THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Quiz 1: Practice Questions

MATH1901/1906: Differential Calculus (Advanced) — Semester 1, 2017

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Some information about the Quiz:

(1) The quiz covers material up to and including Week 5 lectures. That is, material from the tutorials in weeks 2–6.

(2) The quiz runs for 40 minutes.

(3) You must write your answers in pen, not pencil.

(4) The format of the real quiz is mostly "short-answer" questions. There will be answer boxes provided below each question where you should write your final answers. In some questions your working will be considered, so please write neatly.

(5) There will be 10 questions in the real quiz, each worth the same amount. This practice quiz is considerably longer than the real quiz.

(6) The quiz is a closed book examination. No notes or books are allowed.

(7) Non-programmable non-graphics calculators are allowed, but are not needed.

The questions provided here are really just for additional practice (building from the lectures and tutorials). Therefore you should not expect that doing only these practice questions will be adequate preparation – it is strongly advised that you also revise the tutorials from Weeks 2–6, and make your own summary notes of the lecture material.

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1. Let $A = \{-1, 2, 5, 9\}$, $B = \{-2, -1, 2, 9\}$, and $C = \{-1, 0, 5\}$.

(a) Find $A \cap B$ and $B \backslash C$.

Solution: $A \cap B = \{-1, 2, 9\}$ and $B \backslash C = \{-2, 2, 9\}$.

(b) Explain why every injective function $f : A \longrightarrow B$ is necessarily also surjective.

Solution: If $f : A \longrightarrow B$ is injective then $f(a) \neq f(b)$ whenever $a, b \in A$ with $a \neq b$. Thus the elements $f(-1), f(2), f(5), f(9)$ are all different, and so

$$\text{image}(f) = \{f(-1), f(2), f(5), f(9)\} \quad \text{has exactly 4 elements.}$$

However the image is contained in the codomain, and the codomain also has exactly 4 elements, hence the image and codomain are equal and so f is surjective.

More generally: If A and B are and **finite** sets with the same number of elements, then every injective function $f : A \longrightarrow B$ is necessarily also surjective.

2. Let $z = 12 + 5i$ and $w = 2 - 3i$. Calculate the following complex numbers in Cartesian form:

$$5w, \qquad z - 5w, \qquad zw, \qquad z\bar{w}, \qquad |z - w|, \qquad \frac{z}{w}, \qquad w^4, \qquad e^z$$

Solution: We have:

$$\begin{aligned}
5w &= 10 - 15i \\
z - 5w &= (12 + 5i) - (10 - 15i) = 2 + 20i \\
zw &= (12 + 5i)(2 - 3i) = (24 + 15) + (10 - 36)i = 39 - 26i \\
z\bar{w} &= (12 + 5i)(2 + 3i) = (24 - 15) + (10 + 36)i = 9 + 46i \\
|z - w| &= |2(5 + 4i)| = 2\sqrt{5^2 + 4^2} = 2\sqrt{41} \\
\frac{z}{w} &= \frac{z\bar{w}}{w\bar{w}} = \frac{9 + 46i}{13} = \frac{9}{13} + \frac{46}{13}i \\
w^4 &= (2 - 3i)^4 = (-5 - 12i)^2 = -119 + 120i \\
e^z &= e^{12+5i} = e^{12}e^{5i} = e^{12}(\cos 5 + i \sin 5) = e^{12}\cos 5 + ie^{12}\sin 5
\end{aligned}$$

3. Evaluate the following complex numbers. Express your final answers in Cartesian form.

(a) $(1 + i)^{23}$

Solution:

(a) Let $z = 1 + i$. The polar form of z is

$$1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4) = \sqrt{2}e^{\pi i/4}.$$

By De Moivre's Theorem,

$$(1 + i)^n = 2^{n/2}(\cos(n\pi/4) + i \sin(n\pi/4)) = 2^{n/2}e^{\pi i n/4}.$$

In the case $n = 23$, we have

$$(1 + i)^{23} = 2^{23/2}e^{23\pi i/4} = 2^{11}(1 - i).$$

(b) $(-1 + i\sqrt{3}\,)^{23}$

Solution: Let $z = -1 + i\sqrt{3}$. The polar form of z is

$$-1 + i\sqrt{3} = 2e^{2\pi i/3} = 2e^{2\pi i/3}.$$

Then

$$(-1 + i\sqrt{3})^{23} = 2^{23}e^{46\pi i/3} = 2^{23}e^{-2\pi i/3} = -2^{22}(1 + i\sqrt{3}).$$

4. Find all solutions to the following equations:

(a) $z^7 = -1 + \sqrt{3}i$

Solution: The solutions are the 7th roots of $-1 + i\sqrt{3}$. These are $2^{1/7}\exp(2\pi i/21 + 2k\pi i/7)$, where $k = 0, 1, 2, 3, 4, 5, 6$. In the complex plane, they form the vertices of a regular heptagon (seven-sided polygon).

(b) $(2 + i)e^z = 1 + 3i$

Solution: Rewrite the equation as

$$e^z = \frac{1+3i}{2+i} = \frac{(1+3i)(2-i)}{(2+i)(2-i)} = \frac{5+5i}{5} = 1 + i.$$

Thus, writing $z = x + iy$, we have

$$e^x e^{iy} = \sqrt{2}e^{i\pi/4}.$$

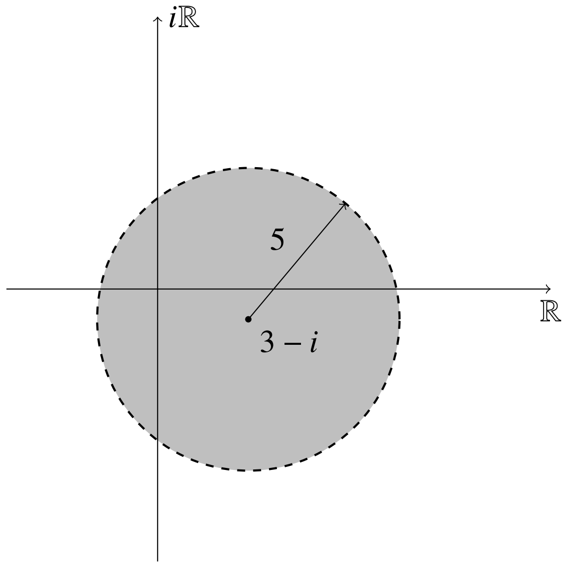
Thus $e^x = \sqrt{2}$, and so $x = \frac{1}{2}\ln 2$, and $y = \pi/4 + 2k\pi$ with $k \in \mathbb{Z}$. Thus the solutions are

$$z = x + iy = \frac{1}{2}\ln 2 + \left(\frac{\pi}{4} + 2k\pi\right)i \quad \text{with } k \in \mathbb{Z}.$$

5. Sketch the following sets in the complex plane:

(a) $\{z \in \mathbb{C} \mid |z - 3 + i| < 5\}$

Solution: The equation $|z - 3 + i| = 5$ is a circle of radius 5 centred at $z = 3 - i$. Thus the sketch is:

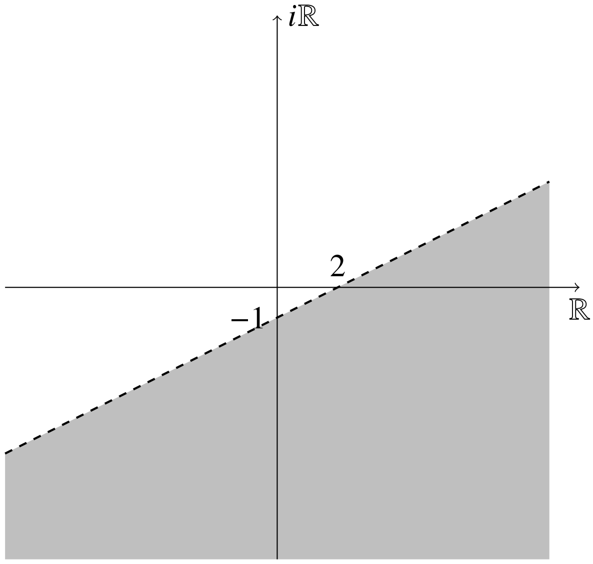


(b) $\{z \in \mathbb{C} \mid \mathrm{Re}((1+2i)z) > 2\}$

Solution: Write $z = x + iy$. Then

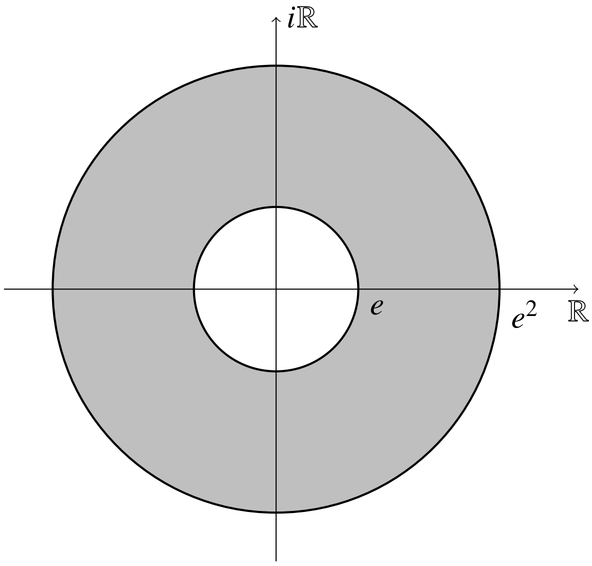
$$\mathrm{Re}((1+2i)z) = \mathrm{Re}((x-2y) + i(2x+y)) = x - 2y.$$

Thus the region is given by the inequality $x - 2y > 2$, and so $y < (x-2)/2$. Thus the sketch is:



6. Sketch the image of the set $A = \{z \in \mathbb{C} \mid 1 \leq \mathrm{Re}(z) \leq 2\}$ under the function $f(z) = e^z$ in the complex plane.

Solution: The set A is the infinite vertical strip bounded on the left by the vertical line $\mathrm{Re}(z) = 1$ and on the right by the vertical line $\mathrm{Re}(z) = 2$, and including these boundary lines. If $\mathrm{Re}(z) = 1$ then we have $z = 1 + iy$ so $e^z = e^1 e^{iy} = e(\cos y + i \sin y)$. Thus as y varies, e^{1+iy} traces out the circle centre at the origin, radius e. That is, the image of the line $\mathrm{Re}(z) = 1$ is the circle centre 0 radius e. Similarly, for any $1 \leq k \leq 2$ the image of the line $\mathrm{Re}(z) = k$ is the circle centre 0 radius e^k. Thus the image of the set A under the function $z \mapsto e^z$ is the annulus centred at 0 with inner radius e and outer radius e^2, and both boundaries included:



7. Given that $z = 3 - i$ and $z = -1 + i$ are roots of the degree 5 equation

$$z^5 - 3z^4 - 4z^3 + 8z^2 + 28z + 20 = 0,$$

find all of the roots.

Solution: Since the polynomial has real coefficients we know that $3 + i$ and $-1 - i$ are also roots, and therefore the polynomial is divisible by

$$(z - (3 - i))(z - (3 + i))(z - (-1 + i))(z - (-1 - i)) = z^4 - 4z^3 + 8z + 20.$$

Performing polynomial long division:

$$\begin{array}{r|l}
 & z \quad +1 \\
\hline
z^4 - 4z^3 + 8z + 20 & z^5 - 3z^4 - 4z^3 + 8z^2 + 28z + 20 \\
 & -z^5 + 4z^4 \qquad\quad - 8z^2 - 20z \\
\cline{2-2}
 & \quad z^4 - 4z^3 \qquad\qquad + 8z + 20 \\
 & \quad -z^4 + 4z^3 \qquad\qquad - 8z - 20 \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

Thus

$$z^5 - 3z^4 - 4z^3 + 8z^2 + 28z + 20 = (z + 1)(z^4 - 4z^3 + 8z + 20),$$

and so $z = -1$ is the final root. So the solutions are

$$z = -1, 3 - i, 3 + i, -1 + i, -1 - i.$$

8. Let

$$A = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0 \text{ and } 0 \leq \mathrm{Im}(z) \leq \pi\}$$
$$B = \{w \in \mathbb{C} \mid \mathrm{Im}(w) \geq 0 \text{ and } |w| \geq 1\}.$$

You are given that the function $f : A \longrightarrow B$ with formula $f(z) = e^z$ is bijective. Find a formula for the inverse function $f^{-1} : B \longrightarrow A$.

Solution: Let $w \in B$. Then $\mathrm{Im}(w) \geq 0$ and $|w| \geq 1$. Thus w is of the form $w = r(\cos\theta + i\sin\theta)$ with $r \geq 1$ and $0 \leq \theta \leq \pi$. Let $z = \ln r + i\theta$. Then $\mathrm{Re}(z) = \ln r \geq 0$ and $\mathrm{Im}(z) = \theta$ satisfies $0 \leq \mathrm{Im}(z) \leq \pi$. Thus $z \in A$, and we have

$$f(z) = e^{\ln r}(\cos\theta + i\sin\theta) = r(\cos\theta + i\sin\theta) = w.$$

Therefore

$$f^{-1}(w) = \ln r + i\theta = \ln|w| + i\mathrm{Arg}(w) \quad \text{is a formula for the inverse function.}$$

9. Find the natural domains of the following functions:

$$\sqrt{4 - x^2}, \quad (4 - x^2)^{-1/2}, \quad \ln x, \quad \sqrt{\ln x}, \quad \frac{\sin x}{x}, \quad \ln(\ln(\ln x))$$

Solution: $f(x) = \sqrt{4 - x^2}$ has natural domain $[-2, 2]$.

$f(x) = (4 - x^2)^{-1/2}$ has natural domain $(-2, 2)$.

$f(x) = \ln x$ has natural domain $(0, \infty)$.

$f(x) = \sqrt{\ln x}$ has natural domain $[1, \infty)$, because we require $\ln x \geq 0$, and so $x \geq 1$.

$f(x) = \frac{\sin x}{x}$ has natural domain $\mathbb{R}\backslash\{0\}$.

$f(x) = \ln(\ln(\ln x))$ has natural domain (e, ∞). To see this, note that we require $\ln(\ln x) > 0$ which is equivalent to $\ln x > 1$, which is equivalent to $x > e$.

10. Decide which of the following functions $f : A \to B$ are surjective, injective, or bijective:

(a) $f : [-1, 1] \to \mathbb{R}$ given by $x \mapsto \sinh x$.

Solution: Since $\sinh x$ is strictly increasing it is injective. The range of f is the interval $[-\sinh 1, \sinh 1]$. Since this does not equal the codomain, the function f is not surjective. Hence it is not bijective.

(b) $f : \mathbb{R} \to [-1, 1]$ given by $x \mapsto \cos x$.

Solution: Since $\cos x$ takes all values in the interval $[-1, 1]$, the range of f is $[-1, 1]$. Since this equals the codomain, f is surjective. However f is not injective because, for example, $f(0) = 1 = f(2\pi)$. Thus f is not bijective.

(c) $f : (0, \infty) \to \mathbb{R}$ given by $x \mapsto \ln\left(x\sqrt{x^2 + 2}\right)$.

Solution: This function is strictly increasing on its domain, because $x(x^2 + 2)$ is clearly strictly increasing on $(0, \infty)$, and the logarithm is also strictly increasing, and the composition of strictly increasing functions is again strictly increasing. Thus f is injective.

For small positive values of x the value of $f(x)$ is a large negative number, and for large positive values of x the value of $f(x)$ is a large positive number. So the range of f is $\mathbb{R}$, and hence the function f is surjective, and hence is bijective.

(d) $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto e^z$.

Solution: The function f is not surjective since there is no complex number z such that $e^z = 0$. The function f is not injective since, for example, $f(0) = f(2\pi i) = 1$. Therefore f is not bijective.

11. Evaluate the following limits or show that the limit does not exist:

(a) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sin 3x}{x} = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3 \times 1 = 3.$$

(b) $\lim\limits_{x \to 0} \dfrac{\cos 3x}{x}$

Solution: This limit does not exist. The limit of the numerator is 1, and the limit of the denominator is 0.

(c) $\lim\limits_{x \to 1} \dfrac{x^3 - 5x + 4}{x^3 - 4x + 3}$

Solution:

$$\begin{aligned}
\lim_{x \to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3} &= \lim_{x \to 1} \frac{(x-1)(x^2 + x - 4)}{(x-1)(x^2 + x - 3)} \\
&= \lim_{x \to 1} \frac{x^2 + x - 4}{x^2 + x - 3} \\
&= \frac{1 + 1 - 4}{1 + 1 - 3} \\
&= 2.
\end{aligned}$$

(d) $\lim\limits_{x \to 0^+} \dfrac{\sin(1/x)}{\ln x}$

Solution: We use the Squeeze Law for limits. For all $x \neq 0$,

$$-1 \leq \sin(1/x) \leq 1,$$

and so for all $x \in (0, 1)$,

$$-\frac{1}{|\ln x|} \leq \frac{\sin(1/x)}{\ln x} \leq \frac{1}{|\ln x|}.$$

Since $\ln x \to -\infty$ as $x \to 0^+$, the upper and lower bounds both tend to zero as $x \to 0^+$. Hence, the squeeze law implies that

$$\lim_{x\to 0^+} \frac{\sin(1/x)}{\ln x} = 0.$$

(e) $\lim_{x\to\infty} x\big(\sqrt{x^2+1} - \sqrt{x^2-1}\big)$

Solution: We have, using the limit laws,

$$\begin{aligned}
x\big(\sqrt{x^2+1} - \sqrt{x^2-1}\big) &= x\frac{\big(\sqrt{x^2+1} - \sqrt{x^2-1}\big)\big(\sqrt{x^2+1} + \sqrt{x^2-1}\big)}{\sqrt{x^2+1} + \sqrt{x^2-1}} \\
&= x\frac{(x^2+1) - (x^2-1)}{\sqrt{x^2+1} + \sqrt{x^2-1}} \\
&= x\frac{2}{\sqrt{1+1/x^2} + \sqrt{1-1/x^2}} \\
&\xrightarrow{x\to\infty} \frac{2}{\sqrt{1+0} + \sqrt{1-0}} = 1.
\end{aligned}$$

(f) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

Solution: After applying a clever trick, this limit is related to the more familiar $(\sin x)/x \to 1$ limit:

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x^2} &= \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} \\
&= \lim_{x\to 0} \frac{1}{1+\cos x} \lim_{x\to 0} \frac{\sin^2 x}{x^2} \\
&= \frac{1}{2}\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \\
&= \frac{1}{2}.
\end{aligned}$$

12. Find all constants $k \in \mathbb{R}$ such that the function

$$f(x) = \begin{cases} k^2\cosh x & \text{if } x < 0 \\ x + 3 & \text{if } x \geq 0. \end{cases}$$

is continuous at every $x \in \mathbb{R}$.

Solution: The function f is continuous at all points $x \neq 0$ since cosh and $x \mapsto x + 3$ are continuous functions. Now f is continuous at 0 if and only if

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0).$$

We have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} k^2 \cosh x = k^2 \lim_{x \to 0^-} \cosh x = k^2 \cosh 0 = k^2$$

and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 3) = 3 = f(0).$$

Therefore the values of k for which f is continuous at $x = 0$ are exactly the k such that $k^2 = 3$. So f is continuous on all of $\mathbb{R}$ exactly when $k = \pm\sqrt{3}$.

13. Suppose that $\lim_{x \to a} f(x) = \ell$ and let $k \neq 0$ be a constant. Prove, using and (ϵ, δ) argument, that

$$\lim_{x \to a} (k f(x)) = k\ell.$$

Solution: This was done in a tutorial.

14. Show, using the squeeze law, that

$$\lim_{x \to a} x^{1/4} = a^{1/4} \quad \text{for all } a > 0.$$

Solution: Let $\epsilon > 0$. Choose $\delta = a^{3/4}\epsilon$. Then

$$\begin{aligned} 0 &\leq |f(x) - \ell| \\ &= |x^{1/4} - a^{1/4}| \\ &= \frac{|x^{1/2} - a^{1/2}|}{x^{1/4} + a^{1/4}} \\ &= \frac{|x - a|}{(x^{1/2} + a^{1/2})(x^{1/4} + a^{1/4})} \\ &\xrightarrow{x \to a} 0 \end{aligned}$$

Thus, by the squeeze law $| \lim_{x \to a} |x^{1/4} - a^{1/4}| = 0$, that is,

$$\lim_{x \to a} x^{1/4} = a^{1/4}.$$

Remark: Note that we have employed a similar algebraic strategy to the case $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ (see tutorials), although here we needed to work a little harder!