CHAPTER **EIGHT**

# Euclidean Geometry

The methods and structures of modern mathematics were established first by the ancient Greeks in their studies of geometry and arithmetic. It was they who realised that mathematics must proceed by rigorous proof and argument, that all definitions must be stated with absolute precision, and that any hidden assumptions, called *axioms*, must be brought out into the open and examined. Their work is extraordinary for their determination to prove details that may seem common sense to the layman, and for their ability to ask the most important questions about the subjects they investigated. Many Greeks, like the mathematician Pythagoras and the philosopher Plato, spoke of mathematics in mystical terms as the highest form of knowledge, and they called their results *theorems* — the Greek word *theorem* means 'a thing to be gazed upon' or 'a thing contemplated by the mind', from θεωρέω 'behold' (our word *theatre* comes from the same root).

Of all the Greek books, Euclid's *Elements* has been the most influential, and was still used as a textbook in nineteenth-century schools. Euclid constructs a large body of theory in geometry and arithmetic beginning from almost nothing — he writes down a handful of initial assumptions and definitions that mostly seem trivial, such as 'Things that are each equal to the same thing are equal to one another'. As is common in Greek mathematics, Euclid introduces geometry first, and then develops arithmetic ideas from it. For example, the product of two numbers is usually understood as the area of a rectangle. Such intertwining of arithmetic and geometry is still characteristic of the most modern mathematics, and has been evident in our treatment of the calculus, which has drawn its intuitions equally from algebraic formulae and from the geometry of curves, tangents and areas.

Geometry done using the methods established in Euclid's book is called *Euclidean geometry*. We have assumed throughout this text that students were familiar from earlier years with the basic methods and results of Euclidean geometry, and we have used these geometric results freely in arguments. This chapter and the next will now review Euclidean geometry from its beginnings and develop it a little further. Our foundations can unfortunately be nothing like as rigorous as Euclid's. For example, we shall assume the four standard congruence tests rather than proving them, and our second theorem is his thirty-second. Nevertheless, the arguments used here are close to those of Euclid, and are strikingly different from those we have used in calculus and algebra. The whole topic is intended to provide a quite different insight into the nature of mathematics.

Constructions with straight edge and compasses are central to Euclid's arguments, and we have therefore included a number of construction problems in an unsystematic fashion. They need to be proven, and they need to be drawn. Their importance lies not in any practical use, but in their logic. For example, three

ISBN: 978-1-107-61604-2  Cambridge University Press

famous constructions unsolved by the Greeks — the trisection of a given angle, the squaring of a given circle (essentially the construction of $\pi$) and the doubling in volume of a given cube (essentially the construction of $\sqrt[3]{2}$) — were an inspiration to mathematicians of the nineteenth century grappling with the problem of defining the real numbers by non-geometric methods. All three constructions were eventually proven to be impossible.

STUDY NOTES: Most of this material will have been covered in Years 9 and 10, but perhaps not in the systematic fashion developed here. Attention should therefore be on careful exposition of the logic of the proofs, on the logical sequence established by the chain of theorems, and on the harder problems. The only entirely new work is in the final Section 8I on intercepts.

Many of the theorems are only stated in the notes, with their proofs left to structured questions in the following exercise. All such questions have been placed at the start of the Development section, even through they may be more difficult than succeeding problems, and are marked 'COURSE THEOREM' — working through these proofs is an essential part of the course.

There are many possible orders in which the theorems of this course could have been developed, but the order given here is that established by the Syllabus. All theorems marked as course theorems may be used in later questions, except where the intention of the question is to provide a proof of the theorem. Students should note carefully that *the large number of further theorems proven in the exercises cannot be used in subsequent questions*.

# 8 A Points, Lines, Parallels and Angles

The elementary objects of geometry are points, lines and planes. Rigorous definitions of these things are possible, but very difficult. Our approach, therefore, will be the same as our approach to the real numbers — we shall describe some of their properties and list some of the assumptions we shall need to make about them.

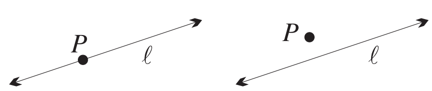
**Points, Lines and Planes:** These simple descriptions should be sufficient.

POINTS: A *point* can be described as having a position but no size. The mark opposite has a definite width, and so is not a point, but it represents a point in our imagination.

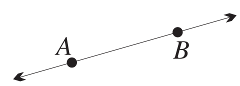

LINES: A *line* has no breadth, but extends infinitely in both directions. The drawing opposite has width and has ends, but it represents a line in our imagination.


PLANES: A *plane* has no thickness, and it extends infinitely in all directions. Almost all our work is two-dimensional, and takes place entirely in a fixed plane.

**Points and Lines in a Plane:** Here are some of the assumptions that we shall be making about the relationships between points and lines in a plane.

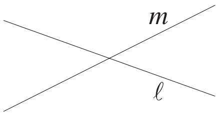


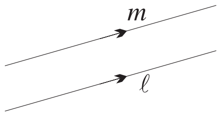

POINT AND LINE: Given a point $P$ and a line $\ell$, the point $P$ may or may not lie on the line $\ell$.

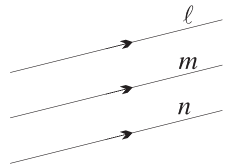



TWO POINTS: Two distinct points $A$ and $B$ lie on one and only one line, which can be named $AB$ or $BA$.

ISBN: 978-1-107-61604-2 



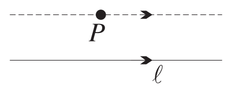




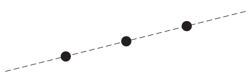
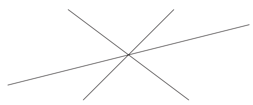

TWO LINES: Given two distinct lines $\ell$ and $m$ in a plane, either the lines intersect in a single point, or the lines have no point in common and are called *parallel lines*, written as $\ell \parallel m$.

THREE PARALLEL LINES: If two lines are each parallel to a third line, then they are parallel to each other.

THE PARALLEL LINE THROUGH A GIVEN POINT: Given a line $\ell$ and a point $P$ not on $\ell$, there is one and only one line through $P$ parallel to $\ell$.



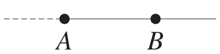

**Collinear Points and Concurrent Lines:** A third point may or may not lie on the line determined by two other points. Similarly, a third line may or may not pass through the point of intersection of two other lines.

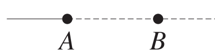



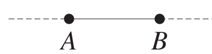
COLLINEAR POINTS: Three or more distinct points are called *collinear* if they all lie on a single line.

CONCURRENT LINES: Three or more distinct lines are called *concurrent* if they all pass through a single point.

**Intervals and Rays:** These definitions rely on the idea that a point on a line divides the rest of the line into two parts. Let $A$ and $B$ be two distinct points on a line $\ell$.








RAYS:
The *ray* $AB$ consists of the *endpoint* $A$ together with $B$ and all the other points of $\ell$ on the same side of $A$ as $B$ is.

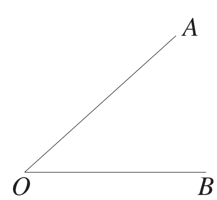
OPPOSITE RAY:
The ray that starts at this same endpoint $A$, but goes in the opposite direction, is called the *opposite ray*.

INTERVALS:
The *interval* $AB$ consists of all the points lying on $\ell$ between $A$ and $B$, including these two endpoints.

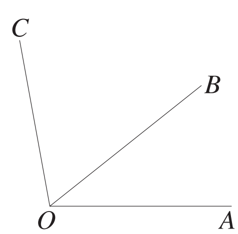
LENGTHS OF INTERVALS: We shall assume that intervals can be measured, and their lengths compared and added and subtracted with compasses.

**Angles:** We need to distinguish between an angle and the size of an angle.

ANGLES: An *angle* consists of two rays with a common endpoint. The two rays $OA$ and $OB$ in the diagram form an angle named either $\angle AOB$ or $\angle BOA$. The common endpoint $O$ is called the *vertex* of the angle, and the rays $OA$ and $OB$ are called the *arms* of the angle.




ADJACENT ANGLES: Two angles are called *adjacent angles* if they have a common vertex and a common arm. In the diagram opposite, $\angle AOB$ and $\angle BOC$ are adjacent angles with common vertex $O$ and common arm $OB$. Also, the overlapping angles $\angle AOC$ and $\angle AOB$ are adjacent angles, having common vertex $O$ and common arm $OA$.

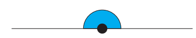



ISBN: 978-1-107-61604-2

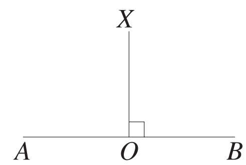
MEASURING ANGLES: The size of an angle is the amount of turning as one arm is rotated about the vertex onto the other arm. The units of degrees are based on the ancient Babylonian system of dividing the revolution into 360 equal parts — there are about 360 days in a year, and so the sun moves about $1^\circ$ against the fixed stars every day. The measurement of angles is based on the obvious assumption that the sizes of adjacent angles can be added and subtracted.

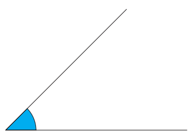
REVOLUTIONS: A *revolution* is the angle formed by rotating a ray about its endpoint once until it comes back onto itself. A revolution is defined to measure $360^\circ$.

STRAIGHT ANGLES: A *straight angle* is the angle formed by a ray and its opposite ray. A straight angle is half a revolution, and so measures $180^\circ$.

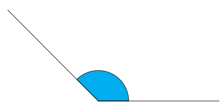


RIGHT ANGLES: Suppose that $AOB$ is a line, and $OX$ is a ray such that $\angle XOA$ is equal to $\angle XOB$. Then $\angle XOA$ is called a *right angle*. A right angle is half a straight angle, and so measures $90^\circ$.

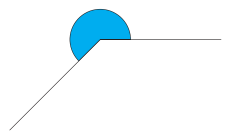


ACUTE ANGLES:
An *acute angle* is an angle greater than $0^\circ$ and less than a right angle.



OBTUSE ANGLES:
An *obtuse angle* is an angle greater than a right angle and less than a straight angle.

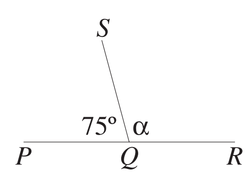


REFLEX ANGLES:
A *reflex angle* is an angle greater than a straight angle and less than a revolution.

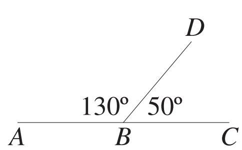
**Angles at a Point:** Two angles are called *complementary* if they add to $90^\circ$. For example, $15^\circ$ is the *complement* of $75^\circ$. Two angles are called *supplementary* if they add to $180^\circ$. For example, $105^\circ$ is the *supplement* of $75^\circ$. Our first theorem relies on the assumption that adjacent angles can be added.

**1**

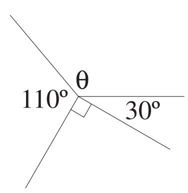
COURSE THEOREM — ANGLES IN A STRAIGHT LINE AND IN A REVOLUTION:
- Two adjacent angles in a straight angle are supplementary.
- Conversely, if adjacent angles are supplementary, they form a straight line.
- Adjacent angles in a revolution add to $360^\circ$.




Given that $PQR$ is a line, $\alpha = 105^\circ$ (angles in a straight angle).




$A$, $B$ and $C$ are collinear (adjacent angles are supplementary).

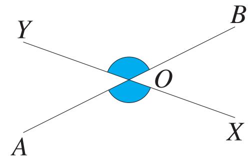



$\theta + 110^\circ + 90^\circ + 30^\circ = 360^\circ$ (angles in a revolution), $\theta = 130^\circ$.

ISBN: 978-1-107-61604-2

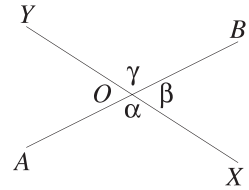
Cambridge University Press

**Vertically Opposite Angles:** Each pair of opposite angles formed when two lines intersect are called *vertically opposite angles*. In the diagram to the right, $AB$ and $XY$ intersect at $O$. The marked angles $\angle AOX$ and $\angle BOY$ are vertically opposite. The unmarked angles $\angle AOY$ and $\angle BOX$ are also vertically opposite.




**2** COURSE THEOREM: Vertically opposite angles are equal.

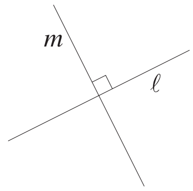
GIVEN: Let the lines $AB$ and $XY$ intersect at $O$.
Let $\alpha = \angle AOX$, let $\beta = \angle BOX$, and let $\gamma = \angle BOY$.




AIM: To prove that $\alpha = \gamma$.

PROOF: $\alpha + \beta = 180^\circ$ (straight angle $\angle AOB$),
and $\gamma + \beta = 180^\circ$ (straight angle $\angle XOY$),
so $\alpha = \gamma$.

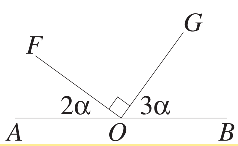
**Perpendicular Lines:** Two lines $\ell$ and $m$ are called *perpendicular*, written as $\ell \perp m$, if they intersect so that one of the angles between them is a right angle. Because adjacent angles on a straight line are supplementary, all four angles must be right angles.



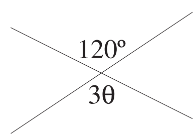

**Using Reasons in Arguments:** Geometrical arguments require reasons to be given for each statement — the whole topic is traditionally regarded as providing training in the writing of mathematical proofs. These reasons can be expressed in ordinary prose, or each reason can be given in brackets after the statement it justifies. All reasons should, wherever possible, give the names of the angles or lines or triangles referred to, otherwise there can be ambiguities about exactly what argument has been used. The authors of this book have boxed the theorems and assumptions that can be quoted as reasons.

**WORKED EXERCISE:** Find $\alpha$ or $\theta$ in each diagram below.

(a)




(b)




SOLUTION:

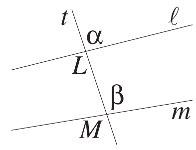
(a) $2\alpha + 90^\circ + 3\alpha = 180^\circ$
(straight angle $\angle AOB$),
$5\alpha = 90^\circ$
$\alpha = 18^\circ$.

(b) $3\theta = 120^\circ$
(vertically opposite angles),
$\theta = 40^\circ$.

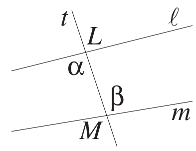
**Angles and Parallel Lines:** The standard results about alternate, corresponding and co-interior angles are taken as assumptions.

TRANSVERSALS: A *transversal* is a line that crosses two other lines (the two other lines may or may not be parallel). In each of the three diagrams below, $t$ is a transversal to the lines $\ell$ and $m$, meeting them at $L$ and $M$ respectively.

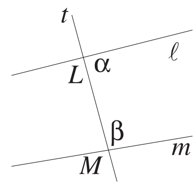
ISBN: 978-1-107-61604-2




CORRESPONDING ANGLES: In the first diagram opposite, the two angles marked $\alpha$ and $\beta$ are called *corresponding angles*, because they are in corresponding positions around the two vertices $L$ and $M$.




ALTERNATE ANGLES: In the second diagram opposite, the two angles marked $\alpha$ and $\beta$ are called *alternate angles*, because they are on alternate sides of the transversal $t$ (they must also be inside the region between the lines $\ell$ and $m$).




CO-INTERIOR ANGLES: In the third diagram opposite, the two angles marked $\alpha$ and $\beta$ are called *co-interior angles*, because they are inside the two lines $\ell$ and $m$, and on the same side of the transversal $t$.

Our assumptions about corresponding, alternate and co-interior angles fall into two groups. The first group are consequences arising when the lines are parallel.

**3**

ASSUMPTION: Suppose that a transversal crosses two lines.

- If the lines are parallel, then any two corresponding angles are equal.
- If the lines are parallel, then any two alternate angles are equal.
- If the lines are parallel, then any two co-interior angles are supplementary.

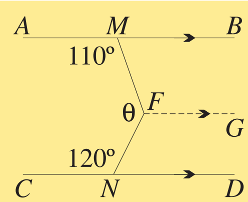
The second group are often neglected. They are the converses of the first group, and give conditions for the two lines to be parallel.

**4**

ASSUMPTION: Suppose that a transversal crosses two lines.

- If any pair of corresponding angles are equal, then the lines are parallel.
- If any pair of alternate angles are equal, then the lines are parallel.
- If any two co-interior angles are supplementary, then the lines are parallel.

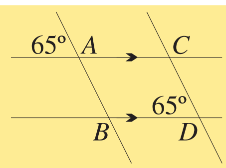
**WORKED EXERCISE:** [A problem requiring a construction]
Find $\theta$ in the diagram opposite.




SOLUTION: Construct $FG \parallel AB$.

Then $\angle MFG = 110^\circ$ (alternate angles, $FG \parallel AB$),
and $\angle NFG = 120^\circ$ (alternate angles, $FG \parallel CD$),
so $\theta + 110^\circ + 120^\circ = 360^\circ$ (angles in a revolution at $F$),
$\theta = 130^\circ$.

**WORKED EXERCISE:** Given that $AC \parallel BD$, prove that $AB \parallel CD$.




SOLUTION: $\angle CAB = 65^\circ$ (vertically opposite at $A$),
so $\angle ABD = 115^\circ$ (co-interior angles, $AC \parallel BD$),
so $AB \parallel CD$ (co-interior angles are supplementary).

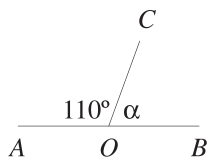
NOTE: A phrase like ‘(co-interior angles)’ alone is never sufficient as a reason. If the two angles are being proven supplementary, the fact that the lines are parallel must also be stated. If the two lines are being proven parallel, the fact that the co-interior angles are supplementary must be stated.

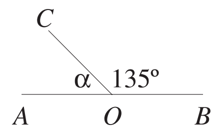
ISBN: 978-1-107-61604-2 

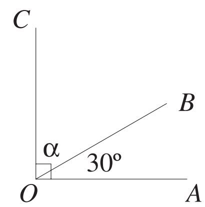
## Exercise 8A

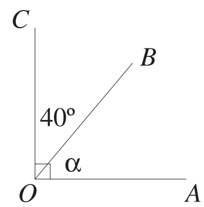
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

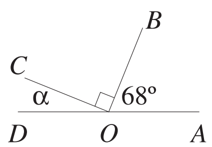
**1.** Find the angles $\alpha$ and $\beta$ in the diagrams below, giving reasons.

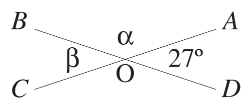
(a)


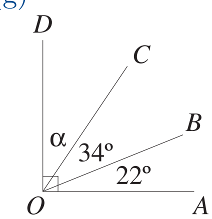

(b)


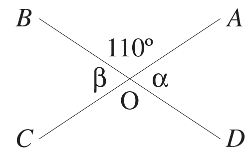

(c)



(d)


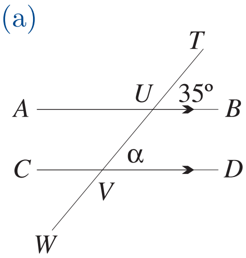

(e)


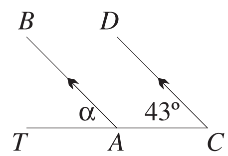

(f)


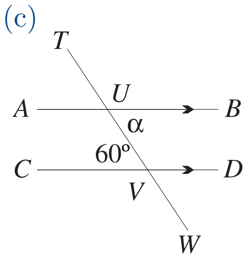

(g)


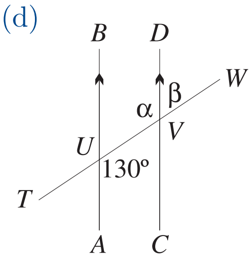

(h)


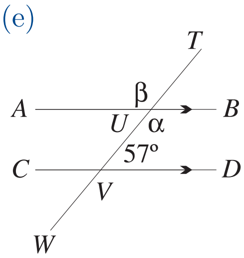

**2.** Find the angles $\alpha$ and $\beta$ in each figure below, giving reasons.

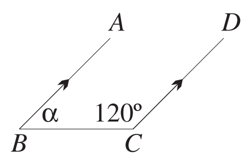
(a)


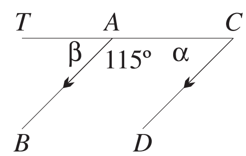

(b)


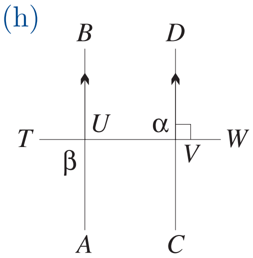

(c)



(d)


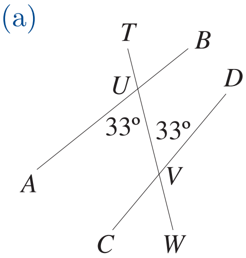

(e)


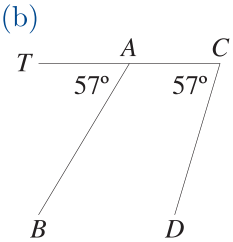

(f)


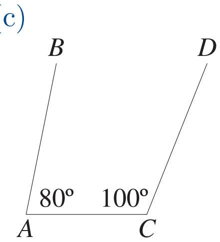

(g)


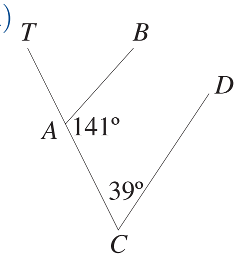

(h)



**3.** Show that $AB \parallel CD$ in the diagrams below, giving all reasons.

(a)



(b)



(c)



(d)


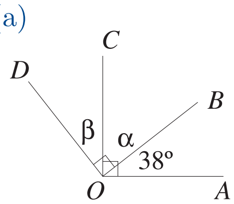

**4.** (a) Sketch a transversal crossing two non-parallel lines so that a pair of alternate angles formed by the transversal are about 45° and 65°.

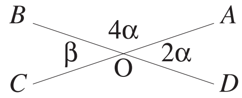
(b) Repeat part (a) so that a pair of corresponding angles are about 90° and 120°.

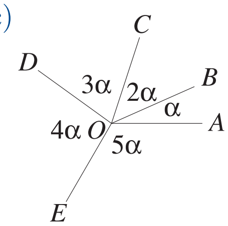
(c) Repeat part (a) so that a pair of co-interior angles are both about 80°.

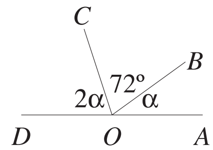
ISBN: 978-1-107-61604-2  Cambridge University Press

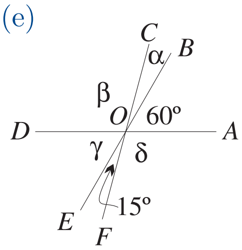
**5.** Find the angles $\alpha$, $\beta$, $\gamma$ and $\delta$ in the diagrams below, giving reasons.

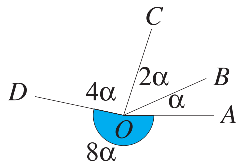
(a)


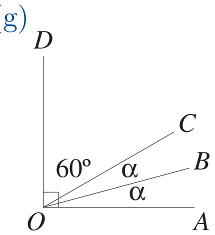

(b)


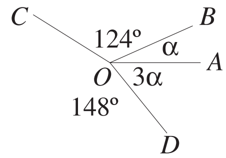

(c)



(d)


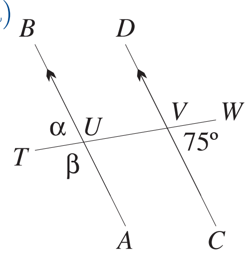

(e)


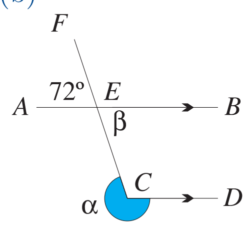

(f)


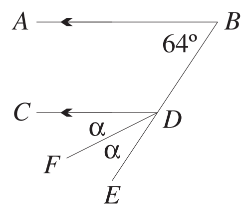

(g)


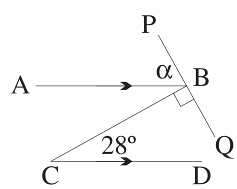

(h)


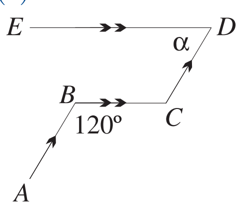

**6.** Find the angles $\alpha$ and $\beta$ in each diagram below. Give all steps in your arguments.

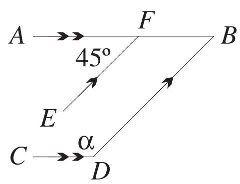
(a)


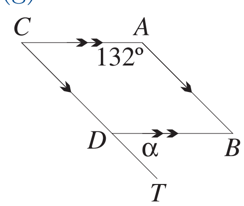

(b)


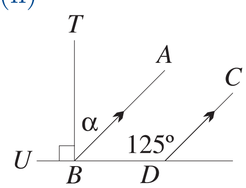

(c)


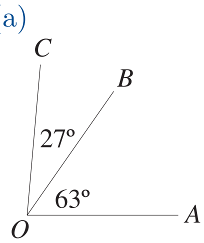

(d)



(e)


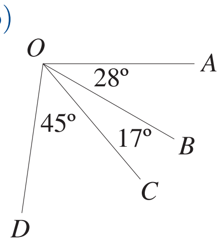

(f)



(g)


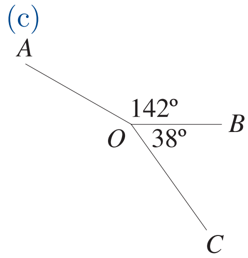

(h)



**7.** (a)


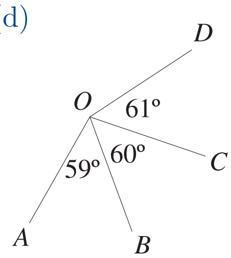

Show that $OC \perp OA$.

(b)



Show that $OD \perp OA$.

(c)


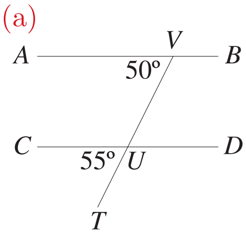

Show that $A$, $O$ and $C$ are collinear.

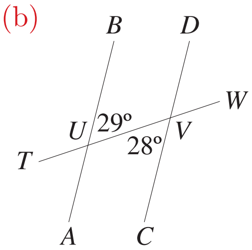
(d)


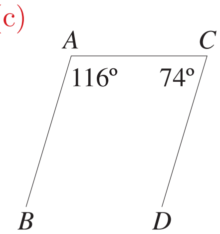

Show that $A$, $O$ and $D$ are collinear.

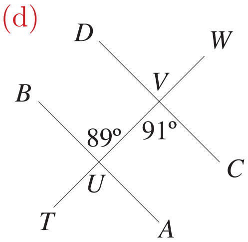
DEVELOPMENT

**8.** Show that $AB$ is not parallel to $CD$ in the diagrams below, giving all reasons.

(a)



(b)


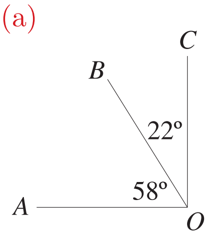

(c)



(d)



ISBN: 978-1-107-61604-2

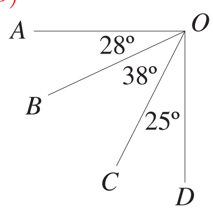
Cambridge University Press

**9.** (a)



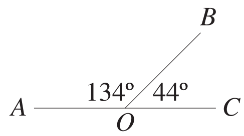

Show that $OC$ is not perpendicular to $OA$.

(b)



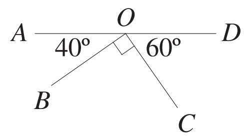

Show that $OD$ is not perpendicular to $OA$.

(c)




Show that $A$, $O$ and $C$ are not collinear.

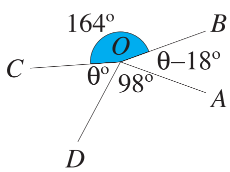
(d)



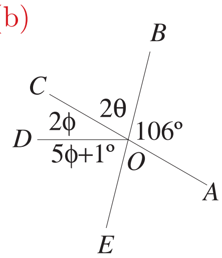

Show that $A$, $O$ and $D$ are not collinear.

**10.** Find $\theta$ and $\phi$ in the diagrams below, giving reasons.

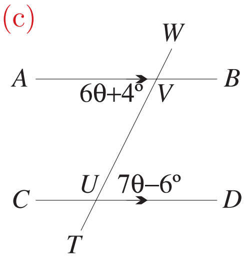
(a)



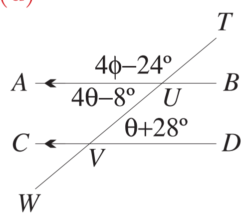

(b)



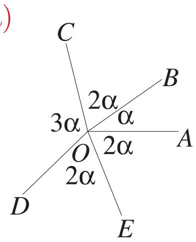

(c)




(d)

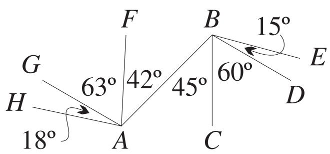



**11.** (a)



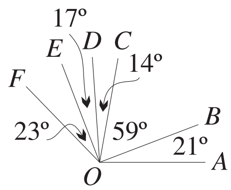

Name all straight angles and vertically opposite angles in the diagram.

(b)




Which two lines in the diagram above are parallel?

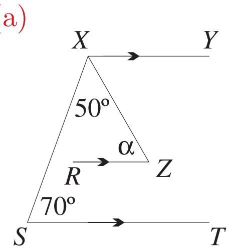
(c)



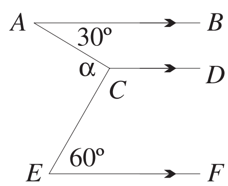

Which two lines in the diagram above form a right angle?

**12.** Find the angle $\alpha$ in each diagram below.

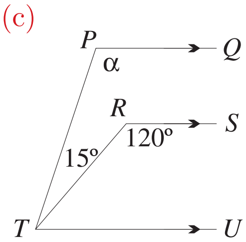
(a)



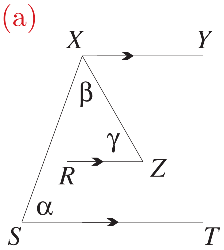

(b)




(c)

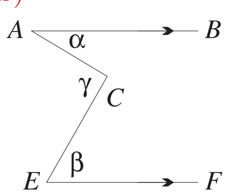



**13.** (a)



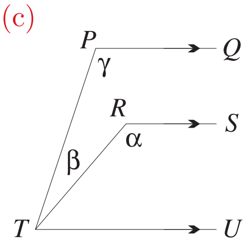

Show that $\gamma = 180° - (\alpha + \beta)$.

(b)



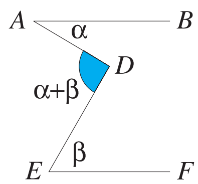

Show that $\gamma = \alpha + \beta$.

(c)




Show that $\gamma = \alpha - \beta$.

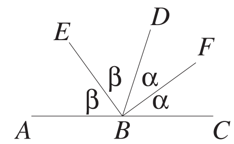
(d)



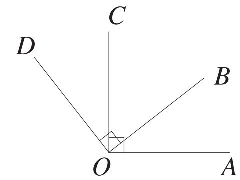

Show that $EF \parallel AB$.

ISBN: 978-1-107-61604-2

**14.** THEOREM: *The bisectors of adjacent supplementary angles form a right angle.* In the diagram to the right, $\angle ABD$ and $\angle DBC$ are adjacent supplementary angles. Given that the line $FB$ bisects $\angle DBC$ and the line $EB$ bisects $\angle ABD$, prove that $\angle FBE = 90°$.

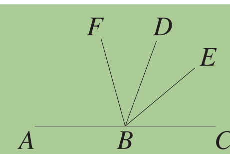



**15.** In the diagram to the right, the line $CO$ is perpendicular to the line $AO$, and the line $DO$ is perpendicular to the line $BO$. Show that the angles $\angle AOD$ and $\angle BOC$ are supplementary.




EXTENSION

**16.** THEOREM: *A generalisation of the result in question 14.* In the diagram opposite, $\angle ABD$ and $\angle DBC$ are adjacent supplementary angles. Suppose that $EB$ divides $\angle DBC$ in the ratio of $k : \ell$, and that $FB$ also divides $\angle DBA$ in the ratio $k : \ell$. Find $\angle FBE$ in terms of $k$ and $\ell$.




**17.** Give concrete examples of the following:

(a) three distinct planes meeting at a point,

(b) three distinct planes meeting at a line,

(c) three distinct parallel planes,

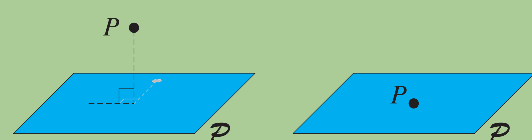
(d) three distinct planes intersecting in three distinct lines,

(e) two distinct parallel planes intersecting with a third plane,

(f) a line parallel to a plane,

(g) a line intersecting a plane.

**18.** There are two possible configurations of a point and a plane. Either the point is in the plane or it is not, as shown in the diagram.

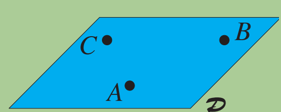



(a) What are the possible configurations of a line and a plane? Draw a diagram of each situation.

(b) What are the possible configurations of two lines? Draw a diagram of each situation.

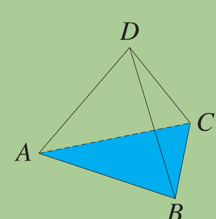
(c) What are the possible configurations of two planes? Draw a diagram of each situation.

**19.** (a)




There is only one plane that passes through any three given non-collinear points. What are three other ways of determining a plane? Draw a diagram of each situation.

(b)

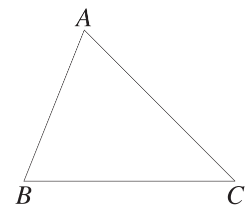



Two lines in space are called *skew* if they neither intersect nor are parallel. Given the tetrahedron $ABCD$ above, name all pairs of skew lines such that each passes through two of its vertices.

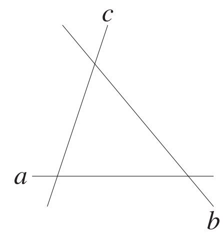
ISBN: 978-1-107-61604-2

# 8 B Angles in Triangles and Polygons

Having introduced angles and intervals, we can now begin to develop the relationships between the sizes of angles and the lengths of intervals. When three intervals are joined into a closed figure, they form a triangle, four such intervals form a quadrilateral, and more generally, an arbitrary number of such intervals form a polygon. Accordingly, this section is a study of angles in polygons. Sections 8C–8E then study the relationships between angles and lengths in triangles and quadrilaterals.




**Triangles:** A *triangle* is formed by taking any three non-collinear points $A$, $B$ and $C$ and constructing the intervals $AB$, $BC$ and $CA$. The three intervals are called the *sides* of the triangle, and the three points are called its *vertices* (the singular is *vertex*).




Alternatively, a triangle can be formed by taking three non-concurrent lines $a$, $b$ and $c$. Provided no two are parallel, the intersections of these lines form the vertices of the triangle.

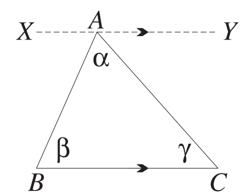
**Interior Angles of a Triangle:** A triangle is a *closed* figure, meaning that it divides the plane into an inside and an outside. The three angles inside the triangle at the vertices are called the *interior angles*, and our first task is to prove that their sum is always 180°.

**5** COURSE THEOREM: The sum of the interior angles of a triangle is a straight angle.

GIVEN: Let $ABC$ be a triangle.
Let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$.

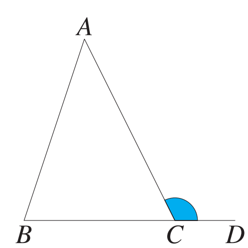
AIM: To prove that $\alpha + \beta + \gamma = 180°$.

CONSTRUCTION: Construct $XAY$ through the vertex $A$ parallel to $BC$.




PROOF: $\angle XAB = \beta$ (alternate angles, $XAY \parallel BC$),
and $\angle YAC = \gamma$ (alternate angles, $XAY \parallel BC$).
Hence $\alpha + \beta + \gamma = 180°$ (straight angle).

**Exterior Angles of a Triangle:** Suppose that $ABC$ is a triangle, and suppose that the side $BC$ is produced to $D$ (the word 'produced' simply means 'extended in the direction $BC$'). Then the angle $\angle ACD$ between the side $AC$ and the extended side $CD$ is called an *exterior angle* of the triangle.

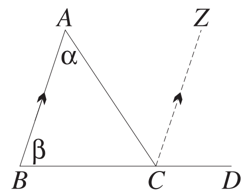



There are two exterior angles at each vertex, and because they are vertically opposite, they must be equal in size. Also, an exterior angle and the interior angle adjacent to it are adjacent angles on a straight line, so they must be supplementary. The exterior angles and interior angles are related as follows.

**6** COURSE THEOREM: An exterior angle of a triangle equals the sum of the interior opposite angles.

GIVEN: Let $ABC$ be a triangle with $BC$ produced to $D$. Let $\angle A = \alpha$ and $\angle B = \beta$.

AIM: To prove that $\angle ACD = \alpha + \beta$.

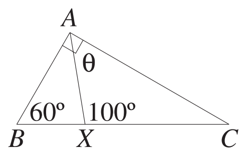



CONSTRUCTION: Construct the ray $CZ$ through the vertex $C$ parallel to $BA$.

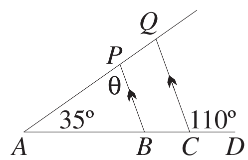
PROOF: $\angle ZCD = \beta$ (corresponding angles, $BA \parallel CZ$),
and $\angle ACZ = \alpha$ (alternate angles, $BA \parallel CZ$).
Hence $\angle ACD = \alpha + \beta$ (adjacent angles).

**WORKED EXERCISE:** Find $\theta$ in each diagram below.

(a)




(b)

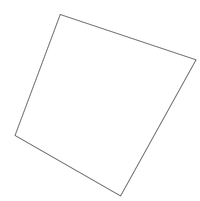



SOLUTION:

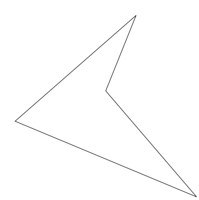
(a) $\angle C = 30°$
(angle sum of $\triangle ABC$),
so $\theta = 50°$
(angle sum of $\triangle ACX$).

(b) $\angle PBC = 110°$
(corresponding angles, $BP \parallel CQ$),
so $\theta = 75°$
(exterior angle of $\triangle ABP$).

**Quadrilaterals:** A *quadrilateral* is a closed plane figure bounded by four intervals. As with triangles, the intervals are called *sides*, and their four endpoints are called *vertices*. (The sides can't cross each other, and no vertex angle can be 180°.)



A quadrilateral may be *convex*, meaning that all its interior angles are less than 180°, or *non-convex*, meaning that one interior angle is greater than 180°. The intervals joining pairs of opposite vertices are called *diagonals* — notice that both diagonals of a convex quadrilateral lie inside the figure, but only one diagonal of a non-convex quadrilateral lies inside it. In both cases, we can prove that the sum of the interior angles is 360°.

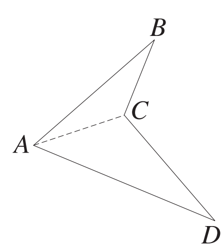


**7** COURSE THEOREM: The sum of the interior angles of a quadrilateral is two straight angles.

GIVEN: Let $ABCD$ be a quadrilateral, labelled so that the diagonal $AC$ lies inside the figure.

AIM: To prove that $\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360°$.

CONSTRUCTION: Join the diagonal $AC$.

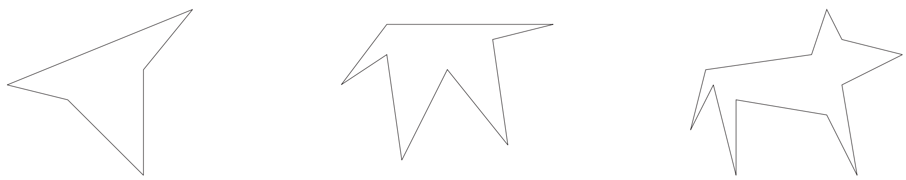



PROOF: The interior angles of $\triangle ABC$ have sum 180°,
and the interior angles of $\triangle ADC$ have sum 180°.
But the interior angles of quadrilateral $ABCD$
are the sums of the interior angles of $\triangle ABC$ and $\triangle ADC$.
Hence the sum of the interior angles of $ABCD$ is 360°.

ISBN: 978-1-107-61604-2  Cambridge University Press

**Polygons:** A *polygon* is a closed figure bounded by any number of straight sides (*polygon* is a Greek word meaning 'many-angled'). A polygon is named according to the number of sides it has, and there must be at least three sides or else there would be no enclosed region. Here are some of the names:

| | | |
|---|---|---|
| 3 sides: triangle | 6 sides: hexagon | 9 sides: nonagon |
| 4 sides: quadrilateral | 7 sides: heptagon | 10 sides: decagon |
| 5 sides: pentagon | 8 sides: octagon | 12 sides: dodecagon |



A pentagon An octagon A dodecagon

Like quadrilaterals, polygons can be *convex*, meaning that every interior angle is less than 180°, or *non-convex*, meaning that at least one interior angle is greater than 180°. A polygon is convex if and only if every one of its diagonals lies inside the figure. Notice that even a non-convex polygon must have at least one diagonal completely inside the figure.

The following theorem generalises the theorems about the interior angles of triangles and quadrilaterals to polygons with any number of sides.

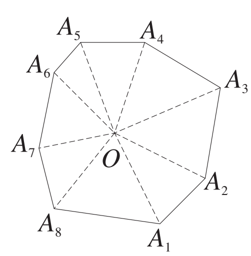
**8** COURSE THEOREM: The interior angles of an $n$-sided polygon have sum $180(n-2)°$.

When the polygon is non-convex, the proof requires mathematical induction because we need to keep chopping off a triangle whose angle sum is 180°— this is carried through in question 23 of the following exercise. The situation is far easier when the polygon is convex, and the following proof is restricted to that case.

GIVEN: Let $A_1A_2 \ldots A_n$ be a convex polygon.

AIM: To prove that $\angle A_1 + \angle A_2 + \ldots + \angle A_n = 180(n-2)°$.

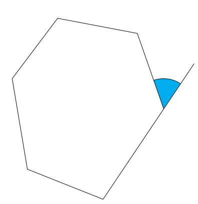
CONSTRUCTION: Choose any point $O$ inside the polygon, and construct the intervals $OA_1$, $OA_2$, ..., $OA_n$, giving $n$ triangles $A_1OA_2$, $A_2OA_3$, ..., $A_nOA_1$.




PROOF: The angle sum of the $n$ triangles is $180n°$.
But the angles at $O$ form a revolution, with size 360°.
Hence for the interior angles of the polygon,

$$\begin{aligned} \text{sum} &= 180n° - 360° \\ &= 180(n-2)°. \end{aligned}$$

**The Exterior Angles of a Polygon:** An exterior angle of a convex polygon at any vertex is the angle between one side produced and the other side, just as in a triangle. We will ignore exterior angles of non-convex polygons, because they would have to involve negative angles. There is a surprisingly simple formula for the sum of the exterior angles.

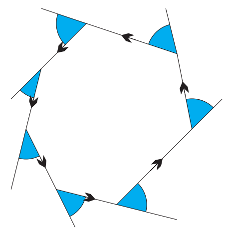


ISBN: 978-1-107-61604-2

**9** COURSE THEOREM: The sum of the exterior angles of a convex polygon is $360°$.

PROOF: At each vertex, the interior and exterior angles add to $180°$, so the sum of all interior and exterior angles is $180n°$.
But the interior angles add to $180(n-2)°$.
Hence the exterior angles must add to $2 \times 180° = 360°$.

**Exterior Angles as the Amount of Turning:** If one walks around a polygon, the exterior angle at each vertex is the angle one turns at that vertex. Thus the sum of all the exterior angles is the amount of turning when one walks right around the polygon. Clearly walking around a polygon involves a total turning of $360°$, and the previous theorem can be interpreted as saying just that. In this way, the theorem can be generalised to say that when one walks around any closed curve, the amount of turning is always $360°$ (provided that the curve doesn't cross itself).



**Regular Polygons:** A *regular polygon* is a polygon in which all sides are equal and all interior angles are equal. Simple division gives:

**10** COURSE THEOREM: In an $n$-sided regular polygon:
- each exterior angle is $\dfrac{360°}{n}$,
- each interior angle is $\dfrac{180(n-2)°}{n}$.

Substitution of $n = 3$ and $n = 4$ gives the familiar results that each angle of an equilateral triangle is $60°$, and each angle of a square is $90°$.

**WORKED EXERCISE:** Find the sizes of each exterior angle and each interior angle in a regular 12-sided polygon.

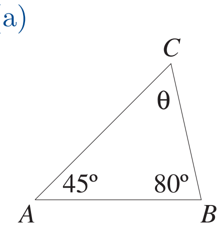
SOLUTION: The exterior angles have sum $360°$, so each exterior angle is $360° \div 12 = 30°$.
Hence each interior angle is $150°$ (angles in a straight angle).
Alternatively, using the formula, each interior angle is $\dfrac{180 \times 10}{12} = 150°$.

## Exercise 8B

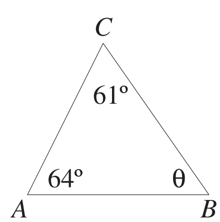
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

**1.** Use the angle sum of a triangle to find $\theta$ in the diagrams below, giving reasons.

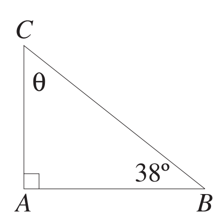
(a)



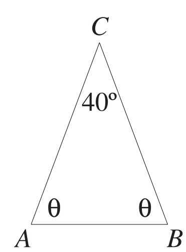

(b)



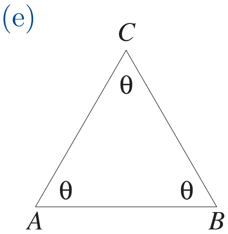

(c)

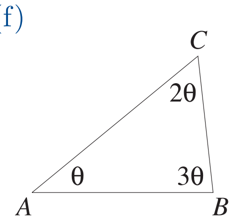


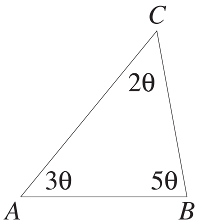

(d)

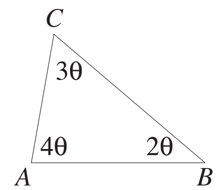



ISBN: 978-1-107-61604-2

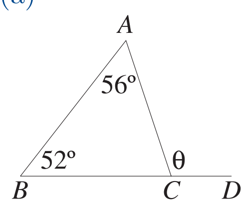
(e) 

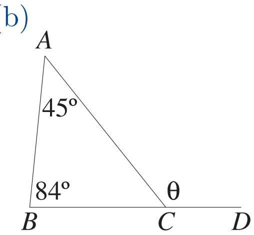

(f) 

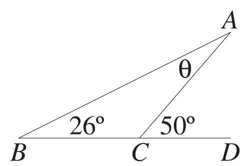

(g) 

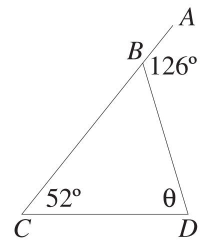

(h) 


**2.** Use the exterior angle of a triangle theorem to find $\theta$, giving reasons.

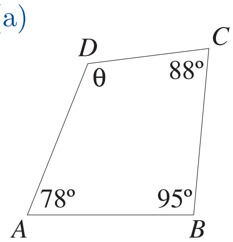
(a) 

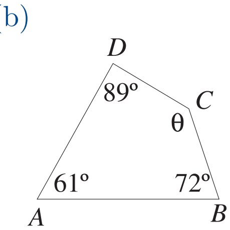

(b) 

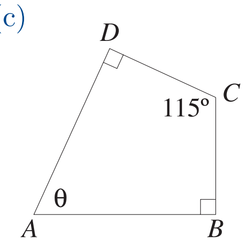

(c) 

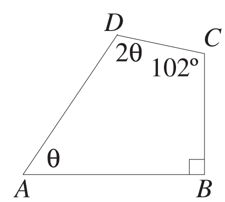

(d) 

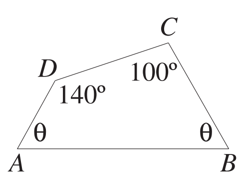

**3.** Use the angle sum of a quadrilateral to find $\theta$ in the diagrams below, giving reasons.

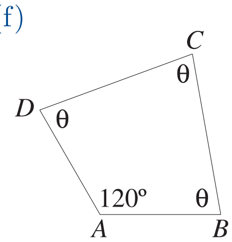
(a) 

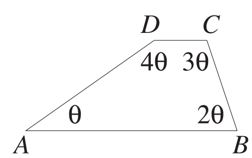

(b) 

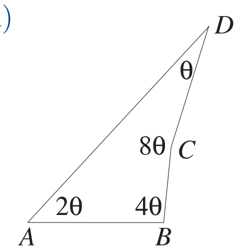

(c) 


(d) 


(e) 


(f) 


(g) 


(h) 


**4.** Demonstrate the formula $180(n-2)^\circ$ for the angle sum of a polygon by drawing examples of the following non-convex polygons and dissecting them into $n-2$ triangles:

(a) a pentagon, (b) a hexagon, (c) an octogon, (d) a dodecagon.

**5.** Find the size of each (i) interior angle, (ii) exterior angle, of a regular polygon with:

(a) 5 sides, (b) 6 sides, (c) 8 sides, (d) 9 sides, (e) 10 sides, (f) 12 sides.

**6.** (a) Find the number of sides of a regular polygon if each interior angle is:

(i) $135^\circ$ (ii) $144^\circ$ (iii) $172^\circ$ (iv) $178^\circ$

(b) Find the number of sides of a regular polygon if its exterior angle is:

(i) $72^\circ$ (ii) $40^\circ$ (iii) $18^\circ$ (iv) $\frac{1}{2}^\circ$

(c) Why is it not possible for a regular polygon to have an interior angle equal to $123^\circ$?

(d) Why is it not possible for a regular polygon to have an exterior angle equal to $71^\circ$?

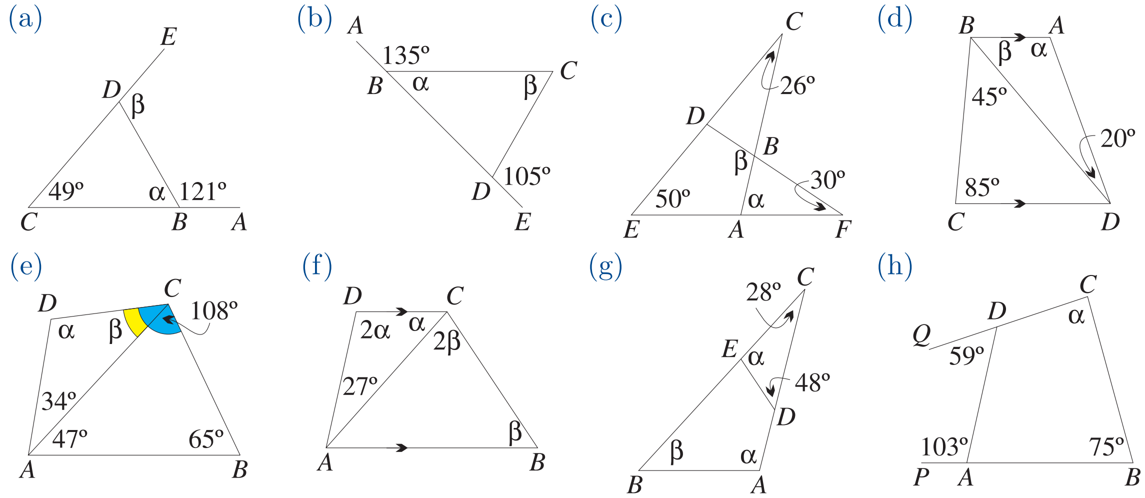
**7.** By drawing a diagram, find the number of diagonals of each polygon, and verify that the number of diagonals of a polygon with $n$ sides is $\frac{1}{2}n(n-3)$:

(a) a convex pentagon, (b) a convex hexagon, (c) a convex octagon.

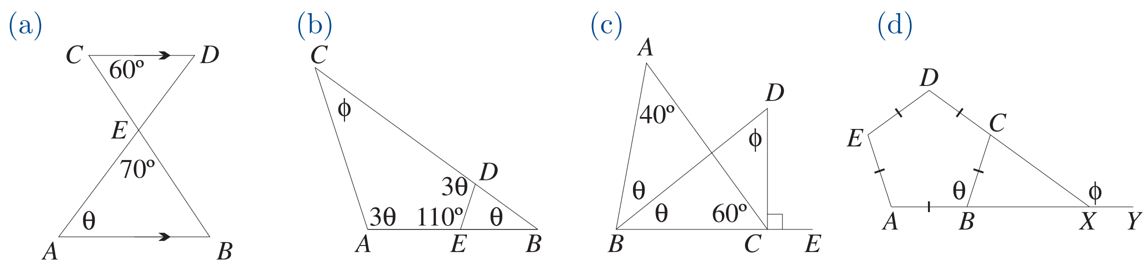
(This will be proven by mathematical induction in question 23.)

ISBN: 978-1-107-61604-2  Cambridge University Press

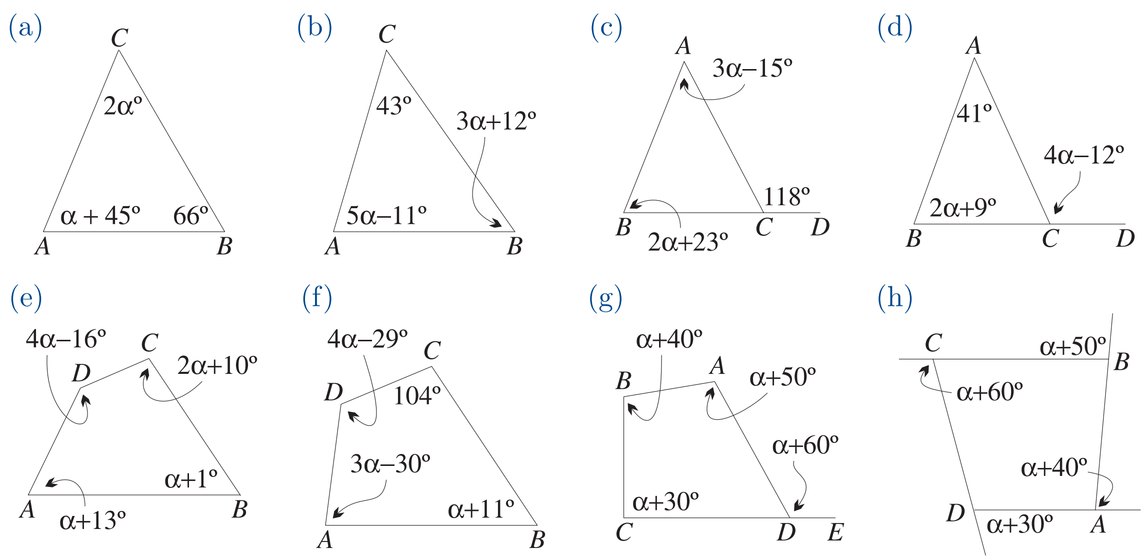
**8.** Find the angles $\alpha$ and $\beta$ in the diagrams below. Give all steps in your argument.



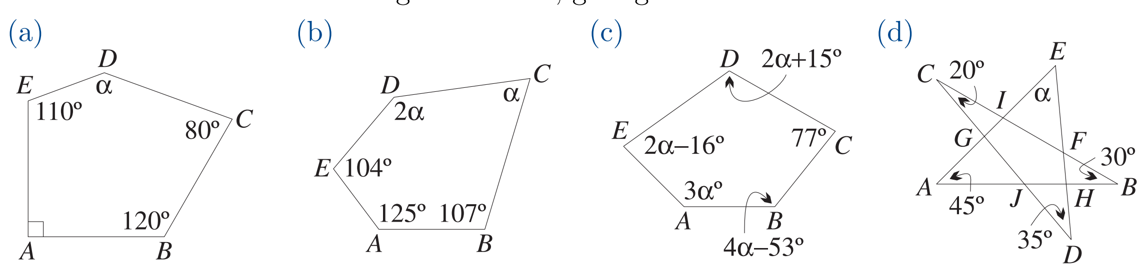

**9.** Find the angles $\theta$ and $\phi$ in the diagrams below, giving all reasons.




**10.** Find the value of $\alpha$ in the diagrams below, giving all reasons.



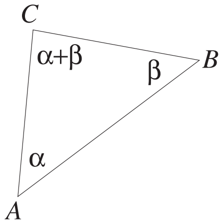

**11.** Find the values of $\alpha$ in the diagrams below, giving all reasons.




ISBN: 978-1-107-61604-2  Cambridge University Press

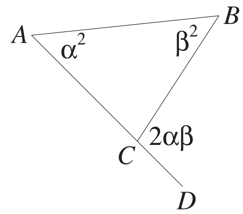
**12.** Prove the given relationships in the diagrams below.

(a)



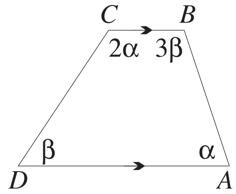

Show that $\alpha + \beta = 90°$.

(b)



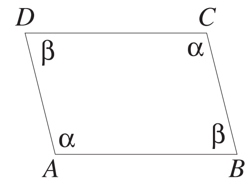

Show that $\alpha = \beta$.

(c)




Show that $\alpha = 72°$ and $\beta = 36°$.

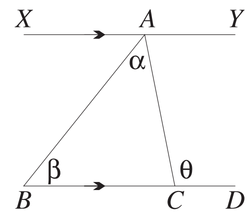
(d)



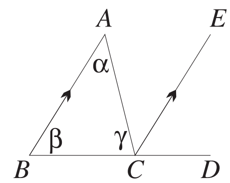

Show that $AB||CD$ and $AD||BC$.

DEVELOPMENT

**13.** COURSE THEOREM: *An alternative proof of the exterior angle theorem.* Given a triangle $ABC$ with $BC$ produced to $D$, construct the line $XY$ through the vertex $A$ parallel to $BD$. Let $\angle CAB = \alpha$ and $\angle ABC = \beta$. Use alternate angles twice to prove that $\angle ACD = \alpha + \beta$.

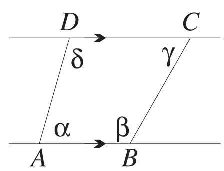



**14.** COURSE THEOREM: *An alternative proof that the angle sum of a triangle is 180°.* Let $ABC$ be a triangle with $BC$ produced to $D$. Construct the line $CE$ through $C$ parallel to $BA$. Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. Prove that $\alpha + \beta + \gamma = 180°$.

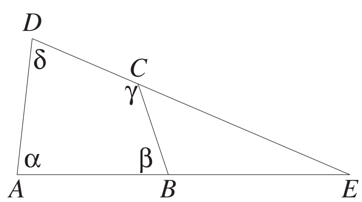



**15.** COURSE THEOREM: *An alternative approach to proving that the angle sum of a quadrilateral is 360°.*

(a) Suppose that a quadrilateral has a pair of parallel sides, and name them $AB$ and $CD$ as shown. Use the assumptions about parallel lines and transversals to prove that the interior angle sum of quadrilateral $ABCD$ is 360°.




(b) Suppose that in quadrilateral $ABCD$ there is no pair of parallel sides. Extend sides $AB$ and $DC$ to meet at $E$ as shown. Use the theorems about angles in triangles to prove that the interior angle sum of quadrilateral $ABCD$ is 360°.



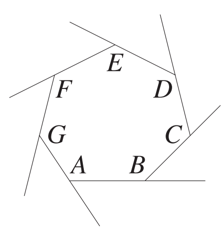

**16.** (a) Determine the ratio of the sum of the interior angles to the sum of the exterior angles in a polygon with $n$ sides.

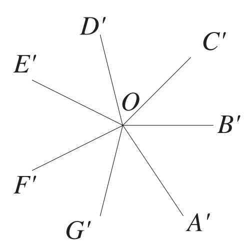
(b) Hence determine if it is possible to have these angles in the ratio: (i) $\frac{8}{3}$ (ii) $\frac{7}{2}$

**17.** Convince yourself that the sum of the exterior angles of a polygon is 360° by carrying out the following constructions. Draw a polygon $ABCD\ldots$ and pick a point $O$ outside the polygon. From $O$ draw $OB'$ in the same direction as $AB$. Next draw $OC'$ in the same direction as $BC$. Then do the same for $CD$ and so on around the polygon. The diagrams show the result for the heptagon $ABCDEFG$.

(a) What is the sum of the angles at $O$?

(b) How are the exterior angles of the polygon related to the angles at $O$?

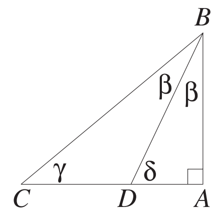



ISBN: 978-1-107-61604-2

Cambridge University Press

**18.** In the right-angled triangle $ABC$ opposite, $\angle CAB = 90^\circ$, and the bisector of $\angle ABC$ meets $AC$ at $D$. Let $\angle ABD = \beta$, $\angle ACB = \gamma$ and $\angle ADB = \delta$. Show that $\delta = 45^\circ + \frac{1}{2}\gamma$.

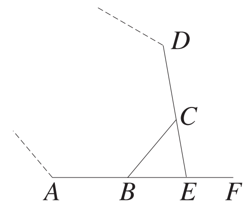



**19.** Three of the angles in a convex quadrilateral are equal. What is:

(a) the smallest possible size, (b) the largest possible size,

of these three equal angles?

**20.** Let $AB$, $BC$ and $CD$ be three consecutive sides of a regular polygon with $n$ sides. Produce $AB$ to $F$, and produce $DC$ to meet $AF$ at $E$.




(a) Find the size of $\angle CEF$ as a function of $n$.

(b) Now suppose that $\angle CEF$ is the interior angle of another regular polygon with $m$ sides. Find $m$ in terms of $n$.

(c) Hence find all pairs of regular polygons that are related in this way.

(d) In each case, if the first polygon has sides of length 1, what is the length of the sides of the second polygon?

**21.** SEQUENCES AND GEOMETRY:

(a) The three angles of a triangle $ABC$ form an arithmetic sequence. Show that the middle-sized angle is $60^\circ$.

(b) The three angles of a triangle $PQR$ form a geometric sequence. Show that the smallest angle and the common ratio cannot both be integers.

**22.** (a) A quadrilateral in which all angles are equal need not have all sides equal (it is in fact a rectangle). Prove, nevertheless, that opposite sides are parallel.

(b) Prove that if all angles of a hexagon are equal, then opposite sides are parallel.

(c) Prove more generally that this holds for polygons with $2n$ sides.

**23.** MATHEMATICAL INDUCTION IN GEOMETRY:

(a) Use mathematical induction to prove that for $n \geq 3$, a polygon with $n$ sides has $\frac{1}{2}n(n-3)$ diagonals. Begin with a triangle, which has no diagonals.

(b) Use mathematical induction to prove that the sum of the interior angles of any polygon with $n \geq 3$ sides, convex or non-convex, is $180(n-2)^\circ$. Begin the induction step by choosing three adjacent vertices $P_k$, $P_{k+1}$ and $P_1$ of the $(k+1)$-gon so that $\angle P_k P_{k+1} P_1$ is acute, and joining the diagonal $P_1 P_k$ to form a triangle and a polygon with $k$ sides.

EXTENSION

**24.** TRIGONOMETRY IN GEOMETRY: Suppose that a regular polygon has $n$ sides of length 1.

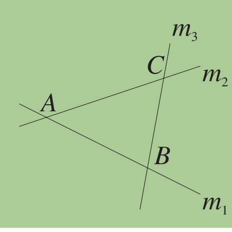
(a) What will be the length of the side of the regular polygon with $2n$ sides that is formed by cutting off the vertices of the given polygon?

(b) Confirm your answer in the case of:

(i) cutting the corners off an equilateral triangle to form a regular hexagon,

(ii) cutting the corners off a square to form a regular octagon.

**25.** TRIGONOMETRY IN GEOMETRY: Three lines with nonzero gradients $m_1$, $m_2$ and $m_3$ intersect at the points $A$, $B$ and $C$. The acute angles $\alpha$, $\beta$ and $\gamma$, between each pair of lines, are found using the usual formula $\tan\alpha = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$.




ISBN: 978-1-107-61604-2  Cambridge University Press

(a) If one of the angles of $\triangle ABC$ is obtuse, explain why one of the acute angles found must be the sum of the other two.

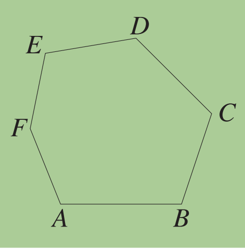
(b) If the signs of $m_1$, $m_2$ and $m_3$ are all the same, what can be deduced about $\triangle ABC$?

(c) If all angles of $\triangle ABC$ are acute, what can be deduced about the sign of $m_1 m_2 m_3$?

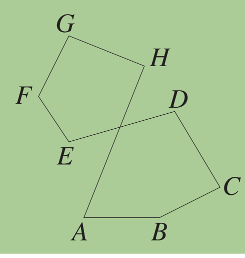
**26.** In a polygon with $n$ sides, none of which are vertical and none horizontal, and all interior angles equal, determine the sign of the product of the gradients of all the sides.

**27.** Counting clockwise turns as negative and anticlockwise turns as positive, through how many revolutions would you turn if you followed the alphabet around the following figures?

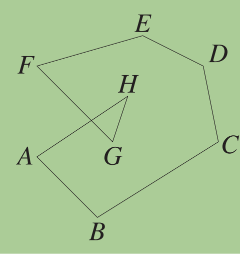
(a)




(b)




(c)

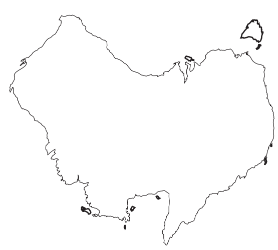



## 8 C Congruence and Special Triangles

As in all branches of mathematics, symmetry is a vital part of geometry. In Euclidean geometry, symmetry is handled by means of congruence, and later through the more general idea of similarity. It is only by these methods that relationships between lengths and angles can be established.

**Congruence:** Two figures are called *congruent* if one figure can be picked up and placed so that it fits exactly on top of the other figure. More precisely, using the language of transformations:

**11** CONGRUENCE: Two figures $\mathcal{S}$ and $\mathcal{T}$ are called *congruent*, written as $\mathcal{S} \equiv \mathcal{T}$, if one figure can be moved to coincide with the other figure by means of a sequence of rotations, reflections and translations.



The congruence sets up a correspondence between the elements of the two figures. In this correspondence, angles, lengths and areas are preserved.

**12** PROPERTIES OF CONGRUENT FIGURES: If two figures are congruent.
- matching angles have the same size,
- matching intervals have the same length,
- matching regions have the same area.

ISBN: 978-1-107-61604-2  Cambridge University Press

**Congruent Triangles:** In practice, almost all of our congruence arguments concern congruent triangles. Euclid's geometry book proves four tests for the congruence of two triangles, but we shall take them as assumptions.

STANDARD CONGRUENCE TESTS FOR TRIANGLES: Two triangles are congruent if:

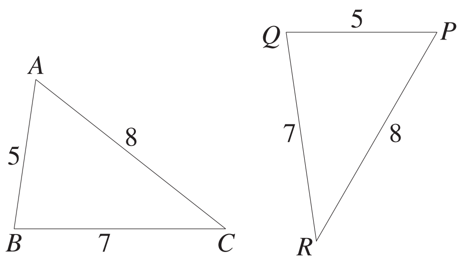
SSS the three sides of one triangle are respectively equal to the three sides of another triangle, or

13 SAS two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, or

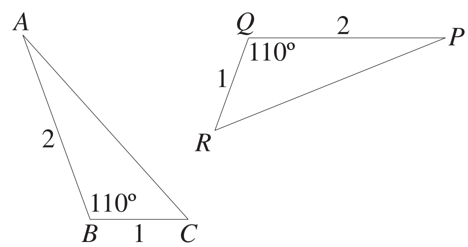
AAS two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, or

RHS the hypotenuse and one side of one right triangle are respectively equal to the hypotenuse and one side of another right triangle.

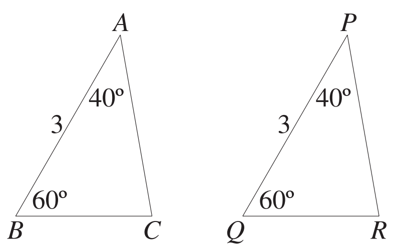
These standard tests are known from earlier years, and have already been discussed in Sections 4H–4J of the Year 11 volume, where they were related to the sine and cosine rules. As mentioned in those sections, there is no ASS test — two sides and a non-included angle — and we constructed two non-congruent triangles with the same ASS specifications. Here are examples of the four tests.



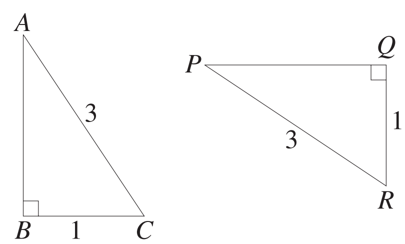

$\triangle ABC \equiv \triangle PQR$ (SSS).
Hence $\angle P = \angle A$, $\angle Q = \angle B$
and $\angle R = \angle C$
(matching angles of congruent triangles).




$\triangle ABC \equiv \triangle PQR$ (SAS).
Hence $\angle P = \angle A$, $\angle R = \angle C$
and $PR = AC$ (matching sides
and angles of congruent triangles).




$\triangle ABC \equiv \triangle PQR$ (AAS).
Hence $QR = BC$ and $RP = CA$
(matching sides of congruent triangles),
and $\angle R = \angle C$ (angle sums of triangles).




$\triangle ABC \equiv \triangle PQR$ (RHS).
Hence $\angle P = \angle A$, $\angle R = \angle C$
and $PQ = AB$ (matching sides
and angles of congruent triangles).

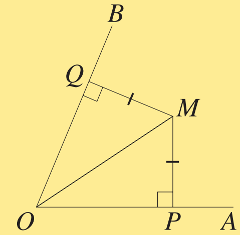
**Using the Congruence Tests:** A fully set-out congruence proof has five lines — the first line introduces the triangles, the next three set out the three pairs of equal sides or angles, and the final line is the conclusion. Subsequent deductions from the congruence follow these five lines. Throughout the congruence proof, all vertices should be named in corresponding order. Each of the four standard congruence tests is used in one of the next four proofs.

ISBN: 978-1-107-61604-2  Cambridge University Press

**WORKED EXERCISE:** The point $M$ lies inside the arms of the acute angle $\angle AOB$. The perpendiculars $MP$ and $MQ$ to $OA$ and $OB$ respectively have equal lengths. Prove that $\triangle POM \equiv \triangle QOM$, and that $OM$ bisects $\angle AOB$.

PROOF: In the triangles $POM$ and $QOM$:
1. $OM = OM$ (common),
2. $PM = QM$ (given),
3. $\angle OPM = \angle OQM = 90°$ (given),

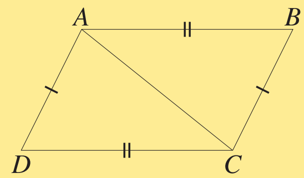
so $\triangle POM \equiv \triangle QOM$ (RHS).
Hence $\angle POM = \angle QOM$ (matching angles).



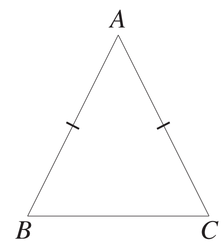

**WORKED EXERCISE:** Prove that $\triangle ABC \equiv \triangle CDA$, and hence that $AD \parallel BC$.

PROOF: In the triangles $ABC$ and $CDA$:
1. $AC = CA$ (common),
2. $AB = CD$ (given),
3. $BC = DA$ (given),

so $\triangle ABC \equiv \triangle CDA$ (SSS).
Hence $\angle BCA = \angle DAC$ (matching angles),
and so $AD \parallel BC$ (alternate angles are equal).




**Isosceles Triangles:** An *isosceles triangle* is a triangle in which two sides are equal. The two equal sides are called the *legs* of the triangle (the Greek word 'isosceles' literally means 'equal legs'), their intersection is called the *apex*, and the side opposite the apex is called the *base*. It is well known that the base angles of an isosceles triangle are equal.




**14** COURSE THEOREM: If two sides of a triangle are equal, then the angles opposite those sides are equal.

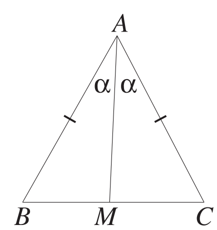
GIVEN: Let $ABC$ be an isosceles triangle with $AB = AC$.

AIM: To prove that $\angle B = \angle C$.

CONSTRUCTION: Let the bisector of $\angle A$ meet $BC$ at $M$.

PROOF: In the triangles $ABM$ and $ACM$:
1. $AM = AM$ (common),
2. $AB = AC$ (given),
3. $\angle BAM = \angle CAM$ (construction),

so $\triangle ABM \equiv \triangle ACM$ (SAS).
Hence $\angle ABM = \angle ACM$ (matching angles of congruent triangles).




**A Test for a Triangle to be Isosceles:** The converse of this result is also true, giving a test for a triangle to be isosceles.

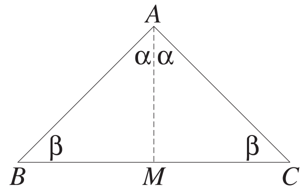
**15** COURSE THEOREM: Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal.

GIVEN: Let $ABC$ be a triangle in which $\angle B = \angle C = \beta$.

ISBN: 978-1-107-61604-2

AIM: To prove that $AB = AC$.

CONSTRUCTION: Let the bisector of $\angle A$ meet $BC$ at $M$.



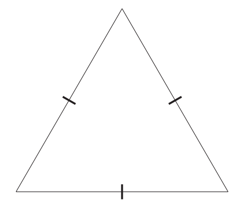

PROOF: In the triangles $ABM$ and $ACM$:

1. $AM = AM$ (common),
2. $\angle B = \angle C$ (given),
3. $\angle BAM = \angle CAM$ (construction),

so $\triangle ABM \equiv \triangle ACM$ (AAS).

Hence $AB = AC$ (matching sides of congruent triangles).

**Equilateral Triangles:** An *equilateral triangle* is a triangle in which all three sides are equal. It is therefore an isosceles triangle in three different ways, and the following property of and test for an equilateral triangle follow easily from the previous theorem and its converse.

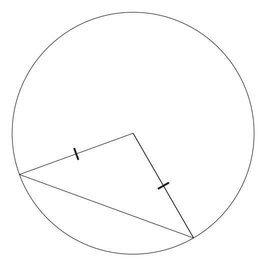


**16** COURSE THEOREM: All angles of an equilateral triangle are equal to $60^\circ$. Conversely, if all angles of a triangle are equal, then it is equilateral.

PROOF: Suppose that the triangle is equilateral, that is, all three sides are equal. Then all three angles are equal, and since their sum is $180^\circ$, they must each be $60^\circ$.

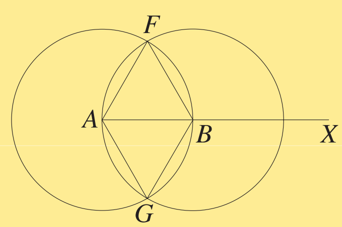
Conversely, suppose that all three angles are equal. Then all three sides are equal, meaning that the triangle is equilateral.

**Circles and Isosceles Triangles:** A *circle* is the set of all points that are a fixed distance (called the *radius*) from a fixed point (called the *centre*). Compasses are used for drawing circles, because the pencil is held at a fixed distance from the centre, where the compass-point is fixed in the paper.



If two points on the circumference are joined to the centre and to each other, then the equal radii mean that the triangle is isosceles. The following worked exercise shows how to construct an angle of $60^\circ$ using straight edge and compasses.

**WORKED EXERCISE:** Construct a circle with centre on the end $A$ of an interval $AX$, meeting the ray $AX$ at $B$. With centre $B$ and the same radius, construct a circle meeting the first circle at $F$ and $G$. Prove that $\angle FAB = \angle GAB = 60^\circ$.

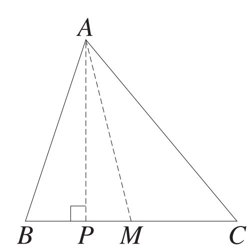



PROOF: Because they are all radii of congruent circles,

$$AF = AB = AG = BF = BG.$$

Hence $\triangle AFB$ and $\triangle AGB$ are both equilateral triangles, and so $\angle FAB = \angle GAB = 60^\circ$.

**Medians and Altitudes:** A *median* of a triangle joins a vertex to the midpoint of the opposite side. An *altitude* of a triangle is the perpendicular from a vertex to the opposite side. These two words are useful when talking about triangles.




In the diagram to the right, $AP$ is one of the three altitudes in $\triangle ABC$. The point $M$ is the midpoint of $BC$, and $AM$ is one of the three medians in $\triangle ABC$.

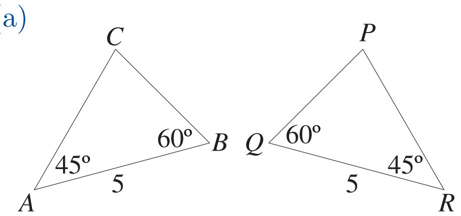
ISBN: 978-1-107-61604-2 

## Exercise 8C

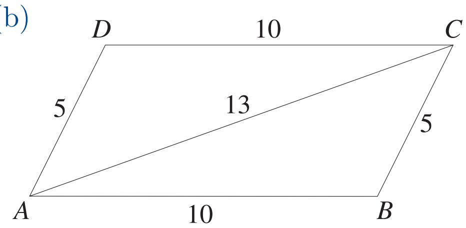
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

**1.** The two triangles in each pair below are congruent. Name the congruent triangles in the correct order and state which test justifies the congruence.

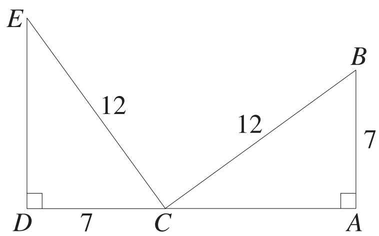
(a)



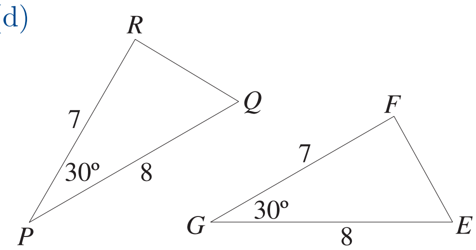

(b)




(c)

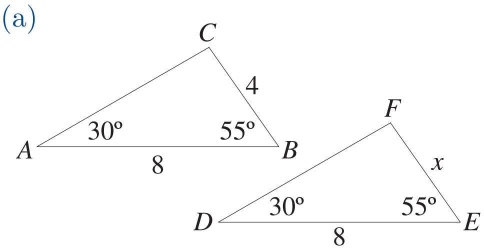



(d)

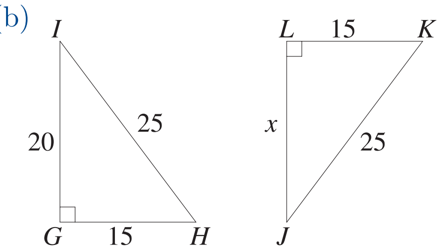



**2.** In each part, identify the congruent triangles, naming the vertices in matching order and giving a reason. Hence deduce the length of the side $x$.

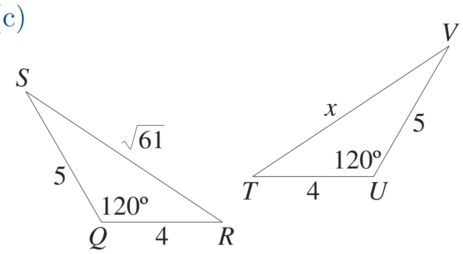
(a)



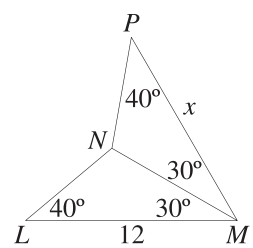

(b)




(c)

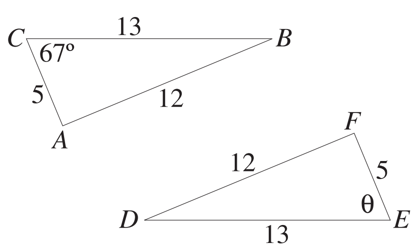



(d)

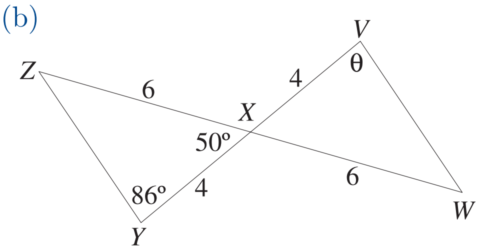



**3.** In each part, identify the congruent triangles, naming the vertices in matching order and giving a reason. Hence deduce the size of the angle $\theta$.

(a)

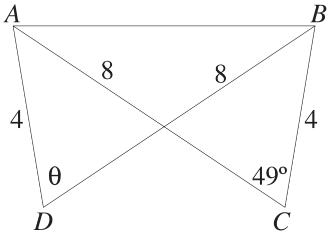



(b)

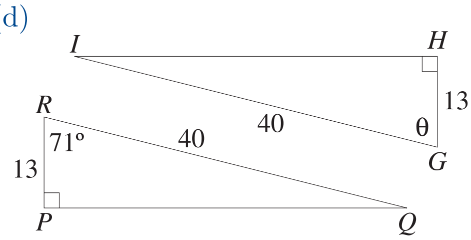



ISBN: 978-1-107-61604-2  

(c)

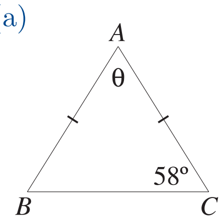



(d)

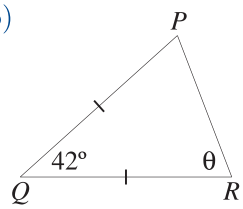



**4.** Find the size of angle $\theta$ in each diagram below, giving reasons.

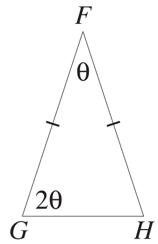
(a)



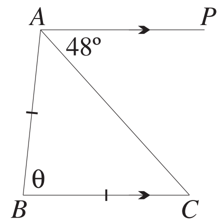

(b)



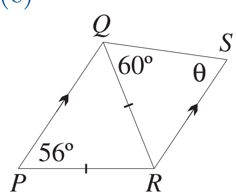

(c)



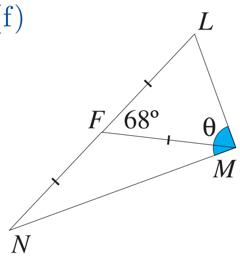

(d)



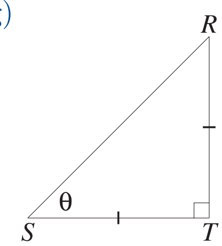

(e)



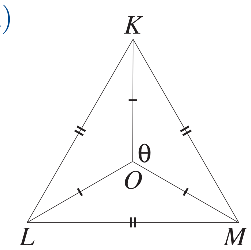

(f)



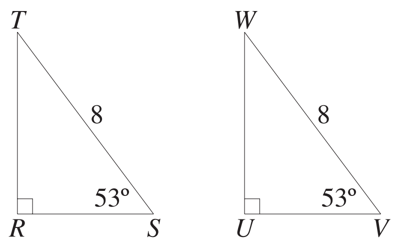

(g)




(h)

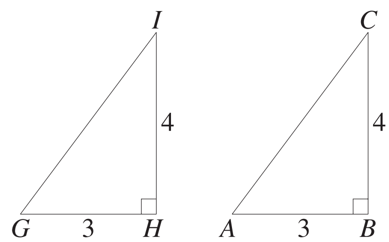



**5.** (a)




When asked to show that the two triangles above were congruent, a student wrote $\triangle RST \equiv \triangle UVW$ (RHS). Although both triangles are indeed right-angled, explain why the reason given is incorrect. What is the correct reason?

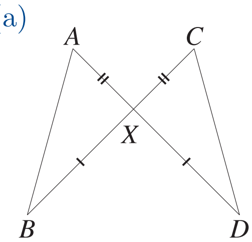
(b)



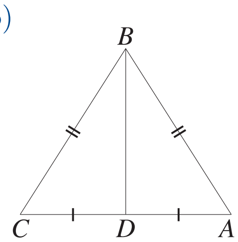

When asked to show that the two triangles above were congruent, another student wrote $\triangle GHI \equiv \triangle ABC$ (RHS). Again, although both triangles are right-angled, explain why the reason given is wrong. What is the correct reason?

**6.** In each part, prove that the two triangles in the diagram are congruent.

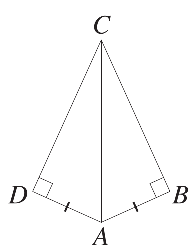
(a)



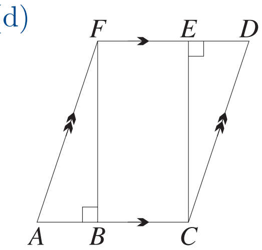

(b)




(c)




(d)



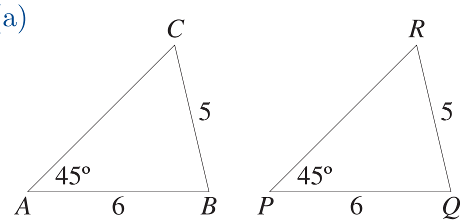

**7.** Let $M$ be any point on the base $BC$ of an isosceles triangle $ABC$. Using the facts that the legs $AB$ and $AC$ are equal, the base angles $\angle B$ and $\angle C$ are equal, and the side $AM$ is common, is it possible to prove that the triangles $ABM$ and $ACM$ are congruent?

ISBN: 978-1-107-61604-2

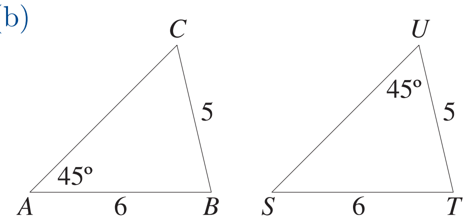
Cambridge University Press

**8.** Explain why the given pairs of triangles cannot be proven to be congruent.

(a)



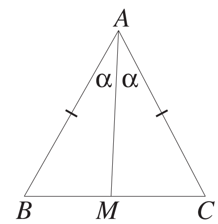

(b)




**9.** (a) What rotational and reflection symmetries does an isosceles triangle have?

(b) What rotational and reflection symmetries does an equilateral triangle have?

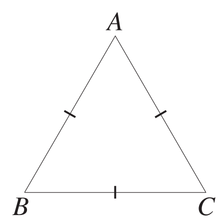
**10.** INTERPRETING THE PROPERTIES OF ISOSCELES AND EQUILATERAL TRIANGLES USING TRANSFORMATIONS:




(a) Sketched on the right is an isosceles triangle $\triangle BAC$ with $AB = AC$. The interval $AM$ bisects $\angle BAC$.

(i) Use the properties of reflections to explain why reflection in $AM$ exchanges $B$ and $C$, and hence explain why $\angle B = \angle C$, why $M$ bisects $BC$, and why $AM \perp BC$.

(ii) Name all the axes of symmetry of $\triangle ABC$.

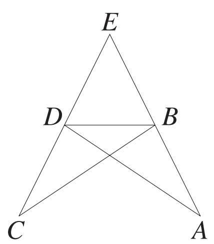



(b) The triangle $\triangle ABC$ on the right is equilateral.

(i) Using part (a), name all the axes of symmetry of the triangle, and hence explain why each interior angle is $60°$.

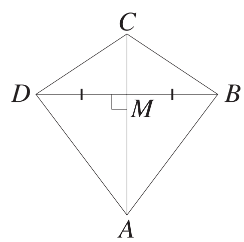
(ii) Describe all rotation symmetries of the triangle.

**11.** (a)




Given that $\triangle ABD \equiv \triangle CDB$ in the diagram above, prove that $\triangle BDE$ is isosceles.

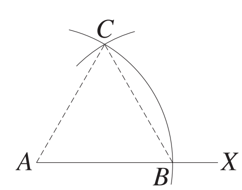
(b)




If $DM = MB$ and $AC \perp DB$, prove that $\triangle ABD$ and $\triangle CBD$ are isosceles.

**DEVELOPMENT**

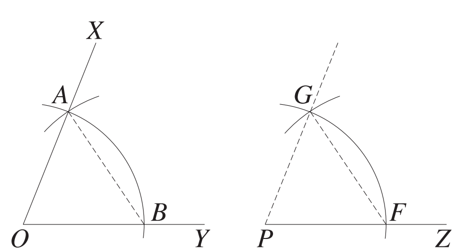
**12.** CONSTRUCTION: *Constructing an angle of 60°.* Let $AX$ be an interval. Construct an arc with centre $A$, meeting the line $AX$ at $B$. With the same radius but with centre $B$, construct a second arc meeting the first one at $C$. Explain why $\triangle ABC$ is equilateral, and hence why $\angle BAC = 60°$.




**13.** CONSTRUCTION: *Copying an angle.* Let $\angle XOY$ be an angle and $PZ$ be an interval. Construct an arc with centre $O$ meeting $OX$ at $A$ and $OY$ at $B$. With the same radius, construct an arc with centre $P$, meeting $PZ$ at $F$. With radius $AB$ and centre $F$, construct an arc meeting the second arc at $G$.

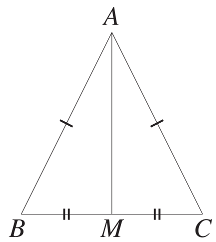
(a) Prove that $\triangle AOB \equiv \triangle FPG$.

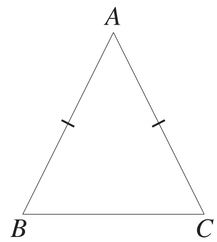
(b) Hence prove that $\angle AOB = \angle FPG$.




ISBN: 978-1-107-61604-2 

**14.** COURSE THEOREM: *Three alternative proofs that the base angles of an isosceles triangle are equal.* Let $ABC$ be an isosceles triangle with $AB = AC$.




(a) In the diagram above, the median $AM$ has been constructed. Prove that the triangles $AMB$ and $AMC$ are congruent, and hence that $\angle B = \angle C$.

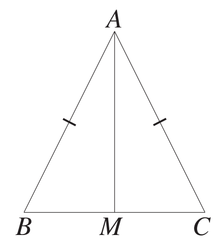
(b) Draw your own triangle $ABC$, and on it construct the altitude $AM$. Prove that $\triangle AMB$ is congruent to $\triangle AMC$, and hence that $\angle B = \angle C$.

(c) This is the most elegant proof, because it uses no construction at all. The two congruent triangles are the same triangle, but with the vertices in a different order.

(i) Prove that $\triangle ABC \equiv \triangle ACB$.

(ii) Hence prove that $\angle B = \angle C$.

**15.** THEOREM: *Properties of isosceles triangles.* In each part you will prove a property of an isosceles triangle. For each proof, use the same diagram, where $\triangle ABC$ is isosceles with $AB = AC$, and begin by proving that $\triangle AMB \equiv \triangle AMC$.

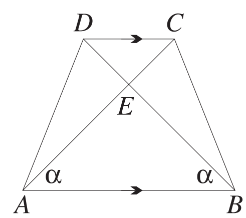



(a) If $AM$ is the *angle bisector* of $\angle A$, show that it is also the perpendicular bisector of $BC$.

(b) If $AM$ is the *altitude* from $A$ perpendicular to $BC$, show that $AM$ bisects $\angle CAB$ and that $BM = MC$.

(c) If $AM$ is the *median* joining $A$ to the midpoint $M$ of $BC$, show that it is also the perpendicular bisector.

**16.** In the diagram, $AB \parallel DC$ and $\angle CAB = \angle ABD = \alpha$.

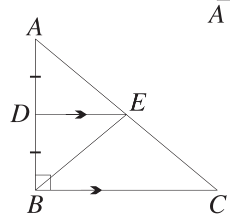



(a) Show that $CE = DE$.

(b) Prove that $\triangle ABC \equiv \triangle BAD$.

(c) Hence show that $\angle DAC = \angle CBD$.

**17.** Triangle $ABC$ has a right angle at $B$, $D$ is the midpoint of $AB$, and $DE$ is parallel to $BC$.

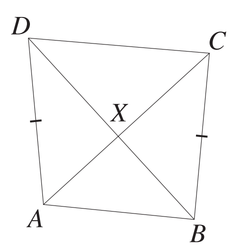



(a) Prove that $\angle ADE$ is a right angle.

(b) Prove that $\triangle AED \equiv \triangle BED$.

(c) Prove that $BE = EC$.

**18.** The diagonals $AC$ and $DB$ of quadrilateral $ABCD$ are equal and intersect at $X$. Also, $AD = BC$.




(a) Show that $\triangle ABC \equiv \triangle BAD$.

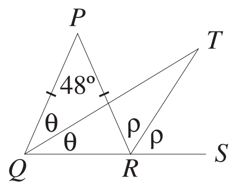
(b) Hence show that $\triangle ABX$ is isosceles.

(c) Thus show that $\triangle CDX$ is also isosceles.

(d) Show that $AB \parallel DC$.

ISBN: 978-1-107-61604-2

**19.** (a)

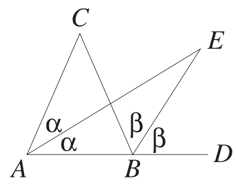



In the diagram, $\triangle PQR$ is isosceles with $PQ = PR$, and $\angle QPR = 48°$. The interval $QR$ is produced to $S$. The bisectors of $\angle PQR$ and $\angle PRS$ meet at the point $T$.

(i) Find $\angle PQR$.

(ii) Find $\angle QTR$.

(b)

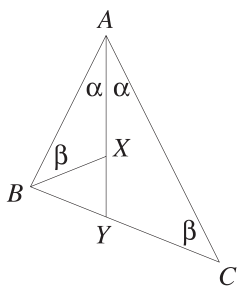



In $\triangle ABC$, $AB$ is produced to $D$. $AE$ bisects $\angle CAB$ and $BE$ bisects $\angle CBD$.

(i) If $\triangle ABE$ is isosceles with $\angle A = \angle E$, show that $\triangle ABC$ is also isosceles.

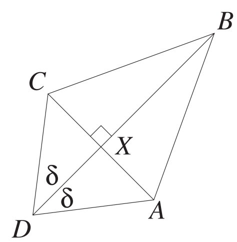
(ii) If $\triangle ABC$ is isosceles with $\angle A = \angle B$, under what circumstances will $\triangle ABE$ be isosceles?

**20.** (a)




The bisector of $\angle BAC$ meets $BC$ at $Y$. The point $X$ is constructed on $AY$ so that $\angle ABX = \angle ACB$. Prove that $\triangle BXY$ is isosceles.

(b)




The diagonals $AC$ and $BD$ of quadrilateral $ABCD$ meet at right angles at $X$. Also, $\angle ADX = \angle CDX$.

(i) Prove that $AD = CD$.

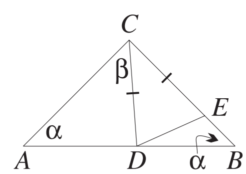
(ii) Hence prove that $AB = CB$.

**21.** In $\triangle ABC$, $\angle CAB = \angle CBA = \alpha$. Construct $D$ on $AB$ and $E$ on $CB$ so that $CD = CE$. Let $\angle ACD = \beta$.

(a) Explain why $\angle CDB = \alpha + \beta$.

(b) Find $\angle DCB$ in terms of $\alpha$ and $\beta$.

(c) Hence find $\angle EDB$ in terms of $\beta$.



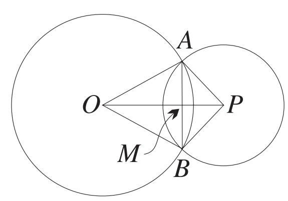

**22.** THEOREM: *The line of centres of two intersecting circles is the perpendicular bisector of the common chord.*
The diagram to the right shows two circles intersecting at $A$ and $B$. The line of centres $OP$ intersects $AB$ at $M$.

(a) Explain why $\triangle ABO$ and $\triangle ABP$ are isosceles.

(b) Show that $\triangle AOP \equiv \triangle BOP$.

(c) Show that $\triangle AMO \equiv \triangle BMO$.

(d) Hence show that $AM = BM$ and $AB \perp OP$.



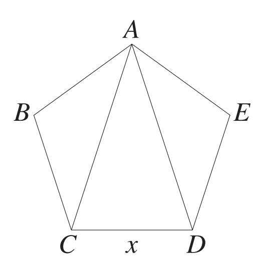

**23.** PENTAGONS AND TRIGONOMETRY: $ABCDE$ is a regular pentagon with side length $x$. Each interior angle is $108°$.

(a) State why $\triangle ABC$ is isosceles and find $\angle CAB$.

(b) Show that $\triangle ABC \equiv \triangle DEA$.

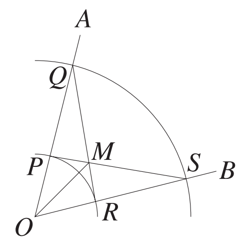
(c) Find $\angle CAD$.

(d) Find an expression for the area of the pentagon in terms of $x$ and trigonometric ratios.




ISBN: 978-1-107-61604-2 

**24.** CONSTRUCTION: *Another construction to bisect an angle.* Given $\angle AOB$, draw two concentric circles with centre $O$, cutting $OA$ at $P$ and $Q$ respectively, and $OB$ at $R$ and $S$ respectively. Let $PS$ and $QR$ meet at $M$.

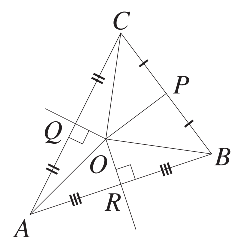



(a) Prove that $\triangle POS \equiv \triangle ROQ$.

(b) Hence prove that $\triangle PMQ \equiv \triangle RMS$.

(c) Hence prove that $OM$ bisects $\angle AOB$.

**25.** THE CIRCUMCENTRE THEOREM: *The perpendicular bisectors of the sides of a triangle are concurrent, and the resulting* circumcentre *is the centre of the* circumcircle *through all three vertices.* Let $P$, $Q$ and $R$ be the midpoints of the sides $BC$, $CA$ and $AB$ of $\triangle ABC$. Let the perpendiculars from $Q$ and $R$ meet at $O$, and join $OA$, $OB$, $OC$ and $OP$.

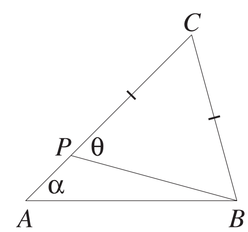



(a) Prove that $\triangle ORA \equiv \triangle ORB$.

(b) Prove that $\triangle OQA \equiv \triangle OQC$.

(c) Hence prove that $OA = OB = OC$, and $OP \perp BC$.

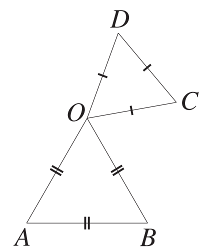
**26.** A GEOMETRIC INEQUALITY: *The angle opposite a longer side of a triangle is larger than the angle opposite a shorter side.* Let $\triangle ABC$ be a triangle in which $CA > CB$. Construct the point $P$ between $C$ and $A$ so that $CP = CB$, and let $\alpha = \angle A$ and $\theta = \angle CPB$.




(a) Explain why $\alpha < \theta$. (b) Explain why $\angle CBP = \theta$.

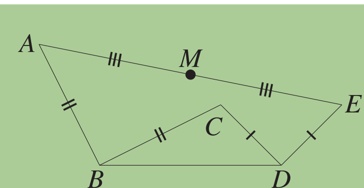
(c) Hence prove that $\alpha < \angle CBA$.

**27.** A ROTATION THEOREM: The triangles $OAB$ and $OCD$ in the figure drawn to the right are both equilateral triangles, and they have a common vertex $O$. Prove that $AC = BD$.

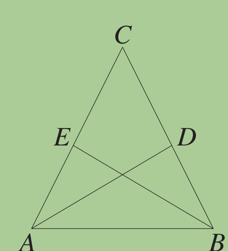



EXTENSION

**28.** In the diagram, $B$ and $D$ are fixed points on a horizontal line. A point $C$ is chosen anywhere in the plane, and $A$ is the image of $C$ after a rotation of $90^\circ$ (anticlockwise) about $B$. $E$ is the image of $C$ after a rotation of $-90^\circ$ (clockwise) about $D$. Find the location of $M$, the midpoint of $AE$, and show that this location is independent of the choice of $C$. [HINT: Let $F$ be the foot of the altitude from $C$ to $BD$. Add the points $G$ and $H$, the two images of $F$ under the two rotations, to the diagram.]




**29.** THREE TESTS FOR ISOSCELES TRIANGLES: Consider the triangle $ABC$, with $D$ on the side $BC$ and $E$ on the side $AC$.




(a) [Straightforward] Suppose that $AD$ and $BE$ are altitudes, and $AD = BE$. Show that $\triangle ABC$ is isosceles.

(b) [More difficult] Suppose that $AD$ and $BE$ are medians, and $AD = BE$. Show that $\triangle ABC$ is isosceles.

(c) [Extremely difficult] Suppose that $AD$ and $BE$ are angle bisectors, and $AD = BE$. Show that $\triangle ABC$ is isosceles.

ISBN: 978-1-107-61604-2 

# 8 D Trapezia and Parallelograms

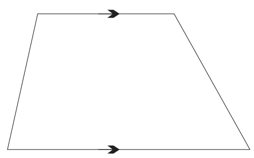
There are a series of important theorems concerning the sides and angles of quadrilaterals. If careful definitions are first given of five special quadrilaterals, these theorems can then be stated very elegantly as properties of these special quadrilaterals and tests for them. This section deals with trapezia and parallelograms, and the following section deals with rhombuses, rectangles and squares.

These theorems have been treated in earlier years, and most proofs have been left to structured questions in the following exercise. The proofs, however, are an essential part of the course, and should be carefully studied.

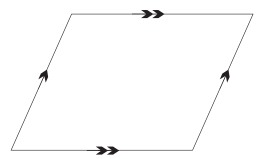
**Definitions of Trapezia and Parallelograms:** These figures are defined in terms of parallel sides. Notice that a parallelogram is a special sort of trapezium.

17

DEFINITIONS:
- A *trapezium* is a quadrilateral with at least one pair of opposite sides parallel.
- A *parallelogram* is a quadrilateral with both pairs of opposite sides parallel.



A trapezium

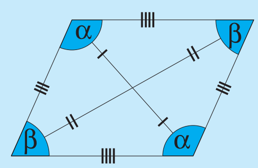


A parallelogram

**Properties of and Tests for Parallelograms:** The standard properties and tests concern the angles, the sides and the diagonals.

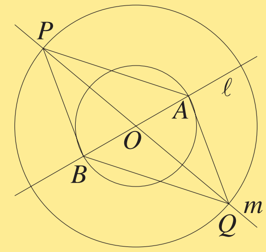
18

COURSE THEOREM: If a quadrilateral is a parallelogram, then:
- adjacent angles are supplementary, and
- opposite angles are equal, and
- opposite sides are equal, and
- the diagonals bisect each other.




Conversely, a quadrilateral is a parallelogram if:
- the opposite angles are equal, or
- the opposite sides are equal, or
- one pair of opposite sides are equal and parallel, or
- the diagonals bisect each other.

**WORKED EXERCISE:** [A construction of a parallelogram] Two lines $\ell$ and $m$ intersect at $O$, and concentric circles are constructed with centre $O$. Let $\ell$ meet the inner circle at $A$ and $B$, and let $m$ meet the outer circle at $P$ and $Q$. Prove that $APBQ$ is a parallelogram.




PROOF: Since the point $O$ is the midpoint of $AB$ and of $PQ$, the diagonals of $APBQ$ bisect each other. Hence $APBQ$ is a parallelogram.

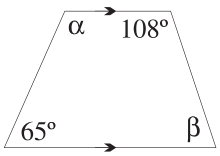
ISBN: 978-1-107-61604-2 

## Exercise **8D**

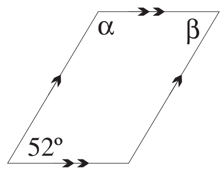
Note: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

**1.** Find the angles $\alpha$ and $\beta$ in the diagrams below, giving reasons.

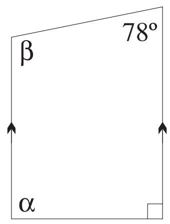
(a)



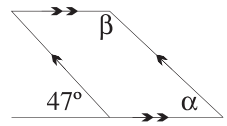

(b)




(c)

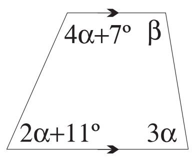



(d)

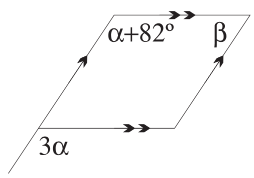



**2.** Write down an equation for $\alpha$ in each diagram below, giving reasons. Solve this equation to find the angles $\alpha$ and $\beta$, giving reasons.

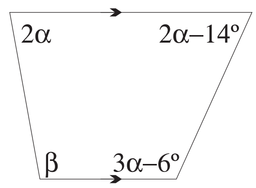
(a)



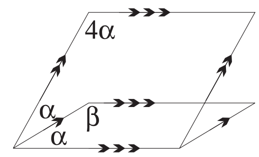

(b)




(c)

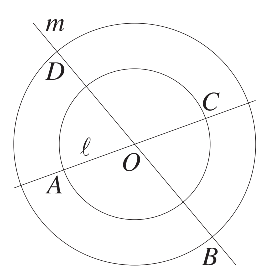



(d)




**3.** Construction: *Constructing a parallelogram from two equal parallel intervals.* Place a ruler with two parallel edges flat on the page, and draw 4 cm intervals $AB$ and $PQ$ on each side of the ruler. What theorem tells us that $ABQP$ is a parallelogram?

**4.** Construction: *Constructing a parallelogram from its diagonals.* Construct two lines $\ell$ and $m$ meeting at $O$. Construct two circles $\mathcal{C}$ and $\mathcal{D}$ with the common centre $O$. Let $\ell$ meet $\mathcal{C}$ at $A$ and $C$, and let $m$ meet $\mathcal{D}$ at $B$ and $D$. Use the tests for a parallelogram to explain why the quadrilateral $ABCD$ is a parallelogram.




**5.** Is it true that if one pair of opposite sides of a quadrilateral are parallel, and the other pair are equal, then the quadrilateral must be a parallelogram?

**6.** (a) What rotation and reflection symmetries does every parallelogram have?

(b) Can a trapezium that is not a parallelogram have any symmetries?

**7.** Trigonometry:

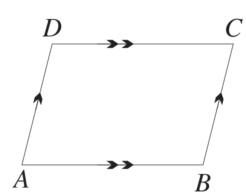
(a) If $ABCD$ is a parallelogram, show that $\sin A = \sin B = \sin C = \sin D$.

(b) Quadrilateral $ABCD$ is a trapezium with $AB \parallel DC$ and with $\angle A = \angle B$. Show that $\sin A = \sin B = \sin C = \sin D$.

DEVELOPMENT

**8.** Properties of a parallelogram: In this question, you must use the definition of a parallelogram as a quadrilateral in which the opposite sides are parallel.

(a) Course theorem: *Adjacent angles of a parallelogram are supplementary, and opposite angles are equal.* The diagram shows a parallelogram $ABCD$. Explain why $\angle A + \angle B = 180°$ and $\angle A = \angle C$.

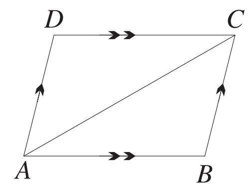



ISBN: 978-1-107-61604-2  Cambridge University Press

(b) COURSE THEOREM: *Opposite sides of a parallelogram are equal.* The diagram shows a parallelogram $ABCD$ with diagonal $AC$.

(i) Prove that $\triangle ACB \equiv \triangle CAD$.

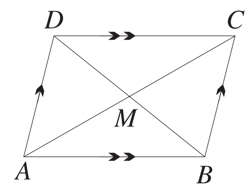
(ii) Hence show that $AB = DC$ and $BC = AD$.




(c) COURSE THEOREM: *The diagonals of a parallelogram bisect each other.* The diagram shows a parallelogram $ABCD$ with diagonals meeting at $M$.

(i) Prove that $\triangle ABM \equiv \triangle CDM$ (use part (b)).

(ii) Hence show that $AM = MC$.

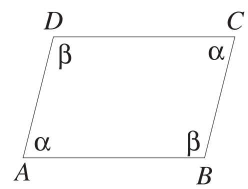



**9.** TESTS FOR A PARALLELOGRAM: These four theorems give the standard tests for a quadrilateral to be a parallelogram.

(a) COURSE THEOREM: *If the opposite angles of a quadrilateral are equal, then it is a parallelogram.*
The diagram opposite shows a quadrilateral $ABCD$ in which $\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$.

(i) Prove that $\alpha + \beta = 180°$.

(ii) Hence show that $AB \parallel DC$ and $AD \parallel BC$.

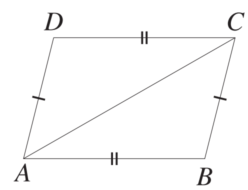



(b) COURSE THEOREM: *If the opposite sides of a quadrilateral are equal, then it is a parallelogram.*
The diagram shows a quadrilateral $ABCD$ in which $AB = DC$ and $AD = BC$, with diagonal $AC$.

(i) Prove that $\triangle ACB \equiv \triangle CAD$.

(ii) Thus prove that $\angle CAB = \angle ACD$, and also that $\angle ACB = \angle CAD$.

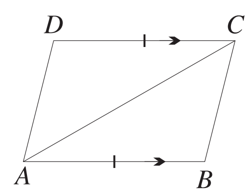
(iii) Hence show that $AB \parallel DC$ and $AD \parallel BC$.




(c) COURSE THEOREM: *If one pair of opposite sides of a quadrilateral are equal and parallel, then it is a parallelogram.* The diagram shows a quadrilateral $ABCD$ in which $AB = DC$ and $AB \parallel DC$, with diagonal $AC$.

(i) Prove that $\triangle ACB \equiv \triangle CAD$.

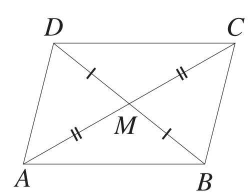
(ii) Hence show that $AD \parallel BC$.




(d) COURSE THEOREM: *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*
In the diagram, $ABCD$ is a quadrilateral in which the diagonals meet at $M$, with $AM = MC$ and $BM = MD$.

(i) Prove that $\triangle ABM \equiv \triangle CDM$.

(ii) Hence use the previous theorem to prove that the quadrilateral $ABCD$ is a parallelogram.



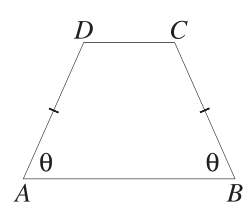

**10.** In quadrilateral $ABCD$, $\angle BAD = \angle ABC$ and $AD = BC$.

(a) Prove that $\triangle BAD \equiv \triangle ABC$.

(b) Why does $\angle ABD = \angle CAB$?

(c) Show that $\angle DAC = \angle DBC$.

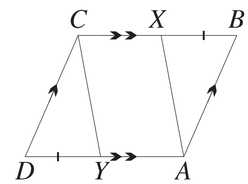
(d) Prove that $ABCD$ is a trapezium.




ISBN: 978-1-107-61604-2

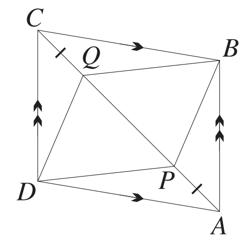
Cambridge University Press

**11.** In the diagram, $ABCD$ is a parallelogram. The points $X$ and $Y$ lie on $BC$ and $AD$ respectively such that $BX = DY$.



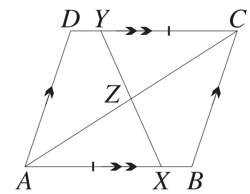

(a) Explain why $\angle ABX = \angle CDY$.
(b) Explain why $AB = CD$.
(c) Show that $\triangle ABX \equiv \triangle CDY$.
(d) Hence prove that $AYCX$ is a parallelogram.

**12.** The diagram shows the parallelogram $ABCD$ with diagonal $AC$. The points $P$ and $Q$ lie on this diagonal in such a way that $AP = CQ$.




(a) Prove that $\triangle ABP \equiv \triangle CDQ$.
(b) Prove that $\triangle ADP \equiv \triangle CBQ$.
(c) Hence prove that $BQDP$ is a parallelogram.

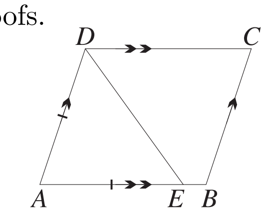
**13.** The diagram shows the parallelogram $ABCD$ and points $X$ and $Y$ on $AB$ and $CD$ respectively, with $AX = CY$. The diagonal $AC$ intersects $XY$ at $Z$.




(a) Prove that $\triangle AXZ \equiv \triangle CYZ$.
(b) Hence prove that $XY$ is concurrent with the diagonals.

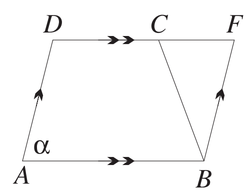
**14.** The previous two questions could have been solved more easily using the standard properties of and tests for a parallelogram. Explain these alternative proofs.

**15.** The diagram to the right shows a parallelogram $ABCD$. The point $E$ is constructed on the side $AB$ in such a way that $AD = AE$. Prove that the interval $DE$ bisects the angle $\angle ADC$. [HINT: Begin by letting $\angle ADE = \theta$.]




**16.** THEOREM: *The base angles of a trapezium are equal if and only if the non-parallel sides are equal.* Let $ABCD$ be a trapezium with $AB \parallel DC$, but $AD$ not parallel to $BC$. Construct $BF \parallel AD$ with $F$ on $DC$, produced if necessary. Let $\angle DAB = \alpha$.

(a) Suppose first that $AD = BC$.
  (i) Prove that $BF = AD$.
  (ii) Hence prove that $\angle ABC = \alpha$.




(b) Conversely, suppose that $\angle ABC = \alpha$.
  (i) Prove that $\angle BFC = \alpha$.
  (ii) Hence prove that $BC = AD$.

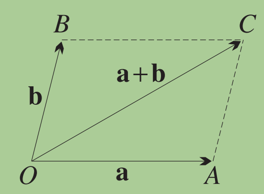
**17.** The diagonals of quadrilateral $ABCD$ meet at $M$, and $\triangle ABM \equiv \triangle DCM$.
(a) Draw a diagram showing this information.
(b) Prove that $ABCD$ is a trapezium with equal base angles.

EXTENSION

**18.** Quadrilateral $ABCD$ is a parallelogram. A point $X$ is chosen on $AB$ and $Y$ is constructed on $DC$ so that $DX = BY$. Note that $DX$ is not perpendicular to $AB$.
(a) Given that $DXBY$ is not a parallelogram, draw a picture of the situation.
(b) What type of quadrilateral is $DXBY$?
(c) What condition needs to be placed on $DX$ in order to guarantee that $DXBY$ is a parallelogram?

ISBN: 978-1-107-61604-2

**19.** PARALLELOGRAMS AND VECTORS: The diagram shows two vectors $\mathbf{a}$ and $\mathbf{b}$ starting from $O$. The parallelogram $OACB$ has been completed so that the diagonal $OC$ represents the vector $\mathbf{a}+\mathbf{b}$. Draw three more parallelograms, each using $O$ as one vertex, so that the diagonals from $O$ represent the vectors: (a) $\mathbf{b}-\mathbf{a}$ (b) $-\mathbf{a}-\mathbf{b}$ (c) $\mathbf{a}-\mathbf{b}$



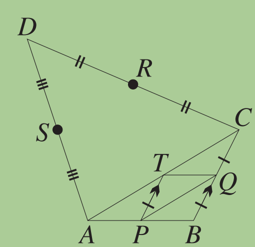

**20.** THEOREM: *The quadrilateral formed by joining the midpoints of the sides of a quadrilateral is a parallelogram.* In quadrilateral $ABCD$, the points $Q$, $R$ and $S$ are the midpoints of $BC$, $CD$ and $DA$ respectively. The two points $P$ and $T$ lie on $AB$ and $AC$ respectively such that $PT = BQ$ and $PT \parallel BQ$.

(a) Explain why $PBQT$ is a parallelogram.

(b) Show that the four triangles $\triangle APT$, $\triangle QPT$, $\triangle PBQ$ and $\triangle TQC$ are all congruent, and that $P$ is the midpoint of $AB$.

(c) Hence show that the line joining the midpoints of two adjacent sides of a quadrilateral is parallel to the diagonal joining those two sides.

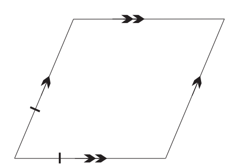
(d) Hence show that $PQRS$ is a parallelogram.




## 8 E Rhombuses, Rectangles and Squares

Rhombuses, rectangles and squares are particular types of parallelograms, and their definitions in this course reflect that understanding. Again, most of the proofs have been encountered in earlier years, and are left to the exercises.

**Rhombuses and their Properties and Tests:** Intuitively, a rhombus is a 'pushed-over square', but its formal definition is:

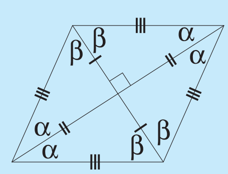


**19** DEFINITION: A *rhombus* is a parallelogram with a pair of adjacent sides equal.

As with the parallelogram, the standard properties and tests concern the sides, the vertex angles and the diagonals.

**20** COURSE THEOREM: If a quadrilateral is a rhombus, then:
- all four sides are equal, and
- the diagonals bisect each other at right angles, and
- the diagonals bisect each vertex angle.

Conversely, a quadrilateral is a rhombus if:
- all sides are equal, or
- the diagonals bisect each other at right angles, or
- the diagonals bisect each vertex angle.



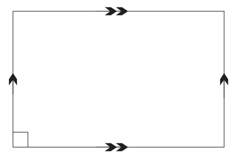

PROOF: Since a rhombus is a parallelogram, its opposite sides are equal. Since also two adjacent sides are equal, all four sides must be equal.

ISBN: 978-1-107-61604-2 

Conversely, suppose that all four sides of a quadrilateral are equal. Since opposite sides are equal, it must be a parallelogram, and since two adjacent sides are equal, it is therefore a rhombus.

This proves the first and fourth points. The remaining proofs are a little more complicated, and are left to the exercises.

**Rectangles and their Properties and Tests:** A rectangle is also defined as a special type of parallelogram.

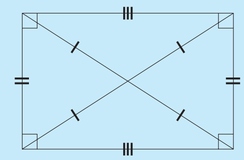


**21** DEFINITION: A *rectangle* is a parallelogram in which one angle is a right angle.

The standard properties and tests for a rectangle are:

**22** COURSE THEOREM: If a quadrilateral is a rectangle, then:
- all four angles are right angles, and
- the diagonals are equal and bisect each other.

Conversely, a quadrilateral is a rectangle if:
- all angles are equal, or
- the diagonals are equal and bisect each other.

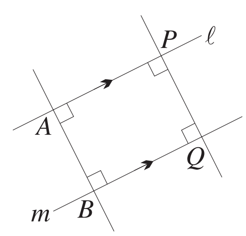


PROOF: Since a rectangle is a parallelogram, its opposite angles are equal and add to 360°. Since one angle is 90°, it follows that all angles are 90°.

Conversely, suppose that all angles of a quadrilateral are equal. Then since they add to 360°, they must each be 90°. Hence the opposite angles are equal, so the quadrilateral must be a parallelogram, and hence is a rectangle.

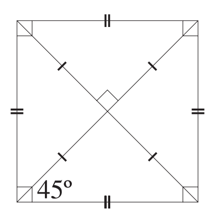
This proves the first and third points. The remaining proofs are left to structured exercises.

**The Distance Between Parallel Lines:** Suppose that $AB$ and $PQ$ are two transversals perpendicular to two parallel lines $\ell$ and $m$. Then $ABQP$ forms a rectangle, because all its vertex angles are right angles. Hence the opposite sides $AB$ and $PQ$ are equal. This allows a formal definition of the distance between two parallel lines.




**23** DEFINITION: The distance between two parallel lines is the length of a perpendicular transversal.

**Squares:** Rhombuses and rectangles are different special sorts of parallelograms. A square is simply a quadrilateral that is both a rhombus and a rectangle.

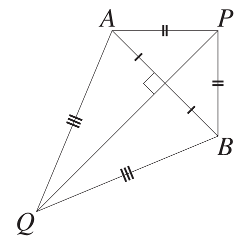



**24** DEFINITION: A *square* is a quadrilateral that is both a rhombus and a rectangle.

It follows then from the previous theorems that all sides of a square are equal, all angles are right angles, and the diagonals bisect each other at right angles and meet each side at 45°.

ISBN: 978-1-107-61604-2 

A NOTE ON KITES: Kites are not part of the course, but they occur frequently in problems. A *kite* is usually defined as a quadrilateral in which two pairs of adjacent sides are equal, as in the diagram to the right, where $AP = BP$ and $AQ = BQ$. A question below develops the straightforward proof that the diagonal $PQ$ is the perpendicular bisector of the diagonal $AB$, and bisects the vertex angles at $P$ and $Q$. Another question deals with tests for kites.



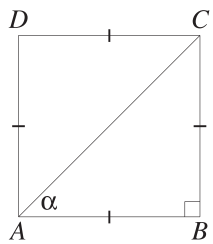

Theorems about kites, however, are not part of the course, and should not be quoted as reasons unless they have been developed earlier in the same question.

## Exercise 8E

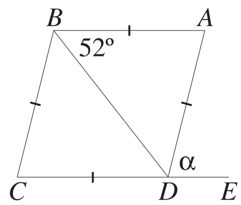
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

**1.** Find $\alpha$ in each of the figures below, giving reasons.

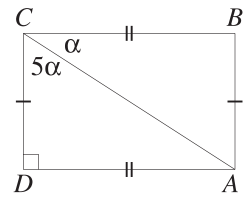
(a)



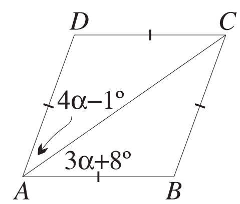

(b)



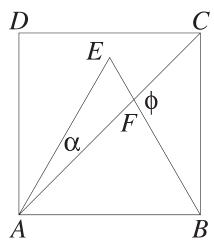

(c)




(d)

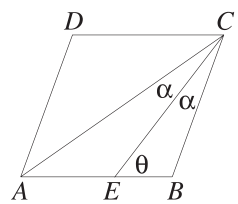



**2.** (a)




Inside the square $ABCD$ is an equilateral $\triangle ABE$. The diagonal $AC$ intersects $BE$ at $F$. Find the sizes of angles $\alpha$ and $\phi$.

(b)



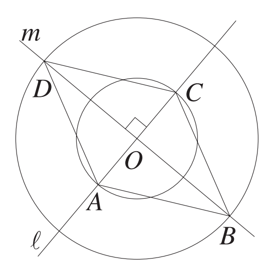

$ABCD$ is a rhombus with the diagonal $AC$ shown. The line $CE$ bisects $\angle ACB$. Show that $\theta = 3\alpha$.

**3.** (a) What rotation and reflection symmetries does:

(i) every rectangle have, (ii) every rhombus have, (iii) every square have?

(b) What rotation and reflection symmetries does a circle have?

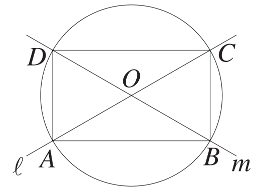
**4.** CONSTRUCTIONS: *Constructing a rectangle, rhombus and square from their diagonals.*



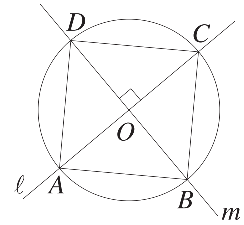

(a) RHOMBUS: Construct any two perpendicular lines $\ell$ and $m$, and let them meet at $O$. Construct two circles $\mathcal{C}$ and $\mathcal{D}$ with the common centre $O$. Let $\ell$ meet $\mathcal{C}$ at $A$ and $C$, and let $m$ meet $\mathcal{D}$ at $B$ and $D$. Use the standard tests for a rhombus to explain why the quadrilateral $ABCD$ is a rhombus.

ISBN: 978-1-107-61604-2 

(b) RECTANGLE: Construct any two non-parallel lines $\ell$ and $m$, and let them meet at $O$. Construct a circle $\mathcal{C}$ with centre $O$ and any radius. Let $\ell$ meet $\mathcal{C}$ at $A$ and $C$, and let $m$ meet $\mathcal{C}$ at $B$ and $D$. Use the standard tests for a rectangle to explain why the quadrilateral $ABCD$ is a rectangle.



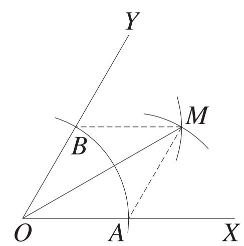

(c) SQUARE: Construct any two perpendicular lines $\ell$ and $m$, and let them meet at $O$. Construct a circle $\mathcal{C}$ with centre $O$ and any radius. Let $\ell$ meet $\mathcal{C}$ at $A$ and $C$, and let $m$ meet $\mathcal{C}$ at $B$ and $D$. Use the standard tests for a square to explain why the quadrilateral $ABCD$ is a square.




**5.** CONSTRUCTION: *The bisector of a given angle.*
Given an angle $\angle XOY$, construct an arc with centre $O$ and any radius meeting the arms $OX$ and $OY$ at $A$ and $B$ respectively. With the same radius and with centres $A$ and $B$, construct two further arcs meeting at $M$.

(a) Why is the quadrilateral $OAMB$ a rhombus?

(b) Hence prove that $OM$ bisects $\angle XOY$.




DEVELOPMENT

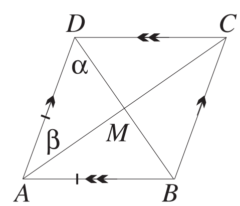
**6.** COURSE THEOREM: *The diagonals of a rhombus bisect each other at right angles, and bisect the vertex angles.* In the diagram, the diagonals of the rhombus $ABCD$ meet at $M$. Since a rhombus is a parallelogram, we already know that the diagonals bisect each other.

(a) Let $\alpha = \angle ADB$. Explain why $\angle ABD = \alpha$.

(b) Hence prove that $\angle CDB = \alpha$.

(c) Let $\beta = \angle DAC$. Prove that $\angle BAC = \beta$.

(d) Hence prove that $AC \perp BD$. (There is no need for congruence in this situation.)

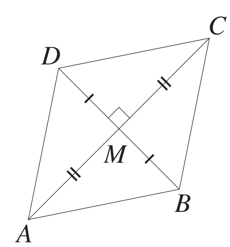



**7.** TESTS FOR A RHOMBUS: The following three parts are structured proofs of the standard tests for a rhombus listed in the notes above.

(a) COURSE THEOREM: *If all sides of a quadrilateral are equal, then it is a rhombus.* Explain, using the previous theorems and the definition of a rhombus, why a quadrilateral with all sides equal must be a rhombus.

(b) COURSE THEOREM: *If the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.* The diagram shows a quadrilateral $ABCD$ in which the diagonals bisect each other at right angles at $M$.

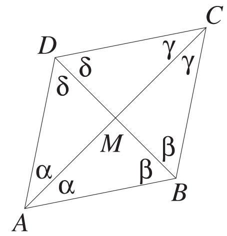
(i) What previous theorem proves that the quadrilateral $ABCD$ is a parallelogram?




(ii) Prove that $\triangle AMD \equiv \triangle AMB$, and hence that $AD = AB$. The quadrilateral $ABCD$ is then a rhombus by definition.

(c) COURSE THEOREM: *If the diagonals of a quadrilateral bisect each other vertex angle, then it is a rhombus.* The diagram shows a quadrilateral $ABCD$ in which the diagonals bisect each vertex angle. Let $\alpha$, $\beta$, $\gamma$ and $\delta$ be as shown.

ISBN: 978-1-107-61604-2  Cambridge University Press




(i) Prove that $\alpha + \beta + \gamma + \delta = 180^\circ$.

(ii) By taking the sum of the angles in $\triangle ABC$ and $\triangle ADC$, prove that $\beta = \delta$.

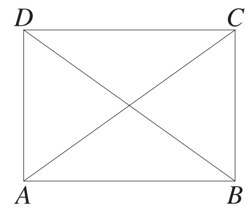
(iii) Similarly, prove that $\alpha = \gamma$, and state why $ABCD$ is a parallelogram.

(iv) Finally, prove that $AB = AD$.

**8.** PROPERTIES OF RECTANGLES: The following two parts are structured proofs of the standard properties of a rectangle listed in the notes above.

(a) COURSE THEOREM: *All the angles in a rectangle are right angles.* Use the definition of a rectangle — as a parallelogram with one angle a right angle — and the properties of a parallelogram to prove that all four angles of a rectangle are right angles.

(b) COURSE THEOREM: *The diagonals of a rectangle are equal and bisect each other.* The diagram shows a rectangle $ABCD$, with diagonals drawn.



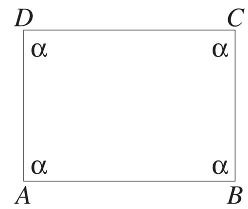

(i) Use the properties of a parallelogram to show that the diagonals bisect each other.

(ii) Prove that $\triangle ABC \equiv \triangle BAD$.

(iii) Hence prove that $AC = BD$.

**9.** TESTS FOR A RECTANGLE: The following two parts are structured proofs of the standard tests for a rectangle listed in the notes above.

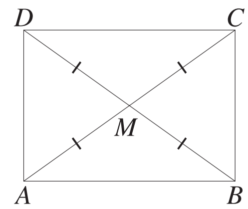
(a) COURSE THEOREM: *If all angles of a quadrilateral are equal, then it is a rectangle.* The diagram shows a quadrilateral $ABCD$ in which all angles are equal.




(i) Prove that all angles are right angles.

(ii) Hence prove that $ABCD$ is a rectangle.

(b) COURSE THEOREM: *If the diagonals of a quadrilateral are equal and bisect each other, then it is a rectangle.* The diagram shows a quadrilateral $ABCD$ in which the diagonals, meeting at $M$, are equal and bisect each other.



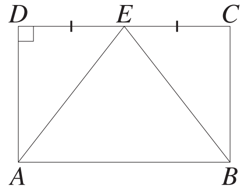

(i) Explain why $ABCD$ is a parallelogram.

(ii) Let $\alpha = \angle BAM$, and explain why $\angle ABM = \alpha$.

(iii) Let $\beta = \angle MBC$, and explain why $\angle MCB = \beta$.

(iv) Using the angle sum of the triangle $ABC$, prove that $\angle ABC = 90^\circ$.

**10.** (a)

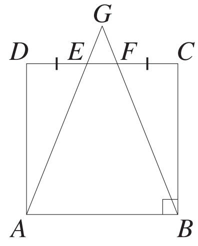



The point $E$ is the midpoint of the side $CD$ of the rectangle $ABCD$.

(i) Prove that $\triangle BCE \equiv \triangle ADE$.

(ii) Hence show that $\triangle ABE$ is isosceles.

(b)




The points $E$ and $F$ are on the side $CD$ in the square $ABCD$, with $CF = DE$. Produce $AE$ and $BF$ to meet at $G$.

(i) Prove that $\triangle BCF \equiv \triangle ADE$.

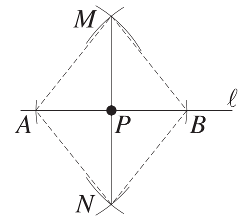
(ii) Hence show that $\triangle ABG$ is isosceles.

ISBN: 978-1-107-61604-2

**11.** CONSTRUCTION: *A right angle at a point on a line.* Given a point $P$ on a line $\ell$, construct an arc with centre $P$ meeting $\ell$ at $A$ and $B$. With increased radius, construct arcs with centres at $A$ and $B$ meeting at $M$ and $N$.

(a) Why is the quadrilateral $AMBN$ a rhombus?

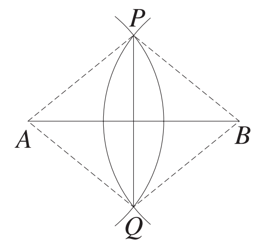
(b) Hence prove that $P$ lies on $MN$ and $MN \perp AB$.




**12.** CONSTRUCTION: *The perpendicular bisector of an interval.* Given an interval $AB$, construct arcs of the same radius, greater than $\frac{1}{2}AB$, with centres at $A$ and $B$. Let the arcs meet at $P$ and $Q$.

(a) Why is the quadrilateral $APBQ$ a rhombus?

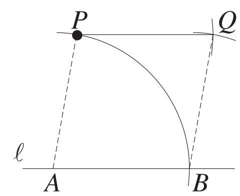
(b) Hence prove that $PQ$ bisects $AB$ and $PQ \perp AB$.




**13.** CONSTRUCTION: *The line parallel to a given line through a given point.* Given a line $\ell$ and a point $P$ not on $\ell$, choose a point $A$ on $\ell$. With centre $A$ and radius $AP$, construct an arc meeting $\ell$ at $B$. With the same radius, draw arcs with centres at $B$ and $P$ meeting at $Q$.

(a) Why is the quadrilateral $APQB$ a rhombus?

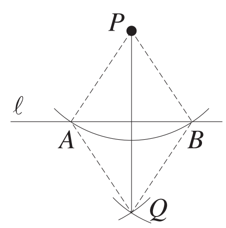
(b) Hence prove that $PQ \parallel \ell$.



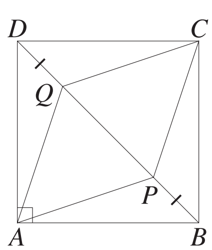

**14.** CONSTRUCTION: *The line perpendicular to a given line through a given point.* Given a line $\ell$ and a point $P$ not on $\ell$, construct an arc with centre $P$ meeting $\ell$ at $A$ and $B$. With the same radius, draw arcs with centres at $A$ and $B$, intersecting at $Q$.

(a) Why is the quadrilateral $APBQ$ a rhombus?

(b) Hence prove that $PQ \perp \ell$.

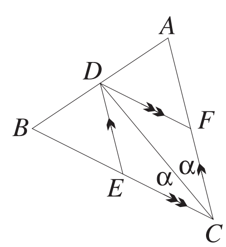



**15.** (a)




$P$ and $Q$ lie on the diagonal $BD$ of square $ABCD$, and $BP = DQ$. (i) Prove that $\triangle ABP \equiv \triangle CBP \equiv \triangle ADQ \equiv \triangle CDQ$. (ii) Hence show that $APCQ$ is a rhombus.

(b)

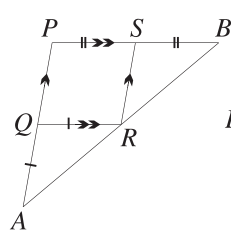



In the triangle $ABC$, $DC$ bisects $\angle BCA$, $DE \parallel AC$ and $DF \parallel BC$.
(i) Explain why is $DECF$ a parallelogram.
(ii) Show that $DECF$ is a rhombus.

**16.** The parallelogram $PQRS$ is inscribed in $\triangle PBA$ with $R$ on $AB$. It is found that $QA = QR$ and $PS = SB$.

(a) Prove that $\triangle BSR \equiv \triangle RQA$.

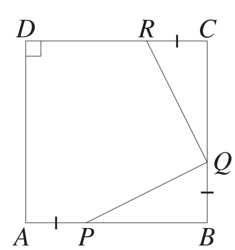
(b) Hence prove that $PQRS$ is a rhombus.




**17.** In the square $ABCD$, $P$ is on $AB$, $Q$ is on $BC$ and $R$ is on $CD$, with $AP = BQ = CR$.

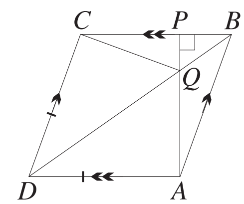
(a) Prove that $\triangle PBQ \equiv \triangle QCR$.

(b) Prove that $\angle PQR$ is a right angle.




ISBN: 978-1-107-61604-2  Cambridge University Press

**18.** In the rhombus $ABCD$, $AP$ is constructed perpendicular to $BC$ and intersects the diagonal $BD$ at $Q$.



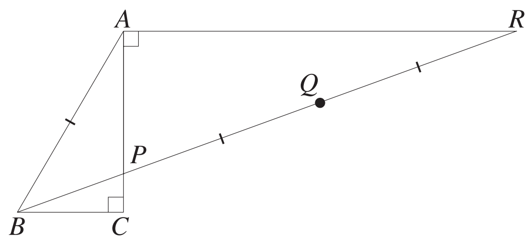

(a) State why $\angle ADB = \angle CDB$.

(b) Prove that $\triangle AQD \equiv \triangle CQD$.

(c) Show that $\angle DAQ$ is a right angle.

(d) Hence find $\angle QCD$.

**19.** The triangles $ABC$ and $APR$ are both right-angled at the vertices marked in the diagram. The midpoint of $PR$ is $Q$, and it is found that $PQ = QR = AB$.



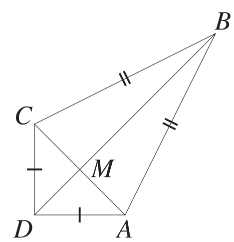

(a) Explain why $\angle PBC = \angle PRA$.

(b) Construct the point $S$ that completes the rectangle $APSR$. Explain why $Q$ is also the midpoint of $AS$ and why $PQ = AQ$.

(c) Hence prove that $\angle PBA = 2 \times \angle PBC$.

**20.** TWO THEOREMS ABOUT KITES: A *kite* is defined to be a quadrilateral in which two pairs of adjacent sides are equal. [NOTE: This definition is not part of the course.]

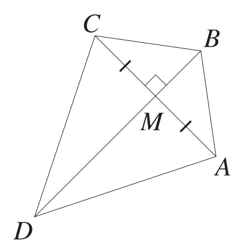
(a) THEOREM: *The diagonals of a kite are perpendicular and one bisects the other.*
The diagram shows a kite $ABCD$ with $AB = BC$ and $AD = DC$. The diagonals $AC$ and $BD$ intersect at $M$.




(i) Prove that $\triangle BAD \equiv \triangle BCD$.

(ii) Use the properties of the isosceles triangle $ABC$ to prove that $DB$ bisects $AC$ at right angles.

(b) THEOREM: *If the diagonals of a quadrilateral are perpendicular, and one is bisected by the other, then the quadrilateral is a kite.*
The diagram shows a quadrilateral with perpendicular diagonals meeting at $M$, and $AM = MC$.




(i) Prove that $\triangle BAM \equiv \triangle BCM$.

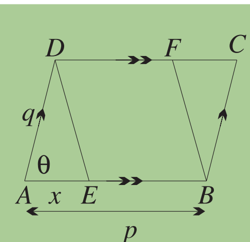
(ii) Hence prove that $BA = BC$.

(iii) Similarly, prove that $DA = DC$.

EXTENSION

**21.** TRIGONOMETRY: The quadrilateral $ABCD$ is a parallelogram with $AB = p$, $AD = q$, $p > q$, and $\angle BAD = \theta$. The points $E$ on $AB$ and $F$ on $CD$ are chosen so that $EBFD$ is a rhombus. Let $AE = x$. Show that

$$x = \frac{p^2 - q^2}{2(p - q\cos\theta)}\,.$$




ISBN: 978-1-107-61604-2  Cambridge University Press

# 8 F Areas of Plane Figures

The standard area formulae are well known. Some of them were used in the development of the definite integral, which extended the idea of area to regions with curved boundaries. The formulae below apply to figures with straight edges, and their proofs by dissection are reviewed below.

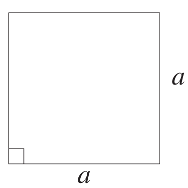
**Course Theorem — Area Formulae for Quadrilaterals and Triangles:** The various area formulae are based on the definition of the area of a rectangle as length times breadth, and on the assumption that area remains constant when regions are dissected and rearranged. The first two formulae below are therefore definitions. The other four formulae can be proven using the diagrams below, which need to be studied until the logic of each dissection becomes clear.

STANDARD AREA FORMULAE:

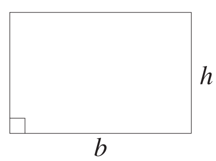
- SQUARE: area = (side length)$^2$
- RECTANGLE: area = (length) $\times$ (breadth)

25
- PARALLELOGRAM: area = (base) $\times$ (perpendicular height)
- TRIANGLE: area = $\frac{1}{2}$ $\times$ (base) $\times$ (perpendicular height)
- RHOMBUS: area = $\frac{1}{2}$ $\times$ (product of the diagonals)
- TRAPEZIUM: area = (average of parallel sides) $\times$ (perpendicular height)

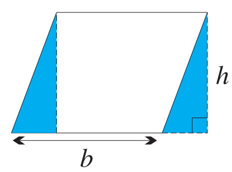
PROOF:



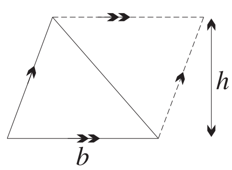

Square: area = $a^2$



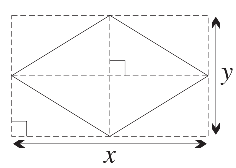

Rectangle: area = $bh$



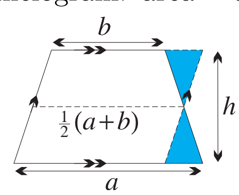

Parallelogram: area = $bh$




Triangle: area = $\frac{1}{2}bh$




Rhombus: area = $\frac{1}{2}xy$




Trapezium: area = $\frac{1}{2}h(a+b)$

NOTE: Because rhombuses are parallelograms, their areas can also be calculated using the formula area = (base) $\times$ (perpendicular height) associated with parallelograms. The formula area = $\frac{1}{2}$ $\times$ (product of the diagonals) gives another, and quite different, approach that is often forgotten in problems.

Because squares are rhombuses, their area can also be calculated using their diagonals. But the diagonals of a square are equal, so the formula becomes

$$\text{area of square} = \tfrac{1}{2} \times (\text{square of the diagonal}).$$

**The Area of the Circle:** The area of a circle is $\pi r^2$, where $r$ is the radius. The proof of this result was discussed in Section 11B of the Year 11 volume as a preliminary to integration — because the boundary is curved, some sort of infinite dissection is necessary, and the proof therefore belongs to the theory of the definite integral.

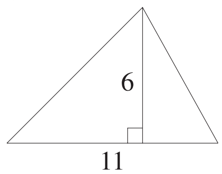
ISBN: 978-1-107-61604-2

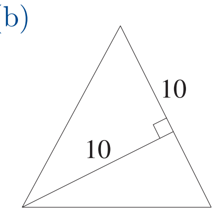
Cambridge University Press

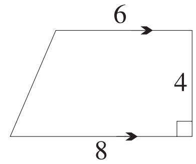
## Exercise 8F

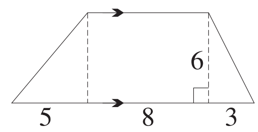
NOTE: The calculation of areas is so linked with Pythagoras' theorem that it is inconvenient to separate them in exercises. Pythagoras' theorem has therefore been used freely in the questions of this exercise, although its formal review is in the next section.

**1.** Find the areas of the following figures:

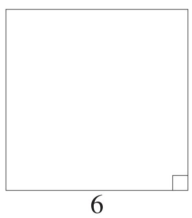
(a)


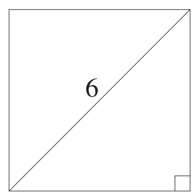

(b)


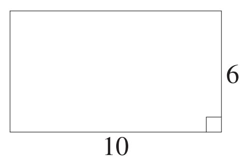

(c)


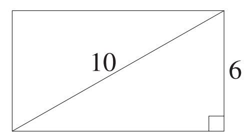

(d)



**2.** Find the area $A$ and the perimeter $P$ of the squares in parts (a) and (b) and the rectangles in parts (c) and (d). Use Pythagoras' theorem to find missing lengths where necessary.

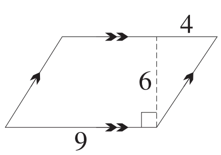
(a)


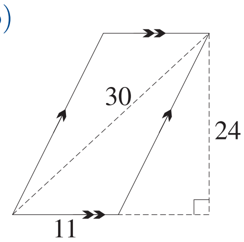

(b)


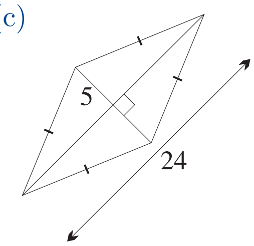

(c)


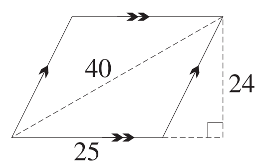

(d)



**3.** Find the area $A$ and the perimeter $P$ of the following figures, using Pythagoras' theorem where necessary. Then find the lengths of any missing diagonals.

(a)



(b)



(c)



(d)



**4.** (a) Explain why the area of a square is half the square of the diagonal.

(b) Show that the area of a rectangle with sides $a$ and $b$ is the same as the area of the square whose side length $s$ is the geometric mean $\sqrt{ab}$ of the sides of the rectangle.

(c) THE TWO AREA FORMULAE FOR TRIANGLES: Let $\triangle ABC$ be right-angled at $C$. Explain why the formula $A = \frac{1}{2} \times (\text{base}) \times (\text{height})$ for the area of the triangle is identical to the trigonometric area formulae $A = \frac{1}{2}ab\sin C$.

DEVELOPMENT

**5.** THEOREM: *A median of a triangle divides the triangle into two triangles of equal area.* Sketch a triangle $ABC$. Let $M$ be the midpoint of $BC$, and join the median $AM$.

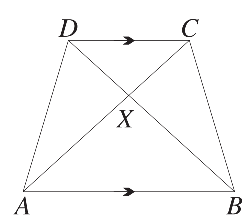
(a) Explain why $\triangle ABM$ and $\triangle ACM$ have the same perpendicular height.

(b) Hence explain why $\triangle ABM$ and $\triangle ACM$ have the same area.

**6.** THEOREM: *The two triangles formed by the diagonals and the non-parallel sides of a trapezium have the same area.* In the trapezium $ABCD$, $AB \parallel DC$ and $AC$ intersects $BD$ at $X$.

(a) Explain why area $\triangle ABC =$ area $\triangle ABD$.

(b) Hence explain why area $\triangle BCX =$ area $\triangle ADX$.

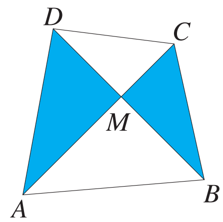



ISBN: 978-1-107-61604-2

**7.** THEOREM: *Conversely, a quadrilateral in which the diagonals form a pair of opposite triangles of equal area is a trapezium.* The diagonals of the quadrilateral $ABCD$ meet at $M$, and $\triangle AMD$ and $\triangle BMC$ have equal areas.

(a) Prove that $\triangle ABD$ and $\triangle ABC$ have equal areas.

(b) Hence prove that $AB \parallel DC$.




**8.** Prove that the four small triangles formed by the two diagonals of a parallelogram all have the same area. Under what circumstances are they all congruent?

**9.** The diagonals of a parallelogram form the diameters of two circles.

(a) Why are they concentric?

(b) If the diagonals are in the ratio $a : b$, what is the ratio of the areas of the circles?

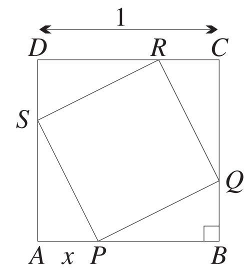
(c) Under what circumstances do the circles coincide?

**10.** In the diagram to the right, $ABCD$ and $PQRS$ are squares, and $AB = 1$ metre. Let $AP = x$.

(a) Find an expression for the area of $PQRS$ in terms of $x$.

(b) What is the minimum area of $PQRS$, and what value of $x$ gives this minimum?

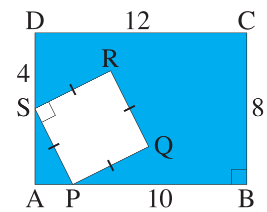
(c) Explain why the result is the same if the total area of the four triangles is maximised.




**11.** The diagram shows a rectangle with a square offset in one corner. All dimensions shown are in metres.

(a) Find the area of the square.

(b) Hence find the shaded area outside the square.




**12.** (a) The diagram shows a regular hexagon inscribed in a circle of radius 1 and centre $O$.

(i) Find the area of $\triangle AOB$.

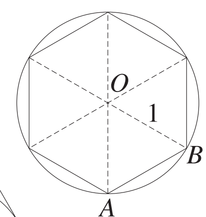
(ii) Hence find the area of the hexagon.

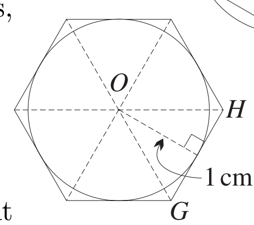
(b) The second diagram on the right shows another regular hexagon escribed around a circle of radius 1, that is, each side is tangent to the circle.

(i) Find the area of $\triangle OGH$.

(ii) Find the area of the hexagon.

(c) Hence explain why $\frac{3}{2}\sqrt{3} < \pi < 2\sqrt{3}$.



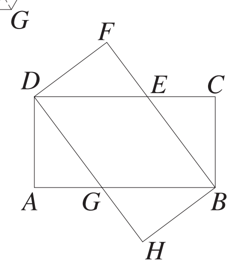




**13.** In the diagram opposite, $ABCD$ and $BFDH$ are congruent rectangles with $AB = 8$ and $BC = 6$.

(a) Explain why $\triangle ADG \equiv \triangle HBG$.

(b) Show that $AG = \dfrac{AB^2 - AD^2}{2AB}$ by using Pythagoras' theorem, and hence find $AG$.

(c) Hence find the area of $BEDG$.




**14.** A parallelogram has sides of length $a$ and $b$, and one vertex angle has size $\theta$.

(a) Show that the area of the parallelogram is $A = ab\sin\theta$.

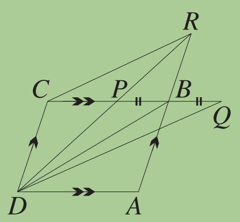
(b) Use the cosine rule to find the squares on the diagonals in terms of $a$, $b$ and $\theta$.

(c) Circles are drawn with the two diagonals as diameters. What is the area of the annulus between the two circles?

ISBN: 978-1-107-61604-2  Cambridge University Press

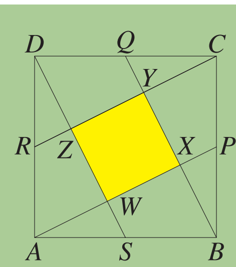
EXTENSION

**15.** (a)



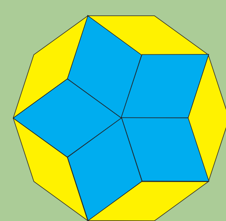

The diagram above shows a parallelogram $ABCD$. The point $P$ lies on the side $BC$, and the side $CB$ is produced to $Q$ so that $BQ = BP$. The intervals $AB$ and $DP$ are produced so that they intersect at $R$. Show that the areas of $\triangle DQB$ and $\triangle CPR$ are equal.

(b)




The diagram above shows a square $ABCD$ with the midpoints of each side being $P$, $Q$, $R$ and $S$ as shown. The intervals $AP$, $BQ$, $CR$ and $DS$ intersect at $W$, $X$, $Y$ and $Z$ as shown. Find the ratio of the areas of the small square $WXYZ$ and the large square $ABCD$.

**16.** The diagram shows the tesselation of a decagon by two types of rhombus, one fat and the other thin. The lengths of the sides of each rhombus and the decagon are all 1 cm.

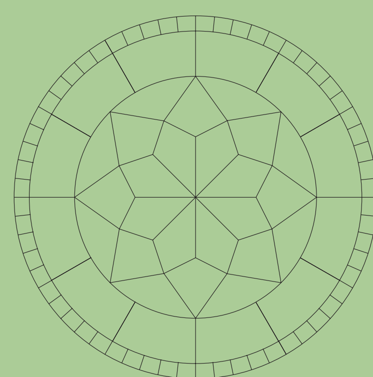


(a) Find both angles in each rhombus, and confirm that the interior angle at each vertex of the decagon is correct.

(b) Hence show that the area of the decagon is

$$A = 5\sin 36^\circ\,(2\cos 36^\circ + 1)\ \text{cm}^2.$$

**17.** A DIFFICULT EQUAL-AREA PROBLEM: The diagram shows the design of the clock-face on the stone towers at Martin Place and Central Railway Station in Sydney. The design within the inner circle seems to be based on dividing it into 24 regions of equal area. Let the inner circle have radius 1.



(a) Show that the radius $OA$ of the circle through the eight points inside the circle where three edges meet is $\frac{1}{2}$.

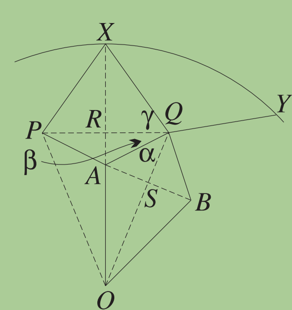
(b) Use the area of the kite $APXQ$ to show that $RQ = \frac{\pi}{12}$.

(c) Use the kite $OAQB$ to find the radius $OQ$ of the circle through the eight points where four edges meet.

(d) Show that $\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$, and hence find $OR$.

(e) Find the lengths $AR$ and $RX$, and hence show that

$$\tan\beta = \frac{\pi(\sqrt{2}+1) - 6}{\pi} \quad\text{and}\quad \tan\gamma = \frac{12 - \pi(\sqrt{2}+1)}{\pi}.$$




(f) Use $AB$ and the area of $\triangle ABQ$ to show that

$$\tan\alpha = \frac{6 - 3\sqrt{2}}{2\pi - 3\sqrt{2}}.$$

(g) Hence find the angle between opposite edges at the eight points where four edges meet, correct to the nearest minute.

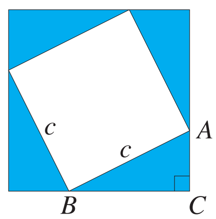
ISBN: 978-1-107-61604-2  Cambridge University Press

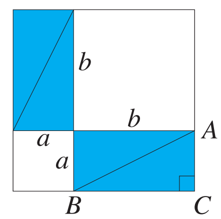
# 8 G Pythagoras' Theorem and its Converse

Pythagoras' theorem hardly needs introduction, having been the basis of so much of the course. But its proof needs attention, and the converse theorem and its interesting proof by congruence will be new for many students.

**Pythagoras' Theorem:** The following proof by dissection of Pythagoras' theorem is very quick, and is one of hundreds of known proofs.

**26** PYTHAGORAS' THEOREM: In a right triangle, the square on the hypotenuse equals the sum of the squares on the other two sides.




GIVEN: Let $\triangle ABC$ be a right triangle with $\angle C = 90°$.

AIM: To prove that $AC^2 + BC^2 = AB^2$.

CONSTRUCTION: As shown.

PROOF: Behold! (To quote an Indian text — is anything further required?)

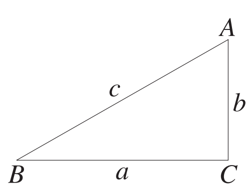
**Pythagorean Triads:** A *Pythagorean triad* consists of three positive integers $a$, $b$ and $c$ such that $a^2 + b^2 = c^2$. For example,

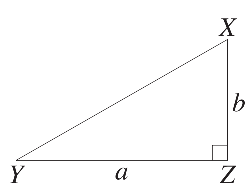
$$3^2 + 4^2 = 5^2 \qquad \text{and} \qquad 5^2 + 12^2 = 13^2,$$

so 3, 4, 5 and 5, 12, 13 are Pythagorean triads. Such triads are very convenient, because they can be the side lengths of a right triangle. An extension question below gives a complete list of Pythagorean triads.

**Converse of Pythagoras' Theorem:** The converse of Pythagoras' theorem is also true, and its proof is an application of congruence. The proof of the converse uses the forward theorem, and is consequently rather subtle.

**27** CONVERSE OF PYTHAGORAS' THEOREM: If the sum of the squares on two sides of a triangle equals the square on the third side, then the angle included by the two sides is a right angle.




GIVEN: Let $ABC$ be a triangle whose sides satisfy the relation $a^2 + b^2 = c^2$.

AIM: To prove that $\angle C = 90°$.

CONSTRUCTION: Construct $\triangle XYZ$ in which $\angle Z = 90°$, $YZ = a$ and $XZ = b$.

ISBN: 978-1-107-61604-2

Cambridge University Press

PROOF: Using Pythagoras' theorem in $\triangle XYZ$,

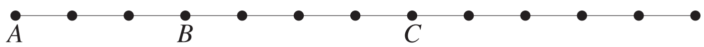
$$XY^2 = a^2 + b^2 \quad \text{(because } XY \text{ is the hypotenuse)}$$
$$= c^2 \quad \text{(given)},$$

so $\quad XY = c$.

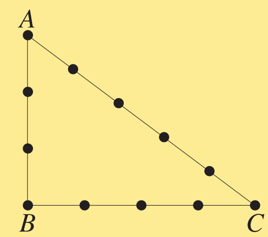
Hence the triangles $ABC$ and $XYZ$ are congruent by the SSS test,

and so $\quad \angle C = \angle Z = 90^\circ \quad$ (matching angles of congruent triangles).

**WORKED EXERCISE:** A long rope is divided into twelve equal sections by knots along its length. Explain how it can be used to construct a right angle.




**SOLUTION:** Let $A$ be one end of the rope. Let $B$ be the point 3 units along, and let $C$ be the point a further 4 units along. Join the two ends of the rope, and stretch the rope into a triangle with vertices $A$, $B$ and $C$. Then since 3, 4, 5 is a Pythagorean triad, the triangle will be right-angled at $B$.




## Exercise 8G

**1.** Which of the following triplets are the sides of a right-angled triangle?

(a) 30, 24, 18 (b) 28, 24, 15 (c) 26, 24, 10 (d) 25, 24, 7 (e) 24, 20, 13

**2.** Find the unknown side of each of the following right-angled triangles with base $b$, altitude $a$ and hypotenuse $c$. Leave your answer in surd form where necessary.

(a) $a = 12$, $b = 5$ (b) $a = 4$, $b = 5$ (c) $b = 15$, $c = 20$ (d) $a = 3$, $c = 7$

**3.** PYTHAGORAS' THEOREM AND THE COSINE RULE: Let $ABC$ be a triangle right-angled at $C$. Write down, with $c^2$ as subject, the cosine rule and Pythagoras' theorem, and explain why they are identical.

**4.** A paddock on level ground is 2 km long and 1 km wide. Answer these questions, correct to the nearest second.

(a) If a farmer walks from one corner to the opposite corner along the fences in 40 minutes, how long will it take him if he walks across the diagonal?

(b) If his assistant jogs along the diagonal in 15 minutes, how long will it take him if he jogs along the fences?

**5.** (a) Use Pythagoras' theorem to find an equation for the altitude $a$ of an isosceles triangle with base $2b$ and equal legs $s$. Hence find the area of an isosceles triangle with:

(i) equal legs 15 cm and base 24 cm, (ii) equal legs 18 cm and base 20 cm.

(b) Write down the altitude in the special case where $s = 2b$. What type of triangle is this and what is its area?

**6.** (a) The diagonals of a rhombus are 16 cm and 30 cm. (i) What are the lengths of the sides? (ii) Use trigonometry to find the vertex angles, correct to the nearest minute.

(b) A rhombus with 20 cm sides has a 12 cm diagonal. How long is the other diagonal?

(c) One diagonal of a rhombus is 20 cm, and its area is 100 cm$^2$.

(i) How long is the other diagonal? (ii) How long are its sides?

**7.** The sides of a rhombus are 5 cm, and its area is 24 cm$^2$.

(a) Let the diagonals have lengths $2x$ and $2y$, and show that $xy = 12$ and $x^2 + y^2 = 25$.

(b) Solve for $x$ and hence find the lengths of the diagonals.

ISBN: 978-1-107-61604-2

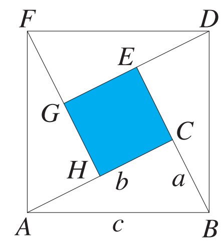
Cambridge University Press

**8.** (a) THE $t$-FORMULAE: Two sides of a right-angled triangle are $2t$ and $t^2 - 1$.
(i) Show that the hypotenuse is $t^2 + 1$. (ii) What are the two possible lengths of the hypotenuse if another side of the triangle is 8 cm?

(b) Show that if $a$ and $b$ are integers with $b < a$, then $a^2 - b^2$, $2ab$, $a^2 + b^2$ is a Pythagorean triad. Then generate and check the Pythagorean triads given by:
(i) $a = 2$, $b = 1$ (ii) $a = 3$, $b = 2$ (iii) $a = 4$, $b = 3$ (iv) $a = 7$, $b = 4$

**DEVELOPMENT**

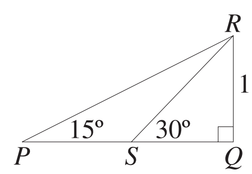
**9.** COURSE THEOREM: *An alternative proof of Pythagoras' theorem.* The triangle $ABC$ is right-angled at $C$. Let the sides be $AB = c$, $BC = a$ and $CA = b$, with $b > a$. The triangles $BDE$, $DFG$ and $FAH$ are congruent to $\triangle ABC$.




(a) Explain why $HC = b - a$.
(b) Find, in terms of the sides $a$, $b$ and $c$, the areas of:
(i) the square $ABDF$, (ii) the square $CEGH$,
(iii) the four triangles.
(c) Hence show that $a^2 + b^2 = c^2$.

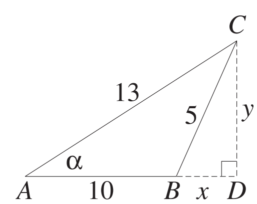
**10.** Let $AD$ be an altitude of $\triangle ABC$, and suppose that $BD = p^2$, $CD = q^2$ and $AD = pq$.
(a) Find $AB^2$ and $AC^2$. (b) Hence show that $\angle A = 90°$.

**11.** In the diagram, $\triangle PQR$ and $\triangle QRS$ are both right-angled at $Q$, with $\angle RPQ = 15°$ and $\angle RSQ = 30°$.



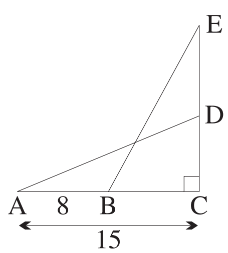

(a) Find $\angle PRS$ and hence show that $PS = RS$.
(b) Given that $QR = 1$ unit, write down the lengths of $QS$ and $RS$ and deduce that $\tan 15° = 2 - \sqrt{3}$.

**12.** (a) In triangle $ABC$, $AB = 10$, $BC = 5$ and $AC = 13$. The altitude is $CD = y$. Let $BD = x$ and $\angle A = \alpha$.



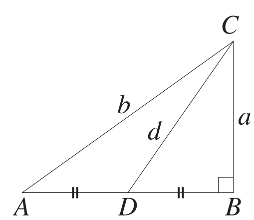

(b) Use Pythagoras' theorem to write down a pair of equations for $x$ and $y$.
(c) Solve for $x$, and hence find $\cos \alpha$ without finding $\alpha$.

**13.** (a)



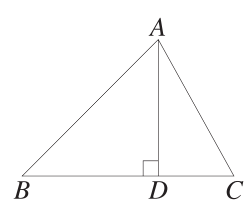

In the diagram, $AD = BE = 25$. $AB = 8$ and $AC = 15$. Find the length of $DE$.

(b)




In the right-angled $\triangle ABC$, the point $D$ bisects the base. Show that $4d^2 = b^2 + 3a^2$.

**14.** The altitude through $A$ in $\triangle ABC$ meets the opposite side $BC$ at $D$. Use Pythagoras' theorem in $\triangle ADB$ and $\triangle ADC$ to show that $AB^2 + DC^2 = AC^2 + BD^2$.



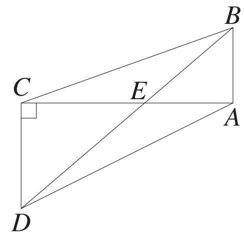

**15.** Triangle $ABC$ is right-angled at $A$. Show that:
(a) $(b + c)^2 - a^2 = a^2 - (b - c)^2$
(b) $(a + b + c)(-a + b + c) = (a - b + c)(a + b - c)$

ISBN: 978-1-107-61604-2  Cambridge University Press

**16.** The quadrilateral $ABCD$ is a parallelogram with diagonal $AC$ perpendicular to $CD$. The two diagonals intersect at $E$. Use Pythagoras' theorem to show that

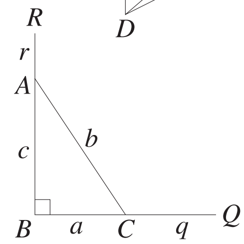
$$DE^2 + 3EA^2 = AD^2\,.$$

[HINT: Begin by letting $CA = 2d$, $CD = a$ and $DA = c$.]




**17.** The triangle $ABC$ has a right angle at $B$ and the sides opposite the respective vertices are $a$, $b$ and $c$. The side $BC$ is produced a distance $q$ to $Q$ while $BA$ is produced a distance $r$ to $R$. Show that

$$QA^2 + RC^2 = QR^2 + AC^2\,.$$




**18.** VARIANTS OF PYTHAGORAS' THEOREM:

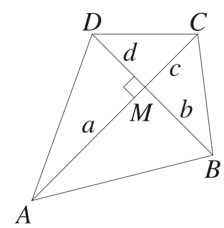
(a) Use Pythagoras' theorem to prove that the semicircle on the hypotenuse of a right-angled triangle equals the sum of the semicircles on the other two sides.

(b) Prove that the equilateral triangle on the hypotenuse of a right-angled triangle equals the sum of the equilateral triangles on the other two sides.

**19.** THEOREM: *If the diagonals of a quadrilateral are perpendicular, then the sums of squares on opposite sides are equal.* Let $ABCD$ be a quadrilateral, with diagonals meeting at right angles at $M$.

(a) Find expressions for $AB^2$, $BC^2$, $CD^2$ and $AD^2$ in terms of $a$, $b$, $c$ and $d$.

(b) Hence show that $AB^2 + CD^2 = BC^2 + AD^2$.

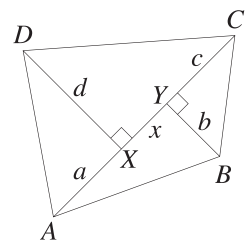



**20.** THEOREM: *Conversely, if the sums of squares of opposite sides of a quadrilateral are equal, then the diagonals are perpendicular.* Let $ABCD$ be a quadrilateral in which $AB^2 + CD^2 = AD^2 + BC^2$. Let $X$ and $Y$ be the feet of the perpendiculars from $B$ and $D$ respectively to $AC$. Let $AX = a$, $BY = b$, $CY = c$, $DX = d$ and $XY = x$.

(a) Use Pythagoras' theorem to show that

$$a^2 + b^2 + c^2 + d^2 = (a + x)^2 + b^2 + (c + x)^2 + d^2.$$

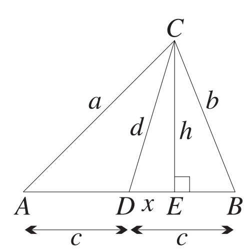
(b) Hence show that $x = 0$ and $AC \perp BD$.




**21.** APOLLONIUS' THEOREM: *The sum of the squares on two sides of a triangle is equal to twice the sum of the square on half the third side and the square on the median to the third side.* The diagram shows $\triangle ABC$ with $AC = a$, $BC = b$ and $AB = 2c$. The median $CD$ has length $d$. Let the altitude $CE$ have length $h$, and let $DE = x$.

(a) Use Pythagoras' theorem to write down three equations.

(b) Eliminate $h$ and $x$ from these equations, and hence show that $a^2 + b^2 = 2(c^2 + d^2)$, as required.



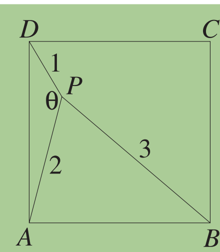

EXTENSION

**22.** (a) Show that $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$.

(b) Hence show that the set of integers that are the sum of two squares is closed under multiplication. (That is, prove that if two integers are each the sum of two squares, then their product is also the sum of two squares.)

ISBN: 978-1-107-61604-2  Cambridge University Press

**23.** The diagram shows a square $ABCD$ with a point $P$ inside it which is 1 unit from $D$, 2 units from $A$ and 3 units from $B$. Let $\angle APD = \theta$.




(a) Show that $\theta = \frac{3\pi}{4}$.

(b) Show that if $P$ is outside the square, then $\theta = \frac{\pi}{4}$.

(c) Is the situation possible if $P$ is 3 units from $C$ instead of from $B$?

**24.** A QUESTION MORE EASILY DONE BY COORDINATE GEOMETRY:

(a) The points $P$ and $Q$ divide a given interval $AB$ internally and externally respectively in the ratio $1 : 2$. The point $X$ lies on the circle with diameter $PQ$. Prove that $AX : XB = 1 : 2$.
[HINT: Drop the perpendicular from $X$ to $AB$, and use Pythagoras' theorem.]

(b) Now suppose that $P$ and $Q$ divide the given interval $AB$ internally and externally respectively in the ratio $1 : \lambda$. Prove that $AX : XB = 1 : \lambda$.

(c) Repeat part (b) using coordinate geometry with the origin at $A$.

**25.** PYTHAGOREAN TRIADS: Suppose that $a^2 + b^2 = c^2$, where $a$, $b$ and $c$ are integers.

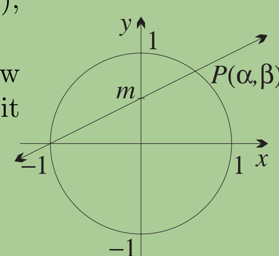
(a) Prove that one of $a$ and $b$ is even, and the other odd. [HINT: Find all possible remainders when the square of each number is divided by 4.]

(b) Prove that one of the three integers is divisible by 5. [HINT: Find all the possible remainders when the square of a number is divided by 5.]

**26.** A LIST OF ALL PYTHAGOREAN TRIADS: A Pythagorean triad $a$, $b$, $c$ is called *primitive* if there is no common factor of $a$, $b$ and $c$.

(a) Show that every Pythagorean triad is a multiple of a primitive Pythagorean triad.

(b) Show that if $a$, $b$, $c$ is a Pythagorean triad, then the point $P(\alpha, \beta)$, where $\alpha = a/c$ and $\beta = b/c$, lies on the unit circle $x^2 + y^2 = 1$.




(c) Let $m = p/q$ be any rational gradient between 0 and 1. Show that the line with gradient $m$ through $M(-1, 0)$ meets the unit circle $x^2 + y^2 = 1$ again at $P(\alpha, \beta)$, where

$$\alpha = \frac{1 - m^2}{1 + m^2} \quad \text{and} \quad \beta = \frac{2m}{1 + m^2} \quad \text{are both rational,}$$

and hence show that $q^2 - p^2$, $2pq$, $q^2 + p^2$ is a Pythagorean triad.

(d) Show that if the integers $p$ and $q$ in part (c) are relatively prime and not both odd, then $q^2 - p^2$, $2pq$, $q^2 + p^2$ is a primitive Pythagorean triad.

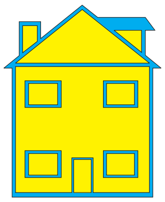
(e) Show that part (d) is a complete list of primitive Pythagorean triads.

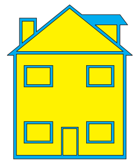
## 8 H Similarity

Similarity generalises the study of congruence to figures that have the same shape but not necessarily the same size. Its formal definition requires the idea of an *enlargement*, which is a stretching in all directions by the same factor.

SIMILARITY: Two figures $\mathcal{S}$ and $\mathcal{T}$ are called *similar*, written as $\mathcal{S} \mathbin{|||} \mathcal{T}$, if one figure can be moved to coincide with the other figure by means of a sequence
**28** of rotations, reflections, translations, and enlargements.
The enlargement ratio involved in these transformations is called the *similarity ratio* of the two figures.

ISBN: 978-1-107-61604-2  Cambridge University Press





Like congruence, similarity sets up a correspondence between the elements of the two figures. In this correspondence, angles are preserved, and the ratio of two matching lengths equals the similarity ratio. Since an area is the product of two lengths, the ratio of the areas of matching regions is the square of the similarity ratio. Likewise, if the idea is extended into three-dimensional space, then the ratio of the volumes of matching solids is the cube of the similarity ratio.

**29** SIMILARITY RATIO: If two similar figures have similarity ratio $1 : k$, then
- matching angles have the same size,
- matching intervals have lengths in the ratio $1 : k$,
- matching regions have areas in the ratio $1 : k^2$,
- matching solids have volumes in the ratio $1 : k^3$.

**Similar Triangles:** As with congruence, most of our arguments concern triangles, and the four standard tests for similarity of triangles will be assumptions. These four tests correspond exactly with the four standard congruence tests, except that equal sides are replaced by proportional sides (the AAS congruence test thus corresponds to the AA similarity test). An example of each test is given below.

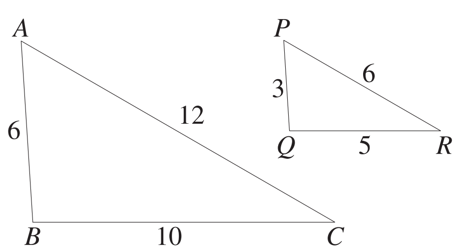
**30** STANDARD SIMILARITY TESTS FOR TRIANGLES: Two triangles are similar if:

SSS the three sides of one triangle are respectively proportional to the three sides of another triangle, or

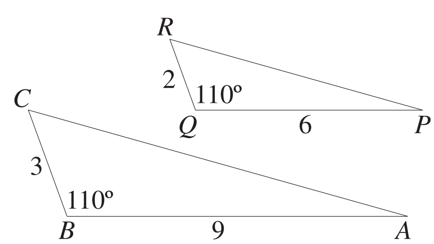
SAS two sides of one triangle are respectively proportional to two sides of another triangle, and the included angles are equal, or

AA two angles of one triangle are respectively equal to two angles of another triangle, or

RHS the hypotenuse and one side of a right triangle are respectively proportional to the hypotenuse and one side of another right triangle.

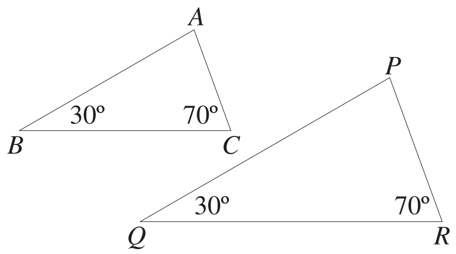



$\triangle ABC \parallel\!| \triangle PQR$ (SSS),
with similarity ratio $2 : 1$.
Hence $\angle P = \angle A$, $\angle Q = \angle B$
and $\angle R = \angle C$
(matching angles of similar triangles).

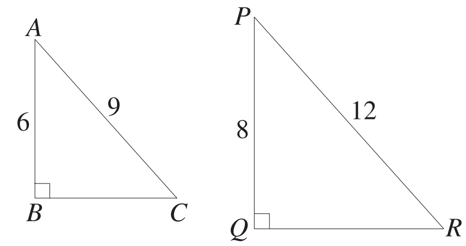



$\triangle ABC \parallel\!| \triangle PQR$ (SAS),
with similarity ratio $3 : 2$.
Hence $\angle P = \angle A$, $\angle R = \angle C$
and $PR = \frac{2}{3}AC$ (matching sides and angles of similar triangles).

ISBN: 978-1-107-61604-2  Cambridge University Press




$\triangle ABC \;|||\; \triangle PQR$ (AA).
Hence $\dfrac{PQ}{AB} = \dfrac{QR}{BC} = \dfrac{RP}{CA}$
(matching sides of similar triangles),
and $\angle P = \angle A$ (angle sums of triangles).




$\triangle ABC \;|||\; \triangle PQR$ (RHS),
with similarity ratio $3:4$.
Hence $\angle P = \angle A$, $\angle R = \angle C$
and $QR = \frac{4}{3}BC$ (matching sides and angles of similar triangles).

**Using the Similarity Tests:** Similarity tests should be set out in exactly the same way as congruence tests — the AA similarity test, however, will need only four lines. The similarity ratio should be mentioned if it is known. Keeping vertices in corresponding order is even more important with similarity, because the corresponding order is needed when writing down the proportionality of sides.

**WORKED EXERCISE:** A tower $TC$ casts a 300-metre shadow $CN$, and a man $RA$ 2 metres tall casts a 2·4-metre shadow $AY$. Show that $\triangle TCN \;|||\; \triangle RAY$, and find the height of the tower and the similarity ratio.

SOLUTION: In the triangles $TCN$ and $RAY$:
1. $\angle TCN = \angle RAY = 90^\circ$ (given),
2. $\angle CNT = \angle AYR =$ angle of elevation of the sun,

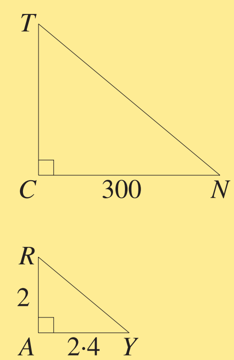
so $\triangle TCN \;|||\; \triangle RAY$ (AA).

Hence $\dfrac{TC}{CN} = \dfrac{RA}{AY}$ (matching sides of similar triangles)

$\dfrac{TC}{300} = \dfrac{2}{2{\cdot}4}$

$TC = 250$ metres.

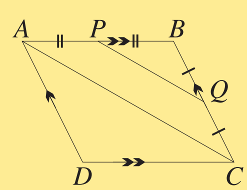
The similarity ratio is $300 : 2{\cdot}4 = 125 : 1$.




**WORKED EXERCISE:** Prove that the interval $PQ$ joining the midpoints of two adjacent sides $AB$ and $BC$ of a parallelogram $ABCD$ is parallel to the diagonal $AC$, and cuts off a triangle of area one eighth the area of the parallelogram.

PROOF: In the triangles $BPQ$ and $BAC$:
1. $\angle PBQ = \angle ABC$ (common),
2. $PB = \frac{1}{2}AB$ (given),
3. $QB = \frac{1}{2}CB$ (given),

so $\triangle BPQ \;|||\; \triangle BAC$ (SAS), with similarity ratio $1:2$.
Hence $\angle BPQ = \angle BAC$ (matching angles of similar triangles),
so $PQ \parallel AC$ (corresponding angles are equal).
Also, area $\triangle BPQ = \frac{1}{4} \times$ area $\triangle BAC$ (matching areas),
and area $\triangle ABC =$ area $\triangle CDA$ (congruent triangles),
so area $\triangle BPQ = \frac{1}{8} \times$ area of parallelogram $ABCD$.




ISBN: 978-1-107-61604-2

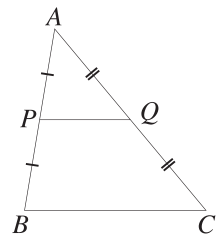
Cambridge University Press

**Midpoints of Sides of Triangles:** Similarity can be applied to configurations involving the midpoints of sides of triangles. The following theorem and its converse are standard results, and will be generalised in Section 8I.

**31** COURSE THEOREM: The interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

GIVEN: Let $P$ and $Q$ be the midpoints of the sides $AB$ and $AC$ of $\triangle ABC$.

AIM: To prove that $PQ \parallel BC$ and $PQ = \frac{1}{2}BC$.




PROOF: In the triangles $APQ$ and $ABC$:
1. $AP = \frac{1}{2}AB$ (given),
2. $AQ = \frac{1}{2}AC$ (given),
3. $\angle A = \angle A$ (common),

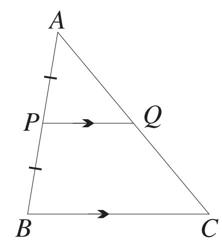
so $\triangle APQ \,|||\, \triangle ABC$ (SAS), with similarity ratio $1:2$.
Hence $\angle APQ = \angle ABC$ (matching angles of similar triangles),
so $PQ \parallel BC$ (corresponding angles are equal).
Also, $PQ = \frac{1}{2}BC$ (matching sides of similar triangles).

**The Converse Theorem:** Since there are two conclusions, there are several different theorems that could be regarded as the converse. The following theorem, however, is standard, and very useful.

**32** COURSE THEOREM: Conversely, the interval through the midpoint of one side of a triangle and parallel to another side bisects the third side.

GIVEN: Let $P$ be the midpoint of the side $AB$ of $\triangle ABC$.
Let the line parallel to $BC$ though $P$ meet $AC$ at $Q$.

AIM: To prove that $AQ = \frac{1}{2}AC$.




PROOF: In the triangles $APQ$ and $ABC$:
1. $\angle PAQ = \angle BAC$ (common),
2. $\angle APQ = \angle ABC$ (corresponding angles, $PQ \parallel BC$),

so $\triangle APQ \,|||\, \triangle ABC$ (AA), and the similarity ratio is $AP : AB = 1:2$.
Hence $AQ = \frac{1}{2}AC$ (matching sides of similar triangles).

**Equal Ratios of Intervals and Equal Products of Intervals:** The fact that the ratios of two pairs of intervals are equal can be just as well expressed by saying that the products of two pairs of intervals are equal:

$$\frac{AB}{BC} = \frac{XY}{YZ} \quad \text{is the same as} \quad AB \times YZ = BC \times XY.$$

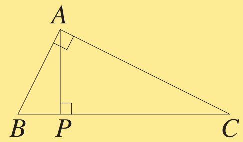
The following worked exercise is one of the best known examples of this.

**WORKED EXERCISE:** Prove that the square on the altitude to the hypotenuse of a right triangle equals the product of the intercepts on the hypotenuse cut off by the altitude.

GIVEN: Let $ABC$ be a triangle with $\angle A = 90°$.
Let $AP$ be the altitude to the hypotenuse.

ISBN: 978-1-107-61604-2  

AIM: To prove that $AP^2 = BP \times CP$.




PROOF: Let $\angle B = \beta$.

Then $\angle BAP = 90^\circ - \beta$ (angle sum of $\triangle BAP$),

so $\angle CAP = \beta$ (adjacent angles in the right angle $\angle BAC$).

In the triangles $PAB$ and $PCA$:

1. $\angle APB = \angle CPA = 90^\circ$ (given),
2. $\angle ABP = \angle CAP$ (proven above),

so $\triangle PAB \mathrel{|||} \triangle PCA$ (AA).

Hence $\dfrac{BP}{AP} = \dfrac{AP}{CP}$ (matching sides of similar triangles),

so $BP \times CP = AP^2$.

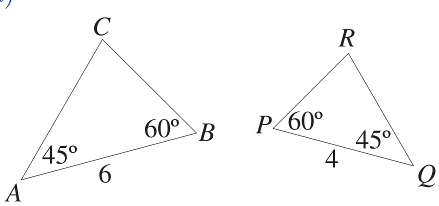
A NOTE ON THE GEOMETRIC MEAN: Recall from Chapter Six of the Year 11 volume that $g$ is a *geometric mean* of $a$ and $b$ if $g^2 = ab$, because then the sequence $a$, $g$, $b$ forms a GP with ratio $\dfrac{g}{a} = \dfrac{b}{g}$. Thus the previous result could be restated in the form of a theorem: *The altitude to the hypotenuse of a right-angled triangle is the geometric mean of the intercepts on the hypotenuse.*

## Exercise 8H

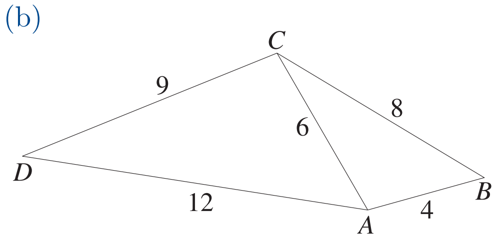
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

**1.** Both triangles in each pair are similar. Name the similar triangles in the correct order and state which test is used.

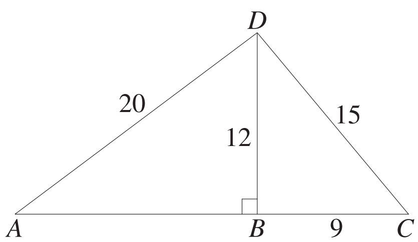
(a)



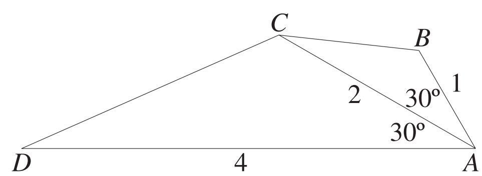

(b)




(c)

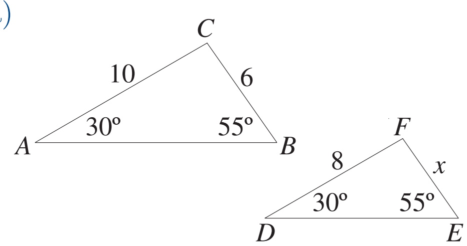



(d)

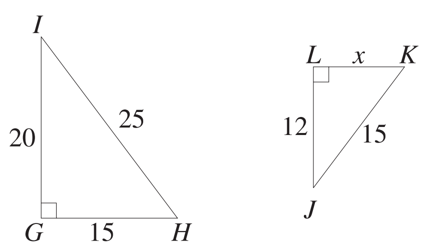



**2.** Identify the similar triangles, giving a reason, and hence deduce the length of the side $x$.

(a)

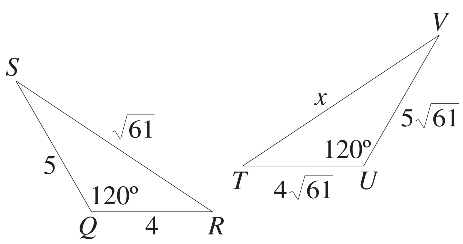



(b)

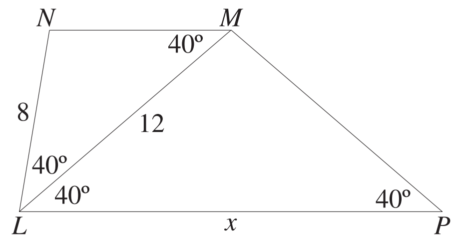



ISBN: 978-1-107-61604-2

(c)

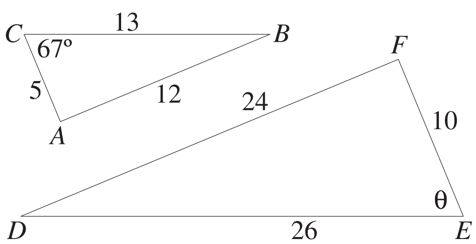



(d)

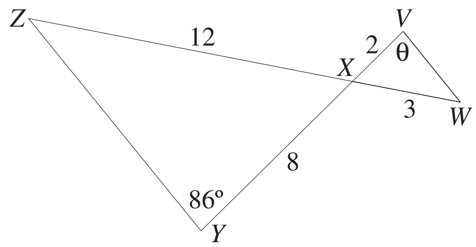



**3.** Identify the similar triangles, giving a reason, and hence deduce the size of the angle $\theta$. In part (b), prove that $VW \parallel ZY$.

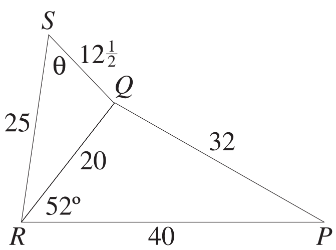
(a)



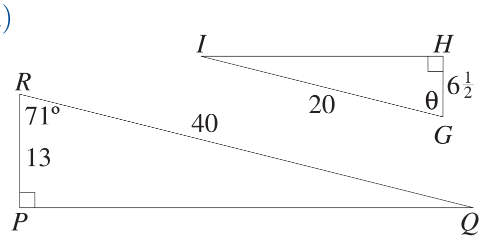

(b)




(c)

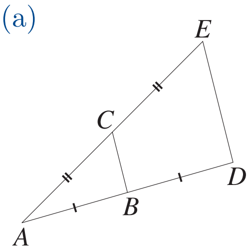



(d)

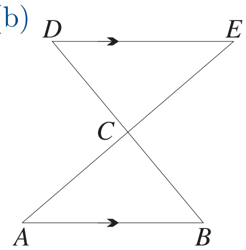



**4.** Prove that the triangles in each pair are similar.

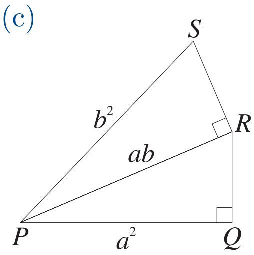
(a)



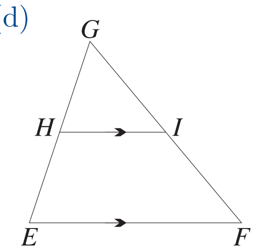

(b)




(c)




(d)

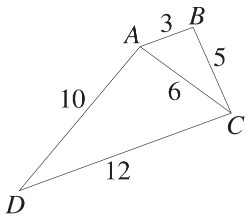



**5.** (a) A building casts a shadow 24 metres long, while a man 1·6 metres tall casts a 0·6-metre shadow. Draw a diagram, and use similarity to find the height of the building.

(b) An architect builds a model of a house to a scale of 1 : 200. The house will have a swimming pool 10 metres long, with surface area 60 $m^2$ and volume 120 $m^3$. What will the length and area of the model pool be, and how much water is needed to fill it?

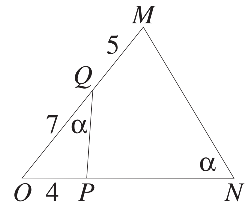
(c) Two coins of the same shape and material but different in size weigh 5 grams and 20 grams. If the larger coin has diameter 2 cm, what is the diameter of the smaller coin?

**6.** (a)




Show that $\triangle ADC \parallel\!| \triangle BCA$, and hence that $AB \parallel DC$.

(b)

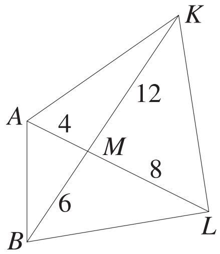



Show that $\triangle OPQ \parallel\!| \triangle OMN$, and hence find $ON$ and $PN$.

ISBN: 978-1-107-61604-2

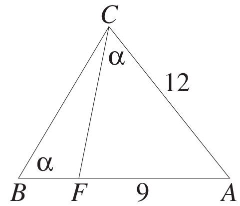
Cambridge University Press

(c)



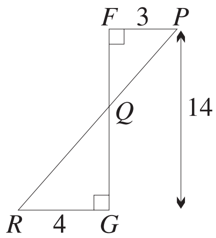

Show that $\triangle AMB \,|||\, \triangle LMK$.
What type of quadrilateral is $ABLK$?

(d)



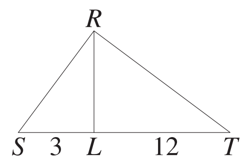

Show that $\triangle ABC \,|||\, \triangle ACF$,
and hence find $AB$ and $FB$.

(e)




Show that $\triangle FPQ \,|||\, \triangle GRQ$,
and hence find $FQ$, $GQ$, $PQ$ and $RQ$.

(f)




Given that $RL \perp ST$ and $SR \perp TR$,
show that $\triangle LSR \,|||\, \triangle LRT$,
and hence find $RL$.

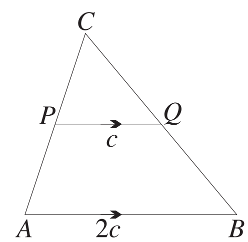
**DEVELOPMENT**

**7.** COURSE THEOREM: *If two triangles are similar in the ratio $1 : k$, then their areas are in the ratio $1 : k^2$.* Suppose that $\triangle ABC \,|||\, \triangle PQR$ (with vertices named in corresponding order) and let the ratio of corresponding sides be $1 : k$.

(a) Write down the area of $\triangle ABC$ in terms of $a$, $b$ and $\angle C$.

(b) Do the same for $\triangle PQR$, and hence show that the areas are in the ratio $1 : k^2$.

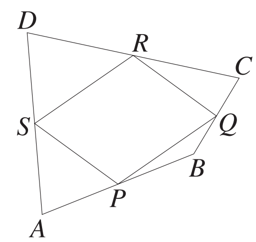
**8.** THEOREM: *The interval parallel to one side of a triangle and half its length bisects the other two sides.* In triangle $ABC$, suppose that $PQ$ is parallel to $AB$ and half its length.




(a) Prove that $\triangle ABC \,|||\, \triangle PQC$.

(b) Hence show that $CP = \frac{1}{2}CA$ and $CQ = \frac{1}{2}CB$.

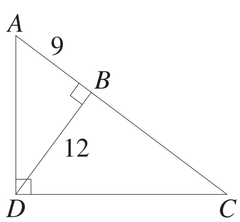
**9.** THEOREM: *The quadrilateral formed by the midpoints of the sides of a quadrilateral is a parallelogram.* Let $ABCD$ be a quadrilateral, and let $P$, $Q$, $R$ and $S$ be the midpoints of the sides $AB$, $BC$, $CD$ and $DA$ respectively.




(a) Prove that $\triangle PBQ \,|||\, \triangle ABC$, and hence that $PQ \parallel AC$.

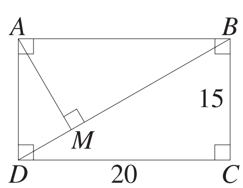
(b) Similarly, prove that $PQ \parallel SR$ and $PS \parallel QR$.

**10.** (a)




Show that $\triangle ABD \,|||\, \triangle ADC$,
and hence find $AD$, $DC$ and $BC$.

(b)



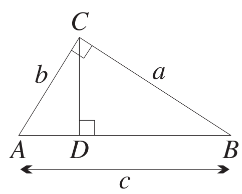

Use Pythagoras' theorem and similarity to find $AM$, $BM$ and $DM$.

ISBN: 978-1-107-61604-2  Cambridge University Press

**11.** THEOREM: Prove that the intervals joining the midpoints of the sides of any triangle dissect the triangle into four congruent triangles, each similar to the original triangle.

NOTE: Many of the hundreds of proofs of Pythagoras' theorem are based on similarity. The next three questions lead you through three of these proofs.

**12.** ALTERNATIVE PROOF OF PYTHAGORAS' THEOREM: In the triangle $ABC$, there is a right angle at $C$, and the sides opposite the respective vertices are $a$, $b$ and $c$. Let $CD$ be the altitude from $C$ to $AB$.

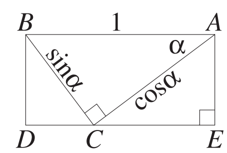



(a) Prove that $\triangle CBD \,|||\, \triangle ABC$, and hence find $BD$ in terms of the given sides.

(b) Similarly, prove that $\triangle ACD \,|||\, \triangle ABC$, and find $AD$.

(c) Hence prove that $a^2 + b^2 = c^2$.

**13.** ALTERNATIVE PROOF OF PYTHAGORAS' THEOREM: In the rectangle $ABDE$, $\triangle ABC$ is right-angled at $C$, and $AB = 1$. Let $\angle BAC = \alpha$, then $BC = \sin\alpha$ and $AC = \cos\alpha$.

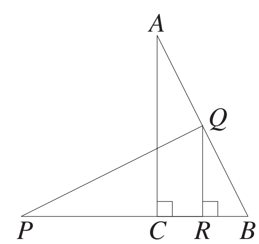



(a) Prove that $\triangle ABC \,|||\, \triangle BCD$, and hence find $DC$.

(b) Similarly, prove that $\triangle ABC \,|||\, \triangle CAE$, and find $EC$.

(c) Hence prove that $\sin^2\alpha + \cos^2\alpha = 1$.

**14.** ALTERNATIVE PROOF OF PYTHAGORAS' THEOREM: In the diagram, $\triangle ABC \equiv \triangle PQR$. The side $PR$ is on the same line as $BC$, and the vertex $Q$ is on $AB$. In $\triangle ABC$, there is a right angle at $C$. Let $AB = c$, $BC = a$ and $AB = c$.




(a) Prove that $\triangle PBQ \,|||\, \triangle ABC$. and hence find $PB$ in terms of the given sides.

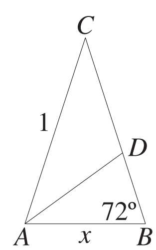
(b) Similarly, prove that $\triangle QBR \,|||\, \triangle ABC$, and hence show that $PB = b + \dfrac{a^2}{b}$.

(c) Hence prove that $a^2 + b^2 = c^2$.

**15.** Explain why the following pairs of figures are, or are not, similar:

(a) two squares, (b) two rectangles, (c) two rhombuses, (d) two equilateral triangles, (e) two isosceles triangles, (f) two circles, (g) two parabolas, (h) two regular hexagons.

**16.** In the diagram, $\triangle ABC$ is isosceles, with $\angle ABC = 72°$, $CB = CA = 1$, and $AB = x$. The bisector of $\angle CAB$ meets $BC$ at $D$.

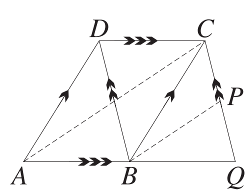



(a) Show that $\triangle ABC \,|||\, \triangle BDA$.

(b) Use part (a) to find the exact value of $x$.

(c) Explain why $\cos 72° = \dfrac{x}{2}$, and hence write down the exact value of $\cos 72°$.

**17.** In the figure, $ABCD$ is a parallelogram. The line $PC$, parallel to $BD$, meets $AB$ produced at $Q$.




(a) Prove that $AB = BQ$.

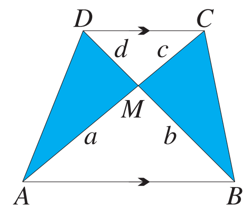
(b) The midpoint of $CQ$ is $P$. Prove that $\triangle PBQ \,|||\, \triangle CAQ$.

(c) Hence prove that $\angle PBQ = \angle CAB$.

ISBN: 978-1-107-61604-2  Cambridge University Press

**18.** THEOREM: *The triangles formed by the diagonals and the parallel sides of a trapezium are similar, and the other two triangles have equal areas.*

(a) In the trapezium $ABCD$, the diagonals intersect at $M$. Let $AM = a$, $BM = b$, $CM = c$ and $DM = d$, and let $\angle AMB = \theta$.




(i) Prove that the unshaded triangles are similar.

(ii) Hence prove that $ad = bc$.

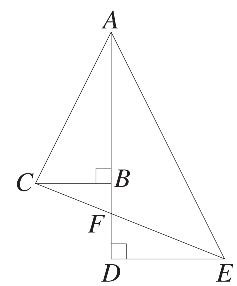
(iii) Prove that the shaded triangles have the same area.

(b) Now suppose that $a = 6$, $b = 4$, $c = 3$ and $d = 2$, with $AB = 8$ and $DC = 4$.

(i) Show that $\cos\theta = -\frac{1}{4}$ and $\sin\theta = \dfrac{\sqrt{15}}{4}$.

(ii) Hence find the area of the trapezium in exact form.

**19.** In the diagram, $\triangle ABC \,|||\, \triangle ADE$ and $\angle B$ is a right angle. The interval $CE$ intersects $BD$ at $F$. Let $AB = a$ and let the ratio of similarity be $AB : AD = k : \ell$.



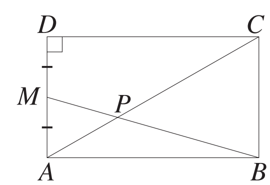

(a) Prove that $\triangle FBC \,|||\, \triangle FDE$.

(b) What is the ratio of the lengths $BF : FD$?

(c) Show that $FD = \dfrac{a\ell(\ell - k)}{k(\ell + k)}$.

(d) Now suppose that $AB : AD = 2 : 3$ and $FD$ is an integer. What are the possible values of $a$?

**20.** In the rectangle $ABCD$, $AB = 2 \times AD$, $M$ is the midpoint of $AD$, and $BM$ intersects $AC$ at $P$.

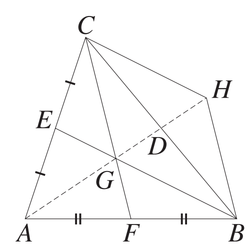



(a) Show that $\triangle APM \,|||\, \triangle CPB$.

(b) Show that $3 \times CP = 2 \times CA$.

(c) Show that $9 \times CP^2 = 5 \times AB^2$.

**21.** THEOREM: *The medians of a triangle are concurrent.*
In the triangle $ABC$, $E$ and $F$ are the midpoints of $AC$ and $AB$ respectively, and $BE$ and $CF$ intersect at $G$. The interval $AG$ is produced to $H$ so that $AG = GH$, and $AH$ intersects $BC$ at $D$.



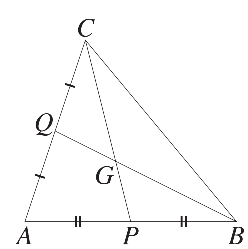

(a) Prove that $\triangle AFG \,|||\, \triangle ABH$.

(b) Hence show that $GC \parallel BH$.

(c) Similarly, prove that $GB \parallel CH$, and hence that $GBHC$ is a parallelogram.

(d) Hence prove that $BD = DC$.

**22.** THEOREM: *The medians of a triangle are concurrent, and the resulting* centroid *trisects each median.*
Concurrency of the medians was proven in the previous question, so it remains to prove that they trisect each other. Let $P$ and $Q$ be the midpoints of the sides $AB$ and $AC$ of $\triangle ABC$. Let the medians $PC$ and $QB$ meet at $G$.




(a) Prove that $\triangle ABC \,|||\, \triangle APQ$.

(b) Hence prove that $PQ = \frac{1}{2}BC$ and $PQ \parallel BC$.

(c) Prove that $\triangle PQG \,|||\, \triangle CBG$, with similarity ratio $1 : 2$.

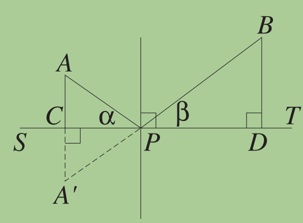
(d) Hence deduce the given theorem.

ISBN: 978-1-107-61604-2  Cambridge University Press

**23.** SEQUENCES AND GEOMETRY: Find the ratio of the sides in a right-angled triangle if:
(a) the sides are in AP, (b) the sides are in GP.

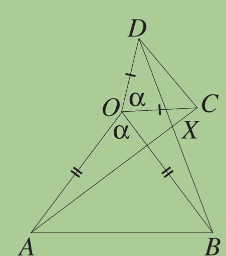
EXTENSION

**24.** FOR REFLECTED LIGHT, THE ANGLE OF INCIDENCE EQUALS THE ANGLE OF REFLECTION: Suppose that a light source is at $A$ above a reflective surface $ST$, and the reflected light is observed at $B$. Further suppose that at the point of reflection $P$, the angle of incidence is $(90° - \alpha)$ and the angle of reflection is $(90° - \beta)$. This means that $\angle APS = \alpha$ and $\angle BPT = \beta$. Let the image of $A$ in the reflecting surface be at $A'$ and let $A'A$ intersect $ST$ at $C$. We will assume that light travels in a straight line and therefore that $A'PB$ is a straight line.



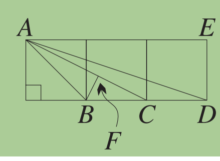

(a) Explain why $AC = A'C$ and $AP = A'P$.
(b) Prove that $\triangle APC \equiv \triangle A'PC$.
(c) Thus prove that $\triangle APC \ ||| \ \triangle BPD$.
(d) Hence prove that the angle of incidence is equal to the angle of reflection.

**25.** Triangles $ABO$ and $CDO$ are similar isosceles triangles with a common vertex $O$. In both triangles, $\angle O = \alpha$ and $\triangle ABO$ is the larger of the two triangles. $AC$ and $DB$ are joined and meet (produced if necessary) at $X$.




(a) Prove that $\triangle ODB \equiv \triangle OCA$.
(b) Show that $\angle BXA = \alpha$.
(c) Now suppose that $\triangle CDO$ is fixed and $\triangle ABO$ rotates about $O$. What is the locus of $X$?
(d) The kite $OAPB$ is completed so that $P$ is on the circumcircle of $\triangle ABO$. Show that $\angle PXB = \frac{1}{2}\alpha$.

**26.** Three equal squares are placed side by side as shown in the diagram, and $AB$, $AC$ and $AD$ are drawn. Prove that $\angle BAC = \angle DAE$. [HINT: Construct $BF \perp AC$ as shown.]




## 8 I Intercepts on Tranversals

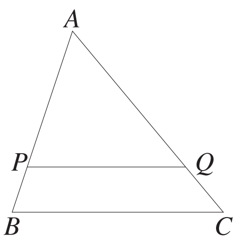
The previous theorem concerning the midpoints of the sides of a triangle can be generalised in two ways. First, the midpoint can be replaced by a point dividing the side in any given ratio. Secondly, the theorem can be applied to the intercepts cut off a transversal by three parallel lines. The word *intercept* needs clarification.

**33** INTERCEPTS: A point $P$ on an interval $AB$ divides the interval into two *intercepts* $AP$ and $PB$.

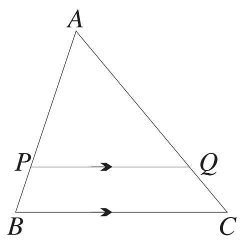

This section, unlike previous sections, will be entirely new for most students.

**Points on the Sides of Triangles:** The proofs of the following theorems are similar to the proofs of the previous two theorems, and are left to the exercise.

34 **COURSE THEOREM:** If two points $P$ and $Q$ divide two sides $AB$ and $AC$ respectively of a triangle in the same ratio $k : \ell$, then the interval $PQ$ is parallel to the third side $BC$, and $PQ : BC = k : k + \ell$.
Conversely, a line parallel to one side of a triangle divides the other two sides in the same ratio.




Given that $AP : PB = AQ : QC = k : \ell$,
it follows that $PQ \parallel BC$
and $PQ : BC = k : k + \ell$ (intercepts).

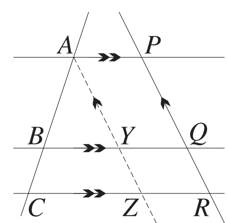



Given that $PQ \parallel BC$,
it follows that
$AP : PB = AQ : QC$ (intercepts).

**Transversals to Three Parallel Lines:** The previous theorems about points on the sides of a triangle can be applied to the intercepts cut off by three parallel lines.

35 **COURSE THEOREM:** If two transversals cross three parallel lines, then the ratio of the intercepts on one transversal is the same as the ratio of the intercepts on the other transversal.
In particular, if three parallel lines cut off equal intercepts on one transversal, then they cut off equal intercepts on all transversals.

The second part follows from the first part with $k : \ell = 1 : 1$, so it will be sufficient to prove only the first part.




GIVEN: Let two transversals $ABC$ and $PQR$ cross three parallel lines, and let $AB : BC = k : \ell$.

AIM: To prove that $PQ : QR = k : \ell$.

CONSTRUCTION: Construct the line through $A$ parallel to the line $PQR$, and let it meet the other two parallel lines at $Y$ and $Z$ respectively.

PROOF: The configuration in $\triangle ACZ$ is the converse part of the previous theorem, and so $AY : YZ = k : \ell$ (intercepts).
But the opposite sides of the parallelograms $APQY$ and $YQRZ$ are equal,
so $\quad AY = PQ$ and $YZ = QR$.
Hence $PQ : QR = k : \ell$.

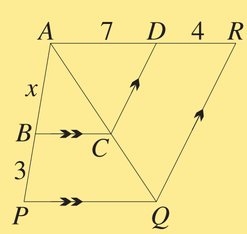
**WORKED EXERCISE:** Find $x$ in the diagram opposite.

**SOLUTION:** $\dfrac{x}{3} = \dfrac{AC}{CQ}$ (intercepts in $\triangle APQ$),

and $\dfrac{AC}{CQ} = \dfrac{7}{4}$ (intercepts in $\triangle AQR$).

Hence $\dfrac{x}{3} = \dfrac{7}{4}$

$x = 5\frac{1}{4}$.

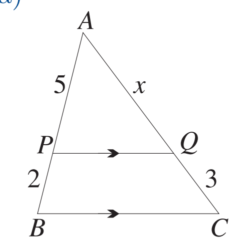


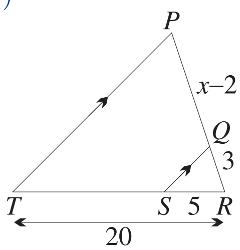

ISBN: 978-1-107-61604-2 

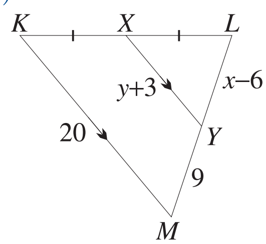
## Exercise 8I

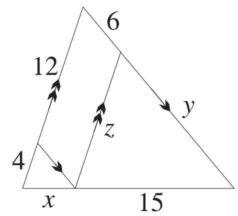
NOTE: In each question, all reasons must always be given. Unless otherwise indicated, lines that are drawn straight are intended to be straight.

**1.** Find the values of $x$, $y$ and $z$ in the following diagrams.

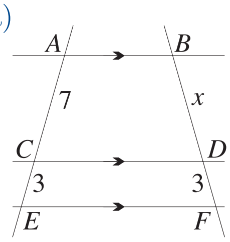
(a)


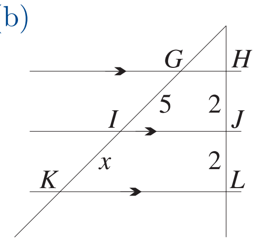

(b)


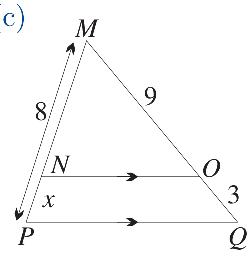

(c)


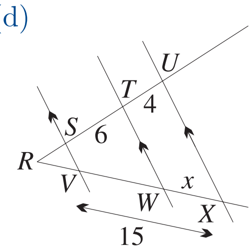

(d)


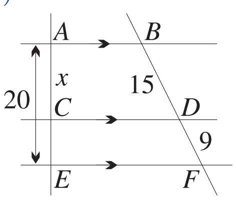

**2.** Find the value of $x$ in each diagram below.

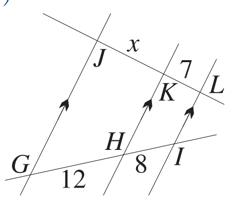
(a)


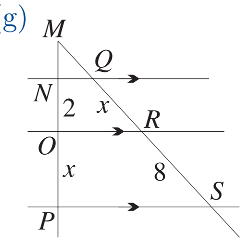

(b)


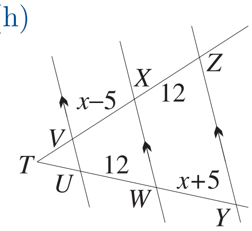

(c)



(d)


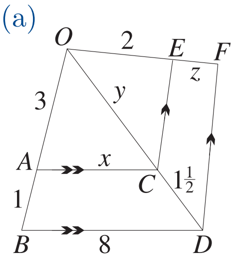

(e)


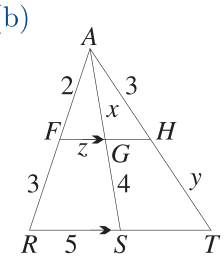

(f)


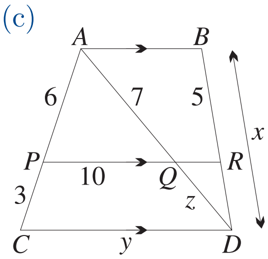

(g)


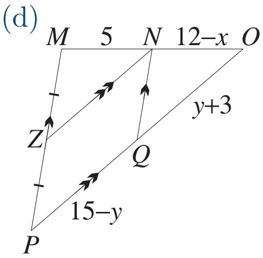

(h)



**3.** Find $x$, $y$ and $z$ in the diagrams below.

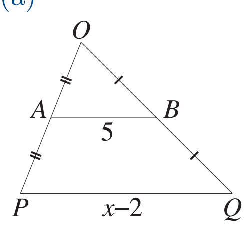
(a)


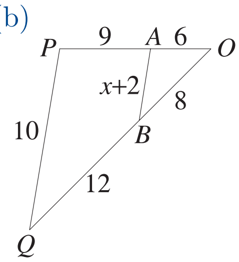

(b)


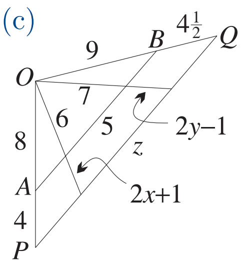

(c)


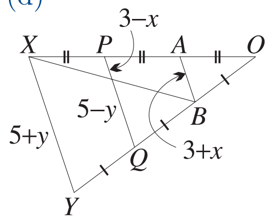

(d)



**4.** Give a reason why $AB \parallel PQ \parallel XY$ as appropriate, then find $x$ and $y$.

(a)



(b)


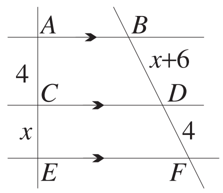

(c)



(d)


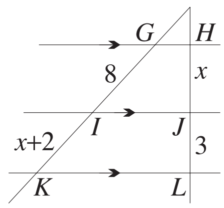

ISBN: 978-1-107-61604-2

**5.** Write down a quadratic equation for $x$ and hence find the value of $x$ in each case.

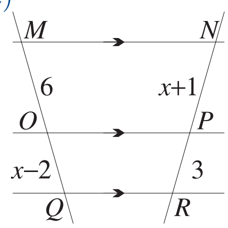
(a)



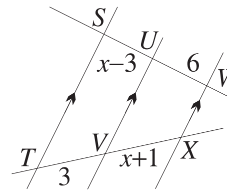

(b)



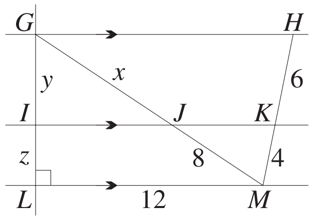

(c)




(d)

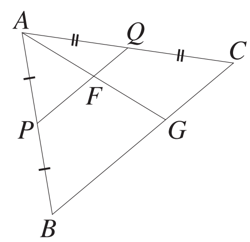



**6.** (a)



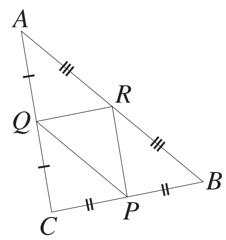

Use Pythagoras' theorem to find $x$, $y$ and $z$.

(b)



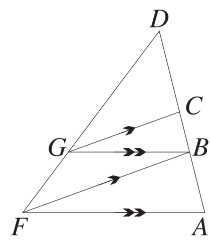

Find $AF : AG$.

(c)




What sort of quadrilateral is $ARPQ$? Find the ratio of areas of $ARPQ$ and $\triangle ABC$.

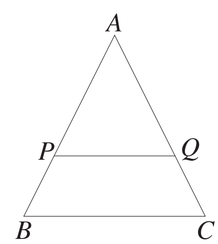
(d)




Show that $FG : GD = BC : CD$ and that $AF : BG = BF : CG$.

**DEVELOPMENT**

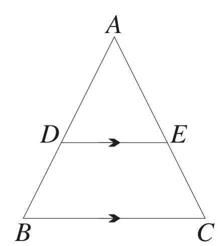
**7.** COURSE THEOREM: *If two points $P$ and $Q$ divide two sides $AB$ and $AC$ respectively of a triangle in the same ratio $k : \ell$, then the interval $PQ$ is parallel to the third side $BC$ and $PQ : BC = k : k+\ell$.* In $\triangle ABC$, the points $P$ and $Q$ divide the sides $AB$ and $AC$ respectively in the ratio $k : \ell$.




(a) Prove that $\triangle ABC \mathbin{|||} \triangle APQ$.

(b) Hence prove that $PQ \parallel BC$ and $PQ : BC = k : k+\ell$.

**8.** COURSE THEOREM: *Conversely, a line parallel to one side of a triangle divides the other two sides in the same ratio.* In $\triangle ABC$, the interval $DE$ is parallel to $BC$.

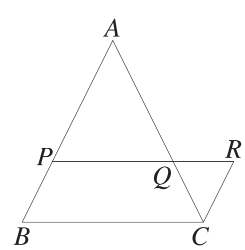



(a) Prove that $\triangle ABC \mathbin{|||} \triangle ADE$.

(b) Let $DE : BC = k : (k+\ell)$. Show that

$$AD : DB = AE : EC = k : \ell.$$

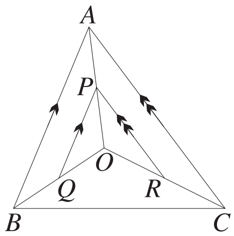
**9.** ALTERNATIVE PROOF OF COURSE THEOREM: *If two points $P$ and $Q$ divide two sides $AB$ and $AC$ respectively of a triangle in the same ratio $k : \ell$, then the interval $PQ$ is parallel to the third side $BC$ and $PQ : BC = k : (k+\ell)$.* In $\triangle ABC$, the points $P$ and $Q$ divide the sides $AB$ and $AC$ respectively in the ratio $k : \ell$. $PQ$ is produced to $R$ so that $PQ : QR = k : \ell$ and $CR$ is joined.




ISBN: 978-1-107-61604-2  Cambridge University Press

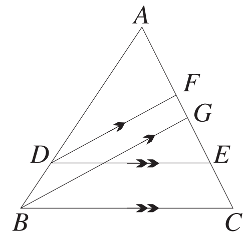
(a) Show that $\triangle APQ \,|||\, \triangle CRQ$.
(b) Hence show that $PBCR$ is a parallelogram.
(c) Hence show that $PQ \parallel BC$ and $PQ : BC = k : (k + \ell)$.

**10.** (a)



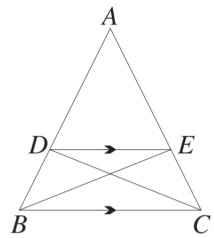

Choose any point $O$ inside $\triangle ABC$ and join $O$ to each vertex. Choose any point $P$ on $OA$ and then construct $PQ \parallel AB$ and $PR \parallel AC$. Prove that $QR \parallel BC$.

(b)

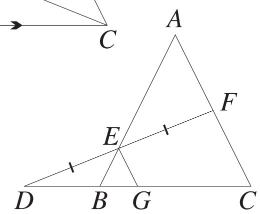



In $\triangle ABC$, the line $DE$ is parallel to the base $BC$. A point $G$ is chosen on $AC$ and then $DF$ is constructed parallel to $BG$. Prove that $AF : AG = DE : BC$.

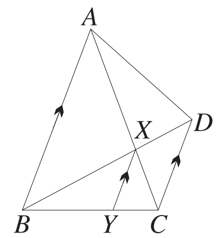
**11.** The triangle $ABC$ is isosceles, with $AB = AC$, and $DE$ is parallel to $BC$.
(a) Use the intercepts theorem to prove that $DB = EC$.
(b) Show that $\triangle BCD \equiv \triangle CBE$.



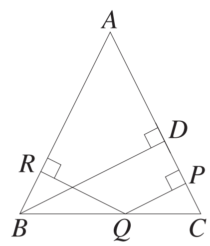

**12.** The triangle $ABC$ is isosceles, with $AB = AC$. The base $CB$ is produced to $D$. The points $E$ on $AB$ and $F$ on $AC$ are chosen so that $E$ is the midpoint of the straight line $DEF$. $G$ is the point on the base such that $CG = GD$.
(a) Prove that $EG \parallel AC$.
(b) Hence show that $FC = 2 \times EB$.




**13.** The diagram shows a trapezium $ABCD$ with $AB \parallel DC$. The diagonals $AC$ and $BD$ intersect at $X$, and $XY$ is constructed parallel to $AB$, intersecting $BC$ at $Y$.
(a) Prove that $AB : CD = BY : YC$.
(b) In a certain trapezium, the length of $AB$ is 18 cm. Given that $BY : BC = 3 : 4$, what is the length of the shorter side?



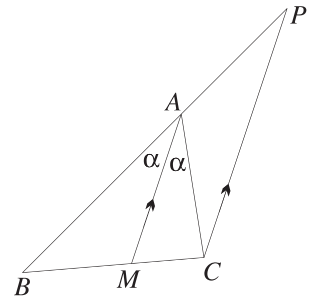

**14.** The triangle $ABC$ is isosceles with $AB = AC$, and $D$ is a point on $AC$ such that $BD \perp AC$. Choose any point $Q$ on the base, and construct the perpendiculars to the equal sides, with $QP \perp AC$ and $QR \perp AB$.
(a) Reflect the triangle $RBQ$ in the line $BQ$, and hence show that $RQ + PQ = BD$.
(b) Construct $CE$ perpendicular to $AB$ at $E$ and use the ratios of intercepts to prove the same result.




**15.** (a) Two vertical poles of height 10 metres and 15 metres are 8 metres apart. Wire stretches from the top of each pole to the foot of the other. Find how high above the ground the wires cross. How would this height change if the poles were 11 metres apart?
(b) In a narrow laneway 2·4 metres wide between two buildings, a 4-metre ladder rests on one wall with its foot against the other wall, and a 3-metre ladder rests on the opposite wall. The ladders touch at their crossover point. How high is that crossover point? [HINT: You will need the height each ladder reaches up the wall, then use similarity.]

ISBN: 978-1-107-61604-2  Cambridge University Press

**16.** THEOREM: *The bisector of the angle at a vertex of a triangle divides the opposite side in the ratio of the including sides.* Let the bisector of $\angle BAC$ in $\triangle ABC$ meet $BC$ at $M$, and let $\angle BAM = \angle CAM = \alpha$. Construct the line through $C$ parallel to $MA$, meeting $BA$ produced at $P$.



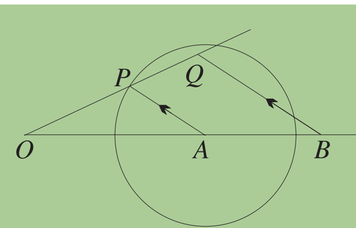

(a) Prove that $\triangle APC$ is isosceles with $AP = AC$.

(b) Hence show that $BM : MC = BA : AC$.

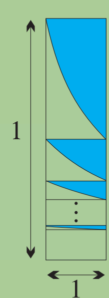
**17.** THEOREM: *Conversely, if the interval joining a vertex of a triangle to a point on the opposite side divides that side in the ratio of the including sides, then the interval bisects the vertex angle.* Prove this using a similar construction.

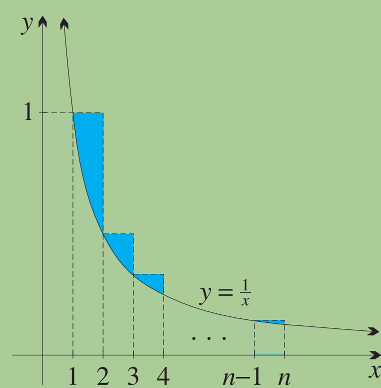
EXTENSION

**18.** In the diagram, $A$ is the centre of a circle with radius $R$. $O$ is a fixed point outside the circle and $B$ is another fixed point on $OA$. For a given point $P$ on the circle, the point $Q$ on the line $OP$ is chosen so that $AP \parallel BQ$. Describe the locus of $Q$ as $P$ moves around the circle. [HINT: Let $A$ divide $OB$ in the ratio $k : \ell$.]




**19.** [The harmonic series and Euler's constant]




The right-hand diagram above shows the curve $y = 1/x$ (not to scale). Upper rectangles have been constructed on the intervals $1 \le x \le 2$, $2 \le x \le 3$, ... and $n-1 \le x \le n$. Let $E_n$ be the total area of the shaded regions inside the rectangles and above the curve.

(a) By considering the difference between the area of the rectangles and the area under the curve, show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n = E_n + \frac{1}{n}$.

(b) The left-hand diagram above shows the shaded regions stacked on top of each other inside a unit square (be careful, the diagrams are not drawn to scale). By drawing appropriate diagonals, show that as $n \to \infty$, $E_n$ converges to a limit $\gamma$ between $\frac{1}{2}$ and 1.

(c) Hence show that $\lim\limits_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = \gamma$. Then use your calculator, or a computer, to get some idea of the value of $\gamma$ by substituting some values of $n$.
[NOTE: The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is called the *harmonic series* — the word comes from music, because if pipes are built of lengths 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... then the notes they sound will be the series of harmonics of the first pipe. The strange number $\gamma \doteqdot 0{\cdot}577$ is called *Euler's constant*. It remains unknown even whether $\gamma$ is rational or irrational.]

*μηδεὶς ἀγεωμέτρητος εἰσίτω* 'Let no-one enter who does not know geometry.' (Inscribed over the doorway to Plato's Academy in Athens.)

GO Online Multiple Choice Quiz

ISBN: 978-1-107-61604-2