THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 4

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Definition of a function $f: A \to B$ and composites, domain, codomain and image/range of a function;
- ☐ Injective, surjective, and bijective functions; inverse functions.
- ☐ The concept of natural domain of a real valued function of a real variable.
- ☐ The graph of a function, and the horizontal line test for injectivity.
- ☐ The hyperbolic sine and cosine functions $\sinh x$ and $\cosh x$.

Outcomes

After completing this tutorial you should

- ☐ understand the concepts of domain, codomain and image/range of functions;
- ☐ be able to calculate the image/range of various functions;
- ☐ be able to prove whether given functions are injective, surjective or bijective and compute inverse functions;
- ☐ identify the natural domain of real valued functions of a real variable;
- ☐ work with the hyperbolic cosine and sine functions, and prove identities involving them.

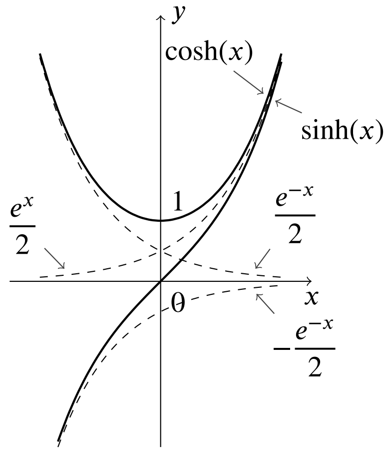
Summary of essential material

The hyperbolic sine and cosine. The *hyperbolic cosine* and *hyperbolic sine* functions are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

for all $x \in \mathbb{R}$. They share many properties with the cosine and sine functions as shown in some questions below.

The graph of the hyperbolic cosine function is the shape of a hanging cable or chain attached at two ends.



Functions. Let A and B be sets. A *function* $f: A \to B$ is a rule which assigns exactly one element of B to each element of A. We write $x \mapsto f(x)$ to indicate the value $f(x)$ assigned to x. The set A is called the *domain* of f, the set B the *codomain* of f. The *image* or *range* of f is $\operatorname{im}(f) = \{f(a) \mid a \in A\} \subseteq B$.

The function f is *surjective* or *onto* if $\operatorname{im}(f) = B$. To show that f is surjective one has to show that for every $y \in B$ there exists $x \in A$ such that $f(x) = y$.

The function f is *injective* or *one-to-one* if every point in the image comes from exactly one element in the domain. To show a function is injective prove

$$\big(x_1, x_2 \in A \text{ and } f(x_1) = f(x_2)\big) \Rightarrow x_1 = x_2$$

(the converse is obvious by definition of a function). The above means for all choices of x_1, x_2 with $f(x_1) = f(x_2)$ the implication has to be true.

The function f is *bijective* or *invertible* if it is both injective and surjective. In that case there exists an *inverse function* is the function $f^{-1}: B \to A$ defined by

$$f^{-1}(y) = \big(\text{the unique element } x \in A \text{ such that } f(x) = y\big).$$

In practice, to find f^{-1} we solve the equation $y = f(x)$ for $x \in A$.

Questions to complete during the tutorial

1. Let $f(x) = x^2$, considered as a function $f : A \to B$ for the various A and B listed below. In each case decide whether f is injective and whether f is surjective.

(a) $f : \mathbb{R} \to \mathbb{R}$ (c) $f : [0,1] \to [0,1]$ (e) $f : \mathbb{N} \to \mathbb{N}$

(b) $f : [-1,2] \to [0,4]$ (d) $f : [0,\infty) \to [0,\infty)$ (f) $f : \mathbb{Q} \to [0,\infty)$

2. (a) Show that $\cosh^2 x - \sinh^2 x = 1$ for all $x \in \mathbb{R}$.

(b) Let $a, b > 0$. Show that $x(t) = a\cosh(t)$, $y(t) = b\sinh(t)$ ($t \in \mathbb{R}$) is a parametric representation of one branch of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(c) Explain, using the graphs, why $\sinh : \mathbb{R} \to \mathbb{R}$ and $\cosh : [0,\infty) \to [1,\infty)$ are bijective. Sketch the graphs of the inverse functions.

3. Let $A = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 2 \text{ and } -\pi < \mathrm{Im}(z) \leq \pi\}$, and let B be the image of A under $f(z) = e^z$.

(a) Sketch A and B, and show that $f : A \to B$ is bijective.

(b) Find a formula for the inverse function $f^{-1} : B \to A$.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is called *strictly increasing* if $x_1 < x_2$ implies that $f(x_1) < f(x_2)$.

(a) Show that if f is strictly increasing then f is injective.

(b) Show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are strictly increasing, then the composition $g \circ f : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Deduce that $g \circ f$ is injective.

(c) Using the result of the previous part, and the fact that e^x is strictly increasing, prove that $\cosh : [0,\infty) \to \mathbb{R}$ is strictly increasing, and hence injective.

5. Each formula below belongs to a function $f : A \to B$ where we take $A \subseteq \mathbb{R}$ to be the natural domain of f, and we take the codomain B to be the image of the natural domain under f. Thus each function is automatically surjective. In each case find A, and decide if the function $f : A \to B$ is a bijection. If so, find a formula for the inverse function.

(a) $f(x) = \dfrac{x-2}{x+2}$, (b) $f(x) = \sqrt{2+5x}$, (c) $f(x) = x|x| + 1$.

6. (a) The function $\cosh : [0,\infty) \to [1,\infty)$ is a bijection, so has an inverse $\cosh^{-1} : [1,\infty) \to [0,\infty)$. Show that $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$.

(b) The function $\cosh : (-\infty, 0] \to [1,\infty)$ is also a bijection. Find a formula for its inverse function.

7. For what values of the constants a, b, c (with $b \neq 0$) is the function with formula

$$f(x) = \frac{x-a}{bx-c} \quad \text{and domain } \{x \in \mathbb{R} \mid x \neq c/b\}$$

equal to its own inverse? (*Hint*: It may help to draw the graph.)

8. Prove the hyperbolic "sum of angles" formulae, for all $x, y \in \mathbb{R}$:

(a) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$ (b) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

Extra questions for further practice

9. Suppose that $f: A \to B$ is bijective. Define what is meant by the inverse function $f^{-1}: B \to A$, and explain why this definition makes sense.

10. Let $A = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 1 \text{ and } 2\pi < \mathrm{Im}(z) \leq 4\pi\}$, and let B be the image of A under $f(z) = e^z$.

(a) Sketch A and B, and show that $f: A \to B$ is bijective.

(b) Find a formula for the inverse function $f^{-1}: B \to A$.

11. Let $f(x) = x^3$, considered as a function $f: A \to B$ for the various A and B listed below. In each case decide whether f is injective and weather f is surjective.

(a) $f: \mathbb{R} \to \mathbb{R}$

(b) $f: \mathbb{Z} \to \mathbb{Z}$

(c) $f: \mathbb{Q} \to \mathbb{Q}$

(d) $f: \{-1, 0, 2\} \to \{-1, 0, 8\}$

(e) $f: [0, 1] \to [-1, 1]$

(f) $f: [0, \infty) \to [0, \infty)$

12. Explain why the functions given by the formulas and domains below are injective. Find their ranges and formulas for their inverses.

(a) $f(x) = x^2 + x,\ x \geq -\frac{1}{2}$.

(b) $g(x) = \sqrt[4]{x},\ x \geq 0$.

(c) $h(x) = \dfrac{1 + e^x}{1 - e^x},\ x \neq 0$.

(d) $f(x) = \ln(3 + \sqrt{x - 4}),\ x \geq 5$.

13. Is the following statement true or false? "A function $f: \mathbb{R} \to \mathbb{R}$ is injective if and only if f is either strictly increasing or strictly decreasing." If you think it is true, give a proof. If you think it is false, give a counterexample.

14. Give an example of functions $f: A \to B$ and $g: B \to C$ such that g is surjective yet the composition function $g \circ f: A \to C$ is not surjective.

15. Last week you proved a closed formula for $1 + 2\cos x + 2\cos 2x + \cdots + 2\cos nx$. Find a corresponding 'hyperbolic' version for $1 + 2\cosh x + 2\cosh 2x + \cdots + 2\cosh nx$.

16. Show that if $f: A \to B$ is bijective, then the inverse function $f^{-1}: B \to A$ is also bijective.

17. Let A, B and C be sets and let $f: A \to B$ and $g: B \to C$ be functions.

(a) Show that if f and g are injective then the composition $g \circ f: A \to C$ is also injective.

(b) Show that if f and g are surjective then the composition $g \circ f: A \to C$ is also surjective.

(c) Deduce that the composition of bijections is again a bijection, and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Challenge questions (optional)

18. We say that the set A has the same *cardinality* as the set B if there exists a bijection $f: A \to B$.

(a) Show that if A has the same cardinality as B, then B has the same cardinality as A. That is, show that if there is a bijection $f: A \to B$ then there is a bijection $g: B \to A$.

(b) Show that if A has the same cardinality as B, and B has the same cardinality as C, then A has the same cardinality as C.

(c) Show that if A and B have finitely many elements then A and B have the same cardinality if and only if A and B have the same number of elements.

19. We say that a set A has the same *cardinality* as the set $\mathbb{N}$ of natural numbers if there is a bijection $f : \mathbb{N} \to A$. In this case we say that A is *countable*. This means that we can write all of the elements of A in a list in which every element occurs exactly once:

$$a_0, a_1, a_2, \dots,$$

where $f(j) = a_j$ is a bijection $f : \mathbb{N} \to A$. Thus, morally, A has the "same size" as $\mathbb{N}$, because the elements of A are paired-up bijectively with the elements of $\mathbb{N}$.

(a) Show that $\mathbb{Z}$ is countable.

(b) Show that the set $\mathbb{N} \times \mathbb{N} = \{(m, n) \mid m \in \mathbb{N} \text{ and } n \in \mathbb{N}\}$ is countable.

(c) Show that if A and B are countable then the set $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ is also countable.

(d) Show that the set $X = \mathbb{Q} \cap [0, 1)$ of all rational numbers in the interval $[0, 1)$ is countable.

(e) Deduce that the set $\mathbb{Q}$ of all rational numbers is countable.
Remark: This is rather surprising, since intuitively $\mathbb{Q}$ feels a lot "bigger" than $\mathbb{N}$.

(f) So perhaps every infinite set is countable? No: Show that the set of real numbers in the interval $[0, 1]$ is *not* countable.
Note: This is tough if you haven't seen something like it before!

(g) The *power set* of a set A is the set $\mathcal{P}(A) = \{B \mid B \subseteq A\}$ consisting of all subsets of A. For example, if $A = \{1, 2, 3\}$ then $\mathcal{A} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, where $\emptyset = \{\}$ is the 'empty set'. Show that for any set A the set $\mathcal{P}(A)$ does not have the same cardinality as A. Hence deduce that there is a set 'bigger' than $\mathbb{R}$, and that in fact there is an infinite number of growing 'sizes' of infinite sets.