# THE UNIVERSITY OF SYDNEY

# PHYS1902 – PHYSICS 1B (ADVANCED)
# SOLUTIONS

## NOVEMBER 2008

### Time allowed: THREE Hours

**MARKS FOR QUESTIONS ARE AS INDICATED**
**TOTAL: 90 marks**

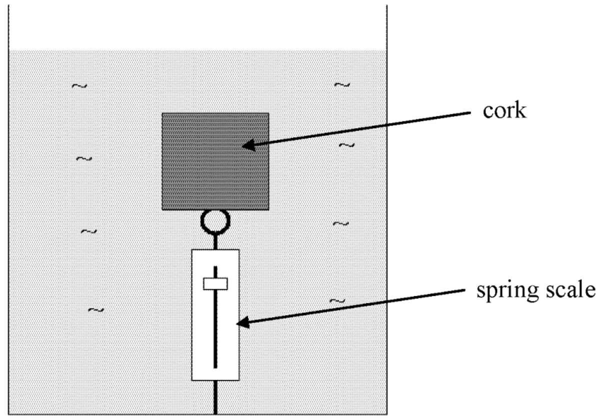
## INSTRUCTIONS

- **All questions are to be answered.**
- **Use a separate answer book for each section.**
- **All answers should include explanations in terms of physical principles.**

## DATA

| | | | |
|---|---|---|---|
| Density of water | $\rho$ | = | $1.00 \times 10^{3}$ kg.m$^{-3}$ |
| Density of air | $\rho$ | = | $1.20$ kg.m$^{-3}$ |
| Atmospheric pressure | 1 atm | = | $1.01 \times 10^{5}$ Pa |
| Magnitude of local gravitational field | $g$ | = | $9.81$ m.s$^{-2}$ |
| Avogadro constant | $N_A$ | = | $6.022 \times 10^{23}$ mol$^{-1}$ |
| Permittivity of free space | $\varepsilon_0$ | = | $8.854 \times 10^{-12}$ F.m$^{-1}$ |
| Permeability of free space | $\mu_0$ | = | $4\pi \times 10^{-7}$ T.m.A$^{-1}$ |
| Elementary charge | $e$ | = | $1.602 \times 10^{-19}$ C |
| Speed of light in vacuum | $c$ | = | $2.998 \times 10^{8}$ m.s$^{-1}$ |
| Planck constant | $h$ | = | $6.626 \times 10^{-34}$ J.s |
| Rest mass of an electron | $m_e$ | = | $9.110 \times 10^{-31}$ kg |
| Rest mass of a neutron | $m_n$ | = | $1.675 \times 10^{-27}$ kg |
| Rest mass of a proton | $m_p$ | = | $1.673 \times 10^{-27}$ kg |
| Rest mass of a hydrogen atom | $m_H$ | = | $1.674 \times 10^{-27}$ kg |
| Boltzmann constant | $k$ | = | $1.381 \times 10^{-23}$ J.K$^{-1}$ |
| Atomic mass unit | u | = | $1.661 \times 10^{-27}$ kg |
| Rydberg constant | R | = | $1.097 \times 10^{7}$ m$^{-1}$ |

**ADV Question 1**




A large piece of cork weighs 0.285 N in air. When held submerged under water by a spring scale as shown, the scale reads 0.855 N.

(a) Draw a free-body diagram for the cork.

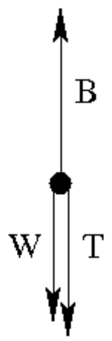
(b) What is the density of the cork?

(c) Now the water is replaced by oil and the scale reads 0.600 N. What is the density of the oil?

**(5 marks)**

**Solution**

(a) Let the density of the cork be $\rho_c$, so the mass of the cork is $m_c = \rho_c V$. There are three forces acting on the cork: the weight force $W = m_c g$, the tension *T*, and the buoyancy $B = \rho_w V$.




**(1 mark)**

(b) We are told the cork weighs 0.285N in air, so

$$\rho_c V g = W = 0.285\,\mathrm{N}$$

The cork is in equilibrium so

$$W + T - \rho_w V g = 0$$

or

$$V = \frac{W + T}{\rho_w g}$$

Hence

$$\rho_c = \frac{W}{V g} = \frac{W}{g} \times \frac{\rho_w g}{W + T} = \frac{W \rho_w}{W + T}$$

which evaluates to

$$\frac{(0.285)(1000)}{0.285 + 0.855} = 250 \text{ kg.m}^{-3}.$$

**(2 marks)**

(c) When the water is replaced by oil, the buoyancy force becomes $B = \rho_o V$ where $\rho_o$ is the density of the oil and T is the tension (now in oil) . So now we have

$$W + T - \rho_o V g = 0$$

so

$$\rho_o = \frac{W + T}{V g} = \frac{W + T}{W / \rho_c} = \rho_c \frac{W + T}{W}$$

$$= 250 \times \frac{0.285 + 0.600}{0.285}$$

$$= 776 \text{ kg.m}^{-3}$$

**(2 marks)**

**ADV Question 2**

A flashlight battery's emf of 1.5 V remains relatively constant over time, but its internal resistance increases substantially with age.

(a) If an (ideal) voltmeter is used to test a very old battery on its own (i.e. not in a circuit), what would be the reading? Explain your answer.

(b) How can you test the 'freshness' of a battery? Describe a testing method and explain why it will reveal if the battery is 'fresh'.

**(5 marks)**

**Solution**

(a) It will read 1.5 V. **(1 mark)**

Justification: the battery can be modeled as an ideal emf of 1.5V in series with an internal resistance. If the battery is not in a circuit, there is no current, and therefore the voltage drop across the internal resistance is 0 V. The voltmeter will therefore read 1.5 V.

**(2 marks)**

(b) The 'freshness' can be tested by using the battery in a simple circuit with some resistance R.

**(1 mark)**

Specifically, such a current will have a total resistance $R + R_{\text{int}}$. Then

$$I = \frac{V}{R + R_{\text{int}}} = \frac{1.5}{R + R_{\text{int}}}$$

If an ammeter is used to measure the current, this formula can be used to determine the internal resistance, which will reveal whether the battery is fresh or old. (Students need not mention it but you do need to have a fresh battery for comparison to know what $R_{\text{int}}$ should be.)

**(1 mark)**

**ADV Question 3**

The wires in a household lamp power cord are typically 3.0 mm apart centre to centre, and carry equal currents in opposite directions.

(a) If the cord carries current to a 100 W light bulb connected across a 240 V potential difference, what force per metre does each wire of the cord exert on the other? (Note that for AC power, the power, voltage and force will all be average values.)

(b) Is the force attractive or repulsive?

(c) Is this force large enough that it should be considered in the design of the lamp cord? Explain your answer.

**(5 marks)**

**Solution**

(a) The (average) current in this circuit is

$I = P/V = (100\text{ W})/(240\text{ V}) = 0.417\text{ A}$

**(1 mark)**

This current is the same magnitude for both wires, but in opposite directions.

**(1 mark)**

Then

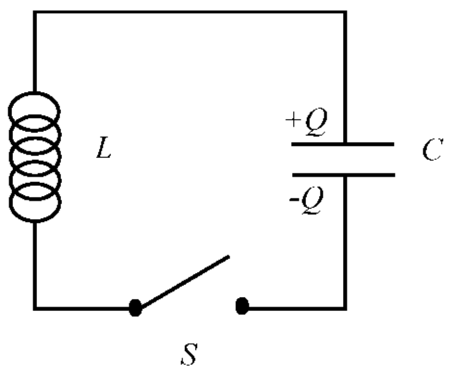
$$\frac{F}{L} = \frac{\mu_0 I I}{2\pi r} = \frac{(4\pi \times 10^{-7})(0.417)^2}{2\pi(0.003)} = 1.2 \times 10^{-5}\text{ N.m}^{-1}$$ **(1 mark)**

(b) Repulsive. **(1 mark)**

(c) No. The force per meter is comparable to the weight of a milligram object.

**(1 mark)**

**ADV Question 4**




A capacitor $C$, an inductor $L$ and a switch $S$ are connected in series. The capacitor is initially charged with a charge $Q$. Assume all connecting wires have zero resistance.

(a) What happens to the charge when the switch is closed? Explain why this occurs.

(b) Sketch a graph of current in the circuit versus time. Label the axes with appropriate values of $C$, $L$, and $Q$.

(c) Explain what happens to the energy initially stored in the capacitor.

**(5 marks)**

**Solution**

(a) The capacitor discharges through the inductor since there is a complete circuit when the switch is closed.

**(½ mark)**

As capacitor discharges, an emf develops in the inductor because of the increasing current, slowing the increase in current

**(½ mark)**

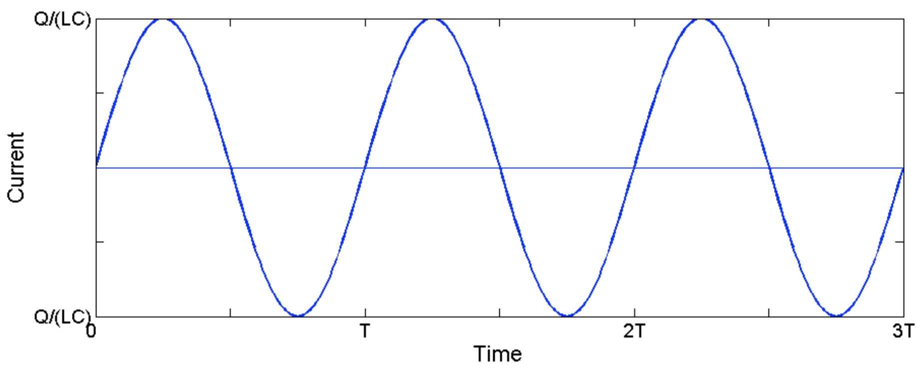
The capacitor recharges in the opposite sense, then the process reverses. The cycle then repeats.

**(½ mark)**

Anything else correct

**(½ mark)**

(b) Sinusoidal graph as below - also accept a decaying sine curve which would result if the coil has some resistance.



$$T = 2\pi\sqrt{LC}$$

$$i_{\max} = \frac{Q}{LC}$$

**(Graph 1 mark; correct labels on graph 1 mark)**

(c) The circuit oscillates at the resonant frequency. Energy stored in the capacitor's electric field when fully charged is stored in the inductor's magnetic field when the capacitor is discharged and current is maximum.

**(1 mark)**

**ADV Question 5**

Light with frequency $\nu$ is incident on a metal with a work function $\phi$ = 1.0 eV. Electrons are observed to be ejected from the metal, up to a maximum kinetic energy $K_{\max}$ described by

$$K_{\max} = h\nu - \phi.$$

(a) What is the maximum kinetic energy of photo-electrons ejected when the light has frequency $\nu = 5.0 \times 10^{14}$ Hz?

(b) What is the maximum kinetic energy of ejected electrons when the metal is illuminated with light with frequency $\nu = 2.0 \times 10^{14}$ Hz?

(c) If the intensity of the incident light is doubled, does the maximum kinetic energy of ejected electrons change? Does the number of ejected electrons change? Explain each of your answers.

(d) Briefly explain how these results are incompatible with classical physics.

**(5 marks)**

**Solution**

(a) The maximum kinetic energy is

$$K_{\max} = h\upsilon - \phi$$
$$= (6.626\times10^{-34})(5.0\times10^{14}) - 1.602\times10^{-19} \text{ J}$$
$$\approx 1.71\times10^{-19} \text{ J}$$
$$\approx 1.1 \text{ eV}$$

The minimum (threshold) frequency to produce electrons corresponds to $K_{\max} = 0$ and is

$$\upsilon_{\min} = \frac{\phi}{h} = \frac{1.602\times10^{-19}\text{ J}}{6.626\times10^{-34}\text{ J.s}} \approx 2.42\times10^{14}\text{ Hz}.$$

Since the frequency $\upsilon = 2.0\times10^{14}$ Hz is less than this no electrons will be produced.

**(2 marks)**

(b) When the intensity of light is doubled the maximum KE of electrons does not change. However, the number of electrons ejected per unit time will double.

**(1 mark)**

(c) In the classical picture the light is a wave which delivers energy continuously to electrons in the metal. The energy in the wave is proportional to the intensity.

In the quantum picture the light consists of photons which give up energy discretely to electrons. The energy of a photon is $E = h\upsilon$, and the number of photons per unit time is proportional to the intensity. The result in part (a) is inconsistent with the classical picture because in the classical picture the energy in the wave is independent of the frequency. Hence, irrespective of $\upsilon$, the wave should be able to supply sufficient energy (in some time) to lead to electron ejection. The result in part (b) is inconsistent with the classical picture because in the classical picture when the intensity is increased the amount of energy supplied per unit time is increased, so the maximum KE of ejected electrons would increase.

**(2 marks)**

**ADV Question 6**

A particle is described at time $t = 0$ by the wave packet

$$\psi(x) = \int_{-\infty}^{+\infty} A(k)\exp(i\,k\,x)\,dk$$

with

$$A(k) = \begin{cases} A_0 / k_0 & \text{for } 0 \le k \le k_0 \\ 0 & \text{for other } k \text{ values.} \end{cases}$$

(a) Show that the wave packet satisfies

$$|\psi(x)|^2 = |A_0|^2 \frac{\sin^2(k_0 x/2)}{(k_0 x/2)^2}.$$

(b) The uncertainty $\Delta x$ in the position of the particle at time $t = 0$ is defined by the smallest positive value of $x$ for which the function $|\psi(x)|^2$ obtained in (a) is zero. Show that this is given by

$$\Delta x = \frac{2\pi}{k_0}.$$

(c) The uncertainty in the momentum $\Delta p_x$ of the particle at time $t = 0$ is given by

$$\Delta p_x = \hbar k_0 .$$

Show that this uncertainty in momentum, together with the uncertainty in position obtained in (b), satisfy the Heisenberg uncertainty principle.

**(5 marks)**

**Solution**

(a) Applying the function $A(k)$ to the description of the particle we have

$$\psi(x) = \frac{A_0}{k_0}\int_0^{k_0} \exp(i\,k\,x)\,dk \quad = \frac{A_0}{k_0}\frac{1}{i\,x}\left[\exp(i\,k\,x)\right]_0^{k_0} \quad = \frac{A_0}{i\,k_0\,x}\left[\exp(i\,k_0 x) - 1\right]$$

$$= \frac{A_0}{k_0}\frac{1}{i\,x}\exp\left(i\,k_0\,x/2\right)\left[\exp(i\,k_0\,x/2) - \exp(-i\,k_0\,x/2)\right]$$

$$= A_0 \exp\left(i\,k_0\,x/2\right)\frac{\sin(k_0\,x/2)}{i\,k_0\,x/2}$$

So

$$|\psi|^2 = |A_0|^2 \frac{\sin^2\left(k_0 x/2\right)}{\left(k_0 x/2\right)^2}.$$

**(1 mark for setting up definite integral; 1 mark for evaluation and manipulation of equation)**

(b) The first zero of the function occurs when

$$\sin(k_0\,x/2) = 0$$

$$\Rightarrow \frac{k_0 x}{2} = \pi$$

$$x = \frac{2\pi}{k_0}$$

Hence

$$\Delta x = \frac{2\pi}{k_0} \text{ at time } t = 0.$$

**(1 mark for zero of function; 1 marks for result)**

(c) The uncertainty in the momentum $\Delta p_x$ of the particle at time $t = 0$ is given by

$$\Delta p_x = \hbar k_0.$$

Hence

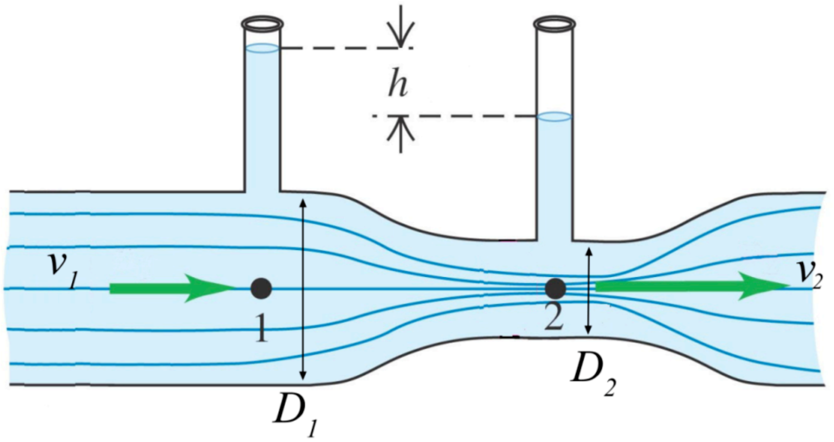
$$\Delta p_x\,\Delta x = (\hbar\,k_0)(2\pi/k_0) = 2\pi\,\hbar$$

This is greater than $\hbar$ and hence the Heisenberg Uncertainty principle is satisfied.

**(1 mark)**

**ADV Question 7**

A Venturi meter (used to measure flow rate in pipes) is installed in a horizontal water pipe, as shown in the diagram. The pipe has circular cross-section, with diameter $D_1$ in the first segment and $D_2$ in the second segment, with $D_2 < D_1$. The density of water is $\rho$. The volume flow rate of the water in the pipe is $R$ (measured in $m^3.s^{-1}$).




(a) Express the speed of flow $v_1$ in the first section of pipe in terms of $R$ and $D_1$.

(b) What is the speed of flow $v_2$ in the second section of pipe?

(c) Using the result in (a) find an expression for the pressure difference between point 1 and point 2.

(d) What is the difference $h$ in the water level in the two tubes expressed in terms of $R$, $D_1$, and $D_2$?

(e) Suppose the volume flow rate through the pipe is $1.0\times10^{-3}$ $m^3.s^{-1}$ and the pipe has diameter 0.10 m at point 1. Is the flow through the pipe turbulent? Note that the density of water is 1000 $kg.m^{-3}$ and the viscosity of water is $1.0\times10^{-3}$ Pa.s.

(f) What change could you make to the system so that the flow is exactly on the boundary between turbulent and laminar flow?

**(10 marks)**

**Solution**

(a) By the equation of continuity, the volume flow rate is constant, so

$$A_1 v_1 = A_2 v_2 = R$$

where $A = \pi\left(\frac{D}{2}\right)^2$ is the cross-sectional area of the pipe and $v$ is the speed of the flow.

Hence

$$v_1 = \frac{R}{A_1} = \frac{4R}{\pi D_1^2}$$

**( 2 marks)**

(b) Similarly,

$$v_2 = \frac{R}{A_2} = \frac{4R}{\pi D_2^2}$$

**(1 mark)**

(c) Apply Bernoulli's equation between points 1 and 2 to find the pressures $P_1$ and $P_2$ in the large and small segments. They are on the same streamline, so Bernoulli's equation applies; they are at the same height so $y_1 = y_2 = 0$. Hence from Bernoulli's equation,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

so

$$\Delta P = P_1 - P_2 = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

or

$$\frac{1}{2}\rho v_1^2\left(\frac{D_1^4}{D_2^4} - 1\right)$$

**(2 marks)**

(d) By Pascal's law, the pressure $P$ at the base of each tube is equal to the pressure exerted by the column of water. The tops are open to the air, so the pressure at the surface is atmospheric pressure $P_0$.

$$P_1 = P_0 + \rho g y_1$$

and

$$P_2 = P_0 + \rho g y_2$$

Hence

$$h = y_1 - y_2 = \frac{P_1 - P_2}{\rho g} = \frac{\Delta P}{\rho g} = \frac{v_2^2 - v_1^2}{g}.$$

From the results for parts (a) and (b) this can be written as

$$h = \frac{8R^2}{\pi^2 g}\left(\frac{1}{D_2^4} - \frac{1}{D_1^4}\right)$$

or

$$\frac{v_1^2}{2g}\left(\frac{D_1^4}{D_2^4} - 1\right)$$

**(3 marks)**

(e) If $R = 10^{-3}\ \text{m}^3.\text{s}^{-1}$ then from part (a)

$$v_1 = \frac{4R}{\pi D_1^2} = \frac{4 \times 10^{-3}}{\pi \times (0.1)^2} = 0.127\ \text{m.s}^{-1}.$$

The Reynolds number $R_e$ for the flow is

$$R_e = \frac{\rho v L}{\eta}$$

where $L$ is a characteristic length scale for the system; for a pipe it is the diameter. Hence for this flow,

$$R_e = \frac{(1000)(0.127)(0.1)}{10^{-3}} = 12700$$

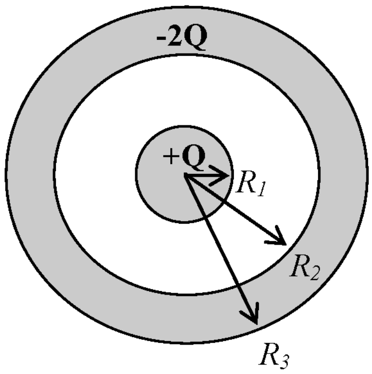
which is much bigger than the critical value of 2000, so the flow is turbulent.

**(1 mark)**

(f) To make the flow critical, you could decrease either the speed of the flow or the diameter of the pipe by a factor of about 6 (or some equivalent combination of the two, so long as the product decreases by 6).

**(1 mark for appropriate suggestion given answer from Reynolds Number – numerical value not required)**

**ADV Question 8**




A conducting sphere of radius $R_1$ has a charge of $+Q$. It is surrounded by a larger conducting concentric spherical shell of inner radius $R_2$ and outer radius $R_3$. The shell has charge $-2Q$.

(a) With the aid of a diagram describe how charge is arranged on the sphere and on the spherical shell.

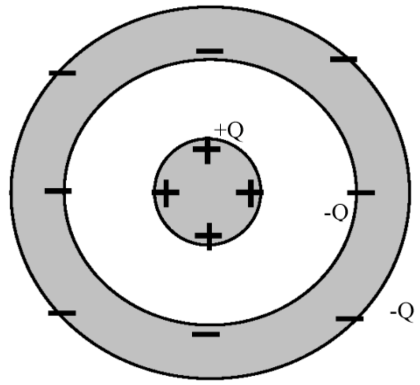
(b) Find expressions for the magnitude of the electric field as a function of the distance $r$ from the centre *and* give its direction:
(i) inside the sphere;
(ii) between the sphere and shell;
(iii) inside the shell;
(iv) outside the shell.
Explain your conclusions.

(c) Plot the *magnitude* of electric field as a function of $r$. Label the axes on your graph and mark in $R_1$, $R_2$ and $R_3$.

**(10 marks)**

**Solution**

(a)




Electric field inside a conductor must be zero **(½ mark)**

A charge of +Q is located on the outer surface of the conducting sphere. **(½ mark)**

A charge of -Q is located on the inside surface of the conducting shell. **(½ mark)**

A charge of -Q is located on the outer surface of the conducting shell. **(½ mark)**

(b) Gauss's Law $\int E.dA = \frac{q_{enclosed}}{\varepsilon_0}$

(i) $E = 0$ because enclosed charge is zero; electrostatic $E$ always zero inside a conductor.

**(1 mark)**

(ii) $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$ outward from the positive charges from Gauss' Law since there is a net positive charge enclosed; field is radial because of symmetry.

**(2 marks)**

(iii) $E = 0$ because net enclosed charge is zero; electrostatic $E$ always zero inside a conductor.

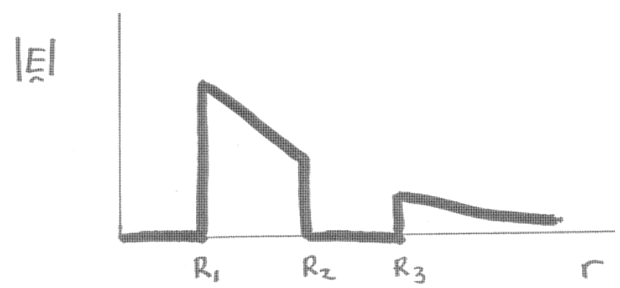
**(1 mark)**

(iv) $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$ inward to the negative charges from Gauss' Law since there is a net negative charge enclosed; field is radial because of symmetry.

**(2 marks)**

Students must give explanations to justify their answers to get full marks, e.g. in terms of Gauss' Law, symmetry and charge distribution.

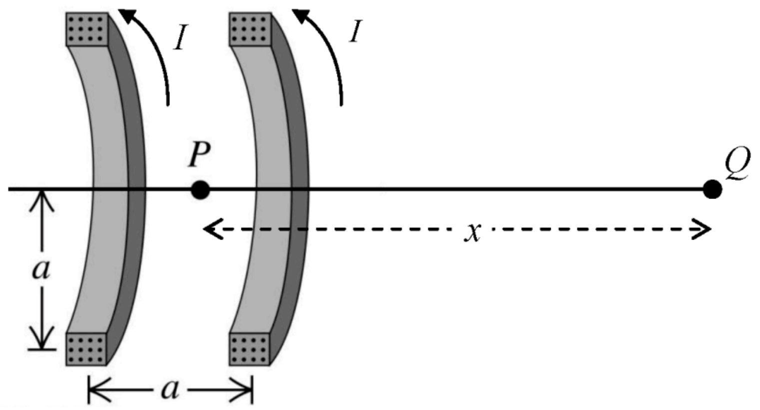
(c)




**(2 marks)**

**ADV Question 9**

The figure below is a cross-sectional view of two circular coils with radius $a$, each wound with $N$ turns of wire carrying current $I$, circulating in the same direction in both coils (as shown in the figure). The coils are separated by a distance $a$ that is equal to their radii. In this configuration, the coils produce a uniform magnetic field in the region between them.




(a) Sketch the magnetic field lines of this configuration.

(b) What is the direction of the magnetic field at point P, which is midway between the coils?

(c) Show that the magnetic field $B$ at a point Q on the axis, a distance $x$ to the right of point P, is given by:

$$B(x) = \frac{\mu_0 N I a^2}{2}\left[\frac{1}{\left[\left(\frac{a}{2}+x\right)^2 + a^2\right]^{3/2}} + \frac{1}{\left[\left(\frac{a}{2}-x\right)^2 + a^2\right]^{3/2}}\right]$$

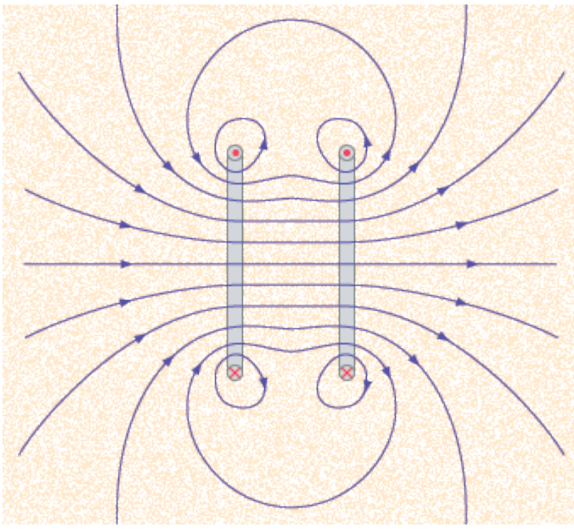
(d) Obtain an expression for the magnitude of the magnetic field at point P.

(e) What is the value of $\frac{dB}{dx}$ at point P ($x = 0$). Justify your answer with reference to the nature of the function $B(x)$ given above.

**(10 marks)**

**Solution**

(a)



**(1 mark)**

(b) To the right (+x direction). **(1 mark)**

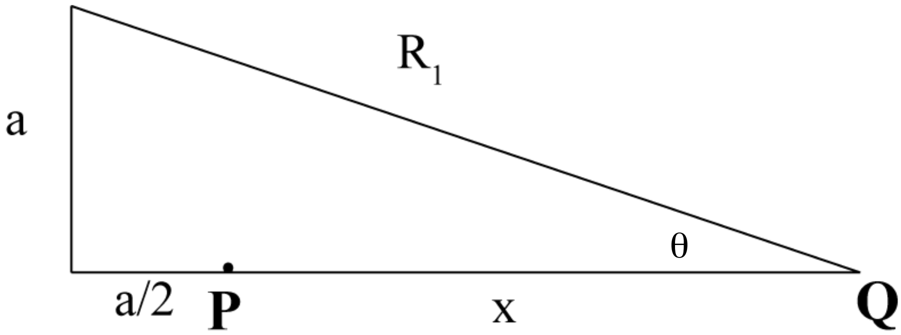
(c) A small segment $dl$ of one wire in the left coil will contribute a magnitude $dB$ of the magnetic field at point Q as follows:

$$dB_l = \frac{\mu_0}{4\pi}\frac{I\,dl}{R_1^2}$$

where

$$R_1^2 = \left(\frac{a}{2} + x\right)^2 + a^2$$

as seen from the diagram below




**(1 mark)**

Only the **component of the magnetic field along the *x* direction will contribute** to the final magnetic field – the magnetic field component in the y direction from this segment of current will be cancelled by an equal, but oppositely directed, magnetic field from another segment on the opposite side of the coil.

**(1 mark)**

Hence the contribution to the resultant magnetic field of this segment is:

$$dB_{l,x} = \frac{\mu_0}{4\pi}\frac{I\,dl}{R_1^2}\sin\theta$$

where $\sin\theta = \dfrac{a}{R_1}$. Note that there is $\sin\theta$ rather than $\cos\theta$ because the B field is directed tangentially to the radial vector between the current segment and point Q.

Hence

$$dB_{l,x} = \frac{\mu_0 I\,dl}{4\pi}\frac{a}{R_1^3}$$

**(1 mark)**

All current segments on the left coil are at the same distance from Q and so we can sum the field over coil segments in a single loop and the *N* loops to get the total field from the left coil. This is given by

$$B_{l,x} = \frac{\mu_0 I\,N\,2\pi a}{4\pi}\frac{a}{R_1^3} = \frac{\mu_0 I\,N}{2}\frac{a^2}{R_1^3}$$

**(1 mark)**

A similar argument for the right coil gives

$$B_{r,x} = \frac{\mu_0 I\,N}{2}\frac{a^2}{R_2^3}$$

where

$$R_2^{\,2} = \left(\frac{a}{2} - x\right)^2 + a^2$$

**(1 mark)**

Hence the total field is given by

$$B_{t,x} = \frac{\mu_0 I\,N}{2}\frac{a^2}{R_1^3} + \frac{\mu_0 I\,N}{2}\frac{a^2}{R_2^3} = \frac{\mu_0 I\,N a^2}{2}\left[\frac{1}{R_1^3} + \frac{1}{R_2^3}\right]$$

$$= \frac{\mu_0 N I a^2}{2}\left[\frac{1}{\left[\left(\frac{a}{2}+x\right)^2 + a^2\right]^{3/2}} + \frac{1}{\left[\left(\frac{a}{2}-x\right)^2 + a^2\right]^{3/2}}\right]$$

**(1 mark)**

(d) At point P $x = 0$ and hence

$$B(0) = \frac{\mu_0 N I a^2}{2}\left[\frac{1}{\left[\left(\frac{a}{2}\right)^2 + a^2\right]^{3/2}} + \frac{1}{\left[\left(\frac{a}{2}\right)^2 + a^2\right]^{3/2}}\right]$$

which gives

$$B(0) = \frac{\mu_0 N I a^2}{2}\left[\frac{1}{\left(\frac{5a}{4}\right)^{3/2}} + \frac{1}{\left(\frac{5a}{4}\right)^{3/2}}\right]$$

$$= \frac{\mu_0 N I a^2}{2}\frac{16}{5^{3/2}a^3} = \frac{8\mu_0 N I}{a5\sqrt{5}}$$

**(1 mark)**

(e) $\dfrac{dB(0)}{dx} = 0$

The equation for the B field is symmetrical around $x = 0$ and so the differential must equal zero at this point.

**(1 mark)**

**ADV Question 10**

The long horizontal straight wire shown in the figures below has constant current $I$ directed to the right.

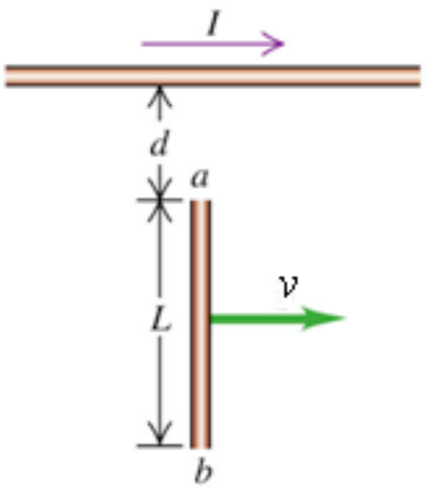



figure (1)

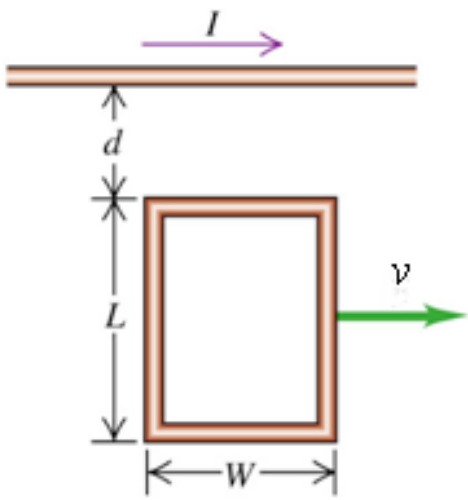



figure (2)

(a) Use Ampère's Law to determine the magnetic field $\vec{B}$ (both magnitude and direction) at a distance $r$ below the current carrying wire.

(b) A metal bar with length $L$ is moving at a constant velocity $v$ to the right in figure (1). Show that the magnitude of the emf, $\varepsilon$, induced in the bar is

$$\varepsilon = \frac{\mu_0 I v}{2\pi} \ln\left(\frac{d+L}{d}\right)$$

(c) Which point, $a$ or $b$ in figure (1), is at the higher electric potential? Explain your answer.

(d) In figure (2) the wire is replaced by a rectangular wire loop of resistance $R$. What is the magnitude of the current induced in the loop? Explain your answer.

**(10 marks)**

**Solution**

(a) We choose a circular path of radius $r$ centred on the wire. Ampère's Law states

$$\int \vec{B} \bullet d\vec{l} = \mu_0 I_{encl}$$

The magnetic field is constant on this path, and is directed tangentially and so Ampère's Law gives

$$B\,2\pi\,r = \mu_0\,I \quad \Rightarrow \quad B = \frac{\mu_0\,I}{2\pi\,r}$$

**(1 mark for magnitude)**

The direction is into the page. **(1 mark for direction)**

(b) The motional emf induced, integrating from $r = d$ to $r = d + L$, is

$$\varepsilon = \int(\vec{v}\times\vec{B})\cdot d\vec{l} = -\int_d^{d+L} v\,B(r)\,dr$$

$$= -\int_d^{d+L} v\ \frac{\mu_0\,I}{2\pi\,r}dr = -\frac{\mu_0\,I\,v}{2\pi}\int_d^{d+L}\frac{1}{r}dr$$

$$= -\frac{\mu_0\,I\,v}{2\pi}\ln\left(\frac{d+L}{d}\right)$$

**(3 marks)**

(c) Point a has a higher potential than point b. (The emf is negative when directed radially outwards.)

**(2 marks)**

(d) Even though the rectangular loop is moving (in a uniform magnetic field), the magnetic flux is constant and therefore

$$\frac{d\Phi_B}{dt} = 0$$

**(2 marks)**

Therefore, there is no current in this loop.

**(1 mark)**

(This answer could also be obtained by noting that the motional emf induced in both the ‘front’ segment and the ‘back’ segment is the same and both pointed upwards.)

**ADV Question 11**

An electron with energy $E$ is incident on a potential step with height $U_0 > E$. The situation may be described by the 1-D time-independent Schrödinger equation

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x),$$

with potential:

$$U(x) = \begin{cases} 0 & \text{for } x < 0 \\ U_0 & \text{for } x \geq 0\ . \end{cases}$$

(a) The wave function in the region $x < 0$ is

$$\psi_{\mathrm{I}}(x) = Ae^{ikx} + Be^{-ikx}.$$

Using the Schrödinger equation, determine the value of $k$.

(b) The wave function in the region $x \geq 0$ is

$$\psi_{\mathrm{II}}(x) = Ce^{-\alpha x}.$$

Using the Schrödinger equation, determine the value of $\alpha$.

(c) The boundary conditions on the wave function at $x = 0$ are

$$\psi_{\mathrm{I}}(0) = \psi_{\mathrm{II}}(0)$$

and

$$\frac{d\psi_I(0)}{dx} = \frac{d\psi_{II}(0)}{dx}.$$

Applying these boundary conditions, show that

$$A = \frac{1}{2}\frac{k + i\alpha}{k}C$$

and

$$B = \frac{1}{2}\frac{k - i\alpha}{k}C$$

(d) Using the results of (c), show that the wave function may be written

$$\psi(x) = \begin{cases} C\left[\cos kx - (\alpha / k)\sin kx\right] & \text{for } x < 0 \\ Ce^{-\alpha x} & \text{for } x \geq 0\ . \end{cases}$$

**(10 marks)**

**Solution**

(a) In region I ($x < 0$) we have $U = 0$, so the Schrodinger equation is

$$\psi'' = -\frac{2mE}{\hbar^2}\psi\ .$$

For the given solution

$$\psi_{\mathrm{I}}(x) = A\, e^{ikx} + B\, e^{-ikx} \qquad (1)$$

we have

$$\psi_I{}' = ik\, Ae^{ikx} - ik\, B\, e^{-ikx} \qquad (2)$$

and

$$\psi_I{}'' = -k^2 Ae^{ikx} - k^2 Be^{-ikx}$$
$$= -k^2\psi_I\ .$$

Comparing this with the Schrodinger equation, we have

$$k^2 = \frac{2mE}{\hbar^2} \quad \text{or} \quad k = \frac{(2mE)^{\frac{1}{2}}}{\hbar}\ .$$

**(2 marks)**

(b) In region II ( $x \ge 0$ ) we have $U = U_0$, so the Schrodinger equation is

$$\psi_{II}'' = \frac{2m}{\hbar^2}(U_0 - E)\psi_{II} \,..$$

For the given solution

$$\psi_{II} = C e^{-\alpha x} \qquad (3)$$

we have

$$\psi_{II}' = -\alpha C e^{-\alpha x} \qquad (4)$$

and

$$\psi_{II}'' = \alpha^2 C e^{-\alpha x}$$
$$= \alpha^2 \psi_{II} \,.$$

Comparing this with the Schrodinger equation we have

$$\alpha^2 = 2m(U_0 - E)/\hbar^2 \,,$$

and so

$$\alpha = \left[2m(U_0 - E)\right]/\hbar \,.$$

**(2 marks)**

(c) Setting $\psi_I(0) = \psi_{II}(0)$ (see (1) and (2)) gives

$$A + B = C \,. \qquad (5)$$

Setting $\psi_I'(0) = \psi_{II}'(0)$ (see (3) and (4)) gives

$$ik(A - B) = -\alpha C \,. \qquad (6)$$

Equation (6) implies that $$A - B = \frac{i\alpha}{k}C \,. \qquad (7)$$

Adding equations (5) and (7) gives

$$A = \frac{1}{2}\left(1 + \frac{i\alpha}{k}\right)C$$
$$= \frac{1}{2}\frac{k + i\alpha}{k}C \,. \qquad (8)$$

**(2 marks)**

Then equation (5) can be written

$$B = C - A$$
$$= C - \frac{1}{2}\frac{k + i\alpha}{k}C \qquad \text{(using equation (8))}$$
$$= \frac{1}{2}\frac{(2k - k - i\alpha)}{k}C$$
$$= \frac{1}{2}\frac{k - i\alpha}{k}C \,, \qquad (9)$$

as required.

**(2 marks)**

(d) In region II ( $x \ge 0$ ) we are given in the question that

$$\psi(x) = \psi_{II}(x) = C e^{-\alpha x}$$

In region I ( $x < 0$ )

$$\psi_1(x) = A\,e^{ikx} + B\,e^{-ikx}$$
$$= \frac{C}{2k}\left[(k+i\alpha)e^{ikx} + (k-i\alpha)e^{-ikx}\right]$$
$$= C\left[\frac{1}{2}\left(e^{ikx} + e^{-ikx}\right) + \frac{i\alpha}{2k}\left(e^{ikx} - e^{-ikx}\right)\right]$$

Using $e^{ikx} = \cos(kx) + i\sin(kx)$ we get

$$\psi_I(x) = C\cos(kx) + \frac{i\alpha}{2k}\left[2i\sin(kx)\right]$$
$$= C\cos(kx) - \frac{\alpha}{k}\sin(kx)$$

Hence in region I $\qquad \psi(x) = \psi_I(x) = C\cos(kx) - \frac{\alpha}{k}\sin(kx)$

**(2 marks)**

**ADV Question 12**

The spectrum of a blackbody is described by the Planck formula

$$I_\lambda(\lambda) = \frac{2\pi h c^2}{\lambda^5\left[e^{(hc)/(\lambda k T)} - 1\right]}.$$

(a) On a single diagram, sketch $I_\lambda(\lambda)$ versus $\lambda$ for two temperatures $T_1$ and $T_2$ such that $T_1 < T_2$. State the Wien displacement law and with reference to your diagram explain what it means.

(b) Using $x = (hc)/(\lambda_p kT)$ show that the peak of the spectrum $\lambda_p$ satisfies the equation

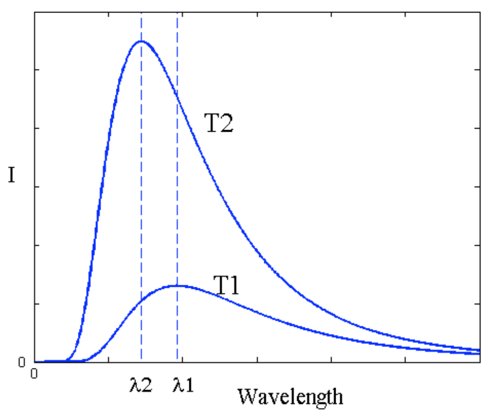
$$5 - x - 5e^{-x} = 0.$$

(c) The solution to the equation in (b) is $x \approx 4.965$. Using this result, derive the value of the constant in the Wien displacement law and determine the peak wavelength corresponding to $T = 6000$ K.

**(10 marks)**

**Solution**

(a)




**(2 marks)**

Wien's law is

$\lambda_{\max} T = b$ where $b$ is a constant. This is shown in the figure where the peak wavelength of the curve moves to shorter wavelengths as the temperature increases, consistent with $\lambda_{\max} = \dfrac{b}{T}$.

**(1 mark)**

(b) Planck's formula is

$$I_\lambda(\lambda) = \frac{2\pi h c^2}{\lambda^5 \left[ e^{(hc)/\left(\lambda k T\right)} - 1 \right]}.$$

If we substitute $x = \dfrac{hc}{\lambda k T}$ into the formula we get

$$I = A \frac{x^5}{e^x - 1}$$

where

$$A = \frac{2\pi h c^2}{1} \left( \frac{kT}{hc} \right)^5 = \frac{2\pi k^5 T^5}{h^4 c^3}$$

**(1 mark)**

For the wavelength of peak intensity we need to find $x$ or $\lambda$ such that

$$\frac{dI}{d\lambda} = \frac{dI}{dx}\frac{dx}{d\lambda} = 0. \qquad (1)$$

**(1 mark for correct condition in $\lambda$ for peak )**

Firstly,

$$\frac{dx}{d\lambda} = -\frac{1}{\lambda^2}\frac{hc}{kT} = -\frac{1}{x^2}\frac{kT}{hc} = -\frac{B}{x^2} \qquad (2)$$

where

$$B = \frac{kT}{hc}.$$

Secondly,

$$\frac{dI}{dx} = A\left[ \frac{5x^4}{e^x - 1} - \frac{x^5}{\left(e^x - 1\right)^2} e^x \right] \qquad (3)$$

**(1 mark)**

Substituting equations (2) and (1) into equation (1) we get

$$A\left[\frac{5x^4}{e^x-1} \quad - \quad \frac{x^5}{\left(e^x-1\right)^2}e^x\right]\left[-\frac{B}{x^2}\right]=0$$

$$\Rightarrow\left[\frac{5x^4}{e^x-1} \quad - \quad \frac{x^5}{\left(e^x-1\right)^2}e^x\right]=0$$

**(1 mark)**

$$\Rightarrow 5\left(e^x-1\right)=x\,e^x$$

$$\Rightarrow 5-x-5e^{-x}=0$$

**(1 mark)**

Note that the values of A, B, and $\dfrac{dx}{d\lambda}$ do not matter (as long as they are non-zero) and need not be worked out.

(c) We are given that solution of equation is $x \quad 4.965$. Hence

$$4.965=\frac{h\,c}{\lambda_{\max}\,k\,T}\Rightarrow$$

$$\lambda_{\max}\ T=b=\frac{h\,c}{4.965\,k\,T}$$

$$=\frac{(6.63\times10^{-34})(3.00\times10^{8})}{(4.965)(1.38\times10^{-23})}=2.90\times10^{-3}\text{ m.K}$$

For

$$T=6000\ K$$

$$\lambda_{\max}=\frac{2.90\times10^{-3}}{6000}=4.84\times10^{-7}\text{ m}$$

$$=484\text{ nm}.$$

**(2 marks)**