The University of Sydney
MATH1902 Linear Algebra (Advanced)

Semester 1 **Board Tutorial for Week 3** 2017

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7. Consider the following points in space:

$$P(1,1,1)\,,\quad Q(-1,-1,0)\,,\quad R(0,1,2)\,,\quad S(2,3,3)$$

(i) Recall from an earlier exercise that $PQRS$ is a rhombus that is not a square. Determine whether the angles $\angle PQR$ and $\angle QRS$ are acute or obtuse (which will be further confirmation that this rhombus is not a square).
(ii) Use a dot product to verify that the diagonals PR and QS are mutually perpendicular.

8. Use the dot product to verify that if $\mathbf{v}$ and $\mathbf{w}$ are any vectors and $\mathbf{w}$ is nonzero, then

$$\mathbf{w} \qquad \text{and} \qquad \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\,\mathbf{w}$$

are mutually perpendicular.

9. Given that $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, find

(i) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$
(ii) the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$
(iii) the scalar component of $\mathbf{v}$ in the direction of $\mathbf{u}$
(iv) the vector projection of $\mathbf{u}$ in the direction of $\mathbf{v}$
(v) the vector projection of $\mathbf{v}$ in the direction of $\mathbf{u}$
(vi) the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$
(vii) the vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$

10. Verify that if $\mathbf{a}$ and $\mathbf{b}$ are vectors of the same length then

$$\mathbf{a} + \mathbf{b} \qquad \text{and} \qquad \mathbf{a} - \mathbf{b}$$

are mutually perpendicular.

11. Use vectors to find the following angles in a cube:

(i) between a major diagonal (between opposite vertices) and an edge,
(ii) between a major diagonal and a face diagonal,
(iii) between diagonals on adjacent faces,
(iv) between major diagonals.

12. Use dot products to show that

(i) if $\mathbf{v}$ is orthogonal to $\mathbf{x}$ and $\mathbf{y}$, then $\mathbf{v}$ is orthogonal to any linear combination $a\mathbf{x} + b\mathbf{y}$ where a and b are scalars;
(ii) conversely, if $\mathbf{v}$ is orthogonal to $\mathbf{x}$ and $a\mathbf{x} + b\mathbf{y}$ where a and b are scalars such that b is nonzero, then $\mathbf{v}$ is orthogonal to $\mathbf{y}$.

13.* Use vectors to show that any angle inscribed in a semicircle is a right angle.

14.* (uniqueness of the dot product) Suppose we have a dot operation on geometric vectors that is commutative and distributes over vector addition, such that

$$\mathbf{i}\cdot\mathbf{j} \;=\; \mathbf{i}\cdot\mathbf{k} \;=\; \mathbf{j}\cdot\mathbf{k} \;=\; 0 \quad\text{and}\quad \mathbf{i}\cdot\mathbf{i} \;=\; \mathbf{j}\cdot\mathbf{j} \;=\; \mathbf{k}\cdot\mathbf{k} \;=\; 1\,.$$

Suppose further that the dot operation is compatible with scalar multiplication in the sense of Useful Fact (viii). Deduce the usual algebraic rule for dot products.

15.* Imagine that you don't know the algebraic formula for the dot product. Starting with the geometric definition for the dot product, apply the Cosine Rule to 'discover' the algebraic formula.

Important Ideas and Useful Facts:

(i) Geometric definition of dot product: If $\mathbf{v}$ and $\mathbf{w}$ are vectors and θ is the angle between them, then

$$\mathbf{v}\cdot\mathbf{w} \;=\; |\mathbf{v}||\mathbf{w}|\cos\theta\,,$$

so that, in the case both vectors are nonzero,

$$\cos\theta \;=\; \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\,.$$

(ii) Algebraic definition of dot product: If $\mathbf{v} = (v_k)_{1\le k\le n}$ and $\mathbf{w} = (w_k)_{1\le k\le n}$, then

$$\mathbf{v}\cdot\mathbf{w} \;=\; \sum_{k=1}^{n} v_k w_k\,.$$

(iii) The angle between two vectors is zero or acute if their dot product is positive. The angle is obtuse or 180° if the dot product is negative. Two vectors are mutually perpendicular (orthogonal) if the dot product is zero.

(iv) Cauchy-Schwarz Inequality: $|\mathbf{v}\cdot\mathbf{w}| \;\le\; |\mathbf{v}||\mathbf{w}|\,.$

(v) Commutativity of dot product: $\mathbf{v}\cdot\mathbf{w} \;=\; \mathbf{w}\cdot\mathbf{v}\,.$

(vi) Distributivity of dot over plus: $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} \;=\; \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}\,.$

(vii) If $\mathbf{v}$ is any vector then $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$, so $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}\,.$

(viii) If $\mathbf{v}$ and $\mathbf{w}$ are vectors and λ is a scalar then $(\lambda\mathbf{v})\cdot\mathbf{w} = \lambda(\mathbf{v}\cdot\mathbf{w}) = \mathbf{v}\cdot(\lambda\mathbf{w})\,.$

(ix) The *vector projection* of $\mathbf{v}$ in the direction of $\mathbf{w}$ is $\dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$, which is the best approximation of $\mathbf{v}$ using a scalar multiple of $\mathbf{w}$.

(x) The *scalar component* of $\mathbf{v}$ in the direction of $\mathbf{w}$ is $\dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|}$, which is plus or minus the magnitude of the vector projection (minus in the case that the angle is obtuse or 180°).

(xi) The *vector component of* $\mathbf{v}$ *orthogonal to* $\mathbf{w}$ is the difference between $\mathbf{v}$ and its vector projection, which is

$$\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\,.$$