The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1	**Longer Solutions to Selected Exercises for Week 8**	2017

11. The matrix equation $A0 = 0A = 0$ makes sense by interpreting the symbol 0 as an abbreviation for zero matrices of compatible dimensions. In particular $A0 = 0$ is an abbreviation for an equation

$$A0_{n\times p} \;=\; 0_{m\times p}$$

for some p, whilst $0A = 0$ is an abbreviation for an equation

$$0_{q\times m}A \;=\; 0_{q\times n}$$

for some q. To assert that these are both equal now forces $p = n$ and $q = m$, so that the original $A0 = 0A = 0$ becomes an abbreviation for

$$A0_{n\times n} \;=\; 0_{m\times m}A \;=\; 0_{m\times n}\ .$$

The assertion $0 \neq 0 \neq 0$ now abbreviates the inequalities

$$0_{n\times n} \;\neq\; 0_{m\times m} \;\neq\; 0_{m\times n}\ ,$$

and the apparent paradox is resolved.

12. Using familiar trigonometric identities the product becomes

$$\begin{aligned} &\begin{bmatrix} r\cos\alpha & -r\sin\alpha \\ r\sin\alpha & r\cos\alpha \end{bmatrix}\begin{bmatrix} s\cos\beta & -s\sin\beta \\ s\sin\beta & s\cos\beta \end{bmatrix} \\ =\ &\begin{bmatrix} \big(r\cos\alpha\big)\big(s\cos\beta\big) - \big(r\sin\alpha\big)\big(s\sin\beta\big) & -\big(r\cos\alpha\big)\big(s\sin\beta\big) - \big(r\sin\alpha\big)\big(s\cos\beta\big) \\ \big(r\sin\alpha\big)\big(s\cos\beta\big) + \big(r\cos\alpha\big)\big(s\sin\beta\big) & -\big(r\sin\alpha\big)\big(s\sin\beta\big) + \big(r\cos\alpha\big)\big(s\cos\beta\big) \end{bmatrix} \\ =\ &\begin{bmatrix} rs\cos(\alpha+\beta) & -rs\sin(\alpha+\beta) \\ rs\sin(\alpha+\beta) & rs\cos(\alpha+\beta) \end{bmatrix} \end{aligned}$$

If we identify the complex number $r\,\mathrm{cis}\ \theta$ (in polar form) with the matrix

$$\begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$$

then the above equation just becomes usual multiplication of complex numbers:

$$(r\,\mathrm{cis}\ \alpha)(s\,\mathrm{cis}\ \beta) \;=\; rs\,\mathrm{cis}\ (\alpha+\beta)\ .$$

But $r\,\mathrm{cis}\ \theta = x + iy$ where $x = r\cos\theta$ and $y = r\sin\theta$, which then gets identified with the matrix

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

so that the usual addition of complex numbers

$$(x_1 + iy_1) + (x_2 + iy_2) \;=\; (x_1 + x_2) + i(y_1 + y_2)$$

corresponds to addition of matrices

$$\begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 & -(y_1+y_2) \\ y_1+y_2 & x_1+x_2 \end{bmatrix} .$$

Thus there is a copy of complex number arithmetic within the arithmetic of 2×2 real matrices, where the real number 1 corresponds to the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the imaginary number i corresponds to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

13. (i) This is the familiar associative law of matrix multiplication, which is always true.

(ii) This is false. For example take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = AB \neq BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

(iii) This is false. For example take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, so that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (AB)^2 \neq A^2B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} .$$

(iv) This is one of the familiar distributive laws, which is always true.

(v) This is always true and follows quickly from properties involving scalars.

(vi) This is always true and follows quickly from the distributive law and properties involving scalars.

(vii) This is false. For example take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so that

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = (A+B)^2 \neq A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} .$$

(viii) This is false. For example take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so that

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = (A+B)(A-B) \neq A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

(ix) This is always true because $AI = IA = A$, so that

$$(A+I)^2 = A^2 + AI + IA + I^2 = A^2 + 2A + I .$$

(x) This is always true because $AI = IA = A$, so that

$$(A+I)(A-I) = A^2 - AI + IA - I^2 = A^2 - A + A - I = A^2 - I .$$

(xi) This is always true by properties of transpose because

$$(A^TB^T)^T = (B^T)^T(A^T)^T = BA .$$

14. Consider an $m \times n$ matrix A, an $n \times p$ matrix B and a $p \times q$ matrix C. Denote the (i,j)-entry of a matrix X by X_{ij}. Then, for $i = 1$ to m and $\ell = 1$ to q,

$$\begin{aligned}
\big((AB)C\big)_{i\ell} &= \sum_{k=1}^{p}(AB)_{ik}C_{k\ell} \;=\; \sum_{k=1}^{p}\Big(\sum_{j=1}^{n}A_{ij}B_{jk}\Big)C_{k\ell} \\
&= \sum_{k=1}^{p}\sum_{j=1}^{n}A_{ij}B_{jk}C_{k\ell} \;=\; \sum_{j=1}^{n}\sum_{k=1}^{p}A_{ij}B_{jk}C_{k\ell} \\
&= \sum_{j=1}^{n}A_{ij}\Big(\sum_{k=1}^{p}B_{jk}C_{k\ell}\Big) \;=\; \sum_{j=1}^{n}A_{ij}(BC)_{j\ell} \\
&= \big(A(BC)\big)_{i\ell}\,.
\end{aligned}$$

Thus $(AB)C = A(BC)$, verifying the associative law for matrix multiplication.

15. (i) Observe that

$$\begin{aligned}
M^2 - 2M + I &= \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} - 2\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ -8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

so that $M^2 = 2M - I$.

(ii) By part (i),

$$\begin{aligned}
M^3 &= M^2M \;=\; (2M - I)M \;=\; 2M^2 - IM \;=\; 2(2M - I) - M \\
&= 4M - 2I - M \;=\; 3M - 2I\,.
\end{aligned}$$

We conjecture that, for any positive integer n,

$$M^n \;=\; nM - (n-1)I\,.$$

Certainly the conjecture is true for $n = 1$, which starts an induction. Suppose the conjecture is true for $n = k$. We verify that it is also true for $n = k + 1$:

$$\begin{aligned}
M^{k+1} &= M^kM \;=\; (kM - (k-1)I)M \;=\; kM^2 - (k-1)IM \\
&= k(2M - I) - (k-1)M \;=\; (k+1)M - kI\,.
\end{aligned}$$

The conjecture follows for all n by mathematical induction.

(iii) $M^5 = 5M - 4I = \begin{bmatrix} 11 & -5 \\ 20 & -9 \end{bmatrix}$, $M^{10} = 10M - 9I = \begin{bmatrix} 21 & -10 \\ 40 & -19 \end{bmatrix}$,

$M^{100} = 100M - 99I = \begin{bmatrix} 201 & -100 \\ 400 & -199 \end{bmatrix}$.

16. (i) Direct substitution produces

$$c \;=\; 3u - 5v \;=\; 3(2x + 3y) - 5(x - 4y) \;=\; x + 29y$$

and

$$d \;=\; 2u + 3v \;=\; 2(2x + 3y) + 3(x - 4y) \;=\; 7x - 6y\,.$$

(ii) The two sets of equations become

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 3 & -5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix},$$

so that

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 29 \\ 7 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+29y \\ 7x-6y \end{bmatrix},$$

yielding $c = x + 29y$ and $d = 7x - 6y$, as before.

17. (i)
$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 15 \\ 4 & -1 & 2 & 29 \\ 0 & 6 & -1 & -10 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 15 \\ 0 & -9 & -10 & -31 \\ 0 & 6 & -1 & -10 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 15 \\ 0 & -18 & -20 & -62 \\ 0 & 18 & -3 & -30 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 15 \\ 0 & -18 & -20 & -62 \\ 0 & 0 & -23 & -92 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 15 \\ 0 & 9 & 10 & 31 \\ 0 & 0 & 1 & 4 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 3 \\ 0 & 9 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right],$$
so that $x = 5$, $y = -1$ and $z = 4$.

(ii)
$$\begin{bmatrix} 2 & -3 & 3 \\ 4 & 9 & -4 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 3 \\ 0 & 15 & -10 \\ 0 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$
so that $x = -t/2$, $y = 2t/3$, $z = t$.

19. Multiplying out, the equation becomes

$$\begin{bmatrix} 3x & -2y \\ 3z & -2w \end{bmatrix} = \begin{bmatrix} -x+4z & -y+4w \\ x+2z & y+2w \end{bmatrix}$$

which quickly yields $x = z$ and $y = -4w$, which can be expressed as a parametric solution:

$$x = s\,,\quad y = -4t\,,\quad z = s\,,\quad w = t \quad (s, t \in \mathbb{R})\,.$$

20. (i) Put $X = \begin{bmatrix} a & b \end{bmatrix}$ and $Y = \begin{bmatrix} c \\ d \end{bmatrix}$, so $\begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix} = YX = \begin{bmatrix} ca & cb \\ da & db \end{bmatrix}$, yielding $XY = ac + bd = -2 + 3 = 1$.

(ii) Put $X = \begin{bmatrix} a & b & c \end{bmatrix}$ and $Y = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$, so

$$\begin{bmatrix} 3 & -3 & 6 \\ 4 & -4 & 8 \\ -2 & 2 & -4 \end{bmatrix} = YX = \begin{bmatrix} da & db & dc \\ ea & eb & ec \\ fa & fb & fc \end{bmatrix},$$

yielding $XY = ad + be + cf = 3 - 4 - 4 = -5$.

21. (i) Matrices A and B commute if and only if

$$\begin{bmatrix} a & b \\ -c & -d \end{bmatrix} = AB = BA = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix},$$

which occurs if and only if $b = -b$ and $c = -c$, that is, $b = c = 0$.

(ii) Matrices A and B commute if and only if

$$\begin{bmatrix} 7c & 7d \\ 7a & 7b \end{bmatrix} = AB = BA = \begin{bmatrix} 7b & 7a \\ 7d & 7c \end{bmatrix},$$

which occurs if and only if $a = d$ and $b = c$.

(iii) Matrices A and B commute if and only if

$$\begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix} = AB = BA = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix},$$

which occurs if and only if $a = d$ and $c = 0$.

22. Consider an $m \times n$ matrix A and an $n \times p$ matrix B. Then AB is $m \times p$ and $(AB)^T$ is $p \times m$. For $k = 1$ to p and $i = 1$ to m, we have

$$\left((AB)^T\right)_{ki} = (AB)_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk} = \sum_{j=1}^{n} B_{jk}A_{ij} = \sum_{j=1}^{n} (B^T)_{kj}(A^T)_{ji} = \left(B^T A^T\right)_{ki},$$

which proves $(AB)^T = B^T A^T$.

23. (i) We claim that

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & nk \\ 0 & 1 \end{bmatrix}$$

for all integers $n \geq 1$. For $n = 1$, the statement holds trivially, which starts an induction. By an inductive hypothesis, for $n > 1$,

$$\begin{aligned} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^n &= \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (n-1)k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & k+(n-1)k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & nk \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

which establishes the inductive step, and the proof is complete.

(ii) We claim that

$$\begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix}^n = \begin{bmatrix} k^n & nk^{n-1} \\ 0 & k^n \end{bmatrix}$$

for all integers $n \geq 1$. For $n = 1$, the statement holds trivially, which starts an induction. By an inductive hypothesis, for $n > 1$,

$$\begin{aligned} \begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix}^n &= \begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix}^{n-1} \begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix} = \begin{bmatrix} k^{n-1} & (n-1)k^{n-2} \\ 0 & k^{n-1} \end{bmatrix} \begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix} \\ &= \begin{bmatrix} k^n & k^{n-1}+(n-1)k^{n-1} \\ 0 & k^n \end{bmatrix} = \begin{bmatrix} k^n & nk^{n-1} \\ 0 & k^n \end{bmatrix}, \end{aligned}$$

which establishes the inductive step, and the proof is complete.

(iii) We claim that

$$\begin{bmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix}^n = \begin{bmatrix} k^n & nk^{n-1} & \frac{n(n-1)}{2}k^{n-2} \\ 0 & k^n & nk^{n-1} \\ 0 & 0 & k^n \end{bmatrix}$$

for all integers $n \geq 1$. For $n = 1$, the statement holds trivially, which starts an induction. By an inductive hypothesis, for $n > 1$,

$$\begin{aligned}
\begin{bmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix}^n &= \begin{bmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix}^{n-1} \begin{bmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix} \\
&= \begin{bmatrix} k^{n-1} & (n-1)k^{n-2} & \frac{(n-1)(n-2)}{2}k^{n-3} \\ 0 & k^{n-1} & (n-1)k^{n-2} \\ 0 & 0 & k^{n-1} \end{bmatrix} \begin{bmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix} \\
&= \begin{bmatrix} k^n & k^{n-1} + (n-1)k^{n-1} & (n-1)k^{n-2} + \frac{(n-1)(n-2)}{2}k^{n-2} \\ 0 & k^n & k^{n-1} + (n-1)k^{n-1} \\ 0 & 0 & k^n \end{bmatrix} \\
&= \begin{bmatrix} k^n & nk^{n-1} & \frac{n(n-1)}{2}k^{n-2} \\ 0 & k^n & nk^{n-1} \\ 0 & 0 & k^n \end{bmatrix},
\end{aligned}$$

after simplifying, which establishes the inductive step, and the proof is complete.

24. Suppose that λ_1 and λ_2 are different scalars. Since $\lambda_1 - \lambda_2 \neq 0$, so we may form the scalar $\dfrac{1}{\lambda_1 - \lambda_2}$. Put

$$\mathbf{s_1} = \mathbf{x}_1 + \lambda_1(\mathbf{x}_1 - \mathbf{x}_2) \qquad \text{and} \qquad \mathbf{s_2} = \mathbf{x}_1 + \lambda_2(\mathbf{x}_1 - \mathbf{x}_2)\,.$$

If $\mathbf{s_1} = \mathbf{s_2}$ then $\lambda_1(\mathbf{x}_1 - \mathbf{x}_2) = \lambda_2(\mathbf{x}_1 - \mathbf{x}_2)$, so that, after rearranging,

$$(\lambda_1 - \lambda_2)(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}\,,$$

yielding

$$\mathbf{x}_1 - \mathbf{x}_2 = \frac{1}{\lambda_1 - \lambda_2}\mathbf{0} = \mathbf{0}\,,$$

and finally $\mathbf{x}_1 = \mathbf{x}_2$, contradicting that $\mathbf{x}_1$ and $\mathbf{x_2}$ are different vectors. Hence $\mathbf{s_1} \neq \mathbf{s_2}$. This proves that the infinitely many different scalars λ produce infinitely many vectors $\mathbf{s}$. This has a natural geometric interpretation: if $\mathbf{x}_1$ and $\mathbf{x}_2$ are position vectors of points P and Q in space, then, as λ varies over $\mathbb{R}$, the vector $\mathbf{s}$ varies over the position vectors of points on the line that passes through P and Q.

25. If A and B are $n \times n$ matrices such that $AB - BA = I$ then

$$\begin{aligned}
n &= \sum_{i=1}^{n} I_{ii} = \sum_{i=1}^{n}(AB - BA)_{ii} = \sum_{i=1}^{n}(AB)_{ii} - \sum_{i=1}^{n}(BA)_{ii} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} - \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}A_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} - \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ji}B_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} - \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} = 0\,,
\end{aligned}$$

which is nonsense, since $n \neq 0$.