THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Solutions Week 7** 2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Differentiate the following (don't worry about the domain of the function or its derivative).

(a) $f(x) = e^{x+5}$

(b) $f(x) = (\ln 4)e^x$

(c) $f(x) = xe^x$

(d) $f(x) = \dfrac{x^2 + 5x + 2}{x + 3}$

(e) $f(x) = (x + 1)^{99}$

(f) $f(x) = xe^{-x^2}$

(g) $f(t) = \tan t$

(h) $f(t) = e^{\cos t}$

(i) $f(t) = e^{t \cos 3t}$

(j) $f(t) = \ln\big(\cos(1 - t^2)\big)$

(k) $f(x) = (x + \sin^5 x)^6$

(l) $f(x) = \sin(\sin(\sin x))$

(m) $f(x) = \sin(6 \cos(6 \sin x))$

Questions for the tutorial

2. For each of the following functions f, find $f(f'(x))$ and $f'(f(x))$.

(a) $f(x) = \frac{1}{x}$,

(b) $f(x) = x^2$,

(c) $f(x) = 2$,

(d) $f(x) = 2x$.

Solution

(a) $f'(x) = -\dfrac{1}{x^2}$, so $f(f'(x)) = -x^2$ and also $f'(f(x)) = -x^2$.

(b) $f'(x) = 2x$, so $f(f'(x)) = (2x)^2 = 4x^2$ and $f'(f(x)) = 2x^2$.

(c) $f'(x) = 0$, so $f(f'(x)) = 2$ and $f'(f(x)) = 0$.

(d) $f'(x) = 2$, so $f(f'(x)) = 4$ and $f'(f(x)) = 2$.

3. For the functions given by the following formulas, find the maximum and minimum values on the indicated intervals.

(a) $f(x) = \dfrac{e^x}{x + 1}$ on $[2, 3]$

(b) $f(x) = \dfrac{x}{x^2 + 1}$ on $[-2, 0]$

(c) $f(x) = e^{x^2-1}$ on $[-1, 1]$

Solution

Observe that in all cases, the given function is continuous on the appropriate closed interval $[a, b]$ and differentiable on the open interval (a, b). The maximum and minimum values occur either at critical points or at the endpoints.

(a) We have $f'(x) = \dfrac{xe^x}{(x+1)^2}$. As this is zero only when $x = 0$, there are no critical points in $[2,3]$. The maximum and minimum values therefore occur at the endpoints. We find that $f(2) = \dfrac{e^2}{3} \approx 2.463$ is the minimum value and $f(3) = \dfrac{e^3}{4} \approx 5.021$ is the maximum value.

(b) We have $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$. There is a critical point at $x = -1$. We find that $f(-2) = -\dfrac{2}{5}$, $f(-1) = -\dfrac{1}{2}$, $f(0) = 0$. Thus the maximum value is 0 and the minimum value is $-\dfrac{1}{2}$.

(c) We have $f'(x) = 2xe^{x^2-1}$ and so the only critical point is at $x = 0$. The minimum value of f is $\dfrac{1}{e}$ at $x = 0$ and the maximum value is 1, at $x = \pm 1$.

4. Consider the function defined by

$$f(x) = \begin{cases} x^2 & \text{for } x \leq 1 \\ e^{ax+b} & \text{for } x > 1. \end{cases}$$

(a) Determine for which values of a and b the function f is continuous at $x = 1$.

(b) Determine for which values of a and b the function f is differentiable at $x = 1$.

Solution

(a) We require that $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$, that is, $1 = e^{a+b}$. So f is continuous at $x = 1$ for all values of a and b such that $b = -a$.

(b) Since a function which is differentiable at $x = 1$ is also continuous at $x = 1$, we certainly must have $a = -b$. The left-hand derivative at $x = 1$ equals 2 and the right-hand derivative at $x = 1$ equals a. Thus f is differentiable at $x = 1$ if and only if $a = -b = 2$.

5. Use Rolle's Theorem and the IVT to show that the equation $x^2 - x\sin x - \cos x = 0$ has exactly 2 solutions.

Solution

Let $f(x) = x^2 - x\sin x - \cos x$, and so

$$f'(x) = 2x - \sin x - x\cos x + \sin x = x(2 - \cos x).$$

Thus the only critical point is at $x = 0$. Now let's assume that there are more than two solutions, that is, there are distinct real numbers $a < b < c$ such that $f(a) = f(b) = f(c) = 0$. By Rolle's Theorem, there exist numbers $d \in (a,b)$ and $e \in (b,c)$ such that $f'(d) = f'(e) = 0$. But this contradicts the fact that there is only one critical point. Therefore there are at most two solutions. Now observe that $f(0) = -1 < 0$ and $f(-\pi) = f(\pi) = \pi^2 + 1 > 0$. By the IVT, there must be a solution in $(-\pi, 0)$ and another solution in $(0, \pi)$. That is, the equation has exactly two solutions.

6. Define a function f by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is differentiable at 0.

Solution

We have $\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$, as $f(0) = 0$. Now for all nonzero h, $-|h| \leq \frac{f(h)}{h} \leq |h|$. Since $\lim_{h\to 0} |h| = 0$, we have $\lim_{h\to 0} \frac{f(h)}{h} = 0$ by the Squeeze Law. Therefore f is differentiable at 0, with $f'(0) = 0$.

7. Prove that if f is differentiable at a and $f(a) \neq 0$, then $|f|$ is also differentiable at a. Give an example to show why the assumption $f(a) \neq 0$ is necessary.

Solution

Suppose $f(a) \neq 0$. Since f is continuous at a, there exists $\delta > 0$ such that for all $x \in (a-\delta, a+\delta)$, $f(x)$ has the same sign as $f(a)$. Therefore on the interval $(a-\delta, a+\delta)$, $|f|$ is either equal to f or $-f$ and so is differentiable at a. A counterexample with $f(a) = 0$ is $f(x) = x$, $a = 0$.

Extra Questions

8. Define a function f by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable everywhere and that f' is not continuous at 0.

Solution

By the definition of derivative,

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x\to 0} x \sin \frac{1}{x} = 0,$$

where the last equality follows from the Squeeze Law, as seen in lectures. At points other than 0, we can simply differentiate f using the product rule and chain rule:

$$f'(x) = 2x \sin \frac{1}{x} + x^2(-\frac{1}{x^2} \cos \frac{1}{x}) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \text{ for } x \neq 0.$$

So f is differentiable everywhere. However, f' is not continuous at 0 because $\lim_{x\to 0} f'(x)$ does not exist. To see this, suppose for a contradiction that $\lim_{x\to 0} f'(x) = \ell$. Then

$$\lim_{x\to 0} \cos \frac{1}{x} = \lim_{x\to 0} 2x \sin \frac{1}{x} - f'(x) = 0 - \ell = -\ell,$$

which is impossible for the same reason as in the proof that $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist.

9. Using Rolle's Theorem, prove that a polynomial of degree $n > 0$ has at most n real roots.

Solution

Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of degree $n > 0$, so $a_n \neq 0$. If $n = 1$ then $p(x) = a_0 + a_1x$ which has the unique root $-\dfrac{a_0}{a_1}$. This starts an induction process. Suppose (as the inductive hypothesis) that $n > 1$ and that the claim holds for any polynomial of degree less than n. In particular, the derivative $p'(x)$ is a polynomial of degree $n-1$, so has at most $n-1$ real roots.
We argue by contradiction. Suppose that $p(x)$ has at least $n+1$ real roots, so there exist

$$x_1 \ < \ x_2 \ < \cdots \ < \ x_n \ < x_{n+1}$$

such that $p(x_i) = 0$ for each $i = 1, \ldots, n+1$. By Rolle's Theorem, for each $i = 1, \ldots, n$ there exists $y_i \in (x_i, x_{i+1})$ such that $p'(y_i) = 0$. But

$$y_1 \ < \ y_2 \ < \cdots \ < \ y_n.$$

That is, all the y_i are distinct real roots of $p'(x) = 0$. This contradicts the assumption that $p'(x)$ has at most $n-1$ real roots. We conclude that $p(x)$ has at most n real roots, which completes the inductive step.

Solution to Question 1

(a) $f'(x) = e^{x+5}$
(b) $f'(x) = (\ln 4)e^x$
(c) $f'(x) = e^x + xe^x = (1+x)e^x$
(d) $f'(x) = \dfrac{(x+3)(2x+5) - (x^2+5x+2)}{(x+3)^2} = \dfrac{x^2+6x+13}{(x+3)^2}$
(e) $f'(x) = 99(x+1)^{98}$
(f) $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2}$
(g) $f'(t) = \dfrac{d}{dt}\left(\dfrac{\sin t}{\cos t}\right) = \dfrac{-\sin t \cdot (-\sin t) + \cos t \cdot \cos t}{\cos t \cdot \cos t} = \dfrac{1}{\cos^2 t} = \sec^2 t$
(h) $f'(t) = (-\sin t)e^{\cos t}$
(i) $f'(t) = (\cos 3t - 3t \sin 3t)e^{t \cos 3t}$
(j) $f'(t) = \dfrac{2t \sin(1-t^2)}{\cos(1-t^2)}$
(k) $f'(x) = 6(x + \sin^5 x)^5(1 + 5\sin^4 x \cos x)$
(l) $f'(x) = \cos(\sin(\sin x)) \cos(\sin x) \cos x$
(m) $f'(x) = -36 \cos(6\cos(6 \sin x)) \sin(6 \sin x) \cos x$