MATH3701: Higher Topology and Differential Geometry Assignment

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1. Consider $d(l\gamma(u)), u \in I$. Note that l is linear and so $dl = l$.

$$\begin{aligned} d(l \circ \gamma(u)) &= d\gamma(u) \cdot dl(\gamma(u)) \\ &= d\gamma(u) \cdot l \circ \gamma(u) \quad \text{as } l \text{ is linear} \\ &= d\gamma(u) \cdot c \ldots (A). \end{aligned}$$

We know that γ sends u into $C \subset P$. For any $x = \gamma(u)$ inside the plane, by definition, $l \circ \gamma(u) = l(x) = c$. Alternatively, we can differentiate as follows.

$$\begin{aligned} \therefore d(l \circ \gamma(u)) &= d(l(x)) \\ &= d(c) \\ &= 0 \ldots (B) \end{aligned}$$

Using both (A) and (B), we arrive at the follwing result.

$$\begin{aligned} \therefore d\gamma(u) \cdot c &= 0 \\ d\gamma(u) &= 0. \end{aligned}$$

Thus all derivatives of γ are zero, so they cannot be linearly independent in $\mathbb{R}^4$. Hence γ is not Frenet. Geometrically speaking, as the curve's derivatives are not linearly independent and all 0, the curve may not move out of the plane it resides in.

2.

$$\begin{aligned} \gamma(u) &= (-u, \sin 2u, \cos 2u)^{\mathsf{T}} \\ \dot{\gamma}(u) &= (-1, 2\cos 2u, -2\sin 2u)^{\mathsf{T}} \\ \ddot{\gamma}(u) &= (0, -4\sin 2u, -4\cos 2u)^{\mathsf{T}}. \end{aligned}$$

Consider $\alpha\dot{\gamma}(u) + \beta\ddot{\gamma}(u) = \mathbf{0}$, where $\alpha, \beta \in \mathbb{R}$. Solving for α and β, we use the following matrix.

$$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 2\cos 2u & -4\sin 2u & 0 \\ -2\sin 2u & -4\cos 2u & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2\sin 2u & -4\cos 2u & 0 \end{array}\right]$$

Clearly, $\alpha = \beta = 0$, thus $\dot{\gamma}(u)$ and $\ddot{\gamma}(u)$ are linearly independent, and so γ is Frenet. For the Frenet frame we get the following results for the ε_i.

$$\varepsilon_1(u) = \frac{\dot{\gamma}(u)}{|\dot{\gamma}(u)|} = \frac{1}{\sqrt{5}}(-1, 2\cos 2u, -2\sin 2u)^{\mathsf{T}}.$$

Notice that $\langle \varepsilon_1(u), \ddot{\gamma}(u) \rangle = 0$ and so $\ddot{\gamma}(u)$ is perpendicular to $\varepsilon_1(u)$. Hence for $\varepsilon_2(u)$ we simply normalise $\ddot{\gamma}(u)$.

$$\varepsilon_2(u) = (0, -\sin 2u, -\cos 2u)^{\mathsf{T}}.$$

For $\varepsilon_3(u)$ we take the cross product of the first two vectors in the distinguished Frenet frame.

$$\begin{aligned} \varepsilon_3(u) &= \varepsilon_1(u) \times \varepsilon_2(u) \\ &= \frac{1}{\sqrt{5}}(-1, 2\cos 2u, -2\sin 2u)^{\mathsf{T}} \times (0, -\sin 2u, -\cos 2u)^{\mathsf{T}} \\ &= \frac{1}{\sqrt{5}}(-2, -\cos 2u, \sin 2u)^{\mathsf{T}}. \end{aligned}$$

Note that $\det(\varepsilon_1(u) \;\; \varepsilon_2(u) \;\; \varepsilon_3(u)) = -1$, so instead choose $\varepsilon_3(u) = \varepsilon_2(u) \times \varepsilon_1(u) = -\varepsilon_1(u) \times \varepsilon 21(u)$, yielding a determinant of $+1$. By the definition of the cross product, and the selections of $\varepsilon_1(u)$, $\varepsilon_2(u)$, and $\varepsilon_3(u)$, they satisfy conditions 1) and 2) for a distinguished Frenet frame. Thus the distinguished Frenet frame for γ is:

$$\begin{aligned}
\varepsilon_1(u) &= \frac{1}{\sqrt{5}}(-1, 2\cos 2u, -2\sin 2u)^{\mathsf{T}} \\
\varepsilon_2(u) &= (0, \sin 2u, \cos 2u)^{\mathsf{T}} \\
\varepsilon_3(u) &= \frac{1}{\sqrt{5}}(2, \cos 2u, -\sin 2u)^{\mathsf{T}}.
\end{aligned}$$

3. **a)** As we have a distinguished Frenet frame for γ, for any i, j, we have $\langle \varepsilon_i(u), \varepsilon_j(u)\rangle = 0$ or 1. In either case we know,

$$\frac{d}{du}\langle \varepsilon_i(u), \varepsilon_j(u)\rangle = 0.$$

Using the product rule, we get the result,

$$\begin{aligned}
\frac{d}{du}\langle \varepsilon_i(u), \varepsilon_j(u)\rangle &= \langle \dot\varepsilon_i(u), \varepsilon_j(u)\rangle + \langle \varepsilon_i(u), \dot\varepsilon_j(u)\rangle \\
\therefore 0 &= \langle \dot\varepsilon_i(u), \varepsilon_j(u)\rangle + \langle \varepsilon_i(u), \dot\varepsilon_j(u)\rangle \\
\therefore \langle \dot\varepsilon_i(u), \varepsilon_j(u)\rangle &= -\langle \varepsilon_i(u), \dot\varepsilon_j(u)\rangle \\
\langle \dot\varepsilon_i(u), \varepsilon_j(u)\rangle &= -\langle \dot\varepsilon_j(u), \varepsilon_i(u)\rangle \\
\therefore w_{ij} &= -w_{ji}.
\end{aligned}$$

b) Part 2) of the proposition indicates that for $1 \le i \le m-1$, then $\mathrm{span}(\varepsilon_1(u), \cdots, \varepsilon_i(u)) = \mathrm{span}(\dot\gamma(u), \ddot\gamma(u), \ldots, \gamma^{(i)}(u))$. Clearly $\varepsilon_i(u) \in \mathrm{span}(\varepsilon_1(u), \cdots, \varepsilon_i(u))$, thus $\varepsilon_i(u) \in \mathrm{span}(\dot\gamma(u), \ddot\gamma(u), \ldots, \gamma^{(i)}(u))$. Hence $\varepsilon_i(u)$ can be written as a linear combination,

$$\varepsilon_i(u) = \alpha_1\dot\gamma(u) + \alpha_2\ddot\gamma(u) + \cdots + \alpha_i\gamma^{(i)}(u).$$

Differentiating yields

$$\begin{aligned}
\dot\varepsilon_i(u) &= \alpha_1\ddot\gamma(u) + \cdots + \alpha_i\gamma^{(i+1)}(u) \\
\therefore \dot\varepsilon_i(u) &\in \mathrm{span}(\ddot\gamma(u), \ldots, \gamma^{(i+1)}(u)).
\end{aligned}$$

As $\mathrm{span}(\ddot\gamma(u), \ldots, \gamma^{(i+1)}(u)) \subseteq \mathrm{span}(\dot\gamma(u), \ddot\gamma(u), \ldots, \gamma^{(i+1)}(u)) = \mathrm{span}(\varepsilon_1(u), \ldots, \varepsilon_{i+1}(u))$, then,

$$\begin{aligned}
\dot\varepsilon_i(u) &\in \mathrm{span}(\dot\gamma(u), \ddot\gamma(u), \ldots, \gamma^{(i+1)}(u)) \\
\therefore \dot\varepsilon_i(u) &\in \mathrm{span}(\varepsilon_1(u), \varepsilon_2(u), \ldots, \varepsilon_{i+1}(u)).
\end{aligned}$$

c) Given $\dot\varepsilon_i(u) \in \mathrm{span}(\varepsilon_1(u), \varepsilon_2(u), \ldots, \varepsilon_{i+1}(u))$ then we can write.

$$\begin{aligned}
\dot\varepsilon_i(u) &= \alpha_1\varepsilon_1(u) + \alpha_2\varepsilon_2(u) + \cdots + \alpha_{i+1}\varepsilon_{i+1}(u). \\
w_{ij} &= \langle \dot\varepsilon_i(u), \varepsilon_j(u)\rangle \\
&= \langle \alpha_1\varepsilon_1(u) + \alpha_2\varepsilon_2(u) + \cdots + \alpha_{i+1}\varepsilon_{i+1}(u), \varepsilon_j(u)\rangle \\
&= \langle \alpha_1\varepsilon_1(u), \varepsilon_j(u)\rangle + \langle \alpha_2\varepsilon_2(u), \varepsilon_j(u)\rangle + \cdots + \langle \alpha_{i+1}\varepsilon_{i+1}(u), \varepsilon_j(u)\rangle \\
&= \begin{cases} 0 & j > i+1 \\ \alpha_j & j \le i+1 \end{cases}
\end{aligned}$$

Similarly for w_{ji},

$$\begin{aligned}
\dot\varepsilon_j(u) &= \beta_1\varepsilon_1(u) + \beta_2\varepsilon_2(u) + \cdots + \beta_{j+1}\varepsilon_{j+1}(u). \\
w_{ji} &= \langle \dot\varepsilon_j(u), \varepsilon_i(u)\rangle \\
&= \langle \beta_1\varepsilon_1(u) + \beta_2\varepsilon_2(u) + \cdots + \beta_{j+1}\varepsilon_{j+1}(u), \varepsilon_i(u)\rangle \\
&= \langle \beta_1\varepsilon_1(u), \varepsilon_i(u)\rangle + \langle \beta_2\varepsilon_2(u), \varepsilon_i(u)\rangle + \cdots + \langle \beta_{j+1}\varepsilon_{j+1}(u), \varepsilon_i(u)\rangle \\
&= \begin{cases} 0 & i > j+1 \\ \beta_i & i \le j+1 \end{cases}
\end{aligned}$$

Furthermore $\alpha_j > 0$, and $\beta_i > 0$, by positivity of the inner product. We also know that $w_{ij} = -w_{ji}$. Hence we have the following cases:

1) $i > j+1$ or $j > i+1$.

In this case either $w_{ij} = 0$ or $w_{ji} = 0$. Either way, as $w_{ij} = -w_{ji}$, we know that $w_{ij} = 0$.

2) $i \leq j+1$ and $j \leq i+1$.

In this case, $i - j \leq 1$, and $i - j \geq -1$. This is equivalent to $|i-j| = 1$.

Thus $w_{ij} = 0$ unless $|i-j| = 1$.

4. Recall we have the following distinguished Frenet frame for γ.

$$\begin{aligned}
\varepsilon_1(u) &= \frac{1}{\sqrt{5}}(-1, 2\cos 2u, -2\sin 2u)^\intercal \\
\varepsilon_2(u) &= (0, -\sin 2u, -\cos 2u)^\intercal \\
\varepsilon_3(u) &= \frac{1}{\sqrt{5}}(2, \cos 2u, -\sin 2u)^\intercal.
\end{aligned}$$

Computing $\kappa_1(u)$ we have,

$$\begin{aligned}
\kappa_1(u) &= \frac{\langle \dot{\varepsilon}_1(u), \varepsilon_2(u) \rangle}{|\dot{\gamma}(u)|} \\
&= \frac{\langle \frac{1}{\sqrt{5}}(0, -4\sin 2u, -4\cos 2u)^\intercal, (0, \sin 2u, \cos 2u)^\intercal \rangle}{\sqrt{5}} \\
&= \frac{0 + 4\sin^2 2u + 4\cos^2 2u}{5} \\
\therefore \kappa_1(u) &= \frac{4}{5}.
\end{aligned}$$

Computing $\kappa_2(u)$ we have,

$$\begin{aligned}
\kappa_2(u) &= \frac{\langle \dot{\varepsilon}_2(u), \varepsilon_3(u) \rangle}{|\dot{\gamma}(u)|} \\
&= \frac{\langle (0, -2\cos 2u, 2\sin 2u)^\intercal, \frac{1}{\sqrt{5}}(2, \cos 2u, -\sin 2u)^\intercal \rangle}{\sqrt{5}} \\
&= \frac{0 - 2\sin^2 2u - 2\cos^2 2u}{5} \\
\therefore \kappa_2(u) &= -\frac{2}{5}.
\end{aligned}$$

5. Given a unit speed Frenet curve $\gamma : I \to \mathbb{R}^m$, we know that $|\dot{\gamma}(u)| = 1$ and,

$$C(u) = (\dot{\gamma}(u), \ddot{\gamma}(u), ..., \gamma^{(m)}(u)).$$

We can also express each column $\gamma^{(i)}(u)$ as a linear combination of the first i vectors in the Frenet frame.

$$\gamma^{(i)}(u) = \alpha_1 \varepsilon_1(u) + \alpha_2 \varepsilon_2(u) + ... + \alpha_i \varepsilon_i(u).$$