The University of Sydney

MATH1901/06 Differential Calculus (Advanced)

A selection of standard Taylor series

It is worthwhile memorising the Taylor series for a selection of elementary functions. Some of the exercises in MATH1901, such as some l'Hôpital-type limits, can be handled by replacing the functions appearing by their standard Taylor series/polynomials (to a few terms) about the origin. Because these series all have a simple pattern in their coefficients, it is enough to just give three or four terms plus dots, from which any Taylor polynomial of any order/degree can be deduced. (Intervals of convergence will be postponed to MATH1903 in Semester 2.)

This table will NOT be included on the exam paper.

- Geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \qquad x \in (-1, 1),$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots, \qquad x \in (-1, 1).$$

- Binomial series:

$$\begin{aligned} (1+x)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots. \end{aligned}$$

  The interval of convergence is $\mathbf{R}$ when α is a nonnegative integer (because the series terminates), $[-1, 1]$ for all other $\alpha > 0$, $(-1, 1]$ for $-1 < \alpha < 0$, and $(-1, 1)$ for all $\alpha \leq -1$. The geometric series is the case $\alpha = -1$.

- Exponential series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots, \qquad x \in \mathbf{R}.$$

- Trigonometric series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \qquad x \in \mathbf{R},$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \qquad x \in \mathbf{R}.$$

  Note that the signs alternate. The cosine, being an even function, has only even powers. The sine has only odd powers.

- Hyperbolic function series:

$$\cosh x \;=\; 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \,\dots\,, \qquad x \in \mathbf{R},$$
$$\sinh x \;=\; x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \,\dots\,, \qquad x \in \mathbf{R}.$$

These differ from the trigonometric series by having all plus signs. In the complex domain ($x \to z$), the series imply

$$\cosh z \;=\; \cos(iz), \qquad \sinh z \;=\; -i\sin(iz), \qquad e^{\pm iz} = \cos z \pm i \sin z.$$

- Logarithm series:

$$\ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \,\dots\,, \qquad x \in (-1,1],$$
$$\ln(1-x) \;=\; -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \,\dots\right), \qquad x \in [-1,1).$$

These extend to the closed disc $|z| \le 1$ except for $z = -1$ and $z = 1$, respectively.

- Inverse tangent series:

$$\tan^{-1} x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \,\dots\,, \qquad x \in [-1,1],$$
$$\tanh^{-1} x \;=\; \frac{1}{2}\ln\!\left(\frac{1+x}{1-x}\right) \;=\; x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \,\dots\,, \qquad x \in (-1,1).$$

The latter series can be deduced from the logarithm series and also from $\tanh^{-1} z = -i\tan^{-1}(iz),\ |z| < 1$.

- There are also reasonably simple series for $\sin^{-1} x$ and $\sinh^{-1} x$, valid for $x \in [-1,1]$, but it is not necessary or desirable to memorise these. They can be obtained when needed by term-by-term integration of the binomial series for $(1-x^2)^{-1/2}$ and $(1+x^2)^{-1/2}$, respectively.