University of Sydney

MATH 1902 - Linear Algebra (Advanced)

Assignment 2

Matrices and Hyper Cubes

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1. Consider the system of equations $A\mathbf{x} = \mathbf{b}$

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -2 & 1 & 2 & -1 \\ -1 & -2 & 1 & 2 \\ 2 & -1 & -2 & x \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} c \\ d \\ e \\ f \end{bmatrix}$$

When the system of equations is solved, we are determining the values of x_1, x_2, x_3, x_4 that satisfy the augmented matrix $M = [A|\mathbf{b}]$.

(a) In order to solve for the variables x_1, x_2, x_3, x_4, we use the augmented matrix $[A|\mathbf{b}]$. By reducing the augmented matrix to its reduced row echelon form, we are able to solve for the desired variables in terms of x, c, d, e, f.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & c \\ -2 & 1 & 2 & -1 & d \\ -1 & -2 & 1 & 2 & e \\ 2 & -1 & -2 & x & f \end{array}\right] \xrightarrow{R_3=R_3+R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & c \\ -2 & 1 & 2 & -1 & d \\ 0 & 0 & 0 & 0 & c+e \\ 2 & -1 & -2 & x & f \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{R_3=R_4} \\ \xrightarrow{R_4=R_3} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & c \\ -2 & 1 & 2 & -1 & d \\ 2 & -1 & -2 & x & f \\ 0 & 0 & 0 & 0 & c+e \end{array}\right] \xrightarrow{R_3=R_2+R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & c \\ -2 & 1 & 2 & -1 & d \\ 0 & 0 & 0 & x-1 & d+f \\ 0 & 0 & 0 & 0 & c+e \end{array}\right]$$

$$\xrightarrow{R_2=2R_1+R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & c \\ 0 & 5 & 0 & -5 & 2c+d \\ 0 & 0 & 0 & x-1 & d+f \\ 0 & 0 & 0 & 0 & c+e \end{array}\right] \xrightarrow{R_2=\frac{1}{5}R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & c \\ 0 & 1 & 0 & -1 & \frac{2c+d}{5} \\ 0 & 0 & 0 & x-1 & d+f \\ 0 & 0 & 0 & 0 & c+e \end{array}\right]$$

$$\xrightarrow{R_1=R_1-2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & c-2(\frac{2c+d}{5}) \\ 0 & 1 & 0 & -1 & \frac{2c+d}{5} \\ 0 & 0 & 0 & x-1 & d+f \\ 0 & 0 & 0 & 0 & c+e \end{array}\right]$$

We have now arrived at the most reduced row echelon form, until we begin to consider the individual cases that can occur. There are 6 different cases to consider, that stem from one of two conditions on x. The two conditions on x are $x = 1$, and $x \neq 1$.

For the first case, $x = 1$, the augmented matrix becomes as follows.

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & c-2(\frac{2c+d}{5}) \\ 0 & 1 & 0 & -1 & \frac{2c+d}{5} \\ 0 & 0 & 0 & 0 & d+f \\ 0 & 0 & 0 & 0 & c+e \end{array}\right] \ldots (A)$$

This case, $x = 1$, has a further 4 possible cases, which we will now examine.

$$x = 1 \implies \begin{cases} c = -e \text{ and } d = -f \implies \text{Consistent, Infinite Solutions} \ldots (1) \\ c = -e \text{ and } d \neq -f \implies \text{Inconsistent} \ldots\ldots\ldots\ldots (2) \\ c \neq -e \text{ and } d = -f \implies \text{Inconsistent} \ldots\ldots\ldots\ldots (3) \\ c \neq -e \text{ and } d \neq -f \implies \text{Inconsistent} \ldots\ldots\ldots\ldots (4) \end{cases}$$

For the second case, $x \neq 1$, the augmented matrix becomes as follows.

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & c-2(\frac{2c+d}{5}) \\ 0 & 1 & 0 & -1 & \frac{2c+d}{5} \\ 0 & 0 & 0 & x-1 & d+f \\ 0 & 0 & 0 & 0 & c+e \end{array}\right] \xrightarrow{R_3=\frac{1}{x-1}R_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & c-2(\frac{2c+d}{5}) \\ 0 & 1 & 0 & -1 & \frac{2c+d}{5} \\ 0 & 0 & 0 & 1 & \frac{d+f}{x-1} \\ 0 & 0 & 0 & 0 & c+e \end{array}\right]$$

$$\xrightarrow{R_2=R_2+R_3}\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & c-2(\frac{2c+d}{5}) \\ 0 & 1 & 0 & 0 & \frac{d+f}{x-1}+\frac{2c+d}{5} \\ 0 & 0 & 0 & 1 & \frac{d+f}{x-1} \\ 0 & 0 & 0 & 0 & c+e \end{array}\right] \ldots (B)$$

This case, $x \neq 1$, has a further 2 possible cases, which we will now examine.

$$x \neq 1 \implies \begin{cases} c=-e \implies \text{Consistent, Infinite Solutions} \ldots (5) \\ c \neq -e \implies \text{Inconsistent} \ldots\ldots\ldots\ldots\ldots\ldots (6) \end{cases}$$

Now examining case (1), as this is the first case that will provide us with solutions to the system of linear equations.

Using row 2 of the augmented matrix (A), we can derive one set of solutions to the variables x_1, x_2, x_3, x_4.

$$\begin{aligned} x_4 - x_2 &= \frac{2c+d}{5} \\ x_2 &= t \text{ where } t \in \mathbb{R} \\ \therefore x_4 - t &= \frac{2c+d}{5} \\ \therefore x_4 &= t + \frac{2c+d}{5} \end{aligned}$$

Now using row 1 of the augmented matrix (A), we can derive the solutions to the remaining variables.

$$\begin{aligned} x_1 - x_3 &= c - 2(\frac{2c+d}{5}) \\ x_2 &= s \text{ where } s \in \mathbb{R} \\ \therefore x_1 - s &= c - 2(\frac{2c+d}{5}) \\ \therefore x_1 &= s + c - 2(\frac{2c+d}{5}) \\ &= s + \frac{5c}{5} - \frac{4c+2d}{5} \\ \therefore x_1 &= s + \frac{c-2d}{5} \end{aligned}$$

Now examining case (5), as this is the first case that will provide us with solutions to the system of linear equations.

Using row 3 of the augmented matrix (B), we can derive a second set of solutions to the variables x_1, x_2, x_3, x_4.

$$x_4 = \frac{d+f}{x-1}$$

Now using row 2 of the augmented matrix (B), we can solve for more variables.

$$x_2 = \frac{d+f}{x-1} + \frac{2c+d}{5}$$

Finally, using row 1 of the augmented matrix (B), we can solve for the remaining variables.

$$\begin{aligned}
x_1 - x_3 &= c - 2(\frac{2c+d}{5}) \\
x_3 &= t \text{ where } t \in \mathbb{R} \\
\therefore x_1 - t &= c - 2(\frac{2c+d}{5}) \\
\therefore x_1 &= t + c - 2(\frac{2c+d}{5}) \\
&= t + \frac{5c}{5} - \frac{4c+2d}{5} \\
\therefore x_1 &= t + \frac{c-2d}{5}
\end{aligned}$$

(b) Now, define the augmented matrix as $M = [A|\mathbf{b}]$. We will now determine the rank of the augmented matrix, M, and the solution dimension of the augmented matrix, M. The rank of the augmented matrix M is defined as the number of pivots in the augmented matrix. Furthermore, the dimension of the solutions to the augmented matrix is defined as the number of parameters used to define the variable's value.

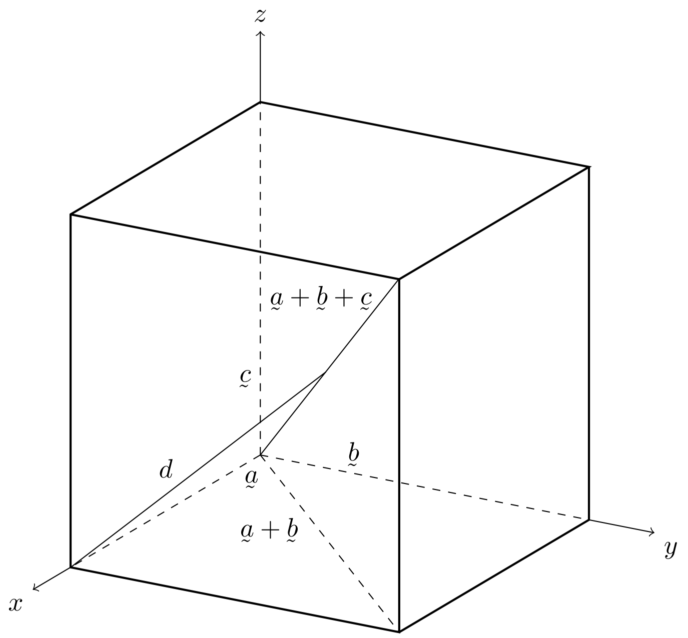
Considering each case separately, we can determine the ranks and dimensions of each potential outcome.

$$\text{Ranks and Dimensions} \implies \begin{cases} (1) \to \text{Rank} = 2, \ \ \text{Dimension} = 2 \\ (2) \to \text{Rank} = 2 \\ (3) \to \text{Rank} = 2 \\ (4) \to \text{Rank} = 2 \\ (5) \to \text{Rank} = 3, \ \ \text{Dimension} = 1 \\ (6) \to \text{Rank} = 3 \end{cases}$$

2. For this question, we are asked to determine the distance between a corner and major diagonal in a cube in $\mathbb{R}^3$. We are then asked to elevate our processes to a hypercube in $\mathbb{R}^4$. For the latter, we will need to use a different method, as the cross product is only defined in $\mathbb{R}^3$. In order to compute the distance between a point P, and a line containing the point Q, and in the direction $\underset{\sim}{v}$, in $\mathbb{R}^3$, we must use the formula:

$$d = \frac{|\underset{\sim}{v} \times \overrightarrow{PQ}|}{|\underset{\sim}{v}|}$$

Furthermore, the following figure shows how the cube is labelled and constructed. It is a unit cube with each of the vectors, $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, a unit vector with value $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$, respectively. The variable d represents the distance from the point $(1, 0, 0)$ to the major diagonal.



(a) In order to solve this distance, we first must derive the value of the major diagonal that runs from the bottom corner to the top corner. To do this we first examine the diagonal that runs along the bottom face of the unit cube. Using vector addition, we can derive its value as:

$$\underset{\sim}{a} + \underset{\sim}{b}$$

Using this value, and vector addition again, we can derive the expression for the major diagonal as:

$$(\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$$

Before solving for the distance betwenn the corner and the diagonal, we need to determine the values of $\underset{\sim}{v}$ and $\overrightarrow{PQ}$. We are using the point $P(1, 0, 0)$, and the point $Q(0, 0, 0)$, which lies on the line with direction $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ Thus the value of $\overrightarrow{PQ}$ is:

$$\begin{aligned} \overrightarrow{PQ} &= Q(0,0,0) - P(1,0,0) \\ &= \underset{\sim}{0} - \underset{\sim}{i} \\ \therefore \overrightarrow{PQ} &= -\underset{\sim}{i} \end{aligned}$$

Now, using the above formula, we will solve for the distance between a corner and the major diagonal of the cube.

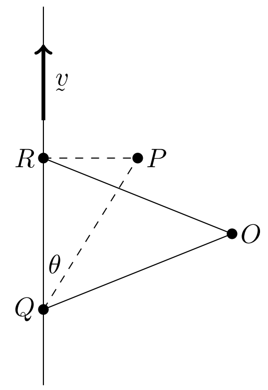
$$
\begin{aligned}
d &= \frac{|\underset{\sim}{v} \times \overrightarrow{PQ}|}{|\underset{\sim}{v}|} \\
&= \frac{|(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) \times (-\underset{\sim}{i})|}{|\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}|} \\
&= \frac{|(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) \times (-\underset{\sim}{i})|}{|\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}|} \\
&= \frac{|(0-0)\underset{\sim}{i} + (-1-0)\underset{\sim}{j} + (0-(-1))\underset{\sim}{k}|}{|\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}|} \\
&= \frac{|-\underset{\sim}{j} + \underset{\sim}{k}|}{|\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}|} \\
&= \frac{\sqrt{(-1)^2 + (1)^2}}{\sqrt{(1)^2 + (1)^2 + (1)^2}} \\
&= \frac{\sqrt{2}}{\sqrt{3}} \\
\therefore d &= \sqrt{\frac{2}{3}}
\end{aligned}
$$

(b) Given the parametric vector form of a line, $r = r_0 + t\underset{\sim}{v}$ where $t \in \mathbb{R}$, and a point P, we are required to show that the point R that is closest to P, is given by:

$$\overrightarrow{OR} = \overrightarrow{OQ} + \frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2}\underset{\sim}{v}$$

where Q is an arbitrary point on the line.

The following diagram will provide the necessary information to establish the proof that will follow.



From vector addition we have the result $\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR}$. Recognising that the required result has the formula for the vector projection of $\overrightarrow{QP}$ in the direction of $\underset{\sim}{v}$, we will examine the rays $\overrightarrow{QP}$ and $\overrightarrow{QR}$.

$$
\begin{aligned}
\cos\theta &= \frac{|\overrightarrow{QR}|}{|\overrightarrow{QP}|} \\
\therefore |\overrightarrow{QR}| &= |\overrightarrow{QP}| \cos\theta \ldots\ldots\ldots\ldots\ldots (1)
\end{aligned}
$$

Now considering the two rays $\overrightarrow{QP}$ and $\overrightarrow{QR}$, we will determine the value of their dot product, to gain a second equation with $\cos\theta$ as a term.

$$\overrightarrow{QP} \cdot \underset{\sim}{v} = |\overrightarrow{QP}||\underset{\sim}{v}| \cos\theta$$
$$\therefore \cos\theta = \frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\overrightarrow{QP}||\underset{\sim}{v}|} \dots\dots\dots\dots\dots (2)$$

Substituting equation (2) into equation (1), we get the following result:

$$|\overrightarrow{QR}| = |\overrightarrow{QP}| \left[\frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\overrightarrow{QP}||\underset{\sim}{v}|} \right]$$
$$\therefore |\overrightarrow{QR}| = \frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|}$$

We want the vector in the direction of $\underset{\sim}{v}$, yet our current result is only the length of $\overrightarrow{QR}$. In order to get the vector projection in the direction of $\underset{\sim}{v}$, we must multiply our current result for $|\overrightarrow{QR}|$ by the direction of $\underset{\sim}{v}$.

$$\therefore |\overrightarrow{QR}| = \frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|}$$
$$\therefore \overrightarrow{QR} = \left[\frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|} \right] \times [\hat{\underset{\sim}{v}}]$$
$$= \left[\frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|} \right] \times \left[\frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} \right]$$
$$\therefore \overrightarrow{QR} = \frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \underset{\sim}{v} \dots\dots\dots\dots\dots (3)$$

Now using the vector addition equation we derived, $\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR}$, and substituting in equation (3), we arrive at the required result.

$$\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR}$$
$$\therefore \overrightarrow{OR} = \overrightarrow{OQ} + \frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \underset{\sim}{v}$$

(c) We are now examining the unit hypercube in $\mathbb{R}^4$, and as a result, are unable to use the cross product, as it is only defined in $\mathbb{R}^3$. As a consequence, we must utilise the above formula that determines the distance between a line and a point, and is defined across $\mathbb{R}^n$, $\forall n \in \mathbb{N}$.

Similar to the unit cube in $\mathbb{R}^3$, we define each side emnating from the origin, $O(0, 0, 0, 0)$, as a vector, $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{d}$. These vectors correspond respectively to the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$, $\underset{\sim}{l}$. The four axes are x, y, z, z_1, where the fourth axis variable will be defined as z_1.

For this proof, we will examine the distance from, obviously the major diagonal of the hypercube in $\mathbb{R}^4$, and the corner defined by the point $P(1, 0, 0, 0)$. Now, in order to compute the distance to the major diagonal, we first must determine the direction of the major diagonal. In order to do this, we can visualise the major diagonal of the unit cube in $\mathbb{R}^3$ to be the diagonal on the bottom face of the unit hypercube in $\mathbb{R}^4$. Thus, by vector addition, we get the following result for the major diagonal of the unit hypercube:

$$(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) + \underset{\sim}{d} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}$$

Now, examining the formula for the distance between a point and a line, we will use the formula to find the point on the major diagonal of the unit hypercube that is closest to the point $P(1,0,0,0)$. Thus, we need the point Q, which will be defined for this proof as $Q(0,0,0,0)$. Furthermore, we need to determine the value of $\overrightarrow{QP}$.

$$\begin{aligned}
\overrightarrow{QP} &= P(1,0,0,0) - Q(0,0,0,0) \\
&= \underset{\sim}{i} - \underset{\sim}{0} \\
\therefore \overrightarrow{QP} &= \underset{\sim}{i}
\end{aligned}$$

Furthermore, the major diagonal must be converted into the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$, $\underset{\sim}{l}$.

$$\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{l}$$

Now using the formula for the distance between a point P and a line, we can determine the closest point on the line, R.

$$\begin{aligned}
\overrightarrow{OR} &= \overrightarrow{OQ} + \frac{\overrightarrow{QP} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2}\underset{\sim}{v} \\
&= \underset{\sim}{0} + \frac{(\underset{\sim}{i}) \cdot (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{l})}{|\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{l}|^2} \times (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{l}) \\
&= \frac{1+0+0+0}{\left(\sqrt{(1)^2 + (1)^2 + (1)^2 + (1)^2}\right)^2} \times (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{l}) \\
\therefore \overrightarrow{OR} &= \frac{1}{4}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{l})
\end{aligned}$$

This gives the position vector of the point R, the closest point on the major diagonal to the point P, one of the corners of the hypercube in $\mathbb{R}^4$. Thus R has the coordinates $R(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Now we will determine the distance between R and P.

$$\begin{aligned}
\overrightarrow{PR} &= R(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) - P(1,0,0,0) \\
&= \frac{1}{4}\underset{\sim}{i} + \frac{1}{4}\underset{\sim}{j} + \frac{1}{4}\underset{\sim}{k} + \frac{1}{4}\underset{\sim}{l} - \underset{\sim}{i} \\
\therefore \overrightarrow{PR} &= -\frac{3}{4}\underset{\sim}{i} + \frac{1}{4}\underset{\sim}{j} + \frac{1}{4}\underset{\sim}{k} + \frac{1}{4}\underset{\sim}{l} \\
\therefore |\overrightarrow{PR}| &= \sqrt{\left(-\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} \\
\therefore |\overrightarrow{PR}| &= \frac{\sqrt{3}}{2}
\end{aligned}$$