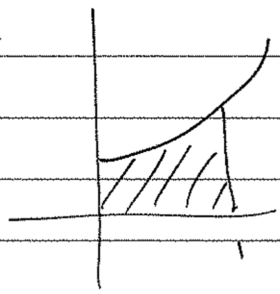
Extended Answer Section

There are **four** *questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

MARKS

1. (a) Let D be the region of the plane bounded by the x-axis, the y-axis, the line $x = 1$, and the curve $y = \cosh x$.

(*i*) Compute the area of D. **2**



$$\text{Area} = \int_0^1 \cosh x \, dx$$

$$= \sinh x \Big|_0^1 = \sinh 1$$

(*ii*) Compute the volume of the solid obtained by rotating D about the y-axis **2**

$u = x \qquad \frac{dv}{dx} = \cosh x$

$\frac{du}{dx} = 1 \qquad v = \sinh x$

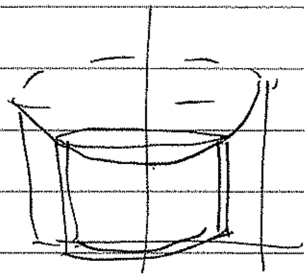
$$\text{Volume} = 2\pi \int_0^1 x \cosh x \, dx$$

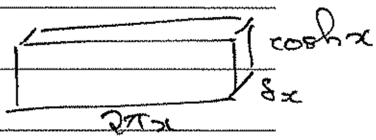
$$= 2\pi x \sinh x \Big|_0^1 - 2\pi \int_0^1 \sinh x \, dx$$

$$= 2\pi \sinh 1 - 2\pi \cosh x \Big|_0^1$$

$$= 2\pi(\sinh(1) - \cosh(1) + 1)$$

$$= 2\pi(1 - e^{-1})$$





MARKS

(b) Let $I(x) = \int_0^x \sqrt{1+t^3}\,dt$. Calculate the integral 2

$$\int_0^1 xI(x)\,dx.$$

Note: The constant $I(1)$ will appear in your answer.

Let $u = I(x)$ $\quad \frac{dv}{dx} = x$

$\frac{du}{dx} = \sqrt{1+x^3}$ $\quad v = \frac{1}{2}x^2$

$$\int_0^1 xI(x)\,dx = \frac{1}{2}x^2 I(x)\Big|_0^1 - \frac{1}{2}\int_0^1 x^2\sqrt{1+x^3}\,dx$$

$u = 1+x^3$

$$= \frac{1}{2}I(1) - \frac{1}{2}\cdot\frac{1}{3}\int_{\phi}^{0} \frac{du}{dx}\sqrt{u}\,dx$$

$$= \frac{1}{2}I(1) + \frac{1}{6}\int_0^1 u^{1/2}\,du$$

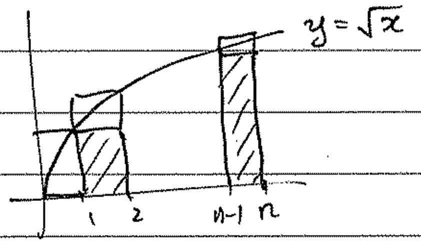
$$= \frac{1}{2}I(1) + \frac{1}{6}\cdot\frac{1}{1+\frac{1}{2}}u^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{1}{2}I(1) + \frac{1}{9}.$$

MARKS

(c) Let $s_n = \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}$.

(*i*) Let P be the partition of $[0, n]$ into n subintervals of length 1. Use the corresponding upper and lower Riemann sums for the integral $\int_0^n \sqrt{x}\,dx$ to find upper and lower bounds for s_n, such that the bounds differ by at most $\sqrt{n}$. **3**



$$L = \sqrt{1} + \sqrt{2} + \dots + \sqrt{n-1}$$
$$= s_n - \sqrt{n}$$
$$U = \sqrt{1} + \sqrt{2} + \dots + \sqrt{n}$$
$$= s_n.$$

$\int_0^n \sqrt{x}\,dx = \frac{2}{3} n^{3/2}$. So

$\boxed{s_n \le \frac{2}{3} n^{3/2} + \sqrt{n}}$ and $\boxed{s_n \geq \frac{2}{3} n^{3/2}}$

These bounds differ by $\sqrt{n}$.

(*ii*) Hence, or otherwise, calculate the limit $\lim_{n \to \infty} \frac{s_n}{n^{3/2}}$. **1**

By above: $\frac{2}{3} \le \frac{s_n}{n^{3/2}} \le \frac{2}{3} + n^{-1} \to \frac{2}{3}$

and so by the squeeze law

$$\lim_{n \to \infty} \frac{s_n}{n^{3/2}} = \frac{2}{3}.$$

MARKS

2. (a) (*i*) Use a comparison test to show that $\displaystyle\int_0^\infty \frac{e^x}{7+2\cosh(2x)}\,dx$ converges. 2

$$0 \leq \frac{e^x}{7+2\cosh(2x)} \leq \frac{e^x}{7+e^{2x}} \quad (\text{since } \cosh(2x) \geq \tfrac{1}{2}e^{2x})$$

$$\leq \frac{e^x}{e^{2x}} = e^{-x}.$$

Since $\int_0^\infty e^{-x}\,dx = 1$ converges, we conclude that $\int_0^\infty \frac{e^x}{7+2\cosh(2x)}\,dx$ converges by the comparison test.

(*ii*) Using an appropriate substitution, or otherwise, calculate the integral 2

$$\int_0^1 \frac{xe^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}\,dx.$$

$u = \sqrt{1+x^2}$, $\frac{du}{dx} = \frac{x}{\sqrt{1+x^2}}$

$$I = \int_1^{\sqrt{2}} e^u\,du$$

$$= e^{\sqrt{2}} - e$$

OR

$x = \sinh y$ $\quad \frac{dx}{dy} = \cosh y.$

$$I = \int_0^{\sinh^{-1}(1)} \frac{\sinh y\, e^{\cosh y}}{\cosh y} \cosh y\,dy$$

$$= \int_0^{\sinh^{-1}(1)} \sinh y\, e^{\cosh y}\,dy$$

$$= e^{\cosh(\sinh^{-1}(1))} - e$$

$$= e^{\sqrt{2}} - e$$

MARKS

(b) (*i*) For integers $m, n \geq 0$ let $I_{m,n} = \int_0^1 x^m (\ln x)^n \, dx$. Show that for $n \geq 1$, **3**

$$I_{m,n} = -\frac{n}{m+1} I_{m,n-1},$$

and hence compute $I_{m,n}$.

[You may use the fact that $\lim_{x \to 0^+} x^\alpha (\ln x)^\beta = 0$ for all $\alpha > 0$ and $\beta \geq 0$.]

Let $u = (\ln x)^n \quad \frac{dv}{dx} = x^m$

$\frac{du}{dx} = \frac{n}{x}(\ln x)^{n-1} \quad v = \frac{1}{m+1} x^{m+1}$

$$I_{m,n} = \frac{x^{m+1}(\ln x)^n}{m+1}\Big|_0^1 - \frac{n}{m+1}\int_0^1 x^m (\ln x)^{n-1} dx$$

$$= -\frac{n}{m+1} I_{m,n-1}.$$

So $I_{m,n} = (-1)^n \frac{n}{m+1} \cdot \frac{n-1}{m+1} \cdot \ldots \cdot \frac{1}{m+1} I_{m,0}$

$$= (-1)^n \frac{n!}{(m+1)^n} \int_0^1 x^m dx$$

$$= (-1)^n \frac{n!}{(m+1)^{n+1}}$$

MARKS

(*ii*) Hence show that 3

$$\int_0^1 x^{-x}\,dx = \sum_{k=1}^{\infty} n^{-n}.$$

n

You may assume that any reasonable series manipulations are valid.

$$\int_0^1 x^{-x}\,dx = \int_0^1 e^{-x\ln x}\,dx$$

$$= \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k (\ln x)^k \right) dx$$

(assuming this is OK) $$= \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{k!} \int_0^1 x^k (\ln x)^k\,dx \right)$$

by (i) $$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (-1)^k \frac{k!}{(k+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{k+1}}$$

$$= \sum_{k=1}^{\infty} n^{-n}$$

MARKS

3. (a) Find the general solution of the differential equation 4

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3e^{-2x}$$

(1) Homogeneous soln: $\lambda^2 + 3\lambda + 2 = 0$

$\lambda = -2, -1$

$y_h(x) = Ae^{-2x} + Be^{-x}$; A, B constants

(2) Particular solution: This is resonant, so try

$y_p(x) = Cxe^{-2x}$

So $\frac{dy}{dx} = Ce^{-2x} - 2Cxe^{-2x} = C(1-2x)e^{-2x}$

$\frac{d^2y}{dx^2} = -2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x}$

$= 4C(x-1)e^{-2x}$

So $4Cx - 4C + 3C - 6Cx + 2Cx = 3$

So $-C = 3$, $C = -3$.

Hence $y_p(x) = -3e^{-2x}x$.

(3) General soln:

$y(x) = Ae^{-2x} + Be^{-x} - 3xe^{-2x}$.

MARKS

(b) Find the general solution of 3

$$\frac{dy}{dx} = \frac{2x+1}{x^2+x+1}(1-y),$$

and show that every solution converges to the equilibrium solution $y = 1$ for $x \to \infty$.

$$\frac{1}{1-y}\frac{dy}{dx} = \frac{2x+1}{x^2+x+1} \quad \text{(separable)}$$

So $\displaystyle\int \frac{1}{1-y}\,dy = \int \frac{2x+1}{x^2+x+1}\,dx$

So $-\ln|1-y| = \ln|x^2+x+1| + C$

$$|1-y| = e^{-\ln|x^2+x+1|}\,e^{-C}$$

$$= e^{-C} \cdot \frac{1}{|x^2+x+1|}$$

$$= e^{-C}\frac{1}{x^2+x+1} \quad (x^2+x+1>0)$$

$$1-y = \frac{\pm e^{-C}}{x^2+x+1}$$

$$\Rightarrow y = 1 + \frac{A}{x^2+x+1} \quad \text{for } A \text{ a constant.}$$

As $x \to \infty$ we have $y \to 1$.

MARKS

(c) Consider the differential equation of the form **3**

$$\frac{dy}{dx} - e^{-x-y} + 1 = 0\,.$$

Introduce a new dependent variable u given by $u = x+y$, and hence find the general solution of the original equation.

$u = x + y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$.

Hence $\frac{du}{dx} - 1 - e^{-u} + 1 = 0$

$\Rightarrow \frac{du}{dx} = e^{-u} \Rightarrow e^{u}\frac{du}{dx} = 1$.

$\Rightarrow \int e^{u}\,du = x + C$

So $e^{u} = x + C$, so $u = \ln|x+C|$

$y = \ln|x+C| - x$.

MARKS

4. (a) A spherical raindrop evaporates at a rate proportional to its surface area, retaining the spherical shape. Derive a differential equation for the radius $r(t)$ of the raindrop and solve it for a raindrop with initial radius r_0 to show that **2**

$$r(t) = r_0 - \alpha t$$

for a constant $\alpha > 0$.

[Note that the volume of a sphere of radius r is $V = 4\pi r^3/3$, and that the surface area is $A = 4\pi r^2$, and assume that the density of water is 1.]

Let $m(t) = \frac{4}{3}\pi r(t)^3$ be the mass of the drop

Then $\frac{dm}{dt} = -\alpha \cdot 4\pi r(t)^2$,

giving $4\pi r'(t) r(t)^2 = -\alpha 4\pi r(t)^2$,

so $\frac{dr}{dt} = -\alpha$.

Thus $r(t) = -\alpha t + C$

at $t=0$, $r(0) = r_0$, so $C = r_0$.

Hence $r(t) = r_0 - \alpha t$ as required.

MARKS

(b) The evaporating raindrop is falling towards the ground. For this type of problem with time-dependent mass the appropriate form of Newton's second law states that the rate of change of the product of mass m with velocity v is equal to the force. The force is given by mg (with positive direction down), where g is the constant gravitational acceleration, with an additional air friction force proportional to the area πr^2 times the velocity. The friction force opposes the velocity. Show that the differential equation for the velocity v of the falling raindrop can be written as **3**

$$\frac{dv}{dt} - \frac{k\alpha}{r(t)} v = g$$

for some constant k.

$\frac{d}{dt}(mv) = mg - \tilde{k}\pi r^2 v \qquad m = \frac{4}{3}\pi r^3$

$\frac{4}{3}\frac{d}{dt}(r^3 v) = \frac{4}{3} g r^3 - \tilde{k} r^2 v$

So $2r^2 v + r^3 \frac{dv}{dt} = g r^3 - \frac{3}{4}\tilde{k} r^2 v$

So $\frac{dv}{dt} + \frac{(\frac{3}{4}\tilde{k} + 2)}{r} v = g$

Hence, let $k\alpha = -2 - \frac{3}{4}\tilde{k}$

$k = \frac{-2 - \frac{3}{4}\tilde{k}}{\alpha}$.

MARKS

(c) Find the particular solution of the differential equation for the falling raindrop for which initially the raindrop is at rest. Assume that $k \neq -1$. **3**

$$\frac{dv}{dt} - \frac{k\alpha}{r(t)} v = g \qquad v(0) = 0$$

$$I(t) = e^{-\int \frac{k\alpha}{r_0 - \alpha t} dt} = e^{k \ln(r_0 - \alpha t)} \qquad \left(\substack{\text{assume} \\ r_0 - \alpha t \geq 0}\right)$$

$$= r(t)^k$$

Thus
$$v = r^{-k} g \int r(t)^k \, dt$$

$$= g r^{-k} \left(-\frac{1}{\alpha} \cdot \frac{1}{k+1} r^{k+1} + C \right)$$

$$= -\frac{g}{\alpha(k+1)} r + C' r^{-k}.$$

$$0 = -\frac{g}{\alpha(k+1)} r_0 + C' r_0^{-k}$$

So
$$C' = \frac{g}{\alpha(k+1)} r_0^{k+1}.$$

$$\Rightarrow \quad v(t) = \frac{g r_0}{\alpha(k+1)} \left[(r_0 r^{-1})^k - 1 \right]$$

MARKS

(d) Assume that $k = -2$. Compute the distance the drop falls from rest until it is completely evaporated. **2**

$$\frac{dv}{dt} = \frac{g r_0}{\alpha(k+1)}\left[\left(\frac{r_0}{r_0-\alpha t}\right)^k - 1\right]$$

$$= \frac{g r_0}{\alpha}\left(1-\left(1-\frac{\alpha}{r_0}t\right)^2\right)$$

$$v = \frac{g r_0}{\alpha}\left(t + \frac{2\alpha}{r_0}\left(1-\frac{\alpha}{r_0}t\right)\right)$$

Drop evaporate when $0 = r(t) = r_0 - \alpha t_\infty \Rightarrow t_\infty = \frac{r_0}{\alpha}$.

Then $v(t_\infty) = \frac{g r_0^2}{\alpha^2}$.