The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 3

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2012

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Topics covered

In lectures last week:

- ☐ The Fundamental Theorem of Calculus.
- ☐ Functions defined using integrals: the logarithm, the error function, the inverse tangent function, the Fresnel integrals, the sine integral, the logarithmic integral.
- ☐ Elementary antiderivatives (Liouville's Theorem).

Objectives

After completing this tutorial sheet you will be able to:

- ☐ Apply the Fundamental Theorem of Calculus in various settings.
- ☐ Quantitatively and qualitatively analyse functions defined by integrals.
- ☐ Decide if certain functions defined by integrals are elementary (challenging!).
- ☐ Use integration and differentiation to prove a beautiful theorem: π is irrational.

Preparation questions to do *before* class

1. Find the derivative of $f(x) = \displaystyle\int_1^{\sqrt{x}} \frac{s^2}{s^2+1}\,ds$

Solution: Let $g(x) = \int_1^x \frac{s^2}{s^2+1}\,ds$. Then $f(x) = g(\sqrt{x})$, and so by the Fundamental Theorem of Calculus and the chain rule we compute

$$f'(x) = g'(\sqrt{x}\,)\,\frac{1}{2\sqrt{x}} = \frac{x}{x+1}\,\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(x+1)}.$$

2. Use integration by parts to calculate $\displaystyle\int_0^1 C(x)\,dx$, where $C(x) = \displaystyle\int_0^x \cos(t^2)\,dt$.

Solution: Let $u = C(x)$ and $\frac{dv}{dx} = 1$. Then $\frac{du}{dx} = \cos(x^2)$ and $v = x$, and so

$$\int_0^1 C(x)\,dx = xC(x)\big|_0^1 - \int_0^1 x\cos(x^2)\,dx = C(1) - \frac{1}{2}\sin(x^2)\big|_0^1 = C(1) - \frac{1}{2}\sin 1.$$

Questions to do in class

3. Find the derivative of the following functions.

(a) $f(x) = \int_x^4 (2+\sqrt{u})^8\,du$

Solution: By the Fundamental Theorem of Calculus,

$$f'(x) = -\frac{d}{dx}\int_4^x (2+\sqrt{u})^8\,du = -(2+\sqrt{x})^8.$$

(b) $f(x) = \int_x^{\cos x} e^{-t^2}\,dt$

Solution: Write

$$f(x) = \int_0^{\cos x} e^{-t^2}\,dt - \int_0^x e^{-t^2}\,dt.$$

By the Fundamental Theorem of Calculus and the chain rule we have

$$\frac{d}{dx}\int_0^{\cos x} e^{-t^2}\,dt = -\sin x\, e^{-\cos^2 x},$$

and so

$$f'(x) = -\sin x\, e^{-\cos^2 x} - e^{-x^2}.$$

4. Recall that the logarithmic integral $\mathrm{Li}(x)$ is defined by $\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$. For $\alpha > 1$, calculate

$$\int_2^\alpha \frac{\mathrm{Li}(x)}{x^2}\,dx.$$

Solution: Using integration by parts, with $u = \mathrm{Li}(x)$ and $\frac{dv}{dx} = \frac{1}{x^2}$, gives

$$\int_2^\alpha \frac{\mathrm{Li}(x)}{x^2}\,dx = -\frac{\mathrm{Li}(x)}{x}\bigg|_2^\alpha + \int_2^\alpha \frac{1}{x\ln x}\,dx = -\frac{\mathrm{Li}(\alpha)}{\alpha} + \int_2^\alpha \frac{1}{x\ln x}\,dx,$$

where we have used $\mathrm{Li}(2) = 0$. Making the change of variable $t = \ln x$ gives

$$\int_2^\alpha \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\ln\alpha} \frac{1}{t}\,dt = \ln\ln\alpha - \ln\ln 2.$$

Therefore

$$\int_2^\alpha \frac{\mathrm{Li}(x)}{x^2}\,dx = \ln\ln\alpha - \ln\ln 2 - \frac{\mathrm{Li}(\alpha)}{\alpha}.$$

5. Let $f(x)$ be a continuous function on $[a, b]$. Apply the Mean Value Theorem to the function

$$F(x) = \int_a^x f(t)\,dt$$

to show that there exists $c \in (a, b)$ such that

$$\frac{1}{b-a}\int_a^b f(t)\,dt = f(c), \qquad \text{and interpret this geometrically.}$$

Solution: Applying the Mean Value Theorem to the function $F(x)$ on the interval $[a, b]$ tells us that there is $c \in (a, b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

But $F'(c) = f(c)$ and $F(a) = 0$, which establishes the required equality. One way to geometrically interpret this, at least in the case when $f(x) \geq 0$, is that the area

$$\int_a^b f(x)\,dx$$

under the curve is equal to the area of a rectangle with base $[a, b]$ and height $f(c)$ for some $c \in (a, b)$. This is 'clear' from drawing a picture.

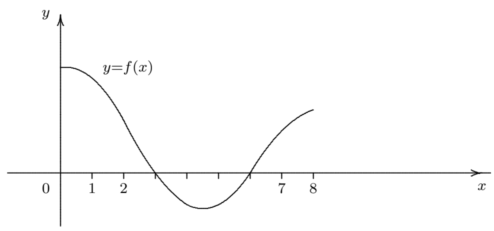
Questions for extra practice

6. Let $f(x) = \displaystyle\int_0^x x\sin(t^2)\,dt$. Find $f''(x)$.

Solution: Since x is constant as far as the integrating variable t is concerned, we can write $f(x) = x\int_0^x \sin(t^2)\,dt$. Now by the product rule and the Fundamental Theorem of Calculus,

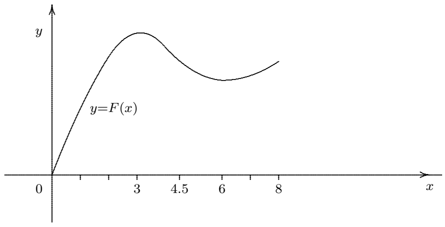
$$\begin{aligned} f'(x) &= x\sin(x^2) + \int_0^x \sin(t^2)\,dt, \\ f''(x) &= \sin(x^2) + x\,\frac{d}{dx}\sin(x^2) + \sin(x^2) = 2\sin(x^2) + 2x^2\cos(x^2). \end{aligned}$$

7. Suppose that a function $y = f(x)$ has the following graph:



Let $F(x)$ be the function defined by $F(x) = \int_0^x f(t)\,dt$ for $0 \leq x \leq 8$. Sketch the graph of $y = F(x)$, indicating points where F has a local maximum or minimum, and any points of inflection.

Solution: $F'(x) = f(x)$ by the Fundamental Theorem of Calculus, and so we see from the graph of $y = f(x)$ that $F'(x) \geq 0$ for $0 \leq x \leq 3$ and for $6 \leq x \leq 8$, while $F'(x) < 0$ for $3 < x < 6$. So $F(x)$ is increasing on $[0, 3]$ and on $[6, 8]$, but decreasing on $[3, 6]$. So $F(x)$ has a local maximum at $x = 3$ and a local minimum at $x = 6$. Also, $F''(x) = f'(x)$, which is negative for $0 < x < 4.5$ and positive for $4.5 < x \leq 8$. Hence $F(x)$ is concave downwards on $[0, 4.5]$, concave upwards on $[4.5, 8]$, and has a point of inflection at $x = 4.5$. Note also that $F(0) = 0$. We can now sketch the graph of $y = F(x)$:



8. If $x \sin(\pi x) = \int_0^{x^2} f(t)\,dt$, find $f(4)$.

Solution: Differentiating both sides of the given equation we get

$$\sin(\pi x) + \pi x \cos(\pi x) = 2x f(x^2).$$

Evaluating both sides of this at $x = 2$, we see that $f(4) = \pi/2$.

9. Suppose that $f(t)$ is continuous on $[a, b]$. Recall the following:

- The *Extreme Value Theorem* says that $f(x)$ attains a global maximum M and a global minimum m on $[a, b]$.
- Then the *Intermediate Value Theorem* implies that if $m \leq A \leq M$ then there exists $c \in [a, b]$ such that $f(c) = A$.

Let $p(t)$ be Riemann integrable on $[a, b]$ with $p(t) \geq 0$ for all $t \in [a, b]$.

(a) Explain why

$$m \int_a^b p(t)\,dt \leq \int_a^b f(t)p(t)\,dt \leq M \int_a^b p(t)\,dt.$$

Solution: It is a general fact that if f and g are Riemann integrable on $[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx. \tag{1}$$

This is obvious if we think in terms of areas. To sketch a proof using Riemann sums, let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, and let $x_j^* \in [x_{j-1}, x_j]$ be a choice of sample points. Since $f(x_j^*) \leq g(x_j^*)$ for each j we have

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j \leq \sum_{j=1}^{n} g(x_j^*)\Delta x_j.$$

As $\|P\| \to 0$ the left hand side approaches $\int_a^b f(x)\,dx$ and the right hand side approaches $\int_a^b g(x)\,dx$. Therefore (1) holds.

So, since $p(t) \geq 0$ we have $mp(t) \leq f(t)p(t) \leq Mp(t)$ for all $t \in [a, b]$, and hence

$$m \int_a^b p(t)\, dt \leq \int_a^b f(t)p(t)\, dt \leq M \int_a^b p(t)\, dt.$$

(b) Deduce that there is $c \in [a, b]$ such that

$$\int_a^b f(t)p(t)\, dt = f(c) \int_a^b p(t)\, dt.$$

This is called the *Mean Value Theorem for integrals.* It is a generalisation of Question 5. We will use it later in the course (§6.2 of the course notes).

Solution: Let $I = \displaystyle\int_a^b p(t)\, dt$. If $I = 0$ then the inequality in (a) shows that

$$\int_a^b f(t)p(t)\, dt = 0,$$

and since $0 = f(c)I$ for *any* $c \in [a, b]$ there is nothing to prove.
If $I \neq 0$ then we have

$$m \leq \frac{1}{I} \int_a^b f(t)p(t)\, dt \leq M.$$

Thus by the second bullet point above, there is $c \in [a, b]$ such that

$$f(c) = \frac{1}{I} \int_a^b f(t)p(t)\, dt,$$

which rearranges to give the desired equality.

Challenging questions

10. Suppose that $f(x)$ and $g(x)$ are rational functions. Recall that Liouville's Theorem says that

$$\int f(x)e^{g(x)}\, dx$$

is an elementary function if and only if there is a rational function $r(x)$ such that $f(x) = r'(x) + g'(x)r(x)$. Is

$$\int e^{1/x}\, dx$$

an elementary function?

Solution: After trying a few changes of variables, and after throwing my whole bag of integration tricks at it, I begin to think that this integral is *not* an elementary function. Let's prove this. Suppose, for a contradiction, that it is an elementary function. Then Liouville tells us that there is a rational function

$$r(x) = \frac{p(x)}{q(x)} \qquad \text{such that} \qquad 1 = r'(x) - \frac{r(x)}{x^2}, \tag{2}$$

where $p(x)$ and $q(x)$ have no factors in common. Rearranging gives

$$x^2 = x^2 r'(x) - r(x). \tag{3}$$

If $q(x)$ is a constant then $x^2 r'(x)$ is a polynomial with degree $\deg(p) + 1$, and $r(x)$ is a polynomial with degree $\deg(p)$, and therefore the degree of $x^2 r'(x) - r(x)$ is $\deg(p) + 1$. Then (3) gives $\deg(p) = 1$, and so $r(x) = p(x) = a + bx$. But then (3) gives $x^2 = bx^2 - bx - a$ for all x, and so $a = b = 0$ and so $r(x) = 0$, a contradiction.

Therefore $q(x)$ is not a constant, and so by the Fundamental Theorem of Algebra there is a number $\alpha \in \mathbb{C}$ such that $q(\alpha) = 0$. If this root of $q(x)$ has multiplicity m, then

$$r(x) = \frac{h(x)}{(x-\alpha)^m}$$

where $h(x)$ is a rational function whose numerator and denominator do not vanish at $x = \alpha$ (here we have used the fact that $p(x)$ and $q(x)$ have no roots in common). Plugging this into (3) gives

$$x^2 = \frac{x^2 h'(x)}{(x-\alpha)^m} - mx^2 \frac{h(x)}{(x-\alpha)^{m+1}} - \frac{h(x)}{(x-\alpha)^m}.$$

Rearranging gives

$$mx^2 \frac{h(x)}{x-\alpha} = x^2 h'(x) - h(x) - x^2 (x-\alpha)^m. \tag{4}$$

If $\alpha \neq 0$ then the left hand side is unbounded as $x \to \alpha$ (we have used the fact that the numerator and denominator of $h(x)$ does not vanish at $x = \alpha$), while the right hand side tends to $\alpha^2 h'(\alpha)$. This forces $\alpha = 0$, in which case (4) becomes

$$mxh(x) = x^2 h'(x) - h(x) - x^{m+2}.$$

Rearranging this we get

$$\frac{h(x)}{x} = xh'(x) - mh(x) - x^{m+1}.$$

Again, the left hand side is unbounded as $x \to 0$ (because the numerator and denominator of $h(x)$ does not vanish at $x = \alpha = 0$), which the right hand side tends to $-mh(0)$. Therefore there is no rational function $r(x)$ satisfying (2), and therefore by Liouville's Theorem the integral is *not* an elementary function.

The following questions use a nice mixture of differentiation and integration to show that π, π^2, and e^r ($r \in \mathbb{Q}\backslash\{0\}$) are irrational. They are adapted from proofs in *Irrational Numbers*, by Ivan Niven (The Carus Mathematical Monographs, No. 11, 1956). The first proof of the irrationality of π (Johann Lambert, 1768) was considerably more complicated.

11. Let $n \geq 0$ be an integer, and let $f_n(x) = \dfrac{x^n(1-x)^n}{n!}$.

(a) Show that $f_n^{(j)}(0)$ and $f_n^{(j)}(1)$ are integers for all $j \in \mathbb{N}$. *Hint: Binomial Theorem to see that $f_n^{(j)}(0)$ is integral. Then use $f_n(1-x) = f_n(x)$.*

Solution: By the Binomial Theorem we have

$$f_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{n+k}}{(n-k)!k!}. \tag{5}$$

If $0 \leq j < n$ then $f_n^{(j)}(0) = 0$ because $f_n(x) = x^n \times$ (a polynomial).
If $n \leq j \leq 2n$ write $j = n + \nu$ where $0 \leq \nu \leq n$. Then by (5) we have

$$f_n^{(j)}(0) = (-1)^\nu \frac{(n+\nu)!}{(n-\nu)!\nu!} = (-1)^\nu \binom{n}{\nu}(n+1)(n+2)\cdots(n+\nu),$$

which is an integer (binomial coefficients are integers).
If $j > 2n$ then $f_n^{(j)}(0) = 0$ (because f_n is a polynomial of degree $2n$).
Thus $f_n^{(j)}(0)$ is an integer for all $j \geq 0$, and since $f_n(1-x) = f_n(x)$ we deduce that $f_n^{(j)}(1-x) = (-1)^j f_n^{(j)}(x)$. Therefore $f^{(j)}(1) = (-1)^j f^{(j)}(0)$ is also an integer for all $j \geq 0$.

(b) Assume that $\pi^2 = \frac{a}{b}$ is rational, with $a, b \in \mathbb{N}\backslash\{0\}$. Let

$$F_n(x) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f_n^{(2k)}(x)\,.$$

Use (a) to show that $F_n(0)$ and $F_n(1)$ are integers.

Solution: Using $\pi^2 = a/b$ gives

$$F_n(x) = \sum_{k=0}^{n} a^{n-k} b^k f_n^{(2k)}(x).$$

Since $a, b, f_n^{(2k)}(0)$ and $f_n^{(2k)}(1)$ are integers it is clear that $F_n(0)$ and $F_n(1)$ are also integers.

(c) Calculate $\frac{d}{dx}\left(F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right)$ and deduce that

$$I_n = \pi a^n \int_0^1 f_n(x)\sin\pi x\,dx \qquad \text{is an integer for all } n.$$

Solution: We have

$$\begin{aligned}\frac{d}{dx}\left(F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right) &= \left(F_n''(x) + \pi^2 F_n(x)\right)\sin\pi x \\ &= b^n\left(\pi^{2n+2} f_n(x) + (-1)^n f_n^{(2n+2)}(x)\right)\sin\pi x.\end{aligned}$$

But $f^{(2n+2)}(x) = 0$ for all x, and therefore

$$\frac{d}{dx}\left(F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right) = b^n \pi^{2n+2} f_n(x)\sin\pi x = \pi^2 a^n f_n(x)\sin\pi x.$$

By the Fundamental Theorem of Calculus this implies that

$$\pi^2 a^n \int_0^1 f_n(x) \sin \pi x \, dx = \pi \left(F_n(0) + F_n(1) \right).$$

The result follows since $F_n(0)$ and $F_n(1)$ are integers.

(d) Obtain a contradiction by noticing that $0 < f_n(x) < \frac{1}{n!}$ for $x \in (0,1)$. Thus π^2 is irrational. Deduce that π is irrational too.

Solution: The inequality $0 < f_n(x) < \frac{1}{n!}$ for $x \in (0,1)$ implies that

$$0 < I_n < \pi \frac{a^n}{n!} \int_0^1 \sin \pi x \, dx = \frac{2a^n}{n!}.$$

Since $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ we obtain a contradiction: Once n is large enough we have $0 < I_n < 1$ which contradicts the fact that I_n is an integer.

Therefore π^2 is irrational. If π is rational then π^2 is also rational, and so π must be irrational.

Remark: A number $\alpha \in \mathbb{R}$ is *algebraic* if it is the root of a (nontrivial) polynomial equation with integer coefficients. For example, $\sqrt{2}$ and $\frac{1+\sqrt{5}}{2}$ are algebraic, being roots of the equations

$$x^2 - 2 = 0 \qquad \text{and} \qquad x^2 - x - 1 = 0$$

respectively. A number which is not algebraic is *transcendental.* In other words, $\alpha \in \mathbb{R}$ is transcendental if there is no equation

$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0 \quad \text{with } \alpha_0, \ldots, \alpha_n \in \mathbb{Z} \text{ not all zero, and } n \in \mathbb{N}.$$

In particular if α is transcendental then α is irrational (because if $\alpha = \frac{p}{q}$ then α satisfies the polynomial equation $q\alpha - p = 0$).

In 1882 Ferdinand von Lindemann proved that π is transcendental. This is a truly sophisticated piece of mathematics. There have been numerous modifications and simplifications of his proof; see Niven's book for a proof along the lines of the above argument. Note that this proves that $\pi^{m/n}$ is irrational for all $m, n \in \mathbb{N}$ (with $m, n \neq 0$), for if $\pi^{m/n}$ is rational then $\pi^{m/n} = p/q$ and so $q^n\pi^m - p^n = 0$ and so π is algebraic, a contradiction.

Lindemann's Theorem also shows that the (at least) 2000 year old problem from antiquity of *squaring the circle* is impossible. You'll discuss this in later mathematics courses when you study *Galois Theory.*

12. Let $f_n(x)$ be as in Question 11.

(a) Let $m \in \mathbb{N}\backslash\{0\}$ and define $G_n(x)$ (depending on n and m) by

$$G_n(x) = \sum_{k=0}^{2n} (-1)^k m^{2n-k} f_n^{(k)}(x).$$

Show that $G_n(0)$ and $G_n(1)$ are integers. Calculate $\frac{d}{dx}\left(e^{mx}G_n(x)\right)$ and deduce that

$$m^{2n+1} \int_0^1 e^{mx} f_n(x) dx = e^m G_n(1) - G_n(0).$$

Solution: By Question 11(a) it is clear that $G_n(0)$ and $G_n(1)$ are integers. We compute

$$\frac{d}{dx}\left(e^{mx}G_n(x)\right) = me^{mx}G_n(x) + e^{mx}G_n'(x) = e^{mx}\left(m^{2n+1}f_n(x) + f_n^{(2n+1)}(x)\right)$$

Since $f^{(2n+1)}(x) = 0$ it follows that

$$m^{2n+1}\int_0^1 e^{mx}f_n(x)\,dx = e^m G_n(1) - e^0 G_n(0) = e^m G_n(1) - G_n(0).$$

(b) Now assume that $e^m = \frac{p}{q}$ is rational. Obtain a contradiction.

Solution: If $e^m = \frac{p}{q}$ (with $p, q \in \mathbb{N}\backslash\{0\}$) is rational then

$$J_n := qm^{2n+1}\int_0^1 e^{mx}f_n(x)\,dx = pG_n(1) - qG_n(0)$$

is an integer. But $0 < f_n(x) < \frac{1}{n!}$ for $x \in (0,1)$ implies that

$$0 < J_n < \frac{qm^{2n+1}}{n!}\int_0^1 e^{mx}\,dx = \frac{qm^{2n}}{n!}(e^m - 1).$$

We obtain a contradiction as before: For large enough n we have $0 < J_n < 1$, which is impossible since J_n is an integer.

(c) Deduce that e^r is irrational for all $r \in \mathbb{Q}\backslash\{0\}$.

Solution: If $r = \frac{m}{k} > 0$ is rational and if e^r is rational, then $e^{rk} = e^m$ is rational. Thus e^r is irrational for all rational $r > 0$, and therefore e^{-r} is irrational too.

Remark: In 1873 Hermite proved that e is transcendental. Hermite's Theorem can be proved using similar techniques to the above irrationality proofs, but it is noticeably more difficult! Here is an outline if you are interested.

Step 0: Let $g(x)$ be a polynomial with integer coefficients, and let

$$f_n(x) = \frac{x^{n-1}}{(n-1)!}g(x) \quad \text{where } n \in \mathbb{N}.$$

Show that (i) $f_n^{(k)}(0)$ is an integer for all $k \geq 0$, and (ii) if $k \neq n-1$ then $f_n^{(k)}(0)$ is divisible by n.

Step 1: Suppose that e is algebraic. Therefore e satisfies an equation

$$a_0 + a_1 e + a_2 e^2 + \cdots + a_N e^N = 0$$

with $\alpha_0, \ldots, \alpha_N \in \mathbb{Z}$ not all zero, and $N \in \mathbb{N}$. We can assume without loss of generality that $a_0 \neq 0$.

Step 2: Let

$$f_n(x) = \frac{x^{n-1}}{(n-1)!}(x-1)^n(x-2)^n\cdots(x-N)^n$$

with N as in Step 1 (f_n is of the form of the function in Step 0). Let

$$F_n(x) = \sum_{k=0}^{(N+1)n-1} f_n^{(k)}(x).$$

Show that $\frac{d}{dx}\big(e^{-x}F_n(x)\big) = -e^{-x}f_n(x)$, and deduce that

$$\int_0^j f_n(x)e^{-x}\,dx = F_n(0) - e^{-j}F_n(j) \qquad \text{for } j \in \mathbb{N}.$$

Step 3: Use Steps 1 and 2 to show that

$$\sum_{j=0}^{N}\left(a_j \int_0^j f_n(x)e^{j-x}\,dx\right) = -\sum_{j=0}^{N}\sum_{k=0}^{(N+1)n-1} a_j f_n^{(k)}(j).$$

Step 4: Use Step 0 to deduce that (i) $f_n^{(k)}(j)$ is an integer for all $0 \leq j \leq N$ and all $k \geq 0$, and (ii) if $0 \leq j \leq N$ and $j \geq 0$ then $f_n^{(k)}(j)$ is divisible by n except possibly for the case $k = n-1$ and $j = 0$.
Step 5: Show that $f_n^{(n-1)}(0) = (-1)^{Nn}N!$
Step 6: Use Steps 3 and 4 to show that

$$\sum_{j=0}^{N}\left(a_j \int_0^j f_n(x)e^{j-x}\,dx\right) = I_n \quad \text{is an integer.}$$

Step 7: Now take n to be a large prime with $n > |a_0|$ and $n > N$. Use divisibility properties from Steps 4 and 5 to explain why $I_n \neq 0$.
Step 8: By Step 7 we have (for large prime n)

$$0 < |I_n| = \left|\sum_{j=0}^{N}\left(a_j \int_0^j f_n(x)e^{j-x}\,dx\right)\right| \leq \sum_{j=0}^{N}\left(|a_j| \int_0^j |f_n(x)|e^{j-x}\,dx\right).$$

Arrive at a contradiction by using bound

$$|f_n(x)| \leq \frac{N^{n-1}(N^n)^N}{(n-1)!} = \frac{N^{(N+1)n-1}}{(n-1)!} \quad \text{for } 0 \leq x \leq N.$$

Working through these beautiful proofs makes me think: "*How on earth did they come up with this!*". These were pretty amazing people to say the least.