THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS

MATH2621 Higher Complex Analysis S2, 2018
Problem Sheet

Good solutions to these problems include reasons. A few problems may require extensions of ideas or definitions in the course; these may be explained in tutorials.

Revision exercises.

1. Show that $|e^z| = e^{\mathrm{Re}(z)}$ for all $z \in \mathbb{C}$.
2. Show that $|e^{i\theta} - 1| = 2\,|\sin(\theta/2)|$ for all $\theta \in \mathbb{R}$
   (a) using geometry;
   (b) using the algebraic formula $e^{i\theta} - 1 = e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2})$.
   Deduce that $|e^{i\theta} - 1| \le |\theta|$ for all $\theta \in \mathbb{R}$.
3. Give a geometric explanation for the inequality
$$|w| \le |w + z| + |z|.$$
4. Suppose that $\alpha = \cos t + i \sin t$ where $0 < t < 2\pi$. Show that
$$\frac{1+\alpha}{1-\alpha} = i \cot \frac{t}{2}.$$
5. Suppose that $\alpha \neq 0$. Show that $\alpha + 1/\alpha$ is real if and only if α is real or $|\alpha| = 1$.
6. Expand $(4 + i)(5 + 3i)$ and hence show that
$$\frac{\pi}{4} = \arctan \frac{1}{4} + \arctan \frac{3}{5}.$$
7. By expanding $(1 + \sqrt{3}i)(1 + i)$, find $\cos(7\pi/12)$ and $\sin(7\pi/12)$ in surd form.
   Answer: $(1 - \sqrt{3})/(2\sqrt{2})$ and $(1 + \sqrt{3})/(2\sqrt{2})$.
8. (a) Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ for all $z_1, z_2 \in \mathbb{C}$.
   (b) Give a geometrical interpretation of this result.

* 9. Show that the triangle in the complex plane whose vertices are z_1, z_2, z_3 is equilateral if and only if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

10. Explain both geometrically and algebraically why $|z+w| = |z|+|w|$ if $\arg z = \arg w$. Does the result extend to arbitrary finite sums of complex numbers?
11. Show that any root z of $z^4 + z + 3 = 0$ satisfies $|z| > 1$ and that any root z of $4z^4 + z + 1 = 0$ satisfies $|z| \le 1$.
12. Suppose that w is an nth root of unity and $w \neq 1$. Show that
$$1 + w + w^2 + \cdots + w^{n-1} = 0;$$
   hence or otherwise, factorise $1 + z + z^2 + z^3 + z^4 + z^5$ over $\mathbb{C}$ and over $\mathbb{R}$.
13. (a) Factorise $z^8 - 15z^4 - 16$ over $\mathbb{C}$ and over $\mathbb{R}$.
    (b) Find the real factorisation of $z^4 + 4$.
    Answer: (b) $(z^2 + 2z + 2)(z^2 - 2z + 2)$

14. (a) Show that for $0 < \theta < 2\pi$:
$$\operatorname{Re}\left(\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right) = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$
(b) Show that for $0 < \theta < 2\pi$:
$$\operatorname{Im}\left(\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right) = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$
(c) Find $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta$ and $\sin\theta + \sin 2\theta + \cdots + \sin n\theta$.

Sets.

15. Are the following sets open, closed, bounded, compact, connected, simply connected, regions or domains?
(a) The set $S_1 = \{z \in \mathbb{C} : |z| < 1\}$
(b) The set $S_2 = \{z \in \mathbb{C} : |z| \leq 1\}$
(c) The set $S_3 = \{z = x + iy \in \mathbb{C} : xy = 1\}$
(d) The set $S_4 = \{z = x + iy \in \mathbb{C} : xy > 1\}$
(e) The set $S_5 = \{z = x + iy \in \mathbb{C} : x \geq 0,\ y > 0\}$
(f) The set $S_6 = \{z \in \mathbb{C} : |z + i| + |z - i| < 4\}$
(g) The set $S_7 = \{z \in \mathbb{C} : |z + i| + |z - i| = 4\}$
(h) The set $S_8 = \mathbb{C}$
(i) The set $S_9 = \emptyset$ (the empty set).

16. If a set S consists of a single point $\{p\}$, then is p an interior point, an exterior point, or a boundary? Is S open, closed, connected, compact, or a region?

† 17. If an open set Ω is simply connected, is its complement connected? If the complement of a bounded open set Ω is connected, is the set simply connected? What if Ω is not bounded?
Answer: Not necessarily. Yes. Perhaps.

Functions.

18. Show that the composition of two polynomials is a polynomial, and the composition of two rational functions is a rational function.

* 19. Suppose that f is a nonconstant rational function. What can you say about the range of f? Hint: Write f as p/q and try to solve $p(z)/q(z) = \lambda$; this is possible if and only if λ is in the range of f.

20. The point $1 + i$ is rotated anticlockwise about the origin through $\pi/6$. Find its image.
Answer: $\frac{1}{2}(\sqrt{3} - 1 + i(\sqrt{3} + 1))$

21. Find the image of (a) the line $y = 2x + 5$ and (b) the circle $|z - 1| = 1$ under the transformation $w = (1 + i)z - 2$.
Answer: (a) $v = -3u - 16$, (b) $|w + 1 - i| = \sqrt{2}$.

22. Find the image of the region $\{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < \pi/2, \operatorname{Im}(z) > 0\}$ under the transformation $w = iz + 2$. Express your answer using set notation.
Answer: $\{(w \in \mathbb{C} : \operatorname{Re} w < 2, 0 < \operatorname{Im} w < \pi/2\}$.

23. Find the image of the following regions under the mapping $w = z^{-1}$:
(a) $x + y = 4$, (b) $|z - 1| = 1$, (c) $|z - 1| \leq 1$, $z \neq 0$.
24. Find the image of
(a) the region $|z - 1| \leq 1$ under the mapping $w = \dfrac{z}{z+2}$,
(b) the line $x + 2y = 2$ under the mapping $w = \dfrac{1}{z+i}$.
25. Let $f(z) = 1/(z - i)$. Find the image of $\{z \in \mathbb{C} : |z| < 1\}$ under f.
Answer: The set $\{w \in \mathbb{C} : \operatorname{Im} w > 1/2\}$.

Fractional linear transformations.

* 26. Suppose that $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Let T be the fractional linear transformation $z \mapsto \dfrac{az+b}{cz+d}$.
(a) Show that if $a, b, c, d \in \mathbb{R}$, then T maps the upper half plane onto the upper half plane. Is the converse true?
(b) Find conditions on a, b, c, d that ensure that T maps the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ onto itself.
(c) Show that if $a = i$, $b = -i$, $c = 1$ and $d = i$, then the corresponding fractional linear transformation maps the upper half plane onto the unit disc. Where does the first quadrant map to?
(d) Use your answer to (c) to connect your answers to (a) and (b).

Estimating the size of functions.

27. Show that if $|z| = R > 1$, then
$$\left|\frac{z}{z^3+1}\right| \leq \frac{R}{R^3-1}.$$
Show also that if $|z| = r < 1$, then
$$\left|\frac{z}{z^3+1}\right| \leq \frac{r}{1-r^3}.$$
28. Show that if $|z| = R > 1$, then
$$\left|\frac{1}{z^4+1}\right| \leq \frac{1}{R^4-1}.$$
Show also that if $|z| = r < 1$, then
$$\left|\frac{1}{z^4+1}\right| \leq \frac{1}{1-r^4}.$$
29. Show that if z is real, then
$$\left|\frac{e^{iz}}{z^2+1}\right| \leq \frac{1}{|z|^2+1}.$$
Is this still true when z is not real? Give two reasons for your answer.
30. Show that if $|z| = R > 2$ and $\operatorname{Im}(z) \geq 0$, then
$$\left|\frac{e^{iz}}{z}\right| \leq \frac{1}{R}.$$

Real and imaginary parts of functions.

31. Suppose that $f(z) = z^3 - z + 1$. Write $f(x)$ in the form $u(x, y) + iv(x, y)$, where $z = x + iy$. Answer: $u(x, y) = x^3 - 3xy{*}2 - x + 1$ and $v(x, y) = 3x^2 y - y{*}3 - y$.
32. Suppose that $f(z) = ze^{-z}$. Write $f(x)$ in the form $u(x, y) + iv(x, y)$, where $z = x + iy$, and in the form $u(r, \theta) + iv(r, \theta)$, where $z = re^{i\theta}$.
33. Suppose that $u(x, y) = x^2 + y^2$ and $v(x, y) = x - y$. Can you write $u(x, y) + iv(x, y)$ as a simple function of z? Or of z and $\bar{z}$?

Graphical representation of complex functions.

* 34. Suppose that $f(z) = az^2 + bz + c$, where $a, b, c \in \mathbb{C}$ and $a \neq 0$.
   (a) Show that the image of a straight line ℓ in $\mathbb{C}$ under f either lies in a straight line or is a parabola.
   (b) Show that the image of a straight line ℓ in $\mathbb{C}$ under f lies in a straight line if and only if ℓ passes through $-b/2a$.
35. Suppose that $f(z) = e^z$. Show that the image of a straight line ℓ in $\mathbb{C}$ under f is a straight line if and only if ℓ is horizontal.

Limits and continuity.

36. Find the following limits, if they exist, or explain why they do not exist.

(a) $\displaystyle\lim_{z\to 0} \frac{e^z - 1}{z}$ (b) $\displaystyle\lim_{z\to 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2}$

(c) $\displaystyle\lim_{z\to 0} e^{1/z}$ (d) $\displaystyle\lim_{z\to 0} e^{1/z^4}$

†(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ †(f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$.

Answer: (a) 1; (b) $1 - i/2$;
(c) the limit does not exist, because we get different limits as we approach $(0, 0)$ along the x axis from the left and from the right;
(d) the limit does not exist, because we get different limits as we approach $(0, 0)$ along different rays $(x, y) = (at, bt)$ as $t \to 0$;
(e) as for (d);
(f) the limit does not exist, because we get different limits as we approach $(0, 0)$ along different curves $(x, y) = (at^2, bt)$ as $t \to 0$;

37. Use the definition of a limit in each of the following.
   (a) Show that $\lim_{z\to\infty} f(z) = \lim_{z\to 0} f(1/z)$.
   (b) Suppose that $\lim_{z\to z_0} g(z)$ exists and is finite. Show that $\lim_{z\to z_0} f(z)$ exists if and only if $\lim_{z\to z_0} f(z) + g(z)$ exists.
   (c) Show that $\lim_{z\to z_0} f(az + b) = \lim_{w\to az_0+b} f(w)$.
38. Prove from the definition that $|z|$ and $\mathrm{Re}(z)$ are continuous.
39. For each of the following functions, state where the function is continuous, and justify your answer:

$$\text{(a)}\ \ f_1(z) = \frac{z^2 + i}{z^2 + 1}; \quad \text{(b)}\ \ f_2(z) = \frac{z^k}{\bar{z}}; \quad \text{(c)}\ \ f_3(z) = (z - i)\,\mathrm{Log}(z^2 + 1).$$

(in (b), k is a positive integer.) Is it possible to extend their definition to a larger domain, maintaining continuity.
Answer: (a) $z \neq \pm i$; (b) $z \neq 0$; if $k \geq 2$, then f_2 can be extended continuously by setting $f_2(0) = 0$; (c) $z \neq iy$, where $y \leq -1$ or $y \geq 1$; f_3 can be extended continuously by setting $f_3(i) = 0$.

Differentiation.

40. Show from the definition, and then using the Cauchy Riemann equations, that $f_1(z) = \operatorname{Im}(z)$ and $f_2(x+iy) = 3x + 4iy$ are nowhere differentiable.
41. Where are the following functions differentiable? Where are they analytic?
    (a) $f_1(z) = z|z|^2$;
    (b) $f_2(x+iy) = x^2 + iy^2$;
    (c) $f_3(x+iy) = (xy^2 + 5) + i(y - x^2y - 3)$;
    (d) $f_4(x+iy) = (3xy^2 + 6x^2 - 4x + 3) + i(3y^2 - 3x^2y + 2y + 15)$;
    (e) $f_5(z) = e^z(z - \bar{z})^2$;
    (f) $f_6(x+iy) = |x| + i|y|$.
    Answer: (a) f_1 is diffentiable at 0;
    (b) f_2 is differentiable on the line $x = y$;
    (c) f_3 is differentiable on the circle $x^2 + y^2 = 1$;
    (d) f_4 is differentiable on the circle $(x+2)^2 + (y-1)^2 = 1$;
    (e) f_5 is differentiable on the x axis, that is, $y = 0$;
    (f) f_6 is differentiable when $xy > 0$;
    f_1 to f_5 are differentiable on sets that are not open, so are not analytic, while f_6 is analytic when $xy > 0$.

* 42. Suppose that

$$f(z) = \begin{cases} e^{-z^{-4}} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0. \end{cases}$$

Show that the Cauchy–Riemann equations hold in $\mathbb{C}$. Is f entire?
Hint: You do not need to compute the partial derivatives, except at 0.
Answer: f is analytic in $\mathbb{C} \setminus \{0\}$, as there it is made up of differentiable functions, and so the Cauchy–Riemann equations hold in $\mathbb{C} \setminus \{0\}$. At 0,

$$\partial u/\partial x = \partial u/\partial y = \partial v/\partial x = \partial v/\partial y = 0;$$

hence the Cauchy–Riemann equations also hold at 0. However, f is not continuous at 0, and hence not differentiable at 0.

43. Show that real-valued entire functions are constant.
44. Show that if f and $g = \bar{f}$ are both differentiable in a domain, then f is constant on that domain.
45. Suppose that f is analytic in Ω, and f' is continuous; let $g(z) = \overline{f(\bar{z})}$. Is g analytic? Where?
    Answer: f satisfies the Cauchy–Riemann equations in Ω, and so g satisfies the Cauchy–Riemann equations in $\bar{\Omega}$. The partial derivatives are continuous, and $\bar{\Omega}$ is open, and so g is analytic in $\bar{\Omega}$
46. Suppose that $a \in \mathbb{R}$. Use the polar form of the Cauchy–Riemann equations to find where the function $f(z) = |z|^a \, e^{ia \operatorname{Arg}(z)}$ is differentiable.

Connections with multivariable calculus†.

47. Suppose that a linear transformation L of $\mathbb{R}^2$ preserves angles between lines passing through the origin. Show that L is the composition of a scalar multiplication and a rotation, and possibly a reflection.
48. A subset S of $\mathbb{C}$ is said to be *arcwise connected* if, given any two points p and q of S, there exists a continuous curve $\gamma : [0,1] \to S$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Show that if an open set is arcwise connected, then the continuous curve γ may be taken to be polygonal.
49. In the area of mathematics called General Topology, a set S is said to be connected if we cannot find two open sets Ω_1 and Ω_2 such that $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega_1 \neq \emptyset$, $\Omega_2 \neq \emptyset$ and $(\Omega_1 \cap S) \cup (\Omega_2 \cap S) = S$.
Show that an open set Ω is connected in this sense if and only if it is arcwise connected. What can you say about regions?

Harmonic functions.

50. Show that the function $u(x,y) = x^4 - 6x^2y^2 + y^4$ is harmonic and find a harmonic conjugate v. Express $f = u + iv$ as a function of z.
Answer: $v(x,y) = 4x^3y - 4xy^3 + C$, where C is a constant; $f(z) = z^4 + iC$.
51. Find all real a, b, c such that $u(x,y) = x^3 + ax^2y + bxy^2 + cy^3$ is harmonic on $\mathbb{R}^2$. For all such a, b, c determine $v(x,y)$ such that $f = u + iv$ is entire and write f as a function of z only.
Answer: $a = -3c$, $b = -3$, $v(x,y) = cx^3 + 3x^2y - 3cxy^2 - y^3$, $f(z) = (1+ic)z^3$.
52. Suppose u, v are harmonic and satisfy the Cauchy-Riemann equations in $\mathbb{R}^2$. Show that $f = u + iv$ satisfies $f'(x) = u_x(x,0) - iu_y(x,0)$ for real x.
53. Using the result of the previous question to guess f, or otherwise, show that the following are harmonic and find analytic functions f of which they are the real parts:
(a) $x - x^3 + 3xy^2$,
(b) $\cos x \cosh y$,
(c) $\dfrac{x}{x^2 + y^2}$, where $(x,y) \neq (0,0)$),
*(d) $\dfrac{x(x^2 + y^2 + 1)}{(x^2 + y^2)^2 + 2(x^2 - y^2) + 1}$, where $(x,y) \neq \pm(0,1)$,
(e) y.
Answer: (a) $z - z^3$; (b) $\cos(z)$; (c) $1/z$; (d) $\dfrac{z}{z^2+1}$; (e) iz.
54. Suppose that v is a harmonic conjugate of u. Show that $-u$ is a harmonic conjugate of v.
55. * Suppose that u is a harmonic function in an open set Ω, and define the function g on Ω by $g(z) = u_x(x,y) - iu_y(x,y)$, where $z = x + iy$. Show that g is holomorphic in Ω. If $f \in H(\Omega)$ and $f' = g$, what can you say about f and u?

Power series.

56. Find the centre and radius of convergence of the following power series; if possible, identify the sum:

(a) $\displaystyle\sum_{n=0}^{\infty}(z-2)^n$ (b) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n}(z-2)^n$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}(z-2)^n$ (d) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n\,n!}(z-2)^n$

(e) $\displaystyle\sum_{n=0}^{\infty}n(z-2)^n$ (f) $\displaystyle\sum_{n=0}^{\infty}n^2(z-2)^n$

(g) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}(z-2)^n$ (h) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}(z-2)^n.$

Exponential and related functions.

57. (a) Show that $|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 - |\sin z|^2 = \cos 2x$.
(b) Show that $|\cos z|^2 + |\sin z|^2 \geq 1$. When does equality hold?

58. Find all $z \in \mathbb{C}$ such that
(a) $e^z = -2$, (b) $\cos z = 2$, (c) $\cosh z = -5$, (d) $\exp(z^2) = 1$.
Answer: (c) $\pm\ln(5+2\sqrt{6}) + (2k+1)\pi i$, where $k \in \mathbb{Z}$.

59. Find the image of $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < \pi/2,\ \mathrm{Im}\, z > 0\}$ under the mappings
(a) $w = \cos z$, (b) $w = \cos^2 z$, (c) $w = \cos^{1/2} z$.
(Use the principal branch of the square root.)

60. Find the image of the region $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1/2,\ -\pi \leq \mathrm{Im}(z) \leq \pi\}$ under the mapping $w = e^z$.

61. Find the maximum value of $|e^{-z^2}|$ for $|z| \geq 5$ and $0 \leq \arg z \leq \pi/8$.
Answer: $\exp(-25/\sqrt{2})$

62. Find all solutions $z \in \mathbb{C}$ of (a) $\cosh z = -1$ (b) $\sin z = 3$ (c) $\sinh z = 2i$.

63. Solve (a) $\sinh z - \cosh z = 2i$, (b) $\cos z + \sin z = i$.
Answer: (a) $-\ln(2) + (2k + \frac{1}{2})\pi i$.

64. Where is the function $f(z) = (z^2+4)^{-1}\,\mathrm{Log}(z+2i)$ analytic?
Answer: f is analytic on the complement of the set $\{2i\} \cup \{x - 2i : x \leq 0\}$.

65. Find all values of (a) i^i, and (b) $\sin^{-1} 10$. Which are the principal values?
Answer: (b) $(2k+\frac{1}{2})\pi \pm i\log(10+3\sqrt{11})$; $\frac{1}{2}\pi - i\log(10+3\sqrt{11})$.

66. Find the principal value of $\left[\left(\dfrac{1+\sqrt{3}i}{2}\right)^{-3}\right]^{1-i}$.

67. Evaluate $\lim\limits_{z\to 0}(\cos z)^{1/z^2}$. (Use the principal branch of the power.)
Answer: $e^{-1/2}$, by l'Hôpital's rule.

68. Write $z = re^{i\theta}$. Find $\mathrm{Re}(z^i)$, $\mathrm{Im}(z^i)$, $|z^i|$ in terms of r and θ. (Use the principal value of z^i.)

69. Define $f(z) = z\,\mathrm{Log}(z)$ when $z \neq 0$, and $f(0) = 0$. Is f continuous or differentiable at 0?
Answer: It is continuous but not differentiable.

70. Define functions f and g by $f(z) = \mathrm{Log}(iz)$ and $g(z) = z^{-1}\,\mathrm{Log}(z+1)$.
(a) Where are f and g analytic?
(b) Find a branch of $\log(iz)$ which is analytic in the region $\{z : \mathrm{Im}(z) > 0\}$.
Answer: (a) f is analytic except when $z = iy$, where $y \geq 0$; while g is analytic except when $z = 0$ or $z = x$, where $x \leq -1$. (b) $i\pi/2 + \mathrm{Log}(z)$.

* 71. In this question, all square roots are the principal branch.
(a) Where is $\sqrt{z+1}$ analytic?
(b) Show that $\sqrt{z+1}\,\sqrt{z-1} = -\sqrt{z^2-1}$ when $\mathrm{Re}\, z < -1$.
(c) Show that $\sqrt{z^2-1}$ is analytic when $\mathrm{Re}\, z < -1$.
(d) Show that $\sqrt{z+1}\,\sqrt{z-1}$ is analytic on the complement of $[-1, 1]$.

* 72. Where is $\mathrm{Log}(z + z^{-1} - 2)$ analytic?
Answer: On the complement of the set $\{z : |z| = 1\} \cup (-\infty, 0]$.

Contour integrals.

73. (a) Evaluate
$$\int_\gamma z\,\mathrm{Im}(z^2)\,dz,$$
where γ is the unit circle traversed once, anticlockwise.
(b) Evaluate
$$\int_\gamma e^{|z|^2}\,\mathrm{Re}(z)\,dz,$$
where γ is the line segment from 0 to $1+i$.
Answer: (a) $-\pi$ (b) $(1+i)(e^2-1)/4$.

74. Evaluate $\displaystyle\int_\gamma \bar{z}\,dz$, where γ is
(a) the straight line from $-1+2i$ to $3+5i$
(b) the upper semicircle of unit radius from -1 to 1.
Answer: (a) $\frac{29}{2} - 11i$ (b) $-i\pi$.

75. (a) Find $\displaystyle\int_\gamma (e^z + z)\,dz$, where $\gamma(t) = 1 + i\pi t$ for $0 \leq t \leq 1$.
(b) Show that $\displaystyle\int_\gamma \mathrm{Log}\, z\,dz = -2i$, where $\gamma(t) = e^{i\pi t}$ for $-\frac{1}{2} \leq t \leq \frac{1}{2}$.

76. Show that $\displaystyle\int_\gamma \sec^2 z\,dz = 2i\,\tanh 1$, for every contour γ in the domain of sec that starts at $-i$ and ends at i.

77. Let γ be any contour from $1-i$ to $1+i$. Evaluate the following integrals:
(a) $\displaystyle\int_\gamma 4z^3\,dz$; (b) $\displaystyle\int_\gamma \cos z\,dz$; (c) $\displaystyle\int_\gamma \sin 2z\,dz$.

78. Let γ be the semi-circle from $2i$ to $-2i$ that passes through -2. Find
$$\int_\gamma z^{-1}\,dz$$

(a) from the definition; (b) by using a suitable branch of log as a primitive.
Answer: πi.

Cauchy's integral formula.

79. Evaluate the following integrals using the Cauchy Integral Formula:

$$\text{(a)} \int_{|z|=2} \frac{e^z\,dz}{z-1}; \qquad \text{(b)} \int_{|z|=4} \frac{e^{2iz}\,dz}{(3z-1)^2}; \qquad \text{(c)} \int_{|z|=2} \frac{(z^2+1)\,dz}{(z-3)(z^2-1)}.$$

(All contours are traversed once anti-clockwise.)
Answer: (a) $2\pi ei$ (b) $-4\pi e^{2i/3}/9$ (c) $-\pi i/2$.

80. Find

$$\int_\gamma \frac{\sin \pi z}{(z-1)^n}\,dz,$$

where $n \in \mathbb{Z}$ and γ is a simple closed anticlockwise contour around 1.
Answer: 0, unless $n = 2, 4, 6, 8, \ldots$; in this case, $(-1)^{n/2}\pi^{n-1}/(n-1)!$.

81. Let γ be the unit circle $\{e^{i\theta}:\ -\pi \le \theta \le \pi\}$. Use the Cauchy integral formula to find

$$\int_\gamma e^z\, z^n\,dz,$$

where $n \in \mathbb{Z}$. Hence evaluate the corresponding real integrals.

82. Define the semicircular arc γ_R by $\gamma_R(t) = Re^{it}$, where $0 \le t \le \pi$ and $R > 1$ is a real constant. Let γ be the join of γ_R and the line segment from $-R$ to R.
(a) Show that, if $z \in \text{Range}(\gamma_R)$, then

$$\left|\frac{e^{iz}}{z^2+1}\right| \le \frac{1}{R^2-1}.$$

(b) Deduce that

$$\left|\int_{\gamma_R} \frac{e^{iz}\,dz}{(z^2+1)}\right| \le \frac{\pi R}{R^2-1}.$$

(c) Evaluate

$$\int_\gamma \frac{e^{iz}\,dz}{(z^2+1)}.$$

(d) Hence find

$$\int_0^\infty \frac{\cos x\,dx}{(x^2+1)}.$$

83. Let $f(z)$ be an entire function that satisfies the inequality $|f(z)| \le 1 + |z|$ for all $z \in \mathbb{C}$. Show that $f(z) = az + b$ for fixed complex numbers a and b.
Hint: Apply the generalized Cauchy integral formula on an arbitrarily large circle to show that $f^{(n)}(0) = 0$ when $n \ge 2$.

* 84. By integrating $f(z) = e^{-z^2}$ round the boundary of the rectangle with corners at a, $a+ib$, $-a+ib$ and $-a$, where $b > 0$, show that

$$\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

The right hand side is $e^{-b^2}\sqrt{\pi}$.

* 85. By integrating $f(z) = (1+z^2)^{-1}$ around the perimeter of the rectangle with corners at 0, a, $a+ib$ and ib and letting $a \to \infty$, where $0 < b < 1$, show that

$$\int_0^\infty \frac{1-b^2+x^2}{(1-b^2+x^2)^2+4b^2x^2}\,dx = \frac{\pi}{2},$$

and

$$\int_0^\infty \frac{x}{(1-b^2+x^2)^2+4b^2x^2}\,dx = \frac{1}{4b}\operatorname{Log}\frac{1+b}{1-b}.$$

* 86. Apply Cauchy's theorem to the function $f(z) = e^{-z^2}$ and the sector of the circle $|z| = R$ bounded by a section of the real axis and the linear segment making an angle $\pi/4$ with the real axis. Show that the integral over the circular boundary tends to zero as $R \to \infty$ and prove that

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Note: These so-called *Fresnel Integrals* are used in diffraction problems.

Taylor series, Laurent series, and residues.

87. Expand each of the following functions in a Taylor series about the given point z_0 and determine the radius of convergence: (a) $f(z) = e^{-z}$, $z_0 = 1$; (b) $f(z) = (z+2)e^{3z}$, $z_0 = 0$; (c) $f(z) = \dfrac{1}{z^2-5z+4}$, $z_0 = 2$.

88. Show that $\dfrac{1}{1+z+z^2} = \displaystyle\sum_{n=0}^{\infty}\left(z^{3n} - z^{3n+1}\right)$ in the disc $|z| < 1$.

89. Without finding the series, state the radius of convergence of the Taylor series for the function f given by $f(z) = \dfrac{\cos z}{z(z+1)}$ about the point $z = 1+i$.

90. Find the Taylor series for $f(z) = (1+z)^{1/2}$ in powers of z in the disc $|z| < 1$. Use the branch of the square root that is equal to $+1$ when $z = 0$.
Answer: $\displaystyle\sum_{n=0}^{\infty} c_n z^n$, where $c_0 = 1$, $c_1 = \frac{1}{2}$ and $c_n = \dfrac{(-1)^{n+1}}{2^n}(2n-3)!!$ when $n > 1$. Here, for instance, $9!! = 9 \times 7 \times 5 \times 3 \times 1$.

* 91. Show that $\tan^{-1} w = \dfrac{1}{2i}\log\dfrac{1+iw}{1-iw}$ (as multi-valued functions). Define

$$\operatorname{pv}\tan^{-1} w = \frac{1}{2i}\operatorname{Log}\frac{1+iw}{1-iw}.$$

Find all points w where $\operatorname{pv}\tan^{-1} w$ is differentiable, and show that, when $|w| < 1$,

$$\operatorname{pv}\tan^{-1} w = w - \frac{w^3}{3} + \frac{w^5}{5} - \cdots.$$

Answer: The function $\operatorname{pv}\tan^{-1}$ is differentiable in $\mathbb{C} \setminus \{it : t \in \mathbb{R}, |t| \geq 1\}$.

* 92. Define the function f by $f(z) = \dfrac{z}{2} + \dfrac{z}{(e^z-1)}$ for $z \neq 0$ and $f(0) = 1$. Show that f has a Maclaurin series of the form $a_0 + a_2z^2 + a_4z^4 + \ldots$, and find a_0, a_2 and a_4.

Note: The numbers B_{2n} such that $a_{2n} = (-1)^{n+1}B_{2n}/n!$ are known as the Bernoulli numbers.

93. Find two distinct Laurent expansions for $\dfrac{1}{z^2(1-z)}$ in powers of z and state the region of convergence of each expansion.
Answer: $\displaystyle\sum_{n=-2}^{\infty} z^n$ when $0 < |z| < 1$, and $\displaystyle\sum_{n=-\infty}^{-3} (-1)z^n$ when $|z| > 1$.

94. Define $f(z) = \dfrac{2z-4}{z^2-4z+3}$. Find the Laurent series for $f(z)$ that converges when $|z-1| > 2$.
Answer: $\displaystyle\sum_{n=-\infty}^{-1} c_n(z-1)^n$, where $c_{-1} = 2$ and $c_n = 2^{-n-1}$ when $n \leq -2$.

95. Define $f(z) = (z-2i)^{-1} - (z+i)^{-1}$. Find the Laurent series for f with centre 1 that converges at 3.

96. Define $f(z) = \operatorname{cosec} z$. Find the Laurent series for f up to terms in z^3 about $z = 0$ that is valid at $\frac{1}{2}$, and state the region of convergence.
Answer: $z^{-1} + \frac{1}{6}z + \frac{7}{360}z^3 + \dots$.

97. Define the function f by $f(z) = \dfrac{3}{(z-2)(z+1)}$. Find all Laurent expansions of f in powers of $z-1$ and state the region of convergence of each expansion.

98. Find the Laurent expansion of $\dfrac{1}{(z-1)(z-2)^2}$ that holds when $0 < |z-1| < 1$.

99. Show that the function $f(z) = \operatorname{cosec}(1/z)$ does *not* have a Laurent series about 0 that converges to f in $B^\circ(0,r)$ for any $r > 0$.

100. Suppose that $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \displaystyle\sum_{n=0}^{\infty} b_n z^n$, and the series converge when $|z| < 2$. By integrating $f(z)\,g(z^{-1})\,z^{-1}$ around the unit circle, show that

$$\sum_{n=0}^{\infty} a_n b_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,g(e^{-i\theta})\,d\theta.$$

Deduce that

$$\sum_{n=0}^{\infty} a_n \bar{b}_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\bar{g}(e^{i\theta})\,d\theta.$$

What if the series are Laurent series that converge when $\frac{1}{2} < |z| < 2$?

101. Find the residues of each of the following functions at each of their singularities:
(a) $\dfrac{e^z}{z^2+1}$; (b) $\dfrac{\sinh z}{(z-i)^3}$; (c) $\dfrac{z^2-3z+1}{(z^2-1)(z-1)}$;
(d) $\dfrac{e^z}{\cosh z}$; (e) $\tan z$.
Answer: (a) $z = i$, $-ie^i/2$; $z = -i$, $ie^{-i}/2$; (b) $z = i$, $(i\sin 1)/2$.
(c) $z = -1$, $\frac{5}{4}$; $z = 1$, $-\frac{1}{4}$; (d) $z = \frac{i\pi}{2} + k\pi i$, 1; (e) $z = \frac{\pi}{2} + k\pi$, -1.

102. Locate and classify the singularities of the following functions:
(a) $\dfrac{1}{z^2 \sin z}$; (b) $\dfrac{z(z-\pi)^2}{\sin^2 z}$; (c) $\exp(z+\dfrac{1}{z})$.
Evaluate the residues at each singularity of these functions (leave your answer to (iii) as a series).
Answer: (a) $z=0$: pole of order 3, residue $1/6$ and $z=k\pi$, $k\neq 0$: simple poles, residue $(-1)^k(k\pi)^{-2}$ (b) $z=0$, simple pole, residue π^2, $z=\pi$, removable singularity, residue 0, and $z=k\pi$, $k\neq 0$ or 1, double poles, residue $(3k^2-4k+1)\pi^2$. (c) $z=0$: essential singularity, residue $\sum_{k=0}^{\infty} 1/(k!\,(k+1)!)$.

103. Find $\displaystyle\int_\Gamma \frac{\cos z}{z^2(z^2+1)}\,dz$ (a) when Γ is the circle $|z|=2$ and (b) when Γ is the circle $|z-\frac{1}{2}i|=1$, both described once anticlockwise.

104. Find $\displaystyle\int_\Gamma \frac{z^3+5}{z(z-1)^3}\,dz$ when Γ is the circle $|z|=2$ described once anticlockwise.
Answer: $2\pi i$

105. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$ when $a,b>0$.

106. Use complex analysis methods to show that
(a) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{\pi}{6}$; (b) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5+3\cos\theta}\,d\theta = \frac{2\pi}{9}$.

107. Find $I_1 = \displaystyle\int_0^{2\pi} \frac{\cos 6\theta\,d\theta}{5+4\sin\theta}$ and $I_2 = \displaystyle\int_0^{2\pi} \frac{\sin 6\theta\,d\theta}{5+4\sin\theta}$, by first computing I_1+iI_2.

108. Evaluate $\displaystyle\int_{|z|=1} \frac{z^2\,dz}{2z+1}$ and hence prove that $\displaystyle\int_0^{\pi} \frac{(2\cos 2\theta+\cos 3\theta)\,d\theta}{5+4\cos\theta} = \frac{\pi}{8}$.

* 109. Suppose that p and q are positive integers and $p>q+1$. By integrating round the boundary of the domain $\{z\in\mathbb{C} : |z|<R,\ 0<\arg(z)<\frac{2\pi}{p}\}$, show that

$$\int_0^{\infty} \frac{x^q}{x^p+1}\,dx = \frac{\pi}{p}\operatorname{cosec}\frac{\pi(q+1)}{p}.$$

* 110. By evaluating suitable contour integrals, and taking suitable limits, show that:
(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\ln(x^2+1)}{x^2+1}dx = \pi\ln 4$; (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx = \pi(1-e^{-1})$;
(c) $\displaystyle\int_0^{\infty} \frac{x^{3/4}}{x^2+x+1}\,dx = \pi\frac{2}{\sqrt{3}}$.

Theory of functions*.

111. Let f be an entire function, and take $a,b\in\mathbb{C}$. For $R>\max\{|a|,|b|\}$, estimate

$$\int_{|z|=R} \frac{f(z)\,dz}{(z-a)(z-b)}.$$

Assuming also that f is bounded, let $R\to\infty$ and show that f is constant.

112. Suppose that f is an entire function, and that there is a (nonempty) open ball $B(w_0,r)$ such that $w\notin \operatorname{Range}(f)$ for all $w\in B(w_0,r)$. Show that f is constant.
Hint: Consider the function $1/(f(z)-w_0)$.

113. Suppose that f is a holomorphic function on $\mathbb{C} \setminus \{z_1, \dots, z_M\}$, that f has a pole of order N_m at z_m, where $1 \le m \le M$, and that $\lim_{z\to\infty} f(z) = 0$.
 (a) Explain why f has a Laurent series of the form

$$f(z) = \sum_{n=-N}^{\infty} c_{m,n}(z - z_m)^n$$

 in a punctured ball $B^\circ(z_m, r_m)$ for some sufficiently small positive r_m. What is the largest possible value for r_m?
 (b) Define $f_m(z) = \sum_{n=-N_m}^{-1} c_{m,n}(z - z_m)^n$. Explain why $f - f_m$ is holomorphic in $B^\circ(z_m, r_m)$.
 (c) Explain why $f - (f_1 + \cdots + f_M)$ is entire and vanishes at infinity.
 (d) Explain why $f = f_1 + \cdots + f_M$.
 (e) Find a formula for the coefficient $c_{m,n}$ similar to the formula for the residue of a function at a pole of order N.
 (f) Can you find a holomorphic function F on $\mathbb{C} \setminus \{z_1, \dots, z_M\}$ such that $F' = f$? What if the domain of F is allowed to be smaller?

114. Suppose that f is holomorphic in an annulus $\mathrm{A}(R) = \{z \in \mathbb{C} : |z| > R\}$, and define $g(z) = f(1/z)$.
 (a) Show that g has an isolated singularity at 0.
 (b) Show that the singularity of g at 0 is a pole if and only if given $R' > R$, there are constants C and N such that $|f(z)| \le C\,|z|^N$ for all $z \in \mathrm{A}(R')$.
 (c) Show that if $|f(z)| \le C\,|z|^N$ for all $z \in \mathrm{A}(R)$, then there is a polynomial p such that $f - p$ is bounded in $\mathrm{A}(R)$.

Counting roots of equations.

115. How many roots of $z^4 - 8z + 10 = 0$ lie in the annulus $1 < |z| < 3$? How many of these lie in the first quadrant?
 Answer: Four; one.

116. How many roots of $z^9 - 2z^6 + z^2 - 8z - 2 = 0$ lie inside the unit circle?
 Answer: One.

117. Let $n \in \mathbb{Z}^+$. Show that $e^z - 4z^n + 1 = 0$ has n roots inside the unit circle.

118. How many roots has $2z^4 - 3z^3 + 3z^2 - z + 1 = 0$ in each quadrant?
 Answer: One in each.

119. How many roots of $z^5 + z^4 + 2z^3 - 8z - 1 = 0$ also satisfy $\operatorname{Re} z > 0$.
 Answer: One.

120. Find the number of solutions of $z^7 - 2z - 5 = 0$ that satisfy $\operatorname{Re} z < 0$.
 Answer: Four.

More on integrals.

* 121. For a complex number z such that $\operatorname{Re}(z) > 1$, consider the improper integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

 (a) Show that the improper integral converges.
 (b) Assuming that it is legitimate to exchange the order of differentiation and integration, show that Γ is holomorphic in $\{z \in \mathbb{C} : \operatorname{Re}(z) > 1\}$.

(c) Show that, if $\mathrm{Re}(z) > 1$, then

$$\Gamma(z+1) = z\,\Gamma(z).$$

(d) Use induction and the formula

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1}$$

to define $\Gamma(z)$ in $\{z \in \mathbb{C} : \mathrm{Re}(z) > -N\}$, where N is a nonnegative integer. Where are the singularities of Γ? What kind of singularites are they?

122. Find the integral $\displaystyle\int_0^\infty \frac{x^q}{x^p+1}\,dx$, where $p, q \in \mathbb{R}^+$ and $p > q+1$, without using complex methods.

Fourier and Laplace transforms.

123. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 - |x|/2 & \text{if } |x| \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the Fourier transform of f, and show that $f, \widehat{f} \in \mathcal{M}(\mathbb{R})$. Hence, using the inversion formula, show that

$$\int_{-\infty}^{\infty} \frac{\sin^2(\xi)}{\xi^2}\,d\xi = 2\pi.$$

124. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the Fourier transform of f. Assuming that the inversion formula holds for f and $\widehat{f}$, compute

$$\int_{-\infty}^{\infty} \frac{\sin(\xi)}{\xi}\,d\xi.$$

125. Compute the Laplace transforms of each of the following functions, stating the domain of the transform:

(a) $f(t) = e^{-at}$ (b) $f(t) = \cos(\omega t)$ (c) $f(t) = t(t^2-1)$,

where $a \in \mathbb{C}$ and $\omega \in \mathbb{R}$.

126. Find the functions whose Laplace transforms are given by

(a) $\dfrac{1}{z(z+1)}$ (b) $\dfrac{z+1}{z^2(z-1)}$ (c) $\dfrac{1}{z^4+1}$ (d) $\dfrac{1}{(z^2+1)^2}$.

127. Use the Laplace transform to solve the differential equation

$$\frac{d^2u}{dt^2} + 4\frac{du}{dt} + 3u = e^{-2t},$$

where $u : [0, \infty) \to \mathbb{R}$ satisfies the initial conditions $u(0) = 0$ and $\dfrac{du}{dt}(0) = 0$.

128. Use the Laplace transform to find the solutions $u, v : [0, \infty) \to \mathbb{R}$ to the integral equations

$$u(t) = e^{-t} + \int_0^t (t-s)\, e^{s-t}\, u(s)\, ds.$$

and

$$v(t) = e^{-t} + \int_0^t (t-s)^2\, v(s)\, ds.$$

Conformal mappings.

* 129. Suppose that $u \in \mathbb{R} \setminus \{0\}$ and $v \in \mathbb{R}$. Show that as y varies between 0 and $\pi/2$, the expression

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} - 1$$

is increasing and changes sign, hence by the Intermediate Value Theorem there exists y for which it is 0. Now consider the mapping $w = \cosh(z)$. Show that the region $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0,\ 0 < \mathrm{Im}(z) < \pi\}$ is mapped bijectively and conformally onto $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Find a formula for the inverse function.

* 130. Show that a bijective conformal mapping defined on the whole complex plane $\mathbb{C}$ is affine, and that a bijective conformal mapping defined on $\mathbb{C} \setminus \{p\}$, where $p \in \mathbb{C}$, is a fractional linear transformation.