The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2 **Solutions to Exercises for Week 5** 2017

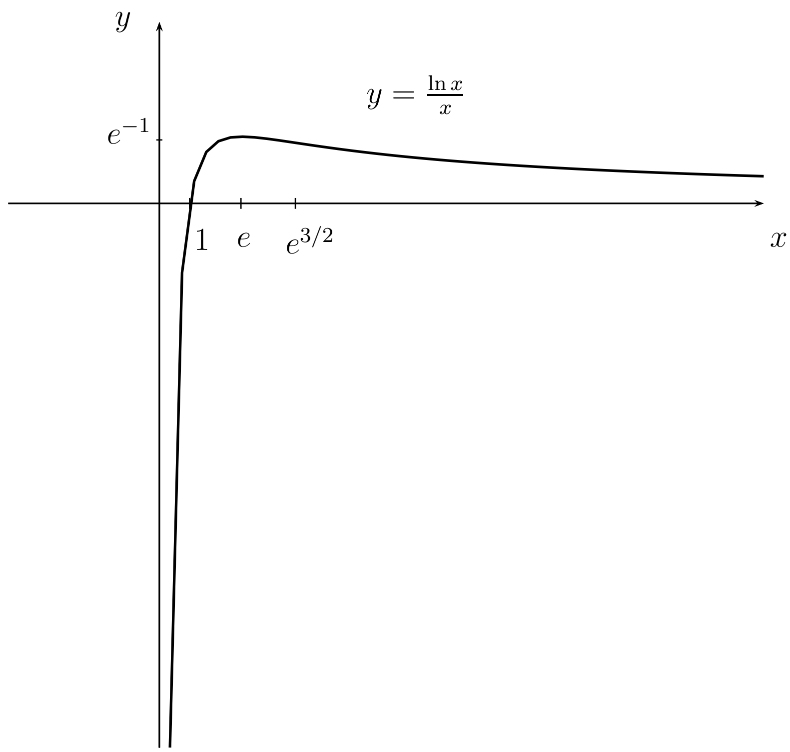
1. For $y = \frac{\ln x}{x}$ $(x > 0)$ we have

$$y' = \frac{1 - \ln x}{x^2} \qquad \text{and} \qquad y'' = \frac{2\ln x - 3}{x^3},$$

so the curve is strictly increasing on $(0, e]$, decreasing on $[e, \infty)$, with a maximum at $x = e$, and concave downwards on $(0, e^{3/2})$, upwards on $(e^{3/2}, \infty)$, with an inflection at $x = e^{3/2}$. The maximum value of y is $\frac{\ln e}{e} = \frac{1}{e}$. Also,

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{x} = 0,$$

by L'Hopital's rule, and $\lim_{x\to 0^+} y = -\infty$.



2. We have $\frac{\pi}{2} < \frac{3.2}{2} = 1.6 < 1.7 < \sqrt{3} < \sqrt{\pi}$, so

$$\sqrt{\pi}^{\,\pi} = e^{\pi \ln \sqrt{\pi}} = e^{\frac{\pi}{2}\ln \pi} < e^{\sqrt{\pi}\ln \pi} = \pi^{\sqrt{\pi}}.$$

3. (i) We have
$$a^0 \;=\; \exp(0\ln a) \;=\; \exp 0 \;=\; 1\,,$$
at the last step, since $\ln 1 = 0$ and $\exp = \ln^{-1}$.

(ii) We have
$$\ln(a^x) \;=\; \ln(\exp(x\ln a)) \;=\; x\ln a\,,$$
since ln and exp are mutually inverse.

(iii) We have
$$a^c a^d = \exp(c\ln a)\exp(d\ln a) = \exp(c\ln a + d\ln a) = \exp((c+d)\ln a) = a^{c+d}\,,$$
in the middle step, by the property that exp turns sums into products.

(iv) We have
$$\begin{aligned}(ab)^c \;&=\; \exp(c\ln(ab)) \;=\; \exp(c(\ln a + \ln b)) \;=\; \exp(c\ln a + c\ln b))\\ &=\; \exp(c\ln a)\exp(c\ln b) \;=\; a^c b^c\,,\end{aligned}$$
in the first middle step, by the property that ln turns products into sums, and in the second middle step, by the property that exp turns sums into products.

(v) We have
$$(a^c)^d \;=\; \exp(d\ln(\exp(c\ln a))) \;=\; \exp(dc\ln a) \;=\; a^{dc} \;=\; a^{cd}\,,$$
in the middle step, since ln and exp are mutually inverse.

(vi) We have, by the Chain Rule, and the derivatives of ln and exp,
$$\frac{d}{dx}\,x^a \;=\; \frac{d}{dx}(\exp(a\ln x)) \;=\; \exp(a\ln x)\left(\frac{a}{x}\right) \;=\; ax^a x^{-1} \;=\; ax^{a-1}\,,$$
using the result of part (iii) in the final step.

4. Observe that
$$\begin{aligned}\log_a x = y \quad &\Longleftrightarrow \quad y = \frac{\ln x}{\ln a} \quad \Longleftrightarrow \quad y\ln a = \ln x\\ &\Longleftrightarrow \quad e^{y\ln a} = e^{\ln x} \quad \Longleftrightarrow \quad a^y = x\,.\end{aligned}$$

(i) $\log_a(xy) \;=\; \frac{\ln(xy)}{\ln a} \;=\; \frac{\ln x + \ln y}{\ln a} \;=\; \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} \;=\; \log_a x + \log_a y\,.$

(ii) $\log_a x^b \;=\; \frac{\ln x^b}{\ln a} \;=\; b\frac{\ln x}{\ln a} \;=\; b\log_a x\,.$

(iii) $\log_a \frac{x}{y} = \log_a xy^{-1} = \log_a x + \log_a y^{-1} = \log_a x + (-1)\log_a y = \log_a x - \log_a y\,.$

5. Suppose $\log_{10} 2$ is rational, say $\log_{10} 2 = \frac{a}{b}$ for some positive integers a, $b \neq 0$. Then $10^{a/b} = 2$, so $10^a = 2^b$. But 5 divides 10^a, so 5 divides 2^b, which is nonsense, as the only prime divisor of 2^b is 2. Hence $\log_{10} 2$ is irrational.

6. From the first exercise, the curve $y = \frac{\ln x}{x}$ is strictly decreasing on $[e,\infty)$, so
$$e \le a < b \quad\Longrightarrow\quad \frac{\ln b}{b} < \frac{\ln a}{a} \quad\Longrightarrow\quad a\ln b < b\ln a \quad\Longrightarrow\quad b^a = e^{a\ln b} < e^{b\ln a} = a^b\,.$$
In particular $\pi^e < e^\pi$, since $e < \pi$.

7. (i) $y = \log_{10}\sqrt{x} = \frac{1}{2}\log_{10}x = \frac{1}{2}\frac{\ln x}{\ln 10}$, so $\frac{dy}{dx} = \frac{1}{2x\ln 10}$.

(ii) $y = x^{\sqrt{x}} = e^{\sqrt{x}\ln x}$, so $\frac{dy}{dx} = e^{\sqrt{x}\ln x}\left(\frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x}\right) = x^{\sqrt{x}}\left(\frac{\ln x+2}{2\sqrt{x}}\right)$.

(iii) $y = e^{x\ln\sin x}$, so $\frac{dy}{dx} = e^{x\ln\sin x}\left(\ln\sin x + \frac{x\cos x}{\sin x}\right) = (\sin x)^x\left(\ln\sin x + \frac{x\cos x}{\sin x}\right)$.

8. (i) $\int(2x+3)e^x\,dx = (2x+3)e^x - \int e^x(2)\,dx = (2x+3)e^x - 2e^x + C$.

(ii) $\int_1^2 t^2\ln t\,dt = \left[\frac{t^3}{3}\ln t\right]_1^2 - \int_1^2 \frac{t^3}{3}\frac{1}{t}\,dt = \frac{8}{3}\ln 2 - \left[\frac{t^3}{9}\right]_1^2 = \frac{8}{3}\ln 2 - \frac{7}{9}$.

(iii) Applying a double angle formula first, we get

$$\int_{-\pi/4}^{\pi/4}\theta\sin\theta\cos\theta\,d\theta = \int_{-\pi/4}^{\pi/4}\frac{\theta}{2}\sin 2\theta\,d\theta = \left[-\frac{\theta}{4}\cos 2\theta\right]_{-\pi/4}^{\pi/4} + \int_{-\pi/4}^{\pi/4}\frac{\cos 2\theta}{4}\,d\theta$$
$$= 0 + \left[\frac{\sin 2\theta}{8}\right]_{-\pi/4}^{\pi/4} = \frac{1}{4} .$$

9. (i) Let A and B be constants such that $\dfrac{1}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$, so that $A(x-2)+B(x+1) = 1$. Putting $x=2$ gives $3B=1$, so $B=\frac{1}{3}$; putting $x=-1$ gives $-3A=1$, so $A=-\frac{1}{3}$. Hence

$$\int\frac{1}{(x+1)(x-2)}\,dx = -\frac{1}{3}\int\frac{dx}{x+1} + \frac{1}{3}\int\frac{dx}{x-2} = \frac{1}{3}\ln\frac{|x-2|}{|x+1|} + C .$$

(ii) Let A and B be constants such that $\dfrac{7}{2x^2+5x-3} = \dfrac{A}{2x-1} + \dfrac{B}{x+3}$, so that $A(x+3)+B(2x-1) = 7$. Putting $x=-3$ gives $-7B=7$, so $B=-1$; putting $x=\frac{1}{2}$ gives $\frac{7}{2}A=7$, so $A=2$. Hence

$$\int\frac{7}{2x^2+5x-3}\,dx = 2\int\frac{dx}{2x-1} - \int\frac{dx}{x+3} = \ln\frac{|2x-1|}{|x+3|} + C .$$

(iii) First note that $\frac{x^4}{x^3-1} = x + \frac{x}{x^3-1}$. Let A, B, C be constants such that

$$\frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} ,$$

so that

$$A(x^2+x+1) + (Bx+C)(x-1) = x .$$

Putting $x=1$ gives $3A=1$, so $A=\frac{1}{3}$; putting $x=0$ gives $A-C=0$ so $C=\frac{1}{3}$; putting $x=2$ gives $7A+2B+C=2$ so $B=-\frac{1}{3}$. Hence

$$\frac{x^4}{x^3-1} = x + \frac{1/3}{x-1} - \frac{1}{3}\frac{x-1}{x^2+x+1} = x + \frac{1/3}{x-1} - \frac{1}{6}\frac{2x+1-3}{x^2+x+1}$$
$$= x + \frac{1/3}{x-1} - \frac{1}{6}\frac{2x+1}{x^2+x+1} + \frac{1/2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

so that

$$\int\frac{x^4}{x^3-1}\,dx = \frac{x^2}{2} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C .$$

10. Integrating by parts we get

$$\int_1^4 xf''(x)\,dx \;=\; \big[xf'(x)\big]_1^4 - \int_1^4 f'(x)\,dx \;=\; 4f'(4) - f'(1) - \big[f(x)\big]_1^4$$
$$= \; 12 - 5 - (f(4) - f(1)) \;=\; 7 - 7 + 2 \;=\; 2.$$

11. Suppose f is continuous and $f(x+y) = f(x)f(y)$ for all x, y. First note that, for all x,

$$f(x) \;=\; f(x/2 + x/2) \;=\; f(x/2)f(x/2) \;=\; f(x/2)^2 \;\geq\; 0\,.$$

If $f(x) = 0$ for some x then, for all y,

$$f(y) \;=\; f(y - x + x) \;=\; f(y-x)f(x) \;=\; f(y-x)0 \;=\; 0\,,$$

so (i) holds.

Suppose (i) does not hold. Certainly then $f(1) > 0$. Observe that

$$f(1) \;=\; f(0+1) \;=\; f(0)f(1)\,,$$

so that $f(0) = 1$. Suppose n and m are positive integers. Then

$$f(1) \;=\; f(1/m + 1/m + \ldots + 1/m) \;=\; f(1/m)f(1/m)\cdots f(1/m) \;=\; f(1/m)^m\,,$$

so that $f(1/m) = f(1)^{1/m}$. Hence

$$f(n/m) \;=\; f(1/m + 1/m + \ldots + 1/m) \;=\; f(1/m)^n \;=\; \Big(f(1)^{1/m}\Big)^n \;=\; f(1)^{n/m}\,.$$

We have verified that $f(q) = f(1)^q$ for all positive rational numbers q. Let x be any positive real number and let $q_1, q_2, \ldots$ be a sequence of rational numbers converging to x (e.g. using the decimal expansion of x). By continuity of f we have

$$f(x) \;=\; \lim_{n\to\infty} f(q_n) \;=\; \lim_{n\to\infty} f(1)^{q_n} \;=\; f(1)^{\lim_{n\to\infty} q_n} \;=\; f(1)^x\,.$$

If x is a negative real number then $-x$ is positive and

$$1 \;=\; f(0) \;=\; f(x-x) \;=\; f(x)f(-x) \;=\; f(x)f(1)^{-x}$$

so $f(x) = f(1)^x$. This completes the proof that (ii) holds.

12. Putting $u = \cos^{n-1} x$ and $v' = \cos x$ so that $v = \sin x$, we can integrate by parts:

$$\begin{aligned}
I_n = \int \cos^n x\,dx &= \int \cos^{n-1} x \frac{d}{dx}(\sin x)\,dx \\
&= \cos^{n-1} x \sin x - \int \sin x \frac{d}{dx}(\cos^{n-1} x)\,dx \\
&= \cos^{n-1} x \sin x - \int \sin x (n-1)\cos^{n-2} x(-\sin x)\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int (1 - \cos^2 x)\cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n\,.
\end{aligned}$$

Rearranging,

$$nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2} \quad \text{or} \quad I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}\,.$$

Now $I_0 = \int \cos^0 x\, dx = \int dx = x + C$, so

$$I_2 = \frac{1}{2}\cos x \sin x + \frac{1}{2} I_0 = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C_1$$

and

$$I_4 = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4} I_2 = \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C_2\,.$$

13. Observe that $\displaystyle\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x = e$ if and only if $\displaystyle\lim_{x\to\infty} x\ln\left(\frac{x+a}{x-a}\right) = 1$ if and only if, by L'Hopital's rule,

$$1 \;=\; \lim_{x\to\infty} \frac{\ln\left(\frac{x+a}{x-a}\right)}{1/x} \;=\; \lim_{x\to\infty} \frac{\frac{x-a}{x+a}\;\frac{x-a-(x+a)}{(x-a)^2}}{-1/x^2} \;=\; \lim_{x\to\infty} \frac{2ax^2}{x^2-a^2} \;=\; \lim_{x\to\infty} \frac{2a}{1-a^2/x^2} \;=\; 2a\,.$$

Hence $a = \frac{1}{2}$.

14. Put $t = \tan\frac{x}{2}$ and observe, by properties of right angled triangles, that

$$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}} \qquad \text{and} \qquad \cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}\,.$$

Hence

$$\sin x \;=\; 2\sin\frac{x}{2}\cos\frac{x}{2} \;=\; \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos x \;=\; \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \;=\; \frac{1-t^2}{1+t^2}\,.$$

Also

$$\frac{dt}{dx} \;=\; \frac{1}{2}\sec^2\frac{x}{2} \;=\; \frac{1}{2}(1+t^2) \qquad \text{so that} \qquad dx \;=\; \frac{2dt}{1+t^2}\,.$$

Hence

$$\begin{aligned}\int \frac{5}{4\sin x + 3\cos x}\,dx \;&=\; \int \frac{5}{\frac{8t}{1+t^2} + \frac{3(1-t^2)}{1+t^2}}\;\frac{2\,dt}{1+t^2} \\ &=\; \int \frac{10}{3+8t-3t^2}\,dt \;=\; \int \frac{10}{(3-t)(1+3t)}\,dt\,.\end{aligned}$$

Let A, B be constants such that

$$\frac{10}{(3-t)(1+3t)} \;=\; \frac{A}{3-t} + \frac{B}{1+3t} \qquad \text{so that} \qquad A(1+3t) + B(3-t) \;=\; 10\,.$$

Putting $t = 3$ gives $10A = 10$, so $A = 1$; putting $t = -1/3$ gives $10B/3 = 10$, so $B = 3$. Hence

$$\begin{aligned}\int \frac{5}{4\sin x + 3\cos x}\,dx \;&=\; \int \frac{dt}{3-t} + \int \frac{3\,dt}{1+3t} \;=\; \ln\frac{|1+3t|}{|3-t|} \;+\; C \\ &=\; \ln\frac{|1+3\tan x/2|}{|3-\tan x/2|} \;+\; C\,.\end{aligned}$$

15. Let $I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$. Put $u = \frac{\pi}{2} - x$ so $du = -dx$ and

$$I = -\int_{\pi/2}^{0} \frac{\sin^n\left(\frac{\pi}{2} - u\right)}{\sin^n\left(\frac{\pi}{2} - u\right) + \cos^n\left(\frac{\pi}{2} - u\right)}\, du \;=\; \int_0^{\pi/2} \frac{\cos^n u}{\cos^n u + \sin^n u}\, du$$

so that

$$2I \;=\; \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx + \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\, dx \;=\; \int_0^{\pi/2} 1\, dx \;=\; \pi/2$$

giving $I = \pi/4$.

16. (i) From the reduction formula in lectures we get immediately, for $n \geq 2$,

$$I_n \;=\; \left[\frac{-\sin^{n-1} x\, \cos x}{n}\right]_0^{\pi/2} \;+\; \frac{n-1}{n} I_{n-2} \;=\; \frac{n-1}{n} I_{n-2}\, .$$

Hence $I_3 = \frac{2}{3} I_1 = \frac{2}{3}$, $I_5 = \frac{4}{5} I_3 = \frac{2}{3}\frac{4}{5}$, and so on, yielding

$$I_{2k+1} \;=\; \frac{2}{3}\,\frac{4}{5}\,\frac{6}{7} \cdots \frac{2k}{2k+1}\, .$$

Also $I_2 = \frac{1}{2} I_0 = \frac{1}{2}\frac{\pi}{2}$, $I_4 = \frac{3}{4} I_3 = \frac{1}{2}\frac{3}{4}\frac{\pi}{2}$, and so on, yielding

$$I_{2k} \;=\; \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6} \cdots \frac{2k-1}{2k}\,\frac{\pi}{2}\, .$$

(ii) Note that, if n is a positive integer, then $\sin^n x \geq \sin^{n+1} x$ for any $x \in [0, \pi/2]$ (since $1 \geq \sin x \geq 0$), so that $I_n \geq I_{n+1}$.

By (i), $I_n > 0$, so for any positive integer k, $1 \;\geq\; \frac{I_{2k+1}}{I_{2k}} \;\geq\; \frac{I_{2k+2}}{I_{2k}}$. But

$$\lim_{k\to\infty} \frac{I_{2k+2}}{I_{2k}} \;=\; \lim_{k\to\infty} \frac{2k+1}{2k+2} \;=\; \lim_{k\to\infty} \frac{2 + \frac{1}{k}}{2 + \frac{2}{k}} \;=\; 1\, ,$$

so, by the Squeeze Limit Law,

$$\lim_{k\to\infty} \frac{I_{2k+1}}{I_{2k}} = 1\, .$$

17. It is a theorem in linear algebra that every vector space has a basis. Let X be a basis for $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$. Certainly $|X| > 1$, since $\mathbb{Q} \neq \mathbb{R}$ (e.g. $\sqrt{2} \notin \mathbb{Q}$). (In fact, $|X|$ is uncountably infinite!) For each $x \in X$ choose $\alpha_x \in \mathbb{R}^+$ and define $f : \mathbb{R} \to \mathbb{R}$ by the rule

$$f(q_1 x_1 + \ldots + q_n x_n) \;=\; \alpha_{x_1}^{q_1} \ldots \alpha_{x_n}^{q_n}$$

where n is a positive integer, $x_1, \ldots, x_n$ are distinct elements of X and $q_1, \ldots, q_n \in \mathbb{Q}$. This rule is sensible because elements of $\mathbb{R}$ are expressed uniquely as linear combinations of elements of the basis X. If $\lambda, \mu \in \mathbb{R}$ then $\lambda = \sum q_x x$, $\mu = \sum r_x x$, so

$$f(\lambda + \mu) \;=\; f\left(\sum (q_x + r_x) x\right) \;=\; \prod \alpha_x^{q_x + r_x} \;=\; \prod \alpha_x^{q_x} \prod \alpha_x^{r_x} \;=\; f(\lambda) f(\mu)\, .$$

This verifies that the algebraic condition holds. We now make f discontinuous. Since $|X| > 1$, choose distinct $x, y \in X$. Note $x, y \neq 0$. Let $\alpha_x = 1$ and $\alpha_y = 2$. We now claim f is discontinuous at y. Let $q_1, q_2, \ldots$ be a sequence of rational numbers such that

$$\lim_{n\to\infty} q_n \;=\; \frac{y}{x}$$

(for example, by using the decimal expansion of y/x), so $\lim_{n\to\infty} q_n x \;=\; y$. If f is continuous at y then

$$2 \;=\; f(y) \;=\; \lim_{n\to\infty} f(q_n x) \;=\; \lim_{n\to\infty} 1^{q_n} \;=\; 1\,,$$

which is impossible. Hence f is discontinuous.