A

MATH 1902 — 2005 exam solutions — 10/6/2010

Q1/ (i) $\underset{\sim}{u} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

(a) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 6 - 2 - 2 = 2$

(b) $$\cos\theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \frac{2}{\sqrt{9+4+1}\sqrt{4+1+1}} = \frac{2}{\sqrt{14}\sqrt{6}}$$

$$= \frac{1}{\sqrt{21}}$$

(c) $$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k}$$

(d) a unit vector perpendicular to $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is

$$\widehat{\underset{\sim}{u} \times \underset{\sim}{v}} = \frac{\underset{\sim}{u} \times \underset{\sim}{v}}{|\underset{\sim}{u} \times \underset{\sim}{v}|} = \frac{1}{\sqrt{1+25+49}}\underset{\sim}{u} \times \underset{\sim}{v} = \frac{1}{\sqrt{75}}\underset{\sim}{u} \times \underset{\sim}{v}$$

$$= \frac{1}{5\sqrt{3}}(\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k}).$$

(e) $$(3\underset{\sim}{u} + 5\underset{\sim}{v}) \times (2\underset{\sim}{u} + 4\underset{\sim}{v}) = 12(\underset{\sim}{u} \times \underset{\sim}{v}) + 10(\underset{\sim}{v} \times \underset{\sim}{u})$$

$$= 2(\underset{\sim}{u} \times \underset{\sim}{v}) = 2\underset{\sim}{i} + 10\underset{\sim}{j} + 14\underset{\sim}{k}.$$

(ii) $\mathcal{P}$ is the plane $\underset{\sim}{r} \cdot \underset{\sim}{v} = c$, $\underset{\sim}{r}_0 = \overrightarrow{OA}$

(a) ℓ perpendicular to $\mathcal{P}$, so parallel to $\underset{\sim}{v}$, and passes through A, so has vector equation

$$\boxed{\underset{\sim}{r} = \underset{\sim}{r}_0 + t\underset{\sim}{v}.}$$

B

Q1/(ii) (b) B is point of intersection between P and l, so want t such that

$$(\underset{\sim}{r}_0 + t\underset{\sim}{v}) \cdot \underset{\sim}{v} = c,$$

i.e. $\underset{\sim}{r}_0 \cdot \underset{\sim}{v} + t|\underset{\sim}{v}|^2 = c$, so $t = \dfrac{c - \underset{\sim}{r}_0 \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2}$.

But $|\underset{\sim}{v}| = 1$ since $\underset{\sim}{v}$ is a unit vector, so

$$t = c - \underset{\sim}{r}_0 \cdot \underset{\sim}{v}$$

and position vector of B is

$$\underset{\sim}{r} = \underset{\sim}{r}_0 + t\underset{\sim}{v} = \boxed{\underset{\sim}{r}_0 + (c - \underset{\sim}{r}_0 \cdot \underset{\sim}{v})\underset{\sim}{v}}.$$

(c) The distance from A to P is

$$|\overrightarrow{AB}| = |(c - \underset{\sim}{r}_0 \cdot \underset{\sim}{v})\underset{\sim}{v}| = |c - \underset{\sim}{r}_0 \cdot \underset{\sim}{v}|\,|\underset{\sim}{v}|$$

$$= |c - \underset{\sim}{r}_0 \cdot \underset{\sim}{v}|, \text{ since } |\underset{\sim}{v}| = 1.$$

(d) Take $A = (1,1,1)$, so $\underset{\sim}{r}_0 = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and

$$\underset{\sim}{v} = \frac{1}{\sqrt{9+4+36}}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 6\underset{\sim}{k}) = \frac{1}{7}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 6\underset{\sim}{k})$$

so $c = \frac{21}{7} = 3$, and distance is

$$|c - \underset{\sim}{r}_0 \cdot \underset{\sim}{v}| = \left|3 - (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) \cdot \tfrac{1}{7}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 6\underset{\sim}{k})\right|$$

$$= \left|3 - \tfrac{1}{7}(3+2-6)\right| = \left|3 + \tfrac{1}{7}\right| = \boxed{\frac{22}{7}}$$

(c)

Q2/ (i) $A = (2,1,-1)$, $B = (1,2,2)$, $C = (3,-1,-1)$,

$\mathcal{H}$ = parallelopiped determined by OA, OB, OC.

(a) volume of $\mathcal{H}$ $= |\vec{OA} \cdot (\vec{OB} \times \vec{OC})|$

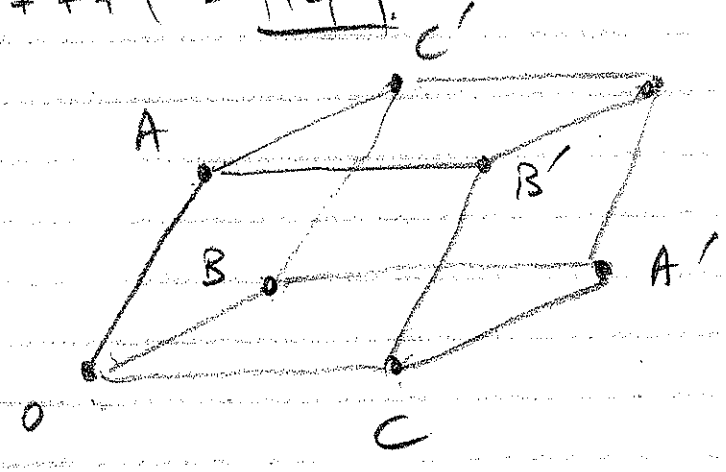
$$= \left| \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 3 & -1 & -1 \end{vmatrix} \right|$$

$$= \left| 2 \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \right|$$

$$= |2(-2+2) - (-1-6) - (-1-6)|$$

$$= |7+7| = \boxed{14}$$

(b)



$OA' = \vec{OC} + \vec{OB} = 3\underline{i} - \underline{j} - \underline{k} + \underline{i} + 2\underline{j} + 2\underline{k} = 4\underline{i} + \underline{j} + \underline{k}$

$OB' = \vec{OC} + \vec{OA} = 3\underline{i} - \underline{j} - \underline{k} + 2\underline{i} + \underline{j} - \underline{k} = 5\underline{i} - 2\underline{k}$

$OC' = \vec{OA} + \vec{OB} = 2\underline{i} + \underline{j} - \underline{k} + \underline{i} + 2\underline{j} + 2\underline{k} = 3\underline{i} + 3\underline{j} + \underline{k}$

so $\boxed{A' = (4,1,1), \quad B' = (5,0,-2), \quad C' = (3,3,1)}$

D

Q4 (c) area of AOBC$'$ $= |\vec{OA} \times \vec{OB}|$

$$= \left| \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} \right| = |4\underline{i} - 5\underline{j} + 3\underline{k}|$$

$$= \sqrt{16+25+9} = \boxed{\sqrt{50}\,.}$$

(d) $M = (2, \tfrac{1}{2}, \tfrac{1}{2})$, $N = (5/2, 0, -1)$, $P = (\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2})$

Let l_1 be line through A & M so has equation

$$\underline{r} = 2\underline{i} + \underline{j} - \underline{k} + t(\tfrac{1}{2}\underline{j} - \tfrac{3}{2}\underline{k})$$

or
$$\begin{cases} x = 2 \\ y = 1 + t/2 \\ z = -1 - 3t/2 \end{cases}$$

Let l_2 be line through B & N so has equation

$$\underline{r} = \underline{i} + 2\underline{j} + 2\underline{k} + s(3/2\,\underline{i} - 2\underline{j} - 3\underline{k})$$

or
$$\begin{cases} x = 1 + 3s/2 \\ y = 2 - 2s \\ z = 2 - 3s \end{cases}$$

These intersect when
$$\begin{cases} 2 = 1 + 3s/2 \\ 1 + t/2 = 2 - 2s \\ -1 - 3t/2 = 2 - 3s \end{cases}$$

So $s = 2/3$, $t = -2/3$ yielding point $\boxed{(2, 2/3, 0)}$

(E)

Q2/(i) (cont.)

<u>Check</u>: let l_3 be line through C & P so has equation

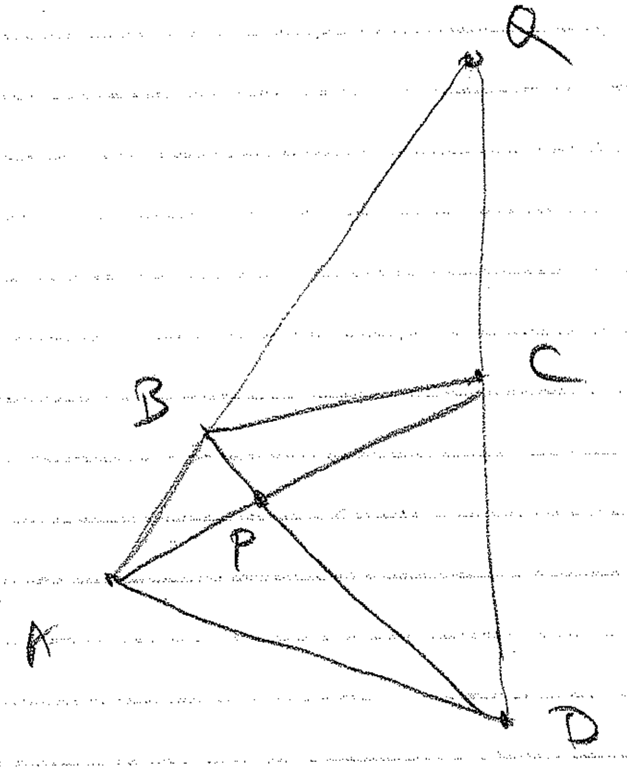
$$\underline{r} = 3\underline{i} - \underline{j} - \underline{k} + u\left(\tfrac{3}{2}\underline{i} - \tfrac{5}{2}\underline{j} - \tfrac{3}{2}\underline{k}\right)$$

or
$$\begin{cases} x = 3 + 3u/2 \\ y = -1 - 5u/2 \\ z = -1 - 3u/2 \end{cases}$$

so need u such that
$$\begin{cases} 2 = 3 + 3u/2 \\ 2/3 = -1 - 5u/2 \\ 0 = -1 - 3u/2 \end{cases}$$

giving $u = -2/3$ in each case. ✓

Q2/(ii)



$\underline{a} = \overrightarrow{OA}$

$\underline{b} = \overrightarrow{OB}$

$\underline{c} = \overrightarrow{OC}$

$\underline{d} = \overrightarrow{OD}$

(a) Given $\overrightarrow{AP} = \alpha\,\overrightarrow{AC}$ and $\overrightarrow{BP} = \beta\,\overrightarrow{BD}$.

Then $(1-\alpha)\underline{a} + \alpha\underline{c} = (1-\alpha)\overrightarrow{OA} + \alpha\overrightarrow{OC} = \overrightarrow{OA} + \alpha(\overrightarrow{AO} + \overrightarrow{OC})$

$= \overrightarrow{OA} + \alpha\overrightarrow{AC} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OP}$, Q2/ (ii) (a) (cont.)

and $(1-\beta)\underline{b} + \beta\underline{d} = (1-\beta)\overrightarrow{OB} + \beta\overrightarrow{OD}$

$= \overrightarrow{OB} + \beta(\overrightarrow{BO} + \overrightarrow{OD}) = \overrightarrow{OB} + \beta\overrightarrow{BD}$

$= \overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OP}$.

Hence $(1-\alpha)\underline{a} + \alpha\underline{c} = \overrightarrow{OP} = (1-\beta)\underline{b} + \beta\underline{d}$,

as required.

(b) Given $\alpha \neq \beta$ and $\alpha, \beta > 0$.

From (a), $(1-\alpha)\underline{a} + \alpha\underline{c} = (1-\beta)\underline{b} + \beta\underline{d}$,

so $(1-\alpha)\underline{a} - (1-\beta)\underline{b} = -\alpha\underline{c} + \beta\underline{d}$,

so, dividing through by $\beta - \alpha \neq 0$,

$$\frac{(1-\alpha)}{\beta-\alpha}\underline{a} - \frac{1-\beta}{\beta-\alpha}\underline{b} = \frac{-\alpha}{\beta-\alpha}\underline{c} + \frac{\beta}{\beta-\alpha}\underline{d},$$

as required.

But Q is the unique point of intersection of lines through AB and CD, so

$$\overrightarrow{OQ} = \left(\frac{1-\alpha}{\beta-\alpha}\right)\underline{a} - \left(\frac{1-\beta}{\beta-\alpha}\right)\underline{b} = \left(\frac{-\alpha}{\beta-\alpha}\right)\underline{c} + \left(\frac{\beta}{\beta-\alpha}\right)\underline{d}$$

So Q divides AB in the ratio $-\left(\frac{1-\beta}{\beta-\alpha}\right) : \frac{1-\alpha}{\beta-\alpha}$

and CD ...... $\frac{\beta}{\beta-\alpha} : \frac{-\alpha}{\beta-\alpha}$.

Q2/(ii) (c) Given P divides AC in ratio 7 : 1

and BD in ratio 5 : 3, then

$$\alpha = 7/8, \quad \beta = 5/8$$

so, from (b), Q divides AB in the ratio

$$-\left(\frac{1-5/8}{5/8-7/8}\right) : \left(\frac{1-7/8}{5/8-7/8}\right) = -\frac{3}{-2} : \frac{1}{-2}$$

$$= 3 : -1$$

and divides CD in the ratio

$$\left(\frac{5/8}{5/8-7/8}\right) : \left(\frac{-7/8}{5/8-7/8}\right) = \frac{5}{-2} : \frac{-7}{-2}$$

$$= -5 : 7$$

(note: error in question?)

Q3/ (i) $\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 2 & 4 & -1 & 0 & -7 \\ -3 & -6 & 1 & 1 & 16 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 1 & -2 & -11 \\ 0 & 0 & -2 & 4 & 22 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & -9 \\ 0 & 0 & 1 & -2 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\quad \begin{cases} x + 2y - w = -9 \\ z - 2w = -11 \end{cases}$

Solution: $\boxed{(x, y, z, w) = (-9 - 2s + t, s, -11 + 2t, t)}$ $(s, t \in \mathbb{R})$

(H)

Q3/ (ii) $R_i \leftrightarrow R_j$ multiplies det by -1

$(\lambda \neq 0)$ $R_i \to \lambda R_i$ " " by λ

$(i \neq j)$ $R_i \to R_i + \lambda R_j$ " " " 1

(iii)
$$\begin{vmatrix} 1 & 6 & 1 & -2 \\ -1 & -3 & -4 & 2 \\ 2 & 12 & 3 & -3 \\ 0 & 1 & -1 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 1 & -2 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -3 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 7 \end{vmatrix} = 3 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 7 \end{vmatrix} = 3 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{vmatrix}$$

$$= 3(7) \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = \boxed{21}.$$

Q4/ (ii) (a) $A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$

so must produce at least one row of zeros when row reduced to echelon form, so there exist elementary matrices $E_1, \dots, E_k$ such that $E_k \cdots E_1 A$ has a row of zeros. then $T = E_k \cdots E_1$ is invertible, since it is a product of invertible matrices.

Q4 (ii) (b) If T had a row of zeros then so would $I = TT^{-1}$, which is nonsense. Hence T does not have a row of zeros.

(c) TA has a row of zeros, so $(TA)B = T(AB)$ also has a row of zeros, since this property persists under multiplication on the right (by any matrix with 3 rows).

Q5/ (i)(a) $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ so $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 - 2 = \lambda^2 - 2\lambda + 1 - 2 = \lambda^2 - 2\lambda - 1 = 0$

when $\lambda = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$, so the eigenvalues are $\lambda_1 = 1+\sqrt{2}$, and $\lambda_2 = 1-\sqrt{2}$.

$A - (1+\sqrt{2})I = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}$

giving eigenspace $\left\{ \begin{bmatrix} \sqrt{2}t \\ t \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$

$A - (1-\sqrt{2})I = \begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$

giving eigenspace $\left\{ \begin{bmatrix} -\sqrt{2}t \\ t \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$

Q5/ (i) (a) (cont.)

Hence we can choose $\underline{v}_1 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$ as an eigenvector for $\lambda_1 = 1 + \sqrt{2}$ and $\underline{v}_2 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$ " " " " $\lambda_2 = 1 - \sqrt{2}$.

(b) Want $\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = s \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} + t \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} \sqrt{2}s - \sqrt{2}t \\ s + t \end{bmatrix}$$

So $\left.\begin{aligned} \sqrt{2}s - \sqrt{2}t &= 1 \\ s + t &= 1 \\ \sqrt{2}s + \sqrt{2}t &= \sqrt{2} \end{aligned}\right\} \Rightarrow 2\sqrt{2}\, s = \sqrt{2} + 1$

So $s = \dfrac{\sqrt{2}+1}{2\sqrt{2}} = \dfrac{2+\sqrt{2}}{4}$

and $t = \dfrac{2-\sqrt{2}}{4}$.

But $\begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} = A^n \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} = A^n (s\underline{v}_1 + t\underline{v}_2)$

$$= sA^n\underline{v}_1 + tA^n\underline{v}_2 = s\lambda_1^n \underline{v}_1 + t\lambda_2^n \underline{v}_2$$

$$= \frac{2+\sqrt{2}}{4}(1+\sqrt{2})^n \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} + \frac{2-\sqrt{2}}{4}(1-\sqrt{2})^n \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} (2+\sqrt{2})\sqrt{2}(1+\sqrt{2})^n - (2-\sqrt{2})\sqrt{2}(1-\sqrt{2})^n \\ (2+\sqrt{2})(1+\sqrt{2})^n + (2-\sqrt{2})(1-\sqrt{2})^n \end{bmatrix}.$$

(K)

Q5/ (i) (c) By (b)

$$\frac{\alpha_n}{\beta_n} = \frac{(2+\sqrt{2})\sqrt{2}(1+\sqrt{2})^n - (2-\sqrt{2})\sqrt{2}(1-\sqrt{2})^n}{(2+\sqrt{2})(1+\sqrt{2})^n + (2-\sqrt{2})(1-\sqrt{2})^n}$$

$$= \sqrt{2}\left(\frac{(2+\sqrt{2})(1+\sqrt{2})^n - (2-\sqrt{2})(1-\sqrt{2})^n}{(2+\sqrt{2})(1+\sqrt{2})^n + (2-\sqrt{2})(1-\sqrt{2})^n}\right)$$

$$= \sqrt{2}\ \frac{2+\sqrt{2} - 2-\sqrt{2}\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^n}{2+\sqrt{2} + 2-\sqrt{2}\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^n}$$

$$\to \sqrt{2}\ \frac{2+\sqrt{2}}{2+\sqrt{2}} = \sqrt{2}$$

as $n \to \infty$, since $\left|\frac{1-\sqrt{2}}{1+\sqrt{2}}\right| < 1$.