MATH1903/1907 Lectures

Week 12, Semester 2, 2017

Daniel Daners

Simple harmonic motion with a forcing term

$$y'' + \omega_0^2 y = \underbrace{\cos\omega t}_{\text{forcing term}}$$

Find a particular solution of the form

$$y_p = A\cos\omega t + B\sin\omega t$$

Note: $y'' + \omega_0 y$ is even [odd] if y is even [odd]

$\cos\omega t$ is even, so we try and find an even particular solution:

$$y_p = A\cos\omega t$$

Substitute into equation:

$$\underbrace{-A\omega^2\cos\omega t}_{y_p''} + \omega_0^2 \underbrace{A\cos\omega t}_{y_p} = \cos\omega t$$

$$-A\omega^2 + \omega_0^2 A = 1$$

$$A(\omega_0^2 - \omega^2) = 1, \text{ so } A = \frac{1}{\omega_0^2 - \omega^2}$$

Hence if $\omega^2 \neq \omega_0^2$, then

$$y_p = \frac{\cos\omega t}{\omega_0^2 - \omega^2}$$

is a particular solution.

Solution of homogeneous equation: $y_h(t) = A\cos\omega_0 t + B\sin\omega_0 t$

We next want a solution of

$$y'' + \omega_0^2 y = \cos\omega_0 t \qquad (\text{case } \omega_0 = \omega)$$

Note $\cos\omega_0 t$ solves the homogeneous equation.

To obtain a solution try

$$y_p(t) = t\left(A\cos\omega_0 t + B\sin\omega_0 t\right)$$

Note: $\cos\omega_0 t$ is an even function

$y'' + \omega_0^2 y$ is even if y is even.

Hence we try the even part of y_p:

$$y_p = tB\sin\omega_0 t \qquad (\text{which is even}).$$

substitute into equation and determine B.

<u>Alternative</u>: obtain the solution as a limit of solutions in case $\omega^2 \neq \omega_0^2$ as $\omega \to \omega_0$.

We deduced

$$y_p = \frac{\cos \omega t}{\omega_0^2 - \omega^2}$$

here we cannot let $\omega \to \omega_0$ as the limit does not exist.

<u>Note</u>: particular solutions are not unique. Adding any solution to the homogeneous problem will given another particular solution.

In our case

solution of homogeneous problem.

$$y_p = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

is another particular solution.

$$y_p = -\frac{1}{\omega + \omega_0} \underbrace{\frac{\cos \omega t - \cos \omega_0 t}{\omega - \omega_0}}_{\text{difference quotient for } \frac{d}{d\omega} \cos \omega t \text{ at } \omega = \omega_0.}$$

$$\xrightarrow[\omega \to \omega_0]{} -\frac{1}{\omega_0 + \omega_0} \frac{d}{d\omega} \cos \omega t \Big|_{\omega = \omega_0}$$

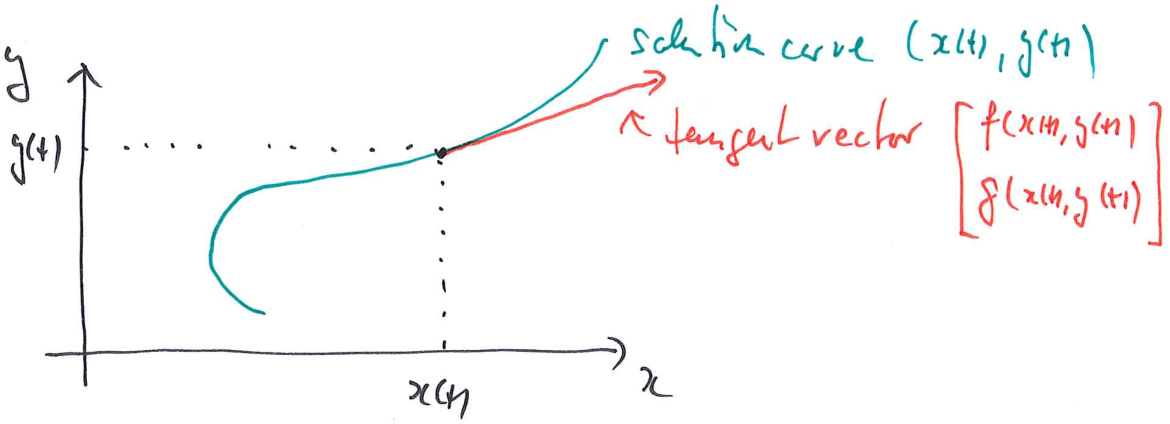
$$= \frac{t \sin \omega_0 t}{2\omega_0}.$$

Consider a system of differential equations

$$x'(t) = f(x(t), y(t))$$
$$y'(t) = g(x(t), y(t))$$

A solution is a curve $(x(t), y(t))$ in the xy-plane parametrised by t. The tangent vector to that curve is given by

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} f(x(t), y(t)) \\ g(x(t), y(t)) \end{bmatrix}$$



Hence at every point (x, y) the solution curve is tangent to the vector $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$

Example: Simple predator-prey system

$x(t)$ size of population of prey at time t (rabbits)

$y(t)$ size of population of predators at time t (foxes)

Equation for $x(t)$	Equation for $y(t)$
$\frac{dx}{dt} = k(y)x$	$\frac{dy}{dt} = h(x)y$
growth rate $k(y)$	Growth rate
$k(y) = a - by$	$h(x) = -d + cx$
a: growth rate in absence of predators; by: reduced growth rate due to predators	d: decay rate in absence of prey (food); cx: growth depending on size of x.

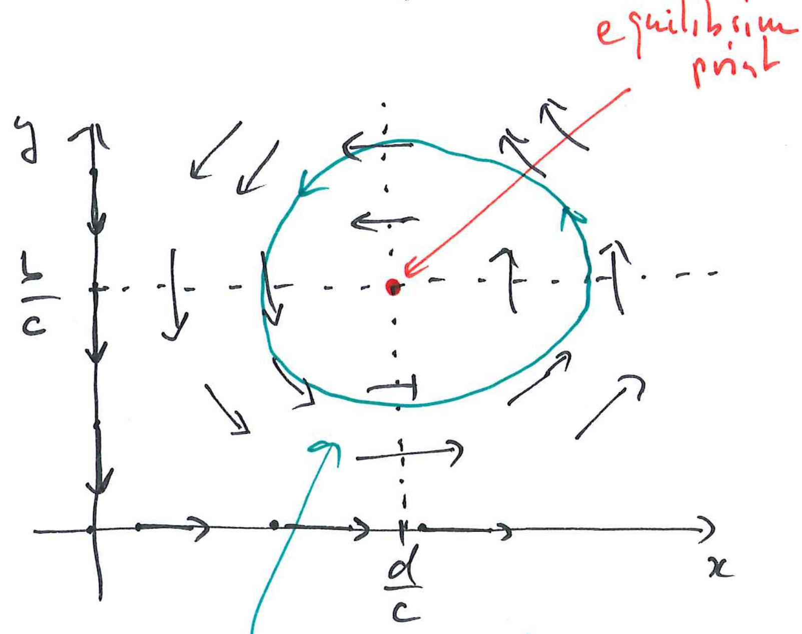
We obtain the system of equations

$$\frac{dx}{dt} = (a - by)x$$

$$\frac{dy}{dt} = (cx - d)y$$

(we assume a, b, c, d are positive constants)

Sketch direction field:



It turns out that populations oscillate.

- check x, y-axes

  $y = 0$: $y' = 0$, $x' \geq 0$

  $x = 0$: $x' = 0$, $y' < 0$

- Find points where field is horizontal

  $\frac{dy}{dt} = 0 = (cx - d)y$

  $y = 0$ or $cx - d = 0$

  $x = \frac{d}{c}$

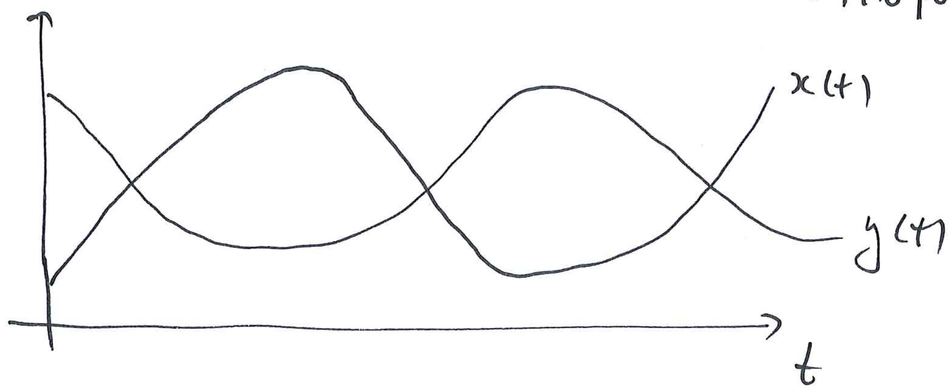
- Find points where field is vertical

  $\frac{dx}{dt} = 0 = (a - by)x = 0$

  $x = 0$ or $a - by = 0$

  $y = \frac{a}{b}$

- interpolate in between



Linear systems of differential equations

$$x' = ax + by \qquad y' = cx + dy$$

a, b, c, d constants

One way to solve: Eliminate one variable, solve for the other one, then substitute back.

Example:

$$x' = 2x + y \quad ①$$
$$y' = 4x - y \quad ②$$

Differentiate ①: $x'' = 2x' + y'$

Substitute ②: $x'' = 2x' + (4x - y)$

Use ① again to eliminate y: $y = x' - 2x$

$$x'' = 2x' + 4x - (x' - 2x)$$
$$= x' + 6x$$

Equation in x: $x'' - x' - 6x = 0$

auxiliary equation $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$

general solution: $x(t) = Ae^{3t} + Be^{-2t}$.

compute y from x using equation ①: $y = x' - 2x$

$$y(t) = 3Ae^{3t} - 2Be^{-2t} - 2(Ae^{3t} + Be^{-2t})$$
$$= Ae^{3t} - 4e^{-2t}$$

In vector form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-2t}$