THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education, Engineering and Science

MATH1901/1906
Differential Calculus (Advanced)

June 2005 Time Allowed: One and a half hours

Lecturer: Jenny Henderson

This Examination has 3 Printed Components.

(1) An Extended Answer Question Paper (this booklet, green 8015A): 3 pages numbered 1 to 3; 6 questions numbered 1 to 6.

(2) A Multiple Choice Question Paper (Yellow 8015B): 4 pages numbered 1 to 4; 15 questions numbered 1 to 15.

(3) A Multiple Choice Answer Sheet (White 8015C): 1 page.

Components 2 and 3 MUST NOT be removed from the examination room.

*This Examination has 2 Sections: **Extended Answer** and **Multiple Choice**.*

*The **Extended Answer Section** is worth 75% of the total marks for the paper: all questions may be attempted; questions are not of equal value; all necessary working must be shown.*

*The **Multiple Choice Section** is worth 25% of the total marks for the paper: all questions may be attempted; questions are of equal value; answers must be coded onto the **Multiple Choice Answer Sheet**.*

1. [12 marks]

(a) Find all complex numbers z satisfying the equation $z^2 + \overline{z} = 2$. (3 *marks*)

(b) Find all complex numbers z satisfying the equation $e^{2z} - 2e^z + 2 = 0$. (3 *marks*)

(c) Let $A = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Find and sketch the image of A under the function $z \mapsto z - \dfrac{1}{z}$. (3 *marks*)

(d) Show that the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$, is neither injective nor surjective. (3 *marks*)

2. [11 marks]

Let f be the function given by $f(x, y) = e^{-(x^2+y^2)}$.

(a) Find the domain and the range of f. (2 *marks*)

(b) Sketch the level curves at heights 1, $1/e$ and $1/e^2$, of the surface given by $z = f(x, y)$. (3 *marks*)

(c) The plane $x + 2y = 2$ intersects the surface $z = f(x, y)$ in a curve in $\mathbb{R}^3$.

(*i*) Find parametric equations for this curve. (3 *marks*)

(*ii*) Find the maximum height of the curve above the xy-plane. (3 *marks*)

3. [14 marks]

(a) Consider the surface $h(x, y) = 1 + 2x\sqrt{y}$.

(*i*) Find $\nabla h(x, y)$ at the point $(3, 4)$. (3 *marks*)

(*ii*) Find the maximum rate of change of h at $(3, 4)$. (2 *marks*)

(*iii*) Find the (two) directions one could begin to move to stay level if one is standing on the surface at $(3, 4, 13)$. (2 *marks*)

(b) You may assume that the Taylor polynomial of order 4 for $\cos x$, about $x = 0$, is

$$T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

with remainder term $R_4(x) = \dfrac{(-\sin c)\, x^5}{5!}$, for some c between 0 and x.

(*i*) Using the information given above, write down the Taylor polynomial of order 8 for $\cos(x^2)$, about $x = 0$. (1 *mark*)

(*ii*) Show that $0.903 < \displaystyle\int_0^1 \cos(x^2)\, dx < 0.906$. (6 *marks*)

4. [13 marks]

(a) (*i*) Complete the following sentence:

We say $\lim_{x \to a} f(x) = \ell$ if, given any $\epsilon > 0$, there exists $\delta > 0$ such that ....

(2 *marks*)

(*ii*) Given $\epsilon > 0$, find a suitable value of δ which establishes the result $\lim_{x \to 1} (2x + 3) = 5$. (2 *marks*)

(b) Find the following limits. (Do not use ϵ, δ methods.)

(*i*) $\lim_{x \to 0} \dfrac{\cos x - 1}{x^2}$ (*ii*) $\lim_{x \to 0^+} x \ln\left(\dfrac{1}{x}\right)$ (*iii*) $\lim_{x \to \infty} (x - \ln(\cosh x))$

(9 *marks*)

5. [15 marks]

(a) State the Intermediate Value Theorem. (2 *marks*)

(b) Let $f(x) = 2x^2 - 3x - \ln x$. Show that there are exactly two solutions of the equation $f(x) = 0$. (5 *marks*)

(c) Let $h(x) = \begin{cases} x^3 & \text{if } x \geq 1, \\ e^{x^2+ax+b} & \text{if } x < 1. \end{cases}$

(*i*) For which a, b is the function h continuous at 1? (2 *marks*)

(*ii*) For which a, b is the function h differentiable at 1? (2 *marks*)

(d) Prove that if a function g is continuous at the point a, then there exists a number $\delta > 0$ such that $g(x)$ is bounded above and below on the interval $(a - \delta, a + \delta)$. (4 *marks*)

6. [13 marks]

Let $F(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

(a) Show that F is continuous at $(0, 0)$. (Hint: use polar coordinates.) (4 *marks*)

(b) Show that $\dfrac{\partial F}{\partial x}(0, 0)$ and $\dfrac{\partial F}{\partial y}(0, 0)$ both exist. (4 *marks*)

(c) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be a unit vector with $u_1 \neq 0$ and $u_2 \neq 0$. Show that $D_{\mathbf{u}}F(0, 0)$ does not exist. (5 *marks*)