Solutions to Tutorial for Week 12

MATH1903: Integral Calculus and Modelling (Advanced) Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/JM/MATH1903/
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Material covered

(1) Homogeneous linear second order differential equations with constant coefficients.

(2) Inhomogeneous linear second order differential equations with constant coefficients.

Outcomes

After completing this tutorial you should

(1) be confident in solving homogeneous second order homogeneous and inhomogeneous differential equations in various contexts.

Questions to do before the tutorial

1. Find the general solution of each of the following.

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 5y = 0.$

Solution: The auxiliary equation $\lambda^2 + 4\lambda - 5 = 0$ has roots $\lambda = -5, 1$, and so the general solution is $y = Ae^{-5x} + Be^x$.

(b) $\dfrac{d^2y}{dt^2} + 9y = 0.$

Solution: The auxiliary equation $\lambda^2 + 9 = 0$ has complex roots $\lambda = \pm 3i$, and so the general solution is $y = C\cos 3t + D\sin 3t$.

2. Consider the second-order non-homogeneous differential equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2$.

(a) Find the general solution of the above differential equation.

Solution: The auxiliary equation $\lambda^2 - 2\lambda + 1 = 0$ has a double root $\lambda = 1$, and so the general solution of the homogeneous equation (also called the complementary equation) is $y_h = Ae^x + Bxe^x$. For a particular solution, try $y_p = ax^2 + bx + c$. Substituting this into the differential equation gives

$$2a - 2(2ax + b) + (ax^2 + bx + c) = x^2.$$

Comparing coefficients of like powers gives $a = 1$, $b - 4a = 0$ and $2a - 2b + c = 0$, and hence $a = 1$, $b = 4$ and $c = 6$. So a particular solution is $y_p = x^2 + 4x + 6$, and the general solution is

$$y = (A + Bx)e^x + x^2 + 4x + 6.$$

(b) Find the particular solution of the above differential equation satisfying the initial conditions $y(0) = y'(0) = 4$.

Solution: The solution above gives $y(0) = A + 6$ and $y'(0) = A + B + 4$. So $y(0) = 4$ and $y'(0) = 4$ imply that $A = -2$ and $B = 2$, and so the required particular solution is $y = 2(x - 1)e^x + x^2 + 4x + 6$.

Questions to complete during the tutorial

3. Find the general solution of each of the following.

(a) $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 9x = 0.$

Solution: The auxiliary equation $\lambda^2 - 6\lambda + 9 = 0$ has repeated roots $\lambda = 3, 3$, and so the general solution is $x = Ae^{3t} + Bte^{3t}$.

(b) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0.$

Solution: The auxiliary equation $\lambda^2 - 6\lambda + 25 = 0$ has complex roots $\lambda = 3 \pm 4i$, and so the general solution is $y = e^{3x}(C\cos 4x + D\sin 4x)$.

4. Solve the following equations, giving the general solution and then the particular solution $y(x)$ satisfying the given boundary or initial conditions.

(a) $y'' + 4y' + 5y = 0, \quad y(0) = 2,\ y'(0) = 4$

Solution: The auxiliary equation $\lambda^2 + 4\lambda + 5 = 0$ has roots $-2 \pm i$, and so the general solution is $y(x) = e^{-2x}(C\cos x + D\sin x)$, which gives $y'(x) = e^{-2x}\{(D-2C)\cos x - (C+2D)\sin x\}$. Hence $y(0) = C$ and $y'(0) = D - 2C$, so the initial conditions imply $C = 2$ and $D = 8$, and the particular solution is $y(x) = 2e^{-2x}(\cos x + 4\sin x)$.

(b) $y'' - 2y' + y = 0, \quad y(2) = 0,\ y'(2) = 1$

Solution: The auxiliary equation $\lambda^2 - 2\lambda + 1 = 0$ has one double root $\lambda = 1$, and so the general solution is $y(x) = (A + Bx)e^x$, which gives $y'(x) = (A + B + Bx)e^x$. Hence $y(2) = (A + 2B)e^2$ and $y'(2) = (A + 3B)e^2$, so the initial conditions imply $A = -2e^{-2}$ and $B = e^{-2}$, and the particular solution is $y(x) = (x-2)e^{x-2}$.

5. First find the general solution of each of the following non-homogeneous second-order differential equations, and then the particular solution for the given initial conditions.

(a) $y'' + 3y' + 2y = 6e^t, \quad y(0) = 1,\ y'(0) = 0.$

Solution: The auxiliary equation $\lambda^2 + 3\lambda + 2 = 0$ has roots $\lambda = -1, -2$, and so the general solution of the homogeneous equation is $y_h = Ce^{-t} + De^{-2t}$. For a particular solution, try $y_p = \alpha e^t$. Substituting this into the differential equation gives $\alpha(e^t + 3e^t + 2e^t) = 6e^t$, which implies $\alpha = 1$. So a particular integral is $y_p = e^t$, and the general solution is

$$y = Ce^{-t} + De^{-2t} + e^t.$$

The solution above gives $y(0) = C + D + 1$ and $\dot{y}(0) = -C - 2D + 1$. So $y(0) = 1$ and $\dot{y}(0) = 0$ imply that $C = -1$ and $D = 1$, and so the required particular solution is $y = -e^{-t} + e^{-2t} + e^t$.

(b) $y'' + 3y' + 2y = 6e^{-t}$, $y(0) = 2$, $y'(0) = 1$.

Solution: The auxiliary equation and hence the general solution of the homogeneous equation are the same as in the last part. In this case, however, the non-homogeneous term is itself a solution of the homogeneous equation and so we will not be able to produce a particular solution of the form αe^{-t}. The standard procedure in this case is to include a factor t. So a suitable trial solution will take the form $y_p = \alpha te^{-t}$. Substitution into the differential equation gives $\alpha(t-2)e^{-t} + 3\alpha(1-t)e^{-t} + 2\alpha te^{-t} = 6e^{-t}$, which implies $\alpha = 6$. So a particular solution is $y_p = 6te^{-t}$, and the general solution is

$$y = (6t + C)e^{-t} + De^{-2t}.$$

The solution above gives $y(0) = C + D$ and $\dot{y}(0) = 6 - C - 2D$. So $y(0) = 2$ and $\dot{y}(0) = 1$ imply that $C = -1$ and $D = 3$, and so the required particular solution is $y = (6t-1)e^{-t} + 3e^{-2t}$.

6. (a) For $\omega \neq 5$, find the general solution of the non-homogeneous differential equation,

$$\frac{d^2y}{dt^2} + 25y = 100 \sin \omega t,$$

and the particular solution subject to the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$.

Solution: The auxiliary equation $\lambda^2 + 25 = 0$ has roots $\lambda = \pm 5i$, and so the general solution of the homogeneous equation is $y_h = C\cos 5t + D\sin 5t$. Since the non-homogeneous term is sinusoidal, we try a particular solution of the form, $y_p = \alpha \sin \omega t + \beta \cos \omega t$. This will work as long as $\omega \neq \pm 5$, which we assume for the present. Now, we can save ourselves some trouble by dropping the $\cos \omega t$ term in y_p. This is permitted because there is no first-order (or any odd-order) derivative term in the differential equation and because only a $\sin \omega t$ term appears on the right-hand side. (If you have any doubt about this, keep the cosine term in y_p and find that its coefficient is zero after a calculation.) Substituting $y_p = \alpha \sin \omega t$ into the differential equation gives $-\alpha\omega^2 \sin \omega t + 25\alpha \sin \omega t = 100 \sin \omega t$, from which it follows that $\alpha = 100/(25 - \omega^2)$. Thus, a particular solution is $y_p = 100(25 - \omega^2)^{-1} \sin \omega t$, and the general solution is

$$y = C\cos 5t + D\sin 5t + \frac{100}{25 - \omega^2} \sin \omega t.$$

We want the particular solution such that $y(0) = \dot{y}(0) = 0$. Differentiation of the general solution gives

$$\dot{y} = -5C \sin 5t + 5D\cos 5t + \frac{100\omega}{25 - \omega^2} \cos \omega t.$$

The initial conditions imply that $C = 0$ and $D = -20\omega/(25 - \omega^2)$. Hence the required particular solution is

$$y = \frac{100 \sin \omega t - 20\omega \sin 5t}{25 - \omega^2}.$$

(b) For $\omega = 5$, find a particular solution of the differential equation. Then determine the particular solution with $y(0) = 0$ and $\dot{y}(0) = 0$.

Solution: In the case $\omega = 5$, a solution of the form $y_p = \alpha \sin \omega t + \beta \cos \omega t$ is a solution of the homogeneous equation. The standard trick in this case is to include a factor t, in which case $y_p = \alpha t \sin 5t + \beta t \cos 5t$. As before, we can simplify the problem by a symmetry argument. Because there is no first-order derivative in the differential equation and because the forcing term is an odd function, we can get away with restricting y_p to be an odd function. Thus $y_p = \beta t \cos 5t$. Its derivatives are $\dot{y}_p = \beta(-5t \sin 5t + \cos 5t)$ and $\ddot{y}_p = \beta(-25t \cos 5t - 10 \sin 5t)$. Substituting into the differential equation and cancelling terms shows that $\beta = -10$. Hence a particular solution is $y_p = -10t \cos 5t$, and the general solution is

$$y = (C - 10t)\cos 5t + D \sin 5t.$$

Its derivative is $\dot{y} = (50t - 5C)\sin 5t + (5D - 10)\cos 5t$. The initial conditions are satisfied by $C = 0$ and $D = 2$. Hence the required particular solution is

$$y = 2\sin 5t - 10t \cos 5t.$$

(c) Find the corresponding particular solution of the differential equation for $\omega = 5$ by fixing t in the result of part (a) and taking the limit as ω approaches its special value.

Solution: If one puts $\omega = 5$ in the result of part (a), the solution becomes a 0/0-type indeterminate form. L'Hôpital's rule can be used to take the limit $\omega \to 5$. Here, we must hold t constant while we take derivatives with respect to ω. Thus, in the case of resonance,

$$\begin{aligned} y = \lim_{\omega \to 5} \frac{100 \sin \omega t - 20\omega \sin 5t}{25 - \omega^2} &= \lim_{\omega \to 5} \frac{(\partial/\partial\omega)(100 \sin \omega t - 20\omega \sin 5t)}{(\partial/\partial\omega)(25 - \omega^2)} \\ &= \left. \frac{100t \cos \omega t - 20 \sin 5t}{-2\omega} \right|_{\omega=5} = \frac{100t \cos 5t - 20 \sin 5t}{-10} = 2\sin 5t - 10t\cos 5t. \end{aligned}$$

Of course, the two methods give the same answer. The factor $10t$ shows that the amplitude grows without bound.

Extra questions for further practice

7. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0,$$

expressing your answer in real form. What is the particular solution satisfying $y(0) = 1$ and $y(\pi/4) = 2$?

Solution: The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0$, which has roots $\lambda = 1 \pm 2i$, and so the general solution is

$$y = e^t(A\cos 2t + B\sin 2t).$$

Hence $y(0) = E$ and $y(\pi/4) = e^{\pi/4}F$. If $y(0) = 1$ and $y(\pi/4) = 2$ then $A = 1$ and $B = 2e^{-\pi/4}$, and hence the particular solution is

$$y = e^t\big(\cos 2t + 2e^{-\pi/4}\sin 2t\big).$$

8. Solve the following equations, giving the general solution and then the particular solution $y(x)$ satisfying the given boundary or initial conditions.

(a) $2y'' - 7y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 1$

Solution: The auxiliary equation $2\lambda^2 - 7\lambda + 5 = 0$ has roots $5/2$ and 1, and so the general solution is $y(x) = Ae^{5x/2} + Be^x$, which gives $y'(x) = (5A/2)e^{5x/2} + Be^x$. Hence $y(0) = A + B$ and $y'(0) = (5A/2) + B$, so the initial conditions imply $A = 0$ and $B = 1$, and the particular solution is $y(x) = e^x$.

(b) $y'' + 4y' + 3y = 0, \quad y(-2) = 1,\ y(2) = 1$

Solution: The auxiliary equation $\lambda^2+4\lambda+3 = 0$ has roots -1 and -3, and so the general solution is $y(x) = Ae^{-x} + Be^{-3x}$. Hence $y(-2) = Ae^2 + Be^6$ and $y(2) = Ae^{-2} + Be^{-6}$, so the boundary conditions imply $Ae^2 + Be^6 = 1$ and $Ae^{-2} + Be^{-6} = 1$. Solving these simultaneous equations gives

$$A = \frac{\sinh 6}{\sinh 4} = 7.3915, \qquad B = -\frac{\sinh 2}{\sinh 4} = -0.1329,$$

and so the particular solution satisfying the boundary conditions is

$$y(x) = 7.3915e^{-x} - 0.1329e^{-3x}.$$

(c) $2y'' - 2y' + 5y = 0, \quad y(0) = 0,\ y(2) = 2$

Solution: The auxiliary equation $2\lambda^2 - 2\lambda + 5 = 0$ has roots $(1 \pm 3i)/2$, and so the general solution is $y(x) = e^{x/2}\{A\cos(3x/2) + B\sin(3x/2)\}$. Hence $y(0) = A$, and the first boundary condition implies $A = 0$. Thus $y(2) = Be\sin 3$, and so the second boundary condition implies $B = 2/(e\sin 3) = 5.2137$, and hence the particular solution satisfying the boundary conditions is $y(x) = 5.2137e^{x/2}\sin(3x/2)$.

(d) $y'' - 4y' + 4y = 0, \quad y(0) = -2,\ y(1) = 0$

Solution: The auxiliary equation $\lambda^2 - 4\lambda + 4 = 0$ has one double root $m = 2$, and so the general solution is $y(x) = (A + Bx)e^{2x}$. Hence $y(0) = A$ and the first boundary condition implies $A = -2$. Thus $y(1) = (-2+B)e^2$, and so the second boundary condition implies $B = 2$, and hence the particular solution satisfying the boundary conditions is $y(x) = 2(x-1)e^{2x}$.

9. Find the particular solution of the differential equation $y'' - 6y' + 9y = e^{3x}$ which satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Solution: In part (a)(v), we have

$$y = \Big(C + Dx + \frac{x^2}{2}\Big)e^{3x},$$
$$y' = \Big(3C + 3Dx + \frac{3x^2}{2} + D + x\Big)e^{3x}.$$

Hence $y(0) = C$ and $y'(0) = 3C + D$. So the conditions $y(0) = 1$ and $y'(0) = 0$ imply that $C = 1$ and $D = -3$. Hence, the required particular solution is

$$y = \Big(1 - 3x + \frac{x^2}{2}\Big)e^{3x}.$$