THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Assignment 1

MATH1901/1906: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: http://sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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The first question is revision/practice using assumed knowledge from your HSC mathematics on manipulating algebraic expressions, inequalities and a proof by induction.

1. (a) Show that for every integer $n \geq 1$ we have

$$x^{n+1} - (n+1)x + n = (x-1)\big[1 + x + x^2 + \cdots + x^n - (n+1)\big]$$

for all $x \in \mathbb{R}$ (no induction required).

Solution: We see that

$$\begin{aligned}(x-1)&\big[1 + x + x^2 + \cdots + x^n - (n+1)\big]\\ &= (x + x^2 + \ldots x^{n+1}) - (1 + x + x^2 + \cdots + x^n) - (x-1)(n+1)\\ &= x^{n+1} - 1 - x(n+1) + n + 1\\ &= x^{n+1} - x(n+1) + n\end{aligned}$$

as required.

(b) Hence show that

$$x^{n+1} - (n+1)x + n \geq 0 \tag{1}$$

for all $x \geq 0$.

Solution: If $x \geq 1$, then $1 + x + x^2 + \cdots + x^n \geq n+1$ since there are $n+1$ terms, all larger or equal to one and hence

$$\big[1 + x + x^2 + \cdots + x^n - (n+1)\big] \geq 0.$$

Moreover, $x \geq 1$ also implies that $x - 1 \geq 0$ and hence from (a)

$$x^{n+1} - (n+1)x + n = (x-1)\big[1 + x + x^2 + \cdots + x^n - (n+1)\big] \geq 0.$$

Here we used that the product of two positive numbers is positive.
If $x \in [0,1)$, then $1 + x + x^2 + \cdots + x^n \geq n+1$ since there are $n+1$ terms, all less or equal to one and hence

$$\big[1 + x + x^2 + \cdots + x^n - (n+1)\big] \leq 0.$$

Moreover, $x \in [0,1)$ also implies that $x - 1 < 0$ and hence from (a)

$$x^{n+1} - (n+1)x + n = (x-1)\big[1 + x + x^2 + \cdots + x^n - (n+1)\big] \geq 0.$$

Here we used that the product of two negative numbers is positive.

(c) Let $x_1, x_2, x_3, \dots$ be a sequence of positive real numbers. For every $n \geq 1$ we consider the averages

$$a_n := \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

By setting $x = \dfrac{a_{n+1}}{a_n}$ in (1), show that for $n \geq 1$

$$a_{n+1}^{n+1} \geq a_n^n x_{n+1}.$$

Solution: If we substitute $x = \dfrac{a_{n+1}}{a_n}$ in (1) we see that

$$\left(\frac{a_{n+1}}{a_n}\right)^{n+1} - (n+1)\frac{a_{n+1}}{a_n} + n \geq 0.$$

Multiplying the equation with a_n^{n+1} yields

$$0 \leq a_{n+1}^{n+1} - (n+1)a_{n+1}a_n^n + na_n^{n+1} = a_{n+1}^{n+1} - a_n^n\big((n+1)a_{n+1} - na_n\big). \tag{2}$$

Using the definition of a_n we have

$$\begin{aligned}(n+1)a_{n+1} - na_n &= (n+1)\frac{x_1 + \cdots + x_n + x_{n+1}}{n+1} - n\frac{x_1 + \cdots + x_n}{n} \\ &= (x_1 + \cdots + x_n + x_{n+1}) - (x_1 + \cdots + x_n) = x_{n+1}.\end{aligned}$$

Substituting this into (2) we obtain

$$0 \leq a_{n+1}^{n+1} - x_{n+1}a_n^n$$

which is equivalent to $a_{n+1}^{n+1} \geq a_n^n x_{n+1}$ as claimed.

(d) Let x_n and a_n be as in the previous part. Using mathematical induction by n, show that

$$a_n^n = \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n \geq x_1 x_2 \dots x_n$$

for all $n \geq 1$.

Solution: For a proof by induction we need to check the base case and then prove the induction step.

Base case $n = 1$: For $n = 1$ we have $a_n = \frac{x_1}{1} = x_1 \geq x_1$ trivially.

Induction Step: Assume the inequality is true for some $n \geq 1$, that is,

$$a_n^n \geq x_1 \dots x_n \qquad \text{(induction assumption)}$$

Using the previous part and the induction assumption we have

$$a_{n+1}^{n+1} \geq a_n^n x_{n+1} \geq (x_1 \dots x_n)x_{n+1} = x_1 \dots x_n x_{n+1}$$

as required. Hence the inequality is true for $n + 1$.

Note. The inequality in part (d) is called the *arithmetic mean – geometric mean inequality*. It is one of the "elementary" inequalities quite hard to prove. The above question guides you through a little known short and elementary proof going back to Bengt Åkerberg. "Classroom Notes: A Proof of the Arithmetic-Geometric Mean Inequality". In: *Amer. Math. Monthly* 70.9 (1963), pp. 997–998. DOI: 10.2307/2313068 and rediscovered by Michael D. Hirschhorn. "The AM–GM Inequality". In: *Math. Intelligencer* 29.4 (2007), p. 7. DOI: 10.1007/BF02986168.

The second question is about mapping properties of functions of a complex variable.

2. Consider the map $f(z) = \dfrac{z+i}{z-i}$ of the complex variable $z \neq i$. Show that the image of the real axis under f lies on a circle centred at the origin of the complex plane. Which points from the circle are missing in the image?

Solution: We are told that the image lies on a circle centred at the origin, hence we compute the modulus. We do that by multiplying with the complex conjugate. We are interested in the image of the real axis, so $z = x + i0 = x$.

$$\left|\frac{x+i}{x-i}\right|^2 = \frac{x+i}{x-i} \times \overline{\frac{x+i}{x-i}} = \frac{x+i}{x-i} \times \frac{x-i}{x+i} = \frac{x^2-i^2}{x^2-i^2} = 1$$

for all $x \in \mathbb{R}$. Hence the image lies on the unit circle (the unit circle is the circle of radius one, centred at the origin).

We need to check which points on the circle are not part of the image. For $x \in \mathbb{R}$ we have, multiplying top and bottom with $x+i$,

$$\frac{x+i}{x-i} = \frac{(x+i)^2}{x^2+1} = \frac{x^2-1}{x^2+1} + \frac{2x}{x^2+1}i$$

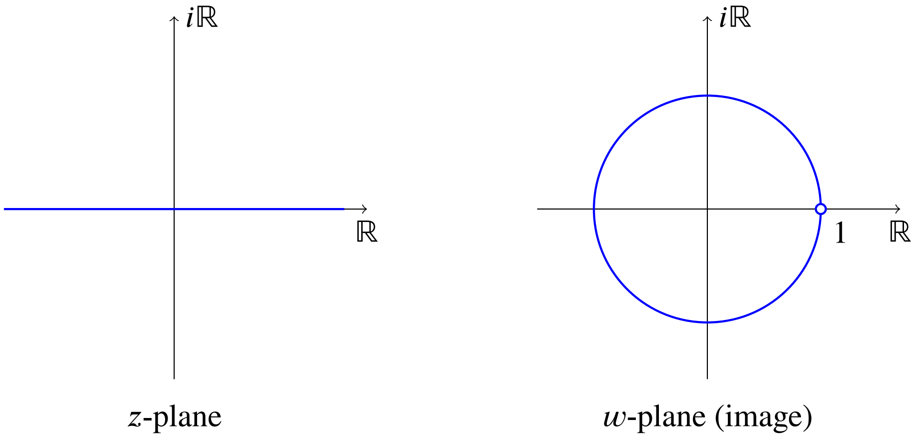
We clearly have

$$-1 \leq \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1} < 1$$

and

$$-1 \leq \frac{2x}{x^2+1} \leq 1.$$

We conclude that $z = 1$ is the only point on the circle not in the image of the real axis. Here is a diagram that illustrates the situation:



Alternative Solution. Here is an alternative solution that only uses complex numbers. If $x \in \mathbb{R}$, then $x+i$ is a complex number with imaginary part equal to one. Hence its modulus–argument form is $x + i = re^{i\theta}$ with $\theta \in (0, \pi)$. If $x \to \infty$, then $\theta \to 0$ and if $x \to -\infty$, then $\theta \to \pi$, but we never reach the limit.

If $x \in \mathbb{R}$, then $x - i = \overline{x+i}$ is the complex conjugate of $x+i$. Hence,

$$\frac{z+i}{z-i} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{2i\theta}$$

with $0 < \theta < \pi$. Hence $0 < 2\theta < 2\pi$ which means the image is the unit circle with 1 removed.