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MATH1903 Lecture 8 Fri 25/8/2017

Recall the definition

$$\boxed{a^x \overset{\text{def}}{=} \exp(x \ln a)}$$

for $a > 0$ and $x \in \mathbb{R}$, where

$$\boxed{\ln x \overset{\text{def}}{=} \int_1^x \frac{1}{t}\,dt} \quad \text{for } x > 0,$$

and

$$\boxed{\exp = \ln^{-1}}.$$

One can then prove exponential laws,

e.g. $\boxed{(ab)^c = a^c b^c}$ ✓

Proof:

$$\begin{aligned}
(ab)^c &\overset{\text{def}}{=} \exp(c \ln(ab)) \\
&= \exp(c(\ln(a) + \ln(b))) \quad \text{(property of ln)} \\
&= \exp(c\ln(a) + c\ln(b)) \\
&= \exp(c\ln(a))\exp(c\ln(b)) \quad \text{(property of exp)} \\
&= a^c b^c
\end{aligned}$$

□

B

Advanced techniques of integration

- integration by parts
- method of partial fractions

reference : pages 2.29 — 2.52

Recall product rule : $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Hence
$$uv = \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right) dx$$
$$= \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$
$$= \int u\,dv + \int v\,du$$

so
$$\boxed{\int u\,dv = uv - \int v\,du}$$

or
$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}$$

— integration by parts formula.

(c)

p. 2.32 $\int xe^x\,dx = \cdots = xe^x - e^x + C$

p. 2.33 $\int x^2 e^x\,dx = \cdots = e^x(x^2 - 2x + 2) + C$.

p. 2.34 $\int \ln x\,dx = \cdots = x\ln x - x + C$.

pp. 2.35, 2.36 $\int e^x \sin x\,dx = \cdots = \frac{e^x}{2}(\sin x - \cos x) + C$

pp 2.39 – 2.41 reduction formula for $\int \sin^n x\,dx$.

difficult extended exercise: prove the Wallis product

$$\boxed{\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}}$$

Theory of partial fractions

– method for integrating rational functions

$$y = \frac{p(x)}{q(x)}$$

where $p(x)$, $q(x)$ are polynomials with $q(x) \neq 0$.

Important facts about polynomials :

Fundamental Theorem of Algebra : Let $p(x)$ be a nonconstant polynomial with coefficient from $\mathbb{C}$ of degree $n \geq 1$. Then $p(x)$ factorises completely as

$$p(x) = \lambda (x-\lambda_1)(x-\lambda_2) \cdots (x-\lambda_n)$$

for some $\lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

(proof difficult)

Corollary : Every real nonconstant polynomial factorises as a product of linear and irreducible quadratic factors.

ax^2+bx+c where $b^2-4ac<0$

$= a(x-\zeta)(x-\bar{\zeta})$ where ζ is one root

E

<u>Proof of Corollary</u>: Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a real polynomial. Then $p(\zeta) = 0$ for some $\zeta \in \mathbb{C}$, by the Fund. Thm. Alg., so that

$$0 = \overline{0} = \overline{p(\zeta)} = \overline{a_0 + a_1 \quad + \cdots + a_n{}^n}$$

$$= \overline{a_0} + \overline{a_1}\,\overline{\zeta} + \cdots + \overline{a_n}(\overline{\zeta})^n$$

$$= a_0 + a_1\overline{\zeta} + \cdots + a_n(\overline{\zeta})^n$$

since $a_0, \ldots, a_n \in \mathbb{R}$

$$= p(\overline{\zeta}),$$

so that $\overline{\zeta}$ is also a root. If $\overline{\zeta} \neq \zeta$, then

$$(x - \zeta)(x - \overline{\zeta}) = x^2 - (\zeta + \overline{\zeta}) + \zeta\overline{\zeta}$$

is an irreducible quadratic factor of $p(x)$.

Thus the linear factors using roots that are not real match up in pairs to give irreducible quadratic factors of $p(x)$, proving the Corollary. $\square$

F

<u>General theory in the arithmetic of rational functions</u>:

If $p(x)$ has degree smaller than that of $q(x)$ then $\frac{p(x)}{q(x)}$ is a sum of rational functions of the form

$$\frac{A}{(x-\lambda)^k}, \quad \frac{Ax+B}{(ax^2+bx+c)^k},$$

using denominators that are linear and irreducible quadratic factors of $q(x)$.

e.g.

$$\frac{x^2+2x+6}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$\exists A, B, C$,

$$\frac{x^2+2x+6}{(x+1)^2(x^2+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

$\exists A, B, C, D, E, F$.

To integrate such a rational function, decompose into pieces of this form and integrate each piece using elementary techniques.

Example: $\int \frac{1}{x^2+1}\,dx = \int \frac{dx}{x^2+1} = ?$

(Answer: $\tan^{-1}x + C$)

Solution using complex numbers & partial fractions:

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}$$

where $A(z+i) + B(z-i) = 1$.

Put $z=i$: $A(2i) = 1$, so $A = \frac{1}{2i} = \frac{-i}{2}$.

Put $z=-i$: $B(-2i) = 1$, so $B = \frac{1}{-2i} = \frac{i}{2}$.

Hence $$\frac{1}{z^2+1} = \frac{-i/2}{z-i} + \frac{i/2}{z+i}$$

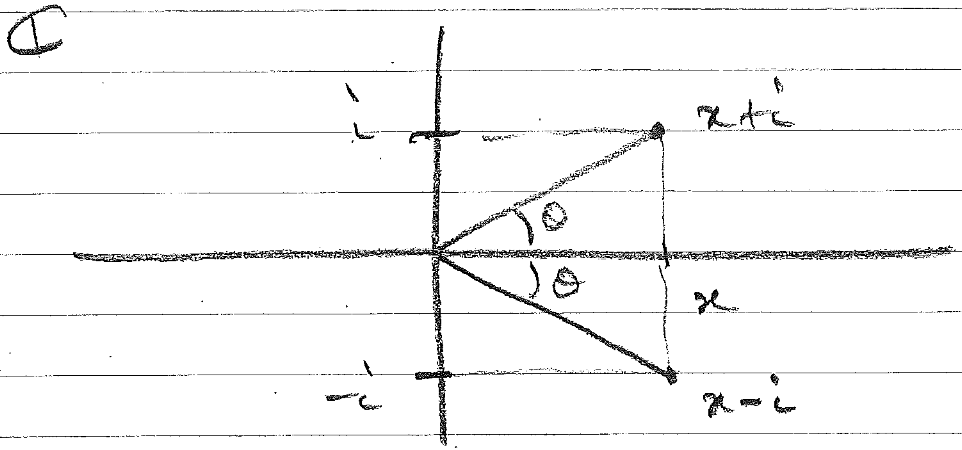
so $$\int \frac{dz}{z^2+1} = \frac{i}{2}\left(\int \frac{dz}{z+i} - \int \frac{dz}{z-i}\right)$$

So $\displaystyle\int \frac{dz}{z^2+1} = \frac{i}{2}\left(\ln(z+i) - \ln(z-i)\right) + C$

using <u>complex</u> logarithms

$$= \frac{i}{2}\ln\left(\frac{z+i}{z-i}\right) + C.$$

Let $z = x$ <u>be real</u>



Recall (from first semester):

$$\boxed{\ln z = \underbrace{\ln|z|}_{\text{real ln}} + i\,\underbrace{\operatorname{Arg} z}_{\text{principal argument}}}$$

Here

$$\ln\left|\frac{x+i}{x-i}\right| = \ln\frac{|x+i|}{|x-i|} = \ln(1) = 0$$

I

and

$$\mathrm{Arg}\left(\frac{x+i}{x-i}\right) = 2\theta$$

$$\left(= \mathrm{Arg}(x+i) - \mathrm{Arg}(x-i)\right).$$

We have

$$\int \frac{dx}{x^2+1} = \frac{i}{2}\ln\left(\frac{x+i}{x-i}\right) + C$$

$$= \frac{i}{2}\left(\ln\left|\frac{x+i}{x-i}\right| + i\,\mathrm{Arg}\left(\frac{x+i}{x-i}\right)\right) + C$$

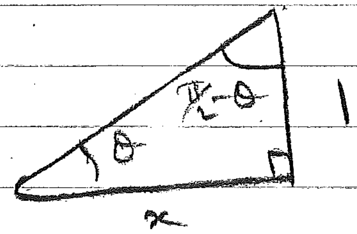
$$= \frac{i}{2}\left(0 + i\,2\theta\right) + C$$

$$= C - \theta \quad = C' + \pi/2 - \theta$$

$$= C' + \tan^{-1} x$$

$$= \tan^{-1} x + C'$$

as expected



$$\tan\left(\pi/2 - \theta\right) = \frac{x}{1}$$

$$= x,$$

so $\frac{\pi}{2} - \theta = \tan^{-1} x$