THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

MATH1902
LINEAR ALGEBRA (ADVANCED)

June 2010 LECTURER: D. Easdown

TIME ALLOWED: **One and a half hours**

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Approved non-programmable calculators may be used.

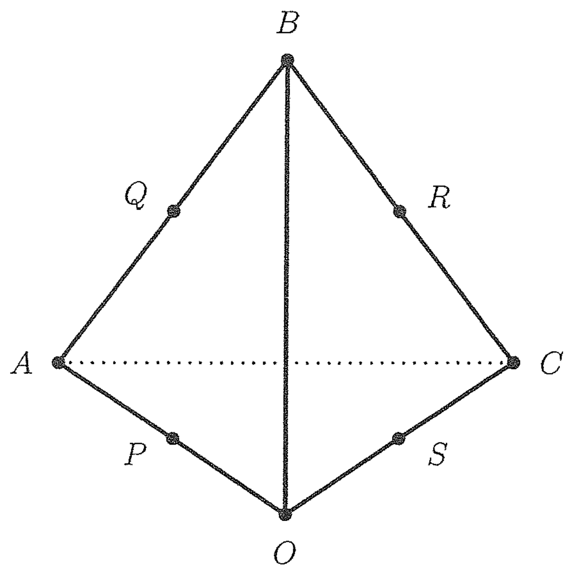
THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

MARKER'S USE ONLY	

Extended Answer Section

There are **four** *questions in this section, each with a number of parts. Write your answers in the answer book(s) provided. Ask for extra books if you need them.*

1. (a) Consider the tetrahedron below, with one vertex at the origin O and other vertices A, B and C. Let P, Q, R and S be the midpoints of OA, AB, BC and CO respectively.



Put $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$. Then

$$\overrightarrow{PR} = \frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}) .$$

(You are not being asked to prove this.)

(*i*) Express $\overrightarrow{QS}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(*ii*) Use vectors to prove that PR and QS intersect and bisect each other. [Hint: Let T be the midpoint of PR and compare $\overrightarrow{QT}$ with $\overrightarrow{QS}$.]

(*iii*) Deduce that the three line segments joining the midpoints of opposite (non-adjacent) edges of a tetrahedron intersect in a common point.

(b) Let $\mathbf{v}$ be any nonzero geometric vector. Suppose that $\mathbf{a}$, $\mathbf{c}$ are vectors parallel to $\mathbf{v}$, and $\mathbf{b}$, $\mathbf{d}$ are vectors perpendicular to $\mathbf{v}$, such that

$$\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d} .$$

Prove that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$. [Hint: Take the dot product with $\mathbf{v}$, expand and simplify.]

[3+4+1+7=15 marks]

2. (a) (*i*) Find A^{-1} where $A = \begin{bmatrix} 1 & 0 & 1 \\ -4 & 1 & -1 \\ 6 & -2 & 1 \end{bmatrix}$.

(*ii*) Use the result of the previous part to solve the following system of equations, where a, b, c are constants:

$$\begin{array}{rcrcrcl} x & & & + & z & = & a \\ -4x & + & y & - & z & = & b \\ 6x & - & 2y & + & z & = & c \end{array}$$

(b) Suppose m and n are positive integers and A and B are matrices such that $AB = I_m$ and $BA = I_n$. Prove that $m = n$ and A and B are both $n \times n$ matrices. [Hint: think about the effect of row reducing a matrix that has more rows than columns.]

[4+3+8=15 marks]

3. (a) State the Fundamental Theorem of Algebra and explain briefly why it implies the existence of eigenvalues for all square matrices with complex number entries.

(b) Criticise the following 'proof' of the Cayley-Hamilton Theorem:

> If $p(\lambda) = \det(M - \lambda I)$ is the characteristic polynomial of a square matrix M, then $p(M) = \det(M - MI) = \det(M - M) = \det(0) = 0$.

Are there any matrices M for which this reasoning is valid?

(c) Let M be a 2×2 matrix and n an integer ≥ 2. Prove that there exist integers a_n and b_n such that

$$M^n = a_n M + b_n I .$$

Find a_5 and b_5 when $M = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

[3+3+9=15 marks]

4. (a) Consider the matrix $M = \begin{bmatrix} -9 & -4 & -4 \\ 12 & 7 & 4 \\ 12 & 4 & 7 \end{bmatrix}$.

(*i*) Given that

$$\det(\lambda I - M) = (\lambda + 1)(\lambda - 3)^2$$

(you do not need to verify this), find the eigenvalues of M and the corresponding eigenspaces.

(*ii*) Write down an invertible matrix P and a diagonal matrix D such that

$$M = PDP^{-1}.$$

(You are not required to calculate P^{-1}.)

(b) (*i*) Let λ be an eigenvalue of $M = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}$. Prove that

$$|\lambda - r| \leq |s| + |t|, \quad \text{or } |\lambda - v| \leq |u| + |w|, \quad \text{or } |\lambda - z| \leq |x| + |y|.$$

Thus eigenvalues of M are never 'too far away' from diagonal entries.

(*ii*) Apply the the previous part to M^T to deduce that all of the eigenvalues λ of

$$M = \begin{bmatrix} 1 & \sqrt{2} & -1 \\ \sqrt{2} & 2 & -\sqrt{2} \\ -\sqrt{2} & -1 & \sqrt{3} \end{bmatrix}$$

have magnitude $< \dfrac{9}{2}$.

[5+2+5+3=15 marks]

End of Extended Answer Section