FRACTALS

"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." (Mandelbrot)

   

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OVERVIEW

A History of Fractals

B What is a Fractal, Examples

C Building (some types of) Fractals

D Dimension of Fractals

With thanks to John Hutchinson (ANU) who provided material on fractals, some being part of this presentation. Acknowledgment: Most of the graphics in these lectures come from the Yale University site
http://classes.yale.edu/math190a/Welcome.html

A History of fractals

- Many fractals go back to classical mathematics & mathematicians:
- George Cantor (1872)
- Giuseppe Peano (1890)
- David Hilbert (1891)
- Helge von Koch (1904)
- Waclaw Sierpinski (1916)
- Gaston Julia (1918)
- Felix Haussdorf (1919) etc.

History of Fractals-cont.

- Many of the early fractals arose in the attempt to fully explore the mathematical content and limits of fundamental notions (e.g., “continuous” or “curve”).
- What we know as the Cantor set, the Koch curve, the Peano curve, the Hilbert curve and the Sierpinski gasket, were regarded as exceptional object, as counter examples, as “mathematical monsters”.
- The *Cantor set*, the *Sierpinski carpet* and the *Menger sponge* played an essential role in the development of early topology.

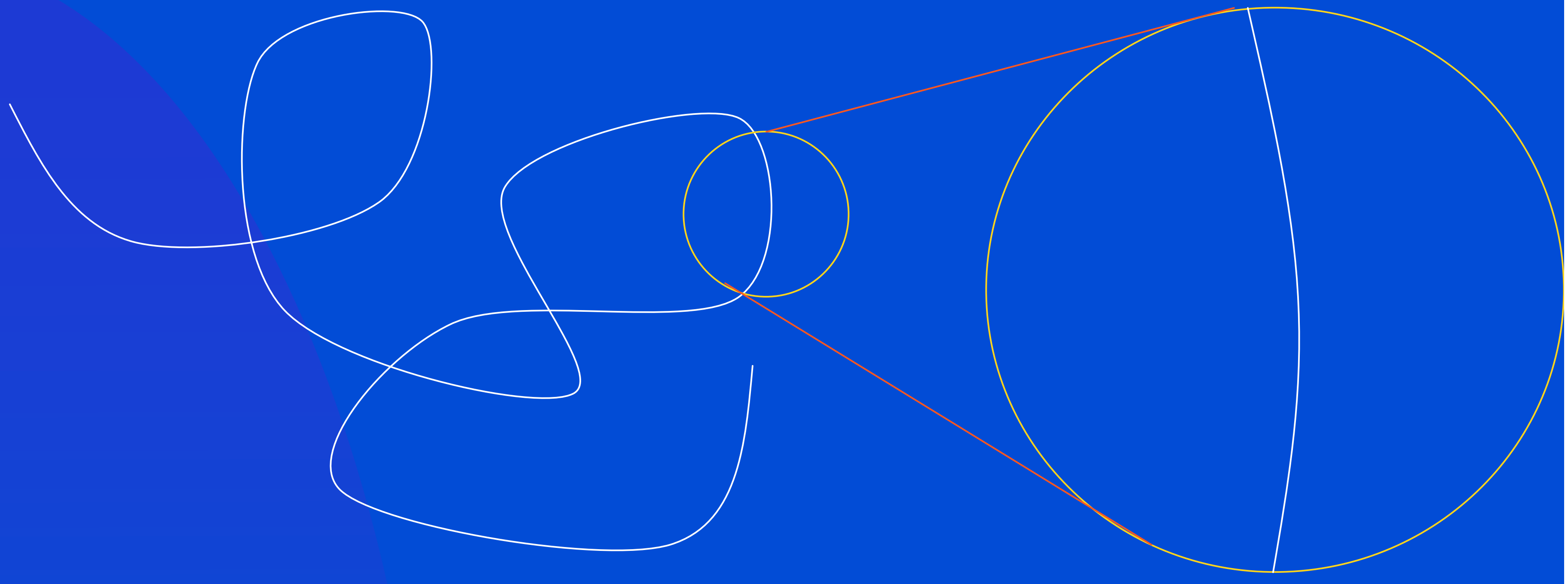
History of Fractals-cont.

- Mandelbrot is the father of fractal geometry and author of the book “The Fractal Geometry of Nature”, 1982. He demonstrated that these early fractals have many features in common with shapes found in nature. Mandelbrot turned upside down the interpretation of fractals proving that they typify the normal rather than being a deviation from the familiar.

B What is a Fractal ?

- A fractal is something which looks “sort of similar” at all scales.
- If you look at smaller and smaller parts of a fractal, perhaps under a microscope, you see similar features at all scales.
- A fractal (in nature) is something which looks similar over a range of scales, say up to four or five doubling magnifications.
- A fractal usually has non integer dimension.

A Squiggle is not a fractal



- Under the microscope, a squiggle looks very much like a straight line
- It looks nothing like the original squiggle
- So a squiggle is not a fractal

Self-similarity

- Is the underlying theme in all fractals.
- The shape is made of smaller copies of itself. The copies are similar to the whole: same shape but different size.
- The cauliflower is not a classical fractal, but it is a natural example, where the self-similarity is readily revealed.

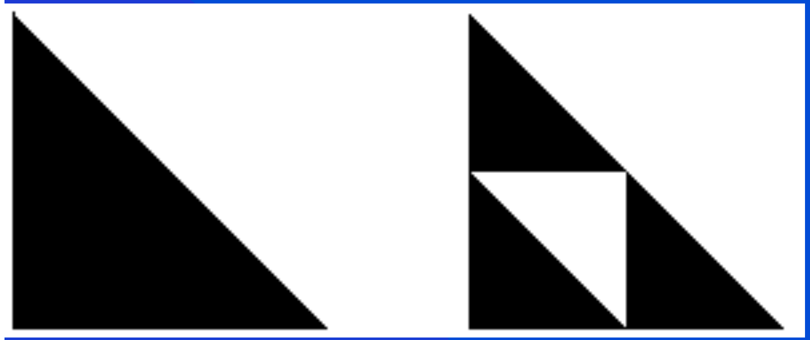
Self-similarity

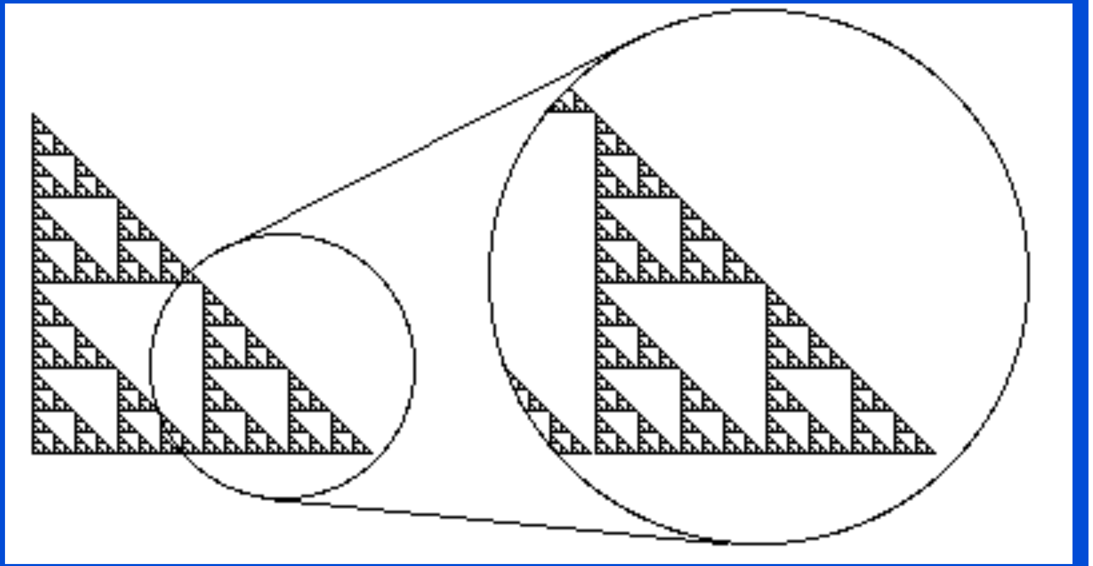
- The cauliflower head contains branches or parts, which when removed and compared with the whole are very much the same, only smaller. These clusters again can be decomposed into smaller ones, which look very similar to the whole as well as to the first generation. The self-similarity carries through for about three or four stages. After that the structures are too small for a further dissection.
- In a mathematical idealization, the self-similarity property of a fractal may be continued infinitely many stages. This leads to new concepts such as fractal dimension which are also useful for natural structures that do not have this “infinite detail”.

Self-similarity

- Self-similarity is only a few decades old, though many constructions make use of it such as the decimal number system.
- All versions of self-similarity imply scale invariance: fractals have no natural size. By contrast, Euclidean objects such as circles, spheres and squares do have a natural size. (Circles and spheres have diameters, squares have side lengths, etc.)

Sierpinski Gasket





The gasket is *self similar*;

it is made up of 3 small scaled copies of itself,

and 9 even smaller copies,

and 27 even smaller copies, …

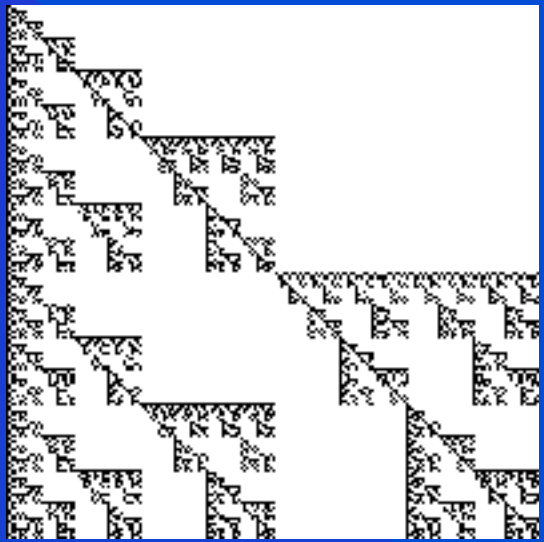
and so ad infinitum

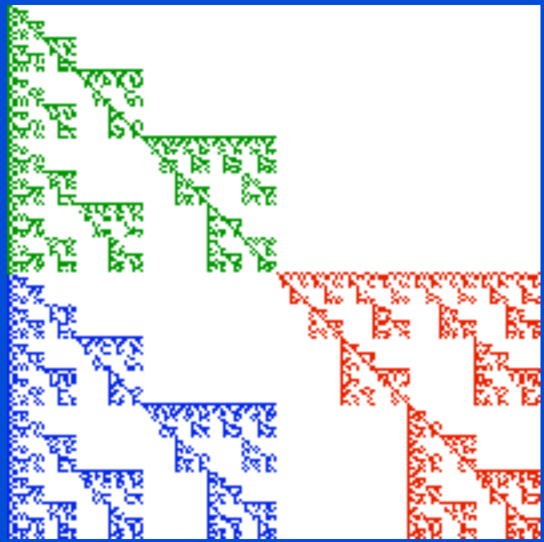
"Big gaskets are made of little gaskets,
The bits into which we slice 'em.
And little gaskets are made of lesser gaskets
And so ad infinitum."

More Examples

- Dendrites and Natural Dendrites
- Fern
- Queen Anne’s Lace
- Different Scalings
- Random Fractal
- Escheresque Fractal
- Mandelbrot Set

Dendrite



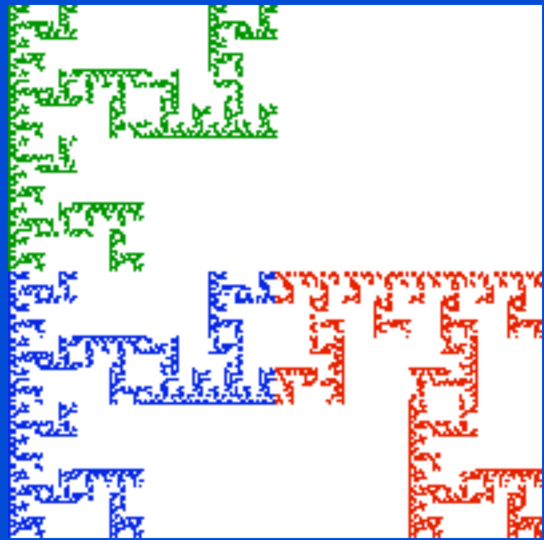


The dendrite is made of three copies of itself, *self-similar*

* each a copy of the original,
* each scaled by a factor of 1/2 in the x- and y-directions,
* each in different orientations and positions.

A Natural Fractal, Mineral Dendrites

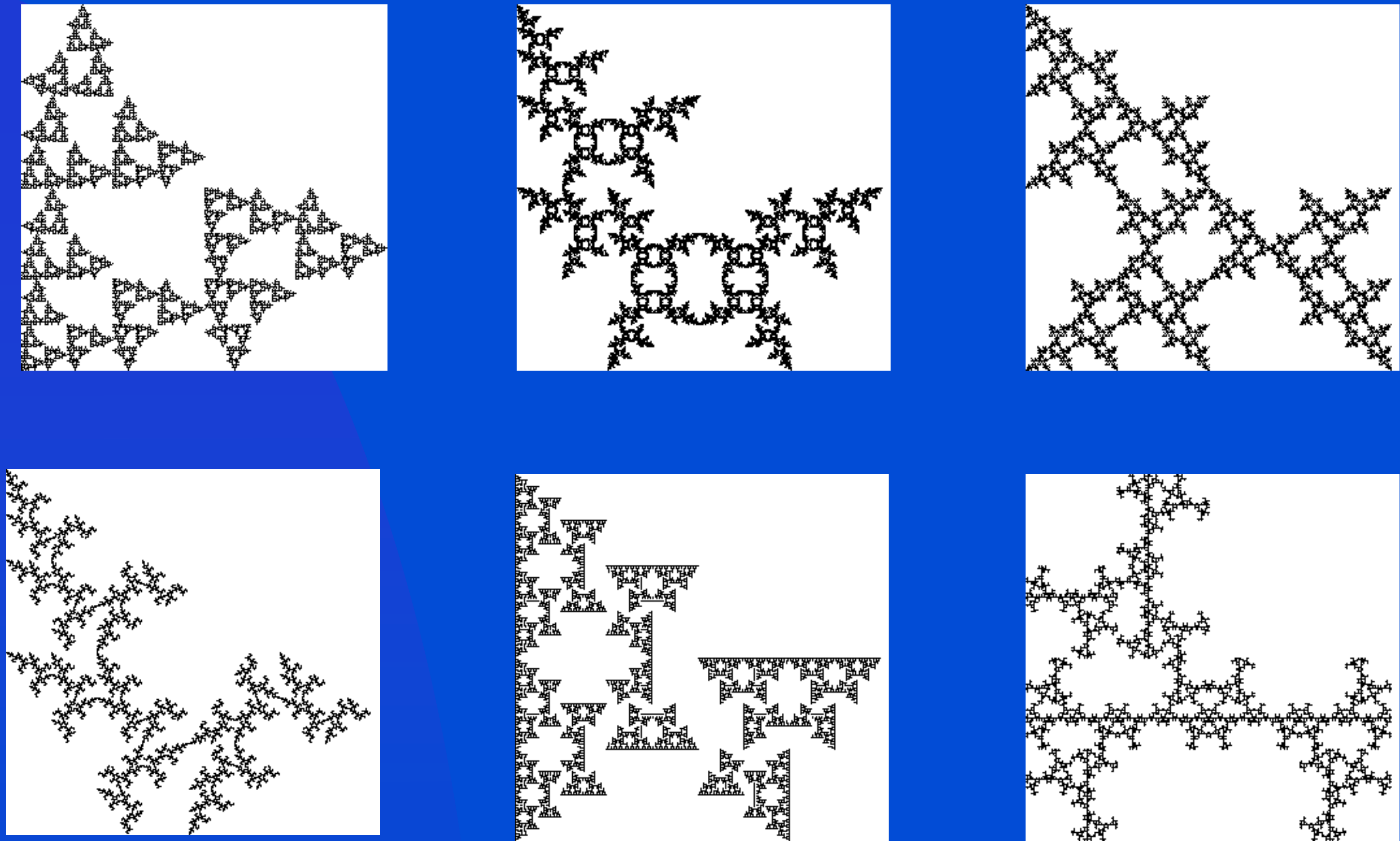
Another Fractal



This also is made of three copies of itself, *self-similar*

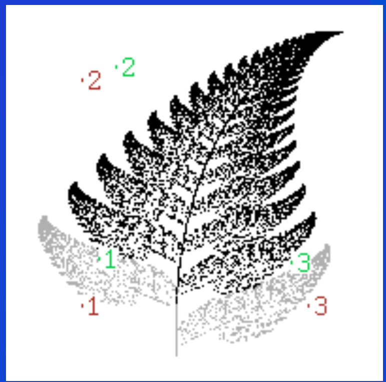
* each a copy of the original,
* each scaled by a factor of 1/2 in the x- and y- directions,
* each in different orientations and positions.

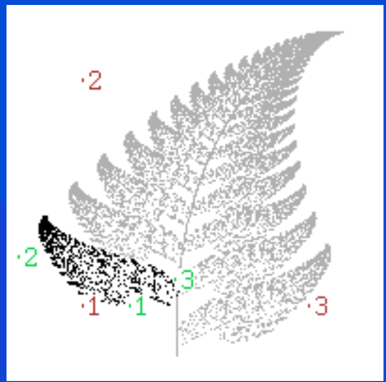
More Examples

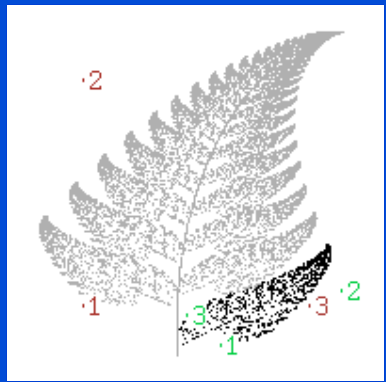


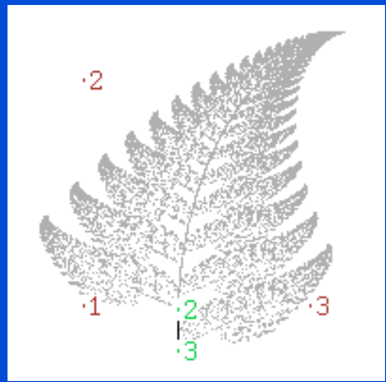
All these *self similar* fractals can be viewed as being made up of three copies of themselves, each scaled by 1/2.

The Fern



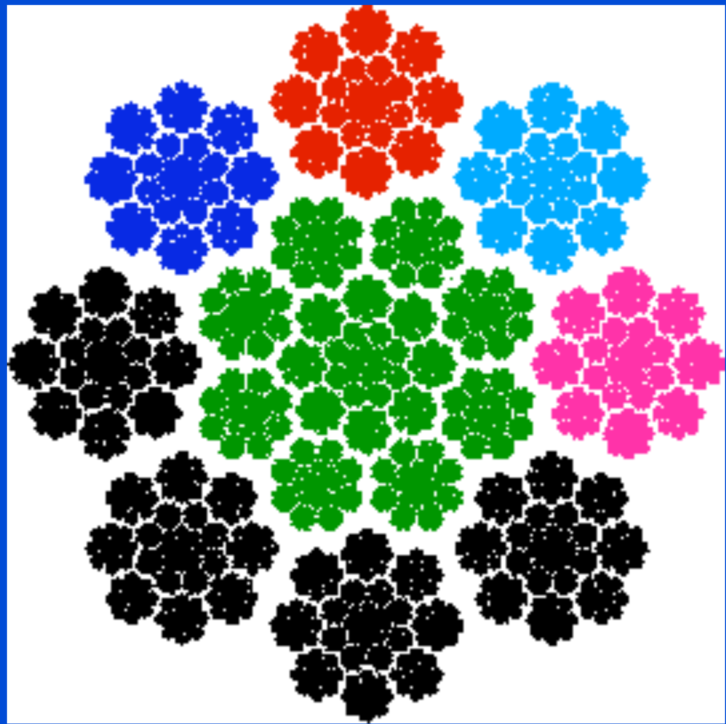






Is made of four scaled copies of itself
(this includes the stem as one part)

Queen Anne's Lace

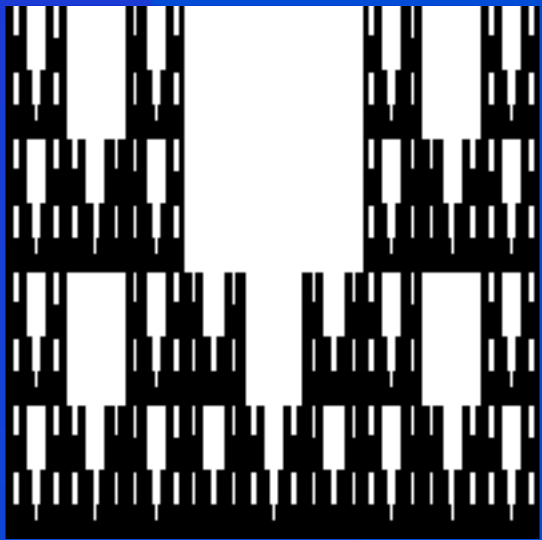


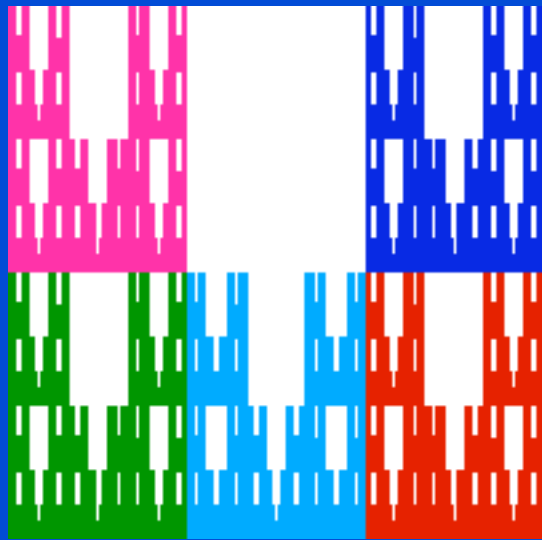
Queen Anne's Lace is made of eight little florets around the perimeter, and one large floret in the middle.

The middle floret is rotated relative to the whole shape.

Each floret is a scaled copy of the whole

Different Scalings in different directions





Each piece is scaled by

1/3 in the x direction and

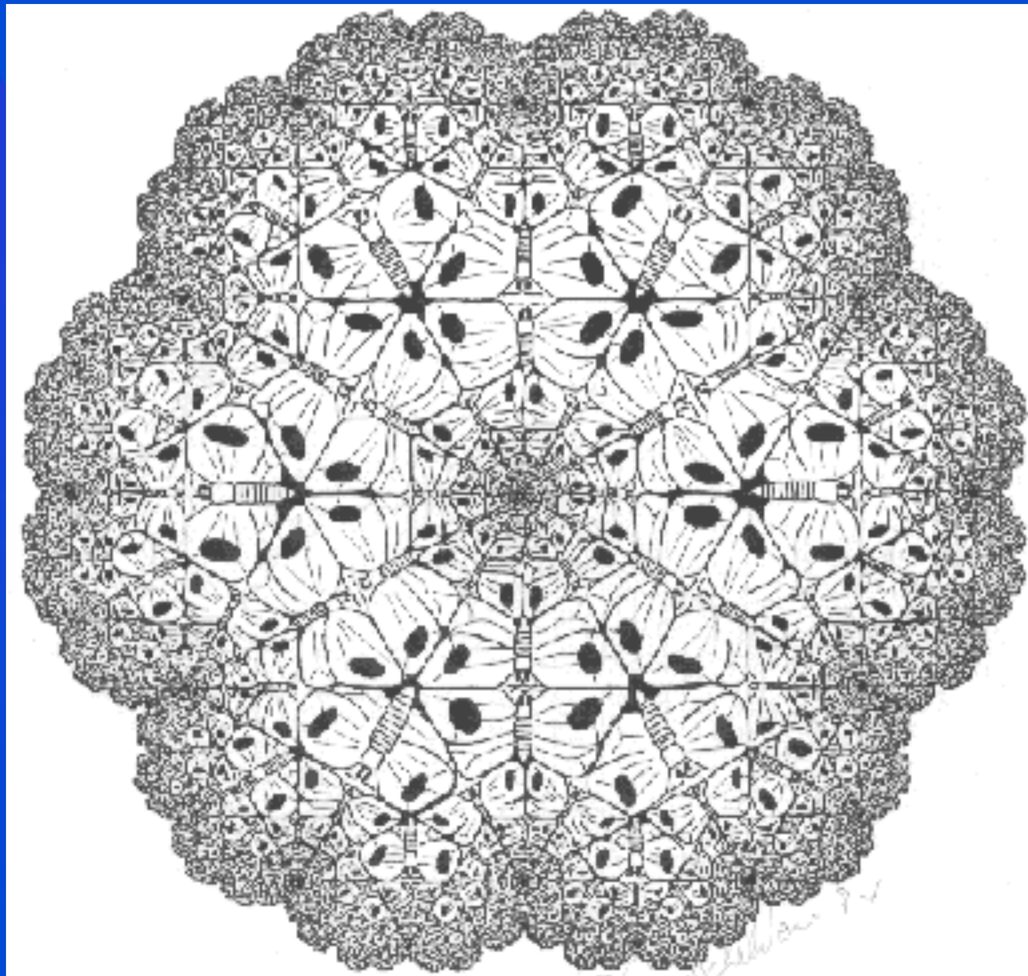
1/2 in the y-direction.

A Random Fractal

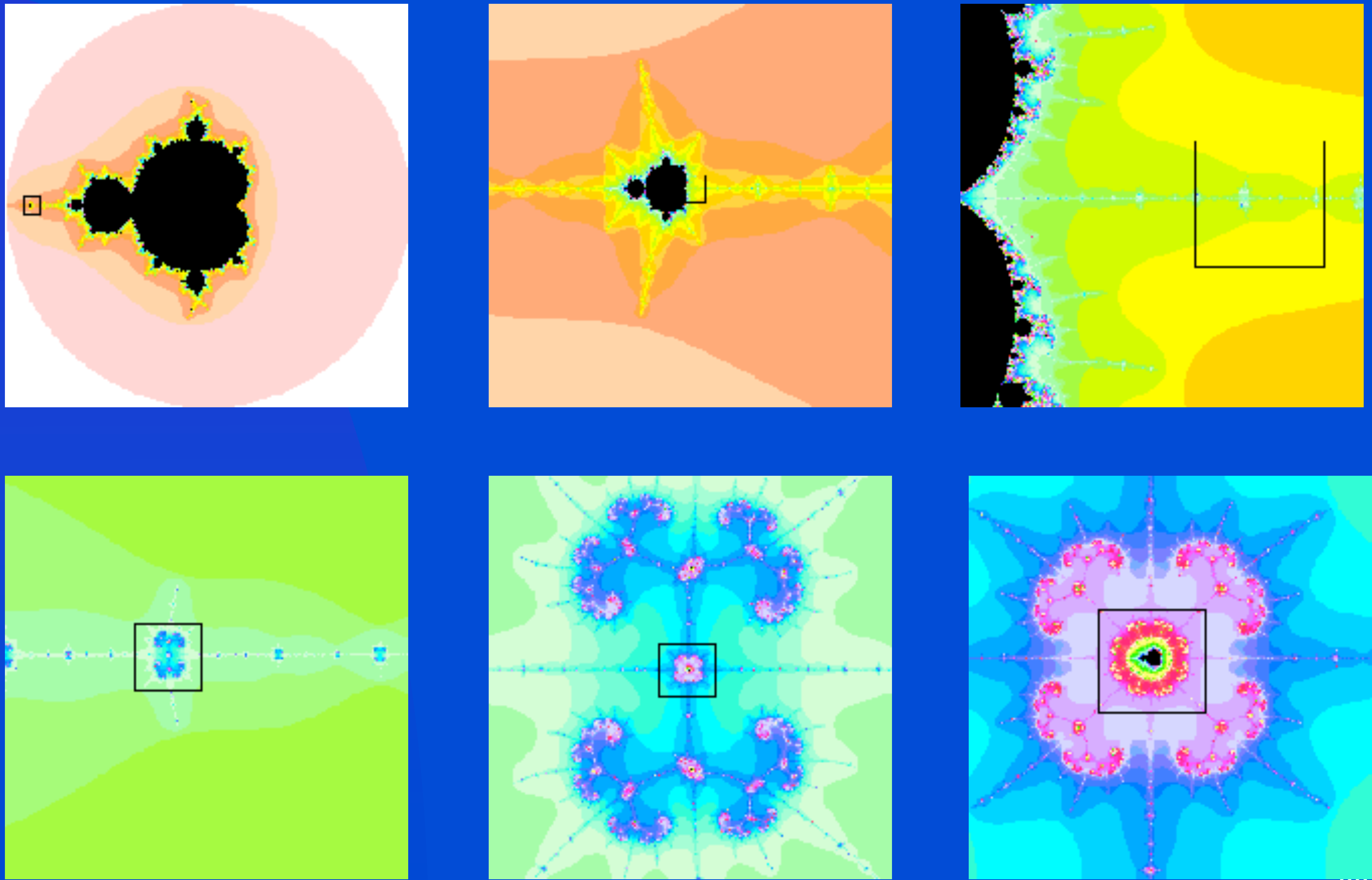
The right window is a rescaling of the x-axis by a factor of 4, and the y-axis by a factor of 2.

The right picture has about the same distribution of jumps as the left.

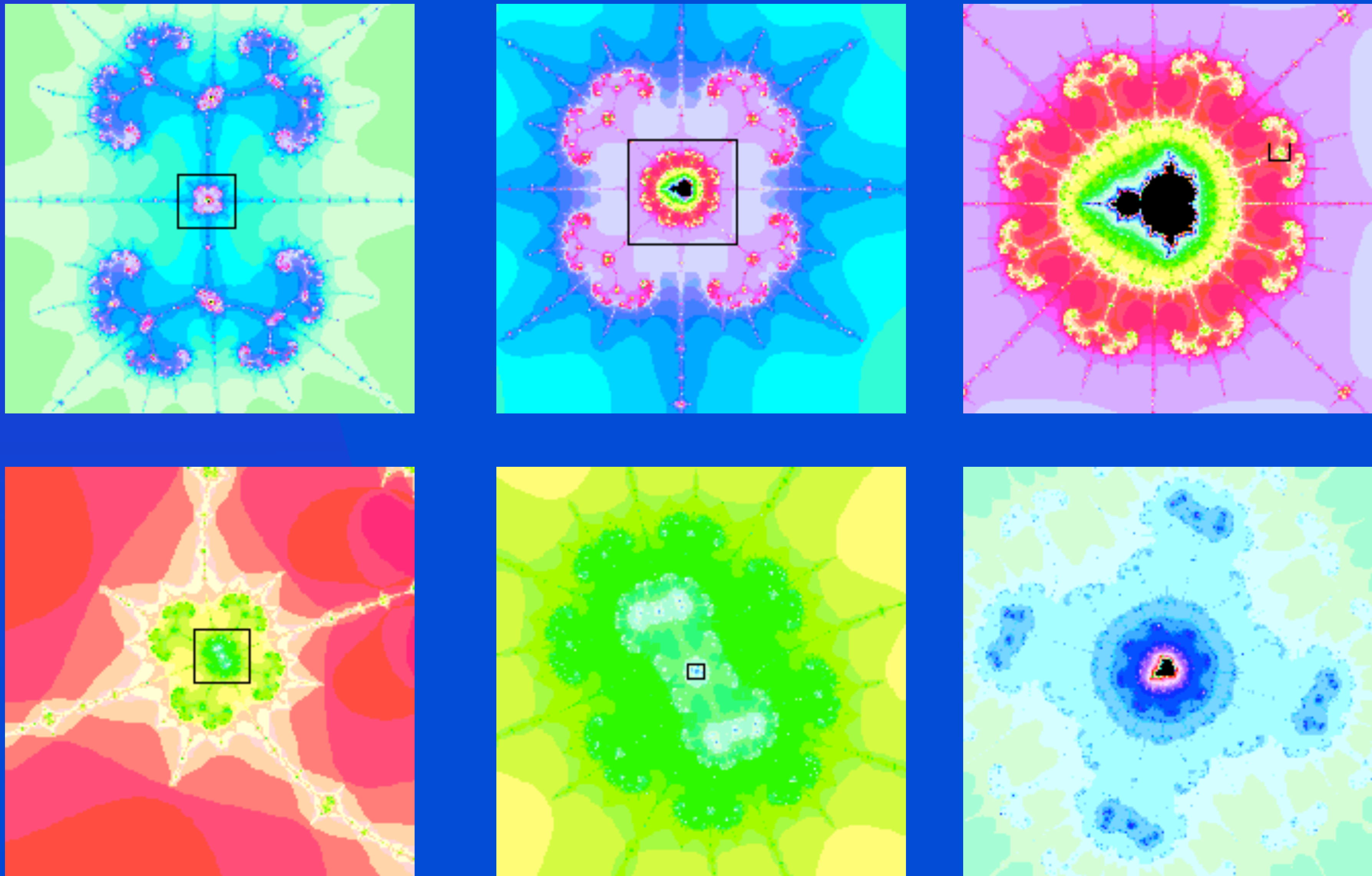
Escheresque fractal



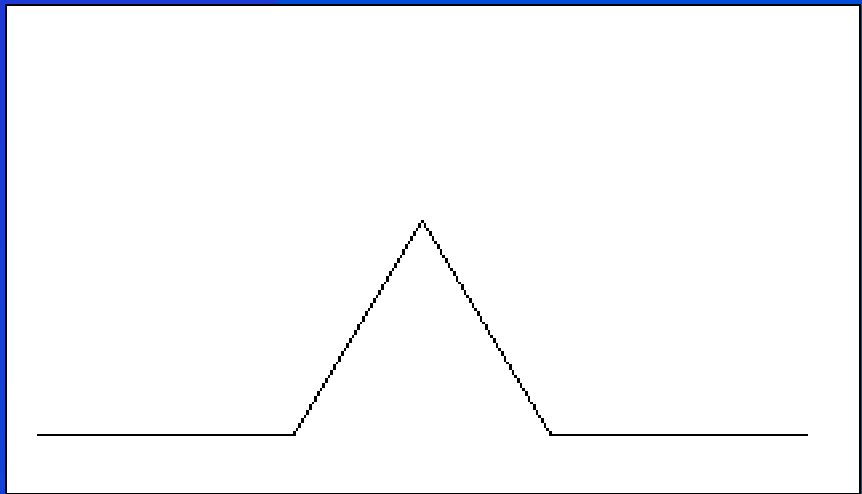
Mandelbrot Set 1



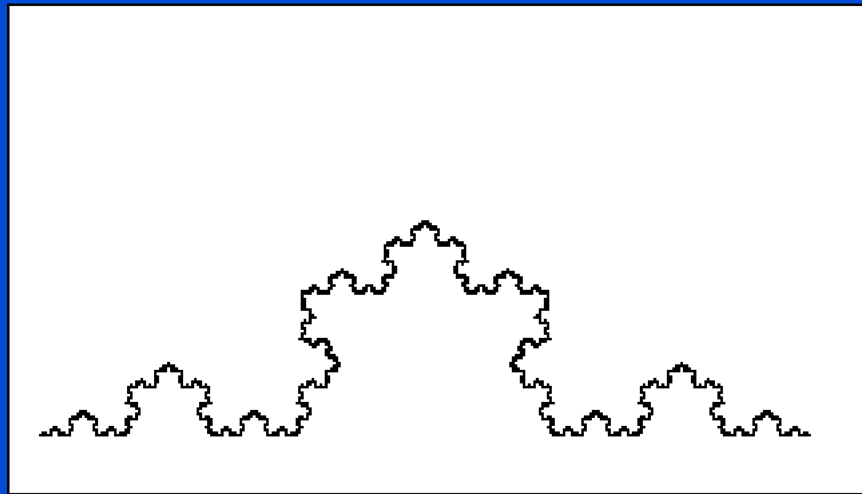
Mandelbrot Set 2



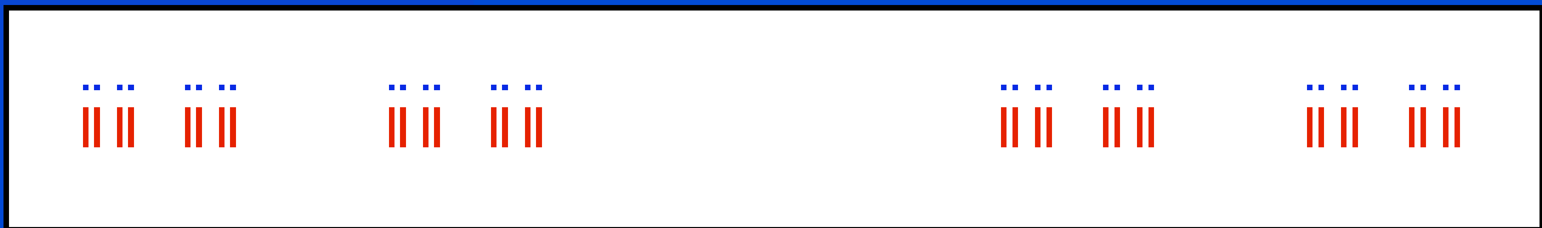
C Building Fractals with Initiators and Generators



The generator of Koch curve



The Koch curve



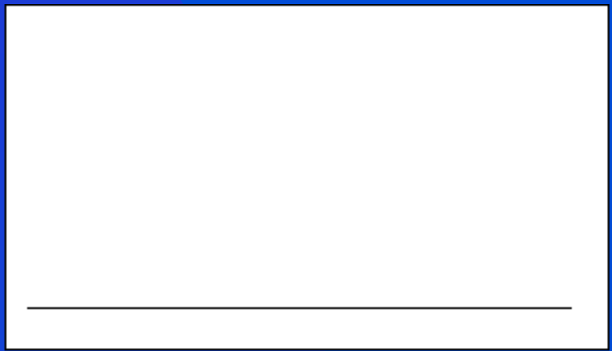
The Cantor Set (after 7 iterations)

Koch curve: Basic construction

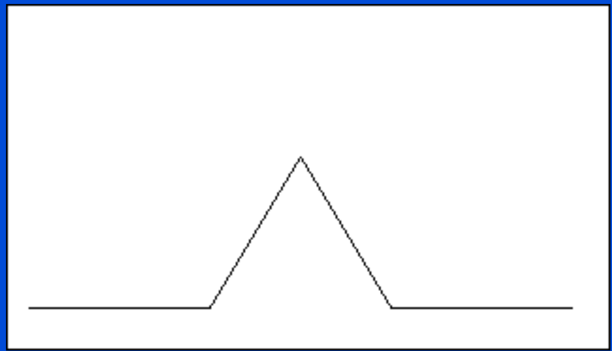
- Begin with a straight line (called the initiator).
- Partition it in 3 equal parts. Then replace the middle third by an equilateral triangle and take away its base. This ends the basic construction step. A reduction of this figure, made of 4 parts, will be reused in the next stages. It is called the generator. We now take each of the 4 line segments and repeat the basic construction.
- Self-similarity is built into the process: using the shape of the generator, we see that the Koch curve is made up of 4 copies of itself, each scaled by a factor of 1/3 horizontally and vertically.

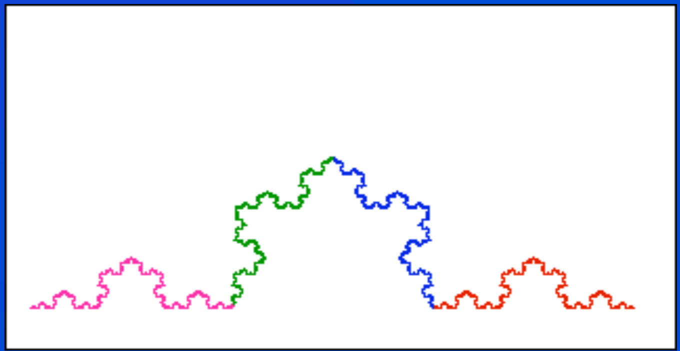
The Koch curve

The initiator



The generator



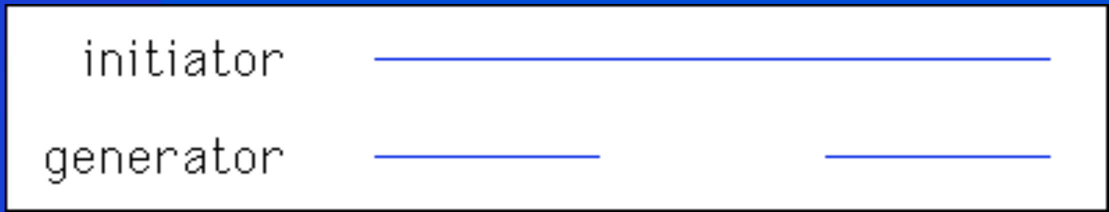


The Koch curve is self similar

It is made up of 4 copies of itself, each scaled by a factor of 1/3.

The Cantor Set

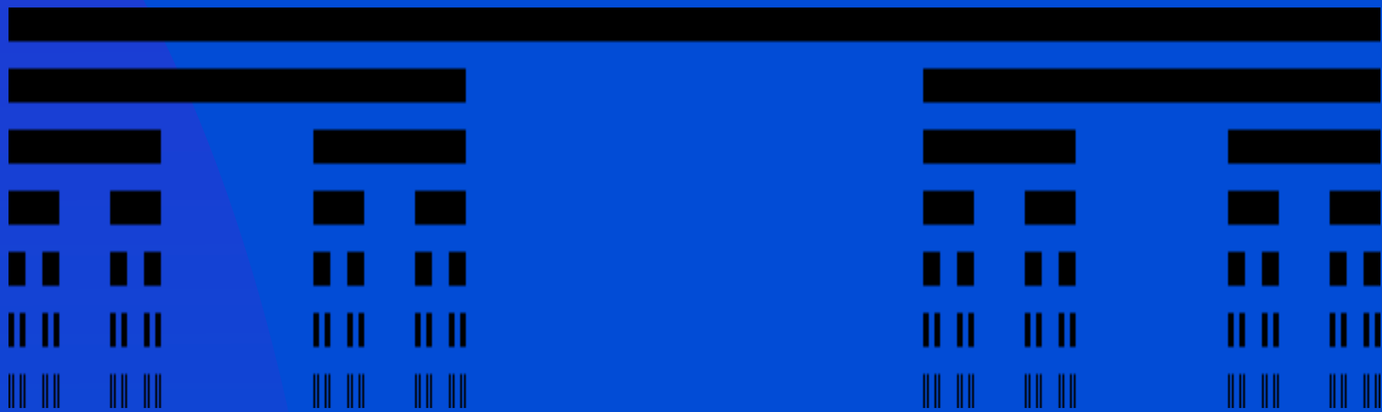
Take as initiator the line segment of length 1, while the generator is the shape shown below.



Start with the interval [0,1]. Now take away the open interval (1/3,2/3). This leaves two intervals [0,1/3] and [2/3,1], each of of length 1/3. This completes the basic construction step. We look at these remaining intervals and remove their middle thirds, which yields 4 intervals of length 1/9.

The Cantor Set

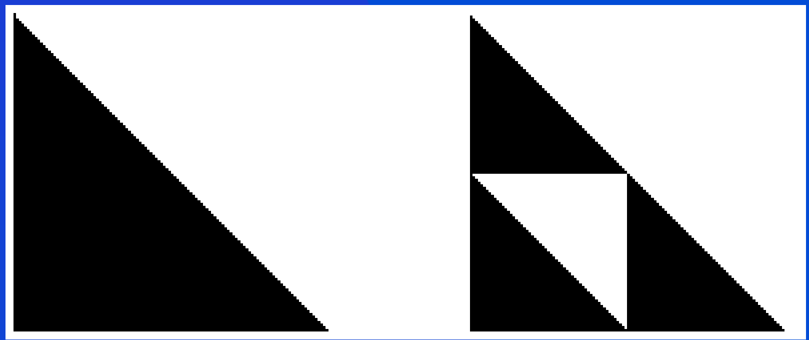
Continue in the same way. The next figure shows the Cantor set after 7 iterations.

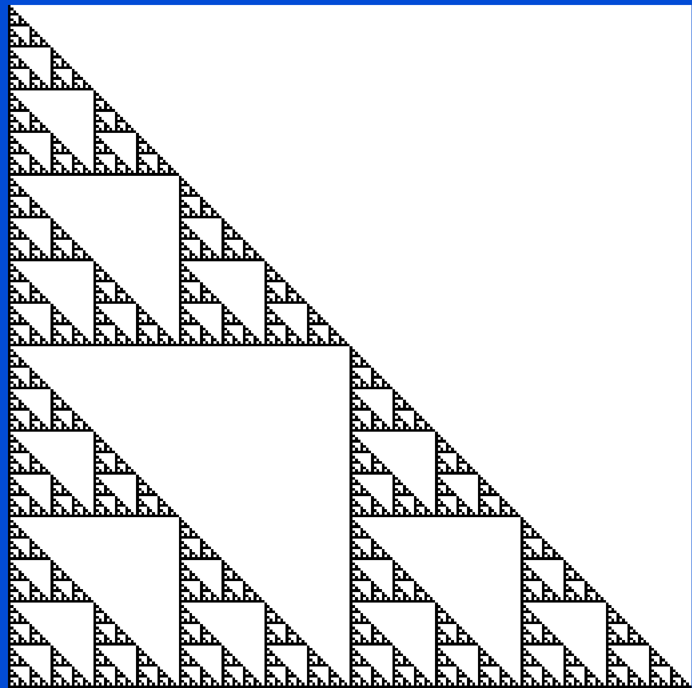


The Cantor set is the set of points which remain if we carry out the removal steps infinitely often. How many points do you think they are in the Cantor set? Can we count them?

Sierpinski Gasket

Self similar: made up of 3 copies of itself, each scaled by a factor of 1/2.





The filled-in triangle is the initiator.
The triangle next to it is the generator.

Sierpinski gasket

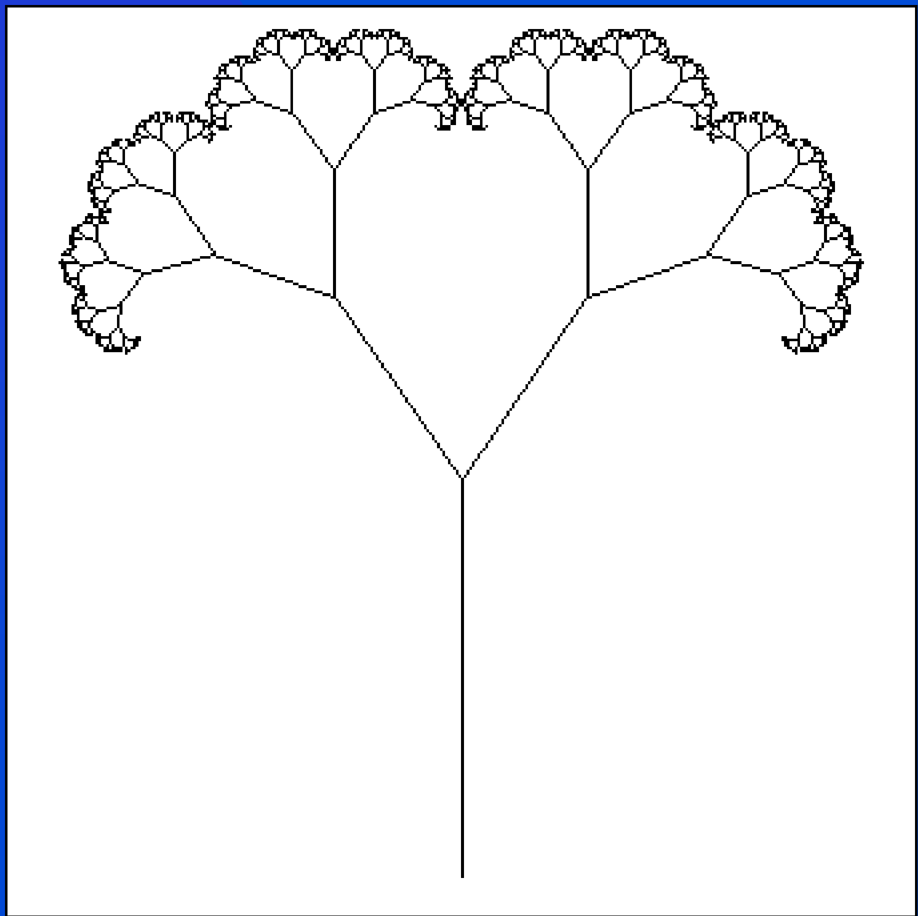
Sierpinski Gasket: Basic Construction

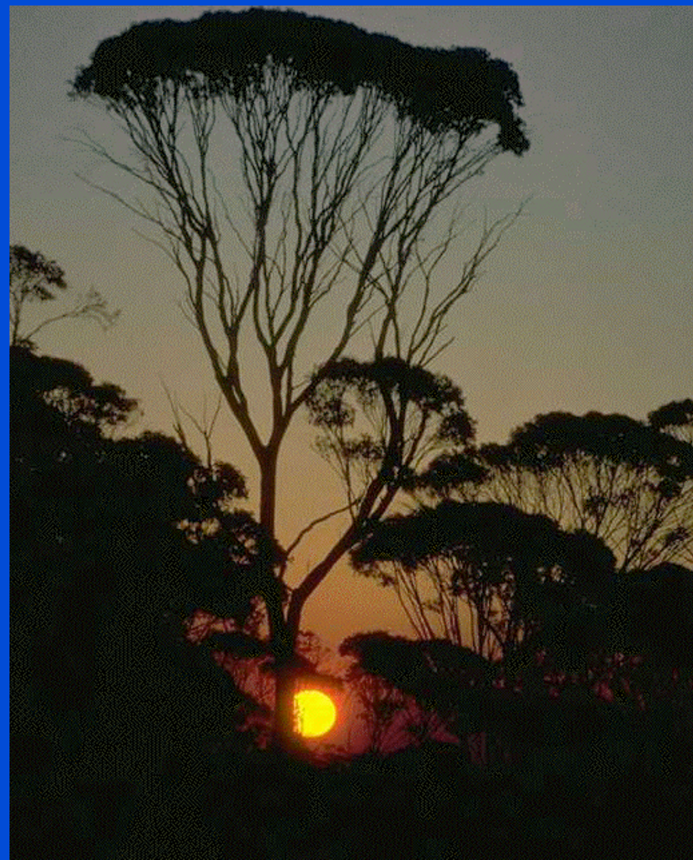
- Begin with a triangle in the plane (a blackened, filled-in triangle), which is called the initiator. Then apply a repetitive scheme of operations.
- Pick the midpoints of its three sides. These midpoints, together with the old vertices of the original triangle, define four congruent triangles of which we drop the center one. The resulting figure is called the generator.
- After this first step, we have three congruent triangles whose sides have exactly half the size of the original triangle.

Sierpinski Gasket:

- With these three remaining triangles we follow the same procedure as in the first step. Repeat the basic step as often as desired.
- We start with one triangle, then produce 3, 9, 27, 81, 243,… triangles, each of which is an exact scaled down version of the triangles in the preceding step.
- The Sierpinski gasket is the set of points in the plane which remain if one carries out this process infinitely often. The sides of each of the triangle in the process are definitely points which belong to the Sierpinski gasket.

Fractal Tree

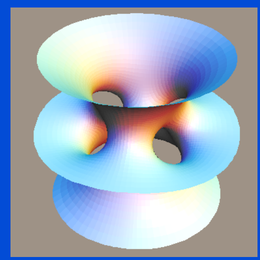




D Dimension of fractals

- Dimension of a line or a squiggle is 1

- Dimension of a “surface” is 2



- Dimension of a “solid object” is 3

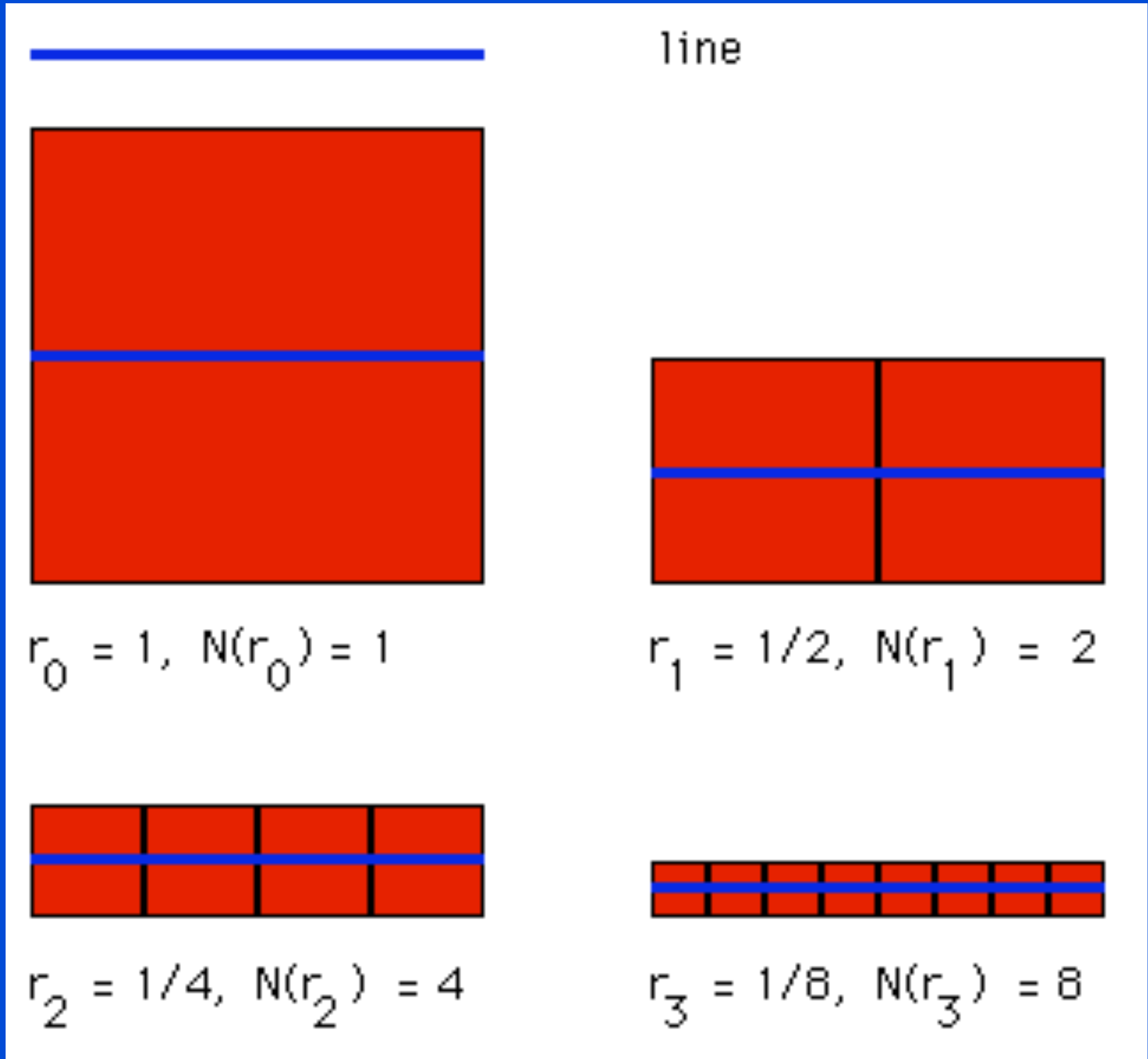
How to find the Dimension

- We will discuss *Box Counting Dimension*
- Then we will discuss *Similarity Dimension*
- There are other notions of Dimension, but they all give the same value in the cases we consider

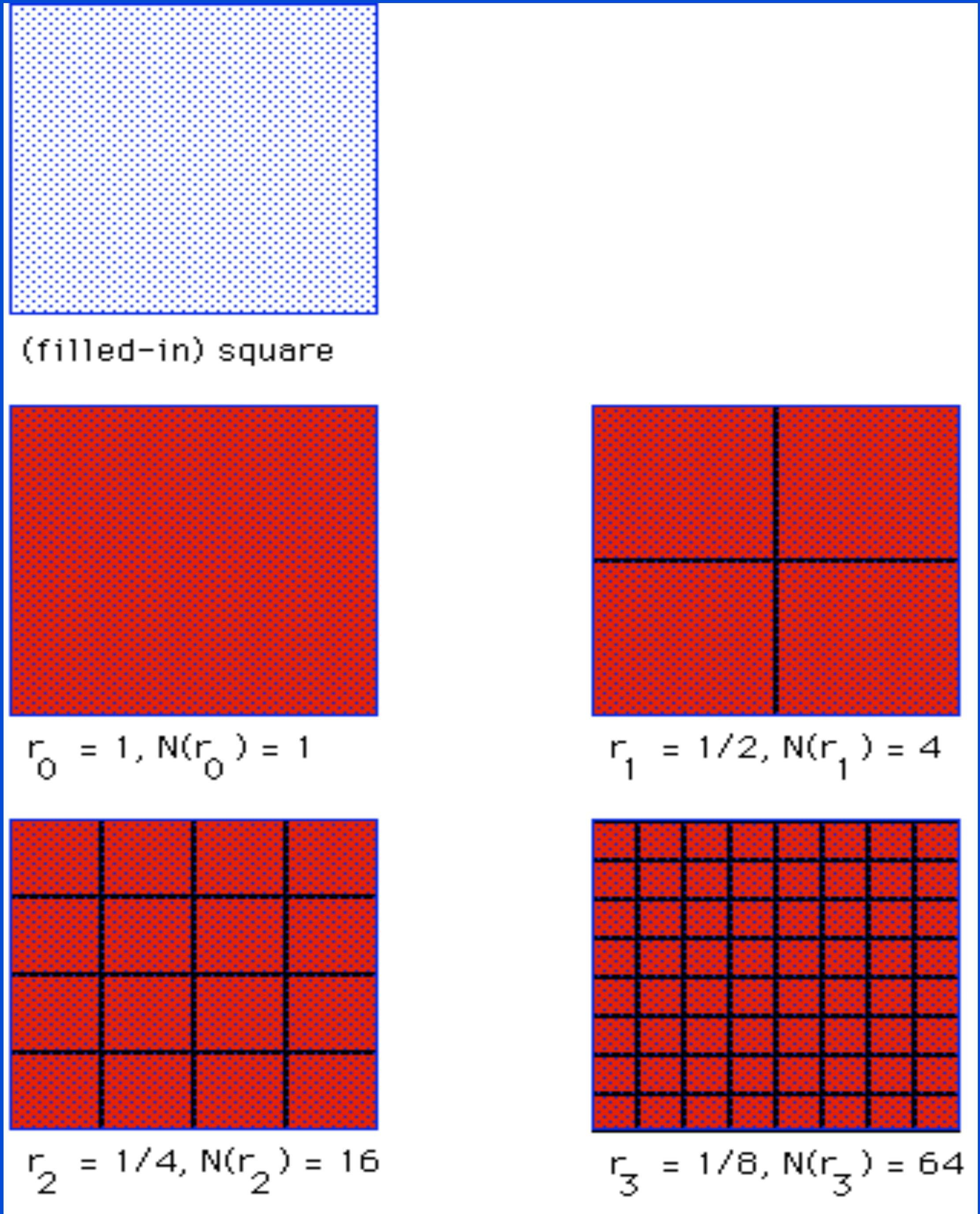
Box Counting Dimension **is computed by covering the object with non overlapping squares of fixed side length r, and then looking for a pattern in how the number N(r) of squares depends on r.**

Box-Counting Dimension of a Unit Line Segment

$N(r) = 1/r$



Dimension of a (filled in) Unit Square



$N(r) = (1/r)^2$

Formula for Box Counting Dimension

- If (approximately) $N(r) = (1/r)^d$ then (approximately) $\log N(r) = d \log(1/r)$,

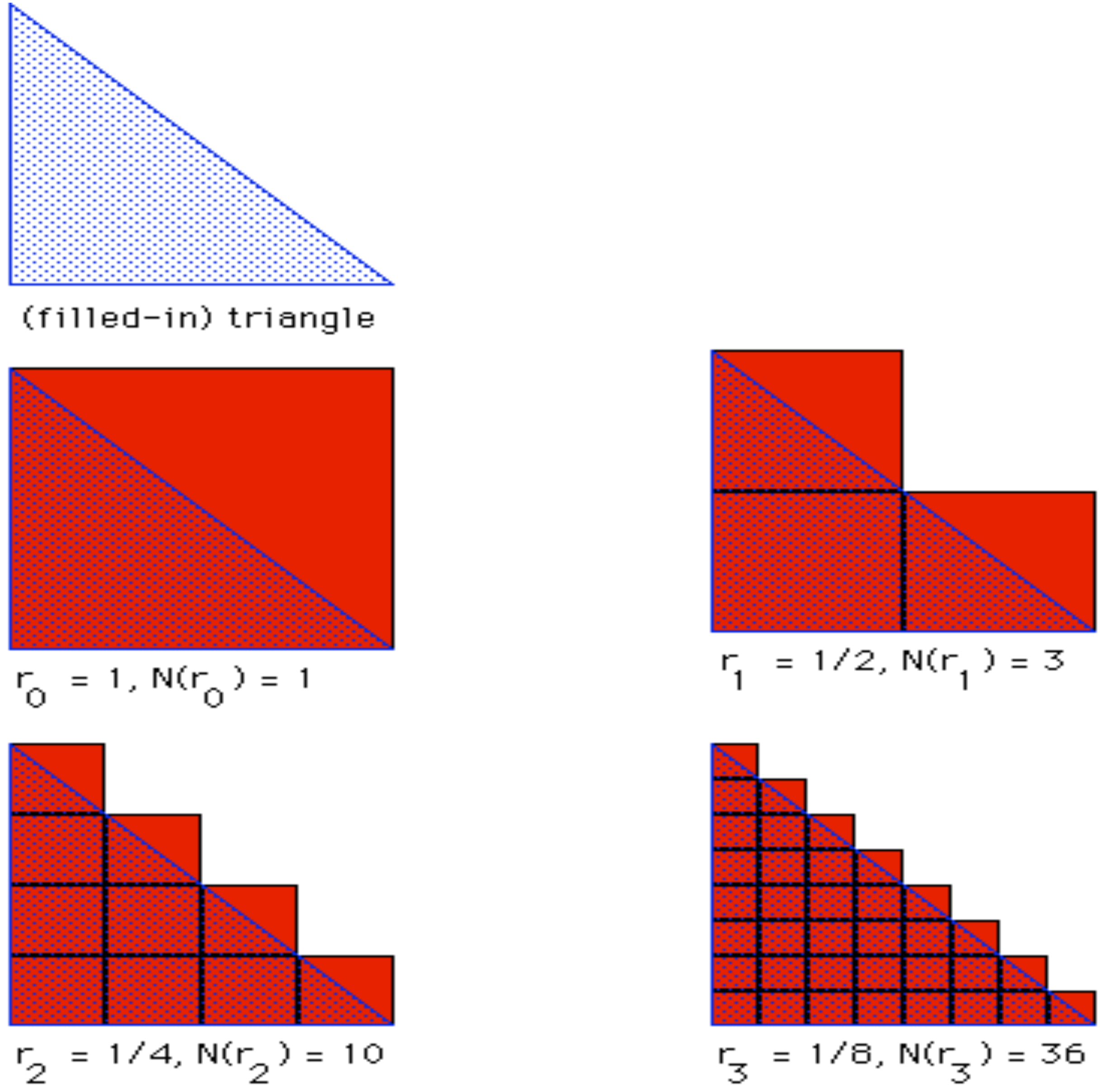
  i.e. $d = \log N(r) \,/\, \log(1/r)$
- We *define* the dimension to be

  $\log N(r) \,/\, \log(1/r)$ for small r.
- More precisely, we *define*

  dimension $= \lim_{r \to 0} \log N(r) \,/\, \log(1/r)$

  if the limit exists.

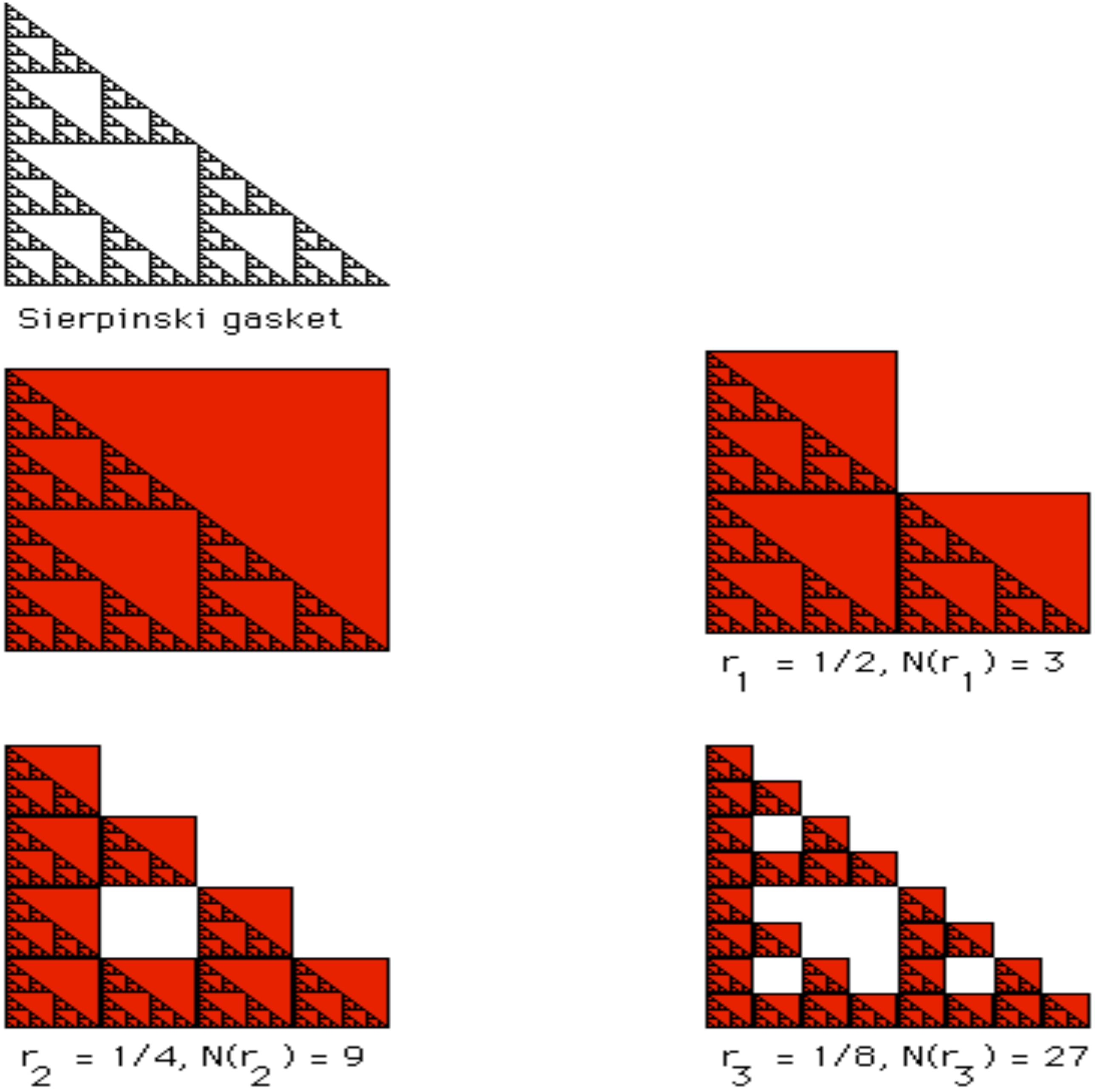
Triangle



$N(1/2^k) = 1+2+3+\ldots+2^k = 2^k(2^k+1)/2$

It follows that the dimension is 2 . *Why?*

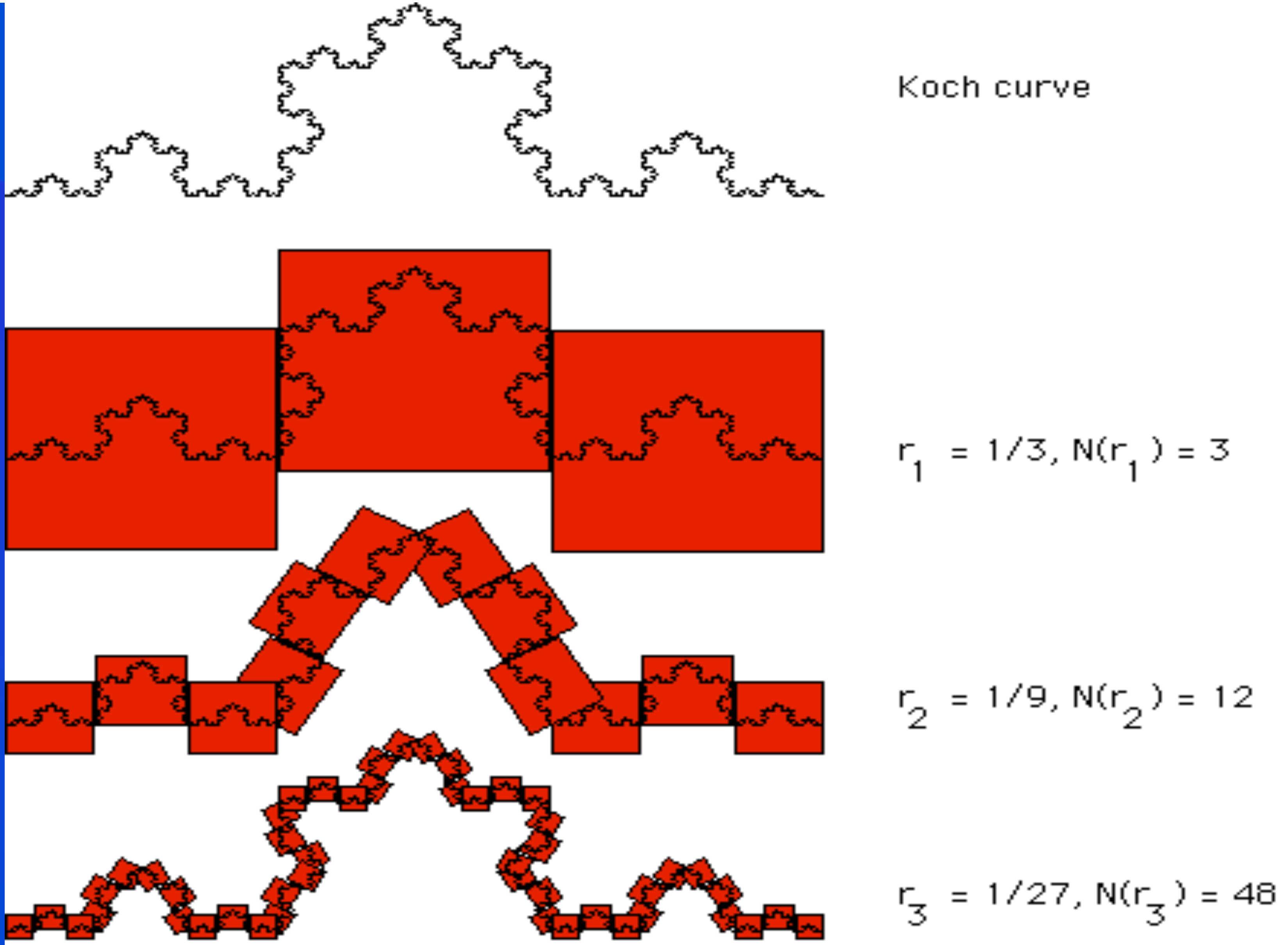
Sierpinski Gasket



$N(1/\ 2^k) = 3^k$

It follows that the dimension is $\log 3 / \log 2 = 1.58996\ldots$

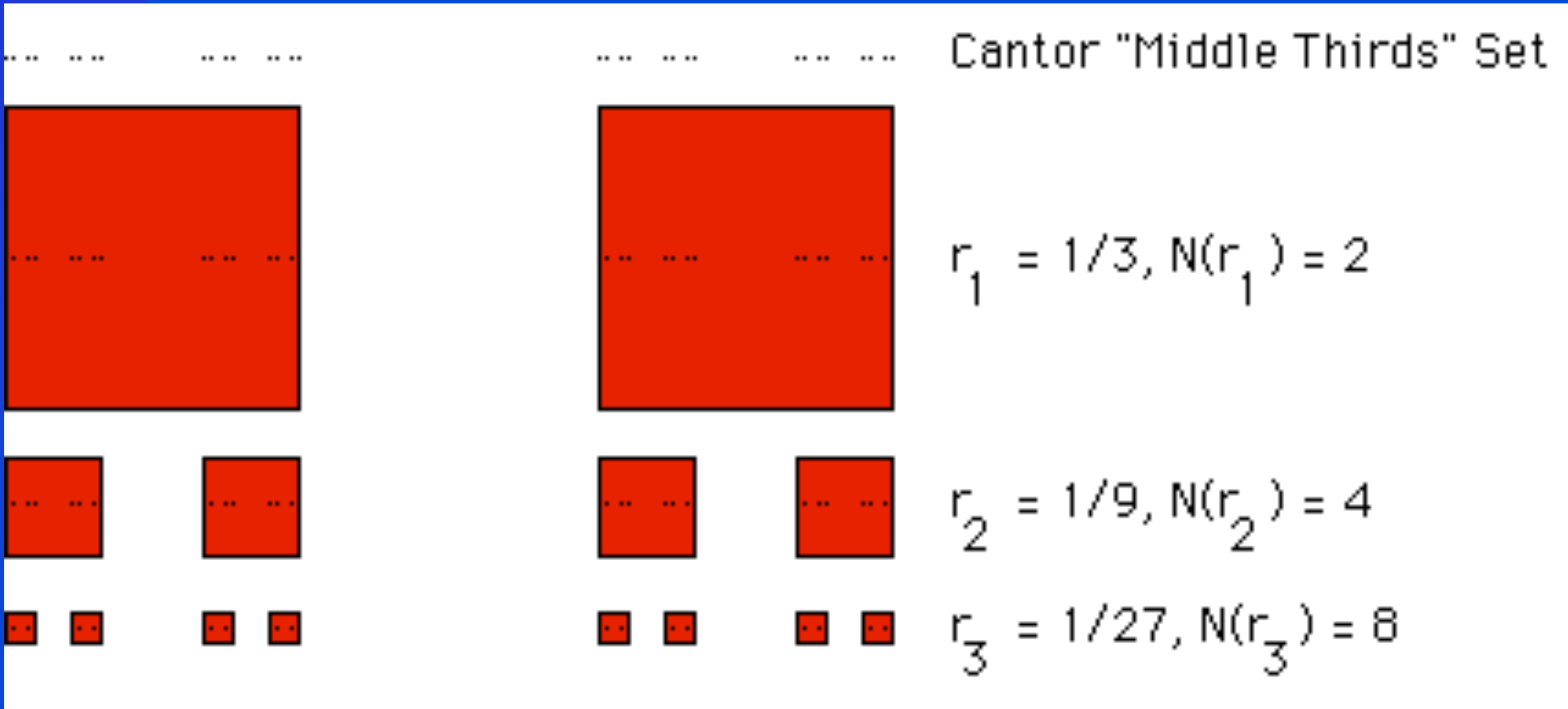
Koch Curve



$N(1/3^k) = 4^{k-1} \times 3$ **(cheating a bit with overlapping boxes!)**

It follows that the dimension is $\log 4 / \log 3 = 1.26186...$

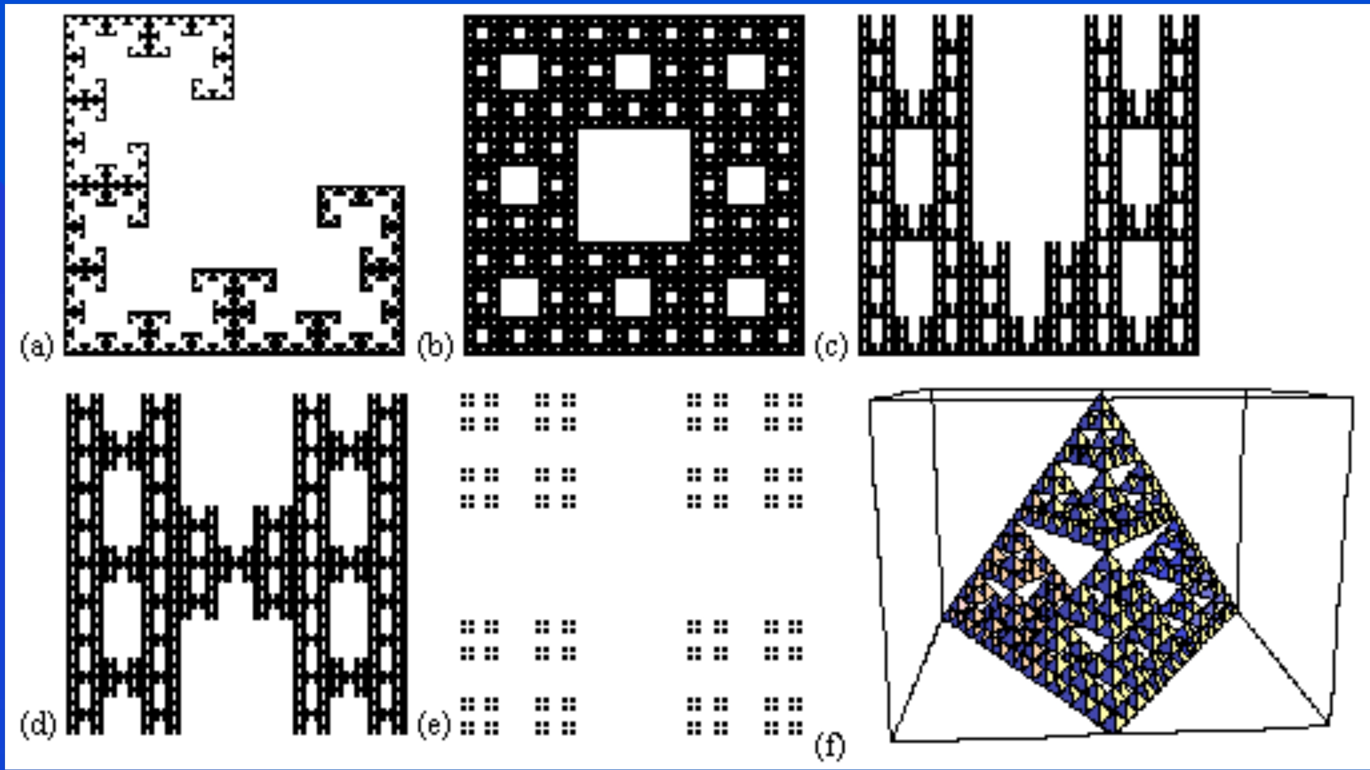
Cantor Set



$N(1/\,3^k) = 2^k$

It follows that the dimension is log 2 / log 3 = .62989…

Similarity Dimension



If a self-similar set is made of

N copies of itself scaled down by r

(and so covered by N boxes of side r in right units)

N^2 copies of itself scaled down by r^2

(and so covered by N boxes of side r^2 in right units)

N^3 copies of itself scaled down by r^3

(and so covered by N^3 boxes of side r^3 in right units)

…………………

N^k copies of itself scaled down by r^k

(and so covered by N^k boxes of side r^k in right units)

…………………

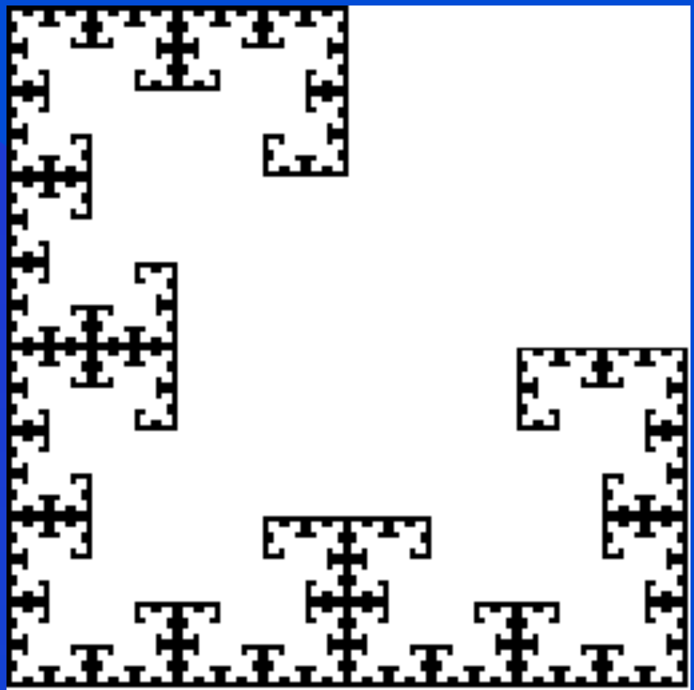
Then the box dimension is

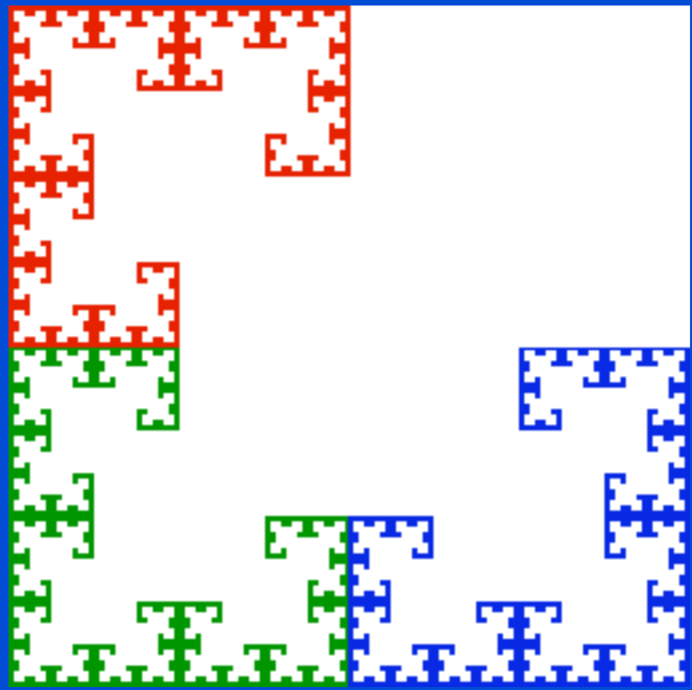
$$\log(N^k)/\log(1/r^k) = \log N / \log (1/r)$$

If a *self similar* set is made of N copies of itself each scaled by the factor r, then we *define* its *similarity dimension* to be

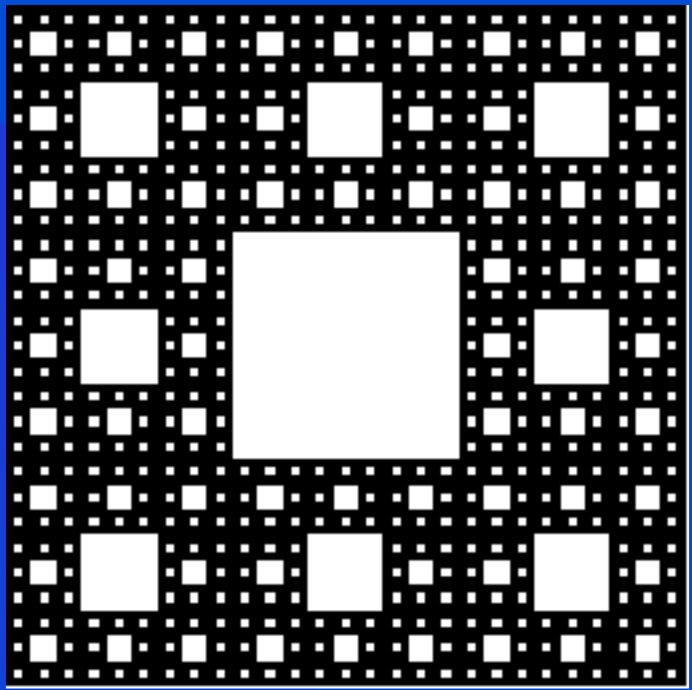
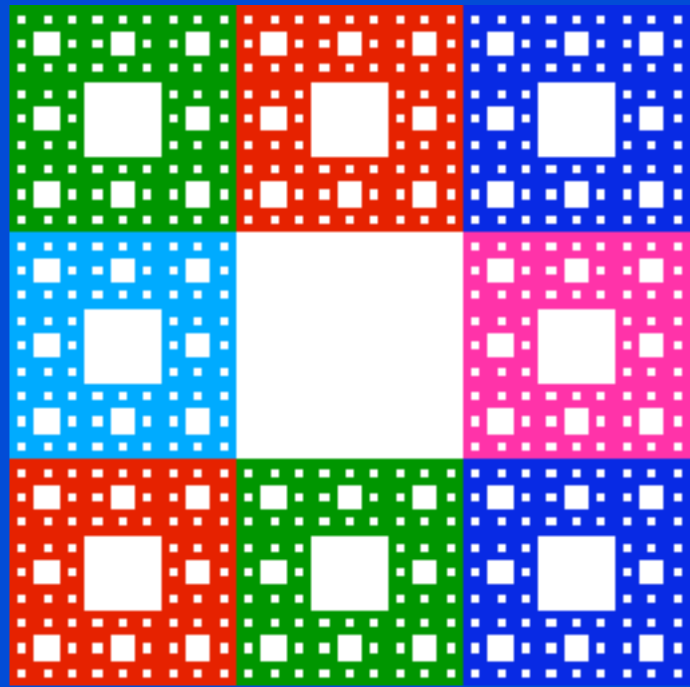
$$\log N / \log(1/r)$$

Agrees with the box dimension, immediate to calculate

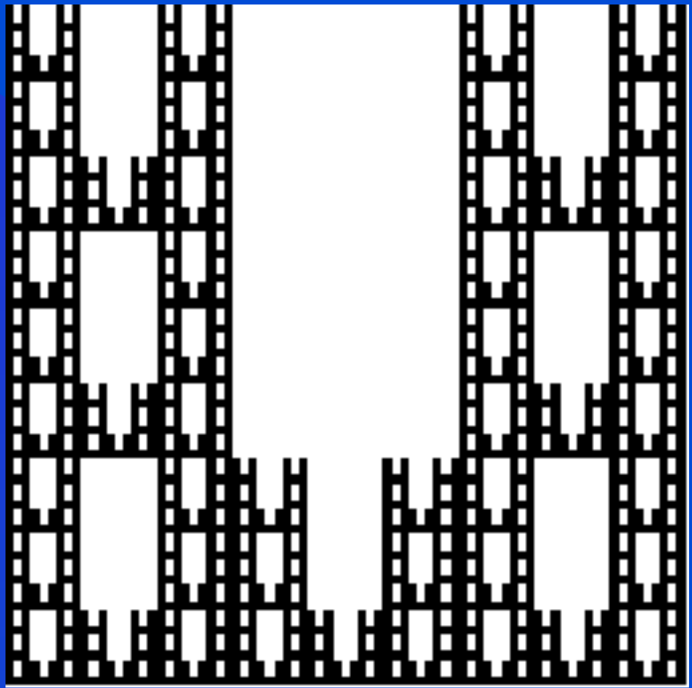


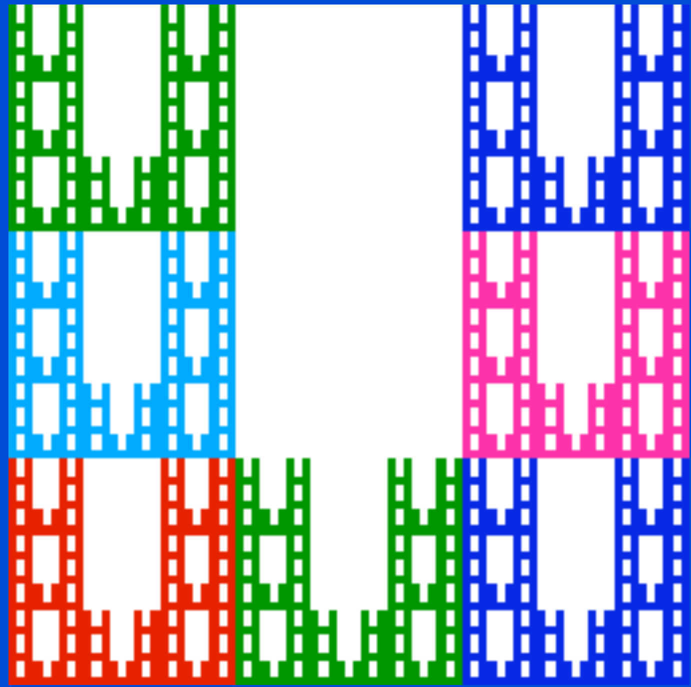


Dimension = log 3 / log 2

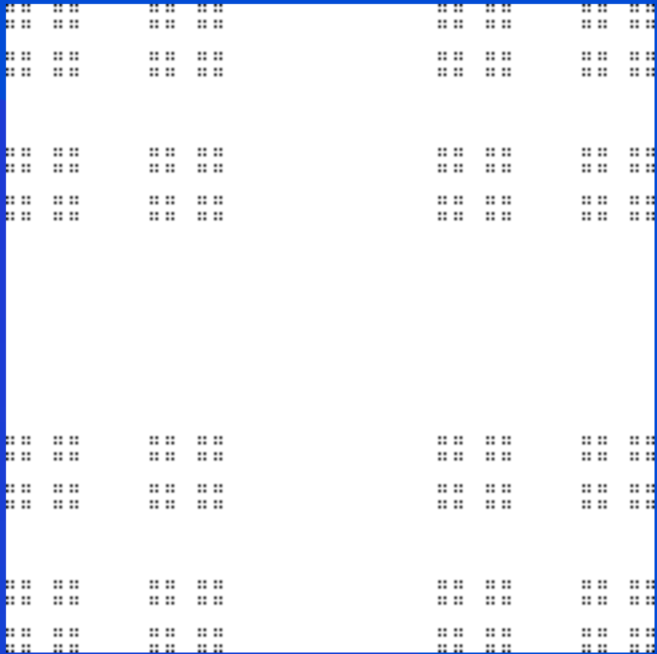
 

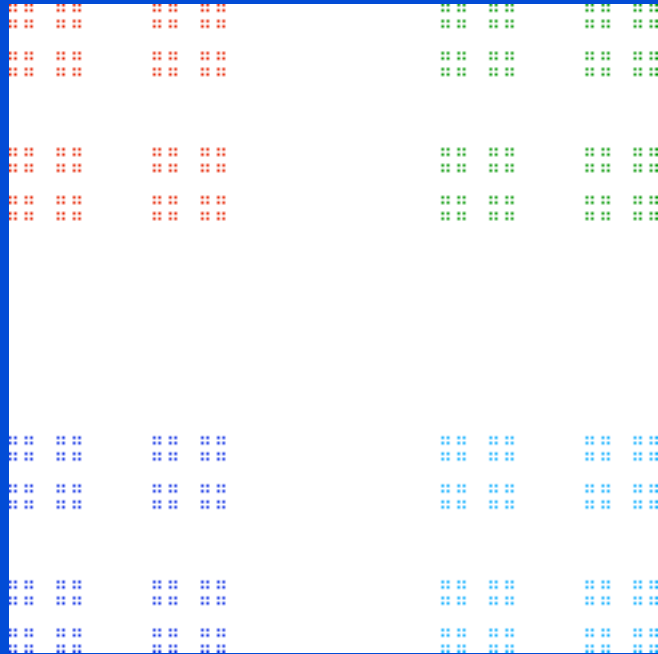
Dimension = log 8 / log 3



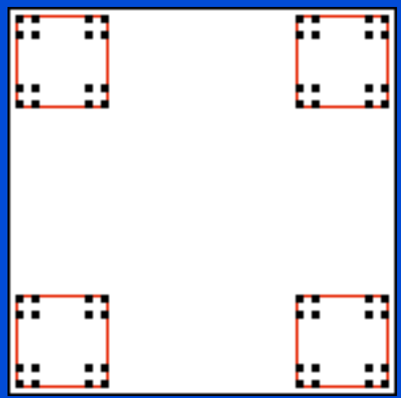


Dimension = log 7 / log 3



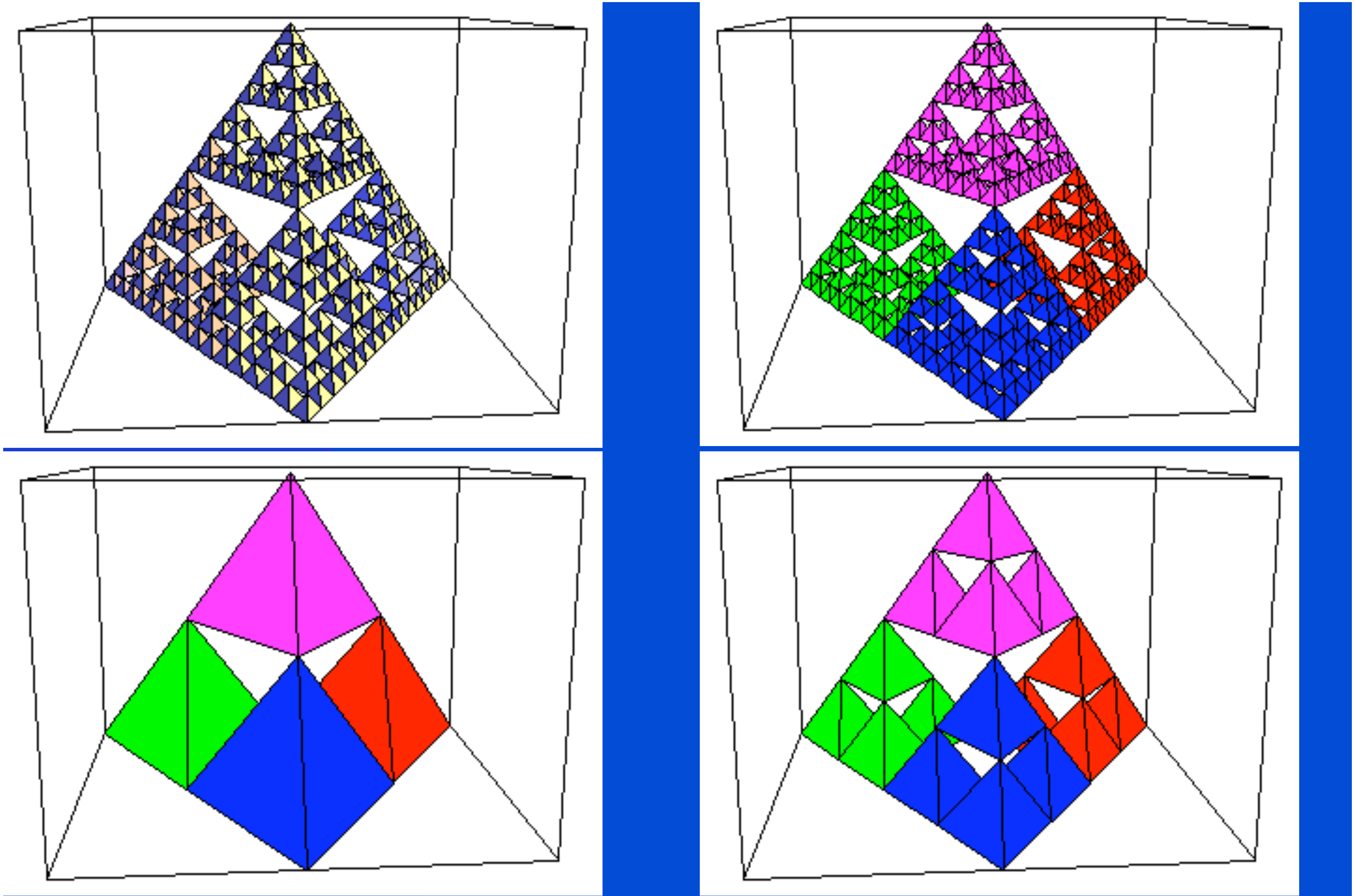


Dimension = log 4 / log 3



Dimension = log 4 / log 4 = 1

But it is nothing like a line!



Dimension = log 4 / log 2 = 2, but it is not a surface!!

What is the Similarity Dimen-sion in each case ?