The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 11

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/
Lecturers: Daniel Daners and David Easdown

Questions marked with * are more difficult questions.

Material covered

- ☐ First order linear and separable differential equations
- ☐ Substitution of variables in differential equations
- ☐ homogeneous second order differential equations with constant coefficients

Outcomes

After completing this tutorial you should

- ☐ consolidate the ability to solve first order linear and separable differential equations by the appropriate methods.
- ☐ be able to solve homogeneous second order differential equations in case of two distinct solutions of the auxiliary equation.

Questions to do before the tutorial

1. Classify each of the following differential equations as separable or linear, and find its general solution.

(a) $\dfrac{dy}{dx} = \dfrac{2xy^2}{1+x^2}$

Solution: The equation is not linear but separable. Separating and integrating we get

$$\int \frac{1}{y^2}dy = \int \frac{2x}{1+x^2}dx$$

and therefore

$$-\frac{1}{y} = \ln(1+x^2) + C.$$

Hence the general solution is

$$y = -\frac{1}{\ln(1+x^2)+C}.$$

(b) $\dfrac{dy}{dx} = x + \dfrac{y}{x}$

Solution: The equation is linear and can be solved by using the method of integrating factors. An integrating factor is a solution to the homogeneous equation

$$\frac{dy}{dx} + \frac{y}{x} = 0,$$

which is given by

$$\exp\Bigl(\int -\frac{1}{x}dx\Bigr) = \exp(-\ln|x|) = |x|^{-1}$$

The absolute value sign can be omitted because any particular solution of the above homogeneous equation is an integrating factor. Hence multiplying the original equation with the integrating factor

$$\frac{d}{dx}\frac{y}{x} = 1$$

and integrating

$$\frac{y}{x} = \int 1dx = x + C.$$

The general solution therefore is

$$y = x(x + C).$$

2. Find the general solutions of the following homogeneous second-order linear differential equations. Then compute the particular solution satisfying the initial conditions $y(0) = \dot{y}(0) = -1$.

(a) $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 6y = 0$

Solution: The auxiliary equation $\lambda^2 - \lambda - 6 = 0$ has roots $\lambda = 3, -2$, and so the general solution is $y = Ae^{3t} + Be^{-2t}$. We next determine the constants A and B using the initial conditions. Differentating the general solution we see that $\dot{y} = 3Ae^{3t} - 2Be^{-2t}$, and so $\dot{y}(0) = 3A - 2B$. The general solution also gives $y(0) = A + B$. Hence the initial conditions imply $A + B = -1$ and $3A - 2B = -1$, and therefore $A = -3/5$ and $B = -2/5$. So the particular solution which satisfies the differential equation is

$$y = -\frac{3}{5}e^{3t} - \frac{2}{5}e^{-2t}$$

(b) $\dfrac{d^2y}{dt^2} + 16y = 0$

Solution: The auxiliary equation $\lambda^2 + 16 = 0$ has complex roots $\lambda = \pm 4i$, and so the general solution is $y = C\cos 4t + D\sin 4t$. Next we determine the constants C and D using the initial conditions. Differentiating the general solution, we see that $\dot{y} = -4C\sin 4t + 4D\cos 4t$, and so $\dot{y}(0) = 4D$. The general solution also gives $y(0) = C$. Hence the initial conditions imply $C = -1$ and $4D = -1$, and therefore $D = -1/4$. So the particular solution which satisfies the differential equation is

$$y = -\cos 4t - \frac{1}{4}\sin 4t.$$

Questions to complete during the tutorial

No questions due to quiz

Extra questions for further practice

3. Find the general solutions of the following homogeneous second-order linear differential equations. Then compute the particular solution satisfying the initial conditions $y(0) = \dot{y}(0) = -1$.

(a) $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 13y = 0.$

Solution: The auxiliary equation $\lambda^2 + 6\lambda + 13 = 0$ has complex roots $\lambda = -3 \pm 2i$, and so the general solution is $y = e^{-3t}(C\cos 2t + D\sin 2t)$.

Differentiating the general solution, we see that

$$\dot{y} = (2D - 3C)e^{-3t}\cos 2t - (2C + 3D)e^{-3t}\sin 2t$$

and so $\dot{y}(0) = 2D - 3C$. The general solution also gives $y(0) = C$. Hence the initial conditions imply $2D - 3C = -1$ and $C = -1$, and therefore $D = -2$. So the particular solution which satisfies the differential equation is

$$y = -e^{-3t}(\cos 2t + 2\sin 2t).$$

4. (a) Classify the differential equation $(x^2+1)\dfrac{dy}{dx} + x = xy^2$ as separable or linear, and find its general solution. Then determine the particular solution for which $y = 0$ when $x = 1$.

Solution: The differential equation is separable. It can be rewritten as

$$\frac{dy}{y^2-1} = \frac{x dx}{x^2+1}.$$

Integrating gives

$$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{1}{2}\ln(x^2+1) + C.$$

Hence,

$$\frac{y-1}{y+1} = \pm e^{2C}(x^2+1) = A(x^2+1).$$

Replacing $\pm e^C$ by a constant A having either sign and solving for y gives the general solution we get

$$y = \frac{1 + A(x^2+1)}{1 - A(x^2+1)}.$$

Next we compute the particular solution. Putting $y = 0$ when $x = 1$ in the general solution gives $A = -1/2$, so the particular solution is

$$y = \frac{1-x^2}{3+x^2}.$$

(b) Use the substitution $v = xw$ to find the general solution of the differential equation

$$2xv\frac{dv}{dx} = 3v^2 - 4x^2.$$

Solution: We find that

$$\frac{dv}{dx} = \frac{d}{dx}(xw) = w + x\frac{dw}{dx},$$

so the differential equation becomes

$$2x(xw)\left(w + x\frac{dw}{dx}\right) = 3(xw)^2 - 4x^2$$

Rewriting we get

$$2x^2w^2 + 2x^3w\frac{dw}{dx} = 3x^2w^2 - 4x^2$$

and finally

$$2xw\frac{dw}{dx} = w^2 - 4.$$

Separating and integrating, we get

$$\int \frac{2w}{w^2-4}dw = \int \frac{1}{x}dx.$$

Hence $\ln|w^2 - 4| = \ln|x| + C$ and so $w^2 - 4 = \pm e^C x = Ax$ if we replace $\pm e^C$ by another constant A of either sign. Hence $w = \pm\sqrt{4 + Ax}$ and so undoing the substitution

$$v = xw = \pm x\sqrt{4 + Ax}.$$

5. (a) Show that a differential equation of the form $y' = f(y/x)$ becomes separable when transformed into a differential equation for $v = y/x$.

Solution: We note that

$$v' = \frac{d}{dx}\frac{y}{x} = \frac{y'}{x} - \frac{y}{x^2}.$$

Using the original differential equation we get

$$v' = \frac{y'}{x} - \frac{y}{x^2} = \frac{1}{x}f\Big(\frac{y}{x}\Big) - \frac{1}{x}\frac{y}{x} = \frac{1}{x}\big(f(v) - v\big).$$

The above is a separable differential equation for v.

(b) Show that the differential equation $\dfrac{dy}{dx} = \dfrac{y^3}{x(x^2+y^2)}$ is of the form $y' = f(y/x)$ and hence solve it by transforming it into a separable equation.

Solution: We can rewrite the equation in the form

$$\frac{dy}{dx} = \frac{y^3}{x^3}\frac{1}{1+\frac{y^2}{x^2}}$$

and hence it is of the form required. Define $v = y/x$. Then $y' = (xv)' = v + xv'$, and so the differential equation implies $v + xv' = v^3/(1+v^2)$ or $xv' = -v/(1+v^2)$. This is a separable equation. Separating and integrating we get

$$-\int \frac{1+v^2}{v}dv = \int \frac{dx}{x}$$

Hence

$$-\ln|v| - \frac{1}{2}v^2 = \ln|x| + C$$

and therefore

$$e^{-v^2/2} = Axv = Ay.$$

Then recalling that $v = y/x$ gives $e^{y^2/(2x^2)} = Ay$. It is not possible to write y explicitly as a function of x.

6. Find the general solution of $\dfrac{dy}{dx} = \dfrac{1}{x+y}$.

Solution: The differential equation is neither linear nor separable. However, the differential equation for the inverse function $x(y)$ is linear:

$$\frac{dx}{dy} = x + y.$$

This equation is best solved by using an integrating factor which is given by e^{-y}. Hence

$$\frac{d}{dy}(xe^{-y}) = ye^{-y}$$

Integrating by parts yields

$$xe^{-y} = \int ye^{-y}\, dy = -ye^{-y} + \int e^{-y}\, dy = -ye^{-y} - e^{-y} + C.$$

Hence

$$x = Ce^{y} - y - 1.$$

Note that this equation is impossible to solve for y, so we leave it in this form.

Alternatively, one could use the substitution $v = x + y$ to turn the equation into a separable equation. This leads to the same expression for the solution as above.

7. Solve the following differential equations by the transformations indicated:

(a) $3x\dfrac{dy}{dx} + y + x^2y^4 = 0$ (a Bernoulli equation, let $w = \dfrac{1}{y^3}$).

Solution: Setting $w = y^{-3}$ implies $y = w^{-1/3}$ and $y' = -(1/3)w^{-4/3}w'$. Substitution into the Bernoulli equation and rearanging gives the linear equation,

$$x\frac{dw}{dx} - w = x^2.$$

Dividing by x puts it in standard form, after which its integrating factor is seen to be $1/x$. Hence $(d/dx)(w/x) = 1$. The general solution therefore is $w = x(x + C)$. In terms of the original variable y, the solution becomes

$$y = \frac{1}{\sqrt[3]{x(x+C)}}.$$

(b) $\dfrac{dy}{dx} + xy^2 + \dfrac{3}{4x^3} = 0$ (a Riccati equation, let $y = \dfrac{1}{2x^2} + \dfrac{1}{w}$).

Solution: If we set $y = 1/(2x^2) + 1/w$, then $y' = -1/x^3 - w'/w^2$. Substitution into the Riccati equation and simplifying the result gives the equation $xw' - w = x^2$ that we met in part (a). Hence $w = x(x + C)$ and so

$$y = \frac{3x + C}{2x^2(x + C)}.$$

*(c) $x\dfrac{dy}{dx} - y = \dfrac{1}{4}\left(\dfrac{dy}{dx}\right)^4$ (a Clairaut equation, differentiate both sides).

Solution: Differentiate both sides of the Clairaut equation $xy' - y = (y')^4/4$ to get

$$\{(y')^3 - x\}y'' = 0.$$

The two factors give solutions of distinct character. The second factor integrates to $y = Cx + C_2$, with two constants. As the original differential equation is of first order we can only have one constant. Note however, that be differentiating the equation we 'lose' a constant. Substitution into the original equation identifies $C_2 = -C^4/4$. Hence the general solution is

$$y = Cx - \frac{1}{4}C^4.$$

This is a one parameter family of straight lines. There is a curved envelope which is tangent to all the lines. This curve also satisfies the Clairaut equation and is the singular integral. It comes from the factor $(y')^3 - x$. Setting this factor to zero gives $y' = x^{1/3}$. We could easily integrate y' to get the solution, but we can also just substitute $y' = x^{1/3}$ into the original diffeernital equation. If we do that we get

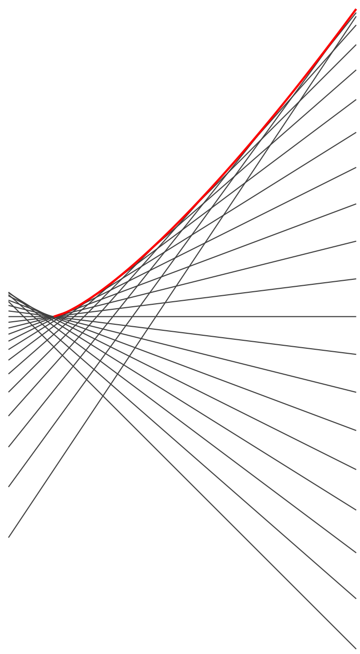
$$y = xy' - \frac{1}{4}(y')^4 = x^{4/3} - \frac{1}{4}x^{4/3} = \frac{3}{4}x^{4/3}.$$

Hence the singular integral (solution) is

$$y = \frac{3}{4}x^{4/3}.$$

The straight line solution $y = Cx - (1/4)C^4$ touches this curved solution tangentially at the point $(C^3, 3C^4/4)$.

The graph below shows the straight line solutions as C varies and the curved solution (plotted red) tangent to the straight lines, forming an "envelope".



Remark Linear differential equations never have singular integrals. Nor do non-linear differential equations in which the highest derivative appears linearly. Suppose $F(x, y, y') = 0$ is a first-order differential equation for y in which the left-hand side is a polynomial in y and y', the degree in y' being at least 2. A singular integral, if one exists, is a function $y = f(x)$ that satisfies both $F = 0$ and $\partial F/\partial y' = 0$. The curve with equation $y = f(x)$ is tangent to each of the curves in the general solution (a one-parameter family depending on a constant C). We call such a curve an *envelope* of the one-parameter family of curves. In the Clairaut equation above, as in most cases, the singular integral is not a member of the family of solutions forming the general solution. Some people assume this is always true. However, in particular cases, it is possible for a singular integral to be a member of the general solution, just as it is possible for a one-parameter family of curves to contain its own envelope.

8. Classify the following differential equations and find the general solution of each.

(a) $\dfrac{dy}{dx} = \dfrac{5-2y}{1+x^2}$

Solution: The differential equation is separable and linear. It is easiest to separate since otherwise we need to introduce an integrating factor. Separating and integrating we get

$$\int (5-2y)^{-1} dy = \int (1+x^2)^{-1}\, dx$$

Hence

$$-\frac{1}{2}\ln|5-2y| = \tan^{-1} x + C$$

and so $\ln|5-2y| = -2C - 2\tan^{-1} x$. Solving for y we get $2y-5 = \pm e^{-2C}\exp(-2\tan^{-1} x)$ and finally

$$y = \frac{5}{2} + A\exp(-2\tan^{-1} x).$$

(b) $\dfrac{dy}{dx} = \dfrac{5-2xy}{1+x^2}$

Solution: The equation is linear and can be solved using an integrating factor. In standard form the equation is

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{5}{1+x^2}.$$

Hence an integrating factor is given by

$$\exp\Big(\int 2x(1+x^2)^{-1}dx\Big) = \exp\big(\ln(1+x^2)\big) = 1+x^2.$$

Therefore

$$\frac{d}{dx}\big((1+x^2)y\big) = 5.$$

If we integrate we obtain

$$(1+x^2)y = 5x + C$$

and so the general solution is

$$y = \frac{5x+C}{1+x^2}.$$

(c) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - 4y = 0$

Solution: This is a homogeneous linear second-order differential equation with constant coefficients. The auxiliary equation is $0 = \lambda^2 + 3\lambda - 4 = (\lambda-1)(\lambda+4)$, which has roots $\lambda = 1$ and $\lambda = -4$. The general solution is therefore

$$y = Ae^t + Be^{-4t}.$$

9. Solve the following differential equations by making suitable substitutions:

(a) $\dfrac{dy}{dx} = \dfrac{x+y}{x+y+2}$

Solution: Define $w = x+y$. Then $w' = 1+y' = 1+w/(w+2) = (2w+2)/(w+2)$, which is separable. We get

$$\begin{aligned}
&\int \frac{(w+2)dw}{2w+2} = \int dx \\
&\Rightarrow \frac{w}{2} + \frac{1}{2}\ln|w+1| = x + C \\
&\Rightarrow (w+1)e^{w} = Ae^{2x} \\
&\Rightarrow x+y+1 = Ae^{x-y}.
\end{aligned}$$

It is not possible to write y explicitly as a function of x.

(b) $\dfrac{dy}{dx} = \dfrac{1}{(x+2y)^2+1}$

Solution: Define $z = x+2y$. Then $z' = 1+2y' = 1+2/(z^2+1) = (z^2+3)/(z^2+1)$, which is separable. We get

$$\begin{aligned}
&\int \frac{z^2+1}{z^2+3}dz = \int dx \\
&\Rightarrow \int \left(1 - \frac{2}{z^2+3}\right)dz = x + C \\
&\Rightarrow z - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right) = x + C \\
&\Rightarrow x + 2y - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}}\right) = x + C \\
&\Rightarrow y = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}}\right) + A.
\end{aligned}$$

where A is an arbitrary constant of integration. It is not possible to write y explicitly as a function of x.