(A)

MATH1902 — 2006 exam solutions — 10/6/2010

Q1/ (a)(i) ℓ is the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{-1}$

and ℓ' is parallel to ℓ, so has direction vector

$\underset{\sim}{v} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, and containing the point $B(3,3,4)$.

Hence the vector equation of ℓ' is

$$\underset{\sim}{r} = 3\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k} + t(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}),$$

yielding parametric equations

$$\boxed{\left.\begin{array}{l} x = 3+2t \\ y = 3+3t \\ z = 4-t \end{array}\right\} \quad t \in \mathbb{R}}$$

(ii) P is perpendicular to ℓ' and contains $B(3,3,4)$, so

has Cartesian equation

$$2x + 3y - z = 2(3) + 3(3) - 4 = 11$$

i.e. $\boxed{2x + 3y - z = 11}$

(iii) ℓ has parametric equations $\left\{\begin{array}{l} x = 1+2t \\ y = -1+3t \\ z = 2-t \end{array}\right.$, so intersects

P when $2(1+2t) + 3(-1+3t) - (2-t) = 11$, i.e. $2 + 4t - 3 + 9t - 2 + t = 11$,

i.e. $14t = 14$, so $t = 1$ and $\boxed{A = (3,2,1)}$.

Q1/(a) (iv) The distance between l and l' is

$$|\vec{AB}| = |\underline{j} + 3\underline{k}| = \sqrt{1+9} = \boxed{\sqrt{10}}.$$

(b) (i) l is the line $\underline{r} = \underline{i} - 3\underline{j} - 3\underline{k} + t(-\underline{i} + \underline{k})$,

so points in the direction $\underline{v} = -\underline{i} + \underline{k}$, whilst

m is the line $\underline{r} = -2\underline{i} + 3\underline{j} + s(4\underline{i} - \underline{j} - \underline{k})$, so

points in the direction $\underline{w} = 4\underline{i} - \underline{j} - \underline{k}$.

Clearly $\underline{v}, \underline{w}$ are not scalar multiples of each other,

so $\boxed{l, m \text{ are not parallel.}}$ The parametric equations are

$$l : \begin{cases} x = 1 - t \\ y = -3 \\ z = -3 + t \end{cases} \qquad m : \begin{cases} x = -2 + 4s \\ y = 3 - s \\ z = -s \end{cases}$$

so if there is an intersection then

$$\begin{aligned} 1 - t &= -2 + 4s && \text{(i)} \\ -3 &= 3 - s && \text{(ii)} \\ -3 + t &= -s && \text{(iii)} \end{aligned}$$

From (ii), $s = 6$, so (i) and (iii) become $\begin{cases} 1 - t = 22 \\ -3 + t = -6 \end{cases}$

so that $t = -21$ and $t = -3$, which is impossible.

Hence $\boxed{l \text{ and } m \text{ are skew.}}$

Q1/ (b) (ii) A vector perpendicular to both l and m is

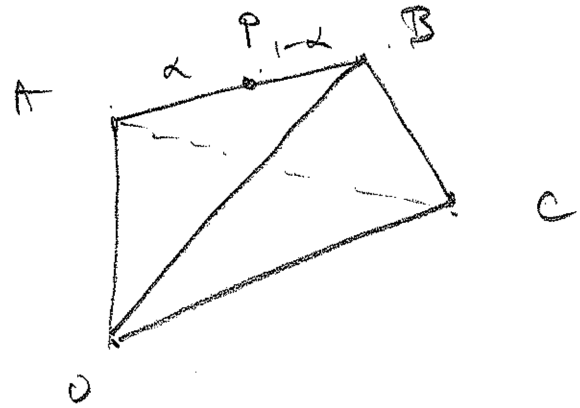
$$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -1 & 0 & 1 \\ 4 & -1 & -1 \end{vmatrix} = \boxed{\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}}$$

(iii) Let θ be the angle between $\underset{\sim}{v}$ and $\underset{\sim}{w}$, so

$$\cos\theta = \frac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{|\underset{\sim}{v}||\underset{\sim}{w}|} = \frac{-4-1}{\sqrt{2}\sqrt{18}} = \frac{-5}{6},$$

so the acute angle between l' and m' is $\boxed{\cos^{-1}\frac{5}{6}}$

Q2/ (a)



(i) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + \alpha\overrightarrow{AB} = \overrightarrow{OA} + \alpha(\overrightarrow{AO} + \overrightarrow{OB})$

$= (1-\alpha)\overrightarrow{OA} + \alpha\overrightarrow{OB} = (1-\alpha)(3\underset{\sim}{i} - \underset{\sim}{j} + 6\underset{\sim}{k}) + \alpha(-\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$

$= (3-4\alpha)\underset{\sim}{i} + (-1+3\alpha)\underset{\sim}{j} + (6-4\alpha)\underset{\sim}{k}$

so $\boxed{P = (3-4\alpha, -1+3\alpha, 6-4\alpha)}$.

(ii) The area of ΔOCP is $\frac{1}{2}|\overrightarrow{OC} \times \overrightarrow{OP}|$

D

Q2/ (a) (ii) (cont.)

$$= \frac{1}{2} \left| \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & 1 \\ 3-4\alpha & -1+3\alpha & 6-4\alpha \end{vmatrix} \right|$$

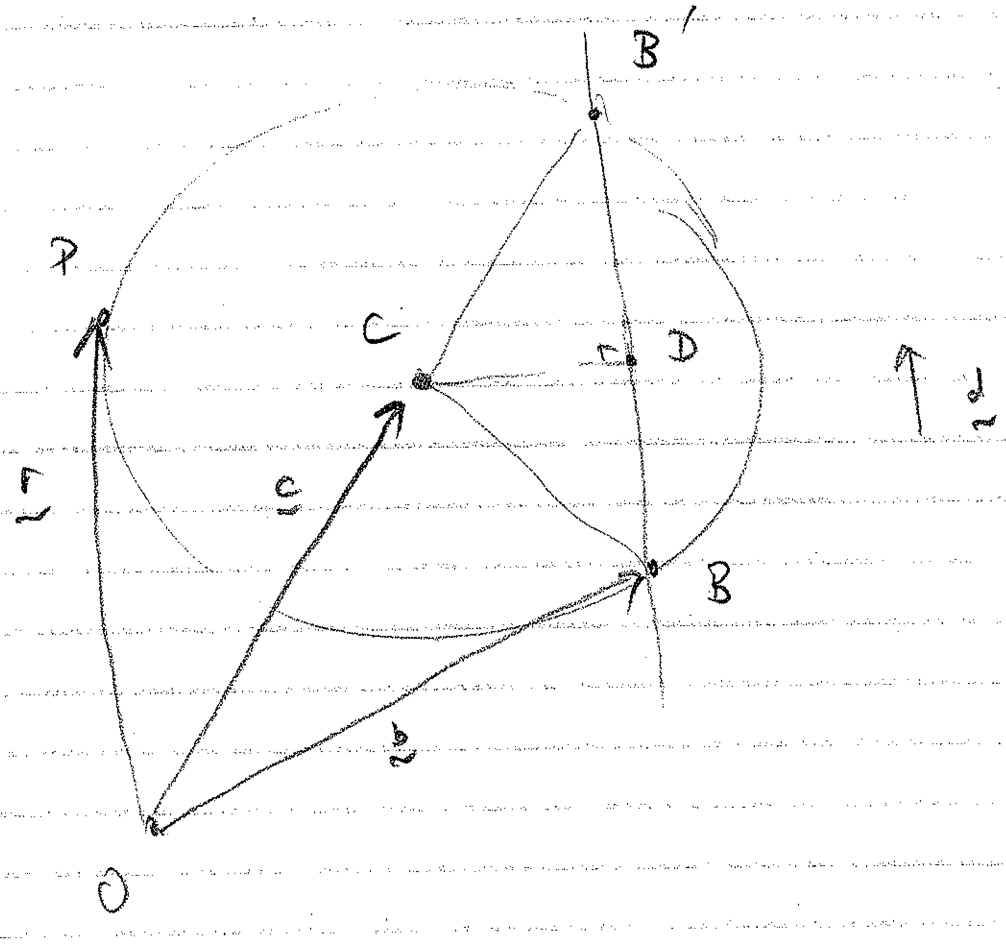
$$= \frac{1}{2} \left| (1-3\alpha)\underset{\sim}{i} - (6-4\alpha-3+4\alpha)\underset{\sim}{j} + (-1+3\alpha)\underset{\sim}{k} \right|$$

$$= \frac{1}{2} \left| (1-3\alpha)\underset{\sim}{i} - 3\underset{\sim}{j} + (-1+3\alpha)\underset{\sim}{k} \right|$$

$$= \frac{1}{2}\sqrt{2(1-3\alpha)^2+9} = \frac{1}{2}\sqrt{2-12\alpha+18\alpha^2+9}$$

$$= \boxed{\frac{1}{2}\sqrt{18\alpha^2-12\alpha+11}}$$

(b)



Q2/ (b) (i) Equation of circle is $|\underline{r}-\underline{c}| = a$, which becomes

$$(\underline{r}-\underline{c})\cdot(\underline{r}-\underline{c}) = |\underline{r}-\underline{c}|^2 = a^2,$$

so $\underline{r}\cdot\underline{r} - \underline{r}\cdot\underline{c} - \underline{c}\cdot\underline{r} + \underline{c}\cdot\underline{c} = a^2$,

so $$\boxed{|\underline{r}|^2 - 2\underline{r}\cdot\underline{c} + |\underline{c}|^2 - a^2 = 0}$$

(ii) Let ℓ be the line $\underline{r} = \underline{b} + t\underline{d}$.

When $t=0$ this becomes $r = \underline{b} + 0\underline{d} = \underline{b}$, the

$\boxed{\text{position vector of } B.}$

Let D be the midpoint of BB' so that $\overrightarrow{BD}$ is

the projection of $\overrightarrow{BC}$ in the direction of $\underline{d}$,

ie, $$\overrightarrow{BD} = \frac{\overrightarrow{BC}\cdot\underline{d}}{|\underline{d}|^2}\underline{d} = \frac{(\underline{c}-\underline{b})\cdot\underline{d}}{|\underline{d}|^2}\underline{d}$$

$$= -\frac{\underline{d}\cdot(\underline{b}-\underline{c})}{|\underline{d}|^2}\underline{d}.$$

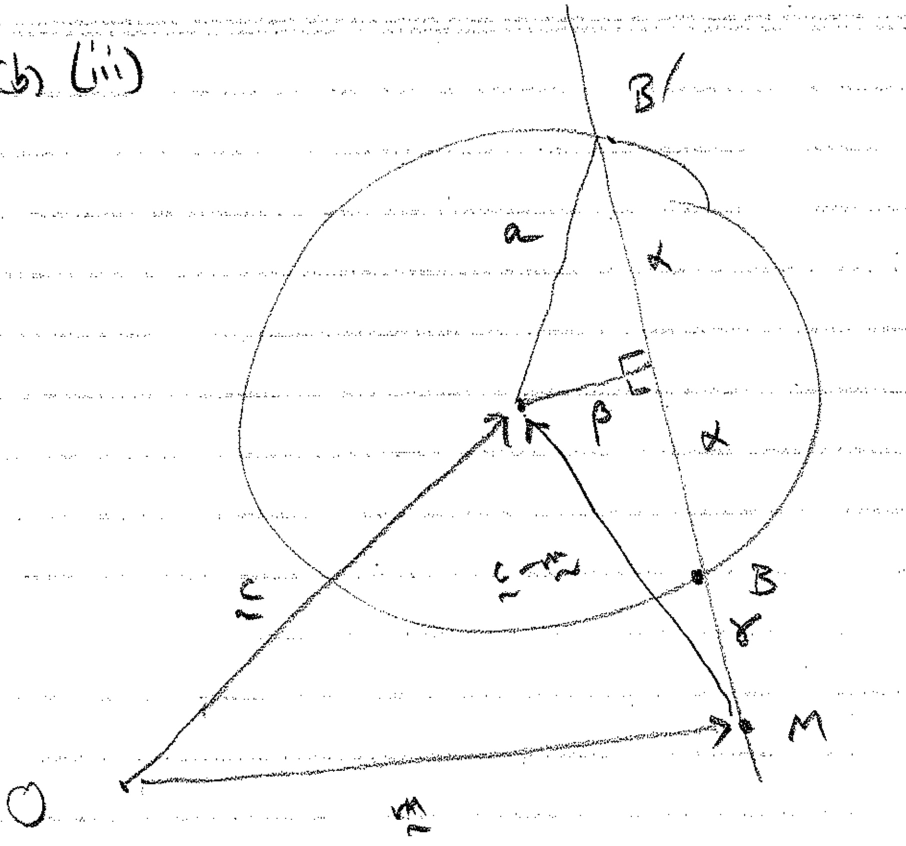
Thus when $t = -2\dfrac{\underline{d}\cdot(\underline{b}-\underline{c})}{|\underline{d}|^2}$ then

$$\underline{r} = \underline{b} + t\underline{d} = \underline{b} - \frac{2\underline{d}\cdot(\underline{b}-\underline{c})}{|\underline{d}|^2}\underline{d} = \underline{b} + 2\overrightarrow{BD}$$

$= \overrightarrow{OB} + \overrightarrow{BB'} = \overrightarrow{OB'}$, $\boxed{\text{the position vector of } B'.}$

(F)

Q2/ (b) (iii)



From the diagram, $\overrightarrow{MB} \cdot \overrightarrow{MB'} = \gamma(\gamma + 2\alpha) = \gamma^2 + 2\gamma\alpha$

$$= (\gamma + \alpha)^2 - \alpha^2$$

$$= |\underline{c} - \underline{m}|^2 - \beta^2 - \alpha^2$$

by Pythagoras

$$= |\underline{c} - \underline{m}|^2 - (\beta^2 + \alpha^2)$$

$$= \boxed{(\underline{c} - \underline{m})^2 - a^2}$$

by Pythagoras again.

The argument is the same for M on the other side of the circle.

Q3/(a) (i) We have the system

$$\begin{cases} x_1 \quad\quad + 3x_3 \quad\quad = 4 \\ -x_1 + x_2 - x_3 \quad\quad = -3 \\ -2x_1 \quad\quad - 5x_3 \quad\quad = -7 \\ -x_1 \quad\quad - 3x_3 + (p^2+p)x_4 = p-4 \end{cases}$$

with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 4 \\ -1 & 1 & -1 & 0 & -3 \\ -2 & 0 & -5 & 0 & -7 \\ -1 & 0 & -3 & p^2+p & p-4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & p^2+p & p \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & p(p+1) & p \end{array}\right]$$

This has a unique solution for $\boxed{p \neq 0, -1}$, in which case it is $(x_1, x_2, x_3, x_4) = (1, -1, 1, 1/(p+1))$

producing $\boxed{\underline{v} = \underline{a} - \underline{b} + \underline{c} + \frac{1}{p+1}\underline{d}}$

(ii) The system is inconsistent if $p \neq 0$ and $p(p+1) = 0$,

i.e., $\boxed{p = -1}$

Q3/ (a) (iii) The system has infinitely many solutions when $p(p+1) = p = 0$, i.e. $\boxed{p = 0}$, in which case we put $x_4 = t$, and still $x_1 = 1$, $x_2 = -1$, $x_3 = 1$, yielding

$$\boxed{\underline{v} = \underline{a} - \underline{b} + \underline{c} + t\,\underline{d}} \quad \text{for } t \in \mathbb{R},$$

Q4/ (a)
$$A = \begin{bmatrix} 3 & 4 & 0 \\ 3 & 7 & 0 \\ -4 & 4 & 7 \end{bmatrix}$$

(i)
$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 & 0 \\ 3 & 7-\lambda & 0 \\ -4 & 4 & 7-\lambda \end{vmatrix} = (7-\lambda) \begin{vmatrix} 3-\lambda & 4 \\ 3 & 7-\lambda \end{vmatrix}$$

$$= (7-\lambda)\left[(3-\lambda)(7-\lambda) - 12\right] = (7-\lambda)\left(\lambda^2 - 10\lambda + 21 - 12\right)$$

$$= (7-\lambda)(\lambda^2 - 10\lambda + 9) = (7-\lambda)(\lambda-1)(\lambda-9),$$

so eigenvalues are $\boxed{\lambda = 7, 1, 9}$.

(ii) $B = -6I + A = A - \lambda I$, for $\lambda = 6$, is invertible because $\lambda = 6$ is not an eigenvalue.

(I)

Q4/ (b) We have A 2×2, $A\underline{v}_1 = \lambda_1 \underline{v}_1$, $A\underline{v}_2 = \lambda_2 \underline{v}_2$,

$\underline{v}_1, \underline{v}_2 \neq \underline{0}$ and $\lambda_1 \neq \lambda_2$. Suppose B is 2×2;

$AB = BA$ and $B\underline{v}_1 = \alpha \underline{v}_2$. WTS $\alpha = 0$

Then $AB\underline{v}_1 = A\alpha \underline{v}_2 = \alpha A \underline{v}_2 = \alpha \lambda_2 \underline{v}_2$

and $BA\underline{v}_1 = B\lambda_1 \underline{v}_1 = \lambda_1 B\underline{v}_1 = \lambda_1 \alpha \underline{v}_2$,

so that

$$\alpha \lambda_2 \underline{v}_2 = \lambda_1 \alpha \underline{v}_2$$

so $(\alpha\lambda_2 - \lambda_1 \alpha)\underline{v}_2 = \underline{0}$,

so $\alpha\lambda_2 - \lambda_1\alpha = 0$ (since $\underline{v}_2 \neq 0$)

so $\alpha(\lambda_2 - \lambda_1) = 0$,

so $\alpha = 0$, since $\lambda_2 - \lambda_1 \neq 0$ (since $\lambda_1 \neq \lambda_2$).

Q5/ (a) (i) If A is skew-symmetric and n is odd

then $\det A = \det(A^T)$ (property of determinants)

$= \det(-A)$ (A being skew-symmetric)

$= (-1)^n \det A$ (property of det)

$= -\det A$ (since n is odd),

so $2\det A = 0$, so $\det A = 0/2 = 0$.

Q5/ (a) (ii) From (i), $0 = \det A = \det(A - \lambda I)$

when $\lambda = 0$, so $\lambda = 0$ is an eigenvalue of A.

(iii)

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 \\ -1 & 0 & 1 & 2 & 2 \\ -1 & 0 & 0 & 1 & 2 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so eigenspace is $\left\{ \begin{bmatrix} t \\ -t \\ t \\ -t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$