The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Exercises for Week 12 (beginning 28 May)** 2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) The *determinant* of a 1×1 matrix $[a]$ is simply the entry a.

(ii) The *determinant* of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

(iii) The *determinant* of a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ is

$$\det A = |A| = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

called the *expansion along the first row*, where the smaller determinant arises by ignoring the row and column of the entry being used as a coefficient.

(iv) Expanding along any row or down any column of a square matrix A produces the same real number, called the *determinant* of A, denoted by $\det A$ or $|A|$, provided one uses adjustment factors given by the chequerboard patterns

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

and so on to higher dimensions. Using sigma notation, if $A = [a_{ij}]$ is an $n\times n$ matrix and A_{ij} denotes the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column of A, then expanding along the ith row (for fixed i) becomes

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} ,$$

and down the jth column (for fixed j) becomes

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} .$$

(v) Determinant method for cross products: If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and $\mathbf{w} = d\,\mathbf{i} + e\,\mathbf{j} + f\,\mathbf{k}$ then

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} .$$

(vi) Multiplicative property: $\det(AB) = (\det A)(\det B)$.

(vii) Invertibility criterion: A square matrix is invertible if and only if its determinant is nonzero.

(viii) If B is obtained from A by swapping two rows or swapping two columns then

$$\det B = -\det A \, .$$

(ix) If B is obtained from A by multiplying a row or column by λ then

$$\det B = \lambda \det A \, .$$

(x) If B is obtained from A by adding a multiple of one row [column] to another row [column] then

$$\det B = \det A \, .$$

(xi) If B is the *transpose* of A, that is, obtained by interchanging rows and columns, then

$$\det B = \det A \, .$$

(xii) If A is *triangular*, that is all entries above or below the diagonal are zero, then $\det A$ is the product of the diagonal elements.

Preparatory Exercises:

1. Find the following determinants:

$$\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix}, \quad \begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}, \quad \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 5 & 4 \\ 3 & 3 \end{vmatrix} .$$

2. Find the determinant $\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix}$ by expanding along the first row.

3. Now find the determinant of the previous exercise by expanding

(i) along the second row (ii) along the third row (iii) down the third column

4. Find the following determinants:

(i) $\begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix}$ (ii) $\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

(v) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{vmatrix}$ (vi) $\begin{vmatrix} 2 & 4 & 6 \\ 7 & 11 & 6 \\ -6 & -6 & 12 \end{vmatrix}$ (vii) $\begin{vmatrix} -4 & 3 & 3 \\ 8 & 7 & 3 \\ 4 & 3 & 3 \end{vmatrix}$

5. Write down quickly the determinants of the following matrices:

$$\text{(i)} \begin{bmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -5 & -1 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 3 & 3 & 8 \\ 0 & -6 & -7 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} -4 & -5 & 11 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

$$\text{(iv)} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{(v)} \begin{bmatrix} 0 & 0 & 5 \\ 6 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix} \quad \text{(vi)} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ -6 & -3 & -7 & -1 \end{bmatrix}$$

Tutorial Exercises:

6. Justify briefly the following calculation:

$$\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 2 & -3 & -2 \\ 3 & -3 & 0 \\ 1 & -14 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ 3 & 0 & 0 \\ 1 & -13 & 0 \end{vmatrix} = -2\begin{vmatrix} 3 & 0 \\ 1 & -13 \end{vmatrix} = 78$$

7. Use elementary row and column operations, or otherwise, to find the following:

$$\text{(i)} \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix} \quad \text{(ii)} \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} \quad \text{(iii)} \begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix}$$

8. Find $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ in each of the following cases:

(i) $\mathbf{v} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $\mathbf{w} = 4\mathbf{i}+5\mathbf{j}+6\mathbf{k}$ (ii) $\mathbf{v} = 2\mathbf{i}-\mathbf{j}+6\mathbf{k}$, $\mathbf{w} = -\mathbf{i}+\mathbf{j}-3\mathbf{k}$

9. Make sense of the expression $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$ where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are geometric vectors. Explain how the formula

$$\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

follows from the formula

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} .$$

10. Calculate $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$ in each of the following cases:

(i) $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{w} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

(ii) $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$, $\mathbf{w} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$

11. Use the multiplicative property of the determinant to verify that if A is an invertible matrix then $\det A \neq 0$ and $\det A^{-1} = (\det A)^{-1}$.

12. Explain briefly, using properties of determinants, why a square matrix with two identical rows or two identical columns has zero determinant.

13. (suitable for group discussion) Decide whether the following statements are true for all 2×2 matrices A and B:

(i) $\det(AB) = (\det B)(\det A)$ (ii) $\det(A + B) = (\det A) + (\det B)$

(iii) $\det(2A) = 2 \det A$ (iv) $\det(-A) = \det A$

14. (suitable for group discussion) Without evaluating it, but using simple properties of determinants, explain why the following determinant is divisible by 17, given that 867, 459 amd 187 are each divisible by 17:

$$\begin{vmatrix} 8 & 6 & 7 \\ 4 & 5 & 9 \\ 1 & 8 & 7 \end{vmatrix}$$

15.* Determine the values of λ for which $\det(A - \lambda I) = 0$ in each case:

(i) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ (iii) $A = \begin{bmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{bmatrix}$

Further Exercises:

16.* Verify directly, by expanding along the first row, that if A is a 2×2 or a 3×3 matrix then $\det A = \det A^T$ where A^T is the transpose of A, obtained by interchanging rows and columns ('reflecting in the diagonal').

17.* Use determinants to give a quick proof that two lines $ax + by = k$ and $cx + dy = \ell$ in the plane intersect in a single point if and only if $ad - bc \neq 0$.

18.* Use induction, and expansion along the first row, to verify that $\det A_n = (-1)^{n-1}$ for $n \geq 2$, where A_n is the $n \times n$ matrix

$$A_n = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix}$$

and it is understood that any blank entry is zero. What happens if you expand down the first column?

19.* Use the multiplicative property of the determinant and elementary matrices to prove that if the determinant of a square matrix is nonzero then the matrix is invertible.

20.* Explain why the following orientation test works for triangles in the plane: If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are points in the plane and

$$\delta \;=\; \delta_{ABC} \;=\; \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \;.$$

then

$$\triangle ABC \text{ is } \begin{cases} \text{anticlockwise} & \text{if } \delta > 0 \\ \text{clockwise} & \text{if } \delta < 0 \\ \text{degenerate} & \text{if } \delta = 0\,. \end{cases}$$

21. Use the test from the previous exercise to decide whether the triangle $\triangle PQR$ is oriented clockwise or anticlockwise in each case:

(i) $P(4,6)\,,\ Q(-7,0)\,,\ R(2,-5)$ (ii) $P(0,1)\,,\ Q(23,24)\,,\ R(-1,-3)$

22.* Let $P = (5,1)$, $Q = (7,9)$ and $R = (1,4)$. In each case, use determinants to deduce whether S lies inside or outside the triangle $\triangle PQR$:

(i) $S(3,3)$ (ii) $S(4,7)$ (iii) $S(6,5)$

23.* This and the following exercise are stepping stones to developing the general recursive theory of determinants, and provide excellent practice that combines proof by induction with sigma notation. Prove by induction, using only expansions along the first row, that any square matrix with a row or column of zeros has zero determinant.

24.** Prove by induction, using expansions along the first row, that if A is a diagonal sum of square matrices B and C, that is, has the shape

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \,,$$

then $\det A = \det B \det C$.

25.** Let X be a finite set with n elements. If $f : X \to X$ is a function then call f a *permutation* if f is one-one and onto, and a *transposition* if f is a permutation that leaves all but two elements of X fixed and interchanges those two elements. It is straightforward to show that any permutation of X is a composite of transpositions (and you can take this fact as granted). We call a permutation *even* if it is a composite of an even number of transpositions, and *odd* if it is a composite of an odd number. Prove that no permutation can be both even and odd. [Hint: relate this to elementary row operations on I_n, and apply determinants.]

Short Answers to Selected Exercises:

1. $1\,,\quad 1\,,\quad -1\,,\quad 2\,,\quad -2\,,\quad 3$

2. $\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = 2\begin{vmatrix} 3 & 4 \\ -2 & 8 \end{vmatrix} + 3\begin{vmatrix} -1 & 4 \\ -7 & 8 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ -7 & -2 \end{vmatrix} = 64 + 60 - 46 = 78$

3. (i) $\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ -2 & 8 \end{vmatrix} + 3\begin{vmatrix} 2 & -2 \\ -7 & 8 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ -7 & -2 \end{vmatrix} = -28 + 6 + 100 = 78$

(ii) $\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = -7\begin{vmatrix} -3 & -2 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ -1 & 4 \end{vmatrix} + 8\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 42 + 12 + 24 = 78$

(iii) $\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = -2\begin{vmatrix} -1 & 3 \\ -7 & -2 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ -7 & -2 \end{vmatrix} + 8\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = -46 + 100 + 24 = 78$

4. (i) -16 (ii) 0 (iii) 1 (iv) -1 (v) 1 (vi) 0 (vii) -96

5. (i) 10 (ii) -36 (iii) 0 (iv) 2 (v) -90 (vi) 16

6. Apply $R_2 \to R_2 + 2R_1$, followed by $R_3 \to R_3 + 4R_1$, followed by $C_2 \to C_2 + C_1$, followed by expansion down the third column and evaluation of 2×2 determinant.

7. (i) -14 (ii) 0 (iii) 32

8. (i) $-3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ (ii) $-3\mathbf{i} + \mathbf{k}$

9. $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ and we have

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

10. (i) -5 (ii) 21

11. Both follow quickly from the observation that $(\det A^{-1})(\det A) = \det(A^{-1}A) = \det I = 1$.

12. The determinant is unchanged by subtracting one row from another, or one column from another. Using identical rows or columns for this subtraction produces a matrix with a zero row or column. Expanding now along that zero row or column yields zero determinant.

13. (i) true (ii) false (iii) false (iv) true

15. (i) $2, -3$ (ii) $2, 3$ (iii) $1, -1, 3$

16. When A is 2×2 the answer is immediate. When A is 3×3 expand along the first rows of A and A^T and manipulate the expressions to see that they are equal.

21. (i) anticlockwise (ii) clockwise

22. (i) inside (ii) outside (iii) on the boundary