The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 13

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/JM/MATH1903/
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Material covered

(1) systems of linear differential equations

(2) matrix exponentials (optional)

Outcomes

After completing this tutorial you should

(1) be able to solve systems of two linear differential equations either by reducing them to a second order equation, or by using the eigenvalues and eigenvectors of the system matrix.

Questions to do before the tutorial

1. Reduce the first order system

$$x' = 7x - 5y \qquad y' = 4x - 5y$$

to a second order equation for either x or y, and then find the general solution of the system.

Solution: Differentiate the first equation and then substitute the second equation to get

$$x'' = 7x' - 5y' = 7x' - 5(4x - 5y)$$

From the first equation we have $5y = 7x - x'$, so

$$x'' = 7x' - 5(4x - 5y)7x' - 5(4x - 7x + x') = 2x' + 15x.$$

Hence the equation for x is $x'' - 2x' - 15x = 0$. Its auxiliary equation is $\lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3) = 0$. The general solution of the second order equation for x therefore is

$$x(t) = Ae^{5t} + Be^{-3t}$$

From the first differential equation we can determine $y(t)$, namely

$$5y = 7x - x' = 7Ae^{5t} + 7Be^{-3t} - 5Ae^{5t} + 3Be^{-3t} = 2Ae^{5t} + 10e^{-3t}$$

Hence the solution of the system is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix} e^{5t} + B \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-3t}.$$

Replacing the constant A by a different constant (again denoted by A) we could write the general solution as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{5t} + B \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-3t}.$$

Questions to complete during the tutorial

2. Find the general solution of the system of differential equations

$$\frac{dx}{dt} = 5x - 3y, \qquad \frac{dy}{dt} = 2x$$

by differentating the first equation and then using the second equation to form a linear second order differential equation for $x(t)$. Then find the particular solutions satisfying the initial conditions $x(0) = 2$ and $y(0) = 1$.

Solution: Differentiating the first equation, and then using the second equation gives

$$\frac{d^2x}{dt^2} = 5\frac{dx}{dt} - 3\frac{dy}{dt} = 5\frac{dx}{dt} - 6x.$$

Hence x satisfies the equation $x'' - 5x' + 6x = 0$. The auxiliary equation is $\lambda^2 - 5\lambda + 6 = 0$. Its roots are $\lambda = 3, 2$. Hence the general solution of the second order equation is

$$x(t) = Ae^{3t} + Be^{2t}$$

for constants A, B. Substituting this solution into the first of the differential equations we get

$$y(t) = \frac{1}{3}\Big(5x - \frac{dx}{dt}\Big) = \frac{1}{3}\Big(5Ae^{3t} + 5Be^{2t} - 3Ae^{3t} + 2Be^{2t}\Big) = \frac{2}{3}Ae^{3t} + Be^{2t}.$$

Using the initial conditions we have $x(0) = 2 = A + B$ and $y(0) = 1 = 2A/3 + B$. Solving we get $A = 3$ and $B = -1$ and so the particular solution is

$$x(t) = 3e^{3t} - e^{2t}, \qquad y(t) = 2e^{3t} - e^{2t}.$$

In vector form we can write

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

3. Find the general solution of the system

$$\dot{x} = 2x - y \qquad \dot{y} = x + 2y$$

by reducing the system to a second order equation.

Solution: If we differentiate the first equation, and then use the second equation we get

$$\ddot{x} = 2\dot{x} - \dot{y} = 2\dot{x} - (x + 2y) = 2\dot{x} - x - 2y.$$

From the first equation $y = 2x - \dot{x}$, so we have

$$\ddot{x} = 2\dot{x} - x - 2(2x - \dot{x}) = 4\dot{x} - 5x.$$

Hence we need to solve the second order equation $\ddot{x} - 4\dot{x} + 5x$. Its auxiliary equation is $\lambda^2 - 4\lambda + 4 = 0$. The solutions turn out to be $\lambda = 2 \pm i$, so the general solution is

$$x(t) = e^{2t}(A\cos t + B\sin t).$$

We then determine $y(t)$ from the first equation

$$\begin{aligned} y(t) &= 2x(t) - \dot{x}(t) \\ &= 2e^{2t}(A\cos t + B\sin t) - 2e^{2t}(A\cos t + B\sin t) - e^{2t}(-A\sin t + B\cos t) \\ &= e^{2t}(B\cos t - A\sin t). \end{aligned}$$

In vector form we can write the solutions as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}\cos t + B\begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{2t}\sin t.$$

4. Consider the pair of differential equations

$$\frac{dx}{dt} = 7x - 2y, \qquad \frac{dy}{dt} = 2x + 3y.$$

(a) Obtain a second-order differential equation for $x(t)$ and find its general solution.

Solution: Differentiating the first equation gives $\ddot{x} = 7\dot{x} - 2\dot{y}$. Then using the second equation to eliminate $\dot{y}$ gives

$$\ddot{x} = 7\dot{x} - 2(2x + 3y) = 7\dot{x} - 4x - 6y.$$

But the first equation implies that $y = (7/2)x - (1/2)\dot{x}$, and so the expression for $\ddot{x}$ becomes

$$\ddot{x} = 7\dot{x} - 4x - 3(7x - \dot{x}) = 10\dot{x} - 25x,$$

which is written conveniently in the form

$$\ddot{x} - 10\dot{x} + 25x = 0.$$

The auxiliary equation $\lambda^2 - 10\lambda + 25 = 0$ has a double root $\lambda = 5$, and so the general solution for $x(t)$ is

$$x(t) = Ae^{5t} + Bte^{5t}.$$

(b) Find the associated general solution for $y(t)$.

Solution: The first differential equation implies that $y = (7/2)x - (1/2)\dot{x}$. Hence, with the general solution for $x(t)$ given above, we find that

$$y(t) = \frac{7}{2}\big(Ae^{5t} + Bte^{5t}\big) - \frac{1}{2}\big(5Ae^{5t} + 5Bte^{5t} + Be^{5t}\big) = \left(A - \frac{B}{2}\right)e^{5t} + Bte^{5t}.$$

(c) Show that, if $x(0) > y(0)$, then $x(t)$ and $y(t)$ increase without limit as $t \to \infty$. Conversely, if $x(0) < y(0)$, show that $x(t)$ and $y(t)$ decrease without limit as $t \to \infty$.

Solution: The solutions above give $x(0) = A$ and $y(0) = A - (1/2)B$, and hence $x(0) - y(0) = (1/2)B$. If $x(0) > y(0)$, then $B > 0$, and so we see from the expressions given above that $x(t)$ and $y(t)$ will increase without limit as $t \to \infty$. On the other hand, if $x(0) < y(0)$, then $B < 0$, and we see from the expressions given above that $x(t)$ and $y(t)$ will decrease without limit as $t \to \infty$. (Note that the value of A is immaterial, since for any choice of A we find that $|Bte^{5t}| > |Ae^{5t}|$ and $|Bte^{5t}| > |(A - B/2)e^{5t}|$ for sufficiently large values of t.)

(d) Find the particular solution for $x(t)$ and $y(t)$ satisfying the initial conditions $x(0) = 2$, $\dot{y}(0) = 1$.

Solution: The expressions above for $x(t)$ and $y(t)$ give $x(0) = A$ and $\dot{y}(0) = 5A - (3/2)B$. Hence the initial conditions $x(0) = 2$, $\dot{y}(0) = 1$ indicate that we must set $A = 2$ and $B = 6$. Thus the particular solutions are

$$x(t) = 2e^{5t} + 6te^{5t} = (6t + 2)e^{5t}, \qquad y(t) = -e^{5t} + 6te^{5t} = (6t - 1)e^{5t}.$$

(e) Confirm explicitly that the expressions you have obtained for $x(t)$ and $y(t)$ obey the first order equations given in part (a).

Solution: Differentiating, we get

$$\dot{x} = (30t + 16)e^{5t}, \qquad \dot{y}(t) = (30t + 1)e^{5t}.$$

Moreover,

$$7x - 2y = (42t + 14)e^{5t} - (12t - 2)e^{5t} = (30t + 16)e^{5t}$$

and

$$2x + 3y = (12t + 4)e^{5t} + (18t - 3)e^{5t} = (30t + 1)e^{5t}.$$

Thus $\dot{x} = 7x - 2y$ and $\dot{y} = 2x + 3y$, as required.

Extra questions for further practice

5. Find the general solution of the pair of differential equations,

$$\frac{dx}{dt} = 5x - 3y, \qquad \frac{dy}{dt} = 2x,$$

by differentiating the first equation and then using the second to obtain a linear second-order differential equation for $x(t)$. (Your solution should have *two* arbitrary constants of integration.) Find the particular solution satisfying the initial conditions $x = 2$, $y = 1$ when $t = 0$.

Solution: Differentiating the first equation gives

$$\frac{d^2x}{dt^2} = 5\frac{dx}{dt} - 3\frac{dy}{dt} = 5\frac{dx}{dt} - 6x,$$

where we have used the fact that $dy/dt = 2x$. Hence $x(t)$ satisfies $\ddot{x} - 5\dot{x} + 6x = 0$. The auxiliary equation is $\lambda^2 - 5\lambda + 6 = 0$, which has roots $\lambda = 3, 2$, and so the general solution for x is

$$x = Ae^{3t} + Be^{2t}.$$

Hence $\dot{x} = 3Ae^{3t} + 2Be^{2t}$. But the first-order equation $\dot{x} = 5x - 3y$ implies that

$$\begin{aligned} y = \frac{1}{3}\big(5x - \dot{x}\big) &= \frac{1}{3}\big(5Ae^{3t} + 5Be^{2t} - 3Ae^{3t} - 2Be^{2t}\big) \\ &= \frac{2}{3}Ae^{3t} + Be^{2t}. \end{aligned}$$

At time $t = 0$ we therefore have $x = A + B$ and $y = (2/3)A + B$. But we are told that $x = 2$ and $y = 1$ when $t = 0$, which implies that $A = 3$ and $B = -1$. So the particular solution is

$$x(t) = 3e^{3t} - e^{2t}, \qquad y = 2e^{3t} - e^{2t}.$$

6. The following is a model for an arms race between two superpowers X and Y. Denote the level of preparation of X for war by $x(t)$ and that of Y by $y(t)$, where t represents time. The model consists of the equations,

$$\frac{dx}{dt} = -ax + by, \qquad \frac{dy}{dt} = cx - dy,$$

where a, b, c and d are positive constants.

(a) Eliminate y to obtain a second-order homogeneous linear differential equation for $x(t)$.

Solution: Differentiate the first equation with respect to time to obtain

$$\frac{d^2x}{dt^2} = -a\frac{dx}{dt} + b\frac{dy}{dt},$$

substitute dy/dt from the second,

$$\frac{d^2x}{dt^2} = -a\frac{dx}{dt} + b(cx - dy),$$

and eliminate by by using the first equation again, to get

$$\frac{d^2x}{dt^2} = -a\frac{dx}{dt} + bcx - d\left(\frac{dx}{dt} + ax\right).$$

Rewriting this differential equation in the standard form, we have

$$\frac{d^2x}{dt^2} + (a+d)\frac{dx}{dt} + (ad - bc)x = 0.$$

(b) Write down the auxiliary equation of this differential equation, and its general solution in terms of its roots λ_1 and λ_2. Show that if $ad > bc$ then both roots of the auxiliary equation must be negative. What does this suggest about the likelihood of war?

Solution: The auxiliary equation is

$$\lambda^2 + (a+d)\lambda + (ad - bc) = 0,$$

with roots

$$\lambda_{1,2} = \frac{-(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{-(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

The second of these expressions makes it clear that both roots are real, so the general solution is

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.$$

From the auxiliary equation we see that the sum of the roots $\lambda_1 + \lambda_2 = -(a+d)$ is negative, while the product of the roots $\lambda_1\lambda_2 = ad - bc$ is positive, if $ad > bc$. Thus both roots must be negative.

The solution therefore predicts that X and Y prepare less and less for war. This suggests that war would be increasingly unlikely.

(c) Suppose $a = d = 1$, $b = c = 3$, and that $x = 5$, $y = 1$ at the initial time $t = 0$. Find the particular solution for x and y. What can you conclude about the likelihood of war in this case?

Solution: With $a = d = 1$, $b = c = 3$, the roots of the auxiliary equation are $2, -4$. Thus the general solution for $x(t)$ is

$$x = Ae^{-4t} + Be^{2t},$$

and so $\dot{x} = -4Ae^{-4t} + 2Be^{2t}$. But the first of the two original equations now reads $\dot{x} = -x + 3y$, and hence

$$\begin{aligned} y = \frac{1}{3}\left(\frac{dx}{dt} + x\right) &= \frac{1}{3}\big(-4Ae^{-4t} + 2Be^{2t} + Ae^{-4t} + Be^{2t}\big) \\ &= -Ae^{-4t} + Be^{2t}. \end{aligned}$$

At time $t = 0$ we therefore have $x = A + B$ and $y = -A + B$. But we are told that $x = 5$ and $y = 1$ when $t = 0$, which implies that $A = 2$ and $B = 3$. So the particular solution is

$$x(t) = 2e^{-4t} + 3e^{2t}, \qquad y = -2e^{-4t} + 3e^{2t}.$$

The exponential growth of the terms involving e^{2t} ensures that both X and Y become ever more prepared for war, and so the danger of war may well increase. (However, without a more detailed model, it is not clear that increased military spending will result in a war; it may instead lead to the economic collapse of one of the superpowers.)

(d) Briefly discuss the assumptions underlying the model. Do you think these assumptions are realistic?

Solution: The model assumes that there is no limit on the rate at which military spending can grow. In reality, an exponentially growing military budget cannot be sustained for very long, as the superpowers only have limited resources. The model will inevitably break down as the level of expenditure becomes unaffordable.

7. An electrical circuit comprises a capacitor, a resistor, and an inductor connected in series to a voltage generator with a prescribed voltage $V(t)$. In terms of the electric charge $Q(t)$ and the current $I(t) = dQ/dt$, the voltage drops across the components are Q/C for a capacitance C,

IR for a resistance R, and LdI/dt for an inductance L. The total voltage drop is equal to that supplied by the generator,

$$\frac{Q}{C} + IR + L\frac{dI}{dt} = V(t).$$

(a) Using $I = dQ/dt$, derive the equation,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}.$$

Solution: Just differentiate the given equation $(Q/C) + IR + LdI/dt = V(t)$, using the tacit hypothesis that C, R and L are constants, and the fact that $dQ/dt = I$, and we get the desired equation.

(b) Find the general solution of the homogeneous equation, distinguishing three possible types of behaviour, depending on the values of $R^2 - 4L/C$. Show that, provided $L, R > 0$, all solutions die out as $t \to \infty$.

Solution: The auxiliary equation is $L\lambda^2 + R\lambda + 1/C = 0$, whose roots are

$$\lambda_1, \lambda_2 = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}.$$

Case 1: $R^2 > 4L/C$. Then $0 < R^2 - 4L/C < R^2$ (assuming $L, C > 0$), and so $\sqrt{R^2 - 4L/C} < R$. Hence the numerator of both λ_1 and λ_2 is negative. Thus $\lambda_1, \lambda_2 < 0$. Hence the general solution of the homogeneous equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0$$

is

$$I(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

which clearly tends to 0 as $t \to \infty$, because $\lambda_1, \lambda_2 < 0$.

Case 2: $R^2 < 4L/C$. Then the two roots λ_1 and λ_2 are

$$\lambda_1, \lambda_2 = -\frac{R}{2L} \pm i\omega, \quad \text{where} \quad \omega = \frac{\sqrt{4L/C - R^2}}{2L}.$$

Now the general solution of the homogeneous equation is

$$I(t) = e^{-Rt/(2L)}\big(c_1 \cos \omega t + c_2 \sin \omega t\big).$$

This clearly tends to 0 as $t \to \infty$, because $-R/(2L) < 0$.

Case 3: $R^2 = 4L/C$. Then the only root of the auxiliary equation is $-R/(2L)$, this being a double root. Now the general solution of the homogeneous equation is

$$I(t) = e^{-Rt/(2L)}\big(c_1 + c_2 t\big).$$

This clearly tends to 0 as $t \to \infty$, because $-R/(2L) < 0$. We use here the fact that $xe^{-x} \to 0$ as $x \to \infty$, as you can see from L'Hopital's rule, for example.

(c) Find a particular solution for an AC voltage source $V(t) = V_0 \cos(\omega_0 t)$.

Solution: As the forcing term is $V_0 \cos(\omega_0 t)$ there will be a particular solution of the form, $I(t) = A\cos\omega_0 t + B \sin \omega_0 t$ for some A and B to be determined. Take this $I(t)$ and calculate $Ld^2I/dt^2 + RdI/dt + I/C$, and set this equal to $V_0 \cos \omega_0 t$. The result is

$$\big\{-L\omega_0^2 A + R\omega_0 B + A/C\big\} \cos \omega_0 t + \big\{-L\omega_0^2 B - R\omega_0 A + B/C\big\} \sin \omega_0 t = V_0 \cos \omega_0 t.$$

Equating the coefficients of $\cos\omega_0 t$ and $\sin\omega_0 t$ and solving for A and B gives

$$A = \frac{C(1 - CL\omega_0^2)V_0}{(1 - CL\omega_0^2)^2 + (CR\omega_0)^2}, \qquad B = \frac{C^2 R\omega_0 V_0}{(1 - CL\omega_0^2)^2 + (CR\omega_0)^2}.$$

Hence the particular solution of the postulated form is

$$I(t) = CV_0 \frac{(1 - CL\omega_0^2)\cos\omega_0 t + CR\omega_0 \sin\omega_0 t}{(1 - CL\omega_0^2)^2 + (CR\omega_0)^2}.$$

Adding the homogeneous solution and giving the integration constants particular values yields an infinite supply of particular solutions. All of these differ from the solution just found by exponentially decaying terms when $R > 0$ and $L > 0$. Physically, this means that, when an AC circuit is switched on, there is, in general, a brief transient current after which the current settles into a steady sinusoidal form with the same frequency as the applied voltage.

Challenge questions (optional)

In lectures we showed that solutions to a linear systems of differential equations can be obtained by using methods of linear algebra, in particular eigenvalues and eigenvectors. In vector notation we saw that a system takes the form

$$\frac{d}{dt}\boldsymbol{u}(t) = M\boldsymbol{u}(t)$$

with M being a matrix. Such an equation looks like the standard equation $x' = mx$ in one dimension. Its solution is

$$x(t) = Ce^{mt}.$$

We can try a similar idea for systems and write a matrix exponential

$$\boldsymbol{u}(t) = e^{Mt}.$$

But what is a matrix exponential? We can define it by means of the *exponential series*

$$e^{Mt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} M^k.$$

It turns out that this series converges. In the following exercises you find examples how to compute theses exponentials using your knowledge from first semester linear algebra MATH1902.

8. The system of differential equations from Question 2 can alternatively be written in vector form

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

If we set $u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and denote the matrix by M the above can be written as $\boldsymbol{u}' = M\boldsymbol{u}$. The solution with initial condition $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \boldsymbol{u}_0$ can be written as the matrix exponential $\boldsymbol{u}(t) = e^{Mt}\boldsymbol{u}_0$. In practice, computing the matrix exponential is often unnecessary, and the general solution is sufficent. If desired the matrix exponential can be obtained from that.

(a) Compute the eigenvalues and eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of the matrix M by first finding the roots λ_1, λ_2 of the characteristic polynomial $\det(M - \lambda I)$.

Solution: The characteristic polynomial is

$$\det(M - \lambda I) = \det\begin{bmatrix} 5-\lambda & -3 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3),$$

so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. The eigenvectors we get by reducing $M - \lambda I$. For $\lambda_1 = 2$ we get

$$\begin{bmatrix} 5-2 & -3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

so $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector. Similarly, for $\lambda_2 = 3$ we get

$$\begin{bmatrix} 5-3 & -3 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \to \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$$

so $\boldsymbol{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector.

(b) From the previous part $Ae^{\lambda_1 t}\boldsymbol{v}_1 + Be^{\lambda_2 t}\boldsymbol{v}_2$ is the general solution of the system of equations. Determine the solution with initial values $x(0) = 2$ and $y(0) = 1$. Compare the result with that in Question 2.

Solution: From the previous part

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + B \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t}$$

is the general solution. Using the initial conditions we get

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + B \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

We solve that system using methods from linear algebra: we row-reduce the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

which means that $A = -1$ and $B = 1$. Substituting into the general solution we get

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t}$$

which is the same as in Question 2.

(c) Here we want to describe a way to get the matrix exponential e^{tM}. Form the matrix $X(t) := \begin{bmatrix} e^{\lambda_1 t}\boldsymbol{v}_1 & e^{\lambda_2 t}\boldsymbol{v}_2 \end{bmatrix}$ with columns given by $e^{\lambda_2 t}\boldsymbol{v}_2$ and $e^{\lambda_t 2}\boldsymbol{v}_2$.

(i) Show that $\dfrac{d}{dt}X(t) = MX(t)$, where differentiation of matrices is done on each entry.

Solution: Differentiating we get

$$\frac{d}{dt}X(t) = \begin{bmatrix} e^{\lambda_1 t}\lambda_1\boldsymbol{v}_1 & e^{\lambda_2 t}\lambda_2\boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} Me^{\lambda_1 t}\boldsymbol{v}_1 & Me^{\lambda_2 t}\boldsymbol{v}_2 \end{bmatrix} = MX(t)$$

as required.

(ii) Assuming that the solution of $\boldsymbol{u}' = M\boldsymbol{u}$ with given initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$ is unique, show that $e^{Mt} = X(t)X_0^{-1}$. (For the uniqueness, see Question 10.)

Solution: For any given initial value $\boldsymbol{u}_0$ both $e^{Mt}\boldsymbol{u}_0$ solves the initial value problem $\boldsymbol{u}' = M\boldsymbol{u}$, $\boldsymbol{u}(0) = \boldsymbol{u}_0$. By the previous part we have

$$\frac{d}{dt}X(t)X(0)^{-1}\boldsymbol{u}_0 = MX(t)X(0)^{-1}\boldsymbol{u}_0$$

and $X(0)X(0)^{-1}\boldsymbol{u}_0 = \boldsymbol{u}_0$. Hence $X(t)X(0)^{-1}$ is a solution to the same equation, and therefore $e^{Mt} = X(t)X_0^{-1}$ as claimed.

(d) Use the above method to compute e^{Mt}.

Solution: In the present case

$$X(t) = \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}$$

and so

$$X(0)^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

and therefore

$$e^{tM} = X(t)X(0)^{-1} = \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{bmatrix}.$$

9. Consider the system of differential equations $x' = -x-y$, $y' = x-3y$. Denote by $M = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ the matrix associated with that system.

(a) Show that M has a double eigenvalue $\lambda = -2$ and one independent eigenvector $\boldsymbol{v}$ only.

Solution: The characteristic polynomial is

$$\det \begin{bmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{bmatrix} = (1+\lambda)(3+\lambda) + 1 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2,$$

so $\lambda = -2$ is a double eigenvalue. The eigenvector is computed by row reduction

$$\begin{bmatrix} -1+2 & -1 \\ 1 & -3+2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

so $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector, and up to scalar multiples the only one.

(b) To find a second solution try a solution of the form $(\boldsymbol{w} + t\boldsymbol{v})e^{-2t}$, where $\boldsymbol{w}$ is to be determined and $\boldsymbol{v}$ is the eigenvector.

Solution: We differentiate the given function and use that $\boldsymbol{v}$ is an eigenvector:

$$\frac{d}{dt}(\boldsymbol{w} + t\boldsymbol{v})e^{-2t} = -2(\boldsymbol{w} + t\boldsymbol{v})e^{-2t} + \boldsymbol{v}e^{-2t} = M(\boldsymbol{w} + t\boldsymbol{v})e^{-2t}$$

Dividing by e^{-2t} and using that -2 is an eigenvalue we get

$$-2\boldsymbol{w} + \boldsymbol{v} = M\boldsymbol{w},$$

so we have to solve the system $(M + 2I)\boldsymbol{w} = \boldsymbol{v}$. Setting $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ we get

$$(M + 2I)\boldsymbol{w} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 - w_2 \\ w_1 - w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence $\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a solution (there are many solutions!), so that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} 1+t \\ t \end{bmatrix} e^{-2t}$$

is a second solution of the given system of differential equations.

(c) Use the idea from Question 2(c) to compute e^{Mt}.

Solution: Form the matrix

$$X(t) := \begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix} e^{-2t}$$

whose columns are the two independent solutions of the system. We know that

$$\begin{aligned} e^{tM} = X(t)X(0)^{-1} = e^{-2t} \begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \\ = e^{-2t} \begin{bmatrix} 1 & t \\ 1 & 1+t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = e^{-2t} \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}. \end{aligned}$$

10. We show the uniqueness of the solution of $\boldsymbol{u}' = M\boldsymbol{u}$, where M is a matrix.

(a) From the definition of the matrix exponential, show that $Me^{-tM} = e^{tM}M$. Differentiate $e^{tM}e^{tM}$ and hence show that $e^{-tM}e^{tM} = I$ is constant.

Solution: From the series definition of the matrix exponential

$$Me^{tM} = M\sum_{k=0}^{\infty} \frac{t^k}{k!} M^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} MMk = \sum_{k=0}^{\infty} \frac{t^k}{k!} M^k M = \Big(\sum_{k=0}^{\infty} \frac{t^k}{k!} M^k\Big) M = e^{tM}M.$$

Hence if we differentiate, using the product rule

$$\frac{d}{dt} e^{-tM}e^{tM} = -Me^{-tM}e^{tM} + e^{-tM}Me^{tM} = -Me^{-tM}e^{tM} + Me^{-tM}e^{tM} = 0.$$

The product rule also applies to matrix multiplication, but we have to be careful to keep the order of the factors because matrices do not necessarily commute. Hence the product is constant and so

$$e^{-tM}e^{tM} = e^{-0M}e^{0M} = I$$

for all $t \in \mathbb{R}$. Hence e^{-tM} is the inverse of the matrix e^{tM}.

(b) Suppose that $X(t)$ is a solution. Differentiate $e^{-tM}X(t)$ and hence show it is constant. Deduce that the solution is unique and that $e^{tM} = X(t)X(0)^{-1}$

Solution: Differentiating similary as in the previous part

$$\frac{d}{dt} e^{-tM}X(t) = -Me^{-tM}X(t) + e^{-tM}MX(t) = -Me^{-tM}X(t) + Me^{-tM}X(t) = 0.$$

Hence the product is constant, that is, $e^{-tM}X(t) = X(0)$ for all $t \in \mathbb{R}$. Using the previous part $X(t) = e^{tM}X(0)$ is unique.