The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Longer Solutions to Selected Exercises for Week 8** 2012

1. (i) $\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 2 & -3 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & -5 & -10 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]$,

so that $x = 4$ and $y = 2$.

(ii) By back substitution, $z = -2$, $y = 3 + z = 1$, $x = 6 - 2y - 3z = 10$.

2. (i) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 9 \\ 1 & 0 & 1 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 9 \\ 0 & -1 & 2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 7 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{array}\right]$, so that $x = 3$, $y = 4$, $z = 7$.

(ii) $\left[\begin{array}{ccc|c} -3 & 2 & 1 & 4 \\ 4 & 1 & 3 & 9 \\ 1 & -1 & -1 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & -4 \\ 0 & -1 & -2 & -8 \\ 0 & 5 & 7 & 25 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -3 & -15 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$, so that $x = -1$, $y = -2$, $z = 5$.

3. By back substitution: (i) $z = t$, $y = 2 - z = 2 - t$, $x = 4 + 2z = 4 + 2t$;

(ii) $z = t$, $y = -1 + 2z = -1 + 2t$, $x = -2y - 3z = -2(-1 + 2t) - 3t = 2 - 7t$.

4. (i) $\left[\begin{array}{cc|c} 4 & -5 & 7 \\ -3 & 8 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -5 & 7 \\ 1 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & -17 & -17 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$,

so that $x = 3$ and $y = 1$.

(ii) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 2 \\ 2 & 3 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & -1 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 12 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$, so that $x = 3$, $y = -3$, $z = 4$.

5. (i) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$,

so that, by back substitution, $z = t$, $y = 1 + t$, $x = 1 - 2t$.

(ii) $\left[\begin{array}{ccc|c} -3 & 2 & 7 & 1 \\ 5 & -3 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & -4 & -14 & -2 \\ 5 & -3 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -12 & 0 \\ 5 & -3 & -2 & -2 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & -12 & 0 \\ 0 & 2 & 58 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 17 & -1 \\ 0 & 1 & 29 & -1 \end{array}\right]$,

so that, by back substitution, $z = t$, $y = -1 - 29t$, $x = -1 - 17t$.

6. (i) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$,

so that, by back substitution, $z = t$, $y = -2t$, $x = t$.

(ii) $\left[\begin{array}{cccc|c} -1 & 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & -2 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 \\ 0 & -1 & 2 & 2 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & -1 & 0 \\ 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 7 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{7} & 0 \\ 0 & 1 & 0 & -\frac{8}{7} & 0 \\ 0 & 0 & 1 & \frac{3}{7} & 0 \end{array}\right]$,

so that, by back substitution, $w = t$, $z = -\frac{3}{7}t$, $y = \frac{8}{7}t$, $x = -\frac{2}{7}t$.

7. $\left[\begin{array}{ccccc} 2 & 3 & 1 & -1 & 4 \\ -2 & -3 & 1 & 2 & -3 \\ 2 & 3 & 2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccccc} 2 & 3 & 1 & -1 & 4 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right]$

$\sim \left[\begin{array}{ccccc} 2 & 3 & 0 & -4 & 6 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & -5 & 5 \end{array}\right] \sim \left[\begin{array}{ccccc} 2 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$,

so that, by back substitution, $x_5 = t$, $x_4 = t$, $x_3 = -t$, $x_2 = s$, $x_1 = -\frac{3}{2}s - t$.

8. If we assign zero to each variable then each equation is satisfied, so that there is at least one solution of any homogeneous system. Hence all homogeneous systems are consistent.

9. (i) $\left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 1 & 1 & 4 & 3 \\ 2 & 3 & 11 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right]$,

so that the system is inconsistent, that is, has no solution.

(ii) $\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 2 & 4 & -1 & 4 & -1 \\ -1 & -2 & 2 & -5 & 5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 0 & 0 & -3 & 6 & -9 \\ 0 & 0 & 3 & -6 & 9 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$,

so that, by back substitution, $w = t$, $z = 3 + 2t$, $y = s$, $x = 1 - 2s - t$.

10. Call the polynomial

$$p(x) = ax^3 + bx^2 + cx + d$$

where a, b, c, d are constants to be determined, so that $p'(x) = 3ax^2 + 2bx + c$. But

$$p(1) = -2\,, \quad p(-1) = -10\,, \quad p'(1) = 0\,, \quad p'(-1) = 12\,,$$

yielding the system

$$\begin{array}{rcrcrcrcr} a & + & b & + & c & + & d & = & -2 \\ -a & + & b & - & c & + & d & = & -10 \\ 3a & + & 2b & + & c & & & = & 0 \\ 3a & - & 2b & + & c & & & = & 12 \end{array}$$

and augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -2 \\ -1 & 1 & -1 & 1 & -10 \\ 3 & 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 12 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -2 \\ 0 & 2 & 0 & 2 & -12 \\ 0 & -1 & -2 & -3 & 6 \\ 0 & -5 & -2 & -3 & 18 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & -6 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & 2 & -12 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 & -6 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 & -12 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & -6 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right],$$

yielding $a = 1$, $b = -3$, $c = 3$, $d = -3$, so that $p(x) = x^3 - 3x^2 + 3x - 3$.

11. If we call the ages of the family members A, B, C, D respectively, then the information tells us that $A+B+C+D = 70$, $B = 3(C+D)$, $A+10 = 2((C+10)+(D+10))-20$, $C-4 = (A-4)-(B-4)$. This becomes the following system of equations:

$$\begin{array}{rcrcrcrcr} A & + & B & + & C & + & D & = & 70 \\ & & B & - & 3C & - & 3D & = & 0 \\ A & & & - & 2C & - & 2D & = & 10 \\ A & - & B & - & C & & & = & -4 \end{array}$$

with augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 70 \\ 0 & 1 & -3 & -3 & 0 \\ 1 & 0 & -2 & -2 & 10 \\ 1 & -1 & -1 & 0 & -4 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 4 & 4 & 70 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & -1 & -3 & -3 & -60 \\ 0 & -2 & -2 & -1 & -74 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 0 & 4 & 4 & 70 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & -6 & -6 & -60 \\ 0 & 0 & -8 & -7 & -74 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 0 & 30 \\ 0 & 0 & 1 & 1 & 10 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 0 & 30 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right],$$

yielding $A = 30$, $B = 30$, $C = 4$, $D = 6$.

12. We have $x^3 = A(x-1)^3 + B(x-1)^2 + C(x-1) + D$ for all x (by continuity). Putting $x = 1$ gives $D = 1$ immediately. Putting $x = 2$, 0, -1 respectively yields the system

$$\begin{array}{rcrcrcr} A & + & B & + & C & = & 7 \\ -A & + & B & - & C & = & -1 \\ -8A & + & 4B & - & 2C & = & -2 \end{array}$$

with augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 7 \\ -1 & 1 & -1 & -1 \\ -8 & 4 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 7 \\ 0 & 2 & 0 & 6 \\ 0 & 12 & 6 & 54 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 6 & 18 \end{array}\right],$$

yielding $A = 1$, $B = 3$, $C = 3$, $D = 1$.

13. Systematically considering cases involving zero, one and two nonzero rows, the following is a catalogue of reduced echelon 2×3 matrices,where α, β, γ, δ, λ, μ are arbitrary real numbers:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & \alpha \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & \beta & \gamma \\ 0 & 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & \mu \end{bmatrix}$$

14. The possible reduced echelon forms for the augmented matrices are precisely the possibilities catalogued in the previous exercise, augmented by a column of zeros. Because there are only two equations and three unknowns, there must be at most two leading variables, so at least one parameter, in writing down the solution. Hence there is always a nontrivial solution to the given homogeneous system. If we write

$$\mathbf{u} \;=\; u_1\mathbf{i} + u_2\mathbf{j}\,, \quad \mathbf{v} \;=\; v_1\mathbf{i} + v_2\mathbf{j}\,, \quad \mathbf{w} \;=\; w_1\mathbf{i} + w_2\mathbf{j}\,,$$

then the vector equation

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} \;=\; \mathbf{0}$$

is equivalent to the given homogeneous system, so has a nontrivial solution, which proves $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent.

15.
$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ -2 & -\lambda & 1 & 2 \\ 1 & 2 & \lambda & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & -\lambda & -5 & -4 \\ 0 & 2 & \lambda+3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{\lambda+3}{2} & 2 \\ 0 & -\lambda & -5 & -4 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{\lambda+3}{2} & 2 \\ 0 & 0 & -5+\frac{\lambda(\lambda+3)}{2} & -4+2\lambda \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 2 & \lambda+3 & 4 \\ 0 & 0 & (\lambda+5)(\lambda-2) & 4(\lambda-2) \end{array}\right]$$

(i) To be inconsistent, we require $(\lambda+5)(\lambda-2) = 0$ and $4(\lambda-2) \neq 0$, so that $\lambda = -5$.

(ii) To have infinitely many solutions, we require $(\lambda+5)(\lambda-2) = 0 = 4(\lambda-2)$, so that $\lambda = 2$.

(iii) To have a unique solution, we require both (i) and (ii) to fail, that is, $\lambda \neq 2, -5$.

16. (i)
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 5 \\ -1 & -4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which has an infinite solution, using one parameter.

(ii)
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 5 \\ -1 & -4 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right],$$

which is inconsistent, that is, has no solution.

(iii)
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ -1 & -2 & 3 & 6 \\ -1 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

which has a unique solution.

(iv) $\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 3 & 1 & -1 \\ -2 & -2 & 4 & 6 & 2 & 0 \\ 0 & 0 & 0 & -3 & -1 & 4 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 12 & 4 & -2 \\ 0 & 0 & 0 & -3 & -1 & 4 \end{array}\right]$

$\sim \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 3 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 14 \end{array}\right],$

which is inconsistent, that is, has no solution.

17. The equations can be rearranged to give the following system:

$$\begin{array}{rcrcrcr} x & - & y & & & = & -2 \\ & & 5y & + & 4z & = & 23 \\ x & - & 7y & & & = & -20 \\ & & 3y & - & z & = & 7 \end{array}$$

with augmented matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & -2 \\ 0 & 5 & 4 & 23 \\ 1 & -7 & 0 & -20 \\ 0 & 3 & -1 & 7 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 0 & -2 \\ 0 & 5 & 4 & 23 \\ 0 & -6 & 0 & -18 \\ 0 & 3 & -1 & 7 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 5 & 4 & 23 \\ 0 & 3 & -1 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

yielding $x = 1$, $y = 3$, $z = 2$, so that the intersection point is $(1, 3, 2)$.

18. The given system is equivalent to

$$\begin{array}{rcrcrcr} x' & + & 2y' & + & 3z' & = & 0 \\ 4x' & + & 5y' & + & 6z' & = & 1 \\ 7x' & + & 10y' & + & 9z' & = & 2 \end{array}$$

where $x' = 1/x$, $y' = 1/y$, $z' = 1/z$, with augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \\ 7 & 10 & 9 & 2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 1 \\ 0 & -4 & -12 & 2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1/3 \\ 0 & 1 & 3 & -1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -1 & 2/3 \\ 0 & 1 & 2 & -1/3 \\ 0 & 0 & 1 & -1/6 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/6 \end{array}\right],$$

with solution $x' = 1/2$, $y' = 0$, $z' = -1/6$. But if the original system is consistent we would have $y' \neq 0$, which is a contradiction. Hence the original system is inconsistent.

19. Counting carbon, hydrogen and oxygen atoms we get the homogeneous system

$$\begin{array}{rcrcrcrcr} 8\,x & & & - & z & & & = & 0 \\ 18\,x & & & & & - & 2\,w & = & 0 \\ & & 2\,y & - & 2\,z & - & w & = & 0 \end{array}$$

with coefficient matrix $\begin{bmatrix} 8 & 0 & -1 & 0 \\ 18 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/8 & 0 \\ 0 & 0 & 9/4 & -2 \\ 0 & 1 & -1 & -1/2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -1/8 & 0 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 1 & -8/9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -25/18 \\ 0 & 0 & 1 & -8/9 \end{bmatrix},$$

with parametric solution $x = t/9$, $y = 25t/18$, $z = 8t/9$, $w = t$, having smallest positive integer solution $x = 2$, $y = 25$, $z = 16$, $w = 18$, yielding

$$2\,C_8H_{18} + 25\,O_2 \longrightarrow 16\,CO_2 + 18\,H_2O\ .$$

20. If $a \neq 0$ then the lines intersect if and only if the augmented matrix

$$\left[\begin{array}{cc|c} a & b & k \\ c & d & \ell \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & k/a \\ c & d & \ell \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & k/a \\ 0 & d - cb/a & \ell - ck/a \end{array}\right]$$

corresponds to a consistent system with a unique solution, which occurs if and only if $d - cb/a = \dfrac{ad - bc}{a} \neq 0$, that is, $ad - bc \neq 0$. If $a = 0$ then the lines intersect if and only if the augmented matrix

$$\left[\begin{array}{cc|c} a & b & k \\ c & d & \ell \end{array}\right] \sim \left[\begin{array}{cc|c} c & d & \ell \\ 0 & b & k \end{array}\right]$$

corresponds to a consistent system with a unique solution, which occurs if and only if $c \neq 0$ and $b \neq 0$, that is, $ad - bc = -bc \neq 0$. In all cases, the lines intersect if and only if $ad - bc \neq 0$.

21. (i) Different systems produce different augmented matrices, and every augmented matrix arises from some system.

(ii) To operate on the system and then form the augmented matrix has the same effect as forming the augmented matrix and then performing the corresponding elementary row operation.

22. The performance of these operations simultaneously cannot be invertible because it yields a process (function) that is not one-one on augmented matrices (so that inversion is ambiguous). For example, this operation applied to both the following augmented matrices

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

yields the zero augmented matrix

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

In fact, if these refer to systems in x and y, then the set of solutions has been enlarged, from the line $y = -x$ for the first augmented matrix, and the line $y = x$ for the second, to the entire xy-plane for the zero augmented matrix. (Elementary row operations applied in series, not in parallel, do not change the set of solutions of a given system.)

23. The asterisks are intended to be arbitrary real numbers chosen independently:

$$\begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&1&*\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&1&*&*\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix},$$
$$\begin{bmatrix} 1&*&*&*\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&1&*&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 1&*&*&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix},$$
$$\begin{bmatrix} 1&*&0&*\\ 0&0&1&*\\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 1&0&*&*\\ 0&1&*&*\\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 1&*&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix},$$
$$\begin{bmatrix} 1&0&*&0\\ 0&1&*&0\\ 0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 1&0&0&*\\ 0&1&0&*\\ 0&0&1&* \end{bmatrix}$$

The system must have a nontrivial solution, since there are more variables than equations (so that there is at least one nonleading variable to which a parameter may be assigned). If we have four vectors in space, say

$$\begin{aligned} \mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}\,, & \mathbf{v} &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}\,, \\ \mathbf{w} &= w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}\,, & \mathbf{t} &= t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}\,, \end{aligned}$$

then the vector equation

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} + \delta\mathbf{t} = \mathbf{0}$$

is equivalent to the given homogeneous system, so has a nontrivial solution, which proves $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{t}$ are linearly dependent.

24. A given 2×3 matrix is the coefficient matrix of some homogeneous system in x, y, z, and the reduced row echelon forms are catalogued in Exercise 13. Gauss-Jordan elimination always produces a reduced row echelon form and these are the only possibilities. Applying elementary row operations does not change the solution set of a system. Therefore, to see that one can reach at most one possible reduced row echelon form, it is sufficient to show the matrices in the catalogue correspond to systems with different sets of solutions. This is evident from the following table, where each reduced row echelon matrix in the catalogue is paired with one or more solutions (x, y, z) for its corresponding system. It is routine to check that for any pair of matrices, at least one of the solutions listed against one of the matrices is not a solution for the other matrix in the pair. Thus the solutions sets of the homogeneous systems are pairwise different, which completes the proof of uniqueness of the reduced row echelon form.

$$\begin{bmatrix} 0&0&0\\ 0&0&0 \end{bmatrix}, \quad (1,0,0)\,,\ (0,1,0)\,,\ (0,0,1)$$
$$\begin{bmatrix} 0&0&1\\ 0&0&0 \end{bmatrix}, \quad (1,0,0)\,,\ (0,1,0)$$
$$\begin{bmatrix} 0&1&\alpha\\ 0&0&0 \end{bmatrix}, \quad (1,0,0)\,,\ (0,-\alpha,1)$$

$$\begin{bmatrix} 1 & \beta & \gamma \\ 0 & 0 & 0 \end{bmatrix}, \quad (-\beta, 1, 0)\,,\ (-\gamma, 0, 1)$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (1, 0, 0)$$
$$\begin{bmatrix} 1 & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (-\delta, 1, 0)$$
$$\begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & \mu \end{bmatrix}, \quad (-\lambda, -\mu, 1)$$

The solutions listed against a given matrix in fact form a *basis*, that is a minimal set of solutions, such that linear combinations produce every solution. To prove uniqueness for reduced row echelon forms in general, it is possible to describe a general basis for the solutions of the corresponding homogeneous system, and then check that given two such bases, at least one solution for one matrix is not a solution for the other. As in the 2×3 case just considered, that is sufficient to prove uniqueness of the reduced row echelon form.

25. (i) The 1st step requires 1 division. The 2nd step requires 1 multiplication (when making the substitution), 1 subtraction and 1 division. The 3rd step requires 2 multiplications, 2 subtractions and 1 division. Continuing, the nth (final) step requires $n-1$ multiplications, $n-1$ subtractions and 1 division. Altogether there are

$$f(n) \;=\; 1 + 3 + 5 + \ldots + 2n - 1 \;=\; n^2$$

operations (using the closed formula for an arithmetic series).

(ii) Dividing through to put 1 in each diagonal position requires

$$(n+1) + n + \ldots + 2 \;=\; \frac{n(n+3)}{2}$$

divisions. Now, producing zeros above the diagonal, clearing out entries in the traditional manner working from top to bottom, requires (suppressing some of the detail)

$$\begin{aligned} 2n + 4(n-1) + 6(n-2) + \ldots + 2(n-1)(2) \;&=\; 2\big(n + 2(n-1) + \ldots + (n-1)(2)\big) \\ &=\; 2\sum_{i=1}^{n-1} i(n-i+1) \;=\; 2\Bigg((n+1)\sum_{i=1}^{n-1} i \;-\; \sum_{i=1}^{n-1} i^2\Bigg) \\ &=\; 2\left(\frac{(n+1)n(n-1)}{2} \;-\; \frac{n(n-1)(2n-1)}{6}\right) \\ &=\; \frac{n(n-1)(n+4)}{3} \end{aligned}$$

multiplications and subtractions. Hence

$$g(n) \;=\; \frac{n(n+3)}{2} + \frac{n(n-1)(n+4)}{3} \;=\; \frac{2n^3 + 9n^2 + n}{6}\,.$$

(iii) From the previous parts

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} \;=\; \lim_{n\to\infty} \frac{2n^3 + 9n^2 + n}{6n^2} \;=\; \lim_{n\to\infty} \frac{2n + 9 + 1/n}{6} \;=\; \infty\,.$$