Recall: Liouville function

$$\lambda(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_d^{\alpha_d}) = (-1)^{\alpha_1+\alpha_2+\dots+\alpha_d}.$$

Definition: Möbius function $\mu$ is defined as follows:

$$\mu(n) = \begin{cases} \lambda(n) & \text{if } n \text{ is square-free} \\ 0 & \text{otherwise.} \end{cases}$$

$n \in \mathbb{Z}^+$ is called <u>square-free</u> if for any prime $p$, $p^2 \nmid n$.

Table of $\mu(n)$ for small $n$:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| factorization | 1 | 2 | 3 | $2^2$ | 5 | $2\cdot3$ | 7 | $2^3$ | $3^2$ | $2\cdot5$ |
| $\mu(n)$ | 1 | -1 | -1 | 0 | -1 | 1 | -1 | 0 | 0 | 1 |

[ Not for assessment: $\lambda(n)$, $\mu(n)$ are closely related with distribution of primes.

Known result: $\frac{1}{N}\sum_{n=1}^{N} \lambda(n) \xrightarrow[N\to\infty]{} 0$ is equivalent to Prime Number Theorem: # primes between 1 and N is $\approx \frac{N}{\log N}$.

is also equivalent to: $\frac{1}{N}\sum_{n=1}^{N} \mu(n) \xrightarrow[N\to\infty]{} 0$

Riemann hypothesis is equivalent to:

$$\forall \varepsilon>0 \quad \frac{1}{N^{1/2+\varepsilon}} \sum_{n=1}^{N} \lambda(n) \xrightarrow[N\to\infty]{} 0$$

$$\frac{1}{N^{1/2+\varepsilon}} \sum_{n=1}^{N} \mu(n) \xrightarrow[N\to\infty]{} 0 .$$

One can show the following: Let $S$ be the set of all square-free numbers. Then

$$\frac{\# S \cap \{1,2,\dots,N\}}{N} \longrightarrow \frac{6}{\pi^2}$$

It is (hard!) Ex for advanced students

## §10.3 Number of divisors and sum of divisors.

Denote by $\tau(n)$ the number of all positive integer divisors of $n$.

In other words, $\tau(n) = \sum_{d \mid n} 1$.

Examples: $\tau(2) = 2$

$\tau(p) = 2$ for prime $p$

$\tau(p^k) = k+1$ (divisors are $1, p, p^2, \dots, p^k$)

$\tau(1100) = ?$

Denote by $\sigma(n)$ the sum of all positive integer divisors of $n$.

I.e. $\sigma(n) = \sum_{d|n} d$.

Examples: $\sigma(2) = 3$

$\sigma(p) = 1 + p$

$\sigma(p^k) = 1 + p + p^2 + \ldots + p^k = \frac{p^{k+1} - 1}{p - 1}$

$\sigma(1100) = ?$

Proposition: Let $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ be multiplicative function. Then $F : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ given by

$$F(n) := \sum_{d|n} f(d)$$

is also multiplicative function.

Proof: is based on the following lemma:

Lemma: The map given by

$$\cancel{d \mapsto (d_1, d_2)}$$

$$(d_1, d_2) \longmapsto d_1 \cdot d_2$$

from

$$\{d_1, d_2 \in \mathbb{Z}^+ : d_1 | m,\ d_2 | m\}$$

to

$$\{d \in \mathbb{Z}^+ : d \,|\, n \cdot m\}$$

is a bijection (one-to-one correspondence).
Here $n, m \in \mathbb{Z}^+$ with $\gcd(n,m)=1$.

Proof of Proposition based on Lemma:

$$F(n\cdot m) = \sum_{d|nm} f(d) \xlongequal[\text{Lemma}]{} \sum_{\substack{d_1|m \\ d_2|n}} f(d_1 d_2)$$

$$= [\text{multiplicativity}] = \sum_{\substack{d_1|m \\ d_2|n}} f(d_1) f(d_2)$$

$$= \sum_{d_1|m} \sum_{d_2|n} f(d_1) f(d_2) = \left(\sum_{d_1|m} f(d_1)\right)\left(\sum_{d_2|n} f(d_2)\right)$$

$$= F(m)\,F(n).$$

Proof of lemma.

Write $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_d^{\alpha_d}$, $n = q_1^{\beta_1} \cdots q_r^{\beta_r}$ where p's and q's are all distinct primes.

Surjection: (Any $d|mn$ has a preimage $(d_1, d_2)$)

$$d \mid p_1^{\alpha_1} \cdots p_d^{\alpha_d} \cdot q_1^{\beta_1} \cdots q_r^{\beta_r}$$

Split $d$ into two parts: $d_1$ contains all powers of $p$ dividing $d$ and $d_2$ contains all powers of $q$.

Then $(d_1, d_2) \longrightarrow d_1 \cdot d_2 = d$.

Injection: Assume we have $d_1, d_1' | m$, $d_2, d_2' | n$ with $d_1 d_2 = d_1' d_2'$.

$d_1 | d_1' d_2' = d_1 d_2$, $\gcd(d_1, d_2') = 1$

$\Longrightarrow d_1 | d_1'$

By symmetry $d_1' | d_1 \Longrightarrow d_1 = d_1'$. Then also $d_2 = d_2'$. $\boxtimes$

Corollary: $\tau(n)$ and $\sigma(n)$ are multiplicative

Indeed $\tau(n) = \sum_{d|n} 1$ and $f(d) = 1$ is mult.

$\sigma(n) = \sum_{d|n} d$ and $f(d) = d$ is mult.

Example: $\tau(1100) = \tau(2^2 \cdot 5^2 \cdot 11) = \tau(2^2) \cdot \tau(5^2) \cdot \tau(11)$
$= 3 \cdot 3 \cdot 2 = 18$

$\sigma(1100) = \sigma(2^2) \sigma(5^2) \sigma(11) = 7 \cdot 31 \cdot 12$
$= 2604$.