CHAPTER **THREE**

Motion

Anyone watching objects in motion can see that they often make patterns with a striking simplicity and predictability. These patterns are related to the simplest objects in geometry and arithmetic. A thrown ball traces out a parabolic path. A cork bobbing in flowing water traces out a sine wave. A rolling billiard ball moves in a straight line, rebounding symmetrically off the table edge. The stars and planets move in more complicated, but highly predictable, paths across the sky. The relationship between physics and mathematics, logically and historically, begins with these and many similar observations.

Mathematics and physics, however, remain quite distinct disciplines. Physics asks questions about the nature of the world and is based on experiment, but mathematics asks questions about logic and logical structures, and proceeds by thought, imagination and argument alone, its results and methods quite independent of the nature of the world. This chapter will begin the application of mathematics to the description of a moving object. But because this is a mathematics course, our attention will not be on the nature of space and time, but on the new insights that the physical world brings to the mathematical objects already developed earlier in the course. We will be applying the well-known linear, quadratic, exponential and trigonometric functions. Our principal goal will be to produce a striking alternative interpretation of the first and second derivatives as the physical notions of velocity and acceleration so well known to our senses.

Study Notes: The first three sections set up the basic relationship between calculus and the three functions for displacement, velocity and acceleration. Simple harmonic motion is then discussed in Section 3D in terms of the time equations. Section 3E deals with situations where velocity or acceleration are known as functions of displacement rather than time, and this allows a second discussion of simple harmonic motion in Section 3F, based on its characteristic differential equation. The last two Sections 3G and 3H pass from motion in one dimension to the two-dimensional motion of a projectile, briefly introducing vectors.

Students without a good background in physics may benefit from some extra experimental work, particularly in simple harmonic motion and projectile motion, so that some of the motions described here can be observed and harmonised with the mathematical description. Although forces and their relationship with acceleration are only introduced in the 4 Unit course, some physical understanding of Newton's second law $F = m\ddot{x}$ would greatly aid understanding of what is happening.

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3 A Average Velocity and Speed

This first section sets up the mathematical description of motion in one dimension, using a function to describe the relationship between time and the position of an object in motion. Average velocity is described as the gradient of the chord on this displacement–time graph. This will lead, in the next section, to the description of instantaneous velocity as the gradient of a tangent.

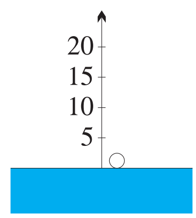
Motion in One Dimension: When a particle is moving in one dimension along a line, its position is varying over time. We can specify that position at any time t by a single number x, called the *displacement*, and the whole motion can be described by giving x as a function of the *time* t.

For example, suppose that a ball is thrown vertically upwards from ground level, and lands 4 seconds later in the same place. Its motion can be described approximately by the following equation and table of values:

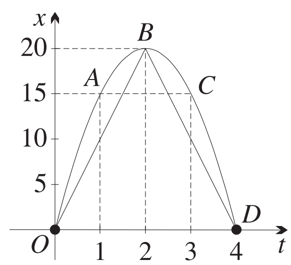
$$x = 5t(4 - t)$$

t	0	1	2	3	4
x	0	15	20	15	0

Here x is the height in metres of the ball above the ground t seconds after it is thrown. The diagram to the right shows the path of the ball up and down along the same vertical line. This vertical line has been made into a number line, with the origin at the ground, upwards as the positive direction, and metres as the units of distance. The origin of time is when the ball is thrown, and the units of time are seconds.



The displacement–time graph is sketched to the right — this graph must not be mistaken as a picture of the ball's path. The curve is a section of a parabola with vertex at $(2, 20)$, which means that the ball achieves a maximum height of 20 metres after 2 seconds. When $t = 4$, the height is zero, and the ball is back on the ground. The equation of motion therefore has quite restricted domain and range:



$$0 \le t \le 4 \qquad \text{and} \qquad 0 \le x \le 20.$$

Most equations of motion have this sort of restriction on the domain of t. In particular, it is a convention of this course that negative values of time are excluded unless the question specifically allows it.

1 MOTION IN ONE DIMENSION: Motion in one dimension is specified by giving the displacement x on the number line as a function of time t after time zero. Negative values of time are excluded unless otherwise stated.

WORKED EXERCISE: In the example above, where $x = 5t(4 - t)$, at what times is the ball $8\frac{3}{4}$ metres above the ground?

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SOLUTION: Put $x = 8\frac{3}{4}$. Then

$$8\tfrac{3}{4} = 5t(4-t)$$
$$\tfrac{35}{4} = 20t - 5t^2$$
$$\boxed{\times 4} \quad 20t^2 - 80t + 35 = 0$$
$$\boxed{\div 5} \quad 4t^2 - 16t + 7 = 0$$
$$(2t-1)(2t-7) = 0$$
$$t = \tfrac{1}{2} \text{ or } 3\tfrac{1}{2}.$$

Hence the ball is $8\frac{3}{4}$ metres high after $\frac{1}{2}$ seconds and again after $3\frac{1}{2}$ seconds.

Average Velocity: During its ascent, the ball in the example above moved 20 metres upwards. This is a change in displacement of $+20$ metres in 2 seconds, giving an average velocity of 10 metres per second. The *average velocity* is thus the gradient of the chord OB on the displacement–time graph (be careful, because there are different scales on the two axes). Hence the formula for average velocity is the familiar gradient formula.

2 AVERAGE VELOCITY: Suppose that a particle has displacement $x = x_1$ at time $t = t_1$, and displacement $x = x_2$ at time $t = t_2$. Then

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}\,.$$

That is, on the displacement–time graph,

$$\text{average velocity} = \text{gradient of the chord.}$$

During its descent, the ball moved 20 metres downwards in 2 seconds, which is a change in displacement of $0 - 20 = -20$ metres. The average velocity is therefore -10 metres per second, and is equal to the gradient of the chord BD.

WORKED EXERCISE: Find the average velocities of the ball during the first second and during the third second.

SOLUTION: Velocity during 1st second

$$= \frac{x_2 - x_1}{t_2 - t_1} = \frac{15 - 0}{1 - 0} = 15\,\text{m/s}.$$

This is the gradient of OA.

Velocity during 3rd second

$$= \frac{x_2 - x_1}{t_2 - t_1} = \frac{15 - 20}{3 - 2} = -5\,\text{m/s}.$$

This is the gradient of BC.

Distance Travelled: The change in displacement can be positive, negative or zero. *Distance*, however, is always positive or zero. In our previous example, the change in displacement during the third and fourth seconds is -20 metres, but the distance travelled is 20 metres.

The *distance travelled* by a particle also takes into account any journey and return. Thus the distance travelled by the ball is $20 + 20 = 40$ metres, even though the ball's change in displacement over the first 4 seconds is zero because the ball is back at its original position.

3 DISTANCE TRAVELLED: The *distance travelled* is always positive or zero, and takes into account any journey and return.

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Average Speed: The *average speed* is the distance travelled divided by the time taken. Speed, unlike velocity, can never be negative.

4 AVERAGE SPEED: $\text{average speed} = \dfrac{\text{distance travelled}}{\text{time taken}}$

During the 4 seconds of its flight, the change in displacement of the ball is zero, but the distance travelled is 40 metres, so

$$\text{average velocity} = \frac{0}{4} = 0\,\text{m/s}, \qquad \text{average speed} = \frac{40}{4} = 10\,\text{m/s}.$$

WORKED EXERCISE: Find the average velocity and average speed of the ball:

(a) during the fourth second, (b) during the last three seconds.

SOLUTION:

(a) During the fourth second,
change in displacement $= -15$ metres,
so average velocity $= -15\,\text{m/s}$.
Distance travelled $= 15$ metres,
so average speed $= 15\,\text{m/s}$.

(b) From $t = 1$ to $t = 4$,
change in displacement $= -15$ metres,
so average velocity $= -5\,\text{m/s}$.
Distance travelled $= 25$ metres,
so average speed $= 8\frac{1}{3}\,\text{m/s}$.

Exercise 3A

1. A particle moves according to the equation $x = t^2 - 4$, where x is the displacement in metres from the origin O at time t seconds after time zero.

(a) Copy and complete the table to the right of values of the displacement at certain times.

t	0	1	2	3
x				

(b) Hence find the average velocity:

(i) during the first second,
(ii) during the first two seconds,
(iii) during the first three seconds,
(iv) during the third second.

(c) Sketch the displacement–time graph, and add the chords corresponding to the average velocities calculated in part (b).

2. A particle moves according to the equation $x = 2\sqrt{t}$, for $t \geq 0$, where distance is in centimetres and time is in seconds.

(a) Copy and complete the table to the right.

t					
x	0	2	4	6	8

(b) Hence find the average velocity as the particle moves:

(i) from $x = 0$ to $x = 2$,
(ii) from $x = 2$ to $x = 4$,
(iii) from $x = 4$ to $x = 6$,
(iv) from $x = 0$ to $x = 6$.

(c) Sketch the displacement–time graph, and add the chords corresponding to the average velocities calculated in part (b). What does the equality of the answers to parts (ii) and (iv) of part (b) tell you about the corresponding chords?

3. A particle moves according to the equation $x = 4t - t^2$, where distance is in metres and time is in seconds.

(a) Copy and complete the table to the right.

t	0	1	2	3	4
x					

(b) Hence find the average velocity as the particle moves:

(i) from $t = 0$ to $t = 2$, (ii) from $t = 2$ to $t = 4$, (iii) from $t = 0$ to $t = 4$.

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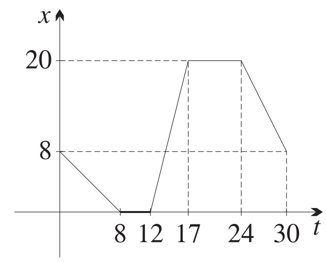
(c) Sketch the displacement–time graph, and add the chords corresponding to the average velocities calculated in part (b).

(d) Find the total distance travelled during the first 4 seconds, and the average speeds over the time intervals specified in part (b).

4. Eleni is practising reversing in her driveway. Starting 8 metres from the gate, she reverses to the gate, and pauses. Then she drives forward 20 metres, and pauses. Then she reverses to her starting point. The graph to the right shows her distance x metres from the front gate after t seconds.



(a) What is her velocity: (i) during the first 8 seconds, (ii) while she is driving forwards, (iii) while she is reversing the second time?

(b) Find the total distance travelled, and the average speed, over the 30 seconds.

(c) Find the change in displacement, and the average velocity, over the 30 seconds.

(d) Find her average speed if she had not paused at the gate and at the garage.

5. Michael the mailman rides 1 km up a hill at a constant speed of 10 km/hr, and then rides 1 km down the other side of the hill at a constant speed of 30km/hr.

(a) How many minutes does he take to ride: (i) up the hill, (ii) down the hill?

(b) Draw a displacement–time graph, with the time axis in minutes.

(c) What is his average speed over the total 2 km journey?

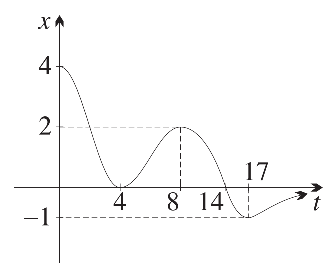
(d) What is the average of the speeds up and down the hill?

6. Sadie the snail is crawling up a 6-metre-high wall. She takes an hour to crawl up 3 metres, then falls asleep for an hour and slides down 2 metres, repeating the cycle until she reaches the top of the wall.

(a) Sketch the displacement–time graph. (b) How long does Sadie take to reach the top?

(c) What is her average speed? (d) Which places does she visit exactly three times?

7. A girl is leaning over a bridge 4 metres above the water, playing with a weight on the end of a spring. The diagram graphs the height x in metres of the weight above the water as a function of time t after she first drops it.



(a) How many times is the weight:

(i) at $x = 3$, (ii) at $x = 1$, (iii) at $x = -\frac{1}{2}$?

(b) At what times is the weight:

(i) at the water surface, (ii) above the water surface?

(c) How far above the water does it rise again after it first touches the water, and when does it reach this greatest height?

(d) What is its greatest depth under the water, and when does it occur?

(e) What happens to the weight eventually?

(f) What is its average velocity:

(i) during the first 4 seconds, (ii) from $t = 4$ to $t = 8$, (iii) from $t = 8$ to $t = 17$?

(g) What distance does it travel: (i) over the first 4 seconds, (ii) over the first 8 seconds, (iii) over the first 17 seconds, (iv) eventually?

(h) What is its average speed over the first: (i) 4, (ii) 8, (iii) 17 seconds?

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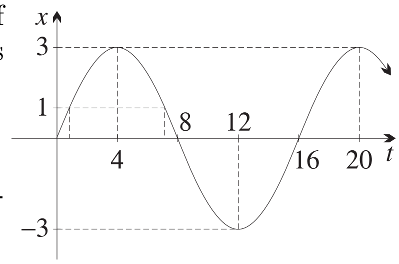
DEVELOPMENT

8. A particle is moving according to $x = 3\sin\frac{\pi}{8}t$, in units of centimetres and seconds. Its displacement–time graph is sketched opposite.



(a) Use $T = \dfrac{2\pi}{n}$ to confirm that the period is 16 seconds.
(b) Find the first two times when the displacement is maximum.
(c) When, during the first 20 seconds, is the particle on the negative side of the origin?
(d) Find the total distance travelled during the first 16 seconds, and the average speed.
(e) (i) Find, correct to three significant figures, the first two positive solutions of the trigonometric equation $\sin\frac{\pi}{8}t = \frac{1}{3}$ [HINT: Use radian mode on the calculator.]
(ii) Hence find, correct to three significant figures, the first two times when $x = 1$. Then find the total distance travelled between these two times, and the average speed during this time.

9. A particle moves according to $x = 10\cos\frac{\pi}{12}t$, in units of metres and seconds.
(a) Find the amplitude and period of the motion.
(b) Sketch the displacement–time graph over the first 60 seconds.
(c) What is the maximum distance the particle reaches from its initial position, and when, during the first minute, is it there?
(d) How far does the particle move during the first minute, and what is its average speed?
(e) When, during the first minute, is the particle 10 metres from its initial position?
(f) Use the fact that $\cos\frac{\pi}{3} = \frac{1}{2}$ to copy and complete this table of values:

t	4	8	12	16	20	24
x						

(g) From the table, find the average velocity during the first 4 seconds, the second 4 seconds, and the third 4 seconds.
(h) Use the graph and the table of values to find when the particle is more than 15 metres from its initial position.

10. A particle is moving on a horizontal number line according to the equation $x = 4\sin\frac{\pi}{6}t$, in units of metres and seconds.
(a) Sketch the displacement–time graph.
(b) How many times does the particle return to the origin by the end of the first minute?
(c) Find at what times it visits $x = 4$ during the first minute.
(d) Find how far it travels during the first 12 seconds, and its average speed in that time.
(e) Find the values of x when $t = 0$, $t = 1$ and $t = 3$. Hence show that its average speed during the first second is twice its average speed during the next 2 seconds.

11. A balloon rises so that its height h in metres after t minutes is $h = 8000(1 - e^{-0{\cdot}06t})$.
(a) What height does it start from, and what happens to the height as $t \to \infty$?
(b) Copy and complete the table to the right, correct to the nearest metre.

t	0	10	20	30
h				

(c) Sketch the displacement–time graph of the motion.

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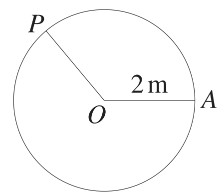
(d) Find the balloon's average velocity during the first 10 minutes, the second 10 minutes and the third 10 minutes, correct to the nearest metre per minute.

(e) Show that the solution of $1 - e^{-0.06t} = 0{\cdot}99$ is $t = \dfrac{\log 100}{0{\cdot}06}$.

(f) Hence find how long (correct to the nearest minute) the balloon takes to reach 99% of its final height.

12. A toy train is travelling anticlockwise on a circular track of radius 2 metres and centre O. At time zero the train is at a point A, and t seconds later it is at the point P distant $x = 4\log(t+1)$ metres around the track.



(a) Sketch the graph of x as a function of t.

(b) Find, when $t = 2$, the position of the point P, the average speed from A to P, the size of $\angle AOP$ and the length of the chord AP (in exact form, then correct to four significant figures).

(c) More generally, find $\angle AOP$ as a function of t. Hence find, in exact form, and then correct to the nearest second, the first three times when the train returns to A.

(d) Explain whether the train will return to A finitely or infinitely many times.

13. Two engines, Thomas and Henry, move on close parallel tracks. They start at the origin, and are together again at time $t = e - 1$. Thomas' displacement–time equation, in units of metres and minutes, is $x = 300\log(t+1)$, and Henry's is $x = kt$, for some constant k.

(a) Sketch the two graphs.

(b) Show that $k = \dfrac{300}{e-1}$.

(c) Use calculus to find the maximum distance between Henry and Thomas during the first $e - 1$ minutes, and the time when it occurs (in exact form, and then correct to the nearest metre or the nearest second).

EXTENSION

14. [The arithmetic mean, the geometric mean and the harmonic mean] The *harmonic mean* of two numbers a and b is defined to be the number h such that

$$\frac{1}{h} \text{ is the arithmetic mean of } \frac{1}{a} \text{ and } \frac{1}{b}.$$

Suppose that town B lies on the road between town A and town C, and that a cyclist rides from A to B at a constant speed U, and then rides from B to C at a constant speed V.

(a) Prove that if town B lies midway between towns A and C, then the cyclist's average speed W over the total distance AC is the harmonic mean of U and V.

(b) Now suppose that the distances AB and BC are not equal.

(i) Show that if W is the arithmetic mean of U and V, then

$$AB : BC = U : V.$$

(ii) Show that if W is the geometric mean of U and V, then

$$AB : BC = \sqrt{U} : \sqrt{V}.$$

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3 B Velocity and Acceleration as Derivatives

If I drive the 160 km from Sydney to Newcastle in 2 hours, my average velocity is 80 km per hour. However, my *instantaneous velocity* during the journey, as displayed on the speedometer, may range from zero at traffic lights to 110 km per hour on expressways. Just as an average velocity corresponds to the gradient of a chord on the displacement–time graph, so an instantaneous velocity corresponds to the gradient of a tangent.

Instantaneous Velocity and Speed: From now on, the words *velocity* and *speed* alone will mean instantaneous velocity and instantaneous speed.

5 INSTANTANEOUS VELOCITY: The *instantaneous velocity* v of the particle is the derivative of the displacement with respect to time:

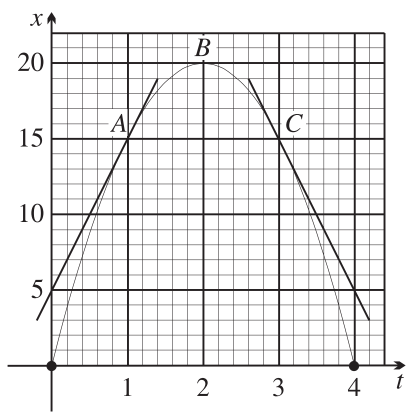
$$v = \frac{dx}{dt} \qquad \text{(This derivative } \frac{dx}{dt} \text{ can also be written as } \dot{x}.)$$

That is, v = gradient of the tangent on the displacement–time graph.

The *instantaneous speed* is the absolute value $|v|$ of the velocity.

The notation $\dot{x}$ is yet another way of writing the derivative. The dot over the x, or over any symbol, stands for differentiation with respect to time t, so that v, dx/dt and $\dot{x}$ are alternative symbols for velocity.

WORKED EXERCISE: Here again is the displacement–time graph of the ball moving with equation $x = 20t - 5t^2$.



(a) Differentiate to find the equation of the velocity v, draw up a table of values at 1-second intervals, and sketch the velocity–time graph.

(b) Measure the gradients of the tangents that have been drawn at A, B and C on the displacement–time graph, and compare your answers with the table of values in part (a).

(c) With what velocity was the ball originally thrown?

(d) What is its impact speed when it hits the ground?

SOLUTION:

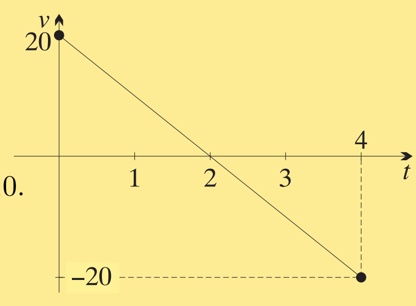
(a) The equation of motion is $x = 5t(4 - t)$

$$x = 20t - 5t^2.$$

Differentiating, $v = 20 - 10t$,

which is linear, with v-intercept 20 and gradient -10.

t	0	1	2	3	4
v	20	10	0	-10	-20



(b) These values agree with the measurements of the gradients of the tangents at A where $x = 1$, at B where $x = 2$, and at C where $x = 3$. (Be careful to take account of the different scales on the two axes.)

(c) When $t = 0$, $v = 20$, so the ball was originally thrown upwards at 20 m/s.

(d) When $t = 4$, $v = -20$, so the ball hits the ground at 20 m/s.

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Vector and Scalar Quantities: Displacement and velocity are *vector quantities*, meaning that they have a direction built into them. In the example above, a negative velocity means the ball is going downwards, and a negative displacement would mean it was below ground level. Distance and speed, however, are called *scalar quantities* — they measure only the magnitude of displacement and velocity respectively, and therefore cannot be negative.

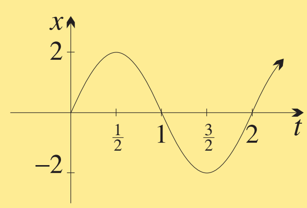
Stationary Points: A particle is stationary when its velocity is zero, that is, when $\dfrac{dx}{dt} = 0$. This is is the origin of the word 'stationary point', introduced in Chapter Ten of the Year 11 volume to describe a point on a graph where the derivative is zero. For example, the thrown ball was stationary for an instant at the top of its flight when $t = 2$, because the velocity was zero at the instant when the motion changed from upwards to downwards.

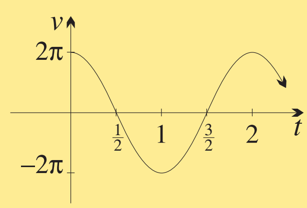
6 STATIONARY POINTS: To find when a particle is *stationary* (meaning momentarily at rest), put $v = 0$ and solve for t.

WORKED EXERCISE: A particle is moving according to the equation $x = 2\sin \pi t$.
(a) Find the equation for its velocity, and graph both equations.
(b) Find when the particle is at the origin, and its speed then.
(c) Find when and where the particle is stationary.
(d) Briefly describe the motion.

SOLUTION: We are given that $x = 2\sin \pi t$.

(a) Differentiating, $v = 2\pi \cos \pi t$,
and the graphs are drawn opposite.





(b) When the particle is at the origin, $x = 0$
$2\sin \pi t = 0$
and since conventionally $t \ge 0$, $t = 0, 1, 2, 3, \ldots$.
When $t = 0,\ 2,\ \ldots$, $v = 2\pi$,
and when $t = 1,\ 3,\ \ldots$, $v = -2\pi$.
Hence the particle is at the origin when $t = 0, 1, 2, \ldots$,
and the speed then is always 2π.

(c) When the particle is stationary, $v = 0$
$2\pi \cos \pi t = 0$
$t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \ldots$.
When $t = \frac{1}{2},\ 2\frac{1}{2},\ \ldots$, $x = 2$,
and when $t = 1\frac{1}{2},\ 3\frac{1}{2},\ \ldots$, $x = -2$.
Hence the particle is stationary when $t = \frac{1}{2},\ 1\frac{1}{2},\ 2\frac{1}{2},\ \ldots$,
and is alternately 2 units right and left of the origin.

(d) The particle oscillates for ever between $x = -2$ and $x = 2$, with period 2, beginning at the origin, and moving first to $x = 2$.

Limiting Values of Displacement and Velocity: Sometimes a question will ask what happens to the particle 'eventually', or 'as time goes on'. This simply means take the limit as $t \to \infty$.

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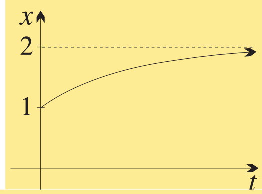
WORKED EXERCISE: A particle moves so that its height x metres above the ground t seconds after time zero is $x = 2 - e^{-3t}$.

(a) Find displacement and velocity initially, and eventually.

(b) Briefly describe the motion and sketch the graphs of displacement and velocity.



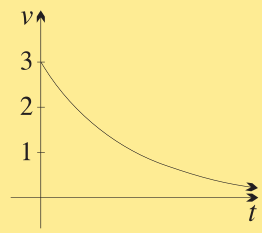
SOLUTION:

(a) We are given that $x = 2 - e^{-3t}$.

Differentiating, $\quad v = 3e^{-3t}$.

When $t = 0$, $\quad x = 1$ and $v = 3$.

As $t \to \infty$, $\quad x \to 2$ and $v \to 0$.



(b) Hence the particle starts 1 metre above the ground with initial velocity $3\,\text{m/s}$ upwards, and moves towards its limiting position at height 2 metres with speed tending to 0.

Acceleration: A particle whose velocity is changing is said to be *accelerating*, and the value of the acceleration is defined to be the rate of change of the velocity. Thus the acceleration is $\dot{v}$, meaning the derivative $\dfrac{dv}{dt}$ with respect to time.

But the velocity is itself the derivative of the displacement, so the acceleration is the second derivative $\dfrac{d^2x}{dt^2}$ of displacement, and can therefore be written as $\ddot{x}$.

7 ACCELERATION AS A SECOND DERIVATIVE: Acceleration is the first derivative of velocity with respect to time, and the second derivative of displacement:

$$\text{acceleration} = \dot{v} = \ddot{x}\,.$$

WORKED EXERCISE: In the previous worked exercise, $x = 2 - e^{-3t}$ and $v = 3e^{-3t}$.

(a) Find the acceleration function, and sketch the acceleration–time graph.

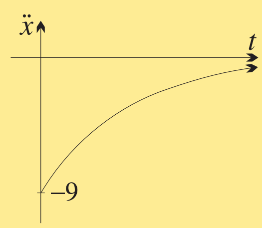
(b) In what direction is the particle accelerating?

(c) What happens to the acceleration eventually?

SOLUTION:

(a) Since $v = 3e^{-3t}$,

$\ddot{x} = -9e^{-3t}$.



(b) The acceleration is always negative, so the particle is accelerating downwards. (Since it is always moving upwards, this means that it is always slowing down.)

(c) Since $e^{-3t} \to 0$ as $t \to \infty$, the acceleration tends to zero as time goes on.

WORKED EXERCISE: The height x of a ball thrown in the air is given by $x = 5t(4 - t)$, in units of metres and seconds.

(a) Show that its acceleration is a constant function, and sketch its graph.

(b) State when the ball is speeding up and when it is slowing down, explaining why this can happen when the acceleration is constant.

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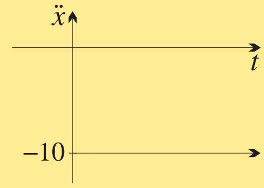
SOLUTION:

(a) Differentiating, $x = 20t - 5t^2$,

$$\dot{x} = 20 - 10t$$
$$\ddot{x} = -10.$$

Hence the acceleration is always $10\,\text{m/s}^2$ downwards.



(b) During the first two seconds, the ball has positive velocity, meaning that it is rising, and the ball is slowing down by $10\,\text{m/s}$ every second.

During the third and fourth seconds, however, the ball has negative velocity, meaning that it is falling, and the ball is speeding up by $10\,\text{m/s}$ every second.

Units of Acceleration: In the previous example, the ball's velocity was decreasing by $10\,\text{m/s}$ every second, and we therefore say that the ball is accelerating at '-10 metres per second per second', written shorthand as $-10\,\text{m/s}^2$ or as $-10\,\text{ms}^{-2}$. The units correspond with the indices of the second derivative $\dfrac{d^2x}{dt^2}$.

Acceleration should normally be regarded as a vector quantity, that is, it has a direction built into it. The ball's acceleration should therefore be given as $-10\,\text{m/s}^2$, or as $10\,\text{m/s}^2$ *downwards* if the question is using the convention of upwards as positive.

Extension — Newton's Second Law of Motion: Newton's second law of motion — a law of physics, not of mathematics — says that when a force is applied to a body free to move, the body accelerates with an acceleration proportional to the force, and inversely proportional to the mass of the body. Written symbolically,

$$F = m\ddot{x},$$

where m is the mass of the body, and F is the force applied. (The units of force are chosen to make the constant of proportionality 1 — in units of kilograms, metres and seconds, the units of force are, appropriately, called *newtons*.) This means that acceleration is felt in our bodies as a force, as we all know when a motor car accelerates away from the lights, or comes to a stop quickly. In this way, the second derivative becomes directly observable to our senses as a force, just as the first derivative, velocity, is observable to our sight.

Although these things are only treated in the 4 Unit course, it is helpful to have an intuitive idea that force and acceleration are closely related.

Exercise 3B

NOTE: Most questions in this exercise are long in order to illustrate how the physical situation of the particle's motion is related to the mathematics and the graph. The mathematics should be well-known, but the physical interpretations can be confusing.

1. A particle moves according to the equation $x = t^2 - 8t$, in units of metres and seconds.
   (a) Differentiate to find the functions v and $\ddot{x}$, and show that the acceleration is constant.
   (b) What are the displacement, velocity and acceleration after 5 seconds?
   (c) When is the particle stationary, and where is it then?

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2. A particle moves on a horizontal line so that its displacement x cm to the right of the origin at time t seconds is $x = t^3 - 6t^2 - t + 2$.
(a) Differentiate to find v and $\ddot{x}$ as functions of t.
(b) Where is the particle initially, and what are its speed and acceleration?
(c) At time $t = 3$: (i) Is the particle left or right of the origin? (ii) Is it travelling to the left or to the right? (iii) In what direction is it accelerating?
(d) When is the particle's acceleration zero, and what is its speed then?

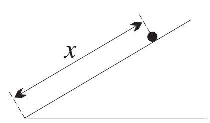
3. Find the functions v and $\ddot{x}$ for a particle P moving according to $x = 2\sin \pi t$.
(a) Show that P is at the origin when $t = 1$, and find its velocity and acceleration then.
(b) In what direction is the particle: (i) moving, (ii) accelerating, when $t = \frac{1}{3}$?

4. If $x = e^{-4t}$, find the functions v and $\ddot{x}$.
(a) Explain why neither x nor v nor $\ddot{x}$ can ever change sign, and state their signs.
(b) Where is the particle: (i) initially, (ii) eventually?
(c) What are the particle's velocity and acceleration: (i) initially, (ii) eventually?

5. A cricket ball is thrown vertically upwards, and its height x in metres at time t seconds after it is thrown is given by $x = 20t - 5t^2$.
(a) Find v and $\ddot{x}$ as functions of t, and show that the ball is always accelerating downwards. Then sketch graphs of x, v and $\ddot{x}$ against t.
(b) Find the speed at which the ball was thrown, find when it returns to the ground, and show that its speed then is equal to the initial speed.
(c) Find its maximum height above the ground, and the time to reach this height.
(d) Find the acceleration at the top of the flight, and explain why the acceleration can be nonzero when the ball is stationary.
(e) When is the ball's height 15 metres, and what are its velocities then?

6. A particle moves according to $x = t^2 - 8t + 7$, in units of metres and seconds.
(a) Find v and $\ddot{x}$ as functions of t, then sketch graphs of x, v and $\ddot{x}$ against t.
(b) When is the particle: (i) at the origin, (ii) stationary?
(c) What is the maximum distance from the origin, and when does it occur: (i) during the first 2 seconds, (ii) during the first 6 seconds, (iii) during the first 10 seconds?
(d) What is the particle's average velocity during the first 7 seconds? When and where is its instantaneous velocity equal to this average?
(e) How far does it travel during the first 7 seconds, and what is its average speed?

7. A smooth piece of ice is projected up a smooth inclined surface, as shown to the right. Its distance x in metres up the surface at time t seconds is $x = 6t - t^2$.



(a) Find the functions v and $\ddot{x}$, and sketch x and v.
(b) In which direction is the ice moving, and in which direction is it accelerating:
(i) when $t = 2$, (ii) when $t = 4$?
(c) When is the ice stationary, for how long is it stationary, where is it then, and is it accelerating then?
(d) Find the average velocity over the first 2 seconds, and the time and place where the instantaneous velocity equals this average velocity.
(e) Show that the average speed during the first 3 seconds, the next 3 seconds and the first 6 seconds are all the same.

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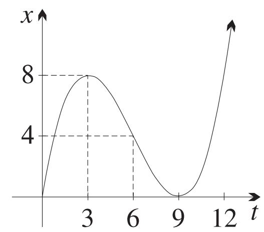
DEVELOPMENT



8. A particle is moving horizontally so that its displacement x metres to the right of the origin at time t seconds is given by the graph to the right.

(a) In the first 10 seconds, what is its maximum distance from the origin, and when does it occur?

(b) When is the particle: (i) stationary, (ii) moving to the right, (iii) moving to the left?

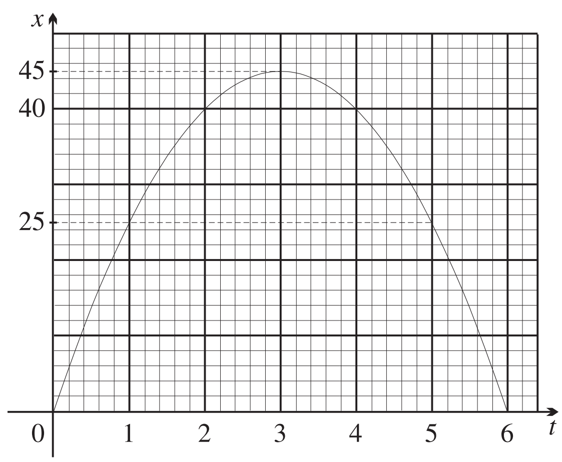
(c) When does it return to the origin, what is its velocity then, and in which direction is it accelerating?

(d) When is its acceleration zero, where is it then, and in what direction is it moving?

(e) During what time is its acceleration negative?

(f) At about what times is: (i) the displacement, (ii) the velocity, (iii) the speed, about the same as at $t = 2$?

(g) Sketch (roughly) the graphs of v and $\ddot{x}$.



9. A stone was thrown vertically upwards, and the graph to the right shows its height x metres at time t seconds after it was thrown.

(a) What was the stone's maximum height, how long did it take to reach it, and what was its average speed during this time?

(b) Draw tangents and measure their gradients to find the velocity of the stone at times $t = 0$, 1, 2, 3, 4, 5 and 6.

(c) For what length of time was the stone stationary at the top of its flight?

(d) The graph is concave down everywhere. How is this relevant to the motion?

(e) Draw a graph of the instantaneous velocity of the stone from $t = 0$ to $t = 6$. What does the graph tell you about what happened to the velocity during these 6 seconds?

10. A particle is moving according to $x = 4\cos\frac{\pi}{4}t$, in units of metres and seconds.

(a) Find v and $\ddot{x}$, and sketch graphs of x, v and $\ddot{x}$ against t, for $0 \le t \le 8$.

(b) What are the particle's maximum displacement, velocity and acceleration, and when, during the first 8 seconds, do they occur?

(c) How far does it travel during the first 20 seconds, and what is its average speed?

(d) When, during the first 8 seconds, is: (i) $x = 2$, (ii) $x < 2$?

(e) When, during the first 8 seconds, is: (i) $v = \frac{\pi}{2}$, (ii) $v > \frac{\pi}{2}$?

11. A particle is oscillating on a spring so that its height is $x = 6\sin 2t$ cm at time t seconds.

(a) Find v and $\ddot{x}$ as function of t, and sketch graphs of x, v and $\ddot{x}$, for $0 \le t \le 2\pi$.

(b) Show that $\ddot{x} = -kx$, for some constant k, and find k.

(c) When, during the first π seconds, is the particle:
(i) at the origin, (ii) stationary, (iii) moving with zero acceleration?

(d) When, during the first π seconds, is the particle:
(i) below the origin, (ii) moving downwards, (iii) accelerating downwards?

(e) Find the first time the particle has: (i) displacement $x = 3$, (ii) speed $|v| = 6$.

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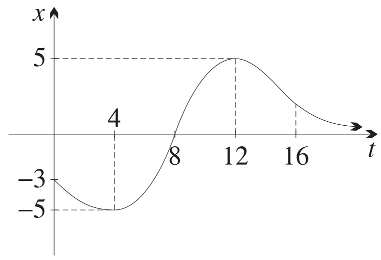
12. A particle is moving vertically according to the graph shown to the right, where upwards has been taken as positive.



(a) At what times is this particle: (i) below the origin, (ii) moving downwards, (iii) accelerating downwards?

(b) At about what time is its speed greatest?

(c) At about what times is: (i) distance from the origin, (ii) velocity, (iii) speed, about the same as at $t = 3$?

(d) How many times between $t = 4$ and $t = 12$ is the instantaneous velocity equal to the average velocity during this time?

(e) How far will the particle eventually travel?

(f) Sketch the graphs of v and $\ddot{x}$ as functions of time.

13. A large stone is falling through a layer of mud, and its depth x metres below ground level at time t minutes is given by $x = 12 - 12e^{-0{\cdot}5t}$.

(a) Find v and $\ddot{x}$ as functions of t, and sketch graphs of x, v and $\ddot{x}$.

(b) In which direction is the stone: (i) travelling, (ii) accelerating?

(c) What happens to the position, velocity and acceleration of the particle as $t \to \infty$?

(d) Find when the stone is halfway between the origin and its final position. Show that its speed is then half its initial speed, and its acceleration is half its initial acceleration.

(e) How long, correct to the nearest minute, will it take for the stone to reach within 1 mm of its final position?

14. Two particles A and B are moving along a horizontal line, with their distances x_A and x_B to the right of the origin O at time t given by $x_A = 4t\,e^{-t}$ and $x_B = -4t^2 e^{-t}$. The particles are joined by a piece of elastic, whose midpoint M has position x_M at time t.

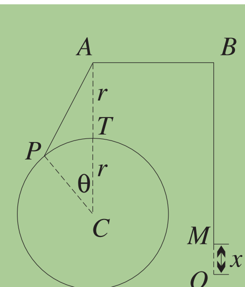
(a) Explain why $x_M = 2e^{-t}(t - t^2)$, find when M returns to the origin, and find its speed and direction at this time.

(b) Find at what times M is furthest right and furthest left of O.

(c) What happens to A, B and M eventually? (d) When are A and B furthest apart?

EXTENSION

15. The diagram to the right shows a point P that is rotating anticlockwise in a circle of radius r and centre C at a steady rate. A string passes over fixed pulleys at A and B, where A is distant r above the top T of the circle, and connects P to a mass M on the end of the string. At time zero, P is at T, and the mass M is at the point O. Let x be the height of the mass above the point O at time t seconds later, and θ be the angle $\angle TCP$ through which P has moved.



(a) Show that $x = -r + r\sqrt{5 - 4\cos\theta}$, and find the range of x.

(b) Find $\dfrac{dx}{d\theta}$, and find for what values of θ the mass M is travelling:

(i) upwards, (ii) downwards.

(c) Show that $\dfrac{d^2x}{d\theta^2} = -\dfrac{2r(2\cos^2\theta - 5\cos\theta + 2)}{(5 - 4\cos\theta)^{\frac{3}{2}}}$. Find for what values of θ the speed of M is maximum, and find $\dfrac{dx}{d\theta}$ at these values of θ.

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(d) Explain geometrically why these values of θ give the maximum speed, and why they give the values of $\dfrac{dx}{d\theta}$ they do.

16. [This question will require resolution of forces.] At what angle α should the surface in question 7 be inclined to the horizontal to produce these equations?

3 C Integrating with Respect to Time

The inverse process of differentiation is integration. Therefore if the acceleration function is known, integration will generate the velocity function, and integration of the velocity function will generate the displacement function.

Initial or Boundary Conditions: Taking the primitive of a function always involves an arbitrary constant. Hence one or more boundary conditions are required to determine the motion completely.

WORKED EXERCISE: The velocity of a particle initially at the origin is $v = \sin \frac{1}{4}t$.
(a) Find the displacement function. (b) Find the acceleration function.
(c) Find the values of displacement, velocity and acceleration when $t = 4\pi$.
(d) Briefly describe the motion, and sketch the displacement–time graph.

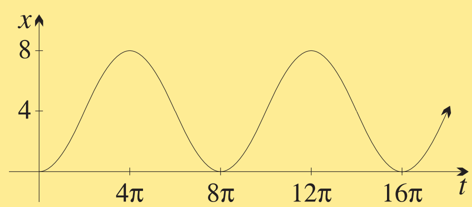
SOLUTION: Given: $v = \sin \frac{1}{4}t$. (1)

(a) Integrating, $x = -4\cos \frac{1}{4}t + C$, for some constant C,
and substituting $x = 0$ when $t = 0$:
$0 = -4 \times 1 + C$,
so $C = 4$, and $x = 4 - 4\cos \frac{1}{4}t$. (2)

(b) Differentiating, $\ddot{x} = \frac{1}{4}\cos \frac{1}{4}t$.

(c) When $t = 4\pi$, $x = 4 - 4 \times \cos \pi = 8$ metres,
$v = \sin \pi = 0\,\text{m/s}$,
and $\ddot{x} = \frac{1}{4}\cos \pi = -\frac{1}{4}\,\text{m/s}^2$. (3)

(d) The particle oscillates between $x = 0$ and $x = 8$ with period 8π seconds.



WORKED EXERCISE: The acceleration of a particle is given by $\ddot{x} = e^{-2t}$, and the particle is initially stationary at the origin.
(a) Find the velocity function. (b) Find the displacement function.
(c) Find the displacement when $t = 10$.
(d) Briefly describe the velocity of the particle as time goes on.

SOLUTION: Given: $\ddot{x} = e^{-2t}$. (1)

(a) Integrating, $v = -\frac{1}{2}e^{-2t} + C$.
When $t = 0$, $v = 0$, so $0 = -\frac{1}{2} + C$,
so $C = \frac{1}{2}$, and $v = -\frac{1}{2}e^{-2t} + \frac{1}{2}$. (2)

(b) Integrating again, $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t + D$.
When $t = 0$, $x = 0$, so $0 = \frac{1}{4} + D$,
so $D = -\frac{1}{4}$, and $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}$. (3)

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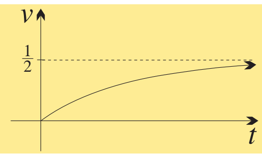
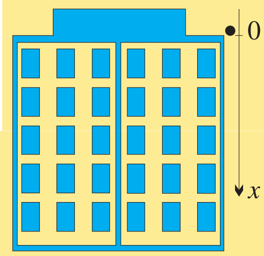
(c) When $t = 10$, $x = \frac{1}{4}e^{-20} + 5 - \frac{1}{4}$
$= 4\frac{3}{4} + \frac{1}{4}e^{-20}$ metres.

(d) The velocity is initially zero, and increases with limit $\frac{1}{2}$ m/s.



The Acceleration Due to Gravity: Since the time of Galileo, it has been known that near the surface of the Earth, a body free to fall accelerates downwards at a constant rate, whatever its mass, and whatever its velocity (neglecting air resistance). This acceleration is called the *acceleration due to gravity*, and is conventionally given the symbol g. The value of this acceleration is about $9{\cdot}8\,\text{m/s}^2$, or in round figures, $10\,\text{m/s}^2$.

WORKED EXERCISE: A stone is dropped from the top of a high building. How far has it travelled, and how fast is it going, after 5 seconds? (Take $g = 9{\cdot}8\,\text{m/s}^2$.)



SOLUTION: Let x be the distance travelled t seconds after the stone is dropped. This puts the origin of space at the top of the building and the origin of time at the instant when the stone is dropped, and makes downwards positive.

Then $\quad \ddot{x} = 9{\cdot}8$ (given). (1)

Integrating, $\quad v = 9{\cdot}8t + C$, for some constant C.

Since the stone was dropped, its initial speed was zero,

and substituting, $\quad 0 = 0 + C$,

so $C = 0$, and $\quad v = 9{\cdot}8t$. (2)

Integrating again, $x = 4{\cdot}9t^2 + D$, for some constant D.

Since the initial displacement of the stone was zero,

$0 = 0 + D$,

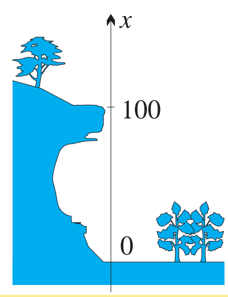
so $D = 0$, and $\quad x = 4{\cdot}9t^2$. (3)

When $t = 5$, $\quad v = 49$ and $x = 122{\cdot}5$.

Hence the stone has fallen $122{\cdot}5$ metres and is moving downwards at $49\,\text{m/s}$.

Making a Convenient Choice of the Origin and the Positive Direction: Physical problems do not come with origins and directions attached, and it is up to us to choose the origins of displacement and time, and the positive direction, so that the arithmetic is as simple as possible. The previous worked exercise made reasonable choices, but the following worked exercise makes quite different choices. In all such problems, the physical interpretation of negatives and displacements is the responsibility of the mathematician, and the final answer should be free of them.

WORKED EXERCISE: A cricketer is standing on a lookout that projects out over the valley floor 100 metres below him. He throws a cricket ball vertically upwards at a speed of $40\,\text{m/s}$, and it falls back past the lookout onto the valley floor below. How long does it take to fall, and with what speed does it strike the ground? (Take $g = 10\,\text{m/s}^2$.)



SOLUTION: Let x be the distance above the valley floor t seconds after the stone is thrown. This puts the origin of space at the valley floor and the origin of time

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at the instant when the stone is thrown. It also makes upwards positive, so that $\ddot{x} = -10$, because the acceleration is downwards.

As discussed, $\ddot{x} = -10$. (1)

Integrating, $v = -10t + C$, for some constant C.

Since $v = 40$ when $t = 0$, $40 = 0 + C$,

so $C = 40$, and $v = -10t + 40$. (2)

Integrating again, $x = -5t^2 + 40t + D$, for some constant D.

Since $x = 100$ when $t = 0$, $100 = 0 + 0 + D$,

so $D = 100$, and $x = -5t^2 + 40t + 100$. (3)

The stone hits the ground when $x = 0$, that is,

$$
\begin{aligned}
-5t^2 + 40t + 100 &= 0 \\
t^2 - 8t - 20 &= 0 \\
(t-10)(t+2) &= 0 \\
t &= 10 \text{ or } -2.
\end{aligned}
$$

Since the ball was not in flight at $t = -2$, the ball hits the ground after 10 seconds. At that time, $v = -100 + 40 = -60$, so it hits the ground at 60 m/s.

Formulae from Physics Cannot be Used: This course requires that even problems where the acceleration is constant, such as the two above, must be solved by integration of the acceleration function. Many readers will know of three very useful equations for motion with constant acceleration a:

$$v = u + at \qquad \text{and} \qquad s = ut + \tfrac{1}{2}at^2 \qquad \text{and} \qquad v^2 = u^2 + 2as.$$

These equations automate the integration process, and so cannot be used in this course. Questions in Exercises 3C and 3E develop proper proofs of these results.

Using Definite Integrals to find Changes of Displacement and Velocity: The change in displacement during some period of time can be found quickly using a definite integral of the velocity. This avoids evaluating the constant of integration, and is therefore useful when no boundary conditions have been given. The disadvantage is that the displacement–time function remains unknown. The change in velocity can be calculated similarly, using a definite integral of the acceleration.

8 USING DEFINITE INTEGRALS TO FIND CHANGES IN DISPLACEMENT AND VELOCITY:

Given the velocity v as a function of time, then from $t = t_1$ to $t = t_2$,

$$\text{change in displacement} = \int_{t_1}^{t_2} v\,dt.$$

Given the acceleration $\ddot{x}$ as a function of time, then from $t = t_1$ to $t = t_2$,

$$\text{change in velocity} = \int_{t_1}^{t_2} \ddot{x}\,dt.$$

WORKED EXERCISE: In these questions, the units are metres and seconds.

(a) Given $v = 4 - e^{4-t}$, find the change in displacement during the third second.

(b) Given $\ddot{x} = 12\sin 2t$, find the change in velocity during the first $\frac{\pi}{2}$ seconds.

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SOLUTION:

(a) Change in displacement

$$= \int_2^3 (4 - e^{4-t})\, dt$$
$$= \Big[4t + e^{4-t}\Big]_2^3$$
$$= (12 + e) - (8 + e^2)$$
$$= 4 + e - e^2 \text{ metres.}$$

(b) Change in velocity

$$= \int_0^{\frac{\pi}{2}} 12 \sin 2t\, dt$$
$$= -6 \Big[\cos 2t\Big]_0^{\frac{\pi}{2}}$$
$$= -6(-1 - 1)$$
$$= 12 \text{ m/s.}$$

Exercise 3C

1. Find the velocity and displacement functions of a particle whose initial velocity and displacement are zero if:

(a) $\ddot{x} = -4$
(b) $\ddot{x} = 6t$
(c) $\ddot{x} = e^{\frac{1}{2}t}$
(d) $\ddot{x} = e^{-3t}$
(e) $\ddot{x} = 8 \sin 2t$
(f) $\ddot{x} = \cos \pi t$
(g) $\ddot{x} = \sqrt{t}$
(h) $\ddot{x} = 12(t + 1)^{-2}$

2. Find the acceleration and displacement functions of a particle whose initial displacement is -2 if:

(a) $v = -4$
(b) $v = 6t$
(c) $v = e^{\frac{1}{2}t}$
(d) $v = e^{-3t}$
(e) $v = 8 \sin 2t$
(f) $v = \cos \pi t$
(g) $v = \sqrt{t}$
(h) $v = 12(t + 1)^{-2}$

3. A stone is dropped from a lookout 80 metres high. Take $g = 10\,\text{m/s}^2$, and downwards as positive, so that $\ddot{x} = 10$.

(a) Using the lookout as the origin, find the velocity and displacement as functions of t. [HINT: When $t = 0$, $v = 0$ and $x = 0$.]
(b) Find: (i) the time the stone takes to fall, (ii) its impact speed.
(c) Where is it, and what is its speed, halfway through its flight time?
(d) How long does it take to go halfway down, and what is its speed then?

4. A stone is thrown downwards from the top of a 120-metre building, with an initial speed of $25\,\text{m/s}$. Take $g = 10\,\text{m/s}^2$, and take upwards as positive, so that $\ddot{x} = -10$.

(a) Using the ground as the origin, find the acceleration, velocity and height x of the stone t seconds after it is thrown. [HINT: When $t = 0$, $v = -25$ and $x = 120$.] Hence find: (i) the time it takes to reach the ground, (ii) the impact speed.
(b) Rework part (a) with the origin at the top of the building, and downwards positive.

5. A particle is moving with acceleration $\ddot{x} = 12t$. Initially it has velocity $-24\,\text{m/s}$, and is 20 metres on the positive side of the origin.

(a) Find the velocity and displacement functions.
(b) When does the particle return to its initial position, and what is its speed then?
(c) What is the minimum displacement, and when does it occur?
(d) Find x when $t = 0$, 1, 2, 3 and 4, and sketch the displacement–time graph.

DEVELOPMENT

6. A car moves along a straight road from its front gate, where it is initially stationary. During the first 10 seconds, it has a constant acceleration of $2\,\text{m/s}^2$, it has zero acceleration during the next 30 seconds, and it decelerates at $1\,\text{m/s}^2$ for the final 20 seconds.

(a) What is the maximum speed, and how far does the car go altogether?
(b) Sketch the graphs of acceleration, velocity and distance from the gate.

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7. Write down a definite integral for each quantity to be calculated below. If possible, evaluate it exactly. Otherwise, use the trapezoidal rule with three function values in part (a), and Simpson's rule with five function values in part (b), giving your answers correct to three significant figures.

(a) Find the change in displacement during the 2nd second of motion of a particle whose velocity is:

(i) $v = \dfrac{4}{t+1}$ (ii) $v = \dfrac{4}{\log(t+1)}$

(b) Find the change in velocity during the 2nd second of motion of a particle whose acceleration is:

(i) $\ddot{x} = \sin \pi t$ (ii) $\ddot{x} = t \sin \pi t$

8. A body is moving with its acceleration proportional to the time elapsed. When $t = 1$, $v = -6$, and when $t = 2$, $v = 3$.

(a) Find the functions $\ddot{x}$ and v. [HINT: Let $\ddot{x} = kt$, where k is the constant of proportionality. Then integrate, using the usual constant C of integration. Then find C and k by substituting the two given values of t.]

(b) When does the body return to its original position?

9. [A proof of three constant-acceleration formulae from physics — not to be used elsewhere]

(a) A particle moves with constant acceleration a. Its initial velocity is u, and at time t it is moving with velocity v and is distant s from its initial position. Show that:

(i) $v = u + at$ (ii) $s = ut + \frac{1}{2}at^2$ (iii) $v^2 = u^2 + 2as$

(b) Solve questions 3 and 4 using formulae (ii) and (i), and again using (iii) and (i).

10. A body falling through air experiences an acceleration $\ddot{x} = -40e^{-2t}\ \text{m/s}^2$ (we are taking upwards as positive). Initially, it is thrown upwards with speed 15 m/s.

(a) Taking the origin at the point where it is thrown, find the functions v and x, and find when the body is stationary.

(b) Find its maximum height, and the acceleration then.

(c) Describe the velocity of the body as $t \to \infty$.

11. If a particle moves from $x = -1$ with velocity $v = \dfrac{1}{t+1}$, how long does it take to get to the origin, and what are its speed and acceleration then? Describe its subsequent motion.

12. A mouse emerges from his hole and moves out and back along a line. His velocity at time t seconds is $v = 4t(t-3)(t-6) = 4t^3 - 36t^2 + 72t$ cm/s.

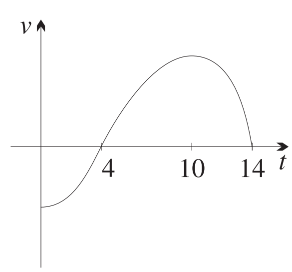
(a) When does he return to his original position, and how fast is he then going?

(b) How far does he travel during this time, and what is his average speed?

(c) What is his maximum speed, and when does it occur?

(d) If a video of these 6 seconds were played backwards, could this be detected?

13. The graph to the right shows a particle's velocity–time graph.



(a) When is the particle moving forwards?

(b) When is the acceleration positive?

(c) When is it furthest from its starting point?

(d) When is it furthest in the negative direction?

(e) About when does it return to its starting point?

(f) Sketch the graphs of acceleration and displacement, assuming that the particle starts at the origin.

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14. A particle is moving with velocity $v = 16 - 4t$ cm/s on a horizontal number line.

(a) Find $\ddot{x}$ and x. (The function x will have a constant of integration.)

(b) When does it return to its original position, and what is its speed then?

(c) When is the particle stationary? Find the maximum distances right and left of the initial position during the first 10 seconds, and the corresponding times and accelerations.

(d) How far does it travel in the first 10 seconds, and what is its average speed?

15. A moving particle is subject to an acceleration of $\ddot{x} = -2\cos t\,\text{m/s}^2$. Initially, it is at $x = 2$, moving with velocity 1 m/s, and it travels for 2π seconds.

(a) Find the functions v and x.

(b) When is the acceleration positive?

(c) When and where is the particle stationary, and when is it moving backwards?

(d) What are the maximum and minimum velocities, and when and where do they occur?

(e) Find the change in displacement and the average velocity.

(f) Sketch the displacement–time graph, and hence find the distance travelled and the average speed.

16. Particles P_1 and P_2 move with velocities $v_1 = 6 + 2t$ and $v_2 = 4 - 2t$, in units of metres and seconds. Initially, P_1 is at $x = 2$ and P_2 is at $x = 1$.

(a) Find x_1, x_2 and the difference $D = x_1 - x_2$.

(b) Prove that the particles never meet, and find the minimum distance between them.

(c) Prove that the midpoint M between the two particles is moving with constant velocity, and find its distance from each particle after 3 seconds.

17. Once again, the trains Thomas and Henry are on parallel tracks, level with each other at time zero. Thomas is moving with velocity $v_T = \dfrac{20}{t+1}$ and Henry with velocity $v_H = 5$.

(a) Who is moving faster initially, and by how much?

(b) Find the displacements x_T and x_H of the two trains, if they start at the origin.

(c) Use your calculator to find during which second the trains are level, and find the speed at which the trains are drawing apart at the end of this second.

(d) When is Henry furthest behind Thomas, and by how much (to the nearest metre)?

18. A ball is dropped from a lookout 180 metres high. At the same time, a stone is fired vertically upwards from the valley floor with speed V m/s. Take $g = 10\,\text{m/s}^2$.

(a) Find for what values of V a collision in the air will occur. Find, in terms of V, the time and the height when collision occurs, and prove that the collision speed is V m/s.

(b) Find the value of V for which they collide halfway up the cliff, and the time taken.

EXTENSION

19. A falling body experiences both the gravitational acceleration g and air resistance that is proportional to its velocity. Thus a typical equation of motion is $\ddot{x} = -10 - 2v\,\text{m/s}^2$. Suppose that the body is dropped from the origin.

(a) By writing $\ddot{x} = \dfrac{dv}{dt}$ and taking reciprocals, find t as a function of v, and hence find v as a function of t. Then find x as a function of t.

(b) Describe the motion of the particle.

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3 D Simple Harmonic Motion — The Time Equations

As has been mentioned before, some of the most common physical phenomena around us fluctuate — sound waves, light waves, tides, heartbeats — and are therefore governed by sine and cosine functions. The simplest such phenomena are governed by a single sine or cosine function, and accordingly, our course makes a detailed study of motion governed by such a function, called *simple harmonic motion*. This section approaches the topic through the displacement–time equation, but the topic will be studied again in Section 3F using the motion's characteristic acceleration–displacement equation.

Simple Harmonic Motion: *Simple harmonic motion* (or *SHM* for short) is any motion whose displacement–time equation, apart from the constants, is a single sine or cosine function. More precisely:

9

SIMPLE HARMONIC MOTION — THE DISPLACEMENT–TIME EQUATION: A particle is said to be moving in *simple harmonic motion with centre the origin* if

$$x = a\sin(nt + \alpha) \qquad \text{or} \qquad x = a\cos(nt + \alpha),$$

where a, n and α are constants, with a and n positive.

- The constant a is called the *amplitude* of the motion, and the particle is confined in the interval $-a \le x \le a$. The origin is called the *centre* of the motion, because it is the midpoint between the two extremes of the motion, $x = -a$ and $x = a$.
- The *period* T of the motion is given by $T = \dfrac{2\pi}{n}$.
- At any time t, the quantity $nt + \alpha$ is called the *phase*. In particular, the phase at time $t = 0$ is α, and therefore α is called the *initial phase*.

Since $\cos\theta = \sin(\theta + \frac{\pi}{2})$, either the sine or the cosine function can be used for any particular motion. If the question allows a choice, it is best to choose the function with zero initial phase, because the algebra is easier when $\alpha = 0$.

Simple Harmonic Motion about Other Centres: The motion of a particle oscillating about the point $x = x_0$ rather than the origin can be described simply by adding the constant x_0.

10

SIMPLE HARMONIC MOTION ABOUT $x = x_0$: A particle is said to be moving in *simple harmonic motion with centre* $x = x_0$ if

$$x = x_0 + a\sin(nt + \alpha) \qquad \text{or} \qquad x = x_0 + a\cos(nt + \alpha).$$

The amplitude of the motion is still a, and the particle is confined to the interval $x_0 - a \le x \le x_0 + a$ with centre at the midpoint $x = x_0$.

WORKED EXERCISE: A particle is moving in simple harmonic motion according to the equation $x = 2 + 4\cos(2t + \frac{\pi}{3})$.

(a) Find the centre, period, amplitude and extremes of the motion.

(b) What is the initial phase, and where is the particle at $t = 0$?

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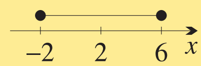
(c) Find the first time when the particle is at:

(i) the centre of motion,
(ii) the origin,
(iii) the maximum displacement,
(iv) the minimum displacement.

SOLUTION:

(a) The equation has the correct form for SHM.
The centre is $x = 2$, the amplitude is 4 and the period is $\frac{2\pi}{2} = \pi$.
The motion therefore lies in the interval $-2 \le x \le 6$.



(b) The initial phase is $\frac{\pi}{3}$. When $t = 0$, $x = 2 + 4\cos\frac{\pi}{3} = 4$.

(c) (i) Put $2 + 4\cos(2t + \frac{\pi}{3}) = 2$.
Then
$$\begin{aligned} \cos(2t + \tfrac{\pi}{3}) &= 0 \\ 2t + \tfrac{\pi}{3} &= \tfrac{\pi}{2} \\ t &= \tfrac{\pi}{12}. \end{aligned}$$

(ii) Put $2 + 4\cos(2t + \frac{\pi}{3}) = 0$.
Then
$$\begin{aligned} \cos(2t + \tfrac{\pi}{3}) &= -\tfrac{1}{2} \\ 2t + \tfrac{\pi}{3} &= \tfrac{2\pi}{3} \\ t &= \tfrac{\pi}{6}. \end{aligned}$$

(iii) Put $2 + 4\cos(2t + \frac{\pi}{3}) = 6$.
Then
$$\begin{aligned} \cos(2t + \tfrac{\pi}{3}) &= 1 \\ 2t + \tfrac{\pi}{3} &= 2\pi \\ t &= \tfrac{5\pi}{6}. \end{aligned}$$

(iv) Put $2 + 4\cos(2t + \frac{\pi}{3}) = -2$.
Then
$$\begin{aligned} \cos(2t + \tfrac{\pi}{3}) &= -1 \\ 2t + \tfrac{\pi}{3} &= \pi \\ t &= \tfrac{\pi}{3}. \end{aligned}$$

Finding Acceleration and Velocity: Velocity and acceleration are found by differentiation in the usual way. Doing this in the case when the centre is at the origin results in a most important relationship between acceleration and displacement:

Let $x = a\sin(nt + \alpha)$.
Then $v = an\cos(nt + \alpha)$
and $\ddot{x} = -an^2\sin(nt + \alpha)$.
Hence $\ddot{x} = -n^2 x$.

Let $x = a\cos(nt + \alpha)$.
Then $v = -an\sin(nt + \alpha)$
and $\ddot{x} = -an^2\cos(nt + \alpha)$.
Hence $\ddot{x} = -n^2 x$.

In both cases, $\ddot{x} = -n^2 x$, meaning that the acceleration is proportional to the displacement, but acts in the opposite direction. This equation is characteristic of simple harmonic motion, and can be used to test whether a given motion is simple harmonic. It is called a *second-order differential equation* because it involves the second derivative of the function.

11 THE DIFFERENTIAL EQUATION FOR SIMPLE HARMONIC MOTION: If a particle is moving in simple harmonic motion with centre the origin and period $\dfrac{2\pi}{n}$, then

$$\ddot{x} = -n^2 x.$$

This means that the acceleration is proportional to the displacement, but acts in the opposite direction.

This equation is usually the most straightforward way to test whether a given motion is simple harmonic with centre the origin.

In Section 3F, we will use this as the starting point for our second discussion of simple harmonic motion.

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WORKED EXERCISE: Suppose that a particle is moving according to $x = 2\sin(3t + \frac{3\pi}{4})$.

(a) Write down the amplitude, centre, period and initial phase of the motion.

(b) Find the times and positions when the velocity is first: (i) zero, (ii) maximum.

(c) Find the times and positions when $\ddot{x}$ is first: (i) zero, (ii) maximum.

(d) Express the acceleration $\ddot{x}$ as a multiple of the displacement x.

SOLUTION:

(a) The amplitude is 2, the centre is $x = 0$, the period is $\frac{2\pi}{3}$, and the initial phase is $\frac{3\pi}{4}$.

(b) Differentiating, $v = 6\cos(3t + \frac{3\pi}{4})$, so the maximum velocity is 6.

(This is because $\cos(3t + \frac{3\pi}{4})$ has a maximum of 1.)

(i) When $v = 0$, $\cos(3t + \frac{3\pi}{4}) = 0$

$$3t + \tfrac{3\pi}{4} = \tfrac{3\pi}{2}$$
$$t = \tfrac{\pi}{4}.$$

When $t = \frac{\pi}{4}$, $x = 2\sin\frac{3\pi}{2}$

$= -2$.

(ii) When $v = 6$, $6\cos(3t + \frac{3\pi}{4}) = 6$

$$3t + \tfrac{3\pi}{4} = 2\pi$$
$$t = \tfrac{5\pi}{12}.$$

When $t = \frac{5\pi}{12}$, $x = 2\sin 2\pi$

$= 0$.

(c) Differentiating, $\ddot{x} = -18\sin(3t + \frac{3\pi}{4})$, so the maximum acceleration is 18.

(This is because $\sin(3t + \frac{3\pi}{4})$ has a maximum of 1.)

(i) When $\ddot{x} = 0$, $\sin(3t + \frac{3\pi}{4}) = 0$

$$3t + \tfrac{3\pi}{4} = \pi$$
$$t = \tfrac{\pi}{12}.$$

When $t = \frac{\pi}{12}$, $x = 2\sin\pi$

$= 0$.

(ii) When $\ddot{x} = 18$, $-18\sin(3t + \frac{3\pi}{4}) = 18$

$$3t + \tfrac{3\pi}{4} = \tfrac{3\pi}{2}$$
$$t = \tfrac{\pi}{4}.$$

When $t = \frac{\pi}{4}$, $x = 2\sin\frac{3\pi}{2}$

$= -2$.

(d) Since $x = 2\sin(3t + \frac{3\pi}{4})$ and $\ddot{x} = -18\sin(3t + \frac{3\pi}{4})$,
it follows that $\ddot{x} = -9x$.
Since $n = 3$, this agrees with the general result $\ddot{x} = -n^2 x$.

Choosing Convenient Origins of Space and Time: Many questions on simple harmonic motion do not specify a choice of axes. In these cases, the reader should set up the axes for displacement and time, and choose the function, to make the equations as simple as possible. First, choose the centre of motion as the origin of displacement — this makes $x_0 = 0$, so that the constant term disappears.

Secondly, the function and the origin of time should, if possible, be chosen so that the initial phase $\alpha = 0$. The key to this is that $\sin t$ is initially zero and rising, and $\cos t$ is initially maximum.

12 CHOOSING THE ORIGIN OF TIME AND THE FUNCTION: Try to make the initial phase zero:

- Use $x = a\cos nt$ if the particle starts at the positive extreme of its motion, and use $x = -a\cos nt$ if the particle starts at the negative extreme.
- Use $x = a\sin nt$ if the particle starts at the middle of its motion with positive velocity, and use $x = -a\sin nt$ if the particle starts at the middle of its motion with negative velocity.
- If the particle starts anywhere else, try to change the origin of time. Otherwise, use $x = a\cos(nt + \alpha)$ or $x = a\sin(nt + \alpha)$, and then substitute the boundary conditions to find α and a.

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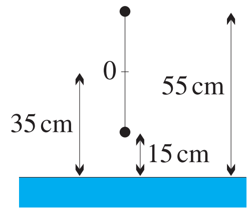
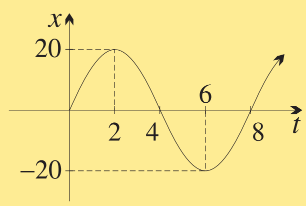
WORKED EXERCISE: A weight hanging from the roof on an elastic string is moving in simple harmonic motion. It takes 4 seconds to move from the bottom of its motion, 15 cm above the floor, to the top of its motion, 55 cm above the floor.



(a) Find where it is 3 seconds after rising through the centre of motion.
(b) Find its speed at the centre of motion.
(c) Find the maximum acceleration.

SOLUTION: Choose the origin at the centre of motion, 35 cm above the ground.
Choose the origin of time at the instant it passes through the centre, moving upwards.
Then the amplitude is 20 cm and the period is $2 \times 4 = 8$ seconds,
so $n = \frac{2\pi}{8} = \frac{\pi}{4}$, and $\qquad x = 20 \sin \frac{\pi}{4} t$.

(a) When $t = 3$, $\qquad x = 20 \sin \frac{3\pi}{4}$
$\qquad = 10\sqrt{2}$,
so the weight is $35 + 10\sqrt{2}$ cm above the ground.



(b) Differentiating, $\qquad v = 5\pi \cos \frac{\pi}{4} t$.
The weight is at the centre when $t = 0$,
and then $\qquad \text{speed} = 5\pi \cos 0$
$\qquad = 5\pi$ cm/s.

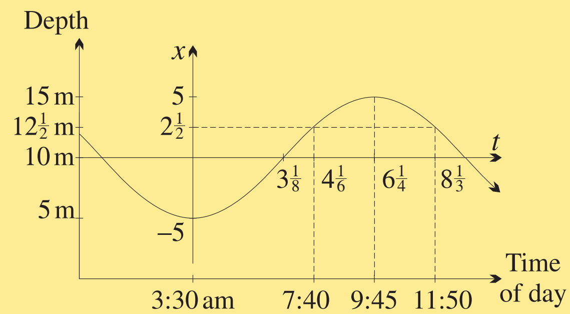
(c) Differentiating again, $\qquad \ddot{x} = -\frac{5\pi^2}{4} \sin \frac{\pi}{4} t$,
so the maximum acceleration is $\frac{5\pi^2}{4}$ cm/s^2.

WORKED EXERCISE: [Tides can be modelled by simple harmonic motion] On a certain day the depth of water in a harbour at low tide at 3:30 am is 5 metres. At the following high tide at 9:45 am the depth is 15 metres. Assuming the rise and fall of the surface of the water to be simple harmonic motion, find between what times during the morning a ship may safely enter the harbour if a minimum depth of $12\frac{1}{2}$ metres of water is required.

SOLUTION: Let x be the number of metres by which the water depth exceeds 10 metres at time t hours after 3:30 am. (This places the origin of displacement at mean tide, and the origin of time at low tide.) The motion is simple harmonic with amplitude 5 metres and period $12\frac{1}{2}$ hours.

Hence $n = \dfrac{2\pi}{T} = \dfrac{2\pi}{12\frac{1}{2}} = \dfrac{4\pi}{25}$,
and so the height is $x = -5 \cos \frac{4\pi}{25} t$.
The ship may enter safely when $x \ge 2\frac{1}{2}$.
Solving first $x = 2\frac{1}{2}$,

$$-5 \cos \tfrac{4\pi}{25} t = 2\tfrac{1}{2}$$
$$\cos \tfrac{4\pi}{25} t = -\tfrac{1}{2}$$
$$\tfrac{4\pi}{25} t = \pi - \tfrac{\pi}{3} \text{ or } \pi + \tfrac{\pi}{3}$$
$$t = 4\tfrac{1}{6} \text{ or } 8\tfrac{1}{3}.$$



Hence from the graph, the ship may enter when $4\frac{1}{6} \le t \le 8\frac{1}{3}$,
that is, between 7:40 am and 11:50 am (remembering that $t = 0$ is 3:30 am).

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The Graphs of $x = a\cos(nt + \alpha)$ and $x = a\sin(nt + \alpha)$: When the initial phase α is nonzero, the graphs can be sketched by shifting the graphs of $x = a\cos nt$ and $x = a\sin nt$. The key step here is to take out the factor of n and write

$$x = a\cos n(t + \alpha/n) \qquad \text{and} \qquad x = a\sin n(t + \alpha/n),$$

These graphs are $x = a\cos nt$ and $x = a\sin nt$ shifted left by α/n.

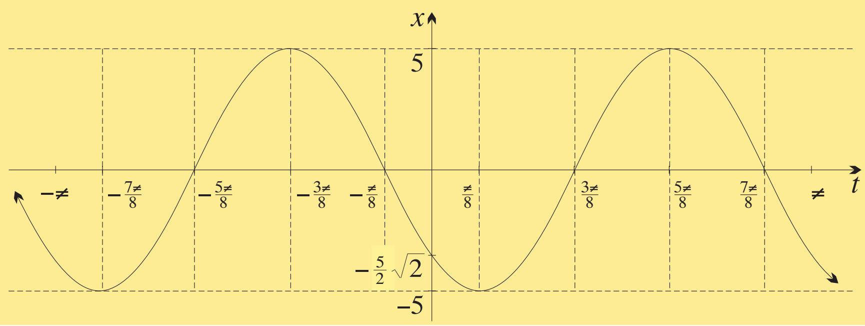
13 THE GRAPHS OF $x = a\cos(nt+\alpha)$ AND $x = a\sin(nt+\alpha)$: Write $nt+\alpha = n(t+\alpha/n)$.
Then the equations become

$$x = a\cos n(t + \alpha/n) \qquad \text{and} \qquad x = a\sin n(t + \alpha/n),$$

which are $x = a\cos nt$ and $x = a\sin nt$ shifted left by α/n.

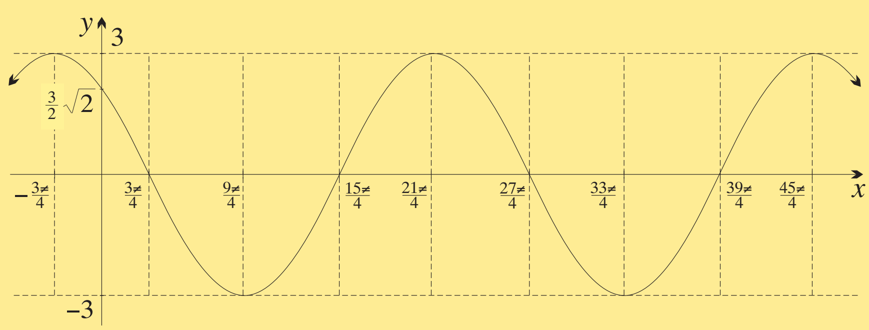
WORKED EXERCISE: Sketch $x = 5\cos\left(2t + \frac{3\pi}{4}\right)$, for $-\pi \le t \le \pi$.

SOLUTION: $x = 5\cos\left(2t + \frac{3\pi}{4}\right) = 5\cos 2\left(t + \frac{3\pi}{8}\right)$
This is a cosine wave with amplitude 5 and period π, shifted left by $\frac{3\pi}{8}$ units.
Also when $t = 0$, $x = 5\cos\frac{3\pi}{4} = -\frac{5}{2}\sqrt{2}$.



WORKED EXERCISE: Sketch $y = -3\sin\left(\frac{1}{3}x - \frac{\pi}{4}\right)$.

SOLUTION: $y = -3\sin\left(\frac{1}{3}x - \frac{\pi}{4}\right) = -3\sin\frac{1}{3}\left(x - \frac{3\pi}{4}\right)$
This is $y = 3\sin\frac{1}{3}\left(x - \frac{3\pi}{4}\right)$ reflected in the x-axis.
This is a sine wave with amplitude 3, and period 6π, shifted right by $\frac{3\pi}{4}$.
Also when $x = 0$, $y = -3\sin\left(-\frac{\pi}{4}\right) = \frac{3}{2}\sqrt{2}$.



WORKED EXERCISE: A weight on a spring is moving in simple harmonic motion with a period of $\frac{2\pi}{5}$ seconds. A laser observation at a certain instant shows it to be 15 cm below the origin, moving upwards at 60 cm/s.

(a) Find the displacement x of the weight above the origin as a function of the time t after the laser observation. Use the form $x = a\sin(nt - \alpha)$.

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(b) Find how long the weight takes to reach the origin (three significant figures).

(c) [A harder question] Find when the weight returns to where it was first observed. Use the $\sin A = \sin B$ approach to solve the trigonometric equation.

SOLUTION:

(a) We know $n = 2\pi/T$ and $T = \frac{2\pi}{5}$, so $n = 5$.

Let $x = a\sin(5t - \alpha)$, where $a > 0$ and $0 \le \alpha < 2\pi$,

then differentiating, $v = 5a\cos(5t - \alpha)$.

When $t = 0$, $x = -15$, so $-15 = a\sin(-\alpha)$,

and since $\sin\theta$ is odd, $15 = a\sin\alpha$. (1)

When $t = 0$, $v = 60$, so $60 = 5a\cos(-\alpha)$,

and since $\cos\theta$ is even, $12 = a\cos\alpha$. (2)

Squaring and adding, $369 = a^2$,

$a = 3\sqrt{41}$ (since $a > 0$).

Substituting, $\sin\alpha = 5/\sqrt{41}$, (1A)

and $\cos\alpha = 4/\sqrt{41}$. (2A)

Hence α is acute with $\alpha = \tan^{-1}\frac{5}{4} \doteqdot 0{\cdot}896\ldots$ (store in memory)

and so $x = 3\sqrt{41}\,\sin(5t - \alpha)$.

(b) When $x = 0$, $\sin(5t - \alpha) = 0$,

so for the first positive solution, $5t - \alpha = 0$

$$t = \tfrac{1}{5}\alpha$$
$$\doteqdot 0{\cdot}179 \text{ seconds.}$$

(c) To find when the weight returns to its starting place,

we need the first positive solution of $x(t) = x(0)$.

That is, $\sin(5t - \alpha) = \sin(-\alpha)$

$5t - \alpha = \pi - (-\alpha)$ (using solutions to $\sin A = \sin B$)

$$5t = \pi + 2\alpha$$
$$t = \tfrac{\pi}{5} + \tfrac{2}{5}\alpha$$
$$\doteqdot 0{\cdot}987 \text{ seconds.}$$

Using the Standard Form $x = b\sin nt + c\cos nt$: We know from Section 2E that functions of the form $x = a\sin(nt+\alpha)$ or $x = \cos(nt+\alpha)$ are equivalent to functions of the form $x = b\sin nt + c\cos nt$. This standard form is often easier to use. First, it avoids the difficulties with the calculation of the auxiliary angle. Secondly, it makes substitution of the initial displacement and velocity particularly easy.

14 THE STANDARD FORM $x = b\sin nt + c\cos nt$ FOR SIMPLE HARMONIC MOTION: When a particle starts neither at the origin nor at one extreme, it may be more convenient to use the standard form

$$x = b\sin nt + c\cos nt.$$

[This becomes $x = x_0 + b\sin nt + c\cos nt$ if the centre is not at the origin.]

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Provided that the centre is at the origin, the displacement still satisfies the differential equation $\ddot{x} = -n^2 x$. To check this:

$$\begin{aligned} x &= b\sin nt + c\cos nt \\ \dot{x} &= nb\cos nt - nc\sin nt \\ \ddot{x} &= -n^2 b\sin nt - n^2 c\cos nt \\ &= -n^2 x, \text{ as required.} \end{aligned}$$

This is hardly surprising, since the function is the same function, but written in a different form.

WORKED EXERCISE: Repeat the previous worked exercise using the standard form $x = b\sin nt + c\cos nt$. Use the t-formulae to solve part (c).

SOLUTION:

(a) Let $x = b\sin 5t + c\cos 5t$.

Differentiating, $v = 5b\cos 5t - 5c\sin 5t$.

When $t = 0$, $x = -15$, so $-15 = 0 + c$
$c = -15$.

When $t = 0$, $v = 60$, so $60 = 5b + 0$
$b = 12$

Hence $x = 12\sin 5t - 15\cos 5t$.

(b) When $x = 0$, $12\sin 5t = 15\cos 5t$
$\tan 5t = \frac{5}{4}$,

so the first positive solution is $t = \frac{1}{5}\tan^{-1}\frac{5}{4}$
$\doteqdot 0{\cdot}179$ seconds.

(c) Put $12\sin 5t - 15\cos 5t = -15$
$4\sin 5t - 5\cos 5t = -5$.

Let $T = \tan\frac{5}{2}t$. (Here $\theta = 5t$, so $\frac{1}{2}\theta = \frac{5}{2}t$.)

Then
$$\begin{aligned} \frac{8T}{1+T^2} - \frac{5(1-T^2)}{1+T^2} &= -5 \\ 8T - 5 + 5T^2 &= -5 - 5T^2 \\ 10T^2 + 8T &= 0 \\ 2T(5T+4) &= 0, \end{aligned}$$

so $\tan\frac{5}{2}t = 0$ or $\tan\frac{5}{2}t = -\frac{4}{5}$.

Hence the first positive solution is $t = \frac{2}{5}(\pi - \tan^{-1}\frac{4}{5})$
$\doteqdot 0{\cdot}987$ seconds.

Exercise 3D

1. A particle is moving in simple harmonic motion with displacement $x = \frac{4}{\pi}\sin \pi t$, in units of metres and seconds.

(a) Differentiate to find v and $\ddot{x}$ as functions of time, and show that $\ddot{x} = -\pi^2 x$.

(b) What are the amplitude, period and centre of the motion?

(c) What are the maximum speed, acceleration and distance from the origin?

(d) Sketch the graphs of x, v and $\ddot{x}$ against time.

(e) Find the next two times the particle is at the origin, and the velocities then.

(f) Find the first two times the particle is stationary, and the accelerations then.

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2. A particle is moving in simple harmonic motion with period 4 seconds and centre the origin, and starts from rest 12 cm on the positive side of the origin.

(a) Find x as a function of t. [HINT: Since it starts at the maximum, this is a cosine function, so put $x = a\cos nt$. Now find a and n from the data.]

(b) Differentiate to find v and $\ddot{x}$ as functions of t, and show that $\ddot{x} = -n^2x$.

(c) How long is it between visits to the origin?

3. A particle moving in simple harmonic motion has speed 12 m/s at the origin. Find the displacement–time equation if it is known that for positive constants a and n:

(a) $x = a\sin 3t$ (b) $x = 2\sin nt$ (c) $x = a\cos 8t$ (d) $x = 16\cos nt$

[HINT: Start by differentiating the given equation to find the equation of v. Then use the fact that the speed at the origin is the maximum value of $|v|$.]

4. [HINT: Since each particle starts from the origin, moving forwards, its displacement–time equation is a sine function. Thus put $x = a\sin nt$, then find a and n from the data.]

(a) A particle moving in simple harmonic motion with centre the origin and period π seconds starts from the origin with velocity 4 m/s. Find x and v as functions of time, and the interval within which it moves.

(b) A particle moving in simple harmonic motion with centre the origin and amplitude 6 metres starts from the origin with velocity 4 m/s. Find x and v as functions of time, and the period of its motion.

5. (a) A particle's displacement is given by $x = b\sin nt + c\cos nt$, where $n > 0$. Find v and $\ddot{x}$ as functions of t. Then show that $\ddot{x} = -n^2x$, and hence that the motion is simple harmonic.

(b) By substituting into the expressions for x and v, find b and c if initially the particle is at rest at $x = 3$.

(c) Find b, c and n, and the first time the particle reaches the origin, if the particle is initially at rest at $x = 5$, and the period is 1 second.

6. A particle's displacement is $x = 12 - 2\cos 3t$, in units of centimetres and seconds.

(a) Differentiate to find v and $\ddot{x}$ as functions of t, show that the particle is initially stationary at $x = 10$, and sketch the displacement–time graph.

(b) What are the amplitude, period and centre of the motion?

(c) In what interval is the particle moving, and how long does it take to go from one end to the other?

(d) Find the first two times after time zero when the particle is closest to the origin, and the speed and acceleration then.

(e) Find the first two times when the particle is at the centre, and the speed and acceleration then.

7. A particle is moving in simple harmonic motion according to $x = 6\sin(2t + \frac{\pi}{2})$.

(a) What are the amplitude, period and initial phase?

(b) Find $\dot{x}$ and $\ddot{x}$, and show that $\ddot{x} = -n^2x$, for some $n > 0$.

(c) Find the first two times when the particle is at the origin, and the velocity then.

(d) Find the first two times when the velocity is maximum, and the position then.

(e) Find the first two times the particle returns to its initial position, and its velocity and acceleration then.

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8. (a) Explain why $\sin(t + \frac{\pi}{2}) = \cos t$, and $\cos(t - \frac{\pi}{2}) = \sin t$:
(i) algebraically, (ii) by shifting.
(b) Simplify $x = \sin(t - \frac{\pi}{2})$ and $x = \cos(t + \frac{\pi}{2})$: (i) algebraically, (ii) by shifting.

DEVELOPMENT

9. A particle is travelling in simple harmonic motion about the origin with period 24 seconds and amplitude 120 metres. Initially it is at the origin, moving forwards.
(a) Write down the functions x and v, and state the maximum speed.
(b) What is the first time when it is 30 metres: (i) to the right of the origin, (ii) to the left of the origin? (Answer correct to four significant figures.)
(c) Find the first two times its speed is half its maximum speed.

10. A particle moves in simple harmonic motion about the origin with period $\frac{\pi}{2}$ seconds. Initially the particle is at rest 4 cm to the right of O.
(a) Write down the displacement–time and velocity–time functions.
(b) Find how long the particle takes to move from its initial position to: (i) a point 2 cm to the right of O, (ii) a point 2 cm on the left of O.
(c) Find the first two times when the speed is half the maximum speed.

11. The equation of motion of a particle is $x = \sin^2 t$. Use trigonometric identities to put the equation in the form $x = x_0 - a\cos nt$, and state the centre, amplitude, range and period of the motion.

12. A particle moves according to $x = 3 - 2\cos^2 2t$, in units of centimetres and seconds.
(a) Use trigonometric identities to put the equation in the form $x = x_0 - a\cos nt$.
(b) Find the centre of motion, the amplitude, the range of the motion and the period.
(c) What is the maximum speed of the particle, and when does it first occur?

13. A particle's displacement is given by $x = b\sin nt + c\cos nt$, where $n > 0$. Find v as a function of t. Then find b, c and n, and the first two times the particle reaches the origin, if:
(a) the period is 4π, the initial displacement is 6 and the initial velocity is 3,
(b) the period is 6, $x(0) = -2$ and $\dot{x}(0) = 3$.

14. By taking out the coefficient of t, state the amplitude, period and natural shift left or right of each graph. Hence sketch the curve in a domain showing at least one full period. Show the coordinates of all intercepts. [HINT: For example, the first function is $x = 4\cos 2(t - \frac{\pi}{6})$, which has amplitude 4, period π, and is $x = 4\cos 2t$ shifted right by $\frac{\pi}{6}$.]
(a) $x = 4\cos(2t - \frac{\pi}{3})$
(b) $x = \frac{1}{3}\sin(\frac{1}{2}t + \frac{\pi}{4})$
(c) $x = -3\cos(\frac{1}{3}t + \pi)$
(d) $x = -2\sin(4t - \pi)$
How many times is each particle at the origin during the first 2π seconds?

15. Use the functions in the previous question to sketch these graphs. Show all intercepts.
(a) $x = 4 + 4\cos(2t - \frac{\pi}{3})$
(b) $x = -1 + \frac{1}{3}\sin(\frac{1}{2}t + \frac{\pi}{4})$
(c) $x = -3 - 3\cos(\frac{1}{3}t + \pi)$
(d) $x = 3 - 2\sin(4t - \pi)$
How many times is each particle at the origin during the first 2π seconds?

16. Given that $x = a\sin(nt + \alpha)$ (in units of metres and seconds), find v as a function of time. Find a, n and α if $a > 0$, $n > 0$, $0 \le \alpha < 2\pi$ and:
(a) the period is 6 seconds, and initially $x = 0$ and $v = 5$,
(b) the period is 3π seconds, and initially $x = -5$ and $v = 0$,
(c) the period is 2π seconds and initially $x = 1$ and $v = -1$.

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17. Given $x = a\cos(2t - \varepsilon)$, find the function v. Find a and ε if $a > 0$, $0 \le \varepsilon < 2\pi$ and:

(a) initially $x = 0$ and $v = 6$, (b) initially $x = 1$ and $v = -2\sqrt{3}$.

18. A particle is moving in simple harmonic motion according to $x = a\cos(\frac{\pi}{8}t + \alpha)$, where $a > 0$ and $0 \le \alpha < 2\pi$. When $t = 2$ it passes through the origin, and when $t = 4$ its velocity is 4 cm/s in the negative direction. Find the amplitude a and the initial phase α.

19. A particle is moving in simple harmonic motion with period 8π seconds according to $x = a\sin(nt + \alpha)$, where x is the displacement in metres, and $a > 0$ and $0 \le \alpha < 2\pi$. When $t = 1$, $x = 3$ and $v = -1$. Find a and α correct to four significant figures.

20. A particle moving in simple harmonic motion has period $\frac{\pi}{2}$ seconds. Initially the particle is at $x = 3$ with velocity $v = 16$ m/s.

(a) Find x as a function of t in the form $x = b\sin nt + c\cos nt$.

(b) Find x as a function of t in the form $x = a\cos(nt - \varepsilon)$, where $a > 0$ and $0 \le \varepsilon < 2\pi$.

(c) Find the amplitude and the maximum speed of the particle.

(d) Find the first time the particle is at the origin, using each of the above displacement functions in turn. Prove that the two answers obtained are the same.

21. A particle moves in simple harmonic motion with period 8π. Initially, it is at the point P where $x = 4$, moving with velocity $v = 6$. Find, correct to three significant figures, how long it takes to return to P:

(a) by expressing the motion in the form $x = b\sin nt + c\cos nt$, and using the t-formulae.

(b) by expressing the motion in the form $x = a\cos(nt - \alpha)$, and using the solutions to $\cos A = \cos B$. [HINT: You will find that $\dfrac{1}{\sqrt{37}} = \cos\alpha$.]

22. A particle moves on a line, and the table below shows some observations of its positions at certain times:

t (in seconds)	0	7	9	11	18
x (in metres)	0			2	0

(a) Complete the table if the particle is moving with constant acceleration.

(b) Complete the table if the particle is moving in simple harmonic motion with centre the origin and period 12 seconds.

23. The temperature at each instant of a day can be modelled by a simple harmonic function oscillating between 9° at 4:00 am and 19° at 4:00 pm. Find, correct to the nearest minute, the times between 4:00 am and 4:00 pm when the temperature is:

(a) 14° (b) 11° (c) 17°

24. The rise and fall in sea level due to tides can be modelled by simple harmonic motion. On a certain day, a channel is 10 metres deep at 9:00 am when it is low tide, and 16 metres deep at 4:00 pm when it is high tide. If a ship needs 12 metres of water to sail down a channel safely, at what times (correct to the nearest minute) between 9:00 am and 9:00 pm can the ship pass through?

25. (a) Express $x = -4\cos 3\pi t + 2\sin 3\pi t$ in the form $x = a\cos(3\pi t - \varepsilon)$, where $a > 0$ and $0 \le \varepsilon < 2\pi$, giving ε to four significant figures. (The units are cm and seconds.)

(b) Hence find, correct to the nearest 0·001 seconds:

(i) when the particle is first 3 cm on the positive side of the origin.

(ii) when the particle is first moving with velocity -1 cm/s.

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26. A particle is moving in simple harmonic motion with period $2\pi/n$, centre the origin, initial position $x(0)$ and initial velocity $v(0)$. Find its displacement–time equation in the form $x = b\sin nt + c\cos nt$, and write down its amplitude.

27. Show that for any particle moving in simple harmonic motion, the ratio of the average speed over one oscillation to the maximum speed is $2:\pi$.

EXTENSION

28. [Sums to products and products to sums are useful in this question.]

(a) Express $\sin nt + \sin(nt+\alpha)$ in the form $a\sin(nt+\varepsilon)$, and hence show that

$$\sin nt + \sin(nt+\pi) \equiv 0.$$

(b) Show that $\sin nt + \sin(nt+\alpha) + \sin(nt+2\alpha) \equiv (1+2\cos\alpha)\sin(nt+\alpha)$, and hence

$$\sin nt + \sin(nt+\tfrac{2\pi}{3}) + \sin(nt+\tfrac{4\pi}{3}) \equiv 0.$$

(c) Prove $\sin nt+\sin(nt+\alpha)+\sin(nt+2\alpha)+\sin(nt+3\alpha) \equiv (2\cos\tfrac{1}{2}\alpha+2\cos\tfrac{3}{2}\alpha)\sin(nt+\tfrac{3}{2}\alpha)$. Hence show that

$$\sin nt + \sin(nt+\tfrac{\pi}{2}) + \sin(nt+\pi) + \sin(nt+\tfrac{3\pi}{2}) \equiv 0.$$

(d) Generalise these results to $\sin nt+\sin(nt+\alpha)+\sin(nt+2\alpha)+\cdots+\sin\big(nt+(k-1)\alpha\big)$, and show that if $\alpha = \dfrac{2\pi}{k}$, then

$$\sin nt + \sin(nt+\alpha) + \sin(nt+2\alpha) + \cdots + \sin\big(nt+(k-1)\alpha\big) \equiv 0.$$

3 E Motion Using Functions of Displacement

In many physical situations, the acceleration or velocity of the particle is more naturally understood as a function of where it is (the displacement x) than of how long it has been travelling (the time t). For example, the acceleration of a body being drawn towards a magnet depends on how far it is from the magnet.

In such situations, the function must be integrated with respect to x rather than t, because x is the variable in the function. This section deals with the necessary mathematical techniques.

Velocity as a Function of Displacement: Suppose that the velocity is given as a function of displacement, for example $v = e^{-x}$. All that is required here is to take reciprocals of both sides, because the reciprocal of $v = \dfrac{dx}{dt}$ is $\dfrac{dt}{dx}$.

15 VELOCITY AS A FUNCTION OF DISPLACEMENT: If the velocity is given as a function of displacement, take the reciprocal to give $\dfrac{dt}{dx}$ as a function of x, and then integrate with respect to x.

WORKED EXERCISE: Suppose that a particle is initially at the origin, and moves according to $v = e^{-x}$ m/s. Find x, v and $\ddot{x}$ in terms of t, and find how long it takes for the particle to travel 1 metre. Briefly describe the subsequent motion.

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SOLUTION:

Given $\dfrac{dx}{dt} = e^{-x}$,

$\dfrac{dt}{dx} = e^{x}$

$t = e^{x} + C$.

When $t = 0$, $x = 0$,

$0 = 1 + C$,

so $C = -1$, and $t = e^{x} - 1$,

and it takes $e - 1$ seconds to go 1 metre.

Solving for x, $e^{x} = t + 1$

$x = \log(t + 1)$.

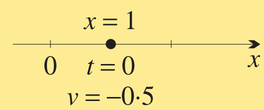
Differentiating, $v = \dfrac{1}{t+1}$

and $\ddot{x} = -\dfrac{1}{(t+1)^2}$.

The particle moves to infinity.
Its velocity remains positive, but decreases with limit zero.

WORKED EXERCISE: A particle moves so that its velocity is proportional to its displacement from the origin O. Initially it is 1 cm to the right of the origin, moving to the left with a speed of 0·5 cm/s. Find the displacement and velocity as functions of time, and briefly describe its motion.

SOLUTION: $v = kx$, for some constant k.



When $x = 1$, $v = -\frac{1}{2}$, so $-\frac{1}{2} = k \times 1$,

so $k = -\frac{1}{2}$, and $v = -\frac{1}{2}x$.

Taking reciprocals, $\dfrac{dt}{dx} = -\dfrac{2}{x}$

and integrating, $t = -2\log x + C$, for some constant C.

When $x = 1$, $t = 0$, so $0 = 0 + C$,

so $C = 0$, and $t = -2\log x$.

Solving for x, $x = e^{-\frac{1}{2}t}$,

and differentiating, $v = -\frac{1}{2}e^{-\frac{1}{2}t}$.

Thus the particle continues to move to the left, its speed decreasing with limit zero, and the origin being its limiting position.

Acceleration as a Function of Displacement: Acceleration has been defined as the rate of change of velocity with respect to time, that is as $\ddot{x} = \dfrac{dv}{dt}$. Dealing with situations where acceleration is a function of displacement requires the following alternative form for acceleration.

16 ACCELERATION AS A DERIVATIVE WITH RESPECT TO DISPLACEMENT:

The acceleration is given by $\ddot{x} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$.

PROOF: [Examinable]

First, using the chain rule:

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \frac{d}{dv}\left(\tfrac{1}{2}v^2\right) \times \frac{dv}{dx} = v\,\frac{dv}{dx}.$$

Secondly, using the chain rule again,

$$v\,\frac{dv}{dx} = \frac{dx}{dt} \times \frac{dv}{dx} = \frac{dv}{dt} = \ddot{x}.$$

The method of solving such problems is now clear:

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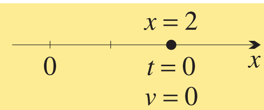
17 **ACCELERATION AS A FUNCTION OF DISPLACEMENT:** If the acceleration is given as a function of displacement, use the form $\ddot{x} = \dfrac{d}{dx}(\frac{1}{2}v^2)$ for acceleration, and then integrate with respect to x.

NOTE: The intermediate step in the proof above shows that $\ddot{x} = v\dfrac{dv}{dx}$ is yet another form of the acceleration. This form is very useful when acceleration is a function of velocity — air resistance is a good example of this, because the resistance offered by the air to a projectile moving through it is a function of the projectile's speed. Such equations are a topic in the 4 Unit course, not the 3 Unit course, but a couple of these questions are offered in the Extension section of the following exercise.

WORKED EXERCISE: Suppose that a ball attached to the ceiling by a long spring will hang at rest at the point $x = 0$. The ball is lifted 2 metres above $x = 0$ and dropped, and subsequently moves according to the equation $\ddot{x} = -4x$. Find its speed as a function of x, and show that it comes to rest 2 metres below $x = 0$. Find its maximum speed and the place where this occurs.

SOLUTION: We know that $\ddot{x} = -4x$,



so $\dfrac{d}{dx}(\frac{1}{2}v^2) = -4x$.

Integrating with respect to x, $\frac{1}{2}v^2 = -2x^2 + \frac{1}{2}C$, for some constant C,

$v^2 = -4x^2 + C$.

(NOTE: It is easier to work with v^2 as the subject, so it is easier to take the constant of integration as $\frac{1}{2}C$ rather than C.)

When $x = 2$, $v = 0$, so $0 = -16 + C$

so $C = 16$, and $v^2 = 16 - 4x^2$.

Hence $v = 0$ when $x = -2$, as required.

The maximum speed is 4 m/s when $x = 0$.

Acceleration as a Function of Displacement — The Second Integration:

Integrating using $\ddot{x} = \dfrac{d}{dx}(\frac{1}{2}v^2)$ will straightforwardly yield v^2 as a function of x. Further integration, however, requires taking the square root of v^2, and this will be blocked or very complicated if the sign of v cannot be determined easily.

18 **ACCELERATION AS A FUNCTION OF DISPLACEMENT — THE SECOND INTEGRATION:** The first integration will give v^2 as a function of x. If the sign of v can be determined, then take square roots to give v as a function of x, and proceed as before.

WORKED EXERCISE: A particle is moving with acceleration function $\ddot{x} = 3x^2$. Initially $x = 1$ and $v = -\sqrt{2}$.

(a) Find v^2 as a function of displacement.

(b) Assuming that v is never positive, find the displacement as a function of time, and briefly describe the motion, mentioning what happens as $t \to \infty$.

(c) [A harder question] Explain why the velocity can never be positive.

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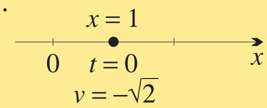
SOLUTION:

(a) Since acceleration is given as a function of displacement, we write:

$$\frac{d}{dx}(\tfrac{1}{2}v^2) = 3x^2$$
$$\tfrac{1}{2}v^2 = x^3 + \tfrac{1}{2}C, \text{ for some constant } C.$$
$$v^2 = 2x^3 + C, \text{ for some constant } C.$$

When $x = 1$, $v = -\sqrt{2}$, so $\quad 2 = 2 + C$,

so $C = 0$, and $\quad v^2 = 2x^3$.



(b) Taking square roots, $\quad v = -\sqrt{2}\,x^{\frac{3}{2}}$, assuming that v is never positive.

Taking reciprocals, $\quad \dfrac{dt}{dx} = -\tfrac{1}{2}\sqrt{2}\,x^{-\frac{3}{2}}$

$$t = \sqrt{2}\,x^{-\frac{1}{2}} + D, \text{ for some constant } D.$$

When $t = 0$, $x = 1$, so $\quad 0 = \sqrt{2} + D$,

so $D = -\sqrt{2}$, and $\quad t = \sqrt{2}\,x^{-\frac{1}{2}} - \sqrt{2}$

$$x^{-\frac{1}{2}} = \frac{t + \sqrt{2}}{\sqrt{2}}$$
$$x = \frac{2}{(t + \sqrt{2})^2}.$$

Hence the particle begins at $x = 1$, and moves backwards towards the origin. As $t \to \infty$, its speed has limit zero, and its limiting position is $x = 0$.

(c) Initially, v is negative. Since $v^2 = 2x^3$, it follows that v can only be zero at the origin; but since $\ddot{x} = 3x^2$ the acceleration at the origin would also be zero. Hence if the particle ever arrived at the origin it would then be permanently at rest. Thus the velocity can never change from negative to positive.

Exercise 3E

1. In each case, v is given as a function of x, and it is known that $x = 1$ when $t = 0$. Express: (i) t in terms of x, (ii) x in terms of t. [HINT: Start by taking reciprocals of both sides, which gives dt/dx as a function of x. Then integrate with respect to x.]

(a) $v = 6$ (b) $v = -6x^{-2}$ (c) $v = 2x - 1$ (d) $v = -6x^2$ (e) $v = -6x^3$ (f) $v = e^{-2x}$ (g) $v = 1 + x^2$ (h) $v = \cos^2 x$

2. In each motion of the previous question, find $\ddot{x}$ using the formula $\ddot{x} = \dfrac{d}{dx}(\tfrac{1}{2}v^2)$.

3. In each case, the acceleration $\ddot{x}$ is given as a function of x. By replacing $\ddot{x}$ by $\dfrac{d}{dx}(\tfrac{1}{2}v^2)$ and integrating, express v^2 in terms of x, given that $v = 0$ when $x = 0$.

(a) $\ddot{x} = 6x^2$ (b) $\ddot{x} = \dfrac{1}{e^x}$ (c) $\ddot{x} = 6$ (d) $\ddot{x} = \dfrac{1}{2x+1}$ (e) $\ddot{x} = \sin 6x$ (f) $\ddot{x} = \dfrac{1}{4+x^2}$

4. A stone is dropped from a lookout 500 metres above the valley floor. Take $g = 10$ m/s^2, ignore air resistance, take downwards as positive, and use the lookout as the origin of displacement — the equation of motion is then $\ddot{x} = 10$.

(a) Replace $\ddot{x}$ by $\dfrac{d}{dx}(\tfrac{1}{2}v^2)$ and show that $v^2 = 20x$. Hence find the impact speed.

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(b) Explain why, during the fall, $v = \sqrt{20x}$ rather than $v = -\sqrt{20x}$.

(c) Integrate to find the displacement–time function, and find how long it takes to fall.

5. [An alternative approach to the worked exercise in Section 3B] A ball is thrown vertically upwards at $20\,\text{m/s}^2$. Take $g = 10\text{m/s}$, ignore air resistance, take upwards as positive, and use the ground as the origin of displacement — the equation of motion is then $\ddot{x} = -10$.

(a) Show that $v^2 = 400 - 20x$, and find the maximum height.

(b) Explain why $v = \sqrt{400 - 20x}$ while the ball is rising.

(c) Integrate to find the displacement–time function, and find how long it takes the ball to reach maximum height.

6. [A formula from physics — not to be used in this course] A particle moves with constant acceleration a, so that its equation of motion is $\ddot{x} = a$. Its initial velocity is u. After t seconds, its velocity is v and its displacement is s.

(a) Use $\dfrac{d}{dx}(\tfrac{1}{2}v^2)$ for acceleration to show that $v^2 = u^2 + 2as$.

(b) Verify the impact speed in the previous question using this formula.

7. The acceleration of a particle P is given by $\ddot{x} = -2x$ (in units of centimetres and seconds), and the particle starts from rest at $x = 2$.

(a) Find the speed of P when it first reaches $x = 1$, and explain whether it must then be moving backwards or forwards.

(b) In what interval is the motion confined, and what is the maximum speed?

8. A particle moves according to $v = \frac{8}{3}x^{-2}$, where $t \ge 1$. When $t = 1$, the particle is at $x = 2$.

(a) Find t as a function of x, and x as a function of t.

(b) Hence find v and $\ddot{x}$ as functions of t.

(c) Use $\ddot{x} = \dfrac{d}{dx}(\tfrac{1}{2}v^2)$ to find $\ddot{x}$ as a function of x.

9. (a) Prove that $\dfrac{d}{dx}(x \log x) = \log x + 1$.

(b) A particle moves according to $\ddot{x} = 1 + \log x$. Initially it is stationary at $x = 1$. Find v^2 as a function of x.

(c) Explain why v is always positive for $t > 0$, and find v when $x = e^2$.

10. A particle moves according to $\ddot{x} = \dfrac{1}{36 + x^2}$, and is initially at rest at O.

(a) Find v^2 as a function of x, and explain why v is always positive for $t > 0$.

(b) Find: (i) the velocity at $x = 6$, (ii) the velocity as $t \to \infty$.

DEVELOPMENT

11. A plane lands on a runway at 100 m/s. It then brakes with a constant deceleration until it stops 2 km down the runway.

(a) Explain why the equation of motion is $\ddot{x} = -k$, for some positive constant k. By integrating with respect to x, find k, and find v^2 as a function of x.

(b) Find: (i) the velocity after 1 km, (ii) where it is when the velocity is 50 m/s.

(c) Explain why, during the braking, $v = \sqrt{10\,000 - 5x}$ rather than $v = -\sqrt{10\,000 - 5x}$.

(d) Integrate to find the displacement–time function, and find how long it takes to stop.

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12. (a) A particle has acceleration $\ddot{x} = e^{-x}$, and initially $v = 2$ and $x = 0$. Find v^2 as a function of x, and explain why v is always positive and at least 2. Then briefly explain what happens as time goes on.

(b) Another particle has the same acceleration $\ddot{x} = e^{-x}$, and initially is also at $x = 0$. Find what the initial velocity V was if the particle first goes backwards, but turns around at $x = -1$. What happens to the velocity as time goes on?

13. The velocity of a particle starting at the origin is $v = \cos^2 2x$.

(a) Explain why the particle can never be in the same place at two different times.

(b) Find x and v as functions of t, and find the limiting position as $t \to \infty$.

(c) Show that $\ddot{x} = -4\cos^3 2x \sin 2x$, and find t, v and $\ddot{x}$ when $x = \frac{\pi}{8}$.

14. Suppose that $v = 6 - 2x$, and that initially, the particle is at the origin.

(a) Find the acceleration at the origin.

(b) Show that $t = -\frac{1}{2}\log(1 - \frac{1}{3}x)$, and find x as a function of t.

(c) Describe the behaviour of the particle as $t \to \infty$.

15. The velocity of a particle at displacement x is given by $v = x^2 e^{-x^2}$, and initially the particle is at $x = \frac{1}{2}$.

(a) Explain why the particle can never be on the negative side of $x = \frac{1}{2}$. Then find the acceleration as a function of x, and hence find the maximum velocity and where it occurs.

(b) Explain why the time T for the particle to travel to $x = 1$ is $T = \displaystyle\int_{\frac{1}{2}}^{1} \frac{e^{x^2}}{x^2}\,dx$. Then use Simpson's rule with three function values to approximate T, giving your answer correct to four significant figures.

16. A particle moves with acceleration $-\frac{1}{2}e^{-x}\ \text{m/s}^2$.

(a) Initially it is at the origin with velocity $1\ \text{m/s}$. Find an expression for v^2.

(b) Explain why v is always positive for $t > 0$. Hence find the displacement as a function of time, and describe what happens to the particle as $t \to \infty$.

17. A particle's acceleration is $\ddot{x} = 2x - 1$, and initially the particle is at rest at $x = 5$.

(a) Find v^2, and explain why the particle can never be at the origin.

(b) Find where $|v| = 2\sqrt{5}$, justifying your answer, and describe the subsequent motion.

18. Two particles A and B are moving towards the origin from the positive side with equations $v_{\text{A}} = -(16 + x^2)$ and $v_{\text{B}} = -4\sqrt{16 - x^2}$. If A is released from $x = 4$, where should B be released from, if they are to be released together and reach the origin together?

19. A particle moves with acceleration function $\ddot{x} = 3x^2$. Initially $x = 1$ and $v = -\sqrt{2}$.

(a) Find v^2 as a function of displacement.

(b) Explain why the velocity can never be positive. Then find the displacement–time function, and briefly describe the motion.

20. For a particle moving on the x-axis, $v^2 = 14x - x^2$.

(a) By completing the square, find the section of the number line where the particle is confined, then find its maximum speed and where this occurs.

(b) Where is $|\ddot{x}| \le 3$?

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21. A particle's acceleration is given by $\ddot{x} = x(3x - 14)$, and initially it is at the origin with velocity $6\sqrt{2}$.

(a) Show that $v^2 = 2(x+2)(x-3)(x-6)$, and sketch the graph of v^2.

(b) Find the velocity and acceleration at $x = 3$. In which direction does the particle move off from $x = 3$?

(c) Find the maximum speed, and where it occurs. Describe the motion of the particle.

22. An electron is fired with initial velocity 10^7 m/s into an alternating force field so that its acceleration x metres from its point of entry is $k\pi \sin \pi x$, for some positive constant k.

(a) Find v^2 in terms of x and k, explain why its velocity never drops below its initial velocity, and find where the electron will have this minimum velocity and where it will have maximum velocity.

(b) If the electron's maximum velocity is 2×10^7 m/s, find k, and hence find the maximum acceleration and where it occurs.

23. The velocity of a particle is proportional to its displacement. When $t = 0$, $x = 2$, and when $t = 10$, $x = 4$. Find the displacement–time function, and find the displacement when $t = 25$.

EXTENSION

24. Newton's law of gravitation says that an object falling towards a planet has acceleration $\ddot{x} = -kx^{-2}$, for some positive constant k, where x is the distance from the centre of the planet. Show that if the body starts from rest at a distance D from the centre, then its speed at a distance x from the centre is $\sqrt{\dfrac{2k(D-x)}{Dx}}$.

25. A projectile is fired vertically upwards with speed V from the surface of the Earth.

(a) Assuming the same equation of motion as in the previous question, and ignoring air resistance, show that $k = gR^2$, where R is the radius of the Earth.

(b) Find v^2 in terms of x and hence find the maximum height of the projectile.

(c) [The escape velocity from the Earth] Given that $R = 6400$ km and $g = 9{\cdot}8$ m/s^2, find the least value of V so that the projectile will never return.

26. Assume that a bullet, fired at 1 km/s, moves through water with deceleration proportional to the square of the velocity, so that $\ddot{x} = -kv^2$, for some positive constant k.

(a) If the velocity after 100 metres is 10 m/s, start with $\ddot{x} = v\dfrac{dv}{dx}$ and find where the bullet is when its velocity is 1 m/s.

(b) If the velocity after 1 second is 10 m/s, use $\ddot{x} = \dfrac{dv}{dt}$ to find at what time the bullet has velocity 1 m/s.

27. Another type of bullet, when fired under water, moves with deceleration proportional to its velocity, so that $\ddot{x} = -kv$, for some positive constant k. Its initial speed is $\frac{1}{2}$ km/s, and its speed after it has gone 50 metres is 250 m/s.

(a) Use $\ddot{x} = v\dfrac{dv}{dx}$ to find v as a function of x, then find x as a function of t.

(b) Show that it takes $\frac{1}{5}\log 2$ seconds to go the first 50 metres, and describe the subsequent motion of the bullet.

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3 F Simple Harmonic Motion — The Differential Equation

In Section 3D, we showed that a particle in simple harmonic motion with the centre of motion at the origin satisfies the differential equation

$$\ddot{x} = -n^2 x.$$

In this section, we shall use this differential equation as the basis of a further study of simple harmonic motion. Since $\ddot{x}$ is now given as a function of displacement rather than time, we will need the techniques of the previous section, which used the identity $\ddot{x} = \dfrac{d}{dx}(\frac{1}{2}v^2)$ before performing the integration.

An Alternative Definition of Simple Harmonic Motion: For a particle moving in simple harmonic motion, the acceleration has a particularly simple form $\ddot{x} = -n^2 x$ when it is expressed as a function of displacement — this is a linear function of x with acceleration proportional to x but oppositely directed. As with many motions, it is this acceleration–displacement function that can be measured accurately by measuring the force at various places on the number line.

For these reasons, it is convenient to introduce an alternative definition of simple harmonic motion as motion satisfying this differential equation.

19 SIMPLE HARMONIC MOTION — THE DIFFERENTIAL EQUATION: The motion of a particle is called *simple harmonic motion* if its displacement from some origin satisfies

$$\ddot{x} = -n^2 x, \quad \text{where } n \text{ is a positive constant.}$$

The acceleration is thus proportional to displacement, but oppositely directed.

This equation is usually the most straightforward way to test whether a given motion is simple harmonic with centre the origin.

Having two definitions of the one thing may be convenient, but it does require a theorem proving that the two definitions are equivalent. First, we proved in Section 3D that motion satisfying $x = a\cos(nt + \alpha)$ or $x = a\sin(nt + \alpha)$ satisfied the differential equation $\ddot{x} = -n^2 x$. Conversely, Extension questions in the following exercise prove that the differential equation has no other solutions. Although this converse is intuitively obvious, its proof is rather difficult and is not required in the course, so the following theorem can be assumed without proof whenever it is required in an exercise.

20 THE SOLUTIONS OF $\ddot{x} = -n^2 x$: If a particle's motion satisfies $\ddot{x} = -n^2 x$, then its displacement–time equation has the form

$$x = a\sin(nt + \alpha) \qquad \text{or} \qquad x = a\cos(nt + \alpha),$$

where a, n and α are constants, with $n > 0$ and $a > 0$. In particular, the period of the motion is $T = \dfrac{2\pi}{n}$.

Alternatively, the solution of the differential equation can be written as

$$x = b\sin nt + c\cos nt, \quad \text{where } b \text{ and } c \text{ are constants.}$$

The following worked exercise extracts the period from the differential equation.

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WORKED EXERCISE: A particle is moving so that $\ddot{x} = -4x$, in units of centimetres and seconds. Initially, it is stationary at $x = 6$.

(a) Write down the period and amplitude, and the displacement–time function.

(b) Find the position, velocity and acceleration of the particle at $t = \frac{\pi}{3}$.

SOLUTION:

(a) Since $\ddot{x} = -4x$, we know $n^2 = 4$, so $n = 2$ and the period is $\dfrac{2\pi}{2} = \pi$.

Since it starts stationary at $x = 6$, we know that $a = 6$.

Hence $\quad x = 6\cos 2t \quad$ (cosine starts at the maximum).

(b) Differentiating, $v = -12\sin 2t$,

and $\quad \ddot{x} = -24\cos 2t$.

When $t = \frac{\pi}{3}$, $\quad x = 6\cos\frac{2\pi}{3} = -3$,

and $\quad v = -12\sin\frac{2\pi}{3} = -6\sqrt{3}\,\text{cm/s}$,

and $\quad \ddot{x} = -24\cos\frac{2\pi}{3} = 12\,\text{cm/s}^2$.

Notice that at $x = \frac{\pi}{3}$, $\ddot{x} = -4x$, as given by the differential equation.

Integrating the Differential Equation: It is quite straightforward to integrate the differential equation once, using the methods of the previous section. This integration gives v^2 as function of x.

WORKED EXERCISE: In the previous worked exercise, $\ddot{x} = -4x$, and the particle was initially stationary at $x = 6$.

(a) Find v^2 as a function of x.

(b) Verify this using the previous expressions for displacement and velocity.

(c) Find the velocity and acceleration when the particle is at $x = 3$.

SOLUTION:

(a) Replacing $\ddot{x}$ by $\dfrac{d}{dx}(\frac{1}{2}v^2)$, $\dfrac{d}{dx}(\frac{1}{2}v^2) = -4x$.

Then integrating, $\quad \frac{1}{2}v^2 = -2x^2 + \frac{1}{2}C$, for some constant C

$$v^2 = -4x^2 + C.$$

When $x = 6$, $v = 0$, so $\quad 0 = -144 + C$,

so $C = 144$, and

$$v^2 = 144 - 4x^2$$

$$v^2 = 4(36 - x^2).$$

(b) This can be confirmed by substituting $x = 6\cos 2t$ and $v = -12\sin 2t$:

$$\text{RHS} = 4(36 - 36\cos^2 2t) = 4 \times 36\sin^2 2t \qquad \text{LHS} = 12^2\sin^2 2t = \text{RHS}$$

(c) When $x = 3$, $\ddot{x} = -4 \times 3 = -12\,\text{cm/s}^2$.

Also, $v^2 = 4(36 - 9) = 4 \times 27$,

so $\quad v = 6\sqrt{3}$ or $-6\sqrt{3}$.

This integration can easily be done in the general case, but the result should be derived by integration each time, and not quoted as a known result. The proof of the following result is left to the exercises.

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21 THE FIRST INTEGRATION OF $\ddot{x} = -n^2x$: If a particle is moving in simple harmonic motion with amplitude a, then

$$v^2 = n^2(a^2 - x^2).$$

This result should be derived by integration each time, and not quoted.

The second integration is blocked — taking the square root of v^2 requires cases, because v is positive half the time and negative the other half — and should not be attempted. Instead, quote the solutions of the differential equation, as explained.

The Five Functions of Simple Harmonic Motion: We now have x, v and $\ddot{x}$ as functions of t, and $\ddot{x}$ and v^2 as functions of x. This gives altogether five functions.

WORKED EXERCISE: A particle P moves so that its acceleration is proportional to its displacement x from a fixed point O and opposite in direction. Initially the particle is at the origin, moving with velocity 12 m/s, and the particle is stationary when $x = 4$.

(a) Find $\ddot{x}$ and v^2 as functions of x.

(b) Find x, v and $\ddot{x}$ as functions of t.

(c) Find the displacement, acceleration and times when the particle is at rest.

(d) Find the velocity, acceleration and times when the displacement is zero.

(e) Find the displacement, velocity and acceleration when $t = \frac{4\pi}{9}$.

(f) Find the acceleration and velocity and times when $x = 2$.

SOLUTION:

(a) We know that $\ddot{x} = -n^2x$,

where $n > 0$ is a constant of proportionality.

Hence $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -n^2x$.

Integrating, $\tfrac{1}{2}v^2 = -\tfrac{1}{2}n^2x^2 + \tfrac{1}{2}C$

$v^2 = -n^2x^2 + C$.

When $x = 0$, $v = 12$, so $144 = 0 + C$, hence $C = 144$.

When $x = 4$, $v = 0$, so $0 = -n^2 \times 16 + 144$,

so $n^2 = 9$ and $n = 3$, since $n > 0$.

Hence $\ddot{x} = -9x$ (1)

and $v^2 = 9(16 - x^2)$. (2)

This can also be written as $\dfrac{x^2}{16} + \dfrac{v^2}{144} = 1$,

which is the unit circle stretched by a factor of 4 in the x-direction, and by a factor of 12 in the v-direction.

(b) Also, since the amplitude a is 4 and $n = 3$,

$x = 4\sin 3t$ (sine starts at the origin, moving up). (3)

Differentiating, $v = 12\cos 3t$, (4)

and $\ddot{x} = -36\sin 3t$. (5)

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(c) Substituting $v = 0$,
from (2), $x = 4$ or -4,
from (1), $\ddot{x} = -36$ or 36,
from (4), $t = \frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \ldots$.

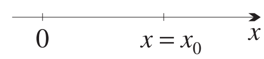
(d) Substituting $x = 0$,
from (2), $v = 12$ or -12,
from (1), $\ddot{x} = 0$,
from (3), $t = 0,\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \ldots$.

(e) Substituting $t = \frac{4\pi}{9}$,
from (3), $x = -2\sqrt{3}$,
from (4), $v = -6$,
from (5), $\ddot{x} = 18\sqrt{3}$.

(f) Substituting $x = 2$,
from (2), $v = 6\sqrt{3}$ or $-6\sqrt{3}$,
from (1), $\ddot{x} = -18$,
from (3), $t = \frac{\pi}{18},\ \frac{5\pi}{18},\ \frac{13\pi}{18},\ \ldots$.

Moving the Origin of Space: So far in this section, only simple harmonic motion with the centre at the origin has been considered. Now both speed and acceleration are independent of what origin is chosen, so the velocity and acceleration functions are unchanged if the centre of motion is shifted from the origin. This means that if the origin is shifted from $x = 0$ to $x = x_0$, then x will be replaced by $x - x_0$ in the equations of motion, but v and $\ddot{x}$ will be unchanged.

Hence the equation of simple harmonic motion with centre $x = x_0$ becomes $\ddot{x} = -n^2(x - x_0)$, and acceleration is now proportional to the displacement from $x = x_0$ but oppositely directed.



22 SIMPLE HARMONIC MOTION WITH CENTRE AT $x = x_0$: A particle is moving in simple harmonic motion about $x = x_0$ if its acceleration is proportional to its displacement $x - x_0$ from $x = x_0$ but oppositely directed, that is, if

$$\ddot{x} = -n^2(x - x_0), \quad \text{where } n \text{ is a positive constant.}$$

WORKED EXERCISE: [This question involves a situation where the centre of motion is at first unknown, and must be found by expressing $\ddot{x}$ in terms of x.] A particle's motion satisfies the equation $v^2 = -x^2 + 7x - 12$.

(a) Show that the motion is simple harmonic, and find the centre, period and amplitude of the motion.

(b) Find where the particle is when its speed is half the maximum speed.

SOLUTION:

(a) Differentiating, $\ddot{x} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$
$= -x + 3\tfrac{1}{2}$,
so $\ddot{x} = -(x - 3\tfrac{1}{2})$,
which is in the form $\ddot{x} = -n^2(x - x_0)$, with $x_0 = 3\tfrac{1}{2}$ and $n = 1$.
Hence the motion is simple harmonic, with centre $x = 3\tfrac{1}{2}$ and period 2π.
Put $v = 0$ (to find where the particle stops at its extremes)
$-x^2 + 7x - 12 = 0$
$x = 3$ or $x = 4$,
so the extremes of the motion are $x = 3$ and $x = 4$, and so the amplitude is $\tfrac{1}{2}$.

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(b) The maximum speed occurs at the centre $x = 3\frac{1}{2}$.
Substituting into $v^2 = -(x-3)(x-4)$, $v^2 = \frac{1}{4}$, and so speed $|v| = \frac{1}{2}$.
To find where the particle is when it has half that speed, put $v = \frac{1}{4}$:

$$-x^2 + 7x - 12 = \tfrac{1}{16}$$
$$16x^2 - 112x + 193 = 0$$

so $\Delta = 8^2 \times 3$, and $\quad x = 3\frac{1}{2} + \frac{1}{4}\sqrt{3}$ or $3\frac{1}{2} - \frac{1}{4}\sqrt{3}$.

Exercise 3F

1. A particle is moving according to $x = 3\cos 2t$ (in units of metres and seconds).
(a) Derive expressions for v and $\ddot{x}$ as functions of t, and for v^2 and $\ddot{x}$ in terms of x.
(b) Find the speed and acceleration of the particle at $x = 2$.

2. A particle is oscillating according to the equation $\ddot{x} = -9x$ (in units of metres and seconds), and is stationary when $x = 5$.
(a) Integrate this equation to find an equation for v^2.
(b) Find the velocity and acceleration when $x = 3$.
(c) What is the speed at the origin, and what is the period?

3. A particle is oscillating according to the equation $\ddot{x} = -16x$ (in units of centimetres and seconds), and its speed at the origin is 24 cm/s.
(a) Integrate this equation to find an equation for v^2.
(b) What are the amplitude and the period?
(c) Find the speed and acceleration when $x = 2$.

4. A particle is moving with amplitude 6 metres according to $\ddot{x} = -4x$ (the units are metres and seconds).
(a) Find the velocity–displacement equation, the period and the maximum speed.
(b) Find the simplest form of the displacement–time equation if initially the particle is:
(i) stationary at $x = 6$,
(ii) stationary at $x = -6$,
(iii) at the origin with positive velocity,
(iv) at the origin with negative velocity.

5. (a) A ball on the end of a spring moves according to $\ddot{x} = -256x$ (in units of centimetres and minutes). The ball is pulled down 2 cm from the origin and released. Find the speed at the centre of motion.
(b) Another ball on a spring moves according to $\ddot{x} + \frac{1}{4}x = 0$ (in units of centimetres and seconds), and its speed at the equilibrium position is 4 cm/s. How far was it pulled down from the origin before it was released?

6. [In these questions, the differential equation will need to be formed first.]
(a) A particle moving in simple harmonic motion has period $\frac{\pi}{2}$ minutes, and it starts from the mean position with velocity 4 m/min. Find the amplitude, then find the displacement and velocity as functions of time.
(b) The motion of a buoy floating on top of the waves can be modelled as simple harmonic motion with period 3 seconds. If the waves rise and fall 2 metres about their mean position, find the buoy's greatest speed and acceleration.

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DEVELOPMENT

7. A particle oscillates between two points A and B 20 cm apart, moving in simple harmonic motion with period 8 seconds. Let O be the midpoint of AB.

(a) Find the maximum speed and acceleration, and the places where they occur.

(b) Find the speed and acceleration when the particle is 6 cm from O.

8. The amplitude of a particle moving in simple harmonic motion is 5 metres, and its acceleration when 2 metres from its mean position is $4\,\text{m/s}^2$. Find the speed of the particle at the mean position and when it is 4 metres from the mean position.

9. (a) A particle is moving with simple harmonic motion of period π seconds and maximum velocity 8 m/s. If the particle started from rest at $x = a$, find a, then find the velocity when the particle is distant 3 metres from the mean position.

(b) A point moves with period π seconds so that its acceleration is proportional to its displacement x from O and oppositely directed. It passes through O with speed 5 m/s. Find its speed and acceleration 1·5 metres from O.

10. (a) A particle moving in simple harmonic motion on a horizontal line has amplitude 2 metres. If its speed passing through the centre O of motion is 15 m/s, find v^2 as a function of the displacement x to the right of O, and find the velocity and the acceleration of the particle when it is $\frac{2}{3}$ metres to the right of O.

(b) A particle moves so that its acceleration is proportional to its displacement x from the origin O. When 4 cm on the positive side of O, its velocity is 20 cm/s and its acceleration is $-6\frac{2}{3}\,\text{cm/s}^2$. Find the amplitude of the motion.

11. [The general integral] Suppose that a particle is moving in simple harmonic motion with amplitude a and equation of motion $\ddot{x} = -n^2 x$, where $n > 0$.

(a) Prove that $v^2 = n^2(a^2 - x^2)$.

(b) Find expressions for: (i) the speed at the origin, (ii) the speed and acceleration halfway between the origin and the maximum displacement.

12. A particle moving in simple harmonic motion starts at the origin with velocity V. Prove that the particle first comes to rest after travelling a distance V/n.

13. A particle moves in simple harmonic motion with centre O, and passes through O with speed $10\sqrt{3}$ cm/s. By integrating $\ddot{x} = -n^2 x$, calculate the speed when the particle is halfway between its mean position and a point of instantaneous rest.

14. (a) A particle moving in a straight line obeys $v^2 = -9x^2 + 18x + 27$. Prove that the motion is simple harmonic, and find the centre of motion, the period and the amplitude.

(b) Repeat part (a) for:

(i) $v^2 = 80 + 64x - 16x^2$

(ii) $v^2 = -9x^2 + 108x - 180$

(iii) $v^2 = -2x^2 - 8x - 6$

(iv) $v^2 = 8 - 10x - 3x^2$

15. (a) Show that the motion $x = \sin^2 5t$ (in units of metres and minutes) is simple harmonic by showing that it satisfies $\ddot{x} = n^2(x_0 - x)$, for some x_0 and some $n > 0$:

(i) by first writing the displacement function as $x = \frac{1}{2} - \frac{1}{2}\cos 10t$,

(ii) by differentiating x directly without any use of double-angle identities.

(b) Find the centre, range and period of the motion, and the next time it visits the origin.

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16. A particle moves in simple harmonic motion according to $\ddot{x} = -9(x - 7)$, in units of centimetres and seconds. Its amplitude is 7 cm.

(a) Find the centre of motion, and hence explain why the velocity at the origin is zero.

(b) Integrate to find v^2 as a function of x, complete the square in this expression, and hence find the maximum speed.

(c) Explain how, although the particle is stationary at the origin, it is nevertheless able to move away from the origin.

17. A particle is moving according to $x = 4\cos 3t - 6\sin 3t$.

(a) Prove that the acceleration is proportional to the displacement but oppositely directed, and hence that the motion is simple harmonic.

(b) Find the period, amplitude and maximum speed of the particle, and find the acceleration when the particle is halfway between its mean position and one of its extreme positions.

18. The motion of a particle is given by $x = 3 + \sin 4t + \sqrt{3}\cos 4t$.

(a) Prove that $\ddot{x} = 16(3 - x)$, and write down the centre and period of the motion.

(b) Express the motion in the form $x = x_0 + a\sin(4t + \alpha)$, where $a > 0$ and $0 \le \alpha < 2\pi$.

(c) At what times is the particle at the centre, and what is its speed there?

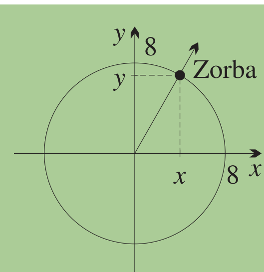
19. A particle moves according to the equation $x = 10 + 8\sin 2t + 6\cos 2t$.

(a) Prove that the motion is simple harmonic, and find the centre of motion, the period and the amplitude.

(b) Find, correct to four significant figures, when the particle first reaches the origin.

EXTENSION

20. [Simple harmonic motion is the projection of circular motion onto a diameter.] A Ferris wheel of radius 8 metres mounted in the north–south plane is turning anticlockwise at 1 revolution per minute. At time zero, Zorba is level with the centre of the wheel and north of it.



(a) Let x and y be Zorba's horizontal distance north of the centre and height above the centre respectively. Show that $x = 8\cos 2\pi t$ and $y = 8\sin 2\pi t$.

(b) Find expressions for $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$, and show that $\ddot{x} = -4\pi^2 x$ and $\ddot{y} = -4\pi^2 y$.

(c) Find how far (in radians) the wheel has turned during the first revolution when:

(i) $x : y = \sqrt{3} : 1$ (ii) $\dot{x} : \dot{y} = -\sqrt{3} : 1$ (iii) $\dot{x} = \dot{y}$

21. A particle moves in simple harmonic motion according to $\ddot{x} = -n^2 x$.

(a) Prove that $v^2 = n^2(a^2 - x^2)$, where a is the amplitude of the motion.

(b) The particle has speeds v_1 and v_2 when the displacements are x_1 and x_2 respectively. Show that the period T is given by

$$T = 2\pi\sqrt{\frac{{x_1}^2 - {x_2}^2}{{v_2}^2 - {v_1}^2}},$$

and find a similar expression for the amplitude.

(c) The particle has speeds of 8 cm/s and 6 cm/s when it is 3 cm and 4 cm respectively from O. Find the amplitude, the period and the maximum speed of the particle.

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22. A particle moving in simple harmonic motion has amplitude a and maximum speed V. Find its velocity when $x = \frac{1}{2}a$, and its displacement when $v = \frac{1}{2}V$. Prove also the more general results

$$|v| = V\sqrt{1 - x^2/a^2} \qquad \text{and} \qquad |x| = a\sqrt{1 - v^2/V^2}\,.$$

23. Two balls on elastic strings are moving vertically in simple harmonic motion with the same period 2π and with centres level with each other. The second ball was set in motion α seconds later, where $0 \le \alpha < 2\pi$, with twice the amplitude, so their equations are

$$x_1 = \sin t \qquad \text{and} \qquad x_2 = 2\sin(t - \alpha).$$

Let $x = \sin t - 2\sin(t - \alpha)$ be the height of the first ball above the second.

(a) Show that $\ddot{x} = -x$, and hence that x is also simple harmonic with period 2π.

(b) Show that the greatest vertical difference A between the balls is $A = \sqrt{5 - 4\cos\alpha}$. What are the maximum and minimum values of A, and what form does x then have?

(c) Show that the balls are level when $\tan t = \dfrac{4T}{1 - 3T^2}$, where $T = \tan\frac{1}{2}\alpha$. How many times are they level in the time interval $0 \le t < 2\pi$?

(d) For what values of α is the vertical distance between the balls maximum at $t = 0$, and what form does x then have?

24. [This is a proof that there are no more solutions of the differential equation $\ddot{x} = -n^2 x$.] Suppose that $\ddot{x} = -n^2 x$, where $n > 0$, and let $a = x(0)$ and $bn = \dot{x}(0)$.

(a) Let $u = x - (a\cos nt + b\sin nt)$. Show that $u(0) = 0$ and $\dot{u}(0) = 0$.

(b) Find $\ddot{u}$, and show that $\ddot{u} = -n^2 u$.

(c) Write $\ddot{u} = \dfrac{d}{du}\left(\frac{1}{2}\dot{u}^2\right)$, then integrate to show that $\dot{u}^2 = -n^2 u^2$.

(d) Hence show that $u = 0$ for all t, and hence that $x = a\cos nt + b\sin nt$.

25. [An alternative proof] Suppose that x and y are functions of t satisfying

$$\ddot{x} = -n^2 x, \qquad \ddot{y} = -n^2 y, \qquad x(0) = y(0), \qquad \text{and} \qquad \dot{x}(0) = \dot{y}(0),$$

where n is a positive constant, and $x(0)$ and $\dot{x}(0)$ are not both zero.

(a) Show that $\dfrac{d}{dt}(\dot{x}y - x\dot{y}) = 0$, and hence that $\dot{x}y = x\dot{y}$, for all t.

(b) Show that $\dfrac{d}{dt}\left(\dfrac{x}{y}\right) = 0$, and hence that $y = x$, for all t.

(c) Hence show that $x = a\cos nt + b\sin nt$, where $a = x(0)$ and $b = \dfrac{\dot{x}(0)}{n}$.

3 G Projectile Motion — The Time Equations

In these final sections, we shall consider one case of motion in two dimensions — the motion of a projectile, like a thrown ball or a shell fired from a gun.

A *projectile* is something that is thrown or fired into the air, and subsequently moves under the influence of gravity alone. Notice that missiles and aeroplanes are not projectiles, because they have motors on them that keep pushing them forwards. We shall ignore any effects of air resistance, so we will not be dealing with things like leaves or pieces of paper where air resistance has a large effect. Everyone can see that a projectile moves in a parabolic path. Our task is to set up the equations that describe this motion.

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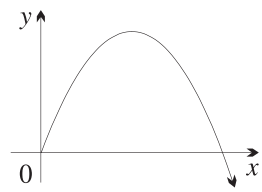
The Coordinates of Displacement and Time: The diagram on the right shows the sort of path we would expect a projectile to move in. The two-dimensional space in which it moves has been made into a number plane by choosing an origin — in this case the point from which the projectile was fired — and measuring horizontal distance x and vertical distance y from this origin. We could put time t on the graph, but this would require a third dimension for the t-axis. But we can treat t as a parameter, because every point on the path corresponds to a unique time after projection.



These pronumerals x, y and t for horizontal distance, vertical distance and time respectively will be used without further introduction in this section.

Velocity and the Resolution of Velocity: When an object is moving through the air, we can describe its velocity by giving its speed and the angle at which it is moving. For example, a ball may at some instant be moving at 12 m/s with an *angle of inclination* of 60° or -60°. This angle of inclination is always measured from the horizontal, and is taken as negative if the object is travelling downwards.

23 SPECIFYING MOTION BY SPEED AND ANGLE OF INCLINATION: The velocity of a projectile can be specified by giving its speed and angle of inclination.
The *angle of inclination* is the acute angle between the path and the horizontal. It is positive if the object is travelling upwards, and negative if the object is travelling downwards.

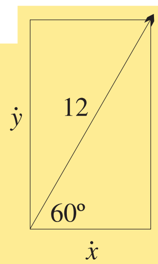
But we can also specify the velocity at that instant by giving the rates $\dot{x}$ and $\dot{y}$ at which the horizontal displacement x and the vertical displacement y are changing. The conversion from one system of measurement to the other requires a *velocity resolution diagram* like those in the worked exercises below.

WORKED EXERCISE: Find the horizontal and vertical components of the velocity of a projectile moving with speed 12 m/s and angle of inclination:

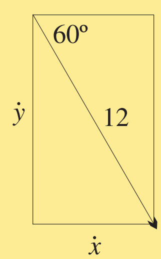
(a) 60° (b) -60°

SOLUTION:

(a) $\dot{x} = 12\cos 60^\circ$
$= 6$ m/s
$\dot{y} = 12\sin 60^\circ$
$= 6\sqrt{3}$ m/s



(b) $\dot{x} = 12\cos 60^\circ$
$= 6$ m/s
$\dot{y} = -12\sin 60^\circ$
$= -6\sqrt{3}$ m/s

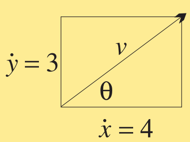


WORKED EXERCISE: Find the speed v and angle of inclination θ (correct to the nearest degree) of a projectile for which:

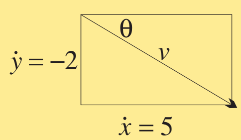
(a) $\dot{x} = 4$ m/s and $\dot{y} = 3$ m/s, (b) $\dot{x} = 5$ m/s and $\dot{y} = -2$ m/s.

SOLUTION:

(a) $v^2 = 4^2 + 3^2$
$v = 5$ m/s
$\tan\theta = \frac{3}{4}$
$\theta \doteqdot 37^\circ$



(b) $v^2 = 5^2 + 2^2$
$v = \sqrt{29}$ m/s
$\tan\theta = -\frac{2}{5}$
$\theta \doteqdot -22^\circ$



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24 **RESOLUTION OF VELOCITY:** To convert between velocity given in terms of speed v and angle of inclination θ, and velocity given in terms of horizontal and vertical components $\dot{x}$ and $\dot{y}$, use a *velocity resolution diagram*.
Alternatively, use the conversion equations

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} v^2 = \dot{x}^2 + \dot{y}^2 \\ \tan\theta = \dot{y}/\dot{x} \end{cases}$$

The Independence of the Vertical and Horizontal Motion: We have already seen that gravity affects every object free to move by accelerating it downwards with the same constant acceleration g, where g is about $9{\cdot}8\,\text{m/s}^2$, or $10\,\text{m/s}^2$ in round figures. Because this acceleration is downwards, it affects the vertical component $\dot{y}$ of the velocity according to $\ddot{y} = -g$. It has no effect, however, on the horizontal component $\dot{x}$, and thus $\ddot{x} = 0$. Every projectile motion is governed by this same pair of equations.

25 **THE FUNDAMENTAL EQUATIONS OF PROJECTILE MOTION:** Every projectile motion is governed by the pair of equations

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g.$$

Unless otherwise indicated, every question on projectile motion should begin with these equations. This will involve four integrations and four substitutions of the boundary conditions.

WORKED EXERCISE: A ball is thrown with initial velocity $40\,\text{m/s}$ and angle of inclination 30° from the top of a stand 25 metres above the ground.

(a) Using the stand as the origin and $g = 10\,\text{m/s}^2$, find the six equations of motion.

(b) Find how high the ball rises, how long it takes to get there, what its speed is then, and how far it is horizontally from the stand.

(c) Find the flight time, the horizontal range, and the impact speed and angle.

SOLUTION: Initially, $x = y = 0$, and $\dot{x} = 40\cos 30^\circ = 20\sqrt{3}\,,$ $\quad \dot{y} = 40\sin 30^\circ = 20.$

(a) To begin, $\ddot{x} = 0.$ (1)
Integrating, $\dot{x} = C_1.$
When $t = 0$, $\dot{x} = 20\sqrt{3}$
$20\sqrt{3} = C_1,$
so $\dot{x} = 20\sqrt{3}.$ (2)
Integrating, $x = 20t\sqrt{3} + C_2.$
When $t = 0$, $x = 0$
$0 = C_2,$
so $x = 20t\sqrt{3}.$ (3)

To begin, $\ddot{y} = -10.$ (4)
Integrating, $\dot{y} = -10t + C_3.$
When $t = 0$, $\dot{y} = 20$
$20 = C_3,$
so $\dot{y} = -10t + 20.$ (5)
Integrating, $y = -5t^2 + 20t + C_4.$
When $t = 0$, $y = 0$
$0 = C_4,$
so $y = -5t^2 + 20t.$ (6)

(b) At the top of its flight, the vertical component of the ball's velocity is zero,
so put $\dot{y} = 0.$
From (5), $-10t + 20 = 0$

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$t = 2$ seconds (the time taken).

When $t = 2$, from (6), $y = -20 + 40$
$= 20$ metres (the maximum height).

When $t = 2$, from (3), $x = 40\sqrt{3}$ metres (the horizontal distance).

Because the vertical component of velocity is zero, the speed there is $\dot{x} = 20\sqrt{3}\,\text{m/s}$.

(c) It hits the ground when it is 25 metres below the stand,

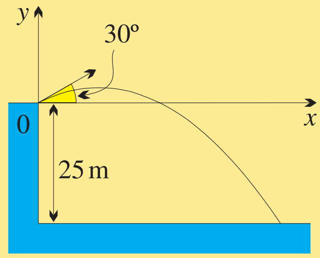
so put $y = -25$.

From (6), $-5t^2 + 20t = -25$

$t^2 - 4t - 5 = 0$

$(t-5)(t+1) = 0$

so it hits the ground when $t = 5$ ($t = -1$ is inadmissable).



When $t = 5$, from (3), $x = 100\sqrt{3}$ metres (the horizontal range).

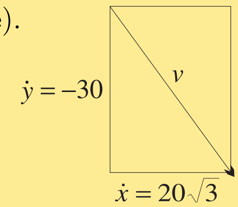
Also, $\dot{x} = 20\sqrt{3}$ and $\dot{y} = -50 + 20 = -30$,

so $v^2 = 1200 + 900$

$v = 10\sqrt{21}\,\text{m/s}$ (the impact speed),

and $\tan\theta = -\frac{30}{20\sqrt{3}}$

$\theta \doteqdot -40°54'$, and the impact angle is about $40°54'$.



Using Pronumerals for Initial Velocity and Angle of Inclination: Many problems in projectile motion require the initial velocity or angle of inclination to be found so that the projectile behaves in some particular fashion. Often the muzzle speed of a gun will be fixed, but the angle at which it is fired can be easily altered — in such situations there are usually two solutions, corresponding to a low-flying shot and a 'lobbed' shot that goes high in the air.

WORKED EXERCISE: A gun at O fires shells with an initial speed of $200\,\text{m/s}$ but a variable angle of inclination α. Take $g = 10\,\text{m/s}^2$.

(a) Find the two possible angles at which the gun can be set so that it will hit a fortress F 2 km away on top of a mountain 1000 metres high.

(b) Show that the two angles are equally inclined to OF and to the vertical.

(c) Find the corresponding flight times and the impact speeds and angles.

SOLUTION: Place the origin at the gun, so that initially, $x = y = 0$.

Resolving the initial velocity, $\dot{x} = 200\cos\alpha$, $\dot{y} = 200\sin\alpha$.

To begin, $\ddot{x} = 0$. (1)

Integrating, $\dot{x} = C_1$.

When $t = 0$, $\dot{x} = 200\cos\alpha$

$200\cos\alpha = C_1$,

so $\dot{x} = 200\cos\alpha$. (2)

Integrating, $x = 200t\cos\alpha + C_2$.

When $t = 0$, $x = 0$

$0 = C_2$,

so $x = 200t\cos\alpha$. (3)

To begin, $\ddot{y} = -10$. (4)

Integrating, $\dot{y} = -10t + C_3$.

When $t = 0$, $\dot{y} = 200\sin\alpha$

$200\sin\alpha = C_1$,

so $\dot{y} = -10t + 200\sin\alpha$. (5)

Integrating, $y = -5t^2 + 200t\sin\alpha + C_4$.

When $t = 0$, $y = 0$

$0 = C_4$,

so $y = -5t^2 + 200t\sin\alpha$. (6)

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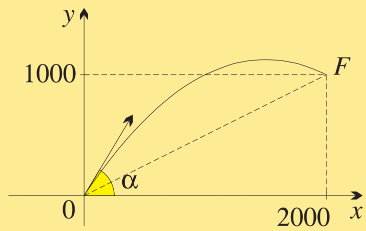
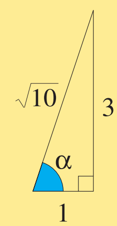
(a) Since the fortress is 2 km away, $x = 2000$
so from (3), $200t\cos\alpha = 2000$
$$t = \frac{10}{\cos\alpha}.$$
Since the mountain is 1000 metres high, $y = 1000$
so from (6), $-5t^2 + 200t\sin\alpha = 1000.$

Hence $$-\frac{500}{\cos^2\alpha} + \frac{2000\sin\alpha}{\cos\alpha} - 1000 = 0$$
$$\sec^2\alpha - 4\tan\alpha + 2 = 0.$$
But $\sec^2\alpha = \tan^2\alpha + 1$,
so $$\tan^2\alpha - 4\tan\alpha + 3 = 0$$
$$(\tan\alpha - 3)(\tan\alpha - 1) = 0$$
$$\tan\alpha = 1 \text{ or } 3$$
$$\alpha = 45^\circ \text{ or } \tan^{-1}3\ [\doteqdot 71^\circ 34'].$$



(b) $\angle OFX = \tan^{-1}\frac{1}{2} \doteqdot 26^\circ 34'$, so the 45° shot is inclined at $18^\circ 26'$ to OF, and the $71^\circ 34'$ shot is inclined at $18^\circ 26'$ to the vertical.
(This calculation can also be done using exact values.)

(c) When $\alpha = 45^\circ$, from (a), $t = \dfrac{10}{\cos\alpha} = 10\sqrt{2}$ seconds,
and when $t = 10\sqrt{2}$, from (5), $\dot{y} = -100\sqrt{2} + 200 \times \frac{1}{2}\sqrt{2} = 0$,
so from (2), the shell hits horizontally at $100\sqrt{2}$ m/s.



When $\alpha = \tan^{-1}3$, $\cos\alpha = \dfrac{1}{\sqrt{10}}$ and $\sin\alpha = \dfrac{3}{\sqrt{10}}$,
so from (a), $t = \dfrac{10}{\cos\alpha} = 10\sqrt{10}$ seconds,
and when $t = 10\sqrt{10}$, from (5), $\dot{y} = -100\sqrt{10} + 60\sqrt{10} = -40\sqrt{10}$,
and from (2), $\dot{x} = 20\sqrt{10}$,
so $$v^2 = 16\,000 + 4000 = 20\,000$$
$$v = 100\sqrt{2}\text{ m/s},$$
and $$\tan\theta = \dot{y}/\dot{x} = -2,$$
$$\theta = -\tan^{-1}2\ [\doteqdot -63^\circ 26'],$$
so the shell hits at $100\sqrt{2}$ m/s at about $63^\circ 26'$ to the horizontal.

Exercise 3G

1. Use a velocity resolution diagram to find $\dot{x}$ and $\dot{y}$, given that the projectile's speed v and angle of inclination θ are:

(a) $v = 12$, $\theta = 30^\circ$ (b) $v = 8$, $\theta = -45^\circ$ (c) $v = 20$, $\theta = \tan^{-1}\frac{4}{3}$

2. Use a velocity resolution diagram to find the speed v and angle of inclination θ of a projectile, given that $\dot{x}$ and $\dot{y}$ are:

(a) $\dot{x} = 6$, $\dot{y} = 6$ (b) $\dot{x} = 7$, $\dot{y} = -7\sqrt{3}$ (c) $\dot{x} = 5$, $\dot{y} = 7$

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3. A stone is projected from a point on level ground with velocity $10\sqrt{2}$ m/s at an angle of elevation of 45°. Let x and y be the respective horizontal and vertical components of the displacement of the stone from the point of projection, and take $g = 10\text{ m/s}^2$.

(a) Use a velocity resolution diagram to determine the initial values of $\dot{x}$ and $\dot{y}$.

(b) Beginning with $\ddot{x} = 0$ and $\ddot{y} = -10$, integrate each equation twice, substituting boundary conditions each time, to find the equations of $\dot{x}$, x, $\dot{y}$ and y in terms of t.

(c) By substituting $\dot{y} = 0$, find the greatest height and the time taken to reach it.

(d) By substituting $y = 0$, find the horizontal distance travelled, and the flight time.

(e) Find $\dot{x}$, x, $\dot{y}$ and y when $t = 0{\cdot}5$.

(i) Hence find how far the stone is from the point of projection when $t = 0{\cdot}5$.

(ii) Use a velocity resolution diagram to find its speed (to the nearest m/s) and its direction of motion (as an angle of elevation to the nearest degree) when $t = 0{\cdot}5$.

4. Steve tosses an apple to Adam who is sitting near him. Adam catches the apple at exactly the same height that Steve released it. Suppose that the initial speed of the apple is $V = 5$ m/s, and the initial angle α of elevation is given by $\tan\alpha = 2$.

(a) Use a velocity resolution diagram to find the initial values of $\dot{x}$ and $\dot{y}$.

(b) Find $\dot{x}$, x, $\dot{y}$ and y by integrating $\ddot{x} = 0$ and $\ddot{y} = -10$, taking the origin at Steve's hands.

(c) Show by substitution into y that the apple is in the air for less than 1 second.

(d) Find the greatest height above the point of release reached by the apple.

(e) Show that the flight time is $\frac{2}{5}\sqrt{5}$ seconds, and hence find the horizontal distance travelled by the apple.

(f) Find $\dot{x}$ and $\dot{y}$ at the time Adam catches the apple. Then use a velocity resolution diagram to show that the final speed equals the initial speed, and the final angle of inclination is the opposite of the initial angle of elevation.

(g) The path of the apple is a parabolic arc. By eliminating t from the equations for x and y, find its equation in Cartesian form.

5. A projectile is fired with velocity $V = 40$ m/s on a horizontal plane at an angle of elevation $\alpha = 60°$. Take $g = 10\text{ m/s}^2$, and let the origin be the point of projection.

(a) Show that $\dot{x} = 20$ and $\dot{y} = -10t + 20\sqrt{3}$, and find x and y.

(b) Find the flight time, and the horizontal range of the projectile.

(c) Find the maximum height reached, and the time taken to reach it.

(d) An observer claims that the projectile would have had a greater horizontal range if its angle of projection had been halved. Investigate this claim by reworking the question with $\alpha = 30°$.

6. A pebble is thrown from the top of a vertical cliff with velocity 20 m/s at an angle of elevation of 30°. The cliff is 75 metres high and overlooks a river.

(a) Derive expressions for the horizontal and vertical components of the displacement of the pebble from the top of the cliff after t seconds. (Take $g = 10\text{ m/s}^2$.)

(b) Find the time it takes for the pebble to hit the water and the distance from the base of the cliff to the point of impact.

(c) Find the greatest height that the pebble reaches above the river.

(d) Find the values of $\dot{x}$ and $\dot{y}$ at the instant when the pebble hits the water. Hence use a velocity resolution diagram to find the speed (to the nearest m/s) and the acute angle (to the nearest degree) at which the pebble hits the water.

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(e) The path of the pebble is a parabolic arc. By eliminating t from the equations for x and y, find its equation in Cartesian form.

7. A plane is flying horizontally at 363·6 km/h and its altitude is 600 metres. It is to drop a food parcel onto a large cross marked on the ground in a remote area.

(a) Convert the speed of the plane into metres per second.

(b) Derive expressions for the horizontal and vertical components of the food parcel's displacement from the point where it was dropped. (Take $g = 10\,\text{m/s}^2$.)

(c) Show that the food parcel will be in the air for $2\sqrt{30}$ seconds.

(d) Find the speed and angle at which the food parcel will hit the ground.

(e) At what horizontal distance from the cross, correct to the nearest metre, should the plane drop the food parcel?

8. Jeffrey the golfer hit a ball which was lying on level ground. Two seconds into its flight, the ball just cleared a 28-metre-tall tree which was exactly $24\sqrt{5}$ metres from where the ball was hit. Let V m/s be the initial velocity of the ball, and let θ be the angle to the horizontal at which the ball was hit. Take $g = 10\,\text{m/s}^2$.

(a) Show that the horizontal and vertical components of the displacement of the ball from its initial position are $x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$.

(b) Show that $V\cos\theta = 12\sqrt{5}$ and $V\sin\theta = 24$.

(c) By squaring and adding, find V. Then find θ, correct to the nearest minute.

(d) Find, correct to the nearest metre, how far Jeffrey hit the ball.

DEVELOPMENT

9. [The general case] A gun at $O(0,0)$ fires a shell across level ground with muzzle speed V and angle α of elevation.

(a) Derive, from $\ddot{x} = 0$ and $\ddot{y} = -g$, the other four equations of motion.

(b) (i) Find the maximum height H, and the time taken to reach it.

(ii) If V is constant and α varies, find the greatest value of H and the corresponding value of α. What value of α gives half this maximum value?

(c) (i) Find the range R and flight time T.

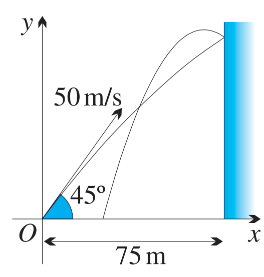
(ii) If V is constant and α varies, find the greatest value of R and the corresponding value of α. What value of α gives half this maximum value?

10. Gee Ming the golfer hits a ball from level ground with an initial speed of 50 m/s and an initial angle of elevation of 45°. The ball rebounds off an advertising hoarding 75 metres away. Take $g = 10\,\text{m/s}^2$.

(a) Show that the ball hits the hoarding after $\frac{3}{2}\sqrt{2}$ seconds at a point 52·5 metres high.

(b) Show that the speed v of the ball when it strikes the hoarding is $5\sqrt{58}$ m/s at an angle of elevation α to the horizontal, where $\alpha = \tan^{-1}\frac{2}{5}$.

(c) Assuming that the ball rebounds off the hoarding at an angle of elevation α with a speed of 20% of v, find how far from Gee Ming the ball lands.



11. Antonina threw a ball with velocity 20 m/s from a point exactly one metre above the level ground she was standing on. The ball travelled towards a wall of a tall building 16 metres away. The plane in which the ball travelled was perpendicular to the wall. The ball struck the wall 16 metres above the ground. Take $g = 10\,\text{m/s}^2$.

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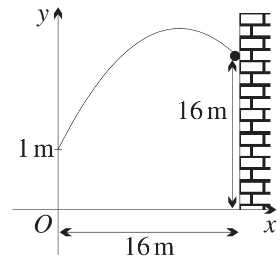


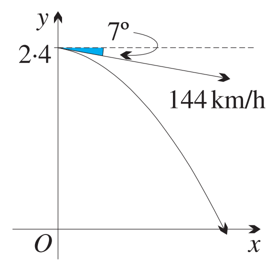
(a) Let the origin be the point on the ground directly below the point from which the ball was released. Show that, t seconds after the ball was thrown, $x = 20t\cos\theta$ and $y = -5t^2 + 20t\sin\theta + 1$, where θ is the angle to the horizontal at which the ball was originally thrown.

(b) The ball hit the wall after T seconds. Show that $4 = 5T\cos\theta$ and $3 = 4T\sin\theta - T^2$.

(c) Hence show that $16\tan^2\theta - 80\tan\theta + 91 = 0$.

(d) Hence find the two possible values of θ, correct to the nearest minute.

12. Glenn the fast bowler runs in to bowl and releases the ball 2·4 metres above the ground with speed 144 km/h at an angle of 7° below the horizontal. Take the origin to be the point where the ball is released, and take $g = 10\,\text{m/s}^2$.



(a) Show that the coordinates of the ball t seconds after its release are given by

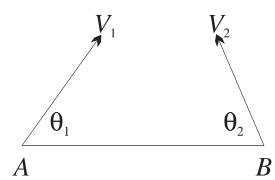
$$x = 40t\cos 7°, \quad y = 2{\cdot}4 - 40t\sin 7° - 5t^2.$$

(b) How long will it be (to the nearest 0·01 seconds) before the ball hits the pitch?

(c) Calculate the angle (to the nearest degree) at which the ball will hit the pitch.

(d) The batsman is standing 19 metres from the point of release. If the ball lands more than 5 metres in front of him, it will be classified as a 'short-pitched' delivery. Is this particular delivery short-pitched?

13. Two particles P_1 and P_2 are projected simultaneously from the points A and B, where AB is horizontal. The motion takes place in the vertical plane through A and B. The initial velocity of P_1 is V_1 at an angle θ_1 to the horizontal, and the initial velocity of P_2 is V_2 at an angle θ_2 to the horizontal. You may assume that the equations of motion of a particle projected with velocity V at an angle θ to the horizontal are $x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$.



(a) Show that the condition for the particles to collide is $V_1\sin\theta_1 = V_2\sin\theta_2$.

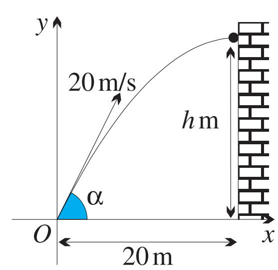
(b) Suppose that $AB = 200$ metres, $V_1 = 30\,\text{m/s}$, $\theta_1 = \sin^{-1}\frac{4}{5}$, $\theta_2 = \sin^{-1}\frac{3}{5}$, $g = 10\,\text{m/s}^2$ and that the particles collide.

(i) Show that $V_2 = 40\,\text{m/s}$, and that the particles collide after 4 seconds.

(ii) Find the height of the point of collision above AB.

(iii) Find, correct to the nearest degree, the obtuse angle between the directions of motion of the particles at the instant they collide.

14. A cricketer hits the ball from ground level with a speed of 20 m/s and an angle of elevation α. It flies towards a high wall 20 metres away on level ground. Take the origin at the point where the ball was hit, and take $g = 10\,\text{m/s}^2$.



(a) Show that the ball hits the wall when $h = 20\tan\alpha - 5\sec^2\alpha$.

(b) Show that $\dfrac{d}{d\alpha}(\sec\alpha) = \sec\alpha\tan\alpha$.

(c) Show that the maximum value of h occurs when $\tan\alpha = 2$.

(d) Find the maximum height.

(e) Find the speed and angle (to the nearest minute) at which the ball hits the wall.

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15. A stone is propelled upwards at an angle θ to the horizontal from the top of a vertical cliff 40 metres above a lake. The speed of propulsion is $20\,\text{m/s}$. Take $g = 10\,\text{m/s}^2$.

(a) Show that $x(t)$ and $y(t)$, the horizontal and vertical components of the stone's displacement from the top of the cliff, are given by

$$x(t) = 20t\cos\theta, \quad y(t) = -5t^2 + 20t\sin\theta.$$

(b) If the stone hits the lake at time T seconds, show that

$$\big(x(T)\big)^2 = 400T^2 - (5T^2 - 40)^2.$$

(c) Hence find, by differentiation, the value of T that maximises $\big(x(T)\big)^2$, and then find the value of θ that maximises the distance between the foot of the cliff and the point where the stone hits the lake.

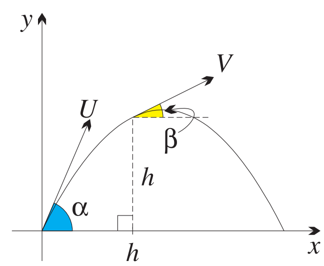
16. A particle P_1 is projected from the origin with velocity V at an angle of elevation θ.

(a) Assuming the usual equations of motion, show that the particle reaches a maximum height of $\dfrac{V^2\sin^2\theta}{2g}$.

(b) A second particle P_2 is projected from the origin with velocity $\frac{3}{2}V$ at an angle $\frac{1}{2}\theta$ to the horizontal. The two particles reach the same maximum height.

(i) Show that $\theta = \cos^{-1}\frac{1}{8}$. (ii) Do the two particles take the same time to reach this maximum height? Justify your answer.

17. A projectile was fired from the origin with velocity U at an angle of α to the horizontal. At time T_1 on its ascent, it passed with velocity V through a point whose horizontal and vertical distances from the origin are equal, and its direction of motion at that time was at an angle of β to the horizontal. At time T_2 the projectile returned to the horizontal plane from which it was fired.



(a) (i) Show that $T_1 = \dfrac{2U}{g}(\sin\alpha - \cos\alpha)$.

(ii) Hence show that $T_2 = \dfrac{T_1}{1 - \cot\alpha}$. (iii) Deduce that $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$.

(b) (i) Explain why $V\cos\beta = U\cos\alpha$.

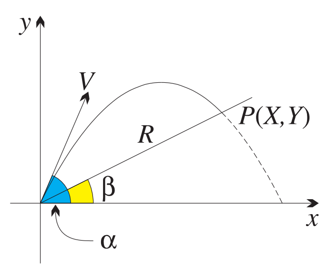
(ii) If $\beta = \frac{1}{2}\alpha$, show that $\beta = \cos^{-1}\left(\dfrac{V + \sqrt{V^2 + 8U^2}}{4U}\right)$.

18. (a) Consider the function $y = 2\sin(x - \theta)\cos x$.

(i) Show that $\dfrac{dy}{dx} = 2\cos(2x - \theta)$. (ii) Hence, or otherwise, show that

$$2\sin(x - \theta)\cos x = \sin(2x - \theta) - \sin\theta.$$

(b) A projectile is fired from the origin with velocity V at an angle of α to the horizontal up a plane inclined at β to the horizontal. Assume that the horizontal and vertical components of the projectile's displacement are given by $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$.



(i) If the projectile strikes the plane at (X, Y), show that

$$X = \frac{2V^2\cos^2\alpha(\tan\alpha - \tan\beta)}{g}.$$

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(ii) Hence show that the range R of the projectile up the plane is given by
$$R = \frac{2V^2 \cos\alpha \sin(\alpha - \beta)}{g \cos^2 \beta}.$$

(iii) Use part (a)(ii) to show that the maximum possible value of R is $\dfrac{V^2}{g(1 + \sin\beta)}$.

(iv) If the angle of inclination of the plane is 14°, at what angle to the horizontal should the projectile be fired in order to attain the maximum possible range?

EXTENSION

19. A tall building stands on level ground. The nozzle of a water sprinkler is positioned at a point P on the ground at a distance d from a wall of the building. Water sprays from the nozzle with speed V and the nozzle can be pointed in any direction from P.
(a) If $V > \sqrt{gd}$, prove that the water can reach the wall above ground level.
(b) Suppose that $V = 2\sqrt{gd}$. Show that the portion of the wall that can be sprayed with water is a parabolic segment of height $\frac{15}{8}d$ and area $\frac{5}{2}d^2\sqrt{15}$.

3 H Projectile Motion — The Equation of Path

The formulae for x and y in terms of t give a parametric equation of the physical path of the projectile through the x–y plane. Eliminating t will give the Cartesian equation of the path, which is simply an upside-down parabola. Many questions are solved more elegantly by consideration of the equation of path. Unless the question gives it, however, the equation of path must be derived each time.

The General Case: The following working derives the equation of path in the general case of a projectile fired from the origin with initial speed V and angle of elevation α.

Resolving the initial velocity, $\dot{x} = V\cos\alpha$ and $\dot{y} = V\sin\alpha$.

To begin, $\ddot{x} = 0$. (1)
Integrating, $\dot{x} = C_1$.
When $t = 0$, $\dot{x} = V\cos\alpha$
$V\cos\alpha = C_1$,
so $\dot{x} = V\cos\alpha$. (2)
Integrating, $x = Vt\cos\alpha + C_2$.
When $t = 0$, $x = 0$
$0 = C_2$,
so $x = Vt\cos\alpha$. (3)

To begin, $\ddot{y} = -g$. (4)
Integrating, $\dot{y} = -gt + C_3$.
When $t = 0$, $\dot{y} = V\sin\alpha$
$V\sin\alpha = C_3$,
so $\dot{y} = -gt + V\sin\alpha$. (5)
Integrating, $y = -\frac{1}{2}gt^2 + Vt\sin\alpha + C_4$.
When $t = 0$, $y = 0$
$0 = C_4$,
so $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$. (6)

From (3), $t = \dfrac{x}{V\cos\alpha}$.

Substituting into (6), $y = -\dfrac{gx^2}{2V^2\cos^2\alpha} + \dfrac{Vx\sin\alpha}{V\cos\alpha}$,

which becomes $y = -\dfrac{gx^2}{2V^2}(1 + \tan^2\alpha) + x\tan\alpha$,

using the Pythagorean identity $\dfrac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$.

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This working must always be shown unless the equation is given in the question.

26 THE EQUATION OF PATH (not to be memorised): The path of a projectile fired from the origin with initial speed V and angle of elevation α is

$$y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha. \quad [\text{NOTE: } \sec^2\alpha = 1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}.]$$

This equation is quadratic in x, $\tan\alpha$ and V, and linear in g and y.

Differentiation of the equation of path gives the gradient of the path for any value of x, and thus is an alternative approach to finding the angle of inclination of a projectile in flight.

WORKED EXERCISE: Use the equation of path above in these questions.

(a) Show that the range on level ground is $\dfrac{V^2}{g}\sin 2\alpha$, and hence find the maximum range for a given initial speed V and variable angle α of elevation.

(b) Arrange the equation of path as a quadratic in $\tan\alpha$, and hence show that with a given initial speed V and variable angle α of elevation, a projectile can be fired through the point $P(x, y)$ if and only if

$$2V^2gy \le V^4 - g^2x^2.$$

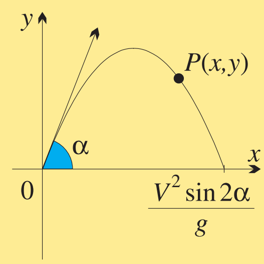
SOLUTION:

(a) Put $y = 0$, then $\dfrac{gx^2\sec^2\alpha}{2V^2} = x\tan\alpha$,

so $x = 0$ or $\dfrac{gx}{2V^2\cos^2\alpha} = \dfrac{\sin\alpha}{\cos\alpha}$

$$x = \frac{2V^2}{g}\cos\alpha\sin\alpha$$

$$x = \frac{V^2}{g}\sin 2\alpha.$$



Hence the projectile lands $\dfrac{V^2}{g}\sin 2\alpha$ away from the origin.

Since $\sin 2\alpha$ has a maximum value of 1 when $\alpha = 45^\circ$,

the maximum range is $\dfrac{V^2}{g}$ when $\alpha = 45^\circ$.

(b) Multiplying both sides of the equation of path by $2V^2$,

$$2V^2y = -gx^2(1 + \tan^2\alpha) + 2V^2x\tan\alpha$$

$$2V^2y + gx^2 + gx^2\tan^2\alpha - 2V^2x\tan\alpha = 0$$

$$gx^2\tan^2\alpha - 2V^2x\tan\alpha + (2V^2y + gx^2) = 0.$$

The equation has now been written as a quadratic in $\tan\alpha$.

It will have a solution for $\tan\alpha$ provided that

$$\Delta \ge 0,$$

that is, $(2V^2x)^2 - 4 \times gx^2 \times (2V^2y + gx^2) \ge 0$

$$4V^4x^2 - 8x^2V^2gy - 4g^2x^4 \ge 0$$

$\boxed{\div 4x^2}$ $\quad 2V^2gy \le V^4 - g^2x^2.$

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Exercise 3H

1. A cricket ball is thrown from the origin on level ground, and the equation of the path of its motion is $y = x - \frac{1}{40}x^2$, where x and y are in metres.

(a) Find the horizontal range of the ball.

(b) Find the greatest height.

(c) Find the gradient of the tangent at $x = 0$, and hence find the angle of projection α.

(d) Find, by substitution, whether the ball goes above or below the point $A(10, 8)$.

(e) The general equation of path is

$$y = -\frac{gx^2}{2V^2 \cos^2 \alpha} + x \tan \alpha,$$

where V is the initial velocity. Taking $g = 10\,\text{m/s}^2$, find V.

2. A stone is fired on a level floor with initial speed $V = 10\,\text{m/s}$ and angle of elevation $45°$.

(a) Find $\dot{x}$, x, $\dot{y}$ and y by integration from $\ddot{x} = 0$ and $\ddot{y} = -10$. Then, by eliminating t, show that the equation of path is $y = -\frac{1}{10}x^2 + x = \frac{1}{10}x(10 - x)$.

(b) Use the theory of quadratics to find the range and the maximum height.

(c) Suppose first that the stone hits a wall 8 metres away.

(i) Find how far up the wall the stone hits.

(ii) Differentiate the equation of path, and hence find the angle of inclination when the stone hits the wall.

(d) Suppose now that the stone hits a ceiling 2·1 metres high.

(i) Find the horizontal distance before impact.

(ii) Find the angle at which the stone hits the ceiling.

3. A particle is projected from the origin at time $t = 0$ seconds and follows a parabolic path with parametric equations $x = 12t$ and $y = 9t - 5t^2$ (where x and y are in metres).

(a) Show that the Cartesian equation of the path is $y = \frac{3}{4}x - \frac{5}{144}x^2$.

(b) Find the horizontal range R and the greatest height H.

(c) Find the gradient at $x = 0$, and hence find the initial angle of projection.

(d) Find $\dot{x}$ and $\dot{y}$ when $t = 0$. Hence use a velocity resolution diagram to find the initial velocity, and to confirm the initial angle of projection.

(e) Find when the particle is 4 metres high, and the horizontal displacement then.

4. A bullet is fired horizontally at $200\,\text{m/s}$ from a window 45 metres above the level ground below. It doesn't hit anything and falls harmlessly to the ground.

(a) Write down the initial values of $\dot{x}$ and $\dot{y}$.

(b) Taking $g = 10\,\text{m/s}^2$ and the origin at the window, find $\dot{x}$, x, $\dot{y}$ and y. Hence find the Cartesian equation of path.

(c) Find the horizontal distance that the bullet travels. [HINT: Put $y = -45$.]

(d) Find, correct to the nearest minute, the angle at which the bullet hits the ground.

DEVELOPMENT

5. A ball is thrown on level ground at an initial speed of V m/s and at an angle of projection α. Assume that, t seconds after release, the horizontal and vertical displacements are given by $x = Vt \cos \alpha$ and $y = Vt \sin \alpha - \frac{1}{2}gt^2$.

(a) Show that the trajectory has Cartesian equation $y = \dfrac{x}{\cos^2 \alpha}\left(\sin \alpha \cos \alpha - \dfrac{gx}{2V^2}\right)$.

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(b) Hence show that the horizontal range is $\dfrac{V^2 \sin 2\alpha}{g}$.

(c) When $V = 30\,\text{m/s}$, the ball lands 45 metres away. Take $g = 10\,\text{m/s}^2$.

(i) Find the two possible values of α.

(ii) A 2-metre-high fence is placed 40 metres from the thrower. Examine each trajectory to see whether the ball will still travel 45 metres.

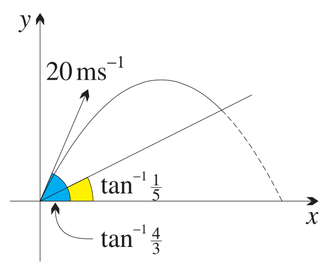
6. A gun can fire a shell with a constant initial speed V and a variable angle of elevation α. Assume that t seconds after being fired, the horizontal and vertical displacements x and y of the shell from the gun are given by the same equations as in the previous question.

(a) Show that the Cartesian equation of the shell's path may be written as

$$gx^2 \tan^2 \alpha - 2xV^2 \tan \alpha + (2yV^2 + gx^2) = 0.$$

(b) Suppose that $V = 200\,\text{m/s}$, $g = 10\,\text{m/s}^2$ and the shell hits a target positioned 3 km horizontally and 0·5 km vertically from the gun. Show that $\tan \alpha = \dfrac{4 \pm \sqrt{3}}{3}$, and hence find the two possible values of α, correct to the nearest minute.

7. A ball is thrown with initial velocity 20 m/s at an angle of elevation of $\tan^{-1} \frac{4}{3}$.



(a) Show that the parabolic path of the ball has parametric equations $x = 12t$ and $y = 16t - 5t^2$.

(b) Hence find the horizontal range of the ball, and its greatest height.

(c) Suppose that, as shown opposite, the ball is thrown up a road inclined at $\tan^{-1} \frac{1}{5}$ to the horizontal. Show that:

(i) the ball is about 9 metres above the road when it reaches its greatest height,

(ii) the time of flight is 2·72 seconds, and find, correct to the nearest tenth of a metre, the distance the ball has been thrown up the road.

8. Talia is holding the garden hose at ground level and pointing it obliquely so that it sprays water in a parabolic path 2 metres high and 8 metres long. Find, using $g = 10\,\text{m/s}^2$, the initial speed and angle of elevation, and the time each droplet of water is in the air. Where is the latus rectum of the parabola?

9. A boy throws a ball with speed V m/s at an angle of 45°to the horizontal.

(a) Derive expressions for the horizontal and vertical components of the displacement of the ball from the point of projection.

(b) Hence show that the Cartesian equation of the path of the ball is $y = x - \dfrac{gx^2}{V^2}$.

(c) The boy is now standing on a hill inclined at an angle θ to the horizontal. He throws the ball at the same angle of elevation of 45° and at the same speed of V m/s. If he can throw the ball 60 metres down the hill but only 30 metres up the hill, use the result in part (b) to show that

$$\tan \theta = 1 - \frac{30g \cos \theta}{V^2} = \frac{60g \cos \theta}{V^2} - 1,$$

and hence that $\theta = \tan^{-1} \frac{1}{3}$.

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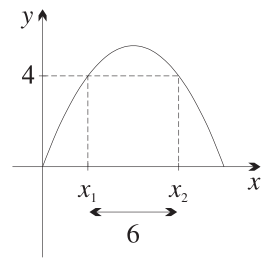
10. A particle is projected from the origin with velocity V m/s at an angle of α to the horizontal.



(a) Assuming that the coordinates of the particle at time t are $(Vt\cos\alpha, Vt\sin\alpha - \frac{1}{2}gt^2)$, prove that the horizontal range R of the particle is $\dfrac{V^2\sin 2\alpha}{g}$.

(b) Hence prove that the path of the particle has equation $y = x\left(1 - \dfrac{x}{R}\right)\tan\alpha$.

(c) Suppose that $\alpha = 45^\circ$ and that the particle passes through two points 6 metres apart and 4 metres above the point of projection, as shown in the diagram. Let x_1 and x_2 be the x-coordinates of the two points.

(i) Show that x_1 and x_2 are the roots of the equation $x^2 - Rx + 4R = 0$.

(ii) Use the identity $(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_2x_1$ to find R.

11. A projectile is fired from the origin with velocity V and angle of elevation α, where α is acute. Assume the usual equations of motion.

(a) Let $k = \dfrac{V^2}{2g}$. Show that the Cartesian equation of the parabolic path of the projectile can be written as

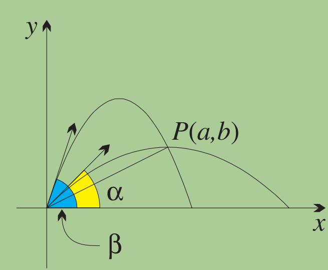
$$x^2\tan^2\alpha - 4kx\tan\alpha + (4ky + x^2) = 0.$$

(b) Show that the projectile can pass through the point (X, Y) in the first quadrant by firing at two different initial angles α_1 and α_2 only if $X^2 < 4k^2 - 4kY$.

(c) Suppose that $\tan\alpha_1$ and $\tan\alpha_2$ are the two real roots of the quadratic equation in $\tan\alpha$ in part (a). Show that $\tan\alpha_1\tan\alpha_2 > 1$, and hence explain why it is impossible for α_1 and α_2 both to be less than 45°.

EXTENSION

12. A gun at $O(0,0)$ has a fixed muzzle speed and a variable angle of elevation.



(a) If the gun can hit a target at $P(a,b)$ with two different angles of elevation α and β, show that the angle between OP and α equals the angle between β and the vertical.

(b) If the gun is firing up a plane of angle of elevation ψ, show that the maximum range is obtained when the gun is fired at the angle that bisects the angle between the plane and vertical.

13. [Some theorems about projectile motion] Ferdinand is feeding his pet bird, Rinaldo, who is sitting on the branch of a tree, by firing pieces of meat to him with a meat-firing device.

(a) Ferdinand aims the device at the bird and fires. At the same instant, Rinaldo drops off his branch and falls under gravity. Prove that Rinaldo will catch the meat.

(b) Rinaldo returns to his perch, and Ferdinand fires a piece of meat so that it will hit the bird. At the same instant, Rinaldo flies off horizontally away from Ferdinand at a constant speed. The meat rises twice the height of the perch, and Rinaldo catches it in flight as it descends. What is the ratio of the horizontal component of the meat's speed to Rinaldo's speed?

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(c) Again Rinaldo returns to his perch, and Ferdinand fires some more meat, intending that the bird will catch it as it descends, having risen twice the height of the perch. At the same instant, Rinaldo flies off horizontally towards Ferdinand at a constant speed, and catches the meat in flight as it ascends. What is the ratio this time?

(d) What is the ratio of Rinaldo's constant speeds in parts (b) and (c)?

14. [The focus and directrix of the path] A gun at $O(0,0)$ has a fixed muzzle speed V and a variable angle α of elevation. Find the vertex, focus and directrix of the parabola by completing the square in the equation of path $y = x\tan\alpha - \dfrac{x^2}{4k\cos^2\alpha}$, where $k = v^2/2g$.

(a) Prove that the directrix is independent of α, and that its height is the maximum height when the projectile is fired vertically upwards.

(b) Find the locus of the focus S and the vertex.

(c) Prove that the initial angle of projection bisects the angle between OS and the vertical.

(d) [A relationship with the definition of the parabola] Let d be a fixed tangent to a fixed circle $\mathcal{C}$, and let $\mathcal{P}$ be any parabola whose focus lies on $\mathcal{C}$ and whose directrix is d. Use the definition of the parabola to show that $\mathcal{P}$ passes through the centre.

15. A projectile is fired up an inclined plane with a fixed muzzle velocity and variable angle of projection. Show that the following four statements are logically equivalent (meaning that if any one of them is true, then the other three are also true).

A. The range up the plane is maximum.
B. The focus of the parabolic path lies on the plane.
C. The angle of projection is at right angles to the angle of flight at impact.
D. The angle of projection bisects the angle between the plane and the vertical.

16. [For those taking physics] Let v be the speed at time t of a projectile fired with initial velocity V and initial angle of elevation α.

(a) Prove that at any time t during the flight, the quantity $gy + \frac{1}{2}v^2$ is independent of time and independent of α.

(b) Explain the interpretation given to this quantity in physics.

GO Online Multiple Choice Quiz

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