The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2 **Exercises for Week 5 (beginning 28 August)** 2017

It might be useful to attempt the Revision and Exploration Exercises before the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Monotone Convergence Theorem: Any bounded infinite monotonic sequence of real numbers converges to a limit.

(ii) Early Transcendental Method: Let $a > 0$. For any integer n, we can define a^n using multiplication and inversion, and guarantee the existence of $a^{1/n}$ using continuity theorems. We then define $a^q = (a^{1/m})^n$ for any rational number $q = n/m$. If $\{q_i\}_{i=1}^{\infty}$ is a monotone sequence of rational numbers approaching a real number x then the Monotone Convergence Theorem guarantees that the sequence $\{a^{q_i}\}_{i=1}^{\infty}$ has a limit, which becomes the definition of a^x. This enables us to define the *exponential* function $y = a^x$ and its inverse *logarithm* function $y = \log_a x$.

(iii) Late Transcendental Method: Define the *natural logarithm* function ln by the rule

$$\ln x \; = \; \int_1^x \frac{1}{t}\, dt$$

for $x > 0$. Then define the *exponential* function $\exp = \ln^{-1}$, and put $e = \exp(1)$, called *Euler's number*. For $a > 0$ and x any real number, now define

$$a^x \; = \; \exp(x \ln a)\ ,$$

and it follows quickly that $e^x = \exp x$.

(iv) Properties of logs and exponentials:

(a) The domain and range of ln are $(0, \infty)$ and $\mathbb{R}$ respectively, and $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

(b) The domain and range of exp are $\mathbb{R}$ and $(0, \infty)$ respectively, and $\frac{d}{dx}(\exp x) = \exp x$.

(c) $\ln(xy) = \ln x + \ln y\,, \quad \ln(x/y) = \ln x - \ln y\,, \ln(x^k) = k \ln x\,.$

(d) $\exp(x+y) = \exp x \exp y\,, a^x a^y = a^{x+y}\,, a^x/a^y = a^{x-y}\,, (a^x)^z = a^{xz}\,, (ab)^x = a^x b^x\,.$

(e) $\log_a(xy) = \log_a x + \log_a y\,, \quad \log_a(x/y) = \log_a x - \log_a y\,, \quad \log_a(x^k) = k \log_a x\,.$

(f) $\log_a(a^x) = x\,, \quad a^{\log_a x} = x\,, \quad A = e^{\ln A}\,, \quad k = \ln e^k\,, \quad \log_a x = \dfrac{\ln x}{\ln a}\,.$

(v) Integration by Parts: The Product Rule $(fg)' = f'g + fg'$ leads to the integration formula

$$\int f(x)g'(x)\; dx \; = \; f(x)g(x) - \int g(x)f'(x)\; dx$$

and its equivalent, in terms of differentials, setting $u = f(x)$ and $v = g(x)$,

$$\int u\; dv \; = \; uv - \int v\; du\ .$$

(vi) Factorisation of real polynomials: Every real polynomial factorises as a product of linear and irreducible quadratic factors.

(vii) Rational functions and their decompositions: A *rational* function is a quotient of two polynomial functions, where the denominator is not the zero function. All real rational functions can be expressed as linear combinations of rational functions of the form p/q where either

(a) $p(x)$ is a constant polynomial and $q(x)$ is a positive power of a linear polynomial, or

(b) $p(x)$ is a linear polynomial and $q(x)$ is a positive power of an irreducible quadratic polynomial.

(viii) Method of partial fractions: To integrate a real rational function f, find appropriate constants that express f as a linear combination of rational functions involving the simple forms (a) and (b) of the previous fact, utilising denominators that are powers of linear and irreducible quadratic factors of the denominator of $f(x)$, and then integrate each piece using appropriate elementary substitutions and identities.

Revision and Exploration:

1. Use calculus to sketch the curve $y = \dfrac{\ln x}{x}$ and find the maximum y-value.

2. Which is bigger: $\sqrt{\pi}^{\,\pi}$ or $\pi^{\sqrt{\pi}}$?

3. Use the definition $a^x = \exp(x \ln a)$ to verify the following, noting any properties of the logarithm and exponential functions on which you rely:

(i) $a^0 = 1$	(ii) $\ln(a^x) = x \ln a$	(iii) $a^c a^d = a^{c+d}$
(iv) $(ab)^c = a^c b^c$	(v) $(a^c)^d = a^{cd}$	(vi) $\frac{d}{dx}\, x^a = a x^{a-1}$

4. For $0 < a \neq 1$ define $\log_a x = \dfrac{\ln x}{\ln a}$. Verify that $\log_a x = y$ if and only if $a^y = x$ (recovering the usual definition in terms of inverse functions), and that the following hold:

(i) $\log_a(xy) = \log_a x + \log_a y$ (ii) $\log_a x^b = b \log_a x$ (iii) $\log_a \frac{x}{y} = \log_a x - \log_a y$

5. Prove that $\log_{10} 2$ is irrational.

Tutorial Exercises:

6. Use the sketch from the first exercise to deduce that if $e \leq a < b$ then $b^a < a^b$, so that, in particular,

$$\pi^e < e^\pi \ .$$

7. Differentiate the following functions of x:

(i) $y = \log_{10} \sqrt{x}$ (ii) $y = x^{\sqrt{x}}$ (iii) $y = (\sin x)^x$

8. Use integration by parts to find the following:

(i) $\displaystyle\int (2x+3)e^x\,dx$ (ii) $\displaystyle\int_1^2 t^2 \ln t\,dt$ *(iii) $\displaystyle\int_{-\pi/4}^{\pi/4} \theta \sin\theta \cos\theta\,d\theta$

9. Use partial fractions to find the following:

(i) $\displaystyle\int \frac{1}{(x+1)(x-2)}\,dx$ (ii) $\displaystyle\int \frac{7}{2x^2+5x-3}\,dx$ *(iii) $\displaystyle\int \frac{x^4}{x^3-1}\,dx$

***10.** Suppose that $f(1)=2$, $f(4)=7$, $f'(1)=5$, $f'(4)=3$ and f'' is continuous. Find

$$\int_1^4 x f''(x)\,dx\ .$$

****11.** (for general discussion) This exercise shows that, for continuous functions, the exponential function is characterized by its algebraic property of turning sums into products. Prove that if f is a continuous function and

$$f(x+y) \ = \ f(x)f(y) \quad (\forall x, y \in \mathbb{R})\ ,$$

then either (i) $f(x)=0$ for all x, or (ii) $f(1)>0$ and $f(x)=[f(1)]^x$ for all x.

Further Exercises:

12. Establish the following reduction formula:

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx\ .$$

Use this formula to find $\int \cos^2 x\,dx$ and $\int \cos^4 x\,dx$.

***13.** Find the value of a for which the following equation is true:

$$\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x \ = \ e\ .$$

***14.** Use the substitution $t = \tan\frac{x}{2}$ to find $\displaystyle\int \frac{5}{4\sin x + 3\cos x}\,dx\ .$

***15.** Prove that

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx \ = \ \frac{\pi}{4}$$

for all positive integers n.

[Hint: make a substitution $u = \frac{\pi}{2} - x$ and rearrange the furniture.]

**16. Let $I_n = \displaystyle\int_0^{\pi/2} \sin^n x\,dx$ for $n \geq 0$, so $I_0 = \dfrac{\pi}{2}$ and $I_1 = 1$.

(i) Use the reduction formula in lectures to verify that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$ and deduce that

$$I_n = \begin{cases} \frac{2}{3}\,\frac{4}{5}\,\frac{6}{7}\,\cdots\,\frac{2k}{2k+1} & \text{if } n = 2k+1 \text{ is odd,} \\[2ex] \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\,\ldots\,\frac{2k-1}{2k}\,\frac{\pi}{2} & \text{if } n = 2k \text{ is even.} \end{cases}$$

(ii) Prove that $\displaystyle\lim_{k\to\infty} \frac{I_{2k+1}}{I_{2k}} = 1$ and deduce the famous *Wallis product* (circa 1656):

$$\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\,\cdots$$

***17. (challenging, for those that know some vector space theory) Find a *discontinuous* function f such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

[Hint: Define f on a basis for $\mathbb{R}$ regarded as a vector space over the field $\mathbb{Q}$.]

Short Answers to Selected Exercises:

1. e^{-1}

2. $\pi^{\sqrt{\pi}}$

7. (i) $\frac{1}{2x\ln 10}$ (ii) $x^{\sqrt{x}}\left(\frac{\ln x+2}{2\sqrt{x}}\right)$ (iii) $(\sin x)^x\left(\ln\sin x + \frac{x\cos x}{\sin x}\right)$

8. (i) $(2x+3)e^x - 2e^x + C$ (ii) $\frac{8}{3}\ln 2 - \frac{7}{9}$ (iii) $\frac{1}{4}$

9. (i) $\frac{1}{3}\ln\frac{|x-2|}{|x+1|} + C$ (ii) $\ln\frac{|2x-1|}{|x+3|} + C$ (iii) $\frac{x^2}{2} + \ln\frac{|x-1|^{1/3}}{(x^2+x+1)^{1/6}} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$

10. 2

12. $\frac{1}{2}\cos x\sin x + \frac{1}{2}x + C$ and $\frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + C$

13. $\frac{1}{2}$

14. $\ln\frac{|1+3\tan x/2|}{|3-\tan x/2|} + C$