University of New South Wales

MATH 2221

Higher Theory and Applications of Differential Equations

Assignment 2

October 12, 2018

1. Consider the following ODEs

$$\frac{du}{dx} = |u|\,, x \in \mathbb{R} \ldots (1) \text{ and } \frac{dv}{dt} = v^{1/2}\,, t \in [0, \infty] \ldots (2)$$

(a) Show that these permit solutions of the form $u = Ae^x$ and $v = Bt^2$ respectively.

Let $u = Ae^x$, then, considering (1),

$$\begin{aligned} \text{LHS} &= \frac{du}{dx} = Ae^x \\ \text{RHS} &= |u| = |A|e^x \qquad \text{as } e^x > 0,\ \forall x \in \mathbb{R} \end{aligned}$$

Thus, (1) permits solutions of the form u for $A \geq 0$.

Let $v = Bt^2$, then, considering (2),

$$\begin{aligned} \text{LHS} &= \frac{dv}{dt} = 2Bt \\ \text{RHS} &= v^{1/2} = \sqrt{B}|t| \\ &= \sqrt{B}t \qquad \text{as } t \in [0, \infty] \end{aligned}$$

Thus, (2) permits solutions of the form v, when $2B = \sqrt{B}$, that is, $B = 0$ or $B = \frac{1}{4}$.

(b) Solve these for the cases $u(0) = 0$ and $v(0) = 0$.

Considering (1) and $u(0) = 0$, we require $Ae^0 = 0$, that is, $A = 0$. So the solution is $u = 0$.

Considering (2) and $v(0) = 0$, we require $B(0)^2 = 0$, which is satisfied $\forall B \in \mathbb{R}$. So the solutions are $v = 0$ and $v = \frac{1}{4}t^2$.

(c) Is the solution unique in either case? Explain your answer.

Considering the solution to (1) and $u(0) = 0$, we get $u = 0$ as the only satisfactory solution, and thus it is unique.

Considering the solution to (2) and $v(0) = 0$, we get $v = 0$ and $v = \frac{1}{4}t^2$ as the satisfactory solutions, and as there is more than one satisfactory solution, the solution is not unique.

2. Consider the function

$$f(x) = 1 - x \text{ for } 0 < x < 1$$

(a) Find both the Fourier Sine and Fourier–Bessel series describing f.

To find the Fourier Sine series for f, f must first be an odd function. Thus, we define g,

$$\begin{aligned} g(x) &= \begin{cases} f(x) & 0 < x < 1 \\ -f(-x) & -1 < x \leq 0 \end{cases} \\ &= \begin{cases} 1 - x & 0 < x < 1 \\ -1 - x & -1 < x \leq 0 \end{cases} \end{aligned}$$

where g is an odd function (the odd extension of f), on $-1 < x < 1$. As a result, g admits a Fourier series of the form $\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$. Thus, using the formulas for Fourier coeffecients, for functions of $2L$-periodicity, with $L = 1$,

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{1}\int_0^1 g(x)\sin\left(\frac{n\pi x}{1}\right)dx \\ &= 2\int_0^1 (1-x)\sin(n\pi x)\,dx \\ &= 2\left[(1-x)\left(\frac{-1}{n\pi}\cos(n\pi x)\right)\Bigg|_0^1 - \int_0^1 \frac{\cos(n\pi x)}{n\pi}dx\right] \\ &= 2\left[(1-x)\left(\frac{-1}{n\pi}\cos(n\pi x)\right)\Bigg|_0^1 - \frac{1}{n^2\pi^2}\sin(n\pi x)\Bigg|_0^1\right] \\ &= 2\left[\frac{1}{n\pi} - 0\right] \\ &= \frac{2}{n\pi} \\ \therefore g(x) &= \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x) \\ \therefore f(x) &= \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x) \end{aligned}$$

as $f(x) = g(x)$ on the interval $0 < x < 1$.

Now, we determine the Fourier Bessel series for f using the general Fourier Bessel form, $\sum_{n=1}^{\infty} A_n J_\nu(k_n x)$. Firstly, the interval of definition is $0 < x < 1$, and thus $l = 1$. Secondly, k_n is defined as the n-th solution to $J_\nu(k_n) = 0$. Lastly, to uniformly fit the Fourier Bessel series to the function $f = 1 - x$, we require, at $x = 0$, the Fourier Bessel series to be equal to 1, as $f(0) = 1$. Therefore, $\nu = 0$ is the only ν that satisifies the restriction. Thus, the general Fourier Bessel form used is instead $\sum_{n=1}^{\infty} A_n J_0(k_n x)$. Using the formula for Fourier Bessel coefficients to derive A_n,

$$
\begin{aligned}
A_n &= \frac{2}{l^2 J_{\nu+1}(k_n l)^2} \int_0^l f(x) J_\nu(k_n x) x dx \\
&= \frac{2}{J_1(k_n)^2} \int_0^1 x(1-x) J_0(k_n x) dx \\
&= \frac{2}{J_1(k_n)^2} \left[\int_0^1 x J_0(k_n x) dx - \int_0^1 x^2 J_0(k_n x) dx \right] \\
&= \frac{2}{J_1(k_n)^2} \left[\frac{1}{k_n} \int_0^1 (k_n x) J_0(k_n x) dx - \frac{1}{k_n} \int_0^1 x (k_n x) J_0(k_n x) dx \right] \\
&= \frac{2}{J_1(k_n)^2} \left[\frac{1}{k_n} (k_n x) J_1(k_n x) \Big|_0^1 - \left(\frac{x}{k_n} (k_n x) J_1(k_n x) \Big|_0^1 - \frac{1}{k_n} \int_0^1 (k_n x) J_1(k_n x) dx \right) \right] \\
&= \frac{2}{J_1(k_n)^2} \left[x J_1(k_n x) \Big|_0^1 - \left(x^2 J_1(k_n x) \Big|_0^1 - \int_0^1 x J_1(k_n x) dx \right) \right] \\
&= \frac{2}{J_1(k_n)^2} \left[J_1(k_n) - J_1(k_n) + \int_0^1 x J_1(k_n x) dx \right] \\
&= \frac{2}{J_1(k_n)^2} \left[\int_0^1 x J_1(k_n x) dx \right] \\
&= \frac{2}{J_1(k_n)^2} \left[\int_0^1 x \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+1+1)} \left(\frac{k_n x}{2} \right)^{2m+1} dx \right] \\
&= \frac{2}{J_1(k_n)^2} \left[\int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{k_n}{2} \right)^{2m+1} x^{2m+2} dx \right] \\
&= \frac{2}{J_1(k_n)^2} \left[\sum_{m=0}^{\infty} \int_0^1 \frac{(-1)^m}{m!(m+1)!} \left(\frac{k_n}{2} \right)^{2m+1} x^{2m+2} dx \right] \\
&= \frac{2}{J_1(k_n)^2} \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{k_n}{2} \right)^{2m+1} \int_0^1 x^{2m+2} dx \right] \\
&= \frac{2}{J_1(k_n)^2} \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{k_n}{2} \right)^{2m+1} \left(\frac{x^{2m+3}}{2m+3} \right) \Bigg|_0^1 \right] \\
&= \frac{2}{J_1(k_n)^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{1}{2m+3} \right) \left(\frac{k_n}{2} \right)^{2m+1}
\end{aligned}
$$

we get $f(x) = \sum_{n=1}^{\infty} A_n J_0(k_n x)$, where A_n is given by the above calculations.

(b) Which has the smallest mean square error when the first 3 terms of each series are used?

To answer this question, the formula $\|e_n\|^2 = \sum_{n=N+1}^{\infty} A_n^2 \|\phi_n\|^2$ will be used.

Considering now the Fourier Sine series for f, where $\phi_n = \sin n\pi x$, and $\|\phi_n\|^2 = \dfrac{1}{2}$. The mean square error for the first three terms of the Fourier Sine series is then as follows.

$$
\begin{aligned}
\|e_n\|^2 &= \sum_{n=N+1}^{\infty} A_n^2 \|\phi_n\|^2 \\
&= \sum_{n=3+1}^{\infty} \left(\frac{2}{n\pi}\right)^2 \cdot \frac{1}{2} \\
&= \frac{2}{\pi^2} \sum_{n=4}^{\infty} \frac{1}{n^2} \\
&= \frac{2}{\pi^2} \left[\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{3} \frac{1}{n^2} \right] \\
&= \frac{2}{\pi^2} \left[\frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9}\right) \right] \\
&= \frac{1}{3} - \frac{2}{\pi^2} \left(1 + \frac{1}{4} + \frac{1}{9}\right) \\
&\approx 0.0575
\end{aligned}
$$

Considering now the Fourier Bessel series for f, where $\phi_n = J_0 k_n x$, and $\|\phi_n\|^2 = J_1(k_n)^2$. From WolphramAlpha, we get $k_1 \approx 2.4048$, $k_2 \approx 5.5201$, and $k_3 \approx 8.6531$. From WolframAlpha, we get $A_1 \approx 0.943296057$, $A_2 \approx 0.189572523$, and $A_3 \approx 0.229550582$.
The mean square error for the first three terms of the Fourier Bessel series is then as follows.

$$
\begin{aligned}
\|e_n\|^2 &= \sum_{n=N+1}^{\infty} A_n^2 \|\phi_n\|^2 \\
&= \|f\|^2 - \sum_{n=1}^{N} A_n^2 \|\phi_n\|^2 \\
&= \int_0^1 (1-x)^2 dx - \sum_{n=1}^{N} \left(\frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} \right)^2 \|\phi_n\|^2 \\
\therefore \|e_3\|^2 &= -\frac{(1-x)^3}{3}\bigg|_0^1 - \sum_{n=1}^{3} \frac{\langle f, \phi_n \rangle^2}{\|\phi_n\|^2} \\
&= \frac{1}{3} - 2\left(0.0239820615 + 0.004160952 + 0.003883353\right) \\
&\approx 0.2693
\end{aligned}
$$

(c) In either case, as each additional term, n, is added and as $n \to \infty$, can a point $0 < a_n < 1$ always be found such that the series Sf differs from f by more than 1/2 (i.e. $|Sf(a_n) - f(a_n)| > 1/2$)? Explain your answer.

Consider first the Fourier Sine series for f. Set up the sequence $= \frac{1}{N}$, which will be used as we take $N \to \infty$. As a result, the Fourier Sine series can be written as $\sum_{n=1}^{N} \frac{2}{n\pi} \sin\left(\frac{n\pi}{N}\right)$.

$$
\begin{aligned}
L_1 &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{2}{n\pi} \sin\left(\frac{n\pi}{N}\right) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{2N}{n\pi N} \sin\left(\frac{n\pi}{N}\right) \\
&= \lim_{N\to\infty} \frac{2}{N} \sum_{n=1}^{N} \frac{N}{n\pi} \sin\left(\frac{n\pi}{N}\right) \\
&= \lim_{N\to\infty} \frac{2}{N} \sum_{n=1}^{N} \frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}} \\
&= \lim_{N\to\infty} \frac{2}{N} \sum_{n=1}^{N} 1 \\
&= \lim_{N\to\infty} \frac{2}{N} N \\
&= \lim_{N\to\infty} 2 \\
\therefore L_1 &= 2 \quad \text{when } \frac{n}{N} \notin \mathbb{Z} \\
\therefore L_1 &= 0 \quad \text{when } \frac{n}{N} \in \mathbb{Z}
\end{aligned}
$$

Applying the same sequence to $f = 1 - x$ we get the following result.

$$
\begin{aligned}
L_2 &= \lim_{N\to\infty} 1 - \frac{1}{N} \\
&= 1 - 0 \\
\therefore L_2 &= 1
\end{aligned}
$$

Consider now $a_n \in (0, 1)$ such that $|a_n - 0| < \delta$, for $\delta > 0$. As δ approaches 0, a_n also approaches 0, and is not an integer, as it can never reach 0 as a consequence of the strict inequalities establishing the domain of a_n. As a result, $Sf_n(a_n) \to 2$, and thus $|Sf_n(a_n) - f(a_n)| = |2 - 1| > 1/2$. This confirms that such an a_n exists to satisfy the constraints.

Consider now the Fourier Bessel series for f. Due to the selection of $\nu = 0$, the Fourier Bessel series uniformly converges to f for all x and for all $n \geq N$. By the uniform convergence theorem,

$$|Sf(x) - f(x)| < \epsilon \qquad 0 < x < 1$$

select $\epsilon > 0$, then for some positive integer M, for all $n > M$ we get,

$$|Sf(x) - f(x)| < \frac{1}{2} \qquad 0 < x < 1$$

which suggests there is not an a_n that exists such that $|Sf(a_n) - f(a_n)| > 1/2$, whenever $n > M$, but not for all n.

3. You are working in collaboration with glaciologists who are storing ice cores. The cores are long and thin and perfectly insulated save a small amount of heating at a rate α at one end. The glaciologist hope to balance this warming with cooling at a rate β at the other end. You have determined that the ice core obeys the following boundary value problem

$$u_t - u_{xx} = 0 \ldots (*) \, , \, u_x(0) = \beta \, , \, u_x(l) = \alpha$$

where u is temperature, t is time and l is the length of the core.

For this question, the method of separation of variables will need to be used, as such, the solution $u(x,t) = X(x)T(t)$. Therefore, the above conditions can be rewritten as

$$\begin{aligned} u_t &= u_{xx} \\ \therefore XT' &= TX'' \\ \therefore \frac{T'}{T} &= \frac{X''}{X} \ldots (**) \\ u_x(0) = \beta &\iff TX'(0) = \beta \\ u_x(l) = \alpha &\iff TX'(l) = \alpha \end{aligned}$$

(a) What should β be such that the ice core's temperature remains stable $(u_t = 0)$?

If $u_t = 0$, then $XT' = 0$, and thus $T = C$, for C constant.
Further, $u_t - u_{xx} = 0$ becomes $u_{xx} = 0$.
Therefore, $TX'' = 0$, and thus, $X = Ax + B$, for A, B constants.
Therefore, $u = X(x)T(t) = (Ax + B)C \ldots (1)$.
Applying the boundary conditions to (1), from $u_x(0) = TX'(0) = CX'(0) = \beta$, we get $\beta = A$.
From $u_x(l) = TX'(l) = CX'(l) = \alpha$, we get $\alpha = A$.
As a result, $\beta = \alpha$ in order to keep the temperature of the ice core stable, that is, the rate of heating should equal the rate of cooling.

(b) Assuming $\alpha = 1$, $l = 10$ and the average temperature of the core is -15, what is the solution for u in the stable case?

To find the solution for $u = (Ax + B)C$ for the average temperature of the core as -15, firstly we must apply any given conditions. As $\alpha = A = 1$, $u = (x + B)C$. Furthermore, with $u_x(0) = 1$, we get $C = 1$. Thus, $u = x + B$. These results are also a consequence of the stable temperature condition. Now, using the simple average value integral formula,
$u_{\text{avg}}(x,t) = \dfrac{1}{b-a} \displaystyle\int_a^b u(x,t)dx$ we get,

$$\begin{aligned} f_{\text{avg}}(x) &= \frac{1}{b-a} \int_a^b f(x)dx \\ &= \frac{1}{10-0} \int_0^{10} (x + B)dx \\ \therefore -15 &= \frac{1}{10} \int_0^{10} (x + B)dx \\ -150 &= \left[\frac{x^2}{2} + Bx \right] \Bigg|_0^{10} \\ &= \left[\frac{100}{2} + 10B - 0 \right] \\ \therefore 10B &= -200 \\ \therefore B &= -20 \end{aligned}$$

In the stable case, $u = x - 20$.

(c) If the cooling mechanism were to fail ($\beta = 0$) how long would it take before the ice core started to melt (i.e. when would u rise above 0 at any point)?

Firstly, to arrive at the solution, we need to solve equation $(*)$. Using the separation of variables method and equation $(**)$, we start with $\frac{T'}{T} = \frac{X''}{X}$, which becomes $\frac{T'}{T} = -\lambda = \frac{X''}{X}$, and can be broken into $\frac{T'}{T} = -\lambda \ldots (1*)$ and $\frac{X''}{X} = -\lambda \ldots (2*)$, for $\lambda \geq 0$.

In the following equations, solutions will be of the form Ce^{kx} or Ce^{kt}. This provides the justification for the use of the characteristic polynomial. Considering first equation $(1*)$,

$$\begin{aligned}
\frac{T'}{T} &= -\lambda \\
T' + \lambda T &= 0 \\
k + \lambda &= 0 \quad \text{considering the characteristic polynomial} \\
k &= -\lambda \\
\therefore T(t) &= Ae^{-\lambda t}
\end{aligned}$$

Considering now equation $(2*)$. Set $\lambda = w^2$. Further, from the boundary conditions, $u_x(0) = 0$, so $X'(0) = 0$. Also, $u_x(10) = \alpha = 1$, so $X'(10) = 1$, as $T(t) = C = 1$. Consider a solution of the form $u = U + V$. U solves the inhomogeneous boundary problem, $u_t - u_{xx} = 0$, $u_x(0) = 0$, and $u_x(l) = \alpha$. V solves the homogeneous boundary value problem, $u_t - u_{xx} = 0$, $u_x(0) = 0$, and $u_x(l) = 0$.

In order to solve this equation, 3 cases must be considered; $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Considering the first case of the homogeneous problem, $\lambda > 0$,

$$\begin{aligned}
\frac{X''}{X} &= -w^2 \\
X'' + w^2 X &= 0 \\
k^2 + w^2 &= 0 \quad \text{considering the characteristic polynomial} \\
\therefore k &= \pm iw \\
X(x) &= C_1 e^{iwx} + C_2 e^{-iwx} \\
\therefore X(x) &= D_1 \cos(wx) + D_2 \sin(wx) \quad \text{using the complex trigonometric identities} \\
X'(x) &= -wD_1 \sin(wx) + wD_2 \cos(wx) \\
\therefore 0 &= wD_2 \implies D_2 = 0 \quad \text{from } X'(0) = 0 \\
\therefore X(x) &= D_1 \cos(wx) \\
\therefore 0 &= -wD_1 \sin(10w) \quad \text{from } X'(10) = 0 \\
\therefore 10w &= n\pi \quad n \in \mathbb{Z} \\
\therefore w &= \frac{n\pi}{10} \quad n \in \mathbb{Z} \\
\therefore X(x) &= D_1 \cos\left(\frac{n\pi x}{10} x\right)
\end{aligned}$$

Considering the case $\lambda = 0$,

$$\begin{aligned}
\frac{X''}{X} &= 0 \\
X'' &= 0 \\
\therefore X(x) &= Ax + B \\
\therefore 0 &= A \quad \text{from } X'(0) = 0 \\
\therefore X(x) &= B
\end{aligned}$$

Considering the case $\lambda < 0$,

$$\begin{aligned}
\frac{X''}{X} &= w^2 \\
X'' - w^2 X &= 0 \\
k^2 - w^2 &= 0 \quad \text{considering the characteristic polynomial} \\
\therefore k &= \pm w \\
X(x) &= C_1 e^{wx} + C_2 e^{-wx} \\
\therefore X(x) &= D_1 \cosh(wx) + D_2 \sinh(wx) \quad \text{using the complex hyperbolic trigonometric identities} \\
X'(x) &= wD_1 \sinh(wx) + wD_2 \cosh(wx) \\
\therefore 0 &= wD_2 \implies D_2 = 0 \quad \text{from } X'(0) = 0 \\
\therefore X(x) &= D_1 \cosh(wx) \\
\therefore 0 &= wD_1 \sinh(10w) \quad \text{from } X'(10) = 0 \\
\therefore D_1 &= 0 \quad \text{as } w > 0 \implies sinh(10w) > 0 \\
\therefore X(x) &= 0
\end{aligned}$$

By definition, the homogeneous solution $V = \sum_{n=1}^{\infty} V_n$, where $V_n(x,t) = X_n(x)T_n(t)$.

Using the results for the three cases on λ, $V_n(x,t) = \left[0 + B + D_n \cos\left(\frac{n\pi x}{10} x\right)\right] e^{-\left(\frac{n\pi}{10}\right)^2 t}$.

Thus, $V(x,t) = \frac{D_0}{2} + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{10}\right) e^{-\left(\frac{n\pi}{10}\right)^2 t}$

From lectures, U is given by $U = \frac{\alpha}{l}\left(t + \frac{x^2}{2}\right) = \frac{1}{10}\left(t + \frac{x^2}{2}\right)$.

From this, $u(x,t) = U + V = \frac{1}{10}\left(t + \frac{x^2}{2}\right) + \frac{D_0}{2} + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{10}\right) e^{-\left(\frac{n\pi}{10}\right)^2 t}$.

Applying the initial condition $u(x,0) = x - 20$, we get the result,

$$x - \frac{x^2}{20} - 20 = \frac{D_0}{2} + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{10}\right)$$

Treating this as a Fourier Cosine series for the function $f(x) = x - \frac{x^2}{20} - 20$, we solve for the coefficients using the appropriate formulas.

$$\begin{aligned}
D_0 &= \frac{2}{10}\int_0^{10}\left(x - \frac{x^2}{20} - 20\right)dx \\
&= \frac{1}{5}\left(\frac{x^2}{2} - \frac{x^3}{60} - 20x\right)\Bigg|_0^{10} \\
&= -\frac{100}{3} \\
D_n &= \frac{2}{10}\int_0^{10}\left(x - \frac{x^2}{20} - 20\right)\cos\left(\frac{n\pi x}{10}\right)dx \\
&= \frac{1}{5}\left[\left(x - \frac{x^2}{20} - 20\right)\frac{10}{n\pi}\sin\left(\frac{n\pi x}{10}\right)\Bigg|_0^{10} - \frac{10}{n\pi}\int_0^{10}\left(1 - \frac{x}{10}\right)\sin\left(\frac{n\pi x}{10}\right)dx\right] \\
&= -\frac{1}{5}\left[\frac{10}{n\pi}\int_0^{10}\left(1 - \frac{x}{10}\right)\sin\left(\frac{n\pi x}{10}\right)dx\right] \\
&= -\frac{2}{n\pi}\left[\frac{10}{n\pi}\left(\frac{x}{10} - 1\right)\cos\left(\frac{n\pi x}{10}\right)\Bigg|_0^{10} + \frac{1}{n\pi}\int_0^{10}\cos\left(\frac{n\pi x}{10}\right)dx\right] \\
&= -\frac{20}{n^2\pi^2}
\end{aligned}$$

Therefore $u(x,t) = \frac{1}{10}\left(t + \frac{x^2}{2}\right) - \frac{100}{6} - \sum_{n=1}^{\infty}\frac{20}{n^2\pi^2}\cos\left(\frac{n\pi x}{10}\right)e^{-\left(\frac{n\pi}{10}\right)^2 t}$.

Now in order to solve for when the temperature is first greater than 0, we look at the warmest part of the rod, that is $x = 10$, where the heating is occuring, and solve for t such that $u(10,t) > 0$. Using desmos to graph the function,

$$u(10,t) = \frac{1}{10}(t + 50) - \frac{100}{6} - \sum_{n=1}^{\infty}\frac{20}{n^2\pi^2}(-1)^n e^{-\left(\frac{n\pi}{10}\right)^2 t}$$

we get $t = \frac{350}{3}$, as the time for when the temperature of one part of the rod rises above 0, that is, the rod begins to melt.