The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 6

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

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Topics covered

In lectures last week:

- ☐ Sequences $a_1, a_2, \dots$ Squeeze law, ratio test for sequences.
- ☐ Asymptotic equivalence of sequences ($a_n \sim b_n$).
- ☐ Series $a_1 + a_2 + \cdots$. The geometric series, harmonic series, and p-series.
- ☐ Comparison test, ratio test and asymptotic comparison test for series.

Objectives

After completing this tutorial sheet you will be able to:

- ☐ Use the ratio test, squeeze law, and limit laws to compute limits of sequences.
- ☐ Use comparison tests to determine the convergence/divergence of a series.
- ☐ Use Riemann sums to decide convergence/divergence of series.
- ☐ Compute the value of some series by computing the limit of partial sums.

Preparation questions to do *before* class

1. Calculate the limit of the following sequences, or show that they diverge.

(a) $a_n = \dfrac{n^4 + 3n^3 \cos n - 2}{3n^4 - n}$

(b) $a_n = \dfrac{n!(2n)!}{(3n)!}$ (ratio test!)

2. Determine if the following series converge or diverge. (Don't forget the ratio test!)

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n n 3^{-n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5\cos(3n) + 2}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 5^n}{(2n)!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 3n - 2}{n^3 + 1}$

Questions to attempt in class

3. Calculate the limit of the following sequences, or show that they diverge.

(a) $a_n = \dfrac{3 + \cos n^2}{\sqrt{n}}$

(b) $a_n = \sqrt[n]{n}$

(c) $a_n = \dfrac{n^2}{3n^2 + 2n - 1}$

(d) $a_n = \dbinom{2n}{n}$

4. Decide if the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$ (g) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^n n!}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{2-\sin\sqrt{n}}{n^3}$ (h) $\displaystyle\sum_{n=1}^{\infty} \sin(n^2)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-n}}{\sqrt{n}}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-2n+5}{n^3+4}$ (i) $\displaystyle\sum_{n=1}^{\infty} \frac{\cosh n}{e^{2n}+n^2}$

5. Let $r \in \mathbb{R}$, and let $s_n(r) = 1 - r^2 + r^4 - \cdots + (-1)^{n-1} r^{2n-2}$.

(a) Find a closed formula for $s_n(r)$, and deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} + (-1)^{n+1} \int_0^1 \frac{r^{2n}}{1+r^2}\, dr.$$

(b) Hence prove *Leibnitz' Formula* $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$.

(c) Adapt this proof to show that $\ln 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{5} - \cdots$.

Discussion question

6. The Prime Number Theorem implies that the nth prime satisfies $p_n \sim n \ln n$. Given this information, discuss the convergence/divergence of the series

$$\sum_{\text{primes } p} \frac{1}{p}.$$

Questions for extra practice

7. Decide if the following sequences converge. If they converge find the limit.

(a) $a_n = \dfrac{1+2+\cdots+n}{n^2}$ (c) $a_n = \left(1+\dfrac{1}{n}\right)^n$

(b) $a_n = e^{-n}\cosh n$ (d) $a_n = \dfrac{\ln n}{n^\epsilon}$, $(\epsilon > 0)$

8. Decide if the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cosh n}{n^4+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$ (d) $\displaystyle\sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n+1}\right]$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+3}}$

9. For which values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{x^{2n+1}}{2n+1}$ converge/diverge?

Challenging problems

10. In this question you derive *Stirling's Asymptotic Formula* for $n!$

(a) Show that $\displaystyle\int_1^n \frac{\{x\}}{x}\,dx = \sum_{k=1}^{n-1}\int_k^{k+1}\frac{x-k}{x}\,dx$, and conclude that

$$\ln n! = n\ln n - n + 1 + \int_1^n \frac{\{x\}}{x}\,dx\,,$$

where $\{x\} \in [0,1)$ is the fractional part of $x \geq 0$.

(b) Integrate by parts (see the relevant question of Tutorial 4) to show that

$$\ln n! = n\ln n - n + 1 + \frac{1}{2}\ln n - \frac{1}{2}\int_1^n \frac{\{x\}-\{x\}^2}{x^2}\,dx.$$

(c) Deduce that $\displaystyle\lim_{n\to\infty}\frac{n!}{\sqrt{n}n^n e^{-n}} = e^C$ for some constant C.

(d) Use the Wallis formula (Tutorial 5) to evaluate C, and deduce that

$$n! \sim \sqrt{2\pi n}\,n^n e^{-n}.$$

11. The Fibonacci sequence is $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(a) Prove that $F_{2n} = F_n^2 + 2F_nF_{n-1}$ and $F_{2n-1} = F_n^2 + F_{n-1}^2$ for $n \geq 1$, and hence deduce that $F_{2n}F_{n-1} - F_{2n-1}F_n = (-1)^n F_n$ for $n \geq 1$.

(b) Show that $\displaystyle\sum_{k=1}^{n}\frac{1}{F_{2^k}} = 2 - \frac{F_{2^n-1}}{F_{2^n}}$ for all $n \geq 1$.

(c) Hence deduce that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{F_{2^k}} = \frac{5-\sqrt{5}}{2}$.

12. (a) The *Riemann-Lebesgue Lemma* says that if $f(x)$ is well behaved, then

$$\lim_{n\to\infty}\int_a^b f(x)\sin(nx)\,dx = \lim_{n\to\infty}\int_a^b f(x)\cos(nx)\,dx = 0.$$

Use integration by parts to prove the Riemann-Lebesgue Lemma under the assumption that $f(x)$ has continuous derivative.

(b) Use the Riemann-Lebesgue Lemma to show that

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sin x} - \frac{1}{x}\right)\sin(2nx)\,dx = 0.$$

Hence deduce that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{\sin x}\,dx.$$

(c) Show that $\displaystyle\frac{\sin(2nx)}{\sin x} = 2\sum_{k=1}^{n}\cos[(2k-1)x]$ for $n \geq 2$, $\sin x \neq 0$, and hence

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$