# MATH1081 - Assignment 1

1. Prove that $\{15m - 7 : m \in \mathbb{Z}\}$ is a proper subset of $\{5n + 3 : n \in \mathbb{Z}\}$.

   **Proof:** We are required to prove that $\{15m - 7 : m \in \mathbb{Z}\}$ is a proper subset of $\{5n + 3 : n \in \mathbb{Z}\}$. Let $S = \{15m - 7 : m \in \mathbb{Z}\}$, and $T = \{5n + 3 : n \in \mathbb{Z}\}$. Consider the set $S$, which can be manipulated as follows.

   $$\begin{aligned} S &= \{15m - 7 : m \in \mathbb{Z}\} \\ &= \{15m - 10 + 3 : m \in \mathbb{Z}\} \\ S &= \{5(3m - 2) + 3 : m \in \mathbb{Z}\} \end{aligned}$$

   Let $x \in S$. Because $m$ is an integer, so too is $3m - 2$. Thus, let $n = 3m - 2$. Therefore, $x \in T$, because $x \in S$. Therefore $S \subseteq T$.

   Assume that $T \subseteq S$. By definition, every element of $T$ must also be an element of $S$. Select $n = 0$. This gives the element $y = 3$, where $y \in T$, clearly. In order for $y \in S$, we must satisfy $15m - 7 = 3$, where $m \in \mathbb{Z}$. Therefore, $m = \frac{4}{5}$, which is clearly not an integer. Thus, $y \notin S$, and hence our assumption that $T \subseteq S$ is incorrect. Hence, $T \nsubseteq S$, and so $S \neq T$.

   A proper subset of a set $A$, is a set $B$, where $B \subseteq A$, and $A \neq B$. Thus, as $S \subseteq T$, and $S \neq T$, then clearly, $S$ is a proper subset of $T$.

   $\square$

2. A relation $\preceq$ is defined on $\mathbb{R}$ by

   $$x \preceq y \text{ if and only if } y = x + k \text{ for some integer } k \geq 0.$$

   Prove that $\preceq$ is a partial order.

   **Proof:** We are required to prove that a relation $\preceq$ defined on $\mathbb{R}$ by $x \preceq y$ if and only if $y = x + k$, for some integer $k \geq 0$. In order to prove $\preceq$ is a partial order, we must prove that $\preceq$ is reflexive, anti-symmetric, and transitive. As a result, the proof will be completed by proving these properties hold for $\preceq$. Select $k = 0$. Therefore $k \in \mathbb{Z}$ and $k \geq 0$.

   <u>Reflexive:</u> A partial order defined on a set $S$ is reflexive if $\forall x \in S$, $x \preceq x$. Let $a \in \mathbb{R}$. For all real numbers, $a = a$, which can be written as $a = a + 0$, which is also equivalent to $a = a + k$, based on the selection of $k$. Thus, $a \preceq a$. Therefore, $\preceq$ is reflexive.

   <u>Anti-Symmetric:</u> A partial order defined on a set $S$ is anti-symmetric if $\forall x, y \in S$, $x \preceq y$ and $y \preceq x$ implies $x = y$. Let $a, b \in \mathbb{R}$, $a \preceq b$, and $b \preceq a$. We can rewrite these statements as $b = a + k$, and $a = b + k$. Based on the selection of $k$, the previous statements become $b = a$, and $a = b$. Thus, $a \preceq b$ and $b \preceq a$ implies $a = b$. Therefore, $\preceq$ is anti-symmetric.

   <u>Transitive:</u> A partial order defined on a set $S$ is transitive if $\forall x, y, z \in S$, $x \preceq y$, and $y \preceq z$ implies $x \preceq z$. Let $a, b, c \in \mathbb{R}$, $a \preceq b$, and $b \preceq c$. These statements can be written as $b = a + k$, and $c = b + k$. With the selection of $k$, $b = a$, and $c = b$. Therefore $c = a$, which can be written as $c = a + k$, from the selection of $k$. Thus, $a \preceq c$, and so $a \preceq b$, and $b \preceq c$ imply $a \preceq c$. Therefore, $\preceq$ is transitive.

   As $\preceq$ is reflexive, anti-symmetric, and transitive, $\preceq$ is a partial order. $\square$

3. Prove that for an integer $k \geq 0$

$$(4(k+1)-1)5^{k+1} - (4k-1)5^k = (16k+16)5^k.$$

Hence simplify

$$\sum_{k=0}^{n-1} (k+1)5^k.$$

**Proof:** Let $k \in \mathbb{Z}$, such that $k \geq 0$. Let $P(k)$ be the predicate

$$(4(k+1)-1)5^{k+1} - (4k-1)5^k = (16k+16)5^k.$$

Consider the LHS of $P(k)$.

$$\begin{aligned}
\text{LHS} &= (4(k+1)-1)5^{k+1} - (4k-1)5^k \\
&= 5^k\left[(4(k+1)-1)5 - (4k-1)\right] \\
&= 5^k\left[(4k+4-1)5 - 4k - 1\right] \\
&= 5^k\left[20k + 15 - 4k + 1\right] \\
&= 5^k\left[16k + 16\right] \\
&= (16k+16)5^k \\
&= \text{RHS of } P(k)
\end{aligned}$$

This clearly verifies that $P(k)$ is true $\forall k \in \mathbb{Z}$ such that $k \geq 0$.

Consider again the predicate $P(k)$, which we have previously proved true, and thus we shall label it now the statement $S(k)$, after swapping the LHS and RHS.

$$\begin{aligned}
\sum_{k=0}^{n-1}\left[(16k+16)5^k\right] &= \sum_{k=0}^{n-1}\left[(4(k+1)-1)5^{k+1} - (4k-1)5^k\right] \quad \text{[Summing from 0 to } n-1\text{]} \\
16\sum_{k=0}^{n-1}\left[(k+1)5^k\right] &= \sum_{k=0}^{n-1}\left[(4(k+1)-1)5^{k+1} - (4k-1)5^k\right] \\
\text{RHS} &= \sum_{k=0}^{n-1}\left[(4(k+1)-1)5^{k+1} - (4k-1)5^k\right] \\
&= \sum_{k=0}^{n-1}\left[(4(k+1)-1)5^{k+1}\right] - \sum_{k=0}^{n-1}\left[(4k-1)5^k\right] \quad \text{[Splitting the summation by term]} \\
&= \sum_{k=1}^{n}\left[(4k-1)5^k\right] - \sum_{k=0}^{n-1}\left[(4k-1)5^k\right] \quad \text{[Changing the summation index]} \\
&= (4n-1)5^n + \sum_{k=1}^{n-1}\left[(4k-1)5^k\right] - \sum_{k=1}^{n-1}\left[(4k-1)5^k\right] - (4(0)-1)5^0 \\
&= (4n-1)5^n + 1
\end{aligned}$$

The statement $S(k)$ now becomes $16\sum_{k=0}^{n-1}\Big[(k+1)5^k\Big] = (4n-1)5^n+1$, and thus the simplification of $\sum_{k=0}^{n-1}(k+1)5^k$ is

$$\sum_{k=0}^{n-1}\Big[(k+1)5^k\Big] = \frac{1}{16}\Big[(4n-1)5^n+1\Big]$$

This completes the proof. $\square$