MATH 1902 SOLUTIONS First Semester 2010

1 (a) (i) $\overrightarrow{QS} = \overrightarrow{QA} + \overrightarrow{AO} + \overrightarrow{OS}$

$= \frac{1}{2}\overrightarrow{BA} - \underset{\sim}{a} + \frac{1}{2}\overrightarrow{OC}$

$= \frac{1}{2}(-\underset{\sim}{b} + \underset{\sim}{a}) - \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$

$= \frac{1}{2}(-\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c})$. (3)

(ii) $\overrightarrow{QT} = \overrightarrow{QP} + \overrightarrow{PT}$

$= -\frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\overrightarrow{PR}$

$= -\frac{1}{2}\underset{\sim}{b} + \frac{1}{4}(-\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$

$= \frac{1}{4}(-\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c})$

$= \frac{1}{2}\overrightarrow{QS}$ (3)

so $QT \parallel QS$ and T is the midpoint of QS, proving PR and QS bisect each other. (1)

(iii) The previous argument works for each pair of skew edges, so the intersection point for each pair must be common. (1)

1 (b) $\underline{a} + \underline{b} = \underline{c} + \underline{d}$

so $(\underline{a} + \underline{b}) \cdot \underline{v} = (\underline{c} + \underline{d}) \cdot \underline{v}$

so $\underline{a} \cdot \underline{v} + \underline{b} \cdot \underline{v} = \underline{c} \cdot \underline{v} + \underline{d} \cdot \underline{v}$

so $\underline{a} \cdot \underline{v} = \underline{c} \cdot \underline{v}$, since $\underline{b} \cdot \underline{v} = \underline{d} \cdot \underline{v} = 0$. (3)

But $\underline{a}, \underline{c} \parallel \underline{v} \neq \underline{0}$, so

$\underline{a} = \lambda \underline{v}$, $\underline{c} = \mu \underline{v}$ $\exists \lambda, \mu$.

Hence

$(\lambda \underline{v}) \cdot \underline{v} = (\mu \underline{v}) \cdot \underline{v}$,

i.e. $\lambda |\underline{v}|^2 = \mu |\underline{v}|^2$,

i.e. $(\lambda - \mu) |\underline{v}|^2 = 0$

But $|\underline{v}|^2 \neq 0$, so $\lambda - \mu = 0$, i.e. $\lambda = \mu$,

so $\underline{a} = \lambda \underline{v} = \mu \underline{v} = \underline{c}$ (2)

Hence $\underline{b} = \underline{c} + \underline{d} - \underline{a}$

$= \underline{d} + \underline{c} - \underline{c}$

$= \underline{d}$. (1)

1 (b) Alternative solution (that might be popular)

$\underline{a} + \underline{b} = \underline{c} + \underline{d}$ (1)

so $\underline{a} - \underline{c} = \underline{d} - \underline{b}$

But $\underline{a}, \underline{c} \parallel \underline{v}$ so $\underline{a} - \underline{c} \parallel \underline{v}$ (1)

and $\underline{b}, \underline{d} \perp \underline{v}$ so $\underline{d} - \underline{b} \perp \underline{v}$ (1)

(1)

since $(\underline{d} - \underline{b}) \cdot \underline{v} = \underline{d} \cdot \underline{v} - \underline{b} \cdot \underline{v}$
$= 0 - 0 = 0$

Hence $\underline{a} - \underline{c} = \underline{d} - \underline{b}$ is both $\parallel$ and $\perp$ to $\underline{v} \neq \underline{0}$,

so

$\underline{a} - \underline{c} = \underline{d} - \underline{b} = \underline{0}$, (2)

So $\underline{a} = \underline{c}$ and $\underline{b} = \underline{d}$. (1)

2(a) (i)
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -4 & 1 & -1 & 0 & 1 & 0 \\ 6 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & -2 & -5 & -6 & 0 & 1 \end{array}\right]$$

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$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & -2 & -5 & -3 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -1 & -2 & -1 \\ -2 & -5 & -3 \\ 2 & 2 & 1 \end{bmatrix}$ 1

(ii)
$$\begin{bmatrix} 1 & 0 & 1 \\ -4 & 1 & -1 \\ 6 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

So
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \\ -2 & -5 & -3 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

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$$= \begin{bmatrix} -a - 2b - c \\ -2a - 5b - 3c \\ 2a + 2b + c \end{bmatrix}$$

2(b) Let A be $p \times q$, B be $r \times s$.

Then $AB = I_m$ gives $q = r$ and $p = s = m$,

whilst $BA = I_n$ gives $s = p$ and $r = q = n$.

Hence A is $m \times n$ and B is $n \times m$. (3)

WLOG suppose $m > n$, so A has more rows than columns. After row reducing A we must have a row of zeros, so EA has a row of zeros, for some invertible $m \times m$ matrix E (product of elementary matrices). But then $E = EI_m = EAB = (EA)B$ must also have a row of zeros, contradicting that E is invertible.

Hence $m = n$ and A, B are $n \times n$. (5)

3. (a) Fund. Thm Alg: Every nonconstant polynomial has a root in $\mathbb{C}$. (1)

If M is a square matrix then $\det(M-\lambda I)$ is a nonconstant polynomial, which therefore has a root in $\mathbb{C}$, which is an eigenvalue of M. (2)

(b) The equality $p(M) = \det(M-MI)$ is false, because of invalid substitution, giving $n \times n$ matrix $p(M)$ equalling a scalar $\det(M-MI)$, (2)

except when $n=1$, when the reasoning is valid. (1)

(c) Let $p(\lambda) = \det(M-\lambda I) = \lambda^2 + k\lambda + l$ $\exists k, l$.

By Cayley-Hamilton, $M^2 + kM + lI = 0$, so

$$M^2 = -kM - lI.$$ (2)

Thus we may take $a_2 = -k$, $b_2 = -l$ and the claim holds for $n=2$, starting an induction. (1)

3(c) (continued)

If $n > 2$ then, by an inductive hypothesis,

$$M^n = M(M^{n-1}) = M(a_{n-1}M + b_{n-1}I)$$

for some integers a_{n-1}, b_{n-1}, so that

$$M^n = a_{n-1}M^2 + b_{n-1}M$$

$$= a_{n-1}(-kM - \ell I) + b_{n-1}M$$

$$= (-a_{n-1}k + b_{n-1})M + (-a_{n-1}\ell)I$$

$$= a_nM + b_nI$$

where $a_n = (-a_{n-1}k + b_{n-1})$ and $b_n = -a_{n-1}\ell$,

and the claim holds in general.

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When $M = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, $p(\lambda) = \lambda(\lambda-2) - 1 = \lambda^2 - 2\lambda - 1$,

so $M^2 = 2M + I$,

$$M^3 = 2M^2 + M = 4M + 2I + M = 5M + 2I$$

$$M^4 = 5M^2 + 2M = 10M + 5I + 2M = 12M + 5I$$

$$M^5 = 12M^2 + 5M = 24M + 12I + 5M = 29M + 12I$$

so $a_5 = 29$, $b_5 = 12$.

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4(a)(i) eigenvalues are $\lambda = -1$, $\lambda = 3$. (1)

$$\begin{bmatrix} -8 & -4 & -4 \\ 12 & 8 & 4 \\ 12 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ (2)

so eigenspace for $\lambda = -1$ is

$$\left\{ \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}$$

$$\begin{bmatrix} -12 & -4 & -4 \\ 12 & 4 & 4 \\ 12 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so eigenspace for $\lambda = 3$ is (2)

$$\left\{ \begin{bmatrix} -s/3 - t/3 \\ s \\ t \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\}$$

(ii) $P = \begin{bmatrix} -1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (2)