Properties The Menger sponge is the union of 20 copies of itself, each scaled by 1/3. So we say that the Menger sponge is *self-similar*. Remember that we can only draw approximations to the actual Menger sponge, as in Figure 4.17.

The nth approximation to the Menger Sponge consists of 20^n cubes each of which is obtained by scaling the original cube by $(1/3)^n$.

Suppose the initial cube has side length equal to one. It follows that for the nth approximation to the Menger Sponge:

$$\begin{aligned}\text{surface area } A_n &= \text{no. of cubes} \times \text{surface area of initial cube}\\ &\quad \times \text{area scaling factor} = 20^n \times 6 \times \left(\frac{1}{3}\right)^{2n} = 6\left(\frac{20}{9}\right)^n\\ \text{volume } V_n &= \text{no. of cubes} \times \text{volume of initial cube}\\ &\quad \times \text{volume scaling factor} = 20^n \times 1 \times \left(\frac{1}{3}\right)^{3n} = \left(\frac{20}{27}\right)^n.\end{aligned}$$

Notice that the surface area A_n approaches infinity as n approaches infinity. On the other hand, the volume V_n approaches zero. We write

$$A_n \to \infty,\ V_n \to 0 \text{ as } n \to \infty.$$

It is possible to give a precise definition of volume and surface area. Then it turns out that the surface area of the Menger Sponge is infinite but the volume is zero. Moreover, the surface of the Menger Sponge is in fact the same as the Menger Sponge.

Once again, none of this follows automatically, but requires careful definition and proof.

Cantor Set

The Cantor set is the simplest fractal set. For this reason it is useful to study it in more detail.

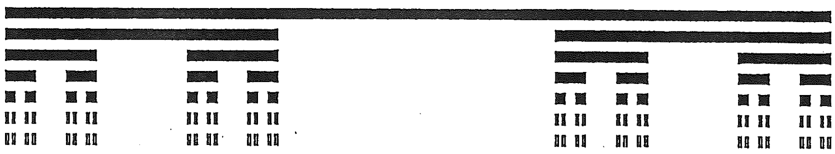

Figure 4.18: Seven approximations to the Cantor Set

Construction Begin with a closed line segment of length one, and replace it by 2 line segments each of length 1/3 as in Figure 4.15.[9] (The middle open segment "removed" from the initial segment has length 1/3.)

Replace each of these 2 new segments by another 2 segments each 1/3 the length of the segment being replaced (i.e. remove the middle open third of each segment). This gives $4 = 2^2$ line segments.

[9]We have drawn a very "fat" line for visual purposes. Of course, a line really has not "thickness".

Next replace each of these new segments by another 2 segments each 1/3 the length of the segment being replaced (i.e. remove the middle open third of each segment). This gives $8 = 2^3$ line segments.

Etc.

Let $C_0, C_1, C_2, C_3, \ldots$ denote the sets in Figure 4.15. That is:

$$C_0 = [0,1]$$

$$C_1 = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right] \tag{4.8}$$

$$C_2 = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{3}{9}\right] \cup \left[\frac{6}{9},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right] \tag{4.9}$$

$$C_3 = \left[0,\frac{1}{27}\right] \cup \left[\frac{2}{27},\frac{3}{27}\right] \cup \left[\frac{6}{27},\frac{7}{27}\right] \cup \left[\frac{8}{27},\frac{9}{27}\right] \cup \left[\frac{18}{27},\frac{19}{27}\right] \cup \left[\frac{20}{27},\frac{21}{27}\right] \cup \left[\frac{24}{27},\frac{25}{27}\right] \cup \left[\frac{26}{27},1\right]$$

etc.

The *Cantor Set* C is the set of points which are in *every* C_n. We write

$$C = \bigcap_{n \geq 0} C_n. \tag{4.10}$$

Notice that C_n consists of 2^n intervals each of length $(1/3)^n$ and that

$$C_0 \supset C_1 \supset C_2 \supset C_3 \supset \cdots \supset C_n \supset \cdots \supset C.$$

The Cantor set is the union of 2 copies of itself, each obtained by scaling by 1/3. For this reason the cantor set is *self-similar*.

Describing the Points in the Cantor Set

Endpoints of Intervals Every endpoint of every interval used in the construction, is itself in the Cantor Set. For example, $0 \in C$ since it is clear that $0 \in C_n$ for every n.

Similarly, $1/3 \in C$ since $1/3 \in C_n$ for every n. Likewise for 8/27 and so on.

One way to see this is to notice that at each stage in the construction we are throwing away the middle open third of each interval. So we always keep any endpoints.

It might seem that the only points in the Cantor set are the endpoints of such intervals. *This is wrong!*

Addresses of Points To better understand what points are in the Cantor set, let us go back and look again at the construction.

Every point x in the Cantor set C is either in the left interval $[0,1/3]$ or the right interval $[2/3,1]$.

For example, suppose $x \in C$ is in the right interval $[2/3,1]$. Then either x is in the left interval $[6/9,7/9]$ or the right interval $[8/9,1]$.

Suppose $x \in C$ is in the left interval $[6/9,7/9]$. Then either x is in the left interval $[18/27,19/27]$ or the right interval $[20/27,21/27]$.

Suppose $x \in C$ is in the left interval $[18/27,19/27]$. Etc.

If we write L for left and R for right, then such a point x will be described by an infinite sequence of the form

$$RLL\ldots.$$

Every point in the Cantor set C can be described by an infinite sequence of L's and R's in this way. Moreover, *every* such infinite sequence describes a point in C.

For example, the infinite sequence

$$LRRLRLLRLRRLLLRLRLRLLLLRRRLLRLLRLLLRLRRRRRL\ldots$$

describes a point x which is in the interval $[0,1]$, also in $[0,1/3]$ (go Left), also in $[2/9,3/9]$ (go Right), also in $[8/27,9/27]$ (go Right), also in $[24/81,25/81]$ (go Left), also in $[74/243,75/243]$ (go Right), also in $[222/729,223/729]$ (go Left), etc.

With this notation, left endpoints of intervals obtained in the construction of the C_n correspond to an infinite sequence ending in L's. Right endpoints of intervals obtained in the construction of the C_n correspond to an infinite sequence ending in R's.

For example, the point $8/9$ corresponds to $RRLLLLL\ldots$, since we went right twice and then forever stay left. In a similar way, $2/27$ corresponds to $LLRLLLLL\ldots$, since we went left twice and then right and then forever stay left. *Mark* $8/9$ *and* $2/27$ *in Figure 4.18.*

Tree Representation A convenient way to represent infinite sequences consisting of the terms L and R is by means of an infinite tree which branches twice at each node. See Figure 4.19, where the first few branches are shown.

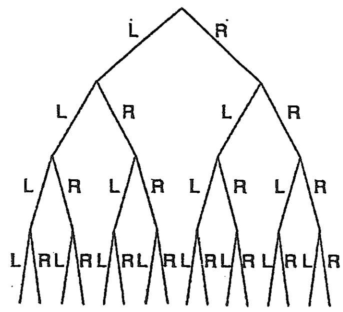

Figure 4.19: First few branches of a two branching tree

Each *infinite* branch corresponds to a point in the Cantor set and each point in the Cantor set corresponds to exactly one *infinite* branch.

Base 3 Representation We know that every point in the interval $[0,1]$ has a decimal expansion. We saw on page 81 that it also has a binary expansion.

Because of the following Theorem, we will be interested here in the base 3, or *ternary*, expansion of numbers $x \in [0,1]$. For this we need just the numerals 0, 1 and 2. We then have the following result.

Theorem 4.3.1. *A number x in the interval $[0, 1]$ is in the Cantor set C if and only if it has a ternary expansion consisting just of 0s and 2s. Moreover, such a number x will have exactly one ternary expansion which consists only of 0s and 2s.*

Proof. **Step A.** Suppose for example the ternary expansion of some $x \in [0, 1]$ is

$$x = .0220202\ldots.$$

We will show that $x \in C$.

If we consider the intervals $[0, 1/3]$, $[1/3, 2/3]$ and $[2/3, 1]$, because the first digit in the ternary expansion of x is 0, it follows that x is in the first of these three intervals, i.e. $[0, 1/3]$. *Draw a diagram.* In particular, $x \in C_1$, see (4.8).

Subdividing $[0, 1/3]$, consider the intervals $[0, 1/9]$, $[1/9, 2/9]$ and $[2/9, 3/9]$. It follows from the ternary expansion for x, because the second digit is 2, that x is in the third of these, i.e. $[2/9, 3/9]$. In particular, $x \in C_2$, see (4.9).

Subdividing $[2/9, 3/9]$ we consider the intervals $[6/27, 7/27]$, $[7/27, 8/27]$ and $[8/27, 9/27]$. It follows from the ternary expansion for x, because the third digit is 2, that x is in the third of these, i.e. $[8/27, 9/27]$. In particular, $x \in C_3$.

Etc.

In this way we see that $x \in C_n$ for every n, and so $x \in C$. See (4.10)

The same argument shows for *any* $x \in [0, 1]$, that if x has a ternary expansion using just the numerals 0 and 2 but not 1, then $x \in C$.

Step B. On the other hand, suppose $x \in C$. *We will now show that x has some ternary expansion which uses the numerals* 0 *and* 2 *but not* 1.

Since $x \in C$ it follows in particular that $x \in C_1$. This implies x is in either the interval $[0, 1/3]$ or the interval $[2/3, 1]$. (It might also be in the interval $[1/3, 2/3]$, but then it must be one of the two endpoints of $[1/3, 2/3]$, and so also is in either $[0, 1/3]$ or $[2/3, 1]$.) So the first digit in the ternary expansion of x can be taken to be 0 or 2.

We subdivide the corresponding interval [0, 1/3] or [2/3, 1] into three subintervals. Because $x \in C$, x is in either the first or third of these subintervals. This means the second digit of its ternary expansion can be taken to be 0 or 2.

We subdivide the relevant subinterval into three subsubintervals. Because $x \in C$, x is in either the first or third of these subsubintervals. This means the third digit of its ternary expansion can also be taken to be 0 or 2.

Etc.

In this way we see that if $x \in C$ then x has a ternary expansion with only 0s and 2s. I

Because we have just one choice, namely left or right interval, at each stage, there is moreover *exactly one* ternary expansion for $x \in C$ which consists just of the numerals 0 or 2. □

How Large is the Cantor Set?

Length Since the approximation C_n consists of 2^n intervals of length $(1/3)^n$, its total length is $(2/3)^n$.

We will not give a precise definition of the length of complicated sets like the Cantor set. But it is possible to do this. All we need here is the property that if $A \subset B$ then the length of A is $\leq$ the length of B.

Since $C \subset C_n$ (*why?*) it follows that the length of C is $\leq (2/3)^n$ for *every* n. So in the sense of "length", C is small.

Cardinality The Cantor set is certainly infinite. This is clear because we have an infinite number of choices to make using the L, R tree representation.

But is the Cantor set countable or uncountable? (The proof of the following Theorem is incomplete, in that it uses a theorem we have not actually proved. See the discussion within the proof itself.)

Theorem 4.3.2. *The Cantor set has cardinality c.*

"Proof". Let C denote the Cantor set. We will prove:

1. There is a one-to-one correspondence between C and some subset of $[0, 1]$.
2. There is a one-to-one correspondence between $[0, 1]$ and some subset of C.

It seems reasonable that from these two facts there should be a one-to-one correspondence between (all of) C and (all of) $[0, 1]$. This is indeed the case, but to show it one needs the Cantor-Schroeder-Bernstein Theorem on page 128, which we have not proved. For this reason I have written *"Proof"*.

From Theorem 3.5.4 on page 123, the cardinality of $[0, 1]$ is c. So once we know there is a one-to-one correspondence between C and $[0, 1]$ it follows that C also has cardinality c.

The first fact (1.) is easy to prove, since C is a subset of $[0, 1]$. The one-to-one correspondence just sends each $x \in C$ to the same $x \in [0, 1]$.

For (2.) we use the fact from page 81 that every $x \in [0, 1]$ has at least one binary expansions of the form $.a_1a_2a_3 \ldots a_n \ldots$, where each a_1 is either 0 or 1. For each $x \in [0, 1]$ choose one such binary expansion, replace each 0 by L and each 1 by R, and so get an infinite sequence of L's and R's which we call $s(x)$. (Notice that if $x \neq y$ then $s(x) \neq s(y)$, *why?*)

Using addresses of points in the Cantor set as on page 155 we can identify each $s(x)$ with a member of the Cantor set C, and in this way we get a one-to-one correspondence between $[0, 1]$ and a subset of C. *Why us it only a subset of C?*

This completes the "Proof" because of the comments above. □

Questions

1 In Figure 4.3 we begin with the line segment from $x = 0$ to $x = 1$ on the x-axis. We then sketch the first four approximations to the Koch curve (each "begins" at the point $x = 0$ on the x-axis and "ends" at the point $x = 1$).

What is the length of the first approximation to the Koch curve? How about the second, third, fourth, nth approximations?