THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Solutions Week 5** 2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Sketch the graph of the function:

$$f(x) = \begin{cases} 0 & x < 0, \\ 1 & x = 0, \\ x + 2 & x > 0. \end{cases}$$

Find $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ (no need for formal proofs). Does $\lim_{x \to 0} f(x)$ exist?

Questions for the tutorial

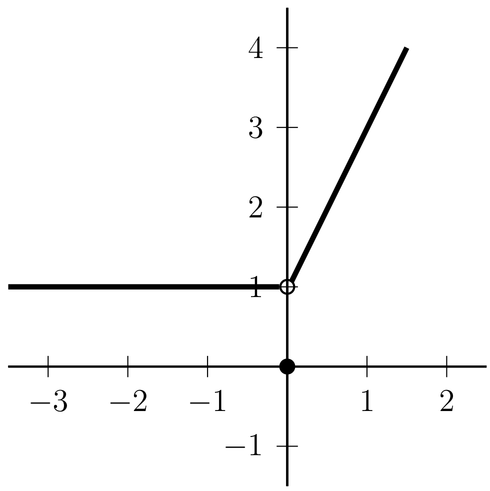
2. Sketch the function with formula

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 2x + 1 & \text{if } x > 0. \end{cases}$$

Find suitable values of δ such that whenever $0 < |x| < \delta$, we have

(a) $|f(x) - 1| < 0.01$, (b) $|f(x) - 1| < 0.001$, (c) $|f(x) - 1| < \epsilon$, where $\epsilon > 0$.

Solution



(a) We want to find δ such that whenever $0 < |x| < \delta$, we have $|f(x) - 1| < 0.01$; that is, $0.99 < f(x) < 1.01$. First observe that if $x < 0$, we have $f(x) = 1$ and so the condition $0.99 < f(x) < 1.01$ is automatically satisfied. If $x > 0$, then $0.99 < f(x) < 1.01$ if and only if $2x + 1 < 1.01$, that is, $x < 0.005$. Therefore if we take $\delta = 0.005$ (or any smaller positive number), then whenever $0 < |x| < \delta$, we are guaranteed that $|f(x) - 1| < 0.01$.

(b) This time, when $x > 0$ we require $0.999 < f(x) < 1.001$, that is, $2x + 1 < 1.001$. If we take $\delta = 0.0005$ (or any smaller positive number), then whenever $0 < |x| < \delta$, we are guaranteed that $|f(x) - 1| < 0.001$.

(c) When $x > 0$ we require $2x + 1 < 1 + \epsilon$, that is, $0 < x < \frac{\epsilon}{2}$. We can take δ to be $\frac{\epsilon}{2}$ (or any smaller positive number).

3. Find the following limits using one or more of the limit laws.

(a) $\lim_{x\to 3} \dfrac{x^2+3x+2}{4x^2-x+1}$ (b) $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$ (c) $\lim_{x\to 1} \dfrac{x^2-3x+2}{x^3-1}$

(d) $\lim_{x\to 0} x^2 \cos\dfrac{2}{x}$ (e) $\lim_{x\to 0} \dfrac{\sqrt{3+2x}-\sqrt{3}}{x}$ (f) $\lim_{x\to\infty} \dfrac{x+\sin^3 x}{2x-1}$

(g) $\lim_{x\to\infty} \sqrt{\dfrac{3-x}{4-x}}$ (h) $\lim_{x\to\infty} \sqrt{\dfrac{3-x}{4-x^2}}$ (i) $\lim_{x\to\infty} (\sqrt{x}-\sqrt{x+1})$

Solution

The solutions below are written with the expectation that all limit laws may be validly applied to each of the simplified expressions.

(a) Using the Addition, Product and Quotient Laws, we have

$$\lim_{x\to 3} \frac{x^2+3x+2}{4x^2-x+1} = \frac{(\lim_{x\to 3}\ x)^2+3(\lim_{x\to 3}\ x)+2}{4(\lim_{x\to 3}\ x)^2-\lim_{x\to 3}\ x+1} = \frac{3^2+3(3)+2}{4(3)^2-3+1} = \frac{10}{17}.$$

(b) For all $x \neq 1$,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1.$$

Using the Addition Law, we see that $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = \lim_{x\to 1}\ x+1 = 1+1 = 2.$

(c) Observe that for all $x \neq 1$,

$$\frac{x^2-3x+2}{x^3-1} = \frac{(x-1)(x-2)}{(x-1)(x^2+x+1)} = \frac{(x-2)}{(x^2+x+1)}.$$

Then, using the Addition, Product and Quotient Laws, we have

$$\lim_{x\to 1} \frac{x^2-3x+2}{x^3-1} = \lim_{x\to 1} \frac{x-2}{x^2+x+1} = \frac{1-2}{1+1+1} = -\tfrac{1}{3}.$$

(d) We use the Squeeze Law. Since $-x^2 \leq x^2\cos\dfrac{2}{x} \leq x^2$ and $\lim_{x\to 0} \pm x^2 = 0$, we have $\lim_{x\to 0} x^2\cos\dfrac{2}{x} = 0.$

(e) We can't use the limit laws with the expression in its present form, so we manipulate it first.

$$\begin{aligned}\frac{\sqrt{3+2x}-\sqrt{3}}{x} &= \frac{(\sqrt{3+2x}-\sqrt{3})(\sqrt{3+2x}+\sqrt{3})}{x(\sqrt{3+2x}+\sqrt{3})}\\ &= \frac{3+2x-3}{x(\sqrt{3+2x}+\sqrt{3})}\\ &= \frac{2}{\sqrt{3+2x}+\sqrt{3}}.\end{aligned}$$

Hence

$$\lim_{x\to 0} \frac{\sqrt{3+2x}-\sqrt{3}}{x} = \lim_{x\to 0} \frac{2}{\sqrt{3+2x}+\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Note that the last step used the Substitution Law to evaluate the limit of the denominator.

(f) $\displaystyle\lim_{x\to\infty} \frac{x+\sin^3 x}{2x-1} = \lim_{x\to\infty} \frac{1+\frac{\sin^3 x}{x}}{2-\frac{1}{x}} = \frac{1}{2}.$

Note: we have used the fact that $\displaystyle\lim_{x\to\infty} \frac{\sin^3 x}{x} = 0$, which follows from an application of the Squeeze Law. Since $-1 \le \sin^3 x \le 1$, we have (for $x > 0$)

$$-\frac{1}{x} \le \frac{\sin^3 x}{x} \le \frac{1}{x} \quad \text{and} \quad \lim_{x\to\infty} \pm\frac{1}{x} = 0.$$

(g) Divide top and bottom inside the square root sign by $-x$. We obtain

$$\sqrt{\frac{3-x}{4-x}} = \sqrt{\frac{-\frac{3}{x}+1}{-\frac{4}{x}+1}}.$$

Now as $x \to \infty$, $-\frac{3}{x}+1 \to 1$ and $-\frac{4}{x}+1 \to 1$. By the Substitution Law, as the square root function is continuous, we see that $\displaystyle\lim_{x\to 0} \sqrt{\frac{3-x}{4-x}} = \sqrt{\frac{1}{1}} = 1.$

(h) This time we divide top and bottom by $-x^2$. We obtain

$$\sqrt{\frac{3-x}{4-x^2}} = \sqrt{\frac{-\frac{3}{x^2}+\frac{1}{x}}{-\frac{4}{x^2}+1}}.$$

Now as $x \to \infty$, $-\frac{3}{x^2}+\frac{1}{x} \to 0$ and $-\frac{4}{x^2}+1 \to 1$. By the Substitution Law, as the square root function is continuous, we see that $\displaystyle\lim_{x\to\infty} \sqrt{\frac{3-x}{4-x^2}} = \sqrt{\frac{0}{1}} = 0.$

(i) $\displaystyle\lim_{x\to\infty} (\sqrt{x}-\sqrt{x+1}) = \lim_{x\to\infty} \frac{(\sqrt{x}-\sqrt{x+1})(\sqrt{x}+\sqrt{x+1})}{\sqrt{x}+\sqrt{x+1}} = \lim_{x\to\infty} \frac{-1}{\sqrt{x}+\sqrt{x+1}} = 0.$

4. Prove the following results using the ϵ, δ definition:

(a) $\displaystyle\lim_{x\to a} c = c$

(b) $\displaystyle\lim_{x\to 4} f(x) = -3$, where $f(x) = \begin{cases} 5-2x & \text{if } x \neq 4, \\ 100 & \text{if } x = 4. \end{cases}$

(c) $\displaystyle\lim_{x\to 0} g(x) = 0$, where $g(x) = \begin{cases} 3x & \text{if } x \text{ is rational}, \\ 7x & \text{if } x \text{ is irrational}. \end{cases}$

Solution

(a) Given any $\epsilon > 0$, we need to show there exists a number $\delta > 0$ so that for any x satisfying $0 < |x-a| < \delta$, we have $|c-c| < \epsilon$. But $|c-c| = 0$, so we can choose any δ (such as $\delta = 57$) for this to be true.

(b) Observe that to find $\displaystyle\lim_{x\to 4} f(x)$, the value of the function at 4 is irrelevant. So for this proof we will need to use the formula $f(x) = 5-2x$. The first step is to investigate $|f(x)-(-3)|$ and if possible to get some idea of which values of x will make this expression less than any given $\epsilon > 0$.
Now $|f(x)-(-3)| = |(5-2x)-(-3)| = |8-2x| = 2|4-x|$. This tells us that the difference between $f(x)$ and -3 can be made as small as we like by making the difference between x and 4 small enough. In particular, to guarantee that $|f(x)-(-3)| < \epsilon$ (which is the same as saying that $2|4-x| < \epsilon$), we need only ensure that the x values satisfy $|x-4| < \epsilon/2$. Thus $|f(x)-(-3)| < \epsilon$ whenever $0 < |x-4| < \delta$, where $\delta = \epsilon/2$.

(c) We must prove that for each $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$, we have $|g(x)| < \epsilon$. But clearly $|g(x)| \leq 7|x|$, so $\delta = \epsilon/7$ has the required property.

5. The function f is defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Using the Squeeze Law, prove that $\lim_{x\to 0} f(x) = 0$.

Solution

We observe that $0 \leq f(x) \leq x^2$ for all x, and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} x^2$. So, by the Squeeze Law, $\lim_{x\to 0} f(x) = 0$.

6. (a) Give an example of a function f for which $\lim_{x\to 0} f(x)^2$ exists but $\lim_{x\to 0} f(x)$ does not.

(b) Give an example of a function f for which $\lim_{x\to 0} f(x^2)$ exists but $\lim_{x\to 0} f(x)$ does not.

Solution

(a) Let $f(x) = 1$ if $x \geq 0$ and $f(x) = -1$ if $x < 0$. Then $f(x)^2 = 1$ for all x and so $\lim_{x\to 0} f(x)^2$ exists and equals 1, whereas $\lim_{x\to 0} f(x)$ does not exist. (Side comment: if $\lim_{x\to 0} f(x)^2 = 0$ then it does follow that $\lim_{x\to 0} f(x) = 0$, because the relevant inequality $|f(x)| < \epsilon$ is equivalent to $|f(x)^2| < \epsilon^2$.)

(b) Again, we can let $f(x) = 1$ if $x \geq 0$ and $f(x) = -1$ for $x < 0$. Then $\lim_{x\to 0} f(x)$ does not exist but $\lim_{x\to 0} f(x^2)$ exists and equals 1, as $f(x^2) = 1$ for all x.

7. Suppose f has domain $\mathbb{R}$. To say that $\lim_{n\to\infty} f(n) = \ell$ (where n takes only integer values) means that for any $\epsilon > 0$, there exists M such that whenever n is an integer and $n > M$, then $|f(n) - \ell| < \epsilon$. Give an example of a function f with domain $\mathbb{R}$ such that $\lim_{n\to\infty} f(n)$ exists in this sense, but $\lim_{x\to\infty} f(x)$ (where x takes real values) does not exist.

Solution

One example is the function given by $f(x) = \sin \pi x$. For this function,

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \sin \pi n = 0,$$

since $\sin \pi n = 0$ for every integer n. However, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin \pi x$ does not exist: whenever $x = n + \dfrac{1}{2}$ for $n \in \mathbb{Z}$, we have $\sin \pi x = \pm 1$, so $f(x)$ takes the values $1, 0, -1$ for arbitrarily large x.

8. Using the ϵ, δ definition of limit, prove that if the limit of a function exists as $x \to a$, then the limit is unique. To be precise, prove that if $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} f(x) = m$, then $\ell = m$. (*Hint*: Assume that $\ell \neq m$ and obtain a contradiction by setting $\epsilon = \dfrac{|\ell - m|}{2}$.)

Solution

Assuming that $\ell \neq m$, let $\epsilon = \dfrac{|\ell - m|}{2}$. Then as $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} f(x) = m$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - \ell| < \frac{|\ell - m|}{2}$$

and

$$0 < |x-a| < \delta_2 \implies |f(x) - m| < \frac{|\ell - m|}{2}.$$

Let $\delta = \min(\delta_1, \delta_2)$, and choose some x such that $0 < |x-a| < \delta$. Then we have $|f(x) - \ell| < \dfrac{|\ell - m|}{2}$ and $|f(x) - m| < \dfrac{|\ell - m|}{2}$, and hence (adding the last two inequalities)

$$|f(x) - \ell| + |f(x) - m| < |\ell - m|.$$

However, for all real numbers a, b, c, we have $|a-b| = |a-c+c-b| \leq |a-c| + |c-b|$. Therefore

$$|\ell - m| \leq |\ell - f(x)| + |f(x) - m| = |f(x) - \ell| + |f(x) - m| < |\ell - m|,$$

which is a contradiction. So the assumption $\ell \neq m$ must have been wrong, and the result is proved.

You may have noticed that the above argument is similar to the proof of the Addition Law. In fact, if you assume the Addition Law (and consequently the Subtraction Law), then there is a quicker argument:

$$\ell - m = \lim_{x\to a} f(x) - \lim_{x\to a} f(x) = \lim_{x\to a}(f(x) - f(x)) = 0.$$

The only reason not to prefer this proof is that it seems a little odd to prove one of the properties of limits (the Addition/Subtraction Law) before proving that limits themselves are uniquely defined.

Extra Questions

9. Students often have difficulty remembering the ϵ, δ definition of the statement $\lim_{x\to a} f(x) = \ell$. For each of the following misremembered versions, work out what it means and why it is not the right definition.

(a) For each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x-a| < \delta$, we have $0 < |f(x) - \ell| < \epsilon$.

(b) For each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x-a| < \delta$, we have $|f(x) - \ell| < \epsilon$.

(c) For each ϵ, there exists $\delta > 0$ such that whenever $0 < |x-a| < \delta$, we have $|f(x) - \ell| < \epsilon$.

(d) For each $\epsilon > 0$, there exists δ such that whenever $0 < |x-a| < \delta$, we have $|f(x) - \ell| < \epsilon$.

(e) For each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x-a| < \delta$, we have $|f(x) - \ell| > \epsilon$.

(f) For each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x-a| < \epsilon$, we have $|f(x) - \ell| < \delta$.

(g) For each $\delta > 0$, there exists $\epsilon > 0$ such that whenever $0 < |x-a| < \delta$, we have $|f(x) - \ell| < \epsilon$.

(h) For each $\delta > 0$, there exists $\epsilon > 0$ such that whenever $0 < |x-a| < \epsilon$, we have $|f(x) - \ell| < \delta$.

Solution

(a) Here the extra inequality $0 < |f(x) - \ell|$ means that $f(x)$ is forbidden to equal ℓ when x is close to a. So this 'definition' would not be able to handle the case where $f(x)$ is the constant function with value ℓ, for instance.

(b) Here the inequality $0 < |x-a|$ has been forgotten, which makes the statement stronger because it now applies to $x = a$ as well: when $x = a$ the 'definition' says that $|f(a)-\ell| < \epsilon$ for each $\epsilon > 0$, which implies $f(a) = \ell$. So this 'definition' is actually equivalent to saying that $\lim_{x\to a} f(x) = \ell$ and $f(a) = \ell$; it would be no good for the many occasions when $f(a)$ is either undefined or not equal to $\lim_{x\to a} f(x)$.

(c) This is wrong because the condition $\epsilon > 0$ has been omitted. If ϵ is a negative number or zero, the inequality $|f(x) - \ell| < \epsilon$ is impossible to satisfy, so this 'definition' would actually rule out everything.

(d) This time the condition $\delta > 0$ has been omitted. If δ is a negative number or zero, the chain of inequalities $0 < |x - a| < \delta$ is impossible, so there are no values of x to which the constraint $|f(x) - \ell| < \epsilon$ applies. So this 'definition' would allow anything at all (it is true for any f, a, ℓ).

(e) Here $|f(x) - \ell| < \epsilon$ has been replaced by $|f(x) - \ell| > \epsilon$. This 'definition' says that no matter how large ϵ is, $f(x)$ is more than ϵ distance away from ℓ as long as x is sufficiently close to a. (This would be true if $\lim_{x\to a} f(x) = \infty$, for instance.) It is practically the opposite of the right definition, because $f(x)$ is getting further from ℓ as x approaches a, rather than closer.

(f) Here δ and ϵ have been swapped in the "whenever ..., we have ..." part. The 'definition' now means that for any $\epsilon > 0$, the set of values $f(x)$ for $0 < |x - a| < \epsilon$ is contained in the open interval $(\ell - \delta, \ell + \delta)$ for some $\delta > 0$. This is the concept one usually expresses by saying that the set of values $f(x)$ for $0 < |x-a| < \epsilon$ is *bounded*. This neither implies nor is implied by $\lim_{x\to a} f(x) = \ell$. We have seen functions whose range is bounded but which do not have well-defined limits at various points in the domain; and conversely, the fact that the set of values $1/x$ for $0 < |x - 1| < 1$ is not bounded does not impair the truth of $\lim_{x\to 1} 1/x = 1$.

(g) This is the same as the previous part but with the letters ϵ and δ interchanged throughout, which makes no difference to the meaning.

(h) This is the right definition but with the letters ϵ and δ interchanged. Logically, the choice of letters makes no difference, so this is not wrong. However, swapping the traditional names of two quantities which play such different roles is horribly confusing.

10. Prove that $\lim_{x\to 0} f(x)$ does not exist, where $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$

Solution

The idea is that any interval (no matter how small) surrounding 0 contains rational and irrational numbers, and so in this interval, f takes the values 1 and 0. Hence $f(x)$ cannot approach a limit as $x \to 0$.

Here is a formal proof by contradiction. Suppose that $\lim_{x\to 0} f(x) = \ell$. Given $\epsilon = \frac{1}{2}$, there exists a $\delta > 0$ such that

$$0 < |x| < \delta \Rightarrow |f(x) - \ell| < \tfrac{1}{2}.$$

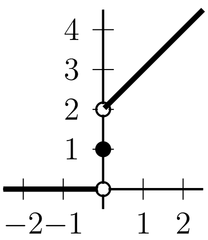
Take any positive rational number r with $0 < r < \delta$. Then $f(r) = 0$, so $|0 - \ell| < \frac{1}{2}$, that is, $-\frac{1}{2} < \ell < \frac{1}{2}$.
Now take any positive irrational number s with $0 < s < \delta$. Then $f(s) = 1$, so $|1 - \ell| < \frac{1}{2}$, that is, $\frac{1}{2} < \ell < \frac{3}{2}$. This contradicts $-\frac{1}{2} < \ell < \frac{1}{2}$, so $\lim_{x \to 0} f(x)$ does not exist.

Postscript: You may be wondering how to prove that you can find rational and irrational numbers r and s with $0 < r < \delta$ and $0 < s < \delta$. Here is one suggestion. To find an appropriate rational number r, we simply find a positive integer N which is greater than $1/\delta$, and then take $r = 1/N$ (as $1/N < \delta$). To find an appropriate irrational number s, first find a positive integer N such that $1/N < \delta$ as before, but with the extra condition that N is not a perfect square. Thus $\dfrac{1}{N^{3/2}} = \dfrac{1}{\sqrt{N}}\dfrac{1}{N}$ is irrational, and

$$\frac{1}{N^{3/2}} = \frac{1}{\sqrt{N}}\frac{1}{N} \leq \frac{\sqrt{N}}{N\sqrt{N}} = \frac{1}{N} < \delta.$$

We may then take $s = \dfrac{1}{N^{3/2}}$.

Solution to Question 1



We have $\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 2$.
Since the limits from the left and the right are not equal, $\lim_{x \to 0} f(x)$ does not exist.