1 Curves and Surfaces

[M] – Maple/Gnuplot; [A] – additional/optional problems; [H] – harder problems.

1.1 Curves in $\mathbb{R}^n$

1: Sketch the curves $(x, y) = (t, t)$ and $(x, y) = (t^2, t + 2)$ for $t \in \mathbb{R}$, and find the two points where they intersect.

2: Sketch the projections of the following curves onto the plane $z = 0$ and onto the plane $y = 0$.

i) $(x, y, z) = (\cos 2t, \sin 2t, \sin t)$, $t \in \mathbb{R}$.

ii) $(x, y, z) = (\cos t, \sin t, \sin 3t)$, $t \in \mathbb{R}$. (Rotate the image until it looks like the ABC logo).

[M] **3**: Sketch the curves in Q2 by the command `spacecurve` in Maple; `plot3` in Matlab; or `splot` in Gnuplot. Rotate the image to see what the curve looks like from different viewpoints.

4: Find the unit tangent vector to the parametrised curve $\boldsymbol{r}(t)$ at $t = a$ and write down a parametric equation for the tangent line to the curve at a.

i) $\boldsymbol{r}(t) = 3 \cos t\, \boldsymbol{i} + 3 \sin t\, \boldsymbol{j} + 4t\boldsymbol{k}$, $a = \pi/4$

ii) $\boldsymbol{r}(t) = t\, \boldsymbol{i} + t^2\, \boldsymbol{j} + t^3\, \boldsymbol{k}$, $a = 1$.

5: Consider the two curves given in parametric form by $\boldsymbol{r}(t) = (t^2 - t, t^2 + t)$ and $\boldsymbol{r}(t) = (t + t^2, t - t^2)$ for $t \in \mathbb{R}$.

i) Find the *two* points of intersection of the curves.

ii) Find the angle between the two curves at each point of intersection.

iii) Find all points on the curves where the tangent is parallel to $\boldsymbol{i}$.

iv) Find all points on the curves where the tangent is parallel to $\boldsymbol{j}$.

v) For $-2 \le t \le 2$, sketch both curves on the same diagram. Show clearly all the points and angles you have found.

6: Sketch the curve given parametrically by $(x, y) = (t^3, t^5)$, $t \in \mathbb{R}$. Show that this parametrisation does not give a tangent vector for the curve at $(0, 0)$. Find a parametrisation of this curve which does give a tangent vector for the curve at $(0, 0)$.

[A] **7**: Suppose that $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^3$, $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^3$ and $\lambda : \mathbb{R} \to \mathbb{R}$. Prove that

i) $\dfrac{d}{dt}(\lambda \boldsymbol{f}) = \lambda \dfrac{d\boldsymbol{f}}{dt} + \dfrac{d\lambda}{dt}\boldsymbol{f}$.

ii) $\dfrac{d}{dt}(\boldsymbol{f} \cdot \boldsymbol{g}) = \dfrac{d\boldsymbol{f}}{dt} \cdot \boldsymbol{g} + \boldsymbol{f} \cdot \dfrac{d\boldsymbol{g}}{dt}$.

iii) $\dfrac{d}{dt}(\boldsymbol{f} \times \boldsymbol{g}) = \dfrac{d\boldsymbol{f}}{dt} \times \boldsymbol{g} + \boldsymbol{f} \times \dfrac{d\boldsymbol{g}}{dt}$.

8: Suppose $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are three differentiable functions from $\mathbb{R}$ to $\mathbb{R}^3$ such that for every $t \in \mathbb{R}$ the vectors $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$, $\boldsymbol{w}(t)$ form an orthonormal basis in $\mathbb{R}^3$.

i) Prove that $\boldsymbol{u}'(t) \perp \boldsymbol{u}(t)$ and $\boldsymbol{u}'(t) \cdot \boldsymbol{v}(t) = -\boldsymbol{u}(t) \cdot \boldsymbol{v}'(t)$ for all t.

ii) Suppose that $\boldsymbol{r}(t) = x(t)\boldsymbol{u}(t) + y(t)\boldsymbol{v}(t) + z(t)\boldsymbol{w}(t)$ for some three functions x, y, z from $\mathbb{R} \to \mathbb{R}$. Show that $x(t) = \boldsymbol{r}(t) \cdot \boldsymbol{u}(t)$ and $y(t) = \boldsymbol{r}(t) \cdot \boldsymbol{v}(t)$ and $z(t) = \boldsymbol{r}(t) \cdot \boldsymbol{w}(t)$ for all t.

9: A particle moves in a plane such that its position at time t is given by $\boldsymbol{r}(t) = (3t^2, t^3 - 9t)$.

i) Find all positions at which the velocity of the particle is perpendicular to its acceleration.

ii) Show that there are no positions where the velocity of the particle is parallel to its acceleration.

10: i) Show that for a particle moving with velocity $\boldsymbol{v}(t)$, if $\boldsymbol{v}(t) \cdot \boldsymbol{v}'(t) = 0$ for all t then the speed v is constant.

ii) A particle of mass m with position vector $\boldsymbol{r}(t)$ at time t is acted on by a total force

$$\boldsymbol{F}(t) = \lambda \boldsymbol{r}(t) \times \boldsymbol{v}(t),$$

where λ is a constant and $\boldsymbol{v}(t)$ is the velocity of the particle. Show that the speed v of the particle is constant. (Note that Newton's second law of motion in its vector form is $\boldsymbol{F} = m\boldsymbol{a}$.)

11: At time t a particle is at position $\boldsymbol{r}(t)$.

i) Show that

$$\boldsymbol{r} \cdot \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}\left(\frac{1}{2}r^2\right),$$

where r is the distance of the particle from the origin.

ii) Show $\dfrac{d(v^2)}{dt} = 2\dfrac{d^2\boldsymbol{r}}{dt^2} \cdot \dfrac{d\boldsymbol{r}}{dt}$

1.2 Surfaces in $\mathbb{R}^3$

[M] **12**: Use Maple/Gnuplot to plot the following functions $f : \mathbb{R}^2 \to \mathbb{R}$.

i) $f(x, y) = \cos(x + y)$.

ii) $f(x, y) = \sin(\sqrt{x^2 + y^2})$.

iii) $f(x, y) = x^3 - x^2 y$.

iv) $f(x, y) = x^4 - 2x^2 y$.

13: i) Sketch level curves $f = \pm 1$ for the functions (i) and (ii) of Q12.

ii) Sketch the section of the function (i) of Q12 by the planes $x = \pm\frac{\pi}{4}; \pm\frac{\pi}{2}$.

iii) Sketch the section of function (ii) of Q12 by the plane $y = x \tan\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

[M] **14**: Sketch the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = y^3 - y - 2x^2$. Use the `plot3d` command in Maple (or `surf` in Matlab; or `splot` of Gnuplot) to sketch the graph for $|x| \le 1.2$, $|y| \le 1.5$. Use the Maple command `contourplot` (or the Matlab command `contour`; or `splot` with `set contour` in Gnuplot, see `help contour` in Gnuplot) to look at some of the contours of f.

[M] **15**: Apply the Maple command `contourplot` (or the Matlab command `contour`; or `splot` with `set contour` in Gnuplot, see `help contour` in Gnuplot) to the function $f(x, y) = x/(1 + x^2 + y^2)$ for $|x| \le 3$, $|y| \le 2$ and then use these contours to sketch the surface. (Note that the contours in the Maple sketch are not labelled with the corresponding values of f, but by default their colour shades from yellow to red as the value of f increases.) Use `plot3d` (or `splot` of Gnuplot) to check your sketch.

[M] **16**: On separate diagrams, sketch the surfaces in $\mathbb{R}^3$ defined by the following equations:

i) $z = x^2 + y^2$;

ii) $2z^2 = x^2 + y^2$;

iii) $x^2 + y^2 + z^2 = 9$;

iv) $x^2 + y^2 = 4$;

v) $x^2 + y^2 - z^2 = 1$;

vi) $x^2 - y^2 - z^2 = 1$.

17: For each function of Q16, sketch the sections of the graph of the function by the planes $x = \alpha$, $y = \alpha$ and $z = \alpha$, where $\alpha = -1, 0, 1$.

18: Consider the region above the cone $z^2 = x^2 + y^2$, $z \ge 0$, and inside the sphere $x^2 + y^2 + z^2 = 2az$, with $a > 0$.

i) Sketch the section of this region by the plane $x = 0$.

ii) Describe the curve of intersection of these surfaces.

iii) What is the projection of the region on the x, y plane?

19: i) Parametrise the curve of intersection of two cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

ii) What is the curve of intersection.

iii) Find the projection of the curve of intersection onto the plane $z = 0$.

20: Let

$$S = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - 1)^2 = 1 \}$$

and

$$T = \{ (x, y, z) \in \mathbb{R}^3 : (z + 1)^2/4 = x^2 + y^2, z \ge -1 \}.$$

i) Find the z-coordinates of the points of intersection of S and T and sketch the projection into the xy-plane of the curves of intersection.

ii) Sketch the section of S and T by the plane $x = 0$ on the same diagram.

iii) For what values of a do the surfaces $x^2 + y^2 + (z - 1)^2 = 1$ and $a(z + 1)^2 = x^2 + y^2$ $(z \ge -1)$ not intersect?

21: Find the projection into the xy-plane of the curve of intersection of the surfaces $2z = x^2 - y^2 + 2x$ and $3z = 4x^2 + y^2 - 2x$ and express its equation in polar co-ordinates.

[M] **22**: Sketch the surfaces given parametrically as follows and use `plot3d` in Maple; or `splot` with `set parametric` in Gnuplot, to check your answers.

i) $(x, y, z) = (\cos u \sin v, \sin u \sin v, \cos v)$, $0 \le u \le 2\pi$, $0 \le v \le \pi/2$.

ii) $(x, y, z) = (\cos u \cosh v,\ \sin u \cosh v,\ \sinh v)$, $0 \le u \le 2\pi$, $v \in \mathbb{R}$.

iii)*

$$\left.\begin{aligned} x &= \big(2 + v\sin(u/2)\big)\cos u \\ y &= \big(2 + v\sin(u/2)\big)\sin u \\ z &= v\cos(u/2) \end{aligned}\right\}$$

for $0 \le u \le 2\pi$, $-1 \le v \le 1$.

[Hints: Let (r, θ) be polar coordinates in the xy-plane, so that points in $\mathbb{R}^3$ are described by cylindrical coordinates (r, θ, z). Show that the intersection of the surface with the half-plane $\theta = u_0$ is the curve $(r, \theta, z) = \big(2 + v\sin(u_0/2), u_0, v\cos(u_0/2)\big)$, $-1 \le v \le 1$. Verify that this is a line segment and work out how its position changes as u_0 varies from 0 to 2π.]

Answers to problems

A1: $(1,1)$ and $(4,4)$.

A3: webnotes[1] **A4**: i) $1/5\left(-3/\sqrt{2}, 3/\sqrt{2}, 4\right)$, $\left(3/\sqrt{2}, 3/\sqrt{2}, \pi\right) + \lambda\left(-3/\sqrt{2}, 3/\sqrt{2}, 4\right)$. ii) $(1,2,3)/\sqrt{14}$, $(1,1,1) + \lambda(1,2,3)$. **A5**: i) ii) $(0,0)$, $\pi/2$; $(2,0)$, $\cos^{-1}(.8)$. iii) $(3/4, -1/4)$, $(3/4, 1/4)$ iv) $(-1/4, 3/4)$, $(-1/4, -3/4)$. **A7**: webnotes[2] **A9**: i) $(3,-8)$, $(0,0)$, $(3,8)$ **A10**: i) Use $v(t)^2 = \|\boldsymbol{v}(t)\|^2 = \boldsymbol{v}(t)\cdot\boldsymbol{v}(t)$. ii) $\boldsymbol{F}(t)\cdot\boldsymbol{v}(t) = m\boldsymbol{v}'(t)\cdot\boldsymbol{v}(t) = (\lambda\boldsymbol{r}(t)\times\boldsymbol{v}(t))\cdot\boldsymbol{v}(t) = 0$. **A17**: i) paraboloid ii) cone iii) sphere iv) cylinder iv) hyperboloid v) hyperboloid of 2 sheets. **A18**: circle $z = a$, $x^2 + y^2 = a^2$, disc $x^2 + y^2 \le a^2$ **A19**: $\boldsymbol{r}(t) = (\cos t, \sin t, \pm\sin t)$; ellipse; circle $\boldsymbol{r}(t) = (\cos t, \sin t, 0)$. **A20**: iii) $a > 1/3$ **A21**: $r = 2\cos\theta$, $0 \le \theta < 2\pi$

[1] http://web.maths.unsw.edu.au/~potapov/2111_2015/Week-1-Lecture-1.html#g_t_0060_0060ABC_0027_0027-curve-in-gnuplot-_0028video_0029

[2] http://web.maths.unsw.edu.au/~potapov/2111_2015/Algebraic-properties-of-derivative-of-curve.html