THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Assignment 2

MATH1902: Linear Algebra (Advanced) — Semester 1, 2017

Web Page: http://sydney.edu.au/science/maths/u/UG/JM/MATH1902/
Lecturer: Holger Dullin

The *rank* of a matrix M is the number of pivots in its (reduced) row echelon form. We write $\operatorname{rank} M$ for the rank of M. When we talk about the dimension of the solution of a system of linear equations we mean the number of parameters in the general solution.

1. Consider the system of equations $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -2 & 1 & 2 & -1 \\ -1 & -2 & 1 & 2 \\ 2 & -1 & -2 & x \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} c \\ d \\ e \\ f \end{bmatrix}$$

(a) Solve the system of equations, carefully treating the various cases depending on the values of the parameters x, c, d, e, f.

Solution: The first and 3rd row of A are negatives of each other, so $R_3 \to R_3 + R_1$. Except for the last entry the 2nd and 4th row are negatives of each other, so $R_4 \to R_4 + R_2$. To eliminate the leading -2 in the 2nd row do $R_2 \to R_2 + 2R_1$. After these three row operations the augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & c \\ 0 & 5 & 0 & -5 & 2c+d \\ 0 & 0 & 0 & 0 & c+e \\ 0 & 0 & 0 & x-1 & d+f \end{array}\right]$$

Now swap R_3 and R_4.

1. When $c + e \neq 0$ the system is inconsistent.

   From now on assume that $e = -c$.

2. If $x - 1 \neq 0$ we can divide by $x - 1$ and find the row-echelon form

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & c \\ 0 & 1 & 0 & -1 & \frac{1}{5}(2c+d) \\ 0 & 0 & 0 & 1 & \frac{d+f}{x-1} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

   The general solution is $x_4 = \frac{d+f}{x-1}$, $x_3 = t \in \mathbb{R}$, $x_2 = \frac{1}{5}(2c+d) + \frac{d+f}{x-1}$, $x_1 = c + t + 2\frac{d+f}{x-1} - 2x_2 = c + t - \frac{2}{5}(2c+d)$.

3. If $x - 1 = 0$ and $d + f \neq 0$ the system is inconsistent.

4. If $x - 1 = 0$ and $d + f = 0$ another row of zeroes appears, and the row-echelon form is

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & c \\ 0 & 1 & 0 & -1 & \frac{1}{5}(2c+d) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is $x_4 = t \in \mathbb{R}$, $x_3 = s \in \mathbb{R}$, $x_2 = \frac{1}{5}(2c+d)+t$, $x_1 = c+2t+s-2x_2 = c+s-\frac{2}{5}(2c+d)$.

(b) Define $M = [A|\mathbf{b}]$, the augmented matrix. For each of the cases found in part (a) state what is rank A, rank M, and (if consistent) the dimension of the solution set. (Note that when computing the rank of M you should ignore the vertical bar.)

Solution: Corresponding to the four cases in part (a) we have:

1. In this case the rank of M is one larger than the rank of A, since there is a row in M of the form $[0000*]$ where the $*$ denotes a non-zero element. When $x = 1$ there may be another such row, but it can be removed by a row-operation, so $\operatorname{rank} A = 2$ and $\operatorname{rank} M = 3$. When $x \neq 1$ we have $\operatorname{rank} A = 3$ and $\operatorname{rank} M = 4$.
2. $\operatorname{rank} A = 3$ and $\operatorname{rank} M = 3$ and $\dim = 1$.
3. $\operatorname{rank} A = 2$ and $\operatorname{rank} M = 3$.
4. $\operatorname{rank} A = 2$ and $\operatorname{rank} M = 2$ and $\dim = 2$.

Remark: The general statement is that the system is consistent when $\operatorname{rank} A = \operatorname{rank} M$, and that $\operatorname{rank} A + \dim$ equals the number of variables.

2. In the unit square the distance of two corners from a diagonal is $1/\sqrt{2}$. This question is about similar distances in the unit cube in $\mathbb{R}^3$ and the unit (hyper) cube in $\mathbb{R}^4$.

In $\mathbb{R}^3$ you may work with the basic unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, or alternatively fixing these basic vectors you may work with column vectors that have the components as entries, e.g. the vector

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \quad \text{may be written as} \quad \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Vectors in $\mathbb{R}^4$ are similarly represented as column vectors with 4 entries.

The unit cube in dimension n has corners with each coordinate either 0 or 1, and hence there are 2^n corners. By the main diagonal of the unit cube we mean a diagonal with the largest possible length, e.g. the one from the origin to the corner which has all coordinates equal to 1.

(a) For the unit cube in $\mathbb{R}^3$ find the distances of corners to the main diagonal. Here you may use our usual formula involving the cross product for the distance between a point and a line.

Solution: The six corners not on the diagonal have coordinates $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$, so they have either one 1 and two 0s or one 0 and two 1s. The components of the corresponding position vectors have the same number of 0s and 1s. The direction vector of the main diagonal is $\mathbf{v} = [111]^T$, and we take the origin O as a point on that line. Thus we need to compute the cross product $\mathbf{v} \times \overrightarrow{OP}$ for each corner P. For all six corners off the diagonal these vectors have the components $0, 1, -1$ in all possible orders, and hence have the same length $\sqrt{2}$. Thus the distance to the main diagonal is $\sqrt{2}/|\mathbf{v}| = \sqrt{2/3}$.

(b) Given a line $\mathbf{r}_0 + t\mathbf{v}, t \in \mathbb{R}$, and a point P, show that the point R on the line that is closest to P is given by

$$\overrightarrow{OR} = \overrightarrow{OQ} + \frac{\overrightarrow{QP} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$$

where Q is an arbitrary point on the line.

Solution: We decompose the vector $\overrightarrow{QP}$ into a component in the direction of $\mathbf{v}$, and a component at a right angle to the direction of $\mathbf{v}$. The component in the direction of $\mathbf{v}$ is given by the usual vector projection formula

$$\frac{\overrightarrow{QP}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\,.$$

To find the position vector of the point R on the line we add this vector to the position vector of Q, and this gives the desired formula.

(c) For the unit cube in $\mathbb{R}^4$ find the distances of corners to the main diagonal. Since the cross product is only defined in $\mathbb{R}^3$ the formula mentioned in part (a) cannot be used here. Instead use the vector projection formula from part (b) to find the distance.

Solution: In the hypercube in $\mathbb{R}^4$ there are sixteen corners, two of which lie on the main diagonal. The remaining corners have either one, two, or three coordinates equal to 1, and all other coordinates equal to 0. To find the point nearest to the main diagonal we need to compute the dot products with the direction vector of the line, which is $\mathbf{v} = [1111]^T$. Denote by l the number of coordinates equal to 1 in a given corner P. Take $Q = O$ the origin as a point on the main diagonal. Then

$$\overrightarrow{OR} = \mathbf{0} + \frac{\overrightarrow{QP}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{l}{4}\mathbf{v}$$

To obtain the distance we compute the length of $\overrightarrow{PR}$. The components of $\overrightarrow{PR}$ take the values $l/4-1$ and $l/4$ only. The value $l/4-1$ is taken on l times, while the values $l/4$ is taken $4-l$ times, so that the length of the vector squared is

$$|\overrightarrow{PR}|^2 = (l/4-1)^2 l + (l/4)^2(4-l) = \frac{l(4-l)}{4}\,.$$

Thus the distance is $\sqrt{3}/2$ for corners with $l = 1, 3$ ccordinates equal to 1, and the distance is 1 for for corners with $l = 2$ coordinates equal to 1.

Remark: It is not hard to generalise to the hypercube in $\mathbb{R}^n$. The distances are $\sqrt{n(1-r)r}$ where $r = l/n$, $l = 1, \ldots, n-1$. In even dimension the largest distance is $\sqrt{n}/2$.