The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2	**First Assignment**	2017

This assignment comprises four questions and is worth 5% of the overall assessment. It should be completed, scanned and uploaded using Turnitin through the MATH1903 Blackboard portal **by 5 pm on Thursday 17 August**.

Your answers should aim to be convincing, concise and a pleasure to read. Please show relevant working, and present your arguments clearly and grammatically, using mathematical precision and appropriate connecting words of explanation.

The first three questions explore decompositions of a real-valued function, on an appropriate domain, into its even and odd components. In the third question, which explores some examples, one might notice some important relationships between well known functions (and later, after gaining fluency with power series expansions, one might notice a connection also with geometric series in the second example).

The fourth question is challenging, and a convincing and complete argument will demonstrate mastery of elementary concepts and properties associated with definite integrals, including the effects of elementary substitutions, their relationships with derivatives and the Fundamental Theorem of Calculus.

Your tutor will give you feedback and allocate an overall letter grade (and mark) using the following criteria:

- *A^+ (10): excellent and scholarly work, answering all parts of all questions, with clear and accurate explanations and working, with appropriate acknowledgement of sources, if appropriate, and at most minor or trivial errors or omissions;*
- *A(9): excellent work, but with one or two substantial omissions, errors or misunderstandings overall;*
- *B^+ (8): very good work, but with three or four substantial omissions, errors or misunderstandings overall;*
- *B(7): very good work, but with five or six substantial omissions, errors or misunderstandings overall;*
- *C^+ (6): good work, with more than six substantial omissions, errors or misunderstandings, but writing answers that include at least eight distinct positive attributes;*
- *C(5): reasonable attempt, making more than six substantial omissions, errors or misunderstandings, but writing answers that include six or seven distinct positive attributes;*
- *D(4): limited progress including just four or five distinct positive attributes;*
- *E(2): limited progress, including just two or three distinct positive attributes;*
- *F(1): some attempt, but including just one positive attribute.*

Consider a nonempty subset Δ of $\mathbb{R}$ that is closed under taking negatives (that is, for all $x \in \Delta$, we have $-x \in \Delta$). For example, $\Delta = \mathbb{R}$, $\Delta = \mathbb{R}\backslash\{\pm 1\}$ and $\Delta = \mathbb{R}\backslash\mathbb{Z}$ are all closed under taking negatives. Now put

$$\mathbb{R}^\Delta \;=\; \{ \text{ functions } f : \Delta \to \mathbb{R} \ \} \,.$$

For $\lambda, \mu \in \mathbb{R}$ and $f, g \in \mathbb{R}^\Delta$, define the *linear combination* $\lambda f + \mu g \in \mathbb{R}^\Delta$ by the rule

$$(\lambda f + \mu g)(x) \;=\; \lambda f(x) + \mu g(x)$$

for all $x \in \Delta$. As special cases, we get the sum $f + g$ (when $\lambda = \mu = 1$) and the scalar multiple λf (when $\mu = 0$).

Then, in fact, $\mathbb{R}^\Delta$ becomes a real vector space of dimension $|\Delta|$ (the cardinality of Δ) with respect to addition and scalar multiplication of functions. (In particular, $\mathbb{R}^\mathbb{R}$ is the infinite dimensional real vector space comprising all real-valued functions of a single unrestricted real variable.)

The zero vector in this context is just the *zero function*, denoted by $\mathbf{0} \in \mathbb{R}^\Delta$, given by the rule $\mathbf{0}(x) = 0$ for all $x \in \Delta$.

Recall that, if $f \in \mathbb{R}^\Delta$, then we say that f is *even* if $f(x) = f(-x)$ for all $x \in \Delta$, and we say that f is *odd* if $f(x) = -f(-x)$ for all $x \in \Delta$.

It is routine to verify that the zero function $\mathbf{0}$ is the only member of $\mathbb{R}^\Delta$ that is both even and odd, and that linear combinations of even and odd functions are even and odd respectively. These facts may be quoted freely.

1. Let $f \in \mathbb{R}^\Delta$ and define $g, h \in \mathbb{R}^\Delta$ by the rules

$$g(x) \;=\; \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad h(x) \;=\; \frac{f(x) - f(-x)}{2}$$

for all $x \in \Delta$. Decide which of g and h is even or odd, and briefly verify any claims.

2. Let $f \in \mathbb{R}^\Delta$. Prove that

$$f \;=\; f_{\text{even}} + f_{\text{odd}}$$

for some unique functions $f_{\text{even}}, f_{\text{odd}} \in \mathbb{R}^\Delta$ such that f_{even} is even and f_{odd} is odd.

(This, in fact, verifies that the direct sum decomposition $R^\Delta = V_1 \oplus V_2$ holds where V_1 and V_2 are the subspaces of $\mathbb{R}^\Delta$ consisting of even and odd functions respectively. The direct sum decomposition of a vector space is a topic students typically explore in Second Year linear algebra.)

3. Find simplified expressions for $f_{\text{even}}(x)$ and $f_{\text{odd}}(x)$ in each of the following special cases:

(a) $\Delta = \mathbb{R}$, and $f(x) = e^x$ for all $x \in \Delta$.

(b) $\Delta = \mathbb{R}\backslash\{\pm 1\}$, and $f(x) = \dfrac{1}{1-x}$ for all $x \in \Delta$.

(c) $\Delta = \mathbb{R}\backslash\mathbb{Z}$, and $f(x) = \lfloor x \rfloor$ (the floor function) for all $x \in \Delta$.

4. Suppose throughout this next exercise that $f \in \mathbb{R}^{\mathbb{R}}$ and f is continuous, so all definite integrals $\displaystyle\int_a^b f(x)\,dx$ exist for all $a, b \in \mathbb{R}$. You may freely quote the Fundamental Theorem of Calculus where relevant.

(a) Use properties of definite integrals and integration by substitution to verify that, for all $a \in \mathbb{R}$,

$$\int_{-a}^{a} f(x)\,dx \;=\; \begin{cases} 2\displaystyle\int_0^a f(x)\,dx & \text{if } f \text{ is even,} \\ 0 & \text{if } f \text{ is odd.} \end{cases}$$

(b) Fix $a \in \mathbb{R}$ and define the area function $A \in \mathbb{R}^{\mathbb{R}}$ by the rule

$$A(x) \;=\; \int_a^x f(t)\,dt$$

for $x \in \mathbb{R}$. Prove the following:

(i) the function A is even if and only if f is odd;

(ii) the function A is odd if and only if f is even and $A(0) = 0$.