8025A Semester 1 2010

The University of Sydney

Faculties of Arts, Economics, Education, Engineering and Science

MATH1901/1906

Differential Calculus (Advanced)

June 2010

Lecturer: C M Cosgrove

Time Allowed: One and a half hours

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

Marker's use only	

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination; there are 20 questions; the questions are of equal value; all questions may be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination; there are 4 questions; the questions are of equal value; all questions may be attempted; working must be shown.

Approved non-programmable non-graphics calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

MARKS

1. (a) In the complex z-plane, $z = x + iy$, sketch the set satisfying the inequality,
$$|z-2| + |z+2| \leq 5.$$
[Note: you may assume that the boundary of the region is an ellipse with equation of the form, $x^2/a^2 + y^2/b^2 = 1$, where a and b can be found without doing a lengthy algebraic calculation by just looking for the x- and y-intercepts of the ellipse.] 4

(b) Factorise the polynomial,
$$P(z) = z^4 - 2z^3 - z^2 + 2z + 10,$$
into linear and/or quadratic factors with real coefficients, given that $2+i$ is one of the roots of the polynomial. 4

(c) Let S denote the closed sector $0 \leq \arg z \leq 2\pi/3$ in the complex z-plane, including the vertex at $z = 0$. Show that the function, $g : S \to \mathbb{C}$, $z \mapsto z^3$, is surjective but not injective. 4

2. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto \tan^{-1}(x^2 + 3y^2)$, and let P be the point $(2, 1)$ in the xy-plane.

(*i*) Calculate the directional derivative $D_{\mathbf{u}}f$ at P in the direction of the vector $\mathbf{u} = 4\mathbf{i} - \mathbf{j}$. 2

(*ii*) Find the equation of the tangent plane to the graph of $z = f(x, y)$ at the point on the graph vertically above P. Express your answer in the form $z = ax + by + c$. 2

(b) Let g denote the function $g : \mathbb{R} \to \mathbb{R}$ given by
$$g(x) = \begin{cases} \dfrac{\sin\sqrt{x}}{\sqrt{x}}, & x > 0 \\ 1, & x = 0 \\ \dfrac{\sinh\sqrt{-x}}{\sqrt{-x}}, & x < 0. \end{cases}$$

(*i*) Use standard series expansions for $\sin x$ and $\sinh x$ to obtain separate one-sided Taylor series for $g(x)$ about $x = 0$ on the right and left, and deduce that $g(x)$ actually has an ordinary (two-sided) Taylor series about $x = 0$. (It will be sufficient for you to give the first four nonzero terms plus dots.) 5

(*ii*) Use the result of the previous part to evaluate $g'(0)$, $g''(0)$ and $g'''(0)$. 3

MARKS

3. (a) Find the following limits, showing the steps of your working clearly, or show that the limit does not exist. (You may use any valid method. Allow $+\infty$ and $-\infty$ as values that a limit can take.)

(*i*) $\displaystyle\lim_{x\to 2} \frac{x^3 - x^2 - 8x + 12}{x^3 - 12x + 16}$. (*ii*) $\displaystyle\lim_{x\to\infty} \frac{\ln(x^2+4)}{\sinh^{-1} x}$. 2,2

(*iii*) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2-y^2}$. (*iv*) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$. 2,2

(b) Use the Mean Value Theorem to prove the following statement (epsilon-delta proof not needed):

Suppose $f(x)$ is continuous on $[a, b)$ and differentiable on (a, b) and that $f'(x)$ tends to a finite limit L as $x \to a^+$. Then $f(x)$ is right-differentiable at $x = a$ and $f'(a) = L$. 4

4. (a) A tractrix is a curve in the first and fourth quadrants of the xy-plane with equation $y = \pm f(x)$, where

$$f(x) = a\cosh^{-1}(a/x) - \sqrt{a^2 - x^2}\,, \qquad 0 < x \le a.$$

It has a horizontal cusp on the positive x-axis at $(a, 0)$ and approaches both ends of the y-axis asymptotically. Here, we are interested only in the part with $y \ge 0$.

(*i*) Show that $f'(x) = -\sqrt{a^2 - x^2}/x$. (You may assume that the derivative of $\cosh^{-1} x$ is $1/\sqrt{x^2-1}$.) 3

(*ii*) Let P denote the point $(c, f(c))$ on the curve. Calculate the equation of the tangent line to the curve at P. 2

(*iii*) Find the point Q where the tangent line in part (*ii*) crosses the y-axis and show that the line segment PQ has constant length (independent of c). 3

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function having three or more continuous derivatives. Define $G : \mathbb{R}^2 \to \mathbb{R}$ by

$$G(x, y) = \begin{cases} \dfrac{f(y) - f(x)}{y - x}, & y \ne x, \\ f'(x), & y = x. \end{cases}$$

Use any method to calculate the partial derivative $\partial G/\partial y$ at (x, x). [Hint: the recommended method is to replace $f(y)$ by its Taylor polynomial $T_3(y)$ about $y = x$. Methods based on l'Hôpital's rule or the chain rule for partial derivatives will also work.] 4

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.