CHAPTER **TEN**

The Geometry of the Derivative

Working out the shape of a curve from its equation is a fundamental concern of this course. Now that we have the derivative, the systematic approach to sketching unfamiliar curves, begun in Chapter Three, can be extended by two further questions:

1. Where is the curve sloping upwards, where is it sloping downwards, and where does it have any maximum or minimum values?
2. Where is the curve concave up, where is it concave down, and where does it change from one concavity to the other?

These will become standard procedures for investigating unfamiliar curves (in this text they will be Steps 5 and 6 of a curve sketching menu). In particular, the algorithm for finding maximum and minimum values of a function can be applied to all sorts of practical and theoretical questions.

Study Notes: Sections 10A–10F develop the standard procedures for dealing with the questions raised above about the shape of a curve. This is an important place where algebraic procedures should be freely supplemented by curve sketching software, so that a number of curves similar to those given here can be quickly drawn to demonstrate the effect of changing a constant or the form of an equation in various ways. Sections 10G–10I apply curve sketching methods to maximisation and minimisation problems, particularly in practical and geometric contexts. The final Section 10J begins to reverse the process of differentiation in preparation for the definite integral in Chapter Eleven.

10 A Increasing, Decreasing and Stationary at a Point

At a point where a curve is sloping upwards, the tangent has positive gradient, and y is increasing as x increases. At a point where it is sloping downwards, the tangent has negative gradient, and y is decreasing as x increases. That is, for a function $f(x)$ defined at $x = a$:

1 Increasing, decreasing and stationary at a point:

If $f'(a) > 0$, then $f(x)$ is called *increasing* at $x = a$.

If $f'(a) < 0$, then $f(x)$ is called *decreasing* at $x = a$.

If $f'(a) = 0$, then $f(x)$ is called *stationary* at $x = a$.

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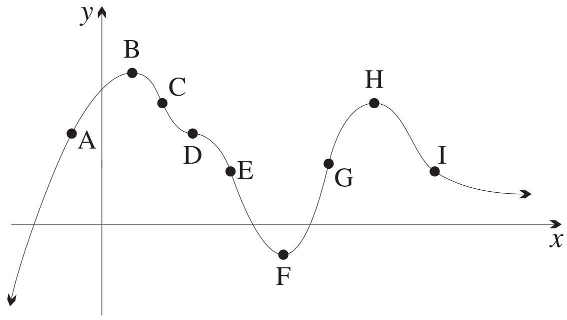
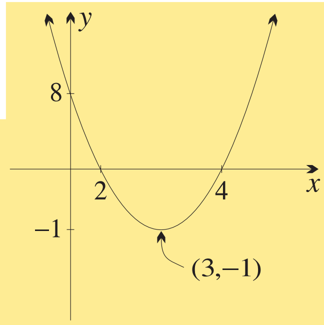
For example, the curve in the diagram to the right is:

- increasing at A and G,
- decreasing at C, E and I,
- stationary at B, D, F and H.



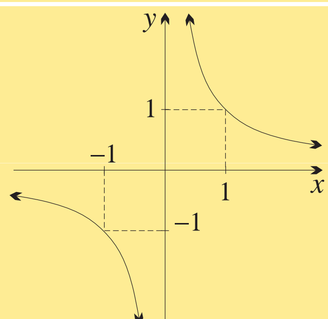
NOTE: These definitions of increasing, decreasing and stationary are *pointwise* definitions, because they concern the behaviour of the function at a point rather than over an interval. In later work on inverse functions and inverse trigonometric functions, we will be considering functions that are increasing or decreasing over an interval rather than at a point.

WORKED EXERCISE: Use the derivative to show that the graph of $f(x) = (x-2)(x-4)$ is stationary at the vertex $V(3,-1)$, decreasing to the left of V, and increasing to the right of V.



SOLUTION: Expanding, $f(x) = x^2 - 6x + 8$,
so $\quad f'(x) = 2x - 6$
$\qquad\quad = 2(x-3)$.
Since $f'(3) = 0$, the curve is stationary at $x = 3$.
Since $f'(x) > 0$ for $x > 3$, the curve is increasing for $x > 3$.
Since $f'(x) < 0$ for $x < 3$, the curve is decreasing for $x < 3$.

WORKED EXERCISE: Use the derivative to show that the graph of $y = \dfrac{1}{x}$ has no stationary points, and is decreasing for all values of x in its domain.

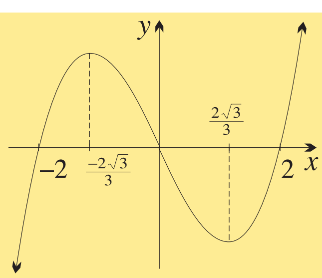


SOLUTION: Differentiating, $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$.
The domain is $x \neq 0$, so for all x in the domain, x^2 is positive, the derivative is negative, and the curve is decreasing.

NOTE: The value $y = \frac{1}{2}$ at $x = 2$ is greater than the value $y = -\frac{1}{2}$ at $x = -2$, despite the fact that the curve is decreasing for all x in the domain. This sort of thing can of course only happen because of the break in the curve at $x = 0$.

WORKED EXERCISE: Find where $y = x^3 - 4x$ is decreasing.

SOLUTION: $y' = 3x^2 - 4$,
so y' has zeroes at $x = \frac{2}{3}\sqrt{3}$ and at $x = -\frac{2}{3}\sqrt{3}$,
and is negative between the two zeroes.
So the curve is decreasing for $-\frac{2}{3}\sqrt{3} < x < \frac{2}{3}\sqrt{3}$.
[To sketch the curve, notice also that the function is odd, with zeroes at $x = 0$, $x = 2$ and $x = -2$.]



WORKED EXERCISE: Show that $f(x) = x^3 + x - 1$ is always increasing. Find $f(0)$ and $f(1)$, and explain why the curve has exactly one x-intercept. (A diagram is not actually needed in this exercise, although a sketch always helps.)

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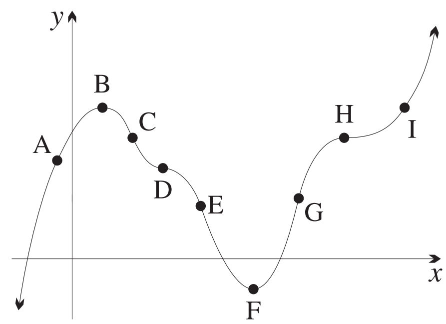
SOLUTION: Differentiating, $f'(x) = 3x^2 + 1$.
Since squares can never be negative, $f'(x)$ can never be less then 1,
so the function is increasing for every value of x.
Because $f(0) = -1$ is negative and $f(1) = 1$ is positive and $f(x)$ is continuous,
the curve must cross the x-axis somewhere between 0 and 1,
and because the function is increasing for every value of x,
it can never go back and cross the x-axis at a second point.

Exercise 10A

1. In the diagram to the right, name the points where:
(a) $f'(x) > 0$
(b) $f'(x) < 0$
(c) $f'(x) = 0$



2. (a) Show that $y = -5x + 2$ is decreasing for all x.
(b) Show that $y = x + 7$ is increasing for all x.
(c) Show that $y = x^3$ is increasing for all values of x, apart from $x = 0$ where it is stationary.
(d) Show that $y = 3x^2$ is increasing for $x > 0$ and decreasing for $x < 0$. What happens at $x = 0$?
(e) Show that the function $y = \sqrt{x}$ is increasing for all values of $x > 0$.
(f) Show that $y = \dfrac{1}{x^2}$ is increasing for $x < 0$ and decreasing for $x > 0$.

3. (a) Find $f'(x)$ for the function $f(x) = 4x - x^2$.
(b) For what values of x is: (i) $f'(x) > 0$, (ii) $f'(x) < 0$, (iii) $f'(x) = 0$?
(c) Find $f(2)$, then, by interpreting these results geometrically, sketch a graph of $f(x)$.

4. (a) Find $f'(x)$ for the function $f(x) = x^3 - 3x^2 + 5$.
(b) For what values of x is: (i) $f'(x) > 0$, (ii) $f'(x) < 0$, (iii) $f'(x) = 0$?
(c) Evaluate $f(0)$ and $f(2)$, then, by interpreting these results geometrically, sketch a graph of $y = f(x)$.

5. (a) Differentiate $f(x) = -\dfrac{3}{x}$, and hence prove that $f(x)$ increases for all x in its domain.
(b) Explain why $f(-1) > f(2)$ despite this fact.

6. Find the derivative of each of the following functions. By solving $dy/dx > 0$, find the values of x for which the function is increasing.
(a) $y = x^2 - 4x + 1$ (b) $y = 7 - 6x - x^2$ (c) $y = 2x^3 - 6x$ (d) $y = x^3 - 3x^2 + 7$

7. (a) Find the values of x for which $y = x^3 + 2x^2 + x + 7$ is an increasing function.
(b) Find the values of x for which $y = x^4 - 8x^2 + 7$ is a decreasing function.

DEVELOPMENT

8. The graphs of four functions (a), (b), (c) and (d) are shown below. The graphs of the derivatives of these functions, in scrambled order, are shown in I, II, III and IV. Match the graph of each function with the graph of its derivative.

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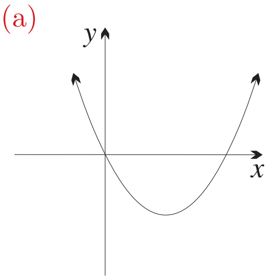
(a)


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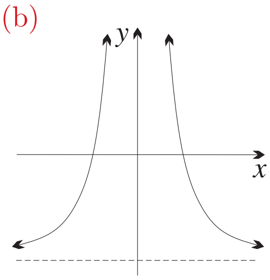

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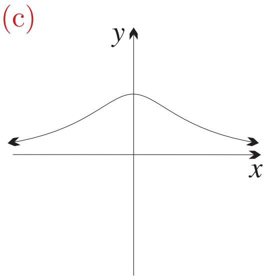

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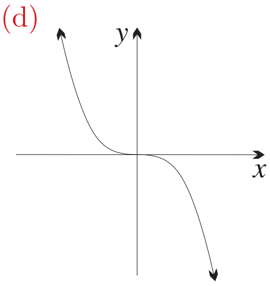
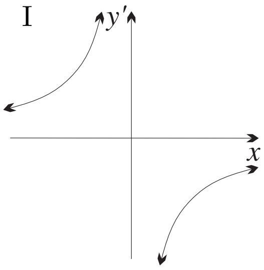


I


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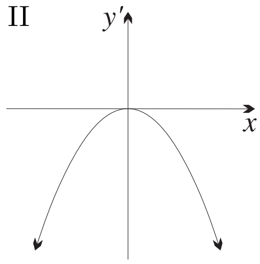

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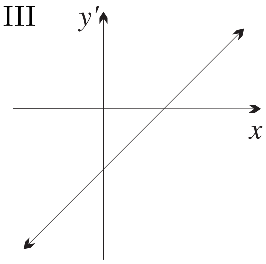
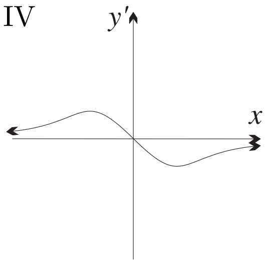
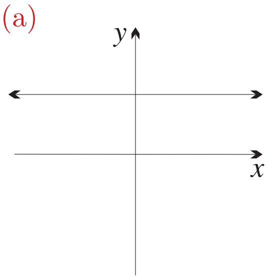


IV


9. Look carefully at each of the functions drawn below to establish where they are increasing, decreasing and stationary. Hence draw a graph of the derivative of each of the functions.

(a)


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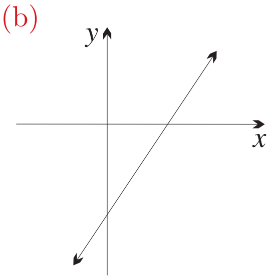

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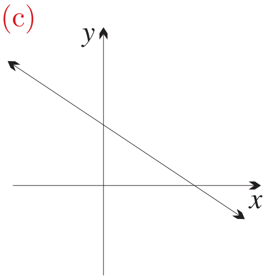

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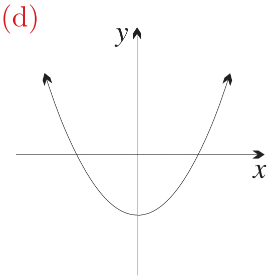

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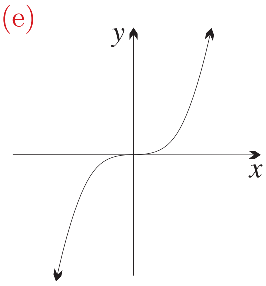

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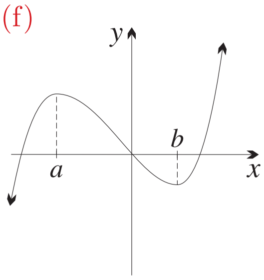

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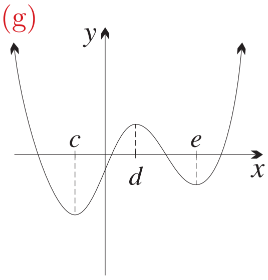

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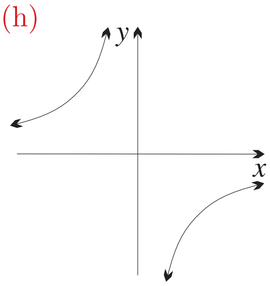

10. By finding $f'(x)$ show that:

(a) $f(x) = \dfrac{2x}{x-3}$ is decreasing for all $x \neq 3$,

(b) $f(x) = \dfrac{x^3}{x^2+1}$ is increasing for all x, apart from $x = 0$ where it is stationary.

11. (a) Find $f'(x)$ for the function $f(x) = \frac{1}{3}x^3 + x^2 + 5x + 7$.

(b) By completing the square, show that $f'(x)$ is always positive, and hence that $f(x)$ is increasing for all x.

12. (a) Prove that $f(x) = 2x^3 - 3x^2 + 5x + 1$ has no stationary points.

(b) Show that $f'(x) > 0$ for all values of x, and hence that the function is always increasing.

(c) Deduce that the equation $f(x) = 0$ has only one real root.

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13. (a) Prove that $y = -x^3 + 2x^2 - 5x + 7$ is decreasing for all values of x.
(b) Hence deduce the number of solutions of the equation $7 - 5x + 2x^2 - x^3 = 0$.

14. (a) Show that $f(x) = \dfrac{x^2+1}{x}$ is an odd function. (b) Find $f'(x)$.
(c) For what values of x is: (i) $f'(x) > 0$, (ii) $f'(x) < 0$, (iii) $f'(x) = 0$?
(d) Evaluate $f(1)$ and $f(-1)$. (e) State the equations of any vertical asymptotes.
(f) By interpreting these results geometrically, sketch a graph of the function.

15. Sketch graphs of continuous curves suggested by the properties below:

(a) $f(1) = f(-3) = 0$,
$f'(-1) = 0$,
$f'(x) > 0$ when $x < -1$,
$f'(x) < 0$ when $x > -1$.

(b) $f(2) = f'(2) = 0$,
$f'(x) > 0$ for all $x \neq 2$.

(c) $f(x)$ is odd,
$f(3) = 0$ and $f'(1) = 0$,
$f'(x) > 0$ for $x > 1$,
$f'(x) < 0$ for $0 \le x < 1$.

(d) $f(x) > 0$ for all x,
$f'(0) = 0$,
$f'(x) < 0$ for $x < 0$,
$f'(x) > 0$ for $x > 0$.

(e) $f(0) = 0$,
$f'(x) < 0$ for all $x < 0$,
$|f'(x_1)| < |f'(x_2)|$ for $x_1 < x_2 < 0$,
$f'(x) > 0$ for all $x > 0$,
$|f'(x_1)| < |f'(x_2)|$ for $x_1 > x_2 > 0$.

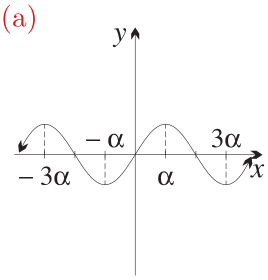
16. A function $f(x)$ has derivative $f'(x) = -x(x+2)(x-1)$.
(a) Draw a graph of $y = f'(x)$, and hence establish where $f(x)$ is increasing, decreasing and stationary.
(b) Draw a possible graph of $y = f(x)$, given that $f(0) = 2$.

17. (a) If $f(x) = \dfrac{x^2-4}{x^2-1}$, find $f'(x)$.
(b) Establish that $f'(x) < 0$ when $x < 0$ $(x \neq -1)$, and $f'(x) > 0$ when $x > 0$ $(x \neq 1)$.
(c) State the equations of any horizontal or vertical asymptotes.
(d) Hence sketch a graph of $y = f(x)$.

18. For what values of x is $y = \dfrac{x^2}{2x^2+x+1}$ decreasing?

19. (a) For $f(x) = \dfrac{1-x^2}{x^2+1}$: (i) find $f'(x)$, (ii) evaluate $f(0)$, (iii) show that $f(x)$ is even.
(b) Hence explain why $f(x) \le 1$ for all x.

20. Look carefully at each of the functions drawn below to establish where they are increasing, decreasing and stationary. Hence draw a graph of the derivative of each of the functions.

(a)


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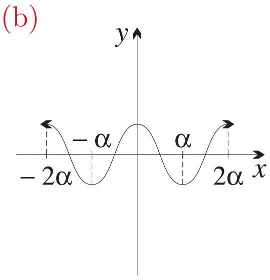

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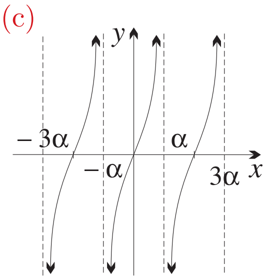

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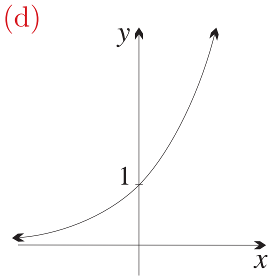

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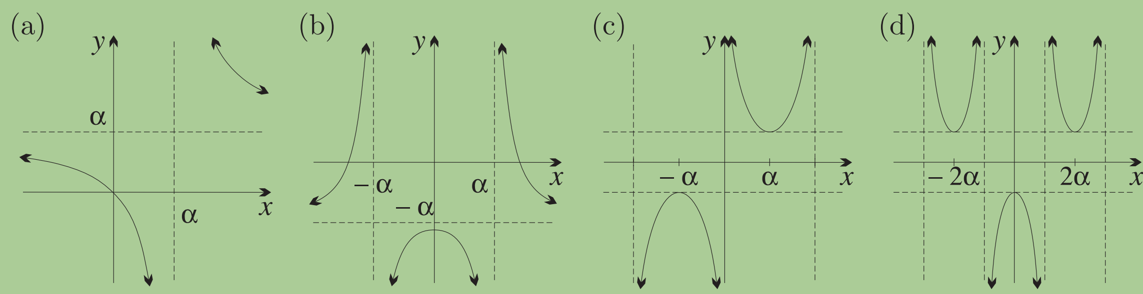
EXTENSION

21. Draw a graph of the derivative of each function graphed below.



22. [This question proves that a differentiable function that is zero at its endpoints must be horizontal somewhere in between.] Suppose that $f(x)$ is continuous in the interval $a \le x \le b$ and differentiable for $a \le x \le b$, and suppose that $f(a) = f(b) = 0$.

(a) Suppose first that $f(x) > 0$ for some x in $a < x < b$, and choose $x = c$ so that $f(c)$ is the maximum value of $f(x)$ in the interval $a < x < b$.

(i) Explain why $\dfrac{f(x) - f(c)}{x - c} \ge 0$ for $a \le x < c$, and $\dfrac{f(x) - f(c)}{x - c} \le 0$ for $c \le x < b$.

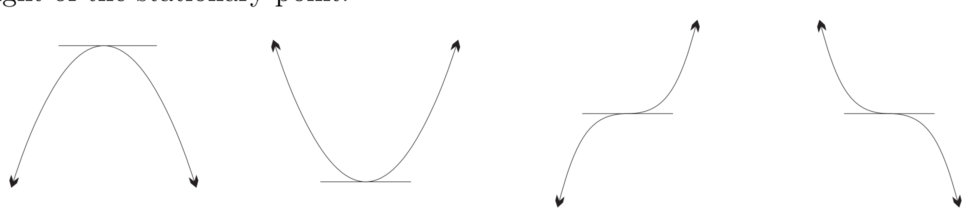
(ii) Hence explain why $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ must be zero.

(b) Complete the proof by considering the other two possible cases:

(i) $f(x) < 0$ for some x in $a \le x \le b$, (ii) $f(x) = 0$ for $a \le x \le b$.

10 B Stationary Points and Turning Points

Stationary points can be classified into four different types, according to whether the curve turns upwards or downwards from the tangent to the left and to the right of the stationary point:



Maximum turning point	Minimum turning point	Stationary point of inflexion	Stationary point of inflexion

The words used in describing this classification need to be properly defined.

Local or Relative Maximum and Minimum: The words *maximum* and *minimum* are usually used for *local* or *relative* maxima and minima, that is, they relate only to the curve in the immediate neighbourhood of the point being considered. Suppose now that $A\big(a, f(a)\big)$ is a point on a curve $y = f(x)$. Then:

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2

LOCAL MAXIMUM: The point A is called a *local* or *relative maximum* if
$$f(x) \le f(a), \quad \text{for all } x \text{ in some small interval around } a.$$
LOCAL MINIMUM: Similarly, A is called a *local* or *relative minimum* if
$$f(x) \ge f(a), \quad \text{for all } x \text{ in some small interval around } a.$$
EXTREMUM: Any local maximum or minimum is called an *extremum*.

Turning Points: A turning point is a stationary point where the curve smoothly turns over from increasing to decreasing or from decreasing to increasing, as in the first and second diagrams above. Such a situation results in a local maximum or minimum.

3

TURNING POINTS: A stationary point is called a *turning point* if the derivative changes sign around the point.
In other words, a turning point is a stationary point that is a local maximum or minimum.

Stationary Points of Inflexion: In the last two diagrams above, there is no turning point, because in the third diagram the curve is increasing on both sides of the stationary point, and in the fourth diagram the curve is decreasing on both sides of the stationary point. Because of the presence of the stationary point, the curve flexes around the stationary point, changing concavity from downwards to upwards, or from upwards to downwards, with the surprising effect that the tangent at the stationary point actually crosses the curve.

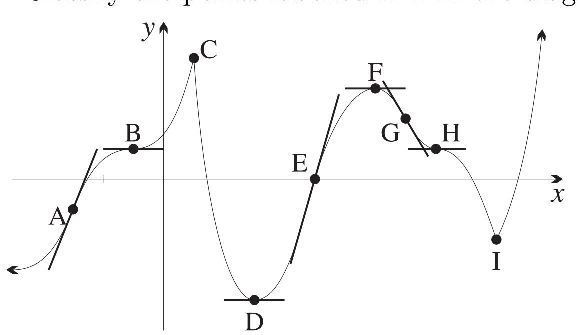
4

POINTS OF INFLEXION: A *point of inflexion* is a point on the curve where the tangent crosses the curve. That is, it is a point where the concavity changes from upwards to downwards or from downwards to upwards.

STATIONARY POINTS OF INFLEXION: A *stationary point of inflexion* is a point of inflexion with horizontal tangent. That is, it is both a point of inflexion and a stationary point.

The diagram below demonstrates the various phenomena described in these definitions:

WORKED EXERCISE: Classify the points labelled A–I in the diagram below.



SOLUTION: C and F are local maxima, with F being a maximum turning point.
D and I are local minima, with D being a minimum turning point.
B and H are stationary points of inflexion.
A, E and G are points of inflexion, but are not stationary points.

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Analysing Stationary Points and Slope: We now appeal to the theorem, discussed in Chapter Three, to the effect that a function can only change sign at a zero or at a discontinuity. But we apply this theorem now not to the function $f(x)$ but to its derivative $f'(x)$.

5 CHANGES BETWEEN INCREASING AND DECREASING: A function can only change from increasing to decreasing, or from decreasing to increasing, at a zero or a discontinuity of the derivative.

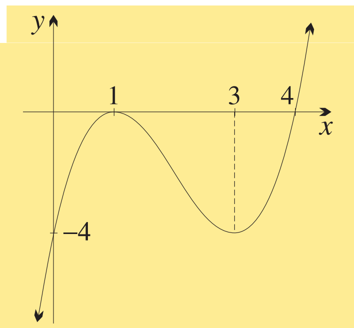
Here then is the method for analysing the stationary points, and also for gaining an overall picture of the whole slope of the function.

6 USING THE DERIVATIVE $f'(x)$ TO ANALYSE STATIONARY POINTS AND SLOPE:
1. Find the zeroes and discontinuities of the derivative $f'(x)$.
2. Draw up a table of test points of the derivative $f'(x)$ around its zeroes and discontinuities, followed by a table of slopes, to see where its sign changes.

The table will show not only the nature of each stationary point, but also where the function is increasing and where it is decreasing across its whole domain.

The table of slopes in the third row of the table gives an outline picture of the shape of the curve, and is a good preparation for a proper sketch.

WORKED EXERCISE: Find the stationary points of the cubic $y = x^3 - 6x^2 + 9x - 4$, determine their nature, and sketch the curve.



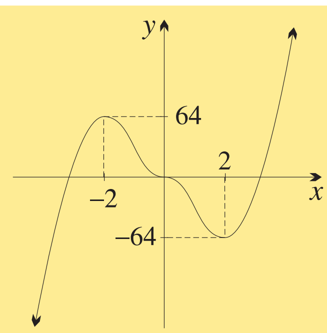
SOLUTION: $y' = 3x^2 - 12x + 9$
$\quad = 3(x-1)(x-3)$,
so y' has zeroes at $x = 1$ and 3, and no discontinuities:

x	0	1	2	3	4
y'	9	0	-3	0	9
	/	—	\	—	/

When $x = 1$, $y = 0$, and when $x = 3$, $y = -4$ (from the original equation), so $(1, 0)$ is a maximum turning point, and $(3, -4)$ is a minimum turning point. [In fact, the function factors as $y = (x-1)^2(x-4)$.]

NOTE: Only the signs of y' are relevant. But if the actual values of y' are not calculated, some other argument should be given as to how the signs were obtained.

WORKED EXERCISE: Find the stationary points of the quintic $f(x) = 3x^5 - 20x^3$, determine their nature, and sketch the curve.



SOLUTION: $f'(x) = 15x^4 - 60x^2$
$\quad = 15x^2(x-2)(x+2)$,
so $f'(x)$ has zeroes at $x = -2$, $x = 0$ and $x = 2$, and has no discontinuities:

x	-3	-2	-1	0	1	2	3
$f'(x)$	675	0	-45	0	-45	0	675
	/	—	\	—	\	—	/

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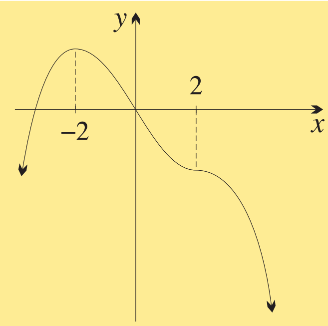
When $x = 0$, $y = 0$, when $x = 2$, $y = -64$, and when $x = -2$, $y = 64$, so $(-2, 64)$ is a maximum turning point, $(2, -64)$ is a minimum turning point, and $(0, 0)$ is a stationary point of inflexion.

NOTE: This function $f(x) = 3x^5 - 20x^3$ is an odd function, and it has as its derivative $f'(x) = 15x^4 - 60x^2$, which is even. In general, the derivative of an even function is odd, and the derivative of an odd function is even — this provides a useful check of the working. The result is obvious for polynomials because the indices reduce by 1, but see the last question in Exercise 10E for a general proof.

WORKED EXERCISE: Given the function sketched on the right, write down a possible equation for the derivative of the function, and a table of values to justify it.

SOLUTION: A possibility is $f'(x) = -(x + 2)(x - 2)^2$. As its table of values shows, this function is zero at $x = 2$ and negative on both sides of it, and it changes sign around $x = -2$.

x	-3	-2	0	2	3
$f'(x)$	25	0	-8	0	-5
	/	—	\	—	\



WORKED EXERCISE: The graph of the cubic $f(x) = x^3 + ax^2 + bx + c$ passes through the origin and has a stationary point at $A(2, 2)$. Find a, b and c.

SOLUTION: To find the three unknown constants, we need three independent equations.

Since $f(0) = 0$, $\quad 0 = 0 + 0 + 0 + c$ and so $c = 0$. (1)

Since $f(2) = 2$, $\quad 2 = 8 + 4a + 2b + c$

and since $c = 0$, $\quad 2a + b = -3$. (2)

Differentiating, $\quad f'(x) = 3x^2 + 2ax + b$

and since $f'(2) = 0$, $\quad 0 = 12 + 4a + b$

$\quad 4a + b = -12$. (3)

Substracting (2) from (3), $\quad 2a = -9$

$\quad a = -4\frac{1}{2}$,

and substituting into (2), $\quad -9 + b = -3$

$\quad b = 6$.

Exercise 10B

1. Find the derivative of each function and complete the given table to determine the nature of the stationary point. Sketch each graph, indicating all important features.

(a) $y = x^2 - 4x + 3$:

x	1	2	3
y'			

(c) $y = 3x^2 + 11x - 4$:

x	-2	$-\frac{11}{6}$	-1
y'			

(b) $y = 12 + 4x - x^2$:

x	1	2	3
y'			

(d) $y = 3 + 5x - 2x^2$:

x	1	$\frac{5}{4}$	2
y'			

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2. Find the stationary point(s) of each function and use a table of values of dy/dx to determine its nature. Sketch each graph, indicating all intercepts with the axes.

(a) $y = 2x^2 - 3x + 5$ (b) $y = 5 - 4x - x^2$ (c) $y = x^3 - 3x^2$ (d) $y = 12x - x^3$

3. Find the stationary points of each of the following functions and use a table of values of dy/dx to determine their nature. Sketch each graph (do not find the x-intercepts).

(a) $y = 2x^3 + 3x^2 - 36x + 15$
(b) $y = x^3 + 4x^2 + 4x$
(c) $y = 16 + 4x^3 - x^4$
(d) $y = 3x^4 - 16x^3 + 24x^2 + 11$

DEVELOPMENT

4. (a) Use the product rule to show that if $y = x(x-2)^3$, then $y' = 2(2x-1)(x-2)^2$.
(b) Find any stationary points and use a table of values of y' to analyse them.
(c) Sketch a graph of the function, indicating all important features.

5. (a) Expand and simplify $(x+2)(x-3)^2$.
(b) If $f(x) = 3x^4 - 16x^3 - 18x^2 + 216x + 40$, find $f'(x)$ in factored form.
(c) Hence find all stationary points and analyse them.
(d) Sketch a graph of $y = f(x)$.

6. (a) If $f(x) = (x-2)^2(x+4)^3$, show that $f'(x) = (x-2)(x+4)^2(5x+2)$.
(b) Find all stationary points and analyse them.
(c) Sketch $y = f'(x)$ and hence determine where $y = f(x)$ is increasing and decreasing.
(d) Sketch a graph of $y = f(x)$, indicating all important features.

7. Using the method outlined in the previous question, sketch graphs of the these functions:

(a) $y = x^2(3-x)^2$
(b) $y = (1-x)^3(x+2)^2$
(c) $y = (x-5)^2(2x+1)$
(d) $y = (3x-2)^2(2x-3)^3$

8. (a) If $f(x) = \dfrac{3x}{x^2+1}$, show that $f'(x) = \dfrac{3(1-x)(1+x)}{(x^2+1)^2}$.
(b) Hence find any stationary points and analyse them.
(c) Sketch a graph of $y = f(x)$, indicating all important features.
(d) Hence state how many roots the equation $\dfrac{3x}{x^2+1} = c$ has for:

(i) $c > \frac{3}{2}$ (ii) $c = \frac{3}{2}$ (iii) $0 < c < \frac{3}{2}$ (iv) $c = 0$

9. The tangent to the curve $y = x^2 + ax - 15$ is horizontal at the point where $x = 4$. Find the value of a.

10. The curve $y = ax^2 + bx + c$ passes through the points $(1, 4)$ and $(-1, 6)$ and obtains its maximum value when $x = -\frac{1}{2}$. Find the values of a, b and c.

11. The curve $y = ax^2 + bx + c$ touches the line $y = 2x$ at the origin and has a maximum point when $x = 1$. Find the values of a, b and c.

12. The function $y = ax^3 + bx^2 + cx + d$ has a relative maximum at the point $(-2, 27)$ and a relative minimum at the point $(1, 0)$. Find the values of a, b, c and d.

13. (a) Sketch graphs of the following functions, clearly indicating any stationary points (but leave the y-coordinates in factored form):

(i) $y = x^4(1-x)^6$ (ii) $y = x^4(1-x)^7$ (iii) $y = x^5(1-x)^6$ (iv) $y = x^5(1-x)^7$

(b) Show that $y = x^a(1-x)^b$ has a turning point whose x-coordinate divides the interval between the points $(0, 0)$ and $(1, 0)$ in the ratio $a : b$.

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EXTENSION

14. Let $f(x) = x^3 + 3bx^2 + 3cx + d$.

(a) Show that $y = f(x)$ has two distinct turning points if and only if $b^2 > c$.

(b) If $b^2 > c$, show that the vertical distance between the turning points is $4(b^2 - c)^{\frac{3}{2}}$. [HINT: Use the sum and product of the roots of the derived function.]

10 C Critical Values

As discussed in the previous section, the derivative of a function can change sign at a zero or a discontinuity of the derived function. Such values are called *critical values*. The examples so far have mostly avoided functions whose derivative has a discontinuity, and this section will deal with them more systematically.

7 CRITICAL VALUES: A zero or discontinuity of the derivative is called a *critical value* of the function.

THE TABLE OF TEST POINTS OF $f'(x)$: Because these critical values are the only places where the derivative can change sign, a table of test points of $f'(x)$ around them will be sufficient to analyse the stationary points and to show where the function is increasing and where it is decreasing.

WORKED EXERCISE:

(a) Find the critical values of $y = \dfrac{1}{x(x-4)}$, then use a table of test points of $\dfrac{dy}{dx}$ to analyse stationary points and find where the function is increasing and decreasing.

(b) Analyse the sign of the function in its domain, find any vertical and horizontal asymptotes, then sketch the curve.

SOLUTION:

(a) The domain of the function is $x \neq 0$ and $x \neq 4$.
Differentiating using the chain rule,

$$\frac{dy}{dx} = \frac{-1}{x^2(x-4)^2} \times (2x-4)$$
$$= \frac{2(2-x)}{x^2(x-4)^2},$$

so $\dfrac{dy}{dx}$ has a zero at $x = 2$

and discontinuities at $x = 0$ and $x = 4$:

x	-1	0	1	2	3	4	5
$\dfrac{dy}{dx}$	$\frac{6}{25}$	$*$	$\frac{2}{9}$	0	$-\frac{2}{9}$	$*$	$-\frac{6}{25}$
	/	*	/	—	\	*	\

So the function has a maximum turning point at $\left(2, -\frac{1}{4}\right)$, it is increasing for $x < 2$ (except at $x = 0$), and it is decreasing for $x > 2$ (except at $x = 4$).

Let $u = x^2 - 4x$,

then $y = \dfrac{1}{u}$.

So $\dfrac{du}{dx} = 2x - 4$

and $\dfrac{dy}{du} = -\dfrac{1}{u^2}$.

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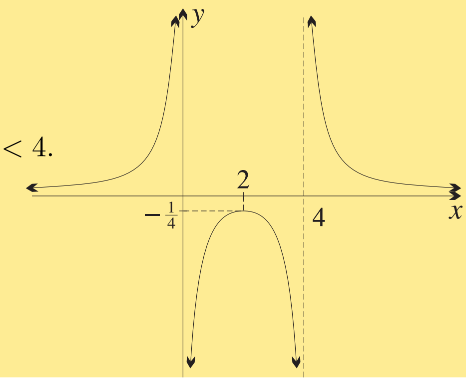
(b) The function itself is never zero, and it has discontinuites at $x = 0$ and $x = 4$:

x	-1	0	2	4	5
y	$\frac{1}{5}$	$*$	$-\frac{1}{4}$	$*$	$\frac{1}{5}$

so $y > 0$ for $x < 0$ or $x > 4$, and $y < 0$ for $0 < x < 4$.
As $x \to 4^+$, $y \to \infty$, and as $x \to 4^-$, $y \to -\infty$,
as $x \to 0^+$, $y \to -\infty$, and as $x \to 0^-$, $y \to \infty$,
so $x = 0$ and $x = 4$ are vertical asymptotes.
Also, $y \to 0$ as $x \to \infty$ and as $x \to -\infty$,
so the x-axis is a horizontal asymptote.



Rational Functions: A *rational function* is a function like $y = \dfrac{x-2}{x^2 - 4x}$, which is the ratio of two polynomials. These functions can be very complicated to sketch. Besides the difficult algebra of the quotient rule, there may be asymptotes, zeroes, turning points and inflexions. The curve $y = \dfrac{1}{x(x-4)}$ above is a rational function, but was a little simpler to handle because of the constant numerator.

Taking the Limit of $f'(x)$ near Critical Values and for Large Values of x: In the previous worked example, there were asymptotes at the two values $x = 0$ and $x = 4$ where $f'(x)$ was undefined, so no further analysis was needed. In other situations, however, the shape of the curve may not be clear near a value $x = a$ where $f'(x)$ is undefined. It may then be necessary to examine the behaviour of $f'(x)$ as $x \to a^+$ and as $x \to a^-$. Furthermore, the shape of the curve as $x \to \infty$ and as $x \to -\infty$ may need examination of the behaviour of both $f'(x)$ and $f(x)$.

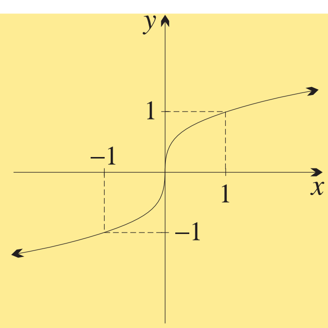
8

BEHAVIOUR NEAR DISCONTINUITIES OF THE DERIVATIVE AND FOR LARGE x:
1. For each discontinuity $x = a$ of $f'(x)$, it may be necessary to examine the behaviour of $f'(x)$ as $x \to a^+$ and as $x \to a^-$.
2. It may also be necessary to examine $f'(x)$ as $x \to \infty$ and as $x \to -\infty$.

WORKED EXERCISE: [Vertical tangents] Analyse the critical values of $y = x^{\frac{1}{3}}$, then sketch the curve. This curve and the next were discussed in Section 7J on differentiability, but the methods of these sections are well suited to them, provided that the behaviour of the derivative is properly analysed near its discontinuities.

SOLUTION: $y = x^{\frac{1}{3}}$ is an odd function, defined everywhere.
It is zero at $x = 0$, positive for $x > 0$, and negative for $x < 0$.
Differentiating, $y' = \frac{1}{3}x^{-\frac{2}{3}}$,
so y' has no zeroes, and has a discontinuity at $x = 0$:

x	-1	0	1
y'	$\frac{1}{3}$	$*$	$\frac{1}{3}$
	/	$*$	/



Since $y' \to \infty$ as $x \to 0^+$ and as $x \to 0^-$, there is a vertical tangent at the origin.
Also $y' \to 0$ as $x \to \infty$ and as $x \to -\infty$, so the curve flattens out away from the origin, but $y \to \infty$ as $x \to \infty$, so there is no horizontal asymptote.

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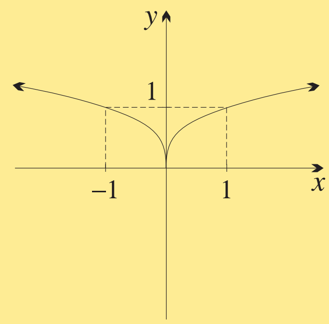
WORKED EXERCISE: [Cusps] Analyse the critical values of $y = x^{\frac{2}{3}}$, then sketch it.

SOLUTION: $y = x^{\frac{2}{3}}$ is an even function, defined everywhere.
It is zero at $x = 0$ and positive elsewhere.
Differentiating, $y' = \frac{2}{3}x^{-\frac{1}{3}}$,
so y' has no zeroes, and has a discontinuity at $x = 0$:

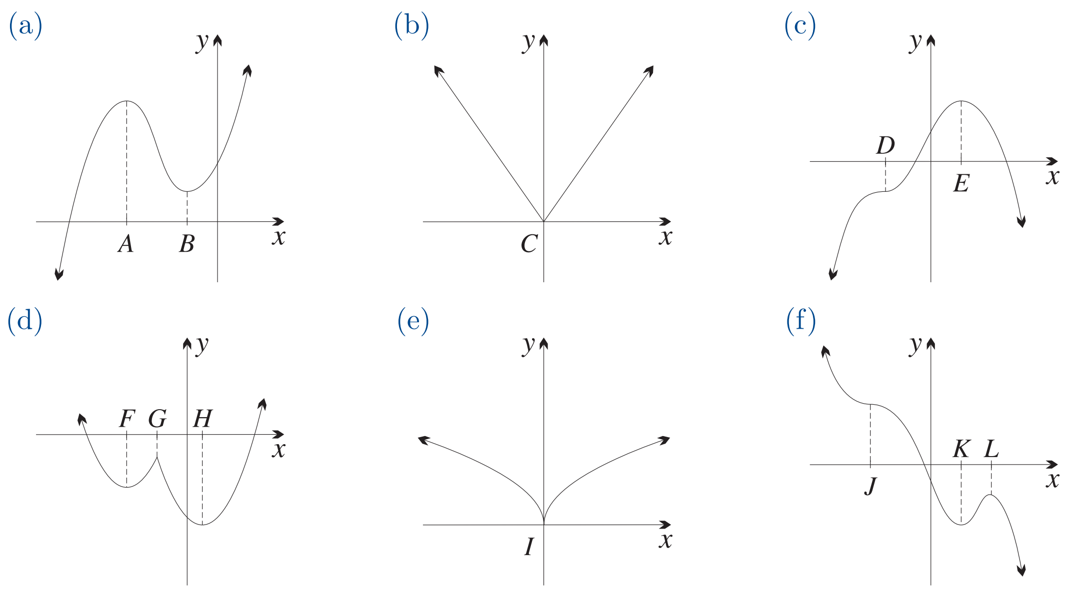
x	-1	0	1
y'	$-\frac{2}{3}$	$*$	$\frac{2}{3}$
	$\backslash$	$*$	$/$



As $x \to 0^+$, $y' \to \infty$, and as $x \to 0^-$, $y' \to -\infty$,
so there is a cusp at the origin.
Again $y' \to 0$ as $x \to \infty$ and as $x \to -\infty$, so the curve flattens out away from the origin, and again $y \to \infty$ as $x \to \infty$, so there is no horizontal asymptote.

Exercise 10C

1. All the critical points have been labelled on the graphs of the functions drawn below. State which of these are relative maxima or minima or horizontal points of inflexion.



2. The derivatives of various functions have been given below. Find the critical values. Use a table of test points of dy/dx to find which critical values give turning points or horizontal points of inflexion:

(a) $\dfrac{dy}{dx} = x - 1$

(b) $\dfrac{dy}{dx} = (x-3)(2x+1)$

(c) $\dfrac{dy}{dx} = x(x-3)^2$

(d) $\dfrac{dy}{dx} = (x+2)^3(x-4)$

(e) $\dfrac{dy}{dx} = \dfrac{x}{x-1}$

(f) $\dfrac{dy}{dx} = \dfrac{x^2}{x-1}$

(g) $\dfrac{dy}{dx} = \dfrac{x}{(x-1)^2}$

(h) $\dfrac{dy}{dx} = \dfrac{x^2}{(x-1)^3}$

(i) $\dfrac{dy}{dx} = x^{-\frac{1}{3}}$

(j) $\dfrac{dy}{dx} = x - \dfrac{1}{x}$

(k) $\dfrac{dy}{dx} = \sqrt{x} - \dfrac{1}{\sqrt{x}}$

(l) $\dfrac{dy}{dx} = \dfrac{2-x}{\sqrt{2+x}(1-x)^3}$

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DEVELOPMENT

3. (a) State the domain of the function $y = x + \dfrac{1}{x}$.

(b) Show that $\dfrac{dy}{dx} = \dfrac{x^2 - 1}{x^2}$, and write down any critical values.

(c) Find and analyse any stationary points, using a table of values of $\dfrac{dy}{dx}$.

(d) Describe what happens to y as $x \to \infty$ and $x \to -\infty$ (and find the oblique asymptote).

(e) Sketch a graph of the function. indicating all important features.

4. Differentiate these functions using the quotient rule. Use a table of values of y' to analyse any stationary points. Find any asymptotes, then sketch each function:

(a) $y = \dfrac{x}{x^2 - 1}$ (b) $y = \dfrac{x^2}{1 + x^2}$ (c) $y = \dfrac{x^2 - 4}{x^2 - 1}$ (d) $y = \dfrac{x^2 + 1}{(x - 1)^2}$

5. (a) State the domain of the function $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$.

(b) Show that $f'(x) = \dfrac{x - 1}{2x\sqrt{x}}$ and write down any critical values.

(c) Find the stationary point, and use a table of values of dy/dx to determine its nature.

(d) Describe what happens to $f(x)$ and to $f'(x)$ as $x \to \infty$.

(e) Sketch a graph of the function, indicating all important features.

6. Use the steps outlined in the previous question to graph the following functions:

(a) $y = x - \dfrac{1}{x}$ (b) $y = x^2 + \dfrac{1}{x^2}$

7. (a) Find the domain and any asymptotes of $y = \dfrac{\sqrt{x}}{\sqrt{9 + x^2}}$.

(b) Show that its derivative is $\dfrac{dy}{dx} = \dfrac{(3 - x)(3 + x)}{2(9 + x^2)^{\frac{3}{2}}\sqrt{x}}$.

(c) Find the critical values, and analyse them with a table of test points.

(d) By examining the limit of the derivative as $x \to 0^+$, determine the shape of the curve near the origin, then sketch the curve.

8. Consider the function $y = |x| + 3$. (a) Find $\dfrac{dy}{dx}$ when $x < 0$ and when $x > 0$.

(b) Hence find any critical values, and sketch a graph of the function.

9. Consider the function $y = |x - 2|$. (a) Find $\dfrac{dy}{dx}$ when $x > 2$ and when $x < 2$.

(b) Hence find any critical values, and sketch a graph of the function.

10. (a) Differentiate $f(x) = (x - 2)^{\frac{1}{5}}$.

(b) Show that there are no stationary points, but that a critical value occurs at $x = 2$.

(c) By considering the sign of $f'(x)$, sketch a graph of $y = f(x)$.

11. (a) Differentiate $f(x) = (x - 1)^{\frac{2}{3}}$.

(b) Show that the critical point $(1, 0)$ is not a stationary point.

(c) By considering the sign of $f'(x)$ when $x < 1$ and when $x > 1$, sketch $y = f(x)$.

(d) Hence sketch: (i) $y = 3 + (x - 1)^{\frac{2}{3}}$ (ii) $y = 1 - (x - 1)^{\frac{2}{3}}$

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EXTENSION

12. (a) Differentiate $y = x^{\frac{1}{2}} - x^{\frac{3}{2}}$. (b) Find those values of x for which $y' = 0$, and hence determine the coordinates of any critical points.
(c) Hence sketch a graph of $y^2 = x(1-x)^2$.

13. Sketch graphs of the following functions, indicating all critical points:
(a) $y = |(x-1)(x-3)|$ (b) $y = |x-2| + |x+1|$ (c) $y = x^2 + |x|$

14. (a) State the domain and range of the function $\sqrt{x} + \sqrt{y} = \sqrt{c}$, where c is a constant.
(b) Use implicit differentiation to show that $y' = -\sqrt{y/x}$.
(c) By considering the behaviour of y' as $x \to 0^+$ and $y \to 0^+$, sketch a graph of the curve, labelling all critical points.

10 D Second and Higher Derivatives

The derivative of the derivative of a function is called the *second derivative* of the function. As for the derivative, there is a variety of notations, including

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f^{(2)}(x) \quad \text{or} \quad f''(x) \quad \text{or} \quad y''.$$

This section is concerned with the algebraic manipulation of the second derivative — the geometric implications are left until the next section.

WORKED EXERCISE: Find the successive derivatives of $y = x^4 + x^3 + x^2 + x + 1$.

SOLUTION: $y = x^4 + x^3 + x^2 + x + 1$

$\frac{dy}{dx} = 4x^3 + 3x^2 + 2x + 1$

$\frac{d^2y}{dx^2} = 12x^2 + 6x + 2$

$\frac{d^3y}{dx^3} = 24x + 6$

$\frac{d^4y}{dx^4} = 24$

$\frac{d^5y}{dx^5} = 0$

NOTE: The degree of the polynomial goes down by one with each differentiation, so that the fifth and all higher derivatives vanish. In general, the $(n+1)$th and higher derivatives of a polynomial of degree n vanish, but the nth derivative does not.

The eventual vanishing of the higher derivatives of polynomials is actually a characteristic property of polynomials, because the converse is also true. If the $(n+1)$th derivative of a polynomial vanishes but the nth does not, then the function is a polynomial of degree n. The result seems clear, but as yet we lack the machinery for a formal proof.

Exercise 10D

1. Find the first, second and third derivatives of the following:
(a) x^{10} (b) $3x^5$ (c) $4 - 3x$ (d) $x^2 - 3x$ (e) $4x^3 - x^2$ (f) $x^{0\cdot 3}$ (g) x^{-1} (h) $\frac{1}{x^2}$ (i) $5x^{-3}$ (j) $x^2 + \frac{1}{x}$

2. Use the chain rule to find the first and second derivatives of the following:
(a) $(x+1)^2$ (b) $(3x-5)^3$ (c) $(1-4x)^2$ (d) $(8-x)^{11}$

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3. By writing them with negative indices, find the first and second derivatives of the following:

(a) $\dfrac{1}{x+2}$ (b) $\dfrac{1}{(3-x)^2}$ (c) $\dfrac{1}{(5x+4)^3}$ (d) $\dfrac{2}{(4-3x)^2}$

4. By writing them with fractional indices, find the first and second derivatives of the following functions:

(a) $\sqrt{x}$ (b) $\sqrt[3]{x}$ (c) $x\sqrt{x}$ (d) $\dfrac{1}{\sqrt{x}}$ (e) $\sqrt{x+2}$ (f) $\sqrt{1-4x}$

5. (a) If $f(x) = 3x + \dfrac{1}{x^3}$, find: (i) $f'(2)$ (ii) $f''(2)$

(b) If $f(x) = (2x-3)^4$, find: (i) $f'(1)$ (ii) $f''(1)$ (iii) $f'''(1)$ (iv) $f''''(1)$

6. If $f(x) = ax^2 + bx + c$ and $f(1) = 5$, $f'(1) = 2$ and $f''(1) = -1$, find a, b and c.

DEVELOPMENT

7. Find the first and second derivatives of the following:

(a) $\dfrac{x}{x+1}$ (b) $\dfrac{x-1}{2x+5}$ (c) $\dfrac{x}{1+x^2}$

8. If $f(x) = x(x-1)^4$, use the product rule to find $f'(x)$ and $f''(x)$.

9. Find the values of x for which $y'' = 0$ if:

(a) $y = x^4 - 6x^2 + 11$ (b) $y = x^3 + x^2 - 5x + 7$ (c) $y = x\sqrt{x+1}$

10. (a) If $y = 3x^2 + 7x + 5$, prove that $\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) = x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}$.

(b) If $y = (2x-1)^4$, prove that $\dfrac{d}{dx}\left(y\dfrac{dy}{dx}\right) = y\dfrac{d^2y}{dx^2} + (\dfrac{dy}{dx})^2$.

(c) If $y = 2x^2 - \dfrac{3}{\sqrt{x}}$, prove that $2x^2\dfrac{d^2y}{dx^2} = x\dfrac{dy}{dx} + 2y$.

11. (a) Find the first, second and third derivatives of x^n.

(b) Find the nth and $(n+1)$th derivatives of x^n.

12. Find y' and y'' in terms of t in each of the following cases:

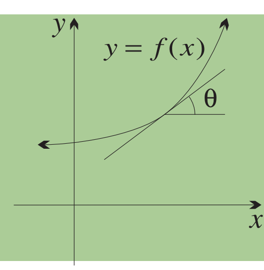
(a) $x = 3t + 1$, $y = 2t$ (c) $x = 1 - 5t$, $y = t^3$ (e) $x = (t-2)^2$, $y = 3t$

(b) $x = 2t$, $y = \dfrac{1}{t}$ (d) $x = t - \dfrac{1}{t}$, $y = t + \dfrac{1}{t}$ (f) $x = \dfrac{1+t}{1-t}$, $y = \dfrac{1-t}{1+t}$

13. Find positive integers a and b such that $x^2y'' + 2xy' = 12y$, where $y = x^a + x^{-b}$.

EXTENSION

14. The curvature C of the graph $y = f(x)$ is defined as the absolute value of the rate of change of the angle θ with respect to arc length. That is, $C = \dfrac{|f''(x)|}{\left(1 + f'(x)^2\right)^{\frac{3}{2}}}$. Compute the curvature of a circle of radius r.



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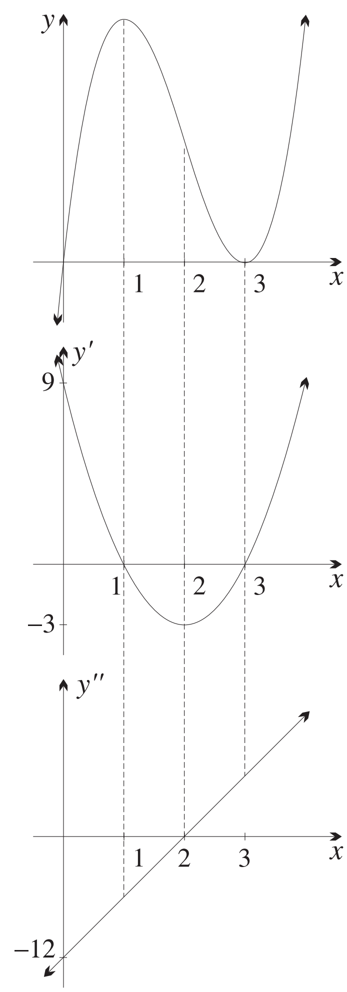
10 E Concavity and Points of Inflexion

Sketched on the right are a cubic function and its first and second derivatives. These sketches are intended to show how the concavity of the original graph is determined by the sign of the second derivative.

$$\begin{aligned} y &= x^3 - 6x^2 + 9x = x(x-3)^2 \\ y' &= 3x^2 - 12x + 9 = 3(x-1)(x-3) \\ y'' &= 6x - 12 \qquad\quad = 6(x-2) \end{aligned}$$



The sign of each derivative tells whether the function above it is increasing or decreasing. So the second graph describes the gradient of the first, and the third graph describes the gradient of the second.

To the right of $x = 2$, the top graph is concave up. This means that as one moves along the curve from left to right, the tangent rotates, with its gradient steadily increasing. Thus for $x > 2$, the gradient function y' is increasing as x increases, as can be seen seen in the middle graph. The bottom graph is the gradient of the middle graph, and accordingly y'' is positive for $x > 2$.

Similarly, to the left of $x = 2$ the top graph is concave down. This means that its gradient function y' is steadily decreasing as x increases. The bottom graph is the derivative of the middle graph, so y'' is negative for $x < 2$.

This example demonstrates that the concavity of a graph $y = f(x)$ at any value $x = a$ is determined by the sign of its second derivative at $x = a$.

9 CONCAVITY AND THE SECOND DERIVATIVE:
If $f''(a)$ is negative, the curve is concave down at $x = a$.
If $f''(a)$ is positive, the curve is concave up at $x = a$.

Points of Inflexion: A point of inflexion is a point where the tangent crosses the curve, as was defined in Section 10B. This means that the curve must curl away from the tangent on opposite sides of the tangent, so the concavity must change around the point. The three diagrams above show how the point of inflexion at $x = 2$ results in a minimum turning point at $x = 2$ in the middle graph of y'. Hence the bottom graph of y'' has a zero at $x = 2$, and changes sign around $x = 2$.

This discussion gives the full method for analysing concavity and finding points of inflexion. Once again, the method uses the fact that y'' can only change sign at a zero or a discontinuity of y''.

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10 USING $f''(x)$ TO ANALYSE CONCAVITY AND FIND POINTS OF INFLEXION:

1. Find the zeroes and discontinuities of the second derivative $f''(x)$.
2. Use a table of test points of the second derivative $f''(x)$ around its zeroes and discontinuities, followed by a table of concavities, to see where its sign changes.

The table will show not only any points of inflexion, but also the concavity of the graph across its whole domain.

Inflexional Tangents: It is often useful in sketching to find the gradient of any *inflexional tangents* (tangents at the point of inflexion). A question will often ask for this before requiring the sketch.

WORKED EXERCISE: Find any points of inflexion of $f(x) = x^5 - 5x^4$ and the gradients of the inflexional tangents, and describe the concavity. Find also any turning points, and sketch the curve.

SOLUTION: Here

$$\begin{aligned} f(x) &= x^5 - 5x^4 &&= x^4(x-5) \\ f'(x) &= 5x^4 - 20x^3 &&= 5x^3(x-4) \\ f''(x) &= 20x^3 - 60x^2 &&= 20x^2(x-3). \end{aligned}$$

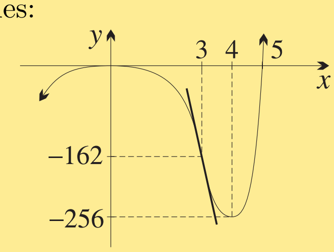
Hence $f'(x)$ has zeroes at $x = 0$ and $x = 4$, and no discontinuities:

x	-1	0	1	4	5
$f'(x)$	25	0	-15	0	625
	/	—	\	—	/

so $(0, 0)$ is a maximum turning point, and $(4, -256)$ is a minimum turning point.

Also, $f''(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities:

x	-1	0	1	3	4
$f''(x)$	-80	0	-40	0	320
	⌢	·	⌢	·	⌣



so $(3, -162)$ is a point of inflexion (but $(0, 0)$ is not).

Since $f'(3) = -135$, the inflexional tangent has gradient -135.

The graph is concave down for $x < 0$ and $0 < x < 3$, and concave up for $x > 3$.

NOTE: The example given above is intended to show that $f''(x) = 0$ is NOT a sufficient condition for a point of inflexion — the sign of $f''(x)$ must also change around the point.

WORKED EXERCISE: [Finding pronumerals in a function] For what values of b is the graph of the quartic $f(x) = x^4 - bx^3 + 5x^2 + 6x - 8$ concave down at the point where $x = 2$?

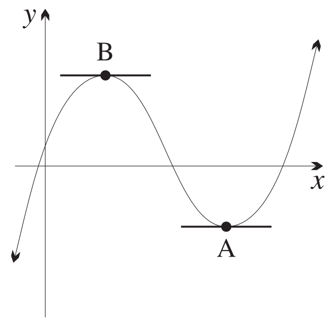
SOLUTION: Differentiating,

$$\begin{aligned} f'(x) &= 4x^3 - 3bx^2 + 10x + 6 \\ f''(x) &= 12x^2 - 6bx + 10. \end{aligned}$$

Put $f''(2) < 0$, then

$$\begin{aligned} 48 - 12b + 10 &< 0 \\ 12b &> 58 \\ b &> 4\tfrac{5}{6}. \end{aligned}$$

Using the Second Derivative to Test Stationary Points: If the curve is concave up at a stationary point, then the point must be a minimum turning point, as in the point A on the diagram to the right. Similarly the curve is concave down at B, which must therefore be a maximum turning point.



This gives an alternative test of the nature of a stationary point. Suppose that $x = a$ is a stationary point of a function $f(x)$. Then:

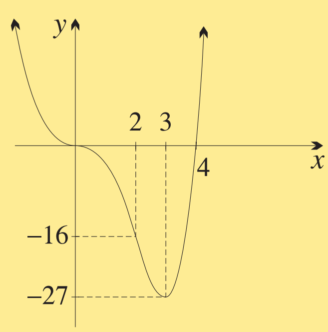
11 USING THE SECOND DERIVATIVE TO TEST A STATIONARY POINT:
- If $f''(a) > 0$, the curve is concave up at $x = a$, and there is a minimum turning point there.
- If $f''(a) < 0$, the curve is concave down at $x = a$, and there is a maximum turning point there.
- If $f''(a) = 0$, more work is needed. Go back to the table of values of $f'(x)$, or else use a table of values of $f''(x)$.

The third point is most important — all four cases are possible for the shape of the curve at $x = a$ when the second derivative vanishes there, and without further work, no conclusion can be made at all. The previous example of $y = x^5 - 5x^4$ shows that such a point can be a turning point. The following worked exercise is an example where such a point turns out to be a point of inflexion.

WORKED EXERCISE: Use the second derivative, if possible, to determine the nature of the stationary points of the graph of $f(x) = x^4 - 4x^3$. Find also any points of inflexion, examine the concavity over the whole domain, and sketch the curve.

SOLUTION: Here
$$\begin{aligned} f(x) &= x^4 - 4x^3 &&= x^3(x-4) \\ f'(x) &= 4x^3 - 12x^2 &&= 4x^2(x-3) \\ f''(x) &= 12x^2 - 24x &&= 12x(x-2), \end{aligned}$$
so $f'(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities.
Since $f''(3) = 36$ is positive, $(3, -27)$ is a minimum turning point,
but $f''(0) = 0$, so no conclusion can be drawn about $x = 0$.

x	-1	0	1	3	4
$f'(x)$	-16	0	-8	0	64
	$\backslash$	—	$\backslash$	—	$/$



so $(0, 0)$ is a stationary point of inflexion.
$f''(x)$ has zeroes at $x = 0$ and $x = 2$, and no discontinuities:

x	-1	0	1	2	3
$f''(x)$	36	0	-12	0	36
	$\smile$	$\cdot$	$\frown$	$\cdot$	$\smile$

so, besides the horizontal inflexion at $(0, 0)$, there is an inflexion at $(2, -16)$, and the inflexional tangent at $(2, -16)$ has gradient -16.
The graph is concave down for $0 < x < 2$, and concave up for $x < 0$ and for $x > 2$.

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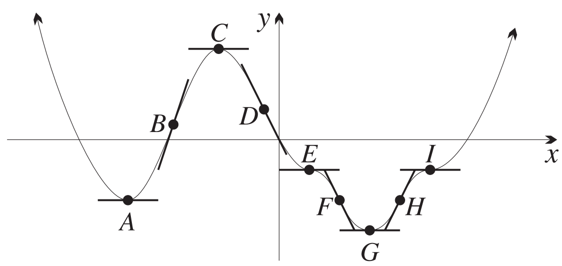
Exercise 10E

1. Complete the table below for the function to the right, stating at each given point whether the first and second derivative would be positive, negative or zero:

Point	A	B	C	D	E	F	G	H	I
y'									
y''									



2. (a) Show that $y = x^2 - 3x + 7$ is concave up for all values of x.
(b) Show that $y = -3x^2 + 2x - 4$ is concave down for all values of x.
(c) Show that $y = \dfrac{1}{x}$ is concave up for $x > 0$ and concave down for $x < 0$.
(d) Show that $y = x^3 - 3x^2 - 5x + 2$ is concave up for $x > 1$ and concave down for $x < 1$.

3. (a) If $f(x) = x^3 - 3x$, find $f'(x)$ and $f''(x)$. (b) Hence find any stationary points and, by examining the sign of $f''(x)$, determine their nature.
(c) Find the coordinates of any points of inflexion.
(d) Sketch a graph of the function, indicating all important features.

4. (a) If $f(x) = x^3 - 6x^2 - 15x + 1$, find $f'(x)$ and $f''(x)$. (b) Hence find any stationary points and, by examining the sign of $f''(x)$, determine their nature.
(c) Find the coordinates of any points of inflexion.
(d) Sketch a graph of the function, indicating all important features.

DEVELOPMENT

5. Find the range of values of x for which the curve $y = 2x^3 - 3x^2 - 12x + 8$ is:
(a) increasing, (b) decreasing, (c) concave up, (d) concave down.

6. Find the x-coordinates of any points of inflexion of the following curves:
(a) $\dfrac{d^2y}{dx^2} = x+5$ (b) $\dfrac{d^2y}{dx^2} = (x+5)^2$ (c) $\dfrac{d^2y}{dx^2} = (x-3)(x+2)$ (d) $\dfrac{d^2y}{dx^2} = (x-3)^2(x+2)$

7. Sketch a small section of the graph of the continuous function f about $x = a$ if:
(a) $f'(a) > 0$ and $f''(a) > 0$
(b) $f'(a) > 0$ and $f''(a) < 0$
(c) $f'(a) < 0$ and $f''(a) > 0$
(d) $f'(a) < 0$ and $f''(a) < 0$

8. Sketch possible graphs of continuous functions with these properties:
(a) $f(-5) = f(0) = f(5) = 0$, and $f'(3) = f'(-3) = 0$, and $f''(x) > 0$ for $x < 0$, and $f''(x) < 0$ for $x > 0$
(b) $f'(2) = f''(2) = 0$, and $f''(1) > 0$, and $f''(3) < 0$

9. By finding the second derivative, explain why the curve $y = ax^2 + bx + c$, $a \neq 0$:
(a) is concave up if $a > 0$, (b) is concave down if $a < 0$, (c) has no points of inflexion.

10. (a) If $f(x) = x^4 - 12x^2$, find $f'(x)$ and $f''(x)$.
(b) Find the coordinates of any stationary points, and use $f''(x)$ to determine their nature.
(c) Find the coordinates of any points of inflexion.
(d) Find the gradient of the curve at the two points of inflexion.
(e) Sketch a graph of the function, showing all important features.

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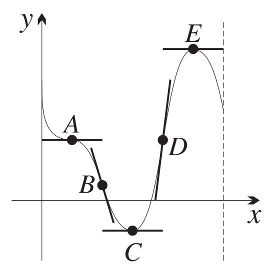
11. (a) If $f(x) = 7 + 5x - x^2 - x^3$, find $f'(x)$ and $f''(x)$.
(b) Find any stationary points and distinguish between them.
(c) Find the coordinates of any points of inflexion.
(d) Sketch a graph, showing all important features.
(e) Find the gradient of the curve at the point of inflexion.
(f) Hence show that the inflexional tangent has equation $144x - 27y + 190 = 0$.

12. (a) If $y = x^3 + 3x^2 - 72x + 14$, find y' and y''.
(b) Show that the curve has a point of inflexion at $(-1, 88)$.
(c) Show that the gradient of the tangent at the point of inflexion is -75.
(d) Hence find the equation of the tangent at the point of inflexion.

13. (a) If $f(x) = x^3$ and $g(x) = x^4$, find $f'(x)$, $f''(x)$, $g'(x)$ and $g''(x)$.
(b) Both $f(x)$ and $g(x)$ have a stationary point at $(0, 0)$. Evaluate $f''(x)$ and $g''(x)$ when $x = 0$. Can you determine the nature of the stationary points from this calculation?
(c) Use a table of values of $f'(x)$ and $g'(x)$ to determine the nature of the stationary points.

14. A curve has equation $y = ax^3 + bx^2 + cx + d$, a turning point at $(0, 5)$, a point of inflexion when $x = \frac{1}{2}$, and crosses the x-axis at $x = -1$. Find the values of a, b, c and d.

15. (a) Show that if $y = \dfrac{x+2}{x-3}$, then $y'' = \dfrac{10}{(x-3)^3}$.
(b) By examining the sign of $(x-3)^3$, determine when the curve is concave up, and when it is concave down.
(c) Hence sketch a graph of $y = \dfrac{x+2}{x-3}$.



16. Given the graph of $y = f(x)$ drawn to the right, on separate axes sketch graphs of:
(a) $y = f'(x)$
(b) $y = f''(x)$

17. (a) Show that if $y = x(x-1)^3$, then $f'(x) = (x-1)^2(4x-1)$ and $f''(x) = 6(x-1)(2x-1)$.
(b) Sketch $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ on the same axes and compare them.

18. (a) Find $f'(x)$ and $f''(x)$ for the function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$.
(b) By completing the square, show that $f'(x) > 0$ for all x, and hence that $f(x)$ is an increasing function.
(c) Find the coordinates of any points of inflexion.
(d) Hence sketch a graph of the function.

19. (a) Find the values of x for which the function $y = x^{\frac{2}{3}}$ is:
(i) increasing, (ii) decreasing, (iii) concave up, (iv) concave down.
(b) Hence sketch a graph of the function, indicating all critical points.

EXTENSION

20. (a) Use the definition of the derivative to show that the derivative of an even function is odd. [HINT: If $f(x)$ is an even function, then $f(-x+h) - f(-x) = f(x-h) - f(x)$.]
(b) Show similarly that the derivative of an odd function is even.

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10 F Curve Sketching using Calculus

Two quite distinct ways have been used so far to sketch a curve from its equation. Familiar equations can often be sketched using the method of 'known curves and their transformations' introduced at the end of Chapter Two. For unfamiliar equations, there is the 'curve sketching menu' introduced at the end of Chapter Three. This section will complete that curve sketching menu by adding Steps 5 and 6, which apply the calculus.

12

A CURVE SKETCHING MENU:

0. PREPARATION: Combine any fractions using a common denominator, and then factor top and bottom as far as possible.
1. DOMAIN: Find the domain of $f(x)$ (*always* do this first).
2. SYMMETRY: Find whether the function is odd or even, or neither.
3. A. INTERCEPTS: Find the y-intercept and all x-intercepts (zeroes).
   B. SIGN: Use a table of test points of $f(x)$ to find where the function is positive, and where it is negative.
4. A. VERTICAL ASYMPTOTES: Examine the behaviour near any discontinuities, noting any vertical asymptotes.
   B. HORIZONTAL ASYMPTOTES: Examine behaviour as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes (and possibly any oblique asymptotes).
5. THE FIRST DERIVATIVE:
   A. Find the zeroes and discontinuities of $f'(x)$.
   B. Use a table of test points of $f'(x)$ to determine the nature of the stationary points, and the slope of the function throughout its domain.
   C. It may be necessary to take limits of $f'(x)$ near discontinuities of $f'(x)$ and as $x \to \infty$ and as $x \to -\infty$.
6. THE SECOND DERIVATIVE:
   A. Find the zeroes and discontinuities of $f''(x)$.
   B. Use a table of test points of $f''(x)$ to find any points of inflexion (it may be useful to find the gradients of the inflexional tangents), and the concavity of the function throughout its domain.
7. ANY OTHER FEATURES.

The final Step 7 is a routine warning that the variety of functions can never be contained in any simple menu. In particular, symmetries about points other than the origin and lines other than the y-axis have not been considered here, yet every parabola has an axis of symmetry, and every cubic has point symmetry in its point of inflexion. Neither has periodicity been included, despite the fact that the trigonometric functions repeat periodically.

It is rare for all these steps to be entirely relevant for any particular function. The domain should always be considered first in every example, and oddness and evenness are always significant, but beyond this, only experience can be a guide as to which steps are essential to bring out the characteristic shape of the curve. The menu should be a checklist and not a rigid prescription. Examination questions usually give a guide as to what is to be done, and the particular arrangement of the menu above belongs to this text and not to the Syllabus.

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Known Curves and their Transformations: It's important to keep in mind that familiar curves can be sketched much more quickly by recognising that the curve is some transformation of a curve that is already well known. Calculus is then quite unnecessary. This approach was discussed in detail in the final section of Chapter Two.

Exercise **10F**

1. Use the steps of the curve sketching menu to sketch the graphs of the following polynomial functions. Indicate the coordinates of any stationary points, points of inflexion, and intercepts with the axes. Do not attempt to find the intercepts with the x-axis in parts (c), (e) and (f):

(a) $y = 2x^3 - 3x^2 + 5$ (c) $y = 12x^3 - 3x^4 + 11$ (e) $y = x^3 - 3x^2 - 24x + 5$
(b) $y = 9x - x^3$ (d) $y = x(x-6)^2$ (f) $y = x^4 - 16x^3 + 72x^2 + 10$

2. (a) If $f(x) = \dfrac{x^2}{1+x^2}$, show that $f'(x) = \dfrac{2x}{(1+x^2)^2}$ and $f''(x) = \dfrac{2-6x^2}{(1+x^2)^3}$.
(b) Hence find the coordinates of any stationary points and determine their nature.
(c) Find the coordinates of any points of inflexion.
(d) State the equation of the horizontal asymptote.
(e) Sketch a graph of the function, indicating all important features.

3. (a) If $f(x) = \dfrac{4x}{x^2+9}$, show that $f'(x) = \dfrac{36-4x^2}{(x^2+9)^2}$ and $f''(x) = \dfrac{8x^3-216x}{(x^2+9)^3}$.
(b) Hence find the coordinates of any stationary points and determine their nature.
(c) Find the coordinates of any points of inflexion.
(d) State the equation of the horizontal asymptote.
(e) Sketch a graph of the function, indicating all important features.

DEVELOPMENT

4. Sketch graphs of the following rational functions, indicating all stationary points, points of inflexion and intercepts with the axes. For each question solve the equation $y = 1$ to see where the graph cuts the horizontal asymptote: (a) $y = \dfrac{x^2-x-2}{x^2}$ (b) $y = \dfrac{x^2-2x}{(x+2)^2}$

5. Without finding inflexions, sketch the graphs of the following functions. Indicate any asymptotes, stationary points and intercepts with the axes (some of them also have oblique asymptotes):

(a) $y = \dfrac{x}{x-1}$ (d) $y = \dfrac{x-x^2-1}{1+x+x^2}$ (g) $y = x^2 + \dfrac{1}{x^2}$
(b) $y = \dfrac{x^2-1}{x^2-4}$ (e) $y = \dfrac{x}{x^2+1}$ (h) $y = \dfrac{x^2+5}{x-2}$
(c) $y = \dfrac{1}{(x-2)(x+1)}$ (f) $y = x + \dfrac{1}{x}$ (i) $y = \dfrac{(x+1)^3}{x}$

6. Write down the domain of each of the following functions and sketch a graph, clearly indicating any stationary points and intercepts with the axes:

(a) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$ (b) $y = x\sqrt{3-x}$ (c) $y = \dfrac{\sqrt{x}}{2+x}$ (d) $y = \dfrac{x}{\sqrt{1+x}}$

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7. By carefully noting their critical points, sketch the graphs of the following functions:

(a) $y = x^{\frac{2}{7}}$ (b) $y = (x-2)^{\frac{2}{3}} + 4$ (c) $y = (1-x)^{\frac{1}{4}} - 2$

8. Using $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, find any stationary points and sketch the graphs of the functions:

(a) $x = 6t,\ y = 3t^2$ (b) $x = \dfrac{1}{t},\ y = \dfrac{1}{t^2}$ (c) $x = t+1,\ y = t^3 - 3t$

EXTENSION

9. (a) Sketch $f(x) = x(4 - x^2)$, clearly indicating all stationary points and intercepts.

(b) What is the relationship between the x-coordinates of the stationary points of the function $y = \big(f(x)\big)^2$ and the information found in part (a)?

(c) Hence sketch $y = \big(f(x)\big)^2$ and $y = \big(f(x)\big)^3$.

10. (a) Sketch $f(x) = (x+5)(x-1)$, clearly indicating the turning point.

(b) Where do the graphs of the functions $y = f(x)$ and $y = \dfrac{1}{f(x)}$ intersect?

(c) Differentiate $y = \dfrac{1}{f(x)}$, and explain why $\dfrac{1}{f(x)}$ increases as $f(x)$ decreases and vice versa.

(d) Using part (a), find and analyse the stationary point of $y = \dfrac{1}{(x+5)(x-1)}$.

(e) Hence sketch a graph of the reciprocal function on the same diagram as part (a).

11. (a) Use implicit differentiation to find dy/dx if $x^3 + y^3 = a^3$ where a is constant.

(b) Hence sketch the graph of $x^3 + y^3 = a^3$, showing all critical points.

12. (a) Sketch a graph of the function $y = |2x-1| + |x+3|$, showing all critical points.

(b) Hence solve the inequality $|2x-1| > 4 - |x+3|$.

10 G Global Maximum and Minimum

Australia has many high mountain peaks, each of which is a *local* or *relative maximum*, because each is the highest point relative to other peaks in its immediate locality. Mount Kosciuszko is the highest of these, but it is still not a *global* or *absolute maximum*, because there are higher peaks on other continents of the globe. Mount Everest in Asia is the global maximum over the whole world.

Suppose now that $f(x)$ is a function defined on some domain, not necessarily the natural domain of the function, and that $A\big(a, f(a)\big)$ is a point on the curve $y = f(x)$ within the domain. Then:

13

GLOBAL MAXIMUM: The point A is called a *global* or *absolute maximum* if

$$f(x) \le f(a), \quad \text{for all } x \text{ in the domain.}$$

GLOBAL MINIMUM: Similarly, A is called a *global* or *absolute minimum* if

$$f(x) \ge f(a), \quad \text{for all } x \text{ in the domain.}$$

The following diagrams illustrate what has to be considered in the general case.

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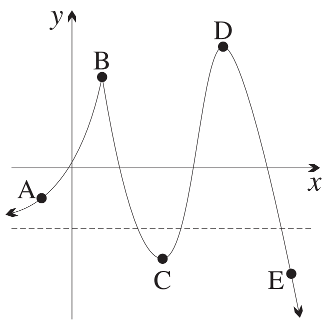
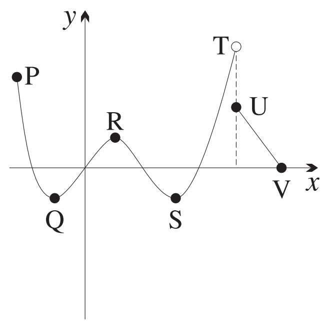


The domain of $f(x)$ is the whole real line.

1. There are local maxima at the point B, where $f'(x)$ is undefined, and at the turning point D. This point D is also the global maximum.
2. There is a local minimum at the turning point C, which is lower than all points on the curve to the left past A. But there is no global minimum, because the curve goes infinitely far downwards to the right of E.



The domain of $f(x)$ is the closed interval on the x-axis from P to V.

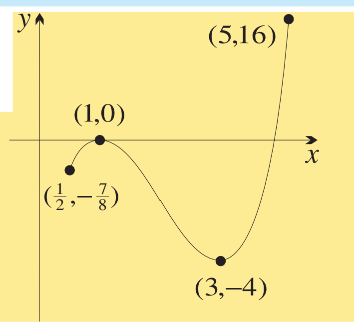
1. There are local maxima at the turning point R and at the endpoint P. But there is no global maximum, because the point T has been omitted from the curve.
2. There are local minima at the two turning points Q and S, and at the endpoint V. These points Q and S have equal heights and are thus both global minima.

Testing for Global Maximum and Minimum: These examples show that there are three types of points that must be considered and compared when finding the global maximum and minimum of a function $f(x)$ defined on some domain.

14 TESTING FOR GLOBAL MAXIMUM AND MINIMUM: Examine and compare:
1. turning points,
2. boundaries of the domain (which may involve behaviour for large x),
3. discontinuities of $f'(x)$ (because they may be local extrema).

More simply, examine and compare the critical values and the boundary values.

WORKED EXERCISE: Find the global maximum and minimum of $f(x) = x^3 - 6x^2 + 9x - 4$, where $\frac{1}{2} \le x \le 5$.



SOLUTION: The unrestricted curve is sketched in Section 10B, and substituting the boundaries, $f(\frac{1}{2}) = -\frac{7}{8}$ and $f(5) = 16$ gives the diagram on the right.
So the global maximum is 16 when $x = 5$,
and the global minimum is -4 when $x = 3$.

WORKED EXERCISE: Find the global maximum and minimum of $y = \dfrac{x+1}{x^2+3}$ for $x \ge 0$.

SOLUTION:
$$y' = \frac{(x^2+3) - 2x(x+1)}{(x^2+3)^2}$$
$$= \frac{-(x^2+2x-3)}{(x^2+3)^2}$$

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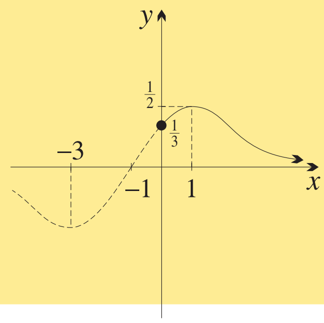
$$= \frac{-(x+3)(x-1)}{(x^2+3)^2},$$

so y' has a zero at $x = 1$ and no discontinuities:

x	0	1	2
y'	$\frac{1}{3}$	0	$-\frac{5}{49}$
	/	—	\

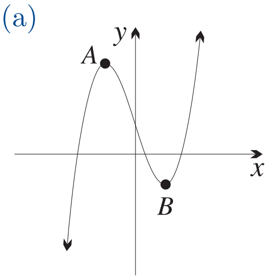


So $(1, \frac{1}{2})$ is a maximum turning point.
When $x = 0$, $y = \frac{1}{3}$, and $y \to 0$ as $x \to \infty$,
so $(1, \frac{1}{2})$ is a global maximum, but there is no global minimum.

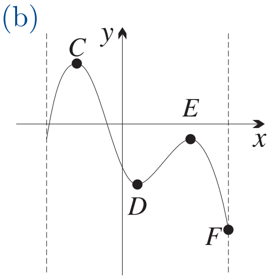
Exercise 10G

1. In each diagram below, name the points that are: (i) absolute maxima, (ii) absolute minima, (iii) relative maxima, (iv) relative minima, (v) horizontal points of inflexion.

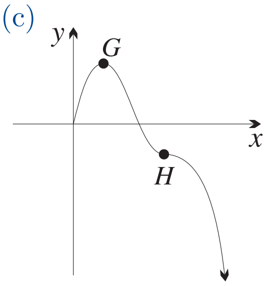
(a)



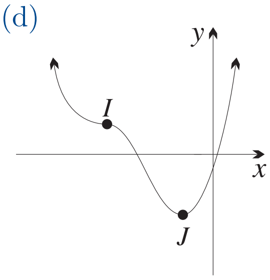
(b)



(c)



(d)



2. Sketch each of the given functions and state the global minimum and global maximum of each function in the specified domain:

(a) $y = x^2$, $-2 \le x \le 2$

(b) $y = 5 - x$, $0 \le x \le 3$

(c) $y = \sqrt{16 - x^2}$, $-4 \le x \le 4$

(d) $y = |x|$, $-5 \le x \le 1$

(e) $y = \sqrt{x}$, $0 \le x \le 8$

(f) $y = \dfrac{1}{x}$, $-4 \le x \le -1$

(g) $y = x^{\frac{1}{3}}$, $1 \le x \le 8$

(h) $y = \begin{cases} -1, & \text{for } x < -2, \\ x+1, & \text{for } -2 \le x < 1, \\ 2, & \text{for } x \ge 1. \end{cases}$

DEVELOPMENT

3. Sketch graphs of the following functions, indicating any stationary points. Determine the absolute minimum and maximum points for each function in the specified domain.

(a) $y = 7$, $-1 \le x \le 4$

(b) $y = \dfrac{1}{x^2}$, $3 \le x \le 4$

(c) $y = x^2 - 4x + 3$, $0 \le x \le 5$

(d) $y = x^3 - 3x^2 + 5$, $-3 \le x \le 2$

(e) $y = 3x^3 - x + 2$, $-1 \le x \le 1$

(f) $y = x^3 - 6x^2 + 12x$, $0 \le x \le 3$

4. Find (i) the local maxima or minima and (ii) the global maximum and minimum of the function $y = x^4 - 8x^2 + 11$ on each of the following domains:

(a) $1 \le x \le 3$ (b) $-4 \le x \le 1$ (c) $-1 \le x \le 0$

5. Use the complete curve sketching menu to sketch the following functions. State the absolute minimum and maximum values of each function on the domain $-2 \le x \le 1$.

(a) $y = x^2 + 1$ (b) $y = \sqrt{x^2 + 1}$ (c) $y = \dfrac{1}{\sqrt{x^2+1}}$ (d) $y = \dfrac{x}{\sqrt{x^2+1}}$

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EXTENSION

6. We have assumed without comment that a function that is continuous on a closed interval has a global maximum in that interval, and that the function reaches that global maximum at some value in the interval (and similarly for minima). Proving this obvious-looking result is beyond the course, but draw sketches, with and without asymptotes, to show that the result is false when either the function is not continuous or the interval is not closed.

7. Consider the even function $y = \sin \frac{360^\circ}{x}$ (and try graphing it on a machine).
(a) How many zeroes are there in the closed intervals $1 \le x \le 10$, $0{\cdot}1 \le x \le 1$ and $0{\cdot}01 \le x \le 0{\cdot}1$? (b) How many zeroes are there in the open interval $0 < x < 1$?
(c) Does the function have a limit as $x \to 0^+$ or as $x \to \infty$?
(d) Does the function have a global maximum or minimum?

10 H Applications of Maximisation and Minimisation

The practical applications of maximisation and minimisation should be obvious — for example, maximise the volume of a box built from a rectangular sheet of cardboard, minimise the fuel used in a flight, maximise the profits from manufacturing and selling an article, minimise the metal used in a can of soft drink.

Maximisation and Minimisation Using Calculus: Many of these problems involve only quadratic functions, and so can be solved by the methods of Chapter Eight without any appeal to calculus. The use of the derivative to find the global maximum and minimum, however, applies to any differentiable function (and may be more convenient for some quadratics).

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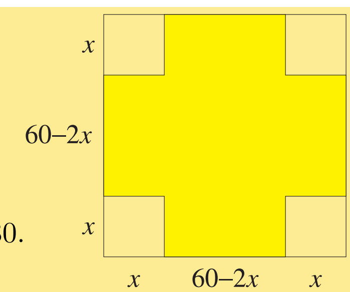
MAXIMISATION AND MINIMISATION PROBLEMS: After drawing a diagram:
1. Introduce the two variables from which the function is to be formed.
   'Let y (or whatever) be the quantity that is to be maximised,
   and let x (or whatever) be the quantity that can be varied.'
2. Form an equation in the two variables, noting any restrictions.
3. Find the global maximum or minimum.
4. Write a careful conclusion.

NOTE: A claim that a stationary point is a maximum or minimum must be justified by a proper analysis of the nature of the stationary point.

WORKED EXERCISE: An open rectangular box is to be made by cutting square corners out of a square piece of cardboard 60 cm × 60 cm and folding up the sides. What is the maximum volume of the box, and what are its dimensions then? What dimensions give the minimum volume?



SOLUTION: Let V be the volume of the box,
and let x be the side lengths of the squares.
Then the box is x cm high,
with base a square of side length $60 - 2x$,
so $V = x(60 - 2x)^2$,
$= 3600x - 240x^2 + 4x^3$, where $0 < x < 30$.
Differentiating, $V' = 3600 - 480x + 12x^2$

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$= 12(x - 30)(x - 10),$

so V' has zeroes at $x = 10$ and $x = 30$, and no discontinuities.

Furthermore, $V'' = -480 + 24x,$

so $V''(10) = -240 < 0$ and $V''(30) = 240 > 0$.

Hence $(10, 16\,000)$ is a maximum turning point,

and the maximum volume is $16\,000\,\text{cm}^3$ when the box is $10\,\text{cm} \times 40\,\text{cm} \times 40\,\text{cm}$.

Also, $V = 0$ at both boundaries $x = 0$ and $x = 30$, so the minimum value is zero when the dimensions are $60\,\text{cm} \times 60\,\text{cm} \times 0\,\text{cm}$ or $0\,\text{cm} \times 0\,\text{cm} \times 30\,\text{cm}$.

Introducing Extra Pronumerals: Some problems require other variable or constant pronumerals to be added during the working, but these other pronumerals must not be confused with the two variables from which the function is to be formed.

WORKED EXERCISE: The cost C, in dollars per hour, of running a boat depends on the speed v km/hr of the boat according to the formula $C = 500 + 40v + 5v^2$. On a trip from Port A to Port B, what speed will minimise the total cost of the trip?

SOLUTION: [NOTE: The introduction of the constant D is the key step here.]

Let T be the total cost of the trip. We seek T as a function of v.

Let D be the distance between the two ports,

then since time $= \dfrac{\text{distance}}{\text{speed}}$, the time for the trip is $\dfrac{D}{v}$,

so the total cost is
$$
\begin{aligned}
T &= (\text{cost per hour}) \times (\text{time for the trip}) \\
&= C \times \frac{D}{v} \\
T &= \frac{500\,D}{v} + 40D + 5Dv, \quad \text{where } v > 0.
\end{aligned}
$$

Differentiating,
$$
\begin{aligned}
\frac{dT}{dv} &= -\frac{500\,D}{v^2} + 5D, \quad \text{since } D \text{ is a constant,} \\
&= \frac{5D}{v^2}(-100 + v^2) \\
&= \frac{5D}{v^2}(v - 10)(v + 10),
\end{aligned}
$$

so $\dfrac{dT}{dv}$ has a zero at $v = 10$, and no discontinuities for $v > 0$.

Differentiating again, $\dfrac{d^2T}{dv^2} = \dfrac{1000\,D}{v^3}$, which is positive for $v = 10$,

so $v = 10$ gives a minimum turning point.

When $v = 10$, $C = 500 + 400 + 500 = 1400$ dollars per hour,

so a speed of 10 km/hr will minimise the cost of the trip.

Exercise 10H

NOTE: You must always prove that any stationary point is a maximum or minimum, either by creating a table of values of the derivative, or by substituting into the second derivative, or perhaps in some other way. It is never acceptable to assume this from the wording of a question.

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1. (a) If $x + y = 8$, express $P = x^2 + y^2$ in terms of x only.

(b) Find $\dfrac{dP}{dx}$ and hence find the value of x for which P obtains its minimum value.

(c) Calculate the minimum value of P (and prove that it is a minimum).

2. (a) If $2x + y = 11$, express $P = xy$ in terms of x only.

(b) Find $\dfrac{dP}{dx}$ and hence the value of x for which P obtains its maximum value.

(c) Calculate the maximum value of P (and prove that it is a maximum).

3. At time t seconds, a particle has height $h = 3 + t - 2t^2$ metres.

(a) Find dh/dt and show that the maximum height occurs after 0·25 seconds.

(b) Find the maximum height.

4. (a) A rectangle has a constant perimeter of 20 cm. If the length of the rectangle is x cm, show that it must have width $(10 - x)$ cm, and hence that its area is $A = 10x - x^2$.

(b) Find dA/dx, and hence find the value of x for which A is maximum.

(c) Hence find the maximum area.

5. A landscaper is constructing a rectangular garden bed. Three of the sides are to be fenced using 40 metres of fencing, while an existing wall will form the fourth side of the rectangle.

(a) If x is the length of the side opposite the wall, show that the remaining two sides each have length $(20 - \frac{1}{2}x)$ m.

(b) Show that the area is $A = 20x - \frac{1}{2}x^2$.

(c) Find dA/dx and hence the value of x for which A obtains its maximum value.

(d) Find the maximum area of the garden bed.

6. The total cost of producing x telescopes per day is given by $C = (\frac{1}{5}x^2 + 15x + 10)$ dollars, and each telescope is sold for a price of $(47 - \frac{1}{3}x)$ dollars.

(a) Find an expression for the revenue R raised from the sale of x telescopes.

(b) Find an expression for the daily profit P made if x telescopes are sold $(P = R - C)$.

(c) How many telescopes should be made daily in order to maximise the profit?

DEVELOPMENT

7. The sum of two positive numbers is 40. [HINT: Let the numbers be x and $40 - x$.] Find the numbers if:

(a) their product is a maximum,

(b) the sum of their squares is a minimum, [HINT: Let $S = x^2 + (40 - x)^2$.]

(c) the product of the cube of one number and the square of the other number is a maximum. [HINT: Let $P = x^3(40 - x)^2$, and show that $P' = 5x^2(x - 24)(x - 40)$.]

8. A rectangle has a constant area of 36 cm^2.

(a) If the length of the rectangle is x, show that its perimeter is $P = 2x + \dfrac{72}{x}$.

(b) Show that $\dfrac{dP}{dx} = \dfrac{2(x-6)(x+6)}{x^2}$ and hence find the minimum possible perimeter.

9. A piece of wire of length 10 metres is cut into two pieces and used to form two squares.

(a) If one piece of wire has length x metres, find the side length of each square.

(b) Show that the combined area of the squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.

(c) Find dA/dx and hence find the value of x that makes A a minimum.

(d) Find the least possible value of the combined areas.

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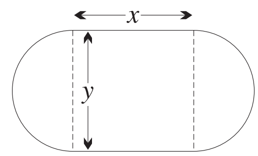
10. A running track of length 400 metres is designed using two sides of a rectangle and two semicircles, as shown. The rectangle has length x metres and the semicircles each have diameter y metres.



(a) Show that $x = \frac{1}{2}(400 - \pi y)$.

(b) The region inside the track will be used for field events. Show that its area is $A = \frac{1}{4}(800y - \pi y^2)$.

(c) Hence find the maximum area that may be enclosed.

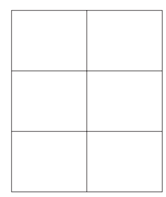
11. A box has a square base and no lid. Let the square base have length x cm and the box have height h cm.

(a) Show that the surface area of the box is given by $S = x^2 + 4xh$.

(b) If the box has a volume of $32\,\text{cm}^3$, show that $h = \dfrac{32}{x^2}$ and hence that $S = x^2 + \dfrac{128}{x}$.

(c) Find dS/dx, and hence find the dimensions of the box that minimise its surface area.

12. A window frame consisting of 6 equal rectangles is shown on the right.

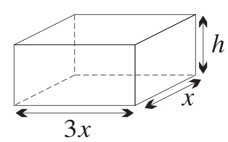


(a) Let the entire frame have height h metres and width y metres. If 12 metres of frame is available, show that $y = \frac{1}{4}(12 - 3h)$.

(b) Show that the area of the window is $A = 3h - \frac{3}{4}h^2$.

(c) Find $\dfrac{dA}{dh}$ and hence find the dimensions of the frame so that the area of the window is maximum.

13. The steel frame of a rectangular prism, as illustrated in the diagram, is three times as long as it is wide. The prism has a volume of $4374\,\text{m}^3$. Find the dimensions of the frame so that the minimum amount of steel is used.



14. A transport company runs a truck from Hobart to Launceston, a distance of 250 km, at a constant speed of v km/hr. For a given speed v, the cost per hour is $6400 + v^2$ cents.

(a) If the trip costs C cents, show that $C = 250\left(\dfrac{6400}{v} + v\right)$.

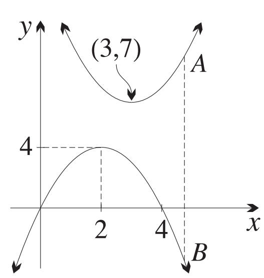
(b) Find the speed for which the cost of the journey is minimised.

(c) Find the minimum cost of the journey.

15. A cardboard box is to have square ends and a volume of $768\,\text{cm}^3$. It is to be be sealed using two pieces of tape, one passing entirely around the length and width of the box and the other passing entirely around the height and width of the box. Find the dimensions of the box so that the least amount of tape is used.

16. Two sides of a rectangle lie along the x and y axes. The vertex opposite the origin is in the first quadrant and lies on $3x + 2y = 6$. What is the maximum area of the rectangle?

17. The diagram to the right shows a point A on the curve $y = (x - 3)^2 + 7$, and a point B with the same x-coordinate as A on the curve $y = x(4 - x)$. Find an expression for the length of AB, and determine its minimum length.



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18. A point A lies on the curve $y = x(5 - x)$, and a point B with the same x-coordinate as A lies on the curve $y = x(x - 3)$. Show this information on a diagram, then find an expression for the length of AB, and determine the maximum length if $0 \le x \le 4$.

19. (a) Sketch the parabola $y = 4 - x^2$ and the tangent at $P(a, 4 - a^2)$ in the first quadrant. (b) Find the equation of the tangent at P. (c) Hence find the value of a for which the area of the triangle formed by the tangent and the coordinate axes will be minimum.

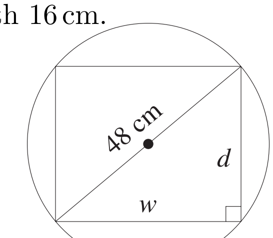
20. The point $P(x, y)$ lies on the parabola $y = x^2$.

(a) Show that the distance from P to the line $x - y - 1 = 0$ is $\frac{1}{2}\sqrt{2}(x^2 - x + 1)$.

(b) Hence find P for which the distance is minimum.

21. Find the maximum area of a right triangle with hypotenuse of length 16 cm.

22. Engineers have determined that the strength s of a rectangular beam varies as the product of the width w and the square of the depth d of the beam, that is, $s = kwd^2$ for some constant k. Find the dimensions of the strongest rectangular beam that can be cut from a cylindrical log with diameter 48 cm.

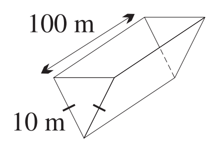


23. (a) Draw a diagram showing the region enclosed between the parabola $y^2 = 4ax$ and its latus rectum $x = a$.

(b) Find the dimensions of the rectangle of maximum area that can be inscribed in this region.

24. The point A lies on the positive half of the x-axis, the point B lies on the positive half of the y-axis, and the interval AB passes through the point $P(5, 3)$. Find the coordinates of A and B so that $\triangle AOB$ has minimum area.

25. The ends of a 100 metre long trough are isosceles triangles whose equal sides have length 10 metres. Find the height of the trough in order that its volume is maximised.



26. A man in a rowing boat is presently 6 km from the nearest point A on the shore. He wants to reach as soon as possible a point B that is a further 20 km down the shore from A. If he can row at 8 km/hr and run at 10 km/hr, how far from A should he land?

27. A page of a book is to have 80 cm^2 of printed material. There is to be a 2 cm margin at the top and bottom and a 1 cm margin on each side of the page. What should be the dimensions of the page in order to use the least amount of paper?

28. (a) An open rectangular box is to be formed by cutting squares of side length x cm from the corners of a rectangular sheet of metal that has length 40 cm and width 15 cm. Find the value of x in order to maximise the volume of the box.

(b) An open rectangular box is to be formed by cutting equal squares from a sheet of tin which has dimensions a metres by b metres. Find the area of the squares to be removed if the box is to have maximum volume. Check your answer to part (a).

EXTENSION

29. If an object is placed u cm in front of a lens of focal length f cm, then the image appears v cm behind the lens, where $\dfrac{1}{v} + \dfrac{1}{u} = \dfrac{1}{f}$. Show that the minimum distance between the image and the object is $4f$ cm.

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30. The method of least squares can be used to find the line of best fit for a series of points obtained experimentally. The line of best fit through the points (x_1, y_1), (x_2, y_2), …, (x_n, y_n) is obtained by selecting the linear function $f(x)$ that minimises the sum $\big(y_1 - f(x_1)\big)^2 + \big(y_2 - f(x_2)\big)^2 + \cdots + \big(y_n - f(x_n)\big)^2$. Find the value of m for which the equation $y = mx + 2$ best fits the points $(1, 6)$, $(2, 7)$, $(3, 13)$ and $(5, 15)$.

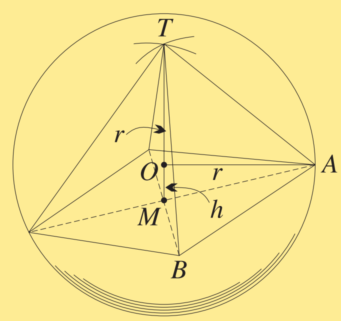
31. Snell's law states that light travels through a homogeneous medium in a straight line at a constant velocity dependent upon the medium. Let the velocity of light in air be v_1 and the velocity of light in water be v_2. Show that light will travel from a point in air to a point in water in the shortest possible time if $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$, where the light ray makes angles of θ_1 and θ_2 in air and water respectively with the normal to the surface.
NOTE: This question can be redone after Chapter Fourteen is completed.

10 I Maximisation and Minimisation in Geometry

Geometrical problems provided the classic situations where maximisation and minimisation problems were first clearly stated and solved. In many of these problems, the answer and the solution both have considerable elegance and clarity, and they make the effectiveness of calculus very obvious.

WORKED EXERCISE: [A difficult example] A square pyramid is inscribed in a sphere (the word 'inscribed' means that all five vertices of the pyramid lie on the surface of the sphere). What is the maximum ratio of the volumes of the pyramid and the sphere, and what are the corresponding proportions of the pyramid?

SOLUTION: Let the volume of the pyramid be V.
Let the height MT of the pyramid be $r + h$, where $-r \le h \le r$,
r being the constant radius of the sphere.



Then from the diagram, $MA^2 = r^2 - h^2$ (Pythagoras' theorem)
and since $MB = MA$, $AB^2 = 2(r^2 - h^2)$.

So
$$\begin{aligned} V &= \tfrac{1}{3} \times AB^2 \times MT \\ &= \tfrac{2}{3}(r^2 - h^2)(r + h) \end{aligned}$$
and the required function is $V = \tfrac{2}{3}(r^3 + r^2h - rh^2 - h^3)$.

Differentiating,
$$\begin{aligned} \frac{dV}{dh} &= \tfrac{2}{3}(r^2 - 2rh - 3h^2) \\ &= \tfrac{2}{3}(r - 3h)(r + h), \end{aligned}$$
so dV/dh has zeroes at $h = \frac{1}{3}r$ or $-r$, and no discontinuities.

Also,
$$\frac{d^2V}{dh^2} = \tfrac{2}{3}(-2r - 6h),$$
which is negative for $h = \frac{1}{3}r$, giving a maximum (it is positive for $h = -r$).

So the volume is maximum when $h = \frac{1}{3}r$, that is, when $MT = \frac{4}{3}r$,
and then
$$\begin{aligned} AB^2 &= 2(r^2 - \tfrac{1}{9}r^2) \\ &= \tfrac{16}{9}r^2. \end{aligned}$$
This means that $AB = MT = \frac{4}{3}r$,
so that the height and base side length of the pyramid are equal.

Then
$$\begin{aligned} \text{ratio of volumes} &= \tfrac{1}{3} \times \tfrac{16}{9}r^2 \times \tfrac{4}{3}r : \tfrac{4}{3}\pi r^3 \\ &= 16 : 27\pi. \end{aligned}$$

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Exercise 10I

1. The sum of the height h of a cylinder and the circumference of its base is 10 metres.
(a) If r is the radius of the cylinder, show that the volume is $V = \pi r^2(10 - 2\pi r)$.
(b) Find the maximum volume of the cylinder.

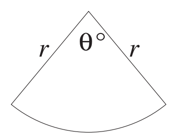
2. A closed cylindrical can is to have a surface area of $60\pi\,\text{cm}^2$.
(a) If the cylinder has height h and radius r, show that $h = \dfrac{30 - r^2}{r}$.
(b) Show that the volume of the can is given by $V = \pi r(30 - r^2)$.
(c) Find the maximum possible volume of the can in terms of π.

3. A rectangle has an area of $121\,\text{m}^2$.
(a) If the length is x metres and the perimeter is P metres, show that $P = 2\left(x + \dfrac{121}{x}\right)$.
(b) Hence show that the perimeter is a minimum when the rectangle is a square.

4. A right circular cone is to have a fixed slant height of s cm.
(a) Let the cone have height h and radius r. Explain why $s^2 = r^2 + h^2$.
(b) Show that the volume of the cone is given by $V = \frac{1}{3}\pi h(s^2 - h^2)$.
(c) Show that the volume of the cone is maximised when $h = \frac{1}{3}\sqrt{3}s$.
(d) Hence find the maximum volume of the cone.

DEVELOPMENT

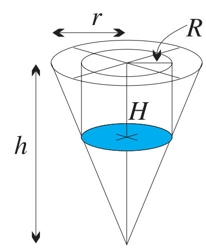
5. A piece of wire of length ℓ is bent to form a sector of a circle of radius r.
(a) If the sector subtends an angle of $\theta°$ at the centre, show that $\ell = 2r + \dfrac{\pi r\theta}{180}$, and find an expression for θ in terms of ℓ and r.
(b) Show that the area of the sector is $A = \frac{1}{2}r(\ell - 2r)$.
(c) Show that the area of the sector is maximised when $r = \frac{1}{4}\ell$.



6. A cylinder of height h metres is inscribed in a sphere of constant radius R metres.
(a) If the cylinder has radius r metres, show that $r^2 = R^2 - \frac{1}{4}h^2$.
(b) Show that the volume of the cylinder is given by $V = \frac{\pi}{4}h(4R^2 - h^2)$.
(c) Show that the volume of the cylinder is maximised when $h = \frac{2}{3}\sqrt{3}R$.
(d) Hence show that the ratio of the volume of sphere to the maximum volume of the cylinder is $\sqrt{3} : 1$.

7. The sum of the radii of two circles is constant, so that $r_1 + r_2 = k$, where k is constant.
(a) Find an expression for the sum of the areas of the circles in terms of one variable only.
(b) Hence show that the sum of the areas is least when the circles are congruent.

8. A cylinder of height H and radius R is inscribed in a cone of constant height h and constant radius r.
(a) Use similar triangles to show that $H = \dfrac{h(r - R)}{r}$.
(b) Find an expression for the volume of the cylinder in terms of the variable R.
(c) Find the maximum possible volume of the cylinder in terms of h and r.



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9. A cylinder is inscribed in a cone of height h and base radius r. Using a method similar to the previous question, find the maximum possible value of the curved surface area of the cylinder.

10. A rectangle is inscribed in a quadrant of a circle of radius r so that two of its sides are along the bounding radii of the quadrant.

(a) If the rectangle has length x and width y, show that $x^2 + y^2 = r^2$, and hence that the area of the rectangle is given by $A = y\sqrt{r^2 - y^2}$.

(b) Use the product rule and chain rule to find dA/dy.

(c) Show that the maximum area of the rectangle is $\frac{1}{2}r^2$.

11. A cylindrical can open at one end is to have a fixed outside surface area S.

(a) Show that if the can has height h and radius r, then $h = \dfrac{S - \pi r^2}{2\pi r}$.

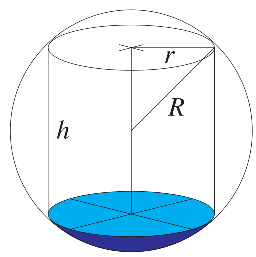
(b) Find an expression for the volume of the cylinder in terms of r, and hence show that the maximum possible volume is attained when the height of the can equals its radius.

12. A right triangle has base 60 cm and height 80 cm. A rectangle is inscribed in the triangle, so that one of its sides lies along the base of the triangle.

(a) Let the rectangle have length y cm and height x cm. Then, by using similar triangles, show that $y = \frac{3}{4}(80 - x)$.

(b) Hence find the dimensions of the rectangle of maximum area.

13. A cylinder, open at both ends, is inscribed in a sphere of constant radius R.

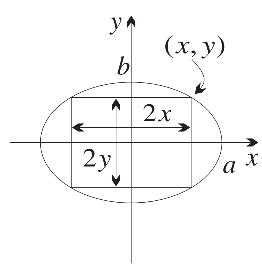


(a) Let the cylinder have height h and radius r as illustrated in the diagram. Show that $h = 2\sqrt{R^2 - r^2}$.

(b) Show that the surface area of the cylinder is given by $S = 4\pi r\sqrt{R^2 - r^2}$.

(c) Hence find the maximum surface area of the cylinder in terms of R.

14. (a) An ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Make y^2 the subject, and hence write down the equation of the upper half of the ellipse.



(b) A rectangle with vertical side of length $2y$ and horizontal side of length $2x$ is inscribed in the ellipse as shown. Find, in terms of x, an expression for the area of the rectangle.

(c) Hence find maximum area of the rectangle, and the value of x for which this occurs.

15. Show that a rectangle with fixed perimeter has its shortest diagonal when it is a square.

16. Show that a closed cylindrical can of fixed volume will have minimum surface area when its height is equal in length to its diameter.

17. A cone of height h cm is inscribed in a sphere of constant radius R cm. Find the ratio $h : R$ when the volume of the cone is maximised.

18. (a) The perimeter of an isosceles triangle is 12 cm. Find its maximum area.

(b) Prove the general result that for all isosceles triangles of constant perimeter, the one with maximum area is equilateral.

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19. Find the maximum area of an isosceles triangle in which the equal sides have a fixed length of a units. NOTE: The equilateral triangle is not the triangle of maximum area.

20. A parallelogram is inscribed in a triangle so that they have one vertex in common. The other vertices of the parallelogram lie on the three different sides of the triangle. Show that the maximum area of the parallelogram is half that of the triangle.

21. An isosceles triangle is to circumscribe a circle of constant radius r. Find the minimum area of such a triangle.

22. A cone with semi-vertical angle 45° is to circumscribe a sphere of radius 12 cm. Is there any maximisation problem here, and what can be said about the height and radius of the cone?

EXTENSION

23. A square of side $2x$ and a circle of variable radius r, where $r < x$, have a common centre. In each corner of the square, a circle is constructed so that it touches two sides of the square and the centre circle. Find the value of r that minimises the sum of the areas of the five circles.

10 J Primitive Functions

This section reverses the process of differentiation, and asks what we can say about a function if we know its derivative. The discussion of these questions here will be needed for the methods of integration established in the next chapter.

Functions with the Same Derivative: A great many different functions can all have the same derivative. For example, all these functions have the same derivative $2x$:

$$x^2, \qquad x^2 + 3, \qquad x^2 - 2, \qquad x^2 + 4\tfrac{1}{2}, \qquad x^2 - \pi.$$

These functions are all the same apart from a constant term. This is true generally — any two functions with the same derivative differ only by a constant.

16 THEOREM:
(a) If a function $f(x)$ has derivative zero in an interval $a < x < b$, then $f(x)$ is a constant function in $a < x < b$.
(b) If $f'(x) = g'(x)$ for all x in an interval $a < x < b$, then $f(x)$ and $g(x)$ differ by a constant in $a < x < b$.

PROOF:

(a) Because the derivative is zero, the gradient of the curve must be zero throughout the interval. The curve must therefore be a horizontal straight line, and $f(x)$ is a constant function.

(b) Take the difference between $f(x)$ and $g(x)$ and apply part (a).

Let $\quad h(x) = f(x) - g(x).$

Then $\quad h'(x) = f'(x) - g'(x)$

$\qquad\qquad = 0, \;$ for all x in the interval $a < x < b$.

Hence by part (a), $h(x) = C$, where C is a constant,

so $\quad f(x) - g(x) = C$, as required.

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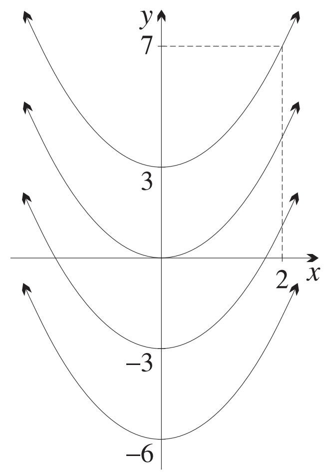
The Family of Curves with the Same Derivative: Continuing with the very first example, the various functions whose derivatives are $2x$ must all be of the form

$$f(x) = x^2 + C, \text{ where } C \text{ is a constant.}$$

By taking different values of the constant C, these functions form an infinite family of curves, each consisting of the parabola $y = x^2$ translated upwards or downwards.



Boundary Conditions: If we know also that the curve must pass through a particular point, say $(2, 7)$, then we can evaluate the constant C by substituting the point into $f(x) = x^2 + C$:

$$7 = 4 + C.$$

So $C = 3$ and hence $f(x) = x^2 + 3$ — in place of the infinite family of functions, there is now a single function. Such an extra condition is called a *boundary condition*. (It is also called an *initial condition* if it involves the value of y when $x = 0$, particularly when x is time.)

Primitives: We need a suitable name for the result of this reverse process:

17 DEFINITION: A function $F(x)$ is called a *primitive* of $f(x)$ if the derivative of $F(x)$ is $f(x)$, that is, if $F'(x) = f(x)$.

A function always has infinitely many different primitives, because if $F(x)$ is any primitive of $f(x)$, then $F(x) + C$ is also a primitive of $f(x)$. To give another example, these functions are all primitives of $x^2 + 1$:

$$\tfrac{1}{3}x^3 + x, \qquad \tfrac{1}{3}x^3 + x + 7, \qquad \tfrac{1}{3}x^3 + x - 13, \qquad \tfrac{1}{3}x^3 + x + 4\pi,$$

and the general primitive of $x^2 + 1$ is $\frac{1}{3}x^3 + x + C$, where C is a constant.

A Rule for Finding Primitives: We have seen that a primitive of x is $\frac{1}{2}x^2$, and a primitive of x^2 is $\frac{1}{3}x^3$. Reversing the formula $\dfrac{d}{dx}(x^{n+1}) = (n+1)x^n$ gives the general rule:

18 FINDING PRIMITIVES: Suppose that $n \neq -1$.

If $\dfrac{dy}{dx} = x^n$, then $y = \dfrac{x^{n+1}}{n+1} + C$, for some constant C.

'Increase the index by 1 and divide by the new index.'

WORKED EXERCISE: Find primitives of: (a) $x^3 + x^2 + x + 1$ (b) $5x^3 + 7$

SOLUTION:

(a) $\dfrac{dy}{dx} = x^3 + x^2 + x + 1,$

$y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C,$
where C is a constant.

(b) $f'(x) = 5x^3 + 7,$

$f(x) = \frac{5}{4}x^4 + 7x + C,$
where C is a constant.

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WORKED EXERCISE: Find primitives of: (a) $\dfrac{1}{x^2}$ (b) $\sqrt{x}$

SOLUTION:

(a) $$\begin{aligned} f'(x) &= \frac{1}{x^2} \\ &= x^{-2}, \\ f(x) &= -x^{-1} + C, \\ &\quad \text{where } C \text{ is a constant,} \\ &= -\frac{1}{x} + C. \end{aligned}$$

(b) $$\begin{aligned} \frac{dy}{dx} &= \sqrt{x} \\ &= x^{\frac{1}{2}}, \\ y &= \tfrac{2}{3}\,x^{\frac{3}{2}} + C, \\ &\quad \text{where } C \text{ is a constant.} \end{aligned}$$

Linear Extension: Reversing the formula $\dfrac{d}{dx}(ax+b)^{n+1} = a(n+1)(ax+b)^n$ gives:

19 EXTENSION TO POWERS OF LINEAR FUNCTIONS: Suppose that $n \neq -1$.
If $\dfrac{dy}{dx} = (ax+b)^n$, then $y = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$, for some constant C.
'Increase the index by 1 and divide by the new index and by the coefficient of x.'

WORKED EXERCISE: Find primitives of:

(a) $(3x+1)^4$ (b) $(1-3x)^6$ (c) $\dfrac{1}{(x+1)^2}$ (d) $\sqrt{x+1}$

SOLUTION:

(a) $$\begin{aligned} \frac{dy}{dx} &= (3x+1)^4, \\ y &= \tfrac{1}{15}(3x+1)^5 + C, \\ &\quad \text{where } C \text{ is a constant.} \end{aligned}$$

(b) $$\begin{aligned} \frac{dy}{dx} &= (1-3x)^6, \\ y &= -\tfrac{1}{21}(1-3x)^7 + C, \\ &\quad \text{where } C \text{ is a constant.} \end{aligned}$$

(c) $$\begin{aligned} \frac{dy}{dx} &= (x+1)^{-2}, \\ y &= -(x+1)^{-1} + C, \\ &\quad \text{where } C \text{ is a constant,} \\ &= -\frac{1}{x+1} + C. \end{aligned}$$

(d) $$\begin{aligned} \frac{dy}{dx} &= (x+1)^{\frac{1}{2}}, \\ y &= \tfrac{2}{3}(x+1)^{\frac{3}{2}} + C, \\ &\quad \text{where } C \text{ is a constant.} \end{aligned}$$

Finding the Primitive, Given a Boundary Condition: Often the derivative and a particular value of a function are known. In this case, first find the general primitive, then substitute the known value of the function to work out the constant.

WORKED EXERCISE: Given that $\dfrac{dy}{dx} = 6x^2 + 1$, and that $y = 12$ when $x = 2$, find y as a function of x.

SOLUTION: Since $\dfrac{dy}{dx} = 6x^2 + 1$,

$y = 2x^3 + x + C$, for some constant C.

Substituting $x = 2$ and $y = 12$, $12 = 16 + 2 + C$,

so $C = -6$, and hence $y = 2x^3 + x - 6$.

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WORKED EXERCISE: Given that $f''(x) = (2x-1)^2$, and $f(0) = f(1) = 0$, find $f(2)$.

SOLUTION: Since $f''(x) = (2x-1)^2$,

$f'(x) = \frac{1}{6}(2x-1)^3 + C$, for some constant C

and $f(x) = \frac{1}{48}(2x-1)^4 + Cx + D$, for some constant D.

Since $f(0) = 0$, $0 = \frac{1}{48} + D$

$D = -\frac{1}{48}$.

Since $f(1) = 0$, $0 = \frac{1}{48} + C - \frac{1}{48}$.

So $C = 0$, and hence $f(2) = \frac{81}{48} - \frac{1}{48}$

$= \frac{5}{3}$.

Primitives of Discontinuous Functions: [This is a subtle point about the primitives of discontinuous functions, which is really beyond the standard course.] When a function is not continuous at some point, the constant may take different values in different parts of the domain. For example, the function $1/x^2$ is not defined at $x = 0$, and its domain disconnects into the two pieces $x > 0$ and $x < 0$. Some primitives of $1/x^2$ are

$$-\frac{1}{x} \quad \text{and} \quad -\frac{1}{x} + 5 \quad \text{and} \quad f(x) = \begin{cases} -\dfrac{1}{x} + 4, & \text{for } x < 0, \\ -\dfrac{1}{x} - 7, & \text{for } x > 0. \end{cases}$$

and the general primitive is $F(x) = \begin{cases} -\dfrac{1}{x} + A, & \text{for } x < 0, \\ -\dfrac{1}{x} + B, & \text{for } x > 0. \end{cases}$ where A and B are unrelated constants. In most applications, however, only one branch of the function has any physical significance.

Exercise 10J

1. Find primitives of each of the following (where a and b are constants):

(a) x^6 (b) $3x$ (c) 5 (d) $5x^9$ (e) $21x^6$ (f) $\frac{1}{3}x^{12}$ (g) 0 (h) $2x^2 + 5x^7$ (i) $3x^2 - 4x^3 - 5x^4$ (j) $ax^3 + bx^2$ (k) x^a (l) $ax^a + bx^b$

2. Find primitive functions of the following by first expanding the products:

(a) $x(x-3)$ (b) $(x^2+1)^2$ (c) $(3x-1)(x+4)$ (d) $x^2(5x^3 - 4x)$ (e) $(2x^2+1)^2$ (f) $x(ax-3)^2$

3. Write these functions using negative powers of x. Find the primitive functions, giving your answers in fractional form without negative indices.

(a) $\dfrac{1}{x^2}$ (b) $\dfrac{2}{x^3}$ (c) $\dfrac{1}{3x^2}$ (d) $-\dfrac{2}{5x^4}$ (e) $\dfrac{1}{x^2} - \dfrac{1}{x^3}$ (f) $\dfrac{a}{bx^2}$ (g) $\dfrac{1}{x^a}$ (h) $\dfrac{x^a + x^b}{x^a}$

4. Write these functions with fractional indices, and hence find the primitive functions:

(a) $\sqrt{x}$ (b) $\dfrac{1}{\sqrt{x}}$ (c) $\sqrt[3]{x}$ (d) $\dfrac{2}{3\sqrt{x}}$ (e) $\sqrt[5]{x^3}$

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5. Find y as a function of x if:

(a) $\dfrac{dy}{dx} = 2x + 3$, and $y = 8$ when $x = 1$,

(b) $\dfrac{dy}{dx} = 9x^2 + 4$, and $y = 1$ when $x = 0$,

(c) $\dfrac{dy}{dx} = \sqrt{x}$, and $y = 2$ when $x = 9$.

6. Box 18 of the text states the rule that the primitive of x^n is $\dfrac{x^{n+1}}{n+1}$, provided that $n \neq -1$. Why can't this rule be used when $n = -1$?

DEVELOPMENT

7. Find each family of curves whose gradient function is given below. Then sketch the family, and find the member of the family passing through $A(1, 2)$.

(a) $\dfrac{dy}{dx} = -4x$ (b) $\dfrac{dy}{dx} = 3$ (c) $\dfrac{dy}{dx} = 3x^2$ (d) $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

8. Find primitive functions of each of the following by recalling that if $y' = (ax + b)^n$, then $y = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$.

(a) $(x + 1)^3$ (c) $(3x - 4)^6$ (e) $(ax - b)^5$

(b) $\dfrac{1}{(x-2)^4}$ (d) $(1 - 7x)^3$ (f) $\dfrac{2}{(1-9x)^{10}}$

9. Find primitive functions of each of the following:

(a) $\sqrt{x+1}$ (c) $\sqrt{2x-7}$ (e) $\sqrt{ax+b}$

(b) $\sqrt{1-x}$ (d) $\dfrac{1}{\sqrt{2-3x}}$ (f) $\dfrac{3}{2\sqrt{4x-1}}$

10. (a) Find y if $y' = (2x + 1)^3$, and $y = -1$ when $x = 0$.

(b) Find y if $y'' = 6x + 4$, and when $x = 1$, $y' = 2$ and $y = 4$.

(c) Find y if $y'' = \sqrt{3-x}$, and when $x = -1$, $y' = \frac{16}{3}$ and $y = 14\frac{8}{15}$.

11. Find the primitive functions of each of the following:

(a) $x^a x^b$ (c) x^{ab} (e) $x\sqrt{x}$ (g) $\sqrt{x}(x + 1)$

(b) $\dfrac{x^a}{x^b}$ (d) $ax^b + bx^a$ (f) $\dfrac{1}{\sqrt{x}} + \sqrt{x}$ (h) $\dfrac{\sqrt{x}-1}{2\sqrt{x}}$

12. (a) Find the equation of the curve through the origin whose gradient is $\dfrac{dy}{dx} = 3x^4 - x^3 + 1$.

(b) Find the curve passing through $(2, 6)$ with gradient function $\dfrac{dy}{dx} = 2 + 3x^2 - x^3$.

(c) Find the curve through the point $(\frac{1}{5}, 1)$ with gradient function $y' = (2 - 5x)^3$.

(d) A curve with gradient function $f'(x) = cx + d$ has a turning point at $(2, 0)$ and crosses the y-axis when $y = 4$. Find c and d, and hence find the equation of the curve.

13. Given that $\dfrac{dy}{dt} = 8t^3 - 6t^2 + 5$, and $y = 4$ when $t = 0$, find y when $t = 2$.

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14. (a) At any point on a curve, $\dfrac{d^2y}{dx^2} = 2x - 10$. The curve passes through the point $(3, -34)$, and at this point the tangent to the curve has a gradient of 20. Find the y-intercept.

(b) Find the curve through the points $(1, 6)$ and $(-1, 8)$ with $y'' = 8 - 6x$.

15. Water is leaking out through a hole in the bottom of a bucket at a rate $\dfrac{dv}{dt} = -8(25 - t)$, where $v\,\text{cm}^3$ is the volume of liquid in the bucket at time t seconds. Initially there was $2\frac{1}{2}$ litres of water in the bucket. Find v in terms of t, and hence find the time taken for the bucket to be emptied.

16. The price of an item on sale is being reduced at a rate given by $\dfrac{dp}{dt} = -\dfrac{30}{t^4}$, where p is the price in dollars and t is the number of days the item has been on sale. After two days the item was retailing for \$5.25. Find the price of the item after 5 days. Why will the item always retail for a price above \$4?

17. (a) The velocity $\dfrac{dx}{dt}$ (rate of change of position x at time t) of a particle is given by $\dfrac{dx}{dt} = t^2 - 3t$ cm/sec. If the particle starts 1 cm to the right of the origin, find its position after 3 seconds.

(b) The acceleration at time t of a particle travelling on the x-axis is given by $\dfrac{d^2x}{dt^2} = 2t - 5$. If the particle is initially at rest at the origin, find its position after 4 seconds.

18. A stone is thrown upwards according to the equation $\dfrac{dv}{dt} = -10$, where v is the velocity in metres per second and t is the time is seconds. Given that the stone is thrown from the top of a building 30 metres high with an initial velocity of 5 m/s, find how long it takes for the stone to hit the ground.

EXTENSION

19. The gradient function of a curve is given by $f'(x) = -\dfrac{1}{x^2}$. Find the equation of the curve, given that $f(1) = f(-1) = 2$. Sketch a graph of the function.

20. (a) Prove that for a polynomial of degree n, the $(n + 1)$th and higher derivatives vanish, but the nth does not.

(b) Prove that if the $(n + 1)$th derivative of a polynomial vanishes but the nth does not, then the polynomial has degree n.

GO Online Multiple Choice Quiz

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