THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Assignment 2

MATH1903/1907: Integral Calculus and Modelling (Advanced) — Semester 2, 2017

Web Page: http://sydney.edu.au/science/maths/u/UG/JM/MATH1903/
Lecturers: Daniel Daners and David Easdown

This assignment is due by **5pm Thursday 5 October 2017**, via Turnitin. A PDF copy of your answers must be uploaded in the Learning Management System (Blackboard) at https://elearning.sydney.edu.au.

The School of Mathematics and Statistics encourages some collaboration between students when working on problems, but students must write up and submit their own version of the solutions.

This assignment is worth 5% of your final assessment for this course. Your answers should be well written, neat, thoughtful, mathematically concise, and a pleasure to read. Please cite any resources used and show all working. Present your arguments clearly using words of explanation and diagrams where relevant. After all, mathematics is about communicating your ideas. This is a worthwhile skill which takes time and effort to master. The marker will give you feedback and allocate an overall letter grade and mark to your assignment using the following criteria:

Mark	Grade	Criterion
10	A+	Excellent and scholarly work, answering all parts of all questions, with clear and accurate explanations and working, with appropriate acknowledgement of sources (if appropriate) and at most minor or trivial errors or omissions;
9	A	Excellent work, but with one or two substantial omissions, errors or misunderstandings overall;
8	B+	Very good work, but with three or four substantial omissions, errors or misunderstandings overall;
7	B	Very good work, but with five or six substantial omissions, errors or misunderstandings overall;
6	C+	Good work, with more than six substantial omissions, errors or misunderstandings, but writing answers that include at least eight distinct positive attributes;
5	C	A reasonable attempt, making more than six substantial omissions, errors or misunderstandings, but writing answers that include six or seven distinct positive attributes;
4	D	Limited progress including just four or five distinct positive attributes;
2	E	Limited progress, including just two or three distinct positive attributes;
1	F	Some attempt, but including just one positive attribute.

1. Consider a differentiable function $f : I \to \mathbb{R}$ on the open interval I. Fix a point $a \in I$. The fundamental theorem of calculus asserts that

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

In the questions below assume all derivatives required exist and are continuous.

(a) Use integration by parts to show that

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t) f''(t)\,dt.$$

(b) Use induction by n to show that for $n \in \mathbb{N}$ we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt.$$

2. Consider the differential equation

$$y' = xy(y-4)/4.$$

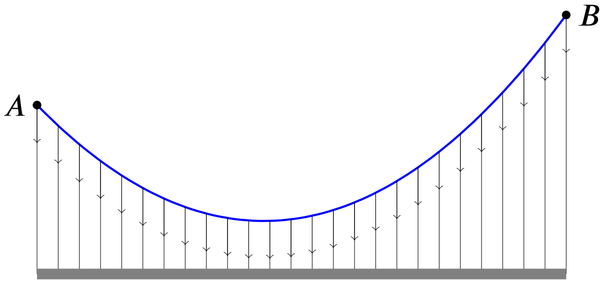
(a) Sketch the direction field of the differential equation between $-4 \le x \le 4$ and $-2 \le y \le 6$. Briefly explain how you found it.

(b) Find the general solution of the differential equation.

(c) Find the particular solution with initial value $y(0) = 2$.

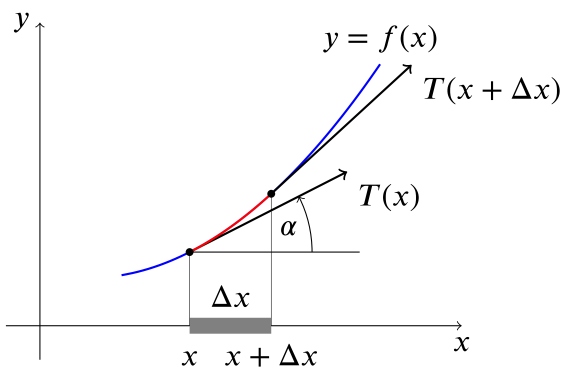
(d) Determine the equilibrium solutions and comment on their stability properties.

3. A perfectly flexible and weightless cable is suspended at points A and B. A uniformly distributed load of density ρ per length is supended from the cable exerting a force in the vertical direction as shown below. A real world example would be a cable suspension bridge.



Uniformly distributed load

The aim of this question is to find the shape of the cable. Assume that the shape can be described by the graph of a smooth function $f : (a, b) \to \mathbb{R}$. To find an equation for the shape we need to look at the balance of forces on the section of the cable between x and $x + \Delta x$. We let $T(x)$ be the magnitude of the tension force of the cable in the direction of the tangent of the cable at $(x, f(x))$.



Let $\alpha = \alpha(x)$ be angle the tangent makes with the horizontal and note that $\tan(\alpha) = f'(x)$.

(a) Explain why the force acting in the horizontal direction on the cable is constant in x and deduce that $T(x) = H\sqrt{1 + (f'(x))^2}$ for some constant H.

(b) Explain why the forces acting in the vertical direction on the segment of the cable between x and $x + \Delta x$ satisfy the equation

$$H(f'(x + \Delta x) - f'(x)) = \rho g \Delta x,$$

where g is the gravitational constant.

(c) Hence determine $f(x)$. What shape does the cable have?

(d) Show that $\int_a^b T(x)\,dx = HL$, where L is the length of the cable.