THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1	**Tutorial Week 10**	2012

(*These preparatory questions should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

1. Sketch the curves given by the following parametric equations (find corresponding cartesian equations if possible).

(a) In $\mathbb{R}^2$, $x = 1 + \cos t$, $y = 2 + \sin t$, $t \in [0, \pi]$. Mark the points corresponding to $t = 0,\ \pi/2,\ \pi$ on your sketch.

(b) In $\mathbb{R}^2$, $x = 1 + 2\cos t$, $y = 2 + \sin t$, $t \in [0, 2\pi]$. Mark the points corresponding to $t = 0,\ \pi/2,\ \pi,\ 3\pi/2,\ 2\pi$ on your sketch.

(c) In $\mathbb{R}^2$, $x = 2t$, $y = 4t^2 + 1$, $t \in [0, 1]$. Mark the points corresponding to $t = 0,\ 1/2,\ 1$.

(d) In $\mathbb{R}^3$, $x = 0$, $y = 3 - 3t$, $z = 2t$, $t \in \mathbb{R}$. Mark the points corresponding to $t = 0,\ 1,\ -1$.

2. What are the natural domains of the functions $f(x, y) = \sqrt{xy}$ and $g(x, y) = \ln(x^2 + y^2 - 1)$?

Questions for the tutorial

3. Show that the curve $\mathcal{C}$ with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$, $t \in \mathbb{R}$, passes through $(1, 4, 0)$ and $(9, -8, 28)$ but not $(4, 7, -6)$.

4. (a) Find the intersection points of the helix whose general point is given parametrically as $(\cos t, \sin t, t)$, $t \in \mathbb{R}$, with the sphere whose cartesian equation is $x^2 + y^2 + z^2 = 4$.

(b) Find all points common to the helices $\mathcal{C}_1$ and $\mathcal{C}_2$, where

$$\mathcal{C}_1(t) = (\cos t, \sin t, t), \quad t \in \mathbb{R}, \qquad \mathcal{C}_2(s) = (\cos s, s, \sin s), \quad s \in \mathbb{R}.$$

5. Describe the curves in $\mathbb{R}^3$ given by the following parametric equations.

(a) $x = t\cos t$, $y = t \sin t$, $z = 5t$, $\quad t \in [0, 100]$.

(b) $x = 2\cos t$, $y = \sin t$, $z = e^{-t}$, $\quad t \geq 0$.

6. Determine the domain and range of the functions whose formulas appear below.

(a) $f(x, y) = \sqrt{x - y}$ (b) $f(x, y) = \tan^{-1}(y/x)$

(c) $G(x, y) = \sqrt{x + y} - \sqrt{x - y}$ (d) $h(x, y) = \sin^{-1}(x + y)$

7. Sketch the level curves at heights $c = 0, \pm 1, \pm 2$ for the functions given by:

(a) $1 - x^2 - y^2$ (b) $4x^2 + y^2$ (c) $y - x^2$ (d) $2x + 3y$

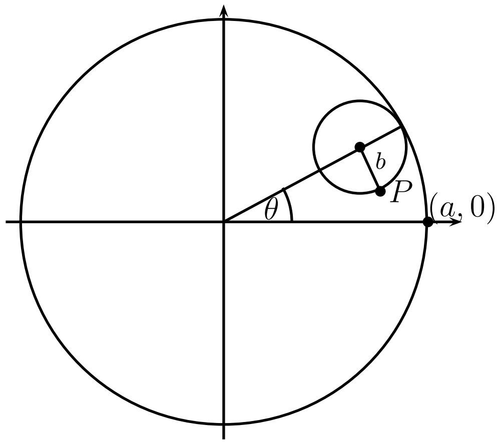
8. Find the domains and ranges, and describe the level curves, of the functions defined by:

(a) $\sqrt{4 - x^2 - y^2}$ (b) $(x-1)(y+1)$ (c) $\dfrac{2xy}{x^2+y^2}$

Extra Questions

9. If we take take a curve $z = f(y)$ in the yz-plane and revolve it about the z-axis, we obtain a *surface of revolution* in space. What do the level curves look like? Prove that the rule for the surface of revolution thus obtained is $z = f(\sqrt{x^2+y^2})$. Use this result to deduce that the equation $x^2 + y^2 + 2x + 2y - z^2 + 2 = 0$ defines a cone in space.

10. A circle of radius b rolls on the inside of a larger circle of radius a. The curve traced out by a fixed point P on the circumference of the smaller circle is called a hypocycloid.



(a) If the initial position of P is $(a, 0)$, and the parameter θ is chosen as in the figure, show that the parametric equations of the hypocycloid are

$$x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\,\theta\right), \qquad y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\,\theta\right).$$

(b) Show that if $b = a/4$, the parametric equations reduce to $x = a\cos^3\theta$, $y = a\sin^3\theta$. Sketch the curve in this case.

Solution to Question 1

(a) The curve is the top half (semi-circle) of the circle $(x-1)^2 + (y-2)^2 = 1$, of radius 1, centred at $(1, 2)$. The value $t = 0$ gives the point $(2, 2)$, $t = \pi/2$ gives $(1, 3)$, $t = \pi$ gives $(0, 2)$. The semi-circle is traced anticlockwise from $(2, 2)$ as t increases from 0 to π.
(b) The curve is an ellipse, $(\frac{x-1}{2})^2 + (y-2)^2 = 1$. The value $t = 0$ gives the point $(3, 2)$, $t = \pi/2$ gives $(1, 3)$, $t = \pi$ gives $(-1, 2)$, $t = 3\pi/2$ gives $(1, 1)$ and $t = 2\pi$ gives $(3, 2)$. The ellipse is traced anticlockwise as t increases from 0 to 2π.
(c) The curve is part of the parabola $y = x^2 + 1$, for $x \in [0, 2]$. The value $t = 0$ gives the point $(0, 1)$, $t = 1/2$ gives $(1, 2)$, $t = 1$ gives $(2, 5)$.
(d) The curve is a line through $(0, 3, 0)$ in the direction of $-3\mathbf{j} + 2\mathbf{k}$. Its cartesian equations are $x = 0$, $\frac{z}{2} = \frac{y-3}{-3}$. The value $t = 0$ gives $(0, 3, 0)$, $t = 1$ gives $(0, 0, 2)$, $t = -1$ gives $(0, 6, -2)$.

Solution to Question 2

The natural domain of f is $\{(x, y) \,|\, xy \geq 0\}$, which is the union of the first and third quadrants of the xy-plane, including both axes. The natural domain of g is $\{(x, y) \,|\, x^2 + y^2 > 1\}$, the set of all points in the xy-plane lying outside the unit circle.