The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 3

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2012

Lecturers: Daniel Daners and James Parkinson

Topics covered

In lectures last week:

- ☐ The Fundamental Theorem of Calculus.
- ☐ Functions defined using integrals: the logarithm, the error function, the inverse tangent function, the Fresnel integrals, the sine integral, the logarithmic integral.
- ☐ Elementary antiderivatives (Liouville's Theorem).

Objectives

After completing this tutorial sheet you will be able to:

- ☐ Apply the Fundamental Theorem of Calculus in various settings.
- ☐ Quantitatively and qualitatively analyse functions defined by integrals.
- ☐ Decide if certain functions defined by integrals are elementary (challenging!).
- ☐ Use integration and differentiation to prove a beautiful theorem: π is irrational.

Preparation questions to do *before* class

1. Find the derivative of $f(x) = \displaystyle\int_1^{\sqrt{x}} \frac{s^2}{s^2+1}\,ds$

2. Use integration by parts to calculate $\displaystyle\int_0^1 C(x)\,dx$, where $C(x) = \displaystyle\int_0^x \cos(t^2)\,dt$.

Questions to do in class

3. Find the derivative of the following functions.

(a) $f(x) = \displaystyle\int_x^4 (2+\sqrt{u})^8\,du$

(b) $f(x) = \displaystyle\int_x^{\cos x} e^{-t^2}\,dt$

4. Recall that the logarithmic integral $\mathrm{Li}(x)$ is defined by $\mathrm{Li}(x) = \displaystyle\int_2^x \frac{dt}{\ln t}$. Suppose that $\alpha > 2$ satisfies $\mathrm{Li}(\alpha) = 1$. Calculate

$$\int_2^\alpha \frac{\mathrm{Li}(x)}{x^2}\,dx.$$

5. Let $f(x)$ be a continuous function on $[a, b]$. Apply the Mean Value Theorem to the function

$$F(x) = \int_a^x f(t)\,dt$$

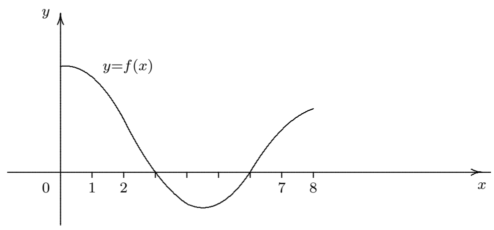
to show that there exists $c \in (a, b)$ such that

$$\frac{1}{b-a}\int_a^b f(t)\,dt = f(c), \qquad \text{and interpret this geometrically.}$$

Questions for extra practice

6. Let $f(x) = \displaystyle\int_0^x x\sin(t^2)\,dt$. Find $f''(x)$.

7. Suppose that a function $y = f(x)$ has the following graph:



Let $F(x)$ be the function defined by $F(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 8$. Sketch the graph of $y = F(x)$, indicating points where F has a local maximum or minimum, and any points of inflection.

8. If $x\sin(\pi x) = \displaystyle\int_0^{x^2} f(t)\,dt$, find $f(4)$.

9. Suppose that $f(t)$ is continuous on $[a, b]$. Recall the following:

- The *Extreme Value Theorem* says that $f(x)$ attains a global maximum M and a global minimum m on $[a, b]$.
- Then the *Intermediate Value Theorem* implies that if $m \le A \le M$ then there exists $c \in [a, b]$ such that $f(c) = A$.

Let $p(t)$ be Riemann integrable on $[a, b]$ with $p(t) \ge 0$ for all $t \in [a, b]$.

(a) Explain why

$$m\int_a^b p(t)\,dt \le \int_a^b f(t)p(t)\,dt \le M\int_a^b p(t)\,dt.$$

(b) Deduce that there is $c \in [a, b]$ such that

$$\int_a^b f(t)p(t)\,dt = f(c)\int_a^b p(t)\,dt.$$

This is called the *Mean Value Theorem for integrals.* It is a generalisation of Question 5. We will use it later in the course (§6.2 of the course notes).

Challenging questions

10. Suppose that $f(x)$ and $g(x)$ are rational functions. Recall that Liouville's Theorem says that

$$\int f(x)e^{g(x)}\,dx$$

is an elementary function if and only if there is a rational function $r(x)$ such that $f(x) = r'(x) + g'(x)r(x)$. Is

$$\int e^{1/x}\,dx$$

an elementary function?

The following questions use a nice mixture of differentiation and integration to show that π, π^2, and e^r ($r \in \mathbb{Q}\backslash\{0\}$) are irrational. They are adapted from proofs in *Irrational Numbers*, by Ivan Niven (The Carus Mathematical Monographs, No. 11, 1956). The first proof of the irrationality of π (Johann Lambert, 1768) was considerably more complicated.

11. Let $n \geq 0$ be an integer, and let $f_n(x) = \dfrac{x^n(1-x)^n}{n!}$.

(a) Show that $f_n^{(j)}(0)$ and $f_n^{(j)}(1)$ are integers for all $j \in \mathbb{N}$. *Hint: Binomial Theorem to see that $f_n^{(j)}(0)$ is integral. Then use $f_n(1-x) = f_n(x)$.*

(b) Assume that $\pi^2 = \frac{a}{b}$ is rational, with $a, b \in \mathbb{N}\backslash\{0\}$. Let

$$F_n(x) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f_n^{(2k)}(x)\,.$$

Use (a) to show that $F_n(0)$ and $F_n(1)$ are integers.

(c) Calculate $\frac{d}{dx}\left(F_n'(x)\sin \pi x - \pi F_n(x)\cos \pi x\right)$ and deduce that

$$I_n = \pi a^n \int_0^1 f_n(x)\sin \pi x\,dx \qquad \text{is an integer for all } n.$$

(d) Obtain a contradiction by noticing that $0 < f_n(x) < \frac{1}{n!}$ for $x \in (0,1)$. Thus π^2 is irrational. Deduce that π is irrational too.

12. Let $f_n(x)$ be as in Question 11.

(a) Let $m \in \mathbb{N}\backslash\{0\}$ and define $G_n(x)$ (depending on n and m) by

$$G_n(x) = \sum_{k=0}^{2n} (-1)^k m^{2n-k} f_n^{(k)}(x)\,.$$

Show that $G_n(0)$ and $G_n(1)$ are integers. Calculate $\frac{d}{dx}\left(e^{mx}G_n(x)\right)$ and deduce that

$$m^{2n+1}\int_0^1 e^{mx} f_n(x)dx = e^m G_n(1) - G_n(0)\,.$$

(b) Now assume that $e^m = \frac{p}{q}$ is rational. Obtain a contradiction.

(c) Deduce that e^r is irrational for all $r \in \mathbb{Q}\backslash\{0\}$.