The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Board Tutorial for Week 9** 2017

(i) The *inverse* of a matrix A is a matrix A^{-1} such that, for some positive integer n,

$$AA^{-1} = A^{-1}A = I_n .$$

Only square matrices have inverses. When it exists, the inverse A^{-1} is unique.

(ii) Only half of the definition needs to be checked, in the following sense: if A is a square matrix and $AB = I$ or $BA = I$ then

$$AB = BA = I$$

so that the inverse A^{-1} exists and equals B.

(iii) A matrix is *invertible* if its inverse exists. If A and B are invertible matrices of the same size then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(iv) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define the *determinant* of A to be $\det A = ad - bc$. Then A is invertible if and only if $\det A \neq 0$, in which case $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(v) Let A be an invertible matrix. If n is an integer define

$$A^n = \begin{cases} I & \text{if } n = 0 \\ \underbrace{AA\ldots A}_{n \text{ times}} & \text{if } n \text{ is positive} \\ \underbrace{A^{-1}A^{-1}\ldots A^{-1}}_{-n \text{ times}} & \text{if } n \text{ is negative} \end{cases}$$

Then, for all integers m, n and all nonzero scalars λ,

$$A^mA^n = A^{m+n} , \qquad (A^{-1})^{-1} = A , \qquad (A^m)^n = A^{mn} , \qquad (\lambda A)^{-1} = \frac{1}{\lambda}A^{-1} .$$

(vi) A square matrix A is invertible if and only if the augmented matrix $\left[\,A \mid I\,\right]$ can be row reduced to $\left[\,I \mid B\,\right]$, in which case $A^{-1} = B$.

(vii) If a system of equations can be expressed in the form $A\mathbf{x} = \mathbf{b}$ where A is invertible, then $\mathbf{x} = A^{-1}\mathbf{b}$.

(viii) An $n \times n$ matrix is called *elementary* if it is the result of applying a single elementary row [column] operation to the identity matrix I_n.

(ix) If E is the elementary matrix obtained by applying the elementary row [column] operation ρ to I_n, and A is any matrix with n rows, then the matrix product EA $[AE]$ is the matrix obtained by applying ρ to A.

(x) The inverse of an elementary matrix is elementary.

(xi) Every invertible matrix is the product of elementary matrices.

6. Explain briefly why the matrix equations $AB = BA = I_n$ imply that A and B are square matrices of the same size.

7. Find the inverse of each of the following matrices when it exists:

(i) $\begin{bmatrix} 5 & 2 \\ 3 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 4 & 6 \\ 7 & 11 & 6 \\ -6 & -6 & 12 \end{bmatrix}$ (vi) $\begin{bmatrix} -4 & 3 & 3 \\ 8 & 7 & 3 \\ 4 & 3 & 3 \end{bmatrix}$

8. Suppose that A and D are invertible matrices. Explain briefly why the matrix equation $ABD = ACD$ implies $B = C$. Does it matter if A and D are of different sizes?

9. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 3 \end{bmatrix}$ and use it to solve for x, y and z where

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 2 \\ 2x & + & 2y & + & 3z & = & 0 \\ 3x & + & 4y & + & 3z & = & 1 \end{array}$$

10. Explain briefly why a square matrix with a row or column of zeros is not invertible.

11. Explain briefly why the inverse of an elementary matrix is elementary. [Hint: think about inverting elementary row operations.]

12. Which of the following are true for all invertible matrices A, B, C of the same size? Find a proof or counterexample in each case.

(i) $(ABC)^{-1} = A^{-1}B^{-1}C^{-1}$ (ii) $(ABA)^{-1} = A^{-1}B^{-1}A^{-1}$

(iii) $(A^{-1})^{-1} = A$ (iv) $-(-A)^{-1} = A^{-1}$

(v) $C^{-1}(ABC^{-1})^{-1}AB = I$ (vi) $(A+B)^{-1} = A^{-1} + B^{-1}$

(vii) $A^{-1}(I+A)A = A + I$ (viii) $(A+I)(A^{-1} - I) = A^{-1} - A$

(ix) $A^2 - 2A + I = 0 \implies A^{-1} = 2I - A$

(x) $A^2 - 2A + I = 0 \implies A = I$

13.* Express each of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ as products of elementary matrices.

14.* Use row reduction to determine the value of λ for which the following matrix is *not* invertible:

$$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 1 & 2 \\ -3 & -4 & \lambda \end{bmatrix}$$

15.* Let m and n be positive integers and suppose that B and C are matrices such that

$$AB = A = CA$$

for all $m \times n$ matrices A. Prove that $B = I_n$ and $C = I_m$.