MATH3611: Higher Analysis
Assignment 2

Name: Keegan Gyoery zID: z5197058

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1. **Claim:** $\text{Int}(S) = \emptyset$.

   Proof: Consider $S = \{\{x_n\}_{n=1}^{\infty} \in \ell^1 : |x_n| < 1/n,\ \forall n\}$, and choose $s \in S$, such that $s = (s_1, s_2, \ldots, s_n, \ldots)$. Choose r, so that for some $\epsilon > 0$, $r_1 = (s_1, s_2, \ldots, s_n + \epsilon/2, \ldots)$ and $r_2 = (s_1, s_2, \ldots, s_n - \epsilon/2, \ldots)$. Examining the epsilon ball around s, we have

$$\mathsf{B}(s, \epsilon) = \{x \in \ell^1 : d_{\|\cdot\|_1}(s, x) < \epsilon\}.$$

   Clearly, $s \in \mathsf{B}(s, \epsilon)$ and $r \in \mathsf{B}(s, \epsilon)$ as

$$\begin{aligned}\|r_1 - s\|_1 &= \frac{\epsilon}{2} < \epsilon,\\ \|r_2 - s\|_1 &= \left|\frac{-\epsilon}{2}\right| = \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

   However, we can always choose n such that

$$\frac{1}{n} < \max\left\{\left|s_n + \frac{\epsilon}{2}\right|, \left|s_n - \frac{\epsilon}{2}\right|\right\}.$$

   Thus, either $r_1 \notin S$ or $r_2 \notin S$. So, for all $s \in S$ and all $\epsilon > 0$, $\mathsf{B}(s, \epsilon)$ will contain points in $S^{\complement}$. Hence, $\text{Int}(S) = \emptyset$.

   Claim: $\text{Bd}(S) = \{\{x_n\}_{n=1}^{\infty} \in \ell^1 : |x_n| \leq 1/n,\ \forall n\}$.

   Proof: From the definition of S, we have $S^{\complement} = \{\{x_n\}_{n=1}^{\infty} \in \ell^1 : \exists n,\ |x_n| \geq 1/n\}$. Consider the sets $T_1 = \{\{x_n\}_{n=1}^{\infty} \in \ell^1 : \exists n,\ |x_n| > 1/n\}$, and $T_2 = \{\{x_n\}_{n=1}^{\infty} \in \ell^1 : |x_n| \leq 1/n,\ \forall n\}$. By construction we have $S = T_1 \cup T_2$.

   Choose $t_1 \in T_1$ such that $t_1 = \{t_n\}_{n=1}^{\infty}$, and let $t_n > 1/n$ for some n. Select $\epsilon = \dfrac{1}{2}\left(|t_n| - 1/n\right)$ and consider a second sequence $x = \{x_n\}_{n=1}^{\infty} \in \mathsf{B}(t_1, \epsilon)$. Examining the distance between the sequences,

$$\begin{aligned}\|t_n - x_n\|_1 = |t_n - x_n| &< \epsilon\\ |t_n| - |x_n| &\leq |t_n - x_n|\\ \therefore |t_n| - |x_n| &< \epsilon\\ \therefore |t_n| - |x_n| &< \frac{1}{2}\left(|t_n| - \frac{1}{n}\right)\\ \therefore |x_n| &> \frac{1}{2}\left(|t_n| + \frac{1}{n}\right)\\ \therefore |x_n| &> \frac{1}{n},\end{aligned}$$

   so $x \in T_1$. Hence, around every sequence $t_1 \in T_1$, there exists $\mathsf{B}(t_1, \epsilon) \subseteq T_1$. Thus, $T_1 \subset \text{Int}(S^{\complement})$.

Choose $t_2 \in T_2$ such that $t_2 = \{t_n\}_{n=1}^{\infty}$, and consider $\mathsf{B}(t_2, \epsilon)$. Again, consider a second sequence $x = \{x_n\}_{n=1}^{\infty}$, defined piecewise by

$$|x_n| = \begin{cases} |t_n| - \epsilon/4^n, & \text{if } t_n \neq 1/n \\ |t_n|, & \text{if } t_n = 1/n. \end{cases}$$

We also construct x such that each x_n has the same sign as the corresponding t_n. It follows that $|x_n| < 1/n$ for all n. Condsider now the set $K = \{n : t_n = 1/n\} \subset \mathbb{N}$. K is a strict subset of the natural numbers, as if we had $K = \mathbb{N}$, this would produce the harmonic series, a sequence that would not be in ℓ^1, and hence not considered.

Examining the distances between the sequences,

$$\begin{aligned} \|t_n - x_n\|_1 &= |t_n - x_n| \\ &= \epsilon \sum_{n \in K} \frac{1}{4^n} \\ &< \epsilon \sum_{n \in \mathbb{N}} \frac{1}{4^n} \\ &= \frac{\epsilon}{3} \\ \therefore \|t_n - x_n\|_1 &< \epsilon. \end{aligned}$$

As $x \in S$, for all $t_2 \in T_2$ and all $\epsilon > 0$, $\mathsf{B}(t_2, \epsilon)$ contains a sequence in S. Thus, $T_2 \cap \mathsf{Int}(S^{\complement}) = \emptyset$. So $\mathsf{Int}(S^{\complement}) = T_1$. By definition, $\mathsf{Bd}(S) = (\ell^1, \|\cdot\|_1) \setminus (\mathsf{Int}(S) \cup \mathsf{Int}(S^{\complement})) = (\ell^1, \|\cdot\|_1) \setminus \mathsf{Int}(S^{\complement})$, as $\mathsf{Int}(S) = \emptyset$. So, $\mathsf{Bd}(S) = \left(\mathsf{Int}(S^{\complement})\right)^{\complement} = T_1^{\complement}$. Thus, $\mathsf{Bd}(S) = \{\{x_n\}_{n=1}^{\infty} \in \ell^1 : |x_n| \leq 1/n,\ \forall n\}$.

Claim: $\mathsf{Cl}(S) = \mathsf{Bd}(S) = \{\{x_n\}_{n=1}^{\infty} \in \ell^1 : |x_n| \leq 1/n,\ \forall n\}$.

Proof: By definition, $\mathsf{Cl}(S) = \mathsf{Int}(S) \sqcup \mathsf{Bd}(S) = \mathsf{Bd}(S)$ as $\mathsf{Int}(S) = \emptyset$.

2. **Claim:** There exists a continuous function $f : [-1, 1] \to \mathbb{R}$ such that

$$\int_{-1}^{1} \frac{f(t)}{\pi + (x-t)^4} dt = f(x) - \pi x, \qquad \forall x \in [-1, 1].$$

Proof: Consider the metric space $(C[-1,1], \|\cdot\|_\infty)$. The above integral equation can be rearranged to the form

$$f(x) = \pi x + \int_{-1}^{1} \frac{f(t)}{\pi + (x-t)^4} dt,$$

which may be written as a fixed point equation $Tf = f$, where the map T is defined as

$$Tf(x) = \pi x + \int_{-1}^{1} \frac{f(t)}{\pi + (x-t)^4} dt.$$

Considering any $f_1, f_2 \in C[-1,1]$, we have

$$\begin{aligned}
\|Tf_1 - Tf_2\|_\infty &= \left\| \pi x + \int_{-1}^{1} \frac{f_1(t)}{\pi + (x-t)^4} dt - \pi x - \int_{-1}^{1} \frac{f_2(t)}{\pi + (x-t)^4} dt \right\|_\infty \\
&= \left\| \int_{-1}^{1} \frac{f_1(t)}{\pi + (x-t)^4} dt - \int_{-1}^{1} \frac{f_2(t)}{\pi + (x-t)^4} dt \right\|_\infty \\
&= \left\| \int_{-1}^{1} \frac{f_1(t) - f_2(t)}{\pi + (x-t)^4} dt \right\|_\infty \\
&= \sup_{-1 \le x \le 1} \left| \int_{-1}^{1} \frac{f_1(t) - f_2(t)}{\pi + (x-t)^4} dt \right| \\
&\le \sup_{-1 \le x \le 1} \int_{-1}^{1} \left| \frac{f_1(t) - f_2(t)}{\pi + (x-t)^4} \right| dt \\
&= \sup_{-1 \le x \le 1} \int_{-1}^{1} \left| \frac{1}{\pi + (x-t)^4} \right| |f_1(t) - f_2(t)| dt \\
&\le \|f_1 - f_2\|_\infty \left\{ \sup_{-1 \le x \le 1} \int_{-1}^{1} \left| \frac{1}{\pi + (x-t)^4} \right| dt \right\} \\
&\le c \|f_1 - f_2\|_\infty ,
\end{aligned}$$

where

$$c = \sup_{-1 \le x \le 1} \left\{ \int_{-1}^{1} \left| \frac{1}{\pi + (x-t)^4} \right| dt \right\}.$$

By showing $c < 1$, we can prove that T is a contraction map. We have

$$\begin{aligned}
c &= \sup_{-1 \le x \le 1} \left\{ \int_{-1}^{1} \left| \frac{1}{\pi + (x-t)^4} \right| dt \right\} \\
&\le \sup_{-1 \le x \le 1} \left\{ \int_{-1}^{1} \left| \frac{1}{\pi} \right| dt \right\} \\
&= \sup_{-1 \le x \le 1} \left\{ \frac{2}{\pi} \right\} \\
&= \frac{2}{\pi} \\
\therefore c &< 1.
\end{aligned}$$

Thus, for any $f_1, f_2 \in C[-1,1]$, $\|Tf_1 - Tf_2\|_\infty \le c \|f_1 - f_2\|_\infty$, where $c < 1$, and hence T is a contraction map. From lectures, we have that $C[-1,1]$ is complete, and thus the sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ converges to the fixed point $f \in C[-1,1]$. Thus there exists a continous function $f : [-1,1] \to \mathbb{R}$.