THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Solutions Week 9** 2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Given the Taylor formula $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots\ldots + (-1)^{n-1}\dfrac{x^n}{n} + R_n(x)$, where $R_n(x) = \dfrac{(-1)^n x^{n+1}}{(n+1)(1+c)^{n+1}}$ for some c between 0 and x,

(a) find the Taylor polynomial of order $n+2$ for $x^2\ln(1+x)$ about the point 0,

(b) find the Taylor polynomial of order n for $\ln(1-x)$ about the point 0.

Questions for the tutorial

2. Find the Taylor polynomial $T_5(x)$ of order five about $x = 0$ for each of the following functions. Write down the remainder term $R_5(x)$ in each case, and estimate the size of the error if $T_5(1)$ is used as an approximation to $f(1)$.

(a) $f(x) = \sqrt{1+x}$ (b) $f(x) = \cosh x$

Solution

(a) Computing the derivatives of $f(x)$ we find that

$$\begin{aligned}
f(x) &= \sqrt{1+x} = (1+x)^{1/2}, & f(0) &= 1\\
f'(x) &= \tfrac{1}{2}(1+x)^{-1/2}, & f'(0) &= \tfrac{1}{2}\\
f''(x) &= -\tfrac{1}{2}.\tfrac{1}{2}(1+x)^{-3/2}, & f''(0) &= -\tfrac{1}{2^2}\\
f^{(3)}(x) &= \tfrac{1}{2}.\tfrac{1}{2}.\tfrac{3}{2}(1+x)^{-5/2}, & f^{(3)}(0) &= \tfrac{3}{2^3}\\
f^{(4)}(x) &= -\tfrac{1}{2}.\tfrac{1}{2}.\tfrac{3}{2}.\tfrac{5}{2}(1+x)^{-7/2}, & f^{(4)}(0) &= -\tfrac{3\times 5}{2^4}\\
f^{(5)}(x) &= \tfrac{1}{2}.\tfrac{1}{2}.\tfrac{3}{2}.\tfrac{5}{2}.\tfrac{7}{2}(1+x)^{-9/2}, & f^{(5)}(0) &= \tfrac{3.5.7}{2^5}
\end{aligned}$$

So the Taylor polynomial of $f(x)$ of order 5 about $x = 0$ is

$$T_5(x) = 1 + \frac{x}{2} - \frac{x^2}{2^2.2!} + \frac{3x^3}{2^3.3!} - \frac{15x^4}{2^4.4!} + \frac{105x^5}{2^5.5!}.$$

The remainder term is given by the formula

$$R_5(x) = \frac{f^{(6)}(c)}{6!}\,x^6$$

for some c between 0 and x. If we take $T_5(1)$ as an approximation to $f(1) = \sqrt{2}$, then

$$R_5(1) = \frac{f^{(6)}(c)}{6!}, \quad \text{where } 0 < c < 1.$$

Now $f^{(6)}(x) = -\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}.\frac{9}{2}(1+x)^{-11/2}$ and so

$$R_5(1) = -\frac{3.5.7.9}{2^6.6!}\,\frac{1}{(1+c)^{11/2}}.$$

As $0 < c < 1$, we see that $\dfrac{1}{(1+c)^{11/2}} < 1$, and hence

$$|R_5(1)| < \frac{3.5.7.9}{2^6.6!} \approx 0.02051.$$

Thus the error in the approximation will not exceed 0.02051.

(b) Note that $\dfrac{d}{dx}\cosh x = \sinh x$, and $\dfrac{d}{dx}\sinh x = \cosh x$. So

$$f^{(n)}(x) = \begin{cases} \sinh x, & \text{if } n \text{ is odd.} \\ \cosh x, & \text{if } n \text{ is even.} \end{cases}$$

Hence $f^{(n)}(0) = \sinh 0 = 0$ for $n = 1, 3, 5$, and $f^{(n)}(0) = \cosh 0 = 1$ for $n = 0, 2, 4$. The Taylor polynomial of order five about $x = 0$ is therefore the quartic polynomial

$$T_5(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}.$$

In this case, we have

$$R_5(x) = \frac{f^{(6)}(c)}{6!}\,x^6 = \frac{\cosh c}{6!}\,x^6,$$

for some c between 0 and x. Therefore

$$R_5(1) = \frac{\cosh c}{6!} = \frac{e^c + e^{-c}}{2 \times 6!},$$

where $0 < c < 1$. Using a crude but simple upper bound for $e^c + e^{-c}$, we have

$$\frac{e^c + e^{-c}}{2 \times 6!} < \frac{e + 1}{2 \times 6!} < \frac{4}{2 \times 6!}$$

on this interval. Hence

$$|R_5(1)| < \frac{4}{2 \times 6!} \approx 0.0056.$$

3. (a) Find the Taylor polynomial of order 4 about $x = 0$ for $\dfrac{1}{1+x}$.

(b) Find the Taylor polynomial of order 5 about $x = 0$ for $\ln(1+x)$.

(c) What relationship can you see between the two polynomials above? Why might you expect such a relationship?

Solution

(a) We calculate the derivatives of $f(x) = \dfrac{1}{1+x}$.

$$\begin{aligned} f(x) &= (1+x)^{-1}, & f(0) &= 1, \\ f'(x) &= -(1+x)^{-2}, & f'(0) &= -1, \\ f''(x) &= 2(1+x)^{-3}, & f''(0) &= 2, \\ f'''(x) &= -6(1+x)^{-4}, & f'''(0) &= -6, \\ f^{(4)}(x) &= 24(1+x)^{-5}, & f^{(4)}(0) &= 24. \end{aligned}$$

Therefore, the Taylor polynomial of $f(x)$ of order 4 about $x = 0$ is

$$T(x) = 1 - x + x^2 - x^3 + x^4.$$

(b) We again calculate derivatives, this time of $g(x) = \ln(1 + x)$.

$$\begin{aligned} g(x) &= \ln(1+x), & g(0) &= 0, \\ g'(x) &= (1+x)^{-1}, & g'(0) &= 1, \\ g''(x) &= -(1+x)^{-2}, & g''(0) &= -1, \\ g'''(x) &= 2(1+x)^{-3}, & g'''(0) &= 2, \\ g^{(4)}(x) &= -6(1+x)^{-4}, & g^{(4)}(0) &= -6, \\ g^{(5)}(x) &= 24(1+x)^{-5}, & g^{(5)}(0) &= 24. \end{aligned}$$

Therefore, the Taylor polynomial for $g(x)$ of order 5 about $x = 0$ is

$$S(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.$$

(c) Notice that $T(x)$ is the derivative of $S(x)$. This happens because $f(x) = g'(x)$.

4. (a) Find the Taylor polynomials of orders 2 and 4 about $x = \frac{\pi}{2}$, for $f(x) = \cos x$. Use these polynomials to estimate $\cos\frac{4\pi}{7}$ and $\cos\frac{5\pi}{7}$. Compare your results with those obtained from a calculator.

(b) Use Taylor polynomials of order 3 about $x = \dfrac{\pi}{2}$ and $x = \pi$ to estimate $\sin 3$. Which is the better approximation?

Solution

(a) We need to calculate the first four derivatives of $f(x)$ and evaluate them at $\dfrac{\pi}{2}$. They are

$$\begin{aligned} f(x) &= \cos x, & f(\tfrac{\pi}{2}) &= 0, \\ f'(x) &= -\sin x, & f'(\tfrac{\pi}{2}) &= -1, \\ f''(x) &= -\cos x, & f''(\tfrac{\pi}{2}) &= 0, \\ f'''(x) &= \sin x, & f'''(\tfrac{\pi}{2}) &= 1, \\ f^{(4)}(x) &= \cos x, & f^{(4)}(\tfrac{\pi}{2}) &= 0. \end{aligned}$$

Therefore, if $T_2(x)$ and $T_4(x)$ are the Taylor polynomials of orders 2 and 4 respectively then $T_2(x) = -(x - \frac{\pi}{2})$ and $T_4(x) = -(x - \frac{\pi}{2}) + \frac{1}{6}(x - \frac{\pi}{2})^3$. Using the Taylor polynomials to estimate $\cos\frac{4\pi}{7}$ and $\cos\frac{5\pi}{7}$ we get (to six decimal places)

$$T_2(\tfrac{4\pi}{7}) = -\tfrac{\pi}{14} = -0.224399$$

and

$$T_2(\tfrac{5\pi}{7}) = -\tfrac{3\pi}{14} = -0.673198.$$

While

$$T_4(\tfrac{4\pi}{7}) = -\tfrac{\pi}{14} + \tfrac{1}{6}(\tfrac{\pi}{14})^3 = -0.222516$$

and

$$T_4(\tfrac{5\pi}{7}) = -\tfrac{3\pi}{14} + \tfrac{1}{6}(\tfrac{3\pi}{14})^3 = -0.6223499.$$

However, using a calculator we find that

$$\cos(\tfrac{4\pi}{7}) = -\tfrac{\pi}{14} = -0.222521$$

and

$$\cos(\tfrac{5\pi}{7}) = -\tfrac{3\pi}{14} = -0.623490.$$

Notice that the Taylor polynomial $T_4(x)$ gives a better approximation to $\cos x$ in both cases.

(b) Expanding about $\dfrac{\pi}{2}$ yields

$$\sin 3 \approx 1 - \frac{1}{2}(3 - \frac{\pi}{2})^2 \approx -0.0213\ldots$$

Expanding about π yields

$$\sin 3 \approx -(3 - \pi) + \frac{(3-\pi)^3}{6} = 0.141\ldots$$

The second approximation is better, which is not surprising since π is much closer to 3 than is $\pi/2$.

5. Find the Taylor polynomial of order 2 for $f(x) = \tan^{-1} x$ about 0, and write down the remainder term. Using this information, show that $\displaystyle\int_0^{0.1} \tan^{-1} x\,dx$ lies between 0.00499 and 0.00501.

Solution

We have

$$f'(x) = \frac{1}{1+x^2}, \quad f''(x) = -\frac{2x}{(1+x^2)^2}, \quad f'''(x) = \frac{6x^2-2}{(1+x^2)^3}.$$

Therefore $T_2(x) = 0 + 1x + 0\dfrac{x^2}{2!} = x$ and $R_2(x) = \left(\dfrac{6c^2-2}{(1+c^2)^3}\right)\dfrac{x^3}{3!}$, where c is between 0 and x. Taylor's formula gives

$$\tan^{-1} x = x + R_2(x),$$

and so

$$\int_0^{0.1} \tan^{-1} x\,dx = \int_0^{0.1} x\,dx + \int_0^{0.1} R_2(x)\,dx,$$

or

$$\int_0^{0.1} \tan^{-1} x\,dx = 0.005 + \int_0^{0.1} R_2(x)\,dx.$$

We now estimate the size of $\displaystyle\int_0^{0.1} R_2(x)\,dx$. First, observe that since x runs from 0 to 0.1 in this problem, we must have $0 < c < 0.1$ as c is between 0 and x. Then

$$|R_2(x)| = \left|\frac{6c^2-2}{(1+c^2)^3}\right| \frac{x^3}{3!} \le |6c^2 - 2| \frac{x^3}{3!} \le 2\frac{x^3}{3!} = \frac{x^3}{3}.$$

Now since $\left|\int_0^{0.1} R_2(x)\,dx\right| \le \int_0^{0.1} |R_2(x)|\,dx$, we obtain

$$\left|\int_0^{0.1} R_2(x)\,dx\right| \le \int_0^{0.1} \frac{x^3}{3}\,dx = \left[\frac{x^4}{12}\right]_0^{0.1} = \frac{10^{-4}}{12} < 0.00001.$$

We conclude that $\int_0^{0.1} \tan^{-1} x\,dx$ lies between $0.005 - 0.00001$ and $0.005 + 0.00001$, that is, between 0.00499 and 0.00501.

6. You are given that the Taylor polynomial $T_3(x)$ of order 3 for $\sqrt{1+x}$, about 0, is $T_3(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$, with $R_3(x) = -\frac{15}{16}(1+c)^{-\frac{7}{2}}\frac{x^4}{4!}$, for some c between 0 and x.

(a) Write down the Taylor polynomial of order 9 about 0 for $\sqrt{1+x^3}$.

(b) Use your answer to the previous part to find an approximation to the integral $\int_0^1 \sqrt{1+x^3}\,dx$. Find an upper bound for the error involved.

Solution

(a) By a theorem proved in lectures, the Taylor polynomial of order 9 about 0 for $\sqrt{1+x^3}$ is given by replacing x by x^3 in the Taylor polynomial of order 3 for $\sqrt{1+x}$.
The Taylor polynomial of order 9 about 0 for $\sqrt{1+x^3}$ is therefore

$$\sqrt{1+x^3} = 1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16}.$$

(b) Our approximation to $\int_0^1 \sqrt{1+x^3}\,dx$ is (to 6 decimal places)

$$\int_0^1 (1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16})\,dx \approx 1.113393.$$

We now find a bound for the error. Observe that when $0 < x \le 1$, we have $0 < c < 1$ and so $0 < (1+c)^{-\frac{7}{2}} < 1$. So

$$|R_3(x)| = \frac{15}{16}(1+c)^{-\frac{7}{2}}\frac{x^4}{4!} < \frac{15x^4}{16 \times 4!}$$

and thus

$$|R_3(x^3)| < \frac{15x^{12}}{16 \times 4!}.$$

Therefore

$$\left|\int_0^1 R_3(x^3)\,dx\right| \le \int_0^1 |R_3(x^3)|\,dx \le \int_0^1 \frac{15x^{12}}{16 \times 4!}\,dx = \frac{5}{1664} \approx 0.003005$$

gives an upper bound for the error.
Hence $\int_0^1 \sqrt{1+x^3}\,dx \approx 1.113 \pm .003$ (to 3 decimal places).

7. Use the Taylor polynomial of order 3 for $\sinh x$ about 0 to estimate $\int_0^1 \sinh x\,dx$. Determine the accuracy of your estimate and compare it to the value of the integral found using your calculator (the integral equals $\cosh 1 - \cosh 0 = \frac{e+e^{-1}}{2} - 1$). What difference would it make to the accuracy if we had used the Taylor polynomial of order 4?

Solution

The Taylor polynomial of order 3 for $\sinh x$ about 0 is

$$x + \frac{x^3}{6}.$$

The remainder term is $R_3(x) = (\sinh c)\dfrac{x^4}{4!}$ for some number c between 0 and x. The Taylor formula gives us

$$\sinh x = x + \frac{x^3}{6} + R_3(x).$$

Integrating both sides between 0 and 1 gives

$$\int_0^1 \sinh x\,dx = \int_0^1 (x + \tfrac{x^3}{6})\,dx + \int_0^1 R_3(x)\,dx = \frac{13}{24} + \int_0^1 R_3(x)\,dx.$$

Observe that when $0 < x \le 1$, we have $0 < c < 1$, and so

$$0 < \sinh c < \sinh 1 = \frac{e - e^{-1}}{2} < \frac{e}{2} < \frac{3}{2}.$$

(We have used a crude but simple upper bound of 3 for e.) So for $0 < x \le 1$,

$$0 < R_3(x) < \frac{3}{2} \times \frac{x^4}{4!} = \frac{x^4}{16}.$$

Thus

$$0 \le \int_0^1 R_3(x)\,dx \le \int_0^1 \frac{x^4}{16}\,dx = \frac{1}{80} = 0.0125.$$

Putting all this together tells us that the required integral lies in the interval $(\frac{13}{24}, \frac{13}{24} + \frac{1}{80})$. That is, $\displaystyle\int_0^1 \sinh x\,dx$ is in the interval $(0.54166, 0.55416)$.

The Taylor polynomial of order 4 for $f(x) = \sinh x$ about $x = 0$ has degree 3 and is the same polynomial as the one we used above. However the remainder is now $R_4(x) = \cosh d\,\dfrac{x^5}{5!}$, where d is a number between 0 and x. Noting that $0 < d < 1$ (as x runs from 0 to 1 in the integral), we have

$$\cosh d = \frac{e^d + e^{-d}}{2} < \frac{e+1}{2} < 2.$$

So $0 < R_4(x) < \dfrac{2x^5}{5!}$ and

$$0 \le \int_0^1 R_4(x)\,dx \le \int_0^1 \frac{2x^5}{5!}\,dx = \frac{1}{360} \approx 0.00277.$$

Now we can be sure that $\displaystyle\int_0^1 \sinh x\,dx$ lies in the interval $(0.54166, 0.54444)$. This is a better result than that obtained using $R_3(x)$. Note that the exact value of the integral is 0.54308 to five decimal places.

Extra Questions

8. (a) The hyperbolic tan function is defined by $\tanh x = \dfrac{\sinh x}{\cosh x}$. It is a bijection from $\mathbb{R}$ to $(-1, 1)$. Find a formula for $\tanh^{-1} x$ in terms of natural logarithms and use it to show that $\ln 2 = 2\tanh^{-1} \frac{1}{3}$.

(b) Find the Taylor polynomial of order $2n$ for $\tanh^{-1} x$ about the point 0 and write down its remainder term. (*Hint*: use the Taylor formulas for $\ln(1 \pm x)$ given in Question 1.)

(c) Use the $n = 8$ case of the previous part to estimate $\ln 2$. Show that the error is less than 5×10^{-7}.

Solution

(a) Write

$$y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

After rearrangement, we obtain $e^{2x} = \dfrac{1+y}{1-y}$ from which we see that $x = \dfrac{1}{2} \ln\left(\dfrac{1+y}{1-y}\right)$.

Therefore $\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$.

When $x = \frac{1}{3}$, we have $2\tanh^{-1} \frac{1}{3} = \ln\left(\dfrac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \ln 2$.

(b) First, note that $\tanh^{-1}(x) = \dfrac{1}{2}(\ln(1+x) - \ln(1-x))$. From Question 1 we replace n by $2n$ to obtain

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots + \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} + \frac{x^{2n+1}}{(2n+1)(1+c)^{2n+1}},$$

where c is between 0 and x, and

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots - \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} - \frac{x^{2n+1}}{(2n+1)(1+c')^{2n+1}},$$

where c' is between 0 and $-x$. Subtracting these two expressions gives

$$\begin{aligned}
&\ln(1+x) - \ln(1-x) \\
&= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots + \frac{x^{2n-1}}{2n-1}\right) + \frac{x^{2n+1}}{(2n+1)}\left(\frac{1}{(1+c)^{2n+1}} + \frac{1}{(1+c')^{2n+1}}\right).
\end{aligned}$$

Therefore the Taylor polynomial of order $2n$ for $\tanh^{-1}(x)$ about 0 is

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots + \frac{x^{2n-1}}{2n-1},$$

a polynomial of degree $2n-1$. The remainder term is

$$\frac{x^{2n+1}}{2(2n+1)}\left(\frac{1}{(1+c)^{2n+1}} + \frac{1}{(1+c')^{2n+1}}\right),$$

where c is between 0 and x and c' is between 0 and $-x$.

(c) Setting $n = 8$ and $x = \frac{1}{3}$ in the previous part, we estimate $\ln 2 = 2\tanh^{-1} \frac{1}{3}$ as

$$2\left(\frac{1}{1 \times 3^1} + \frac{1}{3 \times 3^3} + \frac{1}{5 \times 3^5} + \cdots + \frac{1}{15 \times 3^{15}}\right).$$

The error in this estimate is less than or equal to

$$\frac{(\frac{1}{3})^{17}}{17}\left(\left|\frac{1}{(1+c)^{17}}\right| + \left|\frac{1}{(1+c')^{17}}\right|\right),$$

where c is between 0 and $\frac{1}{3}$ and c' is between 0 and $-\frac{1}{3}$. Now clearly $\dfrac{1}{(1+c)^{17}} < 1$.

Also, as $-\frac{1}{3} < c' < 0$, it is easy to show that $\dfrac{1}{(1+c')^{17}} < \left(\dfrac{3}{2}\right)^{17}$. Thus the error is less than or equal to

$$\frac{(\frac{1}{3})^{17}}{17}\left(1+\left(\frac{3}{2}\right)^{17}\right)=\frac{1}{3^{17}\times 17}+\frac{1}{2^{17}\times 17}<5\times 10^{-7}.$$

Note that this is a very much smaller error than the error associated with using the Taylor polynomial of order 16 for $\ln(1+x)$ with $x=1$, to calculate $\ln 2$.

9. Consider the function given by

$$f(x)=\begin{cases} e^{-1/x^2} & x\neq 0,\\ 0 & x=0.\end{cases}$$

Show that f is differentiable and that $f'(0)=0$. Then show that f' is differentiable and that $f''(0)=0$. In fact, it turns out that f is differentiable any number of times and its derivative at zero is always zero! This means that its Taylor polynomial about 0 of order n, for any n, is the zero polynomial. This function is "all remainder".

Solution

We give the calculation of $f'(0)$ only. Now from the definition of the derivative as a limit $f'(0)=\lim\limits_{x\to 0}\dfrac{e^{-1/x^2}}{x}$ (if this limit exists).

Using l'Hôpital's Rule, we can show that

$$\lim_{x\to\infty}\frac{e^{x^2}}{x}=\infty.$$

Replace x by $\frac{1}{x}$ and let $x\to 0^+$. This gives

$$\lim_{x\to 0^+}\frac{e^{1/x^2}}{1/x}=\infty,$$

that is,

$$\lim_{x\to 0^+} xe^{1/x^2}=\infty,$$

and hence

$$\lim_{x\to 0^+}\frac{1}{xe^{1/x^2}}=0.$$

This can be rearranged to give

$$\lim_{x\to 0^+}\frac{e^{-1/x^2}}{x}=0.$$

Now replace x by $-x$ in the above limit. We obtain

$$\lim_{x\to 0^-}\frac{e^{-1/(-x)^2}}{-x}=0$$

and so

$$\lim_{x\to 0^-}\frac{e^{-1/x^2}}{x}=0.$$

Therefore $f'(0)=\lim\limits_{x\to 0}\dfrac{e^{-1/x^2}}{x}$ exists, f is differentiable at 0 and $f'(0)=0$.
The proof that f' is differentiable and that $f''(0)=0$ is similar to the above.

Solution to Question 1

(a) We multiply the Taylor formula for $\ln(1+x)$ by x^2 to obtain

$$x^2\ln(1+x) = x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \ldots\ldots + (-1)^{n-1}\frac{x^{n+2}}{n} + \frac{(-1)^n x^{n+3}}{(n+1)(1+c)^{n+1}}.$$

This equation shows that the polynomial $T(x) = x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \ldots\ldots + (-1)^{n-1}\frac{x^{n+2}}{n}$ of degree $n+2$ has the property that

$$\lim_{x\to 0}\frac{x^2\ln(1+x) - T(x)}{x^{n+2}} = 0,$$

so it must be the Taylor polynomial of order $n+2$ about 0, for $x^2\ln(1+x)$.

(b) We replace x by $-x$ in the formula for $\ln(1+x)$:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots - \frac{x^n}{n} - \frac{x^{n+1}}{(n+1)(1+c)^{n+1}},$$

for some c between 0 and $-x$. By similar reasoning to part (a), $-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots - \frac{x^n}{n!}$ must be the Taylor polynomial of order n about 0, for $\ln(1-x)$.