The University of Sydney
School of Mathematics and Statistics

MATH1902
Linear Algebra (Advanced)

June 2013 Lecturer: Holger Dullin

Time Allowed: One and a half hours

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Approved non-programmable calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Marker's use only	

Extended Answer Section

There are **four** *questions in this section, each with a number of parts. Write your answers in the answer book(s) provided. Ask for extra books if you need them.*

1. Consider a pyramid with a quadratic base with corners $P(0,0,0)$, $Q(0,1,0)$, $R(1,1,0)$, $S(1,0,0)$, and apex $A(1/2,1/2,h)$ with height $h > 0$.

(a) Find the cartesian equation for the plane through QRA.

(b) Given that the cartesian equation of the plane through PQA is $hx + z/2 = 0$, find the distance of S from this plane.

(c) Find the distance of the line through R and A from the origin.

(d) Find the height h of the apex A of the pyramid for which the acute angle between adjacent faces is $\pi/3$.

[4+3+4+4=15 marks]

2. (a) Find the condition on b_1, b_2, b_3 for which the system of linear equations

$$\begin{aligned} x_1 + x_2 + 3x_3 &= b_1 \\ x_1 + 2x_2 + 2x_3 &= b_2 \\ x_1 + 3x_2 + x_3 &= b_3 \end{aligned}$$

is consistent.

(b) Solve the system with $b_1 = b_2 = b_3 = 0$.

(c) Interpret the three equations in part (a) as cartesian equations for three planes. Describe qualitatively how these planes intersect when the condition from part (a) is satisfied.

(d) Consider the system of m linear equations in n variables given by $A\mathbf{x} = \mathbf{b}$. Denote by $\mathbf{x}_p$ a particular solution of the system $A\mathbf{x} = \mathbf{b}$. Denote by $\mathbf{x}_h$ a solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

(i) Show that $\mathbf{x}_p + \mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{b}$.

(ii) Write down a system of linear equations whose solution is $3\mathbf{x}_p - 8\mathbf{x}_h$.

[4+3+4+4=15 marks]

3. Recall that two $n \times n$ matrices A and B are similar if there is an invertible $n \times n$ matrix P such that $PB = AP$. The matrix P is called a similarity transformation. A matrix is diagonalisable if it is similar to a diagonal matrix.

(a) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$. Thus find a similarity transformation P and a diagonal matrix D such that $PD = AP$.

(b) Suppose B is another matrix that is similar to the matrix D found in the previous part with similarity transformation $Q = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, so that $QD = BQ$. Find a similarity transformation between the matrices A and B.

(c) Show that the matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are not similar, even though they have the same eigenvalues.

(d) Show that if two matrices A and C are similar to diagonal matrices by the same similarity transformation P, then they satisfy $AC = CA$.

[4+3+4+4=15 marks]

4. Given three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, define the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ by

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v}_1, \\ \mathbf{u}_2 &= \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1, \\ \mathbf{u}_3 &= \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \,. \end{aligned}$$

(a) Given that $\mathbf{v}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v}_2 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v}_3 = 3\mathbf{i} + 3\mathbf{k}$, compute $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and show that any two of these vectors are perpendicular.

(b) Compute the determinant $\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{vmatrix}$ and explain why the result implies that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ from part (a) are linearly independent.

(c) Prove that in general any two of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ defined in the beginning of the question are perpendicular.

(d) Show that if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, then the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent and hence are non-zero.

[4+4+4+3=15 marks]

End of Extended Answer Section