MATH 1902 Assignment 1

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Tutorial: Wednesday 3-4 pm
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Q1. a) i $U = \{\underset{\sim}{i} - 3\underset{\sim}{j},\ \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k},\ -2\underset{\sim}{j} + \underset{\sim}{k},\ 0\}$

$\therefore$ Consider the two vectors, $\underset{\sim}{v} = \underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{u} = -2\underset{\sim}{j} + \underset{\sim}{k}$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{i} - 3\underset{\sim}{j} + (-2\underset{\sim}{j} + \underset{\sim}{k})$

$= \underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k}$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} \notin U$, $\therefore U$ is not a subspace of V

ii $U = \{\alpha \underset{\sim}{j} \mid 0 \neq \alpha \in \mathbb{R}\}$

$\therefore$ Consider the two vectors, $\underset{\sim}{u} = \alpha\underset{\sim}{j}$, and $\underset{\sim}{v} = -\alpha\underset{\sim}{j}$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} = \alpha\underset{\sim}{j} + (-\alpha\underset{\sim}{j}) \Rightarrow$ Let $\alpha = 3$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} = 3\underset{\sim}{j} + (-3\underset{\sim}{j})$

$= 0\underset{\sim}{j}$

$= \alpha\underset{\sim}{j} - \alpha\underset{\sim}{j}$

$= 0\underset{\sim}{j}$

As $\alpha \neq 0$, $\therefore \underset{\sim}{u} + \underset{\sim}{v} \notin U$, $\therefore U$ is not a subspace of V

iii $U = \{\alpha(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}) \mid \alpha \in \mathbb{R}\}$

$\therefore$ Consider the two vectors, $\underset{\sim}{u} = \alpha_1(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$ and $\underset{\sim}{v} = \alpha_2(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$, where $\alpha_1, \alpha_2 \in \mathbb{R}$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} = \alpha_1(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}) + \alpha_2(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

$= \alpha_1\underset{\sim}{i} + 2\alpha_1\underset{\sim}{j} + 3\alpha_1\underset{\sim}{k} + \alpha_2\underset{\sim}{i} + 2\alpha_2\underset{\sim}{j} + 3\alpha_2\underset{\sim}{k}$

$= (\alpha_1 + \alpha_2)\underset{\sim}{i} + (\alpha_1 + \alpha_2)2\underset{\sim}{j} + (\alpha_1 + \alpha_2)3\underset{\sim}{k}$

$= (\alpha_1 + \alpha_2)(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

As $\alpha_1, \alpha_2 \in \mathbb{R}$, $\therefore \alpha_1 + \alpha_2 = \beta$, where $\beta \in \mathbb{R}$

$\therefore \ \underset{\sim}{u} + \underset{\sim}{v} = \beta(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

This is in the form of a typical set member of U.

$\therefore \ \underset{\sim}{u} + \underset{\sim}{v} \in U$

Now $\quad \underset{\sim}{u} = \alpha_1(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

$\therefore \lambda \underset{\sim}{u} = \lambda\alpha_1(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$ where $\alpha_1, \lambda \in \mathbb{R}$

$\therefore \lambda\alpha_1 = \beta$ as $\beta \in \mathbb{R}$

$\therefore \lambda\underset{\sim}{u} = \beta(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$, which is in the form of a typical set member of U

$\therefore \lambda\underset{\sim}{u} \in U$

$\therefore$ U satisfies both conditions and is a subspace of V

iv $\quad U = \{\alpha\underset{\sim}{i} + \beta\underset{\sim}{k} \mid \alpha, \beta \in \mathbb{R}\}$

$\therefore$ Consider the two vectors $\underset{\sim}{u} = \alpha_1\underset{\sim}{i} + \beta_1\underset{\sim}{k}$ and $\underset{\sim}{v} = \alpha_2\underset{\sim}{i} + \beta_2\underset{\sim}{k}$ where $\alpha_1, \alpha_2; \beta_1, \beta_2 \in \mathbb{R}$

$$\begin{aligned}
\therefore \underset{\sim}{u} + \underset{\sim}{v} &= \alpha_1\underset{\sim}{i} + \beta_1\underset{\sim}{k} + \alpha_2\underset{\sim}{i} + \beta_2\underset{\sim}{k} \\
&= \alpha_1\underset{\sim}{i} + \alpha_2\underset{\sim}{i} + \beta_1\underset{\sim}{k} + \beta_2\underset{\sim}{k} \\
&= (\alpha_1 + \alpha_2)\underset{\sim}{i} + (\beta_1 + \beta_2)\underset{\sim}{k}
\end{aligned}$$

as $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ $\therefore \alpha_1 + \alpha_2 = \lambda_1$ and $\beta_1 + \beta_2 = \lambda_2$

$= \lambda_1\underset{\sim}{i} + \lambda_2\underset{\sim}{k}$ where $\lambda_1, \lambda_2 \in \mathbb{R}$

which is in the form of the typical member

$\therefore \ \underset{\sim}{u} + \underset{\sim}{v} \in U$

Now $\quad \underset{\sim}{u} = \alpha_1\underset{\sim}{i} + \beta_1\underset{\sim}{k}$

$$\begin{aligned}
\therefore \lambda\underset{\sim}{u} &= \lambda(\alpha_1\underset{\sim}{i} + \beta_1\underset{\sim}{k}) \\
&= \lambda\alpha_1\underset{\sim}{i} + \lambda\beta_1\underset{\sim}{k}
\end{aligned}$$

As $\lambda, \alpha_1, \beta_1 \in \mathbb{R}$, $\therefore \lambda\alpha_1 = \lambda_1$ and $\lambda\beta_1 = \lambda_2$
where $\lambda_1, \lambda_2 \in \mathbb{R}$

$\therefore \lambda \underset{\sim}{u} = \lambda_1 \underset{\sim}{i} + \lambda_2 \underset{\sim}{k}$

This is in the form of a typical member

$\therefore \lambda \underset{\sim}{u} \in U$

$\therefore$ U satisfies the conditions, and is such a subspace of V

v $U = \{ v \mid |v| \leq 1 \}$

Consider the vector $\underset{\sim}{u}$ with $|\underset{\sim}{u}| = 1$

Now consider ~~XXX~~ $\lambda = 2$

$\therefore \lambda \underset{\sim}{u} = \lambda \underset{\sim}{u}$ where $|\underset{\sim}{u}| = 1$, and ~~XXX~~ $\lambda = 2$

$\therefore |\lambda \underset{\sim}{u}| = \lambda |\underset{\sim}{u}|$

$\therefore \lambda |\underset{\sim}{u}| = \lambda \times 1$

~~XXX~~ $= 2$ as ~~XXX~~ $\lambda = 2$

which is greater than 1

$\therefore \lambda \underset{\sim}{u} \notin U$, $\therefore$ U is not a subspace

vi $U = \{0\}$

Consider the two vectors, $\underset{\sim}{u} = \underset{\sim}{0}$, $\underset{\sim}{v} = \underset{\sim}{0}$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0} + \underset{\sim}{0}$

$= \underset{\sim}{0}$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} \in U$

<u>Now</u> $\underset{\sim}{u} = \underset{\sim}{0}$, $\lambda \in \mathbb{R}$

$\therefore \lambda \underset{\sim}{u} = \lambda \times \underset{\sim}{0}$

$= \underset{\sim}{0}$

$\therefore \lambda \underset{\sim}{u} \in U$

$\therefore$ U satisfies the conditions, and is a subspace

Q1b.) The definition of linear independence for any n vectors is:

$$\alpha_1 \underset{\sim}{v_1} + \alpha_2 \underset{\sim}{v_2} + \dots + \alpha_n \underset{\sim}{v_n} = 0 \quad (1)$$

where $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ is the only solution to equation (1) where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$

The definition of linear dependence for any n vectors is:

$$\alpha_1 \underset{\sim}{v_1} + \alpha_2 \underset{\sim}{v_2} + \dots + \alpha_n \underset{\sim}{v_n} = 0 \quad (1)$$

where not all α's are $= 0$, and the equation (1) is still satisfied. where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$

Now Consider the set $U = \{\alpha \underset{\sim}{i} + \beta \underset{\sim}{k} \mid \alpha, \beta \in \mathbb{R}\}$

Let $\underset{\sim}{u} = \alpha \underset{\sim}{i} + 0 \underset{\sim}{k}$

and $\underset{\sim}{v} = 0 \underset{\sim}{i} + \beta \underset{\sim}{k}$ where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$

$\therefore$ To test for linear independence:

$$\lambda_1 \underset{\sim}{u} + \lambda_2 \underset{\sim}{v} = 0 \qquad \text{where } \lambda_1, \lambda_2 \in \mathbb{R} \setminus \cancel{\{0\}}$$

$$\therefore \lambda_1 (\alpha \underset{\sim}{i} + 0 \underset{\sim}{k}) + \lambda_2 (0 \underset{\sim}{i} + \beta \underset{\sim}{k}) = 0$$

$$\therefore \lambda_1 \alpha \underset{\sim}{i} + \lambda_2 \beta \underset{\sim}{k} = 0$$

This has only one solution where $\lambda_1 = \lambda_2 = 0$ as different components cannot be added

$\therefore$ $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly independent.

Now Select a third vector from the set:

$$\underset{\sim}{w} = \alpha_1 \underset{\sim}{i} + \beta_1 \underset{\sim}{k} \qquad \text{where } \alpha_1, \beta_1 \in \mathbb{R} \setminus \{0\}$$

$$\therefore \lambda_1 \underset{\sim}{u} + \lambda_2 \underset{\sim}{v} + \lambda_3 \underset{\sim}{w} = 0 \qquad \text{where } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$$

$$\therefore \lambda_1 \alpha \underset{\sim}{i} + \lambda_2 \beta \underset{\sim}{k} + \lambda_3 \alpha_1 \underset{\sim}{i} + \lambda_3 \beta_1 \underset{\sim}{k} = 0$$

$$\therefore \lambda_1 \alpha \underset{\sim}{i} + \lambda_3 \alpha_1 \underset{\sim}{i} + \lambda_2 \beta \underset{\sim}{k} + \lambda_3 \beta_1 \underset{\sim}{k} = 0 \quad (2)$$

$\therefore$ there exists a $\lambda_3 \alpha_1 = -\lambda_1 \alpha$ and $\lambda_3 \beta_1 = -\lambda_2 \beta$

therefore the equation (2) can be satisfied without all scalar coefficients equalling zero

$\therefore$ the three vectors are linearly dependent

$\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in U$ are linearly dependent

Q1. c) If U is a subspace of V, then U must satisfy the following conditions:

For any two vectors, $\underset{\sim}{u}, \underset{\sim}{v} \in U$:

- $\underset{\sim}{u} + \underset{\sim}{v} \in U$

AND

- $\lambda \underset{\sim}{u} \in U$ where $\lambda \in \mathbb{R}$

Iff U satisfies both conditions, it is a subspace of V.

$\therefore$ examine the condition $\lambda \underset{\sim}{u} \in U$

U is a subspace of V and so this condition must hold for all u.

$\therefore$ consider the case when $\lambda = 0$

$$\lambda \underset{\sim}{u} \in U, \quad \lambda \underset{\sim}{u} = 0 \times \underset{\sim}{u}$$
$$= \underset{\sim}{0} \quad \text{by definition}$$

$\therefore \underset{\sim}{0} \in U$ as $\lambda \underset{\sim}{u} \in U$

$\therefore$ For any subspace, U of V, the zero vector is an element of the set U.

Q2a) Let $p(x) = a + bx + cx^2$ $\quad a, b, c \in \mathbb{R}$

and $p_0(x) = a_0 + b_0 x + c_0 x^2$ $\quad a_0, b_0, c_0 \in \mathbb{R}$

Using polynomial arithmetic that behaves like vector addition as outlined:

Thus:

$$p(x) + p_0(x) = a + bx + cx^2 + a_0 + b_0 x + c_0 x^2$$
$$= (a + a_0) + (b + b_0)x + (c + c_0)x^2$$

Now $p(x) + p_0(x) = p(x) \Rightarrow$ RTP

shows the existence of the zero polynomial that behaves like the zero vector.

For this to occur, we need:

$a + a_0 = a$, $b + b_0 = b$, $c + c_0 = c$

$\therefore a_0 = b_0 = c_0 = 0$

$\therefore p_0(x) = 0 + 0x + 0x^2$
$= 0$

$\therefore p(x) + p_0(x) = p(x)$

which behaves like the zero vector, where $\underset{\sim}{v} + \underset{\sim}{0} = \underset{\sim}{v}$, and satisfies AXIOM 3

Using similar logic, we will determine the existence of a polynomial that satisfies axiom 4.

Let $p(x) = a + bx + cx^2$ $\quad a, b, c \in \mathbb{R}$

$p'(x) = a' + b'x + c'x^2$ $\quad a', b', c' \in \mathbb{R}$

Using polynomial arithmetic that behaves like vector arithmetic as outlined:

$$p(x) + p'(x) = a + bx + cx^2 + a' + b'x + c'x^2$$
$$= (a + a') + (b + b')x + (c + c')x^2$$

RTP $p(x) + p'(x) = 0$

For this to occur, we require:

$(a + a') = 0$, $(b + b') = 0$, $(c + c') = 0$

$\therefore a' = -a$, $b' = -b$, $c' = -c$

$\therefore p'(x) = -a - bx - cx^2$

$\therefore p(x) + p'(x) = 0$

which behaves like the additive inverse, and satisfies Axiom 4

Q2.b) Axiom 1 $\Rightarrow$ Commutative Addition

The addition of ordered pairs is defined as:

$$(a,b)+(c,d)=(a+c,b+d)$$

RTP $(a,b)+(c,d)=(c,d)+(a,b)$

To prove the law of commutivity:

$(a,b)+(c,d)=(a+c,b+d)$ where $a,b,c,d \in \mathbb{R}$

$(c,d)+(a,b)=(c+a,d+b)$

$=(a+c,b+d)$

$=(a,b)+(c,d)$ = LHS

$\therefore$ Axiom 1 holds

Axiom 2 $\Rightarrow$ Associative Addition

RTP $(a,b)+\left[(c,d)+(e,f)\right]=\left[(a,b)+(c,d)\right]+(e,f)$ where $a,b,c,d,e,f \in \mathbb{R}$

LHS $=(a,b)+\left[(c,d)+(e,f)\right]$

$=(a,b)+((c+e),(d+f))$

$=(a+(c+e),b+(d+f))$

$=(a+c+e,b+d+f)$

RHS $=\left[(a,b)+(c,d)\right]+(e,f)$

$=((a+c),(b+d))+(e,f)$

$=((a+c)+e,(b+d)+f)$

$=(a+c+e,b+d+f)$

= LHS

$\therefore (a,b)+\left[(c,d)+(e,f)\right]=\left[(a,b)+(c,d)\right]+(e,f)$

$\therefore$ Axiom 2 Holds

Axiom 7 $\Rightarrow$ Distributive II

RTP $(\lambda+\mu)(a,b)=\lambda(a,b)+\mu(a,b)$ where $a,b,\mu,\lambda \in \mathbb{R}$

~~RH~~

RHS $=\lambda(a,b)+\mu(a,b)$

$=(\lambda a,\lambda b)+(\mu a,\mu b)$ [scalar multiplication definition]

$=(\lambda a+\mu a,\lambda b+\mu b)$

$=((\lambda+\mu)a,(\lambda+\mu)b)$

$=(\lambda+\mu)(a,b)$

= LHS

$\therefore$ Axiom 7 holds

Q2.c) The addition of ordered pairs is defined as:

$$(a,b)+(c,d) = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$

Axiom 1 $\Rightarrow$ Commutative Addition

RTP $(a,b)+(c,d) = (c,d)+(a,b)$ where $a,b,c,d \in \mathbb{R}$

$$LHS = (a,b)+(c,d)$$
$$= \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$

$$RHS = (c,d)+(a,b)$$
$$= \left(\frac{c+a}{2}, \frac{d+b}{2}\right)$$
$$= \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$
$$= LHS$$

$\therefore$ Axiom 1 Holds

Axiom 2 $\Rightarrow$ Associative Addition

RTP $[(a,b)+(c,d)]+(e,f) = (a,b)+[(c,d)+(e,f)]$

where $a,b,c,d,e,f \in \mathbb{R}$

$$LHS = [(a,b)+(c,d)]+(e,f)$$
$$= \left(\frac{a+c}{2}, \frac{b+d}{2}\right)+(e,f)$$
$$= \left(\frac{\frac{a+c}{2}+e}{2}, \frac{\frac{b+d}{2}+f}{2}\right)$$
$$= \left(\frac{a+c}{4}+\frac{e}{2}, \frac{b+d}{4}+\frac{f}{2}\right)$$

$$RHS = (a,b)+[(c,d)+(e,f)]$$
$$= (a,b)+\left(\frac{c+e}{2}, \frac{d+f}{2}\right)$$
$$= \left(\frac{a+\frac{c+e}{2}}{2}, \frac{b+\frac{d+f}{2}}{2}\right)$$
$$= \left(\frac{a}{2}+\frac{c+e}{4}, \frac{b}{2}+\frac{d+f}{4}\right)$$

$$\neq LHS$$

$\therefore$ Axiom 2 does not hold

Axiom 6 $\Rightarrow$ Distributive I

RTP $\lambda((a,b)+(c,d)) = \lambda(a,b)+\lambda(c,d)$
where $a,b,c,d,\lambda \in \mathbb{R}$

$$LHS = \lambda((a,b)+(c,d))$$
$$= \lambda\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$
$$= \left(\frac{\lambda(a+c)}{2}, \frac{\lambda(b+d)}{2}\right) \quad \text{[scalar multiplication definition]}$$

$$RHS = \lambda(a,b)+\lambda(c,d)$$
$$=(\lambda a, \lambda b)+(\lambda c, \lambda d) \quad \text{[scalar multiplication definition]}$$
$$= \left(\frac{\lambda a + \lambda c}{2}, \frac{\lambda b+\lambda d}{2}\right)$$
$$= \left(\frac{\lambda(a+c)}{2}, \frac{\lambda(b+d)}{2}\right)$$
$$= LHS$$

$\therefore$ Axiom 6 holds

Axiom 7 $\Rightarrow$ Distributive II

RTP $(\lambda+\mu)(a,b) = \lambda(a,b)+\mu(a,b)$ where $a,b,\lambda,\mu \in \mathbb{R}$

$$RHS = \lambda(a,b)+\mu(a,b)$$
$$= (\lambda a, \lambda b)+(\mu a, \mu b) \quad \text{[definition of scalar multiplication]}$$
$$= \left(\frac{\lambda a+\mu a}{2}, \frac{\lambda b+\mu b}{2}\right)$$
$$= \left(\frac{(\lambda+\mu)a}{2}, \frac{(\lambda+\mu)b}{2}\right)$$
$$= \frac{(\lambda+\mu)}{2}(a,b)$$
$$\neq LHS$$

$\therefore$ Axiom 7 does not hold