THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION, ENGINEERING AND SCIENCE

MATH1901/1906

DIFFERENTIAL CALCULUS (ADVANCED)

June 2012

LECTURER: C M Cosgrove

TIME ALLOWED: **One and a half hours**

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination; there are 20 questions; the questions are of equal value; all questions may be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination; there are 4 questions; the questions are of equal value; all questions may be attempted; working must be shown.

Approved non-programmable non-graphics calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

MARKER'S USE ONLY	

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

MARKS

1. (a) In the complex z-plane, $z = x + iy$, sketch the set satisfying the inequality, **3**

$$|z + 4 + 3i| \leq 3.$$

(b) Factorise the polynomial,

$$P(z) = z^4 - 3z^3 + 10z^2 + 9z + 13,$$

into linear and/or quadratic factors with real coefficients, given that $2 + 3i$ is one of the roots of the polynomial. **4**

(c) Find all the non-real cube roots of -8, expressing your answers in Cartesian form. **2**

(d) Starting with the standard limit $(\sin x)/x \to 1$ as $x \to 0$, deduce the value of

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}.$$

(Do not use l'Hôpital's rule.) **3**

2. (a) Let $f : \mathbb{R}^2\backslash\{0,0\} \to \mathbb{R}$, $(x, y) \mapsto \ln(x^2 + 3y^2)$, and let P denote the point $(2, 1)$ in the xy-plane.

   (*i*) Calculate the directional derivative $D_{\hat{\mathbf{u}}}f$ of f at P in the direction of the vector $\mathbf{u} = 4\mathbf{i} - \mathbf{j}$. **3**

   (*ii*) Find the unit vector $\hat{\mathbf{v}}$ in the direction in which the directional derivative of f at P is maximised, and give the corresponding value of the maximum directional derivative, that is, $D_{\hat{\mathbf{v}}}f$ at P. **2**

   (*iii*) Find the equation of the tangent plane to the graph of $z = f(x, y)$ at the point on the graph vertically above P. Express your answer in the form $z = ax + by + c$. **3**

(b) Use any method to calculate the Taylor polynomial $T_3(x)$ of order 3 about $x = 0$ of the function, **4**

$$f(x) = e^{2x} \cos x.$$

MARKS

3. (a) Find the following limits, showing the steps of your working clearly, or show that the limit does not exist. (You may use any valid method. Allow $+\infty$ and $-\infty$ as values that a limit can take.)

(i) $\displaystyle\lim_{x\to 2} \frac{x^3+5x^2-32x+36}{x^3-12x+16}$. **2**

(ii) $\displaystyle\lim_{x\to 0} (\cos x)^{\cot^2 x}$. **3**

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy^3}{(x^2+y^2)^2}$. **3**

(b) *Calculator problem.* Show that the transcendental equation,

$$\sinh x = 2x,$$

has one and only one root on the interval $[2.0, 2.5]$, and find an interval of length 0.1 that contains this root. **4**

4. In this question, $f(x) = \sin x$ and $g(x) = \sin(x^3)$.

(a) From the Taylor polynomial $T_4(x)$ of order 4 for $f(x)$ about $x = 0$, deduce the Taylor polynomial $T_{14}(x)$ of order 14 for $g(x)$ about $x = 0$. (The actual degree will be 9. You do not need to explain why this polynomial is a Taylor polynomial.) **3**

(b) From the standard formula for the remainder term $R_4(x)$ for $f(x)$, deduce a suitable formula for the remainder term $R_{14}(x)$ for $g(x)$ about $x = 0$. (You may assume $x > 0$.) Conclude that

$$T_{14}(x) < \sin(x^3) < T_{15}(x)$$

whenever $0 < x < (\pi/2)^{1/3}$. **4**

(c) *Calculator problem.* Use the results of parts (a) and (b) to give a numerical estimate for

$$\int_0^{1/2} \sin(x^3)\, dx$$

to seven decimal places, and prove that it is correct to this level of accuracy. (You will need to calculate upper and lower bounds to a bit more than seven decimal places.) **5**

Standard Derivatives

The following derivatives can be quoted without proof unless a question specifically asks you to show details. These results can be combined with the standard rules of differentiation (not listed here) to differentiate more complicated functions. For example, $(d/dx)\sin(ax+b) = a\cos(ax+b)$. Natural domains common to both sides are assumed.

1. $\dfrac{d}{dx} x^k = kx^{k-1} \quad (k \in \mathbb{R})$
2. $\dfrac{d}{dx} e^x = e^x$
3. $\dfrac{d}{dx} \ln x = \dfrac{1}{x} \quad (x > 0)$
4. $\dfrac{d}{dx} \sin x = \cos x$
5. $\dfrac{d}{dx} \cos x = -\sin x$
6. $\dfrac{d}{dx} \tan x = \sec^2 x$
7. $\dfrac{d}{dx} \cot x = -\operatorname{cosec}^2 x$
8. $\dfrac{d}{dx} \sec x = \sec x \tan x$
9. $\dfrac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$
10. $\dfrac{d}{dx} \sinh x = \cosh x$
11. $\dfrac{d}{dx} \cosh x = \sinh x$
12. $\dfrac{d}{dx} \tanh x = \operatorname{sech}^2 x$
13. $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$
14. $\dfrac{d}{dx} \cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$
15. $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1+x^2}$
16. $\dfrac{d}{dx} \sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$
17. $\dfrac{d}{dx} \cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}} \quad (x > 1)$
18. $\dfrac{d}{dx} \tanh^{-1} x = \dfrac{1}{1-x^2} \quad (|x| < 1)$

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.