8027A SEMESTER 2 2006

THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH1903/1907

INTEGRAL CALCULUS AND MODELLING (ADVANCED)

November 2006

LECTURER: C M Cosgrove

TIME ALLOWED: **One and a half hours**

Name: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 25% of the total examination;
there are 15 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 75% of the total examination;
there are 5 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.
There is a table of integrals after the last question in this booklet.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

MARKS

1. (a) Evaluate the definite integral, **4**

$$\int_0^{\pi/4} x \cos 3x \, dx.$$

(b) Evaluate the indefinite integral, **4**

$$\int \frac{d\theta}{\sin\theta\cos^2\theta}.$$

(c) Calculate the upper Riemann sum for the function $f(x) = x^2$ on the interval $[0, 2]$ using a partition of the interval into n equal subintervals. If desired, you may quote the formula, $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$, without proof. **4**

2. (a) The area under the graph of $y = \ln x$ from $x = 1$ to $x = 3$ is rotated about the x-axis to form a solid of revolution. Write the volume of the solid as a definite integral in two ways:

(i) as an integral with respect to x by the disc method; **2**

(ii) as an integral with respect to y by the shell method. **2**

Do not attempt to evaluate either of these integrals.

(b) An astroid is a closed curve in the xy-plane given by the equation,

$$x^{2/3} + y^{2/3} = a^{2/3}, \qquad a > 0.$$

It is symmetric about both axes. Calculate the arc length of the astroid by the following steps:

(i) In the first quadrant, express y in terms of x; **2**

(ii) Calculate dy/dx and then $1 + (dy/dx)^2$. (As a check on your work, the latter result should be a relatively simple.) **2**

(iii) Integrate $\sqrt{1 + (dy/dx)^2}$ from $x = 0$ to $x = a$. **2**

(iv) Deduce that the arc length of the complete astroid is $6a$. **2**

MARKS

3. (a) Derive an addition theorem for $\sinh(A+B)$. (The recommended method is to transform the corresponding trigonometric addition theorem using the identities $\sin(iA) = i\sinh A$ and $\cos(iA) = \cosh A$.) **3**

(b) Solve the differential equation,

$$\frac{dy}{dx} = \frac{\sqrt{1+y^2}}{\sqrt{1+x^2}},$$

subject to the initial condition $y = K$ when $x = 0$. Express your answer in the form $y = f(x)$, where $f(x)$ is an algebraic function. (If you use integral number 17 in the Table of Standard Integrals, it is recommended that you use the version involving the inverse hyperbolic function, not the logarithm.) **6**

(c) Find the particular solution of the linear differential equation,

$$\frac{dw}{dx} + (\cot 2x)w = \sqrt{\cot x}\,, \qquad 0 < x < \pi/2,$$

such that $w = 0$ when $x = \pi/4$. **3**

4. (a) Use the minimum number of terms of suitable standard power series to evaluate the l'Hôpital-type limit:

$$\lim_{x\to 0} \frac{\cosh(x^2) + \cos(x^2) - 2}{\sin^8 x}.$$

(Do not attempt to use l'Hôpital's rule itself.) **4**

(b) Find the general solution of the second-order differential equation,

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 13x = 0.$$

Express your answer in terms of real-valued functions of t. **4**

(c) Find the particular solution of the differential equation,

$$u'' - 4u' + 3u = 0,$$

for the function $u(x)$, the prime denoting differentiation with respect to x, subject to the constraints, $u(1) = 5e$ and $u'(1) = 7e$. **4**

MARKS

5. It is believed that some species may become extinct if the size of the population falls below a certain number. In order to model the population size, $P(t)$, of such a species, the following modified logistic model may be used:

$$\frac{dP}{dt} = k(P - S)\left(1 - \frac{P}{L}\right),$$

where S, L and k are positive constants, and $S < L$.

(a) What are the equilibrium solutions to the differential equation? **1**

(b) Sketch a graph of $\frac{dP}{dt}$ against P, for $P \geq 0$. **2**

(c) Let P_0 be the initial population size. Using your graph in part (b) (or otherwise), describe what eventually happens to the population size in each of the following cases: **2**

(*i*) $0 < P_0 < S$, (*ii*) $S < P_0 < L$, (*iii*) $P_0 > L$.

(d) Find the particular solution to the differential equation, with $P(0) = P_0$. **5**

(e) Suppose $0 < P_0 < S$. Using your solution in part (d) show that there is a positive value of t for which $P = 0$. (That is, a value of t for which the species becomes extinct.) **2**

Table of Standard Integrals

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

3. $\displaystyle\int e^x\,dx = e^x + C$

4. $\displaystyle\int \sin x\,dx = -\cos x + C$

5. $\displaystyle\int \cos x\,dx = \sin x + C$

6. $\displaystyle\int \tan x\,dx = -\ln\left|\cos x\right| + C$

7. $\displaystyle\int \cot x\,dx = \ln\left|\sin x\right| + C$

8. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

9. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

10. $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

11. $\displaystyle\int \sec x\,dx = \ln\left|\sec x + \tan x\right| + C$

12. $\displaystyle\int \operatorname{cosec} x\,dx = \ln\left|\operatorname{cosec} x - \cot x\right| + C$

13. $\displaystyle\int \sinh x\,dx = \cosh x + C$

14. $\displaystyle\int \cosh x\,dx = \sinh x + C$

15. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (|x| < a)$

16. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C'$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C' \quad (x > a)$

Linearity: $\displaystyle\int \big(\lambda f(x) + \mu g(x)\big)\,dx = \lambda\int f(x)\,dx + \mu\int g(x)\,dx$

Integration by substitution: $\displaystyle\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$

Integration by parts: $\displaystyle\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.