THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 12

MATH1901: Differential Calculus (Advanced) | Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Partial derivatives of functions $f(x, y)$.
- ☐ The formula of the tangent plane to the graph $z = f(x, y)$.
- ☐ The Mixed Derivatives Theorem (also known as Clairaut's Theorem).
- ☐ The Chain Rule for functions $f(x, y)$.

Outcomes

After completing this tutorial you should

- ☐ quickly and efficiently compute partial derivatives and equations of tangent planes;
- ☐ appreciate the statement of the Mixed Derivatives Theorem, and understand its limitations.
- ☐ calculate partial derivatives directly from the limit definition in relevant cases.
- ☐ use the chain rule to compute partial and total derivatives,
- ☐ appreciate the subtleties involved in defining the notion of differentiability for functions $f(x, y)$.

Summary of essential material

Definition of partial derivatives. The *partial derivative* of $f(x, y)$ with respect to x is the derivative of f obtained by fixing y and differentiating with respect to x. By first principles it is the limit

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

provided the limit exists. Similarly, the *partial derivative* of $f(x, y)$ with respect to y is the derivative of f obtained by fixing x and differentiating with respect to x. By first principles it is the limit

$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

provided the limit exists. Geometrically, the partial derivative $f_x(x_0, y_0)$ is the slope of the curve obtained by intersecting the graph of f with the plane parallel to the xz-axis through the point (x_0, y_0) at x_0. A similar interpretation holds for $f_y(x_0, y_0)$.

Calculating partial derivatives. To calculate the x-partial derivative $f_x(x, y)$ we fix y (that is, consider y to be a constant) and differentiate with respect to x as usual. To calculate the y-partial derivative, we fix x and differentiate with respect to y.

Tangent planes to graphs. The graph of a real valued function of two variables is a surface. The equation of the *tangent plane* to the graph $z = f(x, y)$ at the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Questions to complete during the tutorial

No tutorial questions due to quiz.

Extra questions for further practice

1. (a) Find the equation of the tangent plane to $z = \sin(x^2 - y) + 4xy + 3$ at the point $(x, y) = (2, 4)$.

Solution: First compute the partials:

$$\begin{aligned} f_x(x,y) &= 2x\cos(x^2 - y) + 4y & \Rightarrow && f_x(2,4) &= 20 \\ f_y(x,y) &= -\cos(x^2 - y) + 4x & \Rightarrow && f_y(2,4) &= 7. \end{aligned}$$

Since $f(2,4) = 35$, the tangent plane is

$$z = 35 + 20(x - 2) + 7(y - 4).$$

(b) Find the equation of the tangent plane to the surface $z = e^x \ln y$ at $(3, 1, 0)$.

Solution: Put $f(x, y) = e^x \ln y$. Then $f_x(x, y) = e^x \ln y$ and $f_y(x, y) = \frac{e^x}{y}$. So $f_x(3, 1) = 0$ and $f_y(3, 1) = e^3$. Thus the equation of the tangent plane is

$$z = 0 + 0(x - 3) + e^3(y - 1),$$

that is, $z = e^3 y - e^3$.

2. Find all points at which the tangent plane to the surface $z = x^2 + 2xy + 2y^2 - 6x + 8y$ is horizontal.

Solution: At the point corresponding to $x = a,\ y = b$, the tangent plane has equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

This is horizontal (that is, it's of the form $z =$ constant) when $f_x(a, b) = f_y(a, b) = 0$. Now $f_x(a, b) = 2a + 2b - 6$ and$f_y(a, b) = 2a + 4b + 8$. Setting each expression equal to 0 and solving simultaneously gives $a = 10,\ b = -7$. The required point on the surface is then $(10, -7, -58)$.

3. Define a function f of two variables by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Find $f_x(x, y)$ and $f_y(x, y)$. To find $f_x(0, 0)$ and $f_y(0, 0)$ you will need to use the definition of partial derivatives in terms of limits.

Solution: For points $(x, y) \neq (0, 0)$, we find (by usual rules of differentiation) that

$$\begin{aligned} f_x(x,y) &= \frac{4x^2y^3 + x^4y - y^5}{(x^2 + y^2)^2}, \\ f_y(x,y) &= \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}. \end{aligned}$$

At the point $(0, 0)$ we compute

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0/h^2}{h} = 0,$$

and similarly

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0, 0 + h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0/h^2}{h} = 0.$$

(b) Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$. Again, you will need to use the limit definitions.

Solution: At $(0,0)$ we have

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h^5/h^4}{h} = -1,$$

and

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{1h^5/h^4}{h} = 1.$$

(c) Observe that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Why does this not contradict the Mixed Derivatives Theorem?

Solution: Indeed we saw above that $f_{xy}(0,0) \neq f_{yx}(0,0)$. This is not a contradiction to the Mixed Derivatives Theorem (of course, since we proved that theorem!). Instead, the hypotheses of the theorem must not hold for the function $f(x,y)$. We will show that $f_{xy}(x,y)$ is not continuous at $(x,y) = (0,0)$.

For $(x,y) \neq (0,0)$ we use the rules of differentiation to compute

$$f_{xy}(x,y) = \frac{x^6 - y^6 + 9x^4y^2 - 9x^2y^4}{(x^2+y^2)^3}.$$

Now consider the limit as $(x,y) \to (0,0)$ in this expression. Along the x-axis ($y=0$) we have

$$f_{xy}(x,0) = \frac{x^6}{x^6} = 1 \to 1,$$

while along the y-axis ($x=0$) we have

$$f_{xy}(0,y) = \frac{-y^6}{y^6} = -1 \to -1.$$

So the limit

$$\lim_{(x,y)\to(0,0)} f_{xy}(x,y) \quad \text{does not exist.}$$

In particular, $f_{xy}(x,y)$ is not continuous at $(0,0)$.

4. Partial derivatives are functions of x and y again. Hence we can take further partioal derivatives. Let $f(x,y) = 1 + x^2 + 2y^2 + 2y + x^2y$. Calculate

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial y}.$$

Note that the mixed derivatives are equal.

Solution: We have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2x + 2xy \\
\frac{\partial f}{\partial y} &= 4y + 2 + x^2 \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(4y + 2 + x^2\right) = 2x \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x + 2xy\right) = 2x.
\end{aligned}$$

The mixed partial derivatives are equal, which turns out to be the case alwaus in the case for "well behaved" functions.

Challenge questions (optional)

5. In this question we investigate what the definition of *differentiability* should be for a function $f(x, y)$. In class we considered the function $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$g(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Both of the first order partial derivatives of this function exist, indeed $g_x(0, 0) = g_y(0, 0) = 0$. However we certainly do not want to call this function differentiable at $(x, y) = (0, 0)$, it is not even continuous there!!! Thus defining "differentiability" of a function $f(x, y)$ at $(x, y) = (a, b)$ to simply mean that $f_x(a, b)$ and $f_y(a, b)$ exist is not appropriate.

Instead a better approach is to define a function $f(x, y)$ to be differentiable at the point $(x, y) = (a, b)$ if $f(x, y)$ is "well approximated" by a tangent plane at $(x, y) = (a, b)$. That is, there is a plane $z = f(a, b) + m_1(x - a) + m_2(y - b)$ such that

$$f(x, y) - [f(a, b) + m_1(x - a) + m_2(y - b)]$$

is very small for all (x, y) close to (a, b). How small? We will insist that the difference between $f(x, y)$ and the tangent plane is considerably smaller than the distance from (x, y) to (a, b). A way of quantifying this is to require:

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - [f(a, b) + m_1(x - a) + m_2(y - b)]}{\sqrt{(x - a)^2 + (y - b)^2}} = 0. \tag{2}$$

Thus we have arrived at a definition: A function $f(x, y)$ is *differentiable* at $(x, y) = (a, b)$ if there are numbers $m_1, m_2 \in \mathbb{R}$ such that (2) holds.

6. Show that the function $g(x, y)$ in (1) is not differentiable at $(x, y) = (0, 0)$.

Solution: Suppose that there are numbers m_1, m_2 such that (2) holds. Thus the limit exists and equals 0 along all paths of approach to $(0, 0)$. In particular, along $y = x$ we have

$$0 = \lim_{x\to 0} \frac{g(x, x) - [g(0, 0) + m_1 x + m_2 x]}{\sqrt{x^2 + x^2}} = \lim_{x\to 0} \frac{-1 - (m_1 + m_2)x}{\sqrt{2}|x|},$$

however the limit on the right does not exist (since the denominator tends to 0, and the numerator tends to 1). Thus $g(x, y)$ is not differentiable at $(x, y) = (0, 0)$.

7. Show that if $f(x, y)$ is differentiable at $(x, y) = (a, b)$, then it is continuous at $(x, y) = (a, b)$.

Solution: This is similar to the proof that we did in lectures for 1-variable functions. Suppose that $f(x, y)$ is differentiable at $(x, y) = (a, b)$. Hence there are numbers m_1, m_2 such that (2) holds. Then

$$\begin{aligned} &f(x, y) - f(a, b) \\ &= \frac{f(x, y) - [f(a, b) + m_1(x - a) + m_2(y - b)]}{\sqrt{(x - a)^2 + (y - b)^2}} \sqrt{(x - a)^2 + (y - b)^2} + m_1(x - a) + m_2(y - b). \end{aligned}$$

All terms on the right tend to 0 as $(x, y) \to (0, 0)$, and so by Limit Laws,

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

8. Show that if $f(x, y)$ is differentiable at $(x, y) = (a, b)$ then $m_1 = f_x(a, b)$ and $m_2 = f_y(a, b)$. In particular, differentiability implies that the partial derivatives exist. (The example $g(x, y)$ shows that the converse is false).

Solution: If $f(x, y)$ is differentiable, then the limit in (2) exists (by definition). So the limits along any path of approach exist (and they are all equal). Approaching (a, b) along the line $y = b$ we have

$$\begin{aligned} 0 &= \lim_{x \to a} \frac{|f(x, b) - f(a, b) - m_1(x - a)|}{\sqrt{(x - a)^2}} \\ &= \lim_{x \to a} \left| \frac{f(x, b) - f(a, b)}{x - a} - m_1 \right|, \end{aligned}$$

and this implies that

$$\lim_{x \to a} \frac{f(x, b) - f(a, b)}{x - a} = m_1.$$

The left hand side is the definition of $f_x(a, b)$, and so $f_x(a, b)$ exists and equals m_1. A similar argument shows that $f_y(a, b)$ exists and equals m_2 by approaching along $x = a$.

9. *Very challenging!* Show that if $f_x(x, y)$ and $f_y(x, y)$ exist around $(x, y) = (a, b)$ and are continuous at $(x, y) = (a, b)$, then $f(x, y)$ is differentiable at $(x, y) = (a, b)$. So most reasonable functions are differentiable, which is reassuring.

Hint: It helps to write $f(x, y) - f(a, b) = [f(x, y) - f(a, y)] + [f(a, y) - f(a, b)]$.

Solution: Here is the rough idea. We'll tighten it up afterwards. Our plan is to show that

$$\lim_{(x,y) \to (a,b)} \frac{f(x, y) - [f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)]}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

Let

$$E(x, y) = f(x, y) - f(a, b) - f_x(a, b)(x - a) - f_y(a, b)(y - b).$$

Then, being very relaxed with things,

$$\begin{aligned} E(x, y) &= [f(x, y) - f(a, y)] + [f(a, y) - f(a, b)] - f_x(a, b)(x - a) - f_y(a, b)(y - b) \\ &\approx f_x(a, y)(x - a) + f_y(a, b)(y - b) - f_x(a, b)(x - a) - f_y(a, b)(y - b) \\ &= \left[f_x(a, y) - f_x(a, b)\right](x - a). \end{aligned}$$

In the approximation we've used

$$f_x(a, b) \approx \frac{f(x, b) - f(a, b)}{x - a},$$

because equality holds in the limit $x \to a$.

Therefore

$$\left| \frac{E(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} \right| \approx \frac{|f_x(a, y) - f_x(a, b)||x - a|}{\sqrt{(x - a)^2 + (y - b)^2}} \le |f_x(a, y) - f_y(a, b)|.$$

By the continuity of f_x at $(x, y) = (a, b)$ we have that $f_x(a, y) - f_x(a, b) \to 0$ as $y \to b$, and so it looks plausible that

$$\lim_{(x,y) \to (a,b)} \frac{E(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0,$$

and so $f(x, y)$ is differentiable at $(x, y) = (a, b)$.

Of course the above is not a watertight proof, but all we need to do is tighten it up a little. As with the proof of the Mixed Derivatives Theorem, the "tightening up" comes in the form of applications of the Mean Value Theorem to replace the approximations. So we start again from

$$E(x, y) = [f(x, y) - f(a, y)] + [f(a, y) - f(a, b)] - f_x(a, b)(x - a) - f_y(a, b)(y - b). \tag{3}$$

Consider the function $g(x) = f(x, y)$ with y fixed. By hypothesis $g'(x) = f_x(x, y)$ exists for (x, y) near (a, b). By the Mean Value Theorem applied to $g(x)$ on the interval $[a, x]$ (if $a < x$) or $[x, a]$ (if $x < a$) there exists $\tilde{x}$ between a and x such that

$$f_x(\tilde{x}, y) = \frac{f(x, y) - f(a, y)}{x - a}.$$

Therefore

$$[f(x, y) - f(a, y)] = f_x(\tilde{x}, y)(x - a) \quad \text{for some } \tilde{x} \text{ between } a \text{ and } x. \tag{4}$$

A very similar argument, using the function $h(y) = f(a, y)$, shows that

$$[f(a, y) - f(a, b)] = f_y(a, \tilde{y})(y - b) \quad \text{for some } \tilde{y} \text{ between } b \text{ and } y. \tag{5}$$

Using (4) and (5) in (3) gives

$$\begin{aligned} \left| \frac{E(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} \right| &= \frac{|[f_x(\tilde{x}, y) - f_x(a, b)](x - a) + [f_y(a, \tilde{y}) - f_y(a, b)](y - b)|}{\sqrt{(x-a)^2 + (y-b)^2}} \\ &\leq \frac{|f_x(\tilde{x}, y) - f_x(a, b)||x - a|}{\sqrt{(x-a)^2 + (y-b)^2}} + \frac{|f_y(a, \tilde{y}) - f_y(a, b)||y - b|}{\sqrt{(x-a)^2 + (y-b)^2}} \\ &\leq |f_x(\tilde{x}, y) - f_x(a, b)| + |f_y(a, \tilde{y}) - f_y(a, b)|. \end{aligned}$$

Now, since we are assuming that $f_x(x, y)$ and $f_y(x, y)$ are continuous at $(x, y) = (a, b)$, and since $\tilde{x}$ is between a and x and $\tilde{y}$ is between b and y, we have

$$\lim_{(x,y)\to(a,b)} |f_x(\tilde{x}, y) - f_x(a, b)| = \lim_{(x,y)\to(a,b)} |f_y(a, \tilde{y}) - f_y(a, b)| = 0.$$

Thus by the Squeeze Law,

$$\lim_{(x,y)\to(a,b)} \frac{E(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0,$$

and we are done.