THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Quiz 2: Practice Questions

MATH1901/1906: Differential Calculus (Advanced) Semester 1, 2017

Some information for the Quiz:

(1) The quiz covers material up to (and including) week 10 lectures. That is, material from the tutorials in weeks 2–11. The focus is primarily on the material from the lectures in weeks 6–10 (tutorial weeks 7–11), however you need to know the concepts from the earlier weeks too.

(2) The quiz runs for 40 minutes.

(3) You must write your answers in pen, not pencil.

(4) The format of the real quiz is mostly "short-answer" questions. There will be answer boxes provided below each question where you should write your final answers. Your working may be considered, so please write neatly. The format is similar to Quiz 1.

(5) There will be 10 questions in the real quiz, each worth the same amount.

(6) The quiz is a closed book examination. No notes or books are allowed.

(7) Non-programmable non-graphics calculators are allowed, but are not needed.

The questions provided here are for additional practice (building from the lectures and tutorials). It is strongly advised that you also revise the material from lectures and tutorials.

1. Find the third order Taylor polynomial $T_3(x)$ centred at $x = 0$ for the function $f(x) = \sqrt{x+1}$, and use Taylor's Theorem to find a formula for the remainder $R_3(x) = f(x) - T_3(x)$.

2. Let $T_{2n}(x)$ be the Taylor polynomial of the function $f(x) = \cosh x$ of order $2n$ centred at $x = 0$, and let $R_{2n}(x) = f(x) - T_{2n}(x)$ be the corresponding remainder term. According to Taylor's Theorem, there is a number c between 0 and x such that (circle the correct answer):

(a) $R_{2n}(x) = \dfrac{x^{2n}}{(2n)!}\sinh c$

(b) $R_{2n}(x) = \dfrac{x^{2n+1}}{(2n+1)!}\sinh c$

(c) $R_{2n}(x) = \dfrac{c^{2n+1}}{(2n+1)!}\sinh c$

(d) $R_{2n}(x) = \dfrac{c^{2n}}{(2n)!}\sinh c$

3. You are given that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, and that $f(1) = 3$ and $-2 \le f'(x) \le 4$ for all $x \in \mathbb{R}$. Which one of the following statements is necessarily true?

(a) $-1 \le f(3) \le 11$

(b) $-3 \le f(3) \le 6$

(c) $3 \le f(3) \le 13$

(d) $5 \le f(3) \le 14$

4. Calculate the following limits

(a) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3}$

(b) $\displaystyle\lim_{x\to 0} (\cos x)^{1/x^2}$

5. Let $f : (-1, \infty) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions

$$f(x) = |x| \ln(1+x) \qquad \text{and} \qquad g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \ge 0 \end{cases}$$

Which one of the following statements is true?

(a) Neither $f(x)$ nor $g(x)$ are differentiable at $x = 0$.

(b) Both $f(x)$ and $g(x)$ are differentiable once and only once at $x = 0$.

(c) $f''(x)$ is continuous at $x = 0$, and $g'(0) = 0$.

(d) $f'(x)$ is continuous at $x = 0$, and $g''(0) = 2$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the real valued function of 2 real variables given by the formula

$$f(x, y) = \frac{\sin y}{\cosh(x - y^2)}.$$

Which one of the following statements is true?

(a) f is injective, and the range of f is $[-1, 1]$.

(b) f is not injective, and the range of f is $[-1, 1]$.

(c) f is injective, and the range of f is $\mathbb{R}$.

(d) f is not injective, and the range of f is $\mathbb{R}$.

7. Let C_1 and C_2 be the curves in $\mathbb{R}^3$ with parametric equations

$$C_1(t) = (1+t, 1+2t^2, 2+2t^2) \qquad t \in \mathbb{R}$$
$$C_2(s) = (1+s, 3-2s^2, 4-2s^2) \qquad s \in \mathbb{R}.$$

Which one of the following statements is true?

(a) The curves C_1 and C_2 are both contained in a common plane.

(b) There is no plane containing both of the curves C_1 and C_2.

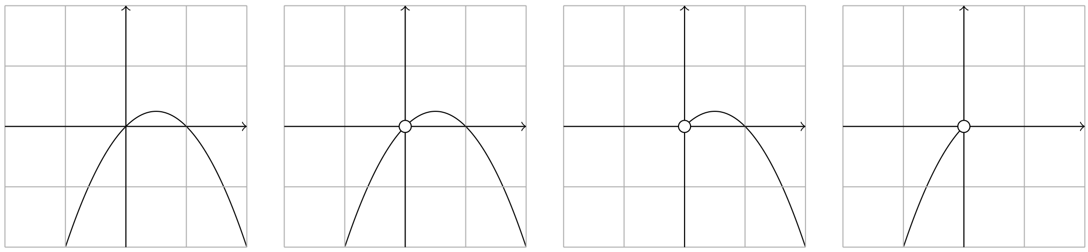
(c) The curves C_1 and C_2 do not intersect.

(d) The curves C_1 and C_2 intersect at exactly one point.

8. Let

$$f(x,y) = \frac{x}{\sqrt{x-y}}.$$

Which of the following best represents the level curve of $f(x,y)$ of height $z = 1$?



(a) (b) (c) (d)

9. For each of the following cases, write down the Taylor polynomial $T_n(x)$ of order n centred at $x = a$ for the functions $f(x)$, and use Taylor's Theorem to write down a formula for the associated remainder term $R_n(x)$.

(a) $f(x) = \cos x$, $n = 2$, and $a = \pi/4$.

(b) $f(x) = \ln(1+x)$, $n = 3$, and $a = 1$.

10. Let $f(x,y) = \ln(1 - \sqrt{x+y})$. Find the natural domain and the corresponding range of f.

11. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = xg(x)$.

(a) Show that f is differentiable at $x = 0$, with $f'(0) = g(0)$.

(b) If g is no longer assumed to be continuous, is it still true that f is differentiable at 0?

12. Calculate the following limits, or show that they do not exist.

(a) $\displaystyle\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3}$

(b) $\displaystyle\lim_{x\to 0^+} (\sinh x)^{1/x}$

(c) $\displaystyle\lim_{x\to\infty} (\sinh x)^{1/x}$

(d) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$

13. Calculate the following limits, or show that they do not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + 2y^3}{x^2 + y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2 + y^2}$

(c) $\displaystyle\lim_{(x,y)\to(\pi,0)} \frac{y^2 + \cos x}{x^2 + y^2}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$.

14. Draw the level curves of heights $c = 0, 1, 2$ of the functions:

(a) $f(x, y) = e^{x^2+y^2}$

(b) $g(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

15. Suppose that f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Which of the following statements are true, and which are false?

(a) There exists a point $c \in (0, 1)$ such that $f'(c) = 0$.

(b) There exists a point $c \in (0, 1)$ such that $f'(c) = f(1) - f(0)$.

(c) There exists a unique point $c \in (0, 1)$ such that $f'(c) = f(1) - f(0)$.

(d) If $f'(x) = 0$ for all $x \in (0, 1)$ then f is a constant function on $[0, 1]$.

16. Let $f(x) = x^3 - x$. Find all points $c \in (-2, 2)$ that satisfy the conclusion of the Mean Value Theorem when it is applied to f on the interval $[-2, 2]$.

17. Let $f(x, y) = x^2 + y^2 + 1$ and consider the surface $z = f(x, y)$. Which of the following is true?

(a) This surface intersects the xz-plane in a straight line.

(b) This surface intersects the plane $x = 3$ in a circle.

(c) Every point on the curve $C(t) = (t \cos t, t \sin t, t^2 + 1)$ lies on this surface.

(d) For all $c > 1$, the level curve of this surface at height $z = c$ is a parabola.