MATH1903/1907 Lectures

Week 7, Semester 2, 2017

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Differential equations

What is it?

Equation involving

- an unknown function
- one or more derivatives of that function.

Simplest form:

$$y'(x) = f(x) \qquad \text{solution} \quad y = \int f(x)\,dx$$

$$y'(x) = y(x) \qquad \text{solution} \quad y(x) = a e^{x}$$

general form of an explicit first order differential equation

$$y'(x) = F(x, y(x)) \qquad (*)$$

Terminology:

- x independent variable
- y dependent variable
- If F does not explicitly depend on x we call $(*)$ an autonomous equation:

$$y'(x) = F(y(x))$$

- If F depends explicitly on x we call $(*)$ a non-autonomous equation

We also consider "higher order" differential equations:

Involve higher order derivatives:

$$ay'' + by' + cy = f(x)$$

Second order differential equation: Highest order derivative involved is second order.

<u>Order</u> of a d.e: order of highest derivative of the unknown function in the equation.

<u>Classification of d.e.'s</u>

- order: 1st order, 2nd order, ...
- unknown function y only depends on one variable $x \in \mathbb{R}$.
  ⇝ involves only "ordinary derivatives",
  Hence the term "ordinary differential eq" (ODE)
- unknown function depends on several variables
  ⇝ involves the "partial derivatives"
  Hence the term "partial differential eq" (PDE)

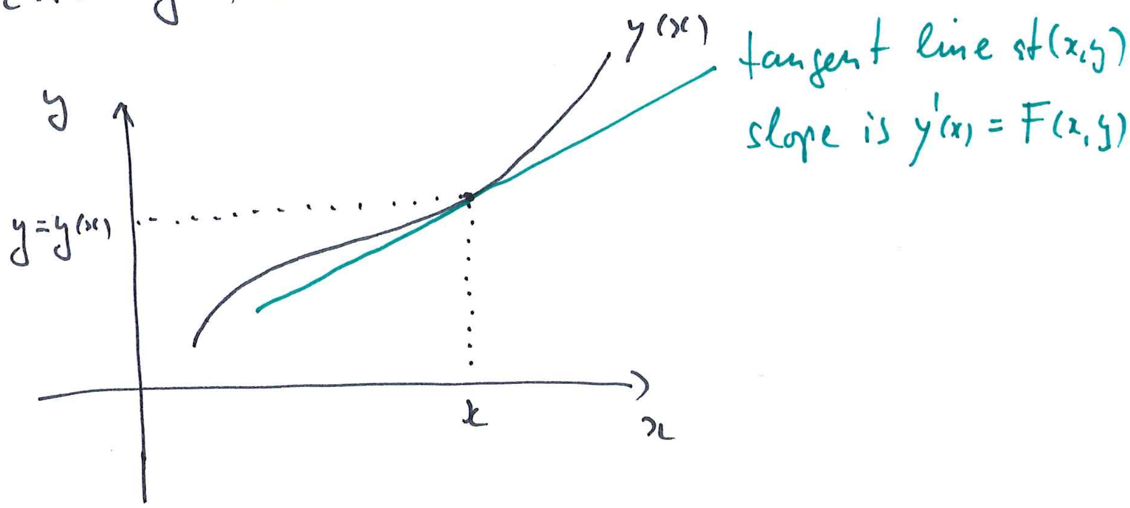
Explicit first order differential equations

$$y' = F(x,y) \quad \text{or} \quad y'(x) = F(x, y(x))$$

Most equations cannot be solved explicitly.
Hence we need tools that allow us to find information about the solution without solving the equation ("qualitative theory of d.e's")

One such tool: direction field

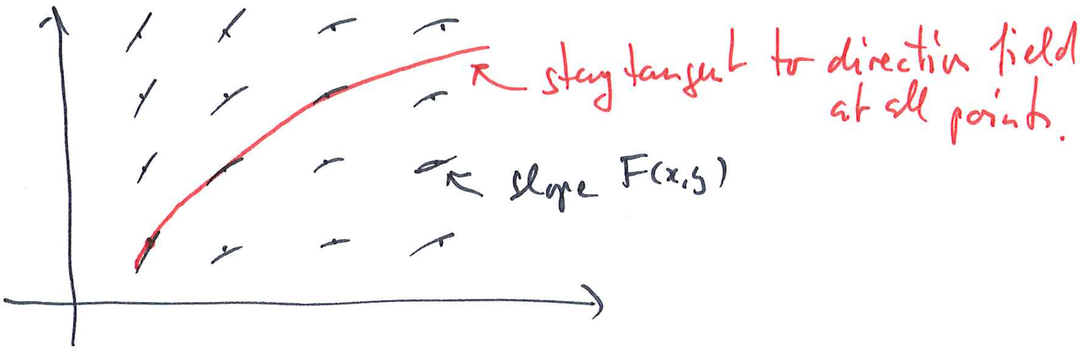
The solution of $y'(x) = F(x, y(x))$ is a function $y(x)$. Consider its graph



The slope of the tangent to $y(x)$ is given by $F(x,y)$ at each point (x,y).

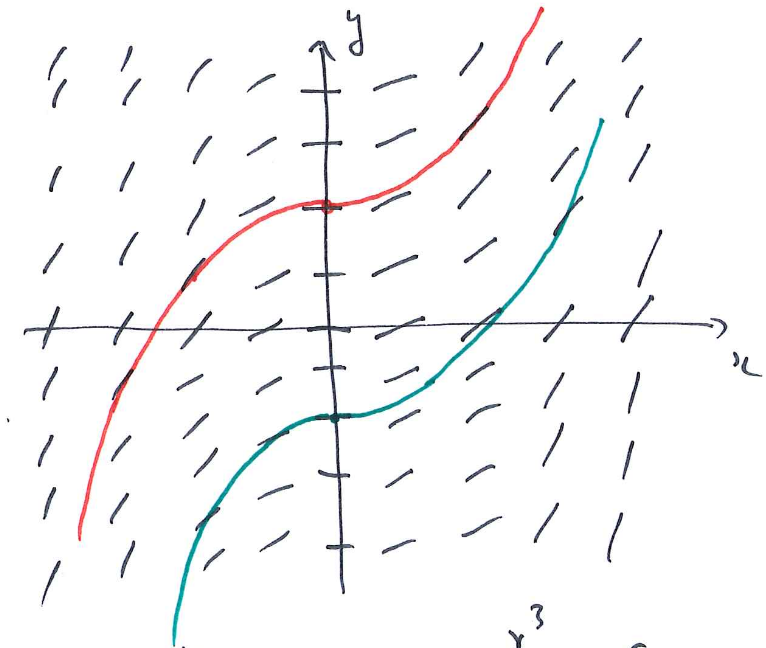
Hence, without solving the d.e we know the slope of solution curve at every point (x,y).

Plot directions:



Example:

$y' = x^2$ Direction field:



Explicit solution: $y(x) = \frac{x^3}{3} + C$

<u>Example:</u> simple population model

Introduce quantities:

$N(t)$: size of population at time t (dep. var)

t : time (indep. var)

Need to look at balance of quantities.

Consider increments during a (small) time interval Δt

$$N(t+\Delta t) - N(t) \approx k\, N(t)\, \Delta t$$

Increase/decrease proportional to current size of the population. k is constant of proportionality.

Divide by Δt

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = k\, N(t)$$

Let $\Delta t \to 0$:

$$\frac{dN}{dt} = k N \qquad \text{diff. eq. for } N(t)$$

Simplest case: k = constant

$\rightsquigarrow$ exponential growth/decay

$$N(t) = N(0)\, e^{kt}$$

More complicated: k may depend on t or N, or both!

Assume: There exists a maximal sustainable population of size M.

Want:

If $N(t) < M$: $k > 0$

If $N(t) > M$: $k < 0$

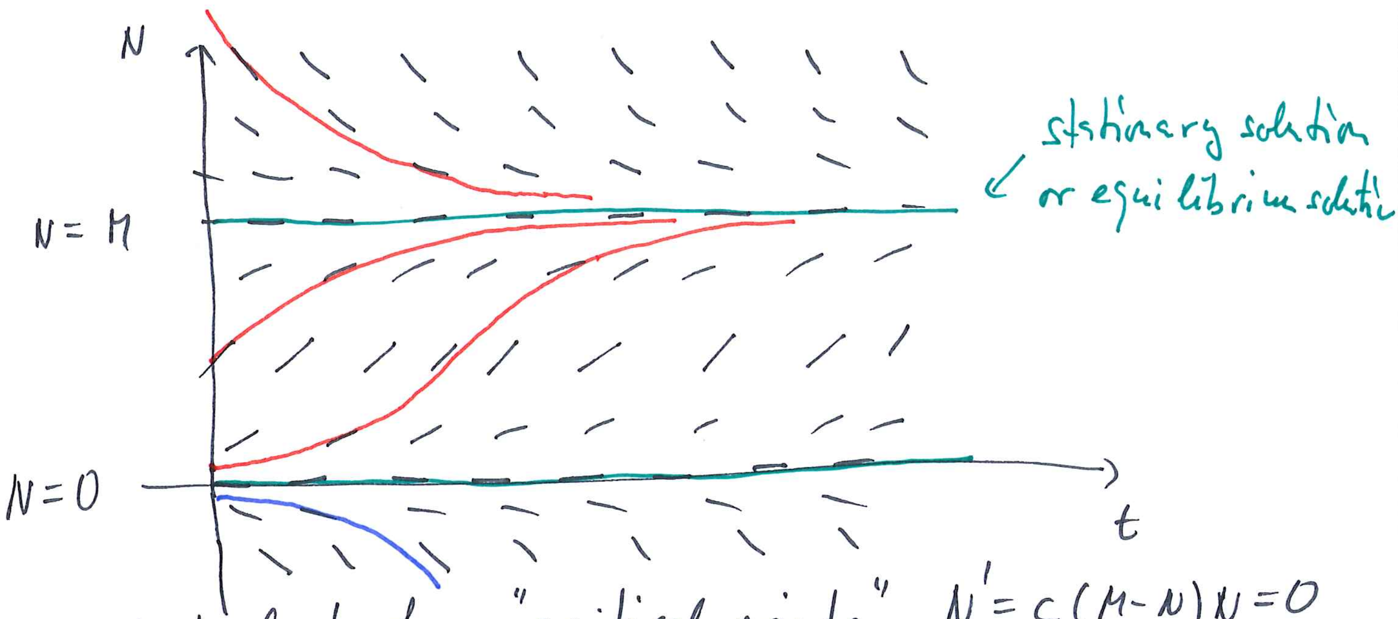
One possibility is

$$k = c\,(M - N(t))$$

This leads to the logistic model:

$$\frac{dN}{dt} = c\,(M - N(t))\,N(t)$$

Direction field for $N' = c(M-N)N$



First look for "critical points" $N' = c(M-N)N = 0$

Solutions: $N=0$, $N=M$.

Equilibrium $N=0$:

For every initial value $N \neq 0$ close to $N=0$, the solution moves away: the equilibrium is unstable

Equilibrium $N=M$:

For every initial value $N \neq M$ close to M, the solution moves towards M: equilibrium is stable or attractive.

Solve $N' = c(M-N)$ explicitly.

Rearrange so that N appears only on one side:

$$\frac{N'(t)}{(M-N(t))N(t)} = c$$

Integrate with respect to t:

$$\int \frac{N'(t)}{(M-N(t))N(t)}dt = \int c\,dt = ct + C$$

$\parallel$ substitution

$$\int \frac{1}{(M-N)N} dN = ct + C$$

Use partial fractions: $\frac{1}{(M-N)N} = \frac{1}{M}\left(\frac{1}{N} + \frac{1}{M-N}\right)$

Hence

$$\int \frac{1}{(M-N)N} dN = \frac{1}{M}\int \frac{1}{N} + \frac{1}{M-N}\, dN = \frac{1}{M}\left(\log|N| - \log|M-N|\right)$$

$$= \frac{1}{M}\log\left|\frac{N}{M-N}\right|$$

Now solve for N:

$$\frac{1}{M} \log \left| \frac{N}{M-N} \right| = ct + C$$

$$\log \left| \frac{N}{M-N} \right| = Mct + MC$$

$$\left| \frac{N}{M-N} \right| = e^{Mct+MC} = e^{MC} e^{Mct}$$

$$\frac{N}{M-N} = \underbrace{\pm e^{MC}}_{\text{constant}} e^{Mct} = A e^{Mct}$$

Hence

$$N = A(M-N) e^{Mct}$$

$$N(t) = \frac{A e^{cMt} M}{1 + A e^{cMt}} = \frac{AM}{e^{-cMt} + A}$$

Note: If $A > 0$, then $N(t) \longrightarrow M$ as $t \to \infty$

consistent with direction field.

Simple model of debt repayment

Introduce quantities:

$D(t)$ debt in \$ at time t

I interest rate in % per annum

R repayment per year

t time in years.

Derive a differential equation for D.

Consider increment of D between t and $t+\Delta t$

$$D(t+\Delta t) - D(t) \approx \underbrace{\underbrace{\frac{I}{100} D(t) \Delta t}_{\text{interest charged}} - \underbrace{R \Delta t}_{\text{repayments}}}_{\text{pro rata during } \Delta t}$$

Divide by Δt and let $\Delta t \to 0$

$$\frac{D(t+\Delta t) - D(t)}{\Delta t} = \frac{I}{100} D(t) - R$$

$$\downarrow$$

$$D'(t) = \frac{I}{100} D(t) - R$$

Note: In general, I and R are functions of t (variable interest & repayments)

Assume I, R are constant, sketch direction field.

First find equilibria: $D'(t) = 0 = \frac{I}{100} D - R$

$D = \frac{100R}{I}$.