THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Solutions Week 10** 2012

(*These preparatory questions should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

1. Sketch the curves given by the following parametric equations (find corresponding cartesian equations if possible).

(a) In $\mathbb{R}^2$, $x = 1 + \cos t$, $y = 2 + \sin t$, $t \in [0, \pi]$. Mark the points corresponding to $t = 0$, $\pi/2$, π on your sketch.

(b) In $\mathbb{R}^2$, $x = 1 + 2\cos t$, $y = 2 + \sin t$, $t \in [0, 2\pi]$. Mark the points corresponding to $t = 0$, $\pi/2$, π, $3\pi/2$, 2π on your sketch.

(c) In $\mathbb{R}^2$, $x = 2t$, $y = 4t^2 + 1$, $t \in [0, 1]$. Mark the points corresponding to $t = 0$, $1/2$, 1.

(d) In $\mathbb{R}^3$, $x = 0$, $y = 3 - 3t$, $z = 2t$, $t \in \mathbb{R}$. Mark the points corresponding to $t = 0$, 1, -1.

2. What are the natural domains of the functions $f(x, y) = \sqrt{xy}$ and $g(x, y) = \ln(x^2 + y^2 - 1)$?

Questions for the tutorial

3. Show that the curve $\mathcal{C}$ with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$, $t \in \mathbb{R}$, passes through $(1, 4, 0)$ and $(9, -8, 28)$ but not $(4, 7, -6)$.

Solution

Let $\mathcal{C}(t) = (t^2, 1 - 3t, 1 + t^3)$. Then $\mathcal{C}(-1) = (1, 4, 0)$ and $\mathcal{C}(3) = (9, -8, 28)$, so $\mathcal{C}$ passes through $(1, 4, 0)$ and $(9, -8, 28)$. If $\mathcal{C}$ passes through $(4, 7, -6)$, then for some t we must have $t^2 = 4$, $1 - 3t = 7$, and $1 + t^3 = -6$, which is impossible; no t can satisfy all equations simultaneously. (The second equation shows that $t = -2$, and while this satisfies the first equation, it doesn't satisfy the third.) Hence $\mathcal{C}$ does not pass through $(4, 7, -6)$.

4. (a) Find the intersection points of the helix whose general point is given parametrically as $(\cos t, \sin t, t)$, $t \in \mathbb{R}$, with the sphere whose cartesian equation is $x^2 + y^2 + z^2 = 4$.

(b) Find all points common to the helices $\mathcal{C}_1$ and $\mathcal{C}_2$, where

$$\mathcal{C}_1(t) = (\cos t, \sin t, t), \quad t \in \mathbb{R}, \qquad \mathcal{C}_2(s) = (\cos s, s, \sin s), \quad s \in \mathbb{R}.$$

Solution

(a) Put $x = \cos t$, $y = \sin t$ and $z = t$ in the equation, $x^2 + y^2 + z^2 = 4$, of the sphere. Then $\cos^2 t + \sin^2 t + t^2 = 4$ gives $t = \pm\sqrt{3}$. That is, the points of intersection are: $(\cos\sqrt{3}, \sin\sqrt{3}, \sqrt{3})$ and $(\cos\sqrt{3}, -\sin\sqrt{3}, -\sqrt{3})$.
The point $(\cos\sqrt{3}, \sin\sqrt{3}, \sqrt{3})$ lies on the circle centre $(0, 0, \sqrt{3})$ and radius 1, in the plane $z = \sqrt{3}$. The point $(\cos\sqrt{3}, -\sin\sqrt{3}, -\sqrt{3})$ lies on the circle centre $(0, 0, -\sqrt{3})$ and radius 1, in the plane $z = -\sqrt{3}$.

(b) If there is a point on both $\mathcal{C}_1$ and $\mathcal{C}_2$, then there exist $t \in \mathbb{R}$ and $s \in \mathbb{R}$ such that

$$\cos t = \cos s \qquad (1)$$
$$\sin t = s \qquad (2)$$
$$t = \sin s \qquad (3)$$

From (1), $t = s + 2k\pi$ or $t = -s + 2k\pi$ (where $k \in \mathbb{Z}$). Substituting into (2) gives $\sin(s + 2k\pi) = s$ or $\sin(-s + 2k\pi) = s$, that is, $\sin s = s$ or $-\sin s = s$. But $\pm \sin s = s$ if and only if $s = 0$ (a quick sketch of the graphs $y = x$ and $y = \sin x$ illustrates this). Then from (3), we must have $t = s = 0$. The unique common point on the two helices is the point

$$(\cos 0, \sin 0, 0) = (\cos 0, 0, \sin 0) = (1, 0, 0).$$

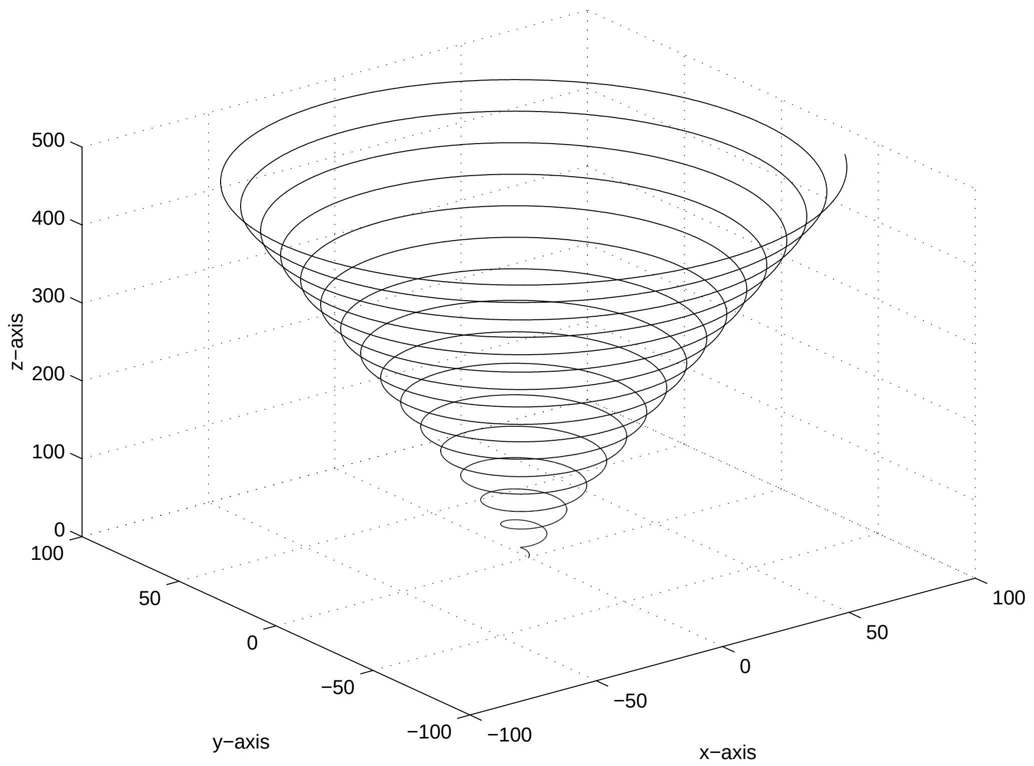
5. Describe the curves in $\mathbb{R}^3$ given by the following parametric equations.

(a) $x = t\cos t,\ y = t\sin t,\ z = 5t, \quad t \in [0, 100]$.

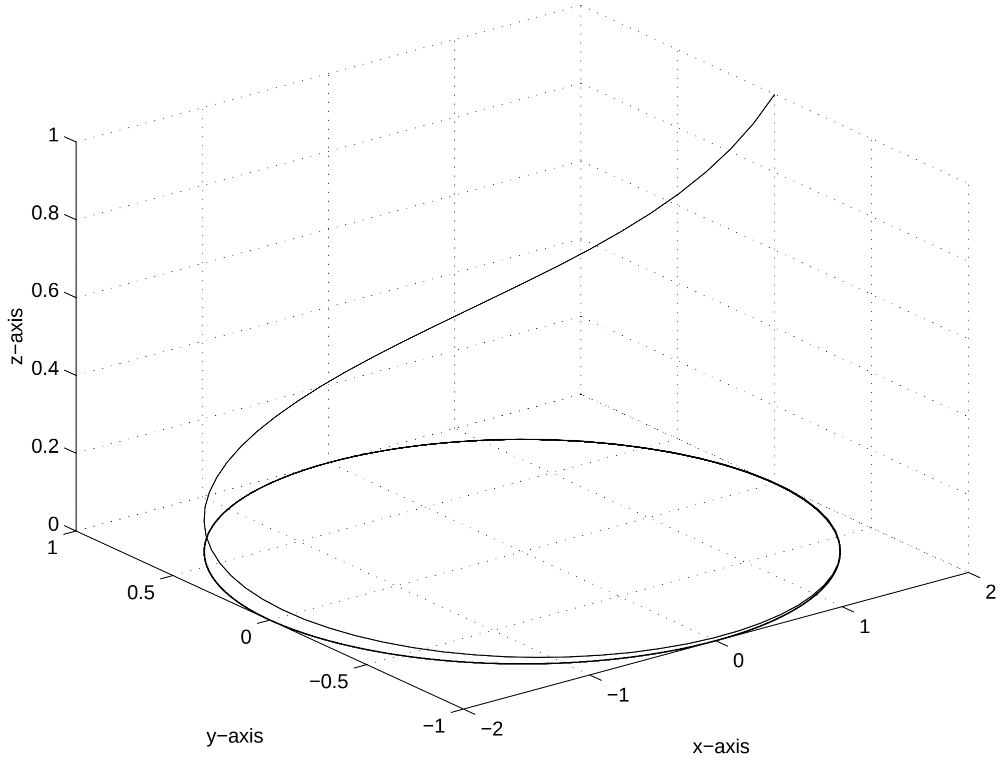
(b) $x = 2\cos t,\ y = \sin t,\ z = e^{-t}, \quad t \geq 0$.

Solution

(a) As t increases, so does z. For any t, we have $x^2 + y^2 = t^2$. Hence the points lie on a circular spiral of increasing radius. The starting point is $(0, 0, 0)$ and the finishing point is $(100\cos 100, 100\sin 100, 500) \approx (86.2, -50.6, 500)$.



(b) The curve is confined to that part of space corresponding to values of z in the interval $(0, 1]$. It is a helix tracing out an elliptical orbit, with starting point $(2, 0, 1)$. As t takes values in any interval of the form $[a, a + 2\pi]$, the curve makes one revolution of the z axis with its height above the xy plane decreasing.



6. Determine the domain and range of the functions whose formulas appear below.

(a) $f(x, y) = \sqrt{x - y}$ (b) $f(x, y) = \tan^{-1}(y/x)$

(c) $G(x, y) = \sqrt{x + y} - \sqrt{x - y}$ (d) $h(x, y) = \sin^{-1}(x + y)$

Solution

(a) Domain $= \{(x, y) \mid y \leq x\}$. (This corresponds to points in the half-plane lying on and below the line $y = x$.) For every $a \geq 0$, $a = \sqrt{a^2 - 0} = f(a^2, 0)$, and so the range is $[0, \infty)$.

(b) Domain $= \{(x, y) \mid x \neq 0\}$. (This is all of $\mathbb{R}^2$ except the y axis.) Range $= (-\frac{\pi}{2}, \frac{\pi}{2})$. To see this, let $x = 1$. Then $f(x, y) = \tan^{-1} y$ which takes all values in $(-\frac{\pi}{2}, \frac{\pi}{2})$ as y varies through $\mathbb{R}$.

(c) Domain $= \{(x, y) \mid x + y \geq 0 \text{ and } x - y \geq 0\} = \{(x, y) \mid -x \leq y \leq x\}$. (These are the points in the first and fourth quadrants on and between the lines $y = x$ and $y = -x$.) Observe that on the line $y = x$ in the domain, $G(x, y) = \sqrt{2x}$, which can take all values in $[0, \infty)$. On the line $y = -x$ in the domain, $G(x, y) = -\sqrt{2x}$, which can take all values in $(-\infty, 0]$. Thus the range is $\mathbb{R}$.

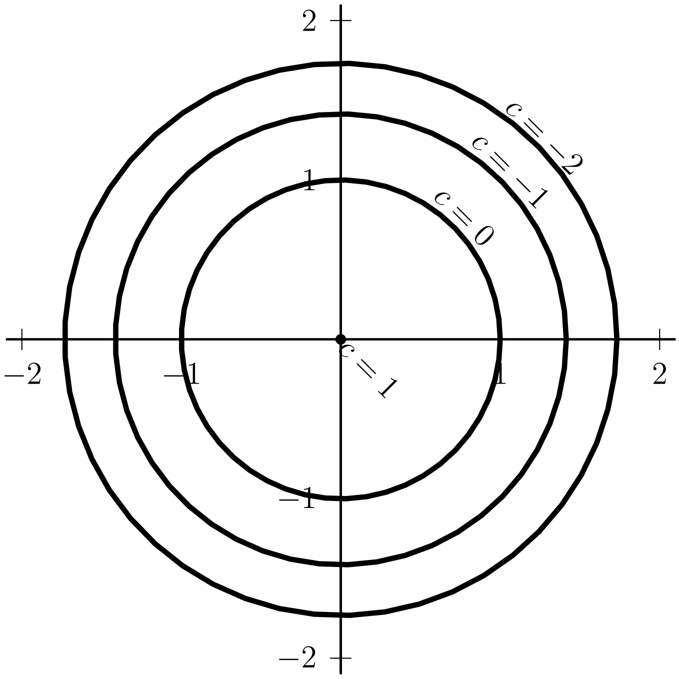
(d) The domain is $\{(x, y) \mid -1 \leq x + y \leq 1\}$. These are the points lying in an infinite strip on and between the lines $y = -1 - x$ and $y = 1 - x$. When $x + y = -1$, $h(x, y) = \sin^{-1}(-1) = -\pi/2$. When $x + y = 1$, $h(x, y) = \sin^{-1}(1) = \pi/2$. The $\sin^{-1}$ function is an increasing function and takes all values between $-\pi/2$ and $\pi/2$ as the value of $x + y$ moves from -1 to 1. Therefore the range is $[-\pi/2, \pi/2]$.

7. Sketch the level curves at heights $c = 0, \pm 1, \pm 2$ for the functions given by:

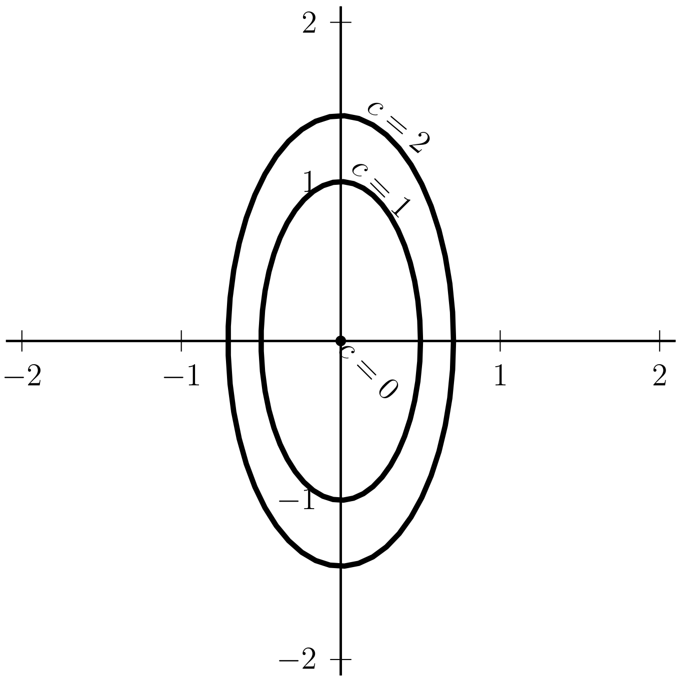
(a) $1 - x^2 - y^2$ (b) $4x^2 + y^2$ (c) $y - x^2$ (d) $2x + 3y$

Solution

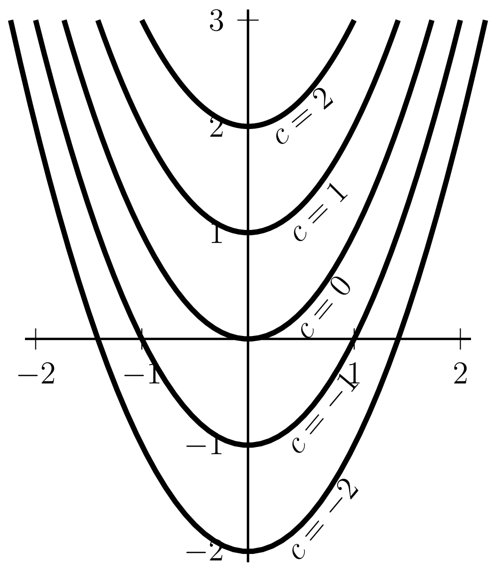
(a) The level curves at heights $c = 0, \pm 1, -2$ of $1 - x^2 - y^2$ are shown in the figure. The level curve at height $c = 2$ is empty, as 2 is not in the range of the function.



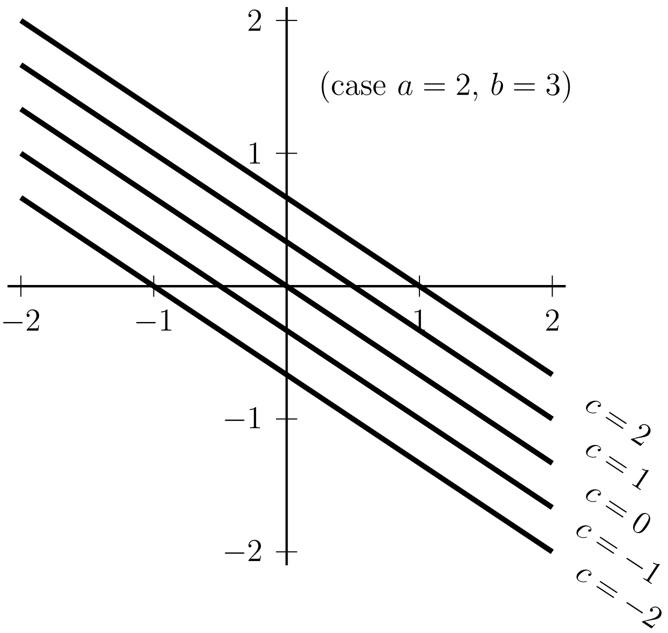
(b) The level curves at heights $c = 0, 1, 2$ of $4x^2 + y^2$ are shown in the figure. The level curves at heights -1 and -2 are empty.



(c) The level curves at heights $c = 0, \pm 1, \pm 2$ of $y - x^2$ are:



(d) The level curves at heights $c = 0, \pm 1, \pm 2$ of $ax + by$ (where a, b are constants) are straight lines. The diagram shows them when $a = 2, b = 3$.



8. Find the domains and ranges, and describe the level curves, of the functions defined by:

(a) $\sqrt{4 - x^2 - y^2}$ (b) $(x - 1)(y + 1)$ (c) $\dfrac{2xy}{x^2 + y^2}$

Solution

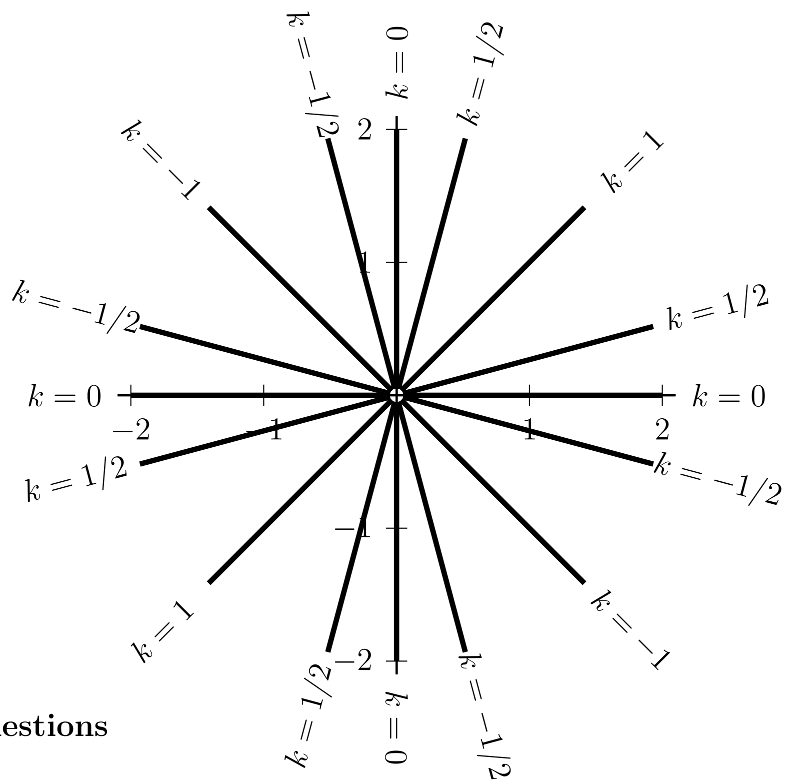
(a) Domain $= \{(x, y) \mid x^2 + y^2 \le 4\}$ (a circle and its interior); range $= [0, 2]$. Level curves are either empty (at height $k < 0$), a single point (at height $k = 2$) or circles (at heights k where $0 \le k < 2$).

(b) Domain $= \mathbb{R}^2$; range $= \mathbb{R}$; level curves are hyperbolas in "shifted" first and third quadrants at positive heights and hyperbolas in "shifted" second and fourth quadrants at negative heights. At height 0, the level curve is the union of the lines $x = 1$ and $y = -1$.

(c) The domain is $\mathbb{R}^2 \setminus \{(0,0)\}$. (This is the plane without the origin.) For all (x, y) in the domain, write $x = r\cos\theta$ and $y = r\sin\theta$, where $r > 0$ and θ can take any value. Then

$$f(x,y) = \frac{2r^2\cos\theta\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \sin 2\theta.$$

As θ can take all real values, we see that the range is $[-1, 1]$. The level curve at height k consists of all (x, y) whose corresponding θ is a solution of $\sin 2\theta = k$. For $k = 1$, the only $\theta \in [0, 2\pi)$ which solve this equation are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$, so the level curve at height 1 is the line $y = x$ (without the origin). Similarly, the level curve at height -1 is the line $y = -x$ (without the origin). When $-1 < k < 1$, there are four solutions of $\sin 2\theta = k$ with $0 \le \theta < 2\pi$, so the level curve consists of two lines. For example, at height 0 the level curve is the union of the x and y axes. The following picture shows some of these level curves.



Extra Questions

9. If we take take a curve $z = f(y)$ in the yz-plane and revolve it about the z-axis, we obtain a *surface of revolution* in space. What do the level curves look like? Prove that the rule for the surface of revolution thus obtained is $z = f(\sqrt{x^2 + y^2})$. Use this result to deduce that the equation $x^2 + y^2 + 2x + 2y - z^2 + 2 = 0$ defines a cone in space.

Solution

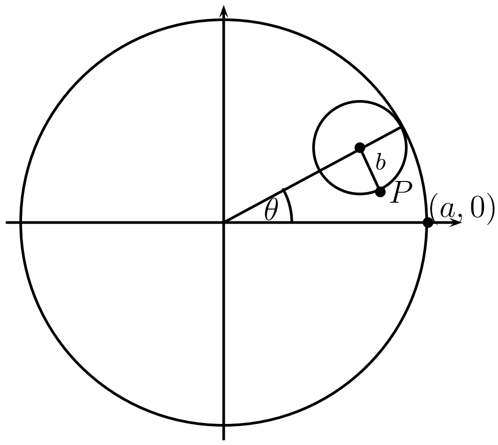
Let's assume that $z = f(y)$ is defined for values of y lying in the half-plane corresponding to $y \ge 0$. In general, the level curve at height c will be a circle centred at the origin in the xy-plane. For particular values of c the level curve could be a degenerate circle containing a single point, or concentric circles (if the curve $z = f(y)$ has the same z value for two different y values).

Let (a, b, c) be a point on the surface; that is, (a, b, c) lies on the horizontal cross section at height c. Projecting the horizontal cross section down onto the xy-plane shows that (a, b) lies on a circle of radius $\sqrt{a^2 + b^2}$ crossing the y-axis at the point $(0, \sqrt{a^2 + b^2})$.

Thus the point $(0, \sqrt{a^2+b^2}, c)$ lies on the surface and also lies on the curve $z = f(y)$ in the yz plane; that is, $c = f(\sqrt{a^2+b^2})$. This proves the rule given in the question.

Observe that $x^2 + y^2 + 2x + 2y - z^2 + 2 = 0$ iff $z^2 = (x+1)^2 + (y+1)^2$. This describes a surface obtained by translating the surface described by $z^2 = x^2 + y^2$, that is, the surface $z = \pm\sqrt{x^2+y^2}$. (Translate 1 unit in the direction of the vector $-\mathbf{i}$ and then one unit in the direction of $-\mathbf{j}$.) This is the union of two surfaces of revolution, one obtained by revolving the line $z = y$ with $y \geq 0$ in the yz-plane about the z-axis, and one obtained by revolving the line $z = -y$ with $y \geq 0$ in the yz-plane about the z-axis. Each is a single cone; the union of both is a double cone, described by the given equation $x^2 + y^2 + 2x + 2y - z^2 + 2 = 0$.

10. A circle of radius b rolls on the inside of a larger circle of radius a. The curve traced out by a fixed point P on the circumference of the smaller circle is called a hypocycloid.



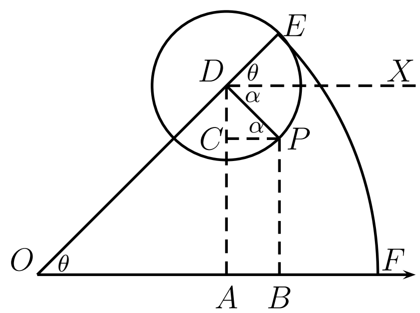
(a) If the initial position of P is $(a, 0)$, and the parameter θ is chosen as in the figure, show that the parametric equations of the hypocycloid are

$$x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\,\theta\right), \qquad y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\,\theta\right).$$

(b) Show that if $b = a/4$, the parametric equations reduce to $x = a\cos^3\theta, \quad y = a\sin^3\theta$. Sketch the curve in this case.

Solution

(a) Mark the points O, A, B, C, D, E, F, X as shown below. The dashed lines DX and CP are horizontal and DA, BP are vertical.

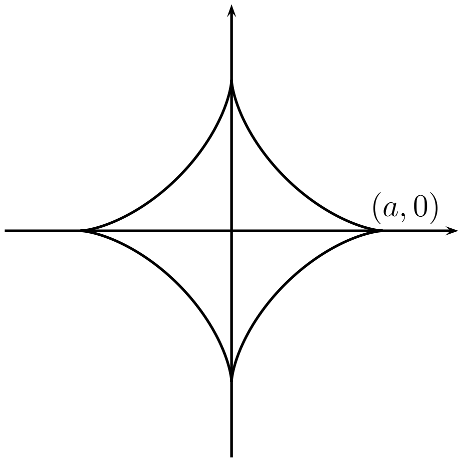


Let $\alpha = \angle XDP$. Observe that $\angle XDE = \theta$ and $\angle DPC = \alpha$. Suppose that P has coordinates (x, y). Noting that $OD = a - b$ and $DP = b$, we see that

$$x = OA + AB = (a - b)\cos\theta + b\cos\alpha,$$
$$y = AD - AC = (a - b)\sin\theta - b\sin\alpha.$$

But arc EF equals arc EP plus k times the circumference of the small circle, where $k \in \mathbb{Z}$. So $a\theta = b(\theta + \alpha + 2k\pi)$. This gives $\alpha = \dfrac{a-b}{b}\,\theta - 2k\pi$, and the result follows.

(b) When $b = a/4$, we find that $x = \dfrac{a}{4}(3\cos\theta + \cos 3\theta)$. But $\cos 3\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$ and this can be further simplified by expanding $\cos 2\theta$ and $\sin 2\theta$. This leads to the result $x = a\cos^3\theta$ and similar working shows that $y = a\sin^3\theta$.
The curve is called an astroid and its sketch is shown below.



Solution to Question 1

(a) The curve is the top half (semi-circle) of the circle $(x-1)^2 + (y-2)^2 = 1$, of radius 1, centred at $(1, 2)$. The value $t = 0$ gives the point $(2, 2)$, $t = \pi/2$ gives $(1, 3)$, $t = \pi$ gives $(0, 2)$. The semi-circle is traced anticlockwise from $(2, 2)$ as t increases from 0 to π.
(b) The curve is an ellipse, $(\frac{x-1}{2})^2 + (y-2)^2 = 1$. The value $t = 0$ gives the point $(3, 2)$, $t = \pi/2$ gives $(1, 3)$, $t = \pi$ gives $(-1, 2)$, $t = 3\pi/2$ gives $(1, 1)$ and $t = 2\pi$ gives $(3, 2)$. The ellipse is traced anticlockwise as t increases from 0 to 2π.
(c) The curve is part of the parabola $y = x^2 + 1$, for $x \in [0, 2]$. The value $t = 0$ gives the point $(0, 1)$, $t = 1/2$ gives $(1, 2)$, $t = 1$ gives $(2, 5)$.
(d) The curve is a line through $(0, 3, 0)$ in the direction of $-3\mathbf{j} + 2\mathbf{k}$. Its cartesian equations are $x = 0$, $\frac{z}{2} = \frac{y-3}{-3}$. The value $t = 0$ gives $(0, 3, 0)$, $t = 1$ gives $(0, 0, 2)$, $t = -1$ gives $(0, 6, -2)$.

Solution to Question 2

The natural domain of f is $\{(x, y) \mid xy \geq 0\}$, which is the union of the first and third quadrants of the xy-plane, including both axes. The natural domain of g is $\{(x, y) \mid x^2 + y^2 > 1\}$, the set of all points in the xy-plane lying outside the unit circle.