CHAPTER **ONE**

Methods in Algebra

Mathematics is the study of structure, pursued using a highly refined form of language in which every word has an exact meaning, and in which the logic is expressed with complete precision. As the structures and the logic of their explanation become more complicated, the language describing them in turn becomes more specialised, and requires systematic study for the meaning to be understood. The symbols and methods of *algebra* are one aspect of that special language, and fluency in algebra is essential for work in all the various topics of the course.

STUDY NOTES: Several topics in this chapter will probably be quite new — the four cubic identities of Section 1E, solving a set of three simultaneous equations in three variables in Section 1G, and the language of sets in Section 1J. The rest of the chapter is a concise review of algebraic work which would normally have been carefully studied in previous years, and needs will therefore vary as to the amount of work required on these exercises.

1 A Terms, Factors and Indices

A *pronumeral* is a symbol that stands for a number. The pronumeral may stand for a known number, or for an unknown number, or it may be a *variable*, standing for any one of a whole set of possible numbers. Pronumerals, being numbers, can therefore be subjected to all the operations that are possible with numbers, such as addition, subtraction, multiplication and division (except by zero).

Like and Unlike Terms: An *algebraic expression* is an expression such as

$$x^2 + 2x + 3x^2 - 4x - 3,$$

in which pronumerals and numbers and operations are combined. The five *terms* in the above expression are x^2, $2x$, $3x^2$, $-4x$ and -3. The two *like terms* x^2 and $3x^2$ can be combined to give $4x^2$, and the like terms $2x$ and $-4x$ can be combined to give $-2x$. This results in three *unlike terms* $4x^2$, $-2x$ and -3, which cannot be combined.

WORKED EXERCISE: $x^2 + 2x + 3x^2 - 4x - 3 = 4x^2 - 2x - 3$

Multiplying Terms: To simplify a product like $3xy \times (-6x^2y) \times \frac{1}{2}y$, it is best to work systematically through the signs, the numerals, and the pronumerals.

WORKED EXERCISE: (a) $4ab \times 7bc = 28ab^2c$ (b) $3xy \times (-6x^2y) \times \frac{1}{2}y = -9x^3y^3$

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Index Laws: Here are the standard laws for dealing with indices (see Chapter Six for more detail).

1

INDEX LAWS:

$$a^x\, a^y = a^{x+y} \qquad (ab)^x = a^x\, b^x$$

$$\frac{a^x}{a^y} = a^{x-y} \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$(a^x)^n = a^{xn}$$

WORKED EXERCISE:

(a) $3x^4 \times 4x^3 = 12x^7$

(b) $(48x^7y^3) \div (16x^5y^3) = 3x^2$

(c) $(3a^4)^3 = 27a^{12}$

(d) $(-5x^2)^3 \times (2xy)^4 = -125x^6 \times 16x^4y^4$
$\qquad = -2000x^{10}y^4$

(e) $\dfrac{(6x^4y)^2}{3(x^2y^3)^3} = \dfrac{36x^8y^2}{3x^6y^9}$
$\qquad = \dfrac{12x^2}{y^7}$

Exercise 1A

1. Simplify:

(a) $3x - 2y + 5x + 6y$
(b) $2a^2 + 7a - 5a^2 - 3a$
(c) $9x^2 - 7x + 4 - 14x^2 - 5x - 7$
(d) $3a - 4b - 2c + 4a + 2b - c + 2a - b - 2c$

2. Find the sum of:

(a) $x + y + z,\ 2x + 3y - 2z$ and $3x - 4y + z$
(b) $2a - 3b + c,\ 15a - 21b - 8c$ and $24b + 7c + 3a$
(c) $5ab + bc - 3ca,\ ab - bc + ca$ and $-ab + 2ca + bc$
(d) $x^3 - 3x^2y + 3xy^2,\ -2x^2y - xy^2 - y^3$ and $x^3 + 4y^3$

3. Subtract:

(a) x from $3x$
(b) $-x$ from $3x$
(c) $2a$ from $-4a$
(d) $-b$ from $-5b$

4. From:

(a) $7x^2 - 5x + 6$ take $5x^2 - 3x + 2$
(b) $4a - 8b + c$ take $a - 3b + 5c$
(c) $3a + b - c - d$ take $6a - b + c - 3d$
(d) $ab - bc - cd$ take $-ab + bc - 3cd$

5. Subtract:

(a) $x^3 - x^2 + x + 1$ from $x^3 + x^2 - x + 1$
(b) $3xy^2 - 3x^2y + x^3 - y^3$ from $x^3 + 3x^2y + 3xy^2 + y^3$
(c) $b^3 + c^3 - 2abc$ from $a^3 + b^3 - 3abc$
(d) $x^4 + 5 + x - 3x^3$ from $5x^4 - 8x^3 - 2x^2 + 7$

6. Multiply:

(a) $5a$ by 2
(b) $6x$ by -3
(c) $-3a$ by a
(d) $-2a^2$ by $-3ab$
(e) $4x^2$ by $-2x^3$
(f) $-3p^2q$ by $2pq^3$

7. Simplify:

(a) $2a^2b^4 \times 3a^3b^2$
(b) $-6ab^5 \times 4a^3b^3$
(c) $(-3a^3)^2$
(d) $(-2a^4b)^3$

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8. If $a = -2$, find the value of: (a) $3a^2 - a + 4$ (b) $a^4 + 3a^3 + 2a^2 - a$

9. If $x = 2$ and $y = -3$, find the value of: (a) $8x^2 - y^3$ (b) $x^2 - 3xy + 2y^2$

10. Simplify: (a) $\dfrac{5x}{x}$ (b) $\dfrac{-7x^3}{x}$ (c) $\dfrac{-12a^2b}{-ab}$ (d) $\dfrac{-27x^6y^7z^2}{9x^3y^3z}$

11. Divide:

(a) $-2x$ by x
(b) $3x^3$ by x^2
(c) x^3y^2 by x^2y
(d) a^6x^3 by $-a^2x^3$
(e) $14a^5b^4$ by $-2a^4b$
(f) $-50a^2b^5c^8$ by $-10ab^3c^2$

DEVELOPMENT

12. Simplify: (a) $\dfrac{3a \times 3a \times 3a}{3a + 3a + 3a}$ (b) $\dfrac{3c \times 4c^2 \times 5c^3}{3c^2 + 4c^2 + 5c^2}$ (c) $\dfrac{ab^2 \times 2b^2c^3 \times 3c^3a^4}{a^3b^3 + 2a^3b^3 + 3a^3b^3}$

13. Simplify: (a) $\dfrac{(-2x^2)^3}{-4x}$ (b) $\dfrac{(3xy^3)^3}{3x^2y^4}$ (c) $\dfrac{(-ab)^3 \times (-ab^2)^2}{-a^5b^3}$ (d) $\dfrac{(-2a^3b^2)^2 \times 16a^7b}{(2a^2b)^5}$

14. What must be added to $4x^3 - 3x^2 + 2$ to give $3x^3 + 7x - 6$?

15. Take the sum of $2a - 3b - 4c$ and $-4a + 7b - 5c$ from the sum of $4c - 2b$ and $5b - 2a - 2c$.

16. If $X = 2b + 3c - 5d$ and $Y = 4d - 7c - b$, take $X - Y$ from $X + Y$.

17. Divide the product of $(-3x^7y^5)^4$ and $(-2xy^6)^3$ by $(-6x^3y^8)^2$.

EXTENSION

18. For what values of x is it true that: (a) $x \times x \leq x + x$? (b) $x \times x \times x \leq x + x + x$?

1 B Expanding Brackets

The laws of arithmetic tell us that $a(x + y) = ax + ay$, whatever the values of a, x and y. This enables expressions with brackets to be *expanded*, meaning that they can be written in a form without brackets.

WORKED EXERCISE:

(a) $3x(x - 2xy) = 3x^2 - 6x^2y$

(b) $a^2(a - b) - b^2(b - a)$
$\quad = a^3 - a^2b - b^3 + ab^2$

(c) $(4x - 2)(4x - 3)$
$\quad = 4x(4x - 3) - 2(4x - 3)$
$\quad = 16x^2 - 12x - 8x + 6$
$\quad = 16x^2 - 20x + 6$

Special Quadratic Identities: These three identities are so important that they need to be memorised rather than worked out each time.

2

SQUARE OF A SUM:	$(A + B)^2 = A^2 + 2AB + B^2$
SQUARE OF A DIFFERENCE:	$(A - B)^2 = A^2 - 2AB + B^2$
DIFFERENCE OF SQUARES:	$(A + B)(A - B) = A^2 - B^2$

WORKED EXERCISE:

(a) $(4x + 5y)^2 = 16x^2 + 40xy + 25y^2$ (square of a sum)

(b) $\left(t - \dfrac{1}{t}\right)^2 = t^2 - 2 + \dfrac{1}{t^2}$ (square of a difference)

(c) $(x^2 + 3y)(x^2 - 3y) = x^4 - 9y^2$ (difference of squares)

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Exercise 1B

1. Expand:

(a) $4(a+2b)$
(b) $x(x-7)$
(c) $-3(x-2y)$
(d) $-a(a+4)$
(e) $5(a+3b-2c)$
(f) $-3(2x-3y+5z)$
(g) $-2x(x^3-2x^2-3x+1)$
(h) $3xy(2x^2y-5x^3)$
(i) $-2a^2b(a^2b^3-2a^3b)$

2. Expand and simplify:

(a) $3(x-2)-2(x-5)$
(b) $-7(2a-3b+c)-6(-a+4b-2c)$
(c) $x^2(x^3-5x^2+6x-1)-2x(x^4+10x^3-2x^2-7x+3)$
(d) $-2x^3y(3x^2y^4-4xy^5+5y^7)-3xy^2(x^2y^6+2x^4y^3-2x^3y^4)$

3. Expand and simplify:

(a) $(x+2)(x+3)$
(b) $(2a+3)(a+5)$
(c) $(x-4)(x+2)$
(d) $(2b-7)(b-3)$
(e) $(3x+8)(4x-5)$
(f) $(6-7x)(5-6x)$

4. (a) By expanding $(A+B)(A+B)$, prove the special expansion $(A+B)^2 = A^2+2AB+B^2$.
(b) Similarly, prove the special expansions:
(i) $(A-B)^2 = A^2-2AB+B^2$
(ii) $(A-B)(A+B) = A^2-B^2$

5. Expand, using the special expansions:

(a) $(x-y)^2$
(b) $(a+3)^2$
(c) $(n-5)^2$
(d) $(c-2)(c+2)$
(e) $(2a+1)^2$
(f) $(3p-2)^2$
(g) $(3x+4y)(3x-4y)$
(h) $(4y-5x)^2$

6. Multiply:

(a) $a-2b$ by $a+2b$
(b) $2-5x$ by $5+4x$
(c) $4x+7$ by itself
(d) x^2+3y by x^2-4y
(e) $a+b-c$ by $a-b$
(f) $9x^2-3x+1$ by $3x+1$

7. Expand and simplify:

(a) $\left(t+\dfrac{1}{t}\right)^2$
(b) $\left(t-\dfrac{1}{t}\right)^2$
(c) $\left(t+\dfrac{1}{t}\right)\left(t-\dfrac{1}{t}\right)$

DEVELOPMENT

8. (a) Subtract $a(b+c-a)$ from the sum of $b(c+a-b)$ and $c(a+b-c)$.
(b) Subtract the sum of $2x^2-3(x-1)$ and $2x+3(x^2-2)$ from the sum of $5x^2-(x-2)$ and $x^2-2(x+1)$.

9. Simplify: (a) $14-\big(10-(3x-7)-8x\big)$ (b) $4\Big(a-2(b-c)-\big(a-(b-2)\big)\Big)$

10. Use the special expansions to find the value of: (a) 102^2 (b) 999^2 (c) 203×197

11. Expand and simplify:

(a) $(a-b)(a+b)-a(a-2b)$
(b) $(x+2)^2-(x+1)^2$
(c) $(a-3)^2-(a-3)(a+3)$
(d) $(p+q)^2-(p-q)^2$
(e) $(2x+3)(x-1)-(x-2)(x+1)$
(f) $3(a-4)(a-2)-2(a-3)(a-5)$

12. If $X=x-a$ and $Y=2x+a$, find the product of $Y-X$ and $X+3Y$ in terms of x and a.

13. Expand and simplify:

(a) $(x-2)^3$
(b) $(x+y+z)^2-2(xy+yz+zx)$
(c) $(x+y-z)(x-y+z)$
(d) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

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14. Prove the identities:
(a) $(a+b+c)(ab+bc+ca)-abc=(a+b)(b+c)(c+a)$
(b) $(ax+by)^2+(ay-bx)^2+c^2(x^2+y^2)=(x^2+y^2)(a^2+b^2+c^2)$

EXTENSION

15. If $2x=a+b+c$, show that $(x-a)^2+(x-b)^2+(x-c)^2+x^2=a^2+b^2+c^2$.

16. If $(a+b)^2+(b+c)^2+(c+d)^2=4(ab+bc+cd)$, prove that $a=b=c=d$.

1 C Factorisation

Factorisation is the reverse process of expanding brackets, and will be needed on a routine basis throughout the course. The various methods of factorisation are listed systematically, but in every situation common factors should always be taken out first.

3 METHODS OF FACTORISATION:
HIGHEST COMMON FACTOR: Always try this first.
DIFFERENCE OF SQUARES: This involves two terms.
QUADRATICS: This involves three terms.
GROUPING: This involves four or more terms.

Factoring should continue until each factor is *irreducible*, meaning that it cannot be factored further.

Factoring by Highest Common Factor and Difference of Squares: In every situation, look for any common factors of all the terms, and then take out the highest common factor.

WORKED EXERCISE: Factor: (a) $18a^2b^4-30b^3$ (b) $80x^4-5y^4$

SOLUTION:
(a) The highest common factor of $18a^2b^4$ and $30b^3$ is $6b^3$,
so $18a^2b^4-30b^3=6b^3(3a^2b-5)$.

(b) $$\begin{aligned}80x^4-5y^4&=5(16x^4-y^4) &&\text{(highest common factor)}\\&=5(4x^2-y^2)(4x^2+y^2) &&\text{(difference of squares)}\\&=5(2x-y)(2x+y)(4x^2+y^2) &&\text{(difference of squares again)}\end{aligned}$$

Factoring Monic Quadratics: A quadratic is called *monic* if the coefficient of x^2 is 1. Suppose that we want to factor a monic quadratic expression like $x^2-13x+36$. We look for two numbers whose sum is -13 (the coefficient of x) and whose product is 36 (the constant).

WORKED EXERCISE: Factor: (a) $x^2-13x+36$ (b) $a^2+12ac-28c^2$

SOLUTION:
(a) The numbers with sum -13 and product 36 are -9 and -4,
so $x^2-13x+36$
$=(x-9)(x-4)$.

(b) The numbers with sum 12 and product -28 are 14 and -2,
so $a^2+12ac-28c^2$
$=(a+14c)(a-2c)$.

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Factoring Non-monic Quadratics: In a *non-monic* quadratic like $2x^2 + 11x + 12$, where the coefficient of x^2 is not 1, we look for two numbers whose sum is 11 (the coefficient of x), and whose product is 24 (the product of the constant term and the coefficient of x^2).

WORKED EXERCISE: Factor: (a) $2x^2 + 11x + 12$ (b) $6s^2 - 11st - 10t^2$

SOLUTION:

(a) The numbers with sum 11 and product 24 are 8 and 3,
so $2x^2 + 11x + 12$
$$= (2x^2 + 8x) + (3x + 12)$$
$$= 2x(x + 4) + 3(x + 4)$$
$$= (2x + 3)(x + 4).$$

(b) The numbers with sum -11 and product -60 are -15 and 4,
so $6s^2 - 11st - 10t^2$
$$= (6s^2 - 15st) + (4st - 10t^2)$$
$$= 3s(2s - 5t) + 2t(2s - 5t)$$
$$= (3s + 2t)(2s - 5t).$$

Factoring by Grouping: When there are four or more terms, it is sometimes possible to split the expression into groups, factor each group in turn, and then factor the whole expression by taking out a common factor or by some other method.

WORKED EXERCISE: Factor: (a) $12xy - 9x - 16y + 12$ (b) $s^2 - t^2 + s - t$

SOLUTION:

(a) $12xy - 9x - 16y + 12 = 3x(4y - 3) - 4(4y - 3)$
$$= (3x - 4)(4y - 3)$$

(b) $s^2 - t^2 + s - t = (s + t)(s - t) + (s - t)$
$$= (s - t)(s + t + 1)$$

Exercise 1C

1. Write as a product of two factors:

(a) $ax - ay$
(b) $x^2 + 3x$
(c) $3a^2 - 6ab$
(d) $12x^2 + 18x$
(e) $6a^3 + 2a^4 + 4a^5$
(f) $7x^3y - 14x^2y^2 + 21xy^2$

2. Factor by grouping in pairs:

(a) $ax - ay + bx - by$
(b) $a^2 + ab + ac + bc$
(c) $x^2 - 3x - xy + 3y$
(d) $2ax - bx - 2ay + by$
(e) $ab + ac - b - c$
(f) $2x^3 - 6x^2 - ax + 3a$

3. Factor each difference of squares:

(a) $x^2 - 9$
(b) $1 - a^2$
(c) $4x^2 - y^2$
(d) $25x^2 - 16$
(e) $1 - 49k^2$
(f) $81a^2b^2 - 64$

4. Factor each of these quadratic expressions:

(a) $x^2 + 8x + 15$
(b) $x^2 - 4x + 3$
(c) $a^2 + 2a - 8$
(d) $y^2 - 3y - 28$
(e) $c^2 - 12c + 27$
(f) $p^2 + 9p - 36$
(g) $u^2 - 16u - 80$
(h) $x^2 - 20x + 51$
(i) $t^2 + 23t - 50$
(j) $x^2 - 9x - 90$
(k) $x^2 - 5xy + 6y^2$
(l) $x^2 + 6xy + 8y^2$
(m) $a^2 - ab - 6b^2$
(n) $p^2 + 3pq - 40q^2$
(o) $c^2 - 24cd + 143d^2$

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5. Write each quadratic expression as a product of two factors:

(a) $2x^2 + 5x + 2$
(b) $3x^2 + 8x + 4$
(c) $6x^2 - 11x + 3$
(d) $3x^2 + 14x - 5$
(e) $9x^2 - 6x - 8$
(f) $6x^2 - 7x - 3$
(g) $6x^2 - 5x + 1$
(h) $3x^2 + 13x - 30$
(i) $12x^2 - 7x - 12$
(j) $12x^2 + 31x - 15$
(k) $24x^2 - 50x + 25$
(l) $2x^2 + xy - y^2$
(m) $4a^2 - 8ab + 3b^2$
(n) $6p^2 + 5pq - 4q^2$
(o) $18u^2 - 19uv - 12v^2$

6. Write each expression as a product of three factors:

(a) $3a^2 - 12$
(b) $x^4 - y^4$
(c) $x^3 - x$
(d) $5x^2 - 5x - 30$
(e) $25y - y^3$
(f) $16 - a^4$
(g) $4x^2 + 14x - 30$
(h) $x^3 - 8x^2 + 7x$
(i) $x^4 - 3x^2 - 4$
(j) $ax^2 - a - 2x^2 + 2$
(k) $16m^3 - mn^2$
(l) $ax^2 - a^2x - 20a^3$

DEVELOPMENT

7. Factor as fully as possible:

(a) $72 + x - x^2$
(b) $(a - b)^2 - c^2$
(c) $a^3 - 10a^2b + 24ab^2$
(d) $a^2 - b^2 - a + b$
(e) $x^4 - 256$
(f) $4p^2 - (q + r)^2$
(g) $6x^4 - x^3 - 2x^2$
(h) $a^2 - bc - b + a^2c$
(i) $9x^2 + 36x - 45$
(j) $4x^4 - 37x^2 + 9$
(k) $x^2y^2 - 13xy - 48$
(l) $x(x - y)^2 - xz^2$
(m) $20 - 9x - 20x^2$
(n) $4x^3 - 12x^2 - x + 3$
(o) $12x^2 - 8xy - 15y^2$
(p) $x^2 + 2ax + a^2 - b^2$
(q) $9x^2 - 18x - 315$
(r) $x^4 - x^2 - 2x - 1$
(s) $10x^3 - 13x^2y - 9xy^2$
(t) $x^2 + 4xy + 4y^2 - a^2 + 2ab - b^2$
(u) $(x + y)^2 - (x - y)^2$

EXTENSION

8. Factor fully:

(a) $a^2 + b(b + 1)a + b^3$
(b) $a(b + c - d) - c(a - b + d)$
(c) $(a^2 - b^2)^2 - (a - b)^4$
(d) $4x^4 - 2x^3y - 3xy^3 - 9y^4$
(e) $(x^2 + xy)^2 - (xy + y^2)^2$
(f) $(a^2 - b^2 - c^2)^2 - 4b^2c^2$
(g) $(ax + by)^2 + (ay - bx)^2 + c^2(x^2 + y^2)$
(h) $x^2 + (a - b)xy - aby^2$
(i) $a^4 + a^2b^2 + b^4$
(j) $a^4 + 4b^4$

1 D Algebraic Fractions

An *algebraic fraction* is a fraction containing pronumerals. They are manipulated in the same way as arithmetic fractions, and factorisation plays a major role.

Addition and Subtraction of Algebraic Fractions: A common denominator is required, but finding the lowest common denominator can involve factoring all the denominators.

4 ADDITION AND SUBTRACTION OF ALGEBRAIC FRACTIONS: First factor all denominators. Then work with the lowest common denominator.

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WORKED EXERCISE:

(a) $\dfrac{1}{x-4} - \dfrac{1}{x}$
$$= \frac{x-(x-4)}{x(x-4)}$$
$$= \frac{4}{x(x-4)}$$

(b) $\dfrac{2}{x^2-x} - \dfrac{5}{x^2-1} = \dfrac{2}{x(x-1)} - \dfrac{5}{(x-1)(x+1)}$
$$= \frac{2(x+1)-5x}{x(x-1)(x+1)}$$
$$= \frac{2-3x}{x(x-1)(x+1)}$$

Multiplication and Division of Algebraic Fractions: The key step here is to factor all numerators and denominators completely before cancelling factors.

5 MULTIPLICATION AND DIVISION OF ALGEBRAIC FRACTIONS: First factor all numerators and denominators completely. Then cancel common factors.

To divide by an algebraic fraction, multiply by its reciprocal in the usual way.

WORKED EXERCISE:

(a) $\dfrac{2a}{9-a^2} \times \dfrac{a-3}{a^3+a} = \dfrac{2a}{(3-a)(3+a)} \times \dfrac{a-3}{a(a^2+1)}$
$$= -\frac{2}{(a+3)(a^2+1)}$$

(b) $\dfrac{6abc}{ab+bc} \div \dfrac{6ac}{a^2+2ac+c^2} = \dfrac{6abc}{b(a+c)} \times \dfrac{(a+c)^2}{6ac}$
$$= a+c$$

Simplifying Compound Fractions: A *compound fraction* is a fraction in which either the numerator or the denominator is itself a fraction.

6 SIMPLIFYING COMPOUND FRACTIONS: Multiply top and bottom by something that will clear fractions from numerator and denominator together.

WORKED EXERCISE:

(a) $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{4}+\frac{1}{6}} = \dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{4}+\frac{1}{6}} \times \dfrac{12}{12}$
$$= \frac{6-4}{3+2}$$
$$= \tfrac{2}{5}$$

(b) $\dfrac{\dfrac{1}{t}+\dfrac{1}{t+1}}{\dfrac{1}{t}-\dfrac{1}{t+1}} = \dfrac{\dfrac{1}{t}+\dfrac{1}{t+1}}{\dfrac{1}{t}-\dfrac{1}{t+1}} \times \dfrac{t(t+1)}{t(t+1)}$
$$= \frac{(t+1)+t}{(t+1)-t}$$
$$= 2t+1$$

Exercise 1D

1. Simplify:

(a) $\dfrac{x}{2x}$ (c) $\dfrac{3x^2}{9xy}$ (e) $\dfrac{12xy^2z}{15x^2yz^2}$

(b) $\dfrac{a}{a^2}$ (d) $\dfrac{12ab}{4a^2b}$ (f) $\dfrac{uvw^2}{u^3v^2w}$

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2. Simplify:

(a) $\frac{x}{3} \times \frac{3}{x}$ (b) $\frac{a}{4} \div \frac{a}{2}$ (c) $x \times \frac{3}{x^2}$

(d) $\frac{a^2}{2b} \times \frac{b^2}{a^2}$ (e) $\frac{3x^2}{4y^2} \times \frac{2y}{x}$ (f) $\frac{x^2}{3ay^3} \div \frac{x^2}{3ay^3}$

(g) $\frac{5}{a} \div 10$ (h) $\frac{2ab}{3c} \times \frac{c^2}{ab^2}$ (i) $\frac{8a^3b}{5} \div \frac{4ab}{15}$

(j) $\frac{2a}{3b} \times \frac{5c^2}{2a^2b} \times \frac{3b^2}{2c}$ (k) $\frac{12x^2yz}{8xy^3} \times \frac{24xy^2}{36yz^2}$ (l) $\frac{3a^2b}{4b^3c} \times \frac{2c^2}{8a^3} \div \frac{6ac}{16b^2}$

3. Write as a single fraction:

(a) $\frac{x}{2} + \frac{x}{5}$ (b) $\frac{a}{3} - \frac{a}{6}$ (c) $\frac{x}{8} - \frac{y}{12}$

(d) $\frac{2a}{3} + \frac{3a}{2}$ (e) $\frac{7b}{10} - \frac{19b}{30}$ (f) $\frac{xy}{30} - \frac{xy}{18}$

(g) $\frac{1}{x} + \frac{1}{2x}$ (h) $\frac{3}{4x} + \frac{4}{3x}$ (i) $\frac{1}{a} - \frac{1}{b}$

(j) $x + \frac{1}{x}$ (k) $a + \frac{b}{a}$ (l) $\frac{1}{2x} - \frac{1}{x^2}$

4. Simplify:

(a) $\frac{x+1}{2} + \frac{x+2}{3}$ (b) $\frac{2x-1}{5} - \frac{x+3}{2}$ (c) $\frac{2x+1}{3} - \frac{x-5}{6} + \frac{x+4}{4}$ (d) $\frac{3x-7}{5} + \frac{4x+3}{2} - \frac{2x-5}{10}$

(e) $\frac{x-5}{3x} - \frac{x-3}{5x}$ (f) $\frac{1}{x} - \frac{1}{x+1}$ (g) $\frac{1}{x+1} - \frac{1}{x+1}$ (h) $\frac{2}{x-3} + \frac{3}{x-2}$

(i) $\frac{2}{x+3} - \frac{2}{x-2}$ (j) $\frac{x}{x+y} + \frac{y}{x-y}$ (k) $\frac{a}{x+a} - \frac{b}{x+b}$ (l) $\frac{x}{x-1} - \frac{x}{x+1}$

5. Factor where possible and then simplify:

(a) $\frac{a}{ax+ay}$ (b) $\frac{3a^2-6ab}{2a^2b-4ab^2}$ (c) $\frac{x^2+2x}{x^2-4}$

(d) $\frac{a^2-9}{a^2+a-12}$ (e) $\frac{x^2+2xy+y^2}{x^2-y^2}$ (f) $\frac{x^2+10x+25}{x^2+9x+20}$

(g) $\frac{ac+ad+bc+bd}{a^2+ab}$ (h) $\frac{y^2-8y+15}{2y^2-5y-3}$ (i) $\frac{9ax+6bx-6ay-4by}{9x^2-4y^2}$

6. Simplify:

(a) $\frac{3x+3}{2x} \times \frac{x^2}{x^2-1}$ (b) $\frac{a^2+a-2}{a+2} \times \frac{a^2-3a}{a^2-4a+3}$ (c) $\frac{c^2+5c+6}{c^2-16} \div \frac{c+3}{c-4}$

(d) $\frac{x^2-x-20}{x^2-25} \times \frac{x^2-x-2}{x^2+2x-8} \div \frac{x+1}{x^2+5x}$ (e) $\frac{ax+bx-2a-2b}{3x^2-5x-2} \times \frac{9x^2-1}{a^2+2ab+b^2}$ (f) $\frac{2x^2+x-15}{x^2+3x-28} \div \frac{x^2+6x+9}{x^2-4x} \div \frac{6x^2-15x}{x^2-49}$

7. Simplify:

(a) $\frac{1}{x^2+x} + \frac{1}{x^2-x}$ (b) $\frac{1}{x^2-4} + \frac{1}{x^2-4x+4}$ (c) $\frac{1}{x-y} + \frac{2x-y}{x^2-y^2}$

(d) $\frac{3}{x^2+2x-8} - \frac{2}{x^2+x-6}$ (e) $\frac{x}{a^2-b^2} - \frac{x}{a^2+ab}$ (f) $\frac{1}{x^2-4x+3} + \frac{1}{x^2-5x+6} - \frac{1}{x^2-3x+2}$

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8. Simplify:

(a) $\dfrac{b-a}{a-b}$

(b) $\dfrac{v^2-u^2}{u-v}$

(c) $\dfrac{x^2-5x+6}{2-x}$

(d) $\dfrac{1}{a-b}-\dfrac{1}{b-a}$

(e) $\dfrac{m}{m-n}+\dfrac{n}{n-m}$

(f) $\dfrac{x-y}{y^2+xy-2x^2}$

DEVELOPMENT

9. Study the worked exercise on compound fractions and then simplify:

(a) $\dfrac{1-\frac{1}{2}}{1+\frac{1}{2}}$

(b) $\dfrac{2+\frac{1}{3}}{5-\frac{2}{3}}$

(c) $\dfrac{\frac{1}{2}-\frac{1}{5}}{1+\frac{1}{10}}$

(d) $\dfrac{\frac{17}{20}-\frac{3}{4}}{\frac{4}{5}-\frac{3}{10}}$

(e) $\dfrac{\frac{1}{x}}{1+\frac{2}{x}}$

(f) $\dfrac{t-\frac{1}{t}}{t+\frac{1}{t}}$

(g) $\dfrac{1}{\frac{1}{b}+\frac{1}{a}}$

(h) $\dfrac{\frac{x}{y}+\frac{y}{x}}{\frac{x}{y}-\frac{y}{x}}$

(i) $\dfrac{1-\frac{1}{x+1}}{\frac{1}{x}+\frac{1}{x+1}}$

(j) $\dfrac{\frac{3}{x+2}-\frac{2}{x+1}}{\frac{5}{x+2}-\frac{4}{x+1}}$

10. If $x=\dfrac{1}{\lambda}$ and $y=\dfrac{1}{1-x}$ and $z=\dfrac{y}{y-1}$, show that $z=\lambda$.

11. Simplify:

(a) $\dfrac{x^4-y^4}{x^2-2xy+y^2}\div\dfrac{x^2+y^2}{x-y}$

(b) $\dfrac{8x^2+14x+3}{8x^2-10x+3}\times\dfrac{12x^2-6x}{4x^2+5x+1}\div\dfrac{18x^2-6x}{4x^2+x-3}$

(c) $\dfrac{(a-b)^2-c^2}{ab-b^2-bc}\times\dfrac{c}{a^2+ab-ac}\div\dfrac{ac-bc+c^2}{a^2-(b-c)^2}$

(d) $\dfrac{x-y}{x}+\dfrac{x^3+y^3}{xy^2}-\dfrac{x^2+y^2}{x^2}$

(e) $\dfrac{x+4}{x-4}-\dfrac{x-4}{x+4}$

(f) $\dfrac{4y}{x^2+2xy}-\dfrac{3x}{xy+2y^2}+\dfrac{3x-2y}{xy}$

(g) $\dfrac{8x}{x^2+5x+6}-\dfrac{5x}{x^2+3x+2}-\dfrac{3x}{x^2+4x+3}$

(h) $\dfrac{1}{x-1}+\dfrac{2}{x+1}-\dfrac{3x-2}{x^2-1}-\dfrac{1}{x^2+2x+1}$

12. (a) Expand $\left(x+\dfrac{1}{x}\right)^2$.

(b) Suppose that $x+\dfrac{1}{x}=3$. Use part (a) to evaluate $x^2+\dfrac{1}{x^2}$ without attempting to find the value of x.

EXTENSION

13. Simplify these algebraic fractions:

(a) $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}$

(b) $\left(1+\dfrac{45}{x-8}-\dfrac{26}{x-6}\right)\left(3-\dfrac{65}{x+7}+\dfrac{8}{x-2}\right)$

(c) $\left(2-\dfrac{3n}{m}+\dfrac{9n^2-2m^2}{m^2+2mn}\right)\div\left(\dfrac{1}{m}-\dfrac{1}{m-2n-\dfrac{4n^2}{m+n}}\right)$

(d) $\dfrac{1}{x+\dfrac{1}{x+2}}\times\dfrac{1}{x+\dfrac{1}{x-2}}\div\dfrac{x-\dfrac{4}{x}}{x^2-2+\dfrac{1}{x^2}}$

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1 E Four Cubic Identities

The three special quadratic identities will be generalised later to any degree. For now, here are the cubic versions of them. They will be new to most people.

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CUBE OF A SUM:	$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
CUBE OF A DIFFERENCE:	$(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$
DIFFERENCE OF CUBES:	$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$
SUM OF CUBES:	$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

The proofs of these identities are left to the first two questions in the following exercise.

WORKED EXERCISE: Here is an example of each identity.

(a) $(x+5)^3 = x^3 + 15x^2 + 75x + 125$

(b) $(2x-3y)^3 = 8x^3 - 36x^2y + 54xy^2 - 27y^3$

(c) $x^3 - 8 = (x-2)(x^2 + 2x + 4)$

(d) $4^3 + 5^3 = (4+5)(16 - 20 + 25) = 9 \times 21 = 3^3 \times 7$

WORKED EXERCISE: (a) Simplify $\dfrac{a^3+1}{a+1}$. (b) Factor $a^3 - b^3 + a - b$.

SOLUTION:

(a) $$\frac{a^3+1}{a+1} = \frac{(a+1)(a^2-a+1)}{a+1} = a^2 - a + 1$$

(b) $$\begin{aligned} &a^3 - b^3 + a - b \\ &= (a-b)(a^2+ab+b^2) + (a-b) \\ &= (a-b)(a^2+ab+b^2+1) \end{aligned}$$

Exercise 1E

1. (a) Prove the factorisation $A^3 - B^3 = (A-B)(A^2 + AB + B^2)$ by expanding the RHS.
(b) Similarly, prove the factorisation $A^3 + B^3 = (A+B)(A^2 - AB + B^2)$.

2. (a) Prove the identity $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ by writing $(A+B)^3 = (A+B)(A^2 + 2AB + B^2)$ and expanding.
(b) Similarly, prove the identity $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$.

3. Expand:
(a) $(a+b)^3$ (b) $(x-y)^3$ (c) $(b-1)^3$ (d) $(p+2)^3$ (e) $(1-c)^3$ (f) $(t-3)^3$ (g) $(2x+5y)^3$ (h) $(3a-4b)^3$

4. Factor:
(a) $x^3 + y^3$ (b) $a^3 - b^3$ (c) $y^3 + 1$ (d) $g^3 - 1$ (e) $b^3 - 8$ (f) $8c^3 + 1$ (g) $27 - t^3$ (h) $125 + a^3$ (i) $27h^3 - 1$ (j) $u^3 - 64v^3$ (k) $a^3b^3c^3 + 1000$ (l) $216x^3 + 125y^3$

5. Write as a product of three factors:
(a) $2x^3 + 16$ (b) $a^4 - ab^3$ (c) $24t^3 + 81$ (d) $x^3y - 125y$ (e) $250p^3 - 432q^3$ (f) $27x^4 + 1000xy^3$ (g) $5x^3y^3 - 5$ (h) $x^6 + x^3y^3$

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6. Simplify:

(a) $\dfrac{x^3-1}{x^2-1}$

(b) $\dfrac{a^2-3a-10}{a^3+8}$

(c) $\dfrac{a^3+1}{6a^2}\times\dfrac{3a}{a^2+a}$

(d) $\dfrac{x^2-9}{x^4-27x}\div\dfrac{x+3}{x^2+3x+9}$

7. Simplify:

(a) $\dfrac{3}{a-2}-\dfrac{3a}{a^2+2a+4}$

(b) $\dfrac{1}{x^3-1}+\dfrac{x+1}{x^2+x+1}$

(c) $\dfrac{1}{x^2-2x-8}-\dfrac{1}{x^3+8}$

(d) $\dfrac{a^2}{a^3+b^3}+\dfrac{a-b}{a^2-ab+b^2}+\dfrac{1}{a+b}$

DEVELOPMENT

8. Factor as fully as possible:

(a) a^3+b^3+a+b

(b) x^6-64

(c) $2a^4-3a^3+16a-24$

(d) $(x+y)^3-(x-y)^3$

(e) $s^3-t^3+s^2-t^2$

(f) $(t-2)^3+(t+2)^3$

(g) $(a-2b)^3+(2a-b)^3$

(h) x^6-7x^3-8

(i) u^7+u^6+u+1

(j) $2+x^3-3x^6$

(k) $x^7-x^3+8x^4-8$

(l) $a^5+a^4+a^3+a^2+a+1$

9. Simplify:

(a) $\dfrac{6a^2+6}{a^2+a+1}\times\dfrac{a^3-1}{a^3-3a^2}\times\dfrac{a^3+a^2}{a^4-1}$

(b) $\dfrac{x^4-8x}{x^2-4x-5}\times\dfrac{x^2+2x+1}{x^3-x^2-2x}\div\dfrac{x^2+2x+4}{x-5}$

(c) $\dfrac{(a+1)^3-(a-1)^3}{3a^3+a}$

(d) $\dfrac{1}{x-3}-\dfrac{8x}{x^3-27}-\dfrac{x-3}{x^2+3x+9}$

(e) $\dfrac{3x^2+2x+4}{x^3-1}-\dfrac{x+1}{x^2+x+1}-\dfrac{2}{x-1}$

(f) $\dfrac{1+x+x^2}{1-x^3}+\dfrac{x-x^2}{(1-x)^3}$

EXTENSION

10. Find the four quartic identities that correspond to the cubic identities in this exercise. That is, find the expansions of $(A+B)^4$ and $(A-B)^4$ and find factorisations of A^4+B^4 and A^4-B^4.

11. Factor as fully as possible: (a) x^7+x (b) $x^{12}-y^{12}$

12. If $x+y=1$ and $x^3+y^3=19$, find the value of x^2+y^2.

13. Simplify $(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x+y)^2(x-y)$.

14. If $a+b+c=0$, show that $(2a-b)^3+(2b-c)^3+(2c-a)^3=3(2a-b)(2b-c)(2c-a)$.

15. Simplify $\dfrac{a^4-b^4}{a^2-2ab+b^2}\div\dfrac{a^2b+b^3}{a^3-b^3}\times\dfrac{a^2b-ab^2+b^3}{a^4+a^2b^2+b^4}$.

16. Simplify $(1+a)^2\div\left(1+\dfrac{a}{1-a+\dfrac{a}{1+a+a^2}}\right)$.

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1 F Linear Equations and Inequations

The rules for solving equations and for solving inequations are the same, except for a qualification about multiplying or dividing an inequation by a negative:

8 LINEAR EQUATIONS: Any number can be added to or subtracted from both sides. Both sides can be multiplied or divided by any nonzero number.

LINEAR INEQUATIONS: The rules for inequations are the same as those for equations, except that when both sides are multiplied or divided by a negative number, the inequality sign is reversed.

WORKED EXERCISE: Solve: (a) $\dfrac{4-7x}{4x-7} = 1$ (b) $x - 12 < 5 + 3x$

SOLUTION:

(a)
$$\frac{4-7x}{4x-7} = 1$$
$\boxed{\times\,(4x-7)}$ $4 - 7x = 4x - 7$

$\boxed{+\,7x}$ $4 = 11x - 7$

$\boxed{+\,7}$ $11 = 11x$

$\boxed{\div\,11}$ $x = 1$

(b)
$$x - 12 < 5 + 3x$$
$\boxed{-\,3x}$ $-2x - 12 < 5$

$\boxed{+\,12}$ $-2x < 17$

$\boxed{\div\,(-2)}$ $x > -8\tfrac{1}{2}$

Because of the division by the negative, the inequality was reversed.

Changing the Subject of a Formula: Similar sequences of operations allow the subject of a formula to be changed from one pronumeral to another.

WORKED EXERCISE: Given the formula $y = \dfrac{x+1}{x+a}$:

(a) change the subject to a, (b) change the subject to x.

SOLUTION:

(a)
$$y = \frac{x+1}{x+a}$$
$\boxed{\times\,(x+a)}$ $xy + ay = x + 1$

$\boxed{-\,xy}$ $ay = x + 1 - xy$

$\boxed{\div\,y}$ $a = \dfrac{x + 1 - xy}{y}$

(b)
$$y = \frac{x+1}{x+a}$$
$\boxed{\times\,(x+a)}$ $xy + ay = x + 1$

$xy - x = 1 - ay$

$x(y-1) = 1 - ay$

$\boxed{\div\,(y-1)}$ $x = \dfrac{1-ay}{y-1}$

Exercise 1F

1. Solve:

(a) $-2x = -20$
(b) $3x > 2$
(c) $-a = 5$
(d) $\dfrac{x}{-4} \le -1$
(e) $-1 - x = 0$
(f) $0{\cdot}1y = 5$
(g) $2t < t$
(h) $-\tfrac{1}{2}x = 8$

2. Solve:

(a) $3x - 5 = 22$
(b) $4x + 7 \ge -13$
(c) $1 - 2x < 9$
(d) $6x = 3x - 21$
(e) $-13 \le 5a - 6$
(f) $-2 > 4 + \dfrac{t}{5}$
(g) $19 = 3 - 7y$
(h) $23 - \dfrac{u}{3} \ge 7$

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3. Solve:

(a) $5x - 2 < 2x + 10$
(b) $5 - x = 27 + x$
(c) $16 + 9a > 10 - 3a$
(d) $13y - 21 \le 20y - 35$
(e) $13 - 12x \ge 6 - 3x$
(f) $3(x + 7) = -2(x - 9)$
(g) $8 + 4(2 - x) > 3 - 2(5 - x)$
(h) $7x - (3x + 11) = 6 - (15 - 9x)$
(i) $4(x + 2) = 4x + 9$
(j) $3(x - 1) < 2(x + 1) + x$
(k) $(x - 3)(x + 6) \le (x - 4)(x - 5)$
(l) $(1 + 2x)(4 + 3x) = (2 - x)(5 - 6x)$
(m) $(x + 3)^2 > (x - 1)^2$
(n) $(2x - 5)(2x + 5) = (2x - 3)^2$

4. Solve:

(a) $\dfrac{x}{8} = \dfrac{1}{2}$
(b) $\dfrac{a}{12} = \dfrac{2}{3}$
(c) $\dfrac{y}{20} < \dfrac{4}{5}$
(d) $\dfrac{1}{x} = 3$
(e) $\dfrac{2}{a} = 5$
(f) $3 = \dfrac{9}{2y}$
(g) $\dfrac{2x + 1}{5} \ge -3$
(h) $\dfrac{5a}{3} - 1 \ge \dfrac{3a}{5} + 1$
(i) $\dfrac{7 - 4x}{6} < 1$
(j) $\dfrac{5 + a}{a} = -3$
(k) $\dfrac{9 - 2t}{t} = 13$
(l) $6 - \dfrac{c}{3} > c$
(m) $\dfrac{1}{a} + 4 = 1 - \dfrac{2}{a}$
(n) $\dfrac{4}{x - 1} = -5$
(o) $\dfrac{3x}{1 - 2x} = 7$
(p) $\dfrac{11t}{8t + 13} = -2$

5. Solve:

(a) $\dfrac{x}{3} - \dfrac{x}{5} \ge 2$
(b) $\dfrac{a}{10} - \dfrac{a}{6} < 1$
(c) $\dfrac{x}{6} + \dfrac{2}{3} = \dfrac{x}{2} - \dfrac{5}{6}$
(d) $\dfrac{1}{x} - 3 = \dfrac{1}{2x}$
(e) $\dfrac{1}{2x} - \dfrac{2}{3} = 1 - \dfrac{1}{3x}$
(f) $\dfrac{x}{3} - 2 < \dfrac{x}{2} - 3$
(g) $\dfrac{x - 2}{3} > \dfrac{x + 4}{4}$
(h) $\dfrac{3}{x - 2} = \dfrac{4}{2x + 5}$
(i) $\dfrac{x + 1}{x + 2} = \dfrac{x - 3}{x + 1}$
(j) $\dfrac{(3x - 2)(3x + 2)}{(3x - 1)^2} = 1$
(k) $\dfrac{a + 5}{2} - \dfrac{a - 1}{3} > 1$
(l) $\dfrac{3}{4} - \dfrac{x + 1}{12} \le \dfrac{2}{3} - \dfrac{x - 1}{6}$
(m) $\dfrac{2x}{5} + \dfrac{2 - 3x}{4} < \dfrac{3}{10} - \dfrac{3 - 5x}{2}$
(n) $\frac{3}{4}(x - 1) - \frac{1}{2}(3x + 2) = 0$
(o) $\dfrac{4x + 1}{6} - \dfrac{2x - 1}{15} = \dfrac{3x - 5}{5} - \dfrac{6x + 1}{10}$
(p) $\dfrac{7(1 - x)}{12} - \dfrac{3 + 2x}{9} \ge \dfrac{5(2 + x)}{6} - \dfrac{4 - 5x}{18}$

6. (a) If $v = u + at$, find a when $t = 4$, $v = 20$ and $u = 8$.

(b) Given that $v^2 = u^2 + 2as$, find the value of s when $u = 6$, $v = 10$ and $a = 2$.

(c) Suppose that $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{t}$. Find v, given that $u = -1$ and $t = 2$.

(d) If $S = -15$, $n = 10$ and $a = -24$, find ℓ, given that $S = \frac{n}{2}(a + \ell)$.

(e) Temperatures in degrees Fahrenheit and degrees Celsius are related by the formula $F = \frac{9}{5}C + 32$. Find the value of C that corresponds to $F = 95$.

(f) Suppose that the variables c and d are related by the formula $\dfrac{3}{c + 1} = \dfrac{5}{d - 1}$. Find c when $d = -2$.

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DEVELOPMENT

7. Solve each of the following inequations for the given domain of the variable, and graph each solution on the real number line:

(a) $2x - 3 < 5$, where x is a positive integer.
(b) $1 - 3x \le 16$, where x is a negative integer.
(c) $4x + 5 > 2x - 3$, where x is a real number.
(d) $7 - 2x \ge x + 1$, where x is a real number.
(e) $4 \le 2x < 14$, where x is an integer.
(f) $-12 < 3x < 9$, where x is an integer.
(g) $1 < 2x + 1 \le 11$, where x is a real number.
(h) $-10 \le 2 - 3x \le -1$, where x is a real number.

8. Solve each of these problems by constructing and then solving a linear equation:

(a) Five more than twice a certain number is one more than the number itself. What is the number?
(b) I have \$175 in my wallet, consisting of \$10 and \$5 notes. If I have twice as many \$10 notes as \$5 notes, how many \$5 notes do I have?
(c) My father is 24 years older than me, and 12 years ago he was double my age. How old am I now?
(d) The fuel tank in my new car was 40% full. I added 28 litres and then found that it was 75% full. How much fuel does the tank hold?
(e) A certain tank has an inlet valve and an outlet valve. The tank can be filled via the inlet valve in 6 minutes and emptied (from full) via the outlet valve in 10 minutes. If both valves are operating, how long would it take to fill the tank if it was empty to begin with?
(f) A basketball player has scored 312 points in 15 games. How many points must he average per game in his next 3 games to take his overall average to 20 points per game?
(g) A cyclist rides for 5 hours at a certain speed and then for 4 hours at a speed 6 km/h greater than her original speed. If she rides 294 km altogether, what was her initial speed?
(h) Two trains travel at speeds of 72 km/h and 48 km/h respectively. If they start at the same time and travel towards each other from two places 600 km apart, how long will it be before they meet?

9. Rearrange each formula so that the pronumeral written in the brackets is the subject:

(a) $a = bc - d$ $[b]$
(b) $t = a + (n-1)d$ $[n]$
(c) $\dfrac{p}{q+r} = t$ $[r]$
(d) $u = 1 + \dfrac{3}{v}$ $[v]$
(e) $\dfrac{a}{2} - \dfrac{b}{3} = a$ $[a]$
(f) $\dfrac{1}{f} + \dfrac{2}{g} = \dfrac{5}{h}$ $[g]$
(g) $x = \dfrac{y}{y+2}$ $[y]$
(h) $a = \dfrac{b+5}{b-4}$ $[b]$
(i) $c = \dfrac{7+2d}{5-3d}$ $[d]$
(j) $u = \dfrac{v+w-1}{v-w+1}$ $[v]$

10. Solve: (a) $\dfrac{x}{x-2} + \dfrac{3}{x-4} = 1$ (b) $\dfrac{3a-2}{2a-3} - \dfrac{a+17}{a+10} = \dfrac{1}{2}$

EXTENSION

11. (a) Show that $\dfrac{x-1}{x-3} = 1 + \dfrac{2}{x-3}$.

(b) Hence solve $\dfrac{x-1}{x-3} - \dfrac{x-3}{x-5} = \dfrac{x-5}{x-7} - \dfrac{x-7}{x-9}$.

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1 G Quadratic Equations

This section reviews the solution of quadratic equations by factorisation and by the quadratic formula. The third method, completing the square, will be reviewed in Section 1I.

Solving a Quadratic by Factorisation: This method is the simplest, but it normally only works when the roots are rational numbers.

9 SOLVING A QUADRATIC BY FACTORING:
1. Get all the terms on the left, then factor the left-hand side.
2. Use the principle that if $AB = 0$, then $A = 0$ or $B = 0$.

WORKED EXERCISE: Solve $5x^2 + 34x - 7 = 0$.

SOLUTION:

$$
\begin{aligned}
5x^2 + 34x - 7 &= 0 \\
(5x - 1)(x + 7) &= 0 \quad \text{(factoring the LHS)} \\
5x - 1 = 0 \ \text{or} \ x + 7 &= 0 \quad \text{(one of the factors must be zero)} \\
x = \tfrac{1}{5} \ \text{or} \ x &= -7
\end{aligned}
$$

Solving a Quadratic by the Formula: This method works whether the solutions are rational numbers or involve surds.

10 THE QUADRATIC FORMULA: The solution of $ax^2 + bx + c = 0$ is

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Always calculate $b^2 - 4ac$ first.

The formula is proven by completing the square, as discussed in Chapter Eight.

WORKED EXERCISE: Use the quadratic formula to solve:

(a) $5x^2 + 2x - 7 = 0$ (b) $3x^2 + 4x - 1 = 0$

SOLUTION:

(a) $5x^2 + 2x - 7 = 0$

Here
$$
\begin{aligned}
b^2 - 4ac &= 2^2 + 140 \\
&= 144 \\
&= 12^2,
\end{aligned}
$$
so
$$
\begin{aligned}
x &= \frac{-2 + 12}{10} \ \text{or} \ \frac{-2 - 12}{10} \\
&= 1 \ \text{or} \ -1\tfrac{2}{5}.
\end{aligned}
$$

(b) $3x^2 + 4x - 1 = 0$

Here
$$
\begin{aligned}
b^2 - 4ac &= 4^2 + 12 \\
&= 28 \\
&= 4 \times 7,
\end{aligned}
$$
so
$$
\begin{aligned}
x &= \frac{-4 + 2\sqrt{7}}{6} \ \text{or} \ \frac{-4 - 2\sqrt{7}}{6} \\
&= \tfrac{1}{3}(-2 + \sqrt{7}) \ \text{or} \ \tfrac{1}{3}(-2 - \sqrt{7}).
\end{aligned}
$$

Exercise 1G

1. Solve:

(a) $x^2 = 9$
(b) $a^2 - 4 = 0$
(c) $1 - t^2 = 0$
(d) $x^2 = \frac{9}{4}$
(e) $4x^2 - 1 = 0$
(f) $25y^2 = 16$

2. Solve by factoring:

(a) $x^2 - 5x = 0$
(b) $c^2 + 2c = 0$
(c) $t^2 = t$
(d) $3a = a^2$
(e) $2b^2 - b = 0$
(f) $3u^2 + u = 0$
(g) $3y^2 = 2y$
(h) $12u + 5u^2 = 0$

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3. Solve by factoring:

(a) $x^2 - 3x + 2 = 0$ (b) $x^2 + 6x + 8 = 0$ (c) $a^2 + 2a - 15 = 0$ (d) $y^2 + 4y = 5$ (e) $p^2 = p + 6$ (f) $a^2 = a + 132$ (g) $c^2 + 18 = 9c$ (h) $8t + 20 = t^2$ (i) $u^2 + u = 56$ (j) $50 + 27h + h^2 = 0$ (k) $k^2 = 60 + 11k$ (l) $\alpha^2 + 20\alpha = 44$

4. Solve by factoring:

(a) $3a^2 - 7a + 2 = 0$ (b) $2x^2 + 11x + 5 = 0$ (c) $3b^2 - 4b - 4 = 0$ (d) $2y^2 + 5y = 12$ (e) $5x^2 - 26x + 5 = 0$ (f) $4t^2 + 9 = 15t$ (g) $t + 15 = 2t^2$ (h) $10u^2 + 3u - 4 = 0$ (i) $25x^2 + 9 = 30x$ (j) $6x^2 + 13x + 6 = 0$ (k) $12b^2 + 3 + 20b = 0$ (l) $6k^2 + 13k = 8$

5. Solve using the quadratic formula, giving exact answers followed by approximations to four significant figures where appropriate:

(a) $x^2 - x - 1 = 0$ (b) $y^2 + y = 3$ (c) $a^2 + 12 = 7a$ (d) $u^2 + 2u - 2 = 0$ (e) $c^2 - 6c + 2 = 0$ (f) $4x^2 + 4x + 1 = 0$ (g) $2a^2 + 1 = 4a$ (h) $5x^2 + 13x - 6 = 0$ (i) $2b^2 + 3b = 1$ (j) $3c^2 = 4c + 3$ (k) $4t^2 = 2t + 1$ (l) $x^2 + x + 1 = 0$

6. Solve by factoring:

(a) $x = \dfrac{x+2}{x}$ (b) $a + \dfrac{10}{a} = 7$ (c) $y + \dfrac{2}{y} = \dfrac{9}{2}$ (d) $(5b - 3)(3b + 1) = 1$ (e) $\dfrac{5k+7}{k-1} = 3k + 2$ (f) $\dfrac{u+3}{2u-7} = \dfrac{2u-1}{u-3}$

7. Find the exact solutions of:

(a) $x = \dfrac{1}{x} + 2$ (b) $\dfrac{4x-1}{x} = x$ (c) $a = \dfrac{a+4}{a-1}$ (d) $\dfrac{5m}{2} = 2 + \dfrac{1}{m}$ (e) $\dfrac{y+1}{y+2} = \dfrac{3-y}{y-4}$ (f) $2(k-1) = \dfrac{4-5k}{k+1}$

8. (a) If $y = px - ap^2$, find p, given that $a = 2$, $x = 3$ and $y = 1$.

(b) Given that $(x - a)(x - b) = c$, find x when $a = -2$, $b = 4$ and $c = 7$.

(c) Suppose that $S = \dfrac{n}{2}\big(2a + (n-1)d\big)$. Find the positive value of n if $S = 80$, $a = 4$ and $d = 6$.

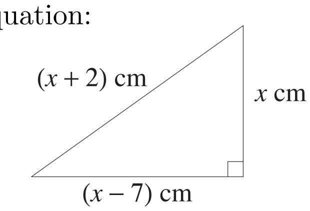
9. Find a in terms of b if: (a) $a^2 - 5ab + 6b^2 = 0$ (b) $3a^2 + 5ab - 2b^2 = 0$

10. Find y in terms of x if: (a) $4x^2 - y^2 = 0$ (b) $x^2 - 9xy - 22y^2 = 0$

DEVELOPMENT

11. Solve each problem by forming and solving a suitable quadratic equation:

(a) Find the value of x in the diagram opposite.



(b) Find a positive integer which when increased by 30 is 12 less than its square.

(c) Two positive numbers differ by 3 and the sum of their squares is 117. Find the numbers.

(d) A rectangular area can be completely tiled with 200 square tiles. If the side length of each tile was increased by 1 cm, it would only take 128 tiles to tile the area. Find the side length of each tile.

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(e) The numerator of a certain fraction is 3 less than its denominator. If 6 is added to the numerator and 5 to the denominator, the value of the fraction is doubled. Find the fraction.

(f) A photograph is 18 cm by 12 cm. It is to be surrounded by a frame of uniform width whose area is equal to that of the photograph. Find the width of the frame.

(g) A certain tank can be filled by two pipes in 80 minutes. The larger pipe by itself can fill the tank in 2 hours less than the smaller pipe by itself. How long does each pipe take to fill the tank on its own?

(h) Two trains each make a journey of 330 km. One of the trains travels 5 km/h faster than the other and takes 30 minutes less time. Find the speeds of the trains.

12. Solve each of these equations:

(a) $\dfrac{2}{a+3} + \dfrac{a+3}{2} = \dfrac{10}{3}$

(b) $\dfrac{k+10}{k-5} - \dfrac{10}{k} = \dfrac{11}{6}$

(c) $\dfrac{3t}{t^2-6} = \sqrt{3}$

(d) $\dfrac{3m+1}{3m-1} - \dfrac{3m-1}{3m+1} = 2$

EXTENSION

13. (a) Find x in terms of c, given that $\dfrac{2}{3x-2c} + \dfrac{3}{2x-3c} = \dfrac{7}{2c}$.

(b) Find x in terms of a and b if $\dfrac{a^2b}{x^2} + \left(1 + \dfrac{b}{x}\right) a = 2b + \dfrac{a^2}{x}$.

1 H Simultaneous Equations

This section will review the two algebraic approaches to simultaneous equations — substitution and elimination (graphical interpretations will be discussed in Chapters Two and Three). Both linear and non-linear simultaneous equations will be reviewed, and the methods extended to systems of three equations in three unknowns.

Solution by Substitution: This method can be applied whenever one of the equations can be solved for one of the variables.

11 SIMULTANEOUS EQUATIONS BY SUBSTITUTION: Solve one of the equations for one of the variables, then substitute it into the other equation.

WORKED EXERCISE: Solve these simultaneous equations by substitution:

(a) $3x - 2y = 29$ (1)
$4x + y = 24$ (2)

(b) $y = x^2$ (1)
$y = x + 2$ (2)

SOLUTION:

(a) Solving (2) for y, $y = 24 - 4x$. (2A)

Substituting (2A) into (1), $3x - 2(24 - 4x) = 29$
$x = 7$.

Substituting $x = 7$ into (1), $21 - 2y = 29$
$y = -4$.

So $x = 7$ and $y = -4$.

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(b) Substituting (1) into (2),
$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \text{ or } -1.$$
From (1), when $x = 2$, $y = 4$, and when $x = -1$, $y = 1$.
So $x = 2$ and $y = 4$, or $x = -1$ and $y = 1$.

Solution by Elimination: This method, when it can be used, is more elegant, and can involve less algebraic manipulation with fractions.

12 SIMULTANEOUS EQUATIONS BY ELIMINATION: Take suitable multiples of the equations so that one variable is eliminated when the equations are added or subtracted.

WORKED EXERCISE: Solve these simultaneous equations by elimination:

(a) $3x - 2y = 29$ (1)
$4x + 5y = 8$ (2)

(b) $x^2 + y^2 = 53$ (1)
$x^2 - y^2 = 45$ (2)

SOLUTION:

(a) Taking $4 \times (1)$ and $3 \times (2)$,
$$12x - 8y = 116 \qquad (1A)$$
$$12x + 15y = 24. \qquad (2A)$$
Subtracting (1A) from (2A),
$$23y = -92$$
$$y = -4.$$
Substituting into (1),
$$3x + 8 = 29$$
$$x = 7.$$
So $x = 7$ and $y = -4$.

(b) Adding (1) and (2),
$$2x^2 = 98$$
$$x^2 = 49.$$
Subtracting (2) from (1),
$$2y^2 = 8$$
$$y^2 = 4.$$
So $x = 7$ and $y = 2$,
or $x = 7$ and $y = -2$,
or $x = -7$ and $y = 2$,
or $x = -7$ and $y = -2$.

Systems of Three Equations in Three Variables: The key step here is to reduce the system to two equations in two variables.

13 SOLVING THREE SIMULTANEOUS EQUATIONS: Using either substitution or elimination, produce two simultaneous equations in two of the variables.

WORKED EXERCISE: Solve simultaneously:
$$3x - 2y - z = -8 \qquad (1)$$
$$5x + y + 3z = 23 \qquad (2)$$
$$4x + y - 5z = -18 \qquad (3)$$

SOLUTION: Subtracting (3) from (2), $\quad x + 8z = 41. \qquad (4)$
Doubling (3), $\quad 8x + 2y - 10z = -36 \qquad (3A)$
and adding (1) and (3A), $\quad 11x - 11z = -44$
$$x - z = -4. \qquad (5)$$
Equations (4) and (5) are now two equations in two unknowns.
Subtracting (5) from (4), $\quad 9z = 45$
$$z = 5.$$

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Substituting $z = 5$ into (5), $x = 1$
and substituting into (2), $y = 3$.
So $x = 1$, $y = 3$ and $z = 5$ (which should be checked in the original equations).

Exercise 1H

1. Solve by substitution:

(a) $y = 2x$ and $3x + 2y = 14$
(b) $y = -3x$ and $2x + 5y = 13$
(c) $y = 4 - x$ and $x + 3y = 8$
(d) $x = 5y + 4$ and $3x - y = 26$
(e) $2x + y = 10$ and $7x + 8y = 53$
(f) $2x - y = 9$ and $3x - 7y = 19$
(g) $4x - 5y = 2$ and $x + 10y = 41$
(h) $2x + 3y = 47$ and $4x - y = 45$

2. Solve by elimination:

(a) $2x + y = 1$ and $x - y = -4$
(b) $2x + 3y = 16$ and $2x + 7y = 24$
(c) $3x + 2y = -6$ and $x - 2y = -10$
(d) $5x - 3y = 28$ and $2x - 3y = 22$
(e) $3x + 2y = 7$ and $5x + y = 7$
(f) $3x + 2y = 0$ and $2x - y = 56$
(g) $15x + 2y = 27$ and $3x + 7y = 45$
(h) $7x - 3y = 41$ and $3x - y = 17$
(i) $2x + 3y = 28$ and $3x + 2y = 27$
(j) $3x - 2y = 11$ and $4x + 3y = 43$
(k) $4x + 6y = 11$ and $17x - 5y = 1$
(l) $8x = 5y$ and $13x = 8y + 1$

3. Solve by substitution:

(a) $y = 2 - x$ and $y = x^2$
(b) $y = 2x - 3$ and $y = x^2 - 4x + 5$
(c) $y = 3x^2$ and $y = 4x - x^2$
(d) $x - y = 5$ and $y = x^2 - 11$
(e) $x - y = 2$ and $xy = 15$
(f) $3x + y = 9$ and $xy = 6$
(g) $x^2 - y^2 = 16$ and $x^2 + y^2 = 34$
(h) $x^2 + y^2 = 117$ and $2x^2 - 3y^2 = 54$

DEVELOPMENT

4. Solve each of these problems by constructing and then solving a pair of simultaneous equations:

(a) If 7 apples and 2 oranges cost \$4, while 5 apples and 4 oranges cost \$4·40, find the cost of each apple and orange.
(b) Twice as many adults as children attended a certain concert. If adult tickets cost \$8 each, child tickets cost \$3 each and the total takings were \$418, find the numbers of adults and children who attended.
(c) A man is 3 times as old as his son. In 12 years time he will be twice as old as his son. How old is each of them now?
(d) At a meeting of the members of a certain club, a proposal was voted on. If 357 members voted and the proposal was carried by a majority of 21, how many voted for and how many voted against the proposal?
(e) The value of a certain fraction becomes $\frac{1}{5}$ if one is added to its numerator. If one is taken from its denominator, its value becomes $\frac{1}{7}$. Find the fraction.
(f) Kathy paid \$320 in cash for a CD player. If she paid in \$20 notes and \$10 notes and there were 23 notes altogether, how many of each type were there?
(g) Two people are 16 km apart on a straight road. They start walking at the same time. If they walk towards each other, they will meet in 2 hours, but if they walk in the same direction (so that the distance between them is decreasing), they will meet in 8 hours. Find their walking speeds.

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(h) A certain integer is between 10 and 100. Its value is 8 times the sum of its digits and if it is reduced by 45, its digits are reversed. Find the integer.

5. Solve simultaneously:

(a) $\frac{y}{4} - \frac{x}{3} = 1$ and $\frac{x}{2} + \frac{y}{5} = 10$

(b) $4x + \frac{y-2}{3} = 12$ and $3y - \frac{x-3}{5} = 6$

6. Solve simultaneously:

(a) $x = 2y$, $y = 3z$, $x + y + z = 10$

(b) $x + 2y - z = -3$, $3x - 4y + z = 13$, $2x + 5y = -1$

(c) $2a - b + c = 10$, $a - b + 2c = 9$, $3a - 4c = 1$

(d) $p + q + r = 6$, $2p - q + r = 1$, $p + q - 2r = -9$

(e) $2x - y - z = 17$, $x + 3y + 4z = -20$, $5x - 2y + 3z = 19$

(f) $3u + v - 4w = -4$, $u - 2v + 7w = -7$, $4u + 3v - w = 9$

7. Solve simultaneously:

(a) $x + y = 15$ and $x^2 + y^2 = 125$

(b) $x - y = 3$ and $x^2 + y^2 = 185$

(c) $2x + y = 5$ and $4x^2 + y^2 = 17$

(d) $x + y = 9$ and $x^2 + xy + y^2 = 61$

(e) $x + 2y = 5$ and $2xy - x^2 = 3$

(f) $3x + 2y = 16$ and $xy = 10$

EXTENSION

8. Solve simultaneously:

(a) $\frac{7}{x} - \frac{5}{y} = 3$ and $\frac{2}{x} + \frac{25}{2y} = 12$

(b) $9x^2 + y^2 = 52$ and $xy = 8$

9. Consider the equations $12x^2 - 4xy + 11y^2 = 64$ and $16x^2 - 9xy + 11y^2 = 78$.

(a) By letting $y = mx$, show that $7m^2 + 12m - 4 = 0$.

(b) Hence, or otherwise, solve the two equations simultaneously.

1 I Completing the Square

We will see in Chapter Eight that completing the square, because it can be done in all situations, is more important for the investigation of quadratics than factoring. For example, the quadratic formula reviewed earlier is proven by completing the square. The review in this section will be restricted to *monic* quadratics, in which the coefficient of x^2 is 1.

Perfect Squares: When the quadratic $(x + \alpha)^2$ is expanded,

$$(x + \alpha)^2 = x^2 + 2\alpha x + \alpha^2,$$

the coefficient of x is twice α and the constant is the square of α. Reversing the process, the constant term in a perfect square can be found by taking half the coefficient of x and squaring it.

14 COMPLETING THE SQUARE IN AN EXPRESSION: To complete the square in a given expression $x^2 + bx + \cdots$, halve the coefficient b of x and square it.

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WORKED EXERCISE: Complete the square in: (a) $a^2+16a+\cdots$ (b) $x^2-3x+\cdots$

SOLUTION:

(a) The coefficient of a is 16, half of 16 is 8, and $8^2=64$,
so $a^2+16a+64=(a+8)^2$.

(b) The coefficient of x is -3, half of -3 is $-1\frac{1}{2}$, and $(-1\frac{1}{2})^2=2\frac{1}{4}$,
so $x^2-3x+2\frac{1}{4}=(x-1\frac{1}{2})^2$.

Solving Quadratic Equations by Completing the Square: This is the process underlying the quadratic formula.

15 SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE:
Complete the square in the quadratic by adding the same to both sides.

WORKED EXERCISE: Solve: (a) $t^2+8t=20$ (b) $x^2-x-1=0$ (c) $x^2+x+1=0$

SOLUTION:

(a)
$$\begin{aligned} t^2+8t&=20\\ t^2+8t+16&=36\\ (t+4)^2&=36\\ t+4=6 \text{ or } t+4&=-6\\ t=2 \text{ or } &-10 \end{aligned}$$

(b)
$$\begin{aligned} x^2-x-1&=0\\ x^2-x+\tfrac{1}{4}&=1\tfrac{1}{4}\\ (x-\tfrac{1}{2})^2&=\tfrac{5}{4}\\ x-\tfrac{1}{2}&=\tfrac{1}{2}\sqrt{5} \text{ or } -\tfrac{1}{2}\sqrt{5}\\ x&=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5} \text{ or } \tfrac{1}{2}-\tfrac{1}{2}\sqrt{5} \end{aligned}$$

(c)
$$\begin{aligned} x^2+x+1&=0\\ x^2+x+\tfrac{1}{4}&=-\tfrac{3}{4}\\ (x+\tfrac{1}{2})^2&=-\tfrac{3}{4} \end{aligned}$$
This is impossible, because a square can't be negative,
so the equation has no solutions.

Exercise 1I

1. Write down the constant which must be added to each expression in order to create a perfect square:

(a) x^2+2x (b) y^2-6y (c) a^2+10a (d) m^2-18m (e) c^2+3c (f) x^2-x (g) b^2+5b (h) t^2-9t

2. Factor:

(a) x^2+4x+4 (b) y^2+2y+1 (c) $p^2+14p+49$ (d) $m^2-12m+36$ (e) $t^2-16t+64$ (f) $400-40u+u^2$ (g) $x^2+20xy+100y^2$ (h) $a^2b^2-24ab+144$

3. Copy and complete:

(a) $x^2+6x+\cdots=(x+\cdots)^2$
(b) $y^2+8y+\cdots=(y+\cdots)^2$
(c) $a^2-20a+\cdots=(a+\cdots)^2$
(d) $b^2-100b+\cdots=(b+\cdots)^2$
(e) $u^2+u+\cdots=(u+\cdots)^2$
(f) $t^2-7t+\cdots=(t+\cdots)^2$
(g) $m^2+50m+\cdots=(m+\cdots)^2$
(h) $c^2-13c+\cdots=(c+\cdots)^2$

4. Solve each of the following quadratic equations by completing the square:

(a) $x^2-2x=3$ (b) $x^2-6x=0$ (c) $a^2+6a+8=0$ (d) $y^2+3y=10$ (e) $b^2-5b-14=0$ (f) $x^2+4x+1=0$ (g) $x^2-10x+20=0$ (h) $y^2-y+2=0$ (i) $a^2+7a+7=0$

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5. Complete the square for each of the given expressions:

(a) $p^2 - 2pq + \cdots$ (c) $x^2 - 6xy + \cdots$ (e) $u^2 - uv + \cdots$
(b) $a^2 + 4ab + \cdots$ (d) $c^2 + 40cd + \cdots$ (f) $m^2 + 11mn + \cdots$

DEVELOPMENT

6. Solve by dividing both sides by the coefficient of x^2 and then completing the square:

(a) $3x^2 - 15x + 18 = 0$ (d) $2x^2 + 8x + 3 = 0$ (g) $3x^2 - 8x - 3 = 0$
(b) $2x^2 - 4x - 1 = 0$ (e) $4x^2 + 4x - 3 = 0$ (h) $2x^2 + x - 15 = 0$
(c) $3x^2 + 6x + 5 = 0$ (f) $4x^2 - 2x - 1 = 0$ (i) $2x^2 - 10x + 7 = 0$

7. (a) If $x^2 + y^2 + 4x - 2y + 1 = 0$, show that $(x+2)^2 + (y-1)^2 = 4$.
(b) Show that the equation $x^2 + y^2 - 6x - 8y = 0$ can be written in the form $(x-a)^2 + (y-b)^2 = c$, where a, b and c are constants. Hence write down the values of a, b and c.
(c) If $x^2 + 1 = 10x + 12y$, show that $(x-5)^2 = 12(y+2)$.
(d) Find values for A, B and C if $y^2 - 6x + 16y + 94 = (y+C)^2 - B(x+A)$.

EXTENSION

8. (a) Write down the expansion of $(x+\alpha)^3$ and hence complete the cube in

$$x^3 + 12x^2 + \cdots = (x + \cdots)^3.$$

(b) Hence use a suitable substitution to change the equation $x^3 + 12x^2 + 30x + 4 = 0$ into a cubic equation of the form $u^3 + cu + d = 0$.

1 J The Language of Sets

We will often want to speak about collections of things such as numbers, points and lines. In mathematics, these collections are called *sets*, and this section will introduce or review some of the language associated with sets. Logic is very close to the surface when talking about sets, and particular attention should be given to the words 'if', 'if and only if', 'and', 'or' and 'not'.

Listing Sets and Describing Sets: A *set* is a collection of things. When a set is specified, it needs to be made absolutely clear what things are its members. This can be done by *listing* the members inside curly brackets:

$$S = \{1, 3, 5, 7, 9\},$$

read as 'S is the set whose members are 1, 3, 5, 7 and 9'. It can also be done by *writing a description* of its members inside curly brackets, for example,

$$T = \{\text{odd integers from 0 to 10}\},$$

read as 'T is the set of odd integers from 0 to 10'.

Equal Sets: Two sets are called *equal* if they have exactly the same members. Hence the sets S and T in the previous paragraph are equal, which is written as $S = T$. The order in which the members are written doesn't matter at all, neither does repetition, so, for example,

$$\{1, 3, 5, 7, 9\} = \{3, 9, 7, 5, 1\} = \{5, 9, 1, 3, 7\} = \{1, 3, 1, 5, 1, 7, 9\}.$$

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Members and Non-members: The symbol $\in$ means 'is a member of', and the symbol $\notin$ means 'is not a member of', so if $A = \{ 3, 4, 5, 6 \}$, then

$$3 \in A \qquad \text{and} \qquad 2 \notin A \qquad \text{and} \qquad 6 \in A \qquad \text{and} \qquad 9 \notin A,$$

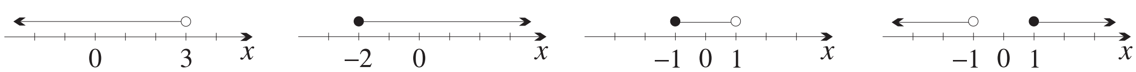
which is read as '3 is a member of A' and '2 is not a member of A', and so on.

Set-builder Notation: A third way to specify a set is to write down, using a colon (:), all the conditions something must fulfil to be a member of the set. For example,

$$\{n : n \text{ is a positive integer, } n < 5\} = \{ 1, 2, 3, 4 \},$$

which is read as 'The set of all n such that n is a positive integer and n is less than 5'.

If the type of number is not specified, real numbers are normally intended. Here are some sets of real numbers, in set-builder notation, with their graphs on the number line:



$\{x : x < 3\}$ $\qquad$ $\{x : x \ge -2\}$ $\qquad$ $\{x : -1 \le x < 1\}$ $\qquad$ $\{x : x < -1 \text{ or } x \ge 1\}$

The Size of a Set: A set may be *finite* or *infinite*. If a set S is finite, then $|S|$ is the symbol for the number of members in S. For example,

$$\{ \text{positive even numbers} \} \text{ is infinite} \qquad \text{and} \qquad |\{ \text{a, e, i, o, u} \}| = 5.$$

Some sets have only one member, for example $\{ 3 \}$ is 'the set whose only member is 3'. The set $\{ 3 \}$ is a different object from the number 3:

$$3 \in \{ 3 \} \qquad \text{and} \qquad 3 \ne \{ 3 \} \qquad \text{and} \qquad |\{ 3 \}| = 1.$$

The Empty Set: The symbol $\emptyset$ represents the *empty set*, which is the set with no members:

$$|\emptyset| = 0 \qquad \text{and} \qquad x \notin \emptyset, \ \text{whatever } x \text{ is.}$$

There is only one empty set, because any two empty sets have the same members (that is, none at all) and so are equal.

Subsets of Sets: A set A is called a *subset* of a set B if every member of A is a member of B. This relation is written as $A \subset B$. For example,

$$\{ \text{men in Australia} \} \subset \{ \text{people in Australia} \}$$
$$\{ 2, 3, 4 \} \not\subset \{ 3, 4, 5 \}.$$

Because of the way subsets have been defined, every set is a subset of itself. Also the empty set is a subset of every set. For example,

$$\{ 1, 3, 5 \} \subset \{ 1, 3, 5 \}, \qquad \emptyset \subset \{ 1, 3, 5 \} \qquad \text{and} \qquad \{ 3 \} \subset \{ 1, 3, 5 \}.$$

'If' means Subset, 'If and Only If' means Equality: The word 'if' and the phrase 'if and only if' are fundamental to mathematical language. They have an important interpretation in the language of sets, the first in terms of subsets of sets, the second in terms of equality of sets:

$A \subset B$ means 'If $x \in A$, then $x \in B$'.

$A = B$ means '$x \in A$ if and only if $x \in B$'.

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Union and Intersection: The *union* $A \cup B$ of two sets A and B is the set of everything belonging to A or to B or to both. Their *intersection* $A \cap B$ is the set of everything belonging to both A and B. For example,

if $A = \{0, 1, 2, 3\}$ and $B = \{1, 3, 6\}$,
then $A \cup B = \{0, 1, 2, 3, 6\}$ and $A \cap B = \{1, 3\}$.

Two sets A and B are called *disjoint* if they have no elements in common, that is, if $A \cap B = \emptyset$. For example, the sets $\{2, 4, 6, 8\}$ and $\{1, 3, 5, 7\}$ are disjoint.

'Or' means Union, 'And' means Intersection: The definitions of union and intersection can be written in set-builder notation using the words 'and' and 'or':

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

This connection between the words 'and' and 'or' and set notation should be carefully considered. The word 'or' in mathematics always means 'and/or', and never means 'either, but not both'.

The Universal Set and the Complement of a Set: A *universal set* is the set of everything under discussion in a particular situation. For example, if $A = \{1, 3, 5, 7, 9\}$, then possible universal sets are the set of all positive integers less than 11, or the set of all real numbers.

Once a universal set E is fixed, then the *complement* $\overline{A}$ of any set A is the set of all members of that universal set which are not in A. For example,

if $A = \{1, 3, 5, 7, 9\}$ and $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
then $\overline{A} = \{2, 4, 6, 8, 10\}$.

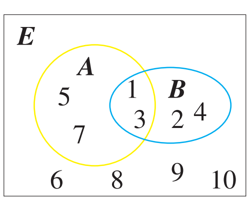
Notice that every member of the universal set is either in A or in $\overline{A}$, but never in both A and $\overline{A}$:

$$A \cup \overline{A} = E \qquad \text{and} \qquad A \cap \overline{A} = \emptyset.$$

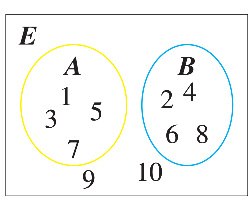
'Not' means Complement: There is an important connection between the word 'not' and the complement of a set. If the definition of the complementary set is written in set-builder notation,

$$\overline{A} = \{x \in E : x \text{ is } \underline{\text{not}} \text{ a member of } A\}.$$

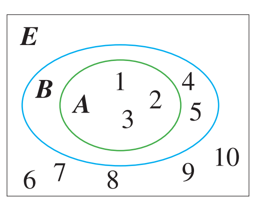
Venn Diagrams: A *Venn diagram* is a diagram used to represent the relationship between sets. For example, the four diagrams below represent the four different possible relationships between two sets A and B. In each case, the universal set is again $E = \{1, 2, 3, \ldots, 10\}$.



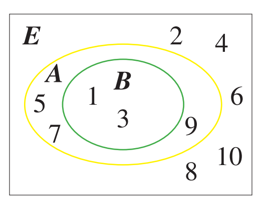
$A = \{1, 3, 5, 7\}$
$B = \{1, 2, 3, 4\}$



$A = \{1, 3, 5, 7\}$
$B = \{2, 4, 6, 8\}$



$A = \{1, 2, 3\}$
$B = \{1, 2, 3, 4, 5\}$



$A = \{1, 3, 5, 7, 9\}$
$B = \{1, 3\}$

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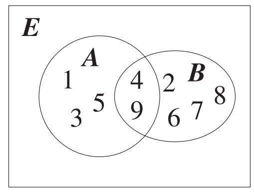
Compound sets such as $A \cup B$, $A \cap B$ and $\overline{A} \cap B$ can be visualised by shading regions the of Venn diagram, as is done in several questions in the following exercise.

The Counting Rule for Sets: To calculate the size of the union $A \cup B$ of two sets, adding the sizes of A and of B will not do, because the members of the intersection $A \cap B$ would be counted twice. Hence $|A \cap B|$ needs to be subtracted again, and the rule is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For example, the Venn diagram on the right shows the two sets:

$$A = \{\,1, 3, 4, 5, 9\,\} \qquad \text{and} \qquad B = \{\,2, 4, 6, 7, 8, 9\,\}.$$



From the diagram, $|A \cup B| = 9$, $|A| = 5$, $|B| = 6$ and $|A \cap B| = 2$, and the formula works because $9 = 5 + 6 - 2$.

When two sets are disjoint, there is no overlap between A and B to cause any double counting. With $A \cap B = \emptyset$ and $|A \cap B| = 0$, the counting rule becomes

$$|A \cup B| = |A| + |B|.$$

Problem Solving Using Venn Diagrams: A Venn diagram is often the most convenient way to sort out problems involving overlapping sets of things. In the following exercise, the number of members of each region is written inside the region, rather than the members themselves.

WORKED EXERCISE: 100 Sydneysiders were surveyed to find out how many of them had visited the cities of Melbourne and Brisbane. 31 people had visited Melbourne, 26 people had visited Brisbane and 12 people had visited both cities. Find how many people had visited:

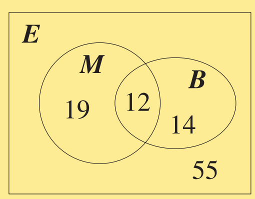
(a) Melbourne or Brisbane,

(b) Brisbane but not Melbourne,

(c) only one of the two cities,

(d) neither city.

SOLUTION: Let M be the set of people who have visited Melbourne, let B be the set of people who have visited Brisbane, and let E be the universal set of all people surveyed. Calculations should begin with the 12 people in the intersection of the two regions. Then the numbers shown in the other three regions of the Venn diagram can easily be found, and so:



(a) $|\{\text{ visited Melbourne or Brisbane }\}| = 19 + 14 + 12 = 45$

(b) $|\{\text{ visited Melbourne only }\}| = 19$

(c) $|\{\text{ visited only one city }\}| = 19 + 14 = 33$

(d) $|\{\text{ visited neither city }\}| = 100 - 45 = 55$

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Exercise 1J

1. State whether each set is finite or infinite. If it is finite, state its number of members:

(a) $\{1, 3, 5, \dots\}$
(b) $\{0, 1, 2, \dots, 9\}$
(c) $\emptyset$
(d) { points on a line }
(e) $\{n : n$ is a positive integer and $1 < n < 20\}$
(f) $\{x : 3 \le x \le 5\}$
(g) { a, l, g, e, b, r, a }
(h) { multiples of 7 that are less than 100 }

2. Decide whether each of the following statements is true or false:

(a) If two sets have the same number of members, then they are equal.
(b) If two sets are equal, then they have the same number of members.
(c) If $A = \{0, 0\}$, then $|A| = 1$.
(d) $|\{0\}| = 0$
(e) $1\,000\,000 \in \{1, 2, 3, \dots\}$
(f) $|\{40, 41, 42, \dots, 60\}| = 20$

3. State in each case whether or not $A \subset B$ (that is, whether A is a subset of B):

(a) $A = \{2, 4, 5\}$, $B = \{n : n$ is an even positive integer and $n < 10\}$
(b) $A = \{2, 3, 5\}$, $B =$ { prime numbers less than 10 }
(c) $A =$ { d, a, n, c, e }, $B =$ { e, d, u, c, a, t, i, o, n }
(d) $A = \emptyset$, $B = \{51, 52, 53, \dots, 99\}$
(e) $A = \{3, 6, 9, \dots\}$, $B = \{6, 12, 18, \dots\}$

4. Answer true or false:

(a) If $A \subset B$ and $B \subset A$, then $A = B$.
(b) If $A \subset B$ and $B \subset C$, then $A \subset C$.

5. List all the subsets of each of these sets:

(a) { a }
(b) { a, b }
(c) { a, b, c }
(d) $\emptyset$

6. Find $A \cup B$ and $A \cap B$ for each pair of sets:

(a) $A =$ { m }, $B =$ { m, n }
(b) $A = \{2, 4, 6\}$, $B = \{4, 6, 8\}$
(c) $A = \{1, 3, 4, 6, 9\}$, $B = \{2, 4, 5, 7, 8, 9\}$
(d) $A =$ { c, o, m, p, u, t, e, r }, $B =$ { s, o, f, t, w, a, r, e }
(e) $A =$ { prime numbers less than 12 }, $B =$ { odd numbers less than 12 }

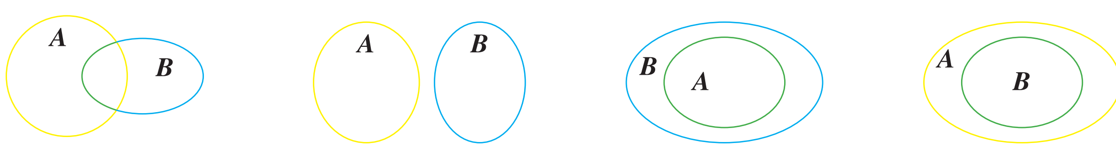
7. If $A =$ { students who study Japanese } and $B =$ { students who study History }, carefully describe each of the following sets: (a) $A \cap B$ (b) $A \cup B$

8. Copy and complete: (a) If $P \subset Q$, then $P \cup Q = \dots$ (b) If $P \subset Q$, then $P \cap Q = \dots$

9. Let $A = \{1, 3, 6, 8\}$ and $B = \{3, 4, 6, 7, 10\}$, with universal set $\{1, 2, 3, \dots, 10\}$. List the members of:

(a) $\overline{A}$
(b) $\overline{B}$
(c) $\overline{A} \cup \overline{B}$
(d) $\overline{A \cup B}$
(e) $\overline{A} \cap \overline{B}$
(f) $\overline{A \cap B}$

10. Select the Venn diagram that best shows the relationship between each pair of sets A and B:



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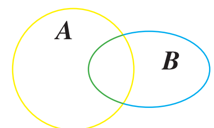
(a) $A = \{\text{positive integers}\}$, $B = \{\text{positive real numbers}\}$
(b) $A = \{7, 1, 4, 8, 3, 5\}$, $B = \{2, 9, 0, 7\}$
(c) $A = \{\text{multiples of } 3\}$, $B = \{\text{multiples of } 5\}$
(d) $A = \{\text{l, e, a, r, n}\}$, $B = \{\text{s, t, u, d, y}\}$
(e) $A = \{\text{politicians in Australia}\}$, $B = \{\text{politicians in NSW}\}$.

11. In each of the following, A and B represent sets of real numbers. For each part, graph on separate number lines: (i) A, (ii) B, (iii) $A \cup B$, (iv) $A \cap B$.
(a) $A = \{x : x > 0\}$, $B = \{x : x \le 3\}$ (b) $A = \{x : x \le -1\}$, $B = \{x : x > 2\}$
(c) $A = \{x : -3 \le x < 1\}$, $B = \{x : -1 \le x \le 4\}$

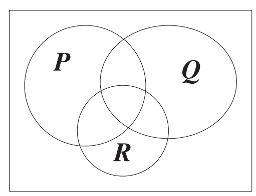
12. (a) Explain the counting rule $|A \cup B| = |A| + |B| - |A \cap B|$ by making reference to the Venn diagram opposite.
(b) If $|A \cup B| = 17$, $|A| = 12$ and $|B| = 10$, find $|A \cap B|$.
(c) Show that the relationship in part (a) is satisfied when $A = \{3, 5, 6, 8, 9\}$ and $B = \{2, 3, 5, 6, 7, 8\}$.



13. Use a Venn diagram to solve each of these problems:
(a) In a group of 20 people, there are 8 who play the piano, 5 who play the violin and 3 who play both. How many people play neither?
(b) Each person in a group of 30 plays either tennis or golf. 17 play tennis, while 9 play both. How many play golf?
(c) In a class of 28 students, there are 19 who like geometry and 16 who like trigonometry. How many like both if there are 5 students who don't like either?

DEVELOPMENT

14. Shade each of the following regions on the given diagram (use a separate diagram for each part).
(a) $P \cap Q \cap R$
(b) $(P \cap R) \cup (Q \cap R)$
(c) $\overline{P} \cup \overline{Q} \cup \overline{R}$ (where $\overline{P}$ denotes the complement of P)



15. A group of 80 people was surveyed about their approaches to keeping fit. It was found that 20 jog, 22 swim and 18 go to the gym on a regular basis. If 10 people both jog and swim, 11 people both jog and go to the gym, 6 people both swim and go to the gym and 43 people do none of these activities on a regular basis, how many people do all three?

EXTENSION

16. (a) Explain why a five-member set has twice as many subsets as a four-member set.
(b) Hence find a formula for the number of subsets of an n-member set.

17. How many different possibilities for shading are there, given a Venn diagram with three overlapping sets within a universal set?

18. Express in words: '$\{\emptyset\} \ne \emptyset$ because $\emptyset \in \{\emptyset\}$'. Is the statement true or false?

19. Decide whether or not the following statement is true:

$$A \subset B \text{ if and only if, if } x \notin B \text{ then } x \notin A.$$

20. Simplify $\big(A \cap (A \cap \overline{B})\big) \cup \big((A \cap B) \cup (B \cap \overline{A})\big)$.

21. The definition $A = \{\text{sets that are not members of themselves}\}$ is impossible. Explain why, by considering whether or not A is a member of itself.

GO Online Multiple Choice Quiz

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