The University of Sydney
School of Mathematics and Statistics

Solutions to Assignment 1

MATH1902: Linear Algebra (Advanced) — Semester 1, 2017

Web Page: http://sydney.edu.au/science/maths/u/UG/JM/MATH1902/
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1. Given any vector space V we can define a subspace U. A non-empty subset U of V is called subspace, if for any two vectors $\mathbf{u}, \mathbf{v} \in U$ and any scalar λ we have $\mathbf{u}+\mathbf{v} \in U$ and $\lambda\mathbf{u} \in U$. So in words, a subset of vectors is a subspace, if the sum of any two vectors in the subset and the scalar multiple of any vector in the subset is also part of the subset. Now let $V = \mathbb{R}^3$ be the space of geometric vectors in three dimensions and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the standard unit vectors.

(a) For each of the following subsets U of V, either show that it is a subspace, or give a counterexample that shows that it is not a subspace.

(i) $U = \{\mathbf{i} - 3\mathbf{j}, \mathbf{i} - \mathbf{j} - \mathbf{k}, -2\mathbf{j} + \mathbf{k}, \mathbf{0}\}$, the set of the 4 given vectors

Solution: This subset is not a subspace because for example
$(\mathbf{i} - 3\mathbf{j}) + (\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2\mathbf{i} - 4\mathbf{j} - \mathbf{k} \notin U$ or $2(\mathbf{i} - 3\mathbf{j}) = 2\mathbf{i} - 6\mathbf{j} \notin U$.

(ii) $U = \{\alpha\mathbf{j} \mid 0 \neq \alpha \in \mathbb{R}\}$

Solution: This subset is not a subspace because, in particular, the zero-vector is not included. So a counterexample could be either $\mathbf{j} + (-\mathbf{j}) = \mathbf{0} \notin U$ or $0\mathbf{j} = \mathbf{0} \notin U$.

(iii) $U = \{\alpha(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \mid \alpha \in \mathbb{R}\}$

Solution: Take two vectors $\mathbf{u}$ and $\mathbf{v}$ from U. So there exist real numbers α_1 and α_2 such that $\mathbf{u} = \alpha_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $\mathbf{v} = \alpha_2(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$. Then

$$\mathbf{u}+\mathbf{v} = \alpha_1(\mathbf{i}+2\mathbf{j}+3\mathbf{k})+\alpha_2(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = (\alpha_1+\alpha_2)(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = \alpha(\mathbf{i}+2\mathbf{j}+3\mathbf{k}),$$

where we use distributivity of the scalar product and have $\alpha = (\alpha_1 + \alpha_2) \in \mathbb{R}$. Thus, $\mathbf{u} + \mathbf{v} \in U$.
Similarly, for any $\lambda \in \mathbb{R}$

$$\lambda\mathbf{u} = \lambda(\alpha_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})) = (\lambda\alpha_1)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = \alpha(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

where we use associativity of the scalar product and have $\alpha = \lambda\alpha_1 \in \mathbb{R}$. Hence, $\lambda\mathbf{u} \in U$.
Both conditions are satisfied and so U is a subspace.

(iv) $U = \{\alpha\mathbf{i} + \beta\mathbf{k} \mid \alpha, \beta \in \mathbb{R}\}$

Solution: Take two vectors $\mathbf{u}$ and $\mathbf{v}$ from U. So there exist real numbers α_1, β_1 and α_2, β_2 such that $\mathbf{u} = \alpha_1\mathbf{i} + \beta_1\mathbf{k}$ and $\mathbf{v} = \alpha_2\mathbf{i} + \beta_2\mathbf{k}$. Then

$$\mathbf{u} + \mathbf{v} = (\alpha_1\mathbf{i} + \beta_1\mathbf{k}) + (\alpha_2\mathbf{i} + \beta_2\mathbf{k}) = (\alpha_1 + \alpha_2)\mathbf{i} + (\beta_1 + \beta_2)\mathbf{k} = \alpha\mathbf{i} + \beta\mathbf{k},$$

where we use distributivity of the scalar product and have $\alpha = (\alpha_1 + \alpha_2), \beta = (\beta_1 + \beta_2) \in \mathbb{R}$. Thus, $\mathbf{u} + \mathbf{v} \in U$.
Similarly, for any $\lambda \in \mathbb{R}$

$$\lambda\mathbf{u} = \lambda(\alpha_1\mathbf{i} + \beta_2\mathbf{k}) = (\lambda\alpha_1)\mathbf{i} + (\lambda\beta_1)\mathbf{k} = \alpha\mathbf{i} + \beta\mathbf{k},$$

where we use distributivity and associativity of the scalar product and have $\alpha = \lambda\alpha_1, \beta = \lambda\beta_1 \in \mathbb{R}$. Hence, $\lambda\mathbf{u} \in U$.
Both conditions are satisfied and so U is a subspace.

(v) $U = \{\mathbf{v} \mid |\mathbf{v}| \leq 1\}$, the set of vectors with length up to 1

Solution: A vector in U is for example the vector $\mathbf{i}$ because it is a unit vector and so $|\mathbf{i}| = 1 \leq 1$ (so in particular any unit vector is an element of the set). But then $\mathbf{i} + \mathbf{i} = 2\mathbf{i}$ is not in the set because

$$|2\mathbf{i}| = |2||\mathbf{i}| = 2 \cdot 1 = 2 \nleq 1.$$

Similarly, we can argue with the scalar multiple of $\mathbf{i}$, so take $\lambda = 2$ then for the same reason as above $2\mathbf{i} \notin U$.

(vi) $U = \{\mathbf{0}\}$, the set just containing the zero vector

Solution: Since there is just one vector in U we have for any two $\mathbf{u}, \mathbf{v} \in U$ that $\mathbf{u} = \mathbf{v} = \mathbf{0}$. So $\mathbf{u} + \mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0} \in U$, where we use the property of the zero-vector as the neutral element. Similarly, for any $\lambda \in \mathbb{R}$ we use the properties of the zero-vector to get $\lambda\mathbf{u} = \lambda\mathbf{0} = \mathbf{0} \in U$. So both conditions are satisfied and U is a subspace. Note that this subspace which just consists of the zero-vector is the smallest possible subspace and the only one with dimension 0.

(b) Consider U from part (iv). Write down two vectors $\mathbf{u}, \mathbf{v} \in U$ that are linearly independent. Show that if you take any third vector $\mathbf{w} \in U$, then the three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent. (This establishes that the dimension of U is 2.)

Solution: A good choice for $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} = 1\mathbf{i} + 0\mathbf{k} = \mathbf{i}$ and $\mathbf{v} = 0\mathbf{i} + 1\mathbf{k} = \mathbf{k}$. These two are linearly independent because given any linear combination $\alpha\mathbf{i} + \beta\mathbf{k}$ only adds up to the zero-vector if $\alpha = \beta = 0$. If at least one of these scalars is non-zero the length of the resulting vector is bigger than zero and so it can't be the zero vector.
Now suppose we take any vector $\mathbf{w}$ from U then there exist scalar α and β such that $\mathbf{w} = \alpha\mathbf{i} + \beta\mathbf{k}$. So looking at a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ which adds up to the zero-vector we get

$$\begin{aligned}\lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \lambda_3\mathbf{w} = \lambda_1\mathbf{i} + \lambda_2\mathbf{k} + \lambda_3(\alpha\mathbf{i} + \beta\mathbf{k}) &= \mathbf{0}\\ (\lambda_1 + \lambda_3\alpha)\mathbf{i} + (\lambda_2 + \lambda_3\beta)\mathbf{k} &= \mathbf{0}\end{aligned}$$

Now $\mathbf{i}$ and $\mathbf{k}$ are linearly independent, so the scalar factors need to be zero. Hence,

$$\begin{aligned}\lambda_1 + \lambda_3\alpha &= 0\\ \lambda_2 + \lambda_3\beta &= 0\end{aligned}$$

But choosing $\lambda_3 = 1 \neq 0$, $\lambda_1 = -\alpha$, and $\lambda_2 = -\beta$ is a combination which satisfies both equations and has at least one non-zero scalar, no matter what α and β are. So

$$-\alpha\mathbf{u} - \beta\mathbf{v} + 1\mathbf{w} = -\alpha\mathbf{i} - \beta\mathbf{k} + (\alpha\mathbf{i} + \beta\mathbf{k}) = \mathbf{0},$$

adds up non-trivially to the zero-vector and so the set of vectors is linearly dependent.

(c) Show that for any subspace U of V, the zero vector $\mathbf{0}$ is part of U.

Solution: Let U be a subspace of V then it is by definition a non-empty subset of V. So there exists at least one vector $\mathbf{u}$ in U. But since U is a subspace $(-1)\mathbf{u}$ also has to be in U. Hence, the sum of both, too. But $\mathbf{u} + (-1)\mathbf{u} = \mathbf{u} - \mathbf{u} = \mathbf{0}$, so $\mathbf{0} \in U$.
An even quicker argument is the following. Given that there has to be at least one vector $\mathbf{u}$ in U, there also needs to be the vector $0\mathbf{u} = \mathbf{0}$ in U.

2. Recall that the eight axioms of a vector space are

1)	$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	commutative addition
2)	$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$	associative addition
3)	$\mathbf{v}+\mathbf{0}=\mathbf{v}$	existence of zero vector
4)	$\mathbf{v}+(-\mathbf{v})=\mathbf{0}$	existence of additive inverse
5)	$\lambda(\mu\mathbf{v})=(\lambda\mu)\mathbf{v}$	associative scalar multiplication
6)	$\lambda(\mathbf{u}+\mathbf{v})=\lambda\mathbf{u}+\lambda\mathbf{v}$	distributive I
7)	$(\lambda+\mu)\mathbf{v}=\lambda\mathbf{v}+\mu\mathbf{v}$	distributive II
8)	$1\mathbf{v}=\mathbf{v}$	scalar 1 is neutral element

where $\mathbf{u},\mathbf{v},\mathbf{w}$ are arbitrary vectors and λ,μ are scalars.

(a) Consider the set of polynomials $a+bx+cx^2$ in the variable x of up to degree 2. Here a,b,c are real numbers and the scalars are real numbers as well. Addition of 'vectors' is the usual addition of polynomials, and scalar multiplication is the usual multiplication of a polynomial by a real number. Show that there is a polynomial that acts like the zero vector and satisfies axiom 3. Show that for every polynomial $\mathbf{v}$ in the space there is a polynomial $\mathbf{u}$ such that $\mathbf{v}+\mathbf{u}=\mathbf{0}$, establishing axiom 4. (In fact all eight axioms hold, so this is a vector space.)

Solution: Axiom 3: We claim that $\mathbf{0}=0+0x+0x^2$. Consider a general polynomial $\mathbf{v}=a+bx+cx^2$ and verify $\mathbf{v}+\mathbf{0}=a+bx+cx^2+0+0x+0x^2=(a+0)+(b+0)x+(c+0)x^2=a+bx+cx^2=\mathbf{v}$.

Axiom 4: Again let $\mathbf{v}=a+bx+cx^2$. Now we claim that the polynomial $\mathbf{u}=(-1)\mathbf{v}=-a-bx-cx^2$ is the additive inverse. In fact, $\mathbf{v}+\mathbf{u}=a+bx+cx^2-a-bx-cx^2=0+0x+0x^2=\mathbf{0}$.

(b) Consider the set of ordered pairs of real numbers (a,b). The operation $+$ for ordered pairs is defined by $(a,b)+(c,d)=(a+c,b+d)$ and scalar multiplication is defined by $\lambda(a,b)=(\lambda a,\lambda b)$. Carefully prove that the axiom 1, 2, and 7 hold. (In fact all eight axioms hold, so this is a vector space.)

Solution: Consider three arbitrary ordered pairs (a,b), (c,d), and (e,f) with $a,b,c,d,e,f\in\mathbb{R}$ and scalars $\lambda,\mu\in\mathbb{R}$. For each proof we write a sequence of P, S, R corresponding to whether the corresponding equal sign in the proof uses the definition of plus, scalar multiplication, or properties of the real numbers, respectively.

Axiom 1: $(a,b)+(c,d)=(a+c,b+d)=(c+a,d+b)=(c,d)+(a,b)$. PRP

Axiom 2: $(a,b)+((c,d)+(e,f))=(a,b)+(c+e,d+f)=(a+(c+e),b+(d+f))=((a+c)+e),(b+d)+f)=(a+c,b+d)+(e,f)=((a,b)+(c,d))+(e,f)$. PPRPP

Axiom 7: $(\lambda+\mu)(a,b)=((\lambda+\mu)a,(\lambda+\mu)b)=(\lambda a+\mu a,\lambda b+\mu b)=(\lambda a,\lambda b)+(\mu a,\mu b)=\lambda(a,b)+\mu(a,b)$. SRPS

(c) Consider the set of ordered pairs of real numbers (a,b). The operation $+$ in this case is defined by $(a,b)+(c,d)=((a+c)/2,(b+d)/2)$ and scalar multiplication is defined by $\lambda(a,b)=(\lambda a,\lambda b)$. Consider axioms 1, 2, 6, and 7. For each one, either prove that the axiom holds, or give an example that shows that it does not hold.

Solution: Axiom 1 holds: $(a,b)+(c,d)=((a+c)/2,(b+d)/2)$ where on the right the $+$ is the ordinary $+$ between real numbers, which is commutative. Now starting with the right hand side $(c,d)+(a,b)=((c+a)/2,(d+b)/2)=((a+c)/2,(b+d)/2)$, which is the same, and hence $(a,b)+(c,d)=(c,d)+(a,b)$.

Axiom 2 does not hold: $(4,4) + ((2,2) + (2,2)) = (4,4) + (2,2) = (3,3)$, while starting on the right hand side gives $((4,4) + (2,2)) + (2,2) = (3,3) + (2,2) = (5/2, 5/2)$, which is different.

Axiom 6 holds: $\lambda((a,b) + (c,d)) = \lambda((a+c)/2, (b+d)/2) = (\lambda(a+c)/2, \lambda(b+d)/2)$ Starting from the right hand side gives $\lambda(a,b) + \lambda(c,d) = (\lambda a, \lambda b) + (\lambda c, \lambda d) = (\lambda(a+c)/2, \lambda(b+d)/2)$, which is the same.

Axiom 7 does not hold: $(1+2)(3,4) = 3(3,4) = (9,12)$, while starting on the right hand side gives $1(3,4) + 2(3,4) = (3,4) + (6,8) = (9/2, 6)$, which is different.