School of Mathematics University of New South Wales

MATH2701: Abstract Algebra and Fundamental Analysis
Short Assignment 1

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1. Let $GL_n(\mathbb{R}) \coloneqq \{A \in M_n(\mathbb{R}) \mid A \text{ is invertible}\}$ be the general linear group. Show that

$$O_n(\mathbb{R}) = \{Q \in GL_n(\mathbb{R}) \mid Q^T Q = I\}$$

is a subgroup.

By the subgroup lemma, we have to show that $O_n(\mathbb{R})$ is a non-empty subset of $GL_n(\mathbb{R})$, and it satisfies the closure conditions. Let $M \in O_n(\mathbb{R})$. By definition of $O_n(\mathbb{R})$, $M \in GL_n(\mathbb{R})$. Thus, $O_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$. Furthermore, clearly $I \in O_n(\mathbb{R})$, so $O_n(\mathbb{R})$ is a non-empty subset of $GL_n(\mathbb{R})$. Considering closure under composition, let $M_1, M_2 \in O_n(\mathbb{R})$, such that

$$M_1^T M_1 = I \ldots\ldots (A)$$
$$M_2^T M_2 = I \ldots\ldots (B).$$

Further, it is clear that $M_1 M_2 \in GL_n(\mathbb{R})$. Consider now

$$\begin{aligned}
(M_1M_2)^T(M_1M_2) &= (M_2^T M_1^T)(M_1M_2) \\
&= M_2^T(M_1^T M_1)M_2 \\
&= M_2^T(I)M_2 \text{ by } (A) \\
&= M_2^T M_2 \\
\therefore (M_1M_2)^T(M_1M_2) &= I \text{ by } (B)
\end{aligned}$$

and so $M_1M_2 \in O_n(\mathbb{R})$, satisfying closure under composition. Examining closure under inverses, let $M \in O_n(\mathbb{R})$, we claim $M^{-1} = M^T$. As $M \in GL_n(\mathbb{R})$, clearly $M^T \in GL_n(\mathbb{R})$, and so $M^{-1} \in GL_n(\mathbb{R})$. As $M \in O_n(\mathbb{R})$, $M^T M = I \ldots\ldots (C)$. Now, consider

$$\begin{aligned}
(M^{-1})^T(M^{-1}) &= (M^T)^{-1}(M^{-1}) \\
&= (MM^T)^{-1} \\
&= \big((M^T M)^T\big)^{-1} \\
&= \big((I)^T\big)-1 \text{ by } (C) \\
&= (I)^{-1} \\
\therefore (M^{-1})^T(M^{-1}) &= I
\end{aligned}$$

thus $M^{-1} = M^T \in O_n(\mathbb{R})$. It is now clear to see that $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

2. Let $\tau : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry and assume that $\tau(\mathbf{0}) = \mathbf{0}$. Show that

(a) τ preserves the dot product on $\mathbb{R}^n$: $\tau(\mathbf{x}) \cdot \tau(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

As τ is an isometry, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have the result $\|\tau(\mathbf{x}) - \tau(\mathbf{y})\|^2 = \|\mathbf{x} - \mathbf{y}\|^2 \ldots (A)$. Considering the case when $\mathbf{y} = \mathbf{0}$, coupled with $\tau(\mathbf{0}) = \mathbf{0}$, the above equation yields $\|\tau(\mathbf{x})\|^2 = \|\mathbf{x}\|^2 \ldots (B)$. Returning to the equation labelled (A), we are able to deduce that τ preserves the dot product, with the aid of equation (B).

$$\begin{aligned}
\|\tau(\mathbf{x}) - \tau(\mathbf{y})\|^2 &= \|\mathbf{x} - \mathbf{y}\|^2 \\
\therefore [\tau(\mathbf{x}) - \tau(\mathbf{y})][\tau(\mathbf{x}) - \tau(\mathbf{y})] &= (\mathbf{x} - \mathbf{y})(\mathbf{x} - \mathbf{y}) \\
\therefore \tau(\mathbf{x}) \cdot \tau(\mathbf{x}) - 2\tau(\mathbf{x}) \cdot \tau(\mathbf{y}) + \tau(\mathbf{y}) \cdot \tau(\mathbf{y}) &= \mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \\
\therefore \|\tau(\mathbf{x})\|^2 - 2\tau(\mathbf{x}) \cdot \tau(\mathbf{y}) + \|\tau(\mathbf{y})\|^2 &= \|\mathbf{x}\|^2 - 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2 \\
\therefore \|\tau(\mathbf{x})\|^2 - 2\tau(\mathbf{x}) \cdot \tau(\mathbf{y}) + \|\tau(\mathbf{y})\|^2 &= \|\tau(\mathbf{x})\|^2 - 2\mathbf{x} \cdot \mathbf{y} + \|\tau(\mathbf{y})\|^2 \text{ by } (B) \\
\therefore -2\tau(\mathbf{x}) \cdot \tau(\mathbf{y}) &= -2\mathbf{x} \cdot \mathbf{y} \\
\therefore \tau(\mathbf{x}) \cdot \tau(\mathbf{y}) &= \mathbf{x} \cdot \mathbf{y}
\end{aligned}$$

(b) if $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$, then the matrix

$$Q = (\tau(\mathbf{e}_1), \tau(\mathbf{e}_2), \ldots, \tau(\mathbf{e}_n))$$

is orthogonal.

Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. As S is the standard basis, S is orthonormal, by definition

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{else} \end{cases} \quad \ldots\ldots (A).$$

For Q to be orthogonal, $Q^T Q = I$. This is equivalent to

$$\tau(\mathbf{e}_i) \cdot \tau(\mathbf{e}_j) = \begin{cases} 0 & \text{if } i \neq j \\ > 0 & \text{else} \end{cases}$$

As we know from the previous question, τ preserves the dot product, and so $\tau(\mathbf{e}_i) \cdot \tau(\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j$ for all i, j. Thus, using result (A), we satisfy the above requirements for Q to be orthogonal, and thus $Q = (\tau(\mathbf{e}_1), \tau(\mathbf{e}_2), \ldots, \tau(\mathbf{e}_n))$ is orthogonal.

(c) $\tau = T_{Q,\mathbf{0}}$ is a linear isomorphism.

From the Theorem in the Lecture Notes, we can decompose any isometry on $\mathbb{R}^n$ into a translation composed with multiplication by an orthogonal matrix. That is, $\tau(\mathbf{x}) = T_{A,\mathbf{b}}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is orthogonal, and $\mathbf{b}$ is the vector of translation. By the construction of Q in the previous part, $Q(\mathbf{e}_i) = \tau(\mathbf{e}_i)$, for all i, and Q is orthogonal. Thus, let $A = Q$. Furthermore, as $\tau(\mathbf{0}) = \mathbf{0}$, we have $\tau(\mathbf{0}) = T_{Q,\mathbf{b}}(\mathbf{0}) = Q(\mathbf{0}) + \mathbf{b} = \mathbf{b} = \mathbf{0}$. As $\mathbf{b} = \mathbf{0}$, we get the result $\tau = T_{Q,\mathbf{0}} = Q$. As Q is a linear map, and invertible, $\tau = T_{Q,\mathbf{0}}$ is a linear isomorphism.

This assignment is completely my own work except where acknowledged
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