MATH 1903 - Revision (Weeks 1-6)

Week 1

- Riemann sums
  - definition
  - examples
  - upper/lower Riemann sums
  - telescoping sums
- Riemann Integral
  - definition
  - Theorem: Continuous fns are Riemann Integrable (proof omitted)

Examples:

(1) Find a closed formula for $\sum_{j=1}^{n} j$ by considering $\sum_{j=1}^{n}\left(j^2-(j-1)^2\right)$.

Similarly, find a closed formula for $\sum_{j=1}^{n} j^2$.

(2) Find a closed formula for the upper Riemann sum for $f(x)=x^2$ over $[0,1]$ using the partition of $[0,1]$ into n equal pieces.

Hence calculate $\int_0^1 x^2\,dx$.

Solution

(1) $\sum_{j=1}^{n}(j^2-(j-1)^2) = (\cancel{1^2}-0^2)+(\cancel{2^2}-\cancel{1^2})+\cancel{\ }\dots\cancel{\ }(n^2-(\cancel{n-1})^2)$

$$= n^2$$

$$\sum_{j=1}^{n}(j^2-(j-1)^2) = \sum_{j=1}^{n}(j^2-j^2+2j-1) = 2\sum_{j=1}^{n} j - n.$$

So $2\sum_{j=1}^{n} j - n = n^2$, so $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$. //

Now consider

$$\sum_{j=1}^{n}(j^3-(j-1)^3) = (\cancel{1^3}-0^3)+(\cancel{2^3}-\cancel{1^3})+\cancel{\ }\dots\cancel{\ }+(n^3-(\cancel{n-1})^3) = n^3$$

$$\sum_{j=1}^{n}(j^3-(j-1)^3) = \sum_{j=1}^{n}(\cancel{j^3}-\cancel{j^3}+3j^2-3j+1)$$

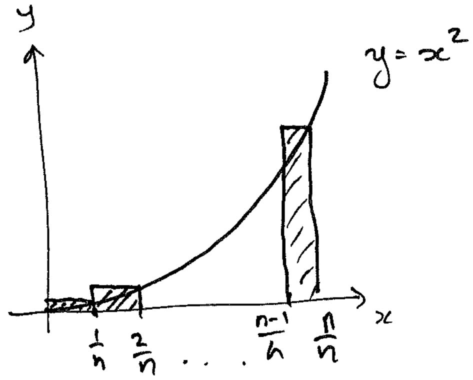
$$= 3\sum_{j=1}^{n} j^2 - 3\sum_{j=1}^{n} j + n$$

$$= 3\sum_{j=1}^{n} j^2 - \frac{3n(n+1)}{2} + n \quad \text{(from above)}.$$

So $3\sum_{j=1}^{n} j^2 - \frac{3n(n+1)}{2} + n = n^3$, which rearranges to

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

(2)



$$U_n = \sum_{j=1}^{n} f(x_j^*)\Delta x_j \qquad x_j^* = \frac{j}{n}, \quad \Delta x_j = \frac{1}{n}$$

$$= \sum_{j=1}^{n} \left(\frac{j}{n}\right)^2 \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{j=1}^{n} j^2$$

$$= \frac{(n+1)(2n+1)}{6n^2} \quad \text{(by previous qn).}$$

Since $f(x) = x^2$ is continuous, it is Riemann integrable on $[0,1]$, and so the Riemann sums tend to the Riemann integral as $\|P\| \to 0$. Therefore

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{3}$$

(as expected).

Week 2

- Fundamental Theorem of Calculus
  - Mean Value Theorem
  - FTC (with proof)
- Functions defined by integrals
  - error function, $C(x) = \int_0^x \cos(t^2)\,dt$, etc.

Examples

(1) Find $\frac{d}{dx}\int_0^{e^x} \cos(t^2)\,dt$

(2) Find $\int_0^1 C(x)\,dx$.

Soln (1) By the chain rule and FTC, we have

$$\frac{d}{dx}\int_0^{e^x} \cos(t^2)\,dt = e^x \cos(e^{2x})$$

(2) Integrate by parts, with $u = C(x)$, $\frac{dv}{dx} = 1$.

$$\frac{du}{dx} = \cos(x^2) \qquad v = x.$$

So

$$\int_0^1 C(x)\,dx = xC(x)\Big|_0^1 - \int_0^1 x\cos(x^2)\,dx$$

$$= C(1) - \tfrac{1}{2}\sin(x^2)\Big|_0^1$$

$$= C(1) - \tfrac{1}{2}\sin(1).$$

Week 3

- Area
- Volume of revolution
  - discs
  - cylindrical shells
- Lengths
  - graphs
  - parametrised curve
- Surface area of revolution
- Hyperbolic substitutions

Example Calculate the length of $f(x) = \frac{1}{2}x^2$ between $x=0$ and $x=1$.

Soln

$$L = \int_0^1 \sqrt{1+x^2}\,dx \qquad \begin{aligned} x &= \sinh u \\ dx &= \cosh u\,du \end{aligned}$$

$$= \int_0^{\sinh^{-1}(1)} \sqrt{1+\sinh^2 u}\,\cosh u\,du$$

$$= \int_0^{\sinh^{-1}(1)} \cosh^2 u\,du \qquad \begin{aligned} \cosh^2 u - \sinh^2 u &= 1 \\ \cosh^2 u + \sinh^2 u &= \cosh(2u) \end{aligned}$$

$$= \int_0^{\sinh^{-1}(1)} \tfrac{1}{2}(1+\cosh 2u)\,du \qquad \Rightarrow \cosh^2 u = \tfrac{1}{2}(1+\cosh 2u)$$

$$= \left(\tfrac{1}{2}u + \tfrac{1}{4}\sinh(2u)\right)_0^{\sinh^{-1}(1)}$$

$$\boxed{L = \tfrac{1}{2}\sinh^{-1}(1) + \tfrac{1}{4}\sinh(2\sinh^{-1}(1)).}$$

(It is fine to leave the answer as above. But for an exercise you might like to show that

$$L = \tfrac{1}{2}\ln(1+\sqrt{2}) + \tfrac{1}{\sqrt{2}}$$
)

Week 4

- Improper integrals
  - Calculating as limits of proper integrals
  - p-integrals
  - Comparison test

Example (1) Does $\int_1^\infty \frac{e^x}{e^{2x}+1}\,dx$ converge?

(2) Use integration by parts to show that
$$\int_1^b \frac{\sin x}{x}\,dx = \cos(1) - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\,dx$$

Hence show that $\int_1^\infty \frac{\sin x}{x}\,dx$ converges.

Soln (1) $\left|\frac{e^x}{e^{2x}+1}\right| \leq \frac{e^x}{e^{2x}} = e^{-x}$, and

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}(e^{-1}-e^{-b}) = \frac{1}{e} \text{ (converges).}$$

Therefore $\int_1^\infty \frac{e^x}{e^{2x}+1}\,dx$ converges by the Comparison Test.

(2) Let $u = \frac{1}{x}$, $\frac{dv}{dx} = \sin x$. So $\frac{du}{dx} = -\frac{1}{x^2}$ and $v = -\cos x$.

So $$\int_1^b \frac{\sin x}{x}\,dx = -\frac{\cos x}{x}\Big|_1^b - \int_1^b \frac{\cos x}{x^2}\,dx$$
$$= \cos(1) - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\,dx.$$

Then: $\lim_{b\to\infty}\frac{\cos b}{b} = 0$ (squeeze Law), and

$\int_1^\infty \frac{\cos x}{x^2}\,dx$ converges (by comparison: $\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$,

and $\int_1^\infty \frac{1}{x^2}\,dx$ converges - p-integral).

Hence $$\int_1^\infty \frac{\sin x}{x}\,dx = \cos(1) - \int_1^\infty \frac{\cos x}{x^2}\,dx \text{ converges.}$$

Week 5

- Sequences
  - Squeeze Law
  - Ratio test
  - Asymptotic sequences
- Series
  - geometric series
  - harmonic series
  - p-series
  - comparison test
  - Ratio test

Examples

(1) Does $\lim_{n\to\infty} \frac{1}{3^{2n}}\binom{2n}{n}$ converge? If so, find the limit.

(2) Decide convergence/divergence of

(a) $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ (b) $\sum_{k=1}^{\infty} \frac{\cosh k}{k^2}$ (c) $\sum_{k=1}^{\infty} 2^{-k} k^2$

Soln (1) Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{3^{2n+2}} \frac{(2n+2)!}{(n+1)!^2} \cdot \frac{3^{2n} n!^2}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{1}{9} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = \frac{4}{9}.$$

Since $\frac{4}{9}<1$, $\lim_{n\to\infty} a_n = 0$ (Ratio test).

(2) (a) $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converges, since

$$\left|\frac{\cos k}{k^2}\right| \le \frac{1}{k^2} \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges (p-series)}$$

(b) $\sum_{k=1}^{\infty} \frac{\cosh k}{k^2}$ diverges, since

$$\lim_{k\to\infty} \frac{\cosh k}{k^2} = \lim_{k\to\infty} \frac{\frac{1}{2}(e^k + e^{-k})}{k^2} = \infty .$$

(c) $\sum_{k=1}^{\infty} 2^{-k} k^2$ converges. We use the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{-(n+1)}(n+1)^2}{2^{-n}\, n^2}$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1,$$

and so the series converges.

Week 6

- Taylor polynomials
  - Examples
  - Remainder term
- Taylor series
  - Convergence (e^x, $\cos x$, $\sin x$...)
  - Examples (binomial, $\ln(1+x)$, ...)

Example Use a third order Taylor polynomial to approximate (with error bounds)

$$\int_0^1 \frac{\sin x}{x}\,dx.$$

Soln: $\sin x = T_3(x) + R_3(x)$

and $T_3(x) = x - \frac{x^3}{3!}$

$$R_3(x) = \frac{f^{(4)}(c)}{4!}x^4 = \frac{\sin c}{4!}x^4$$

for some c between 0 and x.

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \frac{T_3(x)+R_3(x)}{x}\,dx$$

$$= \int_0^1 \left(1 - \frac{x^2}{3!}\right)dx + \int_0^1 \frac{R_3(x)}{x}\,dx$$

$$= 1 - \frac{1}{18} + E = \frac{17}{18} + E, \quad \text{where}$$

$$|E| = \left|\int_0^1 \frac{\sin c}{4!}x^3\,dx\right| \leq \int_0^1 \frac{1}{4!}|\sin c|\,x^3\,dx \leq \frac{1}{4!}\int x^3\,dx = \frac{1}{4\times 4!}$$

Hence $|E| \leq \frac{1}{96}$

So $\int_0^1 \frac{\sin x}{x}\,dx \approx \frac{17}{18}$ with error $\pm\frac{1}{96}$.

ie $0.934027\dot{7} \leq \int_0^1 \frac{\sin x}{x}\,dx \leq 0.95486\dot{1}$