THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

MATH1903/1907

INTEGRAL CALCULUS AND MODELLING (ADVANCED)

November 2011

LECTURERS: D Daners, J Parkinson

TIME ALLOWED: **One and a half hours**

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination; there are 20 questions; the questions are of equal value; all questions may be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination; there are 4 questions; the questions are of equal value; all questions may be attempted; working must be shown.

Approved non-programmable calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

MARKER'S USE ONLY	

Extended Answer Section

*There are **four** questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

MARKS

1. (a) Calculate the volume of the solid obtained by revolving the region of $\mathbb{R}^2$ bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$ and $y = 0$ about the y-axis. **2**

(b) Calculate the length of the curve in $\mathbb{R}^2$ with parametric equations **3**

$$x(t) = 3t^2 + 2, \qquad y(t) = 4 - t^3, \qquad \text{with } t \in [0, 1].$$

MARKS

(c) Calculate the value of the improper integral **3**

$$\int_0^\infty \frac{1}{(x+1)(x+2)}\,dx.$$

(d) Find $\dfrac{d}{dx}\displaystyle\int_x^{e^x} \ln(1+t^2)\,dt.$ **3**

MARKS

2. (a) (*i*) Let m, n be integers with $m < n$, and let $f(x)$ be a monotone decreasing continuous function with $f(x) \geq 0$ for all x. Use upper and lower Riemann sums on the interval $[m, n]$ to show that **3**

$$f(n) \leq \sum_{k=m}^{n} f(k) - \int_{m}^{n} f(x)\,dx \leq f(m).$$

(*ii*) Hence, or otherwise, show that the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges. **2**

Question 2 continues on the next page

MARKS

(b) You are given that the equation

$$ye^y = x$$

implicitly defines a function $y = y(x)$ with domain $x \geq -e^{-1}$ and range $y \geq -1$, and that this function can be differentiated any number of times.

(*i*) Calculate the integral **3**

$$\int_0^e \frac{1}{1 + y(x)}\,dx.$$

(*ii*) Find the second order Taylor polynomial for $y(x)$ about $x = 0$. **3**

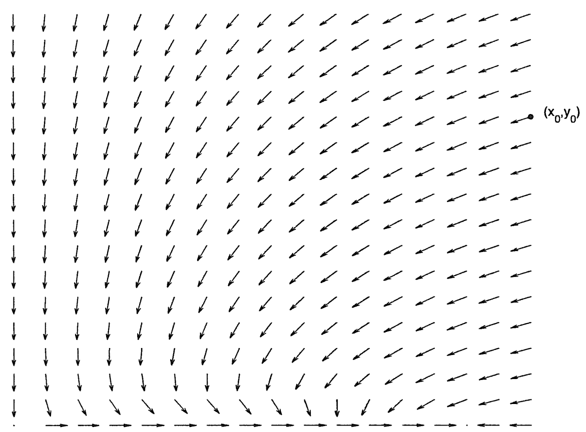
Question 3 begins on the next page

MARKS

3. (a) Find the general solution to the differential equation **2**

$$y' \cos^2 x = y^2(1 - \sin x).$$

(b) The diagram below shows a vector field of a system of two differential equations. In that diagram, draw the trajectory of the solution starting at the point (x_0, y_0) marked in the diagram. **1**



MARKS

(c) (*i*) Find the general solution of homogeneous second order differential equation **2**

$$\ddot{x} + \dot{x} - 6x = 0.$$

(*ii*) Find a particular solution of the inhomogeneous differential equation **3**

$$\ddot{x} + \dot{x} - 6x = e^{2t}.$$

Question 3 continues on the next page

MARKS

(d) Solve the initial value problem 3

$$u' = 2xu + x^3, \qquad u(0) = 2.$$

QUESTION 4 BEGINS ON THE NEXT PAGE

MARKS

4. (a) By infusion, the glucose concentration of blood is increased at a constant rate measured in mg/minute. At the same time, the glucose is converted and excreted from the blood at a rate proportional to the present concentration of the glucose.

   (*i*) Carefully define all dependent and independent variables needed to model the concentration of the glucose in the blood. **1**

   (*ii*) Derive a differential equation describing the concentration of the glucose as a function of time. Use the variables you introduced in (*i*). **2**

MARKS

(b) Consider the nonlinear differential equation

$$xy' = y + ax\sqrt{x^2 + y^2}, \qquad x > 0,$$

where $a > 0$ is a constant.

(i) Show that $v := yx^{-1}$ satisfies the separable differential equation **2**

$$v' = a\sqrt{1 + v^2}$$

(ii) Use the differential equation in part (i) to get the general solution to the original differential equation. (Note the table of standard integrals.) **2**

Question 4 continues on the next page

MARKS

(c) Consider the system of differential equations

$$x' = 2x - y$$
$$y' = x + 2y$$

(*i*) Determine the stability of the zero solution $x = y = 0$. **1**

(*ii*) Find the solution of the system for the initial values $x(0) = 0$ and $y(0) = -1$. **3**