The University of Sydney
School of Mathematics and Statistics

# Solutions to Tutorial for Week 6

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

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## Topics covered

In lectures last week:

- ☐ Sequences $a_1, a_2, \dots$ Squeeze law, ratio test for sequences.
- ☐ Asymptotic equivalence of sequences ($a_n \sim b_n$).
- ☐ Series $a_1 + a_2 + \cdots$. The geometric series, harmonic series, and $p$-series.
- ☐ Comparison test, ratio test and asymptotic comparison test for series.

## Objectives

After completing this tutorial sheet you will be able to:

- ☐ Use the ratio test, squeeze law, and limit laws to compute limits of sequences.
- ☐ Use comparison tests to determine the convergence/divergence of a series.
- ☐ Use Riemann sums to decide convergence/divergence of series.
- ☐ Compute the value of some series by computing the limit of partial sums.

## Preparation questions to do *before* class

**1.** Calculate the limit of the following sequences, or show that they diverge.

(a) $a_n = \dfrac{n^4 + 3n^3 \cos n - 2}{3n^4 - n}$

***Solution:*** Dividing numerator and denominator by the dominant behaviour $n^4$ gives

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1 + 3\frac{\cos n}{n} - \frac{2}{n^4}}{3 - \frac{1}{n^3}} = \frac{1}{3}.$$

(b) $a_n = \dfrac{n!(2n)!}{(3n)!}$ (ratio test!)

***Solution:*** Often when there are lots of factorials the ratio test is the way to go. We have

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)!(2n+2)!}{(3n+3)!} \cdot \frac{(3n)!}{n!(2n)!} \\
&= \lim_{n\to\infty} \frac{(n+1)(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)} \\
&= \frac{4}{27} < 1.
\end{aligned}$$

Therefore $a_n \to 0$ by the ratio test for sequences.

**2.** Determine if the following series converge or diverge. (Don't forget the ratio test!)

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n n3^{-n}$

***Solution:*** We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{3n} = \frac{1}{3} < 1,$$

and so the series converges by the ratio test for series.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{5\cos(3n)+2}{n^2}$

***Solution:*** We guess that this series will converge, because it is more or less like a $p$-series with $p = 2 > 1$. So we apply the comparison test:

$$\left|\frac{5\cos(3n)+2}{n^2}\right| \le \frac{7}{n^2} \qquad \text{for all } n \ge 1,$$

and since $\sum \frac{7}{n^2} = 7\sum\frac{1}{n^2}$ converges we conclude that the series converges.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2 5^n}{(2n)!}$

***Solution:*** Looks like a classic ratio test question, owing to all of the factorials and powers floating around. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5(n+1)(n+1)}{(2n+1)(2n+2)} = \frac{5}{4} > 1,$$

and so the series diverges by the ratio test.

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+3n-2}{n^3+1}$

***Solution:*** The $n$th term of this series is behaving like $\frac{1}{n}$, and so we expect the series to diverge by comparison with the $p$-series with $p = 1$ (the harmonic series). We have

$$a_n = \frac{n^2+3n-2}{n^3+1} \ge \frac{n^2+3n-2n}{n^3+1} = \frac{n^3+n}{n^3+1} \ge \frac{n^2}{2n^3} = \frac{1}{2n},$$

and since $\sum\frac{1}{2n} = \frac{1}{2}\sum\frac{1}{n}$ diverges we see that the series diverges.
Note that finding the inequalities can be a little fiddly. In this example it is cleaner to use the asymptotic comparison test:

$$a_n = \frac{n^2+3n-2}{n^3+1} \sim \frac{1}{n},$$

and so the series diverges by asymptotic comparison with $\sum\frac{1}{n}$.

## Questions to attempt in class

**3.** Calculate the limit of the following sequences, or show that they diverge.

(a) $a_n = \dfrac{3+\cos n^2}{\sqrt{n}}$

***Solution:*** Since

$$0 \leq \left|\frac{3+\cos n^2}{\sqrt{n}}\right| \leq \frac{4}{\sqrt{n}}$$

we have $a_n \to 0$ by the Squeeze Law.

(b) $a_n = \sqrt[n]{n}$

***Solution:*** We have

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim_{n\to\infty} \frac{\ln n}{n}}.$$

Since $\ln n \leq \sqrt{n}$ for large $n$, we have $0 \leq \frac{\ln n}{n} \leq \frac{1}{\sqrt{n}}$ for all large $n$. Thus, by the squeeze law, $\frac{\ln n}{n} \to 0$. So $\sqrt[n]{n} \to e^0 = 1$.

(c) $a_n = \dfrac{n^2}{3n^2+2n-1}$

***Solution:*** Dividing the numerator and denominator by $n^2$ shows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3+\frac{2}{n}-\frac{1}{n}} = \frac{1}{3+0-0} = \frac{1}{3}.$$

(d) $a_n = \dbinom{2n}{n}$

***Solution:*** Since $\dbinom{2n}{n} = \dfrac{(2n)!}{n!^2}$ it seems difficult to compute the limit of $a_n$ directly. Factorials are often best treated using the ratio test for sequences.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(2n+2)!n!^2}{(n+1)!^2(2n)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2} = 4 > 1.$$

Therefore by the ratio test $|a_n|$ diverges to $\infty$, and therefore $a_n \to \infty$.

**4.** Decide if the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

***Solution:*** $\ln n$ grows slower than any power of $n$. Therefore, for example, $\ln n \leq \sqrt{n}$ for sufficiently large $n$. Then

$$\frac{\ln n}{n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}},$$

and so the series converges by comparison with the $p$-series with $p = 3/2$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^n n!}$

***Solution:*** We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)^n = \frac{e}{2},$$

where we have used the (famous) limit $(1+1/n)^n \to e$. Since $e/2 > 1$ we conclude that the series diverges (by the ratio test).

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-n}}{\sqrt{n}}$

***Solution:*** The dominant behaviour here is the $e^{-n}$, which gives extremely rapid convergence. Since

$$\frac{e^{-n}}{\sqrt{n}} \leq e^{-n}$$

we see that the series converges by comparison with the geometric series with $r = e^{-1} < 1$. Alternatively you can use the ratio test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$

***Solution:*** The dominant behaviour here is the $n^2$ on the denominator, and so we want to compare this series with the $p$-series with $p = 2$. We have

$$\left|\frac{\cos n}{n^2+1}\right| \leq \frac{1}{n^2+1} \leq \frac{1}{n^2+n^2} = \frac{1}{2n^2}.$$

Therefore the given series converges by comparison with $\sum \frac{1}{n^2}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{2-\sin\sqrt{n}}{n^3}$

***Solution:*** The $n^3$ on the denominator will ensure convergence:

$$\left|\frac{2-\sin\sqrt{n}}{n^3}\right| \leq \frac{3}{n^3},$$

and so the original series converges too by comparison with $\sum \frac{1}{n^3}$.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-2n+5}{n^3+4}$

***Solution:*** Since

$$\frac{n^2-2n+5}{n^3+4} \sim \frac{1}{n},$$

and since $\sum \frac{1}{n}$ diverges, we see that the original series diverges too by the asymptotic comparison test.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$

***Solution:*** Again, factorials are often best treated using the ration test: Let $a_n = \frac{5^n}{n!}$. Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1,$$

and so by the ratio test for series we see that the series converges.

(h) $\displaystyle\sum_{n=1}^{\infty} \sin(n^2)$

***Solution:*** We saw in class that if a series converges then $a_n \to 0$. Therefore this series cannot converge.

This example shows that we need to be careful when trying to deduce convergence or divergence of $\sum_{k=1}^{\infty} f(k)$ from $\int_1^{\infty} f(x)\,dx$. For as we have seen, $\int_1^{\infty} \sin(x^2)\,dx$ converges (rather surprisingly!).

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\cosh n}{e^{2n} + n^2}$

***Solution:*** The $n$th term satisfies

$$a_n = \frac{\cosh n}{e^{2n} + n^2} = \frac{e^n + e^{-n}}{2(e^{2n} + n^2)} = \frac{1 + e^{-2n}}{2e^n + 2n^2e^{-n}} \sim \frac{1}{2}e^{-n}.$$

Thus the series converges by (asymptotic) comparison with the geometric series with $r = e^{-1} < 1$.

**5.** Let $r \in \mathbb{R}$, and let $s_n(r) = 1 - r^2 + r^4 - \cdots + (-1)^{n-1}r^{2n-2}$.

(a) Find a closed formula for $s_n(r)$, and deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} + (-1)^{n+1} \int_0^1 \frac{r^{2n}}{1+r^2}\,dr.$$

***Solution:*** By the geometric sum formula we have

$$1 - r^2 + r^4 - \cdots + (-1)^{n-1}r^{2n-2} = \frac{1 - (-1)^n r^{2n}}{1 + r^2},$$

and integrating this formula between $r = 0$ and $r = 1$ proves the stated formula, using $\displaystyle\int_0^1 \frac{1}{1+r^2}\,dr = \frac{\pi}{4}$.

(b) Hence prove *Leibnitz' Formula* $\displaystyle\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ .

***Solution:*** We have

$$0 \le \int_0^1 \frac{r^{2n}}{1+r^2}\,dr \le \int_0^1 r^{2n}\,dr = \frac{1}{2n+1}.$$

Thus by the squeeze law we have

$$\lim_{n\to\infty}\int_0^1 \frac{r^{2n}}{1+r^2}\,dr = 0.$$

Thus from the previous part we have

$$\lim_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{5}-\cdots+\frac{(-1)^{n-1}}{2n-1}\right)=\frac{\pi}{4},$$

proving the result.

(c) Adapt this proof to show that $\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$.

***Solution:*** Here we start with the formula

$$1-r+r^2-r^3+\cdots+(-1)^{n-1}r^{n-1}=\frac{1-(-1)^n r^n}{1+r},$$

valid whenever $r\neq -1$. Integrating between 0 and 1 gives

$$1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{n-1}}{n}=\ln 2+(-1)^{n+1}\int_0^1\frac{r^n}{1+r}\,dr.$$

A similar argument to before shows that the integral tends to 0 as $n\to\infty$, hence the formula.

## Discussion question

**6.** The Prime Number Theorem implies that the $n$th prime satisfies $p_n \sim n\ln n$. Given this information, discuss the convergence/divergence of the series

$$\sum_{\text{primes } p}\frac{1}{p}$$

***Solution:*** By the asymptotic comparison test (and, of course, the Prime Number Theorem) the series $\sum\frac{1}{p}$ converges if and only if $\sum\frac{1}{n\ln n}$ converges. Consider the latter series. By considering upper Riemann sums of the function $f(x)=\frac{1}{x\ln x}$ on $[1,N+1]$ using the partition into $N$ equal parts (of length 1) we have

$$\sum_{n=2}^{N}\frac{1}{n\ln n}\geq\int_2^{N+1}\frac{dx}{x\ln x}=\ln(\ln(N+1))-\ln(\ln 2),$$

and this tends to $\infty$ as $N\to\infty$. Thus $\sum\frac{1}{n\ln n}$ diverges, and hence $\sum\frac{1}{p}$ diverges too! In particular, there are 'more' primes than squares, because $\sum\frac{1}{n^2}$ converges.

There is actually an awful lot that one can say about this series. Here are some remarks (for interest only!).

***Remark:*** The series $\sum\frac{1}{p}$ diverges very very very very slowly! The partial sum up the the $n$th prime behaves roughly like $\ln(\ln n)$. In particular, the $n$th partial sum with $n = 100000000000000$ is roughly 3.5, even though the series diverges to $\infty$.

To be more precise, Mertens' Second Theorem says that

$$\lim_{n\to\infty}\left(\sum_{p\le n}\frac{1}{p}-\ln(\ln n)\right)=M,$$

where $M$ is the Meissel-Mertens constant ($M\approx 0.261497212\cdots$), and moreover that

$$\left|\sum_{p\le n}\frac{1}{p}-\ln(\ln n)-M\right|\le\frac{4}{\ln(n+1)}+\frac{2}{n\ln n}\qquad\text{for all } n\ge 4.$$

Using this we calculate that

$$3.6104\le\sum_{p\le 1000000000000000}\frac{1}{p}\le 3.8587.$$

***Remark:*** Appealing to the (difficult!) Prime Number Theorem to show that $\sum\frac{1}{p}$ diverges is shooting sparrows with a cannon. Indeed one can show that the series diverges in a more elementary (but not 'easy'!) way as follows. We begin with the fact that

$$\ln n\le\sum_{j=1}^{n}\frac{1}{j},\tag{1}$$

which is proved using Riemann sums: The sum on the right is the upper Riemann sum for $f(x)=\frac{1}{x}$ on the interval $[1,n{+}1]$ using the partition $P=\{1,2,\ldots,n,n{+}1\}$, and therefore

$$\int_1^{n+1}\frac{1}{x}\,dx\le\sum_{j=1}^{n}\frac{1}{j},\qquad\text{giving}\qquad\ln(n+1)\le\sum_{j=1}^{n}\frac{1}{j},$$

hence the inequality (1) since $\ln n\le\ln(n{+}1)$. Now, each integer $j$ can be expressed in a unique way as a product of a square-free integer and a square. Thus

$$\sum_{j=1}^{n}\frac{1}{j}\le\sum_{s,k}\frac{1}{sk^2}$$

where the sum on the right is over all square-free numbers $1\le s\le n$ and all $1\le k\le n$ (each term $\frac{1}{j}$ with $1\le j\le n$ occurs exactly once in the sum on the right hand side, but this sum contains some extra terms, hence the inequality). Now

$$\sum_{s,k}\frac{1}{sk^2}=\left(\sum_{s}\frac{1}{s}\right)\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)\le\left(\sum_{s}\frac{1}{s}\right)\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)=C\sum_{s}\frac{1}{s},$$

where $C=\sum_{k=1}^{\infty}\frac{1}{k^2}$ ($p$-series, in fact $C=\pi^2/6$) and the $s$ sum is over square-free numbers $1\le s\le n$. Thus

$$\ln n\le\sum_{j=1}^{n}\frac{1}{j}\le C\sum_{s}\frac{1}{s}\le C\prod_{p\le n}\left(1+\frac{1}{p}\right),$$

where the product is over primes $p \le n$ (all square-free reciprocals $\frac{1}{s}$ with $1 \le s \le n$ occur in the expansion of this product, but the expansion of the product has extra terms, hence the inequality). Using the elementary inequality $1 + x \le e^x$ gives

$$\ln n \le C \prod_{p \le n} \left(1 + \frac{1}{p}\right) \le C \prod_{p \le n} e^{\frac{1}{p}} = C e^{\sum_{p \le n} \frac{1}{p}}.$$

Taking logarithms gives

$$\ln(\ln n) \le \ln C + \sum_{p \le n} \frac{1}{p}.$$

Therefore the sum diverges.

***Remark:*** The *Twin Prime Conjecture* asserts that there are infinitely many pairs of primes $(p, p')$ with $p' = p + 2$. For example, $(3,5)$, $(5,7)$ and $(11,13)$, $(17,19)$ are twin primes. This famous conjecture is unsolved, but it is generally believed to be true. Rather remarkably, Viggo Brun showed in 1919 that the series

$$\sum_{\text{twin primes}} \frac{1}{p}$$

**converges**. The value of this series is known as *Brun's constant*. The value of this constant is believed to be about 1.9021605, but none of these digits are known with 100% accuracy (not even the initial 1).

## Questions for extra practice

**7.** Decide if the following sequences converge. If they converge find the limit.

(a) $a_n = \dfrac{1 + 2 + \cdots + n}{n^2}$

***Solution:*** Since $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{2} = \frac{1}{2}.$$

(b) $a_n = e^{-n} \cosh n$

***Solution:*** Using the definition of cosh we have $a_n = \frac{1}{2}\left(1 + e^{-2n}\right)$. Therefore $a_n \to \frac{1}{2}$ as $n \to \infty$.

(c) $a_n = \left(1 + \dfrac{1}{n}\right)^n$

***Solution:*** We have

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x \ln\left(1 + \frac{1}{x}\right)} = e^{\lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right)}.$$

By L'Hôpital's Rule,

$$\lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}} = 1.$$

Therefore $a_n \to e^1 = e$.

(d) $a_n = \dfrac{\ln n}{n^\epsilon}, \quad (\epsilon > 0)$

***Solution:*** By L'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\ln n}{n^\epsilon} = \lim_{n\to\infty} \frac{1}{\epsilon n^\epsilon} = 0.$$

That is; $\ln x$ grows slower than any positive power of $x$. We use this fact quite a lot!

**8.** Decide if the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cosh n}{n^4+1}$

***Solution:*** Using the definition of cosh we see that $a_n \to \infty$ as $n \to \infty$. Therefore this series does not converge.

(b) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

***Solution:*** We have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 e^{-1} = e^{-1} < 1,$$

and so the series converges by the ratio test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

***Solution:*** You may use the ratio test to see that the series converges:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to e^{-1},$$

where we have used Question 7(c). Alternatively,

$$a_n = \frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{3}{n} \times \frac{2}{n} \times \frac{1}{n} \le 1 \times \cdots \times 1 \times \frac{2}{n} \times \frac{1}{n} = \frac{2}{n^2}.$$

Therefore the series converges by comparison with $\sum \frac{1}{n^2}$.

(d) $\displaystyle\sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n+1}\right]$

***Solution:*** The $n$th term of this series is

$$a_n = \frac{n+1-n+1}{n^2-1} = \frac{2}{n^2-1} \sim \frac{2}{n^2},$$

and so the series to converge by (asymptotic) comparison with $2\sum \frac{1}{n^2}$. Alternatively, this is one of the rare cases when we can actually compute the value of the series by computing the partial sums:

$$s_N = \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{3}{2} - \frac{1}{N} - \frac{1}{N-1} \to \frac{3}{2}$$

as $N \to \infty$ (we have used a collapsing sum here).

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$

***Solution:*** Once $n \ge e^2$ we have $\ln n \ge 2$, and so

$$\frac{1}{n^{\ln n}} \le \frac{1}{n^2}.$$

Therefore the series converges by comparison to $\sum \frac{1}{n^2}$.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+3}}$

***Solution:*** The $n$th term is

$$a_n = \frac{2n}{\sqrt{n^5+3}} \sim \frac{2n}{n^{5/2}} = \frac{2}{n^{3/2}}.$$

Therefore the series converges by comparison to the $p$ series with $p = 3/2$.

**9.** For which values of $x$ does the series $\displaystyle\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{x^{2n+1}}{2n+1}$ converge/diverge?

***Solution:*** Let $a_n = \dfrac{\binom{2n}{n}}{2^{2n}} \dfrac{x^{2n+1}}{2n+1} = \dfrac{(2n)!}{2^{2n} n!^2 (2n+1)} x^{2n+1}$. Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(2n+1)(2n+2)(2n+1)}{2^2(n+1)^2(2n+3)} |x|^2 = |x|^2.$$

Therefore by the ratio test the series converges if $|x| < 1$, and diverges if $|x| > 1$. In fact this series converges when $|x| = 1$ too. To see this, apply Stirling's formula (Question 10) to see that

$$\frac{\binom{2n}{n}}{2^{2n}(2n+1)} \sim \frac{1}{2\sqrt{\pi}\, n^{3/2}}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, so does the original series by the asymptotic version of the Comparison Test.

## Challenging problems

**10.** In this question you derive *Stirling's Asymptotic Formula* for $n!$

(a) Show that $\displaystyle\int_1^n \frac{\{x\}}{x}\, dx = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{x-k}{x}\, dx$, and conclude that

$$\ln n! = n \ln n - n + 1 + \int_1^n \frac{\{x\}}{x} dx\,,$$

where $\{x\} \in [0,1)$ is the fractional part of $x \geq 0$.

***Solution:*** Using the hint, we have

$$\begin{aligned}
\int_1^n \frac{\{x\}}{x}\,dx &= \sum_{k=1}^{n-1}\int_k^{k+1}\frac{x-k}{x}\,dx\\
&= \sum_{k=1}^{n-1}(x-k\ln x)\Big|_k^{k+1}\\
&= \sum_{k=1}^{n-1}(1-k\ln(k+1)+k\ln k)\\
&= n-1-(\ln 2+2\ln 3+3\ln 4+\cdots+(n-1)\ln n)\\
&\qquad +(\ln 1+2\ln 2+3\ln 3+\cdots+(n-1)\ln(n-1))\\
&= n-1-n\ln n+(\ln 1+\ln 2+\ln 3+\cdots+\ln n)\\
&= n-1-n\ln n+\ln n!.
\end{aligned}$$

Rearranging gives the desired result.

(b) Integrate by parts (see the relevant question of Tutorial 4) to show that

$$\ln n! = n\ln n - n + 1 + \frac{1}{2}\ln n - \frac{1}{2}\int_1^n \frac{\{x\}-\{x\}^2}{x^2}\,dx.$$

***Solution:*** In the integration by parts formula, let $u=\frac{1}{x}$ and $\frac{dv}{dx}=\{x\}$. Then $\frac{du}{dx}=-\frac{1}{x^2}$. We compute $v=\displaystyle\int\{x\}\,dx$ by an area computation, as in Tutorial 4, to get

$$v=\frac{1}{2}\lfloor x\rfloor+\frac{1}{2}\{x\}^2=\frac{1}{2}\left(x-\{x\}+\{x\}^2\right),$$

where $\lfloor x\rfloor\in\mathbb{Z}$ is the *integer part* of $x$. Therefore

$$\begin{aligned}
\int_1^n\frac{\{x\}}{x}\,dx &= \frac{x-\{x\}+\{x\}^2}{2x}\bigg|_1^n+\frac{1}{2}\int_1^n\frac{x-\{x\}+\{x\}^2}{x^2}\,dx\\
&= \frac{1}{2}\ln n-\frac{1}{2}\int_1^n\frac{\{x\}-\{x\}^2}{x^2}\,dx.
\end{aligned}$$

(c) Deduce that $\displaystyle\lim_{n\to\infty}\frac{n!}{\sqrt{n}n^ne^{-n}}=e^C$ for some constant $C$.

***Solution:*** Let $C_n=1-\displaystyle\frac{1}{2}\int_1^n\frac{\{x\}-\{x\}^2}{x^2}\,dx$. Since

$$\left|\frac{\{x\}-\{x\}^2}{x^2}\right|\le\frac{2}{x^2}$$

we see that $\displaystyle\lim_{n\to\infty}C_n=C$ exists by Comparison to $\displaystyle\int_1^\infty\frac{1}{x^2}\,dx$.
By the previous parts we have

$$\ln n! = n\ln n - n+\frac{1}{2}\ln n+C_n,$$

and therefore $n! = n^n e^{-n}\sqrt{n}e^{C_n}$, and so

$$\frac{n!}{\sqrt{n}n^n e^{-n}} = e^{C_n} \to e^C \qquad \text{as } n \to \infty.$$

(d) Use the Wallis formula (Tutorial 5) to evaluate $C$, and deduce that

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}.$$

***Solution:*** The Wallis Formula from last week gives

$$\pi = \lim_{n\to\infty} \frac{2^{4n} n!^4}{n(2n)!^2}.$$

Plugging $n! = \sqrt{n}n^n e^{-1}e^{C_n}$ into this formula gives

$$\pi = \lim_{n\to\infty} \frac{2^{4n}n^2 n^{4n} e^{-4n} e^{4C_n}}{n(2n)(2n)^{4n}e^{-4n}e^{2C_{2n}}} = \frac{1}{2}\lim_{n\to\infty} e^{4C_n - 2C_{2n}} = \frac{1}{2}e^{2C}.$$

Therefore $C = \frac{1}{2}\ln(2\pi)$. Therefore by (c) we have

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\, n^n e^{-n}} = 1, \qquad \text{and so} \qquad n! \sim \sqrt{2\pi n}\, n^n e^{-n}.$$

***Remark:*** The formulas you derived in (a) and (b) are special cases of the *Euler-Maclaurin summation formula*. You might like to show that

$$\begin{aligned}\sum_{k=m}^{n} f(k) &= \int_m^n f(x)\,dx + f(m) + \int_m^n \{x\}f'(x)\,dx \\ &= \int_m^n f(x)\,dx + \frac{f(m)+f(n)}{2} + \frac{1}{2}\int_m^n \left(\{x\} - \{x\}^2\right) f''(x)\,dx.\end{aligned}$$

You could continue this process by performing more and more integrations by parts. This is a standard technique to extract the asymptotic behaviour of a divergent series, and the general formulation is called the Euler-Maclaurin summation formula. A key thing to note is that these formulas relate sums to integrals. Generally sums a tricky to work with, while the corresponding integrals are often friendlier.

As an application, the first formula immediately gives

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + 1 - \int_1^n \frac{\{x\}}{x^2}\,dx,$$

from which we deduce that

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = \gamma,$$

where $\gamma$ is a constant. This constant is called the *Euler-Mascheroni constant*. As another application, you could show that

$$\lim_{n\to\infty}\left(\sum_{k=2}^{n} \frac{1}{k\ln k} - \ln(\ln n)\right) = \gamma'$$

for some constant $\gamma'$. I'm not sure if this constant has a special name.

**11.** The Fibonacci sequence is $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(a) Prove that $F_{2n} = F_n^2 + 2F_nF_{n-1}$ and $F_{2n-1} = F_n^2 + F_{n-1}^2$ for $n \geq 1$, and hence deduce that $F_{2n}F_{n-1} - F_{2n-1}F_n = (-1)^n F_n$ for $n \geq 1$.

***Solution:*** We prove $F_{2n} = F_n^2 + 2F_nF_{n-1}$ and $F_{2n-1} = F_n^2 + F_{n-1}^2$ simultaneously by induction on $n \geq 1$, with the case $n = 1$ easily verified. By the induction hypothesis

$$\begin{aligned} F_{2n+2} &= F_{2n+1} + F_{2n} = 2F_{2n} + F_{2n-1} = 2(F_n^2 + 2F_nF_{n-1}) + (F_n^2 + F_{n-1}^2) \\ &= 3F_n^2 + 4F_nF_{n-1} + F_{n-1}^2. \end{aligned}$$

Replacing both occurrences of $F_{n-1}$ by $F_{n+1} - F_n$ (using $F_{n+1} = F_n + F_{n-1}$) and rearranging gives

$$F_{2n+2} = F_{n+1}^2 + 2F_nF_{n+1}.$$

Similarly the induction hypothesis shows that $F_{2n+1} = F_{n+1}^2 + F_n^2$, and so both formulae follow by induction.
Using these formulae gives

$$F_{2n}F_{n-1} - F_{2n-1}F_n = F_n(F_{n-1}^2 - F_n^2 + F_nF_{n+1}).$$

We claim that $F_{n-1}^2 - F_n^2 + F_nF_{n+1} = (-1)^n$ for all $n \geq 1$. The case $n = 1$ is easily checked. Then

$$\begin{aligned} F_n^2 - F_{n+1}^2 + F_{n+1}F_{n+2} &= F_n^2 - (F_n + F_{n-1})^2 + F_{n+1}(F_{n+1} + F_n) \\ &= -(F_n^2 - F_{n+1}^2 + F_{n+1}F_{n+2}) = -(-1)^n = (-1)^{n+1}, \end{aligned}$$

proving the claim, and completing the proof of the formula.

(b) Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{F_{2^k}} = 2 - \frac{F_{2^n-1}}{F_{2^n}}$ for all $n \geq 1$.

***Solution:*** Induction, with the case $n = 1$ obvious. By the induction hypothesis

$$\sum_{k=1}^{n+1} \frac{1}{F_{2^k}} = 2 - \frac{F_{2^n-1}}{F_{2^n}} + \frac{1}{F_{2^{n+1}}} = 2 - \frac{F_{2N}F_{N-1} - F_N}{F_NF_{2N}},$$

where $N = 2^n$. Since $N$ is even, part (a) gives $F_{2N}F_{N-1} = F_N + F_{2N-1}F_N$, and so

$$\sum_{k=1}^{n+1} \frac{1}{F_{2^k}} = 2 - \frac{F_{2N-1}}{F_{2N}} = 2 - \frac{F_{2^{n+1}-1}}{F_{2^{n+1}}},$$

completing the induction step.

(c) Hence deduce that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{F_{2^k}} = \frac{5 - \sqrt{5}}{2}$.

***Solution:*** Recall from the course notes that

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1 + \sqrt{5}}{2}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{F_{2^k}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{F_{2^k}} = \lim_{n\to\infty} \left(2 - \frac{F_{2^n-1}}{F_{2^n}}\right) = 2 - \frac{2}{1+\sqrt{5}} = \frac{5-\sqrt{5}}{2}.$$

**12.** (a) The *Riemann-Lebesgue Lemma* says that if $f(x)$ is well behaved, then

$$\lim_{n\to\infty} \int_a^b f(x)\sin(nx)\,dx = \lim_{n\to\infty} \int_a^b f(x)\cos(nx)\,dx = 0.$$

Use integration by parts to prove the Riemann-Lebesgue Lemma under the assumption that $f(x)$ has continuous derivative.

***Solution:*** Let us prove the $\sin(nx)$ result, the $\cos(nx)$ case is similar. Integrating by parts with $u = f(x)$ gives

$$\int_a^b f(x)\sin(nx)\,dx = \frac{f(a)\cos(na) - f(b)\cos(nb)}{n} - \frac{1}{n}\int_a^b f'(x)\cos(nx)\,dx.$$

The first term tends to 0 as $n \to \infty$, and

$$0 \le \frac{1}{n}\left|\int_a^b f'(x)\cos(nx)\,dx\right| \le \frac{1}{n}\int_a^b |f'(x)|\,dx \to 0,$$

and hence the result.

(b) Use the Riemann-Lebesgue Lemma to show that

$$\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin x} - \frac{1}{x}\right)\sin(2nx)\,dx = 0.$$

Hence deduce that

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \frac{\sin(2nx)}{\sin x}\,dx.$$

***Solution:*** Define $f(x) = \frac{1}{\sin x} - \frac{1}{x}$ for $0 < x \le \frac{\pi}{2}$, and $f(0) = 0$. We need to show that $f(x)$ has continuous derivative on $[0, \frac{\pi}{2}]$ so that we can apply the Riemann-Lebesgue Lemma.
If $x \neq 0$ we compute

$$f'(x) = \frac{1}{x^2} - \frac{\cos x}{\sin^2 x}.$$

If $x = 0$ we compute

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h - \sin h}{h^2 \sin h} = \frac{1}{6},$$

where we have used L'Hôpital's Rule. We also have

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left(\frac{1}{x^2} - \frac{\cos x}{\sin^2 x}\right) = \frac{1}{6},$$

where we have again used L'Hôpital's Rule. Thus $f$ has continuous derivative on $[0, \pi/2]$.

Therefore

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{x}\,dx=\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{\sin x}\,dx.$$

Making a change of variable $y=2nx$ in the left hand integral gives

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{x}\,dx=\lim_{n\to\infty}\int_0^{n\pi}\frac{\sin y}{y}\,dy=\int_0^{\infty}\frac{\sin y}{y}\,dy,$$

hence the result.

(c) Show that $\dfrac{\sin(2nx)}{\sin x}=2\displaystyle\sum_{k=1}^{n}\cos[(2k-1)x]$ for $n\geq 2$, $\sin x\neq 0$, and hence

$$\int_0^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}.$$

***Solution:*** Write $\cos(2k-1)x=\frac{1}{2}(e^{(2k-1)ix}-e^{-(2k-1)ix})$ and sum using the geometric summation formula. The details are:

$$\begin{aligned}
2\sum_{k=1}^{n}\cos(2k-1)x &= \sum_{k=1}^{n}(e^{(2k-1)ix}-e^{-(2k-1)ix})\\
&= e^{ix}\sum_{k=1}^{n}e^{(2k-2)ix}-e^{-ix}\sum_{k=1}^{n}e^{-(2k-2)ix}\\
&= e^{ix}\frac{1-e^{2nix}}{1-e^{2ix}}+e^{-ix}\frac{1-e^{-2nix}}{1-e^{-2ix}}\\
&= \frac{1-e^{2nix}}{e^{-ix}-e^{ix}}+\frac{1-e^{-2nix}}{e^{ix}-e^{-ix}}\\
&= \frac{\sin 2nx}{\sin x}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^{\infty}\frac{\sin y}{y}\,dy &= \lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{\sin x}\,dx\\
&= 2\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^{\frac{\pi}{2}}\cos(2k-1)x\,dx\\
&= 2\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1}\\
&= 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}=\frac{\pi}{2},
\end{aligned}$$

where we have used Leibnitz' formula.