THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Week 5** 2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Sketch the graph of the function:

$$f(x) = \begin{cases} 0 & x < 0, \\ 1 & x = 0, \\ x+2 & x > 0. \end{cases}$$

Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ (no need for formal proofs). Does $\lim_{x\to 0} f(x)$ exist?

Questions for the tutorial

2. Sketch the function with formula

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 2x+1 & \text{if } x > 0. \end{cases}$$

Find suitable values of δ such that whenever $0 < |x| < \delta$, we have

(a) $|f(x) - 1| < 0.01$, (b) $|f(x) - 1| < 0.001$, (c) $|f(x) - 1| < \epsilon$, where $\epsilon > 0$.

3. Find the following limits using one or more of the limit laws.

(a) $\displaystyle\lim_{x\to 3} \frac{x^2+3x+2}{4x^2-x+1}$ (b) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x-1}$ (c) $\displaystyle\lim_{x\to 1} \frac{x^2-3x+2}{x^3-1}$

(d) $\displaystyle\lim_{x\to 0} x^2 \cos\frac{2}{x}$ (e) $\displaystyle\lim_{x\to 0} \frac{\sqrt{3+2x}-\sqrt{3}}{x}$ (f) $\displaystyle\lim_{x\to \infty} \frac{x+\sin^3 x}{2x-1}$

(g) $\displaystyle\lim_{x\to \infty} \sqrt{\frac{3-x}{4-x}}$ (h) $\displaystyle\lim_{x\to \infty} \sqrt{\frac{3-x}{4-x^2}}$ (i) $\displaystyle\lim_{x\to \infty} (\sqrt{x}-\sqrt{x+1})$

4. Prove the following results using the ϵ, δ definition:

(a) $\lim_{x\to a} c = c$

(b) $\lim_{x\to 4} f(x) = -3$, where $f(x) = \begin{cases} 5-2x & \text{if } x \neq 4, \\ 100 & \text{if } x = 4. \end{cases}$

(c) $\lim_{x\to 0} g(x) = 0$, where $g(x) = \begin{cases} 3x & \text{if } x \text{ is rational}, \\ 7x & \text{if } x \text{ is irrational}. \end{cases}$

5. The function f is defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

Using the Squeeze Law, prove that $\lim_{x\to 0} f(x) = 0$.

6. (a) Give an example of a function f for which $\lim_{x\to 0} f(x)^2$ exists but $\lim_{x\to 0} f(x)$ does not.

(b) Give an example of a function f for which $\lim_{x\to 0} f(x^2)$ exists but $\lim_{x\to 0} f(x)$ does not.

7. Suppose f has domain $\mathbb{R}$. To say that $\lim_{n\to\infty} f(n) = \ell$ (where n takes only integer values) means that for any $\epsilon > 0$, there exists M such that whenever n is an integer and $n > M$, then $|f(n) - \ell| < \epsilon$. Give an example of a function f with domain $\mathbb{R}$ such that $\lim_{n\to\infty} f(n)$ exists in this sense, but $\lim_{x\to\infty} f(x)$ (where x takes real values) does not exist.

8. Using the ϵ, δ definition of limit, prove that if the limit of a function exists as $x \to a$, then the limit is unique. To be precise, prove that if $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} f(x) = m$, then $\ell = m$. (*Hint*: Assume that $\ell \neq m$ and obtain a contradiction by setting $\epsilon = \dfrac{|\ell - m|}{2}$.)

Extra Questions

9. Students often have difficulty remembering the ϵ, δ definition of the statement $\lim_{x\to a} f(x) = \ell$. For each of the following misremembered versions, work out what it means and why it is not the right definition.

(a) For each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have $0 < |f(x) - \ell| < \epsilon$.

(b) For each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x-a| < \delta$, we have $|f(x)-\ell| < \epsilon$.

(c) For each ϵ, there exists $\delta > 0$ such that whenever $0 < |x-a| < \delta$, we have $|f(x)-\ell| < \epsilon$.

(d) For each $\epsilon > 0$, there exists δ such that whenever $0 < |x-a| < \delta$, we have $|f(x)-\ell| < \epsilon$.

(e) For each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - \ell| > \epsilon$.

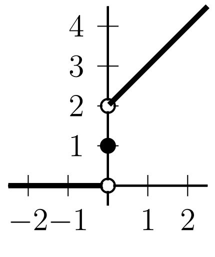
(f) For each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - a| < \epsilon$, we have $|f(x) - \ell| < \delta$.

(g) For each $\delta > 0$, there exists $\epsilon > 0$ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - \ell| < \epsilon$.

(h) For each $\delta > 0$, there exists $\epsilon > 0$ such that whenever $0 < |x - a| < \epsilon$, we have $|f(x) - \ell| < \delta$.

10. Prove that $\lim_{x\to 0} f(x)$ does not exist, where $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$

Solution to Question 1



We have $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = 2$.
Since the limits from the left and the right are not equal, $\lim_{x\to 0} f(x)$ does not exist.