MATH1903/1907 Lectures

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First order linear differential equations

Equations of the form

$$\underbrace{a(x)\,y'(x) + b(x)\,y(x)}_{\text{linear as a function of } y \text{ and } y'.} = f(x)$$

$\uparrow$ inhomogeneity

Inhomogeneous equation: $f(x) \neq 0$

Homogeneous equation: $f(x) \equiv 0$.

$$a(x)\,y'(x) + b(x)\,y(x) = 0$$

Solve by separation of variables

$$\frac{y'(x)}{y(x)} = -\frac{b(x)}{a(x)}$$

Integrate:

$$\ln|y| = -\int \frac{b(x)}{a(x)}\,dx \quad (+C)$$

Properties of linear homogeneous equations:

$$ay' + by = 0$$

Super position principle:

y_1, y_2 are solutions, then

$$c_1 y_1 + c_2 y_2$$

is also a solution for any constants c_1, c_2

check: $y = c_1 y_1 + c_2 y_2$

$$\begin{aligned} ay' + by &= a(c_1 y_1 + c_2 y_2)' + b(c_1 y_1 + c_2 y_2) \\ &= a(c_1 y_1' + c_2 y_2') + b(c_1 y_1 + c_2 y_2) \\ &= c_1 \underbrace{(a y_1' + b y_1)}_{=0} + c_2 \underbrace{(a y_2' + b y_2)}_{=0} = 0 \end{aligned}$$

since y_1 and y_2 are solutions.

Standard form:

original equation: $a(x)y'(x)+b(x)y(x)=0$

If $a(x)\neq 0$ we can divide by $a(x)$:

$$y'(x)+\underbrace{\boxed{\frac{b(x)}{a(x)}}}_{p(x)}y(x)=0$$

Standard form is $\boxed{y'(x)+p(x)y(x)=0}$

The equation is separable:

$$\frac{dy}{y}=-p\,dx$$

Integrate $\quad \ln|y|=-\int p(x)dx+c$

Solve for y $\quad y=\pm e^{c}e^{-\int p(x)dx}=Ae^{-\int p(x)dx}$

Worth remembering: $\boxed{y(x)=Ae^{-\int p(x)dx}}$

Examples

(1) $p = \text{const.}$ $\int p\,dx = px$

Solution of $y' + py = 0$ is $y = A e^{-px}$

(2) $y' + \frac{2}{x} y = 0$. Here $p(x) = \frac{2}{x}$

$$\int p(x)\,dx = \int \frac{2}{x}\,dx = 2 \ln|x| = \ln x^2$$

Hence the general solution is

$$y(x) = A e^{-\int p(x)dx} = A e^{-\ln x^2} = A e^{\ln \frac{1}{x^2}}$$

$$= \frac{A}{x^2}$$

(3) $y' - x^2 y = 0$

Solution is $y(x) = A e^{-\int p(x)dx}$

$$= A e^{+\int x^2 dx} = A e^{\frac{x^3}{3}}.$$

Consider $y' + p(x)y = 0$ with initial cond $y(x_0) = y_0$

$$\frac{y'(x)}{y(x)} = -p(x) \quad , \quad y(x_0) = y_0$$

Integrate:

$$\int_{x_0}^{x} \frac{y'(\xi)}{y(\xi)} d\xi = -\int_{x_0}^{x} p(\xi) d\xi \qquad \text{substitute } z = y(\xi)$$

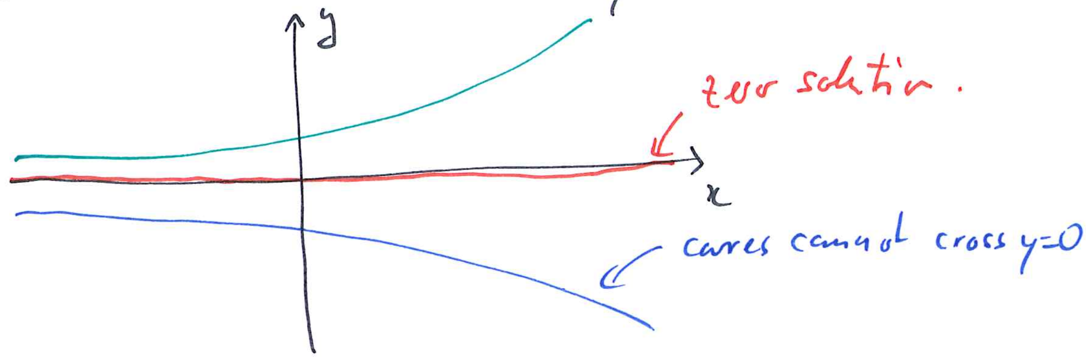
$$\int_{y_0}^{y} \frac{1}{z} dz = -\int_{x_0}^{x} p(\xi) d\xi$$

$$\ln|z| \Big|_{y_0}^{y} = \ln|y| - \ln|y_0| = \ln\left|\frac{y}{y_0}\right| = \int_{x_0}^{x} -p(\xi) d\xi$$

$$\left|\frac{y}{y_0}\right| = e^{-\int_{x_0}^{x} p(\xi) d\xi} > 0$$

$$y(x) = y_0 \, e^{-\int_{x_0}^{x} p(\xi) d\xi} .$$

For this equation we have uniqueness of solutions for any given initial value. Hence solution curves cannot cross. One solution is $y = 0$.



Inhomogeneous linear equations

Examples

- $y' + xy = x$

  $y' = (1-y)x$ separable, so we can solve ....

- $x^2y' + 2xy = \sin x$ not separable!

  $$\underbrace{x^2y' + 2xy}_{(x^2y)'} = \sin x$$

  $(x^2y)' = \sin x$ ← "exact equation"

  Integrate:

  $$\int (x^2y)' dx = x^2y = \int \sin x \, dx + C$$
  $$= -\cos x + C$$

  Solution: $y(x) = \dfrac{C - \cos x}{x^2}$

- $x^3y' + y = \sin x$ not separable, not exact

  how to solve?

Arbitrary inhomogeneous equation in standard form:

$$y' + p(x)y = q(x)$$

Unless we are lucky these equations are not separable and not exact!

<u>Idea:</u> make the equation exact by multiplying it by some function w to be determined.

$$\underbrace{wy' + wpy}_{\overset{\text{want}}{=}(wy)'} = wq$$

$$(wy)' = wq$$

If that is the case, then

$$\boxed{wy'} + \boxed{wpy} = (wy)' = \boxed{wy'} + \boxed{w'y}$$

Hence we require $wpy = w'y$, so that

$$wp = w'$$

or

$$w' - pw = 0 \qquad \text{(linear homogeneous equation!)}$$

Solution: $w = e^{\int p(x)\,dx}$ (no constant since we only want one particular solution)

By construction:
$$(wy)' = wq$$
$$wy = \int wq\,dx + C$$
$$y = \frac{1}{w}\left(\int wq\,dx + C\right)$$

This leads to the following method:

Method of integrating factors

Applies to first order linear inhomogeneous equations in standard form

$$y'(x) + p(x)\,y(x) = g(x)$$

Integrating factor $\quad w(x) = e^{\int p(x)\,dx}$

[A solution of the homogeneous equation $w' - pw = 0$]

Then
$$y = \frac{1}{w}\left(\int w g\,dx + C\right)$$

Example:

Solve $y' + \frac{2}{x} y = x$

Integrating factor: $u(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln|x|} = e^{\ln x^2} = x^2$

Multiply equation by integrating factor:

$$\underbrace{x^2 y' + 2xy}_{\overset{\text{by design!}}{\parallel}} = x^3$$

by design!

$$(x^2 y)' = x^3$$

Integrate

$$x^2 y = \int x^3 dx = \frac{x^4}{4} + C$$

Hence the general solution is

$$\boxed{y(x) = \frac{x^2}{4} + \frac{C}{x^2}}$$

Find the particular solution satisfying $y(-2) = 3$

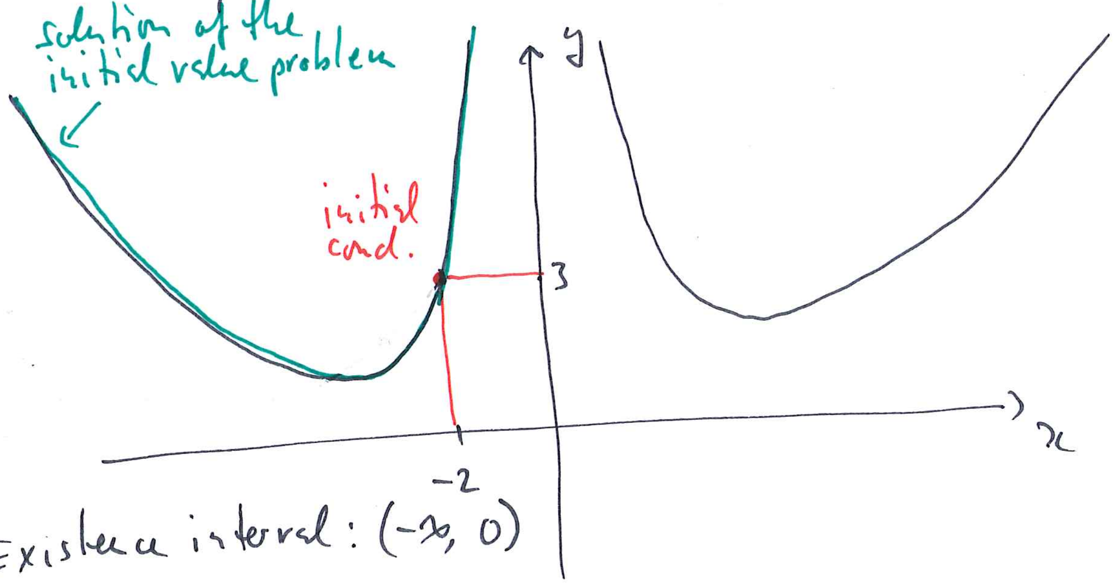
$$y(-2) = \frac{(-2)^2}{4} + \frac{C}{(-2)^2} = 1 + \frac{C}{4} = 3$$

Hence $\frac{C}{4} = 2$, so $C = 8$

The particular solution is

$$y(x) = \frac{x^2}{4} + \frac{8}{x^2}$$

Note: $y(x) \to \infty$ if $|x| \to \infty$ or $x \to 0$:



Existence interval: $(-\infty, 0)$

<u>Remark</u>: The solution of an initial value problem always consists of a <u>connected curve</u>, and does not cross singularities.

We had the equation $y' + \frac{2}{x} y = x$

cannot divide by $x=0$, so we expect the solution to break down at $x=0$.

More efficient way to obtain integration constant.

We had

$$(x^2y)' = x^3$$

Integrate $\quad x^2y = \frac{x^4}{4} + C$ $\leftarrow$ compute C at this stage

Initial cond: $y(-2) = 3$

$$(-2)^2 \cdot 3 = \frac{(-2)^4}{4} + C$$

$$12 = 4 + C$$

$$8 = C$$

More on first order equations

Usually we consider a d.e. for

$$y(x)$$

We can also consider an equation for

$$x(y) \qquad [\text{inverse function}]$$

Derivative of an inverse function:

$f : \mathbb{R} \to \mathbb{R}$, inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$

Then
$$x = f^{-1}(f(x))$$

Assuming f^{-1} is differentiable we apply the chain rule:

$$1 = \frac{dx}{dx} = \frac{d}{dx}\left[f^{-1}(f(x))\right] = (f^{-1})'(f(x))\, f'(x)$$

$$(f^{-1})'(\underbrace{f(x)}_{y}) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(\underbrace{f(x)}_{y}))} = \frac{1}{f'(f^{-1}(y))}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Consequence:

We can rewrite a differential equation in y as a differential equation in x:

$$\frac{dy}{dx} = F(x, y(x)) \iff \frac{1}{\frac{dx}{dy}} = F(x(y), y)$$

or $$\frac{dx}{dy} = \frac{1}{F(x(y), y)}$$

Sometimes we choose the one that is easier to solve.

This is often applied to equations of the form

$$a(x,y) + b(x,y)\frac{dy}{dx} = 0 \qquad \text{[equation for } y\text{]}$$

$$a(x,y)\frac{dx}{dy} + b(x,y) = 0 \qquad \text{[equation for } x\text{]}$$

Often formally written as

$$a(x,y)\,dx + b(x,y)\,dy = 0$$

in "symmetric" form

Some equations are not linear, but they can be transformed into a linear equation.

Example: Bernoulli equation

$$y' + p(x)\,y = q(x)\,y^n$$ non-linear unless $n=1$.

Rewrite this as a d.e for $v := \dfrac{1}{y^{n-1}}$

$$v' = \left(y^{-(n-1)}\right)' = -(n-1)\,y^{-n}\,y' = -(n-1)\,\frac{y'}{y^n}$$

Substitute y' from d.e:

$$v' = -(n-1)\,\frac{-p(x)\,y + q(x)\,y^n}{y^n}$$

$$= -(n-1)\Big(-p(x)\underbrace{\boxed{\frac{1}{y^{n-1}}}}_{=v} + q(x)\Big)$$

$$= -(n-1)\big(-p(x)\,v + q(x)\big)$$ linear inhomogeneous equation.

Then solve for v by integrating factors, then recover y.