80/16

THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION, ENGINEERING AND SCIENCE

MATH191F

ADVANCED LINEAR ALGEBRA

June 1997

TIME ALLOWED: Two Hours

LECTURERS: W Gibson
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THIS EXAMINATION PAPER CONSISTS OF 5 PAGES NUMBERED FROM 1 TO 5
THERE ARE 7 QUESTIONS NUMBERED FROM 1 TO 7

Full marks may not be awarded unless sufficient working is shown.
Answers should be written in the booklets provided.

All 7 questions may be attempted.
Questions are of equal value.

Calculators will be supplied; no other electronic calculators are permitted.

1. (*i*) Given the vectors

$$\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \qquad \mathbf{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

find:

(a) $\mathbf{u} \cdot \mathbf{v}$;

(b) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$;

(c) $\mathbf{u} \times \mathbf{v}$;

(d) a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$;

(e) $(3\mathbf{u} - 2\mathbf{v}) \times (\mathbf{u} + 5\mathbf{v})$.

(*ii*) (a) Find the equation of the plane through the point $A : (2, -1, -1)$ perpendicular to the vector from the origin to A. Give the answer in cartesian form.

(b) Show that the line

$$\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$$

is parallel to the plane $4x + 4y - 5z = 14$.

2. (*i*) Let $ABCD$ be a parallelogram and let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{BC}$. Express the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, and hence prove that in any parallelogram the sum of squares of the diagonals is equal to the sum of the squares of the four sides.

(*ii*) Find the perpendicular distance from the point $A : (1, 2, -1)$ to the line ℓ passing through the points $B : (0, 0, 0)$ and $C : (-1, 0, 2)$. Proceed as follows:

(a) Find the equation of ℓ;

(b) Express, in parametric form, the coordinates of any point N on ℓ;

(c) Now choose N such that $\overrightarrow{AN}$ is perpendicular to ℓ;

(d) Compute the length of $\overrightarrow{AN}$.

3. (*i*) (a) State Euler's formula relating $e^{i\theta}$ to $\cos\theta$ and $\sin\theta$;

(b) Use Euler's formula to derive De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

where n is an integer.

(c) Use De Moivre's theorem to show that

$$\cot 3\theta = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}.$$

(*ii*) (a) Express $1 + i$ in polar form;

(b) Find all the solutions of $z^2 = 1 + i$;

(c) Find all the solutions of

$$z^4 - 2z^2 = -2$$

and plot their positions in the complex plane. (Hint: look upon this expression as a quadratic in z^2.)

4. (*i*) Find the general solution of the system of linear equations

$$\begin{aligned} x + y + z + w &= 2 \\ 2x + 2y + z + 4w &= -7 \\ 7x + 7y + 5z + 11w &= -8. \end{aligned}$$

(*ii*) Use elementary row operations to calculate the determinant of

$$\begin{pmatrix} 1 & 5 & -1 & 4 \\ -1 & -3 & -1 & -4 \\ 2 & 10 & -1 & 9 \\ 0 & 1 & -1 & 6 \end{pmatrix}.$$

5. (*i*) Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

(*ii*) Using your answer to Part (*i*), find the 2×3 matrix X which satisfies the matrix equation

$$X \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(*iii*) Let A be an $n \times n$ matrix, and suppose that both B and C are inverses of A. Prove that $B = C$.

(*iv*) Let A be an $n \times n$ matrix, and suppose that the ith row of A is zero, where i is some integer with $1 \leq i \leq n$. Show that if B is another $n \times n$ matrix then the ith row of AB is also zero, and hence deduce that A does not have an inverse.

6. (*i*) Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$T(\mathbf{e}_1) = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \qquad T(\mathbf{e}_2) = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad T(\mathbf{e}_3) = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}.$$

(a) Find the matrix of T relative to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

(b) Let $\mathcal{P}$ be the plane through the origin with equation $x + z = 0$. Show that if the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in $\mathcal{P}$ then so is $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = T \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(c) Let $\mathbf{e} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Find real numbers a, b, c and d such that

$$T(\mathbf{e}) = a\,\mathbf{e} + b\,\mathbf{f}$$
$$T(\mathbf{f}) = c\,\mathbf{e} + d\,\mathbf{f}.$$

(*ii*) (a) Name three different kinds of geometric transformations of the plane that correspond to linear transformations.

(b) Let $\mathcal{P}$ and T be as in Part (*i*), and consider the transformation of $\mathcal{P}$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto T \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Which kind of geometric transformation is this? (Use the answer to Part (*i*) (c).)

7. Let ω be the complex number $e^{2\pi i/3}$, and let $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$.

(i) Show that $\omega^3 = 1$.

(ii) Show that $\begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}$ is an eigenvector of A, with eigenvalue $a + b\omega + c\omega^2$.

(iii) Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$ are also eigenvectors of A, and determine the corresponding eigenvalues.

(iv) Find a matrix M such that

$$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} M = M \begin{pmatrix} a+b+c & 0 & 0 \\ 0 & a+b\omega+c\omega^2 & 0 \\ 0 & 0 & a+b\omega^2+c\omega \end{pmatrix},$$

and such that all of the columns of M are nonzero.