The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2 **Solutions to Exercises for Week 3** 2017

1. By the Chain Rule,

$$f'(x) \;=\; -\frac{1}{1+x^{-2}}\big(-x^{-2}\big) \;=\; \left(\frac{x^2}{x^2+1}\right)\left(\frac{1}{x^2}\right) \;=\; \frac{1}{x^2+1}\,,$$

which is also the derivative of $\tan^{-1} x$.

2. Suppose that $f : [0,\infty) \to \mathbb{R}$ is a continuous function such that $f'(c) = 0$ for all $c > 0$. If $x > 0$ then, by the Mean Value Theorem applied to the interval $[0,x]$, there exists $c \in (0,x)$ such that

$$0 \;=\; f'(c) \;=\; \frac{f(x)-f(0)}{x-0}\,,$$

so that $f(x) - f(0) = 0$, yielding $f(x) = f(0)$. This verifies that f is a constant function.

3. Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be defined by the rule

$$f(x) \;=\; \frac{|x|}{x} \;=\; \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0\,. \end{cases}$$

Clearly $f'(x) = 0$ for all x in the domain of f. Thus f is differentiable, but f is not a constant function (though is piecewise constant).

4. The curve $y = \frac{1}{x^2}$ is not continuous over $[-1,1]$ (not even defined at $x = 0$), so the hypothesis of the Fundamental Theorem of Calculus does not hold, so Bill is not entitled to quote it. Susan is partly right: the function is positive everywhere that it is defined, so if an area makes sense then at least it should be nonnegative. In fact $\int_{-1}^{1} \frac{1}{x^2}\,dx$ is an improper integral because $\frac{1}{x^2}$ is unbounded. We get

$$\int_0^1 \frac{1}{x^2}\,dx \;=\; \lim_{a\to 0}\int_a^1 \frac{1}{x^2}\,dx \;=\; \lim_{a\to 0}\left[-\frac{1}{x}\right] \;=\; \lim_{a\to 0}\frac{1}{a} - 1 \;=\; \infty\,,$$

and similarly $\int_{-1}^{0} \frac{1}{x^2}\,dx \;=\; \infty$. Thus

$$\int_{-1}^{1} \frac{1}{x^2}\,dx \;=\; \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx \;=\; \infty\,,$$

which is "positive" but not a number at all!

5. (i) Put $u = \cos\theta$, so that $du = -\sin\theta\,d\theta$, and we have

$$\begin{aligned} \int_0^{\pi/2} \cos^4\theta\sin^3\theta\,d\theta \;&=\; \int_0^{\pi/2} (\cos^4\theta - \cos^6\theta)\sin\theta\,d\theta \;=\; -\int_1^0 u^4 - u^6\,du \\ &=\; \int_0^1 u^4 - u^6\,du \;=\; \left[\frac{u^5}{5} - \frac{u^7}{7}\right]_0^1 \;=\; \frac{1}{5} - \frac{1}{7} \;=\; \frac{2}{35}\,. \end{aligned}$$

(ii) Put $u = \sec\theta$, so that $du = \sec\theta\tan\theta\,d\theta$, and we have

$$\begin{aligned}\int_0^{\pi/3} \sec^5\theta\tan^3\theta\,d\theta &= \int_0^{\pi/3} (\sec^6\theta - \sec^4\theta)\sec\theta\tan\theta\,d\theta = \int_1^2 u^6 - u^4\,du \\ &= \left[\frac{u^7}{7} - \frac{u^5}{5}\right]_1^2 = \frac{2^7}{7} - \frac{2^5}{5} - \frac{1}{7} + \frac{1}{5} = \frac{418}{35}\,.\end{aligned}$$

(iii) Put $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2u}\,dx$, and we have

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\sin u}{u} 2u\,du = 2\int \sin u\,du = -2\cos u + C = -2\cos\sqrt{x} + C\,.$$

(iv) Put $u = \sqrt[4]{x+2}$, so that $du = \frac{1}{4}(x+2)^{-3/4}\,dx = \frac{1}{4u^3}\,dx$, and we have

$$\begin{aligned}\int \frac{x}{\sqrt[4]{x+2}}\,dx &= \int \frac{u^4-2}{u} 4u^3\,du = 4\int u^6 - 2u^2\,du \\ &= 4\left(\frac{u^7}{7} - \frac{2u^3}{3}\right) + C = \frac{4}{7}(x+2)^{7/4} - \frac{8}{3}(x+2)^{3/4} + C\,.\end{aligned}$$

(v) Put $u = a^2 - x^2$, so that $du = -2x\,dx$, and we have

$$\begin{aligned}\int_0^a x\sqrt{a^2-x^2}\,dx &= -\frac{1}{2}\int_{a^2}^0 u^{1/2}\,du = \frac{1}{2}\int_0^{a^2} u^{1/2}\,du = \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_0^{a^2} \\ &= \frac{1}{3}(a^2)^{3/2} = \frac{a^3}{3}\,.\end{aligned}$$

(vi) Put $x = a\sin\theta$, so that $dx = a\cos\theta\,d\theta$, and we have

$$\begin{aligned}\int_0^a \sqrt{a^2-x^2}\,dx &= \int_0^{\pi/2} a\cos\theta\; a\cos\theta\,d\theta = a^2\int_0^{\pi/2}\cos^2\theta\,d\theta \\ &= a^2\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi a^2}{4}\,.\end{aligned}$$

6. First note that

$$\int_1^{100} \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^{100} = \frac{2}{3}(1000-1) = 666\,.$$

Using an upper Riemann sum with unit subintervals gives

$$\sqrt{2} + \sqrt{3} + \ldots + \sqrt{100} \geq 666\,.$$

A lower Riemann sum gives

$$1 + \sqrt{2} + \ldots + \sqrt{99} \leq 666\,,$$

so

$$667 \leq 1 + \sqrt{2} + \ldots + \sqrt{100} \leq 676\,.$$

7. (i) By the Fundamental Theorem of Calculus, $f'(x) = \sqrt{x^3+1}$.

(ii) By the Fundamental Theorem of Calculus,

$$f'(x) \;=\; -\frac{d}{dx}\int_4^x (2+\sqrt{u})^8\,du \;=\; -\,(2+\sqrt{x})^8.$$

(iii) Write $f(x) = g(\sqrt{x}\,)$, where $g(x) = \int_1^x \frac{s^2}{s^2+1}\,ds$. By the Fundamental Theorem of Calculus and the chain rule,

$$f'(x) \;=\; g'(\sqrt{x}\,)\,\frac{1}{2\sqrt{x}} \;=\; \frac{x}{x+1}\,\frac{1}{2\sqrt{x}} \;=\; \frac{\sqrt{x}}{2(x+1)}.$$

Alternatively, one could make the substitution $s = \sqrt{t}$, so that $ds = \frac{dt}{2\sqrt{t}}$, to get

$$f(x) \;=\; \int_1^x \frac{\sqrt{t}}{2(t+1)}\,dt.$$

Then the result follows directly from the Fundamental Theorem of Calculus.

8. (i) In both cases (making the substitutions $u = 2x$ and $u = x^2$ respectively), the integrals evaluate to

$$\frac{1}{2}\int_0^4 f(u)\,du \;=\; 5\,.$$

(ii) Making the substitution $u = -x$ gives

$$\int_a^b f(-x)\,dx \;=\; -\int_{-a}^{-b} f(u)\,du \;=\; \int_{-b}^{-a} f(u)\,du \;=\; \int_{-b}^{-a} f(x)\,dx\,,$$

which says that the area under a curve does not change if it is reflected in the y-axis. Making the substitution $u = x + c$ gives

$$\int_a^b f(x+c)\,dx \;=\; \int_{a+c}^{b+c} f(u)\,du \;=\; \int_{a+c}^{b+c} f(x)\,dx\,,$$

which says that an area does not change if it is translated c units in the positive or negative x-direction.

(iii) Observe always that $\sin(\pi-\theta) = \sin\theta$, and $du = -dx$, so we get

$$\begin{aligned}\int_0^\pi x f(\sin x)\,dx \;&=\; -\int_\pi^0 (\pi-u) f(\sin(\pi-u))\,du \;=\; \int_0^\pi (\pi-u) f(\sin u)\,du \\ &=\; \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi x f(\sin x)\,dx\,,\end{aligned}$$

so that

$$2\int_0^\pi x f(\sin x)\,dx \;=\; \pi\int_0^\pi f(\sin x)\,dx\,,$$

yielding finally

$$\int_0^\pi x f(\sin x)\,dx \;=\; \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx\,.$$

9. For $0 < \theta < \pi/2$ and $x = \tan\theta > 0$, we have $\tan(\pi/2 - \theta) = 1/x$, so $\tan^{-1}(1/x) = \pi/2 - \theta$, giving

$$f(x) \;=\; -(\pi/2 - \theta) \;=\; \theta - \pi/2 \;=\; \tan^{-1}(x) - \pi/2\;.$$

For $-\pi/2 < \theta < 0$ and $x = \tan\theta < 0$, we have $\tan(-\pi/2 - \theta) = 1/x$, so $\tan^{-1}(1/x) = -\pi/2 - \theta$, giving

$$f(x) \;=\; -(-\pi/2 - \theta) \;=\; \theta + \pi/2 \;=\; \tan^{-1}(x) + \pi/2\;.$$

Thus f differs from $\tan^{-1}$ by the constant $-\pi/2$ on $(0, \infty)$ and by the constant $\pi/2$ on $(-\infty, 0)$. There is no anomaly, since f is not defined on all of $\mathbb{R}$.

10. (i) We have

$$\int \frac{dx}{x^2+2x+1} \;=\; \int \frac{dx}{(x+1)^2} \;=\; \int (x+1)^{-2}\,dx \;=\; -(x+1)^{-1}+C \;=\; -\frac{1}{x+1}+C\;.$$

(ii) Completing the square and putting $u = x+1$, so that $du = dx$, we have

$$\int \frac{dx}{x^2+2x+2} \;=\; \int \frac{dx}{(x+1)^2+1} \;=\; \int \frac{du}{u^2+1} \;=\; \tan^{-1} u + C \;=\; \tan^{-1}(x+1)+C\;.$$

(iii) Observe that

$$\frac{1}{x^2+2x} \;=\; \frac{1}{x(x+2)} \;=\; \frac{1}{2}\left(\frac{2}{x(x+2)}\right) \;=\; \frac{1}{2}\left(\frac{-x+x+2}{x(x+2)}\right) \;=\; \frac{1}{2}\left(\frac{-1}{x+2}+\frac{1}{x}\right),$$

so that

$$\int \tfrac{dx}{x^2+2x} \;=\; \tfrac{1}{2}\big(\int \tfrac{-dx}{(x+2)} + \int \tfrac{dx}{x}\big) \;=\; \tfrac{1}{2}\big(-\ln|x+2| + \ln|x|\big) + C \;=\; \tfrac{1}{2}\ln\left|\tfrac{x}{x+2}\right| + C\;.$$

11. Introduce a constant terminal, say 0, and rearrange and expand $g(x)$ to become

$$g(x) = \int_0^{\cos x} e^{-t^2}\,dt - \int_0^x e^{-t^2}\,dt.$$

By the Fundamental Theorem of Calculus and the chain rule we have

$$\frac{d}{dx}\int_0^{\cos x} e^{-t^2}\,dt = -\sin x\, e^{-\cos^2 x} \qquad \text{and} \qquad \frac{d}{dx}\int_0^x e^{-t^2}\,dt = e^{-x^2}\,,$$

and so

$$g'(x) = -\sin x\, e^{-\cos^2 x} - e^{-x^2}.$$

12. Since x is constant as far as the integrating variable t is concerned, we can write $f(x) = x\int_0^x \sin(t^2)\,dt$. Now by the product rule and the Fundamental Theorem of Calculus,

$$f'(x) \;=\; x\sin(x^2) + \int_0^x \sin(t^2)\,dt,$$

and

$$f''(x) \;=\; \sin(x^2) + x\frac{d}{dx}\sin(x^2) + \sin(x^2) \;=\; 2\sin(x^2) + 2x^2\cos(x^2)\;.$$

13. Let F be the function defined on the interval $[a, b]$ by the rule

$$F(x) = \int_a^x f(t)\,dt\,.$$

Then, by the Fundamental Theorem of Calculus, F is differentiable and $F'(x) = f(x)$ for each $x \in (a, b)$. By the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a}\int_a^b f(t)\,dt\,.$$

14. If we set $F(x) = \int_0^x f(t)\,dt$, then $\int_0^{x^2} f(t)\,dt = F(x^2)$, and the derivative of this is $F'(x^2) \cdot (2x) = 2xf(x^2)$. So differentiating both sides of the given equation, we get

$$\sin(\pi x) + \pi x \cos(\pi x) = 2xf(x^2).$$

Evaluating both sides of this at $x = 2$, we see that $0 + 2\pi = 4f(4)$. Hence, $f(4) = \pi/2$.

15. This solution relates to the First Assignment so is not included.

16. Put

$$f(x) = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0 \ \ \text{(actually } f(0) \text{ can be anything).} \end{cases}$$

For $x > 0$, we have $\int_0^x f(t)\,dt = \int_0^x (+1)\,dt = x = |x|$. For $x < 0$, we have $\int_0^x f(t)\,dt = \int_0^x (-1)\,dt = \int_x^0 1\,dt = -x = |x|$. Finally, $\int_0^0 f(t)\,dt = 0 = |0|$. This verifies that $\int_0^x f(t)\,dt = |x|$ for all x.

17. A simple calculation reveals that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} > \sum_{n=1}^{7} \frac{1}{n^2} > 1.5$. Observe now that, for any given positive integer $m \geq 3$, $\displaystyle\sum_{n=3}^{m} \frac{1}{n^2}$ is a lower Riemann sum for

$$\int_2^m \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_2^m = -\frac{1}{m} + \frac{1}{2} < \frac{1}{2}$$

so that

$$\sum_{n=1}^{m} \frac{1}{n^2} = 1 + \frac{1}{4} + \sum_{n=3}^{m} \frac{1}{n^2} < 1.75\,,$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1.75\,.$$

Certainly then

$$1.5 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 2\,.$$

To see why the limit exists, put $a_m = \sum_{n=1}^{m} \frac{1}{n^2}$ and $X = \{ a_m \mid m \geq 1 \}$. Observe that

$$a_1 \ < \ a_2 \ < \ a_3 \ < \ \ldots \ < \ a_m \ < \ \ldots \ < \ 1.75 \ .$$

In particular X is bounded above so has a least upper bound L, by completeness of the real numbers. Consider any $\epsilon > 0$. If $a_m \leq L - \epsilon$ for all m then L is not the least upper bound of X, a contradiction. Hence $a_M > L - \epsilon$ for some M, so that $L \geq a_m > L - \epsilon$ for all $m \geq M$. This shows

$$|L - a_m| \ < \epsilon$$

for all $m \geq M$, which proves $\lim_{m \to \infty} a_m$ exists and equals L. (This argument is the substance of the Monotone Convergence Theorem which asserts that all monotonic bounded sequences of real numbers have limits.)