The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Longer Solutions to Selected Exercises for Week 10** 2017

4. (i) $\begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix} = 5(-2) - 2(3) = -16$ (ii) $\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} = 6(1) - 2(3) = 0$

(iii) $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 0 - (-1) = 1$ (iv) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1$

(v) $$\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} = -1 - (-2) = 1$$

(vi) $$\begin{vmatrix} 2 & 4 & 6 \\ 7 & 11 & 6 \\ -6 & -6 & 12 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 6 \\ 5 & 7 & 0 \\ -10 & -14 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 6 \\ 5 & 7 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 - 0 + 0 = 0$$

(vii) $$\begin{vmatrix} -4 & 3 & 3 \\ 8 & 7 & 3 \\ 4 & 3 & 3 \end{vmatrix} = \begin{vmatrix} -4 & 3 & 3 \\ 12 & 4 & 0 \\ 8 & 0 & 0 \end{vmatrix} = 8\begin{vmatrix} 3 & 3 \\ 4 & 0 \end{vmatrix} = 8(0 - 12) = -96$$

5. (i) $\begin{vmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -5 & -1 \end{vmatrix} = 5(-2)(-1) = 10$ (ii) $\begin{vmatrix} 3 & 3 & 8 \\ 0 & -6 & -7 \\ 0 & 0 & 2 \end{vmatrix} = 3(-6)(2) = -36$

(iii) $\begin{vmatrix} -4 & -5 & 11 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{vmatrix} = -0 + 0 - 0 = 0$ (iv) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{vmatrix} = 1(-1)(-2) = 2$

(v) $\begin{vmatrix} 0 & 0 & 5 \\ 6 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = 5(6)(-3) = -90$ (vi) $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ -6 & -3 & -7 & -1 \end{vmatrix} = 4(-2)(2)(-1) = 16$

7. (i) $$\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 3 & 5 \\ 4 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} = -9 - 5 = -14$$

(ii) $$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 0$$

(iii) $$\begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 & 4 \\ 3 & -2 & -5 & 1 \\ 4 & -4 & -7 & 7 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 4 \\ -2 & -5 & 1 \\ -4 & -7 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 9 \\ 0 & 1 & 23 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 9 \\ 1 & 23 \end{vmatrix} = -(-23 - 9) = 32$$

8. (i) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \mathbf{k} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

(ii) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 6 \\ -1 & 1 & -3 \end{vmatrix} = \begin{vmatrix} -1 & 6 \\ 1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 6 \\ -1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} \mathbf{k} = -3\mathbf{i} + \mathbf{k}$

9. Put $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} \cdot \mathbf{w} &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}) \\ &= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right) \cdot (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}) \\ &= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = - \begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} . \end{aligned}$$

10. (i) $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} 1 & -3 & 1 \\ 2 & 3 & -3 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 \\ 0 & 9 & -5 \\ 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 9 & -5 \\ -1 & 0 \end{vmatrix} = -5$

(ii) $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} 2 & -1 & -2 \\ 1 & 5 & 6 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -3 & -2 \\ 7 & 11 & 6 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ 7 & 11 \end{vmatrix} = 21$

11. Suppose A is an invertible matrix. Then $AA^{-1} = I$, so, by the multiplicative property,

$$1 = \det I = \det(AA^{-1}) = (\det A)(\det A^{-1}) .$$

If $\det A = 0$ then $1 = 0(\det A^{-1}) = 0$ which is impossible. Hence $\det A \neq 0$. Dividing through gives

$$\det A^{-1} = \frac{1}{\det A} .$$

13. (i) This is always true, combining the usual multiplicative property with commutativity of scalar multiplication.

(ii) This is false. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\det(A+B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 = 0 + 0 = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = (\det A) + (\det B) .$$

(iii) This is false. In fact, always,

$$\det(2A) = \det(2IA) = \det(2I) \det A = 4 \det A \neq 2 \det A ,$$

except when $\det A = 0$.

(iv) This is true always since

$$\det(-A) = \det(-IA) = \det(-I) \det A = (-1)(-1) \det A = \det A .$$

14. The determinant is unchanged by adding multiples of one column to another, and we may bring out a common factor of any given column:

$$\begin{vmatrix} 8 & 6 & 7 \\ 4 & 5 & 9 \\ 1 & 8 & 7 \end{vmatrix} = \begin{vmatrix} 8 & 6 & 7+6(10)+8(100) \\ 4 & 5 & 9+5(10)+4(100) \\ 1 & 8 & 7+8(10)+100 \end{vmatrix} = \begin{vmatrix} 8 & 6 & 867 \\ 4 & 5 & 459 \\ 1 & 8 & 187 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 6 & 17\alpha \\ 4 & 5 & 17\beta \\ 1 & 8 & 17\gamma \end{vmatrix} = 17 \begin{vmatrix} 8 & 6 & \alpha \\ 4 & 5 & \beta \\ 1 & 8 & \gamma \end{vmatrix}$$

for some integers α, β, γ, which is a multiple of 17, since clearly the determinant of a matrix of integers is an integer.

15. (i) $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & -3-\lambda \end{vmatrix} = (2-\lambda)(-3-\lambda) = 0$ if and only if $\lambda = 2$ or -3.

(ii) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)+2 = \lambda^2-5\lambda+6 = (\lambda-2)(\lambda-3) = 0$ if and only if $\lambda = 2$ or 3.

(iii) $$\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ -4 & -4 & 7-\lambda \end{vmatrix} = \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ 0 & \lambda-3 & 3-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 3-\lambda \\ 0 & \lambda-3 & 0 \end{vmatrix} = -(\lambda-3) \begin{vmatrix} -3-\lambda & 2 \\ -4 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\big((-3-\lambda)(3-\lambda)+8\big) = (3-\lambda)(\lambda^2-1) = (3-\lambda)(\lambda-1)(\lambda+1) = 0$$

if and only if $\lambda = 3$, 1 or -1.

16. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = ad - cb = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \det A^T .$$

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ then $A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}$ and

$$\begin{aligned} \det A &= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ek-fh) - b(dk-fg) + c(dh-eg) \\ &= aek - afh - bdk + bfg + cdh - ceg \\ &= aek - ahf - dbk + dhc + gbf - gec \\ &= a(ek-hf) - d(bk-hc) + g(bf-ec) \\ &= a\begin{vmatrix} e & h \\ f & k \end{vmatrix} - d\begin{vmatrix} b & h \\ c & k \end{vmatrix} + g\begin{vmatrix} b & e \\ c & f \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix} = \det A^T . \end{aligned}$$

17. The lines intersect in a point if and only if the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ \ell \end{bmatrix}$$

has a unique solution, which occurs if and only if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be row reduced to the identity matrix, which in turn occurs if and only if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, and this occurs if and only if its determinant $ad - bd$ is nonzero.

18. When $n = 2$ then $\det A_n = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -1 = (-1)^{2-1}$, which starts an induction. For $n > 2$, expanding along the first row, only the second entry is nonzero, and it is 1, so $\det A_n$ is $(-1)\det B$ where B is the result of deleting the first row and second column of A_n. But, by inspection, $B = A_{n-1}$, so, by an inductive hypothesis, $\det B = (-1)^{n-2}$, and so

$$\det A_n = (-1)\det B = (-1)(-1)^{n-2} = (-1)^{n-1} ,$$

which completes the inductive step and the proof. If we expand down the first column, then the only nonzero entry is the last entry, which is 1, so that $\det A_n = (-1)^{n-1}\det C$, where C is the result of deleting the first column and last row. But, by inspection, $C = I$, so

$$\det A_n = (-1)^{n-1}\det C = (-1)^{n-1}\det I = (-1)^{n-1} ,$$

as before.

19. Suppose that A is a square matrix and $\det A \neq 0$. We may use elementary row operations, say $\rho_1, \ldots, \rho_k$ to transform A into reduced row echelon form B, and then

$$B = E_k \ldots E_1 A$$

where $E_1, \ldots E_k$ are elementary matrices corresponding to $\rho_1, \ldots, \rho_k$ respectively. Hence

$$A = E_1^{-1} \ldots E_k^{-1} B .$$

If $B \neq I$ then B must contain a row of zeros, so that, from an earlier exercise, $\det B = 0$, and then, by the multiplicative property,

$$\det A = \det(E_1^{-1} \ldots E_k^{-1} B) = \det(E_1^{-1} \ldots E_k^{-1})\det B = \det(E_1^{-1} \ldots E_k^{-1})0 = 0 ,$$

contradicting that $\det A \neq 0$. Hence $B = I$ and so $(E_k \ldots E_1)A = I$, which is enough to prove that A is invertible.

20. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be points in the plane and put $\delta = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$.

Then

$$\begin{aligned} \delta &= \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x_1 & x_2 - x_1 & x_3 - x_1 \\ y_1 & y_2 - y_1 & y_3 - y_1 \end{vmatrix} \\ &= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} , \end{aligned}$$

so that, thinking now of the xy-plane in space,

$$\delta\,\mathbf{k} = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = \overrightarrow{AB} \times \overrightarrow{AC}\,.$$

If $\triangle ABC$ is oriented anticlockwise, then, by the Right-Hand Rule, $\delta > 0$. On the other hand, if $\triangle ABC$ is oriented clockwise, then, by the Right-Hand Rule, $\delta < 0$. If $\triangle ABC$ is degenerate then the cross product is the zero vector, so that $\delta = 0$. This completes the proof that the orientation test works.

21. (i) $\begin{vmatrix} 1 & 1 & 1 \\ 4 & -7 & 2 \\ 6 & 0 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -11 & -2 \\ 6 & -6 & -11 \end{vmatrix} = 121 - 12 = 109 > 0$ so the triangle is oriented anticlockwise.

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 23 & -1 \\ 1 & 24 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 23 & -1 \\ 1 & 23 & -4 \end{vmatrix} = -92 + 23 = -69 < 0$ so the triangle is oriented clockwise.

22. (i) Observe that

$$\begin{vmatrix} 1 & 1 & 1 \\ 5 & 7 & 3 \\ 1 & 9 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & 2 & -2 \\ 1 & 8 & 2 \end{vmatrix} = 4 + 16 = 20 > 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 5 & 3 & 1 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & -2 & -4 \\ 1 & 2 & 3 \end{vmatrix} = -6 + 8 = 2 > 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 7 & 1 \\ 3 & 9 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 4 & -2 \\ 3 & 6 & 1 \end{vmatrix} = 4 + 12 = 16 > 0\,,$$

so that $\triangle PQS$, $\triangle PSR$, $\triangle SQR$ are all oriented anticlockwise, which means that S must lie inside $\triangle PQR$.

(ii) Observe that

$$\begin{vmatrix} 1 & 1 & 1 \\ 5 & 7 & 4 \\ 1 & 9 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & 2 & -1 \\ 1 & 8 & 6 \end{vmatrix} = 12 + 8 = 20 > 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 1 \\ 1 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & -1 & -4 \\ 1 & 6 & 3 \end{vmatrix} = -3 + 24 = 21 > 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 7 & 1 \\ 7 & 9 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 3 & -3 \\ 7 & 2 & -3 \end{vmatrix} = -9 + 6 = -3 < 0\,,$$

so that $\triangle PQS$, $\triangle PSR$ are oriented anticlockwise and $\triangle SQR$ clockwise, which means that S must lie outside $\triangle PQR$ (in fact, beyond edge RQ).

(iii) Observe that

$$\begin{vmatrix} 1 & 1 & 1 \\ 5 & 7 & 6 \\ 1 & 9 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & 2 & 1 \\ 1 & 8 & 4 \end{vmatrix} = 8 - 8 = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 5 & 6 & 1 \\ 1 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & 1 & -4 \\ 1 & 4 & 3 \end{vmatrix} = 3 + 16 = 19 > 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 6 & 7 & 1 \\ 5 & 9 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 6 & 1 & -5 \\ 5 & 4 & -1 \end{vmatrix} = -1 + 20 = 19 > 0\,,$$

so that $\triangle PQS$ is degenerate and $\triangle PSR$, $\triangle SQR$ are oriented anticlockwise, which means that S must lies on the boundary of $\triangle PQR$ (in fact, halfway along the edge PQ).

23. Suppose A is an $n \times n$ matrix with first row consisting of $a_1, \ldots, a_n$. Denote the $(n-1) \times (n-1)$ matrix left by deleting the first row and jth column of A by B_j for $j = 1$ to n.

Suppose first that the kth row of A consists of zeros. We show $\det A = 0$ by induction on n. If $n = 0$ then $A = [0]$ so that $\det A = 0$, which starts an induction. Suppose $n > 1$. If $k = 1$ then expanding along the first row gives $\det A = 0$ immediately. If $k > 1$ then

$$\det A = \sum_{j=1}^{n} (-1)^{j-1} a_j \det B_j = \sum_{j=1}^{n} (-1)^{j-1} a_j 0 = 0\,,$$

where we have applied an inductive hypothesis to B_j, whose $(k-1)$th row consists of zeros. This completes the inductive step and the proof that $\det A = 0$.

Suppose now that the kth column of A consists of zeros. We show $\det A = 0$ by induction on n. If $n = 0$ then $A = [0]$ so that $\det A = 0$, which starts an induction. Suppose $n > 1$. Observe that, whenever $j \neq k$, B_j has a column of zeros, so $\det B_j = 0$ by an inductive hypothesis. Hence

$$\det A = \sum_{j=1}^{n} (-1)^{j-1} a_j \det B_j = a_k \det B_k = 0\,,$$

since $a_k = 0$. This completes the inductive step and the proof that $\det A = 0$.

24. Suppose that $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ is a $(m+n) \times (m+n)$ matrix where B is $m \times m$ and C is $n \times n$. We prove that $\det A = (\det B)(\det C)$ by induction on m. If $m = 1$ then $B = [b]$ for some number b and then expanding along the first row of A gives

$$\det A = b \det C = (\det B)(\det C)\,,$$

which starts an induction. Suppose now that $m > 1$. The first row of A consists of elements $b_1, \ldots, b_m, 0, \ldots, 0$, where $b_1, \ldots, b_m$ are the elements in the first row of B. Expanding along the first row of A yields

$$\det A = \sum_{j=1}^{m} (-1)^{j-1} b_j \det A_j\,,$$

where each A_j is the matrix obtained by deleting the first row and jth column of A. But each A_j is a diagonal sum

$$A_j = \begin{bmatrix} B_j & 0 \\ 0 & C \end{bmatrix}$$

where B_j is the result of deleting the first row and jth column of B. But then

$$\det A_j = (\det B_j)(\det C)$$

by an inductive hypothesis (since B_j is $(m-1) \times (m-1)$). Hence

$$\begin{aligned} \det A &= \sum_{j=1}^{m} (-1)^{j-1} b_j \det A_j = \sum_{j=1}^{m} (-1)^{j-1} b_j (\det B_j)(\det C) \\ &= \left(\sum_{j=1}^{m} (-1)^{j-1} b_j \det B_j \right) (\det C) = (\det B)(\det C) , \end{aligned}$$

completing the inductive step and the proof.

25. Suppose that f is a permutation that is both even and odd, so

$$f = \sigma_1 \dots \sigma_k = \tau_1 \dots \tau_\ell$$

for some even integer k and odd integer ℓ and transpositions $\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_\ell$. A transposition that interchanges elements i and j corresponds to an elementary matrix that interchanges the ith and jth rows of an $n \times n$ matrix by pre-multiplication. Denote by E_i and F_j the elementary matrices corresponding to σ_i and τ_j respectively, for each i and j. Then the two decompositions of f yield, by applying elementary row operations successively,

$$E_1 \dots E_k = E_1 \dots E_k I_n = F_1 \dots F_\ell I_n = F_1 \dots F_\ell ,$$

so that

$$\begin{aligned} (-1)^k &= (\det E_1) \dots (\det E_k) = \det(E_1 \dots E_k) \\ &= \det(F_1 \dots F_\ell) = (\det F_1) \dots (\det F_\ell) = (-1)^\ell . \end{aligned}$$

Hence

$$1 = (-1)^k (-1)^\ell = (-1)^{k+\ell} = -1 ,$$

since $k + \ell$ is odd, which is a contradiction.