(A)

Solutions

1. (a) $$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -17 & -4 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -18 & -4 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2/9 & 1/18 & -1/18 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/9 & -2/9 & 2/9 \\ 0 & 1 & 0 & -2/9 & 17/18 & 1/18 \\ 0 & 0 & 1 & 2/9 & 1/18 & -1/18 \end{array}\right]$$

so $$M^{-1} = \frac{1}{18}\begin{bmatrix} 2 & -4 & 4 \\ -4 & 17 & 1 \\ 4 & 1 & -1 \end{bmatrix}$$

(b) $$M\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ so } \begin{bmatrix} r \\ s \\ t \end{bmatrix} = M^{-1}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 2 & -4 & 4 \\ -4 & 17 & 1 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 4 \\ 10 \\ 8 \end{bmatrix} = \begin{bmatrix} 2/9 \\ 5/9 \\ 4/9 \end{bmatrix}$$

so $r = 2/9$, $s = 5/9$, $t = 4/9$.

(B)

1 (c) (i) $\underset{\sim}{v} = \underset{\sim}{i} - 4\underset{\sim}{k}$, $\underset{\sim}{w} = \underset{\sim}{j} + \underset{\sim}{k}$,

$$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & -4 \\ 0 & 1 & 1 \end{vmatrix} = 4\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$$

$$\overrightarrow{PQ} = (2-r)\underset{\sim}{i} + (s-1)\underset{\sim}{j} + (-1+s+4r)\underset{\sim}{k}$$

so $\overrightarrow{PQ} = t(\underset{\sim}{v} \times \underset{\sim}{w})$ means

$$\left\{\begin{array}{l} 2-r = 4t \\ s-1 = -t \\ -1+s+4r = t \end{array}\right\} \text{ so } \left\{\begin{array}{l} r + 4t = 2 \\ s + t = 1 \\ 4r + s - t = 1 \end{array}\right\}$$

which is (b).

(ii) When $r = 2/9$, $P = (11/9, 1, 10/9)$

and when $s = 5/9$, $Q = (3, 5/9, 14/9)$

and these are the closest points on the respective lines.

The distance between the lines is

$$|\overrightarrow{PQ}| = |t||\underset{\sim}{v} \times \underset{\sim}{w}| = \frac{4}{9}\sqrt{16+1+1}$$

$$= \frac{4}{9}\sqrt{18} = \frac{4\sqrt{2}}{3}.$$

(c)

2(a) (i) $\underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{i} \cdot \underset{\sim}{k} = 0$ but $\underset{\sim}{j} \neq \underset{\sim}{k}$,

so take $\underset{\sim}{u} = \underset{\sim}{i}$, $\underset{\sim}{v} = \underset{\sim}{j}$, $\underset{\sim}{w} = \underset{\sim}{k}$.

(ii) If $\underset{\sim}{u} \times \underset{\sim}{v} = \underset{\sim}{u} \times \underset{\sim}{w}$ then $\underset{\sim}{u} \times \underset{\sim}{v} - \underset{\sim}{u} \times \underset{\sim}{w} = \underset{\sim}{0}$,

so $\underset{\sim}{u} \times (\underset{\sim}{v} - \underset{\sim}{w}) = \underset{\sim}{0}$, so $\underset{\sim}{u}$ and $\underset{\sim}{v} - \underset{\sim}{w}$ are parallel.

(iii) Suppose $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\sim}{w}$ and $\underset{\sim}{u} \times \underset{\sim}{v} = \underset{\sim}{u} \times \underset{\sim}{w}$.

From (ii), $\underset{\sim}{u} = \lambda(\underset{\sim}{v} - \underset{\sim}{w})$ for some scalar $\lambda \neq 0$

(since $\underset{\sim}{u} \neq \underset{\sim}{0}$), so $\lambda(\underset{\sim}{v} - \underset{\sim}{w}) \cdot \underset{\sim}{v} = \lambda(\underset{\sim}{v} - \underset{\sim}{w}) \cdot \underset{\sim}{w}$.

Cancelling gives $(\underset{\sim}{v} - \underset{\sim}{w}) \cdot \underset{\sim}{v} = (\underset{\sim}{v} - \underset{\sim}{w}) \cdot \underset{\sim}{w}$, so

$(\underset{\sim}{v} - \underset{\sim}{w}) \cdot (\underset{\sim}{v} - \underset{\sim}{w}) = (\underset{\sim}{v} - \underset{\sim}{w}) \cdot \underset{\sim}{v} - (\underset{\sim}{v} - \underset{\sim}{w}) \cdot \underset{\sim}{w} = 0$,

i.e. $|\underset{\sim}{v} - \underset{\sim}{w}|^2 = 0$, so $|\underset{\sim}{v} - \underset{\sim}{w}| = 0$, so $\underset{\sim}{v} - \underset{\sim}{w} = \underset{\sim}{0}$.

Hence $\underset{\sim}{v} = \underset{\sim}{w}$.

(b) $\det(M - \lambda I) = \begin{vmatrix} 7/10 - \lambda & 3/10 \\ 2/5 & 3/5 - \lambda \end{vmatrix}$

$= (7/10 - \lambda)(3/5 - \lambda) - 6/50 = \lambda^2 - \frac{13}{10}\lambda + \frac{21}{50} - \frac{6}{50}$

$= \lambda^2 - \frac{13}{10}\lambda - \frac{3}{10} = (\lambda - 3/10)(\lambda - 1)$

with roots $\lambda = 1, 3/10$, which are the eigenvalues.

(D)

2(b) $M - I = \begin{bmatrix} -3/10 & 3/10 \\ 2/5 & -2/5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 1$

$M - 3/10\, I = \begin{bmatrix} 4/10 & 3/10 \\ 2/5 & 3/10 \end{bmatrix} \sim \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}$

So $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ is an eigenvector for $\lambda = 3/10$.

Hence $M = PDP^{-1}$ where

$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3/10 \end{bmatrix}$, $P^{-1} = \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$

so $\lim_{n\to\infty} M^n = \lim_{n\to\infty} (PD^nP^{-1})$

$= \lim_{n\to\infty} \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (3/10)^n \end{bmatrix} \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{7}\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{bmatrix}$.

(E)

3 (a) (i) $\vec{PQ} + \vec{RS} = \vec{PS} + \vec{SQ} + \vec{RS} = \vec{PS} + \vec{RS} + \vec{SQ}$
$= \vec{PS} + \vec{RQ} = \vec{RQ} + \vec{PS}$.

(ii) $\vec{SR} = \lambda \vec{PQ}$ and $\vec{PS} = \mu \vec{QR}$ for some λ, μ,

so (i) becomes $\vec{PQ} - \lambda \vec{PQ} = \mu \vec{QR} + \vec{RQ}$, so

$(1-\lambda)\vec{PQ} = (1-\mu)\vec{RQ}$. But $\vec{PQ}$ and $\vec{RQ}$ are not

parallel since P, Q, R do not lie on the same line,

forcing $1-\lambda = 1-\mu = 0$, so $\lambda = \mu = 1$.

In particular $\vec{SR} = \vec{PQ}$.

(b) (i) By symmetry, $\vec{AB} \parallel \vec{EC} \parallel \vec{EF}$ and $\vec{AE} \parallel \vec{BD} \parallel \vec{BF}$,

so ABFE is a parallelogram (by (a)(ii)), and so

is a rhombus since $|\vec{AB}| = |\vec{AE}|$.

(ii) $\lambda \underline{v} = \lambda \vec{DC} = \lambda(\vec{DA} + \vec{AC}) = \lambda \vec{DA} + \lambda \vec{AC} = \vec{CB} + \vec{ED}$
by symmetry

(iii) $\lambda \underline{v} = \vec{CB} + \vec{ED} = \vec{EB} + \vec{CD}$ by (a)(i)

$= \frac{1}{\lambda}\underline{v} - \underline{v} = \left(\frac{1}{\lambda} - 1\right)\underline{v}$, so $\lambda^2 \underline{v} = (1-\lambda)\underline{v}$,

so $\lambda^2 = 1-\lambda$, since $\underline{v} \neq \underline{0}$.

(f)

3 (b) (iv) $\lambda^2+\lambda-1=0$ so $\lambda = \frac{-1+\sqrt{5}}{2}$ $(\lambda>0)$

so $\frac{|\vec{BF}|}{|\vec{BD}|} = \frac{|\vec{AE}|}{|\vec{BE}|} = \frac{|\underline{u}|}{|\vec{EB}|} = \lambda = \frac{-1+\sqrt{5}}{2}$

so F divides BD in the ratio $r:s$ where

$r = -1+\sqrt{5} = \sqrt{5}-1$ and $s = 2-r = 3-\sqrt{5}$.

4. (a) (i) $10M^2 - 11M + I = 0$, so $11M - 10M^2 = I$,

so $M(11I - 10M) = I$, so M^{-1} exists and equals

$11I - 10M$.

(ii) Let the characteristic polynomial of M be

$\lambda^2 + a\lambda + b$, so, by the Cayley-Hamilton Thm,

$M^2 + aM + bI = 0 = M^2 - \frac{11}{10}M + \frac{1}{10}I$,

so $(a+\frac{11}{10})M = (b-\frac{1}{10})I$.

Since M is not a scalar multiple of I,

$a + {}^{11}/_{10} = b - {}^{1}/_{10} = 0$, so $a = -{}^{11}/_{10}$, $b = {}^{1}/_{10}$

so $\lambda^2 - {}^{11}/_{10}\,\lambda + {}^{1}/_{10} = (\lambda-1)(\lambda - {}^{1}/_{10}) = 0$

so $\lambda = 1$ or $\lambda = {}^{1}/_{10}$.

4(b) (i) $\det(\lambda I - M_1) = \det[\lambda + a_0] = \lambda + a_0$.

$$\det(\lambda I - M_2) = \begin{vmatrix} \lambda & -1 \\ a_0 & \lambda + a_1 \end{vmatrix} = \lambda(\lambda + a_1) + a_0 = \lambda^2 + a_1\lambda + a_0$$

(ii) $\det(\lambda I - M_n) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$

Proof: By (i), this holds for $n=1$, starting an induction.

Suppose $n \geq 2$. Then

$$\det(\lambda I - M_n) = \begin{vmatrix} \lambda & -1 & & & \\ 0 & \lambda & -1 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \lambda & -1 \\ a_0 & a_1 & \cdots & a_{n-1} & \lambda + a_n \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda & -1 & & & \\ 0 & \lambda & -1 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \lambda & -1 \\ a_1 & a_2 & \cdots & a_{n-1} & \lambda + a_n \end{vmatrix} + (-1)^n a_0 \begin{vmatrix} -1 & & \\ & -1 & \\ & & \ddots & \\ & & & -1 \end{vmatrix}$$

$$= \lambda(\lambda^{n-1} + a_{n-1}\lambda^{n-2} + \dots + a_2\lambda + a_1) + (-1)^n a_0 (-1)^{n-1}$$
(by an ind. hyp.)

$$= \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2\lambda^2 + a_1\lambda + a_0,$$

and the result follows by induction.

(iii) Let $p(\lambda) = c_0 + c_1\lambda + \dots + c_n\lambda^n$ $(c_n \neq 0)$ and put $a_0 = \frac{c_0}{c_n}, a_1 = \frac{c_1}{c_n}, \dots, a_{n-1} = \frac{c_{n-1}}{c_n}$. Then M_n has an eigenvalue λ, so $a_0 + a_1\lambda + \dots + a_{n-1}\lambda^{n-1} + \lambda^n = \det(\lambda I - M_n) = 0$, i.e. $\frac{p(\lambda)}{c_n} = 0$, i.e. $p(\lambda) = 0$, so $p(\lambda)$ has a root, proving the FTA.