(A)

MATH1902 2007 exam solutions 9/6/2010

Q1/ (a) π has equation $3x-y-2z=-3$, so normal vector $\underline{n} = 3\underline{i}-\underline{j}-2\underline{k}$. The line ℓ passing through $A(1,0,-4)$ perpendicular to π has direction vector $\underline{n}$, so has vector equation $\underline{r} = \underline{i}-4\underline{k}+t(3\underline{i}-\underline{j}-2\underline{k})$, so parametric equations

$$\boxed{\left.\begin{aligned} x &= 1+3t \\ y &= -t \\ z &= -4-2t \end{aligned}\right\} t\in\mathbb{R}}$$

(b) ℓ and π intersect at B where

$$3(1+3t)-(-t)-2(-4-2t)=-3,$$

i.e. $3+9t+t+8+4t=-3$,

i.e. $14t=-14$, so $t=-1$, yielding

$$\boxed{B=(-2,1,-2)}$$

(c) distance from A to π is

$$|\overrightarrow{AB}| = |-3\underline{i}+\underline{j}+2\underline{k}| = \sqrt{9+1+4}$$

$$= \boxed{\sqrt{14}}$$

Q1/ (d) π' contains $A(1,0,-4)$ and is parallel to Π, so also has normal vector $\underset{\sim}{n} = 3\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$, so has Cartesian equation

$$3x - y - 2z = 3(1) - 0 - 2(-4) = 11$$

i.e. $\boxed{3x - y - 2z = 11}$

(e) If the plane $3x - y + cz = -3$ is perpendicular to Π then its normal $\underset{\sim}{v} = 3\underset{\sim}{i} - \underset{\sim}{j} + c\underset{\sim}{k}$ is perpendicular to $\underset{\sim}{n} = 3\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$, so that

$0 = \underset{\sim}{v} \cdot \underset{\sim}{n} = 9 + 1 - 2c$, so $2c = 10$,

yielding $\boxed{c = 5}$

Q2/ (a) (i) The line m is given by $\frac{x}{2} = \frac{y+7}{3} = \frac{z-6}{-11}$ so has direction vector $\underset{\sim}{v} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 11\underset{\sim}{k}$, and the plane p has equation $5x - 2y + z = 9$, so has normal vector $\underset{\sim}{n} = 5\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$. Observe that $\underset{\sim}{v} \cdot \underset{\sim}{n} = 10 - 6 - 11 = -7 \neq 0$, so m is not parallel to p.

(c)

Q2/ (a) (ii)

$$\underset{\sim}{v} \times \underset{\sim}{n} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 3 & -11 \\ 5 & -2 & 1 \end{vmatrix} = -19\underset{\sim}{i} - 57\underset{\sim}{j} - 19\underset{\sim}{k}$$

$$= -19(\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k})$$

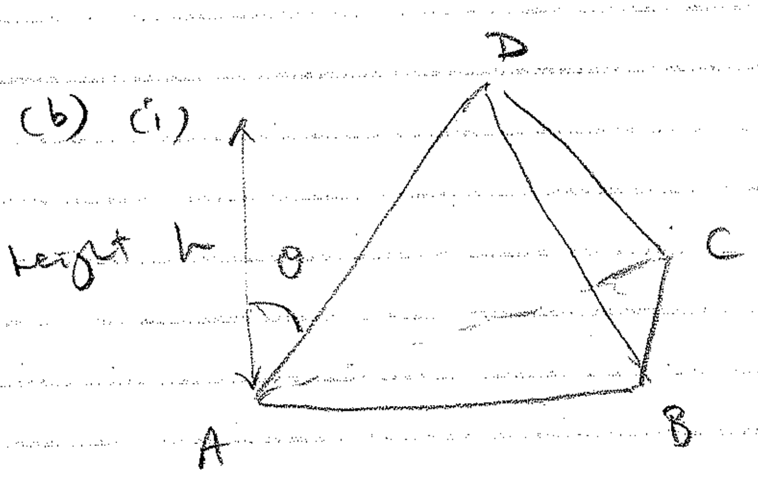
so, for example, $\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ is perpendicular to m and parallel to p.

(iii) We want a plane perpendicular to p and containing m, so the vector $\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ from (ii) will be normal. A point on m is $(0, -7, 6)$ (from setting the numerators $= 0$ in the equations for m), so a Cartesian equation for this plane will be

$$x + 3y + z = 0 + 3(-7) + 6 = -15$$

ie, $\boxed{x + 3y + z = -15}$

(b) (i)



$$V = \frac{1}{3}(\text{area } \Delta ABC)(\text{height})$$

$$= \frac{1}{3}\left(\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right)(h)$$

$$= \frac{1}{6} h \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|.$$

Q2 (b) (i) (cont.)

If θ is the angle between AD and the perpendicular direction of the base ΔABC, then

$$h = |\vec{AD}| \cos\theta = \frac{|\vec{AD}|\,|\vec{AB}\times\vec{AC}| \cos\theta}{|\vec{AB}\times\vec{AC}|}$$

$$= \frac{|\vec{AD}\cdot(\vec{AB}\times\vec{AC})|}{|\vec{AB}\times\vec{AC}|}$$

so

$$V = \frac{1}{6}\frac{|\vec{AD}\cdot(\vec{AB}\times\vec{AC})|}{|\vec{AB}\times\vec{AC}|}|\vec{AB}\times\vec{AC}|$$

$$= \frac{1}{6}|\vec{AD}\cdot(\vec{AB}\times\vec{AC})|.$$

(ii) If $A=(1,2,3)$, $B=(-1,0,5)$, $C=(0,3,1)$, $D=(2,2,2)$

then $\vec{AD} = \underline{i}-\underline{k}$, $\vec{AB} = -2\underline{i}-2\underline{j}+2\underline{k}$, $\vec{AC} = -\underline{i}+\underline{j}-2\underline{k}$

so

$$V = \frac{1}{6}\left|\begin{vmatrix} 1 & 0 & -1 \\ -2 & -2 & 2 \\ -1 & 1 & -2 \end{vmatrix}\right|$$

$$= \frac{1}{6}\left|\begin{vmatrix} -2 & 2 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} -2 & -2 \\ -1 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{6}|4-2-(-2-2)|$$

$$= \frac{1}{6}|6| = 1.$$

(E)

Q3/ (a) $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 4 \\ 3 & 2 & 4 & 2 & 4 \end{array}\right]$

(b) $\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 0 & -1 & -2 & -1 & -2 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & -1 & -2 & -1 & -2 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(c) This corresponds to $\begin{cases} x_1 = 0 \\ x_2 + 2x_3 = 0 \\ x_4 = 2 \end{cases}$

so, putting $x_3 = t$, gives the general solution

$$\boxed{(x_1, x_2, x_3, x_4) = (0, -2t, t, 2) \quad \text{for } t \in \mathbb{R}}$$

(F)

Q3/ (d) Putting $x_1 = 0$, $x_2 = -2t$, $x_3 = t$, $x_4 = 2$,

the new system becomes

$$\left\{\begin{array}{l} -2bt + ct + 2d = 0 \\ -2at + ct = 0 \\ dt + 2a = 2 \end{array}\right\} \text{ for all } t \in \mathbb{R}$$

If $d \neq 0$ then $2a = 2 - dt$ can vary, which is impossible. Hence $d = 0$ and the equations become

$$\left\{\begin{array}{l} -2bt + ct = 0 \\ -2at + ct = 0 \\ 2a = 2 \end{array}\right\} \text{ for all } t \in \mathbb{R}.$$

Dividing through by $t \neq 0$, gives

$$\left\{\begin{array}{l} -2b + c = 0 \\ -2a + c = 0 \\ a = 1 \end{array}\right.$$

so $\boxed{a = 1,\ c = 2,\ b = 1,\ d = 0.}$

Q4/ (a) λ is an <u>eigenvalue</u> of A if there exists a nonzero vector $\underline{v}$ such that $A\underline{v} = \lambda \underline{v}$, in which case $\underline{v}$ is called an <u>eigenvector</u>.

(b) λ is an eigenvalue of A

$\iff$ $A\underline{v} = \lambda \underline{v}$ $\quad (\exists\, \underline{v} \neq \underline{0})$

$\iff$ $A\underline{v} = \lambda I \underline{v}$ $\quad (\exists\, \underline{v} \neq \underline{0})$

$\iff$ $(A - \lambda I)\underline{v} = \underline{0}$ $\quad (\exists\, \underline{v} \neq \underline{0})$

$\iff$ $(A - \lambda I)^{-1}$ does not exist

$\iff$ $\det(A - \lambda I) = 0$

$\iff$ λ is a root of the characteristic polynomial $\det(A - \lambda I)$.

This is a theorem : M^{-1} exists $\iff$ $\det M \neq 0$

($\Rightarrow$) If $(A-\lambda I)^{-1}$ exists then $\underline{v} = (A-\lambda I)^{-1}(A-\lambda I)\underline{v}$

$= (A-\lambda I)^{-1}\underline{0} = \underline{0}$,

contradicting $\underline{v} \neq \underline{0}$

($\Leftarrow$) If $(A-\lambda I)^{-1}$ does not exist then $A-\lambda I$ row reduces to an echelon form with a row of zeros, so the system corresponding to $(A-\lambda I)\underline{v} = \underline{0}$ has a nontrivial solution.

(H)

Q4/ (c) A has a row of zeros so $\det(A) = 0$, so $\det(A - \lambda I) = 0$ when $\lambda = 0$, which shows $\lambda = 0$ is an eigenvalue of A.

(d) $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & 0 & 0 \\ 2 & 2-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ -1 & 4 & 3 & 1-\lambda \end{vmatrix}$

$$= (-\lambda) \begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ -1 & 4 & 1-\lambda \end{vmatrix} = (-\lambda)(1-\lambda) \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix}$$

$$= \lambda(\lambda-1)\left[(2-\lambda)^2 - 4\right] = \lambda(\lambda-1)(\lambda^2 - 4\lambda + 4 - 4)$$

$$= \lambda(\lambda-1)(\lambda^2 - 4\lambda) = \lambda^2(\lambda-1)(\lambda-4),$$

so $\det(A - \lambda I) = 0 \iff \lambda = 0, 1$ or 4.

The eigenvalues are $0, 1$ and 4.

(e) **Claim**: $\underset{\sim}{u} + \underset{\sim}{v}$ is not an eigenvector of B

Proof: Suppose $B(\underset{\sim}{u} + \underset{\sim}{v}) = \alpha(\underset{\sim}{u} + \underset{\sim}{v})$ for some eigenvalue α.

Then $\alpha(\underset{\sim}{u} + \underset{\sim}{v}) = B\underset{\sim}{u} + B\underset{\sim}{v} = \lambda \underset{\sim}{u} + \mu \underset{\sim}{v}$, so $(\lambda - \alpha)\underset{\sim}{u} = (\alpha - \mu)\underset{\sim}{v}$.

Hence $\lambda(\lambda-\alpha)\underset{\sim}{u} = (\lambda-\alpha)\lambda\underset{\sim}{u} = (\lambda-\alpha)B\underset{\sim}{u} = B(\lambda-\alpha)\underset{\sim}{u} = B(\alpha-\mu)\underset{\sim}{v}$

$$= (\alpha-\mu)B\underset{\sim}{v} = (\alpha-\mu)\mu\underset{\sim}{v} = \mu(\alpha-\mu)\underset{\sim}{v}$$

$$= \mu(\lambda-\alpha)\underset{\sim}{u}.$$

Q4/(e) (cont.)

Hence $(\lambda-\mu)(\lambda-\alpha)\underline{u} = \underline{0}$,

so $(\lambda-\mu)(\lambda-\alpha) = 0$, since $\underline{u} \neq \underline{0}$.

Thus $\lambda-\mu = 0$ or $\lambda-\alpha = 0$.

But $\lambda \neq \mu$, so $\lambda - \alpha = 0$, so $\lambda = \alpha$.

A similar argument shows $\mu = \alpha$, contradicting that $\lambda \neq \mu$. This proves α does not exist, so $\underline{u}+\underline{v}$ is not an eigenvector of B, as claimed. $\square$

Q5/(a) Exploring with $n=3$ forces the following:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

suggesting, and confirmed by checking, that

$C = AA^T$ where $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 1 \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$

Q5/(b) $\det C = (\det A)(\det A^T)$

$$= (\det A)^2 = 1^2 = 1$$

since A is triangular with 1's down the diagonal.

(c) Because A is triangular $\lambda = 1$ is the only eigenvalue.

$$A - I = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \; 1 & 0 \end{bmatrix}$$

so the eigenspace is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$