CHAPTER **THREE**

Conics

3A Numerical Problems on the Ellipse

Exercise 3A

1. The ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$.

(a) Find its eccentricity.

(b) Find the coordinates of its foci.

(c) Find the equations of its directrices.

(d) Sketch $\mathcal{E}$, showing the foci and directrices.

(e) Show that the parametric equations $x=5\cos\theta$, $y=4\sin\theta$ represent $\mathcal{E}$.

(f) Find the coordinates of the point on $\mathcal{E}$ corresponding to $\theta=\frac{\pi}{3}$.

2. (a) Show that the ellipse $4x^2+9y^2=16$ has foci $(\pm\frac{2\sqrt{5}}{3},0)$ and directrices $x=\pm\frac{6\sqrt{5}}{5}$.

(b) Write down a pair of parametric equations representing the ellipse.

3. Consider the ellipse $\mathcal{E}$ with equation $5x^2+9y^2=45$.

(a) Find the foci and directrices of $\mathcal{E}$.

(b) Sketch $\mathcal{E}$, showing its foci and directrices.

(c) Draw the auxiliary circle on your diagram.

(d) Show that the parametric equations $x=3\cos\alpha$, $y=\sqrt{5}\sin\alpha$ represent $\mathcal{E}$.

(e) Show on your diagram how to construct the point where $\alpha=\frac{\pi}{3}$.

4. Consider the ellipse $x=5\cos\theta$, $y=3\sin\theta$.

(a) Show that $(-\frac{5\sqrt{2}}{2},\frac{3\sqrt{2}}{2})$ is the point on the ellipse corresponding to $\theta=\frac{3\pi}{4}$.

(b) Find the Cartesian equation of the ellipse.

(c) Find the eccentricity of the ellipse.

(d) Show that each latus rectum is of length $3\frac{3}{5}$ units.

5. (a) An ellipse has foci at $(\pm4,0)$ and its minor axis is of length 6 units. Find its equation.

(b) An ellipse has eccentricity $\frac{2}{3}$ and its major axis (which lies along the x-axis) is of length 12 units. Find its equation.

6. Show that:

(a) the tangent to the ellipse $x^2 + 4y^2 = 100$ at the point $(6, 4)$ has equation $3x + 8y = 50$,

(b) the tangent to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$ at the point $(2, \frac{5}{3})$ has equation $2x + 3y = 9$,

(c) the normal to the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ at the point $(3, \frac{12}{5})$ has equation $100x - 45y = 192$.

7. Show that:

(a) the tangent to the ellipse $x = 2\cos\theta$, $y = \sin\theta$ at the point where $\theta = \frac{\pi}{4}$ has equation $x + 2y = 2\sqrt{2}$,

(b) the tangent to the ellipse $x = 4\cos\theta$, $y = 3\sin\theta$ at the point where $\theta = \frac{2\pi}{3}$ has equation $\sqrt{3}\,x - 4y + 8\sqrt{3} = 0$.

8. Show that:

(a) the chord of contact to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ from the point $(4, -3)$ has equation $3x - 4y = 12$,

(b) the chord of contact to the ellipse $x^2 + 4y^2 = 4$ from the point $(-5, -2)$ has equation $5x + 8y + 4 = 0$,

(c) the chord of contact to the ellipse $x = 5\cos\theta$, $y = 2\sin\theta$ from the point $(3, -3)$ has equation $12x - 75y = 100$.

9. Suppose that $\mathcal{E}$ is the ellipse $\dfrac{x^2}{100} + \dfrac{y^2}{25} = 1$, and let P be the point $(6, 4)$ on $\mathcal{E}$.

(a) Show that the normal at P has equation $8x - 3y = 36$.

(b) If the normal meets the major axis at G, and OH is the perpendicular drawn from the origin to the tangent at P, show that $PG \times OH = 25$.

10. (a) Show that the line $x + 2y + 5 = 0$ is a tangent to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, and show that the point of contact is $(-1\frac{4}{5}, -1\frac{3}{5})$.

(b) Show that the line $2x - 2y + 3 = 0$ is a tangent to the ellipse $2x^2 + 4y^2 = 3$, and show that the point of contact is $(-1, \frac{1}{2})$.

11. (a) Show that the two tangents to the ellipse $\dfrac{x^2}{8} + \dfrac{y^2}{4} = 1$ with gradient 2 have equations $y = 2x \pm 6$.

(b) Show that the two tangents to the ellipse $x^2 + 16y^2 = 25$ which are perpendicular to the line $6x + 2y + 3 = 0$ have equations $4x - 12y = \pm 25$.

12. A variable point P in the number plane moves in such a way that the sum of its distances from two fixed points 6 units apart is always 10 units. Thus, the locus of P is an ellipse. Let the two fixed points be at $(\pm 3, 0)$, so that the major axis of the ellipse lies on the x-axis.

(a) Show, using a diagram, that $a = 5$.

(b) Hence show that the equation of the ellipse is $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

13. The orbit of the earth about the sun is an ellipse with the sun at one of the foci. It is known that the maximum and minimum distances of the earth from the sun are in the ratio 30 : 29. Show that the eccentricity of the earth's orbit is $\frac{1}{59}$.

14. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has foci at $(\pm 5\sqrt{3}, 0)$, and it passes through the point $(8, 3)$.

(a) Show that $a^2 - b^2 = 75$ and that $\dfrac{64}{a^2} + \dfrac{9}{b^2} = 1$.

(b) Hence show that the ellipse has equation $\dfrac{x^2}{100} + \dfrac{y^2}{25} = 1$.

15. The line $y = mx + b$ is a tangent to the ellipse $\dfrac{x^2}{9} + y^2 = 1$.

(a) Show that $b^2 = 9m^2 + 1$.

(b) Hence show that the tangents to the ellipse from the external point $(2, 1)$ have equations $y = 1$ and $4x + 5y = 13$.

16. (a) Determine the values of λ for which the equation $\dfrac{x^2}{4-\lambda} + \dfrac{y^2}{2-\lambda} = 1$ represents an ellipse.

(b) Describe how the shape of the ellipse changes as λ increases from 1 towards 2.

(c) What is the limiting position of the ellipse as $\lambda \to 2$?

3B Theoretical Problems on the Ellipse

Exercise 3B

NOTE: A carefully drawn diagram is essential in each of these problems.

1. P is any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, whose foci are S and S'. Use a diagram and the definition of eccentricity (that is, $\dfrac{PS}{PM} = \dfrac{PS'}{PM'} = e$, where PM and PM' are the perpendiculars from P to the directrices) to prove that $PS + PS' = 2a$.

2. Let $\mathcal{E}$ be the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and let $P(x_1, y_1)$ be a variable point on $\mathcal{E}$.

(a) Show that the tangent to $\mathcal{E}$ at P has equation $\dfrac{x_1 x}{a^2} + \dfrac{y_1 y}{b^2} = 1$.

(b) Show that the normal to $\mathcal{E}$ at P has equation $\dfrac{a^2 x}{x_1} - \dfrac{b^2 y}{y_1} = a^2 - b^2$.

(c) The tangent meets the x-axis at T. Show that T is the point $(\frac{a^2}{x_1}, 0)$.

(d) The normal meets the x-axis at N.
Show that N is the point $\big((\frac{x_1}{a^2}(a^2 - b^2), 0\big)$.

(e) If PF is the perpendicular from P to the x-axis, and O is the origin,
show that $OT \times NF = b^2$.

3. Let $\mathcal{E}$ be the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, and let $P(5\cos\theta, 3\sin\theta)$ be a variable point on $\mathcal{E}$. Also let A and A' be the points $(5, 0)$ and $(-5, 0)$ respectively, and let O be the origin.

(a) Show that the tangent at P has equation $3x\cos\theta + 5y\sin\theta = 15$.

(b) The tangents at A and P meet at R. Show that R has coordinates $\left(5, \frac{3(1-\cos\theta)}{\sin\theta}\right)$.

(c) Show that OR is parallel to $A'P$.

(d) The tangents at A' and P meet at R'. Use gradients to show that $\angle RSR' = 90°$, where S is the focus $(4, 0)$.

4. $\mathcal{E}$ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and $P(a\cos\theta, b\sin\theta)$ is a variable point on $\mathcal{E}$.

(a) Show that the normal at P has equation $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$.

(b) The normal at P cuts the x-axis at A and the y-axis at B. Show that A has coordinates $\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$ and that B has coordinates $\left(0, \frac{b^2-a^2}{b}\sin\theta\right)$.

(c) Show that $\dfrac{PA}{PB} = 1 - e^2$.

5. $S(ae, 0)$ is one of the foci of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The latus rectum through S meets the ellipse in the first quadrant at P.

(a) Show that P has coordinates $\big(ae, a(1 - e^2)\big)$.

(b) Show that the tangent at P has equation $ex + y = a$.

(c) If the tangent at P meets the y-axis at Q, then show that the line QS', where S' is the other focus, is parallel to the normal at P.

6. $\mathcal{E}$ is the ellipse $x = a\cos\theta$, $y = b\sin\theta$. P and Q are the points on $\mathcal{E}$ where $\theta = \alpha + \beta$ and $\theta = \alpha - \beta$ respectively.

(a) Show that the chord PQ has gradient $-\frac{b\cos\alpha}{a\sin\alpha}$.

(b) Hence show that the chord PQ is parallel to the tangent to $\mathcal{E}$ at the point where $\theta = \alpha$.

7. $\mathcal{E}$ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and $P(a\cos\theta, b\sin\theta)$ is a variable point on $\mathcal{E}$. $S(ae, 0)$ is a focus of $\mathcal{E}$, and ST is the perpendicular drawn from S to the tangent at P.

(a) Show that ST has equation $ax\sin\theta - by\cos\theta = a^2 e\sin\theta$.

(b) The line ST meets the directrix $x = \frac{a}{e}$ at R. Show that R has coordinates $\left(\frac{a}{e}, \frac{b\sin\theta}{e\cos\theta}\right)$.

(c) Let O be the centre of $\mathcal{E}$. Show that O, P and R are collinear.

8. The ellipse $\mathcal{E}$ with equation $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ has foci $S(4, 0)$ and $S'(-4, 0)$.

(a) Show that the tangent at $P(x_1, y_1)$ on $\mathcal{E}$ has equation $9x_1x + 25y_1y = 225$.

(b) Suppose that $Q(x_2, y_2)$ is another point on $\mathcal{E}$.
If the line PQ passes through S, show that $4(y_2 - y_1) = x_1y_2 - x_2y_1$.

(c) Show that the tangents at P and Q intersect on the directrix corresponding to S.

(d) Show that the normal at P has equation $25y_1x - 9x_1y = 16x_1y_1$, and decide under what circumstances, if any, it passes through S or S'.

9. $P(a\cos\theta, b\sin\theta)$ is a variable point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. A line is drawn through the centre of the ellipse parallel to the tangent at P. This line intersects the ellipse at R and R'.

(a) Show that the tangent at P has equation $(b\cos\theta)x + (a\sin\theta)y - ab = 0$.

(b) Show that the perpendicular distance from the centre of the ellipse to the tangent at P is $\dfrac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$.

(c) Hence show that the area of $\triangle RPR'$ is independent of where P lies on the ellipse (that is, show that the area does NOT depend on θ).

10. (a) For all real values of θ, the variable point $P(4\cos\theta, 3\sin\theta)$ lies on an ellipse. What is the Cartesian equation of this ellipse?

(b) Find the equation of the tangent to this ellipse at P.

(c) Show that the variable point $Q(-4\sin\theta, 3\cos\theta)$ also lies on the same ellipse for all real values of θ.

(d) The tangents to the ellipse at P and Q intersect at T. Find the coordinates of T, and hence show that T lies on the ellipse $9x^2 + 16y^2 = 288$, for all real values of θ.

11. $\mathcal{E}$ is the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. P is the variable point (x_1, y_1) on $\mathcal{E}$. The tangent to $\mathcal{E}$ at P meets the major axis at T, and it meets the directrix $x=\frac{a}{e}$ at Q. S is the focus $(ae, 0)$, and O is the centre of $\mathcal{E}$. PN is the perpendicular from P to the major axis.

(a) Find the equation of the tangent at P.

(b) Show that $ON \times OT = a^2$.

(c) Show that the y-coordinate of Q is $\frac{b^2(ae-x_1)}{aey_1}$.

(d) Hence show that $SP \perp SQ$.

(e) The line OP and the line through S perpendicular to the tangent at P meet at R. Show that R lies on the directrix $x=\frac{a}{e}$.

12. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\mathcal{E}$ with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, and Q is the point $(a\cos\theta, a\sin\theta)$ on the auxiliary circle $\mathcal{C}$. S is the focus $(ae, 0)$ of the ellipse.

(a) Show that the tangent to $\mathcal{E}$ at P has equation $bx\cos\theta + ay\sin\theta = ab$.

(b) Hence show that the tangent to $\mathcal{C}$ at Q has equation $x\cos\theta + y\sin\theta = a$.

(c) Show that the tangents in parts (a) and (b) above meet on the x-axis.

(d) Use the definition of eccentricity to show that $SP = a(1-e\cos\theta)$ units.

(e) Hence show that $SP = SR$, where SR is the perpendicular from S to the tangent to $\mathcal{C}$ at Q.

13. $\mathcal{E}$ is the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$, and $P(4\cos\theta, 3\sin\theta)$ is a variable point on $\mathcal{E}$. B and B' are the endpoints of the minor axis.

(a) Show that the tangent to $\mathcal{E}$ at P has equation $3x\cos\theta + 4y\sin\theta = 12$.

(b) The tangent at P meets the tangents at B and B' at C and C' respectively. Show that $BC \times B'C' = 16$.

(c) The circle with diameter CC' meets the x-axis at D and D'. Show that $OD \times OD' = 7$.

(d) Let N be a point on the minor axis of $\mathcal{E}$. Prove that $\angle CNC'$ cannot be a right-angle.

14. $\mathcal{E}$ is the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, and $P(a\cos\theta, b\sin\theta)$ is a variable point on $\mathcal{E}$. The eccentricity of $\mathcal{E}$ is e, and $S(ae, 0)$ and $S'(-ae, 0)$ are its foci. ST and $S'T'$ are the perpendiculars from S and S' to the tangent at P, while G and H are the points where the normal at P meets the major and minor axes respectively. R is the point $(0, b\sin\theta)$ on the minor axis, and O is the centre of $\mathcal{E}$.

(a) Prove that $PG : PH = RO : RH$.

(b) Hence show that $PG : PH = 1-e^2 : 1$.

(c) Show that the tangent at P has equation $(b\cos\theta)x + (a\sin\theta)y - ab = 0$.

(d) Hence show that $ST \times S'T' = b^2$.

(e) [The reflection property of the ellipse] Show that $\angle SPT = \angle S'PT'$ by showing that $\sin\angle SPT = \sin\angle S'PT'$.

(f) Deduce that the normal at P bisects $\angle SPS'$.

15. (a) Show that $pq \le \dfrac{p^2+q^2}{2}$ for all real values of p and q.

(b) The ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The point $M(x_0, y_0)$ lies inside $\mathcal{E}$, so that $\dfrac{{x_0}^2}{a^2}+\dfrac{{y_0}^2}{b^2}<1$. The line ℓ has equation $\dfrac{x_0x}{a^2}+\dfrac{y_0y}{b^2}=1$.

(i) Use the result in part (a) to show that the line ℓ lies entirely outside $\mathcal{E}$. That is, show that if $P(x_1, y_1)$ is any point on ℓ, then $\dfrac{{x_1}^2}{a^2}+\dfrac{{y_1}^2}{b^2}>1$.

(ii) The chord of contact to $\mathcal{E}$ from any point $Q(x_2, y_2)$ outside $\mathcal{E}$ has equation

$$\frac{x_2x}{a^2}+\frac{y_2y}{b^2}=1.$$

Show that M lies on the chord of contact to $\mathcal{E}$ from any point on ℓ.

3C Numerical Problems on the Hyperbola

Exercise 3C

1. The hyperbola $\mathcal{H}$ has equation $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$.

(a) Find its eccentricity.
(b) Find the coordinates of its foci.
(c) Find the equations of its directrices.
(d) Write down the equations of its asymptotes.
(e) Sketch $\mathcal{H}$, showing the foci, directrices and asymptotes.
(f) Show that the parametric equations $x = 2\sec\theta$, $y = \sqrt{5}\tan\theta$ represent $\mathcal{H}$.
(g) Find the coordinates of the point on $\mathcal{H}$ corresponding to $\theta = \frac{\pi}{4}$.

2. Consider the hyperbola $\mathcal{H}$ with equation $x^2 - y^2 = 4$.
(a) Find the foci and directrices of $\mathcal{H}$.
(b) Sketch $\mathcal{H}$, showing its foci, directrices and asymptotes.
(c) Show that the parametric equations $x = 2\sec\alpha$, $y = 2\tan\alpha$ represent $\mathcal{H}$.
(d) Show on your diagram how to construct the point where $\alpha = \frac{3\pi}{4}$.

3. The hyperbola $\mathcal{H}$ has equation $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$.

(a) Find its eccentricity.
(b) Find the coordinates of its foci.
(c) Find the equations of its directrices.
(d) Write down the equations of its asymptotes.
(e) Sketch $\mathcal{H}$, showing the foci, directrices and asymptotes.
(f) Write down in terms of θ a pair of parametric equations representing $\mathcal{H}$.
(g) Show that the point on $\mathcal{H}$ corresponding to $\theta = -\frac{\pi}{3}$ is $(8, -3\sqrt{3})$.

4. Consider the hyperbola defined by the parametric equations $x = \sec\theta$, $y = \sqrt{3}\tan\theta$.
(a) Show that $(-\frac{2}{\sqrt{3}}, 1)$ is the point on the hyperbola corresponding to $\theta = -\frac{5\pi}{6}$.
(b) Find the Cartesian equation of the hyperbola.
(c) Find the eccentricity of the hyperbola.
(d) Show that each latus rectum is of length 6 units.

5. (a) A certain hyperbola has foci at $(\pm 4, 0)$ and directrices $x = \pm 1$. Find its equation.
(b) Another hyperbola has eccentricity 3 and its transverse axis lies along the x-axis. Its two x-intercepts are 10 units apart. Find its equation.

6. Show that:
(a) the tangent and normal to the hyperbola $x^2 - 2y^2 = 2$ at the point $(-2, 1)$ have equations $x + y + 1 = 0$ and $x - y + 3 = 0$ respectively.
(b) the tangent and normal to the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ at the point $(4, -3\sqrt{3})$ have equations $\sqrt{3}\,x + y = \sqrt{3}$ and $x - \sqrt{3}\,y = 13$ respectively.

7. Show that:
(a) the tangent to the hyperbola $x = 2\sec\theta$, $y = \tan\theta$ at the point where $\theta = \frac{\pi}{4}$ has equation $x - \sqrt{2}\,y = \sqrt{2}$.
(b) the tangent to the hyperbola $x = \sec\theta$, $y = \tan\theta$ at the point where $\theta = \frac{\pi}{3}$ has equation $2x - \sqrt{3}\,y = 1$.

8. Show that:
(a) the chord of contact to the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ from the point $(1, 1)$ has equation $4x - 9y = 36$,
(b) the chord of contact to the hyperbola $x = 4\sec\theta$, $y = \tan\theta$ from the point $(2, -3)$ has equation $x + 24y = 8$.

9. (a) P is the point $(9, -3)$ on the hyperbola $\dfrac{x^2}{54} - \dfrac{y^2}{18} = 1$. The normal at P meets the hyperbola again at Q. Show that the tangents at P and Q meet at the point $(\frac{9}{2}, \frac{3}{2})$.
(b) The line $y = 2x - 4$ intersects the hyperbola $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$ at P and Q. Show that the tangents at P and Q intersect at the point $(\frac{3}{2}, \frac{1}{2})$.
(c) The tangent at the point $(2, 1)$ on the hyperbola $9x^2 - 4y^2 = 32$ meets the asymptotes of the hyperbola at A and B. Show that the interval AB has length $\frac{2\sqrt{85}}{3}$ units.

10. (a) Show that the line $x - 2y + 1 = 0$ is a tangent to the hyperbola $x^2 - 6y^2 = 3$, and show that the point of contact is $(-3, -1)$.
(b) Show that the line $4x - 3y = 5$ is a tangent to the hyperbola $2x^2 - 3y^2 = 5$, and show that the point of contact is $(2, 1)$.

11. (a) Show that the two tangents to the hyperbola $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$ with gradient -1 have equations $x + y = \pm 1$.
(b) Show that the two tangents to the hyperbola $x^2 - 3y^2 = 6$ which are parallel to the line $2x - y = 7$ have equations $2x - y = \pm\sqrt{22}$.

12. A variable point P in the number plane moves in such a way that the difference of its distances from two fixed points 10 units apart is always 6 units. Thus, the locus of P is a hyperbola. Let the two fixed points be at $(\pm 5, 0)$, so that the transverse axis of the hyperbola lies on the x-axis.
(a) Show, using a diagram, that $a = 3$.
(b) Hence show that the equation of the hyperbola is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

13. The line $y = mx + c$ is a tangent to the hyperbola $2x^2 - y^2 = 1$.
(a) Show that $m^2 = 2(c^2 + 1)$.
(b) Hence show that the tangents to the hyperbola that pass through the point $(2, 3)$ have equations $y = 2x - 1$ and $y = \frac{10}{7}x + \frac{1}{7}$.

14. The line $px + qy + r = 0$ is a tangent to the hyperbola $\dfrac{x^2}{12} - \dfrac{y^2}{3} = 1$.
(a) Show that $r^2 = 12p^2 - 3q^2$.
(b) Hence show that the tangents to the hyperbola with gradient 1 have equations

$$x - y \pm 3 = 0.$$

3D Theoretical Problems on the Hyperbola

Exercise **3D**

NOTE: A carefully drawn diagram is essential in each of these problems.

1. P is any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, whose foci are S and S'. Use a diagram and the definition of eccentricity (that is, $\dfrac{PS}{PM} = \dfrac{PS'}{PM'} = e$, where PM and PM' are the perpendiculars from P to the directrices) to prove that $|PS - PS'| = 2a$.

2. Let $\mathcal{H}$ be the hyperbola $9x^2 - 16y^2 = 144$, and let $P(x_1, y_1)$, where $x_1 > 0$, be a variable point on $\mathcal{H}$.
(a) Show that the foci of $\mathcal{H}$ are at $S(5, 0)$ and $S'(-5, 0)$.
(b) Use the eccentricity of $\mathcal{H}$ to show that $SP = \frac{5x_1 - 16}{4}$ and that $S'P = \frac{5x_1 + 16}{4}$.
(c) Show that the tangent to $\mathcal{H}$ at P has equation $9x_1x - 16y_1y = 144$.
(d) The tangent meets the x-axis at G. Show that G is the point $(\frac{16}{x_1}, 0)$.
(e) Hence prove that $\dfrac{SP}{S'P} = \dfrac{SG}{S'G}$.

3. Let $\mathcal{H}$ be the hyperbola $5x^2 - 4y^2 = 20$, and let $P(2\sec\theta, \sqrt{5}\tan\theta)$ be a variable point on $\mathcal{H}$.
(a) Show that the foci of $\mathcal{H}$ are $(\pm 3, 0)$.
(b) Show that the directrices of $\mathcal{H}$ have equations $x = \pm\frac{4}{3}$.
(c) Show that the asymptotes of $\mathcal{H}$ have equations $y = \pm\frac{\sqrt{5}}{2}x$.
(d) Show that the tangent at P has equation $\sqrt{5}\,x\sec\theta - 2y\tan\theta = 2\sqrt{5}$.
(e) The tangent at P meets a directrix at T. Use gradients to show that PT subtends a right-angle at the corresponding focus.

4. $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and $P(a\sec\theta, b\tan\theta)$ is a variable point on $\mathcal{H}$. O is the centre of $\mathcal{H}$.
(a) Show that the normal at P has equation $ax\sin\theta + by = (a^2 + b^2)\tan\theta$.
(b) The normal meets the x-axis at G. Show that G has coordinates $\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$.
(c) PN is the perpendicular from P to the x-axis. Show that $OG = e^2 \times ON$.

5. The line $x = 1$ is a directrix and the point $(2, 0)$ a focus of a hyperbola of eccentricity $\sqrt{2}$.
(a) Show that the hyperbola has equation $x^2 - y^2 = 2$.
(b) Show that the normal at the point $P(x_1, y_1)$ has equation $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2$.
(c) The normal at P meets the x and y axes at $(n_1, 0)$ and $(0, n_2)$ respectively, and N is the point (n_1, n_2). Show that as P varies on the hyperbola $x^2 - y^2 = 2$, N always lies on the hyperbola $x^2 - y^2 = 8$.

6. Let $\mathcal{H}$ be the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and let S be the focus $(ae, 0)$. $P(a\sec\theta, b\tan\theta)$ is the point on $\mathcal{H}$ in the first quadrant such that PS is parallel to the asymptote $y = -\frac{b}{a}x$.

(a) Show that $\sec\theta + \tan\theta = e$.

(b) Show that the tangent at P has equation $bx\sec\theta - ay\tan\theta = ab$.

(c) Deduce that the tangent at P meets the asymptote $y = -\frac{b}{a}x$ on the directrix corresponding to S.

7. The tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at $P(a\sec\theta, b\tan\theta)$ meets the asymptotes of the hyperbola at A and B.

(a) Show that the tangent at P has equation $bx\sec\theta - ay\tan\theta = ab$.

(b) Show that the points A and B have coordinates $\big(a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta)\big)$ and $\big(a(\sec\theta - \tan\theta), -b(\sec\theta - \tan\theta)\big)$ respectively.

(c) Hence show that P is the midpoint of AB.

8. Let $\mathcal{H}$ be the hyperbola $3x^2 - y^2 = 3$ with centre O.

(a) (i) Show that the foci of $\mathcal{H}$ are $S(2, 0)$ and $S'(-2, 0)$, and that the directrices have equations $x = \pm\frac{1}{2}$.

(ii) Show that the line through S perpendicular to the asymptote with positive gradient has equation $x + \sqrt{3}\,y = 2$.

(iii) The line and the asymptote in part (ii) meet at the point Q. Show that Q lies on the directrix corresponding to S.

(iv) Show also that Q lies on the auxiliary circle of $\mathcal{H}$.

(b) Let $P(x_1, y_1)$ be a variable point on $\mathcal{H}$.

(i) Show that the tangent at P has equation $3x_1x - y_1y = 3$.

(ii) The tangent at P meets the x and y axes at T_1 and T_2 respectively. Show that $\dfrac{OT_1}{OT_2} = \left|\dfrac{y_1}{3x_1}\right|$.

(iii) Let d_1 and d_2 be the perpendicular distances from P to the asymptotes of $\mathcal{H}$. Show that $d_1d_2 = \frac{3}{4}$.

9. $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$, and $P(3\sec\theta, 4\tan\theta)$ is a variable point on $\mathcal{H}$. The tangent at P meets the asymptotes of $\mathcal{H}$ at Q and R, and O is the centre of $\mathcal{H}$.

(a) Show that $QR = 2\sqrt{16 + 25\tan^2\theta}$.

(b) Deduce that $\triangle OQR$ has area 12 square units, regardless of the position of P on $\mathcal{H}$.

10. $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and $P(a\sec\theta, b\tan\theta)$ is a variable point on $\mathcal{H}$. The tangent at P meets a directrix at Q, and S is the corresponding focus. O is the centre of $\mathcal{H}$.

(a) Prove that $SP \perp SQ$.

(b) Deduce that the tangents at the endpoints of a focal chord meet on a directrix.

(c) Prove that the perpendicular from S to the tangent at P meets the line OP on a directrix.

11. $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and $P(a\sec\theta, b\tan\theta)$ is a variable point on $\mathcal{H}$. The vertical line through P meets an aymptote of $\mathcal{H}$ at Q. The tangent to $\mathcal{H}$ at P meets the same asymptote at R. The normal to $\mathcal{H}$ at P meets the x-axis at N. Prove that the angle RQN is a right-angle.

12. $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and $P(a\sec\theta, b\tan\theta)$ is a variable point on $\mathcal{H}$. The tangent and the normal to $\mathcal{H}$ at P cut the y-axis at T and N respectively.

(a) Show that the tangent at P has equation $bx\sec\theta - ay\tan\theta = ab$.

(b) Show that the normal at P has equation $by\sec\theta + ax\tan\theta = a^2e^2\sec\theta\tan\theta$.

(c) Use gradients to prove that the circle with diameter NT passes through the foci of $\mathcal{H}$.

13. $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and $P(a\sec\theta, b\tan\theta)$ is a variable point on $\mathcal{H}$. The tangent to $\mathcal{H}$ at P meets an asymptote of $\mathcal{H}$ at A. AB and AC are the perpendiculars from A to the x and y axes respectively.

(a) Prove that the line BC passes through P.

(b) The tangent and normal at P meet the x-axis at T and N respectively. Prove that the midpoint of the interval TN is never at a focus of $\mathcal{H}$.

14. The points $(r\cos\theta, r\sin\theta)$ and $\big(s\cos(\theta+\frac{\pi}{2}), s\sin(\theta+\frac{\pi}{2})\big)$ lie on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with centre O.

(a) Show that $\dfrac{1}{r^2} + \dfrac{1}{s^2} = \dfrac{1}{a^2} - \dfrac{1}{b^2}$.

(b) P and Q are points on the hyperbola such that $OP \perp OQ$.
Deduce that the value of the expression $\dfrac{1}{OP^2} + \dfrac{1}{OQ^2}$ does not depend on the positions of P and Q on the hyperbola.

15. $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with centre O, and $\mathcal{C}$ is the auxiliary circle $x^2 + y^2 = a^2$. T is a point on $\mathcal{C}$ in the first quadrant, and the tangent to $\mathcal{C}$ at T meets the x-axis at M. MP is perpendicular to the x-axis and P lies on $\mathcal{H}$ in the first quadrant. Let $\angle TOM = \theta$, where $0 < \theta < \frac{\pi}{2}$.

(a) Show that P has coordinates $(a\sec\theta, b\tan\theta)$.

(b) Suppose that the point $Q(a\sec\phi, b\tan\phi)$ is another point on $\mathcal{H}$. If $\theta + \phi = \frac{\pi}{2}$ and $\theta \neq \frac{\pi}{4}$, then show that the chord PQ has equation $ay = b(\cos\theta + \sin\theta)x - ab$.

(c) Show that every such chord PQ passes through a fixed point and write down its coordinates.

(d) Show that as θ approaches $\frac{\pi}{2}$, the chord PQ approaches a line parallel to an asymptote of $\mathcal{H}$.

3E The Rectangular Hyperbola

Exercise **3E**

NOTE: A diagram is essential in each question!

1. $P(ct, \frac{c}{t})$, where c is a positive constant and $t \neq 0$, is a variable point on the hyperbola $xy = c^2$. O is the centre of the hyperbola, PM and PN are the respective perpendiculars from P to the horizontal and vertical asymptotes, and the tangent at P meets the horizontal and vertical asymptotes at A and B respectively.

(a) Show that $PM \times PN$ is constant (that is, independent of t).

(b) Show that the tangent at P has equation $x + t^2y = 2ct$.

(c) Show that P is the midpoint of AB.

(d) Show that the area of $\triangle OAB$ is constant.

2. $\mathcal{R}$ is the rectangular hyperbola $xy = c^2$. $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ are variable points on $\mathcal{R}$. The line PQ cuts the x-axis at A and the y-axis at B.

(a) Show that the line PQ has equation $x + pqy = c(p + q)$.

(b) Show that AB and PQ have the same midpoint.

(c) Hence, or otherwise, show that $AP = BQ$.

3. $\mathcal{R}$ is the rectangular hyperbola $xy = c^2$. $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ are variable points on $\mathcal{R}$. O is the origin.

(a) Show that the tangent at P has equation $x + p^2y = 2cp$, and hence write down the equation of the tangent at Q.

(b) The tangents at P and Q intersect at T. Show that T has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$.

(c) Show that the line OT bisects the chord PQ.

4. $\mathcal{H}$ is the hyperbola $xy = c^2$. $P(ct, \frac{c}{t})$ is a point on $\mathcal{H}$.

(a) Show that the normal at P has equation $t^2x - y = ct^3 - \frac{c}{t}$.

(b) Hence show that this normal meets $\mathcal{H}$ again at the point $Q\left(-\frac{c}{t^3}, -ct^3\right)$.

(c) If PR is a diameter of $\mathcal{H}$, show that $PR \perp QR$.

5. P, Q and R are the points on the rectangular hyperbola $x = ct$, $y = \frac{c}{t}$ where $t = p$, $t = q$ and $t = r$ respectively. The chord PQ subtends a right-angle at R.

(a) Show that $pqr^2 = -1$.

(b) Hence show that PQ is parallel to the normal at R.

6. $\mathcal{H}$ is the hyperbola $xy = c^2$, and $P(ct, \frac{c}{t})$ is a point on $\mathcal{H}$. PN is the perpendicular from P to the x-axis, and M is the point where the tangent at P cuts the y-axis. The line through M parallel to the x-axis meets $\mathcal{H}$ at Q.

(a) Show that the tangent at P has equation $x + t^2y = 2ct$.

(b) Show that Q is the point $\left(\frac{ct}{2}, \frac{2c}{t}\right)$.

(c) Show that the line NQ has equation $4x + t^2y = 4ct$.

(d) Deduce that the line NQ is the tangent to $\mathcal{H}$ at Q.

7. $\mathcal{H}$ is the rectangular hyperbola $xy = c^2$, and $P(ct, \frac{c}{t})$ is a variable point on $\mathcal{H}$. PP' is a diameter of $\mathcal{H}$. The tangent at P meets the horizontal and vertical lines through P' at Q and Q' respectively.
(a) Write down the coordinates of P'.
(b) Show that Q has coordinates $\left(3ct, -\frac{c}{t}\right)$ and that Q' has coordinates $\left(-ct, \frac{3c}{t}\right)$.
(c) Show that P is the midpoint of QQ'.
(d) Show that as P varies on $\mathcal{H}$, both Q and Q' vary on the hyperbola $xy = -3c^2$.

8. $\mathcal{R}$ is the rectangular hyperbola $xy = c^2$. $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ are variable points on $\mathcal{R}$, and PN is the perpendicular from P to the x-axis. The tangent at Q passes through N.
(a) Show that $p = 2q$.
(b) Show that the tangents at P and Q intersect at $T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$.
(c) Show that the locus of T has Cartesian equation $xy = \frac{8c^2}{9}$.

9. $\mathcal{H}$ is the hyperbola $xy = c^2$. $P(ct, \frac{c}{t})$ is a point on $\mathcal{H}$. The normal at P meets the x-axis at A, while the tangent at P meets the y-axis at B. M is the midpoint of AB.
(a) Show that the normal at P has equation $t^3x - ty = c(t^4 - 1)$.
(b) Find the coordinates of M.
(c) Hence show that the locus of M has Cartesian equation $2c^2xy + y^4 = c^4$.

10. $\mathcal{R}$ is the rectangular hyperbola $xy = c^2$, with centre O. $P(ct_1, \frac{c}{t_1})$ and $Q(ct_2, \frac{c}{t_2})$ are variable points on $\mathcal{R}$. The tangents to $\mathcal{R}$ at P and Q intersect at T, and M is the midpoint of PQ.
(a) Show that the tangent at P has equation $x + {t_1}^2y = 2ct_1$, and hence write down the equation of the tangent at Q.
(b) Show that T has coordinates $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.
(c) Show that the points O, T and M are collinear.
(d) Given that the product t_1t_2 is constant as t_1 and t_2 vary, prove that the locus of T is a diameter of $\mathcal{R}$.

11. The hyperbola $\mathcal{H}$ has equation $xy = 16$. $P(4p, \frac{4}{p})$, where $p > 0$, and $Q(4q, \frac{4}{q})$, where $q > 0$, are distinct arbitrary points on $\mathcal{H}$. The tangents to $\mathcal{H}$ at P and Q intersect at T.
(a) Show that the chord PQ has equation $x + pqy = 4(p + q)$.
(b) Show that the tangent at P has equation $x + p^2y = 8p$.
(c) Show that the tangents at P and Q intersect at the point $T\left(\frac{8pq}{p+q}, \frac{8}{p+q}\right)$.
(d) The chord PQ, when extended, passes through the point $(0, 8)$. Deduce that the locus of T is the line $x = 4$, but only for $0 < y < 4$.

12. The line $y = mx + b$ is a tangent to the rectangular hyperbola $xy = c^2$.
(a) Prove that $b^2 = -4mc^2$.
(b) Tangents are drawn to the hyperbola from the point $(-8c, c)$. Show that their equations are $x + 4y + 4c = 0$ and $x + 16y - 8c = 0$.

13. $\mathcal{H}$ is the rectangular hyperbola defined by the parametric equations $x = ct$, $y = \frac{c}{t}$. P, Q and R are the points where $t = t_1$, $t = t_2$ and $t = t_3$ on $\mathcal{H}$. Let A be the point of intersection of the three altitudes of $\triangle PQR$. (A is called the orthocentre of $\triangle PQR$.)
(a) Show that A has coordinates $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.
(b) Hence prove the theorem: "If a rectangular hyperbola passes through the three vertices of a triangle, then it also passes through the orthocentre of the triangle."

14. ℓ is the line $ax + by = 1$, and $\mathcal{H}$ is the rectangular hyperbola $xy = c^2$. The line ℓ cuts $\mathcal{H}$ in two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. $M(x_0, y_0)$ is the midpoint of P_1P_2.

(a) Find a quadratic equation whose roots are x_1 and x_2.

(b) Show that the equation of ℓ may be written $\dfrac{x}{x_0} + \dfrac{y}{y_0} = 2$.

(c) Suppose that $\mathcal{L}$ is a line with gradient m_1 through the centre of a rectangular hyperbola, but not an asymptote. Deduce that $\mathcal{L}$ bisects all chords that have a particular gradient m_2, where $m_2 = -m_1$.

15. $\mathcal{H}$ is the rectangular hyperbola $xy = c^2$. $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ are points on the opposite branches of $\mathcal{H}$. The circle $\mathcal{C}$ has diameter PQ. Let A and B be the points where $\mathcal{C}$ cuts $\mathcal{H}$ again, and let α and β be the respective x coordinates of A and B.

(a) Prove that $\alpha + \beta = 0$.

(b) Hence deduce that AB is a diameter of $\mathcal{H}$.

Chapter Three

Exercise 3A (Page 99)

1(a) $\frac{3}{5}$ **(b)** $(\pm 3, 0)$ **(c)** $x = \pm\frac{25}{3}$ **(f)** $(\frac{5}{2}, 2\sqrt{3})$

2(b) $x = 2\cos\theta,\ y = \frac{4}{3}\sin\theta$

3(a) foci $(\pm 2, 0)$ and directrices $x = \pm 4\frac{1}{2}$

4(b) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ **(c)** $\frac{4}{5}$

5(a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ **(b)** $\frac{x^2}{36} + \frac{y^2}{20} = 1$

16(a) $\lambda < 2$ **(b)** The length of the major axis starts at $2\sqrt{3}$ and approaches $2\sqrt{2}$, while the length of the minor axis stars at 2 and approaches zero. **(c)** When $\lambda = 2$, $b = 0$, so the limiting position is the interval joining $(\pm\sqrt{2}, 0)$.

Exercise 3B (Page 102)

Exercise 3C (Page 106)

1(a) $\frac{3}{2}$ **(b)** $(\pm 3, 0)$ **(c)** $x = \pm\frac{4}{3}$ **(d)** $y = \pm\frac{\sqrt{5}}{2}x$ **(g)** $(2\sqrt{2}, \sqrt{5})$

2(a) foci $(\pm 2\sqrt{2}, 0)$ and directrices $x = \pm\sqrt{2}$

3(a) $\frac{5}{4}$ **(b)** $(\pm 5, 0)$ **(c)** $x = \pm\frac{16}{5}$ **(d)** $y = \pm\frac{3}{4}x$ **(f)** $x = 4\sec\theta,\ y = 3\tan\theta$

4(b) $x^2 - \frac{y^2}{3} = 1$ **(c)** 2

5(a) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ **(b)** $\frac{x^2}{25} - \frac{y^2}{200} = 1$

Exercise 3D (Page 108)

Exercise 3E (Page 111)

