The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Exercises for Week 6** 2017

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

**Important Ideas and Useful Facts:**

(i) A line in space is determined by two points, or by one point and a direction.

(ii) A plane in space is determined either by three non-collinear points, or by one point and a perpendicular (normal) direction.

(iii) If the vector $\mathbf{v}$ points in the direction of a line $\mathcal{L}$ containing the point $P_0$, then the *parametric vector equation* of $\mathcal{L}$ is

$$\mathbf{r} - \mathbf{r}_0 \;=\; t\mathbf{v} \qquad \text{or equivalently} \qquad \mathbf{r} \;=\; \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}$ is the position vector of a typical point on $\mathcal{L}$, $\mathbf{r}_0$ is the position vector of $P_0$ and $t$ is a parameter which varies over all real numbers.

(iv) If the vector $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ points in the direction of a line $\mathcal{L}$ containing the point $P_0(x_0, y_0, z_0)$, then the *parametric scalar equations* of $\mathcal{L}$ are

$$\left.\begin{array}{rcl} x &=& x_0 + ta \\ y &=& y_0 + tb \\ z &=& z_0 + tc \end{array}\right\} \; t \in \mathbb{R}$$

and the *Cartesian equations* are (in the case that $a$, $b$, $c$ are all nonzero):

$$\frac{x - x_0}{a} \;=\; \frac{y - y_0}{b} \;=\; \frac{z - z_0}{c}\;.$$

(v) The shortest distance $d$ from a point $P$ to a line containing the point $Q$ and pointing in the direction of $\mathbf{v}$ is

$$d \;=\; \frac{|\mathbf{v} \times \overrightarrow{PQ}|}{|\mathbf{v}|}\;.$$

(vi) If the vector $\mathbf{n}$ is normal to a plane $\mathcal{P}$ containing the point $P_0$, then the *vector equation* of $\mathcal{P}$ is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} \;=\; 0 \qquad \text{or equivalently} \qquad \mathbf{r} \cdot \mathbf{n} \;=\; \mathbf{r}_0 \cdot \mathbf{n}$$

where $\mathbf{r}$ is the position vector of a typical point and $\mathbf{r}_0$ is the position vector of $P_0$.

(vii) If the vector $\mathbf{n} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is normal to the plane $\mathcal{P}$ containing the point $P_0(x_0, y_0, z_0)$, then the *Cartesian equation* of $\mathcal{P}$ is

$$ax + by + cz \;=\; d$$

where $d = ax_0 + by_0 + cz_0\,.$

(viii) If $P_1$, $P_2$, $P_3$ are non-collinear points on a plane, then a normal vector to the plane is

$$\mathbf{n} \;=\; \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\ .$$

(ix) The shortest distance $d$ from a point $P$ to a plane containing the point $Q$ and with normal vector $\mathbf{n}$ is

$$d \;=\; \frac{|\mathbf{n}\cdot\overrightarrow{PQ}|}{|\mathbf{n}|}\ .$$

**Preparatory Exercises:**

**1.** Find two distinct points on the line $\mathcal{L}_1$ and a vector in the direction of $\mathcal{L}_1$ described by the Cartesian equations

$$\frac{x-2}{3} \;=\; \frac{y+1}{2} \;=\; z-4\ .$$

**2.** Find two distinct points on the line $\mathcal{L}_2$ and a vector in the direction of $\mathcal{L}_2$ described by the parametric equations

$$\left.\begin{array}{rcl} x &=& 8-3t \\ y &=& 3-2t \\ z &=& 6-t \end{array}\right\} \quad t \in \mathbb{R}\ .$$

Is $\mathcal{L}_2$ the same line as $\mathcal{L}_1$ from the previous exercise?

**3.** Find parametric vector, parametric scalar and Cartesian equations of the line passing through the point $(2,3,5)$ in the direction of $\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

**4.** Find a point on the plane $\mathcal{P}$ and a vector normal to $\mathcal{P}$ described by the Cartesian equation

$$4x-3y+6z \;=\; -7\ .$$

**5.** Find vector and Cartesian equations of the plane containing the point $(2,3,5)$ with normal vector $\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

**6.** Let $P=(1,2,3)$, $Q=(-1,-2,-3)$ and $R=(4,-4,4)$.

(i) Express $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in Cartesian form.

(ii) Find the cross product $\overrightarrow{PQ}\times\overrightarrow{PR}$.

(iii) Find a Cartesian equation of the plane containing $P$, $Q$, $R$.

**7.** Find a Cartesian equation for the plane containing

$$P=(6,7,-2)\ , \qquad Q=(0,-8,11)\ , \qquad R=(14,-3,9)\ .$$

**Exercises:**

**16.** Find Cartesian equations of the line passing through $(1, 0, -2)$ and perpendicular to the plane $3x - 4y + z = 6$.

**17.** Verify that the line

$$\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$$

is parallel to the plane $4x + 4y - 5z = 14$.

**18.** Find the Cartesian equation of the plane containing $(1, 1, 1)$ and the line

$$\frac{x-4}{-2} = y+3 = \frac{z-1}{3}\,.$$

**19.** Verify that the line $\mathcal{L}_1$ described by

$$\frac{x-1}{4} = \frac{y-2}{3} = \frac{z-10}{5}$$

is identical to the line $\mathcal{L}_2$ described by

$$\left.\begin{array}{rcl} x &=& -7-4t \\ y &=& -4-3t \\ z &=& -5t \end{array}\right\} \qquad t \in \mathbb{R}\,.$$

**20.*** Find the distance from $P(2, 1, 1)$ to the line $\mathcal{L}$ given by the equations

$$x-1 = \frac{y-1}{3} = \frac{z+4}{-1}\,.$$

Find the closest point to $P$ lying on $\mathcal{L}$.

**21.*** Describe a general technique for finding the shortest distance between two parallel lines in space. Use your method to find the distance between the lines given by the following sets of Cartesian equations:

$$x-1 = \frac{y-2}{4} = \frac{z-3}{-3} \qquad \text{and} \qquad x+1 = \frac{y-3}{4} = \frac{z+1}{-3}\,.$$

**22.**[**] Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lines in space, containing points $P_1$ and $P_2$ respectively, and pointing in non-parallel directions $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Explain why the shortest distance $d$ between $\mathcal{L}_1$ and $\mathcal{L}_2$ is given by the formula

$$d \;=\; \frac{\left|\overrightarrow{P_1P_2}\cdot(\mathbf{v}_1\times\mathbf{v}_2)\right|}{\left|\mathbf{v}_1\times\mathbf{v}_2\right|}\;.$$

Give a general method for finding the closest points $Q_1$ and $Q_2$ on $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Find $d$, $Q_1$ and $Q_2$ for the following lines:

$$\mathcal{L}_1:\;\; x = 2\;,\;\; y \;=\; 1-z\;; \qquad \mathcal{L}_2:\;\; x+1 \;=\; \frac{y-3}{4} \;=\; \frac{z+1}{-3}\;.$$

**23.**[*] A smooth curve $\mathcal{C}$ in space is defined parametrically by the equations

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \\ z = h(t) \end{array}\right\} \qquad t\in\mathbb{R}\,,$$

where $f$, $g$ and $h$ are differentiable functions. Let $\mathbf{r}=\mathbf{r}(t)$ denote the position vector of a point on $\mathcal{C}$ at time $t$. Define the *derivative* of $\mathbf{r}$ to be

$$\mathbf{r}' \;=\; \mathbf{r}'(t) \;=\; \lim_{\delta\to 0}\frac{\mathbf{r}(t+\delta)-\mathbf{r}(t)}{\delta}\;.$$

Give a physical interpretation of $\mathbf{r}'$. Prove that $\mathbf{r}'$ is given parametrically by

$$\left.\begin{array}{l} x = f'(t) \\ y = g'(t) \\ z = h'(t) \end{array}\right\} \qquad t\in\mathbb{R}\,.$$

**24.**[**] Let $z = f(x,y)$ be a real-valued function of two real variables such that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous. It is a theorem of calculus that the tangent plane is a good approximation to the surface $z=f(x,y)$ at a given point $(x_0,y_0,z_0)$. Explain why the vectors

$$\mathbf{i}+\frac{\partial f}{\partial x}(x_0,y_0)\,\mathbf{k} \qquad \text{and} \qquad \mathbf{j}+\frac{\partial f}{\partial y}(x_0,y_0)\,\mathbf{k}$$

are parallel to the tangent plane. Deduce the following equation for the tangent plane to the surface at $(x_0,y_0,z_0)$:

$$z-z_0 \;=\; \frac{\partial f}{\partial x}(x_0,y_0)\,(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)\,(y-y_0)\;.$$

**25.**[**] You are the painter Magritte pondering the horizon, standing on the $xy$-plane, with your eyeball positioned at the point $(10, 0, \sqrt{101})$. You are looking at your canvas, which is the $yz$-plane. You wish to faithfully depict two infinitely thin roads on your canvas, one to the left and one to the right of your vision, represented by the lines

$$\ell_{\mathsf{left}}: \quad x = t\,, \;\; y = -10\,, \;\; z = 0\,, \qquad (t \le 0)$$

and

$$\ell_{\mathsf{right}}: \quad x = t\,, \;\; y = 10\,, \;\; z = 0\,, \qquad (t \le 0)$$

as $t$ heads off towards negative infinity.

For a fixed value of $t$, let $\mathcal{L}_{\mathsf{left}}(t)$ be the line joining your eyeball to the point $(t, -10, 0)$ on $\ell_{\mathsf{left}}$ and $\mathcal{L}_{\mathsf{right}}(t)$ be the line joining your eyeball to $(t, 10, 0)$ on $\ell_{\mathsf{right}}$. Find the point of intersection of $\mathcal{L}_{\mathsf{left}}(t)$ and $\mathcal{L}_{\mathsf{right}}(t)$ with the canvas.

Show that the intersection points form two lines on your canvas which are not parallel, yet do not intersect. What happens to the $z$-values as $t \to -\infty$?

**Selected Short Answers:**

**1.** $(2, -1, 4)\,,\;\; (5, 1, 5)\,,\;\; 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

**2.** $(8, 3, 6)\,,\;\; (5, 1, 5)\,,\;\; -3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ , yes.

**3.** $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} + 3\mathbf{j} - \mathbf{k})\,,\quad \left.\begin{array}{rcl} x &=& 2+t \\ y &=& 3+3t \\ z &=& 5-t \end{array}\right\}\; t \in \mathbb{R}\,,\quad x - 2 = \dfrac{y-3}{3} = \dfrac{z-5}{-1}$

**4.** $(-1, 1, 0)\,,\;\; 4\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

**5.** $\mathbf{r} \cdot \big(\mathbf{i} + 3\mathbf{j} - \mathbf{k}\big) = 6\,,\quad x + 3y - z = 6\,.$

**6.** (i) $-2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}\,,\;\; 3\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ (ii) $-40\mathbf{i} - 16\mathbf{j} + 24\mathbf{k}$ (iii) $5x + 2y - 3z = 0$

**7.** $-7x + 34y + 36z \;=\; 124$

**8.** (i) (b),(n). (ii) (d),(i). (iii) (f),(m). (iv) (a),(k). (v) (g),(l). (vi) (e),(h). (vii) (c),(j).

**9.** (i) $\mathcal{L}_1 : \mathbf{r} = \mathbf{i}+\mathbf{j}+\mathbf{k}+t(5\mathbf{i}-4\mathbf{j}-2\mathbf{k}),\; \left.\begin{array}{rcl} x &=& 1+5t \\ y &=& 1-4t \\ z &=& 1-2t \end{array}\right\} t \in \mathbb{R}\,,\; \dfrac{x-1}{5} = \dfrac{y-1}{-4} = \dfrac{z-1}{-2}$

$\mathcal{L}_2 : \mathbf{r} = 5\mathbf{i}-5\mathbf{j}-3\mathbf{k}+s(-3\mathbf{i}+8\mathbf{j}+6\mathbf{k}),\; \left.\begin{array}{rcl} x &=& 5-3s \\ y &=& -5+8s \\ z &=& -3+6s \end{array}\right\} s \in \mathbb{R}\,,\; \dfrac{x-5}{-3} = \dfrac{y+5}{8} = \dfrac{z+3}{6}$

(ii) $(7/2, -1, 0)$

**10.** $(1, 1, 0)\,,\;\; 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\,,\quad \left.\begin{array}{rcl} x &=& 1+2t \\ y &=& 1-t \\ z &=& -t \end{array}\right\}\; t \in \mathbb{R}\,,\quad \dfrac{x-1}{2} = \dfrac{y-1}{-1} = \dfrac{z}{-1}$

**11.** $-\sqrt{2}/3$

**12.** (i) $0 \le \lambda \le 1$ (ii) $\lambda > 1$ (iii) $\lambda < 0$ (iv) $\lambda = 1/3$ or $\lambda = -1$

**13.** $x_0 x + y_0 y + z_0 z = r^2$

**15.** $\sqrt{21}/3\,, \quad (5/3, -2/3, -2/3)$

**16.** $\dfrac{x-1}{3} = \dfrac{y}{-4} = z + 2$

**17.** $(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \cdot (4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) = 0$

**18.** $12x + 9y + 5z = 26$

**20.** $\sqrt{270}/\sqrt{11}\,, \quad (7/11, -1/11, -40/11)$

**21.** $\sqrt{2275}/13$

**22.** $4/\sqrt{3}\,, \quad (2, 25/3, -22/3)\,, \quad (2/3, 29/3, -6)$

**25.** $\left(0, \dfrac{\pm 100}{t-10}, \dfrac{t\sqrt{101}}{t-10}\right)$, as $t \to -\infty\,,\ z \to \sqrt{101}$