THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH1902

LINEAR ALGEBRA (ADVANCED)

June 2009

LECTURERS: J East, A Molev

TIME ALLOWED: **One and a half hours**

Name: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

1. (10 marks).

(a) Consider the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ described by the equations

$$x + 2y - z = 3 \qquad \text{and} \qquad 2x - y + 8z = 1.$$

(*i*) Find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that $\mathbf{u}_1 \perp \mathcal{P}_1$ and $\mathbf{u}_2 \perp \mathcal{P}_2$. Explain why $\mathcal{P}_1$ and $\mathcal{P}_2$ are not parallel.

(*ii*) Find the parametric vector equation of the line $\mathcal{L}$ which is the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$.

(*iii*) Consider the plane $\mathcal{P}_3$ given by the equation $3x - 2y - z = 5$. Without explicitly calculating the intersection, explain why the intersection of all three planes is a single point.

(b) (*i*) Consider the points $A(2,3)$ and $B(-2,1)$ in the plane. Find the area of the parallelogram that has OA and OB as two of its sides.

(*ii*) Find the angle AOB of the parallelogram in the previous part.

2. (10 marks).

(a) Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero, perpendicular vectors in the plane. Show that if

$$a\,\mathbf{u} + b\,\mathbf{v} = c\,\mathbf{u} + d\,\mathbf{v}$$

for scalars a, b, c, d, then $a = c$ and $b = d$.

(b) Let $ABCD$ be a square, and suppose that M and N divide AB and AD internally, and non-trivially, in the ratios $\alpha : \beta$ and $\gamma : \delta$, respectively, where $\alpha + \beta = \gamma + \delta = 1$. Let P be the point of intersection of DM and BN.

(*i*) Draw a neat diagram displaying this information.

(*ii*) Find scalars p, q, r, s with $p + q = r + s = 1$ such that P divides DM in the ratio $p : q$ and BN in the ratio $r : s$. (Hint: write $\mathbf{p} = \overrightarrow{AP}$ as a linear combination of $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{d} = \overrightarrow{AD}$ in two ways, and apply part (a).)

3. (10 marks).

(a) Give the definition of a left inverse and a right inverse of a matrix A.

(b) Prove that if a matrix A has a left inverse and a right inverse then they are equal.

(c) Find all values of x for which the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & x \end{bmatrix}$$

has a left inverse.

(d) Take a value of x for which the matrix A in the previous part has a left inverse. Explain why the number of left inverse matrices of A is infinite.

4. (10 marks). The matrix C is given by

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Calculate the characteristic polynomial $\det(C - xI)$ of the matrix C.

(b) Find the eigenvalues of C.

(c) Let k be a positive integer. Calculate the eigenvalues of the matrix C^k. Justify your calculation.

(d) Hence, give a formula for the characteristic polynomial of the matrix C^5.

End of Extended Answer Section