Week 4: Inviscid, incompressible, irrotational fluid flow

The continuum hypothesis

We take an elementary volume, however small, to have properties which are the same as for the fluid as a whole. We require a fluid 'particle' to be small compared to the fluid as a whole but large enough to contain a large number of molecules (over which we average).

This hypothesis means that density, momentum, pressure and velocity all become piecewise continuous functions of space and time (bulk properties).

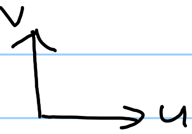
Lagrange specification - follow individual particles

Euler specification - interested in bulk properties
- specify flow in terms of field variables

e.g. velocity at (x,y) is given by $\underline{u}(x,y)$.

The complex potential (Cartesian coordinates (x,y))

If $\underline{u} = (u, v)$ is the velocity field
$= u + iv$



conservation of mass: $\boxed{\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0}$ (1)
(incompressible)
(continuity equation)

$u = u(x,y)$
$v = v(x,y)$

Irrotational: $\boxed{\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0}$ (2)

Define the velocity potential ϕ by

$\boxed{u = \dfrac{\partial \phi}{\partial x} \quad \text{and} \quad v = \dfrac{\partial \phi}{\partial y}}$ then (2) is satisfied.

From (1): $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$. So ϕ satisfies Laplace's equation.

Introduce a stream function $\psi(x,y)$ defined by $\boxed{u = \dfrac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\dfrac{\partial \psi}{\partial x}}$. Then (1) is satisfied

From (2): $\boxed{\dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}=0.}$ So ψ satisfies Laplace's equation.

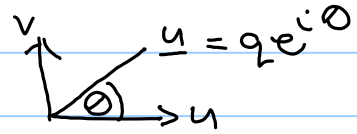
Comparing the velocity components

$\boxed{\dfrac{\partial\phi}{\partial x}=\dfrac{\partial\psi}{\partial y}}$ and $\boxed{\dfrac{\partial\phi}{\partial y}=-\dfrac{\partial\psi}{\partial x}}$ Cauchy-Riemann equations

So the complex function $w=\phi+i\psi$ is an analytic function of $z=x+iy$ in the domain occupied by the fluid.

$w(z)$ is the complex potential for the flow.

$$\frac{dw}{dz}=\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=u-iv=\underline{u}^*$$
$$=qe^{-i\theta}$$



$q=(u^2+v^2)^{1/2}$

Find w to find $\underline{u}$.

Streamlines

Streamlines are the line drawn in the fluid so that the tangent at each point is in the direction of the fluid velocity at that point.
These are the trajectories of the velocity field.
$\psi=$ constant are the equations of the streamlines
The streamlines are orthogonal to the lines of equal potential ϕ.

Stagnation points

Here the velocity is zero. Thus $u=0$ and $v=0$
then $\dfrac{dw}{dz}=0$ at stagnation points.

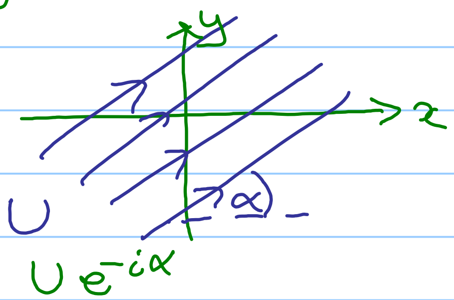
Streamlines only intersect at stagnation points.

Examples

(i) A uniform stream flowing at an angle α to the horizontal has

$u=U\cos\alpha$

$v=U\sin\alpha$



So $\dfrac{dw}{dz}=u-iv=$
$=U\cos\alpha-iU\sin\alpha=Ue^{-i\alpha}$

Then $w = Ue^{-i\alpha}z$

(ii) A point vortex is a system of concentric circular streamlines centred at $z = z_0$, with complex potential

$$w = \frac{-i\Gamma}{2\pi}\log(z - z_0)$$

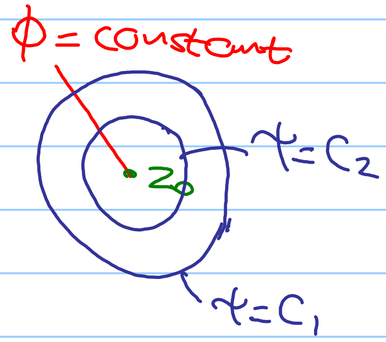
($-\Gamma$ is the circulation - lift $= -\rho U\Gamma \uparrow$)

$\log z = \ln|z| + i\arg z$

Take $z - z_0 = re^{i\theta}$,

$$w = \frac{-i\Gamma}{2\pi}\log(re^{i\theta})$$

$$= \frac{-i\Gamma}{2\pi}(\ln r + i\theta) = \phi + i\psi$$



Take real and imaginary parts

$$\phi = \frac{\Gamma\theta}{2\pi} \quad \text{and} \quad \psi = -\frac{\Gamma\ln r}{2\pi}$$

So the streamlines for $\psi =$ constant correspond to $r =$ constant.

<u>Using conformal transformations to find $w(z)$</u>

We seek $w(z)$, the complex potential in the z-plane, for flow over a rigid body. If the transformation $\zeta = f(z)$ maps the surface of the rigid body and the flow outside it onto the flow outside a half-plane or circle in the ζ-plane, then by defining $W(\zeta) = w(z)$ we have a correspondence between the flow and geometry in both planes.

Any streamline in the z-plane transforms to a streamline in the ζ-plane due to this correspondence in complex potentials.

The complex velocity is $\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz} = \frac{dw}{d\zeta}f'(z)$.