THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH1901/1906

DIFFERENTIAL CALCULUS (ADVANCED)

June 2008

LECTURERS: Anthony Henderson and Charlie Macaskill

TIME ALLOWED: **One and a half hours**

Name: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

1. (a) In the complex plane, sketch the set $\{z \in \mathbb{C} \mid 1 \leq |z - i| \leq 2\}$. (2 marks)

   (b) Find all complex solutions of the equation $z^4 - z^3 - iz + i = 0$. (3 marks)

   (c) Is the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = e^{z^2 - 2z}$ injective? Explain your answer. (2 marks)

   (d) Explain why the function $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = e^{x^2 - 2x}$ is injective, and find its range. (*Note*: this part concerns $\mathbb{R}$ not $\mathbb{C}$.) (3 marks)

2. (a) Find the following limits, showing the steps of your working clearly. You may use any valid method.

   (i) $\displaystyle \lim_{x \to 0} \frac{x^2 \cos \frac{1}{x} + 2}{x + 1}$ (2 marks)

   (ii) $\displaystyle \lim_{x \to \infty} \sqrt{x^2 + 3x} - \sqrt{x^2 - 3x}$ (2 marks)

   (b) Prove that the equation $x \sinh x = \cosh x$ has exactly one solution in the open interval $(1, 2)$. You may assume that $\dfrac{\sinh 2}{\cosh 2} > \dfrac{1}{2}$. (3 marks)

   (c) A special case of the Inequality Law for limits is the statement:

   $$\text{if } f(x) \geq 0 \text{ for all } x, \text{ and } \lim_{x \to a} f(x) = \ell, \text{ then } \ell \geq 0.$$

   Prove this special case, using the ϵ, δ definition of limit. (*Hint*: assume $\ell < 0$ and show this leads to a contradiction.) (3 marks)

3. (a) (*i*) Use l'Hôpital's rule to find $\lim_{x\to 0^+} x\ln x$. (2 marks)

(*ii*) Use the result of part (*i*) to show that

$$\lim_{x\to 0^+} x^n \ln x = 0, \qquad \text{for } n = 1, 2, 3, \dots$$ (1 mark)

(b) Determine the Taylor polynomial of degree 3 for $\tanh x = \dfrac{\sinh x}{\cosh x}$ about 0. You do not need to find the remainder term. (3 marks)

(c) (*i*) Write down the Taylor polynomial $T_n(x)$ of degree n for e^x about 0 and give the remainder term in terms of x, n and c, where c is between 0 and x. (2 marks)

(*ii*) Find n such that

$$\frac{1}{720} < e - T_n(1) < \frac{1}{240}.$$

(You may assume that $e < 3$.) (2 marks)

4. (a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$.

(*i*) Sketch the level curves $z = c$ of the surface $z = f(x, y)$, where $c = 0, \pm 1$. (3 marks)

(*ii*) Find the equation of the tangent plane to $z = f(x, y)$ at the point $(1, 2, -3)$. (2 marks)

(*iii*) Determine the greatest slope of the surface $z = f(x, y)$ at the point $(1, 2, -3)$ and the direction of that greatest slope. (1 mark)

(b) (*i*) Using polar coordinates, or otherwise, show that

$$\lim_{(x,y)\to(0,0)} \frac{x}{(x^2 + y^2)^p} = 0$$

where p is any positive real number such that $p < 1/2$. (2 marks)

(*ii*) Show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2 + y^2}}.$$ (2 marks)

End of Extended Answer Section