4 Differentiation

4.1 Partial derivatives and Jacobians

66: Find all first and second order partial derivatives for the function

$$z = x^5 + y^5 - 3x^3y^3.$$

67: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

i) Calculate

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

first for $(x,y) \neq (0,0)$ (you can use Maple if you like) and then for $(x,y) = (0,0)$.

ii) Show that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

Discuss!

68: Let

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

Does the derivative

$$\frac{\partial^2 f}{\partial x \partial y}(0,0)$$

exist?

69: Find $\dfrac{\partial f}{\partial y}(1,y)$ for the function

$$f(x,y) = x^{x^{x^y}} + (\ln x) \times \tan^{-1}\Big[\tan^{-1}\Big(\sin\big[\cos(xy) - \ln(x+y)\big]\Big)\Big].$$

70: Find a general formula for the Jacobian matrix of the function $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\boldsymbol{f}(x,y,z) = \begin{bmatrix} xy\sin z \\ xy\cos z \\ x^2+y^2+z^2 \end{bmatrix}$$

and find its value at the point $(2,1,0)$.

71: Verify that the equation

$$J(\boldsymbol{f} \cdot \boldsymbol{g}) = \boldsymbol{g}^T \times J\boldsymbol{f} + \boldsymbol{f}^T \times J\boldsymbol{g}$$

holds in the case where

$$\boldsymbol{f}, \boldsymbol{g} : \mathbb{R}^n \mapsto \mathbb{R}^n.$$

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4.2 Definition of differentiability

72: If $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^n$, show that there cannot be two different linear functions

$$\ell : \mathbb{R}^n \to \mathbb{R}$$

satisfying

$$\frac{f(\boldsymbol{a}+\boldsymbol{x}) - f(\boldsymbol{a}) - \ell(\boldsymbol{x})}{\|\boldsymbol{x}\|} \to 0 \quad \text{as} \quad \boldsymbol{x} \to \boldsymbol{0}.$$

73: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x,y,z) = xy + yz + xz.$$

Show, using the definition of differentiability (see these webnotes[19]), that f is differentiable at the point $(1,1,1)$.

74: Let

$$f(x,y) = \sqrt[3]{xy}, \quad x,y \in \mathbb{R}.$$

Find

$$f_x(0,0) \quad \text{and} \quad f_y(0,0).$$

Is this function differentiable at $(0,0)$?

75: Let

$$f(x,y) = \sqrt[3]{x^3+y^3}, \quad x,y \in \mathbb{R}.$$

Find

$$f_x(0,0) \quad \text{and} \quad f_y(0,0).$$

Is this function differentiable at $(0,0)$?

[19] http://web.maths.unsw.edu.au/~potapov/2111_2015/Differentiability-of-vector-map.html

76: Let

$$f(x,y) = \begin{cases} \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{otherwise} \end{cases}$$

Find

$$f_x(0,0) \quad \text{and} \quad f_y(0,0)$$

and show that this function is differentiable at $(0,0)$.

4.3 Best affine approximations

77: What is the best affine approximation to the function $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\boldsymbol{f}(x,y) = \begin{bmatrix} e^{xy^2} \\ x^2 - 3x + y^2 \end{bmatrix}$$

at the point $(1,-1)$.

78: When two resistances r_1 and r_2 are connected in parallel, the total resistance R (measured in ohms) is given by:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

i) Show that $\dfrac{\partial R}{\partial r_1} = \dfrac{R^2}{r_1^2}$.

ii) Use *the best affine approximation* of function $R(r_1, r_2)$, to estimate the maximum possible error in the calculated value of R if the measured values of r_1 and r_2 are $r_1 = 6 \pm 0.1$ ohms and $r_2 = 9 \pm 0.03$ ohms

79: The specific gravity δ of a solid heavier than water is given by

$$\delta = \frac{W}{W - W_1}$$

where W and W_1 are its weight in air and water respectively. W and W_1 are observed to by 17.2 and 9.7 gm. Use *the best affine approximation* of function $\delta(W, W_1)$ to estimate the maximum possible error in the calculated value of δ due to an error of 0.05 gm in each observation.

4.4 Chain Rule, First order

80: Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{g} : \mathbb{R}^m \to \mathbb{R}^p$ and $\boldsymbol{h} = \boldsymbol{g} \circ \boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^p$ and let $\mathbf{a} \in \mathbb{R}^n$. For each of the examples below find the left hand side and the right hand side of the chain rule identity:

$$J_{\mathbf{a}}\boldsymbol{h} = J_{\boldsymbol{f}(\mathbf{a})}\boldsymbol{g} \times J_{\mathbf{a}}\boldsymbol{f}.$$

i)

$$\boldsymbol{f}(x,y,z) = \begin{bmatrix} x^2 - y^2 \\ 2xy \\ z \end{bmatrix}, \quad \boldsymbol{g}(u,v,w) = \begin{bmatrix} u + w^2 \\ u/w \end{bmatrix}, \quad \boldsymbol{a} = (2,1,2).$$

ii)

$$\boldsymbol{f}(x,y) = \begin{bmatrix} x^2 + y \\ x - 2y^2 \end{bmatrix}, \quad \boldsymbol{g}(u,v) = \begin{bmatrix} 2u + v \\ \sin u \\ u + 2v^2 \end{bmatrix}, \quad \boldsymbol{a} = (1,1);$$

iii)

$$g(x,y) = \sqrt{x^2+y^2}, \quad \boldsymbol{f}(s,t) = \begin{bmatrix} e^{st} \\ 1 + s^2 \cos t \end{bmatrix}, \quad \boldsymbol{a} = (1,0).$$

iv)

$$g(x,y) = e^{xy^2}, \quad \boldsymbol{f}(t) = \begin{bmatrix} t\,\cos t \\ t\,\sin t \end{bmatrix}, \quad a = \frac{\pi}{2}.$$

81: A function $f(x,y)$ is said to be *homogeneous* of degree m if $f(tx,ty) = t^m f(x,y)$ for every real number $t > 0$. Euler's theorem states that if f is homogeneous of degree m and if all its partial derivatives of first order exist then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = mf(x,y).$$

i) Verify Euler's theorem for

$$f(x,y) = Ax^2 + Bxy + Cy^2$$

and for
$$g(x,y) = \tan^{-1}\frac{y}{x}, \quad x \neq 0.$$

ii) Prove Euler's theorem.

iii) Generalise the theorem and prove your generalisation.

82: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and
$$z = xy + f\left(\frac{y}{x}\right), \quad (x,y) \in \mathbb{R}^2, \quad x \neq 0.$$
Show that z satisfies the partial differential equation
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2xy.$$

4.5 Directional derivatives

83: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$
Show that for all unit vectors $\boldsymbol{u}$ the directional derivative of f at the origin in the direction $\boldsymbol{u}$ does exist, but f is discontinuous at $(0,0)$. Show that there is no plane which contains all the lines which are tangent to the surface $z = f(x,y)$ at $(0,0,0)$.

84: For each of the following scalar fields

a) find ∇f

b) graph some level curves $f(x,y) =$ constant,

c) indicate ∇f at some points by arrows on these curves.

i) $f(x,y) = xy$

ii) $f(x,y) = x^2 + y^2$

iii) $f(x,y) = \dfrac{y}{x^2}$.

85: Let $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ and $r = \|\boldsymbol{r}\|$.

i) Prove that $\nabla r = \dfrac{\boldsymbol{r}}{r}$ and $\nabla\left(\dfrac{1}{r}\right) = \dfrac{-\boldsymbol{r}}{r^3}$.

ii) Calculate $\nabla(\cos r)$, $\nabla\left(\dfrac{\log r}{r}\right)$.

iii) Prove that $\nabla r^n = nr^{n-2}\boldsymbol{r}$.

86: In each case find ∇f at the point P and use it to find the directional derivative of f at P in the direction of $\boldsymbol{v}$.

i) $f(x,y) = 13x^2 + 7xy + 2y$, $P = (-1,1)$, $\boldsymbol{v} = 5\boldsymbol{i} + 12\boldsymbol{j}$.

ii) $f(x,y,z) = x(x^2+y^2+z^2)$, $P = (1,2,-1)$, $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

87: Suppose $f(x,y)$ is a differentiable function, which has, at the point $\boldsymbol{x}$, directional derivative $1/\sqrt{2}$ in the direction $(1,1)$ and directional derivative $1/5$ in the direction $(3,4)$. Find $\nabla f(\boldsymbol{x})$.

88: A bushwalker is climbing a mountain, of which the equation is $h(x,y) = 400 - (x^2+4y^2)/10000$. Here x, y and h are measured in metres, the x-axis points East and the y-axis points North. The bushwalker is at a point P, 1600 metres West and 400 metres South of the peak.

i) What is the slope of the mountain at P in the direction of the peak?

ii) In which direction at P is the slope greatest?

89: The electrical potential V is given by $V(x,y,z) = x^2 - xy + xyz$.

i) Find the rate of change of the potential V at $(1,1,1)$ in the direction of the vector $\boldsymbol{v} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

ii) In which direction(s) does V change most rapidly at $(1,1,1)$?

iii) What is the maximum rate of change of V at $(1,1,1)$?

90: Skier is on a mountain described by the equation $h(x,y) = 2000 - x^4/10^8 - y^2/10^2$ at the point $(100,1)$. He skis down the mountain, always moving in the direction of steepest descent.

i) In what direction does he start moving?

ii) Describe the curve along which he skis. [You will need to solve a separable first order ODE.]

Answers to problems

A66: $\partial z/\partial x = 5x^4 - 9x^2y^3$, $\partial z/\partial y = 5y^4 - 9x^3y^2$, $\partial^2 z/\partial x^2 = 20x^3 - 18xy^3$, $\partial^2 z/\partial x \partial y = -27x^2y^2$, $\partial^2 z/\partial y^2 = 20y^3 - 18x^3y$. **A67**: See these webnotes[20] for solution **A68**: No **A69**: 0.

A70: $J_{\boldsymbol{x}}\boldsymbol{f} = \begin{bmatrix} y\sin z & x\sin z & xy\cos z \\ y\cos z & x\cos z & -xy\sin z \\ 2x & 2y & 2z \end{bmatrix}$, $J_{(2,1,0)}\boldsymbol{f} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 0 \end{bmatrix}$. **A74**: $f_x = 0$, $f_y = 0$; f is not differentiable, see these webnotes[21] for solution

A75: $f_x = 1$, $f_y = 1$; f is not differentiable, see these webnotes[22] for solution

A76: $f_x = 0$, $f_y = 0$; see these webnotes[23] for solution

A77: $\begin{bmatrix} e \\ -1 \end{bmatrix} + \begin{bmatrix} e & -2e \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x-1 \\ y+1 \end{bmatrix}$

A78: ii) .0408

A79: 0.024

A84: i) $y\,\boldsymbol{i} + x\,\boldsymbol{j}$, $2x\,\boldsymbol{i} + 2y\,\boldsymbol{j}$, $-2y/x^3\,\boldsymbol{i} + 1/x^2\,\boldsymbol{j}$. **A85**: i) $-(\sin r/r)\boldsymbol{r}$, $[(1-\log r)/r^3]\boldsymbol{r}$. **A86**: i) $-155/13$, ii) $10/\sqrt{3}$. **A87**: $(3,-2)$. **A88**: i) $8/5\sqrt{17}$, ii) North East.

A89: i) $\sqrt{3}$, ii) $\pm(2\,\boldsymbol{i} + \boldsymbol{k})$, iii) $\sqrt{5}$.

A90: i) $2\,\boldsymbol{i} + \boldsymbol{j}$, ii) $y = \exp\left[-2.5 \times 10^5/x^2 + 25\right]$.

[20] http://web.maths.unsw.edu.au/~potapov/2111_2015/Clariaut-Theorem.html
[21] http://web.maths.unsw.edu.au/~potapov/2111_2015/Differentiability-Example-II.html
[22] http://web.maths.unsw.edu.au/~potapov/2111_2015/Differentiability-Example-III.html
[23] http://web.maths.unsw.edu.au/~potapov/2111_2015/Differentiability-of-vector-map.html