MATH1081 Discrete Mathematics

UNSW 2019T1

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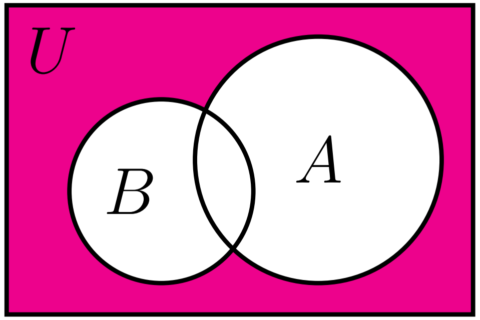

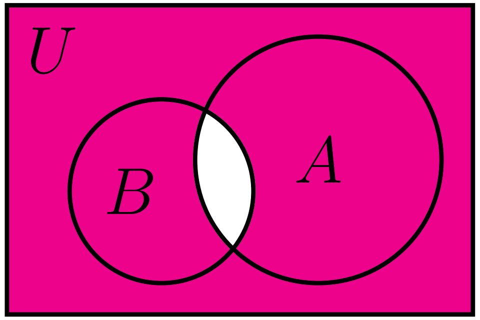

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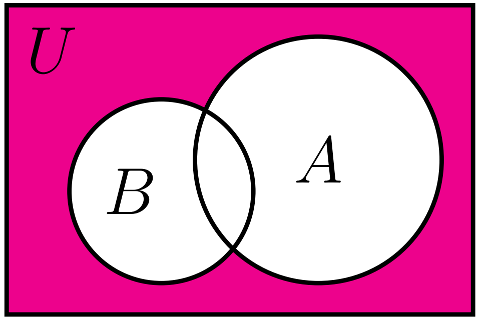

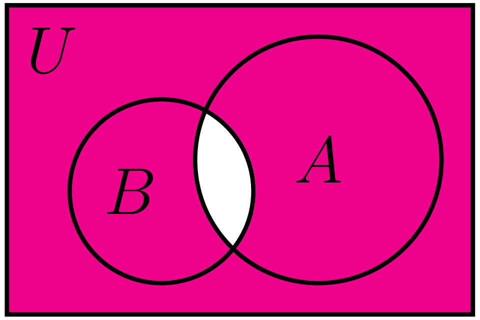

- For a set expression involving only unions, intersections and complements, its *dual* is obtained by replacing $\cap$ with $\cup$, $\cup$ with $\cap$, $\varnothing$ with U, and U with $\varnothing$. The laws of set algebra mostly come in dual pairs.

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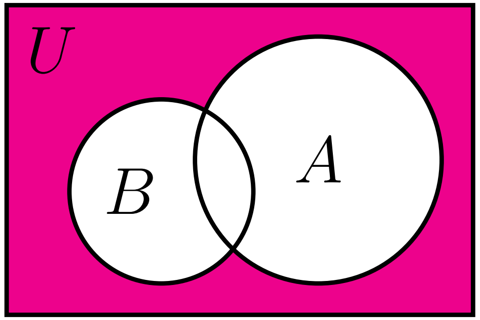

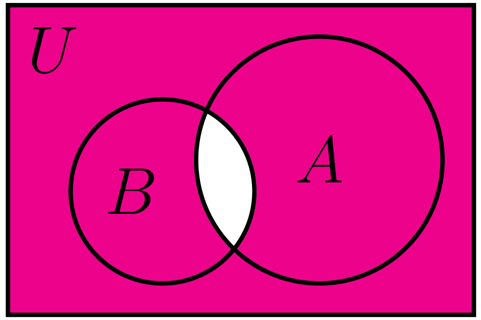

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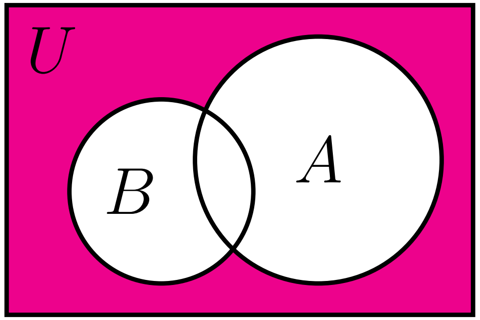

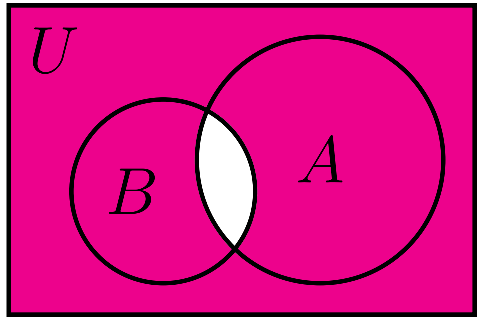

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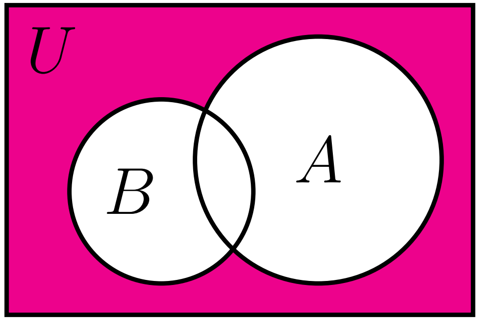

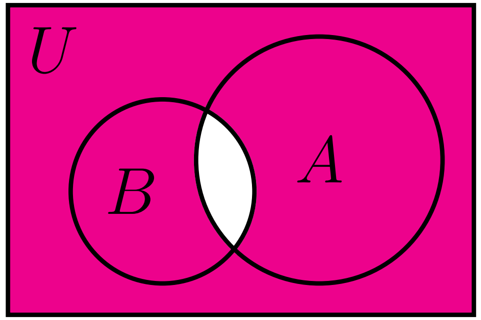

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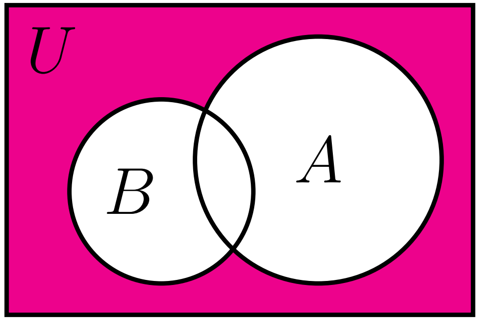

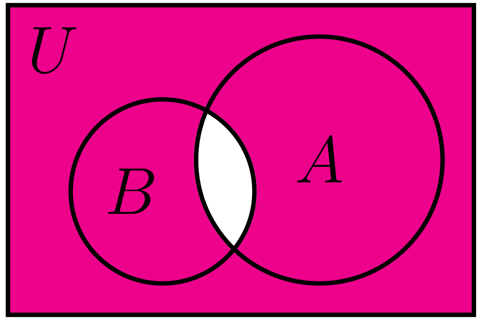

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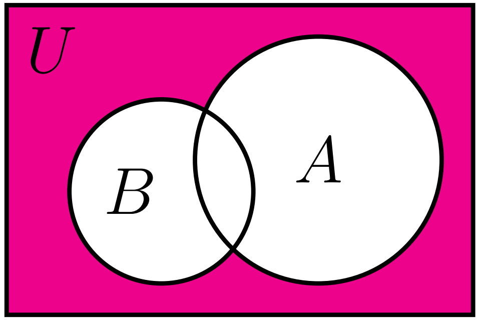

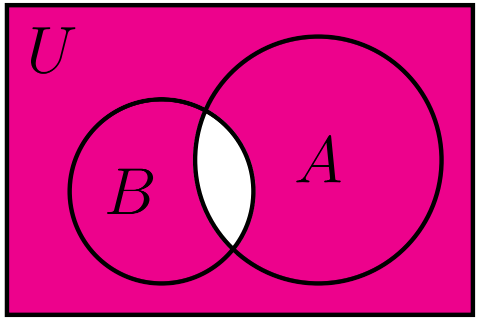

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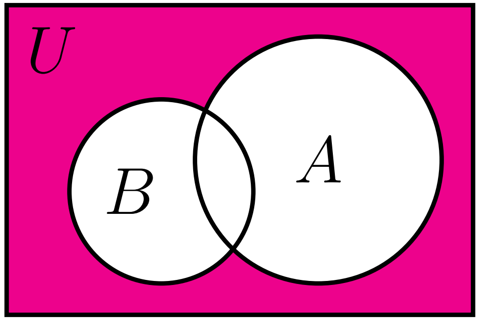

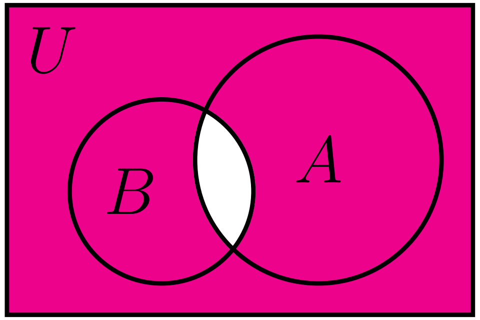

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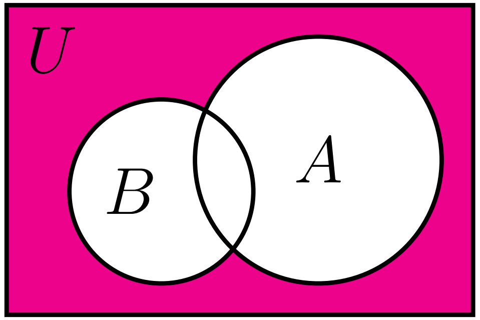

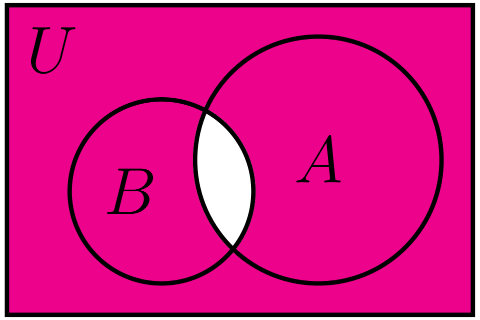

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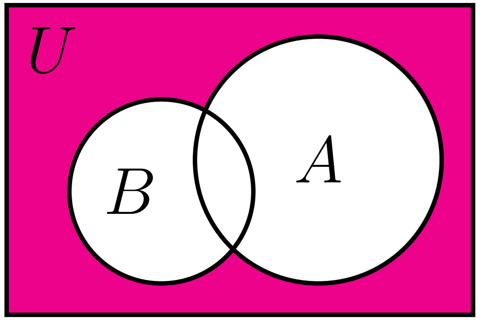

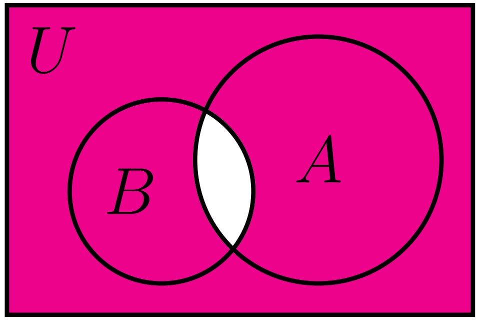

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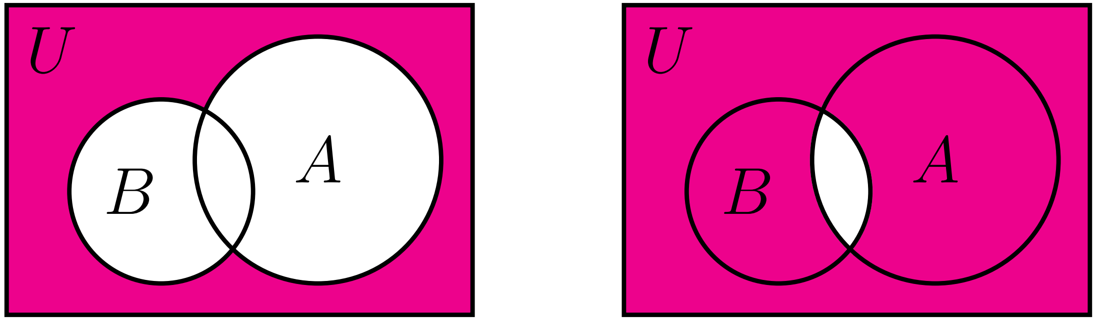

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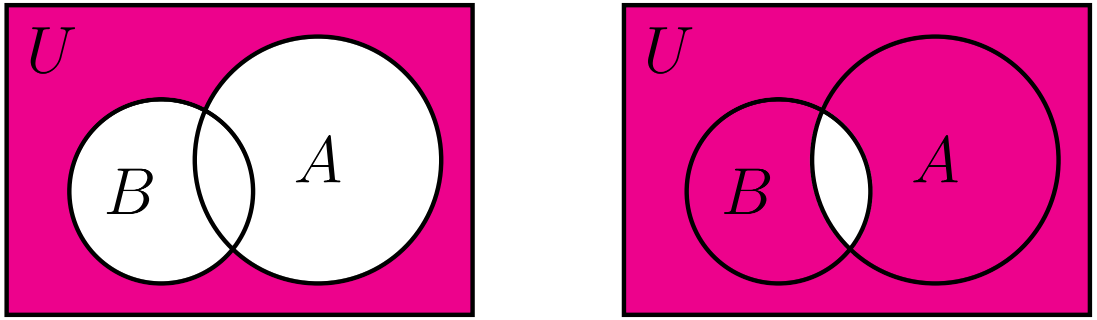

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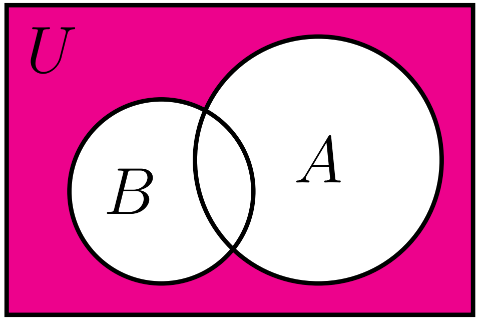

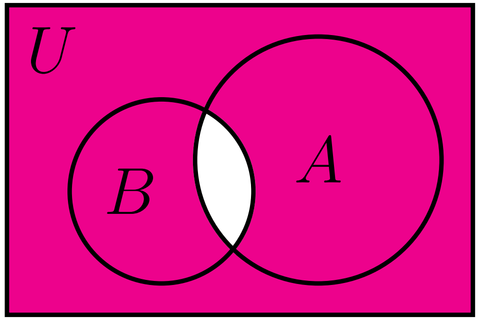

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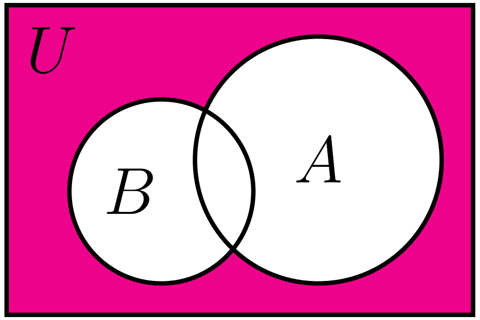

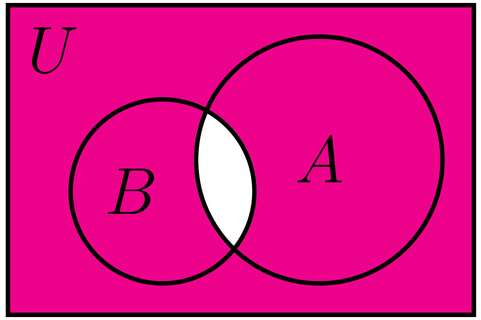

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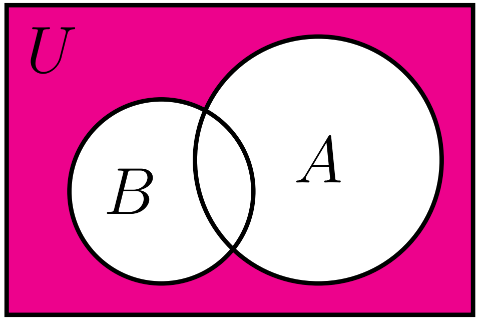
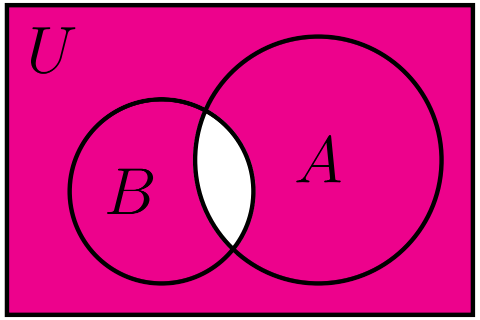
Thus, $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.
This proves that $(A \cup B)^c \subseteq A^c \cap B^c$.

(ii) Suppose now that $x \in A^c \cap B^c$.
Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$.
Thus, $x \notin A \cup B$,

- Laws of set algebra:
  - *De Morgan's Laws* $(A \cup B)^c = A^c \cap B^c$ $\quad (A \cap B)^c = A^c \cup B^c$





Example. Proof of De Morgan's law $(A \cup B)^c = A^c \cap B^c$:

(i) Suppose that $x \in (A \cup B)^c$.
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Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$.
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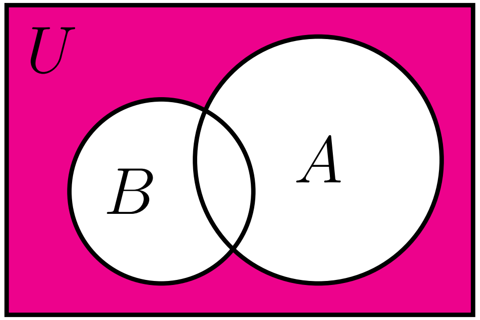

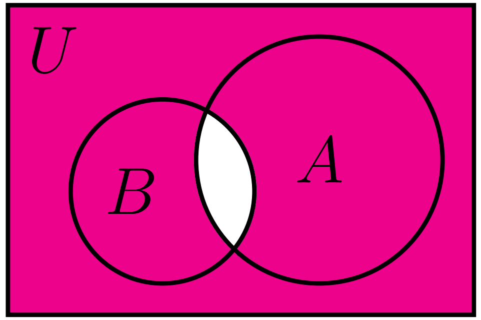

Example. Proof of De Morgan's law $(A \cup B)^c = A^c \cap B^c$:

(i) Suppose that $x \in (A \cup B)^c$.
Then we have $x \notin A \cup B$, so $x \notin A$ and $x \notin B$.
Thus, $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.
This proves that $(A \cup B)^c \subseteq A^c \cap B^c$.

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Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$.
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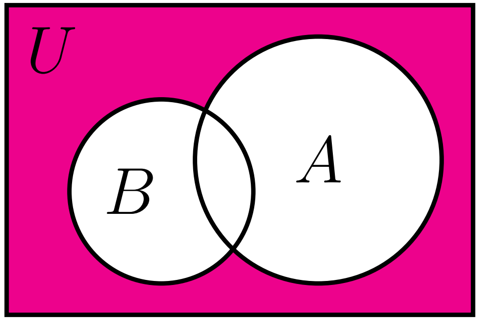

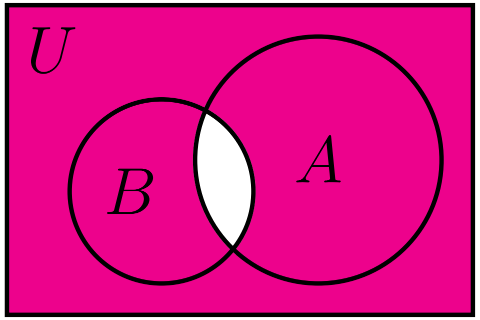

Example. Proof of De Morgan's law $(A \cup B)^c = A^c \cap B^c$:

(i) Suppose that $x \in (A \cup B)^c$.
Then we have $x \notin A \cup B$, so $x \notin A$ and $x \notin B$.
Thus, $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.
This proves that $(A \cup B)^c \subseteq A^c \cap B^c$.

(ii) Suppose now that $x \in A^c \cap B^c$.
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Thus, $x \notin A \cup B$, so $x \in (A \cup B)^c$.
This proves that $A^c \cap B^c \subseteq (A \cup B)^c$.

Combining (i) and (ii), we conclude that $(A \cup B)^c = A^c \cap B^c$.

Example. We can use the laws of set algebra to simplify $(A^c \cap B)^c \cup B$:

Example. We can use the laws of set algebra to simplify $(A^c \cap B)^c \cup B$:

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Example. We can use the laws of set algebra to simplify $(A^c \cap B)^c \cup B$:

$$
\begin{aligned}
(A^c \cap B)^c \cup B &= ((A^c)^c \cup B^c) \cup B && \text{De Morgan's law} \\
&= (A \cup B^c) \cup B && \text{Double complement law}
\end{aligned}
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Exercise. Use the laws of set algebra to simplify $(A \cap (A \cap B)^c) \cup B^c$:

$$
\begin{aligned}
& (A \cap (A \cap B)^c) \cup B^c \\
= {} & (A \cap (A^c \cup B^c)) \cup B^c && \text{De Morgan's law}
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&= (A \cap B^c) \cup B^c && \text{Identity law} \\
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Exercise. Use the laws of set algebra to simplify

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$$\begin{aligned}
& [A \cup (A \cup B^c)] \cap [(A \cup B) \cap (B \cup A^c)] && \\
={}& [(A \cup A) \cup B^c] \cap [(B \cup A) \cap (B \cup A^c)] && \text{Associative and commutative law}
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Exercise. Use the laws of set algebra to simplify $(A \cap (A \cap B)^c) \cup B^c$:

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- Generalized set operations:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \qquad \text{and} \qquad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

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Example. If $A_k = \{k, k+1\}$ for every positive integer k, then

$$\bigcup_{k=1}^{3} A_k =$$

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Exercise. Let $A_k = \{x \in \mathbb{N} \,|\, k \leq x \leq k^2\}$ for every positive integer k. Find

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Example. (The Barber Puzzle) In a certain town there is a barber who shaves all those men, and only those, who do not shave themselves.
Does the barber shave himself?

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- Solution: let U be some known set and define S by

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Hence, we conclude that $S \notin S$ and $S \notin U$.
Thus, the paradox does not occur as long as we have $S \notin U$.

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Hence, we conclude that $S \notin S$ and $S \notin U$.
Thus, the paradox does not occur as long as we have $S \notin U$.
The paradox occurs because our first definition of S refers to itself.

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Example. **(The Barber Puzzle continued)**
Define

$$\begin{aligned} U &= \{\text{all men in town } \textit{except the barber}\} \\ S &= \{A \subseteq U \mid \ A \text{ does not shave himself}\} \\ &= \{A \subseteq U \mid \ A \text{ is shaved by the barber}\} \end{aligned}$$

Then there is no more contradiction.