

8027A SEMESTER 2 2010

THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

MATH1903/1907
INTEGRAL CALCULUS AND MODELLING (ADVANCED)

November 2010

LECTURERS: H Dullin, J Parkinson

TIME ALLOWED: One and a half hours

Family Name:

Other Names:

SID: Seat Number:

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Approved non-programmable calculators may be used.
There is a table of integrals after the last question in this booklet.

**THE QUESTION PAPER MUST NOT BE REMOVED FROM THE
EXAMINATION ROOM.**

MARKER'S USE
ONLY

There are **four** questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.

1. (a) Let D be the region of the plane bounded by the x -axis, the y -axis, the line $x = 1$, and the curve $y = \cosh x$.

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1. The first step in the process of creating a new product is to identify a market need. This involves conducting market research to understand the preferences and behaviors of potential customers.

2. Once a market need is identified, the next step is to develop a concept for the product. This involves brainstorming ideas and creating a prototype to visualize the product.

3. The third step is to conduct a feasibility study. This involves evaluating the technical, financial, and operational aspects of the product to determine if it is viable.

4. If the feasibility study is successful, the next step is to develop a business plan. This involves outlining the marketing, sales, and distribution strategies for the product.

5. The final step is to launch the product. This involves manufacturing the product, setting up distribution channels, and promoting the product to the target market.

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MARKS

2. (a) (i) Use a comparison test to show that $\int_0^\infty \frac{e^x}{7 + 2 \cosh(2x)} dx$ converges.

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- (ii) Using an appropriate substitution, or otherwise, calculate the integral

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$$\int_0^1 \frac{xe^{\sqrt{1+x^2}}}{\sqrt{1+x^2}} dx.$$

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$$I_{m,n} = -\frac{n}{m+1}I_{m,n-1},$$

[You may use the fact that $\lim_{x \rightarrow 0^+} x^\alpha (\ln x)^\beta = 0$ for all $\alpha > 0$ and $\beta \geq 0$.]

MARKS

(ii) Hence show that

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$$\int_0^1 x^{-x} dx = \sum_{k=1}^{\infty} n^{-n}.$$

You may assume that any reasonable series manipulations are valid.

QUESTION 3 BEGINS ON THE NEXT PAGE

3. (a) Find the general solution of the differential equation

4

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3e^{-2x}$$

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$$\frac{dy}{dx} = \frac{2x+1}{x^2+x+1}(1-y),$$

and show that every solution converges to the equilibrium solution $y = 1$ for $x \rightarrow \infty$.

Figure 1 displays the effect of the 2011 earthquake on the relationship between the number of people in a household and the probability of being in a household. The figure is organized into two rows of plots, with the top row representing the 2011 earthquake and the bottom row representing the 2012 earthquake. Each row contains five plots, showing the relationship for different levels of the number of people in a household (0, 1, 2, 3, 4). The x-axis for all plots is 'Number of people in a household' (ranging from 0 to 10), and the y-axis is 'Probability of being in a household' (ranging from 0 to 1). The plots show that the probability of being in a household increases with the number of people in a household, and that the 2011 earthquake had a significant effect on this relationship, particularly for households with 2 or 3 people.

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$$\frac{dy}{dx} - e^{-x-y} + 1 = 0.$$

Introduce a new dependent variable u given by $u = x + y$, and hence find the general solution of the original equation.

QUESTION 4 BEGINS ON THE NEXT PAGE

4. (a) A spherical raindrop evaporates at a rate proportional to its surface area, retaining the spherical shape. Derive a differential equation for the radius $r(t)$ of the raindrop and solve it for a raindrop with initial radius r_0 to show that

$$r(t) = r_0 - \alpha t$$

for a constant $\alpha > 0$.

[Note that the volume of a sphere of radius r is $V = 4\pi r^3/3$, and that the surface area is $A = 4\pi r^2$, and assume that the density of water is 1.]

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MARKS

- (b) The evaporating raindrop is falling towards the ground. For this type of problem with time-dependent mass the appropriate form of Newton's second law states that the rate of change of the product of mass m with velocity v is equal to the force. The force is given by mg (with positive direction down), where g is the constant gravitational acceleration, with an additional air friction force proportional to the area πr^2 times the velocity. The friction force opposes the velocity. Show that the differential equation for the velocity v of the falling raindrop can be written as

$$\frac{dv}{dt} - \frac{k\alpha}{r(t)}v = g$$

for some constant k .

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- (c) Find the particular solution of the differential equation for the falling raindrop for which initially the raindrop is at rest. Assume that $k \neq -1$.

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(d) Assume that $k = -2$. Compute the distance the drop falls from rest until it is completely evaporated.

THERE IS MORE SPACE AVAILABLE ON THE NEXT PAGE

Table of Standard Integrals

$$1. \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$9. \int \sec^2 x dx = \tan x + C$$

$$2. \int \frac{dx}{x} = \ln|x| + C$$

$$10. \int \operatorname{cosec}^2 x dx = -\cot x + C$$

$$3. \int e^x dx = e^x + C$$

$$11. \int \sec x dx = \ln|\sec x + \tan x| + C$$

$$4. \int \sin x dx = -\cos x + C$$

$$12. \int \operatorname{cosec} x dx = \ln|\operatorname{cosec} x - \cot x| + C$$

$$5. \int \cos x dx = \sin x + C$$

$$13. \int \sinh x dx = \cosh x + C$$

$$6. \int \tan x dx = -\ln|\cos x| + C$$

$$14. \int \cosh x dx = \sinh x + C$$

$$7. \int \cot x dx = \ln|\sin x| + C$$

$$15. \int \tanh x dx = \ln \cosh x + C$$

$$8. \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$16. \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (|x| < a)$$

$$17. \int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2 + a^2}\right) + C'$$

$$18. \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2 - a^2}\right) + C' \quad (x > a)$$

$$19. \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \quad (|x| \neq a)$$

End of Extended Answer Section