2 Open and Closed subsets; Limits

[M] – Maple/Gnuplot; [A] – additional/optional problems; [H] – harder problems.

2.1 Open and Closed subsets of $\mathbb{R}^n$

In this subsection, you are only allowed to use definitions of open and closed sets and definition of the boundary of a set.

26: Show that

1) $[a,b]$ is closed, $a,b \in \mathbb{R}$;
2) (a,b) is open, $a,b \in \mathbb{R}$;
3) $\emptyset$ is open and close;
4) $\mathbb{R}$ is open and close;
5) $[a,b)$ is neither closed nor open, $a,b \in \mathbb{R}$;
6) $\mathbb{Q}$ is neither closed nor open;
7) $\Big\{k^{-1}\ :\ k \in \mathbb{Z}, k \neq 0\Big\}$ is neither open and closed;
8) The open ball $B(\mathbf{x},\epsilon)$ is open.

27: Determine whether or not the set

$$\Big\{(m^{-1}, n^{-1}):\quad m,n \in \mathbb{Z},\quad m,n > 0\Big\}$$

is closed.

28: Let

$$\Omega = \Big\{(x,y) \in \mathbb{R}^2 : x + y \neq 0\Big\}.$$

Show that Ω is an open subset of $\mathbb{R}^2$.

[A] **29**: i) If Ω_1 and Ω_2 are open sets in $\mathbb{R}^n$, show that $\Omega_1 \cap \Omega_2$ and $\Omega_1 \cup \Omega_2$ are open.

ii) If Ω_1 and Ω_2 are closed sets in $\mathbb{R}^n$, show that $\Omega_1 \cap \Omega_2$ and $\Omega_1 \cup \Omega_2$ are closed.

30: Show that every point $(0,a)$ with $|a| \leq 1$ is the boundary point of the set

$$S = \Big\{(x,y) \in \mathbb{R}^2 : x > 0,\quad y = \sin(1/x)\Big\}.$$

2.2 Limits

31: Use definition of the limit to show that

i) $\displaystyle\lim_{x\to 2} \frac{x+1}{x+2} = \frac{3}{4}$.

ii) $$\lim_{(x,y)\to(0,0)} \frac{x^4 + x^2 + y^2 + y^4}{x^2 + y^2} = 1.$$

32: Show that the following limits do not exist

i) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2};$$

ii) $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}.$$

33: For the limits below give two proofs: one using *pinching principle* and one using the definition of the limit directly

i) $$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2};$$

ii) $$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2 + x^2 y^2}{x^2 + y^2} = 1.$$

34: Let

$$f(x,y) = \frac{x-y}{x+y}.$$

Show that

$$\lim_{x\to 0}\Big[\lim_{y\to 0} f(x,y)\Big] = 1 \quad\text{and}\quad \lim_{y\to 0}\Big[\lim_{x\to 0} f(x,y)\Big] = -1.$$

Show also that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.

35: Let

$$f(x,y) = \frac{x^2 y^2}{x^2 y^2 + (x-y)^2}.$$

Show that

$$\lim_{x\to 0}\Big[\lim_{y\to 0} f(x,y)\Big] = \lim_{y\to 0}\Big[\lim_{x\to 0} f(x,y)\Big] = 0.$$

Show also that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.

36: Let

$$f(x,y) = (x+y)\,\sin\frac{1}{x}\,\sin\frac{1}{y}, \quad x \neq 0,\ y \neq 0$$

and

$$f(x,y) = 0, \quad x = 0 \text{ or } y = 0.$$

Show that neither

$$\lim_{y\to 0} f(x,y),\ x \neq 0 \quad \text{nor} \quad \lim_{x\to 0} f(x,y),\ y \neq 0$$

exist. Also, use *pinching principle* to show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

37: Use *pinching principle* to show that

$$\lim_{(x,y)\to(0,0)} \frac{xy(x+y)}{x^2 - xy + y^2} = 0.$$

Hint: Prove first that

$$\left|\frac{xy}{x^2 - yx + y^2}\right| \leq 1, \quad \forall (x,y) \neq 0.$$

38: Use *pinching principle* to show that

$$\lim_{(x,y)\to(0,0)} \frac{xy(x+y)}{x^2 + y^2} = 0.$$

Hint: Prove first that

$$\frac{|xy|}{x^2+y^2} \leq \frac{1}{2}.$$

2.3 Limits and Taylor expansions

In the following questions you are allowed to use the known Taylor expansions below. In the expansions below the function $\epsilon(x)$ different from one expansion to another and is such that

$$\epsilon(x): \quad \lim_{x\to 0} \epsilon(x) = 0$$

Taylor expansions

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + x^n\epsilon(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + x^n\epsilon(x)$$

$$\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^k \frac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\epsilon(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + x^{2n+1}\epsilon(x)$$

$$\sinh x = x + \frac{x^3}{3!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\epsilon(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + x^{2n+1}\epsilon(x)$$

$$\cos x = 1 - \frac{x^2}{2} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + x^{2n}\epsilon(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + x^{2n}\epsilon(x)$$

$$\cosh x = 1 + \frac{x^2}{2} + \ldots + \frac{x^{2n}}{(2n)!} + x^{2n}\epsilon(x) = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + x^{2n}\epsilon(x)$$

$$\ln(1+x) = x - \frac{x^2}{2} - \ldots + (-1)^{n-1}\frac{x^n}{n} + x^n\epsilon(x) = \sum_{k=1}^{n} (-1)^{k+1}\frac{x^k}{k} + x^n\epsilon(x)$$

$$(1+x)^\alpha = 1 + \alpha x + \ldots + \binom{\alpha}{n} x^n + x^n\epsilon(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + x^n\epsilon(x)$$

$$\binom{\alpha}{k} = \prod_{s=1}^{k} \frac{\alpha - s + 1}{s} = \frac{\alpha \times (\alpha-1) \times \ldots \times (\alpha - k + 1)}{k!}$$

39: Prove that
$$\lim_{(x,y)\to(0,a)} \frac{\sin(xy)}{x} = a.$$

40: Prove that
$$\lim_{(x,y)\to(a,0)} \frac{1-\cos(xy)}{y^2}.$$

41: Prove that
$$\lim_{(x,y)\to(a,0)} \frac{\ln(1+xy)}{y}.$$

42: Prove that
$$\lim_{(x,y)\to(0,a)} \frac{(x+y)^{2/3}-y^{2/3}}{x}.$$

Answers to problems

A26: Direct argument for part (8) is given in these webnotes[5]. **A34**: See these webnotes[6] for an idea how to show that a limit does not exist **A35**: See answer to Problem 34 **A36**: See these webnotes[7] with an example of argument showing that limit exists **A37**: See the answer to Problem 36 **A38**: See the answer to Problem 36 **A38**: See the answer to Problem 36

[5] http://web.maths.unsw.edu.au/~potapov/2111_2015/A-ball-is-open-subset.html
[6] http://web.maths.unsw.edu.au/~potapov/2111_2015/Limit-by-sequences-_002d_002d-Example.html
[7] http://web.maths.unsw.edu.au/~potapov/2111_2015/Example-of-limit-of-vector-map.html