(A)

MATH1902 2008 exam solutions 9/6/2010

Q1/ (a) M is the intersection point with plane π

with equation $2x-3y-6z=6$, so put $x=z=0$,

giving $-3y=6$, so $y=-2$, so $M=(0,-2,0)$.

Normal to π is $\underset{\sim}{v} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - 6\underset{\sim}{k}$ and normal to π'

is $\underset{\sim}{n} = 4\underset{\sim}{j} + 3\underset{\sim}{k}$, so line ℓ points in direction

$$\underset{\sim}{n} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 4 & 3 \\ 2 & -3 & -6 \end{vmatrix} = -15\underset{\sim}{i} + 6\underset{\sim}{j} - 8\underset{\sim}{k}$$

(Check: $(-15\underset{\sim}{i} + 6\underset{\sim}{j} - 8\underset{\sim}{k}) \cdot (4\underset{\sim}{j} + 3\underset{\sim}{k}) = (-15\underset{\sim}{i} + 6\underset{\sim}{j} - 8\underset{\sim}{k}) \cdot (2\underset{\sim}{i} - 3\underset{\sim}{j} - 6\underset{\sim}{k}) = 0$ ✓)

so has vector equation $\underset{\sim}{r} = -2\underset{\sim}{j} + t(-15\underset{\sim}{i} + 6\underset{\sim}{j} - 8\underset{\sim}{k})$,

which becomes $\begin{cases} x = -15t \\ y = -2+6t \\ z = -8t \end{cases}$

yielding Cartesian equations $\boxed{\dfrac{x}{-15} = \dfrac{y+2}{6} = \dfrac{z}{-8}}$

(b) If θ is the angle between π and π' then it is

also the angle between $\underset{\sim}{v}$ and $\underset{\sim}{n}$ so

$$\cos\theta = \frac{\underset{\sim}{n} \cdot \underset{\sim}{v}}{|\underset{\sim}{n}||\underset{\sim}{v}|} = \frac{-12-18}{\sqrt{25}\,\sqrt{49}} = \frac{-30}{5(7)} = -\frac{6}{7}$$

so acute angle will be $\boxed{\arccos \frac{6}{7}}$

Q1/(c) Line m lies on Π, contains $M(0,-2,0)$ and is parallel to xz-plane, so has direction vector

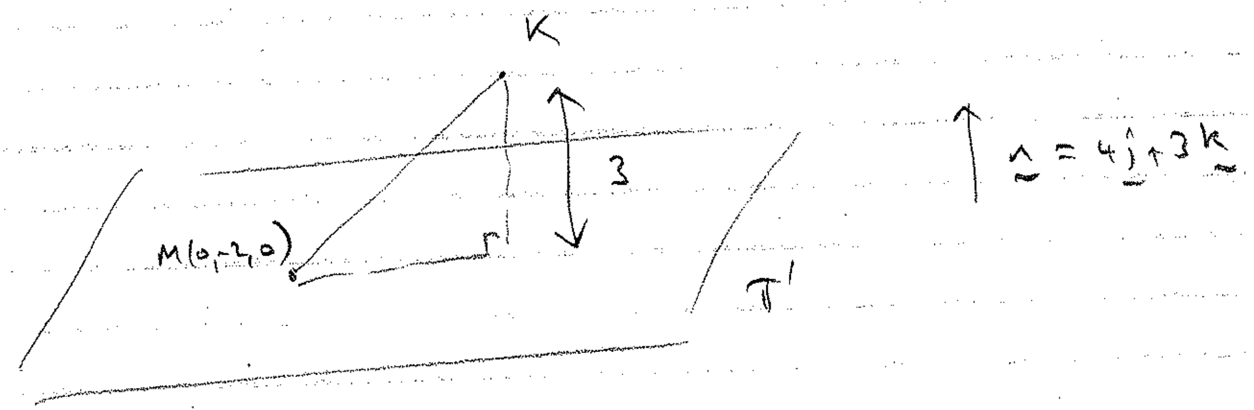
$$\underset{\sim}{n} \times \underset{\sim}{j} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -3 & -6 \\ 0 & 1 & 0 \end{vmatrix} = 6\underset{\sim}{i} + 2\underset{\sim}{k}$$

so has vector equation $\underset{\sim}{r} = -2\underset{\sim}{j} + t(6\underset{\sim}{i} + 2\underset{\sim}{k})$,

so parametric equations

$$\boxed{\left.\begin{aligned} x &= 6t \\ y &= -2 \\ z &= 2t \end{aligned}\right\} t \in \mathbb{R}}$$

(d) From (c), $K = (6t, -2, 2t)$ for some t



and we want $|\overrightarrow{MK} \cdot \hat{\underset{\sim}{n}}| = 3$,

i.e. $\left|(6t\underset{\sim}{i} + 2t\underset{\sim}{k}) \cdot \frac{1}{5}(4\underset{\sim}{j} + 3\underset{\sim}{k})\right| = \left|\frac{6t}{5}\right| = 3$,

i.e. $t = \pm \frac{3(5)}{6} = \pm 5/2$

so $\boxed{K = (\pm 15, -2, \pm 5)}$. (two possible correct answers) c

Q2/ (a)

$$\left[\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 6 \\ -1 & -2 & 4 & 0 & -4 \\ -2 & -4 & 8 & -1 & -10 \\ 1 & 3 & -4 & 1 & 8 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 6 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -4 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 - 4x_3 &= 0 \\ x_2 &= 2 \\ x_4 &= 2 \end{aligned}$$

Put $x_3 = t$, so $x_1 = 4t$ and we have general solution

$$\boxed{(x_1, x_2, x_3, x_4) = (4t, 2, t, 2) \text{ for } t \in \mathbb{R}}$$

(b) Working backwards from an inconsistent system gives:

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -4 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 6 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 6 \\ -1 & -2 & 4 & 0 & -4 \\ -2 & -4 & 8 & -1 & -9 \\ 1 & 3 & -4 & 1 & 8 \end{array}\right]$$

$$\boxed{\text{so take } \underline{u} = \begin{bmatrix} 6 \\ -4 \\ -9 \\ 8 \end{bmatrix}}$$

(infinitely many possible correct answers)

(D)

Q2/ (c) If $A\underset{\sim}{x} = \underset{\sim}{u}$ has a unique solution then after applying our operations in (a) yields an equivalent system $B\underset{\sim}{x} = \underset{\sim}{v}$ for some $\underset{\sim}{v}$

where

$$B = \left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & v_1 \\ 0 & 1 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 1 & v_3 \\ 0 & 0 & 0 & 0 & v_4 \end{array}\right], \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

so that $v_4 = 0$ and $\underset{\sim}{x}$ is described by a family of solutions using one parameter, which is not unique, a contradiction. Hence no $\underset{\sim}{u}$ exists for which $A\underset{\sim}{x} = \underset{\sim}{u}$ has a unique solution.

(d) If system $A\underset{\sim}{x} = \underset{\sim}{u}$ and $A\underset{\sim}{x} = \underset{\sim}{w}$ are consistent then there exist $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ such that

$$A\underset{\sim}{x}_1 = \underset{\sim}{u} \quad \text{and} \quad A\underset{\sim}{x}_2 = \underset{\sim}{w}$$

so that

$$A(\alpha \underset{\sim}{x}_1 + \beta \underset{\sim}{x}_2) = \alpha A\underset{\sim}{x}_1 + \beta A\underset{\sim}{x}_2 = \alpha \underset{\sim}{u} + \beta \underset{\sim}{w},$$

which shows the system $A\underset{\sim}{x} = \alpha \underset{\sim}{u} + \beta \underset{\sim}{w}$ is consistent.

(5)

Q3/ (a) $A\underline{v} = \lambda \underline{v}$ and $B\underline{v} = \mu \underline{v}$ $(\underline{v} \neq 0)$.

Hence

$$(A+B)\underline{v} = A\underline{v} + B\underline{v} = \lambda\underline{v} + \mu\underline{v} = (\lambda+\mu)\underline{v}$$

and

$$(AB)\underline{v} = A(B\underline{v}) = A(\mu\underline{v}) = \mu A\underline{v} = (\mu\lambda)\underline{v},$$

so that $\underline{v}$ is an eigenvector of $A+B$ with respect to eigenvalue $\lambda+\mu$, and an eigenvector of AB with respect to eigenvalue $\mu\lambda$.

(b) Observe that

$$(AB-BA)\underline{v} = AB\underline{v} - BA\underline{v} = \mu\lambda\underline{v} - \lambda\mu\underline{v} = \underline{0}.$$

If $\det(AB-BA) \neq 0$ then $(AB-BA)^{-1}$ exists,

so that
$$\underline{v} = (AB-BA)^{-1}(AB-BA)\underline{v} = (AB-BA)^{-1}\underline{0} = \underline{0},$$

contradicting that $\underline{v} \neq \underline{0}$. Hence $\det(AB-BA) = 0$.

(c) $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$, so

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 4 = \lambda^2 - 7\lambda + 10 - 4$$
$$= \lambda^2 - 7\lambda + 6 = (\lambda-1)(\lambda-6),$$

$$\det(B-\lambda I) = \begin{vmatrix} 2-\lambda & -6 \\ 1 & -3-\lambda \end{vmatrix} = (2-\lambda)(-3-\lambda) + 6 = \lambda^2 + \lambda - 6 + 6$$
$$= \lambda(\lambda+1).$$

F

Q3, (c) (cont.)

So A has eigenvalues $\lambda = 1, 6$

B " " $\lambda = 0, -1$

$$A - I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A - 6I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$B - 0 = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

common eigenspace

$$B + I = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

So a common eigenvector is $\underset{\sim}{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(d) $A^{-3}B^{2}\underline{v} = (6)^{-3}(-1)^{2}\underline{v} = \frac{1}{216}\underset{\sim}{v}$,

so $\underline{v}$ is an eigenvector for $A^{-3}B^{2}$ with eigenvalue $\frac{1}{216}$.

$\underset{\sim}{v}$ eigenvector for A with eigenvalue 6

$\underset{\sim}{v}$ " " B " " -1

Q4/(a) Suppose $(x+y)^2 = x^2 + 2xy + y^2$.

Then $x^2 + xy + yx + y^2 = x^2 + xy + xy + y^2$,

so subtracting yields $yx = xy$, that is, x & y commute. Hence powers of x and y commute, so

$$(x+y)^3 = (x+y)^2(x+y) = (x^2+2xy+y^2)(x+y)$$

$$= x^3 + 2xyx + y^2x + x^2y + 2xy^2 + y^3$$

$$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3.$$

(b) This is immediate from (c).

(c) **Claim**: If x^{-1} or y^{-1} exists and $(xy)^2 = x^2y^2$ then $(xy)^3 = x^3y^3$.

Proof: If x^{-1} exists and $(xy)^2 = x^2y^2$ then we may cancel x to get $yxy = xy^2$, so $(xy)^3 = xyxyxy = x(yxy)xy$

$= x(xy^2)xy = x^2y(yxy) = x^2y(xy^2) = x^2(yxy)y = x^2xy^2y = x^3y^3$.

The result when y^{-1} exists is exactly similar, by left-right symmetry. $\square$