§10.4 Application of multiplicativity of σ: classification of even perfect numbers.

Definition: n is called <u>perfect</u> if it equals to the sum of all its proper divisors.

i.e. $n = \sigma(n) - n$ or $2n = \sigma(n)$.

Examples: $6 = 1+2+3$

$$\sigma(28) = \sigma(2^2 \cdot 7) = \sigma(2^2) \cdot \sigma(7) = 7 \cdot 8 = 2 \cdot 28.$$

So 6, 28 are perfect.

Remark: not known if there are inf. many perfect numbers.

not known if there exists an odd perfect number.

Theorem: n is an even perfect number $\Longleftrightarrow$ $n = 2^k \cdot (2^{k+1} - 1)$ where $k \in \mathbb{Z}^+$ and $2^{k+1} - 1$ is prime.

Remark: primes of the form $2^k - 1$ are called <u>Mersenne primes</u>.

Proof. Write $n = 2^k \cdot m$ where $k \in \mathbb{Z}^+$, m is odd.

$$2n = \sigma(2^k \cdot m) = \sigma(2^k) \cdot \sigma(m) = (2^{k+1} - 1) \cdot \sigma(m).$$

$2^{k+1} - 1 \mid 2n = 2^{k+1} m \Rightarrow 2^{k+1} - 1 \mid m$ or

$m = (2^{k+1} - 1) l$

Rewrite: $2^{k+1} l = \sigma(m) = \sigma((2^{k+1}-1)l)$

Assume $l > 1$. Then

$$\sigma((2^{k+1}-1)l) \geq (2^{k+1}-1) + l + (2^{k+1}-1)l > 2^{k+1} l$$

Contradiction.

(What if $l = 2^{k+1}-1$? - Ex).

$\Rightarrow l = 1$.

Rewrite: $2^{k+1} = \sigma(2^{k+1}-1) \Rightarrow 2^{k+1}-1$ is prime.

Check that any number $n = 2^k(2^{k+1}-1)$, $k \in \mathbb{Z}^+$ ~~$2^{k+1}-1$ is prime~~ with prime $2^{k+1}-1$ is perfect. $\boxtimes$

Up to now we know only 49 Mersenne primes, the largest one is

$$2^{74\,207\,281}-1.$$

§10.5. More on Euler phi-function.

Every multiplicative function is determined by its values at powers of primes:

if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_d^{\alpha_d}$ is the factorization of n with $p_1, p_2, \dots, p_d$ distinct primes then

$$f(n) = \underbrace{f(p_1^{\alpha_1})}\, f(p_2^{\alpha_2}) \dots f(p_d^{\alpha_d})$$

Warning: in general can not write this as $f(p_1)^{\alpha_1}$

Examples: (1) for Euler phi-function: $\varphi(p^k) = p^k - p^{k-1}$

(2) For number of divisors: $\tau(p^k) = k+1$

(3) For sum of divisors : $\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$

(4) For Liouville finction : $\lambda(p^k) = (-1)^k$.

We know how to construct new multiplicative functions $F(n)$ from know one's $f(n)$:

$$F(n) = \sum_{d|n} f(d).$$

Examples: (1) For $f(d)=1$ we have $F(n) = \tau(n)$

(2) For $f(d)=d$ we have $F(n) = \sigma(n)$.

Q: What happens if $f(d) = \varphi(d)$

Try small n: $F(1) = \varphi(1) = 1$

$F(2) = \varphi(1) + \varphi(2) = 2$

$F(3) = \varphi(1) + \varphi(3) = 1+2 = 3$

$F(4) = \varphi(1) + \varphi(2) + \varphi(4) = 1+1+2 = 4$

Theorem: For any $n \in \mathbb{Z}^+$, $\sum_{d|n} \varphi(d) = n$.

Proof 1: LHS and RHS are both multiplicative $\Rightarrow$ it is sufficient to check the equality for $n = p^k$, p is prime.

All divisors of $p^k = n$ are: $1, p, p^2, p^3, \dots, p^k$

$$\begin{aligned} LHS &= \varphi(1) + \varphi(p) + \varphi(p^2) + \dots + \varphi(p^k) \\ &= \cancel{1} + (\cancel{p} - \cancel{1}) + (\cancel{p^2} - \cancel{p}) + \dots + (p^k - \cancel{p^{k-1}}) \\ &= p^k = n \end{aligned}$$

$\boxtimes$

Let work out the table of $\gcd(a,15)$ for a from 0 to 15

a	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\gcd(a,15)$	15	1	1	3	1	5	3	1	1	3	5	1	3	1	1

We have $\gcd(a,15)=1$ 8 times ($\varphi(15)=8$)

$\gcd(a,15)=3$ 4 times. Observe, $\varphi(15/3)=4$

$\gcd(a,15)=5$ 2 times. Observe, $\varphi(15/5)=2$

$\gcd(a,15)=15$ 1 time. Observe, $\varphi(15/15)=1$.