THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1	**Tutorial Solutions Week 3**	2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Express the following complex numbers in Cartesian form:

(a) $2\,\mathrm{cis}\,\frac{\pi}{4}$

(b) $-4\,\mathrm{cis}\,\frac{\pi}{3}$

(c) $\mathrm{cis}\,\frac{\pi}{2}\,\mathrm{cis}\,\frac{\pi}{3}\,\mathrm{cis}\,\frac{\pi}{6}$

(d) $e^{-i\pi}$

(e) $e^{\ln 2+i\pi}$

(f) $e^{1+i}e^{1-i}e^{-2-i\pi}$

Questions for the tutorial

2. Solve the following equations (leaving your answers in polar form) and plot the solutions in the complex plane:

(a) $z^5 = 1$

(b) $z^6 = -1$

(c) $z^3 + i = 0$

(d) $z^4 = 8\sqrt{2} + 8\sqrt{2}\,i$

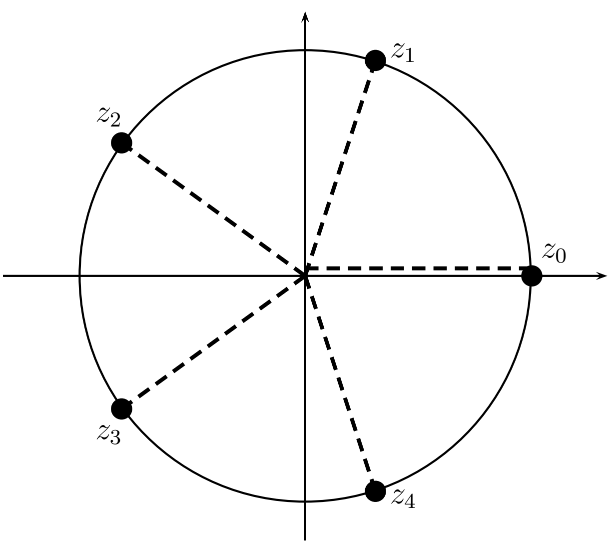
(e) $z^5 + z^3 - z^2 - 1 = 0$, given that $z = i$ is a solution.

Solution

(a) Write $z = r\,\mathrm{cis}\,\theta$ and write 1 in polar form, $1 = 1\,\mathrm{cis}\,0$. Then $z^5 = 1 \iff r^5\,\mathrm{cis}\,5\theta = 1\,\mathrm{cis}\,0$. Equating moduli, we find that $r^5 = 1$, so $r = 1$. Comparing arguments, we find that $\theta = \dfrac{2k\pi}{5}$, for some $k \in \mathbb{Z}$. As k takes the five values 0, 1, 2, 3, 4, we get all five 5th roots of unity, namely

$$z_0 = 1\ ,\ \ z_1 = \mathrm{cis}\,(2\pi/5)\ ,\ \ z_2 = \mathrm{cis}\,(4\pi/5)\ ,\ \ z_3 = \mathrm{cis}\,(6\pi/5)\ ,\ \ z_4 = \mathrm{cis}\,(8\pi/5).$$

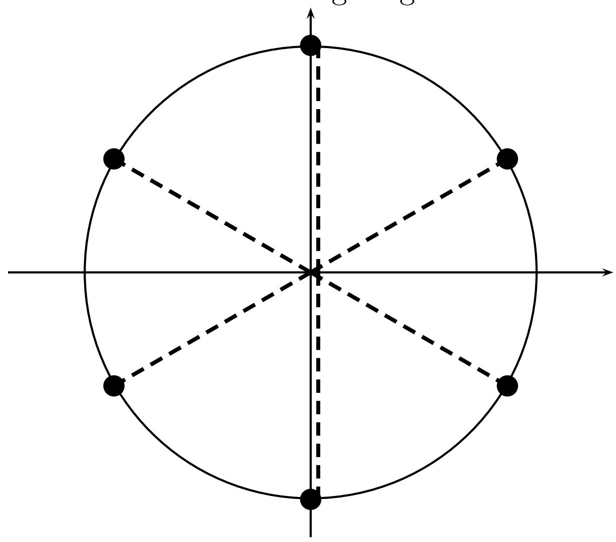
All solutions have modulus 1 and so lie on a unit circle, centred at the origin. The solutions (represented by the dots on the circle) are equally spaced at angles of $2\pi/5$.



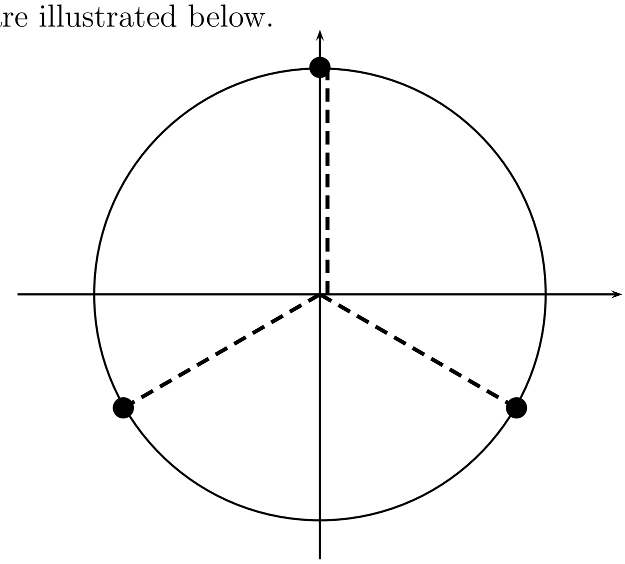
(b) Observe that $z^6 = -1 = \text{cis}\ \pi$. Therefore the six roots are:

$$z = \text{cis}\ (\pi/6)\ ,\quad \text{cis}\ (\pi/2)\ ,\quad \text{cis}\ (5\pi/6)\ , \text{cis}\ (-5\pi/6)\ ,\quad \text{cis}\ (-\pi/2)\ ,\quad \text{cis}\ (-\pi/6)\ .$$

They are illustrated on the following diagram.

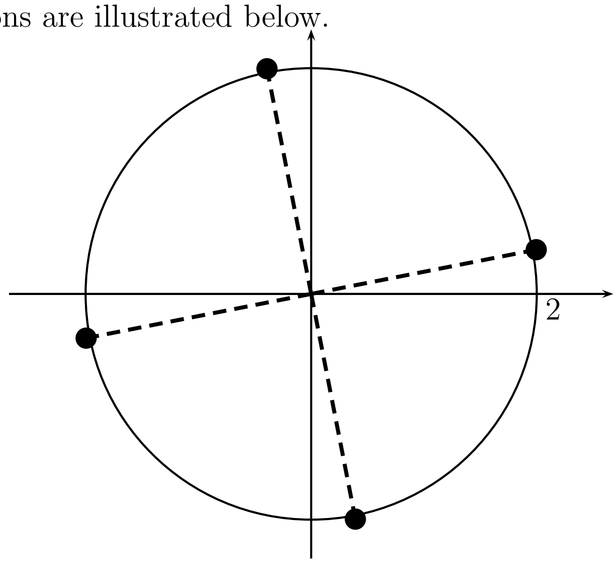


(c) $z^3 = -i = \text{cis}\ (-\pi/2) \iff z = \text{cis}\ (-\pi/6)\ ,\quad \text{cis}\ (\pi/2)\ ,\quad \text{cis}\ (-5\pi/6)$. The solutions are illustrated below.



(d) $$z^4 = 8\sqrt{2} + 8\sqrt{2}\,i = 16\,\text{cis}\ \pi/4 \iff z = 2\,\text{cis}\ (\pi/16)\ ,\quad 2\,\text{cis}\ (9\pi/16)\ ,\quad 2\,\text{cis}\ (-15\pi/16)\ ,\quad 2\,\text{cis}\ (-7\pi/16).$$

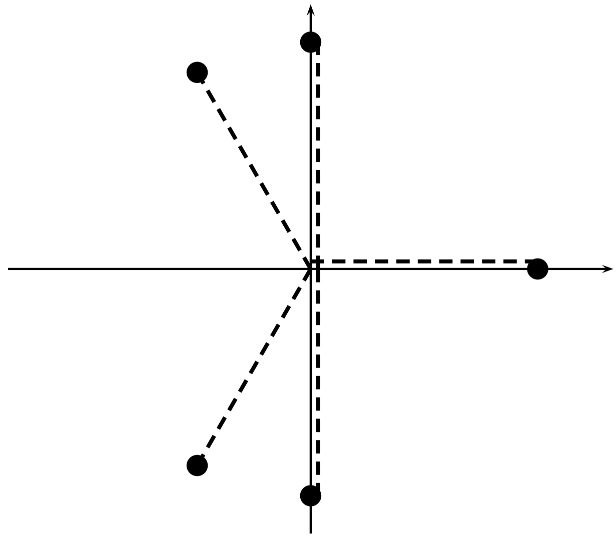
The solutions are illustrated below.



(e) Given that $z = i$ is one solution, $z^5 + z^3 - z^2 - 1 = 0$ has the complex conjugate pair $z = \pm i$ as solutions, so $(z-i)(z+i) = z^2 + 1$ is a factor of the left hand side. Observe that

$$z^5 + z^3 - z^2 - 1 = z^3(z^2+1) - (z^2+1) = (z^2+1)(z^3-1) = 0.$$

Hence the solutions of the original polynomial equation are $\pm i = \operatorname{cis}(\pm\pi/2)$ together with the three cube roots of 1 (which are $\operatorname{cis} 0$ and $\operatorname{cis}(\pm 2\pi/3)$). The solutions all have modulus 1 and are illustrated below.



3. The complex sine and cosine functions are defined by the formulas

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}.$$

(a) Show that when z is real ($z = x$), $\sin z$ and $\cos z$ reduce to the familiar real sine and cosine functions.

(b) Show that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.

(c) Is it true that $|\sin z| \le 1$ and $|\cos z| \le 1$, for all $z \in \mathbb{C}$? (*Hint*: You know these are true when z is real. See what happens when z is purely imaginary, $z = iy$.)

Solution

(a) If $z = x$, then

$$\begin{aligned}\sin z &= \frac{e^{ix} - e^{-ix}}{2i}\\ &= \frac{\cos x + i\sin x - (\cos x - i\sin x)}{2i}\\ &= \frac{2i\sin x}{2i}\\ &= \sin x.\end{aligned}$$

A similar result hold for $\cos z$.

(b)

$$\sin^2 z + \cos^2 z = \frac{e^{2iz} - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} = 1.$$

(c) These are not true statements. When $z = iy$, we have

$$|\sin(iy)| = \left|\frac{e^{-y} - e^{y}}{2i}\right| = \tfrac{1}{2}\left|e^{y} - e^{-y}\right|.$$

In particular, $|\sin(i)| = \dfrac{e - e^{-1}}{2} > 1$. Actually, $|\sin(iy)|$ can be made as large as we please by taking sufficiently large positive y (because e^y becomes arbitrarily large and dominates e^{-y}). A similar result holds for the complex cosine function.

4. Find all solutions of the following equations:

(a) $e^z = i$ (b) $e^z = -10$

(c) $e^z = -1 - i\sqrt{3}$ (d) $e^{2z} = -i$

Solution

(a) Let $z = x + iy$. Recall that by definition of the complex exponential,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \text{ cis } (y).$$

So the equation we want to solve is

$$e^x \text{ cis } (y) = i = 1 \text{ cis } (\pi/2).$$

Equating moduli, we find that $e^x = 1$, and so $x = 0$. Comparing arguments, we find that

$$y = \arg(e^z) = \arg(i) = \pi/2 + 2k\pi,$$

for some $k \in \mathbb{Z}$. Hence there are infinitely many solutions: $z = i(\frac{\pi}{2} + 2k\pi)$, $k \in \mathbb{Z}$.

(b) There are infinitely many solutions: $z = \ln 10 + i(\pi + 2k\pi)$, $k \in \mathbb{Z}$.

(c) There are infinitely many solutions: $z = \ln 2 + i(-\frac{2\pi}{3} + 2k\pi)$, $k \in \mathbb{Z}$.

(d) There are infinitely many solutions: $z = i(-\frac{\pi}{4} + k\pi)$, $k \in \mathbb{Z}$.

5. Sketch and describe the following sets and their images under the function $z \mapsto z^2$.

(a) The set of all points of the form $z = x + 2i$.

(b) The set of all points of the form $z = x + 2xi$.

(c) The set of all points on the upper half of the unit circle centred at the origin, that is, points z with polar coordinates (r, θ) such that $r = 1$ and $0 \leq \theta \leq \pi$.

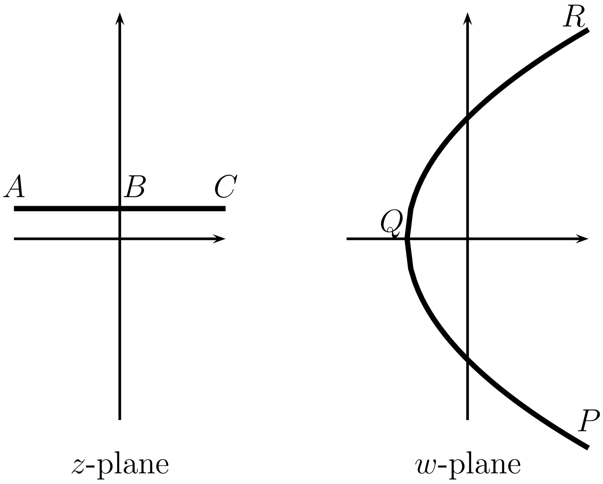
(d) The set of all points on the unit circle centred at the origin, that is, points z with polar coordinates (r, θ) such that $r = 1$ and $-\pi < \theta \leq \pi$.

Solution

(a) Let $w = z^2 = (x + 2i)^2 = x^2 - 4 + 4xi$. Remember that x can take any real value. If we write $w = u + iv$ then $u = x^2 - 4$ and $v = 4x$. We eliminate x from these two equations to obtain

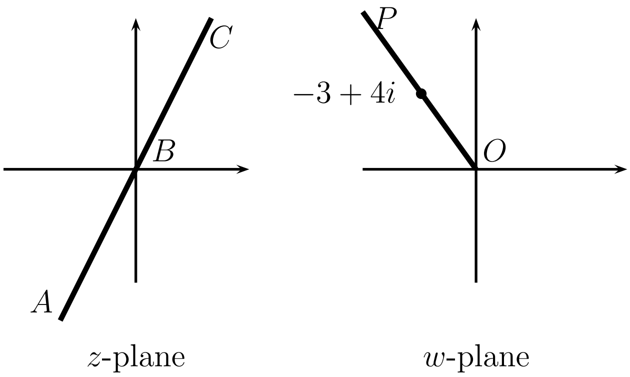
$$u = \left(\frac{v}{4}\right)^2 - 4 = \frac{v^2}{16} - 4.$$

Thus the image set in the w-plane is a parabola, symmetrical about the real axis, passing through the point -4 (denoted by Q in the diagram) and opening to the right.

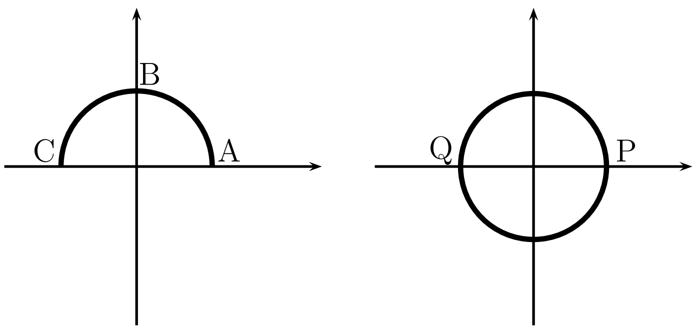


To see how the parabola is traced out as z varies, first think of x as taking very large negative values. Then v is also large and negative, while u is large and positive (we are at the lower right part of the parabola). Now let x increase. When $x = 0$, $v = 0$ and $u = -4$, corresponding to the apex of the parabola. When x is large and positive, so are both v and u (we are now at the upper right part of the parabola). That is, as z moves from A to B to C, w moves from P to Q to R.

(b) Let $w = z^2 = (x + 2xi)^2 = x^2(-3 + 4i)$. All values of w have the same principal argument. As x varies from very large negative numbers through zero to very large positive numbers, the modulus of w (namely $5x^2$) varies from very large, to zero, to very large again. (As z moves from A to B to C, w moves from P to O to P.) The image of the given set in the z-plane is a half-line traced twice in the w-plane, as shown in the following diagram.

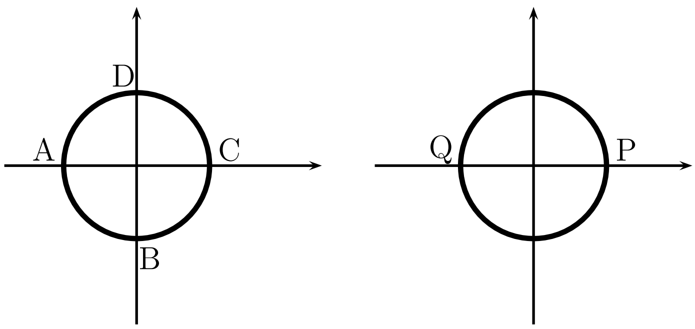


(c) As z has polar coordinates $(1, \theta)$, where $0 \leq \theta \leq \pi$, we see that $w = z^2$ has polar coordinates $(1, \phi)$, where $0 \leq \phi \leq 2\pi$. The image of the semi-circle in the z-plane is therefore the full unit circle in the w-plane. As z moves anticlockwise from A to B to C, w moves anticlockwise from P to Q (along the upper half circle) and then from Q to P (along the lower half circle).



z-plane w-plane

(d) This time, the polar coordinates of z are $(1, \theta)$, where $-\pi < \theta \leq \pi$. Therefore the polar coordinates of w are $(1, \phi)$, where $-2\pi < \phi \leq 2\pi$. The image is again the unit circle. As z moves anticlockwise around the unit circle once (from just under A to B to C to D to A), w moves anticlockwise around the unit circle twice (from just above P to Q (upper half), Q to P (lower half), P to Q (upper half) and Q to P (lower half)).



z-plane w-plane

6. Sketch the following sets and their images under the function $z \mapsto e^z$:

(a) $\{z = x + iy \in \mathbb{C} \mid 0 < x < 2,\ y = \frac{\pi}{2}\}$;

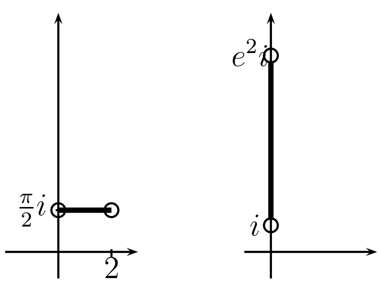
(b) $\{z \in \mathbb{C} \mid x = 1,\ |y| < \frac{\pi}{2}\}$;

(c) $\{z \in \mathbb{C} \mid x < 0,\ \frac{\pi}{3} < y < \pi\}$;

(d) $\{z = (1 + i)t \mid t \in \mathbb{R}\}$.

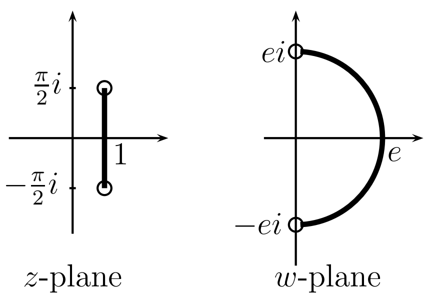
Solution

(a) Let $w = e^z$. As $z = x + i\frac{\pi}{2}$, we have $|w| = |e^z| = e^x$ (hence $1 < |w| < e^2$) and $\mathrm{Arg}(w) = \frac{\pi}{2}$. The image of the given set is the set of points on the imaginary axis of the w-plane, as shown.

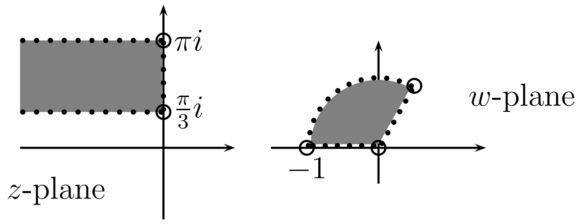


z-plane w-plane

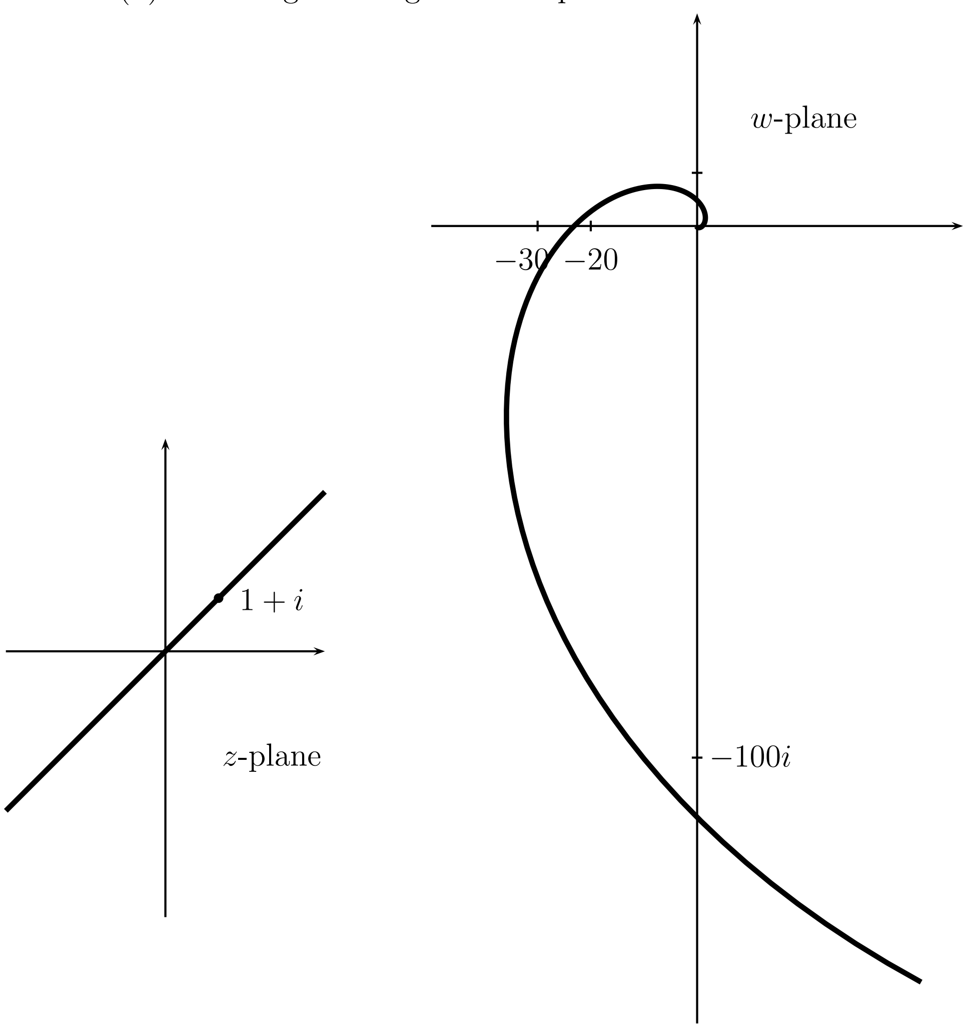
(b) We know that $|w| = |e^z| = e^x = e$ (constant) and $\text{Arg}(w)$ lies between $-\pi/2$ and $\pi/2$. Hence the set of image points is a semi-circle.



(c) Since $x < 0$ and $|w| = e^x$, we have $0 < |w| < 1$. Also, $\pi/3 < \text{Arg}(w) < \pi$. The image set is the interior of the sector of the unit circle, shown.



(d) The image is a logarithmic spiral.



7. (a) Sketch the set $\{z \in \mathbb{C} \mid \frac{1}{2} < |z| < 4,\ 0 \le \operatorname{Arg}(z) \le \frac{\pi}{4}\}$.

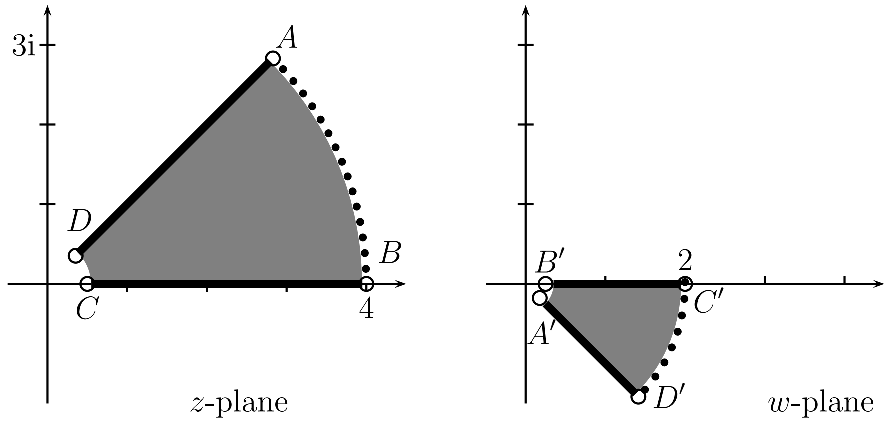
(b) Sketch the image of the set in the w-plane under the function $z \mapsto w = \frac{1}{z}$.

(c) An insect is crawling clockwise around the boundary of the set in the z-plane. Is its image crawling clockwise, or anticlockwise, in the w-plane? (If clockwise, we say the transformation is *orientation-preserving*; if anticlockwise, we say it is *orientation-reversing.*)

(d) Now consider the function $z \mapsto w = \bar{z}$, the complex conjugate of z. Is it orientation-preserving or orientation-reversing?

Solution

(a) The sketch of the original set in the z-plane is given below, on the left.



(b) The sketch of the image of the set in the w-plane is given above, on the right.

(c) The mapping is orientation-preserving. As the insect crawls from A to B to C to D, its image crawls from A' to B' to C' to D'.

(d) The complex conjugate transformation can be thought of as reflection in the real axis and is orientation-reversing.

8. Find all solutions of the equation $e^{2z} - (1 + 3i)e^z + i - 2 = 0$.

Solution

Using the quadratic formula we find that this equation is equivalent to

$$e^z = \frac{(1 + 3i) \pm \sqrt{(1 + 3i)^2 - 4(i - 2)}}{2},$$

which simplifies to

$$e^z = \frac{1 + 3i \pm \sqrt{2i}}{2}.$$

As $(1 + i)^2 = 2i$, we see that the two numbers whose square is $2i$ are $\pm(1 + i)$, so

$$e^z = \frac{1 + 3i \pm (1 + i)}{2} = 1 + 2i \text{ or } i.$$

In polar form, $1 + 2i = \sqrt{5}e^{i\theta}$, where $\theta = \tan^{-1}(2) \approx 1.11$ radians, and $i = e^{i\pi/2}$. If $z = x + iy$ then

$$e^z = e^x e^{iy} = \sqrt{5}e^{i\theta} \text{ or } e^{i\pi/2}.$$

Hence $x = \frac{1}{2}\ln 5$ and $y = \theta + 2k\pi$, or $x = 0$ and $y = \pi/2 + 2k\pi$, for $k \in \mathbb{Z}$. There are infinitely many solutions, of the form

$$z = \frac{1}{2}\ln\,5 + i(\tan^{-1}(2) + 2k\pi) \;\text{ or }\; z = i(\pi/2 + 2k\pi).$$

Extra Questions

9. This question demonstrates that complex numbers can be useful in solving cubic equations, even when all the solutions are real.

(a) Show that for any complex number w, there exists a nonzero complex number z such that $z + \dfrac{1}{z} = w$.

(b) Use this substitution to solve the equation $w^3 - 3w - 1 = 0$.

Solution

(a) The equation $z+\dfrac{1}{z} = w$ becomes $z^2 - wz + 1 = 0$ after multiplying by z and rearranging. This equation is a quadratic in z and has solutions $z = \dfrac{w \pm \sqrt{w^2-4}}{2}$. These solutions are nonzero, because $0^2 - w0 + 1 = 1 \neq 0$. Hence we can divide by z to conclude that we have found solutions to the initial equation $z + \dfrac{1}{z} = w$.

(b) We know from (a) that any w can be written as $z + \dfrac{1}{z}$ for some nonzero $z \in \mathbb{C}$. When this substitution is made,

$$\begin{aligned} w^3 - 3w - 1 &= \left(z + \frac{1}{z}\right)^3 - 3\left(z + \frac{1}{z}\right) - 1 \\ &= z^3 + 3z^2\frac{1}{z} + 3z\frac{1}{z^2} + \frac{1}{z^3} - 3\left(z + \frac{1}{z}\right) - 1 \\ &= z^3 + \frac{1}{z^3} - 1, \end{aligned}$$

so the equation $w^3 - 3w - 1 = 0$ is equivalent to $z^3 + \dfrac{1}{z^3} - 1 = 0$, which is equivalent to $z^6 - z^3 + 1 = 0$. This is a quadratic in z^3 with solutions

$$z^3 = \frac{1+\sqrt{3}i}{2} = \text{cis}\,(\tfrac{\pi}{3}) \;\text{ or }\; z^3 = \frac{1-\sqrt{3}i}{2} = \text{cis}\,(-\tfrac{\pi}{3}).$$

Since the solutions of $z^3 = \text{cis}\,(-\frac{\pi}{3})$ are the inverses of the solutions of $z^3 = \text{cis}\,(\frac{\pi}{3})$, and $z + \dfrac{1}{z}$ doesn't change when you replace z with its inverse, we need only consider the possibility that $z^3 = \text{cis}\,(\frac{\pi}{3})$. As in Q2, we find that the three cube roots of $\text{cis}\,(\frac{\pi}{3})$ are $\text{cis}\,(\frac{\pi}{9})$, $\text{cis}\,(\frac{7\pi}{9})$, and $\text{cis}\,(-\frac{5\pi}{9})$. Now if $z = \text{cis}\,\theta$, then

$$z + \frac{1}{z} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta,$$

so we have proved that the solutions to the original equation $w^3 - 3w - 1 = 0$ are $2\cos\frac{\pi}{9}$, $2\cos\frac{5\pi}{9}$, and $2\cos\frac{7\pi}{9}$. (Notice that these are all real numbers, but we needed the complex numbers to find them.)

10. Let n be a given positive integer. By a *primitive nth root of unity* we mean a solution η of $z^n = 1$ which has the property that its powers $\eta, \cdots, \eta^{n-1}, \eta^n (= 1)$ are exactly the solutions of this equation in $\mathbb{C}$. For example, $e^{i\frac{2\pi}{n}}$ is a primitive nth root of unity.

(a) Find all primitive 6th roots of unity.

(b) Find all primitive 5th roots of unity.

(c) For which values of k, $0 \le k \le n-1$, is $e^{i\frac{2\pi k}{n}}$ a primitive nth root of unity?

Solution

(a) Clearly, $\eta = e^{i\frac{2\pi}{6}}$ is a primitive 6-th root of unity, as its powers $\eta, \eta^2, \eta^3, \eta^4, \eta^5, \eta^6 (= 1)$ give the complete list of 6-th roots of unity, namely

$$\eta = e^{i\frac{2\pi}{6}}, \quad \eta^2 = e^{i\frac{4\pi}{6}}, \quad \eta^3 = e^{i\frac{6\pi}{6}}, \quad \eta^4 = e^{i\frac{8\pi}{6}}, \quad \eta^5 = e^{i\frac{10\pi}{6}}, \quad \eta^6 = e^{i\frac{12\pi}{6}} = 1.$$

However $\eta^2 = e^{i\frac{4\pi}{6}}$ is not a primitive 6-th root of unity: its distinct powers are η^2, $(\eta^2)^2 = \eta^4$ and $(\eta^2)^3 = \eta^6 = 1$. Similarly, η^3 is not a primitive root (its distinct powers are η^3, $\eta^6 = 1$) and η^4 is not a primitive root (its distinct powers are η^4, η^2 and 1). However, η^5 is a primitive root. Its powers are

$$\eta^5, \quad (\eta^5)^2 = \eta^4, \quad (\eta^5)^3 = \eta^3, \quad (\eta^5)^4 = \eta^2, \quad (\eta^5)^5 = \eta, \quad (\eta^5)^6 = 1.$$

Thus the only primitive 6th roots of unity are η and η^5.

(b) With $\eta = e^{i\frac{2\pi}{5}}$, we find that the primitive 5th roots of unity are

$$\eta, \quad \eta^2, \quad \eta^3, \quad \eta^4.$$

(c) The nth root of unity $e^{i\frac{2\pi k}{n}}$ is a primitive nth root if, and only if, k is *relatively prime* to n. (Can you prove this?)

Solution to Question 1

1. (a) $2 \operatorname{cis} \pi/4 \;=\; \sqrt{2} + \sqrt{2}\, i$ (b) $-4 \operatorname{cis} \pi/3 \;=\; -2 - 2\sqrt{3}\, i$

(c) $\operatorname{cis}(\pi/2) \operatorname{cis}(\pi/3) \operatorname{cis}(\pi/6) \;=\; \operatorname{cis} \pi \;=\; -1$ (d) $e^{-i\pi} \;=\; \operatorname{cis}(-\pi) \;=\; -1$

(e) $e^{\ln 2 + i\pi} \;=\; e^{\ln 2} \operatorname{cis} \pi \;=\; -2$ (f) $e^{1+i} e^{1-i} e^{-2-i\pi} \;=\; e^{1+i+1-i-2-i\pi} \;=\; e^{-i\pi} \;=\; -1$