The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2 **Exercises for Week 3 (beginning 14 August)** 2017

It might be useful to attempt the Revision and Exploration Exercises before the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Antiderivatives and Uniqueness up to a Constant: If f and g are functions such that $f' = g$ then we call g the *derivative* of f and f an *antiderivative* of g. If f_1 and f_2 are antiderivatives of g, and g is continuous on an interval then the values of f_1 and f_2 differ by a constant function on this interval.

(ii) The Fundamental Theorem of Calculus Part I: If $a < b$ and f is continuous on $[a, b]$ then the function F defined by

$$F(x) \ = \ \int_a^x f(t)\ dt$$

is differentiable on (a, b) and $F'(x) = f(x)$. Thus F is an antiderivative of f on (a, b).

(iii) The Fundamental Theorem of Calculus Part II: If $a < b$ and f is continuous on $[a, b]$, and F is any antiderivative of f on (a, b), then

$$\int_a^b f(x)\ dx \ = \ F(b) - F(a)\ .$$

Common notations for $F(b) - F(a)$ are $[F(x)]_a^b$, $F(x)]_a^b$ and $F(x)\Big|_a^b$.

(iv) The Indefinite Integral: If f is a continuous function then we write

$$\int f(x)\ dx$$

for any antiderivative of f, and call this the *indefinite integral of* f. Thus if we put $F(x) = \displaystyle\int f(x)\ dx$, for some choice of antiderivative, then $F'(x) = f(x)$. Choices of antiderivatives differ by a constant.

(v) Some Properties and Standard Indefinite Integrals:

(a) $\displaystyle\int kf(x)\ dx = k\int f(x)\ dx$ and $\displaystyle\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$

(b) $\displaystyle\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$, and $\displaystyle\int \frac{1}{x}\ dx = \ \ln|x| + C$

(c) $\displaystyle\int e^x\ dx = \ e^x + C$, $\displaystyle\int \cosh x\ dx = \ \sinh x + C$, $\displaystyle\int \sinh x\ dx = \ \cosh x + C$

(d) $\displaystyle\int \cos x\ dx = \ \sin x + C$, $\displaystyle\int \sin x\ dx = \ -\cos x + C$, $\displaystyle\int \sec^2 x\ dx = \ \tan x + C$

(vi) Substitution Rules: Under appropriate conditions, using the substitution $u = g(x)$, we may manipulate the relationship

$$du = g'(x)dx$$

between differentials, as though they represent actual quantities, obtaining

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \qquad \text{and} \qquad \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\,.$$

Revision and Exploration:

1. Use the Chain Rule to differentiate $f(x) = -\tan^{-1}\left(\frac{1}{x}\right)$. Do you recognise your answer as a well-known derivative?

2. Use the Mean Value Theorem to prove that if $f : [0, \infty) \to \mathbb{R}$ is a continuous function such that $f'(x) = 0$ is zero for all $x > 0$, then f is constant.

3. Can you find a differentiable function that is not constant but for which the derivative is the zero function on the same domain?

Tutorial Exercises:

4. (for general discussion) Bill used the Fundamental Theorem of Calculus to do the following calculation:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -\frac{1}{1} - \left(-\frac{1}{-1}\right) = -2\,.$$

Susan said: "But $\frac{1}{x^2}$ is always positive, so its graph is above the x-axis, and the answer should be a positive number." Was either Bill or Susan correct?

5. Find the following definite and indefinite integrals (where a is a positive constant in the last two parts):

(i) $\displaystyle\int_0^{\pi/2} \cos^4\theta \sin^3\theta\,d\theta$ $\quad$ *(ii) $\displaystyle\int_0^{\pi/3} \sec^5\theta \tan^3\theta\,d\theta$ $\quad$ (iii) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(iv) $\displaystyle\int \frac{x}{\sqrt[4]{x+2}}\,dx$ $\quad$ (v) $\displaystyle\int_0^a x\sqrt{a^2 - x^2}\,dx$ $\quad$ *(vi) $\displaystyle\int_0^a \sqrt{a^2 - x^2}\,dx$

***6.** Use an appropriate integral and Riemann sums to estimate $1 + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{100}$.

7. Find the derivative of f in each case:

(i) $\displaystyle f(x) = \int_{-1}^{x} \sqrt{t^3 + 1}\,dt$ $\quad$ (ii) $\displaystyle f(x) = \int_x^4 (2 + \sqrt{u})^8\,du$ $\quad$ *(iii) $\displaystyle f(x) = \int_1^{\sqrt{x}} \frac{s^2}{s^2 + 1}\,ds$

8. Suppose f is continuous throughout.

(i) Find $\displaystyle\int_0^2 f(2x)\,dx$ and $\displaystyle\int_0^2 xf(x^2)\,dx$ given that $\displaystyle\int_0^4 f(x)\,dx = 10$.

(ii) Verify that

$$\int_a^b f(-x)\,dx \;=\; \int_{-b}^{-a} f(x)\,dx$$

and

$$\int_a^b f(x+c)\,dx \;=\; \int_{a+c}^{b+c} f(x)\,dx\ ,$$

and interpret these results geometrically.

*(iii) Use the substitution $u = \pi - x$ to verify that

$$\int_0^\pi xf(\sin x)\,dx \;=\; \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx\ .$$

9. (for general discussion) Let $f(x) \;=\; -\tan^{-1}\left(\dfrac{1}{x}\right)$. From the first exercise you will have noticed that f and $\tan^{-1}$ are both antiderivatives of the same function, so differ by a constant function, right? But

$$f(1) - \tan^{-1}(1) \;=\; -\frac{\pi}{2} \qquad \text{whilst} \qquad f(-1) - \tan^{-1}(-1) \;=\; \frac{\pi}{2}\ ,$$

so we get different constants! Explain this apparent anomaly.

Further Exercises:

10. Find the following antiderivatives:

(i) $\displaystyle\int \frac{dx}{x^2+2x+1}$ (ii) $\displaystyle\int \frac{dx}{x^2+2x+2}$ *(iii) $\displaystyle\int \frac{dx}{x^2+2x}$

***11.** Find the derivative $g'(x)$ given that $g(x) = \displaystyle\int_x^{\cos x} e^{-t^2}\,dt$.

***12.** Find the second derivative $h''(x)$ given that $h(x) = \displaystyle\int_0^x x\sin(t^2)\,dt$.

***13.** Let f be a continuous function with domain $[a,b]$. Apply the Mean Value Theorem to show that there exists $c \in (a,b)$ such that

$$f(c) \;=\; \frac{1}{b-a}\int_a^b f(t)\,dt\ .$$

***14.** Find $f(4)$ given that f is a function such that

$$x\sin(\pi x) \;=\; \int_0^{x^2} f(t)\,dt\ .$$

***15.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that the function F defined by

$$F(x) = \int_0^x f(t)\,dt$$

is odd if f is even, and even if f is odd.

***16.** Find a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|x| = \int_0^x f(t)\,dt$$

for all $x \in \mathbb{R}$. (Note: by the Fundamental Theorem of Calculus, such a function f cannot be continuous at 0.)

****17.** The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is defined to be $\displaystyle\lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n^2}$. Given that this limit exists, verify that

$$1.5 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 2\,.$$

Can you explain why this limit should exist?

Short Answers to Selected Exercises:

1. $\frac{1}{1+x^2}$

3. A natural example would be f defined by the rule $f(x) = \frac{|x|}{x}$ with domain $\mathbb{R}\backslash\{0\}$.

5. (i) $\frac{2}{35}$ (ii) $\frac{418}{35}$ (iii) $-2\cos\sqrt{x} + C$ (iv) $\frac{4}{7}(x+2)^{7/4} - \frac{8}{3}(x+2)^{3/4} + C$ (v) $\frac{a^3}{3}$ (vi) $\frac{\pi a^2}{4}$

6. lower bound of 667 and upper bound of 676.

7. (i) $\sqrt{x^3+1}$ (ii) $-(2+\sqrt{x})^8$ (iii) $\frac{\sqrt{x}}{2(x+1)}$

8. (i) 5 in both cases

10. (i) $-\frac{1}{x+1} + C$ (ii) $\tan^{-1}(x+1) + C$ (iii) $\frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$

11. $-\sin x\, e^{-\cos^2 x} - e^{-x^2}$

12. $2\sin(x^2) + 2x^2\cos(x^2)$

14. $\pi/2$