THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 3

MATH1901: Differential Calculus (Advanced) Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Roots of complex numbers;
- ☐ Complex exponential function $e^z = e^x(\cos y + i \sin y)$;
- ☐ Functions of a complex variable;
- ☐ Sketching the image of a region.

Outcomes

After completing this tutorial you should

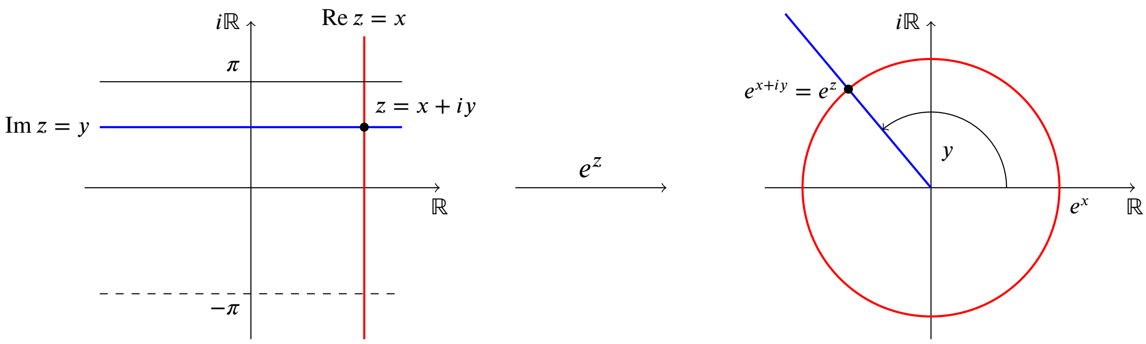
- ☐ find the roots of a complex number;
- ☐ understand the definition of the complex exponential function;
- ☐ manipulate the complex exponential function;
- ☐ solve equations involving the complex exponential function;
- ☐ understand what is meant by a function of a complex variable;
- ☐ sketch the image of a region in the complex plane under a function.

Summary of essential material

The complex exponential function. For any complex number $z = x + iy$, $x, y \in \mathbb{R}$ we define the *complex exponential function* by

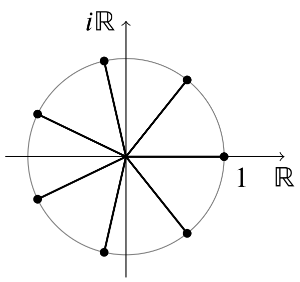
$$e^z := e^x(\cos y + i \sin y),$$

The sketch below shows the mapping properties of the exponential function.



It is an invertible map from $\{z \in \mathbb{C} \mid -\pi < \operatorname{Im} z \leq \pi\}$ onto $\mathbb{C} \setminus \{0\}$. The above mapping properties together with the argument modulus form are useful to determine the images of sets under certain complex functions.

Roots of complex numbers. Given $n \in \mathbb{N}$, the *n-th roots* of a complex number α are the solutions to the equation $z^n = \alpha$. If $\alpha = 1$ we talk about the *roots of unity*. By De Moivre's theorem the roots of unity are equally spaced on the unit circle:



By De Moivre's theorem we have $\left(e^{2\pi ki/n}\right)^n = e^{2\pi ki} = 1$, so $e^{2\pi ki/n}$, $k = 0, \dots, n-1$, are the roots of unity. If $z = re^{i\theta}$ is given in polar form, then again by de Moivre's theorem the n-th roots of z are given by

$$\alpha_k = r^{1/n} e^{\theta i/n + 2\pi ki/n} \qquad k = 0, 1, \dots, n-1.$$

Again the n roots lie on a circel. Its radius in $r^{1/n}$ and they are equally spaced starting from $r^{1/n}e^{i\theta/n}$.

Hints for determining images. Let $f : \mathbb{C} \to \mathbb{C}$ be a function, and let $D \subseteq \mathbb{C}$. The *image of D under f* is

$$\mathrm{im}(D) = \{f(z) \mid z \in D\}.$$

If you need to determine the image of a set under a complex map there are several approaches:

- Write $z = x + iy$ with $x, y \in \mathbb{R}$, then do a computation. This is sometimes useful, but a lot of the time inefficient. The method should only be applied as a last step after using some of the techniques below.
- Use that $z\bar{z} = |z|^2$
- Use geometric properties, in particular $|z_1 - z_2|$ is the distance between z_1 and z_2 on the complex plane.
- Write $z = re^{i\theta}$ in modulus-argument form, in particular if powers of z are involved.

Questions to complete during the tutorial

1. Express the following complex numbers in Cartesian form:

(a) $e^{2\pi i/3}$

Solution: $e^{2\pi i/3} = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$

(b) $e^{i\frac{\pi}{12}}e^{i\frac{2\pi}{3}}e^{i\frac{\pi}{4}}$

Solution: $e^{i\frac{\pi}{12}}e^{i\frac{2\pi}{3}}e^{i\frac{\pi}{4}} = e^{i(\frac{\pi}{12}+\frac{2\pi}{3}+\frac{\pi}{4})} = e^{i\pi} = -1 + 0i.$

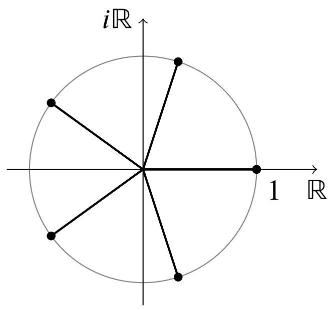
2. Solve the following equations and plot the solutions in the complex plane:

(a) $z^5 = 1$

Solution: Write $z = r(\cos\theta + i\sin\theta)$ and write 1 in polar form, $1 = 1(\cos 0 + i\sin 0)$. Then $z^5 = 1 \iff r^5(\cos(5\theta) + i\sin(5\theta)) = 1(\cos 0 + i\sin 0)$. Equating moduli, we find that $r^5 = 1$, so $r = 1$. Comparing arguments, we find that $\theta = 2k\pi/5$, for some $k \in \mathbb{Z}$. As k takes the five values $0, 1, 2, 3, 4$, we get all five 5th roots of unity, namely

$$z_0 = 1, \quad z_1 = e^{i2\pi/5}, \quad z_2 = e^{i4\pi/5}, \quad z_3 = e^{i6\pi/5}, \quad z_4 = e^{i8\pi/5}.$$

All solutions have modulus 1 and so lie on a unit circle, centred at the origin. The solutions (represented by the dots on the circle) are equally spaced at angles of $2\pi/5$.

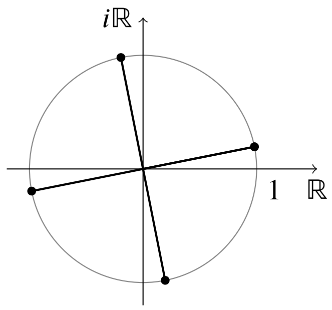


(b) $z^4 = 8\sqrt{2}(1 + i)$

Solution: $z^4 = r^4e^{i4\theta} = 8\sqrt{2} + 8\sqrt{2}i = 16e^{i\pi/4}$, and so

$$z = 2e^{i\pi/16}, \quad 2e^{i9\pi/16}, \quad 2e^{i-15\pi/16}, \quad 2e^{i-7\pi/16}.$$

The solutions are illustrated below.



3. Find all solutions of the following equations:

(a) $e^z = i$

Solution: Let $z = x + iy$. Recall that by definition of the complex exponential,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x e^{iy}.$$

So the equation we want to solve is

$$e^x e^{iy} = i = 1e^{i\pi/2}.$$

Equating moduli, we find that $e^x = 1$, and so $x = 0$. Comparing arguments, we find that

$$y = \arg(e^z) = \arg(i) = \pi/2 + 2k\pi,$$

for some $k \in \mathbb{Z}$. Hence there are infinitely many solutions:

$$z = i\left(\frac{\pi}{2} + 2k\pi\right) \qquad \text{with} \quad k \in \mathbb{Z}.$$

(b) $e^z = -10$

Solution: There are infinitely many solutions:

$$z = \ln 10 + i\,(\pi + 2k\pi) \qquad \text{with} \quad k \in \mathbb{Z}.$$

(c) $e^z = -1 - i\sqrt{3}$

Solution: There are infinitely many solutions:

$$z = \ln 2 + i\left(-\frac{2\pi}{3} + 2k\pi\right) \qquad \text{with} \quad k \in \mathbb{Z}.$$

(d) $e^{2z} = -i$

Solution: There are infinitely many solutions:

$$z = i\left(-\frac{\pi}{4} + k\pi\right) \qquad \text{with} \quad k \in \mathbb{Z}.$$

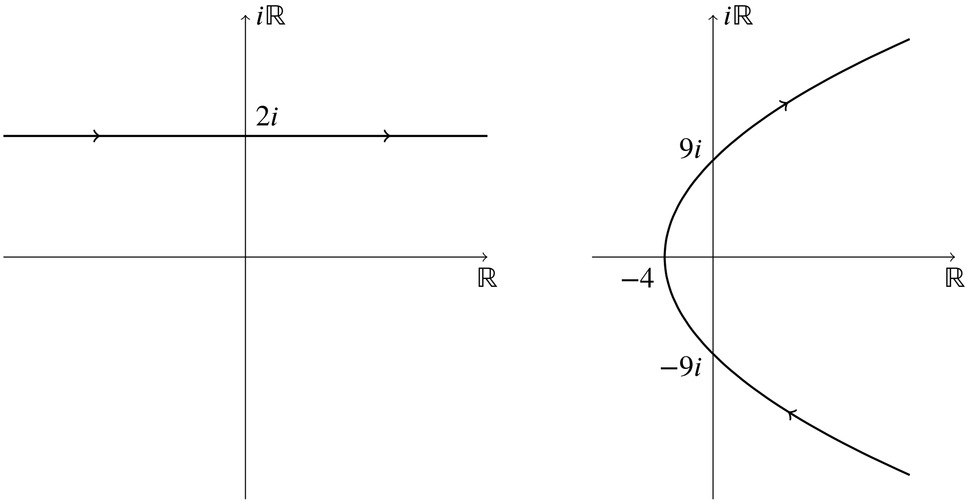
4. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $z \mapsto z^2$. Sketch the following sets, and then sketch their images under the function f.

(a) $A = \{z \in \mathbb{C} \mid \operatorname{Im}(z) = 2\}$

Solution: The set A is a straight line in the z-plane, parallel to the real axis, and cutting the imaginary axis at $2i$. Points $z \in A$ are of the form $z = x + 2i$, where $x \in \mathbb{R}$. Let $w = z^2 = (x+2i)^2 = (x^2-4) + 4xi$. Writing $w = u + iv$ we obtain $u = x^2 - 4$ and $v = 4x$. We eliminate x from these two equations to obtain

$$u = \left(\frac{v}{4}\right)^2 - 4 = \frac{v^2}{16} - 4$$

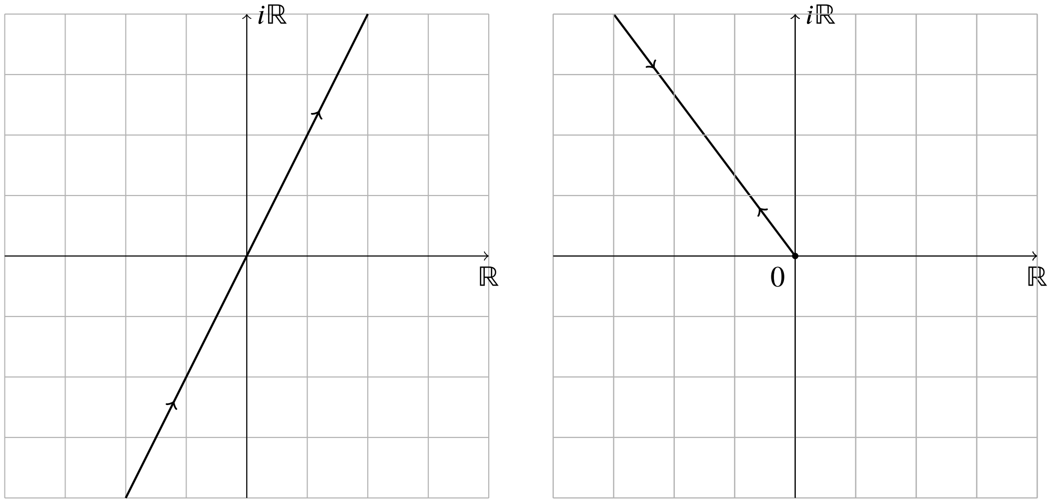
Thus the image of A is a parabola in the w-plane, as illustrated. The parabola is traced out so that if the arrows on A are followed, then the arrows on the parabola are followed.



z-plane $\qquad$ w-plane

(b) $B = \{z \in \mathbb{C} \mid \operatorname{Im}(z) = 2\operatorname{Re}(z)\}$

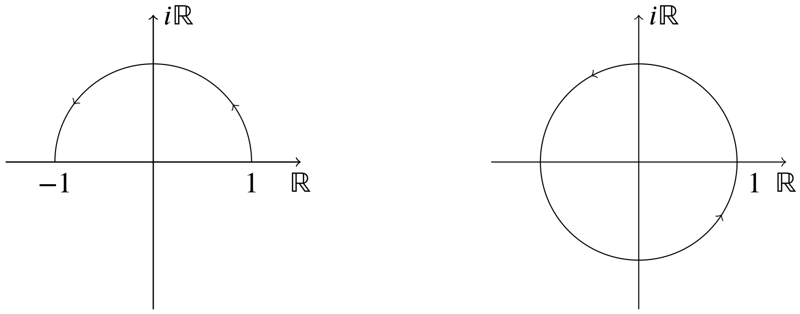
Solution: The points $z = x + iy \in B$ have $y = 2x$. Thus $z = x + 2xi$, and so $z = x(1 + 2i)$. Thus the set B is a straight line in the z-plane. Let $w = z^2 = (x + 2xi)^2 = x^2(-3 + 4i)$. All values of w have the same principal argument, and thus lie on a half line as shown below. As z moves along the arrows shown on B the corresponding image point moves from infinity towards zero, then returns to infinity.



z-plane $\qquad$ w-plane

(c) $C = \{z \in \mathbb{C} \mid |z| = 1 \text{ and } \operatorname{Im}(z) \geq 0\}$

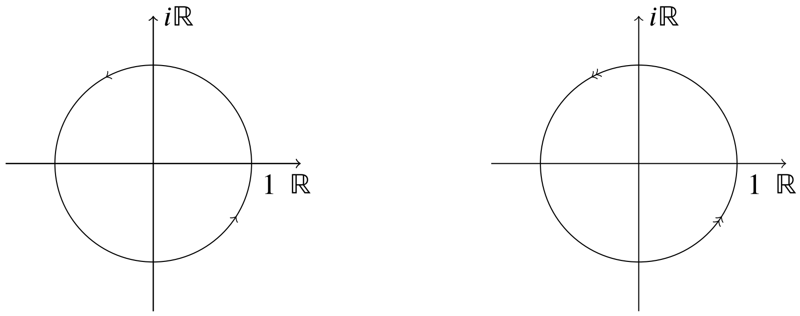
Solution: The set C is the top half of the unit circle in the z-plane. If $z \in C$ then $z = e^{i\theta}$ with $0 \leq \theta \leq \pi$, and thus $w = z^2 = e^{2i\theta}$. As $0 \leq \theta \leq \pi$, the argument 2θ takes all values between 0 and 2π. We conclude that the image of C is the unit circle in the w-plane. As z moves anticlockwise from 1 to -1, w moves anticlockwise from along the full circle starting from 1 and ending at 1.



z-plane $\qquad$ w-plane

(d) $D = \{z \in \mathbb{C} \mid |z| = 1\}$

Solution: Now D is the full unit circle (in the z-plane), and the image of D is again the full unit circle in the w-plane. This image circle is 'traced out' twice: As z moves anticlockwise around the circle in the z-plane from 1 back to 1, the image point $w = z^2$ moves around the circle in the w-plane twice.



z-plane $\qquad$ w-plane

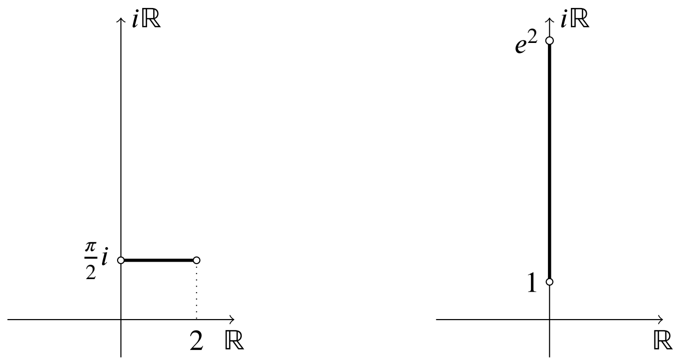
5. Sketch the following sets and their images under the function $z \mapsto e^z$.

(a) $A = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < 2 \text{ and } \mathrm{Im}(z) = \frac{\pi}{2}\}$

Solution: Let $w = e^z$. As $z = x + i\pi/2$ we have

$$w = e^x e^{i\pi/2} = e^x e^{i\pi/2} = ie^x.$$

Since $0 < x < 2$ we have $1 < e^x < e^2$, and so the image of A is as shown below.

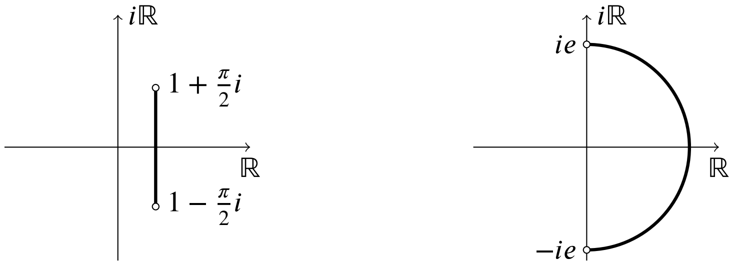


z-plane $\qquad$ w-plane

w-plane

(b) $B = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = 1 \text{ and } |\mathrm{Im}(z)| < \pi/2\}$

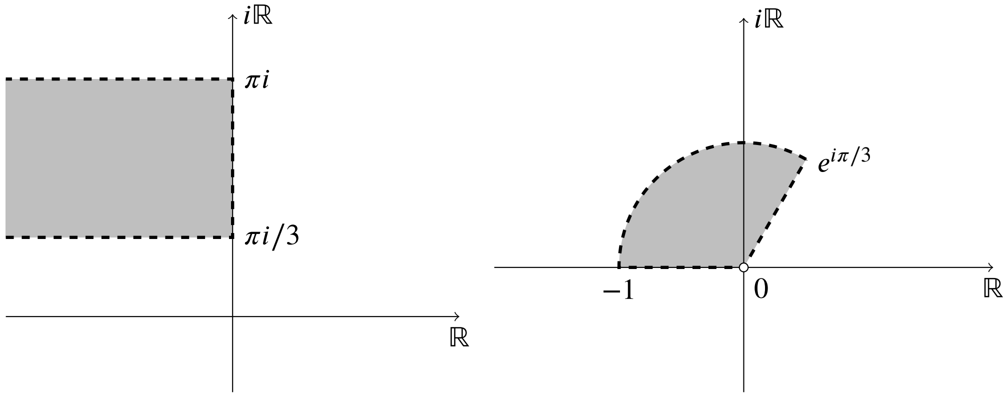
Solution: If $z = x + iy \in B$ then $|w| = |e^z| = e^x = e$ (constant) and $\mathrm{Arg}(w) = y$ lies between $-\pi/2$ and $\pi/2$. Hence the set of image points is a semi-circle.



z-plane $\quad\quad$ w-plane

(c) $C = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0 \text{ and } \pi/3 < \mathrm{Im}(z) < \pi\}$

Solution: Let $z = x + iy \in C$. Since $x < 0$ and $|w| = e^x$, we have $0 < |w| < 1$. Also, $\pi/3 < \mathrm{Arg}(w) < \pi$. Thus the image set is the interior of a sector of the unit circle, as shown (all boundaries are 'dotted', and so are not included in the set).

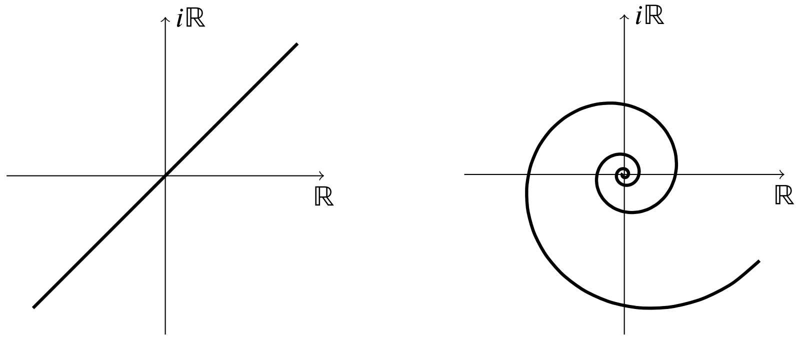


z-plane $\quad\quad$ w-plane

w-plane

(d) $D = \{z = (1+i)t \mid t \in \mathbb{R}\}$

Solution: If $z = (1+i)t$ then $w = e^z = e^t(\cos t + i \sin t)$. Thus, writing $w = u + iv$, we have $u = e^t \cos t$ and $v = e^t \sin t$. These are the parametric equations for a *logarithmic spiral*. Note that the modulus of w is e^t, and so as t gets large $|w|$ gets large, and as $t \to -\infty$ we have $|w| \to 0$.



z-plane $\quad\quad$ w-plane

w-plane

6. Find all solutions of the equation $e^{2z} - (1+3i)e^z + i - 2 = 0$.

Solution: Using the quadratic formula we find that this equation is equivalent to

$$e^z = \frac{(1+3i) \pm \sqrt{(1+3i)^2 - 4(i-2)}}{2},$$

which simplifies to

$$e^z = \frac{1 + 3i \pm \sqrt{2i}}{2}.$$

As $(1+i)^2 = 2i$, we see that the two numbers whose square is $2i$ are $\pm(1+i)$, so

$$e^z = \frac{1+3i \pm (1+i)}{2} = 1+2i \text{ or } i.$$

In polar form, $1+2i = \sqrt{5}e^{i\theta}$, where $\theta = \tan^{-1}(2) \approx 1.11$ radians, and $i = e^{i\pi/2}$. If $z = x+iy$ then

$$e^z = e^x e^{iy} = \sqrt{5}e^{i\theta} \text{ or } e^{i\pi/2}.$$

Hence $x = \frac{1}{2}\ln 5$ and $y = \theta + 2k\pi$, or $x = 0$ and $y = \pi/2 + 2k\pi$, for $k \in \mathbb{Z}$. There are infinitely many solutions, of the form

$$z = \frac{1}{2}\ln 5 + i(\tan^{-1}(2) + 2k\pi) \quad \text{or} \quad z = i(\pi/2 + 2k\pi).$$

7. (a) Use the definition of the complex exponential function to show that for all $n \in \mathbb{N}$ and all $\theta \in \mathbb{R}$

$$\sum_{k=-n}^{n} \left(e^{i\theta}\right)^k = 1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos n\theta.$$

Solution: Since $(e^{i\theta})^k = e^{ik\theta} = \cos k\theta + i\sin k\theta$ we see that

$$\begin{aligned}
\sum_{k=-n}^{n} \left(e^{i\theta}\right)^k &= 1 + \sum_{k=1}^{n} e^{-ik\theta} + \sum_{k=1}^{n} e^{ik\theta} \\
&= 1 + \sum_{k=1}^{n}(\cos k\theta - i\sin k\theta) + \sum_{k=1}^{n}(\cos k\theta + i\sin k\theta) \\
&= 1 + \sum_{k=1}^{n} \cos k\theta + \sum_{k=1}^{n} \cos k\theta \\
&= 1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos n\theta
\end{aligned}$$

since all the sine terms cancel out.

(b) Hence, use the formula for a geometric series to show that

$$1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos n\theta = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} \qquad \text{whenever } \theta \notin 2\pi\mathbb{Z}.$$

The expression is called the n-th *Dirichlet kernel* and appears in the summation of Fourier series.

Solution: The sum $\sum_{k=-n}^{n}\left(e^{i\theta}\right)^k$ is a geometric series with $2n+1$ terms, common ratio $e^{i\theta}$ and first term $e^{-in\theta}$. Hence, using the usual formula for the sum of a geometric series we obtain

$$\begin{aligned}
\sum_{k=-n}^{n} \left(e^{i\theta}\right)^k = e^{-in\theta}\frac{\left(e^{i\theta}\right)^{2n+1} - 1}{e^{i\theta} - 1} &= e^{-i(n+1/2)\theta}\frac{e^{i(2n+1)\theta} - 1}{e^{i\theta/2} - e^{-i\theta/2}} \\
&= \frac{e^{i(2+1/2)\theta} - e^{-i(n+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}} = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}},
\end{aligned}$$

where we use the definition of the complex exponential function for the last equality sign. The method shows the power of going into the complex plane for solving problems in the real line.

Extra questions for further practice

8. Express the following complex numbers in Cartesian form:

(a) $e^{-i\pi}$

Solution: $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1$

(b) $e^{\ln 7 + 2\pi i}$

Solution: $e^{\ln 7+2\pi i} = 7e^{2\pi i} = 7.$

(c) $\sin(i\pi)$

Solution: $\sin(i\pi) = \frac{1}{2i}(e^{i(i\pi)} - e^{-i(i\pi)}) = \frac{1}{2i}(e^{-\pi} - e^{\pi}) = 0 + \frac{e^{\pi}-e^{-\pi}}{2}i$

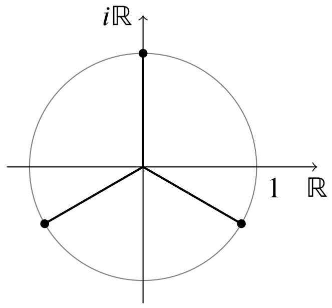
9. Solve the following equations and plot the solutions in the complex plane:

(a) $z^3 = -i$

Solution: $z^3 = r^3 e^{i3\theta} = -i = e^{-i\pi/2}$, and thus the solutions are

$$z = e^{-i\pi/6},\ e^{i\pi/2},\ e^{-5i\pi/6}.$$

The solutions are illustrated below.

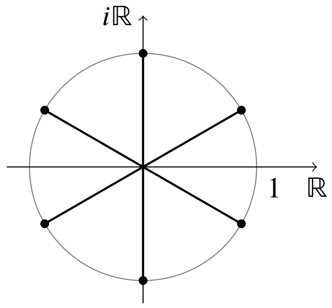


(b) $z^6 = -1$

Solution: Write $z^6 = r^6(\cos(6\theta) + i\sin(6\theta)) = -1 = 1(\cos\pi + i\sin\pi)$. Equating modulii gives $r = 1$, and considering arguments gives $6\theta = \pi + 2k\pi$ with $k \in \mathbb{Z}$. Therefore the six roots are:

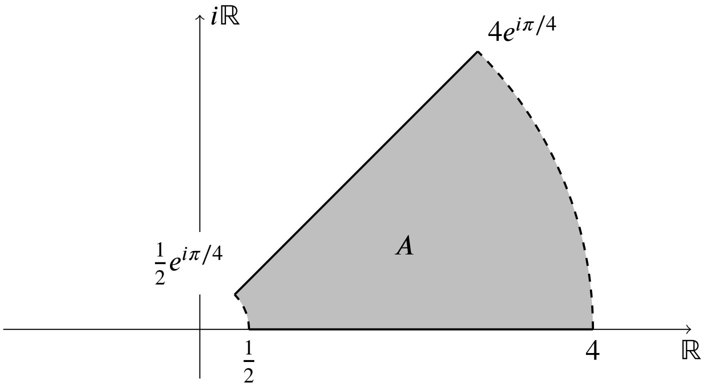
$$z = e^{i\pi/6}, \quad e^{i\pi/2}, \quad e^{5i\pi/6}, \quad e^{-5i\pi/6}, \quad e^{-i\pi/2}, \quad e^{-i\pi/6}.$$

They are illustrated on the following diagram.



10. (a) Sketch the set $A = \{z \in \mathbb{C} \mid 1/2 < |z| < 4,\ 0 \le \operatorname{Arg}(z) \le \pi/4\}$.

Solution: The sketch of A in the z-plane is given below. It is best obtained by looking at z given in modulus–argument form.



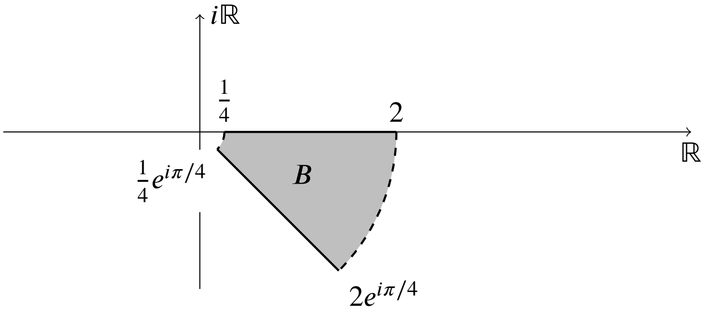
z-plane

(b) Sketch the image B of A in the w-plane under the function $z \mapsto 1/z$.

Solution: If $z = re^{i\theta}$ is given in modulus–argument form, then we have

$$w = \frac{1}{z} = \frac{1}{r}e^{-i\theta}.$$

Hence $B = \{z \in \mathbb{C} \mid 1/4 < |z| < 2,\ 0 \geq \mathrm{Arg}(z) \geq -\pi/4\}$ as sketched below.



w-plane

11. (a) Show that every complex number $z \in \mathbb{C}$ can be expressed as $z = w + 1/w$ for some $w \in \mathbb{C}$.

Solution: The equation $w + 1/w = z$ becomes $w^2 - zw + 1 = 0$ after multiplying by w and rearranging. This equation is a quadratic in w and has solutions $w = \frac{z \pm \sqrt{z^2-4}}{2}$. These solutions are nonzero, because $0^2 - z0 + 1 = 1 \neq 0$. Hence we can divide by w to conclude that we have found solutions to the initial equation $w + 1/w = z$.

Remark: Note it is not true that every $x \in \mathbb{R}$ can be expressed as $x = y + 1/y$ for some $y \in \mathbb{R}$. For example, we can't write $x = 0$ in this form, because such a y would satisfy $y^2 + 1 = 0$. Thus there was really something to prove in this question.

(b) Use this substitution to solve the equation $z^3 - 3z - 1 = 0$.

Solution: We know from (a) that any z can be written as $w + 1/w$ for some nonzero $w \in \mathbb{C}$. When this substitution is made,

$$\begin{aligned} z^3 - 3z - 1 &= \left(w + \frac{1}{w}\right)^3 - 3\left(w + \frac{1}{w}\right) - 1 \\ &= w^3 + 3w^2\frac{1}{w} + 3w\frac{1}{w^2} + \frac{1}{w^3} - 3\left(w + \frac{1}{w}\right) - 1 \\ &= w^3 + \frac{1}{w^3} - 1, \end{aligned}$$

so the equation $z^3 - 3z - 1 = 0$ is equivalent to $w^3 + \frac{1}{w^3} - 1 = 0$, which is equivalent to $(w^3)^2 - (w^3) + 1 = 0$. This is a quadratic in w^3 with solutions

$$w^3 = \frac{1 + \sqrt{3}i}{2} = e^{i\frac{\pi}{3}} \text{ or } w^3 = \frac{1 - \sqrt{3}i}{2} = e^{-i\frac{\pi}{3}}.$$

Since the solutions of $w^3 = e^{-i\frac{\pi}{3}}$ are the inverses of the solutions of $w^3 = e^{i\frac{\pi}{3}}$, and $w + \frac{1}{w}$ doesn't change when you replace w with its inverse, we need only consider the possibility that $w^3 = e^{i\frac{\pi}{3}}$. We find that the three cube roots of $e^{i\frac{\pi}{3}}$ are $e^{i\frac{\pi}{9}}$, $e^{i\frac{7\pi}{9}}$, and $e^{-i\frac{5\pi}{9}}$. Now if $w = e^{i\theta}$, then

$$w + \frac{1}{w} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta,$$

so we have proved that the solutions to the original equation $z^3 - 3z - 1 = 0$ are $2\cos\frac{\pi}{9}$, $2\cos\frac{5\pi}{9}$, and $2\cos\frac{7\pi}{9}$. (Notice that these are all real numbers, but we needed the complex numbers to find them – this phenomenon really confused some of the early pioneers in the area, because at the time complex numbers were not well understood).

12. Solve, using a completion of squares, the general quadratic equation

$$az^2 + bz + c = 0 \quad \text{with } a, b, c \in \mathbb{C}.$$

In other words, prove the standard formula to solve a quadratic equation.

Solution: We may suppose that $a \neq 0$, for otherwise the equation is $bz + c = 0$, which is easy to solve. We have

$$\begin{aligned} az^2 + bz + c = 0 \quad &\Longleftrightarrow \quad z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \\ &\Longleftrightarrow \quad \left(z^2 + \frac{b}{a}z + \frac{b^2}{4a^2}\right) + \frac{c}{a} - \frac{b^2}{4a^2} = 0 \\ &\Longleftrightarrow \quad \left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\ &\Longleftrightarrow \quad z + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\ &\Longleftrightarrow \quad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \end{aligned}$$

where $\pm\sqrt{b^2 - 4ac}$ denote the two square roots of the complex number $b^2 - 4ac$.

Challenge questions (optional)

13. From the definition of $e^{i\theta} = \cos\theta + i\sin\theta$ we deduce that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We replace $\theta \in \mathbb{R}$ by any complex number $z \in \mathbb{C}$ and define

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

and call them the complex cosine and sine functions.

(a) Show that when z is real, $\cos z$ and $\sin z$ reduce to the familiar real functions.

Solution: If $z = x$, then

$$\sin z = \frac{e^{ix} - e^{-ix}}{2i} = \frac{\cos x + i\sin x - (\cos x - i\sin x)}{2i} = \frac{2i\sin x}{2i} = \sin x.$$

A similar result hold for $\cos z$.

(b) Show that $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

Solution: We have

$$\cos^2 z + \sin^2 z = \frac{e^{2iz} - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} = 1.$$

(c) Show that $\cos(z+w) = \cos z \cos w - \sin z \sin w$ for all $z, w \in \mathbb{C}$.

Solution: We have

$$\begin{aligned}
\cos z \cos w - \sin z \sin w &= \frac{1}{4}(e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) + \frac{1}{4}(e^{iz} - e^{-iz})(e^{iw} - e^{-iw}) \\
&= \frac{1}{4}\big(e^{i(z+w)} + e^{i(z-w)} + e^{-i(z-w)} + e^{-i(z+w)} \\
&\qquad + e^{i(z+w)} - e^{i(z-w)} - e^{-i(z-w)} + e^{-i(z+w)}\big) \\
&= \frac{1}{2}\left(e^{i(z+w)} + e^{-i(z+w)}\right) \\
&= \cos(z+w).
\end{aligned}$$

Similarly you could check the formula $\sin(z+w) = \sin z \cos w + \cos z \sin w$.

(d) Is it true that $|\sin z| \leq 1$ and $|\cos z| \leq 1$ for all $z \in \mathbb{C}$?

Solution: These are not true statements. When $z = iy$, we have

$$|\sin(iy)| = \left|\frac{e^{-y} - e^{y}}{2i}\right| = \tfrac{1}{2}\left|e^{y} - e^{-y}\right|.$$

In particular, $|\sin(i)| = (e - e^{-1})/2 > 1$. Actually, $|\sin(iy)|$ can be made as large as we please by taking sufficiently large positive y (because e^{y} becomes arbitrarily large and dominates e^{-y}). A similar result holds for the complex cosine function.

14. There is a 'cubic formula' analogous to the much loved quadratic formula, although it is a lot more complicated. In this question you solve the general cubic equation

$$az^3 + bz^2 + cz + d = 0 \quad \text{with } a, b, c, d \in \mathbb{C},$$

generalising the method of Question 11. Here is an outline of the strategy:

(a) Make a substitution of the form $z = u + \alpha$, with α to be determined, to reduce the equation to the form $u^3 + pu - q = 0$.

Solution: Firstly, we may assume that $a \neq 0$ (otherwise we have a quadratic, and we know how to solve these). Thus after dividing out the a we might as well assume that $a = 1$. Plugging in $z = u + \alpha$ (as the hint suggests) and expanding, we see that the choice $\alpha = -b/3$ makes the coefficient of u^2 equal 0. Then setting $z = u - b/3$ gives, after some algebra,

$$u^3 + pu - q = 0, \quad \text{where} \quad p = c - \frac{b^2}{3} \quad \text{and} \quad q = \frac{bc}{3} - d - \frac{2b^3}{27}.$$

(b) Now attempt a substitution of the form $u = w + \beta/w$ with a cleverly chosen β to reduce the equation to a quadratic in w^3.

Solution: Making a substitution $u = w + \beta/w$ ad expanding we see that the choice $\beta = -p/3$ simplifies things to give

$$w^3 - \frac{p^3}{27}w^{-3} - q = 0.$$

After multiplying through by w^3 we get a quadratic in w^3, specifically

$$(w^3)^2 - q(w^3) - \frac{p^3}{27} = 0.$$

(c) You can now solve this quadratic equation for w, hence back-track to find z.

Solution: By the quadratic formula we obtain

$$w^3 = \frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

A similar argument to that used in Question 11(b) shows that it is sufficient to consider only the $+$ sign here. Thus we find three solutions,

$$w = e^{2\pi i k/3}\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \qquad\qquad k = 0, 1, 2.$$

(here $\sqrt[3]{\ }$ is any choice of cube root). But

$$z = u - \frac{b}{3} = w - \frac{p}{3w} - \frac{b}{3},$$

and so we have found the solutions to the original equation.

15. Let n be a given positive integer. A *primitive nth root of unity* is a solution $z = \alpha$ of the equation $z^n = 1$ with the property that the powers $\alpha, \alpha^2, \cdots, \alpha^{n-1}, \alpha^n$ give all of the nth roots of unity. For example, the 4th root of unity $\alpha = i$ is primitive, since $i, i^2 = -1, i^3 = -i, i^4 = 1$ gives us all 4th roots of unity, while the 4th root $\alpha = -1$ is not primitive, since $-1, (-1)^2 = 1, (-1)^3 = -1, (-1)^4 = 1$ fails to give us all of the 4th roots of unity.

(a) Find all primitive 6th roots of unity.

Solution: Clearly, $\alpha = e^{i\frac{2\pi}{6}}$ is a primitive 6-th root of unity, as its powers $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6 (= 1)$ give the complete list of 6-th roots of unity, namely

$$\alpha = e^{i\frac{2\pi}{6}}, \quad \alpha^2 = e^{i\frac{4\pi}{6}}, \quad \alpha^3 = e^{i\frac{6\pi}{6}}, \quad \alpha^4 = e^{i\frac{8\pi}{6}}, \quad \alpha^5 = e^{i\frac{10\pi}{6}}, \quad \alpha^6 = e^{i\frac{12\pi}{6}} = 1.$$

However $\alpha^2 = e^{i\frac{4\pi}{6}}$ is not a primitive 6-th root of unity: its distinct powers are α^2, $(\alpha^2)^2 = \alpha^4$ and $(\alpha^2)^3 = \alpha^6 = 1$. Similarly, α^3 is not a primitive root (its distinct powers are α^3, $\alpha^6 = 1$) and α^4 is not a primitive root (its distinct powers are α^4, α^2 and 1). However, α^5 is a primitive root. Its powers are

$$\alpha^5, \quad (\alpha^5)^2 = \alpha^4, \quad (\alpha^5)^3 = \alpha^3, \quad (\alpha^5)^4 = \alpha^2, \quad (\alpha^5)^5 = \alpha, \quad (\alpha^5)^6 = 1.$$

Thus the only primitive 6th roots of unity are α and α^5.

(b) Find all primitive 5th roots of unity.

Solution: With $\alpha = e^{i\frac{2\pi}{5}}$, we find that the primitive 5th roots of unity are

$$\alpha, \quad \alpha^2, \quad \alpha^3, \quad \alpha^4.$$

(c) For which values of k, $0 \le k \le n-1$, is $e^{i\frac{2\pi k}{n}}$ a primitive nth root of unity?

Solution: We claim that for $0 < k < n$, the nth root of unity $e^{i\frac{2\pi k}{n}}$ is a primitive nth root if and only if k is *relatively prime* to n (that is, k and n have no factors in common). Let's write out a careful proof of this statement: We need to prove two things:

(1) If $\alpha = e^{i\frac{2\pi k}{n}}$ is a primitive nth root of unity then k and n are relatively prime.

(2) If k and n are relatively prime then $\alpha = e^{i\frac{2\pi k}{n}}$ is a primitive nth root of unity.

To prove (1), let $\alpha = e^{i\frac{2\pi k}{n}}$ be a primitive nth root. Suppose that k and n are not relatively prime, and so k and n have a common factor, d, say. Thus $k = ad$ and $n = bd$ for some integers a, b, with $1 < b < n$. Thus $k = an/b$, and hence $\alpha = e^{i\frac{2\pi a}{b}}$. So $\alpha^b = 1$, and since $1 < b < n$ this implies that list $\alpha, \alpha^2, \ldots, \alpha^n$ does not contain all n of the nth roots of unity (as $\alpha^b = \alpha^n = 1$), a contradiction. Thus k and n are relatively prime.

To prove (2), let k and n be relatively prime. Suppose that $\alpha = e^{i\frac{2\pi k}{n}}$ is not a primitive nth root of unity. Thus there are integers $1 \le q < r \le n$ such that $\alpha^q = \alpha^r$. Thus $\alpha^{r-q} = 1$, and so $e^{i\frac{2\pi k(r-q)}{n}} = 1$. Thus $\frac{2\pi k(r-q)}{n} = 2s\pi$ for some integer s, and since $r \ne q$ we have $s \ne 0$. So $k(r-q) = sn$. Since k and n are relatively prime it must be that k divides s. Thus $s = kt$ for some integer t, and so $k(r-q) = ktn$, giving $r - q = tn \ge n$. This contradicts the fact that $r - q < n$, and hence α is a primitive nth root of unity after all.

16. We introduced the complex numbers by saying something along the lines of: "Append a solution i of the equation $x^2 + 1 = 0$ to the real numbers $\mathbb{R}$". This is a bit mysterious, and you might ask: "What is this magical element i? Where does it live?". Here is a more formal approach. Let $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$ and define an addition operation and a multiplication operation on $\mathbb{R}^2$ by:

$$
\begin{aligned}
(a, b) + (c, d) &= (a + c, b + d) && \text{for all } (a, b) \in \mathbb{R}^2 \\
(a, b)(c, d) &= (ac - bd, bc + ad) && \text{for all } (a, b) \in \mathbb{R}^2.
\end{aligned}
$$

Let $\mathbf{0} = (0, 0)$, $\mathbf{1} = (1, 0)$, and $\mathbf{i} = (0, 1)$.

(a) Show that $\mathbf{0} + (a, b) = (a, b)$ for all $(a, b) \in \mathbb{R}^2$.

Solution: This is obvious: $(0, 0) + (a, b) = (0 + a, 0 + b) = (a, b)$.

(b) Show that $\mathbf{1}(a, b) = (a, b)$ for all $(a, b) \in \mathbb{R}^2$.

Solution: This is also easy: $(1, 0)(a, b) = (a - 0, b + 0) = (a, b)$ for all $a, b \in \mathbb{R}$.

(c) Show that $\mathbf{i}^2 + \mathbf{1} = \mathbf{0}$.

Solution: Again, this is a simple direct calculation using the definitions:

$$\mathbf{i}^2 + \mathbf{1} = (0, 1)(0, 1) + (1, 0) = (-1, 0) + (1, 0) = (0, 0) = \mathbf{0}.$$

(d) Explain why $\mathbb{R}^2$ with the above operations is really just $\mathbb{C}$ in disguise. Thus it is possible to introduce $\mathbb{C}$ without ever talking about the 'imaginary' number i.

Solution: If we we identify $(a, b) \in \mathbb{R}^2$ with $a + ib \in \mathbb{C}$ then the definition of addition and multiplication in $\mathbb{R}^2$ agree with those in $\mathbb{C}$. Under this identification, $(0, 1) \leftrightarrow i$.