CHAPTER **FIVE**

Polynomials

This chapter is an extension of the work already done on polynomials in the Mathematics Extension 1 course. That work is assumed knowledge though some parts of the theory are repeated here for the sake of convenience. The focus of the chapter is on polynomials with real coefficients and the relationships with the zeroes, particularly when they are either complex, or real and repeated.

The pinnacle of the chapter is in Section 5B where the Fundamental Theorem of Algebra is presented along with some of its consequences. The theorem is left unproven as any proof is beyond the scope of the course.

The chapter concludes with harder questions on the relationships between the zeroes and coefficients of a polynomial, and simple examples of how the zeroes may be transformed by use of suitable substitutions.

5A Zeroes and Remainders

Polynomials with Integer Coefficients: If a polynomial with integer coefficients has an integer zero $x = k$, then k is a factor of the constant term. This is a significant aid in factorising a polynomial.

Worked Exercise: It is known that the polynomial $P(x) = x^3 - x^2 - 8x - 6$ has only one integer zero. Find it and hence factorise $P(x)$ completely.

Solution: Since the zero is a factor of 6, the possible values are: $\pm 1, \pm 2, \pm 3$. Testing these one by one:

$$P(1) = -14\,, \qquad P(-1) = 0\,,$$

and there is no need to continue further. By the factor theorem, $(x + 1)$ is a factor of $P(x)$. Performing the long division:

$$\begin{array}{r|l}
 & \; x^2 - 2x - 6 \\ \hline
(x+1) & x^3 - \;\, x^2 - 8x - 6 \\
 & \underline{x^3 + \;\, x^2} \\
 & \quad - 2x^2 - 8x - 6 \\
 & \quad \underline{- 2x^2 - 2x} \\
 & \qquad\quad - 6x - 6 \\
 & \qquad\quad \underline{- 6x - 6} \\
 & \qquad\qquad\qquad 0
\end{array}$$

Thus $P(x) = (x+1)(x^2 - 2x - 6)$
$= (x+1)\Big((x-1)^2 - 7\Big)$ (completing the square)
$= (x+1)(x - 1 - \sqrt{7})(x - 1 + \sqrt{7})$ (difference of two squares.)

1 INTEGER COEFFICIENTS AND ZEROES: If the polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

with integer coefficients a_0, a_1, $a_2, \ldots, a_n$, has an integer zero $x = k$, then k is a factor of the constant term a_0.

PROOF: Since $P(k) = 0$, it follows that

$$a_0 + a_1 k + a_2 k^2 + \ldots + a_n k^n = 0$$

so $a_1 k + a_2 k^2 + \ldots + a_n k^n = -a_0$

thus $k \times (a_1 + a_2 k + \ldots + a_n k^{n-1}) = -a_0\,.$

Since all the terms in the brackets are integers, it follows that the result is also an integer. Thus the left hand side is the product of two integers. Hence, as asserted, k is a factor of a_0.

Polynomials and Complex Numbers: Consider the general polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n\,.$$

Each term in this expression involves an integer power and multiplication by a constant. The terms are then simply added. Since integer powers, multiplication and addition are all natural operations with complex numbers, it follows that the polynomial can be evaluated when x is a complex number. For example if $P(x) = x^2 - 2x + 4$ then at $x = i$ its value is

$$\begin{aligned} P(i) &= i^2 - 2i + 4 \\ &= 3 - 2i\,. \end{aligned}$$

In some examples the polynomial will be written as a function of z in order to emphasise the fact that complex numbers may be substituted. Thus the above example may be written as $P(z) = z^2 - 2z + 4\,.$

It will be necessary in this course to occasionally consider polynomials where the coefficients are also complex numbers. For example, $P(z) = 2z^2 + (1+i)z + 3i\,$, for which

$$\begin{aligned} P(i) &= 2i^2 + (1+i)i + 3i \\ &= -3 + 4i\,. \end{aligned}$$

Remainders and Factors: Here is a quick summary of certain important results from the Mathematics Extension 1 course. In the usual notation, let $P(x)$ and $D(x)$ be any pair of polynomials, where $D(x) \neq 0$. There is a unique pair of polynomials $Q(x)$ and $R(x)$, such that

$$P(x) = D(x) \times Q(x) + R(x)\,,$$

and where either

$$\deg(D) > \deg(R) \quad \text{or} \quad R(x) = 0\,.$$

This is known as the division algorithm. As a consequence, if $D(x) = (x - \alpha)$ then $R(x)$ must be a constant, either zero or non-zero. Let this constant be r. Re-writing the division algorithm:

$$P(x) = (x - \alpha) \times Q(x) + r\,,$$

whence $\quad P(\alpha) = r\,,$

which is known as the remainder theorem.

If $R(x) = 0$ then from the division algorithm we may write

$$P(x) = D(x) \times Q(x)\,,$$

so that $P(x)$ is a product of the factors $D(x)$ and $Q(x)$. In particular, $x - \alpha$ is a factor of $P(x)$ if and only if $P(\alpha) = 0$. This is known as the factor theorem.

The division algorithm, the remainder theorem and the factor theorem are valid for complex numbers as well as real numbers. Though these claims will not be proven here, the results may be freely applied to solve problems.

WORKED EXERCISE: Let $P(x) = x^3 - 2x^2 - x + k$, where k is real.

(a) Show that $P(i) = (2 + k) - 2i$.

(b) When $P(x)$ is divided by $x^2 + 1$ the remainder is $4 - 2x$. Find the value of k.

SOLUTION: (a)
$$\begin{aligned} P(i) &= i^3 - 2i^2 - i + k \\ &= -i + 2 - i + k \\ &= (2 + k) - 2i\,. \end{aligned}$$

(b) By the division algorithm we may write

$$P(x) = (x^2 + 1) \times Q(x) + 4 - 2x\,.$$

Thus $\quad P(i) = 4 - 2i$

whence $\quad (2 + k) - 2i = 4 - 2i\,.$

Equating the real parts gives $k = 2$.

Real Coefficients and Remainders: Suppose that the polynomial $P(z)$ has real coefficients. If the remainder when $P(z)$ is divided by $(z - \alpha)$ is β then the remainder when $P(z)$ is divided by $(z - \overline{\alpha})$ is $\overline{\beta}$. Using the remainder theorem, this is equivalent to the statement that if $P(\alpha) = \beta$ then $P(\overline{\alpha}) = \overline{\beta}$.

WORKED EXERCISE:

(a) Use the remainder theorem to find the remainder when $P(z) = z^3 - 2z^2 + 3z - 1$ is divided by $(z - i)$.

(b) Hence find the remainder when $P(z)$ is divided by $(z + i)$.

SOLUTION:

(a) The remainder is:

$$\begin{aligned} P(i) &= i^3 - 2i^2 + 3i - 1 \\ &= 1 + 2i\,. \end{aligned}$$

(b) It is:
$$\begin{aligned} P(-i) &= P(\,\overline{i}\,) \\ &= \overline{1 + 2i} \\ &= 1 - 2i\,. \end{aligned}$$

2 | REAL COEFFICIENTS AND REMAINDERS: If the polynomial $P(z)$ has real coefficients and if $P(\alpha) = \beta$ then $P(\overline{\alpha}) = \overline{\beta}$.

The proof is not too difficult and is dealt with in a question of the exercise.

Real Coefficients and Complex Zeroes: Suppose that the polynomial $P(z)$ has real coefficients. If $P(z)$ has a complex zero $z = \alpha$ then it is guaranteed to have a second complex zero $z = \overline{\alpha}$. Further, by the factor theorem, there exists another polynomial $Q(z)$ such that:

$$\begin{aligned} P(z) &= (z-\alpha)(z-\overline{\alpha}) \times Q(z) \\ &= (z^2 - (\alpha+\overline{\alpha})z + \alpha\overline{\alpha}) \times Q(z) \\ &= (z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2) \times Q(z)\,. \end{aligned}$$

Thus $P(z)$ has a quadratic factor with real coefficients: $(z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2)$.

WORKED EXERCISE: Consider the polynomial $P(z) = 2z^3 - 3z^2 + 18z + 10$.

(a) Given that $1 - 3i$ is a zero of $P(z)$, explain why $1 + 3i$ is another zero.

(b) Find the third zero of the polynomial.

(c) Hence write $P(z)$ as a product of:

  (i) linear factors,

  (ii) a linear factor and a quadratic factor, both with real coefficients.

SOLUTION: (a) Since $P(z)$ has real coefficients, $\overline{(1-3i)} = 1 + 3i$ is also a zero.

(b) Let the third zero be a, then by the sum of the roots:

$$\begin{aligned} a + (1-3i) + (1+3i) &= \tfrac{3}{2} \\ \text{so} \qquad a + 2 &= \tfrac{3}{2} \\ \text{and} \qquad a &= -\tfrac{1}{2}\,. \end{aligned}$$

(c) (i) By the factor theorem:

$$\begin{aligned} P(z) &= 2(z+\tfrac{1}{2})(z-1+3i)(z-1-3i) \\ &= (2z+1)(z-1+3i)(z-1-3i)\,. \end{aligned}$$

(ii) $P(z) = (2z+1)(z^2 - 2z + 10)$.

> **3** REAL COEFFICIENTS AND ZEROES: If the polynomial $P(z)$ has real coefficients and a complex zero $z = \alpha$ then it is guaranteed to have a second complex zero $z = \overline{\alpha}$.
>
> Consequently $P(z)$ has $(z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2)$ as a factor, which is a quadratic with real coefficients.

PROOF: Suppose that the complex number $z = \alpha$ is a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n\,,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are all real. That is $P(\alpha) = 0$. Then

$$\begin{aligned} P(\overline{\alpha}) &= a_0 + a_1\overline{\alpha} + a_2\overline{\alpha}^{\,2} + \ldots + a_n\overline{\alpha}^{\,n} \\ &= a_0 + a_1\overline{\alpha} + a_2\overline{\alpha^2} + \ldots + a_n\overline{\alpha^n} \quad &&(\text{since } \overline{z}^{\,n} = \overline{z^n}) \\ &= \overline{a_0} + \overline{a_1\alpha} + \overline{a_2\alpha^2} + \ldots + \overline{a_n\alpha^n} &&(\text{since } c\,\overline{z} = \overline{cz} \text{ for real } c) \\ &= \overline{a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_n\alpha^n} &&(\text{since } \overline{w} + \overline{z} = \overline{w+z}) \\ &= \overline{P(\alpha)} \\ &= \overline{0} \\ &= 0\,. \end{aligned}$$

Hence $z = \overline{\alpha}$ is also a zero of the polynomial $P(z)$. Further, as shown above:

$$P(z) = (z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2) \times Q(z)\,.$$

Exercise **5A**

1. It is known that in each case the given polynomial $P(x)$ has only one integer zero. Find it and hence factorise $P(x)$ completely.

(a) $P(x) = x^3 - 6x + 4$ (b) $P(x) = x^3 + 3x^2 - 2x - 2$ (c) $P(x) = x^3 - 3x^2 - 2x + 4$

2. It is known that $1 + i$ is a zero of the polynomial $P(x) = x^3 - 8x^2 + 14x - 12$.

(a) Why is $1 - i$ also a zero of $P(x)$?

(b) Use the sum of the zeroes to find the third zero of $P(x)$.

3. It is known that $1 - 2i$ is a zero of the polynomial $P(x) = x^3 + x + 10$.

(a) Write down another complex zero of $P(x)$, and give a reason for your answer.

(b) Hence show that $x^2 - 2x + 5$ is a factor of $P(x)$.

(c) Find the third zero, and hence write $P(x)$ as a product of factors with real coefficients.

4. It is known that $-3i$ is a zero of the polynomial $P(z) = 2z^3 + 3z^2 + 18z + 27$.

(a) Write down another complex zero of $P(z)$. Justify your answer.

(b) Hence write down a quadratic factor of $P(z)$ with real coefficients.

(c) Write $P(x)$ as a product of factors with real coefficients.

5. Let $P(z) = 2z^3 - 13z^2 + 26z - 10$.

(a) Show that $P(3 + i) = 0$.

(b) State the value of $P(3 - i)$, and give a reason for your answer.

(c) Hence write $P(z)$ as a product of:

(i) linear factors,

(ii) a linear factor and a quadratic factor, both with real coefficients.

DEVELOPMENT

6. Consider the polynomial $Q(x) = x^4 - 6x^3 + 8x^2 - 24x + 16$.

(a) It is known that $Q(2i) = 0$. Why does it follow immediately that $Q(-2i) = 0$?

(b) By using the sum and the product of the zeroes of $Q(x)$, or otherwise, find the other two zeroes of $Q(x)$.

(c) Hence write $Q(x)$ as a product of:

(i) four linear factors,

(ii) three factors with real coefficients,

(iii) two factors with integer coefficients.

7. (a) Solve the equation $x^4 - 3x^3 + 6x^2 + 2x - 60 = 0$ given that $x = 1 + 3i$ is a root.

(b) Solve the equation $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$ given that $x = 1 - i$ is a root.

8. Two of the zeroes of $P(z) = z^4 - 12z^3 + 59z^2 - 138z + 130$ are $a + ib$ and $a + 2ib$, where a and b are real and $b > 0$.

(a) Find the value of a by considering the sum of the zeroes.

(b) Use the product of the zeroes to show that $4b^4 + 45b^2 - 49 = 0$, and hence find b.

(c) Hence express $P(z)$ as the product of quadratic factors with real coefficients.

9. Suppose that $P(x) = x^3 + kx^2 + 6$, where k is real.

(a) Show that $P(2i) = (6 - 4k) - 8i$.

(b) When $P(x)$ is divided by $x^2 + 4$ the remainder is $-4x - 6$. Find the value of k.

10. Let $P(x) = x^3 - x^2 + mx + n$, where both m and n are integers.
(a) Show that $P(-i) = (1 + n) + i(1 - m)$.
(b) When $P(x)$ is divided by $x^2 + 1$ the remainder is $6x - 3$. Find the values of m and n.

11. Suppose that $P(x) = x^3 + x^2 + 6x - 3$.
(a) Use the remainder theorem to find the remainder when $P(x)$ is divided by $x + 2i$.
(b) Hence find the remainder when $P(x)$ is divided by: (i) $x - 2i$, (ii) $x^2 + 4$.

12. Let $P(z) = z^8 - \frac{5}{2}z^4 + 1$. Suppose that w is a root of $P(z) = 0$.
(a) Show that iw and $\frac{1}{w}$ are also roots of $P(z) = 0$.
(b) Find one of the roots of $P(z) = 0$ in exact form.
(c) Hence find all the roots of $P(z) = 0$.

13. Suppose that $P(x) = x^4 + Ax^2 + B$, where A and B are positive real numbers.
(a) Explain why $P(x)$ has no real zeroes.
(b) Given that ic and id, where c and d are real, are zeroes of $P(x)$, write down the other two zeroes of $P(x)$, and give a reason.
(c) Prove that $c^4 + d^4 = A^2 - 2B$.

14. The polynomial $P(x) = x^3 + cx + d$, where c and d are real and non-zero, has a negative real zero k, and two complex zeroes. The graph of $y = P(x)$ has two turning points.
(a) What can be said about the two complex zeroes of $P(x)$, and why?
(b) By considering $P'(x)$, show that $c < 0$.
(c) Sketch the graph of $y = P(x)$.
(d) If $a \pm ib$, where a and b are real, are the complex zeroes of $P(x)$, deduce that $a > 0$.
(e) Prove that $d = 8a^3 + 2ac$.

15. Consider the polynomial function $f(x) = x^3 - 3x + k$, where k is an integer greater than 2.
(a) Show that $f(x)$ has exactly one real zero r, and explain why $r < -1$.
(b) Give a reason why the two complex zeroes of $f(x)$ form a conjugate pair.
(c) If the complex zeroes are $a + ib$ and $a - ib$, use the result for the sum of the roots two at a time to show that $b^2 = 3(a^2 - 1)$.
(d) Find the three zeroes of $f(x)$ given that $k = 2702$, and that a and b are integers.

EXTENSION

16. In the text it was proven that if $P(z)$ is a polynomial with real coefficients and if $P(\alpha) = 0$ then $P(\overline{\alpha}) = 0$. Use a similar approach to prove that if $P(\alpha) = \beta$ then $P(\overline{\alpha}) = \overline{\beta}$.

17. Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial with integer coefficients. Suppose that $P(x)$ has a rational zero $x = \frac{p}{q}$ where p and q have highest common factor 1. Show that p is a factor of a_0 and that q is a factor of a_n.

18. (a) Let u and v be two numbers of the form $u = a + b\sqrt{c}$, where a, b and c are rational numbers but where $\sqrt{c}$ is irrational. Let the notation u^* indicate the value of u when the sign of b is reversed. That is, $u^* = a - b\sqrt{c}$.
(i) Show that $u^* + v^* = (u + v)^*$.
(ii) Show that $\lambda u^* = (\lambda u)^*$ whenever λ is a rational number.
(iii) Prove by induction that $(u^n)^* = (u^*)^n$ for positive integers n.
(b) Suppose that $u = a + b\sqrt{c}$ is a zero of a certain polynomial with rational coefficients. Use the results of part (a) to show that $u^* = a - b\sqrt{c}$ is also a zero of this polynomial.

5B Multiple Zeroes

Multiple zeroes of a polynomial were encountered in the Mathematics Extension 1 course. Recall that for the polynomial $P(x) = (x+2)^3x^2(x-2)$ the value $x = -2$ is called a *triple zero*, the value $x = 0$ is called a *double zero*, and the value $x = 2$ is called a *simple zero*. The general situation is summarised here.

4 MULTIPLE ZEROES: Suppose that the polynomial $P(x)$ may be factored as

$$P(x) = (x-\alpha)^m Q(x)\,, \text{ where } Q(\alpha) \neq 0\,.$$

Then $x = \alpha$ is called a *zero of multiplicity* m.

A zero of multiplicity 1 is called a *simple zero*, and a zero of multiplicity greater than 1 is called a *multiple zero*.

In the special case of a polynomial $P(x)$ with real coefficients which has a real zero $x = \alpha$ with multiplicity $m > 1$ it can be shown that the derivative $P'(x)$ has the same real zero $x = \alpha$ but with multiplicity $(m-1)$.

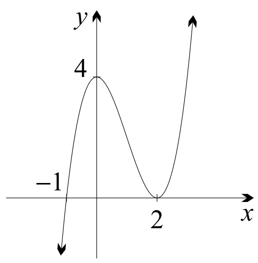
WORKED EXERCISE: The polynomial $P(x) = x^3 - 3x^2 + 4$ has a double zero.
(a) Find the double zero and hence factor $P(x)$.
(b) Sketch the graph of $y = P(x)$.

SOLUTION:
(a) Since $P(x)$ has a double zero, it is a solution of $P'(x) = 0$, that is:

$$3x^2 - 6x = 0$$

so $3x(x-2) = 0$.
Thus $x = 0$ or 2.
Now $P(0) = 4$ and $P(2) = 0$.
Thus $x = 2$ is the double zero.
By the product of roots, the third zero is $x = -1$.
Hence $P(x) = (x-2)^2(x+1)$.

(b) The graph is shown on the right.



5 MULTIPLE ZEROES AND THE DERIVATIVE: Suppose that the polynomial $P(x)$ with real coefficients has a real zero $x = \alpha$ with multiplicity $m > 1$.

Then $x = \alpha$ is a zero of $P'(x)$ with multiplicity $(m-1)$.

PROOF: Let $P(x) = (x-\alpha)^m Q_0(x)$, where $Q_0(\alpha) \neq 0$ and where $m > 1$. Then

$$\begin{aligned} P'(x) &= m(x-\alpha)^{m-1}Q_0(x) + (x-\alpha)^m Q_0'(x) \qquad \text{(by the product rule)} \\ &= (x-\alpha)^{m-1}\Big(mQ_0(x) - (x-\alpha)Q_0'(x)\Big)\,. \end{aligned}$$

Let $Q_1(x)$ be the term in brackets involving Q_0 and Q_0'. Then we may write:

$$P'(x) = (x-\alpha)^{m-1}Q_1(x)\,,$$

where $Q_1(x) = mQ_0(x) - (x-\alpha)Q_0'(x)$.
Now $Q_1(\alpha) = mQ_0(\alpha)$
$\neq 0$ (since $Q_0(\alpha) \neq 0$ and $m > 1$.)
Hence $x = \alpha$ is a zero of $P'(x)$ with multiplicity exactly equal to $(m-1)$.

In fact this result is also true for polynomials with complex zeroes but the proof is beyond the scope of this course.

Multiple Zeroes and Higher Derivatives: Suppose that the polynomial $P(x)$ with real coefficients has a triple zero $x = \alpha$ which is real. Applying the above theorem repeatedly gives:
$x = \alpha$ is a double zero of $P'(x)$, and
$x = \alpha$ is a simple zero of $P''(x)$.

WORKED EXERCISE: It is known that the polynomial $P(x) = x^4 - 6x^2 - 8x - 3$ has a triple zero. (a) Find the triple zero. (b) Hence factorise $P(x)$.

SOLUTION: (a) Differentiating:

$$\begin{aligned} P'(x) &= 4x^3 - 12x - 8 \\ \text{and} \quad P''(x) &= 12x^2 - 12 \\ &= 12(x-1)(x+1)\,. \end{aligned}$$

Thus the possible values of the triple zero are $x = 1$ or $x = -1$.
Since $P'(-1) = 0$ and $P(-1) = 0$ it follows that $x = -1$ is the triple zero.

(b) Let $x = \alpha$ be the remaining zero, then by the sum of the zeroes,

$$\begin{aligned} 3 \times (-1) + \alpha &= 0 \\ \text{thus} \qquad \alpha &= 3\,. \\ \text{Hence} \qquad P(x) &= (x+1)^3(x-3)\,. \end{aligned}$$

This example of a triple zero can be extended to the general case of a polynomial $P(x)$ with real coefficients which has a real zero $x = \alpha$ of multiplicity m. The value $x = \alpha$ is also a zero of each of the derivatives $P^{(j)}(x)$, for $j = 1, \ldots, (m-1)$.

6 MULTIPLE ZEROES AND HIGHER DERIVATIVES: Suppose that the polynomial $P(x)$ with real coefficients has a real zero $x = \alpha$ with multiplicity $m > 1$.

Then $x = \alpha$ is a zero of each of the derivatives $P^{(j)}(x)$, for $j = 1, \ldots, (m-1)$.

This result can be proved relatively easily by induction and is left as an exercise. It is also true for polynomials with complex zeroes but again the proof is beyond the scope of this course.

The Fundamental Theorem of Algebra: All the work encountered so far in this chapter deals with finding the zeroes of various polynomials. Up to this point it has been possible to sidestep an important question: does every polynomial have a zero? For there is no point in searching for one if none exists.

In order to emphasise this point, consider the polynomial $P(x) = x^2 + 1$. Clearly this function has no real zero, and there is no point in searching for one. Yet the polynomial does indeed have two zeroes, both of which happen to be complex numbers: namely i and $-i$. Could it be that there is another polynomial which has neither real nor complex zeroes?

The answer to this question is: every polynomial with degree ≥ 1 has at least one zero, though that zero may be complex. This is such an important and basic fact in the study of mathematics that it is given a title — *The Fundamental Theroem of Algebra.*

7 | The fundamental theorem of algebra: Every polynomial with degree ≥ 1 has at least one zero, though that zero may be complex.

Several eminent mathematicians worked on this theorem including Leibniz, Euler and Argand. But credit is usually given to Gauss for the first proof, which he presented in his doctoral thesis in 1799. This, or any other proof of the theorem, is beyond the scope of this course.

Although the wording given in the box above is imprecise, it is usually sufficent for the problems encountered at this level. Those who have read more widely will know that this theorem may be formally stated in a number of different ways, including: *every polynomial with complex coefficients and degree ≥ 1 has at least one complex zero.*

The Degree and the Number of Zeroes: Although the Fundamental Theorem of Algebra cannot be proven here, it is possible to prove two significant consequences of the theorem. The first is that every polynomial of degree $n \geq 1$ with complex coefficents has precisely n zeroes, as counted by their multiplicities.

This is also true for polynomials with real coefficients. To demonstrate the result, recall that the cubic $P(x) = x^3 - 3x^2 + 4$ encountered in the first worked exercise has three zeroes: the simple zero $x = -1$ and the double zero $x = 2$.

8 | The degree and the number of zeroes: Every polynomial of degree $n \geq 1$ with complex coefficents has precisely n zeroes, as counted by their multiplicities.

Proof: This proof uses induction.

A. Consider the general polynomial of degree one with complex coefficients:

$$P_1(x) = a_0 + a_1 x \,, \qquad \text{where } a_1 \neq 0 \,.$$

Clearly this polynomial has one zero $x = \alpha_1$, where

$$\alpha_1 = -\frac{a_0 \overline{a_1}}{|a_1|^2} \,.$$

Thus the result is true for $n = 1$.

B. Suppose that the result is true for some integer $k \geq 1$. That is, suppose that every polynomial of degree k with complex coefficients

$$P_k(x) = a_0 + a_1 x + \ldots + a_k x^k \,, \qquad \text{where } a_k \neq 0 \,,$$

has k zeroes, $x = \alpha_1, \ldots, \alpha_k$, as counted by their multiplicities. $(**)$

We now prove the statement is true for $n = k + 1$, that is, we prove that every polynomial of degree $k + 1$ with complex coefficents

$$P_{k+1}(x) = a_0 + a_1 x + \ldots + a_{k+1} x^{k+1} \,, \qquad \text{where } a_{k+1} \neq 0 \,,$$

has $k + 1$ zeroes.

Now for any particular polynomial $P_{k+1}(x)$, that polynomial has at least one zero by the Fundamental Theorem of Algebra. Let this zero be $x = \alpha_{k+1}$. Then, by the factor theorem, we may write

$$P_{k+1}(x) = (x - \alpha_{k+1}) Q_k(x)$$

for some polynomial $Q_k(x)$ of degree k. But by the induction hypothesis above $(**)$, $Q_k(x)$ has k zeroes, all of which are thus inherited by $P_{k+1}(x)$.

Hence $P_{k+1}(x)$ has $k+1$ zeroes, $x = \alpha_1, \ldots, \alpha_k, \alpha_{k+1}$, as counted by their multiplicities. Clearly this follows for each and every polynomial $P_{k+1}(x)$.

C. It follows from parts A and B by mathematical induction that the statement is true for all integers $n \geq 1$.

Real Linear and Quadratic Factors: The second significant consequence of the Fundamental Theorem of Algebra is that every polynomial of degree $n \geq 1$ with real coefficients can be written as a product of factors which are either linear or *irreducible* quadratics, each with real coefficients. In this context the word *irreducible* is used to indicate that the quadratic has no real zero.

In order to demonstrate the result, notice that the polynomial $P(x) = x^3 - 1$ can be written as the product

$$P(x) = (x-1)(x^2+x+1)\,.$$

The quadratic factor (x^2+x+1) is irreducible since it has no real zero.

9	REAL LINEAR AND QUADRATIC FACTORS: Every polynomial of degree $n \geq 1$ which has real coefficents can be written as a product of factors which are either linear or irreducible quadratics, each with real coefficients.

PROOF: Let $P_n(x) = a_0 + a_1 x + \ldots + a_n x^n$ be a polynomial with degree $n \geq 1$ which has real coefficients. By the previous result, this polynomial has n zeroes. Let these zeroes be $x = \alpha_1, \ldots, \alpha_n$. There are three cases to consider.

(a) If all the zeroes are real then by the factor theorem we may write:

$$P_n(x) = a_n \times \prod_{k=1}^{n} (x - \alpha_k)\,,$$

which is a product of linear factors with real coefficients.

(b) If all the zeroes are complex numbers then they occur as conjugate pairs, since the coefficients of $P_n(x)$ are real. Hence n is even, and we may put $n = 2j$ for some integer j. Now re-order and re-label the zeroes in conjugate pairs. Thus the first conjugate pair is $x = \alpha_1, \overline{\alpha_1}$, and the last conjugate pair is $x = \alpha_j, \overline{\alpha_j}$. Then by the factor theorem we may write:

$$\begin{aligned} P_n(x) &= a_n \times \prod_{k=1}^{j} \Big((x-\alpha_k)(x-\overline{\alpha_k}) \Big) \\ &= a_n \times \prod_{k=1}^{j} \Big(x^2 - 2\,\mathrm{Re}(\alpha_k)x + |\alpha_k|^2 \Big)\,, \end{aligned}$$

which is a product of irreducible quadratic factors with real coefficients.

(c) If some of the zeroes are complex numbers then they occur as conjugate pairs, since the coefficients of $P_n(x)$ are real. Let the number of conjugate pairs be j, where $1 < 2j < n$. The other zeroes are real. Now re-order and re-label the zeroes with the conjugate pairs listed first. Thus the first conjugate pair is $x = \alpha_1, \overline{\alpha_1}$, and the last conjugate pair is $x = \alpha_j, \overline{\alpha_j}$. The first of the remaining real zeroes is then $x = \alpha_{2j+1}$ and the last is $x = \alpha_n$. Then by the factor theorem:

$$P_n(x) = a_n \times \prod_{k=1}^{j} \Big((x-\alpha_k)(x-\overline{\alpha_k}) \Big) \times \prod_{\ell=2j+1}^{n} (x - \alpha_\ell)$$

$$= a_n \times \prod_{k=1}^{j} \left(x^2 - 2\,\mathrm{Re}(\alpha_k)x + |\alpha_k|^2 \right) \times \prod_{\ell=2j+1}^{n} (x - \alpha_\ell)\,,$$

which, as required, is a product of linear and irreducible quadratic factors with real coefficients. The proof is now complete.

Exercise **5B**

1. Consider the polynomial $P(x) = x^3 - 4x^2 - 3x + 18$.
(a) (i) Show that $P(3)$ and $P'(3)$ are both zero.
(ii) What can be deduced from the results in part (i)?
(b) Use part (a) and the sum of zeroes to find all the zeroes of $P(x)$.
(c) Hence factorise $P(x)$.

2. Consider the polynomial $P(x) = x^4 + 8x^3 + 18x^2 + 16x + 5$.
(a) (i) Show that $P(-1)$, $P'(-1)$ and $P''(-1)$ are all zero.
(ii) What can be deduced from the results in part (i)?
(b) Use part (a) and the product of zeroes to find all the zeroes of $P(x)$.
(c) Hence factorise $P(x)$.

3. The polynomial $P(x) = x^3 - 27x + 54$ has a double zero.
(a) Find the zeroes of $P'(x)$.
(b) Determine which of the zeroes of $P'(x)$ is the double zero of $P(x)$.
(c) Find the remaining simple zero of $P(x)$.

4. The polynomial $P(x) = x^4 + 5x^3 - 75x^2 - 625x - 1250$ has a triple zero.
(a) Find the zeroes of $P''(x)$.
(b) Determine which of the zeroes of $P''(x)$ is the triple zero of $P(x)$.
(c) Find the remaining simple zero of $P(x)$.

5. The polynomial $P(x) = 2x^3 + 5x^2 - 4x - 12$ has a double zero.
(a) Find the double zero.
(b) Find the remaining simple zero, and hence factorise $P(x)$.

6. The polynomial $P(x) = 8x^4 - 28x^3 + 30x^2 - 13x + 2$ has a triple zero.
(a) Find the triple zero.
(b) Find the remaining simple zero, and hence factorise $P(x)$.

DEVELOPMENT

7. Consider the polynomial equation $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$.
(a) Show that $x = 3$ is a double root of the equation.
(b) Hence solve the equation.

8. The polynomial $P(x) = x^3 - 3x^2 - 9x + k$ has a double zero.
(a) Find the two possible values of k.
(b) For each of the possible values of k, factorise $P(x)$.

9. The coefficients of the polynomial $P(x) = ax^3 + bx + c$ are real and $P(x)$ has a multiple zero at $x = 1$. When $P(x)$ is divided by $x + 1$ the remainder is 4. Find the values of a, b and c.

10. The polynomial $P(x) = x^4 + 7x^3 + 9x^2 - 27x + c$ has a triple zero.
(a) Determine the value of the triple zero.
(b) Hence find the value of c. (c) Factorise $P(x)$.

11. (a) Find the values of b and c if $x = 1$ is a double root of the equation
$$x^4 + bx^3 + cx^2 - 5x + 1 = 0\,.$$
(b) Find the other roots of the equation.

12. Consider the constant polynomial $P(x) = 1$. Clearly $P(x)$ has no zero, which may appear to contradict the Fundamental Theorem of Algebra. Explain why it does not.

13. It is known that $(x-1)^2$ is a factor of the polynomial $P(x) = ax^{n+1} + bx^n + 1$. Show that $a = n$ and $b = -(1+n)$.

14. The equation $Ax^3+Bx^2+D = 0$ has a double root. If $D \neq 0$, prove that $27A^2D+4B^3 = 0$.

15. Prove that $P(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$, where $n \geq 2$, has no multiple zeroes.

16. (a) Prove that the polynomial equation $x^4 + mx^2 + n = 0$, where $m \neq 0$, cannot have a root of multiplicity greater than 2.
(b) Let $x = \alpha$ be a double root of the equation in part (a).
(i) Prove that $x = -\alpha$ is also a double root. (ii) For what values of m is α real?
(iii) Prove that $n = \frac{1}{4}m^2$. Hence write down the roots of the equation in terms of m.

EXTENSION

17. The polynomial $P(x)$ with real coefficients has a real zero $x = \alpha$ of multiplicity $m > 1$. Use induction on the value of j to prove that $x = \alpha$ is a zero of each of the derivatives $P^{(j)}(x)$, for $j = 0, 1, \ldots, (m-1)$.

18. Use the Fundamental Theorem of Algebra to carefully explain why every polynomial of odd degree with real coefficients has at least one real zero.

19. The polynomial $P(x) = x^3 + 3px^2 + 3qx + r$ has a double zero.
(a) Prove that the double zero is $\alpha = \dfrac{pq - r}{2(q - p^2)}$.
(b) Hence, or otherwise, prove that $4(p^2 - q)(q^2 - pr) = (pq - r)^2$.

5C The Zeroes and The Coefficients

The relationships between the zeroes and the coefficients of quadratics, cubics and quartics with real coefficients were encountered in the Mathematics Extension 1 course. It is not difficult to prove that those relationships are also valid for polynomials with complex coefficients or complex zeroes, or both. As a matter of convenience, those relationships are repeated here. In each case, let the zeroes be $\alpha_1, \ldots, \alpha_n$ and the coefficients be $a_0, \ldots, a_n$, where n is the degree.

10 ZEROES AND COEFFICIENTS OF A QUADRATIC:

$$\alpha_1 + \alpha_2 = -\frac{a_1}{a_2} \qquad \text{and} \qquad \alpha_1\alpha_2 = \frac{a_0}{a_2}\,.$$

11 ZEROES AND COEFFICIENTS OF A CUBIC:

$$\alpha_1 + \alpha_2 + \alpha_3 = -\frac{a_2}{a_3}\,, \quad \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = \frac{a_1}{a_3} \quad \text{and} \quad \alpha_1\alpha_2\alpha_3 = -\frac{a_0}{a_3}$$

12 ZEROES AND COEFFICIENTS OF A QUARTIC:

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= -\frac{a_3}{a_4} \\
\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 &= +\frac{a_2}{a_4} \\
\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 &= -\frac{a_1}{a_4} \\
\alpha_1\alpha_2\alpha_3\alpha_4 &= +\frac{a_0}{a_4}
\end{aligned}$$

In practice, often only the first and last formulae of each box are required.

The General Case: Looking carefully at these results, it is evident that in each box the left hand side of successive equations is the sum of the zeroes taken one at a time, then two at a time, then three at a time and so on. Thus it is possible to generalise the formulae using sigma notation, as follows.

13 ZEROES AND COEFFICIENTS OF A POLYNOMIAL:

$$\begin{aligned}
\alpha_1 + \alpha_2 + \ldots + \alpha_n &= -\frac{a_{n-1}}{a_n} \\
\sum_{i<j} \alpha_i\alpha_j &= +\frac{a_{n-2}}{a_n} \\
\sum_{i<j<k} \alpha_i\alpha_j\alpha_k &= -\frac{a_{n-3}}{a_n} \\
&\vdots \\
\alpha_1\alpha_2\ldots\alpha_n &= (-1)^n\,\frac{a_0}{a_n}
\end{aligned}$$

It is sometimes convenient to summarise all these formulae in words.

14 ZEROES AND COEFFICIENTS OF A POLYNOMIAL:

$$\text{sum of roots taken } j \text{ at a time} = (-1)^j \times \frac{a_{n-j}}{a_n} \qquad \text{for } 1 \le j \le n\,.$$

WORKED EXERCISE: The polynomial equation $3x^3 + 7x^2 + 11x + 51 = 0$ has roots α, β and γ.

(a) Evaluate $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$.

(b) (i) Find $\alpha^2 + \beta^2 + \gamma^2$.

(ii) Use part (i) to determine how many of the roots are real.

(c) Determine the value of $3(\alpha^3 + \beta^3 + \gamma^3)$.

SOLUTION: (a) Using the sum and product of roots:

$$\alpha+\beta+\gamma = -\tfrac{7}{3}$$

and

$$\begin{aligned}\alpha\beta\gamma &= -\tfrac{51}{3}\\ &= -17\,.\end{aligned}$$

Thus

$$\begin{aligned}\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2 &= \alpha\beta\gamma(\alpha+\beta+\gamma)\\ &= (-\tfrac{7}{3})\times(-17)\\ &= \tfrac{119}{3}\,.\end{aligned}$$

(b) (i)

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2 &= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)\\ &= (-\tfrac{7}{3})^2-2(\tfrac{11}{3})\\ &= -\tfrac{17}{9}\,.\end{aligned}$$

(ii) Since the sum of the squares of the roots is negative, at least one of them is complex.

Since the coefficients of the polynomial equation are real, complex roots come in conjugate pairs, and hence exactly two of them are complex.

Thus there is precisely one real root of the polynomial equation.

(c) Re-arranging the given equation

$$3x^3 = -7x^2-11x-51$$

which is true for each of the three roots. Thus:

$$\begin{aligned}3\alpha^3 &= -7\alpha^2-11\alpha-51\,,\\ 3\beta^3 &= -7\beta^2-11\beta-51\,,\\ 3\gamma^3 &= -7\gamma^2-11\gamma-51\,.\end{aligned}$$

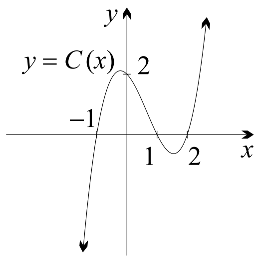
Adding these three yields:

$$\begin{aligned}3(\alpha^3+\beta^3+\gamma^3) &= -7(\alpha^2+\beta^2+\gamma^2)-11(\alpha+\beta+\gamma)-3\times 51\\ &= -7\times(-\tfrac{17}{9})-11\times(-\tfrac{7}{3})-3\times 51\\ &= -114\tfrac{1}{9}\,.\end{aligned}$$

Transforming Roots: Let the polynomial $P(x)$ have real zeroes and real coefficients. If the graph of $y=P(x)$ is transformed by a horizontal shift or by a horizontal enlargement then the zeroes will be similarly transformed. To demonstrate this characteristic, consider the cubic polynomial

$$\begin{aligned}C(x) &= x^3-2x^2-x+2\\ &= (x+1)(x-1)(x-2)\end{aligned}$$

which has zeroes at $x=-1,1$ and 2.



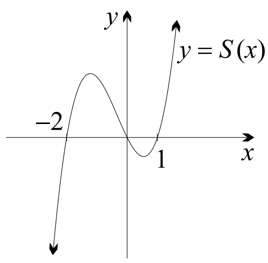
Shifting $y=C(x)$ by 1 unit to the left yields:

$$\begin{aligned}S(x) &= C(x+1)\\ &= (x+1)^3-2(x+1)^2-(x+1)+2\\ &= x^3+x^2-2x\,.\end{aligned}$$

The new polynomial may be factored as

$$S(x) = (x+2)x(x-1)\,,$$

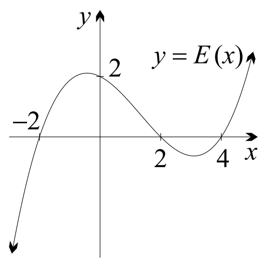
which clearly has zeroes at $x=-2,0$ and 1. This verifies that the zeroes have been shifted to the left by the same amount. The graph on the right also confirms the fact.



Enlarging $y = C(x)$ horizontally by a factor of 2 yields:

$$E(x) = C(\tfrac{1}{2}x)$$
$$= (\tfrac{1}{2}x + 1)(\tfrac{1}{2}x - 1)(\tfrac{1}{2}x - 2)$$

or $\quad E(x) = \frac{1}{8}(x + 2)(x - 2)(x - 4)$.



The zeroes of $E(x)$ are $x = -2, 2$ and 4. This verifies that the zeroes of $C(x)$ have been enlarged by the same factor. The graph on the right also confirms this fact.

Although it will not be proven here, the same procedures may be applied to transform complex zeroes of a polynomial.

15 TRANSFORMED ROOTS:

SHIFTING: To shift the zeroes by an amount k, replace x with $(x - k)$.

STRETCHING: To enlarge the zeroes by a factor of a, replace x with $\dfrac{x}{a}$.

WORKED EXERCISE: The polynomial $P(x) = x^3 - 5x + 3$ has zeroes α, β and γ. Find a cubic polynomial with integer coefficients which has zeroes 3α, 3β and 3γ.

SOLUTION: A polynomial with the required zeroes is:

$$P(\tfrac{1}{3}x) = \tfrac{1}{27}x^3 - \tfrac{5}{3}x + 3\,.$$

Thus a polynomial with integer coefficients is:

$$I(x) = 27P(\tfrac{1}{3}x),$$

so $\quad I(x) = x^3 - 45x + 81$.

Reciprocal of the Roots: In the worked exercise above, the result of the substitution is a new polynomial. Some substitutions, however, lead to a new function instead of a polynomial. Nevertheless, the corresponding equation can be re-arranged into a polynomial equation. The precise effect on the roots of the equation will depend on the particular substitution. One special transformation is the reciprocal of the roots.

For the sake of simplicity, we begin by considering the monic quadratic

$$P(x) = (x - \alpha)(x - \beta)\,.$$

It should be clear that the roots of the corresponding quadratic equation $P(x) = 0$ are $x = \alpha$ and β. What happens to the roots of this equation if we substitute $\frac{1}{x}$ for x?

When written out in full, the equation $P(\frac{1}{x}) = 0$ becomes:

$$(\tfrac{1}{x} - \alpha)(\tfrac{1}{x} - \beta) = 0\,.$$

Although this is not a polynomial equation, it is equivalent to one since we may multiply by x^2 to get:

$$(1 - \alpha x)(1 - \beta x) = 0\,.$$

Thus the roots of the equation are $x = \frac{1}{\alpha}$ and $\frac{1}{\beta}$. Hence the substitution yields a new polynomial equation with roots which are the reciprocal of the original.

The same result is obtained for all polynomials, not just quadratics. There is of course a problem if one of the roots is zero, since the reciprocal of zero is undefined. This issue is dealt with in one of the exercise questions.

16 RECIPROCAL OF THE ROOTS: To get a new polynomial equation with roots which are the reciprocal of the original, replace x with $\frac{1}{x}$ and rearrange.

WORKED EXERCISE: The equation $x^3 - x^2 - 7x + 15 = 0$ has roots α, β and γ.

(a) Find a polynomial equation with integer coefficients which has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

(b) Hence write down the value of $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$.

SOLUTION: (a) An equation with reciprocal roots is:

$$\frac{1}{x^3} - \frac{1}{x^2} - \frac{7}{x} + 15 = 0\,.$$

Multiply this by x^3 to get:

$$1 - x - 7x^2 + 15x^3 = 0\,.$$

(b) The required expression is the sum of the reciprocal roots taken two at a time. Thus, from the polynomial equation in part (a), its value is:

$$\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = \frac{-1}{15}\,.$$

Square of the Roots: The other special substitution encountered in this is course is when x is replaced by $\sqrt{x}$. Again, for the sake of simplicity, we begin by considering the quadratic

$$P(x) = (x - \alpha)(x - \beta)\,.$$

Writing out $P(\sqrt{x}) = 0$ in full:

$$(\sqrt{x} - \alpha)(\sqrt{x} - \beta) = 0\,.$$

Expand this to get:

$$x - (\alpha + \beta)\sqrt{x} + \alpha\beta = 0$$

or

$$x + \alpha\beta = (\alpha + \beta)\sqrt{x}\,.$$

Squaring yields:

$$x^2 + 2\alpha\beta x + \alpha^2\beta^2 = (\alpha^2 + 2\alpha\beta + \beta^2)x$$

so

$$x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$$

or

$$(x - \alpha^2)(x - \beta^2) = 0\,.$$

Thus the roots of the equation are $x = \alpha^2$ and β^2. Hence the substitution yields a new polynomial equation with roots which are the square of the original.

The same result is obtained for all polynomials, not just quadratics.

17 SQUARE OF THE ROOTS: To get a new polynomial equation with roots which are the square of the original, replace x with $\sqrt{x}$ and rearrange.

WORKED EXERCISE: The equation $3x^3 + 7x^2 + 11x + 51 = 0$ has roots α, β and γ.

(a) Find a polynomial equation with integer coefficients that has roots α^2, β^2 and γ^2.

(b) Hence evaluate $\alpha^2 + \beta^2 + \gamma^2$.

Solution:

(a) Substitute $\sqrt{x}$ for x to get:

$$\sqrt{x}(3x+11)+7x+51=0$$

or $$\sqrt{x}(3x+11)=-(7x+51)\,.$$

Squaring both sides yields:

$$x(3x+11)^2=(7x+51)^2$$

so $$9x^3+66x^2+121x=49x^2+714x+2601$$

or $$9x^3+17x^2-593x-2601=0\,.$$

(b) From the coefficients of x^2 and x^3 in part (a) we get $\alpha^2+\beta^2+\gamma^2=-\frac{17}{9}$.

Extension — Other Transformations: Some readers will have already discerned an apparent relationship between the substitution and the effect on the roots. They are inverse functions, and this is born out in the above worked exercises. Thus the substitution $\frac{1}{3}x$ trebled the roots, whilst the substitution $\sqrt{x}$ squared the roots. The relationship holds for all substitutions and can be stated as follows.

Let the polynomial equation $P(x)=0$ have a root $x=\alpha$. Given a function $g(x)$ which has an inverse function $g^{-1}(x)$, the corresponding root of the equation $P(g(x))=0$ is $g^{-1}(\alpha)$, provided that $g^{-1}(\alpha)$ exists.

18 Transformed roots: Let the polynomial equation $P(x)=0$ have a root $x=\alpha$. The corresponding root of the equation $P(g(x))=0$ is $g^{-1}(\alpha)$, provided that $g^{-1}(\alpha)$ exists.

Proof: Since $P(x)=0$ has a root $x=\alpha$ it follows from the Fundamental Theorem of Algebra that we may write

$$P(x)=(x-\alpha)\times Q(x)\,.$$

The equation $P(g(x))=0$ thus becomes

$$(g(x)-\alpha)\times Q(g(x))=0\,.$$

One solution of this equation is:

$$g(x)=\alpha$$

whence $x=g^{-1}(\alpha)$ (if it exists.)

Worked Exercise: The equation $2x^2-3x-2=0$ has roots $x=-\frac{1}{2}$ and 2. Find the roots of $2\cos^2 x-3\cos x-2=0$ for $0\le x\le \pi$.

Solution: Clearly, x has been replaced with $\cos x$. Since there is no real value for $\cos^{-1} 2$, there is only one real solution, viz:

$$\begin{aligned} x &= \cos^{-1}(-\tfrac{1}{2}) \\ &= \tfrac{2\pi}{3} \end{aligned}$$

Exercise **5C**

1. Consider the polynomial $P(x)=x^2+9$, and let $Q(x)=P(x-2)$.

(a) Write down the zeroes of $P(x)$.

(b) Hence write down the zeroes of $Q(x)$. (c) Find $Q(x)$.

(d) By solving the equation $Q(x)=0$, confirm your answer to part (b).

2. Consider the polynomial $P(x) = x^2 + 8x + 20$, and let $Q(x) = P(2x)$.
(a) Find the zeroes of $P(x)$.
(b) Hence write down the zeroes of $Q(x)$. (c) Find $Q(x)$.
(d) By solving the equation $Q(x) = 0$, confirm your answers to part (b).

3. The polynomial $P(x) = x^3 - 5x + 3$ has zeroes α, β and γ. By replacing x with $\frac{1}{2}x$, find a polynomial with integer coefficients which has zeroes 2α, 2β and 2γ.

4. Let α, β and γ be the roots of the equation $x^3 - 5x^2 + 5 = 0$. By replacing x with $(x+1)$, find a polynomial equation with integer coefficients which has roots $(\alpha - 1)$, $(\beta - 1)$ and $(\gamma - 1)$.

5. Consider the polynomial equation $x^3 - 4x^2 + 6x - 8 = 0$ with roots α, β and γ.
Find the value of:
(a) $\alpha + \beta + \gamma$
(b) $\alpha^2 + \beta^2 + \gamma^2$
(c) $\alpha^3 + \beta^3 + \gamma^3$
(d) $\alpha^4 + \beta^4 + \gamma^4$
(e) $\alpha^5 + \beta^5 + \gamma^5$

DEVELOPMENT

6. Suppose that the roots of the polynomial equation $x^3 - 3x + 1 = 0$ are α, β and γ.
(a) Replace x with $\dfrac{1}{x}$ to find a polynomial equation with roots $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$.
(b) Replace x with $\sqrt{x}$ to find a polynomial equation with roots α^2, β^2 and γ^2.

7. Suppose that α, β, γ and δ are the roots of the polynomial equation

$$x^4 + 2x^3 - x^2 + 4x - 3 = 0\,.$$

(a) (i) Find a polynomial equation with roots $\dfrac{\alpha}{3}$, $\dfrac{\beta}{3}$, $\dfrac{\gamma}{3}$ and $\dfrac{\delta}{3}$.
(ii) Hence find a polynomial equation with the reciprocal roots $\dfrac{3}{\alpha}$, $\dfrac{3}{\beta}$, $\dfrac{3}{\gamma}$ and $\dfrac{3}{\delta}$.
(b) (i) Find a polynomial equation with roots $(\alpha - 3)$, $(\beta - 3)$, $(\gamma - 3)$ and $(\delta - 3)$.
(ii) Hence find a polynomial equation with the opposite roots $(3-\alpha)$, $(3-\beta)$, $(3-\gamma)$ and $(3-\delta)$.

8. Let the roots of the polynomial equation $x^3 + mx + n = 0$ be α, β and γ.
(a) Find a cubic polynomial equation, with coefficients in terms of m and n, which has roots α^2, β^2 and γ^2,
(b) Use part (a) and a suitable substitution to find a cubic polynomial equation which has roots $\dfrac{1}{\alpha^2}$, $\dfrac{1}{\beta^2}$ and $\dfrac{1}{\gamma^2}$.

9. (a) Expand $(\sqrt{3} + 1)^2$.
(b) The polynomial equation $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$ has roots α, β, γ and δ. Find the polynomial equation with roots $(\alpha + 1)$, $(\beta + 1)$, $(\gamma + 1)$ and $(\delta + 1)$.
(c) Hence, or otherwise, solve the equation $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$.

10. (a) Find the quartic equation whose roots exceed by 3 the roots of the equation $x^4 + 12x^3 + 49x^2 + 78x + 42 = 0$.
(b) Hence or otherwise solve the equation given in part (a).

11. The polynomial equation $2x^3 + 8x^2 + 3x - 6 = 0$ has roots α, β and γ.
(a) Evaluate: (i) $\alpha + \beta + \gamma$ (ii) $\alpha\beta\gamma$
(b) Hence find a polynomial equation with roots:
(i) $2\alpha+\beta+\gamma$, $\alpha+2\beta+\gamma$ and $\alpha+\beta+2\gamma$ (ii) $\alpha^2\beta\gamma$, $\alpha\beta^2\gamma$ and $\alpha\beta\gamma^2$

12. The polynomial equation $x^4 - ax^3 + bx^2 - abx + 1 = 0$ has roots α, β, γ and δ.
(a) Explain why $\alpha + \beta + \gamma = a - \delta$.
(b) Hence show that $(\alpha + \beta + \gamma)(\alpha + \beta + \delta)(\alpha + \gamma + \delta)(\beta + \gamma + \delta) = 1$.

13. The numbers α, β and γ satisfy the three equations

$$\begin{aligned} \alpha + \beta + \gamma &= 5 \\ \alpha^2 + \beta^2 + \gamma^2 &= 9 \\ \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \tfrac{4}{3} \end{aligned}$$

(a) Find the value of: (i) $\alpha\beta + \alpha\gamma + \beta\gamma$ (ii) $\alpha\beta\gamma$
(b) Explain why α, β and γ are the roots of $x^3 - 5x^2 + 8x - 6 = 0$.
(c) Find the values of α, β and γ.

14. The equation $x^4 + px^3 + qx^2 + rx + s = 0$ has roots α, β, γ and δ.
(a) Find the values of the following in terms of p, q, r and s.
(i) $\alpha + \beta + \gamma + \delta$ (ii) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$
(b) Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = p^2 - 2q$.
(c) (i) Let $P(x) = x^4 - 3x^3 + 5x^2 + 7x - 8$. Use part (b) to show that $P(x) = 0$ cannot have four real roots.
(ii) Evaluate $P(0)$ and $P(1)$, and hence explain why $P(x) = 0$ has exactly two real roots.

15. Suppose that the polynomial $P(z) = z^3 + fz^2 + gz + h$ has zeroes α, $-\alpha$ and β.
The numbers f, g and h are real.
(a) Prove that $fg = h$.
(b) It is known that $P(z)$ does not have three real zeroes.
Explain why two of them are purely imaginary.
[A complex number z is purely imaginary if it has the form $z = iy$ where y is real.]

16. Let α, β and γ be the roots of the cubic equation $x^3 - px - q = 0$, and define S_n by

$$S_n = \alpha^n + \beta^n + \gamma^n \qquad \text{for } n = 1, 2, 3, \ldots$$

(a) Explain why $S_1 = 0$, and show that $S_2 = 2p$ and $S_3 = 3q$.
(b) Prove that for $n > 3$, $S_n = pS_{n-2} + qS_{n-3}$.
(c) Deduce that $\dfrac{\alpha^5 + \beta^5 + \gamma^5}{5} = \left(\dfrac{\alpha^2 + \beta^2 + \gamma^2}{2}\right)\left(\dfrac{\alpha^3 + \beta^3 + \gamma^3}{3}\right)$.

17. Consider the polynomial $P(x) = x^3 + qx^2 + qx + 1$, where q is real. It is known that -1 is a zero of $P(x)$.
(a) Show that if α is a zero of $P(x)$ then so too is $\dfrac{1}{\alpha}$.
(b) Let α be a complex number where $\text{Im}(\alpha) \neq 0$ and $P(\alpha) = 0$.
(i) Show that $|\alpha| = 1$. (ii) Show that $\text{Re}(\alpha) = \dfrac{1-q}{2}$.

EXTENSION

18. The equation $3x^3 - 5x^2 - 2x = 0$ has roots $x = 2, 0$ and $-\frac{1}{3}$. Investigate what happens to these roots when x is replaced by $\frac{1}{x}$. Can you generalise your conclusions?

19. The equation $x^3 + px^2 + qx + r = 0$ has roots a, b and c.

(a) Find a cubic polynomial equation with roots $\dfrac{a+b}{c}$, $\dfrac{b+c}{a}$ and $\dfrac{c+a}{b}$.

(b) Hence or otherwise show that $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} + \dfrac{b}{a} + \dfrac{c}{b} + \dfrac{a}{c} = \dfrac{pq}{r} - 3$.

20. The monic degree n polynomial $P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ has zeroes $1, 2, 3, \ldots, n$.

(a) Find expressions for: (i) a_{n-1} (ii) a_0

(b) (i) Prove by induction that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integer values of n.

(ii) Hence prove that $a_{n-2} = \frac{1}{24}n(n-1)(n+1)(3n+2)$.

21. Let $f(x) = x^3 + cx + d$ have three distinct zeroes t_1, t_2 and t_3.

(a) By considering the graph of $y = f(x)$, explain why $f'(t_1)f'(t_2)f'(t_3) < 0$.

(b) Show that a cubic with zeroes ${t_1}^2$, ${t_2}^2$ and ${t_3}^2$ is $g(x) = x^3 + 2cx^2 + c^2x - d^2$.

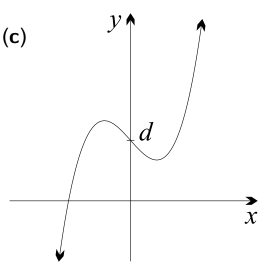
(c) Show that $f'(t_1)f'(t_2)f'(t_3) = -27 \times g(-\frac{c}{3})$.

(d) Hence show that $4c^3 + 27d^2 < 0$.

Chapter Five

Exercise 5A (Page 4)

1(a) $(x-2)(x+1-\sqrt{3})(x+1+\sqrt{3})$
(b) $(x-1)(x+2-\sqrt{2})(x+2+\sqrt{2})$
(c) $(x-1)(x-1-\sqrt{5})(x-1+\sqrt{5})$
2(a) The coefficients of $P(x)$ are real, so complex zeroes occur in conjugate pairs. **(b)** 6
3(a) $1+2i$; the coefficients of $P(x)$ are real, so complex zeroes occur in conjugate pairs.
(c) $P(x)=(x+2)(x^2-2x+5)$
4(a) $3i$; the coefficients of $P(z)$ are real, so complex zeroes occur in conjugate pairs. **(b)** z^2+9
(c) $P(z)=(2z+3)(z^2+9)$
5(b) 0; the coefficients of $P(z)$ are real, so complex zeroes occur in conjugate pairs.
(c)(i) $P(z)=(2z-1)(z-3-i)(z-3+i)$
(ii) $P(z)=(2z-1)(z^2-6z+10)$
6(a) The coefficients of $Q(x)$ are real, so complex zeroes occur in conjugate pairs. **(b)** $3+\sqrt{5}$, $3-\sqrt{5}$
(c)(i) $(x-2i)(x+2i)(x-3-\sqrt{5})(x-3+\sqrt{5})$
(ii) $(x^2+4)(x-3-\sqrt{5})(x-3+\sqrt{5})$
(iii) $(x^2+4)(x^2-6x+4)$
7(a) $x=1\pm 3i$, 3 or -2 **(b)** $x=1\pm i$ or $2\pm i$
8(a) $a=-3$ **(b)** $b=1$
(c) $(x^2-6x+10)(x^2-6x+13)$
9(b) $k=3$
10(b) $m=7$, $n=-4$
11(a) $-7-4i$ **(b)(i)** $-7+4i$ **(ii)** $2x-7$
12(b) $P(z)=\frac{1}{2}(z^4-2)(2z^4-1)$ so one root is $z=\sqrt[4]{2}$. **(c)** $\sqrt[4]{2}$, $\frac{1}{\sqrt[4]{2}}$, $-\sqrt[4]{2}$, $-\frac{1}{\sqrt[4]{2}}$, and $i\sqrt[4]{2}$, $\frac{1}{\sqrt[4]{2}}\,i$, $-i\sqrt[4]{2}$, $-\frac{1}{\sqrt[4]{2}}\,i$
13(a) $P(x)$ has minimum value B, when $x=0$. Since $B>0$, it follows that $P(x)>0$ for all real values of x. **(b)** $-ic$, $-id$; the coefficients of $P(x)$ are real, so complex zeroes occur in conjugate pairs.
14(a) They form a conjugate pair, since $P(x)$ has real coefficients. **(c)**



15(a) The minimum stationary point is at $x=1$. $f(1)=k-2>0$. Hence the graph of $f(x)$ has only one x-intercept which lies to the left of the maximum stationary point at $x=-1$.
(b) $f(x)$ has real coefficients **(d)** -14, $7\pm 12i$

Exercise 5B (Page 11)

1(a)(ii) 3 is a double zero of $P(x)$ **(b)** 3, 3, -2
(c) $P(x)=(x-3)^2(x+2)$
2(a)(ii) -1 is a triple zero of $P(x)$
(b) -1, -1, -1, -5 **(c)** $P(x)=(x+1)^3(x+5)$
3(a) -3 and 3 **(b)** 3 **(c)** -6
4(a) $\frac{5}{2}$ and -5 **(b)** -5 **(c)** 10
5(a) -2 **(b)** $\frac{3}{2}$, $P(x)=(x+2)^2(2x-3)$
6(a) $\frac{1}{2}$ **(b)** 2, $P(x)=(2x-1)^3(x-2)$
7(b) $x=3$, $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$
8(a) $k=27$ or -5
(b) When $k=27$, $P(x)=(x-3)^2(x+3)$ and when $k=-5$, $P(x)=(x+1)^2(x-5)$.
9 $a=1$, $b=-3$, $c=2$
10(a) -3 **(b)** $c=-54$ **(c)** $P(x)=(x+3)^3(x-2)$
11(a) $b=-5$ and $c=8$
(b) $x=\frac{1}{2}(3-\sqrt{5})$ or $\frac{1}{2}(3+\sqrt{5})$
12 The Fundamental Theroem of Algebra only applies to polynomials of degree ≥ 1.
15 HINT: consider $P(x)-P'(x)$
16(b)(ii) $m<0$ **(iii)** $x=-\sqrt{-\frac{m}{2}}$ or $\sqrt{-\frac{m}{2}}$
19(a) HINT: $x^2=-(2px+q)$
(b) HINT: $P'(\alpha)=0$.

Exercise 5C (Page 17)

1(a) $\pm 3i$ **(b)** $2\pm 3i$ **(c)** $Q(x)=x^2-4x+13$
2(a) $-4\pm 2i$ **(b)** $-2\pm i$ **(c)** $Q(x)=4x^2+16x+20$
3 $8P(\frac{x}{2})=x^3-20x+24$
4 $x^3-2x^2-7x+1=0$
5(a) 4 **(b)** 4 **(c)** 16 **(d)** 72 **(e)** 224
6(a) $x^3-3x^2+1=0$ **(b)** $x^3-6x^2+9x-1=0$
7(a)(i) $27x^4+18x^3-3x^2+4x-1=0$
(ii) $x^4-4x^3+3x^2-18x-27=0$
(b)(i) $x^4+14x^3+71x^2+160x+135=0$
(ii) $x^4-14x^3+71x^2-160x+135=0$
8(a) $x^3+2mx^2+m^2x-n^2=0$
(b) $n^2x^3-m^2x^2-2mx-1=0$
9(a) $4+2\sqrt{3}$ **(b)** $x^4-8x^2+4=0$
(c) $x=\sqrt{3}$, $-\sqrt{3}$, $-2+\sqrt{3}$ or $-2-\sqrt{3}$
10(a) $x^4-5x^2+6=0$ **(b)** $x=-3\pm\sqrt{2}$ or $-3\pm\sqrt{3}$
11(a)(i) -4 **(ii)** 3 **(b)(i)** $2x^3+32x^2+163x+262=0$
(ii) $2x^3+24x^2+27x-162=0$
12(a) Use the sum of roots.

13(a)(i) 8 **(ii)** 6 **(c)** $3,\ 1+i,\ 1-i$
14(a)(i) $-p$ **(ii)** $-r$ **(c)(ii)** $P(0)=-8,\ P(1)=2$
19(a) Replace x with $-\frac{p}{x+1}$ to get
$rx^3+(3r-pq)x^2+(p^3-2pq+3r)x+(r-pq)=0$
20(a)(i) $-\frac{1}{2}n(n+1)$ **(ii)** $(-1)^n\,n!$