The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Exercises for Week 8** 2017

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) A *matrix* is an array of numbers, called *entries*. The plural of matrix is *matrices*. If a matrix M has m rows and n columns then we say that M is $m \times n$. We call a matrix M *square* if M is $n \times n$ for some n. The entry lying in the ith row and jth column is called the (i, j)-*entry*.

(ii) A matrix consisting of one row is called a *row vector*. A matrix consisting of one column is called a *column vector*.

(iii) To *add* or *subtract* using two matrices of the same size, simply add or subtract the corresponding entries. To form the *negative* of a matrix, simply take the negatives of its entries. To *multiply a matrix by a scalar*, simply multiply its entries by the scalar.

(iv) The *zero matrix* has all of its entries equal to 0, and is denoted by 0 or $0_{m\times n}$ if the size needs to be emphasised.

(v) The *identity matrix* is a square matrix with *diagonal* entries equal to 1 and all entries off the diagonal equal to 0. The identity matrix is denoted by I or I_n if it is $n \times n$ and the size needs to be emphasised.

(vi) If A is an $m \times n$ matrix and B is an $n \times p$ matrix then the *matrix product* AB is defined and is an $m \times p$ matrix. The (i, k)-entry of AB is the "dot product" of the ith row of A with the kth column of B, which can be expressed using sigma notation:

$$\sum_{j=1}^{n} a_{ij}b_{jk} \;=\; a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}$$

where a_{ij}, b_{jk} denote typical (i, j) and (j, k)-entries of A and B respectively.

(vii) If A, B, C are matrices of appropriate sizes for which the expressions make sense, and λ and μ are scalars, then the following properties hold:

$$A + B = B + A\ , \qquad (A + B) + C = A + (B + C)\ , \qquad A + 0 = 0 + A = A\ ,$$

$$-(-A) = A\ , \qquad A + (-A) = A - A = 0\ , \qquad \lambda(\mu A) = (\lambda\mu)A\ ,$$

$$\lambda(A + B) = \lambda A + \lambda B\ , \qquad (\lambda + \mu)A = \lambda A + \mu A\ , \qquad IA = AI = A\ ,$$

$$(AB)C = A(BC)\ , \qquad A(B + C) = AB + AC\ , \qquad (A + B)C = AC + BC\ ,$$

$$\lambda(BC) = (\lambda B)C = B(\lambda C)\ , \qquad 0A = 0 = A0\ .$$

(viii) Warning: Matrix multiplication is not in general commutative. Most of the time

$$AB \neq BA\,.$$

(ix) Transpose: The *transpose* of a matrix A is the matrix A^T obtained by interchanging rows and columns, that is, the (i,j)-entry of A becomes the (j,i)-entry of A^T. The following hold for matrices A and B of dimensions for which the expressions make sense:

$$(A^T)^T = A\,, \qquad (A+B)^T = A^T + B^T\,, \qquad (AB)^T = B^T A^T$$

Preparatory Exercises:

1. Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 & 0 \\ -1 & -2 & 6 \end{bmatrix}$, $C = \begin{bmatrix} -3 & 5 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 4 \\ 2 & 0 \\ 0 & -7 \\ 1 & -3 \end{bmatrix}$.

Write down the sizes of A, B, C and D.

2. For the matrices of the previous exercise, write out A^T, B^T, C^T, D^T and their sizes.

3. For the matrix

$$M = \begin{bmatrix} 6 & 0 & 3 & -5 \\ 0 & 7 & 2 & 4 \\ 1 & 3 & -2 & 0 \end{bmatrix},$$

locate the

(i) $(2,2)$-entry (ii) $(3,3)$-entry (iii) $(1,4)$-entry (iv) $(3,2)$-entry (v) $(3,4)$-entry.

4. Write out M^T for the matrix M of the previous exercise. Locate within M^T the

(i) $(1,3)$-entry (ii) $(3,1)$-entry (iii) $(4,2)$-entry (iv) $(3,2)$-entry (v) $(4,3)$-entry.

5. Consider the following 2×2 matrices:

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 4 & -3 \end{bmatrix}, \quad D = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

Find

(i) $A+B$ (ii) $A-B$ (iii) $B-C$ (iv) $D+C$ (v) $2A$ (vi) $-B$ (vii) $\frac{1}{5}D$
(viii) AB (ix) BA (x) CD (xi) BC (xii) $A(BC)$ (xiii) $(AB)C$
(xiv) $ABCD$ (xv) A^2 (xvi) B^2 (xvii) $A^2 - B^2$ (xviii) $(A+B)(A-B)$

6. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}.$$

(i) Find AB, BA, CD, DC and $BA + DC$.

(ii) Explain briefly why A^2, B^2, C^2, D^2 and $AB + CD$ do not exist.

Exercises:

16. This exercise is an illustration of the correspondence between multiplication of matrices and composition of compatible systems of equations. Suppose

$$\begin{array}{rcrcl} 2x & + & 3y & = & u \\ x & - & 4y & = & v\,, \end{array}$$

and further that

$$\begin{array}{rcrcl} 3u & - & 5v & = & c \\ 2u & + & 3v & = & d\,. \end{array}$$

Express c and d in terms of x and y by

(i) direct substitution, (ii) matrix multiplication.

17. Solve each of the following for x, y and z:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \\ 0 & 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 29 \\ -10 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 3 \\ 4 & 9 & -4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

18. A square matrix is called *diagonal* if all entries away from the main diagonal are zero. Find a simple rule for multiplying diagonal matrices. More generally, describe in words as simply as you can what happens if you multiply any square matrix (of the same size) (i) on the left by a diagonal matrix, or (ii) on the right by a diagonal matrix.

19. Find all x, y, z and w such that the following matrix equation holds:

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

20.* Find XY in each case, given that X is a row matrix and Y is a column matrix, both with the same number of entries:

(i) $YX = \begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix}$ (ii) $YX = \begin{bmatrix} 3 & -3 & 6 \\ 4 & -4 & 8 \\ -2 & 2 & -4 \end{bmatrix}$

21.* Find necessary and sufficient conditions on a, b, c and d such that the matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with the matrix A in each of the following cases:

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & 7 \\ 7 & 0 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

22.* Use sigma notation to verify that if A and B are matrices such that AB is defined then $(AB)^T = B^T A^T$.

23.* Let k be a constant and n a positive integer which varies. Find formulae for the following. Verify them using mathematical induction.

(i) $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^n$ (ii) $\begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix}^n$ (iii) $\begin{bmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix}^n$

24.* Verify that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are fixed but different column vectors with the same number of entries, and λ is a scalar that varies, then there are infinitely many different column vectors of the form

$$\mathbf{s} = \mathbf{x}_1 + \lambda(\mathbf{x}_1 - \mathbf{x}_2).$$

Interpret your answer geometrically.

25.** Prove that there are no square matrices A and B of the same size such that $AB - BA = I$.

Short Answers to Selected Exercises:

1. 2×2, 2×3, 1×3, 4×2

2. $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, 2×2, $\begin{bmatrix} 3 & -1 \\ 4 & -2 \\ 0 & 6 \end{bmatrix}$, 3×2, $\begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$, 3×1, $\begin{bmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & -7 & -3 \end{bmatrix}$, 2×4

3. (i) 7 (ii) -2 (iii) -5 (iv) 3 (v) 0

4. $\begin{bmatrix} 6 & 0 & 1 \\ 0 & 7 & 3 \\ 3 & 2 & -2 \\ -5 & 4 & 0 \end{bmatrix}$ (i) 1 (ii) 3 (iii) 4 (iv) 2 (v) 0

5. (i) $\begin{bmatrix} 4 & 2 \\ -2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -6 \\ 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 10 & 2 \\ 4 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & -4 \\ -2 & 6 \end{bmatrix}$

(vi) $\begin{bmatrix} -3 & -4 \\ 1 & 2 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (viii) $\begin{bmatrix} 5 & 8 \\ -6 & -10 \end{bmatrix}$ (ix) $\begin{bmatrix} -1 & 6 \\ 1 & -4 \end{bmatrix}$ (x) $\begin{bmatrix} 0 & 10 \\ 40 & -15 \end{bmatrix}$

(xi) $\begin{bmatrix} 16 & -6 \\ -8 & 4 \end{bmatrix}$ (xii) (xiii) $\begin{bmatrix} 32 & -14 \\ -40 & 18 \end{bmatrix}$ (xiv) $\begin{bmatrix} 320 & -70 \\ -400 & 90 \end{bmatrix}$ (xv) $\begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$

(xvi) $\begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix}$ (xvii) $\begin{bmatrix} -2 & -12 \\ -3 & 11 \end{bmatrix}$ (xviii) $\begin{bmatrix} -8 & -14 \\ 4 & 17 \end{bmatrix}$

6. (i) $\begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & 2 \\ 5 & 2 & -1 \end{bmatrix}$, $[\,3\,] = 3$, $\begin{bmatrix} -3 & -4 & -2 \\ -3 & -4 & -2 \\ 15 & 20 & 10 \end{bmatrix}$, $\begin{bmatrix} 0 & -2 & -1 \\ -5 & -4 & 0 \\ 20 & 22 & 9 \end{bmatrix}$

(ii) None of A, B, C, D are square. The matrices AB and CD have different sizes.

7. (i) I_2, I_2, I_3, I_3 (ii) I_2, I_3

8. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 12 \\ 30 \end{bmatrix}$, $\begin{bmatrix} -5 & 2 \end{bmatrix}$, $\begin{bmatrix} 6 & 6 \end{bmatrix}$, 7, -7, 0, 0, 0, 0, 7, 42

9. $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

10. $AA^2 = A(AA) = (AA)A = A^2A$

11. $A0_{n\times n} = 0_{m\times m}A = 0_{m\times n}$ and $0_{n\times n} \neq 0_{m\times m} \neq 0_{m\times n}$

12. $\begin{bmatrix} rs\cos(\alpha+\beta) & -rs\sin(\alpha+\beta) \\ rs\sin(\alpha+\beta) & rs\cos(\alpha+\beta) \end{bmatrix}$

13. The following are always true: (i), (iv), (v), (vi), (ix), (x), (xi)

15. (iii) $\begin{bmatrix} 11 & -5 \\ 20 & -9 \end{bmatrix}$, $\begin{bmatrix} 21 & -10 \\ 40 & -19 \end{bmatrix}$, $\begin{bmatrix} 201 & -100 \\ 400 & -199 \end{bmatrix}$

17. (i) $x = 5$, $y = -1$, $z = 4$ (ii) $x = -t/2$, $y = 2t/3$, $z = t$

18. Multiply corresponding diagonal elements. (i) Multiply each row by the corresponding diagonal element. (ii) Multiply each column by the corresponding diagonal element.

19. $x = s$, $y = -4t$, $z = s$, $w = t$

20. (i) 1 (ii) -5

21. (i) $b = c = 0$ (ii) $a = d$ and $b = c$ (iii) $a = d$ and $c = 0$

23. (i) $\begin{bmatrix} 1 & nk \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} k^n & nk^{n-1} \\ 0 & k^n \end{bmatrix}$ (iii) $\begin{bmatrix} k^n & nk^{n-1} & \frac{n(n-1)}{2}k^{n-2} \\ 0 & k^n & nk^{n-1} \\ 0 & 0 & k^n \end{bmatrix}$