MATH3611: Higher Analysis
Assignment 2

Name: Keegan Gyoery zID: z5197058

1. i) Let $f : [-1, 1] \to [0, 1]$, where $f(x) = |x|$. Extend f to $\mathbb{R}$ by $f(x+2) = f(x)$, $\forall x \in \mathbb{R}$. Consider the series

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n f\left(4^n x\right).$$

Consider the n-th term of the series,

$$g_n(x) = \left(\frac{3}{4}\right)^n f\left(4^n x\right) \leq \left(\frac{3}{4}\right)^n.$$

Let $M_n = \left(\frac{3}{4}\right)^n$. Therefore $g_n(x) \leq M_n$, $\forall n$. Thus,

$$g(x) = \sum_{n=0}^{\infty} g_n(x) \leq \sum_{n=0}^{\infty} M_n = 4.$$

Thus, $g_n \to g$ uniformly on $\mathbb{R}$. From lectures, we proved that uniform convergence of a series of functions implies the limit function is continuous. Thus, $g(x)$ is continuous.

ii) For any x, we have the results

$$|f(x+h) - f(x)| = h, \quad \forall\, 0 \leq h \leq \tfrac{1}{2}, \text{ or}$$
$$|f(x-h) - f(x)| = h, \quad \forall\, 0 \leq h \leq \tfrac{1}{2}.$$

Therefore we also have the results

$$|f(4^n x + 4^n h) - f(4^n x)| = 4^n h, \quad \forall\, 0 \leq 4^n h \leq \tfrac{1}{2}, \text{ or}$$
$$|f(4^n x - 4^n h) - f(4^n x)| = 4^n h, \quad \forall\, 0 \leq 4^n h \leq \tfrac{1}{2}.$$

Fix $k \in \mathbb{N}$, let $h_k = \frac{1}{2}4^{-k}$, and let $h = h_k$, or $h = -h_k$. The above results may be rewritten as

$$|f(4^n x + 4^n h_k) - f(4^n x)| = 4^n h_k, \quad \forall\, 0 \leq 4^n h_k \leq \tfrac{1}{2}, \text{ or}$$
$$|f(4^n x - 4^n h_k) - f(4^n x)| = 4^n h_k, \quad \forall\, 0 \leq 4^n h_k \leq \tfrac{1}{2}.$$

Combining these two results, we have

$$|f(4^n x + 4^n h) - f(4^n x)| = 4^n |h|, \quad \forall\, 0 \leq 4^n h_k \leq \tfrac{1}{2}.$$

Clearly, $4^n h_k = \frac{1}{2}4^{n-k} \geq 0$. Furthermore, when $n \leq k$, $4^n h_k \leq \frac{1}{2}$. Thus,

$$|f(4^n x + 4^n h_k) - f(4^n x)| = 4^n |h|, \text{ when } n \leq k.$$

If $n > k$, we have $4^n h_k = \frac{1}{2}4^{n-k} = 2^{2n-2k-1} = 2^m$, for some $m \in \mathbb{N}$. Therefore, $f(4^n x + 4^n h_k) = f(4^n x + 2^m) = f(4^n x)$. Thus,

$$|f(4^n x + 4^n h_k) - f(4^n x)| = 0, \text{ when } n > k.$$

Clearly, the result follows,

$$|f(4^n(x+h)) - f(4^n x)| = \begin{cases} 0 & n > k \\ 4^n|h| & n \leq k \end{cases}.$$

iii) Fix $k \in \mathbb{N}$.

$$\begin{aligned}
\left|\frac{g(x+h)-g(x)}{h}\right| &= \left|\frac{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n f(4^n(x+h)) - \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n f(4^n x)}{h}\right| \\
&= \left|\frac{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n [f(4^n(x+h)) - f(4^n x)]}{h}\right| \\
&= \left|\frac{\sum_{n=0}^{k}\left(\frac{3}{4}\right)^n [f(4^n(x+h)) - f(4^n x)]}{h}\right| \\
&\geq \left|\frac{\left|\left(\frac{3}{4}\right)^k \left[f(4^k(x+h)) - f(4^k x)\right]\right| - \left|\sum_{n=0}^{k-1}\left(\frac{3}{4}\right)^n [f(4^n(x+h)) - f(4^n x)]\right|}{h}\right| \\
&\geq \left|\frac{\left|\left(\frac{3}{4}\right)^k 4^k|h|\right| - \sum_{n=0}^{k-1}\left|\left(\frac{3}{4}\right)^n [f(4^n(x+h)) - f(4^n x)]\right|}{h}\right| \\
&= \left|\frac{3^k|h| - \sum_{n=0}^{k-1}\left(\frac{3}{4}\right)^n 4^n|h|}{h}\right| \\
&= \left|3^k - \sum_{n=0}^{k-1} 3^n\right| \\
&= \left|3^k - \left(\frac{3^k-1}{2}\right)\right| \\
&= \frac{3^k+1}{2}
\end{aligned}$$

$$\therefore \left|\frac{g(x+h)-g(x)}{h}\right| \geq \frac{3^k+1}{2}.$$

iv) By the definition of the derivative we have

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{k\to\infty}\frac{g\left(x+\frac{1}{2}4^{-k}\right)-g(x)}{\frac{1}{2}4^{-k}}.
\end{aligned}$$

From the above inequality result, we have either

$$\lim_{k\to\infty}\frac{g\left(x+\frac{1}{2}4^{-k}\right)-g(x)}{\frac{1}{2}4^{-k}} \geq \lim_{k\to\infty}\frac{3^k+1}{2}, \text{ or}$$
$$\lim_{k\to\infty}\frac{g\left(x+\frac{1}{2}4^{-k}\right)-g(x)}{\frac{1}{2}4^{-k}} \leq -\lim_{k\to\infty}\frac{3^k+1}{2}.$$

As in either case the RHS diverges as $k \to \infty$, the limit on the LHS does not exist, and thus the derivative $g'(x)$ does not exist, and thus $g(x)$ is not differentiable on $\mathbb{R}$.

2. i) Consider the set of intervals in $\mathbb{R}$, $S = \{(a,b]\}_{a<b\in\mathbb{R}}$. For S to be a base for a topology τ on $\mathbb{R}$, S must be a subset of τ, and the following two conditions must be met:

- $\mathbb{R} = \bigcup_{B\in S} B$,
- $\forall B_1, B_2 \in S$, $\forall x \in B_1 \cap B_2$, $\exists B \in S$ s.t. $B \subseteq B_1 \cap B_2$, and $x \in B$.

Clearly, $\mathbb{R} = \bigcup_{a<b\in\mathbb{R}} (a,b]$. For the second condition, let $B_1, B_2 \in S$ such that $B_1 = (a,b]$, and $B_2 = (c,d]$. Consider the exhaustive cases for the intersection $B_1 \cap B_2$:

Case 1: $B_1 \cap B_2 \neq \emptyset$
Choose $B = (\max\{a,c\}, \min\{b,d\}] \in S$. Clearly, $B = B_1 \cap B_2$, so $B \subseteq B_1 \cap B_2$. Therefore $\forall x \in B_1 \cap B_2$, $B \subseteq B_1 \cap B_2$, and $x \in B$.

Case 2: $B_1 \cap B_2 = \emptyset$
This is vacuously true since there is no $x \in \emptyset$ for which it fails the second condition.

Thus, S is a base for the topology τ.

ii) Let some function $f : \mathbb{R} \to \mathbb{R}$, with topology τ on both domain and codomain. f is continuous if for every $V \in \tau$, $f^{-1}(V) \in \tau$.

a) Define $f(x) = x^2$. Consider the interval $(-1,1] \in \tau$. Thus, $f^{-1}((-1,1]) = [-1,1] \notin \tau$. Therefore, $f(x) = x^2$ is not continuous on τ.

b) Define $g(x) = x^3$. Consider the interval $(a,b] \in \tau$. Thus, $g^{-1}((a,b]) = \left(\sqrt[3]{a}, \sqrt[3]{b}\right] \in \tau$. Therefore $g(x) = x^3$ is continuous on τ.

c) Define $h(x) = \begin{cases} x & x \leq 1 \\ x+1 & x > 1 \end{cases}$. Consider the exhaustive cases for the pre-image of $(a,b] \in \tau$:

Case 1: $1 < a < b \implies h^{-1}((a,b]) = (\max\{a-1,1\}, \max\{b-1,1\}] \in \tau$.

Case 2: $a < b \leq 2 \implies h^{-1}((a,b]) = (\min\{a,1\}, \min\{b,1\}] \in \tau$.

Case 3: $a \leq 1$ and $b > 2 \implies h^{-1}((a,b]) = (a, b-1] \in \tau$.

Therefore $h(x)$ is continuous on τ.

3. i) Construct the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_n = \begin{cases} \frac{1}{m} & \text{if } n = m^3 \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, there are infinitely many n such that $n = m^3$, and $\frac{1}{k} = \left(k^3\right)^{-\frac{1}{3}} = n^{-\frac{1}{3}}$. Thus, $x_n = n^{-\frac{1}{3}}$ for infinitely many n. As $\{x_n\}_{n=1}^{\infty}$ is a subsequence of the harmonic series, it converges in ℓ^2.

ii) Consider the sequence $\left\{n^{\frac{1}{3}}\mathbf{e}_n\right\}_{n=1}^{\infty}$, and the sequence defined above. Examining the inner product of the two sequences, we have

$$\left\langle \left\{n^{\frac{1}{3}}\mathbf{e}_n\right\}_{n=1}^{\infty}, \{x_n\}_{n=1}^{\infty} \right\rangle = \sum_{m=1}^{\infty} 1.$$

However, taking the inner product of $\mathbf{0}$ and the sequence defined in the previous part, we obviously get 0. Thus,

$$\left\langle \left\{ n^{\frac{1}{3}} \mathbf{e}_n \right\}_{n=1}^{\infty}, \{x_n\}_{n=1}^{\infty} \right\rangle neq \left\langle \mathbf{0}, \{x_n\}_{n=1}^{\infty} \right\rangle .$$

Hence, by definition, the sequence $\left\{ n^{\frac{1}{3}} \mathbf{e}_n \right\}_{n=1}^{\infty}$ does not converge weakly to $\mathbf{0}$.

iii) Unsure of how to complete this question.

iv) From the previous part, we have for any sequence $\mathbf{x}_i \in \ell^2$, the distance from $\mathbf{0}$ is at most $\epsilon \cdot k^{-\frac{1}{3}}$. Let the sequence $\mathbf{x}_i = \{\mathbf{e}_n\}_{n=1}^{\infty}$. Examining distances, we have

$$\begin{gathered} \left| \{\mathbf{e}_n\}_{n=1}^{\infty} - \mathbf{0} \right| < \epsilon \cdot k^{-\frac{1}{3}} \\ \therefore \left| \left\{ n^{\frac{1}{3}} \mathbf{e}_n \right\}_{n=1}^{\infty} - \mathbf{0} \right| < \epsilon. \end{gathered}$$

Thus, $\mathbf{0}$ is in the weak closure of the set $\left\{ n^{\frac{1}{3}} \mathbf{e}_n \right\}_{n \in \mathbb{Z}^+}$.