(A)

MATH 1902 2003 exam solutions 11/6/2010

Q1/ $P_1: \; 3x+4y-2z=5$
$P_2: \; 2x-3y+4z=3$ $\Big\}$ ℓ intersection of P_1 & P_2

(i) $\underset{\sim}{n}_1 = 3\underset{\sim}{i}+4\underset{\sim}{j}-2\underset{\sim}{k}$ is perpendicular to P_1

$\underset{\sim}{n}_2 = 2\underset{\sim}{i}-3\underset{\sim}{j}+4\underset{\sim}{k}$ " " " P_2

$$\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 4 & -2 \\ 2 & -3 & 4 \end{vmatrix} = 10\underset{\sim}{i} - 16\underset{\sim}{j} - 17\underset{\sim}{k}$$

(ii) $$\cos\theta = \frac{\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2}{|\underset{\sim}{n}_1||\underset{\sim}{n}_2|} = \frac{6-12-8}{\sqrt{9+16+4}\,\sqrt{4+9+16}} = -\frac{14}{29}.$$

(iii) $(1,1,1)$ lies on ℓ because equations for P_1, P_2 are satisfied:

$$3+4-2 = 7-2 = 5 \checkmark$$
$$2-3+4 = 6-3 = 3 \checkmark$$

(iv) (a) $\underset{\sim}{r} = \underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k} + t(10\underset{\sim}{i}-16\underset{\sim}{j}-17\underset{\sim}{k})$

(b) $$\left.\begin{aligned} x &= 1+10t \\ y &= 1-16t \\ z &= 1-17t \end{aligned}\right\} \; t \in \mathbb{R}$$

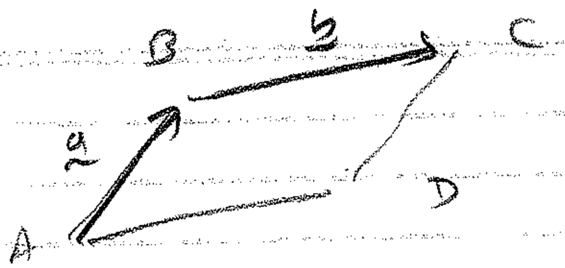
(c) $$\frac{x-1}{10} = \frac{y-1}{-16} = \frac{z-1}{-17}$$

(v) Plane contains $(1,-1,2)$ & has normal $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$ so

has equation $10x-16y-17z = 10+16-34 = -8$.

B

Q2/ (i)



(a) $\vec{AC} = a + b$, $\vec{BD} = b - a$.

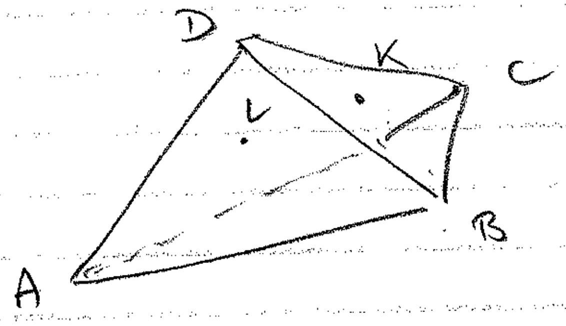
(b) $|\vec{AC}|^2 + |\vec{BD}|^2 = (a+b)\cdot(a+b) + (b-a)\cdot(b-a)$

$$= a\cdot a + \cancel{2a\cdot b} + b\cdot b + b\cdot b - \cancel{2b\cdot a} + a\cdot a$$

$$= 2a\cdot a + 2b\cdot b = 2(|a|^2 + |b|^2),$$

as required.

(ii)



$k = \frac{1}{3}(b + c + d)$

$l = \frac{1}{3}(a + c + d)$

(a) $\vec{KL} = \vec{KO} + \vec{OL} = -k + l = -\frac{1}{3}(b+c+d) + \frac{1}{3}(a+c+d)$

$$= \frac{1}{3}(a + \cancel{c} + \cancel{d} - b - \cancel{c} - \cancel{d}) = \frac{1}{3}(a - b)$$

$$= \frac{1}{3}(\vec{OA} - \vec{OB}) = \frac{1}{3}(\vec{BO} + \vec{OA}) = \frac{1}{3}\vec{BA}$$

$$= -\frac{1}{3}\vec{AB},$$ so KL is $\parallel$ to AB.

Let P be located on AK such that AP : PK = 3 : 1.

Then $\vec{BP} = \vec{BA} + \vec{AP} = -b + a + \frac{3}{4}\vec{AK} = -b + a + \frac{3}{4}(-a + k)$

$$= -b + a + \frac{3}{4}\left(-a + \frac{1}{3}(b+c+d)\right) = -b + a + \frac{1}{4}(-3a + b + c + d)$$

$$= \frac{1}{4}(-4b + 4a - 3a + b + c + d) = \frac{1}{4}(a + c + d - 3b).$$

Q2/ (ii) (a) (cont.)

But $\vec{BL} = -\underline{b} + \underline{l} = -\underline{b} + \frac{1}{3}(\underline{a} + \underline{c} + \underline{d})$

$$= \frac{1}{3}(\underline{a} + \underline{c} + \underline{d} - 3\underline{b}) = \frac{4}{3}\left[\frac{1}{4}(\underline{a} + \underline{c} + \underline{d} - 3\underline{b})\right]$$

$$= 4/3 \ \vec{BP},$$

so $\vec{BP} = 3/4 \ \vec{BL}$, proving $BP : PL = 3 : 1$.

Thus AK and BL meet at P such that

$$AP : PK = 3 : 1 = BP : PL,$$

as required.

(b) By symmetry any two spatial medians intersect at a point P dividing them in the ratio $3:1$, and this point P is the same for each pair, so is the common intersection,

Q3/ (i)

$$\begin{aligned} x + 2y + z &= a \\ 2x + 5y + 3z &= b \\ 2y + 2z &= c \end{aligned}$$

(a) $$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 2 & 5 & 3 & b \\ 0 & 2 & 2 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 2 & 2 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & 0 & c-2b+4a \end{array}\right]$$

To have a solution require $\boxed{c - 2b + 4a = 0}$

Q3/ (i) (a) (cont.)

and then go further:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5a-2b \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $\begin{cases} x \quad\; - z = 5a-2b \\ \quad y + z = b-2a \end{cases}$

Put $z = t$, so $y = b-2a-t$, $x = 5a-2b+t$

and the general solution is

$$\boxed{(x,y,z) = (5a-2b+t,\; b-2a-t,\; t)} \quad t \in \mathbb{R}.$$

(b) A solution exists iff $c-2b+4a=0$

iff $\underset{\sim}{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 2b-4a \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ -4a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 2b \end{bmatrix}$

$$= a\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

as required.

(ii) (a) false, e.g. $\left\{\begin{matrix} x=1 \\ x=1 \end{matrix}\right\}$ has a unique solution

but $r = 2 \neq n = 1$.

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Q3/ (ii) (b) false, e.g. $\left\{\begin{matrix} x=1 \\ x=1 \end{matrix}\right\}$ has a solution

yet $r=2 > n=1$,

(c) false, e.g. $0x=1$ where $A=[0]$, $\underline{x}=[x]$, $\underline{b}=[1]$.

(d) true: $A(B\underline{b}) = (AB)\underline{b} = I_n\underline{b} = \underline{b}$. ✓

Q4/ (i) $a,b,c,d \in \mathbb{Z}$ and $a+b=c+d$, $\lambda_1 = a+b$, $\lambda_2 = a-c$.

$$\det(A-\lambda_1 I) = \begin{vmatrix} a-a-b & b \\ c & d-c-d \end{vmatrix} = \begin{vmatrix} -b & b \\ c & -c \end{vmatrix} = bc-bc=0,$$

$$\det(A-\lambda_2 I) = \begin{vmatrix} a-a+c & b \\ c & d-d+b \end{vmatrix} = \begin{vmatrix} c & b \\ c & b \end{vmatrix} = cb-cb=0,$$

so λ_1, λ_2 are eigenvalues.

$\begin{bmatrix} -b & b \\ c & -c \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for λ_1

$\begin{bmatrix} c & b \\ c & b \end{bmatrix} \sim \begin{bmatrix} c & b \\ 0 & 0 \end{bmatrix}$ so $\begin{bmatrix} -b/c \\ 1 \end{bmatrix}$ " " " " λ_2

(assuming throughout $a,b,c,d \neq 0$)

(otherwise there are a myriad of special cases)

P

Q4/ (ii) (a) $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$

$$AX = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a+2c & -b+2d \\ 2a+2c & 2b+2d \end{bmatrix}$$

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -a+2b & 2a+2b \\ -c+2d & 2c+2d \end{bmatrix}$$

so $-a+2c = -a+2b$, so $\boxed{b=c}$

$$\left.\begin{aligned} -b+2d &= 2a+2b \\ 2a+2c &= -c+2d \end{aligned}\right\} \boxed{3b = 2(d-a)}$$

$$2b+2d = 2c+2d$$

so $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3a & 3b \\ 3b & 3d \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 3a & 2(d-a) \\ 2(d-a) & 3d \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1(d-a)+d+2a & 2(d-a) \\ 2(d-a) & 2(d-a)+d+2a \end{bmatrix}$$

$$= \frac{d-a}{3}\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} + \frac{d+2a}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$= \alpha A + \beta I$ where $\alpha = \frac{d-a}{3}$, $\beta = \frac{d+2a}{3}$.

Clearly any matrix of the form $\alpha A + \beta I$ commutes with A (since A, I commute with A), so this characterises all such possible X, as required.

Q4/ (ii) (b) $A^2 = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

$= \alpha A + \beta I$ where $\boxed{\alpha = 1, \beta = 6.}$

(iii) $Y = \begin{bmatrix} 0 & 5 & 10 \\ 1 & * & * \\ 2 & * & * \end{bmatrix}$ cannot be a polynomial

in $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$ because if $p(\lambda)$ is a

polynomial then $p(B) = \begin{bmatrix} p(3) & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$

and, for example, $5 \neq 0$.