The University of Sydney

MATH1901/06 Differential Calculus (Advanced)

Semester 1 **Practice Questions for Quiz 1** 2012

> **Quiz 1** will be held in the tutorials in **Week 7** (MATH1901 on Monday 23 April 2012, MATH1906 on Thursday 26 April 2012).
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> The quiz questions will be based on material covered in the lectures during Weeks 1–5, which corresponds to material covered in tutorials in Weeks 2–6.
>
> Topics to be tested include complex numbers, functions and their associated technical terms, limits, continuity, epsilon-delta proofs, and theorems obeyed by continuous functions (intermediate value theorem, extreme value theorem).
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> The quiz will run for 40 minutes under exam conditions. It is worth 10% of the assessment for MATH1901. Use a pen, not a pencil. You may use a non-programmable calculator. At the start of the quiz, place all materials on the floor except your pen, your calculator (optional) and your student ID card face up on your desk. Switch off mobile phones.

Solutions to these problems appear below after Question 10.

The actual quiz questions will be considerably shorter than these practice questions and will not have multiple parts. Some quiz questions will be multiple choice.

1. Let $z = 12 + 5i$ and $w = 2 - 3i$. Calculate the following complex numbers in Cartesian form:

$$5w, \qquad z - 5w, \qquad zw, \qquad z\bar{w}, \qquad |z - w|,$$

$$\frac{z}{w}, \qquad w^4, \qquad \sqrt{z}, \qquad \sqrt{w},$$

$$e^z, \qquad \log(z), \qquad \log(w), \qquad z^w.$$

In the case of the square roots, logarithms and non-integer powers, give the principal value only. The power z^w is defined to be $e^{w \log z}$, and the principal value of the power is defined by taking the principal value of $\log z$.

2. Use the polar form and De Moivre's theorem to evaluate the following powers and roots of complex numbers. Express all answers first in strict polar form $re^{i\theta}$ or $r \operatorname{cis} \theta$ with $r > 0$ and $-\pi < \theta \leq \pi$, and then rewrite your answers in a neater form if possible (e.g., Cartesian or cis form). In the case of nth roots, give all n values and identify the principal value.

(a) $(1+i)^{23}$.
(b) $(-1+i\sqrt{3}\,)^{23}$.
(c) $(-1+i\sqrt{3}\,)^{1/7}$.
(d) $(-1)^{1/6}$.
(e) $(1+3i)^{12}$. (Use a calculator.)

3. (a) Sketch the following sets in the complex plane (the solutions below will have descriptions of the sketches, not the sketches themselves):

$$|z-3+i| \;<\; 16, \qquad \operatorname{Re}\big((1+2i)z\big) \;>\; 2, \qquad |z-3|+|z+3| \;=\; 10.$$

(b) Find the image in the complex w-plane of the unit circle $|z| = 1$ under the map $w = (2z-1)/(z-2)$.

4. Give the natural domains on the x-axis or xy-plane of the following functions of one or two real variables (the codomains are either $\mathbf{R}$ or $\mathbf{C}$):

$$\sqrt{4-x^2}\,, \qquad (4-x^2)^{-1/2}, \qquad \ln x, \qquad \sqrt{\ln x}\,,$$

$$\frac{\sin x}{x}, \qquad \cos^{-1}(x^2+y^2), \qquad \ln(\ln(\ln x)),$$

$$\sqrt{x+iy} \quad \text{(principal value)}, \qquad \log(x+iy) \quad \text{(principal value)},$$

$$(-1)^x \quad (\text{codomain } \mathbf{R}), \qquad (-1)^x \quad (\text{codomain } \mathbf{C}).$$

5. Decide which of the following functions $f : A \to B$ are surjective, injective, or bijective:

(a) $f : [-1,1] \to \mathbf{R}, \quad x \mapsto \sinh x$.
(b) $f : (-\infty,\infty) \to [-1,1], \quad x \mapsto \cos x$.
(c) $f : (0,\infty) \to \mathbf{R}, \quad x \mapsto \ln\big(x\sqrt{x^2+2}\,\big)$.

In the cases where f is not surjective, give the unique codomain that makes f surjective. In the cases where f is not injective, break the domain up into parts on which f is injective separately. In the cases where f is bijective, give the inverse function $f^{-1}(x)$.

6. Evaluate the following limits (allow $+\infty$ and $-\infty$ as values that a limit can take) or prove that the limit is undefined:

(a) $\displaystyle\lim_{\theta\to 0} \frac{\sin 3\theta}{\theta}$.

(b) $\displaystyle\lim_{\theta\to 0} \frac{\cos 3\theta}{\theta}$.

(c) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+c} - \sqrt{x+d}}, \qquad a, b, c, d \in \mathbf{R}, \qquad c \neq d.$

(d) $\displaystyle\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3}$.

(e) $\displaystyle\lim_{x\to 0^+} \frac{\sin(1/x)}{\ln x}$.

(f) $\displaystyle\lim_{x\to 0^+} (\ln x)\sin(1/x)$.

(g) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

(h) $\displaystyle\lim_{x\to 0^+} g(x)$, where $g(x) = \begin{cases} \sin x, & x \text{ irrational} \\ x\cos x, & x \text{ rational.} \end{cases}$

7. Identify the discontinuities of the following functions $f : A \to \mathbf{R}$, where A is a subset of the reals, and classify them as removable discontinuities, jump discontinuities, infinite discontinuities, or none of the above. (Note that the named discontinuities are not concerned about whether or not the limiting values of x are in the domain of the function. See the remark at the end of the solution to Question 3 on Tutorial 6.)

(a) $f(x) \;=\; \ln|x| + \dfrac{\sin \pi x}{x-1}$.

(b) $f(x) \;=\; \begin{cases} k\cosh x, & x < 0 \\ (x^2-9)/(x-3), & 0 < x < 3 \text{ and } x > 3. \end{cases}$

In part (b), find k such that f can be extended to a continuous function $g : \mathbf{R} \to \mathbf{R}$ and express g as a function given by two rules.

8. Use the Intermediate Value Theorem to prove that the transcendental equation,

$$\cos x \;=\; x\sin x,$$

has at least one root in the interval $(0, \pi/2)$. Show that this root lies in the interval $(\pi/4, \pi/3)$.

9. Prove the Quotient Law for limits.

10. Show that a nondecreasing function $f : \mathbf{R} \to \mathbf{R}$ has either a finite or a countable number of discontinuities. (It may be helpful to consider, first, the case $f : [0,1] \to [0,1]$.)

Solutions:

1. Label the thirteen parts (a)–(m). We are given $z = 12 + 5i$ and $w = 2 - 3i$.

(a) $5w$ in Cartesian form can be written either $10 - 15i$ or $5(2 - 3i)$.

(b) $z - 5w = (12 + 5i) - (10 - 15i) = 2 + 20i = 2(1 + 10i)$.

(c) $zw = (12 + 5i)(2 - 3i) = (24 + 15) + (10 - 36)i = 39 - 26i = 13(3 - 2i)$.

(d) $z\bar{w} = (12 + 5i)(2 + 3i) = (24 - 15) + (10 + 36)i = 9 + 46i$.

(e) $|z - w| = |2(5 + 4i)| = 2\sqrt{5^2 + 4^2} = 2\sqrt{41}$.

(f) $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{9 + 46i}{13}$. Another (less elegant) way to write it is $\dfrac{9}{13} + \dfrac{46}{13}i$.

(g) $w^4 = (2 - 3i)^4 = (-5 - 12i)^2 = -119 + 120i$.

(h) Let $a + ib = \sqrt{z}$. The principal value has $a \geq 0$. Squaring gives $a^2 - b^2 + 2abi = 12 + 5i$. Equating real and imaginary parts gives two equations,

$$a^2 - b^2 = 12, \qquad 2ab = 5.$$

Then $b = 5/(2a)$. Substituting this b into the first equation and rearranging gives the even quartic equation,

$$4a^4 - 48a^2 - 25 = (2a^2 + 1)(2a^2 - 25) = 0.$$

Because a is real, the first factor cannot vanish. Hence $a^2 = 25/2$ and $a = 5/\sqrt{2}$ (positive value only). Then $b = 5/(2a) = 1/\sqrt{2}$. The principal value of $\sqrt{z}$ is therefore

$$\sqrt{12 + 5i} = \frac{5 + i}{\sqrt{2}}.$$

(i) The same method applied to w gives the principal value,

$$\sqrt{2 - 3i} = \frac{\sqrt{2 + \sqrt{13}} - i\sqrt{-2 + \sqrt{13}}}{\sqrt{2}}.$$

This cannot be further simplified. (A nested square root $\sqrt{a + b\sqrt{c}}$ simplifies when $a^2 - b^2c$ is a perfect square.)

(j) $e^z = e^{12+5i} = e^{12}\, e^{5i} = e^{12}\operatorname{cis}(5)$. The answer can be left as $e^{12}\operatorname{cis}(5)$ or expanded as $e^{12}(\cos(5) + i\sin(5))$. (The cis notation, usually intended as a polar form, can serve also as an abbreviation for a Cartesian form when the trig functions do not have elementary evaluations.)

(k) The formula for the logarithm of a complex number is

$$\log z \;=\; \ln|z| + i\arg(z).$$

The argument takes infinitely many values, differing by multiples of 2π. The principal value of the logarithm is determined by making $\arg z$ take its principal value in the range $(-\pi, \pi]$. For $z = 12 + 5i$, we have

$$\log(12 + 5i) \;=\; \ln 13 + i\tan^{-1}\tfrac{5}{12}\,.$$

Equivalent versions are $\ln 13 + i\cos^{-1}\frac{12}{13}$ and $\ln 13 + i\sin^{-1}\frac{5}{13}$.

(l) Similarly,

$$\log(2 - 3i) \;=\; \tfrac{1}{2}\ln 13 - i\tan^{-1}\tfrac{3}{2}\,.$$

(m) When $z \neq 0$, complex powers are defined by $z^w = e^{w\log z}$. When w is irrational (which includes all non-real complex numbers as well as the real irrationals), the power takes infinitely many distinct values. When w is rational, $w = m/n$ ($m, n \in \mathbf{Z}$, no common factor, $n \geq 1$), the power takes n distinct values. Either way, the principal value of the power is defined by making $\log z$ take its principal value. (An exceptional case is e^w, which is defined as a single-valued function of w, also called $\exp w$. Of course, it agrees with the principal value of e to the power of w.) For the values of z and w here, we have

$$\begin{aligned}
z^w \;=\; (12+5i)^{2-3i} &\;=\; \exp\big\{(2-3i)\log(12+5i)\big\} \\
&\;=\; \exp\big\{(2-3i)(\ln 13 + i\tan^{-1}\tfrac{5}{12}\,)\big\} \\
&\;=\; \exp\big\{2\ln 13 + 3\tan^{-1}\tfrac{5}{12} \;+\; i(-3\ln 13 + 2\tan^{-1}\tfrac{5}{12})\big\} \\
&\;=\; \exp\big(2\ln 13 + 3\tan^{-1}\tfrac{5}{12}\big)\operatorname{cis}\big(-3\ln 13 + 2\tan^{-1}\tfrac{5}{12}\big) \\
&\;=\; 169\exp\big(3\tan^{-1}\tfrac{5}{12}\big)\operatorname{cis}\big(-3\ln 13 + 2\tan^{-1}\tfrac{5}{12}\big).
\end{aligned}$$

This is a good place to stop. It is possible to eliminate the inverse tangent from the cis term, but not everybody would agree that that would be an improvement. (A calculator gives $z^w = 448.917557 - 321.898392i$.)

2. (a) $(1+i)^{23}$. Let $z = 1 + i$. Then $r = |z| = \sqrt{2}$ and $\theta = \arg z = \tan^{-1} 1 = \pi/4$. The polar form of z is

$$1 + i \;=\; re^{i\theta} \;=\; \sqrt{2}\, e^{i\pi/4},$$

or, equivalently,

$$1 + i \;=\; r\operatorname{cis}\theta \;=\; \sqrt{2}\operatorname{cis}(\pi/4) \;=\; \sqrt{2}\,\big(\cos(\pi/4) + i\sin(\pi/4)\big).$$

According to De Moivre's theorem,

$$(1+i)^n \;=\; r^n \operatorname{cis}(n\theta) \;=\; 2^{n/2} \operatorname{cis}(n\pi/4).$$

In the case $n = 23$, we have

$$(1+i)^{23} \;=\; 2^{23/2} \operatorname{cis}(23\pi/4) \;=\; 2^{23/2} \operatorname{cis}(-\pi/4).$$

The right-hand side is the strict polar form. The factor $2^{23/2}$ can also be written $2^{11}\sqrt{2}$ or $2048\sqrt{2}$. The cis term can also be written $e^{-\pi i/4}$ or $\cos(-\pi/4) + i\sin(-\pi/4)$. So there are nine correct ways to express $(1+i)^{23}$ in strict polar form.

A neater way to write $(1+i)^{23}$ is in Cartesian form. First, $\operatorname{cis}(-\pi/4) = \cos(\pi/4) - i\sin(\pi/4) = (1-i)/\sqrt{2}$. Hence,

$$(1+i)^{23} = 2048(1-i).$$

(There are quick ways to evaluate this power by direct multiplication: $(1+i)^2 = 2i$, $(1+i)^4 = (2i)^2 = -4$, $(1+i)^{24} = (-4)^6 = 4096$, $(1+i)^{23} = 4096/(1+i) = 2048(1-i)$.)

(b) $(-1+i\sqrt{3})^{23}$. Let $z = -1+i\sqrt{3}$. Here, $r = |z| = 2$ and θ is in the second quadrant with $\cos\theta = -1/2$ and $\sin\theta = \sqrt{3}/2$. This gives $\theta = \arg z = 2\pi/3$. The polar form of z is

$$-1+i\sqrt{3} \;=\; 2e^{2\pi i/3} \;=\; 2\operatorname{cis}(2\pi/3).$$

Then

$$(-1+i\sqrt{3})^{23} \;=\; 2^{23}\operatorname{cis}(46\pi/3) \;=\; 2^{23}\operatorname{cis}(-2\pi/3).$$

This is the required polar form. The Cartesian form is

$$(-1+i\sqrt{3})^{23} \;=\; -2^{22}(1+i\sqrt{3}).$$

(c) $(-1+i\sqrt{3})^{1/7}$. In polar form with principal argument, we have $-1+i\sqrt{3} = 2\operatorname{cis}(2\pi/3)$. Hence the principal seventh root is $2^{1/7}\operatorname{cis}(2\pi/21)$. The full set of seven distinct seventh roots is $2^{1/7}\operatorname{cis}(2\pi/21 + 2k\pi/7)$, where $k = 0, 1, 2, 3, 4, 5, 6$, or, equivalently, $k = -3, -2, -1, 0, 1, 2, 3$. The latter choice keeps all arguments in their principal range. Hence the seven seventh roots of $-1+i\sqrt{3}$ are

$$2^{1/7}\operatorname{cis}(-16\pi/21), \qquad 2^{1/7}\operatorname{cis}(-10\pi/21), \qquad 2^{1/7}\operatorname{cis}(-4\pi/21), \qquad 2^{1/7}\operatorname{cis}(2\pi/21),$$

$$2^{1/7}\operatorname{cis}(8\pi/21), \qquad 2^{1/7}\operatorname{cis}(2\pi/3), \qquad 2^{1/7}\operatorname{cis}(20\pi/21).$$

In the complex plane, they form the vertices of a regular heptagon (seven-sided polygon). The fourth member of this list is the principal seventh root. The sixth member has the elementary Cartesian form $2^{-6/7}(-1+i\sqrt{3})$. The others are best left in cis form.

(d) $(-1)^{1/6}$. The polar form of -1 is $e^{i\pi}$ or $\operatorname{cis}\pi$. Hence, the principal value of $(-1)^{1/6}$ is

$$\operatorname{cis}(\pi/6) \;=\; (\sqrt{3}+i)/2.$$

The six distinct sixth roots of -1 have the polar forms,

$$\operatorname{cis}(-5\pi/6), \quad \operatorname{cis}(-\pi/2), \quad \operatorname{cis}(-\pi/6), \quad \operatorname{cis}(\pi/6), \quad \operatorname{cis}(\pi/2), \quad \operatorname{cis}(5\pi/6).$$

The corresponding Cartesian forms are

$$-(\sqrt{3}+i)/2, \quad -i, \quad (\sqrt{3}-i)/2, \quad (\sqrt{3}+i)/2, \quad i, \quad (-\sqrt{3}+i)/2.$$

They form the vertices of a regular hexagon. The principal sixth root of -1 is the fourth member of this list.

(e) $(1+3i)^{12}$. We know that the real and imaginary parts must be exact integers. So if we use a calculator to construct the polar form to, say, ten significant figures, then the corresponding Cartesian form can be rounded off to find the required integers. First, let us get the exact polar form. Let $z = 1+3i$. Then

$$r = |z| = \sqrt{10}, \qquad \theta = \arg z = \tan^{-1} 3.$$

The polar form of z is $\sqrt{10}\operatorname{cis}(\tan^{-1} 3)$. The 12th power of z is

$$(1+3i)^{12} = 10^6 \operatorname{cis}(12\tan^{-1} 3) = 10^6 \operatorname{cis}(12\tan^{-1} 3 - 4\pi).$$

The expression on the right is the required polar form with the argument in the principal range (determined with the help of a calculator). Now, use a 10-digit calculator to give the following decimal approximations (with possibly different round-off errors in the last digits):

$$\theta = \tan^{-1} 3 \approx 1.249045772, \qquad 12\theta \approx 14.98854926,$$

$$\cos(12\theta) \approx -0.7521919942, \qquad \sin(12\theta) \approx 0.6589440066,$$

$$10^6 \cos(12\theta) \approx -752191.9942, \qquad 10^6 \sin(12\theta) \approx 658944.0066.$$

(Different calculators may differ in the last one or two digits, depending on how they handle round-off errors. Some calculators hold an eleventh digit internally to slow down the accumulation of round-off errors. Some calculators may just truncate $\cos(12\theta)$, for example, to six digits, giving -0.752192.) Here, we know that the answer has integer real and imaginary parts. So, rounding off to the nearest integers gives the exact value,

$$(1+3i)^{12} = -752192 + 658944i = 64(-11753 + 10296i).$$

A better way to get the exact value of $(1+3i)^{12}$ with a 10-digit calculator is to observe, first, that $(1+3i)^2 = 2(-4+3i)$. This implies

$$(1+3i)^{12} = 64(-4+3i)^6,$$

which is easier to calculate by the above method (or directly). This is left as an exercise.

3. (a) (i). $|z-3+i|<16$. This region is the interior of a circle, boundary not included, with centre at $3-i$ and radius 16. In Cartesian coordinates, the inequality reads,

$$(x-3)^2+(y+1)^2 \;<\; 16^2.$$

Use a dashed (or dotted) line for the circular boundary and shade the interior.

(ii). $\mathrm{Re}\big((1+2i)z\big) \;>\; 2$. Let $z=x+iy$. Then

$$\mathrm{Re}\big((1+2i)(x+iy)\big) \;=\; x-2y>2.$$

Draw the dashed (or dotted) straight line $y \;=\; \frac{1}{2}x-1$ and shade the open half-plane underneath this line.

(iii). $|z-3|+|z+3| \;=\; 10$. This is the equation of an ellipse, centred at the origin. If you know that already, then it is easy to find its axes. Put $z=x$ (real) and get $x=\pm 5$. Put $z=iy$ (pure-imaginary) and get $y=\pm 4$. So the semimajor axis is 5 and the semiminor axis is 4.

If you do not recognize that this is the equation of an ellipse, then you can find out what it is by doing a calculation. Let $z=x+iy$. Then,

$$\begin{aligned}
\sqrt{(x-3)^2+y^2} \;+\; \sqrt{(x+3)^2+y^2} \;&=\; 10,\\
\sqrt{(x+3)^2+y^2} \;&=\; 10 \;-\; \sqrt{(x-3)^2+y^2}\,,\\
(x+3)^2+y^2 \;&=\; 100-20\sqrt{(x-3)^2+y^2}+(x-3)^2+y^2,\\
20\sqrt{(x-3)^2+y^2} \;&=\; 100-12x,\\
5\sqrt{(x-3)^2+y^2} \;&=\; 25-3x,\\
25\big((x-3)^2+y^2\big) \;&=\; (25-3x)^2,\\
25x^2-150x+225+25y^2 \;&=\; 625-150x+9x^2,\\
16x^2+25y^2 \;&=\; 400,\\
\frac{x^2}{25}+\frac{y^2}{16} \;&=\; 1.
\end{aligned}$$

We have arrived at the equation of an ellipse in standard Cartesian form,

$$\frac{x^2}{a^2}\;+\;\frac{y^2}{b^2} \;=\; 1, \qquad a=5, \qquad b=4.$$

The semimajor axis is 5 and the semiminor axis is 4.

(b) First, invert the map $w=f(z)=(2z-1)/(z-2)$ to get

$$z \;=\; f^{-1}(w) \;=\; \frac{2w-1}{w-2}\,.$$

Notice that $f^{-1}(z)=f(z)$. Let $w=u+iv$. The image of the unit circle $|z|=1$ in the complex w-plane has equation,

$$|2w-1| \;=\; |w-2|,$$

$$\begin{aligned}
|2u-1+2iv| &= |u-2+iv|, \\
(2u-1)^2+4v^2 &= (u-2)^2+v^2, \\
4u^2-4u+1+4v^2 &= u^2-4u+4+v^2, \\
3u^2+3v^2-3 &= 0, \\
u^2+v^2 &= 1, \\
|w| &= 1.
\end{aligned}$$

We see that the image of the unit circle $|z| = 1$ is the unit circle $|w| = 1$ in the complex w-plane. (In addition, the orientation of the circle is preserved and the interior maps to the interior, the exterior to the exterior. However, the centre does not map to the centre.)

4. Label the eleven parts (a)–(k). In the real domain, only nonnegative numbers have square roots, only positive numbers have logarithms, and only numbers in the interval $[-1, 1]$ have inverse cosines or sines. Places where a denominator is zero cannot belong to the domain of a function (unless, of course, the function is separately defined at such points).

(a) $\sqrt{4-x^2}$. The natural domain is $-2 \le x \le 2$, or, in interval notation, $[-2, 2]$.

(b) $(4-x^2)^{-1/2}$. The natural domain is $-2 < x < 2$, or $(-2, 2)$.

(c) $\ln x$. The natural domain is $x > 0$, or $(0, \infty)$.

(d) $\sqrt{\ln x}$. We require $\ln x \ge 0$. The natural domain is $x \ge 1$, or $[1, \infty)$.

(e) $\dfrac{\sin x}{x}$. The natural domain is $\mathbf{R}\backslash\{0\}$, or $(-\infty, 0) \cup (0, \infty)$, in other words, all real numbers except zero. (The missing point $x = 0$ in the domain is an example of a removable discontinuity.)

(f) $\cos^{-1}(x^2+y^2)$. The natural domain is the closed unit disc $x^2+y^2 \le 1$ in the xy-plane.

(g) $\ln(\ln(\ln x))$. We require $\ln(\ln x) > 0$, which, in turn, implies $\ln x > 1$ and $x > e$. So the natural domain is $x > e$, or (e, ∞).

(h) $\sqrt{x+iy}$ has a well-defined principal value for all real x and y, so the natural domain is $\mathbf{R}^2$.

(i) $\log(x+iy)$ has a well-defined principal value for every $(x, y) \in \mathbf{R}^2$ except the origin $(0, 0)$. Hence, the natural domain is $\mathbf{R}^2\backslash\{(0, 0)\}$.

(j) $(-1)^x$ with codomain $\mathbf{R}$ is only defined at integer values of x and at rational values of x with odd denominator. (The integers are, of course, included in the rationals, having odd denominator 1.) So the domain of $(-1)^x$ is

$$\Big\{\, x \in \mathbf{R} \;\Big|\; x = p/q,\ p, q \in \mathbf{Z},\ \text{no common factor},\ q \ge 1,\ q \text{ odd} \,\Big\}.$$

Specifically,

$$(-1)^x = \begin{cases} 1, & x = p/q,\ p \text{ even},\ q \text{ odd}, \\ -1, & x = p/q,\ p \text{ odd},\ q \text{ odd}, \\ \text{undefined}, & \text{all other real } x. \end{cases}$$

(k) $(-1)^x$ with codomain $\mathbf{C}$ is well defined for all real x provided we have a rule for choosing one value out of possibly infinitely many complex values. (If we do not impose such a rule, $(-1)^x$ fails the vertical line test for a single-valued function.) The values of $(-1)^x$ are $e^{(2n+1)i\pi x}$, where n runs through the integers. There are q distinct values when $x = p/q$, $p, q \in \mathbf{Z}$, no common factor, $q \geq 1$, and infinitely many values when x is irrational. The principal value of $(-1)^x$ is $e^{i\pi x}$, but we are not necessarily committed to choosing the principal value. (The values ± 1 in part (j), for example, are non-principal except when x is an integer.) So, as long as we select exactly one particular value of $(-1)^x$ for every real x, we have a function whose domain is $\mathbf{R}$.

5. (a) $f : [-1, 1] \to \mathbf{R}$, $x \mapsto \sinh x$. Since $\sinh x$ is strictly increasing, it is injective (passes the horizontal line test). The image of the domain $[-1, 1]$ is the interval $[-\sinh 1, \sinh 1]$. This is the range of f. Since the codomain $\mathbf{R}$ is a bigger set, the function f is not surjective (onto). It can be made surjective (and hence also bijective) by restricting the codomain to the range $[-\sinh 1, \sinh 1]$. (Another way to make f bijective is to lift the artificial restriction on the domain and extend f to a function from $\mathbf{R}$ to $\mathbf{R}$.)

(b) $f : (-\infty, \infty) \to [-1, 1]$, $x \mapsto \cos x$. Since $\cos x$ takes all values in the interval $[-1, 1]$, this interval is both the range and the codomain of f. This means that f is surjective (onto). However, f is not injective because it fails the horizontal line test. To make f injective, restrict the domain to any one of the closed intervals $[n\pi, (n+1)\pi]$, $n \in \mathbf{Z}$. Then f is either strictly increasing or decreasing on each of these intervals. In fact, f is bijective on each of these intervals. (Of course, there are infinitely many other subsets of the original domain in which f is injective or bijective.)

(c) $f : (0, \infty) \to \mathbf{R}$, $x \mapsto \ln\big(x\sqrt{x^2+2}\,\big)$. This function is strictly increasing on its domain and takes all real values. Hence it is both injective and surjective, which means that it is bijective. To get the inverse function, let $y = f(x)$, $0 < x < \infty$. Then

$$
\begin{aligned}
y &= \ln\big(x\sqrt{x^2+2}\,\big), \\
e^y &= x\sqrt{x^2+2}\,, \\
e^{2y} &= x^2(x^2+2), \\
e^{2y} + 1 &= (x^2+1)^2, \\
\sqrt{e^{2y}+1} &= x^2 + 1, \\
\sqrt{\sqrt{e^{2y}+1} - 1} &= x.
\end{aligned}
$$

All square roots are nonnegative (no $\pm$ signs). From the last line, we read off the inverse function,

$$f^{-1}(x) = \sqrt{\sqrt{e^{2x}+1} - 1}\,.$$

Its domain is $\mathbf{R}$ and its range is $(0, \infty)$.

6. Some of these limits depend on the standard limit, $(\sin\theta)/\theta \to 1$ as $\theta \to 0$, proved in lectures.

(a) $\displaystyle\lim_{\theta\to 0} \frac{\sin 3\theta}{\theta} = 3 \lim_{\theta\to 0} \frac{\sin 3\theta}{3\theta} = 3 \lim_{x\to 0} \frac{\sin x}{x} = 3.$

(b) In the case of $(\cos 3\theta)/\theta$ at $\theta = 0$, the best we can do is

$$\lim_{\theta\to 0^+} \frac{\cos 3\theta}{\theta} = +\infty, \qquad \lim_{\theta\to 0^-} \frac{\cos 3\theta}{\theta} = -\infty.$$

Since these one-sided limits are different, the required two-sided limit does not exist.

(c)

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+c} - \sqrt{x+d}} \\
&= \lim_{x\to\infty} \frac{(x+a) - (x+b)}{\sqrt{x+a} + \sqrt{x+b}} \, \frac{\sqrt{x+c} + \sqrt{x+d}}{(x+c) - (x+d)} \\
&= \frac{a-b}{c-d} \lim_{x\to\infty} \frac{\sqrt{x+c} + \sqrt{x+d}}{\sqrt{x+a} + \sqrt{x+b}} \\
&= \frac{a-b}{c-d} \lim_{x\to\infty} \frac{\sqrt{1+c/x} + \sqrt{1+d/x}}{\sqrt{1+a/x} + \sqrt{1+b/x}} \\
&= \frac{a-b}{c-d} \frac{\sqrt{1+0} + \sqrt{1+0}}{\sqrt{1+0} + \sqrt{1+0}} \\
&= \frac{a-b}{c-d}, \qquad c \neq d.
\end{aligned}$$

(d)

$$\begin{aligned}
\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3} &= \lim_{x\to 1} \frac{(x-1)(x^2+x-4)}{(x-1)(x^2+x-3)} \\
&= \lim_{x\to 1} \frac{x^2+x-4}{x^2+x-3} \\
&= \frac{1+1-4}{1+1-3} \\
&= 2.
\end{aligned}$$

(e) $\displaystyle\lim_{x\to 0^+} \frac{\sin(1/x)}{\ln x}$. We use the Squeeze Law for limits. For all $x \neq 0$,

$$-1 \le \sin(1/x) \le 1.$$

Thence, for all $x \in (0, 1)$,

$$-\frac{1}{|\ln x|} \le \frac{\sin(1/x)}{\ln x} \le \frac{1}{|\ln x|}.$$

Since $\ln x \to -\infty$ as $x \to 0^+$, the upper and lower bounds both tend to zero as $x \to 0^+$. Hence, the Squeeze Law implies

$$\lim_{x\to 0^+} \frac{\sin(1/x)}{\ln x} = 0.$$

(f) $\lim_{x\to 0^+} (\ln x)\sin(1/x)$. Let $f(x) = (\ln x)\sin(1/x)$ for $x > 0$. In any interval $(0, \delta)$, $\delta > 0$, the graph $y = f(x)$ oscillates between the curves $y = \ln x$ and $y = -\ln x$. Since $\ln x \to -\infty$ as $x \to 0^+$, the function $f(x)$ oscillates with unbounded amplitude as $x \to 0^+$. In other words, $f(x)$ is unbounded above and below as $x \to 0^+$. Hence, the required one-sided limit does not exist, whether or not infinite values are allowed.

(g) This is an important limit related to $(\sin x)/x \to 1$ as $x \to 0$.

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x^2} &= \lim_{x\to 0} \frac{1}{x^2}\,\frac{1-\cos^2 x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{1}{1+\cos x} \lim_{x\to 0} \frac{\sin^2 x}{x^2} \\
&= \frac{1}{2}\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \\
&= \frac{1}{2}.
\end{aligned}$$

(h) To evaluate the one-sided limit $\lim_{x\to 0^+} g(x)$, where

$$g(x) = \begin{cases} \sin x, & x \text{ irrational} \\ x\cos x, & x \text{ rational}, \end{cases}$$

we use the Squeeze Law. For values of $x \in (0, \pi/2)$, it was shown in lectures that $\sin x < x < \tan x$. In particular, this implies that, for $x \in (0, \pi/2)$,

$$0 \le g(x) \le x,$$

regardless of whether x is rational or irrational. The Squeeze Law immediately implies,

$$\lim_{x\to 0^+} g(x) = 0.$$

Since $g(x)$ is an odd function, the limit from the left is also zero, and so the limit is actually two-sided.

7. (a) The function $f(x) = \ln|x| + \dfrac{\sin \pi x}{x-1}$ has the domain $\mathbf{R}\backslash\{0, 1\}$. It has an obvious infinite discontinuity at $x = 0$ because

$$\lim_{x\to 0} \ln|x| = -\infty.$$

The discontinuity at $x = 1$ is removable because

$$\lim_{x\to 1} \frac{\sin \pi x}{x-1} = \lim_{q\to 0} \frac{\sin \pi(q+1)}{q} = -\pi \lim_{q\to 0} \frac{\sin \pi q}{\pi q} = -\pi,$$

where we let $x = 1 + q$. Hence, $f(x)$ has a finite two-sided limit as $x \to 1$. This discontinuity can be removed by defining $f(1) = -\pi$.

(b) The function,

$$f(x) = \begin{cases} k \cosh x, & x < 0 \\ (x^2 - 9)/(x-3), & 0 < x < 3 \text{ and } x > 3, \end{cases}$$

needs to be examined at the points $x = 0$ and $x = 3$, which are missing from the domain. At $x = 3$, $f(x)$ has the two-sided limit,

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3} (x+3) = 6.$$

So the discontinuity at $x = 3$ is removable, and it can be removed by defining $f(3) = 6$. At $x = 0$, consider the one-sided limits from each side:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} k\cosh x = k,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+3) = 3.$$

When $k \neq 3$, $f(x)$ has a jump discontinuity at $x = 0$. When $k = 3$, $f(x)$ has a removable discontinuity at $x = 0$. So, when $k = 3$, both discontinuities are removable and we can extend the domain of f from $\mathbf{R}\backslash\{0, 3\}$ to all of $\mathbf{R}$. Use a new symbol g to denote the extended function. The formula for g involving two rules is

$$g(x) = \begin{cases} 3\cosh x, & x < 0 \\ x + 3, & x \geq 0. \end{cases}$$

8. Define the function $g(x) = \cos x - x \sin x$. We want to show that $g(x)$ has a zero on the open interval $(0, \pi/2)$, which is a way of saying that the equation $g(x) = 0$ has a root on that interval. The function $g(x)$ is defined and continuous on $\mathbf{R}$, and, in particular, on the closed interval $[0, \pi/2]$. At the endpoints,

$$g(0) = 1, \qquad g(\pi/2) = -\pi/2.$$

So $g(x)$ is positive at one endpoint and negative at the other. The Intermediate Value Theorem implies that $g(x)$ has a zero at at least one interior point of the interval.

Next, consider $x = \pi/4$ and $x = \pi/3$:

$$g(\pi/4) = \cos(\pi/4) - (\pi/4)\sin(\pi/4) = (4-\pi)/(4\sqrt{2}\,) > 0,$$

$$g(\pi/3) = \cos(\pi/3) - (\pi/3)\sin(\pi/3) = (3 - \pi\sqrt{3}\,)/6 < 0.$$

Again $g(x)$ has opposite signs at the endpoints of the interval $[\pi/4, \pi/3]$. Hence, it has a zero at at least one interior point.

Remark. The function $g(x)$ is monotonically decreasing on $[0, \pi/2]$. This implies that there is one, and only one, zero of $g(x)$ on the interval. A little experimenting with a pocket calculator (especially if you know Newton's method) identifies its value as $x = 0.8603335890$.

9. The Quotient Law for limits may be stated as follows:

> Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ with $M \neq 0$. Then $f(x)/g(x)$ tends to a limit as $x \to a$, and
> $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

To prove this statement, we need to be able to find $\delta > 0$ such that, whenever $\epsilon > 0$ is given,

$$\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| < \epsilon$$

for all x such that $0 < |x - a| < \delta$. By changing the sign of $g(x)$, if necessary, we can make the convenient restriction that $M > 0$ without losing any generality.

Let $\epsilon_1 > 0$ be given. (The relationship between ϵ_1 and ϵ will be decided later.) The existence of the limit L means that there exists $\delta_1 > 0$, depending on ϵ_1, such that

$$\left| f(x) - L \right| < \epsilon_1$$

for all x such that $0 < |x - a| < \delta_1$. Similarly, the existence of the limit M means that there exists $\delta_2 > 0$ such that

$$\left| g(x) - M \right| < \epsilon_1$$

for all x such that $0 < |x - a| < \delta_2$. A particular choice of ϵ_1 here is $M/2$, which is positive by hypothesis. So there is a number δ_3 such that

$$M/2 < g(x) < 3M/2$$

for all x such that $0 < |x - a| < \delta_3$. This makes sure that we have a neighbourhood of $x = a$ (possibly excluding $x = a$ itself) on which $g(x)$ is bounded away from zero. Specifically, we know that $0 < 1/g(x) < 2/M$ for $0 < |x - a| < \delta_3$. Now define

$$\delta = \min(\delta_1, \delta_2, \delta_3).$$

This δ depends on ϵ_1 and guarantees that all the inequalities,

$$\left| f(x) - L \right| < \epsilon_1, \qquad \left| g(x) - M \right| < \epsilon_1, \qquad 0 < 1/g(x) < 2/M,$$

hold for all x such that $0 < |x - a| < \delta$.

The next stage of the proof is to decompose the expression $f(x)/g(x) - L/M$ into known small quantities related to ϵ_1. First, we have

$$\begin{aligned}\frac{f(x)}{g(x)} - \frac{L}{M} &= \frac{f(x)-L}{g(x)} + L\left(\frac{1}{g(x)} - \frac{1}{M}\right) \\ &= \frac{f(x)-L}{g(x)} + \frac{L(M-g(x))}{Mg(x)}.\end{aligned}$$

Next suppose that $0 < |x-a| < \delta$. The triangle inequality and the known bounds give

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &\le \frac{|f(x)-L|}{|g(x)|} + \frac{|L|\,|M-g(x)|}{M|g(x)|} \\ &< \frac{2\epsilon_1}{M} + \frac{2|L|\epsilon_1}{M^2} \\ &= \frac{2(|L|+M)}{M^2}\,\epsilon_1.\end{aligned}$$

Now, let arbitrary $\epsilon > 0$ be given and choose ϵ_1 such that

$$\epsilon_1 < \frac{M^2}{2(|L|+M)}\,\epsilon.$$

This makes δ depend on ϵ, and δ now has the property that

$$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| < \epsilon$$

for all x such that $0 < |x-a| < \delta$. This completes the proof that $f(x)/g(x)$ tends to a limit as $x \to a$ and that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

10. **Preamble**. A set A is **countable** if its elements can be put in a one-to-one correspondence with the positive integers. In other words, they can be listed as a sequence in some order. This is equivalent to saying that we can produce a function,

$$f : A \to \mathbf{Z}^+,$$

that is bijective (both injective and surjective). However, all subsets of a countable set are either countable or finite. Consequently, a set A must be countable or finite if we just have an injective function $g : A \to B$, where B is countable.

An exercise on Tutorial 4 showed that the set of all rational numbers is countable. Specifically, the rational numbers in the interval $[0,1]$ can be counted by gathering them into finite blocks as follows:

$$\begin{gathered}0,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{1}{6},\ \frac{5}{6}, \\ \frac{1}{7},\ \frac{2}{7},\ \frac{3}{7},\ \frac{4}{7},\ \frac{5}{7},\ \frac{6}{7},\ \frac{1}{8},\ \frac{3}{8},\ \frac{5}{8},\ \frac{7}{8},\ \frac{1}{9},\ \ldots .\end{gathered}$$

A similar construction will count the rationals on any finite interval. To count all the rationals, a different way of gathering into blocks is needed. Let the rationals be written p/q with $p, q \in \mathbf{Z}$, $q > 0$, and p and q having no common factor. Next, let p/q be assigned the index $N = |p| + q$. Only finitely many rationals have any given index. So they can be counted by listing all the rationals with index 1, then index 2, then index 3, and so on.

The same method can be applied to integer ordered pairs, and then to rational ordered pairs, ordered triples, and so on. More generally, if $\{r_n\}$ and $\{s_n\}$, $n = 1, 2, 3, \ldots$, are countable sets, then so is the set of ordered pairs $\{(r_m, s_n)\}$, because we can arrange these pairs in a sequence as follows:

$$(r_1, s_1), \quad (r_1, s_2), \quad (r_2, s_1), \quad (r_1, s_3), \quad (r_2, s_2), \quad (r_3, s_1),$$

$$(r_1, s_4), \quad (r_2, s_3), \quad (r_3, s_2), \quad (r_4, s_1), \quad (r_1, s_5), \quad \ldots .$$

The set of algebraic numbers is also countable. Algebraic numbers are real or complex roots of polynomial equations with integer coefficients. (The roots of polynomial equations with algebraic coefficients are still algebraic, since they are contained among the roots of a higher-degree polynomial with integer coefficients.) To see that the algebraic numbers are countable, define an index N of an algebraic number to be the sum of the degree of the simplest polynomial of which it is a root and of the absolute values of all the integer coefficients of that polynomial. Only finitely many algebraic numbers have a given index. Starting at $N = 2$ (the lowest index possible), list all algebraic numbers with index 2, then index 3, then index 4, and so on. Eventually, all algebraic numbers, both real and complex, will be listed in a countable sequence.

Real or complex numbers that are not algebraic are called transcendental. Since the reals or not countable, neither are the real transcendentals, and then neither are the complex transcendentals. In a sense, almost all real and complex numbers are transcendental. Known examples of transcendental numbers include e, π, e^π, $2^{\sqrt{3}}$, $\ln 2$, $\log_{10} 2$ and $\cos 1$. (However, note that trig identities imply that $\cos(\pi r)$ and $\sin(\pi r)$ are algebraic whenever r is rational.) The numbers $\pi + e$, πe, π^e and $\pi^{\sqrt{2}}$ are presumed to be transcendental, but have not even been proved irrational, except that at least one of $\pi + e$ and πe must be transcendental. [End of preamble.]

We wish to show that nondecreasing functions $f : \mathbf{R} \to \mathbf{R}$ have either a finite or a countable number of discontinuities. Finite includes zero, so nondecreasing functions may be continuous everywhere. Let A denote any interval of positive or infinite length. The function $f : A \to \mathbf{R}$ is nondecreasing on A if $a, b \in A$ and $b > a$ implies $f(b) \geq f(a)$. It is strictly increasing if $f(b) > f(a)$ always. To be not strictly increasing, a nondecreasing function must be constant somewhere on a subinterval. (Strictly increasing functions can have horizontal tangents, as in the cases x^3 and $x - \sin x$.)

The first step is to show that all discontinuities, if any, of a nondecreasing function are **jump** discontinuities. $f(x)$ has a jump discontinuity at $x = c$ if the one-sided limits $f(c-)$ and $f(c+)$ both exist and are different. (The definition does not require c to be in the domain of f, but c must be in the domain of f in the present context.) Suppose $f(x)$ is nondecreasing on an interval A, and that c is in A minus its left endpoint, if it has one. This means that we can approach c from the left. Let $\{c_n\}$ be any increasing sequence in A whose limit is c. Then $\{f(c_n\}$ is a nondecreasing sequence of real numbers that is bounded above by $f(c)$. Hence the sequence $\{f(c_n)\}$ tends to a finite limit. This limit is independent of the choice of the c_n, and its unique value is $f(c-)$. The existence of $f(c+)$ is proved similarly when c can be approached from the right. So, when c is an interior point of A, we have

$$f(c-) \leq f(c) \leq f(c+).$$

The only two possibilities are

- $f(c+) = f(c) = f(c-)$, in which case $f(x)$ is continuous at $x = c$.
- $f(c+) > f(c-)$, in which case $f(x)$ has a jump discontinuity at $x = c$ with $f(c-) \leq f(c) \leq f(c+)$.

There are obvious modifications when A is a closed or half-closed interval and c is an included endpoint of A. If c is a right endpoint of A, then either $f(x)$ is continuous from the left at c or else $f(c-)$ exists and $f(c-) < f(c)$. Similarly, if c is a left endpoint of A, then either $f(x)$ is continuous from the right at c or else $f(c+)$ exists and $f(c+) > f(c)$. Removable discontinuities are not possible in a nondecreasing function, except for the one-sided jumps just described. Similarly, infinite discontinuities do not occur in a nondecreasing function, except that such a function may run off to $\pm\infty$ approaching the non-included endpoints of an open or half-open interval.

We have established that all discontinuities of a nondecreasing function are jump discontinuities with positive jumps. To show that the set of such discontinuities is countable, it is convenient to restrict attention first to the case,

$$f : [0, 1] \to [0, 1].$$

The biggest possible jump is 1, and this can only occur once, if at all. In fact, a jump J satisfying $1/2 < J \leq 1$ can only occur once, if at all. A jump J satisfying $1/3 < J \leq 1/2$ can occur at most twice. A jump J satisfying $1/4 < J \leq 1/3$ can occur at most three times, and so on.

If $c \in [0, 1]$ is any jump discontinuity of f, define

$$J(c) = \begin{cases} f(c+) - f(c-), & c \in (0, 1), \\ f(0+) - f(0), & c = 0, \\ f(1) - f(1-), & c = 1. \end{cases}$$

For $k = 1, 2, 3, \ldots$, there are at most k values of $c \in [0, 1]$ such that $1/(k+1) < J(c) \leq 1/k$. This allows us to list all the jump discontinuities of $f(x)$, if any, in a finite or infinite sequence of finite blocks. So the number of discontinuities of a nondecreasing function $f : [0, 1] \to [0, 1]$ is either finite or countable.

Next, consider $f : [0, 1] \to \mathbf{R}$. Since $f(0)$ and $f(1)$ are finite real numbers (because 0 and 1 are included in the domain) and f is nondecreasing, we see that the range of f is a subset of the closed interval $[f(0), f(1)]$. Set aside the trivial case where f is constant (which is continuous everywhere), so that we can assume $f(1) > f(0)$. Define $g(x) = (f(x) - f(0))/(f(1) - f(0))$. Then $g(0) = 0$ and $g(1) = 1$ and $g : [0, 1] \to [0, 1]$. Since g

has a finite or countable number of discontinuities, so also does f at the same locations. Similarly, any nondecreasing function $f : [a, b] \to \mathbf{R}$ has a finite or countable number of discontinuities.

Finally, consider $f : \mathbf{R} \to \mathbf{R}$, with f nondecreasing. The restrictions of f to each of the closed subintervals,

$$[0, 1], \ \ [-1, 0], \ \ [1, 2], \ \ [-2, -1], \ \ [2, 3], \ \ [-3, -2], \ \ \ldots ,$$

has a finite or countable number of discontinuities. (If a jump at an integer endpoint is being counted twice, just keep one and ignore the other.) Since there are a countable number of subintervals, we conclude that $f : \mathbf{R} \to \mathbf{R}$ has a finite or countable number of discontinuities.

We have just shown that nondecreasing functions are continuous almost everywhere in a sense. Much deeper theorems assert that nondecreasing functions are differentiable almost everywhere and concave-up functions are twice-differentiable almost everywhere. In these cases, the definition of "almost everywhere" allows exceptional sets that are not necessarily countable, but are still vanishingly small in a precisely defined sense.