The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 8

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/JM/MATH1903/
Lecturers: Daniel Daners and James Parkinson

Material covered

(1) Explicit first order differential equations for y only depending on y'

(2) Separation of variables

(3) Direction fields

(4) Aspects of modelling

Outcomes

After completing this tutorial you should

(1) be able to solve the simplest differential equations

(2) be able to solve equations by separation of variables

(3) be able to sketch simple direction fields and corresponding solutions of simple differential equations.

(4) be able to determine the asymptotic behaviour of solutions by looking at the explicit solution of a differential equation

Questions to do before the tutorial

1. Find the general solution of the following differential equations.

(a) $\dfrac{dy}{dx} = 1 + \sin x + \sin^2 x$, (b) $x^3 \dfrac{dy}{dx} = 2x^2 + 5,\ x > 0$, (c) $\dfrac{dy}{dx} = \dfrac{1}{\cosh y}$,

Questions to complete during the tutorial

2. Find the particular solutions of the differential equations satisfying the given conditions:

(a) $\dfrac{dy}{dx} = 1 - 2x - 3x^2$, $y(1) = -1$.

(b) $e^{2x}\dfrac{dy}{dx} + 1 = 0$, $y(x) \to 2$ as $x \to \infty$.

(c) $\dfrac{dy}{dx} = \dfrac{y}{2} + \dfrac{1}{2y}$, $y(0) = 2$.

3. According to the Gompertz model, the population N of a colony of animals grows according to the differential equation,

$$\frac{dN}{dt} = -\beta N \ln\Big(\frac{N}{M}\Big),$$

where M is the maximum sustainable population size and β is a positive constant.

(a) Sketch the direction field of the differential equation and some possible solutions.

(b) Set $v := \ln(N/M)$. Show that v satisfies the differential equation $v' = -\beta v$.

(c) Solve the differential equation $v' = -\beta v$ and hence find $N(t)$.

(d) Find $\lim_{t\to\infty} N(t)$.

(e) Find the particular solution for which $N(0) = M/2$.

4. Let y be the number of people in a stable economy who have an income of x or more. The economist Vilfredo Pareto (1848–1923) discovered that the rate at which y decreases with increasing x is directly proportional to the number of people with income x or more and inversely proportional to the income x.

   (a) Derive a differential equation for $y(x)$.

   (b) Find the general solution y in terms of x.

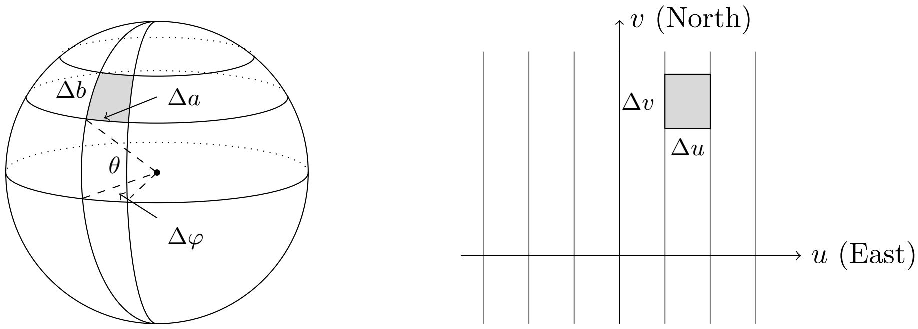
   (c) Find the particular solution of the differential equation given that the minimum income is x_0 and the total population is N.

Extra questions for further practice

5. Find an equation of the curve that passes through $(1, 1)$ and whose slope at (x, y) is y^2/x^3.

   Hint: The curve is tangent to the direction field of the differential equation $y' = y^2/x^3$.

6. The Mercator map is one of the most frequently used maps of the earth. It displays the earth such that the parallels and meridians form a rectangular grid. If φ denotes longitude and θ denotes latitude the coordinates of the map are therefore given by $u = u(\varphi)$ and $v = v(\theta)$.



   Consider a small rectangle on the sphere of side lengths Δa between longitude φ and $\varphi + \Delta\varphi$, and Δb between latitude θ and $\theta + \Delta\theta$, as shown in the figure. The spacing of the parallels on the map is such that the north-south distortion of length is the same as the east-west distortion of length on the map, that is,

$$\frac{\Delta u}{\Delta a} = \frac{\Delta v}{\Delta b}.$$

   Use this condition to derive a differential equation for $v(\theta)$ and solve it. What initial condition should be assumed?

7. Consider a particle of mass m in free fall from height h. Let $x(t)$ be its displacement from the initial position and $v(t) = dx/dt$ its velocity at time t.

   (a) If we neglect any friction forces, according to Newton's law, v satisfies the differential equation

$$m\frac{dv}{dt} = -mg.$$

      (i) Find the solution with initial condition $v(0) = 0$.

      (ii) Find the displacement $x(t)$ with initial condition $x(0) = h$.

(b) Assume now that there is a friction force proportional to the velocity. Then by Newton's law,

$$m\frac{dv}{dt} = -mg - cv$$

for some constant $c > 0$. The negative sign comes from the fact that the force acts in the direction opposite to v.

(i) Find the solution with initial condition $v(0) = 0$.

(ii) Find the terminal speed $v_\infty = |\lim_{t\to\infty} v(t)|$. Express the constant of proportionality c in terms of v_∞ and write down the solution from the previous part.

(iii) Find the displacement $x(t)$ with initial condition $x(0) = h$.

(c) Compute the Taylor polynomials $T_3(t)$ of $x(t)$ for the solutions without and with friction. Verify that for small times they are close to each other.

(d) Denote the solution without friction by $v_n(t)$ and the solution with friction by $v_f(t)$. Show that $|v_n(t)| > |v_f(t)|$ for all $t > 0$.