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THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education, Engineering and Science

MATH1791

Advanced Linear Algebra

June 1998

Time Allowed: Two Hours

Lecturers: W Gibson
R Howlett

This examination paper consists of 5 pages numbered from 1 to 5
There are 7 questions numbered from 1 to 7

Full marks may not be awarded unless sufficient working is shown.
Answers should be written in the booklets provided.

All 7 questions may be attempted.
Questions are of equal value.

Calculators will be supplied; no other
electronic calculators are permitted.

1. Given the vectors

$$\mathbf{u} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}, \qquad \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

find:

(*i*) $\mathbf{u} \cdot \mathbf{v}$;

(*ii*) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$;

(*iii*) $\mathbf{u} \times \mathbf{v}$;

(*iv*) a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$;

(*v*) $(3\mathbf{u} - 2\mathbf{v}) \times (\mathbf{u} + 5\mathbf{v})$.

(*vi*) Show that the vector projection of a vector $\mathbf{p}$ in the direction of a vector $\mathbf{q}$ is given by $(\mathbf{p} \cdot \mathbf{q}/|\mathbf{q}|^2)\mathbf{q}$. Hence find the projection of $\mathbf{u}$ in the direction of $\mathbf{v}$.

(*vii*) Find vectors $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathbf{u} = \mathbf{a} + \mathbf{b}$$

and $\mathbf{a}$ is parallel to $\mathbf{v}$ and $\mathbf{b}$ is perpendicular to $\mathbf{v}$.

(*viii*) Show that $\mathbf{a}$ and $\mathbf{b}$ given in part (*vii*) are unique.

2. (*i*) Let ABC be an isosceles triangle with $AB = AC$ and let M be the midpoint of side BC. Use vector methods to prove that AM is perpendicular to BC.

(*ii*) Let ℓ_1 be the line with equation

$$\mathbf{r} - (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = t(\mathbf{i} - \mathbf{j} + 3\mathbf{k}),$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and t is a scalar parameter. Let ℓ_2 be the line with equation

$$x - 1 = 4 - y = 2z + 8.$$

(a) Show that ℓ_1 and ℓ_2 intersect at a point Q and find the coordinates of Q.

(b) Find the equation of the plane containing the lines ℓ_1 and ℓ_2. (Give the answer in cartesian form: $Ax + By + Cz + D = 0$.)

3. (*i*) State Euler's formula relating $e^{i\theta}$ to $\cos\theta$ and $\sin\theta$. Hence derive the expressions for $\cos\theta$ and $\sin\theta$ in terms of complex exponentials.

(*ii*) Express $1 - i\sqrt{3}$ in polar form and hence find all the solutions of

$$z^3 + i\sqrt{3} = 1.$$

(*iii*) Given that

$$\frac{1+z}{1-z} = e^{i\alpha},$$

where α is real, show that

$$z = i\tan(\alpha/2).$$

Find the corresponding result if

$$\frac{1+z}{1-z} = i\,e^{i\alpha}.$$

4. (*i*) Find a reduced echelon matrix which is row equivalent to

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 3 & 3 & 4 & 4 & 5 & 5 \\ -3 & -3 & -3 & -1 & -1 & 1 & 1 \end{array}\right].$$

(*ii*) Using Part (*i*), or otherwise, find the general solution of the system of linear equations

$$\begin{aligned} u + v + w + x + y + z &= 1 \\ 2u + 2v + 2w + 3x + 3y + 4z &= 4 \\ 3u + 3v + 3w + 4x + 4y + 5z &= 5 \\ -3u - 3v - 3w - x - y + z &= 1. \end{aligned}$$

(*iii*) Suppose that M is a row echelon matrix which has five rows and nine columns, and let

$$M\mathbf{x} = \mathbf{0}$$

be the corresponding homogeneous system of five linear equations in nine unknowns.

If the general solution of this system involves six arbitrary parameters, how many zero rows does M have?

5. (*i*) What does it mean to say that a matrix C is an *inverse* of a matrix A?

(*ii*) Let A and B be $n \times n$ matrices, and suppose that C and D are $n \times n$ matrices such that C is an inverse of A and D is an inverse of B.

Show that DC is an inverse of AB.

(*iii*) Explain why a system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n &= 0 \end{aligned}$$

in which r, the number of equations, is less than n, the number of unknowns, must have infinitely many solutions.

(*iv*) Use Part (*iii*) to explain why the matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

cannot have an inverse.

6. (*i*) Express the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ 3 & 3 & 6 \end{bmatrix}$$

as a product of elementary matrices.

(*ii*) Using the result of Part (*i*), or otherwise, find the determinant of the matrix A of Part (*i*).

(*iii*) Let σ and ρ be the permutations of $\{1, 2, 3, 4\}$ given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}.$$

(a) Draw diagrams representing σ and ρ, and hence determine $\operatorname{sgn}(\sigma)$ and $\operatorname{sgn}(\rho)$.

(b) Calculate the composite $\rho \circ \sigma$.

7. (*i*) Let M be an $r \times r$ matrix, $\mathbf{v}$ an r-component column vector, and λ a scalar.

(a) What does it mean to say that $\mathbf{v}$ is an eigenvector for M and λ the corresponding eigenvalue?

(b) Suppose that $\mathbf{v}$ is an eigenvector for M with corresponding eigenvalue λ. Use induction on n to prove that if n is a positive integer then $\mathbf{v}$ is an eigenvector for M^n with corresponding eigenvalue λ^n.

(c) Suppose that M is invertible and λ is an eigenvalue of M. Prove that $\lambda \neq 0$, and prove that $1/\lambda$ is an eigenvalue of M^{-1}.

(d) Let P be an $r \times r$ matrix that commutes with M. Prove that if $\mathbf{v}$ is an eigenvector of M then $P\mathbf{v}$ is also an eigenvector of M.

(*ii*) Determine all (possibly complex) numbers which are eigenvalues of the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(*iii*) Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find a matrix T such that $T^{-1}AT$ is diagonal. (You are not required to compute T^{-1}).