The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Exercises for Week 5 (beginning 02 April)** 2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Algebraic definition of cross product: If $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$ then

$$\mathbf{v}\times\mathbf{w} \;=\; (v_2w_3 - v_3w_2)\,\mathbf{i} + (v_3w_1 - v_1w_3)\,\mathbf{j} + (v_1w_2 - v_2w_1)\,\mathbf{k}\,.$$

which can be evaluated by

(a) using the "up-and-down-diagonal" method;

(b) using the "expanding brackets" method and the facts that

$$\mathbf{i}\times\mathbf{j} \;=\; \mathbf{k} \;=\; -\big(\mathbf{j}\times\mathbf{i}\big)\,,\quad \mathbf{j}\times\mathbf{k} \;=\; \mathbf{i} \;=\; -\big(\mathbf{k}\times\mathbf{j}\big)\,,\quad \mathbf{k}\times\mathbf{i} \;=\; \mathbf{j} \;=\; -\big(\mathbf{i}\times\mathbf{k}\big)\,,$$

$$\mathbf{i}\times\mathbf{i} \;=\; \mathbf{j}\times\mathbf{j} \;=\; \mathbf{k}\times\mathbf{k} \;=\; 0\,;$$

(c) evaluating a 3×3 determinant (explained later): $\mathbf{v}\times\mathbf{w} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\,.$

(ii) The cross product $\mathbf{v}\times\mathbf{w}$ is always perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ so that

$$\big(\mathbf{v}\times\mathbf{w}\big)\cdot\mathbf{v} \;=\; \big(\mathbf{v}\times\mathbf{w}\big)\cdot\mathbf{w} \;=\; 0\,.$$

(iii) Anti-commutativity of cross product: $\mathbf{v}\times\mathbf{w} \;=\; -\big(\mathbf{w}\times\mathbf{v}\big)\,.$

(iv) Distributivity of cross over plus: $(\mathbf{u}+\mathbf{v})\times\mathbf{w} \;=\; \mathbf{u}\times\mathbf{w} \,+\, \mathbf{v}\times\mathbf{w}\,.$

(v) If $\mathbf{v}$ and $\mathbf{w}$ are vectors and λ is a scalar then

$$(\lambda\mathbf{v})\times\mathbf{w} = \lambda(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\times(\lambda\mathbf{w}) \qquad \text{and} \qquad \mathbf{v}\times\mathbf{v} = \mathbf{0}\,.$$

(vi) The area of the parallelogram inscribed by $\mathbf{v}$ and $\mathbf{w}$ is $|\mathbf{v}\times\mathbf{w}|\,.$

(vii) The area of the triangle inscribed by $\mathbf{v}$ and $\mathbf{w}$ is $\dfrac{|\mathbf{v}\times\mathbf{w}|}{2}\,.$

(viii) Geometric formula for cross product: if θ is the angle between vectors $\mathbf{v}$ and $\mathbf{w}$ chosen so that $0\le\theta\le\pi$ then

$$\mathbf{v}\times\mathbf{w} \;=\; |\mathbf{v}||\mathbf{w}|\sin\theta\;\;\mathbf{u}\,,$$

where $\mathbf{u}$ is the unit vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ such that the triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is right-handed. In particular

$$|\mathbf{v}\times\mathbf{w}| \;=\; |\mathbf{v}||\mathbf{w}|\sin\theta\,.$$

(ix) Triple product: If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \;=\; (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

and its magnitude is the volume of the parallelopiped spanned by the three vectors, when placed tail-to-tail in space. If nonzero, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is positive if and only if the triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is right-handed.

Preparatory Exercises:

1. Write down

(i) $\mathbf{i} \times \mathbf{j}$ (ii) $2\mathbf{i} \times 3\mathbf{j}$ (iii) $\mathbf{i} \times \big(-4\mathbf{j}\big)$ (iv) $\mathbf{j} \times \mathbf{i}$ (v) $\mathbf{j} \times \big(-4\mathbf{i}\big)$
(vi) $\mathbf{j} \times \mathbf{k}$ (vii) $\mathbf{k} \times \mathbf{k}$ (viii) $\mathbf{k} \times \big(-\mathbf{k}\big)$ (ix) $\big(-\mathbf{k}\big) \times \mathbf{i}$ (x) $\big(-\mathbf{k}\big) \times \big(-\mathbf{j}\big)$
(xi) $\mathbf{k} \times \big(\mathbf{i} + \mathbf{k}\big)$ (xii) $\big(3\mathbf{j} - \mathbf{k}\big) \times 2\mathbf{j}$ (xiii) $\big(\mathbf{j} - \mathbf{k}\big) \times \big(\mathbf{k} + \mathbf{j}\big)$

2. Evaluate

(i) $\mathbf{i} \times \big(\mathbf{i} + \mathbf{j} + \mathbf{k}\big)$ (ii) $\big(\mathbf{i} + \mathbf{j} + \mathbf{k}\big) \times \big(2\mathbf{i} + \mathbf{k}\big)$
(iii) $\big(2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}\big) \times \big(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}\big)$ (iv) $\big(\mathbf{i} - \mathbf{j} + 3\mathbf{k}\big) \times \big(3\mathbf{i} + \mathbf{j} - \mathbf{k}\big)$

3. Given that

$$\mathbf{a} \;=\; 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\,, \qquad \mathbf{b} \;=\; \mathbf{i} + \mathbf{j} - \mathbf{k}\,,$$

find

(i) $|\mathbf{a}|$ (ii) $|\mathbf{b}|$ (iii) $\mathbf{a} \times \mathbf{b}$ (iv) $|\mathbf{a} \times \mathbf{b}|$
(v) the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$.

4. Evaluate

(i) $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$ (ii) $\big((\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k})\big) \times (\mathbf{k} + \mathbf{i})$
(iii) $\mathbf{i} \times (\mathbf{j} \times \mathbf{k})$ (iv) $(\mathbf{i} + \mathbf{j}) \times \big((\mathbf{j} + \mathbf{k}) \times (\mathbf{k} + \mathbf{i})\big)$

5. Given that $P = (8, 4, -1)$, $Q = (6, 3, -4)$ and $R = (7, 5, -5)$, find

$$\overrightarrow{QP} \times \overrightarrow{QR}$$

and the area of the triangle $\triangle PQR$.

6. Consider the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(i) Verify by direct calculation that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \;=\; -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}\,.$$

(This identity holds in general, the verification of which is an exercise below.)

(ii) Find the volume of the parallelopiped inscribed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Tutorial Exercises:

7. Given that $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = -2\mathbf{i} + \mathbf{k}$ find

(i) $\mathbf{a} \times \mathbf{b}$ (ii) $\mathbf{a} \times \mathbf{c}$ (iii) $\mathbf{b} \times \mathbf{c}$ (iv) $\mathbf{a} \times \big(\mathbf{b} \times \mathbf{c}\big)$ (v) $\big(\mathbf{a} \times \mathbf{b}\big) \times \mathbf{c}$

(vi) $\mathbf{a} \times \big(\mathbf{a} \times \mathbf{c}\big)$ (vii) $\mathbf{a} \times \big(\mathbf{a} + \mathbf{c}\big)$ (viii) $\big(\mathbf{a} \times \mathbf{a}\big) \times \mathbf{c}$ (ix) $\mathbf{a} \times \big(\mathbf{b} - 2\mathbf{c}\big)$

(x) the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$

(xi) the area of the parallelogram inscribed by $\mathbf{a}$ and $\mathbf{c}$

(xii) the area of the triangle inscribed by $\mathbf{b}$ and $\mathbf{c}$

(xiii) the volume of the parallelopiped inscribed by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

8. Given that $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $\mathbf{v} \times \mathbf{w} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ find

(i) $\mathbf{w} \times \mathbf{v}$ (ii) $\big(\mathbf{v} + 3\mathbf{w}\big) \times \big(2\mathbf{w} - \mathbf{v}\big)$

9. Calculate $|\mathbf{a} \times \mathbf{b}|$ given that $|\mathbf{a}| = 7$, $|\mathbf{b}| = 4$ and $\mathbf{a} \cdot \mathbf{b} = -21$.

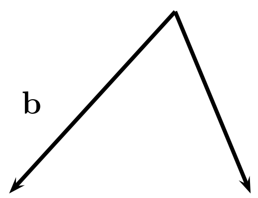
10. Use the algebraic definition of the cross product to verify the following properties for any vectors $\mathbf{v}$ and $\mathbf{w}$:

(i) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} = 0$ (ii) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{w} = 0$

(iii) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$ (iv) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

11. Use the parallelogram property of the cross product to deduce quickly that vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

12. (suitable for group discussion) Let $\mathbf{a}$ and $\mathbf{b}$ be the following vectors in the page:



True or false:

(i) $\mathbf{a} \times \mathbf{b}$ points upwards, away from the page, towards the ceiling

(ii) $\mathbf{b} \times \big(\mathbf{a} - \mathbf{b}\big)$ points downwards, away from the page, towards the floor

(iii) $\mathbf{a} \times \big(\mathbf{b} \times \mathbf{a}\big)$ is perpendicular to $\mathbf{a}$ but not to $\mathbf{b}$

(iv) $\mathbf{b} \times \big(\mathbf{b} \times \mathbf{a}\big)$ is the zero vector

13. (suitable for group discussion) Does the expression $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ make sense? Does the equation $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ imply $\mathbf{v} = \mathbf{w}$ whenever $\mathbf{u} \neq \mathbf{0}$?

14.* A tetrahedron has four faces. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ be vectors perpendicular to the faces, pointing outwards, of length equal to the respective areas of the faces. Verify that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 \;=\; \mathbf{0}\,.$$

15.* Verify that, for any geometric vectors **a**, **b**, **c**,

$$\mathbf{a}\cdot\big(\mathbf{b}\times\mathbf{c}\big) \;=\; \big(\mathbf{a}\times\mathbf{b}\big)\cdot\mathbf{c}\,.$$

Give both algebraic and geometric verifications. Use anticommutativity of the cross-product to deduce that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \;=\; -(\mathbf{b}\times\mathbf{a})\cdot\mathbf{c}\,.$$

Further Exercises:

16. Find two unit vectors perpendicular to both **v** and **w** where

$$\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 7\mathbf{k} \qquad \text{and} \qquad \mathbf{w} = 5\mathbf{i} + \mathbf{j} + \mathbf{k}\,.$$

17. Find the areas of the triangles having vertices

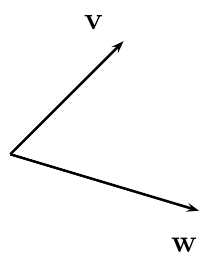
(i) $(0,0,0)\,,\ (2,2,-1)\,,\ (3,-4,2)$ (ii) $(3,-1,2)\,,\ (1,-1,-3)\,,\ (4,-3,1)$

18. Consider the points $P(1,1,1)\,,\ Q(-1,-1,0)\ \ R(0,1,2)\,,\ S(2,3,3)$ in space.

(i) Use cross products to find the areas of the triangle PQR and QRS. Are you surprised? (What type of geometric figure is formed by $PQRS$?)

(ii) Find the distance d_1 from P to R, the distance d_2 from Q to S, and evaluate and interpret $\dfrac{d_1 d_2}{4}$. Are you surprised? (What is the relationship between diagonals of $PQRS$?)

19. Suppose **v** and **w** are the following vectors lying in this page:



Decide which of

(i) $\mathbf{v}\times(\mathbf{v}-\mathbf{w})$ (ii) $(\mathbf{v}-\mathbf{w})\times\mathbf{w}$ (iii) $(\mathbf{v}-\mathbf{w})\times(\mathbf{w}-\mathbf{v})$

(iv) $(\mathbf{v}\times\mathbf{w})\times\mathbf{w}$ (v) $\mathbf{v}\times(\mathbf{v}\times\mathbf{w})$

(a) is perpendicular to **v** but not to **w**.

(b) is perpendicular to **w** but not to **v**.

(c) points upwards, away from the page, towards the ceiling.

(d) points downwards, away from the page, towards the floor.

(e) is the zero vector.

20. Use the cross product to find

(i) a unit vector perpendicular to both $-\mathbf{i} + 2\mathbf{j}$ and $\mathbf{j} + 3\mathbf{k}$,

(ii)* a unit vector which points in a direction which is perpendicular to the triangle with vertices

$$A(0,0,-1)\,,\quad B(1,-2,-1)\,,\quad C(1,-3,-4)$$

such that looking backwards along the vector (from tip to tail) towards the triangle, the vertices A, B, C rotate anticlockwise (in that order).

21.* Verify that if $\mathbf{a}$ and $\mathbf{b}$ are geometric vectors then the following "correction" to the Cauchy-Schwarz Inequality holds:

$$\sqrt{|\mathbf{a}\cdot\mathbf{b}|^2 + |\mathbf{a}\times\mathbf{b}|^2} \;=\; |\mathbf{a}|\;|\mathbf{b}|\;.$$

22.* Carefully verify one of the distributivity laws from the algebraic definition, say the law

$$(\mathbf{u}+\mathbf{v})\times\mathbf{w} \;=\; \mathbf{u}\times\mathbf{w} + \mathbf{v}\times\mathbf{w}\;,$$

and deduce the other distributivity law using anti-commutativity.

23.* Verify that, for any geometric vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,

$$\big(\mathbf{a}\times\mathbf{b}\big)\times\mathbf{c} \;=\; \big(\mathbf{a}\cdot\mathbf{c}\big)\mathbf{b} - \big(\mathbf{b}\cdot\mathbf{c}\big)\mathbf{a}\;.$$

Use anti-commutativity to deduce that

$$\mathbf{a}\times\big(\mathbf{b}\times\mathbf{c}\big) \;=\; \big(\mathbf{a}\cdot\mathbf{c}\big)\mathbf{b} - \big(\mathbf{a}\cdot\mathbf{b}\big)\mathbf{c}\;.$$

24.* Verify the *Jacobi identity*:

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} \;+\; (\mathbf{v}\times\mathbf{w})\times\mathbf{u} \;+\; (\mathbf{w}\times\mathbf{u})\times\mathbf{v} \;\;=\;\; \mathbf{0}\;.$$

25.** Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in space. Prove that the equation

$$\big(\mathbf{u}\times\mathbf{v}\big)\times\mathbf{w} \;=\; \mathbf{u}\times\big(\mathbf{v}\times\mathbf{w}\big)$$

holds if and only if $\mathbf{u}$ and $\mathbf{w}$ are parallel or $\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{w}$. Thus associativity of the cross product fails almost all of the time.

Short Answers to Selected Exercises:

1. (i) $\mathbf{k}$ (ii) $6\mathbf{k}$ (iii) $-4\mathbf{k}$ (iv) $-\mathbf{k}$ (v) $4\mathbf{k}$ (vi) $\mathbf{i}$ (vii) $\mathbf{0}$ (viii) $\mathbf{0}$

(ix) $-\mathbf{j}$ (x) $-\mathbf{i}$ (xi) $\mathbf{j}$ (xii) $2\mathbf{i}$ (xiii) $2\mathbf{i}$

2. (i) $\mathbf{k}-\mathbf{j}$ (ii) $\mathbf{i}+\mathbf{j}-2\mathbf{k}$ (iii) $7\mathbf{i}-14\mathbf{j}-14\mathbf{k}$ (iv) $-2\mathbf{i}+10\mathbf{j}+4\mathbf{k}$

3. (i) 3 (ii) $\sqrt{3}$ (iii) $-\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ (iv) $\sqrt{26}$ (v) $\dfrac{\sqrt{78}}{9}$

4. (i) $\mathbf{0}$ (ii) $-\mathbf{i}+\mathbf{k}$ (iii) $\mathbf{0}$ (iv) $-\mathbf{i}+\mathbf{j}$

5. $-7\mathbf{i}+5\mathbf{j}+3\mathbf{k}$, $\dfrac{\sqrt{83}}{2}$ **6.** 1

7. (i) $-\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ (ii) $-\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ (iii) $\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ (iv) $-2\mathbf{i}-4\mathbf{j}-5\mathbf{k}$

(v) $-2\mathbf{i}-5\mathbf{j}-4\mathbf{k}$ (vi) $2\mathbf{i}+4\mathbf{j}-5\mathbf{k}$ (vii) $-\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ (viii) $\mathbf{0}$ (ix) $\mathbf{i}+2\mathbf{j}+7\mathbf{k}$

(x) $\sqrt{14}/\sqrt{15}$ (xi) 3 (xii) $\sqrt{14}/2$ (xiii) 5

8. (i) $-2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ (ii) $10\mathbf{i}-5\mathbf{j}+15\mathbf{k}$ **9.** $7\sqrt{7}$

12. (i) False (ii) False (iii) True (iv) False **16.** $\pm\dfrac{\sqrt{2}}{6}\big(\mathbf{i}-4\mathbf{j}-\mathbf{k}\big)$

17. (i) $7\sqrt{5}/2$ (ii) $\sqrt{165}/2$

18. (i) $\sqrt{17}/2$, $\sqrt{17}/2$ (ii) $\sqrt{17}/2$

19. (i) (c) (ii) (d) (iii) (c), (d), (e) (iv) (b) (v) (a)

20. (i) $\pm\dfrac{1}{\sqrt{46}}(6\mathbf{i}+3\mathbf{j}-\mathbf{k})$ (ii) $\dfrac{1}{\sqrt{46}}(6\mathbf{i}+3\mathbf{j}-\mathbf{k})$