The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2 **Exercises for Week 6 (beginning 4 September)** 2017

The First Quiz will be conducted during the usual tutorial time. You should attempt these exercises in your own time. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Infinite series: An *infinite series* (or just *series*) has the form

$$a_0 + a_1 + a_2 + \ldots + a_k + \ldots$$

which may be abbreviated to

$$\sum_{k=0}^{\infty} a_k \quad \text{or just} \quad \sum a_k$$

and represents

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k \;=\; \lim_{n\to\infty} (a_0 + a_1 + \ldots + a_n)\ .$$

We call $\displaystyle\sum_{k=0}^{n} a_k = a_0 + a_1 + \ldots + a_n$ a *partial sum*. If the limit $\sum a_k$ exists and is finite then we say that the series *converges*. If the limit $\sum a_k$ does not exist, or is ∞ or $-\infty$, then we say that the series *diverges*.

(ii) Finite geometric series: A finite *geometric* series has the form

$$\sum_{i=0}^{n} ar^i \;=\; a + ar + \ldots + ar^n \;=\; \frac{a(1-r^{n+1})}{1-r}$$

for some starting value a and common ratio r, and the formula on the right always holds.

(iii) Infinite geometric series: An infinite *geometric* series has the form

$$\sum_{i=0}^{\infty} ar^i \;=\; a + ar + \ldots + ar^n + \ldots \;=\; \frac{a}{1-r}\ ,$$

where the formula on the right only holds for $|r| < 1$. The infinite geometric series diverges if $|r| \geq 1$.

(iv) Harmonic series: The *harmonic* series is

$$\sum_{i=0}^{\infty} \frac{1}{i} \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots \;=\; \infty\ ,$$

and diverges.

(v) Improper integrals (unbounded half-interval): The area under the curve $y = f(x)$ for $x \geq a$ is the *improper integral*

$$\int_a^\infty f(x)\,dx \;=\; \lim_{b\to\infty} \int_a^b f(x)\,dx\ ,$$

and is said to *converge* if this limit exists and is finite, and to *diverge* otherwise.

(vi) Improper integrals (unbounded integrand): The area under the curve $y = f(x)$ for $a \leq x < b$, where $x = b$ is a vertical asymptote (from the left), is the *improper integral*

$$\int_a^b f(x)\,dx \;=\; \lim_{\ell\to b^-} \int_a^\ell f(x)\,dx \;=\; \lim_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)\,dx\ ,$$

and is said to *converge* if this limit exists and is finite, and to *diverge* otherwise.

(vii) Comparison Test for improper integrals (unbounded half-interval): Let f and g be continuous functions over the unbounded half-interval $[a, \infty)$ and suppose $f(x) \geq |g(x)|$ for all $x \in [a, \infty)$. Then convergence of the improper integral $\displaystyle\int_a^\infty f(x)\,dx$ implies convergence of $\displaystyle\int_a^\infty g(x)\,dx$.

(viii) Comparison Test for improper integrals (unbounded integrand): Let f and g be continuous functions over the interval $[a, b)$, with a common vertical asymptote $x = b$, and suppose $f(x) \geq |g(x)|$ for all $x \in [a, b)$. Then convergence of the improper integral $\displaystyle\int_a^b f(x)\,dx$ implies convergence of $\displaystyle\int_a^b g(x)\,dx$.

Revision and Exploration:

1. Verify the formula for a finite geometric series:

$$a + ar + \ldots + ar^n \;=\; \frac{a(1 - r^{n+1})}{1 - r}$$

2. Make sense of the following statement using limit notation, and identify for which r the statement holds, that is, the series on the right-hand side converges:

$$\frac{a}{1 - r} \;=\; a + ar + \ldots + ar^n + \ldots$$

Identify different types of possible divergence for the right-hand side.

3. Use a geometric series to evaluate $0.999\ldots = 0.\dot{9}$, where the 9s are forever repeating.

4. Express each of the following geometric series as a ratio of two integers:

(i) $0.009999\ldots$ (ii) $0.1101101\ldots$ (iii) $0.1102102\ldots$

5. Evaluate the following geometric series:

(i) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{11}\right)^n$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{11}{5^n}$ (iii) $\displaystyle\sum_{n=2}^{\infty} \left(-\frac{3}{8}\right)^{n-1}$

Further Exercises:

6. The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges, but grows up reluctantly. The first few consecutive partial sums are (to 2 d.p.)

$$1.00\,,\quad 1.50\,,\quad 1.83\,,\quad 2.08\,,\quad 2.28\,,\quad 2.45\,,\quad 2.59\,,\quad 2.72\,,\quad 2.83\,,\quad 2.93\,,\quad 3.02$$

Suppose that you continue to calculate consecutive partial sums once per second non-stop (yes, no breaks!) for 21 years. Show that your final partial sum still hasn't exceeded the integer 21.

7. Decide for which values of x each of the following geometric series converge, and then find the rule for the function of x that it represents:

(i) $1 + 2x + 4x^2 + 8x^3 + \cdots$ (ii) $1 - 2x + 4x^2 - 8x^3 + \cdots$

***8.** Decide when each of the following series converges and evaluate (by finding a closed formula):

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n}$ (iii) $\displaystyle\sum_{n=0}^{\infty} \tanh^{2n} x$

9. Evaluate the following improper integrals:

(i) $\displaystyle\int_1^{\infty} \frac{dx}{(3x+1)^2}$ (ii) $\displaystyle\int_2^{5} \frac{dx}{\sqrt{x-2}}$ (iii) $\displaystyle\int_{\pi/4}^{\pi/2} \sec^2 x \; dx$

(iv) $\displaystyle\int_0^{1} \ln x \; dx$ (v) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2} \; dx$ (vi) $\displaystyle\int_0^{1} \frac{\ln x}{x^2} \; dx$

***10.** Use the Comparison Test to decide convergence for

(i) $\displaystyle\int_1^{\infty} \frac{\cos^2 x}{x^2} \; dx$ (ii) $\displaystyle\int_1^{\infty} \frac{\cos x}{1+x^2} \; dx$ (iii) $\displaystyle\int_1^{\infty} \frac{dx}{x + e^{-x}}$

(iv) $\displaystyle\int_1^{\infty} \frac{e^{-x}}{x} \; dx$ (v) $\displaystyle\int_0^{1} \frac{e^{-x}}{x} \; dx$ (vi) $\displaystyle\int_0^{\infty} e^{-x^2} \; dx$

11. *Gabriel's horn* is the geometric configuration in space that results from rotating the curve $y = \frac{1}{x}$ about the x-axis for $x \geq 1$. Verify that Gabriel's horn has an infinite surface area but occupies a finite volume.

***12.** Verify that $\displaystyle\int_0^{\infty} x^2 e^{-x^2} \; dx \;=\; \frac{1}{2} \int_0^{\infty} e^{-x^2} \; dx\,.$

***13.** Prove that $\displaystyle\int_0^{1} (\ln x)^n \; dx = (-1)^n n!$ for all positive integers n.

***14.** Prove that $\displaystyle\int_1^\infty \sin(\pi x)\,dx$ does not exist.

***15.** Prove that $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ converges.

*****16.** Prove the Comparison Test for continuous functions over the interval $[1,\infty)$.

*****17.** Functions $f_1,\ldots,f_n$ are called *linearly independent* if, for all real numbers $\lambda_1,\ldots,\lambda_n$, the equation $\lambda_1 f_1(x)+\cdots+\lambda_n f_n(x) = 0$ for all relevant x implies $\lambda_1=\cdots=\lambda_n=0$.

(i) Verify that $\frac{1}{x-1}, \frac{1}{x+1}, \frac{1}{x^2+1}, \frac{x}{x^2+1}$ are linearly independent rational functions lying in

$$V = \left\{ \frac{p(x)}{x^4-1} \;\middle|\; p(x) \text{ is a polynomial of degree} < 4 \right\}.$$

(ii) Verify that $\frac{1}{x-a}\,,\ \frac{1}{(x-a)^2}\,,\ \ldots\ ,\ \frac{1}{(x-a)^n}$ are linearly independent rational functions lying in

$$W = \left\{ \frac{p(x)}{(x-a)^n} \;\middle|\; p(x) \text{ is a polynomial of degree} < n \right\}.$$

(iii) It is an easy fact that V and W are vector spaces of dimension 4 and n respectively. It is a difficult fact (see 2nd year linear algebra) that a linearly independent subset of size d in a vector space of dimension d spans the vector space. How is this and parts (i) and (ii) relevant to partial fraction decompositions?

Short Answers to Selected Exercises:

3. 1

4. (i) $\frac{1}{100}$ (ii) $\frac{110}{999}$ (iii) $\frac{367}{3330}$

5. (i) $\frac{11}{6}$ (ii) $\frac{11}{4}$ (iii) $-\frac{3}{11}$

7. (i) $\frac{1}{1-2x}$ for $-\frac12 < x < \frac12$ (ii) $\frac{1}{1+2x}$ for $-\frac12 < x < \frac12$

8. (i) $\frac{2}{2-x}$ for $-2 < x < 2$ (ii) $\frac{x+2}{1-x}$ for $-5 < x < 1$ (iii) $\cosh^2 x$ for all x

9. (i) $\frac{1}{12}$ (ii) $2\sqrt{3}$ (iii) ∞ (iv) -1 (v) 1 (vi) $-\infty$

10. (i) converges (ii) converges (iii) diverges (iv) converges (v) diverges (vi) converges