# ADV_Q01

## Semester 2, Y2011

**Question 1**

A mass spectrometer uses electric and magnetic fields to measure the masses of ions.

(a) First, positive ions are created, but with a distribution of speeds. Using only the Lorentz force law, explain how electric and magnetic fields can be used as a *velocity selector* to pick out specific speeds of the ions. Draw a diagram to support your explanation.

(b) Next, with all the ions possessing the same speed, a magnetic field is used to separate and measure different masses. The ions move on circular arcs, the radius of which depends on their masses. Again, using only the Lorentz force law, express the mass of an ion in terms of its velocity $v$, the radius of its motion $R$, its charge $q$, and the magnetic field $B$.

**(5 marks)**

**Solution**

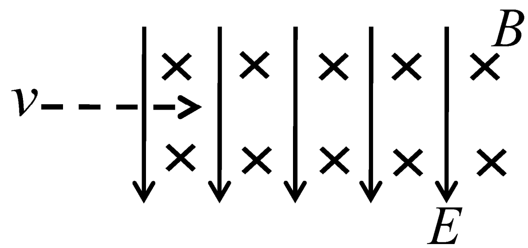
(a)

An electric field is created perpendicular to the direction of the ions, and a magnetic field perpendicular to both of these directions following the right-hand rule. A positive charge feels an electric force $qE$ downwards, and a magnetic force $qvB$ upwards.

**(1 mark)**

These forces will balance if $v = E/B$, regardless of the charge. Only ions with $v = E/B$ will pass through the velocity selector.

**(1 mark)**




**(1 mark)**

(b)

In a magnetic field $B$, a particle with charge $q$ and velocity $v$ experiences a force $F = qvB$ perpendicular to its direction of motion. This force is then the centripetal force $F = mv^2/R$.

**(1 mark)**

Equating these expressions and solving for m yields

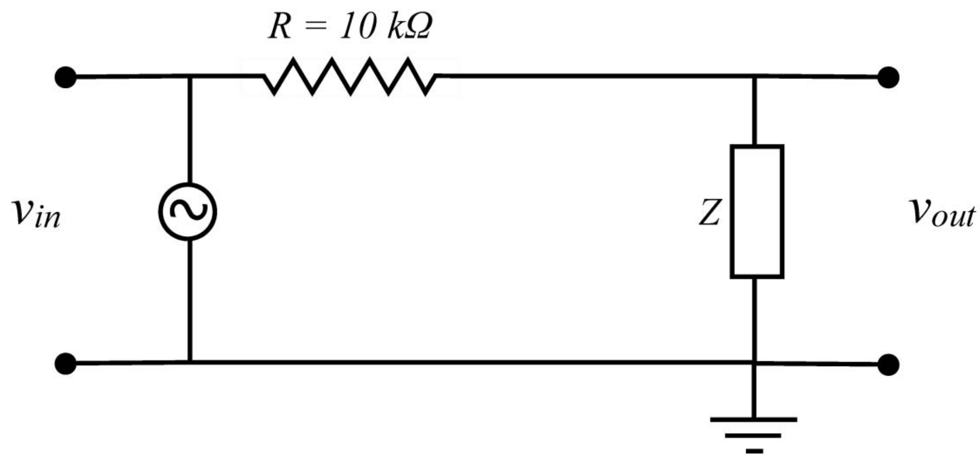
$$m = \frac{qRB}{v}.$$

**(1 mark)**

# ADV_Q02=TEC_Q02

# Semester 2, Y2011

**Question 2**




A simple voltage divider is shown in the circuit above, in which one component, $R$, is a $10\,\mathrm{k}\Omega$ resistor and the other, $Z$, is an unknown impedance (either an ideal capacitor or inductor).

(a) Derive an expression for $v_{out}/v_{in}$ in terms of $R$ and $Z$.

(b) A measurement at a frequency of $10\,\mathrm{kHz}$ gives $|v_{out}|/|v_{in}| = 0.5$. Explain how a further measurement, at say $1\,\mathrm{kHz}$, would allow you to determine whether $Z$ is a capacitor or inductor.

(c) Based on the measurement in part (b) show that at $10\,\mathrm{kHz}$ the reactance, $X$, of $Z$ is given by

$$X = \frac{R}{\sqrt{3}}.$$

**(5 marks)**

**Solution**

(a)

Apply Ohm's law to the circuit.

$$v_{in} = i\left(R+Z\right)$$

The voltage across impedance $Z$ is therefore

$$v_{out} = i\,Z = v_{in}\,\frac{Z}{R+Z} \Rightarrow$$

$$\frac{v_{out}}{v_{in}} = \frac{Z}{R+Z}$$

**(1 mark)**

(b)

We are told that $Z$ is either a capacitor or an inductor. If it is an capacitor then its impedance at $1\,\text{kHz}$ will be larger than at $10\,\text{kHz}$ and $|v_{out}|/|v_{in}|$ will have increased from its value at $1\,\text{kHz}$. If it is an inductor then its impedance at $1\,\text{kHz}$ will be smaller than at $10\,\text{kHz}$ and $|v_{out}|/|v_{in}|$ will have decreased from its value at $1\,\text{kHz}$. So a measurement of $|v_{out}|/|v_{in}|$ will determine which is correct.

**(2 marks)**

(c)

The circuit is a voltage divider and therefore

$$v_{out} = v_{in}\,\frac{Z}{R+Z} \Rightarrow |v_{out}| = |v_{in}|\left|\frac{Z}{R+Z}\right| \Rightarrow \frac{|v_{out}|}{|v_{in}|} = \left|\frac{Z}{R+Z}\right| = 0.5\,.$$

Consider the term involving $R$ and $Z$ and set $Z = j\,X$

$$\left|\frac{Z}{R+Z}\right| = \frac{|Z|}{|R+Z|} = \sqrt{\frac{X^2}{\left(R^2+X^2\right)}}$$

Equating this to a value of $0.5$ we get

$$\frac{1}{4} = \frac{X^2}{R^2+X^2} \Rightarrow R^2 + X^2 = 4X^2 \Rightarrow$$

$$X = \frac{R}{\sqrt{3}}.$$

**(2 marks)**

# ADV_Q03

**Question 3**

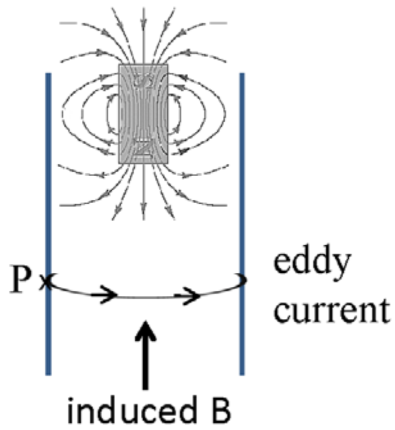
A magnet dropped through a hollow copper pipe is observed to fall very slowly. By considering what happens as the magnet moves past a fixed point P on the pipe, carefully explain this observation.

Assume the magnet falls without spinning, with its north pole downwards.

Your explanation should include a diagram and be given in terms of physical principles but without using equations. Your answer should be no more than one page long, excluding the diagram.

**(5 marks)**

**Solution**




As the magnet approaches point P the magnetic flux through the pipe at the level of P increases. Faraday's law says that an emf is generated inducing eddy currents which in turn create a magnetic field (by Ampere's law).

Lenz's law tells us that the induced current opposes the *change* in the flux that caused it, so that the induced magnetic field when the north pole approaches is *increasing upwards* and acts to repel the falling magnet. This means that the magnet slows down.

After the south pole of the magnet has gone past point P the magnetic flux at the level of P decreases, the eddy currents are reversed in direction. The magnetic field induced by this diminishing flux is thus downwards and acts to attract the magnet. This means that the magnet falls more slowly.

Extra Note: The induced eddy currents around the tube do take many different paths other than the circumferential one shown in the diagram. The main point is that there should at least be a vertical component of the induced magnetic field.

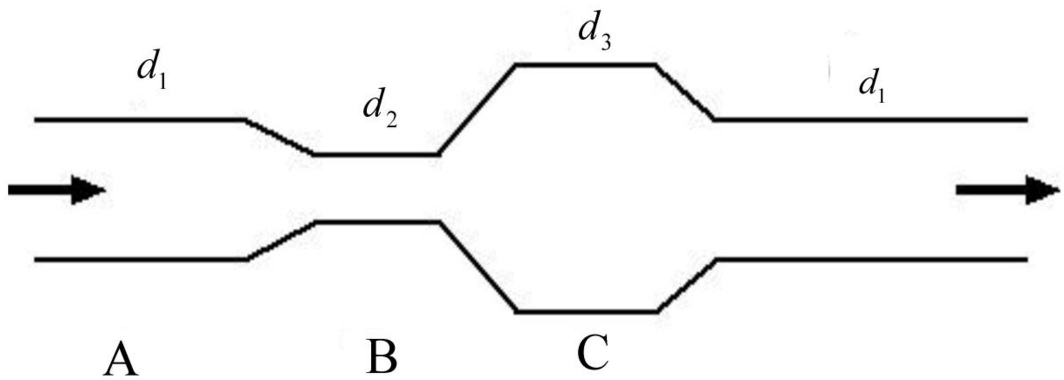
**(1 mark for sensible diagram)**
**(1 mark for emf generated by Faraday's law)**
**(1 mark for induced B field by Ampere's law)**
**(1 mark for Lenz's Law)**
**(1 mark for B to be upwards acting to repel magnet as it approaches)**

# ADV_Q04

## Semester 2, Y2011

**Question 4**

A non-viscous, incompressible fluid flows through a horizontal pipe of circular cross-section. The initial diameter of the pipe is $d_1$, the pipe then shrinks to a diameter $d_2 < d_1$, expands to a diameter $d_3 > d_1$, and then returns to its initial diameter $d_1$ before exiting to the atmosphere.




(a) A thief wants to drill a small hole so some fluid leaks out of the pipe before the exit. Will he be successful if he drills the hole at Point A, Point B, or Point C? Briefly explain your answer.

(b) Suppose the pipe contains alcohol, with density $800\,\text{kg.m}^{-3}$ and viscosity $\eta = 2.0\times10^{-3}\,\text{Pa.s}$, and that the exit pipe has a diameter $d_1 = 0.10\,\text{m}$. If the volume flow rate is $8.0\times10^{-4}\,\text{m}^3.\text{s}^{-1}$, will the flow through the last section of pipe be turbulent?

**(5 marks)**

**Solution**

(a)
By the equation of continuity,

$$A_A v_A = A_B v_B = A_C v_C$$

so since $A_B < A_A < A_C$, then

$$v_B > v_A > v_C$$

Since it is an ideal fluid, we can use Bernoulli's equation: $p + \frac{1}{2}\rho v^2 = \text{constant}$ so

$$p_C > p_A > p_B$$

and the pressure at the exit is equal to atmospheric pressure, so $p_A = p_{exit} = p_{atm}$

**(2 marks)**

Fluid will leak out only if the pressure inside the pipe at the position of the hole is greater than atmospheric pressure

**(1 mark)**

Hence the fluid will leak from a hole drilled at point C, but not holes at points A or B.

**(1 mark)**

(b)
The volume flow rate

$$Q = dV / dt = Av$$

So

$$\begin{aligned} v &= Q / A \\ &= \frac{\left(8.0\times10^{-4}\right)}{\pi\left(0.050\right)^2} = 0.10\,\text{m.s}^{-1}. \end{aligned}$$

Calculate the Reynolds number of the flow

$$\rho v L / \eta = \frac{(800)(0.10)(0.10)}{2.0\times10^{-3}} = 4000.$$

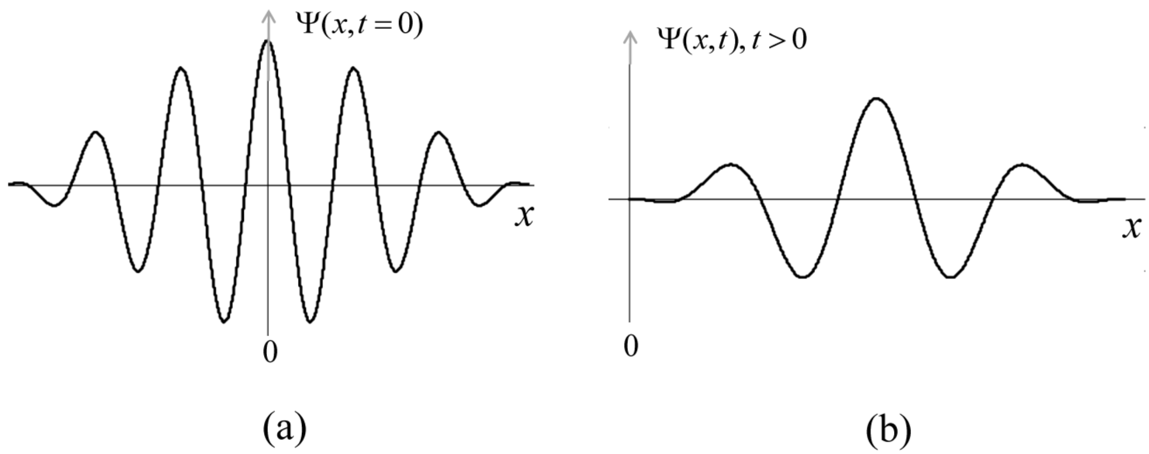
where the length scale $L$ is taken to be the pipe diameter. This is above 2000, so the flow will be turbulent.

**(1 mark)**

# ADV_Q05

# Semester 2, Y2011

**Question 5**




The wave function for an electron free to move in the $x$ direction at time $t = 0$ is shown in panel (a) of the figure above. The wave function for the electron at a later time $t$ is shown in panel (b) of the figure above.

(a) Does this electron have a uniquely defined position at time $t = 0$? Briefly explain.

(b) Does this electron have a uniquely defined momentum at time $t = 0$? Briefly explain.

(c) Does this particle have a uniquely defined energy at time $t = 0$? Briefly explain.

**(5 marks)**

**Solution**

(a)

The electron does not have a well defined position since $\left|\psi\left(x,0\right)\right|^2$ is non-zero over a finite range in $x,$ which defines an uncertainty $\Delta x$ in position.

**(½ mark for answer, 1 mark for reasonable justification)**

(b)

The electron does not have a well defined momentum since $\Delta x > 0,$ from the Heisenberg Uncertainty principle $\Delta p\, \Delta x \geq \hbar$ so there is a non-zero uncertainty in momentum, $\Delta p \geq \hbar / \Delta x$.

**(½ mark for answer, 1 mark for reasonable justification)**

(c)

The electron does not have a well-defined energy at time $t = 0$: diagram (b) shows that the wave function changes with time, so this is not a stationary state. In terms of the energy state the time Uncertainty Principle $\Delta E\, \Delta t \geq \hbar$ we have $\Delta t < \infty$ so $\Delta E > 0,$ i.e. there is uncertainty in energy because the lifetime of the initial state is finite.

**(½ mark for answer "energy not well-defined"; ½ mark for "wave function changes with time"; 1 mark for reasonable justification for why energy not well defined)**

# ADV_Q06

## Semester 2, Y2011

**Question 6**

In quantum physics the orbital angular momentum $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$ of an electron, where $\vec{\boldsymbol{r}}$ is the electron position and $\vec{\boldsymbol{p}}$ is its momentum, is quantised. The magnitude $L$ of the momentum and a component $L_z$ of the momentum with respect to a chosen axis can only take the values

$$L = \sqrt{\ell(\ell+1)}\,\hbar \quad \text{with} \quad \ell = 0, 1, 2, \ldots, n-1,$$

and

$$L_z = m_\ell \hbar \quad \text{with} \quad m_\ell = 0, \pm 1, \pm 2, \ldots, \pm \ell,$$

where $n$ is the principal quantum number, $\ell$ is the orbital quantum number, and $m_\ell$ is the magnetic quantum number.

(a) The Bohr model for the hydrogen atom always has a non-zero orbital angular momentum. Does the quantum model for the hydrogen atom, in its ground state, have a non-zero orbital angular momentum? Briefly explain your answer in terms of the allowed values of $L$ and $L_z$.

(b) The allowed values of $L$ and $L_z$ require that $|L_z| < L$. Explain why this inequality is required for consistency with the Heisenberg Uncertainty Principle.

**(5 marks)**

**Solution**

(a)

The ground state means that the principal quantum has a value $n = 1$. In this case $\ell = 0$ is the only value allowed in the quantum model. Hence $L = \sqrt{\ell(\ell + 1)}\hbar$ with $\ell = 0$ means that $L = 0$, i.e. the magnitude of the orbital angular momentum is zero.

This shows the quantum hydrogen atom has zero orbital angular momentum in its ground state. The wave function is spherically symmetric in contrast to the planar electron orbit for the Bohr model.

**(2 marks)**

(b)

If $|L| = L_z$ then $\vec{L} = L_z \hat{z}$ and the $\hat{z}$ direction is perpendicular to the position and momentum vectors of the electron, according to $\vec{L} = \vec{r} \times \vec{p}$. This implies that the electron is moving in the $x - y$ plane.

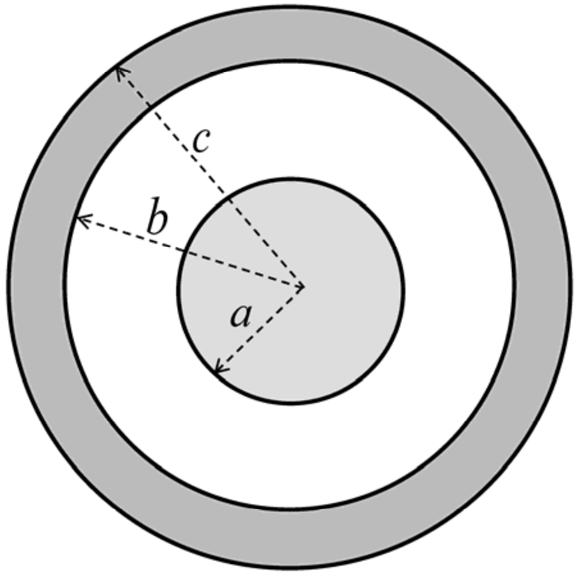
**(1 mark)**

However, if the electron is moving in the *x-y* plane, $\Delta z = 0$ (there is no uncertainty in the position in the $\hat{z}$ direction), and $\Delta p_z = 0$ since $p_z = 0$ i.e. the electron is not moving in the $\hat{z}$ direction. **(1 mark)**. However, then $\Delta p_z \, \Delta z = 0$, which contradicts the Heisenberg Uncertainty Principle. **(1 mark)**

# ADV_Q07=TEC_Q07

# Semester 2, Y2011

**Question 7**




A long coaxial cable consists of an inner cylindrical conductor with radius $a$ and an outer coaxial cylinder with inner radius $b$ and outer radius $c$. The outer cylinder is insulated and has no net charge. The inner cylinder has a uniform positive charge per unit length $\lambda$.

(a) In a cross-sectional diagram, draw where the charges (positive and negative) will reside on the conductors.

(b) Calculate the magnitude and direction of the electric field for a point between the cylinders at a distance $r$ from the axis $(a < r < b)$.

(c) Calculate the magnitude and direction of the electric field for a point outside the outer cylinder, at a distance $r$ from the axis $(r > c)$.

(d) In a cross-sectional diagram, sketch the electric field lines in all regions.

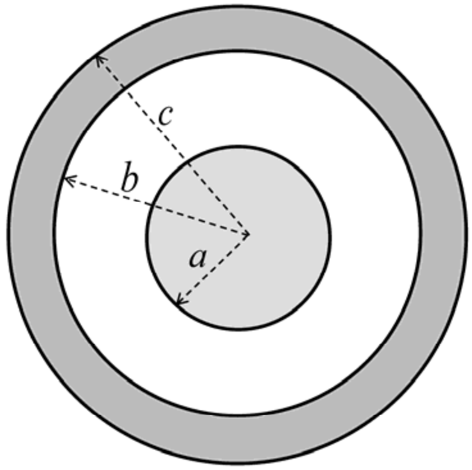
Some damage to the coaxial cable allowed a conducting strip to connect the inner cylindrical conductor to the inside of the outer cylindrical conductor.

(e) In a cross-sectional diagram, draw where the charges (positive and negative) will now reside.

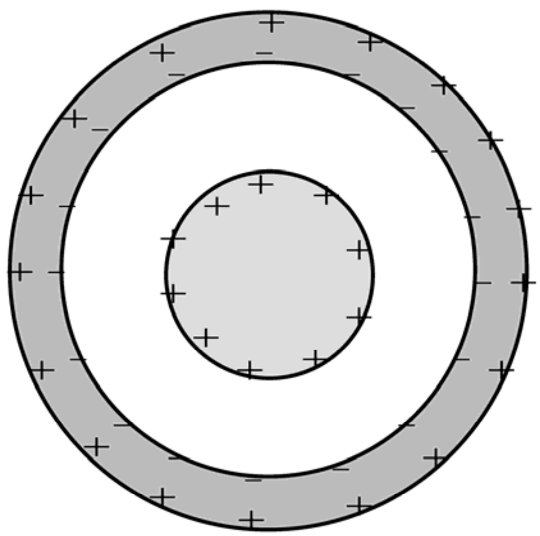
(f) In a cross-sectional diagram, sketch the electric field lines for this new configuration in all regions.

**(10 marks)**

**Solution**
Question setup:




(a)



There is positive charge on the surface of the inside cylinder, negative charge on the inner surface of the outer cylinder and positive charge on the outer surface of the outer cylinder.

**(2 marks)**

(b)
Use Gauss's law, choosing a Gaussian surface that is a cylinder of radius $r$ and length $l$. The electric field will be perpendicular to the side of the cylinder and of constant magnitude, and the electric field will be at right angles to the ends of the cylinder.

$$\oint E \cdot dA = Q_{enc} / \varepsilon_0$$

$$E_r 2\pi r l = \lambda l / \varepsilon_0$$

$$E_r = \lambda / 2\pi \varepsilon_0 r$$

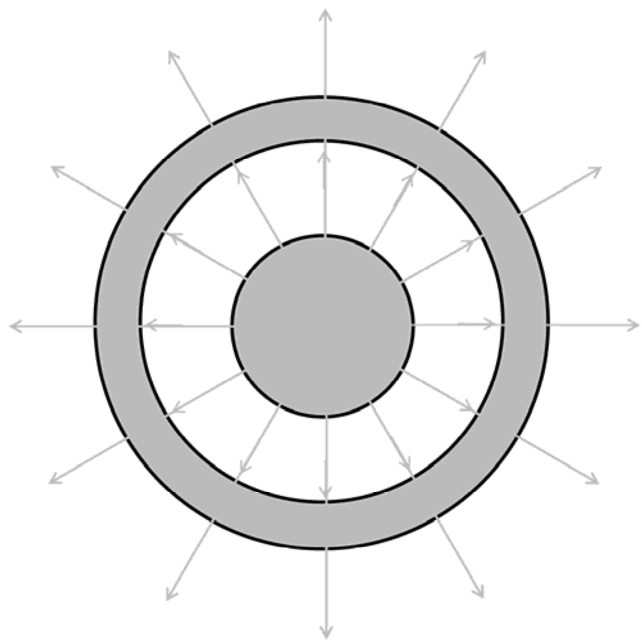
**(3 marks)**

(c)
In the region $r > c$, the calculation proceeds identically as for part (b). There is the same total charge enclosed.

$$E_r = \lambda / 2\pi\varepsilon_0 r$$

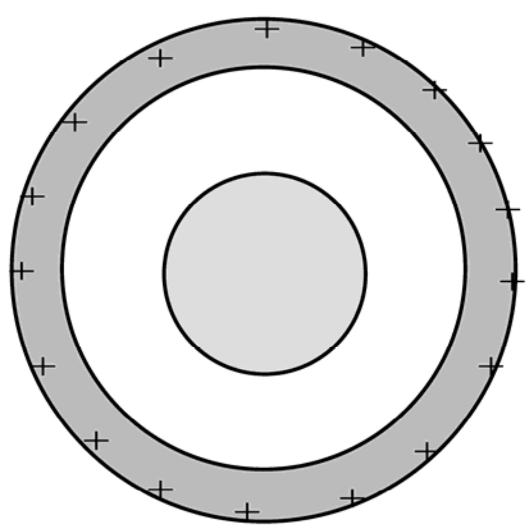
**(2 marks)**

(d)



**(1 mark)**

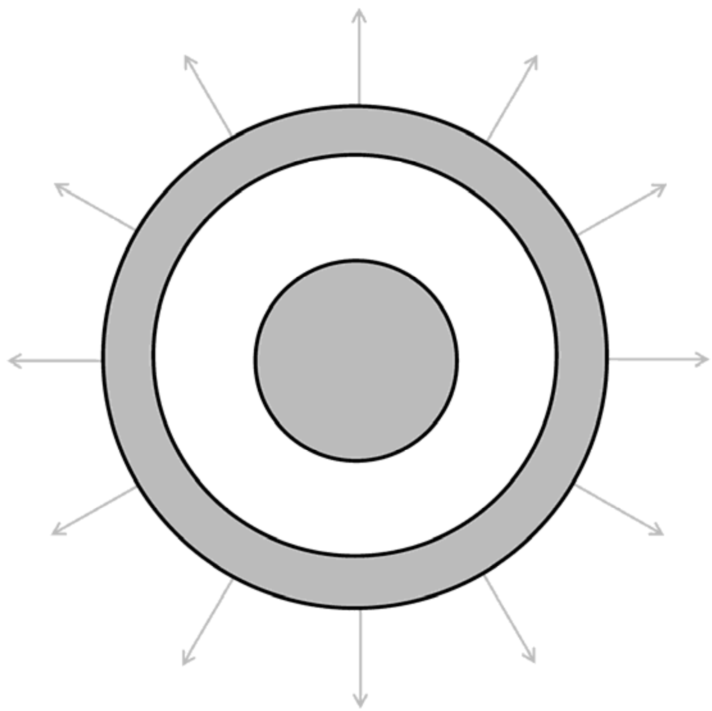
(e)



Positive charge only located on the outside of the outer cylinder.

**(1 mark)**

(f)



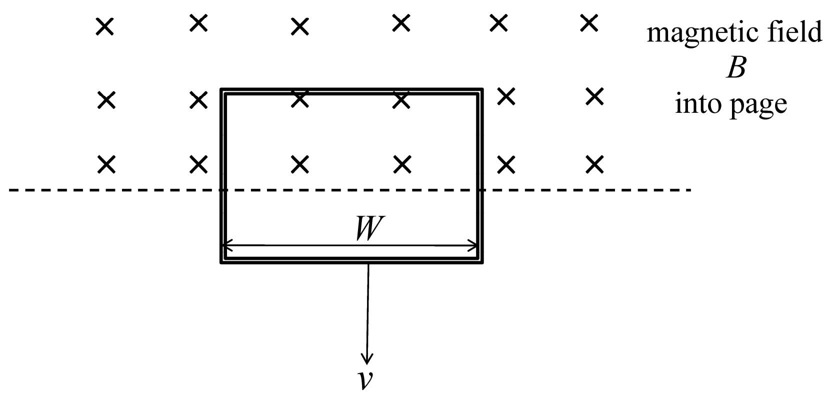
Electric field is non-zero only outside of the outer cylinder.

**(1 mark)**

# TEC_Q08=ADV_Q08

## Semester 2, Y2011

**Question 8**




A rectangular conducting loop of wire has width $W$ and mass $m$. It is initially at rest in a region of uniform magnetic field $B$ directed into the page. The loop is oriented perpendicular to the direction of the field. The loop is allowed to fall out of the region at uniform speed $v$ as shown in the figure above.

(a) Show that the emf induced in the loop as it falls out of the region of the magnetic field is given by

$$\varepsilon = BW\,v.$$

(b) Derive an expression for the magnitude of the induced current flowing in the loop if it has a resistance of $R$.

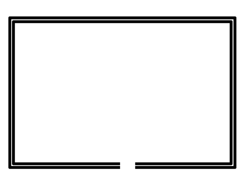
(c) What is the direction of the current flow in the loop? Briefly explain the reason for your answer.

(d) Find an expression for the force exerted by the magnetic field on the loop. What is the direction of the force?

(e) Show that the constant speed $v$ of the loop falling out of the region of magnetic field is given by

$$v = \frac{m\,g\,R}{W^2B^2}.$$

(f) If the loop had been cut as shown in the diagram below, describe what its motion would be as it falls out of the region of the magnetic field.



**(10 marks)**

**Solution**

(a)

Emf as the loop falls out of the magnetic field depends on the rate of change of magnetic flux through the loop which is given by:

$$\varepsilon = -\frac{d\Phi_B}{dt}.$$

The magnetic flux through the loop is decreasing as a smaller fraction of the total area of the loop remains in the magnetic field. If the loop falls a distance $dx$ then the change in the area remaining in the field is given by:

$$dA = -W\,dx \Rightarrow \; d\Phi_B = -BWdx.$$

Hence

$$\varepsilon = BW\frac{dx}{dt} = BW\,v.$$

**(2 marks for correct derivation)**

(b)

If the loop has a resistance $R$ then the current flowing around the loop is given by

$$\varepsilon = i\,R \Rightarrow i = \frac{\varepsilon}{R} = \frac{BW\,v}{R}.$$

**(1 mark)**

(c)

The direction of current flow is clockwise to resist the reduction in $\Phi_B$.

**(½ mark for direction and ½ mark for reason)**

(d)

There is no force on the bottom horizontal section of the loop (no magnetic field).

There is no vertical force on the sides of the loop (force is horizontal).

The force on the top section of the loop (still in the magnetic field) is given by:

$$F = i\,BW = \frac{B^2W^2v}{R}.$$

The force is directed upwards.

**(1 mark for derivation of force; ½ mark for correct result; ½ mark for direction of force)**

(e)

If the loop has constant velocity then there is no net force acting on the loop. Hence

$$mg = \frac{B^2W^2v}{R} \Rightarrow v = \frac{m\,g\,R}{B^2\,W^2}.$$

**(1 mark for no net force; 1 mark for correct derivation of the formula)**

(f)

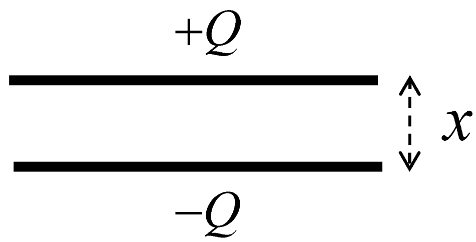
If the loop is cut then there is no induced current and hence no magnetic force acting on the loop. Hence the loop falls under gravity with acceleration $g$.

**(1 mark for no magnetic force; 1 mark for motion under gravity)**

# ADV_Q09

## Semester 2, Y2011

**Question 9**




A parallel-plate capacitor with plate area $A$ and separation $x$ has charges $+Q$ and $-Q$ on its plates. The capacitor is disconnected from the voltage source that placed this charge, so that the charge on each plate now remains fixed.

(a) Show that the total energy stored in this capacitor is given by:

$$U = \frac{Q^2 x}{2\varepsilon_0 A}.$$

(b) The plates are pulled apart an additional distance $dx$. Obtain an expression for the change in the stored energy.

(c) The change in stored energy must equal the work $dW = F\,dx$ done in pulling the plates apart, where $F$ is the force with which the plates attract each other. Find an expression for $F$.

(d) Determine the strength of the electric field $E$ between the two plates, in terms of the plate area $A$, the separation $x$, the charge $Q$, and fundamental constants.

(e) Is the force $F$ determined from part (c) equal to $QE$, where $E$ is the electric field between the plates? Explain why or why not?

**(10 marks)**

**Solution**

(a)

The capacitance $C$ of a parallel plate capacitor with plates of area $A$ separated by distance $x$ is given by

$$C = \varepsilon_0 A / x$$

**(1 mark)**

The potential energy stored in this capacitor with charge $Q$ is

$$U = \frac{Q^2}{2C} = \frac{Q^2 x}{2\varepsilon_0 A}$$

**(1 mark)**

(b)

The change in potential energy $dU$ as the separation changes from $x$ to $x + dx$ is

$$\begin{aligned} dU &= U(x+dx) - U(x) \\ &= \frac{Q^2(x+dx)}{2\varepsilon_0 A} - \frac{Q^2 x}{2\varepsilon_0 A} \\ &= \frac{Q^2 dx}{2\varepsilon_0 A} \end{aligned}$$

**(2 marks)**

(c)

Solving for the force

$$dU = dW = Fdx$$

$$\frac{Q^2 dx}{2\varepsilon_0 A} = Fdx$$

$$F = \frac{Q^2}{2\varepsilon_0 A}$$

**(2 marks)**

(d)

The electric field strength $E$ between the plates is given by

$$E = \frac{V}{x} = \frac{Q}{Cx} = \frac{Q}{\varepsilon_0 A}$$

**(1 mark)**

(e)

Using this expression for the electric field strength $E$ between the plates,

$$QE = \frac{Q^2}{\varepsilon_0 A}$$

Therefore, the force $F$ is equal to $QE/2$, not $QE$.

**(1 mark)**

The electric field $E$ between the plates of the capacitor is due, in equal parts, to the positive charge $+Q$ on one plate and the negative charge $-Q$ on the other plate. The force experienced on one plate (say the positively-charged one) is due only to the electric field contribution from the other plate (in this case, the negatively-charged one).

**(2 marks)**

# ADV_Q10

## Semester 2, Y2011

**Question 10**
A small ball has a density $250\,\text{kg.m}^{-3}$ and volume of $1.0\times10^{-4}\,\text{m}^3$.


(a) When floating on the surface of the water, what fraction of the ball's volume is submerged?

(b) You now push the ball underwater. Draw a free-body diagram of the ball when it is held so that its centre is submerged by 15 cm under the surface of the water.

(c) Calculate the work that you do to push the ball to this depth at constant velocity.

(d) You now release the ball. Briefly describe its motion (qualitative only).

(e) Using energy considerations, calculate the maximum height above the surface of the water the ball will reach. Ignore any drag forces.

**(10 marks)**

**Solution**

(a)

The ball floats with a fraction

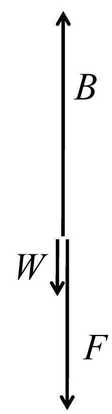
$$\rho / \rho_w = 250/1000 = 25\%$$

of the ball submerged

**(1 mark)**

(b)

Free-body diagram:




**(1 mark)**

(c)

The density of the ball is $250\,\text{kg.m}^{-3}$ so its mass is

$$M = \rho V = (250)(1.0\times10^{-4}) = 0.025\,\text{kg}.$$

so

$$W = M\,g = (0.025)(9.8) = 0.245\text{N}.$$

The buoyancy force $B$ is

$$B = \rho_w V\,g = (1000)(1.0\times10^{-4})(9.8) = 0.98\,\text{N}.$$

When the ball is moving at constant velocity the net force is zero, so the applied force $F$ is

$$F = B - W = 0.980 - 0.245 = 0.735\text{N}.$$

So the work done in moving the ball $0.15\,\text{m}$ is

$$E = Fd = (0.735)(0.15) = 0.110\,\text{J}.$$

**(1 mark for mass, 1 mark for buoyancy, 1 mark for net force, 1 mark for answer)**

(d)

When it is released, the buoyancy force will provide a net upwards force, so the ball will accelerate upwards. When it breaks the surface of the water, the upwards force will reduce, being provided only by that portion of the ball which is underwater. Once it has completely cleared the water, it will be in free-fall, so it will pop out of the water then fall back down again, where it will float as before.

**(2 marks)**

(e)
When it leaves the surface of the water, the ball will have an amount of kinetic energy equal to the work done in pushing it to a depth of $0.15\,\mathrm{m}$. This will then be converted into potential energy as it pops up to a height $h$ (when $K = 0$), then falls. So

$$U = 0.110\mathrm{J} = M\ g\,h \Rightarrow$$

$$h = \frac{(0.110)}{(0.025)(9.8)} = 0.45\,\mathrm{m}.$$

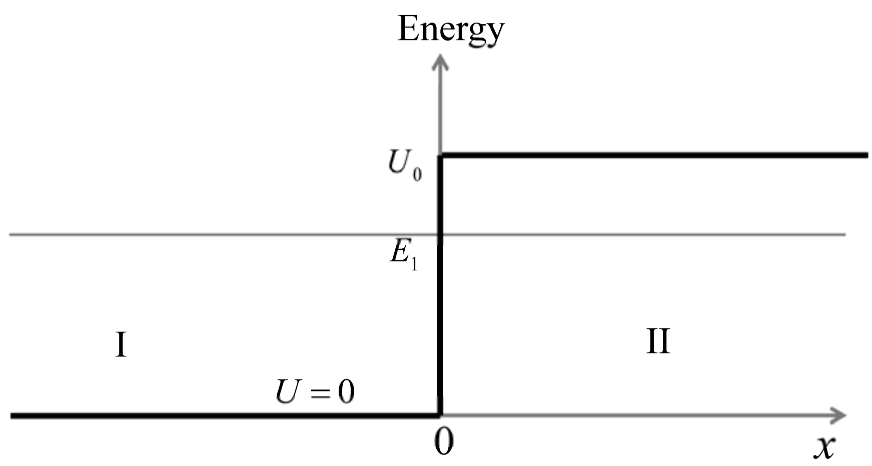
i.e. the ball would pop up to a height of $0.45\,\mathrm{m}$.

**(2 marks)**

# ADV_Q11

# Semester 2, Y2011

**Question 11**




An electron with energy $E_1$ is free to move in $x$ direction. It is located in a region of space with the potential energy configuration shown in the figure. In the region $x < 0$ (denoted region I) the potential energy is $U = 0$, and in the region $x > 0$ (denoted region II) the potential energy is $U = U_0 > E_1$. This situation may be interpreted physically as describing an electron incident from the left on a step barrier potential of height $U_0$. The stationary state of the electron is described by the $1-\mathrm{D}$ time-independent Schrödinger equation

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x).$$

(a) The stationary state wave function of the electron in region I may be written

$$\psi_{\mathrm{I}}(x) = A\mathrm{e}^{ikx} + B\mathrm{e}^{-ikx}.$$

Using the Schrödinger equation, determine the value of $k$.

(b) The stationary state wave function of the electron in region II may be written

$$\psi_{\mathrm{II}}(x) = C\mathrm{e}^{-ax} + D\mathrm{e}^{ax}.$$

Explain why $D = 0$ is required, and using the Schrödinger equation, determine the value of $a$.

(c) By applying continuity of the wave function and its derivative at $x = 0$, show that

$$B = \frac{a + ik}{-a + ik}A \quad \text{and} \quad C = \frac{2ik}{-a + ik}A.$$

(d) Show that the result in part (c) implies $|B|^2 = |A|^2$. Give a brief physical interpretation for this result.

**(10 marks)**

**Solution**

(a)

The Schrodinger equation with $U = 0$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

or

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi. \qquad (1)$$

Differentiating $\psi_I = A\mathrm{e}^{ikx} + B\mathrm{e}^{-ikx}$ twice gives

$$\frac{d^2\psi_I}{dx^2} = -k^2\psi_I$$

and comparing with equation (1) gives

$$k^2 = \frac{2mE}{\hbar^2} \Rightarrow k = \frac{(2mE)^{\frac{1}{2}}}{\hbar}.$$

**(2 marks)**

(b)

The choice $D = 0$ is required or else $\psi_{II} \to \infty$ as $x \to \infty$, which would have no interpretation.

**(1 mark)**

The Schrodinger equation in region II is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} + U_0\psi_{II} = E\psi_{II}$$

or

$$\frac{d^2\psi_{II}}{dx^2} = +\frac{2m(U_0 - E)}{\hbar^2}\psi_{II}\,. \qquad (2)$$

Differentiating $\psi_{II} = C\,\mathrm{e}^{-ax}$ twice gives

$$\frac{d^2\psi_{II}}{dx^2} = a^2\psi_{II}\,.$$

Comparing this with equation (2) gives

$$a^2 = \frac{2m}{\hbar^2}(U_0 - E)$$

i.e.

$$a = \sqrt{2m(U_0 - E)}\;/\;\hbar.$$

**(2 marks)**

(c)

Differentiating $\psi_I(x)$ and $\psi_{II}(x)$ we have

$$\psi_I'(x) = ik\left(A\mathrm{e}^{ikx} - B\mathrm{e}^{-ikx}\right)$$

and

$$\psi_{II}'(x) = -aC\,\mathrm{e}^{-ax}.$$

So the requirements $\psi_I(0) = \psi_{II}(0)$ and $\psi_I'(0) = \psi_{II}'(0)$

**(1 mark)**

give

$$A + B = C \tag{3}$$

and $$ik\left(A - B\right) = -aC. \tag{4}$$

Dividing equation (4) by equation (3) gives

$$ik\frac{A-B}{A+B} = -a \quad \Rightarrow \quad B\left(ik - a\right) = A\left(ik + a\right) \quad \Rightarrow B = \frac{a+ik}{-a+ik}A$$

**(1 mark)**

as required, and substituting this into equation (3) gives

$$C = A\left(1 + \frac{a+ik}{-a+ik}\right) = A\left(\frac{-a+ik+a+ik}{-a+ik}\right) \Rightarrow$$

$$C = \frac{2ik}{-a+ik}A$$

as required.

**(1 mark)**

(d)

We have

$$\left|B\right|^2 = \left|\frac{a+ik}{-a+ik}\right|^2 \left|A\right|^2$$

$$= \frac{\left|a+ik\right|^2}{\left|-a+ik\right|^2}\left|A\right|^2$$

$$= \frac{a^2+k^2}{a^2+k^2}\left|A\right|^2 = \left|A\right|^2$$

as required.

**(1 mark)**

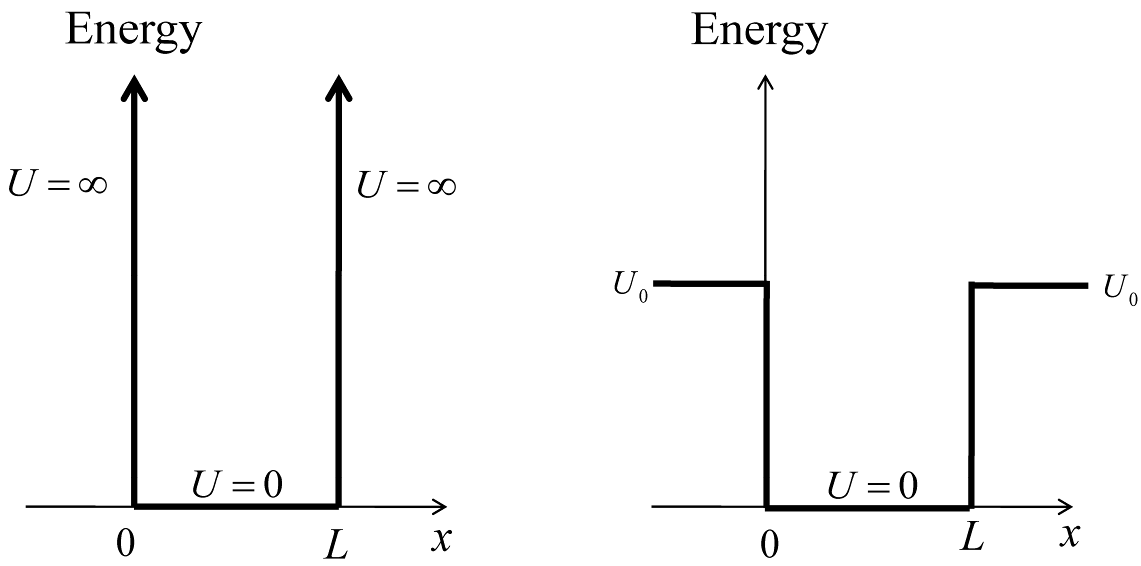
The term $A\mathrm{e}^{ikx}$ may be interpreted as a plane wave electron incident on the step barrier, and $B\mathrm{e}^{-ikx}$ as the plane wave electron reflected from the barrier. The reflection coefficient is $R = \left|B\right|^2 / \left|A\right|^2 = 1$, i.e. the electron is always reflected. This occurs because the "barrier" has infinite length – there is no transmission of the electron to $x > 0$.

**(1 mark)**

# ADV_Q12

## Semester 2, Y2011

**Question 12**




(a) (b)

Consider an electron in an infinite square well extending from $x=0$ to $x=L$ (a "particle in a box"), as shown in panel (a) of the figure above, and also consider an electron in a finite square well extending from $x=0$ to $x=L$, as shown in panel (b) of the figure above. The potential inside each well is $U=0$. The infinite square well has $U=\infty$ in the regions $x<0$ and $x>L$, and the finite square well has $U=U_0$ in these regions.

(a) Sketch the wave functions $\psi_{\text{inf},1}(x)$ and $\psi_{\text{inf},2}(x)$ of the two lowest energy stationary states for the infinite square well shown in panel (a) of the figure above.

(b) Sketch $|\psi_{\text{inf},1}(x)|^2$ and $|\psi_{\text{inf},2}(x)|^2$ for the two states in part (a).

(c) Sketch the wave functions $\psi_{\text{fin},1}(x)$ and $\psi_{\text{fin},2}(x)$ of the two lowest energy stationary states for the finite square well shown in panel (b) of the figure above (assuming two such states exist).

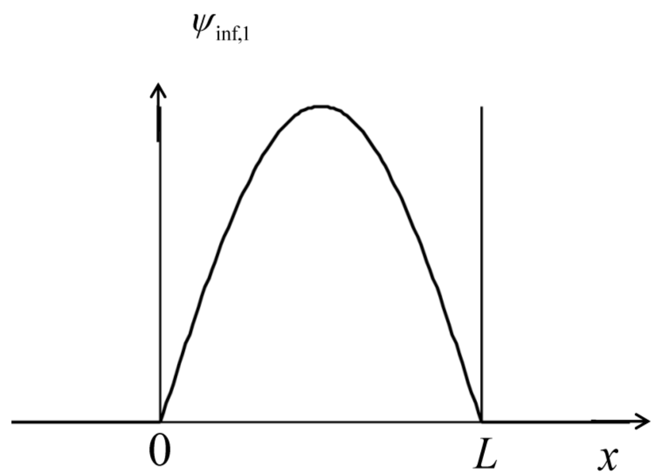
(d) Sketch $|\psi_{\text{fin},1}(x)|^2$ and $|\psi_{\text{fin},2}(x)|^2$ for the two states in part (c).

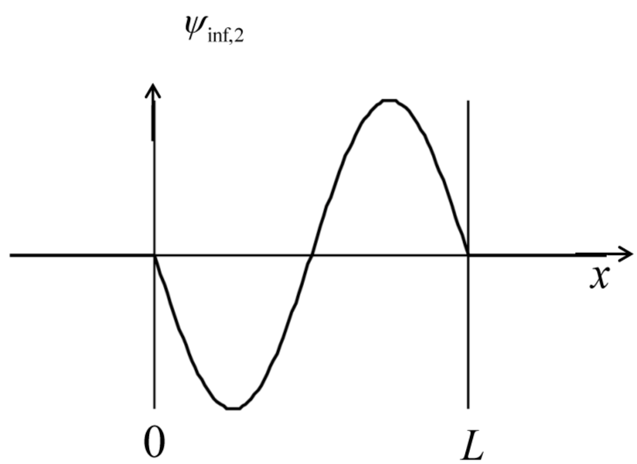
(e) Compare the energies $E_{\text{inf},1}$ and $E_{\text{fin},1}$ of the lowest energy stationary states for the infinite square well as shown in panel (a) and for the finite square well as shown in panel (b). Briefly explain your answer.

**(10 marks)**

**Solution**

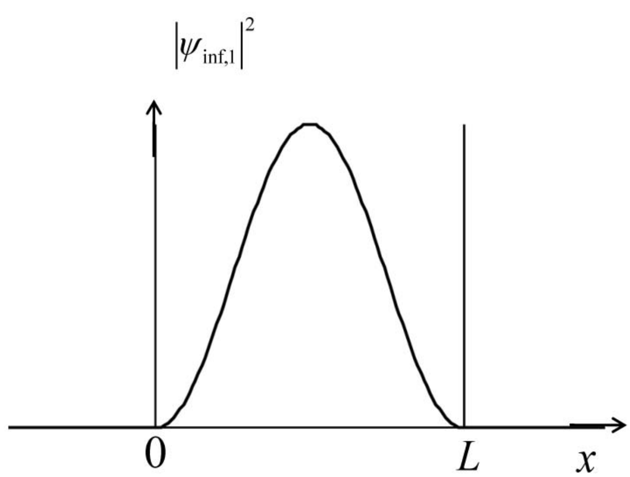
(a)

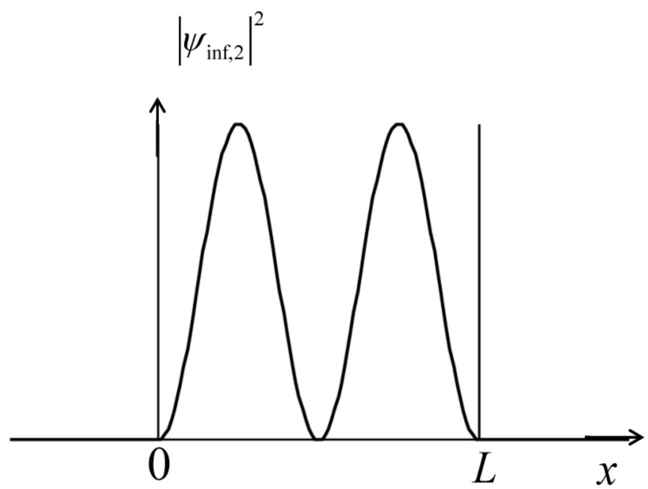



**(2 marks)**

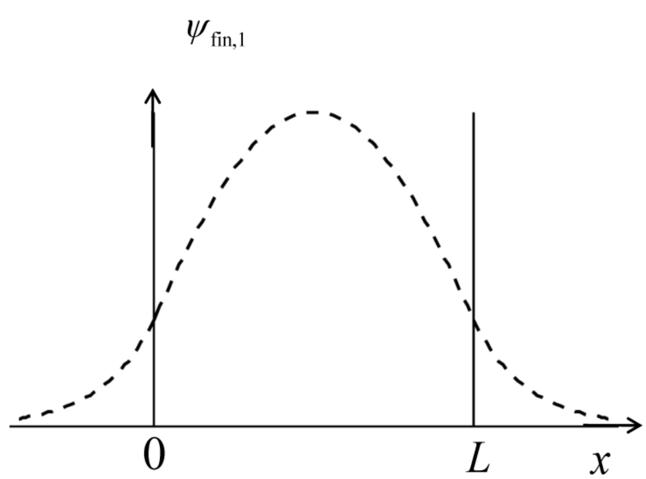
(b)

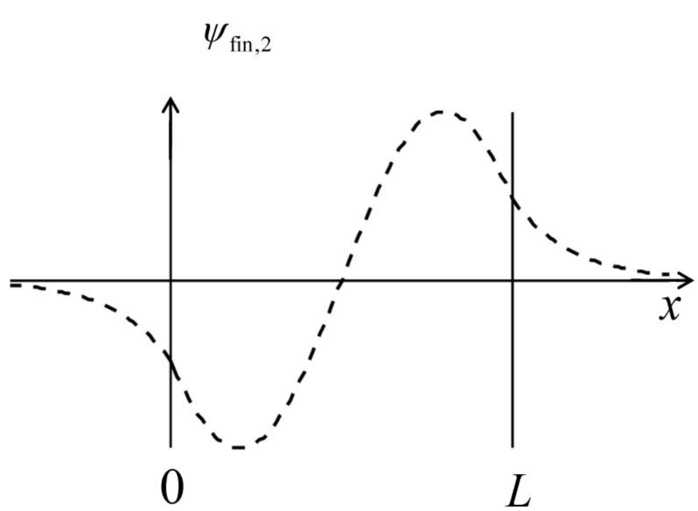





**(2 marks)**

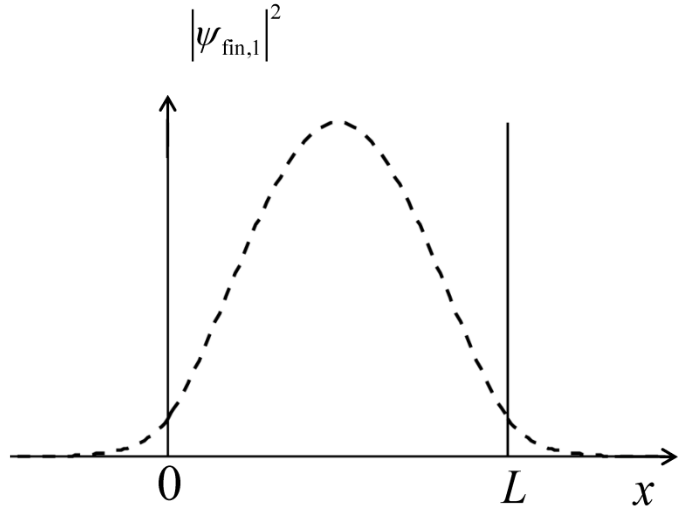
(c)

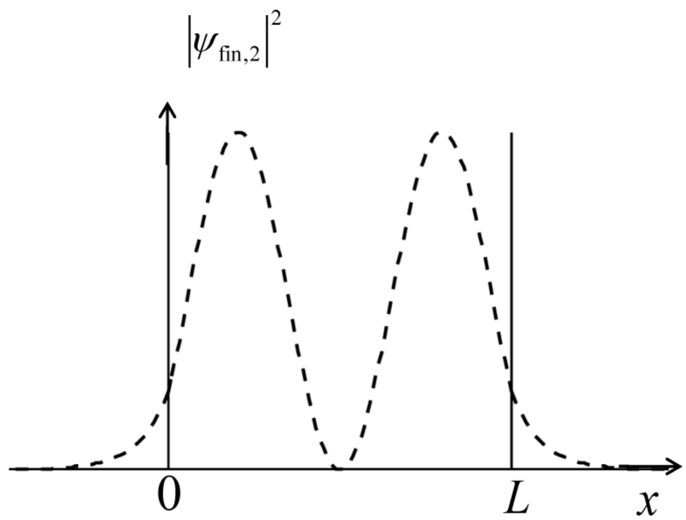



**(2 marks)**

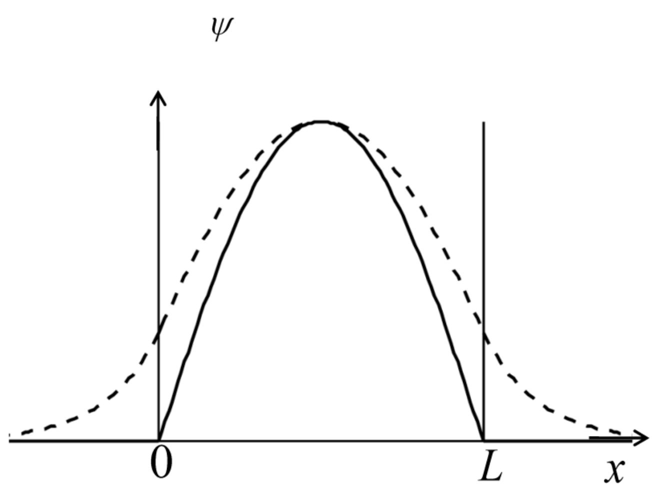
(d)




**(2 marks)**

(e)



We have $E_{\inf,1} > E_{\mathrm{fin},1}$. This may be explained as follows. The two wave functions we have that $\lambda_{\mathrm{fin},1} > \lambda_{\inf,1}$ inside the well, so from the de Broglie wavelength relation $p = \frac{h}{\lambda}$ we have $p_{\mathrm{fin},1} < p_{\inf,1}$. Hence using $E = \frac{p^2}{2m}$ we have $E_{\mathrm{fin},1} < E_{\inf,1}$.

**(2 marks)**