Limits and Calculus for Curves

Definition

For an interval $I \subset \mathbb{R}$ and curve $\mathbf{c} : I \to \mathbb{R}^n$ with

$$\mathbf{c}(t) = \Big(c_1(t), c_2(t), \ldots, c_n(t)\Big),$$

the functions $c_i : I \to \mathbb{R}$, $i = 1, 2, \ldots, n$ are called the components of $\mathbf{c}$.

Define limits, derivatives and integrals component by component.

Definition

- $\lim\limits_{t\to a} \mathbf{c}(t) = \Big(\lim\limits_{t\to a} c_1(t), \lim\limits_{t\to a} c_2(t), \ldots, \lim\limits_{t\to a} c_n(t)\Big)$
- $\dfrac{d\mathbf{c}(t)}{dt} = \dot{\mathbf{c}}(t) = \mathbf{c}'(t) = (c_1'(t), c_2'(t), \ldots, c_n'(t))$
- $\displaystyle\int_a^b \mathbf{c}(t)dt = \left(\int_a^b c_1(t)dt, \int_a^b c_2(t)dt, \ldots, \int_a^b c_n(t)dt\right)$

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Definition

A curve $\mathbf{c} : I \to R^n$ is

- continuous if its component functions are continuous.
- simple if it is continuous and has no multiple points (other than the end points if it is closed).
- smooth if its components are differentiable and their derivatives do not simultaneouly vanish.
- piecewise smooth if it is made up of a finite number of smooth curves.

A curve has an orientation — the direction of increasing t.

We will revisit continuity in the analysis section and give a different definition which we will show is equivalent.

Limits and Calculus for Curves

Example: $\mathbf{r} : [-2, 2] \to \mathbb{R}^2$ with $\mathbf{r}(t) = (t^2 + 1, t^3 + 1)$ is not smooth.

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> plot([t^2+1,t^3+1,t=-2..2], -2..4, -2..4);
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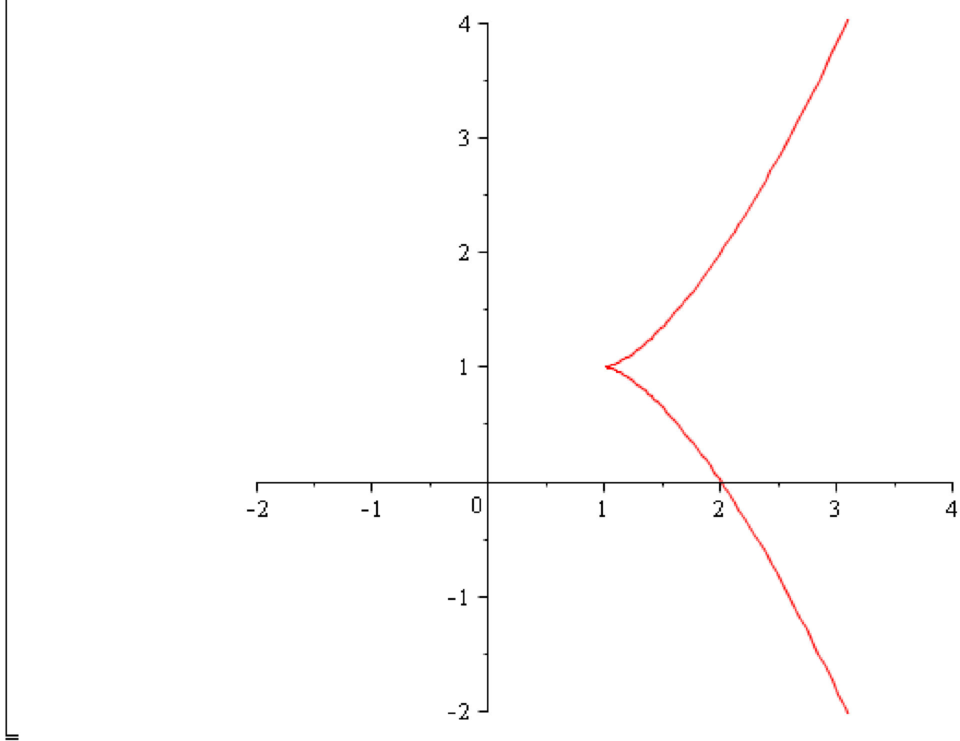


$$\mathbf{r}'(t) = (2t, 3t^2).$$

$$\Downarrow$$

$$\mathbf{r}'(0) = (0, 0)$$

Hence $\mathbf{r}$ is not smooth.

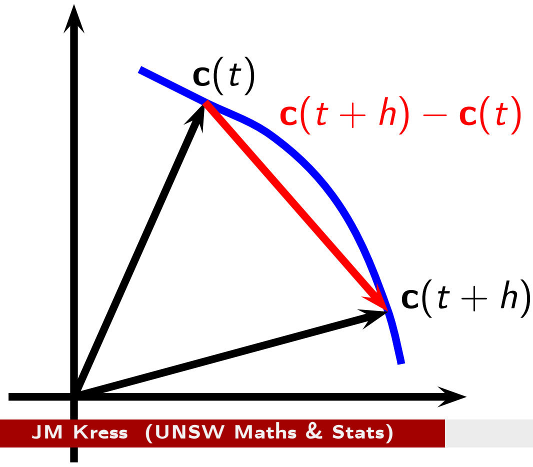
Differentiation Rules for Curves

Working component by component we can prove the following rules from their one variable counterparts.

$$\begin{aligned}
\frac{d}{dt}\Big(\mathbf{c}_1(t) + \mathbf{c}_2(t)\Big) &= \frac{d\mathbf{c}_1(t)}{dt} + \frac{d\mathbf{c}_2(t)}{dt} \\
\frac{d}{dt}\Big(\lambda \mathbf{c}(t)\Big) &= \lambda \frac{d\mathbf{c}(t)}{dt} \\
\frac{d}{dt}\Big(f(t)\mathbf{c}(t)\Big) &= \frac{df(t)}{dt}\mathbf{c}(t) + f(t)\frac{d\mathbf{c}(t)}{dt} \\
\frac{d}{dt}\Big(\mathbf{c}_1(t) \cdot \mathbf{c}_2(t)\Big) &= \frac{d\mathbf{c}_1(t)}{dt} \cdot \mathbf{c}_2(t) + \mathbf{c}_1(t) \cdot \frac{d\mathbf{c}_2(t)}{dt} \\
\frac{d}{dt}\Big(\mathbf{c}_1(t) \times \mathbf{c}_2(t)\Big) &= \frac{d\mathbf{c}_1(t)}{dt} \times \mathbf{c}_2(t) + \mathbf{c}_1(t) \times \frac{d\mathbf{c}_2(t)}{dt} \\
\frac{d}{dt}\Big(\mathbf{c}(f(t))\Big) &= \mathbf{c}'(f(t))\; f'(t)
\end{aligned}$$

Interpretation of the Derivative

$$
\begin{aligned}
\frac{d\mathbf{c}(t)}{dt} &= (c_1'(t), \ldots, c_n'(t)) \\
&= \left(\lim_{h \to 0} \frac{c_1(t+h) - c_1(t)}{h}, \ldots, \lim_{h \to 0} \frac{c_n(t+h) - c_n(t)}{h} \right) \\
&= \lim_{h \to 0} \left(\frac{c_1(t+h) - c_1(t)}{h}, \ldots, \frac{c_n(t+h) - c_n(t)}{h} \right) \\
&= \lim_{h \to 0} \frac{\mathbf{c}(t+h) - \mathbf{c}(t)}{h}
\end{aligned}
$$



As h gets smaller, the direction of $\mathbf{c}(t+h) - \mathbf{c}(t)$ approaches the direction of the tangent to the curve's image. If $\mathbf{c}'(t)$ exists and is non-zero, it is called the tangent vector to $\mathbf{c}$ at t, or the velocity of $\mathbf{c}$ at t. Ie, $\mathbf{v}(t) = \mathbf{c}'(t)$. The speed of $\mathbf{c}$ at t is $|\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)}$.
The second derivative $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{c}''(t)$ is called the acceleration.

Tangent Vector Example

Consider the curve $\mathbf{r} : I \to \mathbb{R}^3$ for an interval $I \subset \mathbb{R}$ given by

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{j} + \frac{\sqrt{5}}{2}\cos 2t\mathbf{k}.$$

a) Find the velocity and acceleration vectors.
b) Show that the velocity and acceleration vectors are perpendicular at $t = \frac{n\pi}{2}$, $n \in \mathbb{Z}$.
c) Find the length of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(2\pi)$.
[Recall: length $= \int_a^b ||\mathbf{r}'(t)||dt$.]
d) Find the unit tangent vector at $t = \dfrac{\pi}{6}$.
e) Sketch the curve and indicate the unit tangent vector found in (d).