(A)

MATH1903 — Lecture 7 — Thurs 24/8/2017

Logs & exponentials

"early" versus "late" transcendental methods

"early method" : $a > 0$

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}, \quad n \in \mathbb{Z}^+$$

$$a^{-n} = \frac{1}{a^n},$$

$$a^{m/n} = \left(a^{1/n}\right)^m$$

where $a^{1/n}$, the nth root of a, exists by continuity theorems.

Let $x \in \mathbb{R}$, approximated by rationals $q_1, q_2, \ldots, q_n, \ldots$

so

$$\boxed{x = \lim_{i \to \infty} q_i}$$

e.g. 3.1, 3.14, 3.141, 3.1415, 3.14159, ...

starts a sequence of rationals approaching π.

We may define

$$\boxed{a^x = \lim_{i \to \infty} a^{q_i}},$$

which exists by limits machinery.

pp 2.3 — 2.7 including Monotone Convergence

Call

$$\boxed{y = a^x}$$

an *exponential* function, and its inverse function

is

$$\boxed{y = \log_a x},$$

called the *logarithm to the base a*,

defined by

$$\boxed{b = \log_a c \iff c = a^b}$$

"late method" : define the natural logarithm function ln in terms of integrals, and then the "natural" exponential function exp as

$$\exp = \ln^{-1},$$

the inverse of ln, and then subsequently define

$$\boxed{y = a^x = \exp(a \ln x)}$$

p2.9 : antiderivative of $\frac{1}{x} = x^{-1}$??

p2.10 : $$\frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}$$

Define $$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt = \int_1^x \frac{dt}{t}}$$

for $x > 0$.

p2.11 : $\ln 1 = 0$, $\ln x > 0$ if $x > 1$,

$\ln x < 0$ if $0 < x < 1$.

D

$$\boxed{\ln(ab) = \ln a + \ln b}$$

"ln turns products into sums"

Proof: $\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt$

$= \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt$

$= \ln a + \int_1^{b} \frac{1}{\cancel{a}u}(\cancel{a})\,du$

$= \ln a + \int_1^{b} \frac{1}{u}\,du$

$= \ln a + \ln b.$ $\square$

Put $t = au$,
so $dt = a\,du$,
$t = a$ when $u = 1$,
$t = ab$ when $u = b$

p2.15 : graph of $\ln x$ and its inverse

$$\boxed{\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0^+} \ln x = -\infty}$$

(E)

Proofs: $\lim_{n\to\infty} \ln(2^n) = \lim_{n\to\infty} \underbrace{\ln 2 + \ln 2 + \dots + \ln 2}_{n \text{ times}}$

$= \lim_{n\to\infty} (n \ln 2)$

$= (\ln 2) \lim_{n\to\infty} n = \infty$,

so $\lim_{x\to\infty} \ln x = \infty$.

Hence $\lim_{x\to 0^+} \ln x = \lim_{y\to\infty} \ln\left(\frac{1}{y}\right)$ $\quad (y = \frac{1}{x})$

$= \lim_{y\to\infty} \ln(y^{-1})$

$= \lim_{y\to\infty} -\ln y$

$= -\lim_{y\to\infty} \ln y = -\infty$.

$\ln(y^{-1}) + \ln(y) = \ln(y^{-1}y) = \ln 1 = 0$,

so $\ln(y^{-1}) = -\ln y$

F

p2.19 : definition of $\exp = \ln^{-1}$.

pp 2.20 – 2.22 : properties of exp.

$$\boxed{\exp(a+b) = (\exp(a))(\exp(b))} \qquad \boxed{\frac{d}{dx}(\exp(x)) = \exp(x)}$$

p2.25
$$e \overset{\text{def}}{=} \exp(1)$$ Euler's number

$$\ln e = 1.$$

p2.26 $\ln(a^n) = n\ln(a)$ for $n \in \mathbb{Z}^+$

so $\boxed{a^n = \exp(n \ln a)}$

suggesting general definition

$$\boxed{a^x \overset{\text{det}}{=} \exp(x \ln a)}$$ p2.27

In particular,

$$e^x = \exp(x \ln e) = \exp(x)$$ ✓

as we hoped.

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p2.28 : summary of properties.

e.g. $\boxed{(ab)^c = a^c b^c}$

Proof: $(ab)^c = \exp(c \ln(ab))$

$= \exp(c(\ln a + \ln b))$

$= \exp(c \ln a + c \ln b)$

$= \exp(c \ln a) \exp(c \ln b)$

$= a^c b^c$. $\square$