CHAPTER **TWO**

Numbers and Functions

The principal purpose of this course is the study of functions of real numbers, and the first task is therefore to make it clear what numbers are and what functions are. The first five sections of this chapter review the four number systems, with particular attention to the arithmetic of surds. The last five sections develop the idea of functions and relations and their graphs, with a review of known graphs, and a discussion of various ways in which the graph of a known function or relation can be transformed, allowing a wide variety of new graphs to be obtained.

Study Notes: Although much of the detail here may be familiar, the systematic exposition of numbers and functions in this chapter will be new and demanding for most pupils. Understanding is vital, and the few proofs that do occur are worth emphasising. In the work on surds, the exact value of the number must constantly be distinguished from its decimal approximation produced on the calculator. In the work on functions, computer sketching can make routine the understanding that a function has a graph — an understanding fundamental for the whole course but surprisingly elusive — and computers are particularly helpful in understanding transformations of graphs and how they can be effected algebraically, because a large number of similar examples can be examined in a short time. Nevertheless, pupils must eventually be able to construct a graph from its equation on their own.

2 A Cardinals, Integers and Rational Numbers

Our experience of numbers arises from the two quite distinct fields of counting and geometry, and we shall need to organise these contrasting insights into a unified view. This section concerns the cardinal numbers, the integers and the rational numbers, which are all based on counting.

The Cardinal Numbers: Counting things requires the numbers 0, 1, 2, 3, ... These numbers are called the *cardinal numbers*, and the symbol $\mathbf{N}$ is conventionally used for the set of all cardinal numbers.

1 Definition: $\mathbf{N} = \{\,\text{cardinal numbers}\,\} = \{\,0,\ 1,\ 2,\ 3,\ \ldots\,\}$

This is an infinite set, because no matter how many cardinal numbers are listed, there will always be more. The number 0 is the smallest cardinal, but there is no largest cardinal, because given any cardinal n, the cardinal $n + 1$ is bigger.

Closure of N: If two cardinals a and b are added, the sum $a + b$ and the product ab are still cardinals. We therefore say that the set $\mathbf{N}$ of cardinals is *closed* under

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addition and multiplication. But the set of cardinals is not closed under either subtraction or division. For example,

$$5 - 7 \text{ is not a cardinal} \qquad \text{and} \qquad 6 \div 10 \text{ is not a cardinal.}$$

Divisibility — HCF and LCM: Division of cardinals sometimes does result in a cardinal. A cardinal a is called a *divisor* of the cardinal b if the quotient $b \div a$ is a cardinal. For example,

$$\{\text{ divisors of } 24\,\} = \{\,1, 2, 3, 4, 6, 8, 12, 24\,\}$$
$$\{\text{ divisors of } 30\,\} = \{\,1, 2, 3, 5, 6, 10, 15, 30\,\}$$

The *highest common factor* or HCF of two or more cardinals is the largest cardinal that is a divisor of each of them, so the HCF of 24 and 30 is 6. The key to cancelling a fraction down to its *lowest terms* is dividing the numerator and denominator by their HCF:

$$\frac{24}{30} = \frac{24 \div 6}{30 \div 6} = \frac{4}{5}$$

If a is a divisor of b, then b is a *multiple* of a. For example,

$$\{\text{ multiples of } 24\,\} = \{\,24, 48, 72, 96, 120, 144, \ldots\,\}$$
$$\{\text{ multiples of } 30\,\} = \{\,30, 60, 90, 120, 150, \ldots\,\}$$

The *lowest common multiple* or LCM of two or more cardinals is the smallest positive cardinal that is a multiple of each of them, so the LCM of 24 and 30 is 120. The key to adding and subtracting fractions is finding the LCM of their denominators, called the *lowest common denominator*:

$$\frac{5}{24} + \frac{7}{30} = \frac{5 \times 5}{120} + \frac{7 \times 4}{120} = \frac{53}{120}$$

Prime Numbers: A *prime number* is a cardinal number greater than 1 whose only divisors are itself and 1. The primes form a sequence whose distinctive pattern has confused every mathematician since Greek times:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, \ldots$$

A cardinal greater than 1 that has factors other than itself and 1 is called a *composite number*, and without giving its rather difficult proof, we shall assume the 'unique factorisation theorem':

2 THEOREM: Every positive cardinal number can be written as a product of prime numbers in one and only one way, apart from the order of the factors.

So, for example, $24 = 2^3 \times 3$, and $30 = 2 \times 3 \times 5$. This theorem means that as far as multiplication is concerned, the prime numbers are the building blocks for all cardinal numbers, no matter how big or complicated they might be. (No primes divide 1, and so the factorisation of 1 into primes requires the qualification that a product of no factors is 1.)

The Greeks were able to prove that there are infinitely many prime numbers, and the proof of this interesting result is given here because it is a clear example of 'proof by contradiction', where one assumes the theorem to be false and then works towards a contradiction.

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3 THEOREM: There are infinitely many prime numbers.

PROOF: Suppose, by way of contradiction, that the theorem were false.
Then there would be a finite list $p_1, p_2, p_3, \ldots p_n$ of all the primes.
Form the product $N = p_1 p_2 p_3 \ldots p_n$ of all of them.
Then $N + 1$ has remainder 1 after division by each of the primes $p_1, p_2, p_3, \ldots p_n$.
So $N + 1$ is not divisible by any of the primes $p_1, p_2, p_3, \ldots p_n$.
So the prime factorisation of $N + 1$ must involve primes other than $p_1, p_2, p_3, \ldots p_n$.
But $p_1, p_2, p_3, \ldots p_n$ is supposed to be a complete list of primes.
This is a contradiction, so the theorem is true.

In case you are tempted to think that all theorems about primes are so easily proven, here is the beginning of the list of *prime pairs*, which are pairs of prime numbers differing by 2:

3, 5; 5, 7; 11, 13; 17, 19; 29, 31; 41, 43; 59, 61; 71, 73; ....

No-one has yet been able to prove either that this list of prime pairs is finite, or that it is infinite. Computers cannot answer this question, because no computer search for prime pairs could possibly establish whether this list of prime pairs terminates or not.

The Integers: The desire to give a meaning to calculations like $5 - 7$ leads to *negative numbers* $-1, -2, -3, -4, \ldots$, and the positive and negative numbers together with zero are called the *integers*, from 'integral' meaning 'whole'. The symbol **Z** (from the German word *zahlen* meaning *numbers*) is conventionally used for the set of integers.

4 DEFINITION: $\mathbf{Z} = \{\text{integers}\} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$

This set **Z** is another infinite set containing the set **N** of cardinal numbers. There is neither a greatest nor a least integer, because given any integer n, the integer $n + 1$ is greater than n, and the integer $n - 1$ is less than n.

Closure of Z: The set **Z** of integers is closed not only under addition and multiplication, but also under subtraction. For example,

$$7 + (-11) = -4 \qquad (-8) \times 3 = -24 \qquad (-16) - (-13) = -3$$

but the set is still not closed under division. For example, $12 \div 10$ is not an integer.

The Rational Numbers: The desire to give meaning to a calculation like 'divide 7 into 3 equal parts' leads naturally to fractions and the system of *rational numbers*. Positive rational numbers were highly developed by the Greeks, for whom ratio was central to their mathematical ideas.

5 DEFINITION: A *rational number* is a number that can be written as a *ratio* or fraction a/b, where a and b are integers and $b \neq 0$:

$$\mathbf{Q} = \{\text{rational numbers}\} \quad \text{(Q stands for quotient.)}$$

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For example, $2\frac{1}{2} = \dfrac{5}{2}$, $-\dfrac{1}{3} = \dfrac{-1}{3}$, $30 \div 24 = \dfrac{5}{4}$, $4 = \dfrac{4}{1}$ and $-7 = \dfrac{-7}{1}$.
Because every integer a is also a fraction $a/1$, the set $\mathbf{Q}$ of rational numbers contains the set $\mathbf{Z}$ of integers. So we now have three successively larger systems of numbers, $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q}$.

Lowest Terms: Multiplication or division of the numerator and denominator by the same nonzero number doesn't change the value of a fraction. So division of the numerator and denominator by their HCF always cancels a fraction down to lowest terms, in which the numerator and denominator have no common factor greater than 1. Multiplying the numerator and denominator by -1 reverses both their signs, so the denominator can always be made positive.

6 A STANDARD FORM: Every rational number can be written in the form a/b, where a and b are integers with highest common factor 1, and $b \geq 1$.

Closure of Q: The rational numbers are closed under all four operations of addition, multiplication, subtraction and division (except by 0). The *opposite* of a rational number a/b is obtained by taking the opposite of either the numerator or denominator. The sum of a number and its opposite is zero:

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b} \qquad \text{and} \qquad \frac{a}{b} + \left(\frac{-a}{b}\right) = 0.$$

The *reciprocal* (inverse is not the correct word) of a nonzero rational number a/b is obtained by exchanging the numerator and denominator. The product of a number and its reciprocal is 1:

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \quad \left(\text{or } \frac{1}{a/b} = \frac{b}{a}\right) \qquad \text{and} \qquad \frac{a}{b} \times \frac{b}{a} = 1.$$

The reciprocal x^{-1} of a rational number x is analogous to its *opposite* $-x$:

$$x \times x^{-1} = 1 \qquad \text{and} \qquad x + (-x) = 0.$$

Terminating and Recurring Decimals: A *terminating decimal* is an alternative notation for a rational number that can be written as a fraction with a power of 10 as the denominator:

$$1\tfrac{2}{5} = \tfrac{14}{10} = 1{\cdot}4 \qquad 3\tfrac{1}{8} = \frac{3125}{1000} = 3{\cdot}125 \qquad 578\tfrac{3}{50} = 578 + \tfrac{6}{100} = 578{\cdot}06$$

Other than this rather narrow purpose, decimals are useful for approximating numbers so that they can be compared or placed roughly on a number line. For this reason, decimals are used when a quantity like distance or time is being physically measured — the act of measuring can never produce an exact answer.

A rational number that cannot be written with a power of 10 as its denominator can, however, be written as a *recurring decimal*, in which the digits to the right of a certain point cycle endlessly. In the division process, the cycling begins when a remainder occurs which has occurred before.

$$\tfrac{2}{3} = 0{\cdot}666\,666\ldots = 0{\cdot}\dot{6} \qquad 6\tfrac{3}{7} = 6{\cdot}428\,571\,428\,571\ldots = 6{\cdot}\dot{4}2857\dot{1}$$
$$13\tfrac{10}{11} = 13{\cdot}909\,09\ldots = 13{\cdot}\dot{9}\dot{0} \qquad 24\tfrac{35}{44} = 24{\cdot}795\,454\,545\,45\ldots = 24{\cdot}79\dot{5}\dot{4}$$

Conversely, every recurring decimal can be written as a fraction, by the following method.

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WORKED EXERCISE: Write each recurring decimal as a fraction in lowest terms:

(a) $0{\cdot}\dot{5}\dot{1}$ (b) $7{\cdot}3\dot{4}8\dot{6}$

SOLUTION:

(a) Let $x = 0{\cdot}\dot{5}\dot{1}$.

Then $x = 0{\cdot}515\,151\ldots$

$\boxed{\times 100}$ $100x = 51{\cdot}515\,151\ldots$

Subtracting the last two lines,

$$99x = 51$$

$$x = \tfrac{17}{33}.$$

So $0{\cdot}\dot{5}\dot{1} = \tfrac{17}{33}$.

(b) Let $x = 7{\cdot}3\dot{4}8\dot{6}$.

Then $x = 7{\cdot}348\,648\ldots$

$\boxed{\times 1000}$ $1000x = 7348{\cdot}648\,648\ldots$

Subtracting the last two lines,

$$999x = 7341{\cdot}3$$

$$x = \tfrac{73413}{9990}.$$

So $7{\cdot}3\dot{4}8\dot{6} = \tfrac{2719}{370}$.

7 **METHOD:** If the cycle length is n, multiply by 10^n and subtract.

NOTE: Some examples in the exercises below show that every terminating decimal has an alternative representation as a recurring decimal with endlessly cycling 9s. For example,

$$1 = 0{\cdot}\dot{9} \qquad 7 = 6{\cdot}\dot{9} \qquad 5{\cdot}2 = 5{\cdot}1\dot{9} \qquad 11{\cdot}372 = 11{\cdot}371\dot{9}$$

Exercise 2A

1. Find all primes: (a) less than 100, (b) between 150 and 200.

2. Find the prime factorisations of:

(a) 24 (b) 60 (c) 72 (d) 126 (e) 104 (f) 135 (g) 189 (h) 294 (i) 315 (j) 605

3. Find the HCF of the numerator and denominator of each fraction, then express the fraction in lowest terms:

(a) $\dfrac{72}{64}$ (b) $\dfrac{84}{90}$ (c) $\dfrac{78}{104}$ (d) $\dfrac{112}{144}$ (e) $\dfrac{168}{216}$ (f) $\dfrac{294}{315}$

4. Find the LCM of the two denominators, and hence express as a single fraction:

(a) $\dfrac{1}{8} + \dfrac{1}{12}$ (b) $\dfrac{5}{18} - \dfrac{2}{15}$ (c) $\dfrac{3}{8} - \dfrac{13}{36}$ (d) $\dfrac{37}{42} + \dfrac{23}{30}$ (e) $\dfrac{55}{72} - \dfrac{75}{108}$ (f) $\dfrac{7}{60} + \dfrac{31}{78}$

5. Express each number as a recurring or terminating decimal. Do not use a calculator.

(a) $\frac{5}{8}$ (b) $\frac{2}{3}$ (c) $\frac{7}{16}$ (d) $\frac{5}{9}$ (e) $\frac{3}{20}$ (f) $\frac{7}{12}$ (g) $4\frac{16}{25}$ (h) $5\frac{4}{11}$ (i) $\frac{23}{8}$ (j) $\frac{17}{6}$

6. Express each decimal as a rational number in lowest terms:

(a) $0{\cdot}15$ (b) $0{\cdot}\dot{7}$ (c) $0{\cdot}108$ (d) $0{\cdot}\dot{1}\dot{8}$ (e) $3{\cdot}12$ (f) $5{\cdot}\dot{4}\dot{5}$ (g) $1{\cdot}\dot{6}$ (h) $1{\cdot}\dot{2}\dot{1}$ (i) $0{\cdot}2\dot{1}$ (j) $6{\cdot}5\dot{3}$

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DEVELOPMENT

7. Express each of the following as a recurring or terminating decimal:

(a) $\frac{11}{6}$ (b) $\frac{13}{12}$ (c) $\frac{7}{15}$ (d) $\frac{16}{37}$ (e) $\frac{2}{27}$ (f) $\frac{13}{24}$ (g) $3\frac{1}{7}$ (h) $1\frac{3}{14}$ (i) $\frac{27}{13}$ (j) $1\frac{5}{21}$

8. Express each decimal as a rational number in lowest terms:

(a) $0{\cdot}\dot{7}\dot{5}$ (b) $1{\cdot}\dot{0}3\dot{7}$ (c) $4{\cdot}\dot{5}6\dot{7}$ (d) $0{\cdot}\dot{4}35\dot{6}$ (e) $1{\cdot}\dot{9}$ (f) $2{\cdot}4\dot{9}$ (g) $1{\cdot}5\dot{2}$ (h) $2{\cdot}3\dot{4}\dot{5}$ (i) $7{\cdot}13\dot{8}$ (j) $0{\cdot}11\dot{3}\dot{6}$

9. Write down the recurring decimals for $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$. What is the pattern?

10. Find the prime factorisation of the following numbers and hence determine the square root of each by halving the indices:

(a) 256 (b) 576 (c) 1225 (d) 1936

11. Write the HCF and LCM of each pair of numbers in prime factor form:

(a) 792 and 1188 (b) 1183 and 1456 (c) 2646 and 3087 (d) 3150 and 5600

12. (a) In order to determine whether a given number is prime, it must be tested for divisibility by smaller primes. Given a number between 200 and 250, which primes need to be tested?

(b) A student finds that none of the primes less than 22 is a factor of 457. What can be said about the number 457?

(c) Which of 247, 329, 451, 503, 727 and 1001 are primes?

EXTENSION

13. Prove that if x is the HCF of a and b, then x must be a factor of $a - b$.

14. The factors of a perfect number, other than itself, add up to that number.
(a) Show that 28 is a perfect number. (b) Euclid knew that if $2^n - 1$ is a prime number, then $2^{n-1}(2^n - 1)$ is a perfect number. Test this proposition for $n = 2$, 3, 4 and 5.

15. (a) Evaluate $\frac{1}{3}$ as a decimal on your calculator.

(b) Subtract $0{\cdot}333\,333\,33$ from this, multiply the result by 10^8 and then take the reciprocal.

(c) Show arithmetically that the final answer in part (b) is 3. Is the answer on your calculator also equal to 3? What does this tell you about the way fractions are stored on a calculator?

16. Two numbers m and n are called *relatively prime* if the HCF of m and n is 1. The Euler function $\phi(n)$ of n is the number of integers less than or equal to n that are relatively prime to n.

(a) Confirm the following by listing the integers that are relatively prime to the given number:

(i) $\phi(9) = 6$ (ii) $\phi(25) = 20$ (iii) $\phi(32) = 16$ (iv) $\phi(45) = 24$

(b) It is known that $\phi(p^k) = p^k - p^{k-1}$ for a prime p and a positive integer k. Show that this is true for $p = 2$ and $k = 1, 2, 3, 4$.

(c) Prove that $\phi(3^n) = 2 \times 3^{n-1}$. Generalise this result to $\phi(p^n)$, where p is prime and n is a positive integer.

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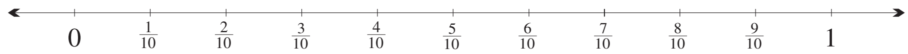
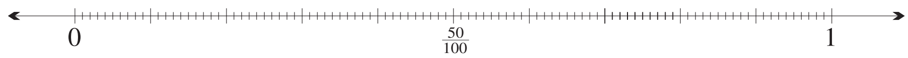
2 B The Real Numbers

Whereas the integers are regularly spaced 1 unit apart, the rational numbers are packed infinitely closely together. For example, between 0 and 1 there are 9 rationals with denominator 10:



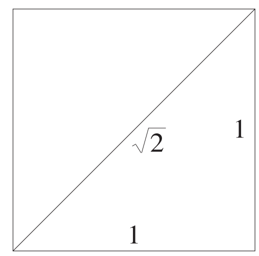
and there are 99 rationals with denominator 100 between 0 and 1:



and so on, until the rationals are spread 'as finely as we like' along the whole number line.

8 THE RATIONAL NUMBERS ARE DENSE: This means that within any small interval on the number line, there are infinitely many rational numbers.

There are Numbers that are Not Rational: Although the rational numbers are dense, there are many more numbers on the number line. In fact 'most numbers' are not rational. In particular, some of the most important numbers you will meet in this course are *irrational*, like $\sqrt{2}$, π and the number e, which we will define later. The proof by contradiction that $\sqrt{2}$ is irrational was found by the Greeks, and the result is particularly important, since $\sqrt{2}$ arises so easily in geometry from the very simple and important process of constructing the diagonal of the unit square. It is a surprising result, and shows very clearly how fractions cannot form a number system sufficient for studying geometry.



9 THEOREM: The number $\sqrt{2}$ is irrational.

PROOF: Suppose, by way of contradiction, that $\sqrt{2}$ were rational.
Then $\sqrt{2}$ could be written as

$$\sqrt{2} = \frac{a}{b}, \quad \text{where } a \text{ and } b \text{ are integers with no common factors, and } b \geq 1.$$

Multiplying both sides by b and then squaring both sides gives

$$a^2 = 2b^2.$$

Since $2b^2$ is even, then the left-hand side a^2 must also be even.
Hence a must be even, because if a were odd, then a^2 would be odd.
So $a = 2k$ for some integer k, and so $a^2 = 4k^2$ is divisible by 4.
So the right-hand side $2b^2$ is divisible by 4, and so b^2 is even.
Hence b must be even, because if b were odd, then b^2 would be odd.
But now both a and b are even, and so have a common factor 2.
This is a contradiction, so the theorem is true.

It now follows immediately that every multiple of $\sqrt{2}$ by a rational number must be irrational. The exercises ask for similar proofs by contradiction that numbers like $\sqrt{3}$, $\sqrt[3]{2}$ and $\log_2 3$ are irrational, and the following worked example shows that $\log_2 5$ is irrational. Unfortunately the proofs that π and e are irrational are considerably more difficult.

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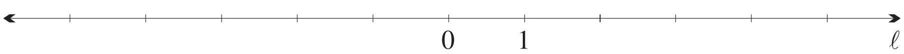
WORKED EXERCISE: Show that $\log_2 5$ is irrational (remember that $x = \log_2 5$ means that $2^x = 5$).

SOLUTION: Suppose, by way of contradiction, that $\log_2 5$ were rational.

Then $\log_2 5$ could be written as $\log_2 5 = \dfrac{a}{b}$,
where a and b are integers with no common factors, and $b \geq 1$.
Writing this using powers, $2^{\frac{a}{b}} = 5$,
and taking the bth power of both sides, $2^a = 5^b$.
Now the LHS is even, being a positive integer power of 2,
and the RHS is odd, being a positive integer power of 5.
This is a contradiction, so the theorem is true.

The Real Numbers and the Number Line: The existence of irrational numbers means that the arithmetic of Section 2A, based on the cardinals and their successive extension to the integers and the rationals, is inadequate, and we require a more general idea of number. It is at this point that we turn away from counting and make an appeal to geometry to define a still larger system called the *real numbers* as the points on the number line. Take a line ℓ and turn it into a number line by choosing two points on it called 0 and 1:



The exercises review the standard methods of using ruler and compasses to construct a point on ℓ corresponding to any rational number, and the construction of further points on ℓ corresponding to the square roots $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, ... Even the number π can notionally be placed on the line by rolling a circle of diameter 1 unit along the line. It seems reasonable therefore to make the following definition.

10 **DEFINITION:** The *real numbers* are the points on the number line:

$$\mathbf{R} = \{\,\text{real numbers}\,\}$$

The real numbers are often referred to as the *continuum*, because the rationals, despite being dense, are in a sense scattered along the number line like specks of dust, but do not 'join up'. For example, the rational multiples of $\sqrt{2}$, which are all irrational, are just as dense on the number line as the rational numbers. It is only the real line itself which is completely joined up, to be the continuous line of geometry rather than falling apart into an infinitude of discrete points.

NOTE: There is, as one might expect, a great deal more to be done here. First, the operations of addition, subtraction, multiplication and division need to be defined, and shown to be consistent with these operations in the rationals. Secondly, one needs to explain why all other irrationals, like $\log_2 5$ and $\sqrt[3]{2}$, really do have their place amongst the real numbers. And thirdly and most fundamentally, our present notion of a line is far too naive and undeveloped as yet to carry the rigorous development of our definitions. These are very difficult questions, which were resolved only towards the end of the 19th century, and then incompletely. Interested readers may like to pursue these questions in a more advanced text.

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The Real Numbers and Infinite Decimals: The convenience of approximating a real number by a terminating decimal, for example $\pi \doteqdot 3{\cdot}141\,59$, leads to the intuitive idea of representing a real number by an infinite decimal.

11 REAL NUMBERS AND DECIMALS:

(a) Every real number can be represented by one and only one infinite decimal, and every infinite decimal corresponds to one and only one real number (excluding decimals with recurring 9s).

(b) A decimal represents a rational number if and only if it is either terminating or recurring.

Exercise 2B

1. Copy the proof that $\log_2 5$ is irrational, given above, and modify it to prove that $\log_2 3$, $\log_2 7$ and $\log_3 5$ are irrational.

2. State which numbers are rational, and express those that are rational as fractions $\frac{p}{q}$ in lowest terms:

(a) $4\frac{1}{2}$, 5, $-5\frac{3}{4}$, 0, π, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$

(b) $\sqrt[3]{27}$, $\sqrt[3]{14}$, $\sqrt{\frac{4}{9}}$, -3, $16^{\frac{1}{2}}$, $7^{\frac{1}{2}}$

3. Given that a, b, c and d are integers, with b and c nonzero, simplify the average of $\frac{a}{b}$ and $\frac{c}{d}$ and explain why it is rational.

DEVELOPMENT

4. Use the proof that $\sqrt{2}$ is irrational as a guide to prove that $\sqrt{3}$, $\sqrt{5}$ and $\sqrt[3]{2}$ are irrational.

5. Why does it follow from the previous question that $2+\sqrt{3}$ is irrational? [HINT: Begin by writing, 'Suppose that $x = 2+\sqrt{3}$ were rational', then subtract 2 from both sides.]

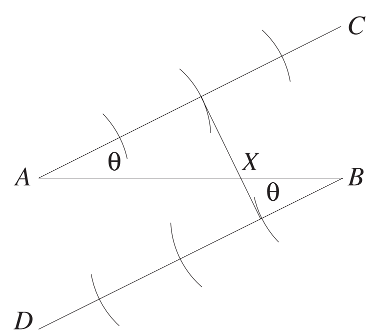
6. [This is a ruler and compasses construction to divide a given interval in the ratio 2 : 1. The method is easily generalised to any ratio.]

(a) About half way down a fresh page construct a horizontal line segment AB of length about 10 cm.

(b) At A construct a ray AC at an acute angle to AB (about 45° will do) and about 15 cm long.

(c) At B construct a second ray BD, parallel with the first and on the opposite side of AB, by copying $\angle BAC$ to $\angle ABD$.

(d) Set the compasses to a fixed radius of about 4 cm and mark off three equal lengths starting from A along AC. Do the same on the other ray starting at B.



(e) Join the second mark on AC with the first mark on BD, which intersects with AB at X. The point X now divides AB in the ratio 2 : 1, or, to put it another way, X is $\frac{2}{3}$ of the way along AB. Confirm this by measurement.

7. Use similar constructions to the one described in the previous question to find the point X on AB that represents the rational number: (a) $\frac{3}{5}$ (b) $\frac{5}{6}$

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8. Lengths representing each of the surds may be constructed using right triangles.

(a) Use a scale of 4 cm = 1 unit to construct a right-angled triangle with base 3 units and altitude 1 unit. Pythagoras' theorem asserts that the hypotenuse has length $\sqrt{10}$ units. Measure the hypotenuse to one decimal place to confirm this.

(b) Use this hypotenuse as the base of another right-angled triangle with altitude 1 unit. What will the length of the hypotenuse of this second triangle be? Measure it to one decimal place to confirm your answer.

9. Use compasses and ruler to construct lengths representing $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{6}$.

EXTENSION

10. (a) Prove that $\sqrt{6}$ is irrational. (b) Hence prove that $\sqrt{2}+\sqrt{3}$ is irrational.
[HINT: Begin, 'Suppose that $x=\sqrt{2}+\sqrt{3}$ were rational', then square both sides.]

11. [Continued fractions and approximations for π]
While π is irrational, we can find good approximations that are rational using the continued fraction expansion of π on the right. The first step is to calculate the first few terms of the continued fraction.

$$\pi = 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cdots}}}}$$

(a) Let $\pi = 3+\dfrac{1}{a_1}$. Then use the calculator to obtain the value of $a_1 = 7\cdot\ldots$ by subtracting 3 from π and taking the reciprocal. Now let $a_1 = 7+\dfrac{1}{a_2}$, and obtain the value of $a_2 = 15\cdot\ldots$ by a similar sequence of operations. Then continue the process twice more (the calculator's approximation to π may not be good enough to obtain 292).

(b) Truncating the continued fraction at 7 yields the familiar result, $\pi \doteqdot 3+\frac{1}{7} = \frac{22}{7}$. Show that this approximation is accurate to two decimal places.

(c) Truncate the continued fraction one step further, simplify the resulting fraction, and find how many decimal places it is accurate to.

(d) Truncate one step further again. Show that the resulting fraction is $\frac{355}{113} = 3\frac{16}{113}$, and that this approximation differs from π by less than 3×10^{-7}.

12. Use the calculator to find the continued fractions for $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$.

13. The series $1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\ldots$ is known to converge to $\dfrac{\pi^2}{8}$. Use your calculator to add the first twelve terms, and hence approximate π to three significant figures.

14. Suppose that a and b are positive irrational numbers, where $a<b$. Choose any positive integer n such that $\dfrac{1}{n} < b-a$, and let p be the greatest integer such that $\dfrac{p}{n} < a$.

(a) Prove that the rational number $\dfrac{p+1}{n}$ lies between a and b.

(b) If $a=\dfrac{1}{\sqrt{1001}}$ and $b=\dfrac{1}{\sqrt{1000}}$, find the least possible value of n and the corresponding value of p.

(c) Hence use part (a) to construct a rational number between $\dfrac{1}{\sqrt{1001}}$ and $\dfrac{1}{\sqrt{1000}}$.

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2 C Surds and their Arithmetic

Numbers like $\sqrt{2}$ and $\sqrt{3}$ occur constantly in our work, because they are required for the solution of quadratic equations. This section and the next two review the various methods of dealing with them.

Square Roots and Positive Square Roots: The square of any real number is positive, except that $0^2 = 0$. This means that negative numbers cannot have square roots, and that the only square root of 0 is 0 itself. Positive numbers, however, have two square roots which are the opposites of each other; for example, the square roots of 9 are 3 and -3. Consequently the well-known symbol $\sqrt{}$ *does not mean square root, but is defined to mean the* positive *square root* (or zero, if $x = 0$).

12 DEFINITION: For $x > 0$, $\sqrt{x}$ means the positive square root of x.
For $x = 0$, $\sqrt{0} = 0$.
For $x < 0$, $\sqrt{x}$ is undefined.

For example, $\sqrt{25} = 5$, even though 25 has two square roots, -5 and 5. The symbol for the negative square root of 25 is $-\sqrt{25}$.

On the other hand every number, positive or negative or zero, has exactly one cube root, and so the symbol $\sqrt[3]{}$ simply means cube root. For example, $\sqrt[3]{8} = 2$ and $\sqrt[3]{-8} = -2$.

What is a Surd: The word *surd* is often used to refer to any expression involving a square or higher root. It is better, however, to use a definition that excludes expressions like $\sqrt{\frac{4}{9}}$ and $\sqrt[3]{8}$, which can be simplified to rational numbers.

13 DEFINITION: An expression $\sqrt[n]{x}$, where x is a rational number and $n \geq 2$ is an integer, is called a *surd* if it is not itself a rational number.

It was proven in the last section that $\sqrt{2}$ was irrational, and in the same way, most roots of rational numbers are irrational. Here is the precise result for square roots, which won't be proven formally, and which is easily generalised to higher roots:

> 'If a and b are positive integers with no common factor, then $\sqrt{a/b}$ is rational if and only if both a and b are squares of integers.'

Simplifying Expressions Involving Surds: Here are some laws from earlier years for simplifying expressions involving square roots. The first pair restate the definition of square root, and the second pair are easily proven by squaring.

14 LAWS CONCERNING SURDS: Suppose that a and b are non-negative real numbers:

(a) $\sqrt{a^2} = a$ (c) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

(b) $\left(\sqrt{a}\right)^2 = a$ (d) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ (provided $b \neq 0$)

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Taking Out Square Divisors: A surd like $\sqrt{50}$, in which 50 is divisible by the square 25, is not regarded as being simplified, because it can be expressed as

$$\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}.$$

15 METHOD: Always check the number inside the square root for divisibility by one of the squares 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, ...

If there is a square divisor, then it is quickest to divide by the largest possible square divisor.

WORKED EXERCISE: Simplify: (a) $\sqrt{72} - \sqrt{50} + \sqrt{12}$ (b) $\left(\sqrt{15} - \sqrt{6}\right)^2$

SOLUTION:

(a)
$$\begin{aligned} &\sqrt{72} - \sqrt{50} + \sqrt{12} \\ &= \sqrt{36 \times 2} - \sqrt{25 \times 2} + \sqrt{4 \times 3} \\ &= 6\sqrt{2} - 5\sqrt{2} + 2\sqrt{3} \\ &= \sqrt{2} + 2\sqrt{3} \end{aligned}$$

(b)
$$\begin{aligned} \left(\sqrt{15} - \sqrt{6}\right)^2 &= 15 - 2\sqrt{90} + 6 \\ &= 21 - 2\sqrt{9 \times 10} \\ &= 21 - 6\sqrt{10} \end{aligned}$$

Exercise 2C

1. Complete the following table of values for $y = \sqrt{x}$ correct to two decimal places, and graph the points. Use a scale of 1 unit = 4 cm on both axes. Join the points with a smooth curve to obtain a graph of the function $y = \sqrt{x}$.

x	0	0·1	0·2	0·3	0·4	0·6	0·8	1	1·5	2
y										

Why were more points chosen near $x = 0$?

2. Simplify the following (assume all pronumerals are positive):

(a) $\sqrt{16}$ (b) $\sqrt{81}$ (c) $\sqrt{36}$ (d) $\sqrt{12}$
(e) $\sqrt{27}$ (f) $\sqrt{20}$ (g) $\sqrt{6x^2}$ (h) $\sqrt{8y^2}$
(i) $2\sqrt{121}$ (j) $5\sqrt{x^2}$ (k) $2\sqrt{18}$ (l) $\sqrt{y^3}$
(m) $\sqrt[3]{64}$ (n) $\sqrt[3]{343}$ (o) $\sqrt[3]{8x^3}$ (p) $\sqrt[3]{4y^3}$

3. Express the following in simplest form without the use of a calculator:

(a) $\sqrt{2} \times \sqrt{3}$ (b) $\sqrt{6} \times \sqrt{2}$ (c) $\sqrt{3} \times \sqrt{15}$ (d) $(\sqrt{5}\,)^2$ (e) $2\sqrt{3} \times 3\sqrt{5}$
(f) $4\sqrt{7} \times 3\sqrt{7}$ (g) $3\sqrt{5} \times \sqrt{15}$ (h) $(2\sqrt{3}\,)^2$ (i) $3\sqrt{6} \times \sqrt{10}$ (j) $\sqrt{56} \times 5\sqrt{6}$
(k) $3\sqrt{12} \times 2\sqrt{18}$ (l) $7\sqrt{24} \times 5\sqrt{18}$ (m) $\sqrt{\pi^2} \times \sqrt{72}$ (n) $\sqrt{2a^4} \times \sqrt{2\pi^3}$ (o) $6\sqrt{44} \times 7\sqrt{48x^4}$

4. Rewrite each value as a single surd, that is, in the form $\sqrt{n}$:

(a) $2\sqrt{5}$ (b) $3\sqrt{3}$ (c) $6\sqrt{2}$
(d) $5\sqrt{6}$ (e) $4\sqrt{3}$ (f) $2\sqrt{8}$
(g) $9\sqrt{7}$ (h) $2\sqrt{17}$ (i) $5x\sqrt{11}$
(j) $6\pi\sqrt{6}$ (k) $3y\sqrt{13y}$ (l) $12a^2\sqrt{6}$

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5. Simplify the following:

(a) $\sqrt{\frac{9}{4}}$ (b) $\sqrt{\frac{9}{16}}$ (c) $\sqrt{\frac{7}{25}}$ (d) $\sqrt{6\frac{1}{4}}$ (e) $\sqrt{5\frac{4}{9}}$ (f) $\sqrt[3]{\frac{7}{64}}$ (g) $\sqrt[3]{\frac{27}{8}}$ (h) $\sqrt[3]{2\frac{10}{27}}$

6. Use the result $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ to simplify these fractions:

(a) $\dfrac{\sqrt{6}}{\sqrt{3}}$ (b) $\dfrac{\sqrt{42}}{\sqrt{7}}$ (c) $\dfrac{\sqrt{60}}{\sqrt{12}}$ (d) $\dfrac{\sqrt{156}}{\sqrt{13}}$ (e) $\dfrac{\sqrt{10}}{\sqrt{40}}$ (f) $\dfrac{\sqrt{8}}{\sqrt{50}}$ (g) $\dfrac{\sqrt{28}}{\sqrt{63}}$ (h) $\dfrac{\sqrt{72}}{\sqrt{98}}$ (i) $\dfrac{\sqrt{96}}{\sqrt{12}}$ (j) $\dfrac{\sqrt{21}}{\sqrt{75}}$ (k) $\dfrac{\sqrt{66}}{\sqrt{54}}$ (l) $\dfrac{\sqrt{91}}{\sqrt{52}}$

7. Simplify each surd, then use the approximations $\sqrt{2} = 1{\cdot}41$, $\sqrt{3} = 1{\cdot}73$ and $\sqrt{5} = 2{\cdot}24$ to evaluate the following to two decimal places:

(a) $\sqrt{8}$ (b) $\sqrt{12}$ (c) $\sqrt{20}$ (d) $\sqrt{18}$ (e) $\sqrt{27}$ (f) $\sqrt{45}$ (g) $\sqrt{50}$ (h) $\sqrt{75}$

DEVELOPMENT

8. Find a pair of values a and b for which $\sqrt{a^2+b^2} \neq a+b$. Are there any values that make the LHS and RHS equal?

9. Simplify each expression, then collect like terms:

(a) $\sqrt{50} - \sqrt{18}$ (b) $3\sqrt{75} + 5\sqrt{3}$ (c) $\sqrt{7} + \sqrt{28}$ (d) $\sqrt{54} - \sqrt{24}$

10. Simplify each surdic expression completely:

(a) $\sqrt{12} + \sqrt{49} - \sqrt{64}$ (b) $\sqrt{96} - \sqrt{24} - \sqrt{54}$ (c) $\sqrt{18} + \sqrt{8} - \sqrt{50}$ (d) $\sqrt{90} - \sqrt{40} + \sqrt{10}$ (e) $\sqrt{45} + \sqrt{80} - \sqrt{125}$ (f) $\sqrt{27} - \sqrt{50} + \sqrt{3}$ (g) $\sqrt{6} + \sqrt{24} + \sqrt{72}$ (h) $\sqrt{27} - \sqrt{117} + \sqrt{52}$ (i) $\sqrt{63} + 2\sqrt{18} - 5\sqrt{7}$

11. Find the value of each pronumeral by first simplifying the surdic terms:

(a) $\sqrt{75} + \sqrt{27} = \sqrt{a}$
(b) $\sqrt{44} + \sqrt{99} = \sqrt{x}$
(c) $\sqrt{240} - \sqrt{135} = \sqrt{y}$
(d) $\sqrt{150} + \sqrt{54} - \sqrt{216} = \sqrt{m}$

12. Expand the following, expressing your answers in simplest form:

(a) $\sqrt{3}(\sqrt{2} + \sqrt{3})$
(b) $\sqrt{5}(\sqrt{5} + \sqrt{15})$
(c) $3\sqrt{2}(\sqrt{6} - \sqrt{8})$
(d) $\sqrt{7}(3\sqrt{3} - \sqrt{14})$
(e) $\sqrt{a}(\sqrt{a} + \sqrt{b})$
(f) $4\sqrt{a}(1 - \sqrt{a})$
(g) $\sqrt{x}(\sqrt{x+2} + \sqrt{x})$
(h) $\sqrt{x-1}(\sqrt{x-1} + \sqrt{x+1})$

13. Expand and simplify:

(a) $(\sqrt{5} + \sqrt{2})(\sqrt{3} - \sqrt{2})$
(b) $(\sqrt{2} + \sqrt{3})(\sqrt{5} + 1)$
(c) $(\sqrt{3} - 1)(\sqrt{2} - 1)$
(d) $(\sqrt{6} - \sqrt{2})(\sqrt{3} + \sqrt{2})$
(e) $(2\sqrt{6} + 1)(2\sqrt{6} + 2)$
(f) $(3\sqrt{7} - 2)(\sqrt{7} + 1)$
(g) $(2\sqrt{5} + \sqrt{3})(2 - \sqrt{3})$
(h) $(3\sqrt{2} - \sqrt{5})(\sqrt{6} - \sqrt{5})$

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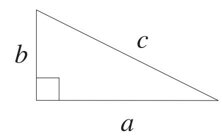
14. Expand and simplify:

(a) $(\sqrt{2}+1)^2$
(b) $(\sqrt{3}+1)(\sqrt{3}-1)$
(c) $(1-\sqrt{5})(1+\sqrt{5})$
(d) $(1-\sqrt{3})^2$
(e) $(\sqrt{3}+\sqrt{2})^2$
(f) $(\sqrt{5}+\sqrt{7})(\sqrt{5}-\sqrt{7})$
(g) $(2\sqrt{6}-5)(2\sqrt{6}+5)$
(h) $(\sqrt{6}-2\sqrt{2})^2$
(i) $(2\sqrt{a}-1)^2$
(j) $(2+\sqrt{a+2})^2$
(k) $(\sqrt{x-1}-1)(\sqrt{x-1}+1)$
(l) $(\sqrt{x+1}+\sqrt{x-2})^2$
(m) $(\frac{1}{2}+\frac{1}{2}\sqrt{5})^2$
(n) $(\frac{1}{2}-\frac{1}{2}\sqrt{5})^2$

15. Fully simplify these fractions:

(a) $\dfrac{\sqrt{3}\times 2\sqrt{5}}{\sqrt{15}}$
(b) $\dfrac{2\sqrt{6}\times\sqrt{5}}{\sqrt{10}}$
(c) $\dfrac{5\sqrt{7}\times\sqrt{3}}{\sqrt{28}}$
(d) $\dfrac{\sqrt{10}\times 3\sqrt{5}}{5\sqrt{2}}$
(e) $\dfrac{\sqrt{15}\times\sqrt{20}}{\sqrt{12}}$
(f) $\dfrac{6\sqrt{3}\times 5\sqrt{2}}{\sqrt{12}\times\sqrt{18}}$
(g) $\dfrac{\sqrt{8}\times 3\sqrt{7}}{\sqrt{21}\times\sqrt{12}}$
(h) $\dfrac{5\sqrt{44}\times\sqrt{14}}{\sqrt{24}\times 3\sqrt{33}}$

16. In the right triangle opposite, find the third side if:



(a) $a=\sqrt{2}$ and $b=\sqrt{7}$
(b) $b=\sqrt{5}$ and $c=2\sqrt{5}$
(c) $a=\sqrt{7}+1$ and $b=\sqrt{7}-1$
(d) $a=2\sqrt{3}+3\sqrt{2}$ and $b=2\sqrt{3}-3\sqrt{2}$

17. The roots of the quadratic equation $ax^2+bx+c=0$ are known to be $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$. Find: (a) the sum of the roots, (b) the product of the roots.

18. Given that x and y are positive, simplify:

(a) $\sqrt{x^2y^3}$
(b) $x\sqrt{x^2y^6}$
(c) $\sqrt{x^2+6x+9}$
(d) $\sqrt{x^3+2x^2+x}$
(e) $\sqrt{x^2y^4(x^2+2x+1)}$
(f) $\sqrt{x^4+2x^3+x^2}$

EXTENSION

19. (a) Show that if $a=1+\sqrt{2}$, then $a^2-2a-1=0$. (b) Hence show that $a=2+\dfrac{1}{a}$ and $\sqrt{2}=1+\dfrac{1}{a}$. (c) Show how these results can be used to construct the continued fraction for $\sqrt{2}$ found in question 12 of Exercise 2B.

2 D Rationalising the Denominator

When dealing with surdic expressions, it is usual to remove any surds from the denominator, a process called *rationalising the denominator*. There are two quite distinct cases.

The Denominator has a Single Term: In the first case, the denominator is a surd or a multiple of a surd.

16 METHOD: In an expression like $\dfrac{5\sqrt{7}}{2\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$.

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WORKED EXERCISE:

(a) $\dfrac{5\sqrt{7}}{2\sqrt{3}} = \dfrac{5\sqrt{7}}{2\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$
$= \dfrac{5\sqrt{21}}{6}$

(b) $\dfrac{55}{\sqrt{11}} = \dfrac{55}{\sqrt{11}} \times \dfrac{\sqrt{11}}{\sqrt{11}}$
$= 5\sqrt{11}$

The Denominator has Two Terms: The second case involves a denominator with two terms, one or both of which contain a surd.

17 METHOD: In an expression like $\dfrac{\sqrt{3}}{5+2\sqrt{3}}$, multiply top and bottom by $5-2\sqrt{3}$.

WORKED EXERCISE:

(a) $\dfrac{\sqrt{3}}{5+2\sqrt{3}}$
$= \dfrac{\sqrt{3}}{5+2\sqrt{3}} \times \dfrac{5-2\sqrt{3}}{5-2\sqrt{3}}$
$= \dfrac{5\sqrt{3}-6}{25-4\times 3}$
$= \dfrac{5\sqrt{3}-6}{13}$

(b) $\dfrac{1}{2\sqrt{3}-3\sqrt{2}}$
$= \dfrac{1}{2\sqrt{3}-3\sqrt{2}} \times \dfrac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$
$= \dfrac{2\sqrt{3}+3\sqrt{2}}{4\times 3 - 9\times 2}$
$= -\dfrac{2\sqrt{3}+3\sqrt{2}}{6}$

The method works because the identity $(x-y)(x+y) = x^2 - y^2$ (the difference of squares) guarantees that all surds will disappear from the denominator. The examples above involved

$$(5+2\sqrt{3})\times(5-2\sqrt{3}) = 25-12 \quad\text{and}\quad (2\sqrt{3}-3\sqrt{2})\times(2\sqrt{3}+3\sqrt{2}) = 12-18.$$

Comparing Expressions Involving Surds: When comparing two compound expressions, find whether the difference between them is positive or negative.

WORKED EXERCISE: Compare: (a) $4\sqrt{6}$ and $\dfrac{14}{\sqrt{2}}$ (b) $5-3\sqrt{2}$ and $6\sqrt{2}-8$

SOLUTION:

(a) $4\sqrt{6} = \sqrt{16\times 6}$
$= \sqrt{96}$
and $\dfrac{14}{\sqrt{2}} = 7\sqrt{2}$
$= \sqrt{98}$.
Hence $4\sqrt{6} < \dfrac{14}{\sqrt{2}}$.

(b) $\left(5-3\sqrt{2}\right) - \left(6\sqrt{2}-8\right)$
$= 13-9\sqrt{2}$
$= \sqrt{169}-\sqrt{162}$
> 0.
Hence $5-3\sqrt{2} > 6\sqrt{2}-8$.

Exercise 2D

1. Express the following with rational denominators, in lowest terms:

(a) $\dfrac{1}{\sqrt{3}}$ (c) $\dfrac{5}{\sqrt{11}}$ (e) $\dfrac{2}{\sqrt{8}}$ (g) $\dfrac{5}{\sqrt{15}}$

(b) $\dfrac{2}{\sqrt{7}}$ (d) $\dfrac{5}{\sqrt{5}}$ (f) $\dfrac{3}{\sqrt{6}}$ (h) $\dfrac{14}{\sqrt{10}}$

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2. Find the sum and the product of each pair:

(a) $2+\sqrt{3}$ and $2-\sqrt{3}$
(b) $3-\sqrt{5}$ and $3+\sqrt{5}$
(c) $1-\sqrt{2}$ and $1+\sqrt{2}$
(d) $5+2\sqrt{6}$ and $5-2\sqrt{6}$
(e) $1+2\sqrt{3}$ and $1-2\sqrt{3}$
(f) $\sqrt{7}-1$ and $-\sqrt{7}-1$

3. Rewrite each fraction with an integer denominator:

(a) $\dfrac{1}{\sqrt{2}-1}$
(b) $\dfrac{1}{1-\sqrt{2}}$
(c) $\dfrac{3}{\sqrt{5}-1}$
(d) $\dfrac{3}{1-\sqrt{5}}$
(e) $\dfrac{1}{\sqrt{3}+\sqrt{2}}$
(f) $\dfrac{1}{\sqrt{5}+\sqrt{3}}$
(g) $\dfrac{3}{4-\sqrt{7}}$
(h) $\dfrac{3}{\sqrt{11}-\sqrt{6}}$

DEVELOPMENT

4. Express each of the following as a fraction with rational denominator:

(a) $\dfrac{2}{3\sqrt{2}}$
(b) $\dfrac{5}{2\sqrt{5}}$
(c) $\dfrac{2\sqrt{3}}{\sqrt{2}}$
(d) $\sqrt{\frac{4}{3}}$
(e) $\sqrt{\frac{3}{7}}$
(f) $\dfrac{\sqrt{2}}{5\sqrt{7}}$
(g) $\sqrt{4\frac{1}{4}}$
(h) $\sqrt{3\frac{3}{11}}$

5. Simplify the following by rationalising the denominator:

(a) $\dfrac{3}{2\sqrt{2}-\sqrt{5}}$
(b) $\dfrac{2}{3\sqrt{2}-4}$
(c) $\dfrac{4}{\sqrt{5}-\sqrt{3}}$
(d) $\dfrac{3\sqrt{3}}{\sqrt{5}+\sqrt{3}}$
(e) $\dfrac{2\sqrt{7}}{5+2\sqrt{7}}$
(f) $\dfrac{1-\sqrt{2}}{1+\sqrt{2}}$
(g) $\dfrac{\sqrt{3}-1}{2-\sqrt{3}}$
(h) $\dfrac{\sqrt{3}-\sqrt{7}}{3\sqrt{6}-\sqrt{7}}$
(i) $\dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$
(j) $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$
(k) $\dfrac{3}{\sqrt{x}+2}$
(l) $\dfrac{1}{q-\sqrt{p}}$
(m) $\dfrac{2}{\sqrt{x+1}+\sqrt{x-1}}$
(n) $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$
(o) $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$
(p) $\dfrac{4}{\sqrt{2}(\sqrt{5}-1)}$
(q) $\dfrac{6}{\sqrt{3}(\sqrt{7}+\sqrt{5})}$
(r) $\dfrac{2x}{\sqrt{x}(\sqrt{x+2}+\sqrt{x})}$

6. Show that each expression is rational: (a) $\dfrac{3}{2+\sqrt{2}}+\dfrac{3}{\sqrt{2}}$ (b) $\dfrac{1}{3+\sqrt{6}}+\dfrac{2}{\sqrt{6}}$

7. Simplify: (a) $\dfrac{1}{1+\sqrt{3}}-\dfrac{1}{1-\sqrt{3}}$ (b) $\dfrac{1}{2(\sqrt{5}+1)}+\dfrac{1}{2(\sqrt{5}-1)}$ (c) $\dfrac{1}{3\sqrt{2}+1}+\dfrac{1}{1-3\sqrt{2}}$

8. Rationalise the denominator of $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$.

9. Evaluate $a+\dfrac{1}{a}$ for these values of a:

(a) $1+\sqrt{2}$
(b) $2-\sqrt{3}$
(c) $\dfrac{3-\sqrt{3}}{3+\sqrt{3}}$
(d) $\dfrac{\sqrt{x}+\sqrt{2-x}}{\sqrt{x}-\sqrt{2-x}}$

10. (a) Show that $\left(x+\dfrac{1}{x}\right)^2=x^2+\dfrac{1}{x^2}+2$. (b) Given that $y=2+\sqrt{5}$, simplify $y+\dfrac{1}{y}$.

(c) Use the result in part (a) to evaluate $y^2+\dfrac{1}{y^2}$ without determining y^2.

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(d) Similarly find $y^2 + \dfrac{1}{y^2}$ for: (i) $y = 1 + \sqrt{2}$ (ii) $y = 2 - \sqrt{3}$

11. Determine, without using a calculator, which is the greater number in each pair:
(a) $2\sqrt{3}$ or $\sqrt{11}$ (b) $7\sqrt{2}$ or $3\sqrt{11}$ (c) $3 + 2\sqrt{2}$ or $15 - 7\sqrt{2}$ (d) $2\sqrt{6} - 3$ or $7 - 2\sqrt{6}$

EXTENSION

12. Express with integer denominator: (a) $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ (b) $\dfrac{1}{\sqrt[3]{2}}$ (c) $\dfrac{1}{\sqrt[3]{2}-1}$

13. (a) If $\sqrt{5} = 2 + \dfrac{1}{a}$, show that $a = 4 + \dfrac{1}{a}$.

(b) Hence deduce the continued fraction for $\sqrt{5}$ as found in question 12 of Exercise 2B.

14. The value of $\sqrt{17}$ is approximately 4·12 to two decimal places.

(a) Substitute this value to determine an approximation for $\dfrac{1}{\sqrt{17}-4}$.

(b) Show that $\dfrac{1}{\sqrt{17}-4} = \sqrt{17} + 4$, and that this last result gives a more accurate value for the approximation than that found in part (a).

15. It is given that $\dfrac{a}{b+\sqrt{c}} + \dfrac{d}{\sqrt{c}}$ is rational, where a, b, c and d are positive integers and c is not a square. Show that as a consequence $db^2 = c(a+d)$. Use this result to show that $\dfrac{a}{1+\sqrt{c}} + \dfrac{d}{\sqrt{c}}$ is not rational.

16. (a) Let $x = \sqrt{2}$, show that $x = \dfrac{x^2+2}{2x}$.

(b) Let $\sqrt{2}$ be approximated by the slightly larger fraction $\dfrac{p}{q}$, that is $\dfrac{p}{q} = \sqrt{2} + \varepsilon$, where ε is small and positive. Show that $\dfrac{p^2+2q^2}{2pq} = \dfrac{\sqrt{2}+\varepsilon}{2} + \dfrac{1}{\sqrt{2}+\varepsilon}$ and hence show that $\dfrac{p^2+2q^2}{2pq} > \sqrt{2}$.

(c) Show that $\dfrac{p^2+2q^2}{2pq} - \sqrt{2}$ is smaller than ε, that is, $\dfrac{p^2+2q^2}{2pq}$ is a better approximation for $\sqrt{2}$. NOTE: These results come from Newton's method for solving $x^2 = 2$ by approximation.

2 E Equality of Surdic Expressions

There is only one way to write an expression like $3 + \sqrt{7}$ as the sum of a rational number and a surd. Although this may seem obvious, the result is surprising in that it generates two equations in rational numbers from just one surdic equation. Here are the precise statements:

18

THEOREM:
(a) Suppose that $a + b\sqrt{x} = A + B\sqrt{x}$, where a, b, A, B and x are rational, and $x \ge 0$ is not the square of a rational. Then $a = A$ and $b = B$.
(b) Suppose that $a + \sqrt{b} = A + \sqrt{B}$, where a, b, A and B are rational with $b \ge 0$ and $B \ge 0$, and b is not the square of a rational. Then $a = A$ and $b = B$.

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PROOF:

(a) Rearranging, $(b - B)\sqrt{x} = A - a$.

Now if $b \neq B$, then $\sqrt{x} = \dfrac{A - a}{b - B}$, which is rational.

This contradicts $\sqrt{x}$ being irrational, so $b = B$, and hence $a = A$.

(b) Given that $a + \sqrt{b} = A + \sqrt{B}$,

then $\sqrt{b} - \sqrt{B} = A - a$, which is a rational number. (1)

Multiplying both sides by $\sqrt{b} + \sqrt{B}$,

$$b - B = (A - a)(\sqrt{b} + \sqrt{B}).$$

Now if $a \neq A$, then we could divide both sides by $A - a$, and

$$\sqrt{b} + \sqrt{B} = \frac{b - B}{A - a}, \text{ which is also rational,} \qquad (2)$$

but then adding (1) and (2), $2\sqrt{b}$ would be a rational number, contradicting the fact that b is not the square of a rational.

Hence $a = A$, and so also $b = B$.

WORKED EXERCISE: Find rational values of x and y if:

(a) $x + y\sqrt{7} = (3 - 2\sqrt{7})^2$

(b) $(x + y\sqrt{5})^2 = 14 - 6\sqrt{5}$

SOLUTION:

(a) $\text{RHS} = 9 - 12\sqrt{7} + 28$

$= 37 - 12\sqrt{7}$

Using part (a) of the theorem above,

$x = 37$ and $y = -12$.

(b) $\text{LHS} = x^2 + 5y^2 + 2xy\sqrt{5}$,

so $x^2 + 5y^2 = 14$

and $xy = -3$.

By inspection, $x = 3$ and $y = -1$,

or $x = -3$ and $y = 1$.

Closure of Sets of Surdic Expressions: The last question in Exercise 2E proves that

$$F = \{\, a + b\sqrt{2} : a \text{ and } b \text{ are rational} \,\}$$

is closed under the four operations of addition, multiplication, subtraction and division. In this way the set F forms a self-contained system of numbers that is larger than the set of rationals. More generally, replacing 2 by any non-square positive integer produces a similar system of numbers.

Exercise 2E

1. Find the values of the pronumerals a and b, given that they are rational:

(a) $a + b\sqrt{5} = 7 - 2\sqrt{5}$

(b) $a - b\sqrt{7} = 2 - 3\sqrt{7}$

(c) $-a + b\sqrt{3} = 7 - 4\sqrt{3}$

(d) $a - b\sqrt{x} = 3 + 2\sqrt{x}$

(e) $a + b\sqrt{x} = \frac{5}{7} + \frac{1}{2}\sqrt{x}$

(f) $a + b\sqrt{x} = \frac{2}{3} + 3\sqrt{x}$

2. Determine the rational numbers a and b:

(a) $2 + \sqrt{b} = a + 3\sqrt{2}$

(b) $a + \sqrt{12} = -1 + b\sqrt{3}$

(c) $a + b\sqrt{7} = 3 - \sqrt{28}$

(d) $-a + 2\sqrt{5} = -5 + \sqrt{b}$

(e) $\frac{4}{3} - b\sqrt{3} = a - \sqrt{\frac{3}{4}}$

(f) $a + b\sqrt{x} = -\frac{1}{5} - \sqrt{\dfrac{9x}{16}}$

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3. Simplify the right-hand expressions in order to determine the rational numbers x and y:

(a) $x + \sqrt{y} = \sqrt{7}\,(\sqrt{7} + 2)$
(b) $x + \sqrt{y} = (1 + \sqrt{3}\,)^2$
(c) $x + y\sqrt{3} = (6 + \sqrt{3}\,)^2$
(d) $x + y\sqrt{5} = (\sqrt{5} + 2)(\sqrt{5} + 3)$
(e) $x - y\sqrt{2} = (\sqrt{6} - \sqrt{3}\,)^2$
(f) $x - \sqrt{y} = (3 - \sqrt{5}\,)^2$

DEVELOPMENT

4. Find the values of the integers x, y and z, given that z has no squares as factors:

(a) $x + y\sqrt{z} = (\sqrt{6} - 2\sqrt{3}\,)(\sqrt{6} + \sqrt{3}\,)$
(b) $x + y\sqrt{z} = (\sqrt{15} + \sqrt{5}\,)^2$
(c) $x + y\sqrt{z} = (\sqrt{5} - 2\sqrt{2}\,)(\sqrt{5} + 3\sqrt{2}\,)$
(d) $x + y\sqrt{z} = (\sqrt{15} - \sqrt{5}\,)^2$

5. Find rational numbers a and b such that:

(a) $a + b\sqrt{3} = \dfrac{1}{2 - \sqrt{3}}$
(b) $a + b\sqrt{5} = \dfrac{1}{2 + \sqrt{5}}$
(c) $a + b\sqrt{5} = \dfrac{2}{\sqrt{5} - 1}$
(d) $a + b\sqrt{3} = \dfrac{\sqrt{3}}{3 + \sqrt{3}}$
(e) $a + b\sqrt{2} = \dfrac{\sqrt{2} + 1}{\sqrt{2} - 1}$
(f) $a + b\sqrt{6} = \dfrac{2\sqrt{6} + 1}{2\sqrt{6} - 3}$

6. Form a pair of simultaneous equations and solve them to find x and y, given that they are rational:

(a) $x - 3 + \sqrt{y + 2} = -1 + \sqrt{5}$
(b) $x + 1 + \sqrt{7} = \frac{7}{2} + \sqrt{y - 1}$
(c) $x - y + \sqrt{x + y} = 3 + \sqrt{6}$
(d) $6 + \sqrt{x - y} = x + y + 3\sqrt{2}$
(e) $x - 2y + \sqrt{x + y} = \frac{1}{2}\sqrt{6}$
(f) $1\frac{1}{2} + \sqrt{x + 2y} = 3x + y + \frac{2}{3}\sqrt{3}$
(g) $xy + \sqrt{3} = 10 + \sqrt{x - y}$
(h) $xy + \sqrt{x + y} = \frac{5}{4} + \sqrt{3}$

7. Find the rational values of a and b, with $a > 0$, by forming two simultaneous equations and solving them by inspection (part (d) may need substitution):

(a) $(a + b\sqrt{2}\,)^2 = 3 + 2\sqrt{2}$
(b) $(a + b\sqrt{5}\,)^2 = 9 - 4\sqrt{5}$
(c) $(a + b\sqrt{3}\,)^2 = 13 - 4\sqrt{3}$
(d) $(a + b\sqrt{7}\,)^2 = 9\frac{1}{4} + 3\sqrt{7}$

EXTENSION

8. (a) Let $\sqrt{15 - 6\sqrt{6}} = x - \sqrt{y}$. Square both sides and form a pair of simultaneous equations to find x and y, given that they are rational. Hence find $\sqrt{15 - 6\sqrt{6}}$.

(b) Similarly simplify: (i) $\sqrt{28 - 10\sqrt{3}}$ (ii) $\sqrt{66 + 14\sqrt{17}}$ (iii) $\sqrt{\frac{7}{12} - \frac{\sqrt{3}}{3}}$

9. Define the set $F = \{x + y\sqrt{2} : x, y \in \mathbf{Q}\}$. The parts of this question demonstrate that F is *closed* under the four algebraic operations. Let $a + b\sqrt{2}$ and $c + d\sqrt{2}$ be members of F.

(a) Show that $(a + b\sqrt{2}\,) + (c + d\sqrt{2}\,)$ has the form $x + y\sqrt{2}$, where $x, y \in \mathbf{Q}$. Thus F is closed under addition.

(b) Show that $(a + b\sqrt{2}\,) - (c + d\sqrt{2}\,)$ has the form $x + y\sqrt{2}$, where $x, y \in \mathbf{Q}$. Thus F is closed under subtraction.

(c) Show that $(a + b\sqrt{2}\,) \times (c + d\sqrt{2}\,)$ has the form $x + y\sqrt{2}$, where $x, y \in \mathbf{Q}$. Thus F is closed under multiplication.

(d) Show that $(a + b\sqrt{2}\,) \div (c + d\sqrt{2}\,)$, where c and d are not both zero, has the form $x + y\sqrt{2}$, where $x, y \in \mathbf{Q}$. Thus F is closed under division.

10. Prove that it is impossible to have $a + \sqrt{b} = A - \sqrt{B}$, where a, b, A and B are rational, with $b \geq 0$ and $B \geq 0$, and b not the square of a rational.

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2 F Relations and Functions

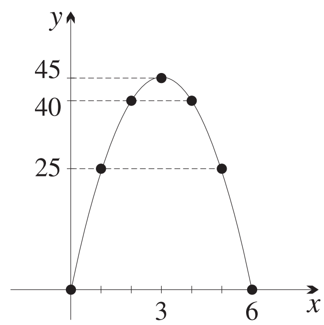
Having clarified some ideas about numbers, we turn now to the functions that will be the central objects of study in this course.

A Function and its Graph: When a quantity y is completely determined by some other quantity x as a result of any rule whatsoever, we say that *y is a function of x*. For example, the height y of a ball thrown vertically upwards will be a function of the time x after the ball is thrown. In units of metres and seconds, a possible such rule is

$$y = 5x(6 - x).$$

We can construct a *table of values* of this function by choosing just a few values of the time x and calculating the corresponding height y:

x	0	1	2	3	4	5	6
y	0	25	40	45	40	25	0



Each x-value and its corresponding y-value can then be put into an ordered pair ready to plot on a graph of the function. The seven ordered pairs calculated here are:

$$(0, 0),\ (1, 25),\ (2, 40),\ (3, 45),\ (4, 40),\ (5, 25),\ (6, 0),$$

and the graph is sketched opposite. The seven representative points have been drawn, but there are infinitely many such ordered pairs, and they join up to make the nice smooth curve drawn in the graph.

We can take a more abstract approach to all this, and identify the function completely with the ordered pairs generated by the function rule. Notice that a y-value can occur twice: for example, the ordered pairs $(1, 25)$ and $(5, 25)$ show us that the ball is 25 metres high after 1 second, and again after 5 seconds when it is coming back down. But no x-value can occur twice because at any one time the ball can only be in one position. So the more abstract definition of a function is:

19 DEFINITION: A *function* is a set of ordered pairs in which no two ordered pairs have the same x-coordinate.

In this way the function is completely identified with its graph, and the rule and the graph are now regarded only as alternative representations of the set of ordered pairs.

Domain and Range: The time variable in our example cannot be negative, because the ball had not been thrown then, and cannot be greater than 6, because the ball hits the ground again after 6 seconds. The *domain* is the set of possible x-values, so the domain is the closed interval $0 \le x \le 6$. Again, the height of the ball never exceeds 45 metres and is never negative. The *range* is the set of possible y-values, so the range is the closed interval $0 \le y \le 45$:

20 DEFINITION: The *domain* of a function is the set of all x-coordinates of the ordered pairs.
The *range* of a function is the set of all y-coordinates.

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The Natural Domain: Any restriction on the domain is part of the function, so the example of the ball thrown into the air is more correctly written as

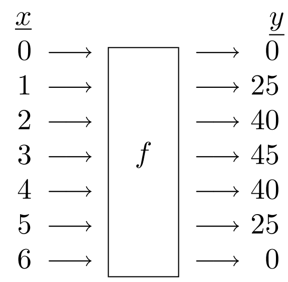
$$y = 5x(6 - x), \text{ where } 0 \le x \le 6.$$

When the equation of a function is given with no restriction, we assume by convention that the domain is as large as possible, consisting of all x-values that can validly be substituted into the equation. So, for example, 'the function $y = \sqrt{4 - x^2}$' means

'the function $y = \sqrt{4 - x^2}$, where $-2 \le x \le 2$',

because one cannot take square roots of negative numbers. Again, 'the function $y = \dfrac{1}{x - 2}$' implies the restriction $x \ne 2$, because division by 0 is impossible. This implied domain is called the *natural domain* of the function.

The Function Machine and the Function Rule: A function can be regarded as a 'machine' with inputs and outputs. For example, on the right are the outputs from the function $y = 5x(6 - x)$ when the seven numbers 0, 1, 2, 3, 4, 5 and 6 are the inputs. This sort of model for a function has of course become far more important in the last few decades because computers and calculators routinely produce output from a given input. If the name f is given to our function, we can write the results of the input/output routines as follows:



$$f(0) = 0, \quad f(1) = 25, \quad f(2) = 40, \quad f(3) = 45, \quad f(4) = 40, \quad \ldots$$

and since the output when x is input is $5x(6-x)$, we can write the *function rule*, using the well-known notation introduced by Leonhard Euler in 1735, as

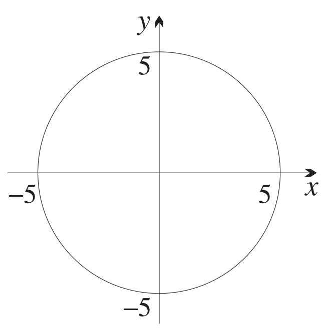
$$f(x) = 5x(6 - x), \text{ where } 0 \le x \le 6.$$

The pronumeral y is lost when the function is written this way, so a hybrid notation is sometimes used to express the fact that y is a function of x:

$$y(x) = 5x(6 - x), \text{ where } 0 \le x \le 6.$$

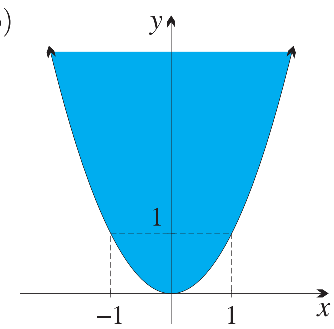
Relations: We shall often be dealing with graphs such as the following:

(a)



$x^2 + y^2 = 25$

(b)



$y \ge x^2$

In case (a), the input $x = 3$ would result in the two outputs $y = 4$ and $y = -4$, because the vertical line $x = 3$ meets the graph at $(3, 4)$ and at $(3, -4)$. In case (b), the input $x = 1$ would give as output $y = 1$ and all numbers greater than 1. Such

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objects are sometimes called ‘multi-valued functions’, but we will use the word *relation* to describe any curve or region in the plane, whether a function or not.

21 DEFINITION: Any set of ordered pairs is called a *relation*.

Once a relation is graphed, it is then quite straightforward to decide whether or not it is a function.

22 VERTICAL LINE TEST: A relation is a function if and only if there is no vertical line that crosses the graph more than once.

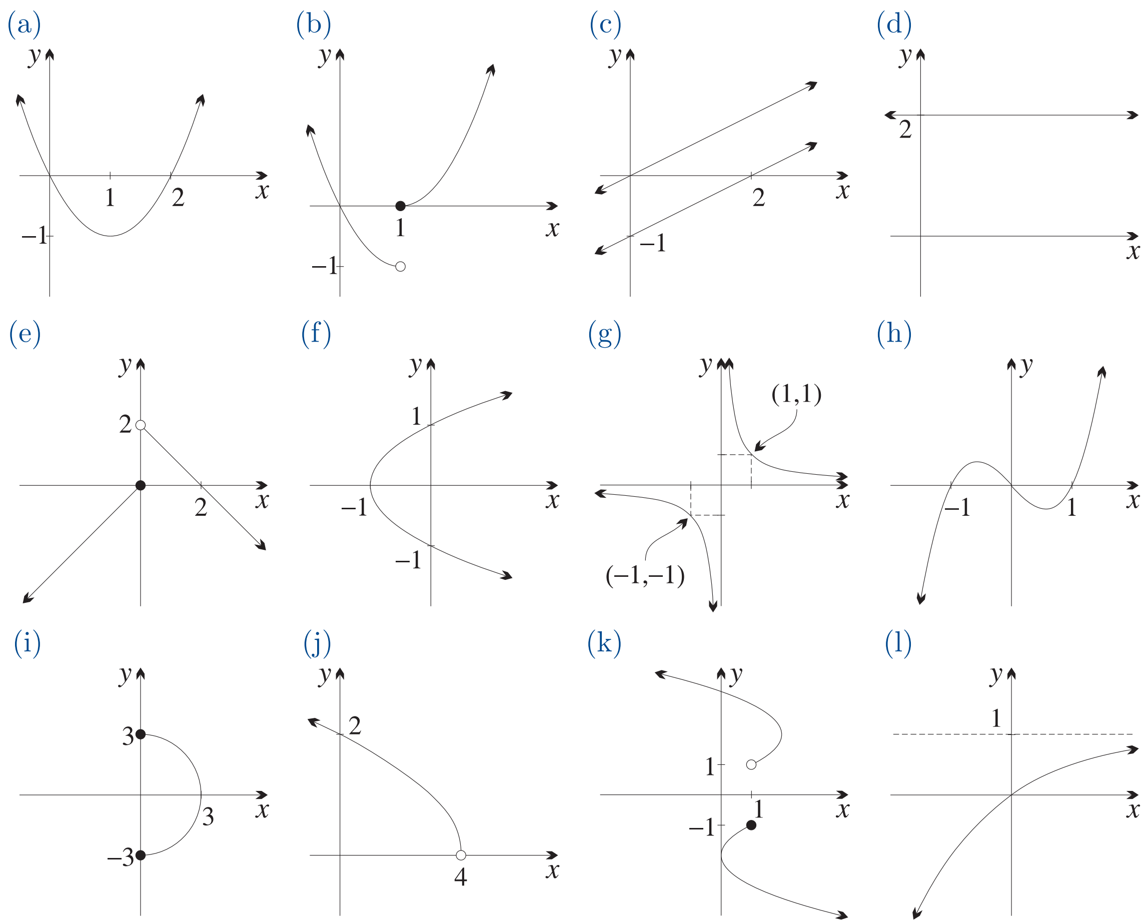
The ideas of domain and range apply not just to functions, but more generally to relations. In the examples given above:

(a) For $x^2 + y^2 = 25$, the domain is $-5 \le x \le 5$, and the range is $-5 \le y \le 5$.

(b) For $y \ge x^2$, the domain is the set $\mathbf{R}$ of all real numbers, and the range is $y \ge 0$.

Exercise 2F

1. Use the vertical line test to determine which of the following graphs represent functions:



2. What are the domain and range of each of the relations in question 1?

3. If $h(x) = x^2 - 2$, find:

(a) $h(2)$ (c) $h(a)$ (e) $h(a+2)$ (g) $h(\frac{1}{2})$ (i) $h(t^2)$

(b) $h(\sqrt{2}\,)$ (d) $h(-a)$ (f) $h(x-1)$ (h) $h(3t+2)$ (j) $h(t+\frac{1}{t})$

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4. If $g(x) = x^2 - 2x$, find:

(a) $g(0)$ (c) $g(-2)$ (e) $g(t)$ (g) $g(w-1)$ (i) $g(1-w)$

(b) $g(1)$ (d) $g(2)$ (f) $g(-t)$ (h) $g(w) - 1$ (j) $g(2-x)$

5. (a) Let $f(x) = \begin{cases} x, & \text{for } x \le 0, \\ 2 - x, & \text{for } x > 0. \end{cases}$ Create a table of values for $-3 \le x \le 3$, and confirm that the graph is that in question 1(e) above.

(b) Let $f(x) = \begin{cases} (x-1)^2 - 1, & \text{for } x < 1, \\ (x-1)^2, & \text{for } x \ge 1. \end{cases}$ Create a table of values for $-1 \le x \le 3$, and confirm that the graph is that in question 1(b) above.

6. Find the natural domains of:

(a) $\ell(x) = x - 3$ (c) $s(x) = \sqrt{x}$ (e) $p(x) = \sqrt{2-x}$

(b) $r(x) = \dfrac{1}{x-3}$ (d) $S(x) = \dfrac{1}{\sqrt{x}}$ (f) $P(x) = \dfrac{1}{\sqrt{2-x}}$

DEVELOPMENT

7. Let $Q(x) = x^2 - 2x - 4$. Show that: (a) $Q(1 - \sqrt{5}\,) = 0$ (b) $Q(1 + \sqrt{5}\,) = 0$

8. Given that $f(x) = x^3 - x + 1$, evaluate and simplify:

(a) $f(h)$ (c) $\dfrac{f(h) - f(0)}{h}$ (e) $\frac{1}{4}\big(f(0) + 2f(\frac{1}{2}) + f(1)\big)$

(b) $f(-h)$ (d) $\dfrac{f(h) - f(-h)}{2h}$ (f) $\frac{1}{6}\big(f(0) + 4f(\frac{1}{2}) + f(1)\big)$

9. Create tables of values for these functions for $n = 1, 2, 3, 4, 5, 6$:

(a) $S(n)$ = the sum of the positive numbers less than or equal to n

(b) $d(n)$ = the number of positive divisors of n

(c) $\sigma(n)$ = the sum of the positive divisors of n

10. Find the natural domains of:

(a) $c(x) = \sqrt{9 - x^2}$ (c) $\ell(x) = \log_3 x$ (e) $p(x) = \dfrac{1}{x^2 - 5x + 6}$

(b) $h(x) = \sqrt{x^2 - 4}$ (d) $q(x) = \dfrac{x+2}{x+1}$ (f) $r(x) = \dfrac{x-3}{x^2-9}$

11. Given the functions $f(x) = x^2$, $F(x) = x + 3$, $g(x) = 2^x$ and $G(x) = 3x$, find:

(a) $f\big(F(5)\big)$ (c) $f\big(F(x)\big)$ (e) $g\big(G(2)\big)$ (g) $g\big(G(x)\big)$

(b) $F\big(f(5)\big)$ (d) $F\big(f(x)\big)$ (f) $G\big(g(2)\big)$ (h) $G\big(g(x)\big)$

12. (a) If $f(x) = 2^x$, show that $f(-x) = \dfrac{1}{f(x)}$.

(b) If $g(x) = \dfrac{x}{x^2+1}$, show that $g(\frac{1}{x}) = g(x)$ for $x \ne 0$.

(c) If $h(x) = \dfrac{x}{x^2-1}$, show that $h(\frac{1}{x}) = -h(x)$ for $x \ne 0$.

(d) If $f(x) = x + \frac{1}{x}$, show that $f(x) \times f(x + \frac{1}{x}) = f(x^2) + 3$.

13. For each of the following functions write out the equation $f(a) + f(b) = f(a+b)$. Then determine if there are any values of a and b for which $f(a) + f(b) = f(a+b)$.

(a) $f(x) = x$ (b) $f(x) = 2x$ (c) $f(x) = x + 1$ (d) $f(x) = 2x + 1$ (e) $f(x) = x^2 + 3x$

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EXTENSION

14. Evaluate $e(x) = \left(1 + \dfrac{1}{x}\right)^x$ on your calculator for $x = 1$, 10, 100, 1000 and 10000, giving your answer to two decimal places. What do you notice happens as x gets large?

15. Let $c(x) = \dfrac{3^x + 3^{-x}}{2}$ and $s(x) = \dfrac{3^x - 3^{-x}}{2}$.

(a) Show that $\big(c(x)\big)^2 = \frac{1}{2}\big(c(2x) + 1\big)$.

(b) Find a similar result for $\big(s(x)\big)^2$.

(c) Hence show that $\big(c(x)\big)^2 - \big(s(x)\big)^2 = 1$.

16. Given that $\operatorname{ath}(x) = \log_2\left(\dfrac{1+x}{1-x}\right)$, show that $\operatorname{ath}\left(\dfrac{2x}{1+x^2}\right) = 2\operatorname{ath}(x)$.

NOTE: A function name does not have to be a single letter. In this case the function has been given the name 'ath' since it is related to the arc hyperbolic tangent which is studied in some university courses.

2 G Review of Known Functions and Relations

This section will briefly review graphs that have been studied in previous years — linear graphs, quadratic functions, higher powers of x, circles and semicircles, half-parabolas, rectangular hyperbolas, exponential functions and log functions.

Linear Functions and Relations: Any equation that can be written in the form

$$ax + by + c = 0, \text{ where } a, b \text{ and } c \text{ are constants (and } a \text{ and } b \text{ are not both zero)},$$

is called a *linear relation*, because its graph is a straight line. Unless $b = 0$, the equation can be solved for y and is therefore a *linear function*.

23 SKETCHING LINEAR FUNCTIONS: Find the x-intercept by putting $y = 0$, and find the y-intercept by putting $x = 0$.

This method won't work when any of the three constants a, b and c is zero:

24 SPECIAL CASES OF LINEAR GRAPHS:

(a) If $a = 0$, then the equation has the form $y = k$, and its graph is a horizontal line with y-intercept k.

(b) If $b = 0$, then the equation has the form $x = \ell$, and its graph is a vertical line with x-intercept ℓ.

(c) If $c = 0$, both intercepts are zero and the graph passes through the origin. Find one more point on it, usually by putting $x = 1$.

WORKED EXERCISE: Sketch the following four lines:

(a) ℓ_1: $x + 2y = 6$ (b) ℓ_2: $x + 2y = 0$ (c) ℓ_3: $y = 2$ (d) ℓ_4: $x = -3$

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SOLUTION:

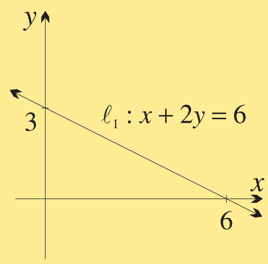
(a) The line ℓ_1: $x + 2y = 6$ has y-intercept $y = 3$ and x-intercept $x = 6$.

(b) The line ℓ_2: $x + 2y = 0$ passes through the origin, and $y = -\frac{1}{2}$ when $x = 1$.

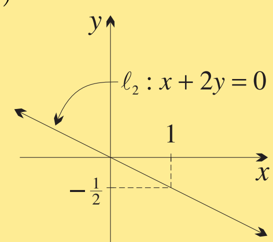
(c) The line ℓ_3 is horizontal with y-intercept 2.

(d) The line ℓ_4 is vertical with x-intercept -3.

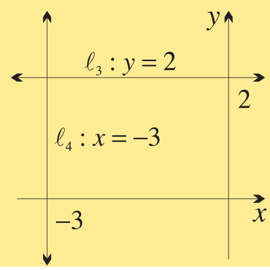
(a)



(b)



(c), (d)



Quadratic Functions: Sketches of quadratic functions will be required before their systematic treatment in Chapters 8 and 9. A *quadratic* is a function of the form

$$f(x) = ax^2 + bx + c, \quad \text{where } a,\ b \text{ and } c \text{ are constants, and } a \neq 0.$$

The graph of a quadratic function is a parabola with axis of symmetry parallel to the y-axis. Normally, four points should be shown on any sketch — the y-intercept, the two x-intercepts (which may coincide or may not exist), and the vertex. There are four steps for finding these points.

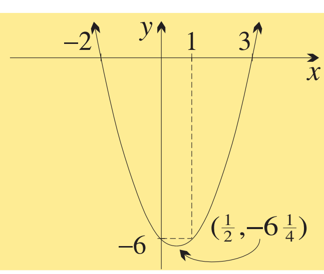
THE FOUR STEPS IN SKETCHING A QUADRATIC FUNCTION:

1. If a is positive, the parabola is concave up.
   If a is negative, the parabola is concave down.
2. To find the y-intercept, put $x = 0$.
3. To find the x-intercepts:
   (a) factor $f(x)$ and write down the x-intercepts, or
   (b) complete the square, or
   25 (c) use the formula, $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.
4. To find the vertex, first find the axis of symmetry:
   (a) by finding the average of the x-intercepts, or
   (b) by completing the square, or
   (c) by using the formula for the axis of symmetry, $x = \dfrac{-b}{2a}$.
   Then find the y-coordinate of the vertex by substituting back into $f(x)$.

WORKED EXERCISE: Sketch the graph of $y = x^2 - x - 6$, using the method of factoring.

SOLUTION: Factoring, $y = (x - 3)(x + 2)$.

1. Since $a = 1$, the parabola is concave up.
2. The y-intercept is -6.
3. The x-intercepts are $x = 3$ and $x = -2$.
4. The axis of symmetry is $x = \frac{1}{2}$ (average of zeroes),
   and when $x = \frac{1}{2}$, $y = -6\frac{1}{4}$, so the vertex is $(\frac{1}{2}, -6\frac{1}{4})$.



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WORKED EXERCISE: Sketch the graph of $y = x^2 + 2x - 3$, using the method of completing the square.

SOLUTION: The curve is concave up, with y-intercept $y = -3$.

Completing the square, $y = (x^2 + 2x + 1) - 1 - 3$

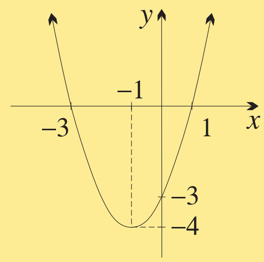
$$y = (x + 1)^2 - 4.$$

So the axis of symmetry is $x = -1$,
and the vertex is $(-1, -4)$.

Putting $y = 0$, $(x + 1)^2 = 4$

$$x + 1 = 2 \text{ or } -2,$$

so the x-intercepts are $x = 1$ and $x = -3$.



WORKED EXERCISE: Sketch the graph of $y = -x^2 + 4x - 5$, using the formulae for the zeroes and the axis of symmetry.

SOLUTION: The curve is concave down, with y-intercept $y = -5$.

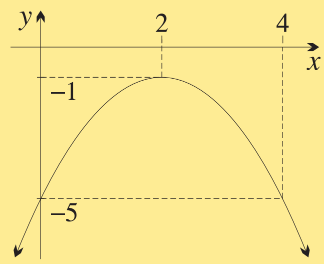
Since $b^2 - 4ac = -4$ is negative, there are no x-intercepts.

The axis of symmetry is $x = -\dfrac{b}{2a}$

$$x = 2.$$

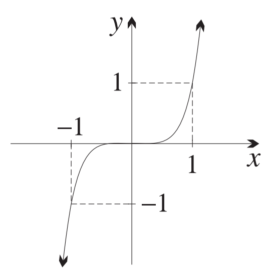
When $x = 2$, $y = -1$, so the vertex is $(2, -1)$.

By reflecting $(0, -5)$ about the axis $x = 2$, when $x = 4$, $y = -5$.



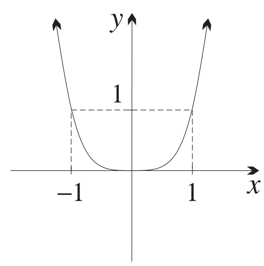
Higher Powers of x: On the right is the graph of $y = x^3$. All odd powers look similar, becoming flatter near the origin as the index increases, and steeper further away.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8



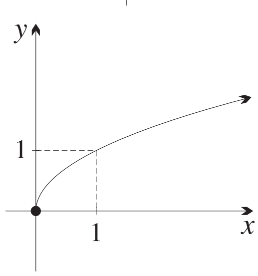
On the right is the graph of $y = x^4$. All even powers look similar — they are always positive, and become flatter near the origin as the index increases, and steeper further away.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	16	1	$\frac{1}{16}$	0	$\frac{1}{16}$	1	16



The Function $y = \sqrt{x}$: The graph of $y = \sqrt{x}$ is the upper half of a parabola on its side, as can be seen by squaring both sides to give $y^2 = x$. Remember that the symbol $\sqrt{x}$ means the *positive* square root of x, so the lower half is excluded:

x	0	$\frac{1}{4}$	1	2	4
y	0	$\frac{1}{2}$	1	$\sqrt{2}$	2



Circles and Semicircles: The graph of $x^2 + y^2 = a^2$ is a circle with radius $a > 0$ and centre the origin, as sketched on the left below. This graph is not a function — solving for y yields

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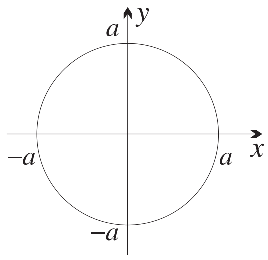
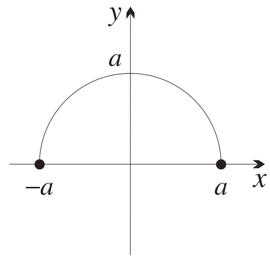
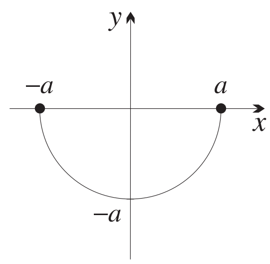
$$y = \sqrt{a^2 - x^2} \quad \text{or} \quad y = -\sqrt{a^2 - x^2}\,,$$

which means there are two values of y for some values of x. The positive square root $y = \sqrt{a^2 - x^2}$, however, is a function, whose graph is the upper semicircle below. Similarly, the negative square root $y = -\sqrt{a^2 - x^2}$ is also a function, whose graph is the lower semicircle below:



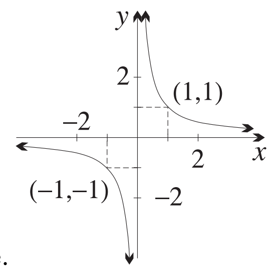




The Rectangular Hyperbola: The *reciprocal function* $y = 1/x$ is well known, but it is worth careful attention because it is the best place to introduce some important ideas about limits and asymptotes. Here is a table of values and a sketch of the graph, which is called a *rectangular hyperbola*:

x	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	2	5	10
y	$*$	10	5	2	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$

x	-10	-5	-2	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{1}{10}$
y	$-\frac{1}{10}$	$-\frac{1}{5}$	$-\frac{1}{2}$	-1	-2	-5	-10



The star ($*$) at $x = 0$ indicates that the function is not defined there.

Limits and Asymptotes Associated with the Rectangular Hyperbola: Here is the necessary language and notation for describing the behaviour of $y = 1/x$ near $x = 0$ and for large x.

1. The domain is $x \neq 0$, because the reciprocal of 0 is not defined. The range can be read off the graph — it is $y \neq 0$.

2. (a) As x becomes very large positive, y becomes very small indeed. We can make y 'as close as we like' to 0 by choosing x sufficiently large. The formal notation for this is

$$y \to 0 \text{ as } x \to \infty \qquad \text{or} \qquad \lim_{x\to\infty} y = 0.$$

(b) On the left, as x becomes very large negative, y also becomes very small:

$$y \to 0 \text{ as } x \to -\infty \qquad \text{or} \qquad \lim_{x\to-\infty} y = 0.$$

(c) The x-axis is called an *asymptote* of the curve (from the Greek word *asymptotos*, meaning 'apt to fall together'), because the curve gets 'as close as we like' to the x-axis for sufficiently large x and for sufficiently large negative x.

3. (a) When x is a very small positive number, y becomes very large, because the reciprocal of a very small number is very large. We can make y 'as large as we like' by taking sufficiently small but still positive values of x. The formal notation is

$$y \to \infty \text{ as } x \to 0^+.$$

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(b) On the left-hand side of the origin, y is negative and can be made 'as large negative as we like' by taking sufficiently small negative values of x:

$$y \to -\infty \text{ as } x \to 0^-.$$

(c) The y-axis is also an asymptote of the curve, because the curve gets 'as close as we like' to the y-axis when x is sufficiently close to zero.

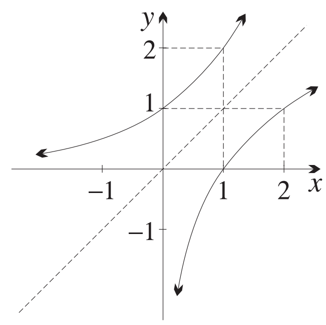
Exponential and Logarithmic Functions: Functions of the form $y = a^x$, where the base a of the power is positive and not equal to 1, are called *exponential functions*, because the variable x is in the *exponent* or *index*. Functions which are of the form $y = \log_a x$ are called *logarithmic functions*. Here are the graphs of the two functions $y = 2^x$ and $y = \log_2 x$.

$y = 2^x$

x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

$y = \log_2 x$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	-2	-1	0	1	2



The two graphs are reflections of each other in the line $y = x$. This is because the second table is just the first table turned upside down, which simply swaps the coordinates of each ordered pair in the function. The two functions are therefore *inverse functions* of each other, and inverse functions in general will be the subject of the next section. Corresponding to the reflection, the x-axis is an asymptote for $y = 2^x$, and the y-axis is an asymptote for $y = \log_2 x$.

Exercise 2G

1. Sketch the following special cases of linear graphs:

(a) $x = 1$ (c) $x = -1{\cdot}5$ (e) $y = 2x$ (g) $x - y = 0$

(b) $y = -2$ (d) $y = 3$ (f) $y = -\frac{1}{2}x$ (h) $3x + 2y = 0$

2. For each linear function, find the y-intercept by putting $x = 0$, and the x-intercept by putting $y = 0$. Then sketch each curve.

(a) $y = x + 1$ (e) $x + y - 1 = 0$ (i) $2x - 3y - 12 = 0$

(b) $y = 4 - 2x$ (f) $2x - y + 2 = 0$ (j) $x + 4y + 6 = 0$

(c) $y = \frac{1}{2}x - 3$ (g) $x - 3y - 3 = 0$ (k) $5x + 2y - 10 = 0$

(d) $y = -3x - 6$ (h) $x - 2y - 4 = 0$ (l) $-5x + 2y + 15 = 0$

3. Determine the main features of each parabola — the vertex, intercepts and, where necessary, any additional points symmetric about the axis. Then sketch a graph showing these features.

(a) $y = x^2$ (c) $y = \frac{1}{2}x^2$ (e) $y = x^2 - 4$ (g) $y = 2 - \frac{1}{2}x^2$

(b) $y = -x^2$ (d) $y = x^2 + 1$ (f) $y = 9 - x^2$ (h) $y = -1 - x^2$

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4. These quadratics are already factored. Find the x-intercepts and y-intercept. Then find the vertex by finding the average of the x-intercepts and substituting. Sketch the graph, then write down the range:

(a) $y = (x-4)(x-2)$
(b) $y = (x-4)(x+2)$
(c) $y = -(x+2)(x+6)$
(d) $y = -(x+2)(x-6)$
(e) $y = x(x-3)$
(f) $y = (x+1)(x+4)$

5. Factor these quadratics. Then find the x-intercepts, y-intercept and vertex, and sketch:

(a) $y = x^2 + 6x + 8$
(b) $y = x^2 - 4x + 4$
(c) $y = x^2 - 10x - 11$
(d) $y = -x^2 + 2x$
(e) $y = -x^2 + 2x + 3$
(f) $y = -x^2 - 2x - 1$
(g) $y = x^2 - 9$
(h) $y = x^2 + 9x + 14$
(i) $y = 4 - x^2$

6. (a) Use a calculator and a table of values to plot the graphs of $y = x$, $y = x^2$, $y = x^3$ and $y = x^4$ on the same graph, for $-1{\cdot}25 \le x \le 1{\cdot}25$. Use a scale of 2 cm to 1 unit and plot points every $0{\cdot}25$ units along the x-axis.

(b) Use a calculator and a table of values to plot the graph of $y = \sqrt{x}$ for $0 \le x \le 4$. Use a scale of 2 cm to 1 unit and plot points every $0{\cdot}5$ units along the x-axis. On the same number plane graph, plot $y = x^2$, for $0 \le x \le 2$, and confirm that the two curves are reflections of each other in the line $y = x$.

7. Identify the centre and radius of each of these circles and semicircles. Then sketch its graph, and write down its domain and range:

(a) $x^2 + y^2 = 1$
(b) $x^2 + y^2 = 9$
(c) $y^2 + x^2 = \frac{1}{4}$
(d) $y^2 + x^2 = \frac{9}{4}$
(e) $y = \sqrt{4 - x^2}$
(f) $y = -\sqrt{1 - x^2}$
(g) $y = \sqrt{\frac{25}{4} - x^2}$
(h) $x = \sqrt{9 - y^2}$
(i) $x = -\sqrt{\frac{1}{4} - y^2}$

8. Use tables of values to sketch these hyperbolas. Then write down the domain and range of each:

(a) $y = \frac{2}{x}$
(b) $y = -\frac{1}{x}$
(c) $xy = 3$
(d) $xy = -2$

9. Use tables of values to sketch these exponential and logarithmic functions. Then write down the domain and range of each:

(a) $y = 3^x$
(b) $y = 3^{-x}$
(c) $y = 10^x$
(d) $y = 10^{-x}$
(e) $y = (\frac{1}{2})^x$
(f) $y = \log_3 x$
(g) $y = \log_4 x$
(h) $y = \log_{10} x$

DEVELOPMENT

10. Factor these quadratics where necessary. Then find the intercepts and vertex and sketch:

(a) $y = (2x-1)(2x-7)$
(b) $y = -x(2x+9)$
(c) $y = 9x^2 - 18x - 7$
(d) $y = 9x^2 - 30x + 25$
(e) $y = -4x^2 + 12x + 7$
(f) $y = -5x^2 + 2x + 3$

11. Complete the square in each quadratic expression and hence find the coordinates of the vertex of the parabola. Use the completed square to find the x-intercepts, then graph:

(a) $y = x^2 - 4x + 3$
(b) $y = x^2 + 2x - 8$
(c) $y = x^2 + 3x + 2$
(d) $y = x^2 - x + 1$

12. Use the formula to find the x-intercepts. Then use the formula for the axis of symmetry, and substitute to find the vertex. Sketch the graphs:

(a) $y = x^2 + 2x - 5$
(b) $y = x^2 - 7x + 3$
(c) $y = 3x^2 - 4x - 1$
(d) $y = 4 + x - 2x^2$

13. Each equation below represents a half-parabola. Draw up a table of values and sketch them, then write down the domain and range of each:

(a) $y = \sqrt{x} + 1$
(b) $y = 1 - \sqrt{x}$
(c) $y = \sqrt{x - 4}$
(d) $y = \sqrt{4 - x}$
(e) $x = \sqrt{y}$
(f) $x = -\sqrt{y}$

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14. Carefully graph each pair of equations on the same number plane. Hence find the intersection points, given that they have integer coordinates:

(a) $y = x$, $y = x^2$
(b) $y = -x$, $y = -x^2 + 2x$
(c) $y = 2x$, $y = x^2 - x$
(d) $y = 1 - 2x$, $y = x^2 - 4x - 2$
(e) $x^2 + y^2 = 1$, $x + y = 1$
(f) $xy = -2$, $y = x - 3$
(g) $y = \frac{12}{x}$, $x^2 + y^2 = 25$
(h) $y = -x^2 + x + 1$, $y = \frac{1}{x}$

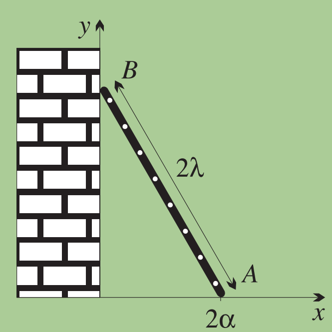
15. Write down the radius of each circle or semicircle and graph it. Also state any points on each curve whose coordinates are both integers:

(a) $x^2 + y^2 = 5$
(b) $y = -\sqrt{2 - x^2}$
(c) $x = \sqrt{10 - y^2}$
(d) $x^2 + y^2 = 17$

16. (a) Show that $(x + y)^2 - (x - y)^2 = 4$ is the equation of a hyperbola. Sketch it.
(b) Show that $(x + y)^2 + (x - y)^2 = 4$ is the equation of a circle. Sketch it.
(c) Solve these two equations simultaneously. Begin by subtracting the equation of the hyperbola from the equation of the circle.
(d) Sketch both curves on the same number plane, showing the points of intersection.

EXTENSION

17. The diagram shows a ladder of length 2λ leaning against a wall so that the foot of the ladder is distant 2α from the wall.



(a) Find the coordinates of B.
(b) Show that the midpoint P of the ladder lies on a circle with centre at the origin. What is the radius of this circle?

18. (a) The line $y = -\frac{1}{4}b^2x + b$ has intercepts at A and B. Find the coordinates of P, the midpoint of AB.
(b) Show that P lies on the hyperbola $y = \dfrac{1}{x}$.
(c) Show that the area of $\triangle OAB$, where O is the origin, is independent of the value of b.

19. The curve $y = 2^x$ is approximated by the parabola $y = ax^2 + bx + c$ for $-1 \le x \le 1$. The values of the constants a, b and c are chosen so that the two curves intersect at $x = -1, 0, 1$.
(a) Find the values of the constant coefficients.
(b) Use this parabola to estimate the values of $\sqrt{2}$ and $1/\sqrt{2}$.
(c) Compare the values found in part (b) with the values obtained by a calculator. Show that the percentage errors are approximately 1·6% and 2·8% respectively.

20. Consider the relation $y^2 = (1 - x^2)(4x^2 - 1)^2$.
(a) Write down a pair of alternative expressions for y that are functions of x.
(b) Find the natural domains of these functions.
(c) Find any intercepts with the axes.
(d) Create a table of values for each function. Select x values every 0·1 units in the domain. Plot the points so found. What is the familiar shape of the original relation? (A graphics calculator or computer may help simplify this task.)

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2 H Inverse Relations and Functions

At the end of the last section, the pair of inverse functions $y = 2^x$ and $y = \log_2 x$ were sketched from a table of values. We saw then how the two curves were reflections of each other in the diagonal line $y = x$, and how the two tables of values were the same except that the rows were reversed. Many functions similarly have a well-defined *inverse function* that sends any output back to the original input. For example, the inverse of the cubing function $y = x^3$ is the cube root function $y = \sqrt[3]{x}$.

$$\begin{array}{cccc}
x & & & y \\
2 & \longrightarrow & & \longrightarrow & 8 \\
1 & \longrightarrow & & \longrightarrow & 1 \\
\frac{1}{2} & \longrightarrow & \text{Cubing} & \longrightarrow & \frac{1}{8} \\
0 & \longrightarrow & y = x^3 & \longrightarrow & 0 \\
-\frac{1}{2} & \longrightarrow & & \longrightarrow & -\frac{1}{8} \\
-1 & \longrightarrow & & \longrightarrow & -1 \\
-2 & \longrightarrow & & \longrightarrow & -8
\end{array}
\qquad
\begin{array}{ccccc}
x & & & & y \\
8 & \longrightarrow & & \longrightarrow & 2 \\
1 & \longrightarrow & & \longrightarrow & 1 \\
\frac{1}{8} & \longrightarrow & \text{Cube Root} & \longrightarrow & \frac{1}{2} \\
0 & \longrightarrow & y = \sqrt[3]{x} & \longrightarrow & 0 \\
-\frac{1}{8} & \longrightarrow & & \longrightarrow & -\frac{1}{2} \\
-1 & \longrightarrow & & \longrightarrow & -1 \\
-8 & \longrightarrow & & \longrightarrow & -2
\end{array}$$

The exchanging of input and output can also be seen in the two tables of values, where the two rows are interchanged:

x	2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	-2
x^3	8	1	$\frac{1}{8}$	0	$-\frac{1}{8}$	-1	-8

x	8	1	$\frac{1}{8}$	0	$-\frac{1}{8}$	-1	-8
$\sqrt[3]{x}$	2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	-2

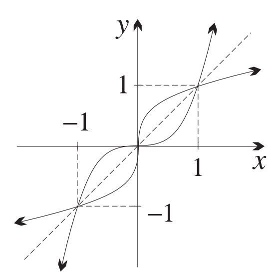
This exchanging of input and output means that the coordinates of each ordered pair are exchanged, so we are led to a definition that can be applied to any relation, whether it is a function or not:

26 DEFINITION: The inverse relation is obtained by reversing each ordered pair.

The exchanging of first and second components means that the domain and range are exchanged:

27 DOMAIN AND RANGE OF THE INVERSE: The domain of the inverse is the range of the relation, and the range of the inverse is the domain of the relation.

Graphing the Inverse Relation: Reversing an ordered pair means that the original first coordinate is read off the vertical axis, and the original second coordinate is read off the horizontal axis. Geometrically, this exchanging can be done by reflecting the point in the diagonal line $y = x$, as can be seen by comparing the graphs of $y = x^3$ and $y = \sqrt[3]{x}$, which are drawn here on the same pair of axes.



28 THE GRAPH OF THE INVERSE: The graph of the inverse relation is obtained by reflecting the original graph in the diagonal line $y = x$.

Finding the Equations and Conditions of the Inverse Relation: When the coordinates are exchanged, the x-variable becomes the y-variable and the y-variable becomes the x-variable, so the method for finding the equation and conditions of the inverse is:

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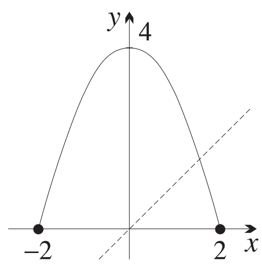
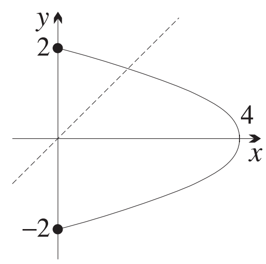
29 THE EQUATION OF THE INVERSE: To find the equations and conditions of the inverse relation, write x for y and y for x every time each variable occurs.

For example, the inverse of $y = x^3$ is $x = y^3$, which can then be solved for y to give $y = \sqrt[3]{x}$.

Testing whether the Inverse Relation is a Function: It is not true in general that the inverse of a function is a function. For example, the sketches below show the graphs of another function and its inverse:



$y = 4 - x^2$, where $-2 \le x \le 2$



$x = 4 - y^2$, where $-2 \le y \le 2$

The first is clearly a function, but the second fails the vertical line test, since the vertical line $x = 3$ crosses the graph at the two points $(3, 1)$ and $(3, -1)$. Before the second graph was even drawn, however, it was obvious from the first graph that the inverse would not be a function because the *horizontal line* $y = 3$ crossed the graph twice (and notice that reflection in $y = x$ exchanges horizontal and vertical lines).

30 HORIZONTAL LINE TEST: The inverse relation of a given relation is a function if and only if no horizontal line crosses the original graph more than once.

Inverse Function Notation: If $f(x)$ is a function whose inverse is also a function, that function is written as $f^{-1}(x)$. The index -1 used here means 'inverse function' and is not to be confused with its more common use for the reciprocal. To return to the original example,

$$\text{if } f(x) = x^3, \text{ then } f^{-1}(x) = \sqrt[3]{x}.$$

The inverse function sends each number back where it came from. Hence if the function and the inverse function are applied successively to a number, in either order, the result is the original number. For example, using cubes and cube roots,

$$\left(\sqrt[3]{8}\right)^3 = 2^3 = 8 \qquad \text{and} \qquad \sqrt[3]{8^3} = \sqrt[3]{512} = 8.$$

31 INVERSE FUNCTIONS: If the inverse relation of a function $f(x)$ is also a function, the inverse function is written as $f^{-1}(x)$. The composition of the function and its inverse sends every number back to itself:

$$f^{-1}\big(f(x)\big) = x \qquad \text{and} \qquad f\big(f^{-1}(x)\big) = x.$$

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WORKED EXERCISE: Find the equations of the inverse relations of these functions. If the inverse is a function, find an expression for $f^{-1}(x)$, and then verify that $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(x)\big) = x$.

(a) $f(x) = 6 - 2x$, where $x > 0$

(b) $f(x) = x^3 + 2$

(c) $f(x) = \dfrac{1-x}{1+x}$

(d) $f(x) = x^2 - 9$

SOLUTION:

(a) Let $y = 6 - 2x$, where $x > 0$.

Then the inverse has the equation $x = 6 - 2y$, where $y > 0$

$$y = 3 - \tfrac{1}{2}x, \text{ where } y > 0.$$

The condition $y > 0$ means that $x < 6$,

so $f^{-1}(x) = 3 - \tfrac{1}{2}x$, where $x < 6$.

Verifying, $f^{-1}\big(f(x)\big) = f^{-1}(6-2x) = 3 - \tfrac{1}{2}(6-2x) = 3 - 3 + x = x$

and $f\big(f^{-1}(x)\big) = f(3 - \tfrac{1}{2}x) = 6 - 2(3 - \tfrac{1}{2}x) = 6 - 6 + x = x$.

(b) Let $y = x^3 + 2$.

Then the inverse has the equation $x = y^3 + 2$

$$y = \sqrt[3]{x-2}\,.$$

So $f^{-1}(x) = \sqrt[3]{x-2}\,$.

Verifying, $f^{-1}\big(f(x)\big) = \sqrt[3]{(x^3+2)-2} = x$

and $f\big(f^{-1}(x)\big) = \big(\sqrt[3]{x-2}\big)^3 + 2 = x$.

(c) Let $y = \dfrac{1-x}{1+x}$.

Then the inverse has equation $x = \dfrac{1-y}{1+y}$.

$\boxed{\times\,(1+y)}$

$$
\begin{aligned}
x + xy &= 1 - y \\
y + xy &= 1 - x \quad \text{(terms in } y \text{ on one side)} \\
y(1+x) &= 1 - x \quad \text{(the key step)} \\
y &= \frac{1-x}{1+x}.
\end{aligned}
$$

So $f^{-1}(x) = \dfrac{1-x}{1+x}$.

Notice that this function and its inverse are identical, so that if the function is applied twice, each number is sent back to itself.

For example, $f\big(f(2)\big) = f\left(-\dfrac{1}{3}\right) = \dfrac{1\frac{1}{3}}{\frac{2}{3}} = 2$.

In general, $f\big(f(x)\big) = \dfrac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = \dfrac{(1+x)-(1-x)}{(1+x)+(1-x)} = \dfrac{2x}{2} = x$.

(d) The function $f(x) = x^2 - 9$ fails the horizontal line test. For example, $f(3) = f(-3) = 0$, which means that the x-axis meets the graph twice. So the inverse relation of $f(x)$ is not a function. Alternatively, one can say that the inverse relation is $x = y^2 - 9$, which on solving for y gives

$$y = \sqrt{x+9} \;\text{ or }\; -\sqrt{x+9},$$

which is not unique, and so the inverse relation is not a function.

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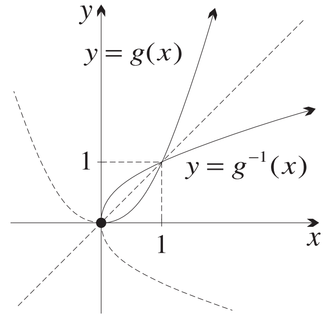
Restricting the Domain so the Inverse is a Function: When the inverse of a function is not a function, it is often convenient to restrict the domain of the function so that this restricted function has an inverse function. The example of taking squares and square roots should already be well known. The function $y = x^2$ does not have an inverse function, because, for example, 49 has two square roots, 7 and -7. If, however, we restrict the domain to $x \ge 0$ and define a new restricted function

$$g(x) = x^2, \quad \text{where } x \ge 0,$$

then the inverse function is $g^{-1}(x) = \sqrt{x}$, where as explained earlier, the symbol $\sqrt{\ }$ means 'take the positive square root (or zero)'. On the right are the graphs of the restricted function and its inverse function, with the unrestricted function and its inverse relation shown dotted. These ideas will be developed in more general situations in the Year 12 Volume.



Exercise 2H

1. Draw the inverse relation for each of the following relations by reflecting in the line $y = x$:

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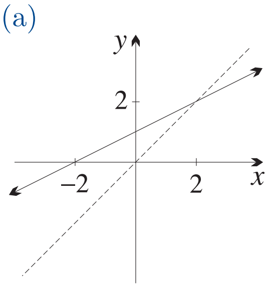

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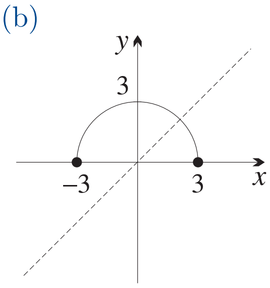

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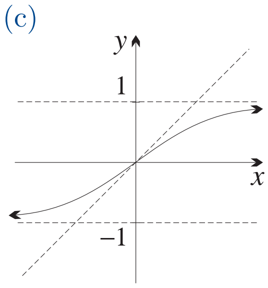

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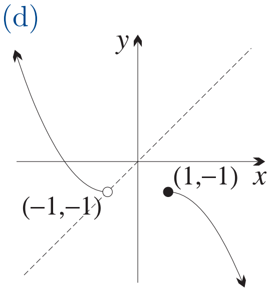

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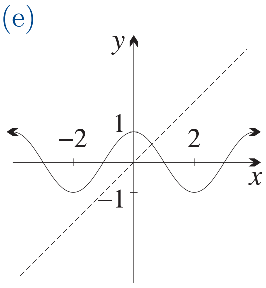

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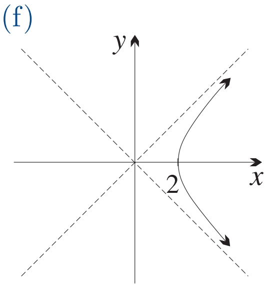
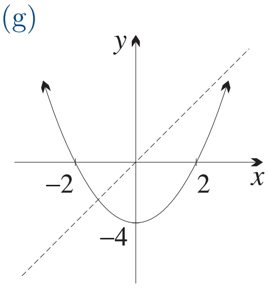


(g)


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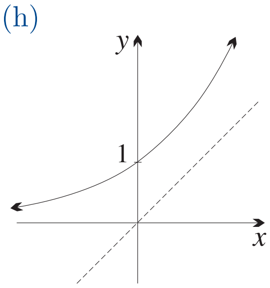

2. Use the vertical and horizontal line tests to determine which relations and which inverse relations drawn in the previous question are also functions.

3. Determine the inverse algebraically by swapping x and y and then making y the subject:

(a) $y = 3x - 2$
(b) $y = \frac{1}{2}x + 1$
(c) $y = 3 - \frac{1}{2}x$
(d) $x - y + 1 = 0$
(e) $2x + 5y - 10 = 0$
(f) $y = 2$

4. For each function in the previous question, draw a graph of the function and its inverse on the same number plane to verify the reflection property. Draw a separate number plane for each part.

5. Find the inverse algebraically by swapping x and y and then making y the subject:

(a) $y = \dfrac{1}{x} + 1$
(b) $y = \dfrac{1}{x+1}$
(c) $y = \dfrac{x+2}{x-2}$
(d) $y = \dfrac{3x}{x+2}$

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6. Swap x and y and solve for y to find the inverse of each of the following functions. What do you notice, and what is the geometric significance of this?

(a) $y = \dfrac{1}{x}$ (b) $y = \dfrac{2x-2}{x-2}$ (c) $y = \dfrac{-3x-5}{x+3}$ (d) $y = -x$

DEVELOPMENT

7. Each pair of functions $f(x)$ and $g(x)$ are mutual inverses. Verify in each case by substitution that: (i) $f\big(g(2)\big) = 2$, and (ii) $g\big(f(2)\big) = 2$.

(a) $f(x) = x + 13$ and $g(x) = x - 13$
(b) $f(x) = 7x$ and $g(x) = \frac{1}{7}x$
(c) $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}(x-6)$
(d) $f(x) = x^3 - 6$ and $g(x) = \sqrt[3]{x+6}$

Verify more generally in each case that: (iii) $f\big(g(x)\big) = x$, and (iv) $g\big(f(x)\big) = x$.

8. Graph each relation and its inverse. Find the equation of the inverse relation. In the cases where the inverse is a function, make y the subject of this equation:

(a) $(x-3)^2 + y^2 = 4$
(b) $y = 2^{-x}$
(c) $(x+1)^2 + (y+1)^2 = 9$
(d) $y = x^2 - 4$
(e) $y = x^2 + 1$
(f) $y = \log_3 x$

9. Write down the inverse of each function, solving for y if it is a function. Sketch the function and the inverse on the same graph and observe the symmetry in the line $y = x$:

(a) $y = x^2$
(b) $y = 2x - x^2$
(c) $y = -\sqrt{x}$
(d) $y = 2^x$
(e) $y = -\sqrt{4 - x^2}$
(f) $y = \left(\frac{3}{2}\right)^x$

10. Explain whether the inverse relation is a function. If it is a function, find $f^{-1}(x)$ and verify the two identities $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(x)\big) = x$.

(a) $f(x) = x^2$
(b) $f(x) = \sqrt{x}$
(c) $f(x) = x^4$
(d) $f(x) = x^3 + 1$
(e) $f(x) = 9 - x^2$
(f) $f(x) = 9 - x^2$, $x \ge 0$
(g) $f(x) = 3^{-x^2}$
(h) $f(x) = \dfrac{1-x}{3+x}$
(i) $f(x) = x^2$, $x \le 0$
(j) $f(x) = x^2 - 2x$, $x \ge 1$
(k) $f(x) = x^2 - 2x$, $x \le 1$
(l) $f(x) = \dfrac{x+1}{x-1}$

11. (a) Show that the inverse function of $y = \dfrac{ax+b}{x+c}$ is $y = \dfrac{b-cx}{x-a}$.

(b) Hence show that $y = \dfrac{ax+b}{x+c}$ is its own inverse if and only if $a + c = 0$.

12. Sketch on separate graphs: (a) $y = -x^2$ (b) $y = -x^2$, for $x \ge 0$
Draw the inverse of each on the same graph, then comment on the similarities and differences between parts (a) and (b).

EXTENSION

13. Suggest restrictions on the domains of the following in order that each have an inverse that is a function (there may be more than one answer). Draw the modified function and its inverse:

(a) $y = -\sqrt{4 - x^2}$
(b) $y = \dfrac{1}{x^2}$
(c) $y = x^3 - x$
(d) $y = \sin(90x^\circ)$
(e) $y = \sqrt{x^2}$
(f) $y = \tan(90x^\circ)$

14. The logarithm laws indicate that $\log_3(x^n) = n\log_3(x)$. Explain why $y = \log_3(x^2)$ does not have an inverse that is a function, yet $y = 2\log_3(x)$ does.

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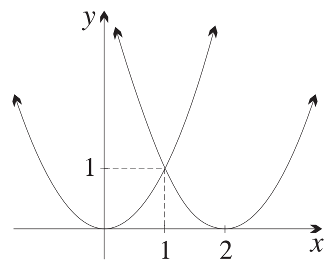
2 I Shifting and Reflecting Known Graphs

There are various standard ways to manipulate given graphs to produce further graphs. For example, a graph can be shifted or stretched or reflected, or two graphs can be combined. Using these processes on known graphs can extend considerably the range of functions and relations whose graphs can be quickly recognised and drawn. This section deals with shifting and reflecting, and the next section deals with some further transformations.

Shifting Left and Right: The graphs of $y = x^2$ and $y = (x-2)^2$ are sketched from their tables of values. They make it clear that the graph of $y = (x-2)^2$ is obtained by shifting the graph of $y = x^2$ to the right by 2 units.

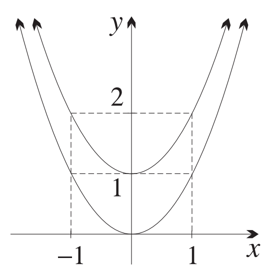
x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$(x-2)^2$	16	9	4	1	0	1	4



32 SHIFTING LEFT AND RIGHT: To shift k units to the right, replace x by $x - k$. Alternatively, if the graph is a function, the new function rule is $y = f(x-k)$.

Shifting Up and Down: The graph of $y = x^2 + 1$ is produced by shifting the graph of $y = x^2$ upwards 1 unit, because the values in the table for $y = x^2 + 1$ are all 1 more than the values in the table for $y = x^2$:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
x^2+1	10	5	2	1	2	5	10



Writing the transformed graph as $y - 1 = x^2$ makes it clear that the shifting has been obtained by replacing y by $y - 1$, giving a rule completely analogous to that for horizontal shifting.

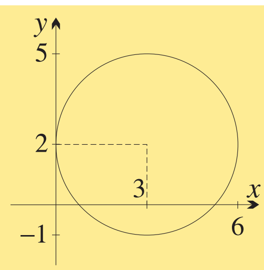
33 SHIFTING UP AND DOWN: To shift ℓ units upwards, replace y by $y - \ell$. Alternatively, if the graph is a function, the new function rule is $y = f(x) + \ell$.

WORKED EXERCISE: Find the centre and radius of the circle $x^2 + y^2 - 6x - 4y + 4 = 0$, then sketch it.

SOLUTION: Completing the square in both x and y,

$$(x^2 - 6x + 9) + (y^2 - 4y + 4) + 4 = 9 + 4$$
$$(x-3)^2 + (y-2)^2 = 9.$$

This is just $x^2 + y^2 = 9$ shifted right 3 and up 2, so the centre is $(3, 2)$ and the radius is 3.



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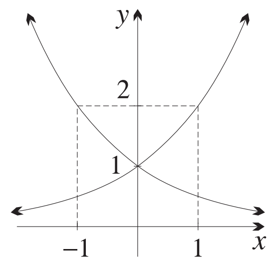
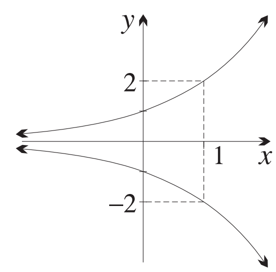
Reflection in the y-axis: When the tables of values for $y = 2^x$ and $y = 2^{-x}$ are both written down, it is clear that the graphs of $y = 2^x$ and $y = 2^{-x}$ must be reflections of each other in the y-axis.



x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

34 REFLECTION IN THE y-AXIS: To reflect in the y-axis, replace x by $-x$. Alternatively, if the graph is a function, the new function rule is $y = f(-x)$.

Reflection in the x-axis: All the values in the table below for $y = -2^x$ are the opposites of the values in the table for $y = 2^x$. This means that the graphs of $y = -2^x$ and $y = 2^x$ are reflections of each other in the x-axis.



x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
-2^x	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-8

Writing the transformed graph as $-y = 2^x$ makes it clear that the reflection has been obtained by replacing y by $-y$, giving a rule completely analogous to that for reflection in the y-axis.

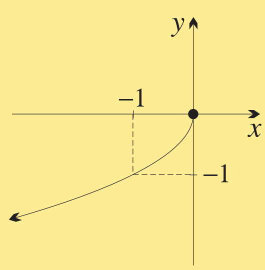
35 REFLECTION IN THE x-AXIS: To reflect in the x-axis, replace y by $-y$. Alternatively, if the graph is a function, the new function rule is $y = -f(x)$.

WORKED EXERCISE: From the graph of $y = \sqrt{x}$, deduce the graph of $y = -\sqrt{-x}$.

SOLUTION: The equation can be rewritten as

$$-y = \sqrt{-x}$$

so the graph is reflected successively in both axes.



NOTE: Reflection in both the x-axis and the y-axis is the same as a rotation of $180°$ about the origin, and the order in which the reflections are done does not matter. This rotation of $180°$ about the origin is sometimes called the *reflection in the origin*, because every point in the plane is transformed along a line through the origin to a point on the opposite side of the origin and the same distance away.

Reflection in the Line $y = x$: The graphs of a relation and its inverse relation are reflections of each other in the diagonal line $y = x$, as discussed earlier in Section 2H.

36 REFLECTION IN $y = x$: To reflect in the line $y = x$, replace x by y and y by x.

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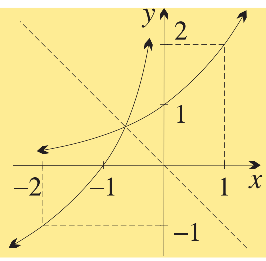
WORKED EXERCISE: [A harder example] What is the equation of the curve obtained when $y = 2^x$ is reflected in the line $y = -x$? Solve the resulting equation for y, and sketch the curves.

SOLUTION: Reflection in $y = -x$ is obtained by reflecting successively in $y = x$, in the x-axis, and in the y-axis (verify this with a square piece of paper). So the successive equations are

$$y = 2^x \longrightarrow x = 2^y \longrightarrow x = 2^{-y} \longrightarrow -x = 2^{-y}.$$

Solving the last equation for y gives $y = -\log_2(-x)$.



Exercise 2I

1. Write down the new equation for each function or relation after the given shift has been applied. Draw a graph of the image after the shift:

(a) $y = x^2$: right 1 unit
(b) $y = 2^x$: down 3 units
(c) $y = \log_2 x$: left 2 units
(d) $y = \frac{1}{x}$: right 3 units
(e) $x^2 + y^2 = 9$: up 1 unit
(f) $y = x^2 - 4$: left 1 unit
(g) $xy = 1$: down 1 unit
(h) $y = \sqrt{x}$: up 2 units

2. Repeat the previous question for a reflection in the given line:

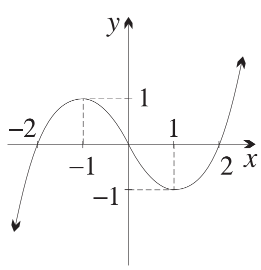
(a) x-axis
(b) y-axis
(c) $y = x$
(d) x-axis
(e) $y = x$
(f) $y = x$
(g) x-axis
(h) y-axis

3. Use the shifting results and completion of the square where necessary to determine the centre and radius of each circle:

(a) $(x + 1)^2 + y^2 = 4$
(b) $(x - 1)^2 + (y - 2)^2 = 1$
(c) $x^2 - 2x + y^2 - 4y - 4 = 0$
(d) $x^2 + 6x + y^2 - 8y = 0$
(e) $x^2 - 10x + y^2 + 8y + 32 = 0$
(f) $x^2 + 14x + 14 + y^2 - 2y = 0$

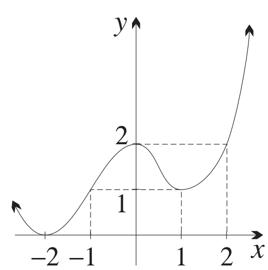
4. In each case an unknown function has been drawn. Draw the functions specified below:

(a)



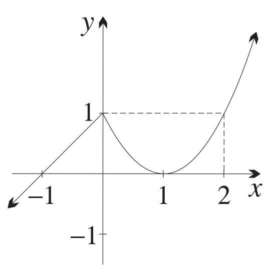
(i) $y = f(x - 2)$ (ii) $y = f(x + 1)$

(b)



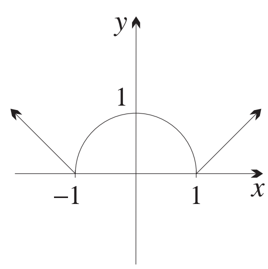
(i) $y = P(x + 2)$ (ii) $y = P(x + 1)$

(c)



(i) $y - 1 = h(x)$ (ii) $y = h(x) - 1$

(d)



(i) $y - 1 = g(x)$ (ii) $y = g(x - 1)$

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DEVELOPMENT

5. In each part explain how the graph of each subsequent equation is a transformation of the first graph (there may be more than one answer), then sketch each curve:

(a) From $y = 2x$: (i) $y = 2x + 4$ (ii) $y = 2x - 4$ (iii) $y = -2x + 4$

(b) From $y = x^2$: (i) $y = x^2 + 9$ (ii) $y = x^2 - 9$ (iii) $y = (x - 3)^2$

(c) From $y = x^2$: (i) $y = (x + 1)^2$ (ii) $y = -(x + 1)^2$ (iii) $y = -(x + 1)^2 + 2$

(d) From $y = \sqrt{x}$: (i) $y = \sqrt{x + 4}$ (ii) $y = -\sqrt{x + 4}$ (iii) $y = -\sqrt{x}$

(e) From $y = \dfrac{1}{x}$: (i) $y = \dfrac{1}{x} + 1$ (ii) $y = \dfrac{1}{x + 2} + 1$ (iii) $y = -\dfrac{1}{x}$

6. Sketch $y = \dfrac{1}{x}$, then use the shifting procedures to sketch the following graphs. Find any x-intercepts and y-intercepts, and mark them on your graphs:

(a) $y = \dfrac{1}{x - 2}$

(b) $y = 1 + \dfrac{1}{x - 2}$

(c) $y = \dfrac{1}{x - 2} - 2$

(d) $y = \dfrac{1}{x + 1} - 1$

(e) $y = 3 + \dfrac{1}{x + 2}$

(f) $y = \dfrac{1}{x - 3} + 4$

7. Complete squares, then sketch each of these circles, stating the centre and radius. By substituting $x = 0$ and then $y = 0$, find any intercepts with the axes:

(a) $x^2 - 4x + y^2 - 10y = -20$

(b) $x^2 + y^2 + 6y - 1 = 0$

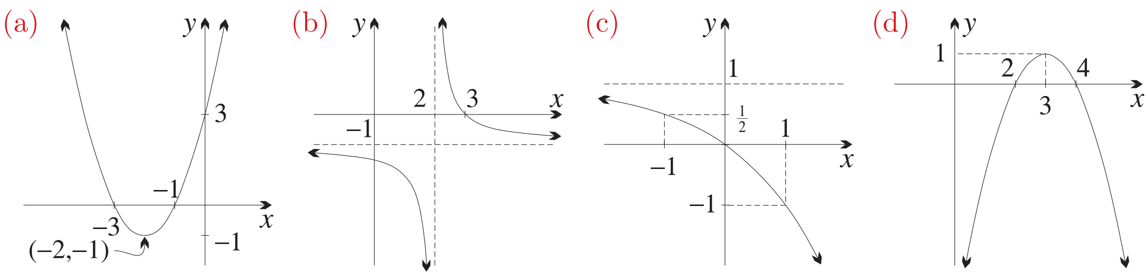
(c) $x^2 + 4x + y^2 - 8y = 0$

(d) $x^2 - 2x + y^2 + 4y = 1$

8. (a) Find the equation of the image of the circle $(x - 2)^2 + (y + 1)^2 = 25$ after it has been reflected in the line $y = x$.

(b) Find the coordinates of the points where the two circles intersect.

9. Describe each graph below as a standard curve transformed by shifts and reflections, and hence write down its equation:



10. [Revision — A medley of curve sketches] Sketch each set of graphs on a single pair of axes, showing all significant points. Use transformations, tables of values, or any other convenient method.

(a) $y = 2x$, $y = 2x + 3$, $y = 2x - 1$.

(b) $y = -\frac{1}{2}x$, $y = -\frac{1}{2}x + 1$, $y = -\frac{1}{2}x - 2$.

(c) $y = x^2$, $y = (x + 2)^2$, $y = (x - 1)^2$.

(d) $x + y = 0$, $x + y = 2$, $x + y = -3$.

(e) $y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$.

(f) $x - y = 0$, $x - y = 1$, $x - y = -2$.

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(g) $x^2 + y^2 = 4$, $x^2 = 1 - y^2$, $y^2 = 25 - x^2$.
(h) $y = 3x$, $x = 3y$, $y = 3x + 1$, $x = 3y + 1$.
(i) $y = 2^x$, $y = 3^x$, $y = 4^x$.
(j) $y = -x$, $y = 4 - x$, $y = x - 4$, $x = -4 - y$.
(k) $y = x^2 - x$, $y = x^2 - 4x$, $y = x^2 + 3x$.
(l) $(x-1)^2 + y^2 = 1$, $(x+1)^2 + y^2 = 1$, $x^2 + (y-1)^2 = 1$.
(m) $y = x^2 - 1$, $y = 1 - x^2$, $y = 4 - x^2$, $y = -1 - x^2$.
(n) $y = (x+2)^2$, $y = (x+2)^2 - 4$, $y = (x+2)^2 + 1$.
(o) $y = x^2 - 1$, $y = x^2 - 4x + 3$, $y = x^2 - 8x + 15$.
(p) $y = \sqrt{9 - x^2}$, $x = -\sqrt{4 - y^2}$, $y = -\sqrt{1 - x^2}$.
(q) $y = \dfrac{1}{x}$, $y = 1 + \dfrac{1}{x}$, $y = -\dfrac{1}{x}$.
(r) $y = \sqrt{x}$, $y = \sqrt{x} + 1$, $y = \sqrt{x + 1}$.
(s) $y = 2^x$, $y = 2^x - 1$, $y = 2^{x-1}$.
(t) $y = \dfrac{1}{x}$, $y = \dfrac{1}{x - 2}$, $y = \dfrac{1}{x + 1}$.
(u) $y = x^3$, $y = x^3 + 1$, $y = (x+1)^3$.
(v) $y = x^4$, $y = (x-1)^4$, $y = x^4 - 1$.
(w) $y = \sqrt{x}$, $y = -\sqrt{x}$, $y = 4 - \sqrt{x}$.
(x) $y = 2^{-x}$, $y = 2^{-x} - 2$, $y = 3 + 2^{-x}$.
(y) $y = x^2$, $x = y^2$, $x = y^2 - 1$.

11. Consider the straight line equation $x + 2y - 4 = 0$.
(a) The line is shifted 2 units left. Find the equation of the new line.
(b) The original line is shifted 1 unit down. Find the equation of this third line.
(c) Comment on your answers, and draw the lines on the same number plane.

12. Explain the point–gradient formula $y - y_1 = m(x - x_1)$ for a straight line in terms of shifts of the line $y = mx$.

EXTENSION

13. Suppose that $y = f(x)$ is a function whose graph has been drawn.
(a) Let $\mathcal{U}$ be shifting upwards a units and $\mathcal{H}$ be reflection in $y = 0$. Write down the equations of the successive graphs obtained by applying $\mathcal{U}$, then $\mathcal{H}$, then $\mathcal{U}$, then $\mathcal{H}$, and prove that the final graph is the same as the first. Confirm the equations by applying these operations successively to a square book or piece of paper.
(b) Let $\mathcal{R}$ be shifting right by a units. Write down the equations of the successive graphs obtained by applying $\mathcal{R}$, then $\mathcal{H}$, then $\mathcal{R}$, then $\mathcal{H}$, and describe the final graph. Confirm using the square book.
(c) Let $\mathcal{V}$ be reflection in $x = 0$ and $\mathcal{I}$ be reflection in $y = x$. Write down the equations of the successive graphs obtained by applying $\mathcal{I}$, then $\mathcal{V}$, then $\mathcal{I}$, then $\mathcal{H}$, and show that the final graph is the same as the first. Confirm using the square book.
(d) Write down the equations of the successive graphs obtained by applying the combination $\mathcal{I}$-followed-by-$\mathcal{V}$ once, twice, …, until the original graph returns. Confirm using the square book.

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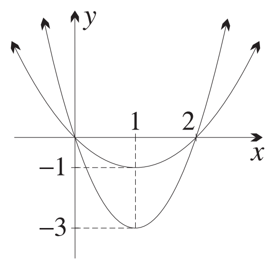
2 J Further Transformations of Known Graphs

This section deals with three further transformations of known graphs: stretching horizontally or vertically, graphing of the reciprocal of a function, and graphing of the sum or difference of two functions. These transformations are a little more difficult than shifting and reflecting, but they will prove very useful later on.

Stretching the Graph Vertically: Each value in the table below for $y = 3x(x-2)$ is three times the corresponding value in the table for $y = x(x-2)$. This means that the graph of $y = 3x(x-2)$ is obtained from the graph of $y = x(x-2)$ by stretching in the vertical direction by a factor of 3:



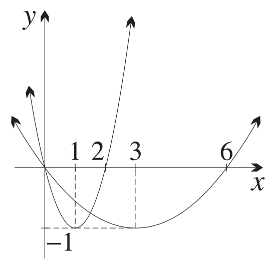
x	-2	-1	0	1	2	3	4
$x(x-2)$	8	3	0	-1	0	3	8
$3x(x-2)$	24	9	0	-3	0	9	24

Writing the transformed graph as $\frac{1}{3}y = x(x-2)$ makes it clear that the stretching has been obtained by replacing y by $\frac{1}{3}y$.

37 STRETCHING VERTICALLY: To stretch the graph in a vertical direction by a factor of a, replace y by y/a.
Alternatively, if the graph is a function, the new function rule is $y = a\,f(x)$.

Stretching the Graph Horizontally: By analogy with the previous example, the graph of $y = x(x-2)$ can be stretched horizontally by a factor of 3 by replacing x by $\frac{1}{3}x$, giving the new function

$$y = \tfrac{1}{3}x\left(\tfrac{1}{3}x - 2\right) = \tfrac{1}{9}x(x-6).$$



The following table of values should make this clear:

x	-3	0	3	6	9
$\frac{1}{3}x$	-1	0	1	2	3
$\frac{1}{3}x\left(\frac{1}{3}x-2\right)$	3	0	-1	0	3

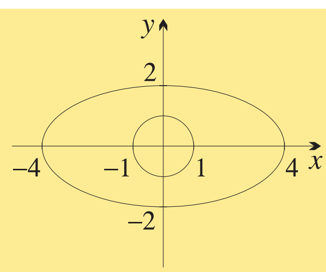
38 STRETCHING HORIZONTALLY: To stretch the graph in a horizontal direction by a factor of a, replace x by x/a.
Alternatively, if the graph is a function, the new function rule is $y = f(x/a)$.

WORKED EXERCISE: Obtain the graph of $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$ from the graph of the circle $x^2 + y^2 = 1$.

SOLUTION: The equation can be rewritten as

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1,$$

which is the unit circle stretched vertically by a factor of 2 and horizontally by a factor of 4.



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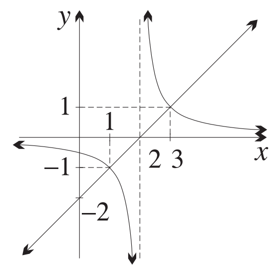
NOTE: Any curve of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is called an *ellipse*. It can be obtained from the unit circle $x^2 + y^2 = 1$ by stretching horizontally by a factor of a and vertically by a factor of b, so that its x-intercepts are a and $-a$ and its y-intercepts are b and $-b$.

The Graph of the Reciprocal Function $y = 1/f(x)$: The graph of $y = x - 2$ is a line with x-intercept 2 and y-intercept -2. The graph of the reciprocal function $y = \dfrac{1}{x-2}$ can be constructed from it by making a series of observations about the reciprocals of numbers:

1. When $x - 2 = 0$ (that is, when $x = 2$), then y is undefined.
2. When $x - 2 \neq 0$, then y has the same sign as $x - 2$.
3. (a) When $x - 2 = 1$, then $y = 1$.
   (b) When $x - 2 = -1$, then $y = -1$.
4. (a) When $x - 2 \to \infty$, then $y \to 0^+$.
   (b) When $x - 2 \to -\infty$, then $y \to 0^-$.
5. (a) When $x - 2 \to 0^+$, then $y \to \infty$.
   (b) When $x - 2 \to 0^-$, then $y \to -\infty$.



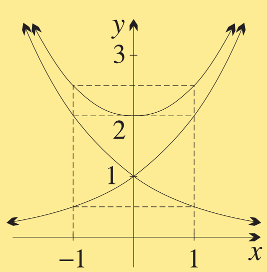
x	0	1	2	3	4
$x - 2$	-2	-1	0	1	2
$\dfrac{1}{x-2}$	$-\frac{1}{2}$	-1	$*$	1	$\frac{1}{2}$

Adding and Subtracting Two Known Functions: Many functions can be written as the sum or difference of two simpler functions:

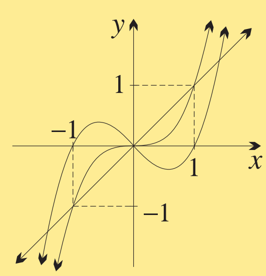
39 SUM AND DIFFERENCE: The graph of the sum or difference of two functions can be obtained from the graphs of the two functions by adding or subtracting the heights at each value of x.

Particularly significant are places where the heights are equal, where the heights are opposite, and where one of the heights is zero.

WORKED EXERCISE: Sketch, on one set of axes, the curves $y = 2^x$ and $y = 2^{-x}$. Hence sketch $y = 2^x + 2^{-x}$.



WORKED EXERCISE: Sketch, on one set of axes, $y = x^3$ and $y = x$. Hence sketch $y = x^3 - x$.



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Exercise 2J

1. On one sets of axes, graph:
   (a) $y = x(4+x)$, $y = 2x(4+x)$, and $y = \frac{x}{2}(4+\frac{x}{2})$.
   (b) $x^2 + y^2 = 36$, $(\frac{x}{2})^2 + (\frac{y}{3})^2 = 36$, and $(2x)^2 + (3y)^2 = 36$.

2. In question 4 of Exercise 2I, the graphs of four unknown functions were drawn. Use stretching procedures to draw these new functions:
   (a) (i) $y = f(2x)$ (ii) $y = 2f(x)$
   (b) (i) $y = P\left(\frac{x}{2}\right)$ (ii) $y = \frac{1}{2}P(x)$
   (c) (i) $\frac{y}{2} = h(x)$ (ii) $y = h\left(\frac{x}{2}\right)$
   (d) (i) $2y = g(x)$ (ii) $y = g(2x)$

3. In each part draw the first two functions on the same number plane, then add or subtract the heights as appropriate to sketch the third function. A table of values may be helpful:
   (a) $y = x$, $y = x^3$, $y = x - x^3$
   (b) $y = x^2$, $y = 2 - x$, $y = x^2 + x - 2$
   (c) $y = -x$, $y = 2^x$, $y = 2^x - x$
   (d) $y = x$, $y = \frac{1}{x}$, $y = x + \frac{1}{x}$

DEVELOPMENT

4. Sketch $x + y = 1$. Then explain how each graph below may be obtained by stretchings of the first graph (there may be more than one answer), and sketch it:
   (a) $\frac{x}{2} + y = 1$ (b) $\frac{x}{2} + \frac{y}{4} = 1$ (c) $2x + y = 1$

5. By considering the reciprocals of numbers, graph: (a) $y = x - 1$, and hence $y = \dfrac{1}{x-1}$,
   (b) $y = x + 2$, and hence $y = \dfrac{1}{x+2}$, (c) $y = 2x - 3$, and hence $y = \dfrac{1}{2x-3}$.

6. Sketch the first two functions on the same number plane and then add or subtract heights to sketch the remaining curves:
   (a) $y = x$, $y = \frac{1}{x}$, $y = x + \frac{1}{x}$, $y = x - \frac{1}{x}$
   (b) $y = x^2$, $y = \frac{1}{x}$, $y = x^2 + \frac{1}{x}$, $y = x^2 - \frac{1}{x}$
   (c) $y = x$, $y = \sqrt{x}$, $y = \sqrt{x} - x$
   (d) $y = \frac{1}{x}$, $y = \sqrt{x}$, $y = \frac{1}{x} - \sqrt{x}$

7. Describe the transformations that have been applied to the curve $y = \sqrt{x}$ in order to obtain each equation, then sketch the given curve:
   (a) $y = 2 - \sqrt{x}$ (b) $y = 2\sqrt{x} - 2$ (c) $y = \sqrt{4 - x}$

8. (a) Sketch a graph of the parabola with equation $y = x^2 - 1$, showing the coordinates of the points where $y = -1$, $y = 0$ and $y = 1$.
   (b) Use the answers to part (a) and other observations about the reciprocals of numbers in order to graph $y = \dfrac{1}{x^2 - 1}$.

9. Carefully graph $y = \sqrt{x+1}$, and hence sketch $y = \dfrac{1}{\sqrt{x+1}}$.

EXTENSION

10. (a) Suggest two simple and distinct transformations for each of the following pairs of curves, by which the second equation may be obtained from the first:
    (i) $y = 2^x$, $y = 2^{x+1}$ (ii) $y = \dfrac{1}{x}$, $y = \dfrac{k^2}{x}$ (iii) $y = 3^x$, $y = 3^{-x}$
    (b) Investigate other combinations of curves and transformations with similar ambiguity.

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11. Determine how the curve $y = x^3 - x$ must be transformed in order to obtain the graph of $y = x^3 - 3x$. [HINT: Only stretchings are involved.]

12. Consider the relation for the *Lemniscate of Bernoulli*, $(x^2+y^2)^2 = x^2-y^2$. Upon expansion this yields $y^4 + (2x^2+1)y^2 + (x^4 - x^2) = 0$, which is a quadratic equation in y^2.

(a) Use the quadratic formula to find y^2 and hence show that y is one of the two functions

$$y = \sqrt{\frac{-(2x^2+1)+\sqrt{8x^2+1}}{2}} \quad \text{or} \quad y = -\sqrt{\frac{-(2x^2+1)+\sqrt{8x^2+1}}{2}}\,.$$

(b) Explain why the domain is restricted to $-1 \le x \le 1$. (c) Find any x or y intercepts.

(d) Create a table of values for each function. Select x values every 0·1 units in the domain. Plot the points so found. What is the shape of the original relation? (A graphics calculator or computer may help simplify this task.)

(e) Write down the equation of this lemniscate if it is shifted right 1 unit and stretched vertically by a factor of 2, and then sketch it.

GO Online Multiple Choice Quiz

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