**MATH562: Continuous Optimisation**
**Homework 1**

Name: Keegan Gyoery UM-ID: 31799451

1. a) Let $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$. The domain of $f$ is the set $D = \mathbb{R}^2$. The range of $f$ is the set $S = \{f(\mathbf{x}) \in \mathbb{R} : f(\mathbf{x}) \geq 0, \forall \mathbf{x} \in D\}$. Considering the Gradient ($\nabla$) and Hessian ($H$) of $f(\mathbf{x})$,

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2(x_1 - 3) \\ 2(x_2 - 3) \end{bmatrix},$$
$$Hf(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Both the Gradient and Hessian of $f(\mathbf{x})$ have domain $D = \mathbb{R}^2$. The Gradient has range $S = M_{2,1}(\mathbb{R})$, the set of all $2 \times 1$ matrices over the real numbers. The Hessian has range

$$S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

b) Let $f(x_1, x_2) = 4x_1^2 + 9x_2^3 - 36$. The domain of $f$ is the set $D = \mathbb{R}^2$. The range of $f$ is the set $S = \mathbb{R}$. Considering the Gradient ($\nabla$) and Hessian ($H$) of $f(\mathbf{x})$,

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 8x_1 \\ 27x_2^2 \end{bmatrix},$$
$$Hf(\mathbf{x}) = \begin{bmatrix} 8 & 0 \\ 0 & 54x_2 \end{bmatrix}.$$

Both the Gradient and Hessian of $f(\mathbf{x})$ have domain $D = \mathbb{R}^2$. The Gradient has range

$$S = \left\{ A \in M_{2,1}(\mathbb{R}) : A = \begin{bmatrix} a \\ b \end{bmatrix}, a \in \mathbb{R}, b \geq 0 \right\}.$$

The Hessian has range

$$S = \left\{ A \in M_{2,2}(\mathbb{R}) : A = \begin{bmatrix} 8 & 0 \\ 0 & a \end{bmatrix}, a \in \mathbb{R} \right\}.$$

c) Let $f(x_1, x_2) = x_1^2 + x_2 + 6$. The domain of $f$ is the set $D = \mathbb{R}^2$. The range of $f$ is the set $S = \mathbb{R}$. Considering the Gradient ($\nabla$) and Hessian ($H$) of $f(\mathbf{x})$,

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 \\ 1 \end{bmatrix},$$
$$Hf(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Both the Gradient and Hessian of $f(\mathbf{x})$ have domain $D = \mathbb{R}^2$. The Gradient has range

$$S = \left\{ A \in M_{2,1}(\mathbb{R}) : A = \begin{bmatrix} a \\ 1 \end{bmatrix}, a \in \mathbb{R} \right\}.$$

The Hessian has range

$$S = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

d) Let $f(x_1, x_2) = x_1x_2 + x_1^3 - 3$. The domain of $f$ is the set $D = \mathbb{R}^2$. The range of $f$ is the set $S = \mathbb{R}$. Considering the Gradient $(\nabla)$ and Hessian $(H)$ of $f(\mathbf{x})$,

$$\nabla f(\mathbf{x}) = \begin{bmatrix} x_2 + 3x_1^2 \\ x_1 \end{bmatrix},$$
$$Hf(\mathbf{x}) = \begin{bmatrix} 6x_1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Both the Gradient and Hessian of $f(\mathbf{x})$ have domain $D = \mathbb{R}^2$. The Gradient has range

$$S = \left\{ A \in M_{2,1}(\mathbb{R}) : A = \begin{bmatrix} a \\ b \end{bmatrix}, a, b \in \mathbb{R} \right\}.$$

The Hessian has range

$$S = \left\{ A \in M_{2,2}(\mathbb{R}) : A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}, a \in \mathbb{R} \right\}.$$

2. Consider $f : \mathbb{R}^3 \to \mathbb{R}^3$, where

$$f(\mathbf{x}) = \begin{bmatrix} x_1^2 + x_2^3 - x_3^4 \\ x_1x_2x_3 \\ 2x_1x_2 - 3x_2x_3 + x_1x_3 \end{bmatrix}.$$

The Jacobian $(J)$ of $f(\mathbf{x})$ is

$$Jf(\mathbf{x}) = \begin{bmatrix} 2x_1 & 3x_2^2 & -4x_3^3 \\ x_2x_3 & x_1x_3 & x_1x_2 \\ 2x_1 + x_3 & 2x_1 - 3x_3 & -3x_2 + x_1 \end{bmatrix}.$$

3. Consider $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(\mathbf{x}) = x_1^3x_2^4 + \dfrac{x_2}{x_1}$. The Gradient $(\nabla)$ and Hessian $(H)$ of $f(\mathbf{x})$ are

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 3x_1^2x_2^4 - \frac{x_2}{x_1^2} \\ \\ 4x_2^3x_1^3 + \frac{1}{x_1} \end{bmatrix},$$

$$Hf(\mathbf{x}) = \begin{bmatrix} 6x_1x_2^4 - \frac{2x_2}{x_1^3} & 12x_1^2x_2^3 - \frac{1}{x_1^2} \\ \\ 12x_1^2x_2^3 - \frac{1}{x_1^2} & 12x_2^2x_1^3 \end{bmatrix}.$$

Both the Gradient and Hessian of $f(\mathbf{x})$ have domain $D = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (x_1, x_2), x_1 \neq 0\}$. The Gradient has range

$$S = \left\{ A \in M_{2,1}(\mathbb{R}) : A = \begin{bmatrix} a \\ b \end{bmatrix}, a, b \in \mathbb{R} \right\}.$$

The Hessian has range

$$S = \left\{ A \in M_{2,2}(\mathbb{R}) : A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in \mathbb{R} \right\}.$$

4. a) Let $\mathbf{x} = (H, L, W)$. The box then has a surface area of $A(\mathbf{x}) = 2HL + 2LW + 2HW$, and a volume of $V(\mathbf{x}) = HLW$. Thus, we can rewrite the formulas given as

$$\begin{aligned} Q(\mathbf{x}) &= KHLW(T - T_a), \\ h_c(\mathbf{x}) &= k_c(2HL + 2LW + 2HW)(T - T_a), \\ h_r(\mathbf{x}) &= k_r(2HL + 2LW + 2HW)(T - T_a^4). \end{aligned}$$

Furthermore, total heat loss, $T(\mathbf{x})$, is found by multiplying rate of heat loss per unit of time, by the amount of time elapsed. Thus, we have the following equation for total heat loss, over $S$ units of time,

$$\begin{aligned} T(\mathbf{x}) &= S(h_c + h_r) \\ \therefore T(\mathbf{x}) &= S(2HL + 2LW + 2HW)\left[k_c(T - T_a) + k_r(T - T_a^4)\right]. \end{aligned}$$

Thus, to formulate the minimisation problem, we must minimise $T(\mathbf{x})$, subject to the conditions

$$\begin{aligned} Q(\mathbf{x}) &\geq Q', \\ 0 \leq H &\leq H', \\ 0 \leq L &\leq L', \\ 0 \leq W &\leq W'. \end{aligned}$$

b) Again, let $\mathbf{x} = (H, L, W)$. Considering an insulation thickness of $t$, on the outside of the box, the box now has the dimensions $\mathbf{x} + 2t = (H + 2t, L + 2t, W + 2t)$. Calculating the volume of the insulation, $V_i(\mathbf{x})$, requires us to calculate the new volume of the box, with the insulation on the outside, and then subtract the original volume of the box, yielding

$$\begin{aligned} V_i(\mathbf{x}) &= V(\mathbf{x} + 2t) - V(\mathbf{x}) \\ &= (H + 2t)(L + 2t)(W + 2t) - HLW \\ \therefore V_i(\mathbf{x}) &= 2t(HL + LW + HW) + 4t^2(H + L + W) + 8t^3. \end{aligned}$$

As cost is proportional to the volume of the insulation, we have $C_i(\mathbf{x}) = \alpha V_i(\mathbf{x})$, for some constant $\alpha$. Thus, for the cost of the insulation, we have

$$C_i(\mathbf{x}) = \alpha\left[2t(HL + LW + HW) + 4t^2(H + L + W) + 8t^3\right].$$

As we may consider the insulation to have negligible effect when computing the surface area and volume for heat loss and heat storage, we may use the equation for $Q(\mathbf{X})$ in part $a$), for the conditions. Thus, in formulating the minimisation problem, we must minimise $C_i(\mathbf{x})$, subject to the conditions

$$\begin{aligned} Q(\mathbf{x}) &\geq Q', \\ 0 \leq H &\leq H', \\ 0 \leq L &\leq L', \\ 0 \leq W &\leq W'. \end{aligned}$$