$
$$a + b = 4. \qquad (1)$$

Also
$$P(-1) = -5$$
$$1 + 2 - a + b = -5$$
$$-a + b = -8. \qquad (2)$$

Adding (1) and (2), $2b = -4$,
subtracting them, $2a = 12$.
Hence $a = 6$ and $b = -2$.

The Factor Theorem: The remainder theorem tells us that the number $P(\alpha)$ is just the remainder after division by $x - \alpha$. But $x - \alpha$ being a factor means that the remainder after division by $x - \alpha$ is zero, so:

12 THE FACTOR THEOREM: Suppose that $P(x)$ is a polynomial and α is a constant. Then $x - \alpha$ is a factor if and only if $f(\alpha) = 0$.

This is a very quick and easy test as to whether $x - \alpha$ is a factor of $P(x)$ or not.

WORKED EXERCISE: Show that $x - 3$ is a factor of $P(x) = x^3 - 2x^2 + x - 12$, and $x + 1$ is not. Then use long division to factor the polynomial completely.

SOLUTION: $P(3) = 27 - 18 + 3 - 12 = 0$, so $x - 3$ is a factor.
$P(-1) = -1 - 2 - 1 - 12 = -16 \neq 0$, so $x + 1$ is not a factor.
Long division of $P(x) = x^3 - 2x^2 + x - 12$ by $x - 3$ (which we omit) gives
$$P(x) = (x - 3)(x^2 + x + 4),$$
and since $\Delta = 1 - 16 = -15 < 0$ for the quadratic, this factorisation is complete.

Factoring Polynomials — The Initial Approach: The factor theorem gives us the beginnings of an approach to factoring polynomials. This approach will be further refined in the next two sections.

13 FACTORING POLYNOMIALS — THE INITIAL APPROACH:
- Use trial and error to find an integer zero $x = \alpha$ of $P(x)$.
- Then use long division to factor $P(x)$ in the form $P(x) = (x - \alpha)Q(x)$.

If the coefficients of $P(x)$ are all integers, then all the integer zeroes of $P(x)$ are divisors of the constant term.

PROOF: We must prove the claim that if the coefficients of $P(x)$ are integers, then every integer zero of $P(x)$ is a divisor of the constant term.
Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$,
where the coefficients $a_n, a_{n-1}, \ldots a_1, a_0$ are all integers,
and let $x = \alpha$ be an integer zero of $P(x)$.
Substituting into $P(\alpha) = 0$ gives $a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$
$$\begin{aligned} a_0 &= -a_n\alpha^n - a_{n-1}\alpha^{n-1} - \cdots - a_1\alpha \\ &= \alpha(-a_n\alpha^{n-1} - a_{n-1}\alpha^{n-2} - \cdots - a_1), \end{aligned}$$
and so a_0 is an integer multiple of α.

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WORKED EXERCISE: Factor $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

SOLUTION: Since all the coefficients are integers, the only integer zeroes are the divisors of the constant term -4, that is 1, 2, 4, -1, -2 and -4.

$$P(1) = 1 + 1 - 9 + 11 - 4$$
$$= 0, \text{ so } x - 1 \text{ is a factor.}$$

After long division (omitted), $P(x) = (x-1)(x^3 + 2x^2 - 7x + 4)$.

Let $Q(x) = x^3 + 2x^2 - 7x + 4$, then $Q(1) = 1 + 2 - 7 + 4$
$= 0$, so $x - 1$ is a factor.

Again after long division (omitted), $P(x) = (x-1)(x-1)(x^2 + 3x - 4)$.

Factoring the quadratic, $P(x) = (x-1)^3(x+4)$.

NOTE: In the next two sections we will develop methods that will often allow long division to be avoided.

Exercise 4D

1. Without division, find the remainder when $P(x) = x^3 - x^2 + 2x + 1$ is divided by:

(a) $x - 1$ (c) $x + 2$ (e) $x - 5$
(b) $x - 3$ (d) $x + 1$ (f) $x + 3$

2. Without division, find which of the following are factors of $F(x) = x^3 + 4x^2 + x - 6$.

(a) $x - 1$ (c) $x - 2$ (e) $x - 3$
(b) $x + 1$ (d) $x + 2$ (f) $x + 3$

3. Without division, find the remainder when $P(x) = x^3 + 2x^2 - 4x + 5$ is divided by:

(a) $2x - 1$ (b) $2x + 3$ (c) $3x - 2$

4. (a) Find k, if $x - 1$ is a factor of $P(x) = x^3 - 3x^2 + kx - 2$.

(b) Find m, if -2 is a zero of the function $F(x) = x^3 + mx^2 - 3x + 4$.

(c) When the polynomial $P(x) = 2x^3 - x^2 + px - 1$ is divided by $x - 3$, the remainder is 2. Find p.

(d) For what value of a is $3x^4 + ax^2 - 2$ divisible by $x + 1$?

DEVELOPMENT

5. (a) Show that $P(x) = x^3 - 8x^2 + 9x + 18$ is divisible by $x - 3$ and $x + 1$.

(b) By considering the leading term and constant term, express $P(x)$ in terms of three linear factors and hence solve $P(x) \geq 0$.

6. (a) Show that $P(x) = 2x^3 - x^2 - 13x - 6$ is divisible by $x - 3$ and $2x + 1$.

(b) By considering the leading term and constant term, express $P(x)$ in terms of three linear factors and hence solve $P(x) \leq 0$.

7. Factor each of the following polynomials and sketch a graph, indicating all intercepts with the axes. You do not need to find any other turning points.

(a) $P(x) = x^3 + 2x^2 - 5x - 6$ (d) $P(x) = x^4 - x^3 - 19x^2 - 11x + 30$
(b) $P(x) = x^3 + 3x^2 - 25x + 21$ (e) $P(x) = 2x^3 + 11x^2 + 10x - 8$
(c) $P(x) = -x^3 + x^2 + 5x + 3$ (f) $P(x) = 3x^4 + 4x^3 - 35x^2 - 12x$

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8. Solve the equations by first factoring the LHS:

(a) $x^3 + 3x^2 - 6x - 8 = 0$

(b) $x^3 - 4x^2 - 3x + 18 = 0$

(c) $x^3 + x^2 - 7x + 2 = 0$

(d) $x^3 - 2x^2 - 2x - 3 = 0$

(e) $6x^3 - 5x^2 - 12x - 4 = 0$

(f) $2x^4 + 11x^3 + 19x^2 + 8x - 4 = 0$

9. (a) If $P(x) = 2x^3 + x^2 - 13x + 6$, evaluate $P\left(\frac{1}{2}\right)$. Use long division to express $P(x)$ in factored form.

(b) If $P(x) = 6x^3 + x^2 - 5x - 2$, evaluate $P\left(-\frac{2}{3}\right)$. Express $P(x)$ in factored form.

10. (a) Factor each of the polynomials $P(x) = x^3 - 3x^2 + 4$, $Q(x) = x^3 + 2x^2 - 5x - 6$ and $R(x) = x^3 - 3x - 2$.

(b) Hence determine the highest monic common factor of $P(x)$, $Q(x)$ and $R(x)$.

(c) What is the monic polynomial of least degree that is exactly divisible by $P(x)$, $Q(x)$ and $R(x)$? Write the answer in factored form.

11. At time t, the position of a particle moving along the x-axis is given by the equation $x = t^4 - \frac{17}{3}t^3 + 8t^2 - 3t + 5$. Find the times at which the particle is stationary.

12. (a) The polynomial $2x^3 - x^2 + ax + b$ has a remainder of 16 on division by $x - 1$ and a remainder of -17 on division by $x + 2$. Find a and b.

(b) A polynomial is given by $P(x) = x^3 + ax^2 + bx - 18$. Find a and b, if $x + 2$ is a factor and -24 is the remainder when $P(x)$ is divided by $x - 1$.

(c) $P(x)$ is an odd polynomial of degree 3. It has $x + 4$ as a factor, and when it is divided by $x - 3$ the remainder is 21. Find $P(x)$.

(d) Find p such that $x - p$ is a factor of $4x^3 - (10p - 1)x^2 + (6p^2 - 5)x + 6$.

13. (a) The polynomial $P(x)$ is divided by $(x - 1)(x + 2)$. Find the remainder, given that $P(1) = 2$ and $P(-2) = 5$. [HINT: The remainder may have degree 1.]

(b) The polynomial $U(x)$ is divided by $(x + 4)(x - 3)$. Find the remainder, given that $U(-4) = 11$ and $U(3) = -3$.

14. (a) The polynomial $P(x) = x^3 + bx^2 + cx + d$ has zeroes at 0, 3 and -3. Find b, c and d.

(b) Sketch a graph of $y = P(x)$.

(c) Hence solve $\dfrac{x^2 - 9}{x} > 0$.

15. (a) Show that the equation of the normal to the curve $x^2 = 4y$ at the point $(2t, t^2)$ is $x + ty - 2t - t^3 = 0$.

(b) If the normal passes through the point $(-2, 5)$, find the value of t.

16. (a) Is either $x + 1$ or $x - 1$ a factor of $x^n + 1$, where n is a positive integer?

(b) Is either $x + a$ or $x - a$ a factor of $x^n + a^n$, where n is a positive integer?

17. When a polynomial is divided by $(x - 1)(x + 3)$, the remainder is $2x - 1$.

(a) Express this in terms of a division identity statement.

(b) Hence, by evaluating $P(1)$, find the remainder when the polynomial is divided by $x - 1$.

18. (a) When a polynomial is divided by $(2x + 1)(x - 3)$, the remainder is $3x - 1$. What is the remainder when the polynomial is divided by $2x + 1$?

(b) When $x^5 + 3x^3 + ax + b$ is divided by $x^2 - 1$, the remainder is $2x - 7$. Find a and b.

(c) When a polynomial $P(x)$ is divided by $x^2 - 5$, the remainder is $x + 4$. Find the remainder when $P(x) + P(-x)$ is divided by $x^2 - 5$. [HINT: Write down the division identity statement.]

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19. Each term of an arithmetic sequence a, $a+d$, $a+2d$, ... is added to the corresponding term of the geometric sequence b, ba, ba^2, ... to form a third sequence S, whose first three terms are -1, -2 and 6. (Note that the common ratio of the geometric sequence is equal to the first term of the arithmetic sequence.)

(a) Show that $a^3 - a^2 - a + 10 = 0$.

(b) Find a, given that a is real.

(c) Hence show that the nth term of S is given by $T_n = 2n - 4 + (-2)^{n-1}$.

EXTENSION

20. When a polynomial is divided by $x - p$, the remainder is p^3. When the polynomial is divided by $x - q$, the remainder is q^3. Find the remainder when the polynomial is divided by $(x-p)(x-q)$.

21. [Finding the equation of a cubic, given its two stationary points] Let $y_1 = f(x)$ be a cubic polynomial with stationary points at $(6, 12)$ and $(12, 4)$. Let $y_2 = g(x) = f(x) - 4$.

(a) Write down the coordinates of the minimum turning point of $g(x)$.

(b) Hence write down the general form of the equation of $g(x)$ in factored form.

(c) Find the value of $g(6)$.

(d) In Exercise 4B, you proved that a cubic has odd symmetry in its point of inflexion. Use this fact to show that $g(9) = 4$.

(e) Hence use simultaneous equations to find a and k and the equation of $g(x)$.

(f) Hence find the equation of the cubic through the stationary points $(6, 12)$ and $(12, 4)$.

(g) In Chapter Ten of the Year 11 volume, you solved this type of question by letting $f(x) = ax^3 + bx^2 + cx + d$ and forming four equations in the four unknowns. Check your answer by this method.

22. (a) If all the coefficients of a monic polynomial are integers, prove that all the rational zeroes are integers. [HINT: Look carefully at the proof under Box 13.]

(b) If all the coefficients of a polynomial are integers, prove that the denominators of all the rational zeroes (in lowest terms) are divisors of the leading coefficient.

23. (a) Use the remainder theorem to prove that $a + b + c$ is a factor of $a^3 + b^3 + c^3 - 3abc$. Then find the other factor. [HINT: Regard it as a polynomial in a.]

(b) Factor $ab^3 - ac^3 + bc^3 - ba^3 + ca^3 - cb^3$.

4 E Consequences of the Factor Theorem

The factor theorem has a number of fairly obvious but very useful consequences, which are presented here as six successive theorems.

A. Several Distinct Zeroes: Suppose that several distinct zeroes of a polynomial have been found, probably using test substitutions into the polynomial.

14 DISTINCT ZEROES: Suppose that α_1, α_2, ... α_s are distinct zeroes of a polynomial $P(x)$. Then $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_s)$ is a factor of $P(x)$.

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PROOF: Since α_1 is a zero, $x - \alpha_1$ is a factor, and $P(x) = (x - \alpha_1)p_1(x)$.
Since $P(\alpha_2) = 0$ but $\alpha_2 - \alpha_1 \neq 0$, $p_1(\alpha_2)$ must be zero.
Hence $x - \alpha_2$ is a factor of $p_1(x)$, and $p_1(x) = (x - \alpha_2)p_2(x)$,
and thus $P(x) = (x - \alpha_1)(x - \alpha_1)p_2(x)$.
Continuing similarly for s steps, $(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_s)$ is a factor of $P(x)$.

B. All Distinct Zeroes: If n distinct zeroes of a polynomial of degree n can be found, then the factorisation is complete, and the polynomial is the product of distinct linear factors.

15 ALL DISTINCT ZEROES: Suppose that $\alpha_1, \alpha_2, \ldots \alpha_n$ are n distinct zeroes of a polynomial $P(x)$ of degree n. Then

$$P(x) = a(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n),$$

where a is the leading coefficient of $P(x)$.

PROOF: By the previous theorem, $(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$ is a factor of $P(x)$,
so $P(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)Q(x)$, for some polynomial $Q(x)$.
But $P(x)$ and $(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$ both have degree n, so $Q(x)$ is a constant.
Equating coefficients of x^n, the constant $Q(x)$ must be the leading coefficient.

Factoring Polynomials — Finding Several Zeroes First: If we can find more than one zero of a polynomial, then we have found a quadratic or cubic factor, and the long divisions required can be reduced or even avoided completely.

16 FACTORING POLYNOMIALS — FINDING SEVERAL ZEROES FIRST:

- Use trial and error to find as many integer zeroes of $P(x)$ as possible.
- Using long division, divide $P(x)$ by the product of the known factors.

If the coefficients of $P(x)$ are all integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

When this procedure is applied to the polynomial factored in the previous section, one rather than two long divisions is required.

WORKED EXERCISE: Factor $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

SOLUTION: As before, all the coefficients are integers, so the only integer zeroes are the divisors of the constant term -4, that is 1, 2, 4, -1, -2 and -4.

$$P(1) = 1 + 1 - 9 + 11 - 4 = 0, \text{ so } x - 1 \text{ is a factor.}$$
$$P(-4) = 256 - 64 - 144 - 44 - 4 = 0, \text{ so } x + 4 \text{ is a factor.}$$

After long division by $(x - 1)(x + 4) = x^2 + 3x - 4$ (omitted),

$$P(x) = (x^2 + 3x - 4)(x^2 - 2x + 1).$$

Factoring the quadratic, $P(x) = (x - 1)(x - 4) \times (x - 1)^2$
$$= (x - 1)^3(x + 4).$$

NOTE: The methods of the next section will allow this particular factoring to be done with no long divisions. The following worked exercise involves a polynomial that factors into distinct linear factors, so that nothing more than the factor theorem is required to complete the task.

WORKED EXERCISE: Factor $P(x) = x^4 - x^3 - 7x^2 + x + 6$ completely.

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SOLUTION: The divisors of the constant term 6 are 1, 2, 3, 6, −1, −2, −3 and −6.

$P(1) = 1 - 1 - 7 + 1 + 6 = 0,$ so $x - 1$ is a factor.
$P(-1) = 1 + 1 - 7 - 1 + 6 = 0,$ so $x + 1$ is a factor.
$P(2) = 16 - 8 - 28 + 2 + 6 = -12 \neq 0,$ so $x - 2$ is not a factor.
$P(-2) = 16 + 8 - 28 - 2 + 6 = 0,$ so $x + 2$ is a factor.
$P(3) = 81 - 27 - 63 + 3 + 6 = 0,$ so $x - 3$ is a factor.

We now have four distinct zeroes of a polynomial of degree 4.
Hence $P(x) = (x - 1)(x + 1)(x + 2)(x - 3)$ (notice that $P(x)$ is monic).

C. The Maximum Number of Zeroes: If a polynomial of degree n were to have $n + 1$ zeroes, then by the first theorem above, it would be divisible by a polynomial of degree $n + 1$, which is impossible.

17 MAXIMUM NUMBER OF ZEROES: A polynomial of degree n has at most n zeroes.

D. A Vanishing Condition: The previous theorem translates easily into a condition for a polynomial to be the zero polynomial.

18 A VANISHING CONDITION: Suppose that $P(x)$ is a polynomial in which no terms have degree more than n, yet which is zero for at least $n + 1$ distinct values of x. Then $P(x)$ is the zero polynomial.

PROOF: Suppose that $P(x)$ had a degree. This degree must be at most n since there is no term of degree more than n. But the degree must also be at least $n+1$ since there are $n+1$ distinct zeroes. This is a contradiction, so $P(x)$ has no degree, and is therefore the zero polynomial.

NOTE: Once again, the zero polynomial $Z(x) = 0$ is seen to be quite different in nature from all other polynomials. It is the only polynomial with an infinite number of zeroes; in fact every real number is a zero of $Z(x)$. Associated with this is the fact that $x - \alpha$ is a factor of $Z(x)$ for all real values of α, since $Z(x) = (x - \alpha)Z(x)$ (which is trivially true, because both sides are zero for all x). It is no wonder then that the zero polynomial does not have a degree.

E. A Condition for Two Polynomials to be Identically Equal: A most important consequence of this last theorem is a condition for two polynomials $P(x)$ and $Q(x)$ to be identically equal — written as $P(x) \equiv Q(x)$, and meaning that $P(x) = Q(x)$ for all values of x.

19 AN IDENTICALLY EQUAL CONDITION: Suppose that $P(x)$ and $Q(x)$ are polynomials of degree n which have the same values for at least $n + 1$ values of x. Then the polynomials $P(x)$ and $Q(x)$ are identically equal, written as $P(x) \equiv Q(x)$, that is, they are equal for all values of x.

PROOF: Let $F(x) = P(x) - Q(x)$.
Since $F(x)$ is zero whenever $P(x)$ and $Q(x)$ have the same value,
it follows that $F(x)$ is zero for at least $n + 1$ values of x,
so by the previous theorem, $F(x)$ is the zero polynomial, and $P(x) \equiv Q(x)$.

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WORKED EXERCISE: Find a, b, c and d, if $x^3 - x = a(x-2)^3 + b(x-2)^2 + c(x-2) + d$ for at least four values of x.

SOLUTION: Since they are equal for four values of x, they are identically equal.
Substituting $x = 2$, $6 = d$.
Equating coefficients of x^3, $1 = a$.
Substituting $x = 0$, $0 = -8 + 4b - 2c + 6$
$2b - c = 1$.
Substituting $x = 1$, $0 = -1 + b - c + 6$
$b - c = -5$.
Hence $b = 6$ and $c = 11$.

F. Geometrical Implications of the Factor Theorem: Here are some of the geometrical versions of the factor theorem — they are translations of the consequences given above into the language of coordinate geometry. They are simply generalisations of the similar remarks about the graphs of quadratics in Box 25 of Section 8I of the Year 11 volume.

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GEOMETRICAL IMPLICATIONS OF THE FACTOR THEOREM:
1. The graph of a polynomial function of degree n is completely determined by any $n + 1$ points on the curve.
2. The graphs of two distinct polynomial functions cannot intersect in more points than the maximum of the two degrees.
3. A line cannot intersect the graph of a polynomial of degree n in more than n points.

In parts (2) and (3), points where the two curves are tangent to each other count according to their multiplicity.

WORKED EXERCISE: By factoring the difference $F(x) = P(x) - Q(x)$, describe the intersections between the curves $P(x) = x^4 + 4x^3 + 2$ and $Q(x) = x^4 + 3x^3 + 3x$, and find where $P(x)$ is above $Q(x)$.

SOLUTION: Subtracting, $F(x) = x^3 - 3x + 2$.
Substituting, $F(1) = 1 - 3 + 2 = 0$, so $x - 1$ is a factor.
$F(-2) = -8 + 6 + 2 = 0$, so $x + 2$ is a factor.
After long division by $(x-1)(x+2) = x^2 + x - 2$,
$F(x) = (x-1)^2(x+2)$.
Hence $y = P(x)$ and $y = Q(x)$ are tangent at $x = 1$, but do not cross there, and intersect also at $x = -2$, where they cross at an angle.
Since $F(x)$ is positive for $-2 < x < 1$ or $x > 1$, and negative for $x < -2$, $P(x)$ is above $Q(x)$ for $-2 < x < 1$ or $x > 1$, and below it for $x < -2$.

A NOTE FOR 4 UNIT STUDENTS: The *fundamental theorem of algebra* is stated, but cannot be proven, in the 4 Unit course. It tells us that the graph of a polynomial of degree n intersects every line in exactly n points, provided first that points where the curves are tangent are counted according to their multiplicity, and secondly that complex points of intersection are also counted. As its name implies, this most important theorem provides the fundamental link between the algebra of polynomials and the geometry of their graphs, and allows the

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degree of a polynomial to be defined either algebraically, as the highest index, or geometrically, as the number of times every line crosses it.

G. Behaviour at Simple and Multiple Zeroes — A Proof: We can now give a satisfactory proof of the theorem stated in Box 7 of Section 4B:

'Suppose that $x = \alpha$ is a zero of multiplicity $m \geq 1$ of a polynomial $P(x)$.

- If $x = \alpha$ has even multiplicity, the curve is tangent to the x-axis at $x = \alpha$, and does not cross the x-axis there.
- If $x = \alpha$ has odd multiplicity at least 3, the curve is tangent to the x-axis at $x = \alpha$, and crosses the x-axis there at a point of inflexion.
- If $x = \alpha$ is a simple zero, then the curve crosses the x-axis at $x = \alpha$ and is not tangent to the x-axis there.'

PROOF: [The proof here is more suited to those taking the 4 Unit course.]

A. Differentiation is required since tangents are involved.

Let $\quad P(x) = (x - \alpha)^m Q(x)$, where $Q(\alpha) \neq 0$,

that is, where $(x - \alpha)$ is not a factor of $Q(x)$.

Using the product rule,
$$\begin{aligned} P'(x) &= m(x - \alpha)^{m-1} Q(x) + (x - \alpha)^m Q'(x) \\ &= (x - \alpha)^{m-1}\big(mQ(x) + (x - \alpha)Q'(x)\big). \end{aligned}$$

When $m = 1$, $\quad P'(\alpha) = Q(\alpha)$, which is not zero since $Q(\alpha) \neq 0$,

but when $m \geq 2$, $\quad P'(\alpha) = 0$.

Hence $x = \alpha$ is a zero of $P'(x)$ if and only if $m \geq 2$,

That is, the curve is tangent to the x-axis at $x = \alpha$ if and only if $m \geq 2$.

B. Since $Q(\alpha) \neq 0$, $P(x) = (x - \alpha)^m Q(x)$ will change sign around $x = \alpha$ when m is odd, and will not change sign around $x = \alpha$ when m is even. This completes the proof.

Exercise 4E

1. Use the factor theorem to write down in factored form:

(a) a monic cubic polynomial with zeroes -1, 3 and 4.

(b) a monic quartic polynomial with zeroes 0, -2, 3 and 1.

(c) a cubic polynomial with leading coefficient 6 and zeroes at $\frac{1}{3}$, $-\frac{1}{2}$ and 1.

2. (a) Show that 2 and 5 are zeroes of $P(x) = x^4 - 3x^3 - 15x^2 + 19x + 30$.

(b) Hence explain why $(x - 2)(x - 5)$ is a factor of $P(x)$.

(c) Divide $P(x)$ by $(x - 2)(x - 5)$ and hence express $P(x)$ as the product of four linear factors.

3. Use trial and error to find as many integer zeroes of $P(x)$ as possible. Use long division to divide $P(x)$ by the product of the known factors and hence express $P(x)$ in factored form.

(a) $P(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$

(b) $P(x) = 2x^4 - 5x^3 - 5x^2 + 20x - 12$

(c) $P(x) = 6x^4 - 25x^3 + 17x^2 + 28x - 20$

(d) $P(x) = 9x^4 - 51x^3 + 85x^2 - 41x + 6$

4. (a) The polynomial $(a - 2)x^2 + (1 - 3b)x + (5 - 2c)$ has three zeroes. What are the values of a, b and c?

(b) The polynomial $(a + 1)x^3 + (b - 3)x^2 + (2c - 1)x + (5 - 4d)$ has four zeroes. What are the values of a, b, c and d?

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DEVELOPMENT

5. (a) If $3x^2 - 4x + 7 \equiv a(x+2)^2 + b(x+2) + c$, find a, b and c.
(b) If $2x^3 - 8x^2 + 3x - 4 \equiv a(x-1)^3 + b(x-1)^2 + c(x-1) + d$, find a, b, c and d.
(c) Use similar methods to express $x^3 + 2x^2 - 3x + 1$ as a polynomial in $(x+1)$.
(d) If the polynomials $2x^2 + 4x + 4$ and $a(x+1)^2 + b(x+2)^2 + c(x+3)^2$ are equal for three values of x, find a, b and c.

6. (a) A polynomial of degree 3 has a double zero at 2. When $x = 1$ it takes the value 6 and when $x = 3$ it takes the value 8. Find the polynomial.
(b) Two zeroes of a polynomial of degree 3 are 1 and -3. When $x = 2$ it takes the value -15 and when $x = -1$ it takes the value 36. Find the polynomial.

7. Show that $x^2 - 3x + 2$ is a factor of $P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + (2^n - 2^m)$, where m and n are positive integers.

8. If two polynomials have degrees m and n respectively, what is the maximum number of points that their graphs can have in common?

9. Explain why a cubic with three distinct zeroes must have two turning points.

10. The line $y = k$ meets the curve $y = ax^3 + bx^2 + cx + d$ four times. Find the values of a, b, c, d and k.

11. Find the turning points of the following polynomials and hence state how many zeroes they have.
(a) $12x - x^3 + 3$
(b) $7 + 5x - x^2 - x^3$
(c) $4x^3 - x^4 - 1$
(d) $3x^4 - 4x^3 + 2$

12. By factoring the difference $F(x) = P(x) - Q(x)$, describe the intersections between the curves $P(x)$ and $Q(x)$.
(a) $P(x) = 2x^3 - 4x^2 + 3x + 1$, $Q(x) = x^3 + x^2 - 8$
(b) $P(x) = x^4 + x^3 + 10x - 4$, $Q(x) = x^4 + 7x^2 - 6x + 8$
(c) $P(x) = -2x^3 + 3x^2 - 25$, $Q(x) = -3x^3 - x^2 + 11x + 5$
(d) $P(x) = x^4 - 3x^2 - 2$, $Q(x) = x^3 - 5x$
(e) $P(x) = x^4 + 4x^3 - x + 5$, $Q(x) = x^3 - 3x^2 - 2x + 5$

13. Suppose that the polynomial equation $P(x) = 0$ has a double root at $x = \alpha$. That is, $P(x) = (x-\alpha)^2 Q(x)$, for some polynomial $Q(x)$, where $Q(\alpha) \neq 0$.
(a) Find $P'(x)$ and show that $P'(\alpha) = 0$.
(b) If $x = 1$ is a double root of $x^4 + ax^3 + bx^2 - 5x + 1 = 0$, find a and b.
(c) Given that $P(x) = 2x^4 - 20x^3 + 74x^2 - 120x + 72$, find $P'(x)$ and hence show that 2 and 3 are double roots of $P(x)$. Factor $P(x)$ completely.

EXTENSION

14. Show that if the polynomials $x^3 + ax^2 - x + b$ and $x^3 + bx^2 - x + a$ have a common factor of degree 2, then $a + b = 0$.

15. [Wallis' product and sine as an infinite product] This extraordinary expression of $\sin \pi x$ as an infinite product result is not possible for us to prove rigorously:

$$\sin \pi x = \pi x \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\left(1 - \frac{x^2}{4^2}\right)\cdots .$$

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(a) Here is a wildly invalid, but still interesting, justification inspired by the factor theorem for polynomials. First, the function $\sin \pi x$ is zero at every integer value of x, so regarding $\sin \pi x$ as a sort of polynomial of infinite degree, $(x-n)$ must be a factor for all n. Writing this another way, x is a factor, and $\left(1-\dfrac{x^2}{n^2}\right)$ is a factor for all n. Secondly, the constant multiple π can be justified (invalidly again) because $\sin \pi x \to \pi x$ as $x \to 0$, and each of the other factors on the RHS has limit 1 as $x \to 0$.

(b) By substituting $x = \frac{1}{2}$ into the expression in part (a), derive from it the identity called *Wallis' product* (which is, in contrast, accessible by methods of the 4 Unit course):

$$\frac{\pi}{2} = \frac{2^2}{1\times 3} \times \frac{4^2}{3\times 5} \times \frac{6^2}{5\times 7} \times \frac{8^2}{7\times 9} \times \cdots = \frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \times \cdots,$$

and use a calculator or computer to investigate the speed of convergence.

(c) By other substitutions into part (a), and using part (b), prove that

$$\sqrt{2} = \frac{2^2}{1\times 3} \times \frac{6^2}{5\times 7} \times \frac{10^2}{9\times 11} \times \frac{14^2}{13\times 15} \times \cdots = \frac{4}{3} \times \frac{36}{35} \times \frac{100}{99} \times \frac{196}{195} \times \cdots$$

$$\frac{3}{2} = \frac{2^2}{1\times 3} \times \frac{4^2}{3\times 5} \times \frac{8^2}{7\times 9} \times \frac{10^2}{9\times 11} \times \frac{14^2}{13\times 15} \times \cdots = \frac{4}{3} \times \frac{16}{15} \times \frac{64}{63} \times \frac{100}{99} \times \frac{196}{195} \times \cdots$$

4 F The Zeroes and the Coefficients

We have already shown in Chapter Eight of the Year 11 volume that if a quadratic $P(x) = ax^2 + bx + c$ has zeroes α and β, then their sum $\alpha + \beta$ and their product $\alpha\beta$ can easily be calculated from the coefficients without ever finding α or β themselves.

21 SUM AND PRODUCT OF ZEROES OF A QUADRATIC:

$$\alpha + \beta = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta = +\frac{c}{a}.$$

This section will generalise these results to polynomials of arbitrary degree. The general result is a little messy to state, so we shall deal with quadratic, cubic and quartic polynomials first.

The Zeroes of a Quadratic: Reviewing the work in Chapter Eight of the Year 11 volume, suppose that α and β are the zeroes of a quadratic $P(x) = ax^2 + bx + c$. By the factor theorem (see Box 15), $P(x)$ is a multiple of the product $(x-\alpha)(x-\beta)$:

$$\begin{aligned} P(x) &= a(x-\alpha)(x-\beta) \\ &= ax^2 \; - \; a(\alpha+\beta)x \; + \; a\alpha\beta \end{aligned}$$

Now equating terms in x and constants gives the results obtained before:

$$\begin{aligned} -a(\alpha+\beta) &= b \\ \alpha + \beta &= -\frac{b}{a} \end{aligned} \qquad \text{and} \qquad \begin{aligned} a(\alpha\beta) &= c \\ \alpha\beta &= \frac{c}{a} \end{aligned}$$

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The Zeroes of a Cubic: Suppose now that the cubic polynomial

$$P(x) = ax^3 + bx^2 + cx + d$$

has zeroes α, β and γ. Again by the factor theorem (see Box 15), $P(x)$ is a multiple of the product $(x-\alpha)(x-\beta)(x-\gamma)$:

$$\begin{aligned} P(x) &= a(x-\alpha)(x-\beta)(x-\gamma) \\ &= ax^3 \;-\; a(\alpha+\beta+\gamma)x^2 \;+\; a(\alpha\beta+\beta\gamma+\gamma\alpha)x \;-\; a\alpha\beta\gamma \end{aligned}$$

Now equating coefficients of terms in x^2, x and constants gives the new results:

22 ZEROES AND COEFFICIENTS OF A CUBIC:

$$\begin{aligned} \alpha+\beta+\gamma &= -\frac{b}{a} \\ \alpha\beta+\beta\gamma+\gamma\alpha &= +\frac{c}{a} \\ \alpha\beta\gamma &= -\frac{d}{a} \end{aligned}$$

The middle formula is best read as 'the sum of the products of pairs of zeroes'.

The Zeroes of a Quartic: Suppose that the four zeroes of the quartic polynomial

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e$$

are α, β, γ and δ. By the factor theorem (see Box 15), $P(x)$ is a multiple of the product $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$:

$$\begin{aligned} P(x) &= a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \\ &= ax^4 \;-\; a(\alpha+\beta+\gamma+\delta)x^3 \;+\; a(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2 \\ &\qquad -\; a(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)x \;+\; a\alpha\beta\gamma\delta. \end{aligned}$$

Equating coefficients of terms in x^3, x^2, x and constants now gives:

23 ZEROES AND COEFFICIENTS OF A QUARTIC:

$$\begin{aligned} \alpha+\beta+\gamma+\delta &= -\frac{b}{a} \\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= +\frac{c}{a} \\ \alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta &= -\frac{d}{a} \\ \alpha\beta\gamma\delta &= +\frac{e}{a} \end{aligned}$$

The second formula gives 'the sum of the products of pairs of zeroes', and the third formula gives 'the sum of the products of triples of zeroes'.

The General Case: Apart from the sum and product of zeroes, notation is a major difficulty here, and the results are better written in sigma notation. Suppose that the n zeroes of the degree n polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are α_1, α_2, ... α_n. Using similar methods gives:

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ZEROES AND COEFFICIENTS OF A POLYNOMIAL:

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}$$

$$\sum_{i<j} \alpha_i \alpha_j = +\frac{a_{n-2}}{a_n}$$

$$\sum_{i<j<k} \alpha_i \alpha_j \alpha_k = -\frac{a_{n-3}}{a_n}$$

$$\cdots\cdots$$

$$\alpha_1 \alpha_2 \cdots \alpha_n = (-1)^n \frac{a_0}{a_n}$$

It is unlikely that anything apart from the first and last formulae would be required.

WORKED EXERCISE: Let α, β and γ be the roots of the cubic equation $x^3 - 3x + 2 = 0$. Use the formulae above to find:

(a) $\alpha + \beta + \gamma$ (c) $\alpha\beta + \beta\gamma + \gamma\alpha$ (e) $\alpha^2 + \beta^2 + \gamma^2$

(b) $\alpha\beta\gamma$ (d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ (f) $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$

Check the result with the factorisation $x^3 - 3x + 2 = (x-1)^2(x+2)$ obtained in the last worked exercise of the previous section.

SOLUTION:

(a) $\alpha + \beta + \gamma = \frac{-0}{1} = 0$

(b) $\alpha\beta\gamma = -\frac{2}{1} = -2$

(c) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-3}{1} = -3$

(d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma}$
$\qquad = \frac{-3}{-2} = \frac{3}{2}$

(e) $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha,$
so $\quad 0^2 = \alpha^2 + \beta^2 + \gamma^2 + 2 \times (-3)$
$\alpha^2 + \beta^2 + \gamma^2 = 6.$

(f) $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$
$\quad = \alpha\beta(\alpha + \beta + \gamma) + \beta\gamma(\beta + \gamma + \alpha) + \gamma\alpha(\gamma + \alpha + \beta) - 3\alpha\beta\gamma$
$\quad = (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$
$\quad = (-3) \times 0 - 3 \times (-2)$
$\quad = 6$

Since $x^3 - 3x + 2 = (x-1)^2(x+2)$, the actual roots are 1, 1 and -2, hence

(a) $\alpha + \beta + \gamma = 1 + 1 - 2 = 0$

(b) $\alpha\beta\gamma = 1 \times 1 \times (-2) = -2$

(c) $\alpha\beta + \beta\gamma + \gamma\alpha = 1 - 2 - 2 = -3$

(d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = 1 + 1 - \frac{1}{2} = 1\frac{1}{2}$

(e) $\alpha^2 + \beta^2 + \gamma^2 = 1 + 1 + 4 = 6$

(f) $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$
$\quad = 1 \times 1 + 1 \times 1 + 1 \times (-2)$
$\qquad + 1 \times 4 + 4 \times 1 + (-2) \times 1$
$\quad = 6,$

all of which agree with the previous calculations.

Factoring Polynomials Using the Factor Theorem and the Sum and Product of Zeroes:

Long division can be avoided in many situations by applying the sum and product of zeroes formulae after one or more zeroes have been found. The full menu for the 3 Unit course now runs as follows:

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FACTORING POLYNOMIALS — THE FULL 3 UNIT MENU:
- Use trial and error to find as many integer zeroes of $P(x)$ as possible.
- Use sum and product of zeroes to find the other zeroes.
- Alternatively, use long division of $P(x)$ by the product of the known factors.

If the coefficients of $P(x)$ are all integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

In the following worked exercise, we factor a polynomial factored twice already, but this time there is no need for any long division.

WORKED EXERCISE: Factor $F(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

SOLUTION: As before, $F(1) = 1 + 1 - 9 + 11 - 4 = 0$,
and $F(-4) = 256 - 64 - 144 - 44 - 4 = 0$.
Let the zeroes be 1, -4, α and β.
Then $\alpha + \beta + 1 - 4 = -1$
$$\alpha + \beta = 2. \qquad (1)$$
Also $\alpha\beta \times 1 \times (-4) = -4$
$$\alpha\beta = 1. \qquad (2)$$
From (1) and (2), $\alpha = \beta = 1$, and so $F(x) = (x-1)^3(x+4)$.

WORKED EXERCISE: Factor completely the cubic $G(x) = x^3 - x^2 - 4$.

SOLUTION: First, $G(2) = 8 - 4 - 4 = 0$.
Let the zeroes be 2, α and β.
Then $2 + \alpha + \beta = 1$
$$\alpha + \beta = -1, \qquad (1)$$
and $2 \times \alpha \times \beta = 4$
$$\alpha\beta = 4. \qquad (2)$$
Substituting (1) into (2), $\alpha(-1-\alpha) = 2$
$$\alpha^2 + \alpha + 2 = 0$$
This is an irreducible quadratic, because $\Delta = -7$,
so the complete factorisation is $G(x) = (x-2)(x^2 + x + 2)$.

NOTE: This procedure — developing the irreducible quadratic factor from the sum and product of zeroes — is really little easier than the long division it avoids.

Forming Identities with the Coefficients: If some information can be gained about the roots of a polynomial equation, it may be possible to form an identity with the coefficients of the polynomial.

WORKED EXERCISE: If one root of the cubic $f(x) = ax^3 + bx^2 + cx + d$ is the opposite of another, prove that $ad = bc$.

SOLUTION: Let the zeroes be α, $-\alpha$ and β.
First, $\alpha - \alpha + \beta = -\dfrac{b}{a}$
$$a\beta = -b. \qquad (1)$$
Secondly, $-\alpha^2 - \alpha\beta + \beta\alpha = \dfrac{c}{a}$
$$a\alpha^2 = -c. \qquad (2)$$

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$$\text{Thirdly,}\qquad -\alpha^2\beta = -\frac{d}{a}$$
$$a\alpha^2\beta = d. \qquad (3)$$
$$\text{Taking } (1)\times(2)\div(3),\qquad a = \frac{bc}{d}$$
$$ad = bc,\quad \text{as required.}$$

A NOTE FOR 4 UNIT STUDENTS: The 4 Unit course develops one further technique for factoring polynomials. It is proven that if α is a zero of $P(x)$, then α is a zero of $P'(x)$ if and only if it is at least a double zero of $P(x)$. Thus multiple zeroes can be uncovered by testing whether a known zero is also a zero of the first derivative. Question 14 in Exercise 4E presented this idea, and it was implicit in paragraph G of Section 4E, but it is not required at 3 Unit level.

Exercise 4F

1. If α and β are the roots of the quadratic equation $x^2 - 4x + 2 = 0$, find:

(a) $\alpha + \beta$
(b) $\alpha\beta$
(c) $\alpha^2\beta + \alpha\beta^2$
(d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$
(e) $(\alpha + 2)(\beta + 2)$
(f) $\alpha^2 + \beta^2$
(g) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$
(h) $\alpha\beta^3 + \alpha^3\beta$
(i) $\left(\alpha + \dfrac{1}{\alpha}\right)\left(\beta + \dfrac{1}{\beta}\right)$

2. If α, β and γ are the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$, find:

(a) $\alpha + \beta + \gamma$
(b) $\alpha\beta + \alpha\gamma + \beta\gamma$
(c) $\alpha\beta\gamma$
(d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$
(e) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$
(f) $(\alpha + 1)(\beta + 1)(\gamma + 1)$
(g) $(\alpha\beta)^2\gamma + (\alpha\gamma)^2\beta + (\beta\gamma)^2\alpha$
(h) $\alpha^2 + \beta^2 + \gamma^2$
(i) $(\alpha\beta)^{-2} + (\alpha\gamma)^{-2} + (\beta\gamma)^{-2}$

Now find the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$ by factoring the LHS. Hence check your answers for expressions (a)–(i).

3. If a_1, a_2, a_3 and a_4 are the roots of the equation $x^4 - 5x^3 + 2x^2 - 4x - 3 = 0$, find:

(a) $\displaystyle\sum_i a_i$
(b) $\displaystyle\sum_{i<j} a_i a_j$
(c) $\displaystyle\sum_{i<j<k} a_i a_j a_k$
(d) $a_1 a_2 a_3 a_4$
(e) $\displaystyle\sum_i (a_i)^{-1}$
(f) $\displaystyle\sum_{i<j} (a_i a_j)^{-1}$
(g) $\displaystyle\sum_{i<j<k} (a_i a_j a_k)^{-1}$
(h) $\displaystyle\sum_i (a_i)^2$

4. In each of the following questions, find each coefficient in turn by considering the sums and products of the roots.

(a) Form a quadratic equation with roots -3 and 2.
(b) Form a cubic equation with roots -3, 2 and 1.
(c) Form a quartic equation with roots -3, 2, 1 and -1.

5. Show that $x = 1$ and $x = -2$ are zeroes of $P(x)$, and use the sum and product of zeroes to find the other one or two zeroes. Note any multiple zeroes.

(a) $P(x) = x^3 - 2x^2 - 5x + 6$
(b) $P(x) = 2x^3 + 3x^2 - 3x - 2$
(c) $P(x) = x^4 + 3x^3 - 3x^2 - 7x + 6$
(d) $P(x) = 3x^4 - 5x^3 - 10x^2 + 20x - 8$

6. Use trial and error to find two integer zeroes of $F(x)$. Then use the sum and product of zeroes to find any other zeroes. Note any multiple zeroes.

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(a) $F(x) = x^4 - 6x^2 - 8x - 3$
(b) $F(x) = x^4 - 15x^2 + 10x + 24$
(c) $F(x) = x^4 - 8x^3 + 6x^2 + 40x + 25$
(d) $F(x) = x^4 + x^3 - 3x^2 - 4x - 4$

DEVELOPMENT

7. If α and β are the roots of the equation $2x^2 + 5x - 4 = 0$, find:
(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2 + \beta^2$ (d) $\alpha^3 + \beta^3$ (e) $|\alpha - \beta|$

8. Consider the polynomial $P(x) = x^3 - x^2 - x + 10$.
(a) Show that -2 is a zero of $P(x)$.
(b) Given that the zeroes of $P(x)$ are -2, α and β, show that $\alpha + \beta = 3$ and $\alpha\beta = 5$.
(c) Solve simultaneously the two equations in part (b) (you will need to form a quadratic in α), and hence show that there are no such real numbers α and β.
(d) Hence state how many times the graph of the cubic crosses the x-axis.

9. (a) Suppose that $x - 3$ and $x + 1$ are factors of $x^3 - 6x^2 + ax + b$. Find a and b, and hence use sum and product of zeroes to factor the polynomial.
(b) Suppose that $2x^3 + ax^2 - 14x + b$ has zeroes at -2 and 4. Find a and b, and hence use sum and product of zeroes to find the other zero.

10. (a) Find values of a and b for which $x^3 + ax^2 - 10x + b$ is exactly divisible by $x^2 + x - 12$, and then factor the cubic.
(b) Find values of a and b for which $x^2 - x - 20$ is a factor of $x^4 + ax^3 - 23x^2 + bx + 60$, and then find all the zeroes.

11. The polynomial $P(x) = x^3 - Lx^2 + Lx - M$ has zeroes α, $\dfrac{1}{\alpha}$ and β.
(a) Show that: (i) $\alpha + \dfrac{1}{\alpha} + \beta = L$ (ii) $1 + \alpha\beta + \dfrac{\beta}{\alpha} = L$ (iii) $\beta = M$
(b) Show that either $M = 1$ or $M = L - 1$.

12. The cubic equation $x^3 - Ax^2 + 3A = 0$, where $A > 0$, has roots α, β and $\alpha + \beta$.
(a) Use the sum of the roots to show that $\alpha + \beta = \frac{1}{2}A$.
(b) Use the sum of the products of pairs of roots to show that $\alpha\beta = -\frac{1}{4}A^2$.
(c) Show that $A = 2\sqrt{6}$.

13. (a) Find the roots of the equation $4x^3 - 8x^2 - 3x + 9 = 0$, given that two of the roots are equal. [HINT: Let the roots be α, α and β.]
(b) Find the roots of the equation $3x^3 - x^2 - 48x + 16 = 0$, given that the sum of two of the roots is zero. [HINT: Let the roots be α, $-\alpha$ and β.]
(c) Find the roots of the equation $2x^3 - 5x^2 - 46x + 24 = 0$, given that the product of two of the roots is 3. [HINT: Let the roots be α, $\dfrac{3}{\alpha}$ and β.]
(d) Find the zeroes of the polynomial $P(x) = 2x^3 - 13x^2 + 22x - 8$, given that one zero is the product of the other two. [HINT: Let the zeroes be α, β and $\alpha\beta$.]

14. (a) Find the roots of the equation $9x^3 - 27x^2 + 11x + 7 = 0$, if the roots form an arithmetic sequence. [HINT: Let the roots be $\alpha - d$, α and $\alpha + d$.] Then find the point of inflexion of $y = 9x^3 - 27x^2 + 11x + 7$, and show that its x-coordinate is one of the roots.
(b) Find the zeroes of the polynomial $P(x) = 8x^3 - 14x^2 + 7x - 1$, if the zeroes form a geometric sequence. [HINT: Let the zeroes be $\dfrac{\alpha}{r}$, α and αr.]

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(c) Solve the equation $2x^3 - 3x^2 - 3x + 2 = 0$, given that the roots are in geometric progression.

15. (a) Two of the roots of the equation $x^3 + 3x^2 - 4x + a = 0$ are opposites. Find the value of a and the three roots.

(b) Two of the roots of the equation $4x^3 + ax^2 - 47x + 12 = 0$ are reciprocals. Find the value of a and the three roots.

(c) Find α and β if the zeroes of the polynomial $x^4 - 3x^3 - 8x^2 + 12x + 16 = 0$ are α, 2α, β and 2β.

16. (a) If one root of the equation $x^3 - bx^2 + cx - d = 0$ is equal to the product of the other two, show that $(c + d)^2 = d(b + 1)^2$.

(b) If the roots of the equation $x^3 + ax^2 + bx + c = 0$ form an arithmetic sequence, show that $9ab = 2a^3 + 27c$, and that one of the roots is $-\frac{1}{3}a$.

(c) If the roots of the equation $x^3 + ax^2 + bx + c = 0$ form a geometric sequence, show that $b^3 = a^3c$, and that one of the roots is $-\sqrt[3]{c}$.

(d) The polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ has two zeroes which are opposites and two zeroes which are reciprocals. Show that:

(i) $b = 1 + d$ (ii) $c = ad$

(e) If the zeroes of the cubic $y = x^3 + ax^2 + bx + c = 0$ form an arithmetic sequence, show that the point of inflexion lies on the x-axis.

17. Consider the cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$, which has zeroes α, β and γ such that $\gamma = \alpha + \beta$.

(a) Show that $\alpha + \beta = -\dfrac{b}{2a}$.

(b) Show that $\alpha\beta = \dfrac{2d}{b}$.

(c) Hence show that α and β are also the roots of the equation $2abx^2 + b^2x + 4ad = 0$.

18. (a) The cubic equation $2x^3 - x^2 + x - 1 = 0$ has roots α, β and γ. Evaluate:

(i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$

(iii) $\alpha\beta\gamma$

(iv) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

(b) The equation $2\cos^3\theta - \cos^2\theta + \cos\theta - 1 = 0$ has roots $\cos a$, $\cos b$ and $\cos c$. Using the results in (a), prove that $\sec a + \sec b + \sec c = 1$.

19. (a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(b) Show that the cubic equation $8x^3 - 6x + 1 = 0$ reduces to the form $\cos 3\theta = -\frac{1}{2}$ by substituting $x = \cos\theta$.

(c) Hence find the three solutions to the cubic equation.

(d) Use the sum and product of roots to evaluate:

(i) $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9}$

(ii) $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{\pi}{9}$

(iii) $\sec\frac{2\pi}{9} + \sec\frac{4\pi}{9} - \sec\frac{\pi}{9}$

(iv) $\cos^2\frac{2\pi}{9} + \cos^2\frac{4\pi}{9} + \cos^2\frac{\pi}{9}$

EXTENSION

20. If α, β and γ are the roots of the equation $x^3 + 5x - 4 = 0$, evaluate $\alpha^3 + \beta^3 + \gamma^3$.

21. If $x^n - 1 = 0$ has n roots, $\alpha_1, \alpha_2, \ldots, \alpha_n$, what is $(1 - \alpha_1)(1 - \alpha_2)\ldots(1 - \alpha_n)$?

22. Suppose that the equation $x^3 + ax^2 + bx + c = 0$ has roots α, β and γ. If $\gamma = \alpha + \beta$, show that $a^3 - 4ab + 8c = 0$. [HINT: Use your work in question 17.]

23. If $y = x^4 + bx^3 + cx^2 + dx + e$ has two double zeroes, express d and e in terms of b and c.

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24. Suppose that $P(x) = x^3 + cx + d$ has zeroes α, β and γ. Show that:
(a) $\alpha\beta = c + \gamma^2$ (b) $(\alpha - \beta)^2 = -3\gamma^2 - 4c$ (c) $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 = -4c^3 - 27d^2$
Check these results for the polynomial $P(x) = (x-1)(x-2)(x+3)$.
NOTE: For any monic cubic, the expression $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ is called the *discriminant*. Students taking the 4 Unit course may like to prove that the cubic has three distinct real zeroes if and only if the discriminant is positive.

4 G Geometry using Polynomial Techniques

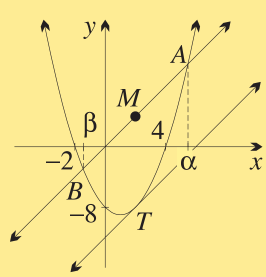
This final section adds the methods of the preceding sections, particularly the sum and product of roots, to the available techniques for studying the geometry of various curves. The standard technique is to examine the roots of the equation formed in the process of solving two curves simultaneously.

Midpoints and Tangents: When two curves intersect, we can form the equation whose solutions are the x- or y-coordinates of points of intersection of the two curves. The midpoint of two points of intersection can then be found using the average of the roots. Tangents can be identified as corresponding to double roots.

The following worked exercise could also be done using quadratic equations, but it is a very clear example of the use of sum and product of roots.

WORKED EXERCISE: The line $y = 2x$ meets the parabola $y = x^2 - 2x - 8$ at the two points $A(\alpha, 2\alpha)$ and $B(\beta, 2\beta)$.
(a) Show that α and β are roots of $x^2 - 4x - 8 = 0$, and hence find the coordinates of the midpoint M of AB.
(b) Use the identity $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$ to find the horizontal distance $|\alpha - \beta|$ from A to B. Then use Pythagoras' theorem and the gradient of the line to find the length of AB.
(c) Find the value of b for which $y = 2x + b$ is a tangent to the parabola, and find the point T of contact.

SOLUTION:



(a) Solving the line and the parabola simultaneously,
$$x^2 - 2x - 8 = 2x$$
$$x^2 - 4x - 8 = 0.$$
Hence $\alpha + \beta = 4$,
and $\alpha\beta = -8$.
Averaging the roots, M has x-coordinate $x = 2$,
and substituting into the line, $M = (2, 4)$.

(b) We know that
$$\begin{aligned}(\alpha-\beta)^2 &= (\alpha+\beta)^2 - 4\alpha\beta\\ &= 16 + 32\\ &= 48\end{aligned}$$
and so $|\alpha - \beta| = 4\sqrt{3}$.
Since the line has gradient 2, the vertical distance is $8\sqrt{3}$,
so using Pythagoras,
$$\begin{aligned}AB^2 &= (4\sqrt{3})^2 + (8\sqrt{3})^2\\ &= 16 \times 15\\ AB &= 4\sqrt{15}.\end{aligned}$$

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(c) Solving $y = x^2 - 2x - 8$ and $y = 2x + b$ simultaneously,
$$x^2 - 4x - (8 + b) = 0$$
Since the line is a tangent, let the roots be θ and θ.
Then using the sum of roots, $\theta + \theta = 4$,
so $\theta = 2$,
Using the product of roots, $\theta^2 = -8 - b$
and since $\theta = 2$, $b = -12$.
So the line $y = 2x - 12$ is a tangent at $T(2, -8)$.

Locus Problems Using Sum and Product of Roots: The preceding theory can make some rather obscure-looking locus problems quite straightforward.

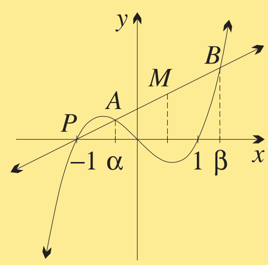
WORKED EXERCISE: A line through the point $P(-1, 0)$ crosses the cubic $y = x^3 - x$ at two further points A and B.

(a) Sketch the situation, and find the locus of the midpoint M of AB.

(b) Find the line through P tangent to the cubic at a point distinct from P.

SOLUTION:

(a) Let $y = m(x + 1)$ be a general line through $P(-1, 0)$.
Solving the line simultaneously with the cubic,
$$x^3 - x = mx + m$$
$$x^3 - (m + 1)x - m = 0.$$
Let the x-coordinates of A and B be α and β respectively.
Then $\alpha + \beta + (-1) = 0$, using the sum of roots,
$$\alpha + \beta = 1.$$
Hence the midpoint M of AB has x-coordinate $\frac{1}{2}(\alpha + \beta) = \frac{1}{2}$,
and the locus of M is therefore the line $x = \frac{1}{2}$.
But the line does not extend below the cubic, so $y \geq -\frac{3}{8}$.



(b) For the line to be a tangent, $\alpha = \beta$,
and since $\alpha + \beta = 1$, we must have $\alpha = \beta = \frac{1}{2}$.
Using the product of roots, $\alpha \times \beta \times (-1) = m$,
so $m = -\frac{1}{4}$, and the line is $y = -\frac{1}{4}(x + 1)$.

Exercise 4G

NOTE: These are geometrical questions, and sketches should be drawn every time — the algebraic result should look reasonable on the diagram. Questions 1–11 have been carefully structured to indicate the intended methods, and questions 12–15 should be done by similar methods.

1. (a) Show that the x-coordinates of the points of intersection of the parabola $y = x^2 - 6x$ and the line $y = 2x - 16$ satisfy the equation $x^2 - 8x + 16 = 0$.

(b) Solve this equation, and hence show that the line is a tangent to the parabola. Find the point T of contact.

2. (a) Show that the x-coordinates of the points of intersection of the line $y = b - 2x$ and the parabola $y = x^2 - 6x$ satisfy the equation $x^2 - 4x - b = 0$.

(b) Suppose now that the line is a tangent to the parabola, so that the roots are α and α.

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(i) Using the sum of roots, show that $\alpha = 2$.

(ii) Using the product of roots, show that $\alpha^2 = -b$, and hence find b.

(iii) Find the equation of the tangent and its point T of contact.

3. The line $y = x + 1$ meets the parabola $y = x^2 - 3x$ at A and B.

(a) Show that the x-coordinates α and β of A and B satisfy the equation $x^2 - 4x - 1 = 0$.

(b) Find $\alpha + \beta$, and hence find the coordinates of the midpoint M of AB.

(c) Use the identity $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to show that the horizontal distance $|\alpha - \beta|$ between A and B is $2\sqrt{5}$.

(d) Use the gradient to explain why the vertical distance between A and B is also $2\sqrt{5}$, and hence use Pythagoras' theorem to find the length AB.

NOTE: Sketches in question 4–6 require the factorisation $x^3 - 5x^2 + 6x = x(x-2)(x-3)$.

4. (a) Show that the x-coordinates of the points of intersection of the line $y = 3 - x$ and the cubic $y = x^3 - 5x^2 + 6x$ satisfy the equation $x^3 - 5x^2 + 7x - 3 = 0$.

(b) Show that $x = 1$ and $x = 3$ are roots, and use the sum of roots to find the third root.

(c) Explain why the line is a tangent to the cubic, find the point of contact and the other point of intersection.

5. (a) Show that the x-coordinates of the points of intersection of the line $y = mx$ and the cubic $y = x^3 - 5x^2 + 6x$ satisfy the equation $x^3 - 5x^2 + (6 - m)x = 0$.

(b) Suppose now that the line is a tangent to the cubic at a point other than the origin, so that the roots are 0, α and α.

(i) Using the sum of roots, show that $\alpha = 2\frac{1}{2}$.

(ii) Using the product of pairs of roots, show that $\alpha^2 = 6 - m$, and hence find m.

(iii) Find the equation of the tangent and its point T of contact.

6. The line $y = x - 2$ meets the cubic $y = x^3 - 5x^2 + 6x$ at $F(2, 0)$, and also at A and B.

(a) Show that the x-coordinates α and β of A and B satisfy $x^3 - 5x^2 + 5x + 2 = 0$.

(b) Find $\alpha + \beta$, and hence find the coordinates of the midpoint M of AB.

(c) Show that $\alpha\beta = -1$, then use the identity $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to show that the horizontal distance $|\alpha - \beta|$ between A and B is $\sqrt{13}$.

(d) Hence use Pythagoras' theorem to find the length AB.

DEVELOPMENT

7. Suppose that the cubic $F(x) = x^3 + ax^2 + bx + c$ has a relative minimum at $x = \alpha$ and a relative maximum at $x = \beta$.

(a) By examining the zeroes of $F'(x)$, prove that $\alpha + \beta = -\frac{2}{3}a$.

(b) Deduce that the point of inflexion occurs at $x = \frac{1}{2}(\alpha + \beta)$.

8. A line is drawn from the point $A(-1, -7)$ on the curve $y = x^3 - 3x^2 + 4x + 1$ to touch the curve again at P.

(a) Write down the equation of the line, given that it has gradient m.

(b) Find the cubic equation whose roots represent the x-coordinates of the points of intersection of the line and the curve.

(c) Explain why the roots of this equation are -1, α and α, and hence find the point T of contact and the value of m.

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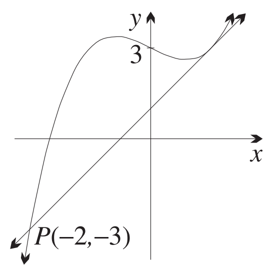
9. The point $P(p, p^3)$ lies on the curve $y = x^3$. A straight line through P cuts the curve again at A and B.
(a) Find the equation of the straight line through P if it has gradient m.
(b) Show that the x-coordinates of A and B satisfy the equation $x^3 - mx + mp - p^3 = 0$.
(c) Hence find the x-coordinate of the midpoint M of AB, and show that for fixed p, M always lies on a line that is parallel to the y-axis.

10. (a) The cubic $x^3 - (m+1)x + (6 - 2m) = 0$ has a root at $x = -2$ and a double root at $x = \alpha$. Find m and α.
(b) Write down the equation of the line ℓ passing through the point $P(-2, -3)$ with gradient m.
(c) The diagram shows the curve $y = x^3 - x + 3$ and the point $P(-2, -3)$ on the curve. The line ℓ cuts the curve at P, and is tangent to the curve at another point A on the curve. Find the equation of the line ℓ.



11. (a) Use the factor theorem to factor the polynomial $y = x^4 - 4x^3 - 9x^2 + 16x + 20$, given that there are four distinct zeroes, then sketch the curve.
(b) The line $\ell: y = mx + b$ touches the quartic $y = x^4 - 4x^3 - 9x^2 + 16x + 20$ at two distinct points A and B. Explain why the x-coordinates α and β of A and B are double roots of $x^4 - 4x^3 - 9x^2 + (16 - m)x + (20 - b) = 0$.
(c) Use the theory of the sum and product of roots to write down four equations involving α, β, m and b.
(d) Hence find m and b, and write down the equation of ℓ.

12. (a) Find k and the points of contact if the parabola $y = x^2 - k$ touches the quartic $y = x^4$ at two points.
(b) Find $c > 0$ and the points of contact if the hyperbola $xy = c^2$ touches the cubic $y = -x^3 + x$ at two points.
(c) Find k and the point T of contact if the parabola $y = x^2 - k$ touches the cubic $y = x^3$.
(d) Find α if the quadratic $y = ax(x-1)$ is tangent to the circle $x^2 + y^2 = 1$ at $x = \alpha$. [HINT: The curves always intersect when $x = 1$.]

13. (a) The variable line $y = 3x + b$ with gradient 3 meets the circle $x^2 + y^2 = 16$ at A and B. Find the locus of the midpoint M of AB.
(b) The fixed point $F(0, 2)$ lies inside the circle $x^2 + y^2 = 16$. A variable line ℓ through F meets the circle at A and B. Find and describe the locus of the midpoint M of AB.
(c) The parabola $y = x(x - a)$ meets the cubic $y = x^3 - 3x^2 + 2x$ at $O(0, 0)$, A and B. Find, including any restrictions, the locus of the midpoint M of AB as a varies.
(d) The line $y = mx + b$ meets the hyperbola $xy = 1$ at A and B. Find the locus of the midpoint M of AB if: (i) m is constant, (ii) b is constant.
(e) The parabola with vertex at $F(-1, -1)$ meets the hyperbola $xy = 1$ again at A and B. Find the locus of the midpoint M of AB.

14. (a) Find a and the points of contact if the parabola $y = x^2 - a$ touches the circle $x^2 + y^2 = 1$ at two distinct points.
(b) Find a and any other points of intersection if the parabola $y = x^2 - a$ touches the circle $x^2 + y^2 = 1$ at exactly one point.
(c) Find a if the circle $x^2 + (y - a)^2 = a^2$ intersects the parabola $y = x^2$ at a point which is a fourfold zero of the quartic formed when solving the two equations simultaneously.

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(d) By solving the line $y = mx + b$ simultaneously with the cubic $y = x^3 - 6x^2 - 2x + 1$ and insisting that there be a triple root, find the point of inflexion of the cubic without using calculus.

(e) Find the line which touches the quartic $y = x^2(x-2)(x-6)$ at two distinct points A and B, and find the distance AB.

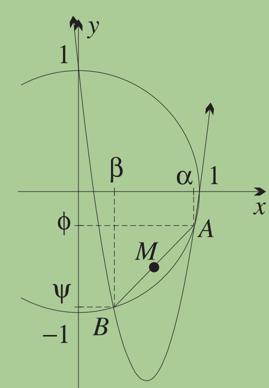
15. [A cubic has odd symmetry in its point of inflexion.] The line $y = mx + n$ meets the cubic $y = ax^3 + bx^2 + cx + d$ in three distinct points A, B and C. Show that if $AB = BC$, then B is the point of inflexion.

EXTENSION

16. A circle passing through the origin O is tangent to the hyperbola $xy = 1$ at A, and intersects the hyperbola again at two distinct points B and C. Prove that $OA \perp BC$.

17. The diagram to the right shows the circle $x^2 + y^2 = 1$ and the parabola $y = (\lambda x - 1)(x - 1)$, where λ is a constant. The circle and parabola meet in the four points

$$P(1,0), \qquad Q(0,1), \qquad A(\alpha,\phi), \qquad B(\beta,\psi).$$



The point M is the midpoint of the chord AB.

(a) Show that the x-coordinates of the points of intersection of the two curves satisfy the equation

$$\lambda^2 x^4 - 2\lambda(1+\lambda)x^3 + (\lambda^2 + 4\lambda + 2)x^2 - 2(1+\lambda)x = 0.$$

(b) Use the formula for the sum of the roots to show that the x-coordinate of M is $\dfrac{\lambda + 2}{2\lambda}$.

(c) Use a similar method to find the y-coordinate of M, and hence show that the locus of M is the line through the origin O parallel to PQ.

(d) For what values of λ is the parabola tangent to the circle in the fourth quadrant?

(e) For what values of λ are the four points P, Q, A and B distinct, with real numbers as coordinates.

18. [Harmonic conjugates] The line $\ell\colon y = mx - mb$ through the point $P(b,0)$ outside the circle $x^2 + y^2 = 1$ meets the circle at the points A and B with x-coordinates α and β.

(a) Show that α and β satisfy the equation $(m^2+1)x^2 - 2m^2bx + (m^2b^2 - 1) = 0$.

(b) Show that if ℓ is a tangent to the circle, then $m^2(b^2 - 1) = 1$. Hence find the equation of the line ST joining the points S and T of the tangents to the circle from P.

(c) The general line ℓ meets ST at Q. Prove that Q divides AB internally in the same ratio as P divides AB externally.

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