The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2 **Exercises for Week 7 (beginning 11 September)** 2017

*It might be useful to attempt the Revision and Exploration Exercises before the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.*

**Important Ideas and Useful Facts:**

(i) Ratio Test for convergence: If the limit $L = \lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right|$ exists then the series $\sum_{k=0}^{\infty} a_k$ converges if $L < 1$, and diverges if $L > 1$. (If the limit $L = 1$ then the Ratio Test gives no information about convergence.)

(ii) Power series: A *power series about $x = a$* is an expression of the form

$$\sum_{k=0}^{\infty} a_k(x-a)^k \;=\; a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n + \ldots \,.$$

When evaluated as a series, convergence or divergence depends on the value of $x$.

(iii) Radius and interval of convergence: There are three possibilities for convergence of a given power series $\sum_{k=0}^{\infty} a_k(x-a)^k$ :

(a) There is a positive constant $R$, called *the radius of convergence*, such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(b) The series converges for all $x$ (in which case we think of the radius of convergence as infinite).

(c) The series converges only for $x = a$ (in which case we think of the radius of convergence as zero).

The *interval of convergence* is the set of all $x$ for which the series converges. In the first case, this could be any of $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$ or $[a-R, a+R]$.

(iv) Taylor and Maclaurin series: The *Taylor series expansion about $x = a$* of an infinitely differentiable function $y = f(x)$ has the form

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \,, \end{aligned}$$

which is a valid representation of the rule for $f$ on the interval of convergence of the power series. When $a = 0$, this expansion is also called a *Maclaurin* series.

(v) Uniqueness of a power series expansion of a function: If a power series can be used to represent the rule of an infinitely differentiable function on its interval of convergence, then it coincides with the Taylor series expansion of the function.

(vi) Differentiating and integrating power series: Assume the radius of convergence $R$ is positive. The function $f$ defined by the rule

$$f(x) \ = \ \sum_{k=0}^{\infty} a_k (x-a)^k$$

is differentiable (and hence continuous) on the interval $(a-R, a+R)$. Its derivative and antiderivative have the same radius of convergence and

$$f'(x) \ = \ \sum_{k=1}^{\infty} k a_k (x-a)^{k-1} \, , \qquad \int f(x)\, dx \ = \ C + \sum_{k=0}^{\infty} a_k \frac{(x-a)^{k+1}}{k+1} \, .$$

(vii) Some common power series expansions: The following converge for all $x$:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^k \frac{x^{2k}}{(2k)!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2k}}{(2k)!} + \ldots
\end{aligned}$$

**Revision and Exploration:**

**1.** Consider a sequence $\{a_n\}_{n=0}^{\infty}$. Suppose $k$ is a positive integer and that $b_n = a_{n+k}$ for each $n \geq 0$. Suppose that $\lim_{n\to\infty} a_n = L$ exists. Write out carefully what this means and verify that $\lim_{n\to\infty} b_n = L$ also.

**2.** Write out Taylor's Theorem from first semester. What is the connection between the Taylor polynomials of a function and the Taylor series expansion of a function?

**3.** Why is the definite integral $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx$ technically improper? How can you make a minor adjustment so that it becomes a proper definite integral?

**4.** Write out the Taylor polynomial of degree 6 for $f(x) = \sin x$ about $x = 0$ and the remainder term predicted by Taylor's Theorem. Deduce that, for $0 < x \leq 1$,

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \ \leq \ \frac{\sin x}{x} \ \leq \ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} \, .$$

**5.** Use the previous exercise to show that $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx = 0.946$ to three decimal places.

**Tutorial Exercises:**

**6.** (for general discussion) Verify that if $\sum_{n=0}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$ . Is the converse true?

**7.** Use the ratio test to decide which of the following series converge:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{2^n}$

**8.** Employ a geometric series to write down a Taylor series about $x = 1$ for $f(x) = \dfrac{1}{x}$.

**9.** Find the Maclaurin series for $\tan^{-1} x$ by first writing down the geometric series for $\dfrac{1}{1+x^2}$ and then antidifferentiating.

*__10.__ Manipulate power series to find the first three nonzero terms in the Maclaurin series for the following functions:

(i) $f(x) = e^{-x^2} \sinh x$ (ii) $g(x) = \dfrac{\ln(1-x)}{e^x}$

**11.** (for general discussion) For any complex number $z$ define

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

which converges always (just as in the real case). Verify Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

for all real numbers $\theta$. (In first semester, Euler's formula was a definition. Now it becomes a theorem!)

**Further Exercises:**

*__12.__ Make sense of and find a power series representation about $x = 0$ for the function $f$ given by the rule

$$f(x) = \int_0^x \frac{e^t - 1}{t}\, dt\ .$$

*__13.__ Apply Taylor's Theorem to the function $f(x) = \ln(1+x)$ to prove that the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to $\ln 2$.

**__14.__ Prove directly (without invoking Taylor's Theorem) that the alternating harmonic series converges. (If you do this successfully, then your proof can be modified to prove a general result known as the *Alternating Series Test*.)

***15.** The *limit comparison test* says that if $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a positive real number, then $\sum a_n$ and $\sum b_n$ converge or diverge together. Use the limit comparison test to decide which of the following series converge:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 2}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n - 2}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n}$

****16.** Prove the limit comparison test for series.

****17.** Make sense of the claim

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} \quad = \quad \sqrt{2\times\sqrt{2\times\sqrt{2\times\sqrt{2\times\cdots}}}}\,.$$

**Short Answers to Selected Exercises:**

**1.** $(\forall\epsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N) \quad |a_n - L| < \epsilon$ .

**6.** The harmonic series is a counterexample to the converse.

**7.** (i) converges (ii) diverges (iii) converges

**8.** $\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$

**9.** $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

**10.** (i) $e^{-x^2} \sinh x = x - \frac{5}{6}x^3 + \frac{41}{120}x^5 + \cdots$ (ii) $e^{-x} \ln(1-x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots$

**15.** (i) converges (ii) diverges (iii) diverges

**17.** Both evaluate to 2.