THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1	**Tutorial Week 6**	2012

(*These preparatory questions should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

1. Let $f(x) = \lfloor x \rfloor$, the largest integer less than or equal to x. Sketch the graph of f. At which points is f continuous? Right-continuous? Left-continuous?

2. Use the Intermediate Value Theorem to show that there is a solution of the equation $x^2 = \sqrt{x+1}$ in the open interval $(1, 2)$.

Questions for the tutorial

3. Determine whether the functions given by the following formulas are continuous at a.

(a) $h(x) = x^2 + \sqrt{7-x}, \quad a = 4$

(b) $k(x) = \dfrac{x^2-1}{x+1}, \quad a = -1$

(c) $F(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x > 0 \\ 1 - x & \text{if } x \le 0 \end{cases} \quad a = 0$

(d) $K(x) = \begin{cases} \dfrac{x^2-1}{x+1} & \text{if } x \ne -1 \\ 6 & \text{if } x = -1 \end{cases} \quad a = -1$

4. (a) Find a constant c so that g is continuous everywhere, where g is defined by:

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4, \\ cx + 20 & \text{if } x \ge 4. \end{cases}$$

(b) Repeat part (a) when

$$g(x) = \begin{cases} -c + \sqrt{x-4} & \text{if } x \ge 4, \\ |x^2 - c^2| & \text{if } x < 4. \end{cases}$$

5. Prove that the equation $\sin x = 2 - x$ has at least one real solution, say α. Use your calculator to find an interval of length 0.01 which contains α.

6. Use the Intermediate Value Theorem to prove that if $f : [0,1] \to [0,1]$ is continuous, there is some c in $[0,1]$ such that $f(c) = c$.

7. A mountaineer leaves home at 7am and walks to the top of the mountain, arriving at 7pm. The following morning, he starts out at 7am from the top of the mountain and takes the same path back, arriving home at 7pm. Use the IVT to show that there is a point on the path that he will cross at exactly the same time of day on both days.

8. (a) Show that if f is continuous on the closed interval $[a,b]$, then there is a function g which is continuous on $\mathbb{R}$, and which satisfies $g(x) = f(x)$ for all x in $[a,b]$.

(b) Give an example to show that the previous assertion is false if $[a,b]$ is replaced by the open interval (a,b).

9. Prove that if f is continuous at a, then $|f|$ is continuous at a. Is the converse true?

Extra Questions

10. Prove the substitution law for limits. That is, assuming that $\lim_{x\to a} g(x) = b$ and that f is continuous at b, prove (using the ϵ, δ definition) that $\lim_{x\to a} f(g(x)) = f(b)$.

11. Suppose that f is continuous on the closed interval $[a, b]$. A theorem mentioned in lectures says that the set of values $\{f(x) \,|\, a \le x \le b\}$ has an upper bound; therefore it has a least upper bound, say α. Assuming this, prove that there is some $c \in [a, b]$ such that $f(c) = \alpha$, and deduce the Extreme Value Theorem. (*Hint*: assume that no such c exists and derive a contradiction by considering the function $g(x) = \dfrac{1}{\alpha - f(x)}$.)

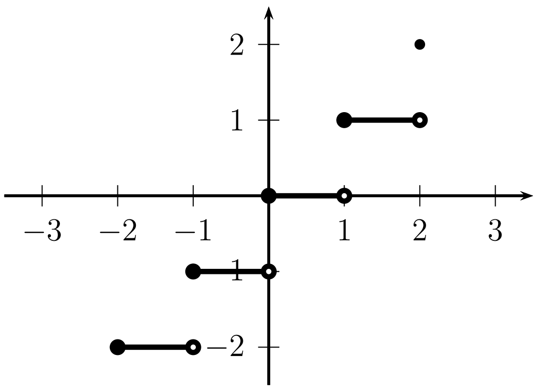
12. Consider the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \dfrac{1}{q} & \text{if } x = \dfrac{p}{q},\ q > 0, \text{ for } p \text{ and } q \text{ relatively prime integers.} \\ & \text{(That is, } \dfrac{p}{q} \text{ is the expression of } x \text{ in lowest terms.)} \end{cases}$$

(In particular, $f(0) = 1$ since 0 in lowest terms is $\dfrac{0}{1}$.) Prove that f is discontinuous at 0, and at every other rational number.

Solution to Question 1

We can write $f(x) = n$ for $x \in [n, n+1), n \in \mathbb{Z}$. The graph is a step function.



Since f is a constant function on each open interval $(n, n+1)$, it is continuous at all points which are not integers. At any integer point n, we have $\lim_{x\to n^+} f(x) = n = f(n)$, so f is also right-continuous at all integer points. Note that f is not left-continuous at integer points n since $\lim_{x\to n^-} f(x) = n - 1 \neq f(n)$.

Solution to Question 2

Let $f(x) = x^2 - \sqrt{x+1}$. The function f is continuous on the interval $[1, 2]$, $f(1) = 1 - \sqrt{2}$ and $f(2) = 4 - \sqrt{3}$. Since $1 - \sqrt{2} < 0 < 4 - \sqrt{3}$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the IVT. Thus there is a solution of $x^2 = \sqrt{x+1}$ in the interval $(1, 2)$.