THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 11

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Real valued functions $f(x, y)$ of 2 real variables.
- ☐ Natural domain and corresponding range of $f(x, y)$.
- ☐ The graph of $f(x, y)$.
- ☐ Level curves (more generally, cross-sections).
- ☐ Limits of functions of 2 variables.

Outcomes

After completing this tutorial you should

- ☐ find the natural domain and corresponding range of a function $f(x, y)$;
- ☐ draw level curves, and sketch graphs of functions $f(x, y)$ in simple cases;
- ☐ calculate limits of functions $f(x, y)$, or show that they don't exist.

Summary of essential material

Functions of two variables. Consider a function $f : D \to \mathbb{R}$ where the domain D is a subset of $\mathbb{R}$. The set of points

$$\operatorname{graph}(f) = \Big\{\big(x, y, f(x, y)\big) \mid (x, y) \in D\Big\} \subseteq \mathbb{R}^3$$

is called the *graph* of f and is a surface in $\mathbb{R}^3$. Given $c \in \mathbb{R}$ the set

$$\Big\{(x, y) \in D \mid f(x, y) = c\Big\}$$

is called the *level set* or *level curve of f at height c*. It is typically a proper curve, but can be empty (if c is not in the image of f) or "degenerate" such as a point or crossing curves.

Limits of functions of two variables. We say that $\lim_{(x,y)\to(a,b)} f(x, y) = \ell$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \Rightarrow \quad |f(x, y) - \ell| < \varepsilon.$$

Analogues of the one-variable limit laws, including the Squeeze Law, hold for functions of two-variables.
Useful techniques to determine limits

- Use simple inequalities: $|xy| \leq x^2 + y^2$, $|\sin a| \leq |a|$, $\log x \leq 1 - x$ $(x > 0)$ and others.
- Convert to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, limit to $(0, 0)$ is the same as $r \to 0$ *independently* of θ.

Useful ways to show that a limit does *not* exist:

- Find two "paths of approach" giving two different limits. Often along the axes, the diagonals or some parabola, depending of the structure of the expression of interest.
- The limit along one particular path does not exist.

Continuity of functions of two variables. A real valued function $f(x, y)$ of two real variables is *continuous* at a point (a, b) in the domain D of f if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

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Questions to complete during the tutorial

1. The Taylor polynomial of order n for the exponential function e^x is

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Substitute $x = i\theta$ into this polynomial and show that $T_n(i\theta) = C_n(\theta) + iS_n(\theta)$, where $C_n(\theta)$ and $S_n(\theta)$ are the Taylor polynomials of order n of the cosine and sine functions respectively.

2. Determine the natural domain and corresponding range of the following functions.

(a) $f(x, y) = \sqrt{x - y}$

(b) $f(x, y) = \tan^{-1}(y/x)$

(c) $f(x, y) = \sqrt{x + y} - \sqrt{x - y}$

(d) $f(x, y) = \sin^{-1}(x + y)$

3. Sketch the level curves at heights $c = 0, \pm 1, \pm 2$ for the functions $f(x, y)$:

(a) $1 - x^2 - y^2$ (b) $4x^2 + y^2$ (c) $y - x^2$ (d) $2x + 3y$

4. Sketch the level curve of height $z = 1$ for the function $f(x, y) = \dfrac{2xy^2}{x^2 + y^4}$.

5. Consider the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}, \text{ defined for } (x, y) \neq (0, 0).$$

Is it possible to define $f(0, 0)$ so that f is continuous at $(0, 0)$? Continuous, as in case of a function of one variable, means that the limit exists and is equal to the function value.

6. Find the limit, if it exists, or show that the limit does not exist.

(a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ (b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x - y}{x^2 + y^2}$ (c) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2}{x^2 + y^2}$

Extra questions for further practice

7. Find the domains and ranges, and describe the level curves, of the functions defined by:

(a) $\sqrt{4 - x^2 - y^2}$

(b) $(x-1)(y+1)$

8. Decide whether or not the following limits exist.

(a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

(b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} \sin \frac{1}{x^2+y^4}$

(c) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

(d) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$

Challenge questions (optional)

9. Use the ϵ, δ definition of the limit of a function of two variables to show that

$$\lim_{(x,y)\to(1,2)} (x^2 + y) = 3.$$