CHAPTER **FOUR**

Trigonometry

One of the major reasons why trigonometry is important is that the graphs of the sine and cosine functions are waves. Waves appear everywhere in the natural world, for example as water waves, as sound waves, or as the various electromagnetic waves responsible for radio, heat, light, ultraviolet radiation, X-rays and gamma rays. In quantum mechanics, a wave is associated with every particle. Trigonometry began, however, in classical times as the study of the relationships between angles and lengths in geometrical figures. Its name, from the Greek words *trigos* meaning 'land' and *metros* meaning 'measurement', reminds us that trigonometry is fundamental to surveying and navigation. This introductory chapter establishes the geometric context of the trigonometric functions and their graphs, developing them from the geometry of triangles and circles.

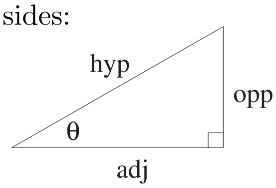
Study Notes: Trigonometric problems involving right triangles (Sections 4A and 4B) and the sine, cosine and area rules (Sections 4H, 4I and 4J) should be familiar. On the other hand, the extension of the trigonometric functions to angles of any magnitude and the graphs of these functions (Sections 4C, 4D and 4E), and the work on trigonometric identities and trigonometric equations (Sections 4F and 4G) will mostly be new. Machine drawing of a variety of trigonometric graphs could be helpful in establishing familiarity with the graphs.

4 A Trigonometry with Right Triangles

This section and the next will review the earlier definitions, based on triangles, of the six trigonometric functions for acute angles, and apply them to problems involving right triangles.

The Definition of the Trigonometric Functions: Suppose that θ is any acute angle (this means that $0 < \theta < 90°$ — angles of $0°$ or $90°$ are not acute angles). Construct a right triangle with θ as one of the other two angles, and label the sides:

hyp — the *hypotenuse*, the side opposite the right angle,
opp — the side *opposite* the angle θ,
adj — the third side, *adjacent* to θ but not the hypotenuse.



1 Definition: $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$

$\operatorname{cosec}\theta = \dfrac{\text{hyp}}{\text{opp}}$ $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$ $\cot\theta = \dfrac{\text{adj}}{\text{opp}}$

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Any two triangles with angles of 90° and θ are similar, because they have the same three angles (this is the AA similarity test), and so their sides are in the same ratio. Hence the values of the six trigonometric functions at θ, defined above, are the same, whatever the size of the triangle. The full names of the six functions are:

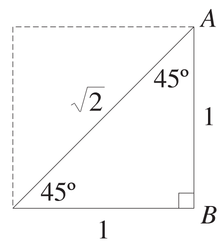
sine, cosine, tangent, cosecant, secant, cotangent.

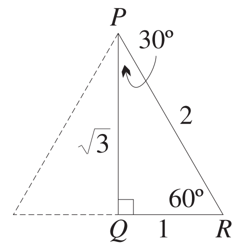
Special Angles: The values of the six trigonometric functions can be calculated exactly for the three acute angles 30°, 45° and 60°. The right triangle $\triangle ABC$ below, with two 45° angles, is formed by taking half of a square with side length 1. The right triangle $\triangle PQR$, whose other angles are 60° and 30°, is half of an equilateral triangle with side length 2. The third sides can then be calculated using Pythagoras' theorem, giving the exact values in the table below.

2

A TABLE OF EXACT VALUES

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\operatorname{cosec}\theta$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$





Trigonometric Functions of Other Angles: The values of the trigonometric functions of other angles are rather complicated, but the calculator can be used to find approximations to them. Make sure you know how to use your particular machine to enter angles in degrees and minutes, and how to change angles given in decimals of a degree to angles given to the nearest minute. Here are two examples to try on your own calculator:

$\sin 53^\circ 47' \doteqdot 0{\cdot}8068$ and $\sin\theta = \frac{5}{8}$, so $\theta \doteqdot 38^\circ 41'$.

Finding an Unknown Side of a Triangle: The calculator does not have the secant, cosecant and cotangent functions, so it is best to use only sine, cosine and tangent in these problems.

3

TO FIND AN UNKNOWN SIDE OF A RIGHT TRIANGLE:

1. Start by writing $\dfrac{\text{unknown side}}{\text{known side}} = \ldots$ (place the unknown top left).
2. Complete the RHS with sin, cos or tan, or the reciprocal of one of these.

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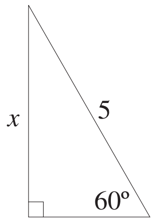
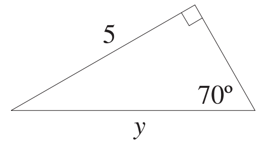
WORKED EXERCISE: Find the sides marked with pronumerals in these triangles, in exact form if possible, or else correct to five significant figures.

(a)



(b)



SOLUTION:

(a)
$$\frac{x}{5} = \sin 60^\circ$$
$$\boxed{\times 5} \quad x = 5\sin 60^\circ$$
$$= \frac{5\sqrt{3}}{2}$$

(b)
$$\frac{y}{5} = \frac{1}{\sin 70^\circ}$$
$$\boxed{\times 5} \quad y = \frac{5}{\sin 70^\circ}$$
$$\doteqdot 5{\cdot}3209$$

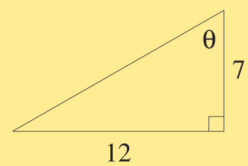
Finding an Unknown Angle: As before, use only sine, cosine and tangent.

4 FINDING AN UNKNOWN ANGLE: To find an angle when given two sides of a right triangle, work out which one of $\cos\theta$, $\sin\theta$ or $\tan\theta$ is known.

WORKED EXERCISE: Find θ in the given triangle.

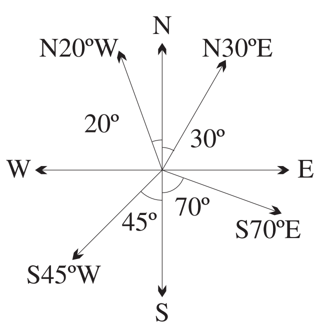
SOLUTION: The given sides are the opposite and the adjacent sides, so $\tan\theta$ is known.

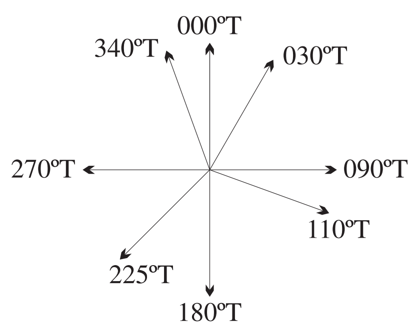
$$\tan\theta = \frac{12}{7}$$
$$\theta \doteqdot 59^\circ 45'.$$



Compass Bearings and True Bearings: *Compass bearings* specify direction in terms of the four cardinal directions north, south, east and west. Any other direction is given by indicating the deviation from north or south towards the east or west. The diagram on the left below gives four examples: N30°E, N20°W, S70°E and S45°W (which can also be written simply as SW).

True bearings are measured clockwise from north. The diagram on the right below gives the same four directions expressed as true bearings: 030°T, 340°T, 110°T and 225°T. It is usual for three digits to be used even for numbers of degrees under 100.





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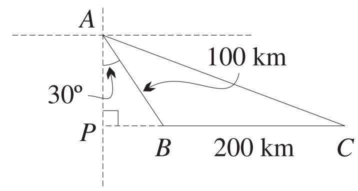
WORKED EXERCISE: [Compass bearings and true bearings]
A plane flying at 400 km per hour flies from A to B in a direction S30°E for 15 minutes, then turns sharply to fly due east for 30 minutes to C.

(a) Find how far south and east of A the point B is.

(b) Find the true bearing of C from A, to the nearest degree.



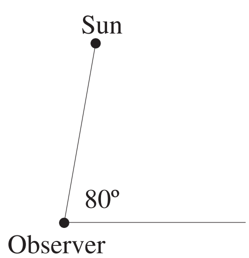
SOLUTION:

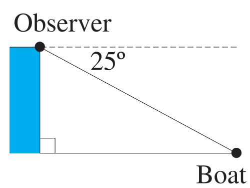
(a) The distances AB and BC are 100 km and 200 km respectively.
From the diagram on the right,

$$PB = 100\cos 60^\circ$$
$$= 50\,\text{km},$$
and $$AP = 100\sin 60^\circ$$
$$= 50\sqrt{3}\,\text{km}.$$

(b) $$\tan\angle PAC = \frac{PC}{AP}$$
$$= \frac{50+200}{50\sqrt{3}}$$
$$= \frac{5}{\sqrt{3}}$$
$$\angle PAC \doteqdot 71^\circ,$$
so the bearing of C from A is about 109°T.

Angles of Elevation and Depression: Angles of elevation and depression are always measured from the horizontal. They are always acute angles.





The *angle of elevation* of the sun in the diagram above is 80°, because the angle at the observer between the sun and the horizontal is 80°.

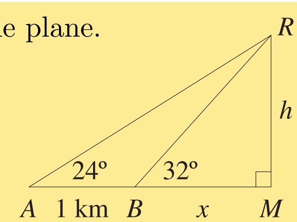
For an observer on top of the cliff, the *angle of depression* of the boat is 25°, because the angle at the observer between boat and horizontal is 25°.

WORKED EXERCISE: [An example with two triangles] A walker walks on a flat plane directly towards a distant high rocky outcrop R. At point A the angle of elevation of the outcrop is 24°, and a kilometre closer at B the angle of elevation is 32°.

(a) Find the horizontal distance from B to the outcrop, to the nearest metre.

(b) Find the height of the outcrop above the plane, to the nearest metre.

SOLUTION: Let M be the point directly below R and level with the plane.



Let $x = BM$, and $h = RM$.
From $\triangle BMR$, $h = x\tan 32^\circ$,
and from $\triangle AMR$, $h = (x+1)\tan 24^\circ$.

(a) Equating these expressions for h,

$$x\tan 32^\circ = (x+1)\tan 24^\circ$$
$$x(\tan 32^\circ - \tan 24^\circ) = \tan 24^\circ$$
$$x = \frac{\tan 24^\circ}{\tan 32^\circ - \tan 24^\circ}$$
$$BM \doteqdot 2{\cdot}478\,\text{km}.$$

(b) Substituting,

$$h = \frac{\tan 24^\circ\,\tan 32^\circ}{\tan 32^\circ - \tan 24^\circ}$$
$$RM \doteqdot 1{\cdot}549\,\text{km}.$$

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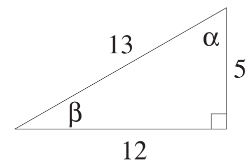
Exercise 4A

1. Use your calculator to find, correct to four decimal places:
(a) $\sin 24°$
(b) $\cos 61°$
(c) $\tan 35°$
(d) $\cos 32°24'$
(e) $\tan 78°40'$
(f) $\cos 16°51'$
(g) $\operatorname{cosec} 20°$
(h) $\sec 48°$
(i) $\cot 56°$
(j) $\cot 28°30'$
(k) $\sec 67°43'$
(l) $\operatorname{cosec} 81°13'$

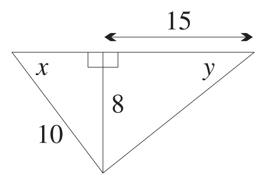
2. Use your calculator to find the acute angle θ correct to the nearest degree if:
(a) $\tan\theta = 4$
(b) $\sin\theta = 0{\cdot}456$
(c) $\cos\theta = \frac{7}{9}$
(d) $\sec\theta = 3$
(e) $\operatorname{cosec}\theta = 5{\cdot}963$
(f) $\cot\theta = 2\frac{4}{7}$

3. Use your calculator to find the acute angle α correct to the nearest minute if:
(a) $\cos\alpha = \frac{3}{4}$
(b) $\tan\alpha = 0{\cdot}3$
(c) $\sin\alpha = 0{\cdot}7251$
(d) $\cot\alpha = 0{\cdot}23$
(e) $\operatorname{cosec}\alpha = \frac{20}{13}$
(f) $\sec\alpha = 3{\cdot}967$

4. From the diagram opposite, write down the value of:
(a) $\sin\alpha$
(b) $\tan\beta$
(c) $\sec\beta$
(d) $\cot\alpha$
(e) $\operatorname{cosec}\alpha$
(f) $\sec\alpha$



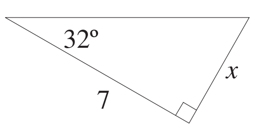
5. (a) Use Pythagoras' theorem to find the third side in each of the right triangles in the diagram opposite.
(b) Write down the value of:
(i) $\cos y$
(ii) $\sin x$
(iii) $\cot x$
(iv) $\operatorname{cosec} y$
(v) $\sec x$
(vi) $\cot y$

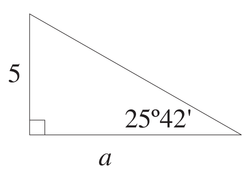


6. Draw the two special triangles containing the acute angles 30°, 60° and 45°. Hence write down the exact value of:
(a) $\sin 60°$ (b) $\tan 30°$ (c) $\cos 45°$ (d) $\sec 60°$ (e) $\operatorname{cosec} 45°$ (f) $\cot 30°$

7. Find, without using a calculator, the value of:
(a) $\sin 45° \cos 45° + \sin 30°$
(b) $\sin 60° \cos 30° - \cos 60° \sin 30°$
(c) $1 + \tan^2 60°$
(d) $\operatorname{cosec}^2 30° - \cot^2 30°$

8. Find, correct to one decimal place, the lengths of the sides marked with pronumerals:

(a)


(b)


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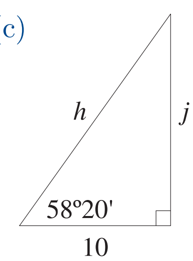
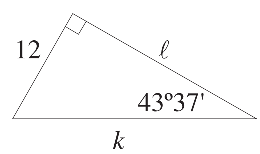
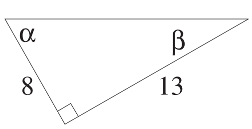


(d)


9. Find the sizes of the angles marked with pronumerals, correct to the nearest minute:

(a)


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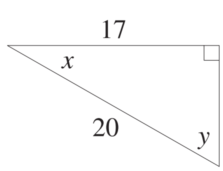

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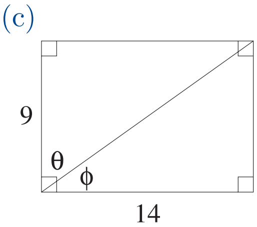

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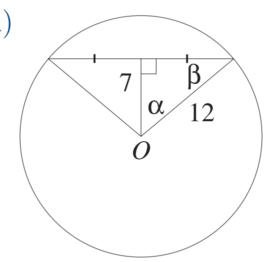

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DEVELOPMENT

10. If $A = 17°25'$ and $B = 31°49'$, use your calculator to find, correct to two decimal places:

(a) $\cos 3A$ (c) $\tan(B - A)$ (e) $\operatorname{cosec}(2A + B)$

(b) $3\cos A$ (d) $\tan B - \tan A$ (f) $\operatorname{cosec} 2A + \operatorname{cosec} B$

11. It is given that α is an acute angle and that $\tan\alpha = \frac{\sqrt{5}}{2}$.

(a) Draw a right-angled triangle showing this information.

(b) Use Pythagoras' theorem to find the length of the unknown side.

(c) Hence: (i) write down the exact values of $\sin\alpha$ and $\cos\alpha$,
(ii) show that $\sin^2\alpha + \cos^2\alpha = 1$.

12. Suppose that β is an acute angle and $\sec\beta = \frac{\sqrt{11}}{3}$.

(a) Find the exact value of: (i) $\operatorname{cosec}\beta$, (ii) $\cot\beta$. (b) Show that $\operatorname{cosec}^2\beta - \cot^2\beta = 1$.

13. Without using a calculator, show that:

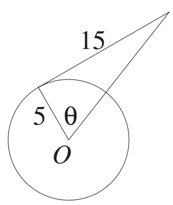
(a) $1 + \tan^2 45° = \sec^2 45°$ (d) $\dfrac{2\tan 30°}{1 - \tan^2 30°} = \tan 60°$

(b) $2\sin 30°\cos 30° = \sin 60°$ (e) $\sin 60°\tan 45°\tan 30° = \cos 30°\cot 45°\cot 60°$

(c) $\cos^2 60° - \cos^2 30° = -\frac{1}{2}$ (f) $\sec^2 30°\cot^2 60° - 8\cos^2 45°\sin^2 60° = -\frac{23}{9}$

14. Find each pronumeral, correct to four significant figures, or to the nearest minute:

(a)



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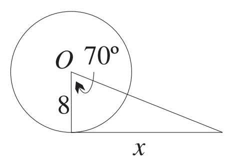

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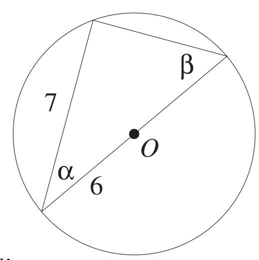
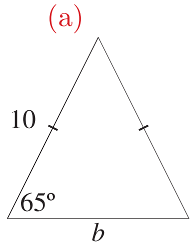


15. Find the value of each pronumeral, correct to three decimal places:

(a)



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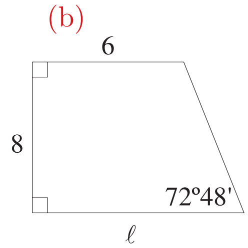

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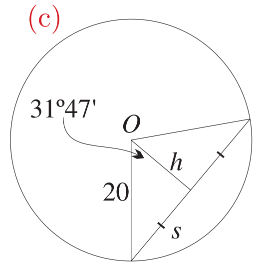

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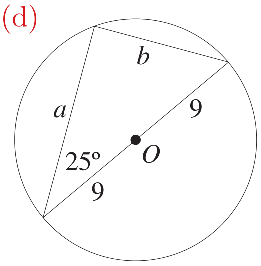


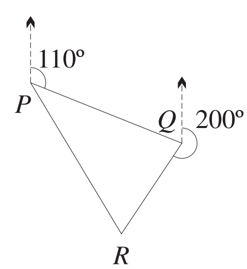
16. A ladder of length 5 metres is placed on level ground against a vertical wall. If the foot of the ladder is 1·5 metres from the base of the wall, find, to the nearest degree, the angle at which the ladder is inclined to the ground.

17. Find, to the nearest degree, the angle of depression of a boat 200 metres out to sea, from the top of a vertical cliff of height 40 metres.

18. A ship leaves port P and travels 150 nautical miles to port Q on a bearing of 110°. It then travels 120 nautical miles to port R on a bearing of 200°.

(a) Explain why $\angle PQR = 90°$.

(b) Find, to the nearest degree, the bearing of port R from port P.



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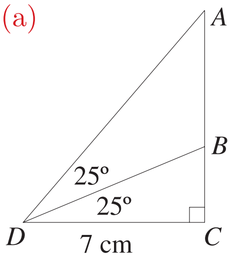
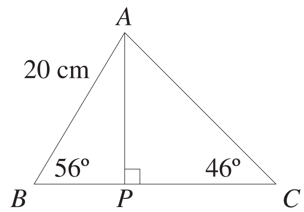
19. (a)



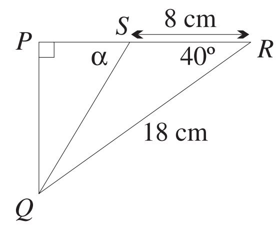
Show that $AC = 7\tan 50°$ and $BC = 7\tan 25°$, and hence find the length AB correct to 1 mm.

(b)



Show that $AP = 20\sin 56°$, and hence find the length of PC, giving your answer correct to 1 cm.

(c)



Show that $PR = 18\cos 40°$, find an expression for PQ, and hence find the angle α to the nearest minute.

20. Answer to four significant figures, or to the nearest minute:

(a) A triangle has sides of 7 cm, 7 cm and 5 cm. What are the sizes of its angles?

(b) An isosceles triangle has base angles of 76°. What is the ratio of base to side length?

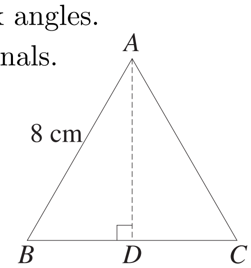
(c) A rectangle has dimensions 7 cm × 12 cm. At what acute angle do the diagonals meet?

(d) The diagonals of a rectangle meet at 35°. Find the ratio of the length and breadth.

(e) The diagonals of a rhombus are 16 cm and 10 cm. Find the vertex angles.

(f) One vertex angle of a rhombus is 25°. Find the ratio of the diagonals.

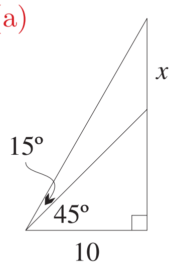
21. In the figure drawn on the right, $\triangle ABC$ is an equilateral triangle with side length 8 cm.



(a) Show that the perpendicular height AD is $4\sqrt{3}$ cm.

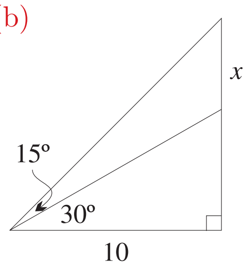
(b) Hence find the exact area of the triangle.

22. (a)



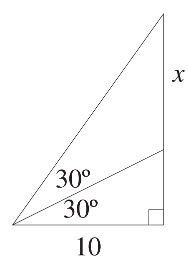
Show that $x = 10(\sqrt{3} - 1)$.

(b)



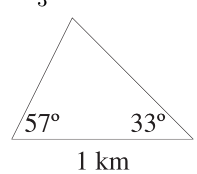
Show that $x = \frac{10}{3}(3 - \sqrt{3})$.

(c)



Show that $x = \frac{20}{3}\sqrt{3}$.

23. From the ends of a straight horizontal road 1 km long, a balloon directly above the road is observed to have angles of elevation of 57° and 33° respectively. Find, correct to the nearest metre, the height of the balloon above the road.

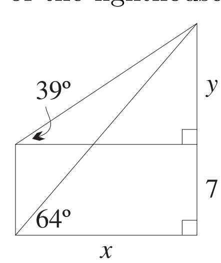


24. From a ship sailing due north, a lighthouse is observed to be on a bearing of 42°. Later, when the ship is 2 nautical miles from the lighthouse, the bearing of the lighthouse from the ship is 148°. Find, correct to three significant figures, the distance of the lighthouse from the initial point of observation.

25. (a) Use two right triangles in the diagram to write down two equations involving x and y.

(b) By solving the equations simultaneously, show that

$$x = \frac{7}{\tan 64° - \tan 39°}.$$



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EXTENSION

26. [The regular pentagon and the exact value of $\sin 18°$] The regular pentagon $ABCDE$ has sides of length 1 unit. The diagonals AD, BD and AC have been drawn, and the diagonals BD and AC meet at P.

(a) Find the size of each interior angle of the pentagon.

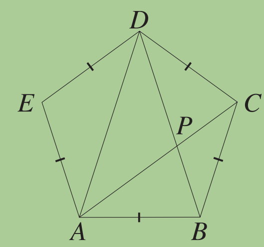
(b) Show that $\angle DAB = 72°$ and $\angle DAP = \angle BAP = 36°$.

(c) Show that the triangles DAB and ABP are similar.

(d) Let $BP = x$, and show that $AB = AP = DP = 1$ and $DA = \dfrac{1}{x}$.

(e) Show that $AD = \frac{1}{2}\left(\sqrt{5}+1\right)$.

(f) Hence show that $\sin 18° = \cos 72° = \frac{1}{4}\left(\sqrt{5}-1\right)$.



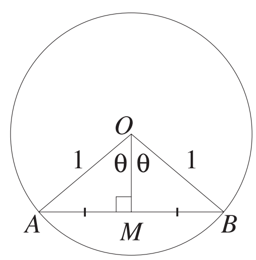
4 B Theoretical Exercises on Right Triangles

Many problems in trigonometry involve diagrams in which the sides and angles are given in terms of pronumerals. The two worked examples given here have been chosen because they give the classical definition of the sine function (first example), and explain the reason why the words 'secant' and 'tangent' are used (second example). They also show how close the connection is between the trigonometric functions and circles.

An Earlier Definition of the Sine Function: An earlier interpretation of $\sin\theta$ defined it as the length of the 'semichord' subtending an angle θ at the centre of a circle of radius 1. Suppose that a chord AB of a circle of radius 1 subtends an angle 2θ at the centre O. We need to prove that $\sin\theta = \frac{1}{2}AB$.

PROOF: Let M be the midpoint of AB,
then by circle geometry, $OM \perp AB$ and $\angle AOM = \theta$,
so in the right triangle $\triangle AMO$,

$$\frac{AM}{AO} = \sin\theta$$
$$AM = \sin\theta$$
$$\sin\theta = \tfrac{1}{2}AB.$$



The Origin of the Words Secant and Tangent: The word 'tangent' comes from the Latin *tangens* meaning 'touching', and a tangent to a circle is a line touching it at one point. The word 'secant' comes from the Latin *secans* meaning 'cutting', and a secant to a circle is a line cutting it at two points. The following construction shows how an angle θ at the centre of a circle of radius 1 is associated with an interval on a tangent of length $\tan\theta$, and an interval on a secant of length $\sec\theta$.

Suppose that P is a point outside a circle of radius 1. Let one of the tangents from P touch the circle at T, and let PT subtend an angle θ at the centre O. Construct the secant through P and O, and join the radius OT. Then

$$PT = \tan\theta \qquad \text{and} \qquad PO = \sec\theta.$$

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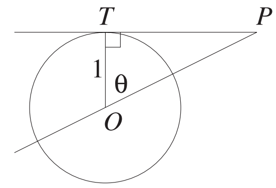
PROOF: By the radius and tangent theorem, the radius OT and the tangent PT are perpendicular, so in the right triangle $\triangle PTO$,

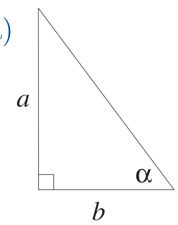
$$\frac{PT}{OT} = \tan\theta \quad \text{and} \quad \frac{PO}{TO} = \sec\theta.$$

Thus $PT = \tan\theta$ and $PO = \sec\theta$.



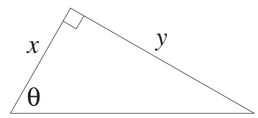
Exercise 4B

1. (a)



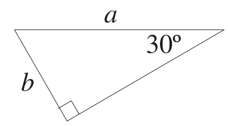
Show that $a = b\tan\alpha$.

(b)



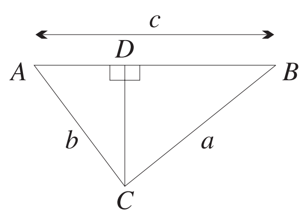
Show that $\sin^2\theta = \dfrac{y^2}{x^2+y^2}$.

(c)



Show that $a = 2b$.

2. (a)

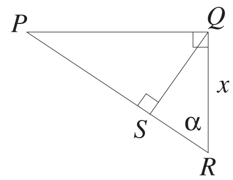


(i) Show that $AD = b\cos A$ and find a similar expression for BD.

(ii) Hence show that

$$c = a\cos B + b\cos A.$$

(b)



(i) Show that $PR = x\sec\alpha$.

(ii) Show that $SR = x\cos\alpha$.

(iii) Hence show that

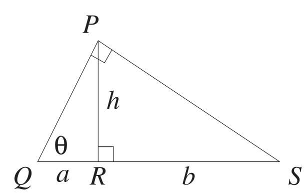
$$PS = x(\sec\alpha - \cos\alpha).$$

3. In the diagram opposite, $\triangle PQS$ is a right triangle, and PR is the altitude to the hypotenuse QS.

(a) Explain why $\angle RPS = \theta$.

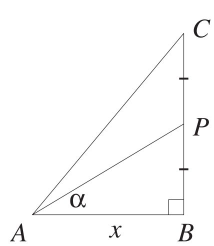
(b) Find two expressions for $\tan\theta$.

(c) Hence show that $ab = h^2$.



DEVELOPMENT

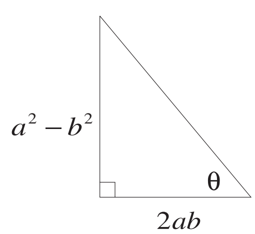
4. (a)



In the diagram above, $\triangle ABC$ is a right triangle and P is the midpoint of BC. If $\angle PAB = \alpha$, show that

$$BC = 2x\tan\alpha.$$

(b)



Prove the algebraic identity

$$(a^2-b^2)^2 + (2ab)^2 = (a^2+b^2)^2.$$

Hence show that $\sin\theta = \dfrac{a^2-b^2}{a^2+b^2}$.

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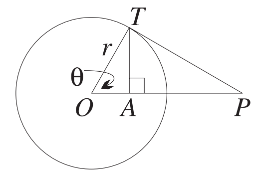
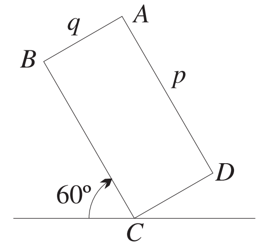
5. The tangent PT in the diagram opposite is perpendicular to the radius OT of a circle of radius r and centre O. Let A be the foot of the perpendicular from T to OP, and let $\angle TOP = \theta$.

(a) Explain why $\angle ATP = \theta$.

(b) Show that $AP = r\sin\theta\tan\theta$.



6. A rectangle $ABCD$ with length p and breadth q starts off lying flat on a horizontal plane. It is then rotated 60° clockwise about C until it reaches the position shown in the diagram opposite. Find the final heights of D, B and A above the plane.



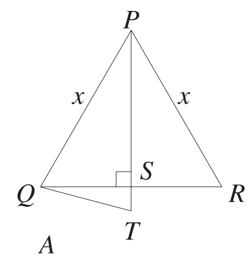
7. PQR is an equilateral triangle of side length x, and PS is the perpendicular from P to QR. PS is produced to T so that $PT = x$.

(a) Show that $\angle PQT = 75^\circ$ and hence that $\angle SQT = 15^\circ$.

(b) Show that $QS = \frac{1}{2}x$ and that $PS = \frac{1}{2}x\sqrt{3}$.

(c) Show that $ST = \frac{1}{2}x(2-\sqrt{3}\,)$.

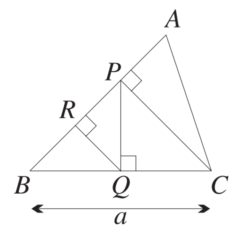
(d) Hence show that $\tan 15^\circ = 2-\sqrt{3}$.



8. In triangle ABC, lines CP, PQ and QR are drawn perpendicular to AB, BC and AB respectively.

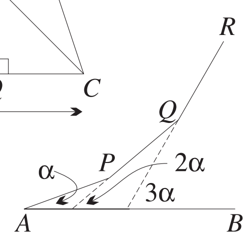
(a) Explain why $\angle RBQ = \angle RQP = \angle QPC$.

(b) Show that $QR = a\sin B\cos^2 B$.

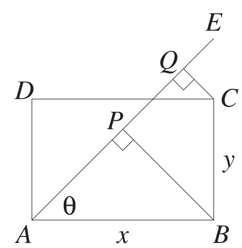


9. AP, PQ and QR are three equal intervals inclined at angles α, 2α and 3α respectively to interval AB. Show that:

$$\tan\angle BAR = \frac{\sin\alpha+\sin 2\alpha+\sin 3\alpha}{\cos\alpha+\cos 2\alpha+\cos 3\alpha}.$$



10. In the diagram opposite, $ABCD$ is a rectangle in which $AB = x$ and $BC = y$. BP and CQ are drawn perpendicular to the interval AE, which is inclined at an angle θ to AB. Show that $AQ = x\cos\theta + y\sin\theta$.



EXTENSION

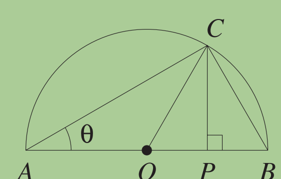
11. In the diagram, O is the centre of the semicircle ACB, and P is the foot of the perpendicular from C to the diameter AB. Let $\angle OAC = \theta$.

(a) Show that $\angle POC = 2\theta$ and that $\angle PCB = \theta$.

(b) Using the two triangles $\triangle APC$ and $\triangle ABC$, show that

$$\sin\theta\cos\theta = \frac{PC}{AB}.$$

(c) Hence show that $2\sin\theta\cos\theta = \sin 2\theta$.



12. Using the same diagram as the previous question:

(a) Explain why $AP - PB = 2 \times OP$.

(b) Show that $\cos^2\theta - \sin^2\theta = \frac{AP}{AC}\times\frac{AC}{AB} - \frac{PB}{CB}\times\frac{CB}{AB}$.

(c) Hence show that $\cos^2\theta - \sin^2\theta = \cos 2\theta$.

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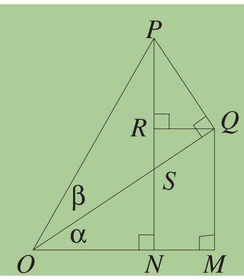
13. In the given diagram, $\angle MOQ = \alpha$ and $\angle QOP = \beta$. Also, $PN \perp OM$, $PQ \perp OQ$ and $QR \perp PN$.
(a) Explain why: (i) $\angle RPQ = \alpha$, (ii) $NP = MQ + RP$.
(b) Hence use triangles OPN, MOQ, RPQ and POQ to show that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.



4 C Trigonometric Functions of a General Angle

The definitions of the trigonometric functions given in Section 4A only work for acute angles, because only an angle between $0°$ and $90°$ can be put into a right triangle. This section introduces a set of more general definitions based on circles in the coordinate plane. The new definitions will apply to any angle, but will, of course, give the same values at acute angles as the previous definitions.

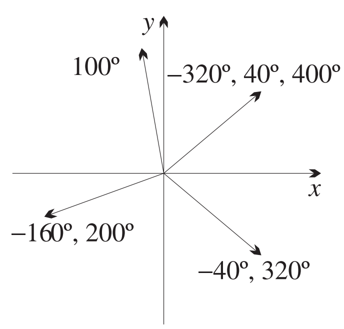
Putting a General Angle on the Cartesian Plane: Suppose that θ is any angle — possibly negative, possibly obtuse or reflex, possibly greater than $360°$. Our first task is to establish a geometrical representation of the angle θ on the Cartesian plane so that we can work with the angle. We shall associate with θ a ray with vertex at the origin.

5 DEFINITION: To find the *ray corresponding to* θ, rotate the positive half of the x-axis through an angle θ in the anticlockwise direction.

Here are some examples of angles and the rays corresponding to them — notice how the angle is written at the end of the arrow representing the ray. If the angle is negative, then the ray is rotated backwards, which means clockwise. Hence one ray can correspond to many angles. For example, all the following angles have the same ray as $40°$:

$$\ldots,\ -680°, -320°,\ 400°,\ 760°, \ldots$$

A given ray thus corresponds to infinitely many angles, all differing by multiples of $360°$.



6 ANGLES AND RAYS: To each angle, there corresponds exactly one ray.
To each ray, there correspond infinitely many angles, all differing from each other by multiples of $360°$.

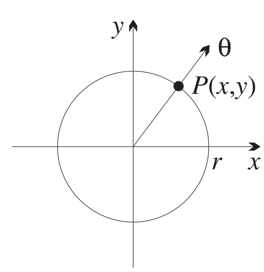
The Definitions of the Trigonometric Functions: Suppose that θ is any angle. Construct the ray corresponding to θ, and construct a circle with centre the origin and any positive radius r. Let the ray and the circle intersect at the point $P(x, y)$. We now define the six trigonometric functions by:

7 DEFINITION:

$$\sin\theta = \frac{y}{r} \qquad \operatorname{cosec}\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



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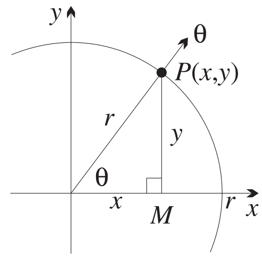
NOTE: We chose r to be 'any positive radius'. If a different radius had been chosen, x, y and r would change, but the two figures would be similar. Since the definitions depend only on the ratios of the lengths, the values of the trigonometric functions would not change.

Agreement with the Earlier Definition: Suppose that θ is an acute angle ($0° < \theta < 90°$), and construct the ray corresponding to θ. Drop the perpendicular from P to meet the x-axis at M; then $\theta = \angle POM$. Relating the sides to the angle θ,



$$\text{hyp} = OP = r, \qquad \text{opp} = PM = y, \qquad \text{adj} = OM = x,$$

and so the old and the new definitions coincide.

NOTE: Most people find that the diagram above is the easiest way to learn the new definitions of the trigonometric functions. Take the old definitions in terms of hypotenuse, opposite and adjacent sides, and make the replacements

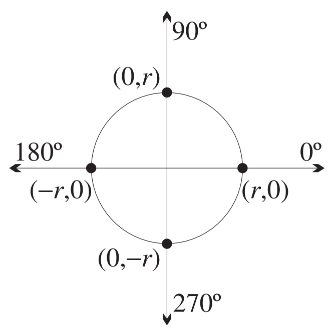
$$\text{hyp} \longrightarrow r, \qquad \text{opp} \longrightarrow y, \qquad \text{adj} \longrightarrow x.$$

Boundary Angles: Integer multiples of $90°$, that is $\dots, -90°, 0°, 90°, 180°, \dots$, are called *boundary angles* because they lie on the boundaries between quadrants. The values of the trigonometric functions at these boundary angles are not always defined, and are 0, 1 or -1 when they are defined. The accompanying diagram can be used to calculate them, and the results are shown in the table below. A star ($*$) means that the function is undefined at that value.

8

THE BOUNDARY ANGLES:

θ	$0°$	$90°$	$180°$	$270°$
x	r	0	$-r$	0
y	0	r	0	$-r$
r	r	r	r	r
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\tan\theta$	0	$*$	0	$*$
$\operatorname{cosec}\theta$	$*$	1	$*$	-1
$\sec\theta$	1	$*$	-1	$*$
$\cot\theta$	$*$	0	$*$	0



In practice, the answer to any question about the values of the trigonometric functions at these boundary angles should be read off the graphs of the functions, and these graphs need to be known very well indeed.

The Domains of the Trigonometric Functions: The trigonometric functions are defined everywhere except where the denominator is zero. Since y is zero at the angles $\dots, -180°, 0°, 180°, 360°, \dots$ and x is zero at $\dots, -90°, 90°, 270°, 540°, \dots$:

9

DOMAINS OF THE TRIGONOMETRIC FUNCTIONS:

$\sin\theta$ and $\cos\theta$ are defined for all angles θ.

$\tan\theta$ and $\sec\theta$ are undefined for $\theta = \dots, -90°, 90°, 270°, 540°, \dots$.

$\cot\theta$ and $\operatorname{cosec}\theta$ are undefined for $\theta = \dots, -180°, 0°, 180°, 360°, \dots$.

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Exercise 4C

1. On a number plane, draw rays representing the following angles:

(a) $40°$ (b) $110°$ (c) $190°$ (d) $290°$ (e) $420°$ (f) $500°$

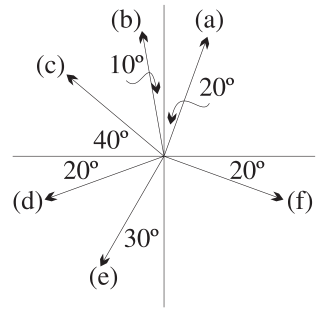
2. Repeat the previous question for these angles:

(a) $-50°$ (b) $-130°$ (c) $-250°$ (d) $-350°$ (e) $-440°$ (f) $-550°$

3. For each of the angles in question 1, name the negative angle between $-360°$ and $0°$ that is represented by the same ray.

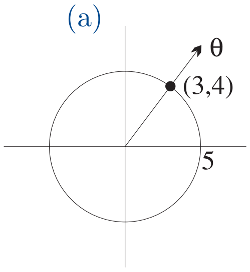
4. For each of the angles in question 2, name the positive angle between $0°$ and $360°$ that is represented by the same ray.

5. Write down two positive angles between $0°$ and $720°$ and two negative angles between $-720°$ and $0°$ that are represented by each of the rays in the diagram on the right.

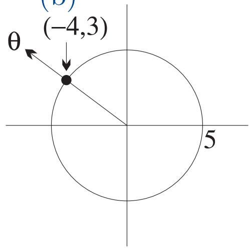


6. Write down the values of the six trigonometric ratios of the angle θ in each diagram:

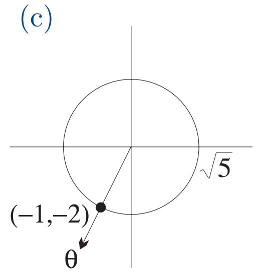
(a)



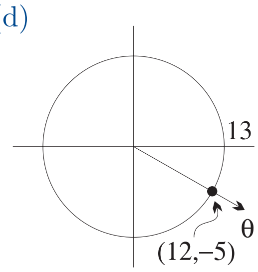
(b)



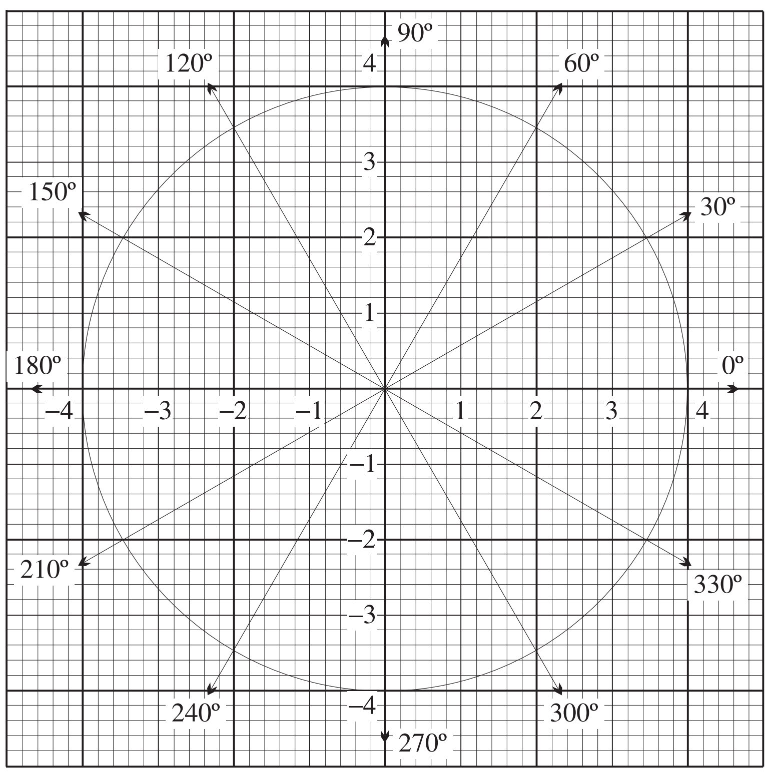
(c)



(d)



7. [The graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$] The diagram shows angles from $0°$ to $360°$ at $30°$ intervals. The circle has radius 4 units.



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(a) Use the diagram and the definitions of the three trigonometric ratios to complete the following table. Measure values of x and y correct to two decimal places, and use your calculator only to perform the necessary divisions.

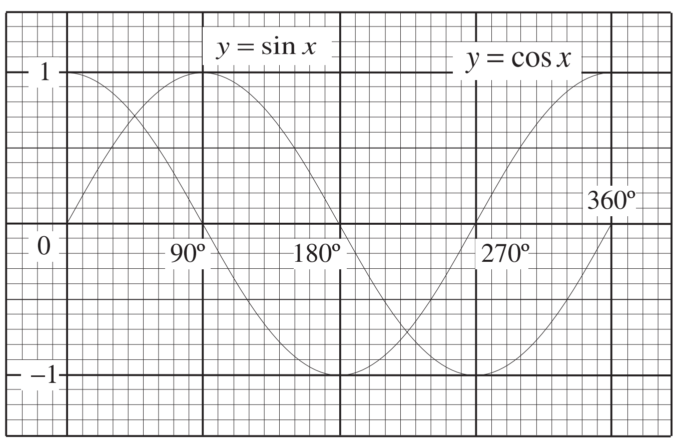
θ	$-30°$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$	$390°$
x															
y															
r															
$\sin\theta$															
$\cos\theta$															
$\tan\theta$															

(b) Use your calculator to check the accuracy of the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ that you obtained in part (a).

(c) Using the table of values in part (a), graph the curves $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ as accurately as possible on graph paper. Use the following scales: 2 mm represents $10°$ on the horizontal axis and 2 cm represents 1 unit on the vertical axis.

DEVELOPMENT

8.



(a) Read off the diagram above the value of:

(i) $\cos 60°$ (ii) $\sin 210°$ (iii) $\sin 72°$ (iv) $\cos 18°$ (v) $\sin 144°$ (vi) $\cos 36°$ (vii) $\cos 153°$ (viii) $\sin 27°$ (ix) $\sin 234°$ (x) $\cos 306°$

(b) Find from the graphs two values of x between $0°$ and $360°$ for which:

(i) $\sin x = 0{\cdot}5$ (ii) $\cos x = -0{\cdot}5$ (iii) $\sin x = 0{\cdot}9$ (iv) $\cos x = 0{\cdot}6$ (v) $\sin x = 0{\cdot}8$ (vi) $\cos x = -0{\cdot}8$ (vii) $\sin x = -0{\cdot}4$ (viii) $\cos x = -0{\cdot}3$

(c) Find two values of x between $0°$ and $360°$ for which $\sin x = \cos x$.

9. [The graphs of $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$] From the definitions of the trigonometric functions,

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

(a) Explain why the graph of $y = \operatorname{cosec}\theta$ has vertical asymptotes wherever $\sin\theta = 0$. Explain why the upper branches of $y = \operatorname{cosec}\theta$ have a minimum of 1 wherever $y = \sin\theta$ has a maximum of 1, and the lower branches have a maximum of -1 wherever $y = \sin\theta$ has a minimum of -1. Hence sketch the graph of $y = \operatorname{cosec}\theta$.

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(b) Use similar methods to produce the graph of $y = \sec\theta$ from the graph of $y = \cos\theta$, and the graph of $y = \cot\theta$ from the graph of $y = \tan\theta$.

EXTENSION

10. [The equation of a cone] The equation behind the definition of all the trigonometric functions is $x^2 + y^2 = r^2$, which is the equation of the circle, and is also Pythagoras' theorem. A third interpretation of this equation comes from regarding x, y and r all as variables, and plotting the resulting surface on a three-dimensional coordinate system with axes labelled x, y and r.

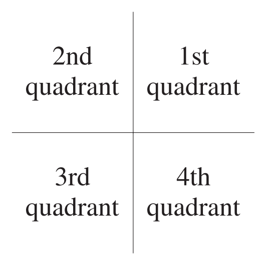
(a) Explain why the surface obtained in this way is a double cone, with vertex at the origin, and with a right angle at the vertex.

(b) What sort of curve is obtained by fixing r at some nonzero value r_0 and letting x and y vary (that is, by cutting the surface with the plane $r = r_0$)?

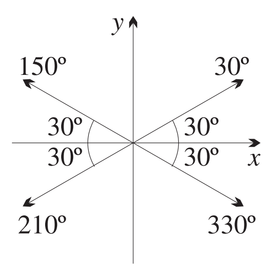
(c) What sort of curve is obtained by fixing x at some nonzero value x_0 and letting y and r vary (that is, by cutting the surface with the plane $x = x_0$)?

4 D The Quadrant, the Related Angle and the Sign

It would have been obvious from the calculations in the previous exercise that symmetry in the x-axis and the y-axis plays a large role in the values taken by the trigonometric functions. This section examines that symmetry, and explains how the values of the trigonometric functions of any angle can easily be expressed in terms of the values of the trigonometric functions of acute angles. The diagram shows the conventional anticlockwise numbering of the four quadrants of the coordinate plane — acute angles are in the first quadrant and obtuse angles are in the second quadrant.



The Quadrant and the Related Angle: The diagram opposite shows the four rays corresponding to the four angles 30°, 150°, 210°, 330°. These four rays lie in each of the four quadrants of the plane, and they each make the same acute angle 30° with the x-axis. Consequently, the four rays are just the reflections of each other in the two axes.



10 QUADRANT AND RELATED ANGLE: Suppose that θ is any angle.
The *quadrant* of θ is the quadrant (1, 2, 3 or 4) in which the ray lies.
The *related angle* of θ is the acute angle between the ray and the x-axis.

So each of the four angles in the diagram has the same related angle, 30°. The only time when θ and its related angle are the same is when θ is an acute angle, that is an angle between 0° and 90°.

The Signs of the Trigonometric Functions: The signs of the trigonometric functions depend only on the signs of x and y (the radius r is a positive constant). The signs of x and y depend in turn only on the quadrant in which the ray lies. Thus we can easily compute the signs of the trigonometric functions from the accompanying diagram and the definitions:

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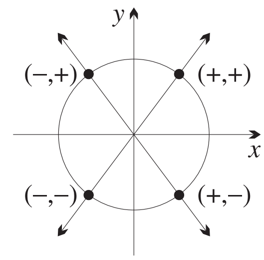
quadrant	1st	2nd	3rd	4th
x	$+$	$-$	$-$	$+$
y	$+$	$+$	$-$	$-$
r	$+$	$+$	$+$	$+$
$\sin\theta$	$+$	$+$	$-$	$-$
$\cos\theta$	$+$	$-$	$-$	$+$
$\tan\theta$	$+$	$-$	$+$	$-$
$\operatorname{cosec}\theta$	$+$	$+$	$-$	$-$
$\sec\theta$	$+$	$-$	$-$	$+$
$\cot\theta$	$+$	$-$	$+$	$-$

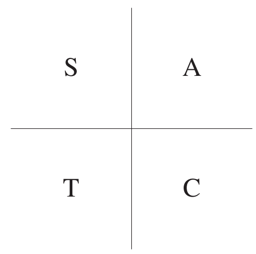


In NSW, these results are usually remembered by the phrase:

11 SIGNS OF THE TRIGONOMETRIC FUNCTIONS: '<u>A</u>ll <u>S</u>tations <u>T</u>o <u>C</u>entral'

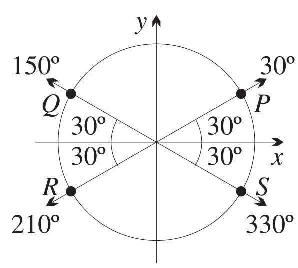
indicating that the four letters A, S, T and C are placed successively in the four quadrants as shown. The significance of the letters is:

A means all six functions are positive,
S means only sine (and cosecant) are positive,
T means only tangent (and cotangent) are positive,
C means only cosine (and secant) are positive.



Study each of the graphs constructed in the previous exercise to see how the table of signs above, and the ASTC rule, agree with your observations about when the graph is above the x-axis and when it is below.

The Angle and the Related Angle: In the diagram on the right, a circle of radius r has been added to the earlier diagram that showed the four angles $30°$, $150°$, $210°$ and $330°$ all with the same related angle of $30°$.



The four points P, Q, R and S where the rays meet the circle are all reflections of each other in the x and y axes. Because of this symmetry, the coordinates of these four points are identical apart from their sign. Hence the various trigonometric functions on these angles will all be the same too, except that the signs may be different.

12 ANGLE AND RELATED ANGLE: The trigonometric functions of any angle θ are the same as the trigonometric functions of its related angle, apart from a possible change of sign. (NOTE: The sign is found using the ASTC diagram.)

Evaluating the Trigonometric Functions at Any Angle: This gives a straightforward way of evaluating the trigonometric functions of any angle, and later, a very clear way of solving trigonometric equations.

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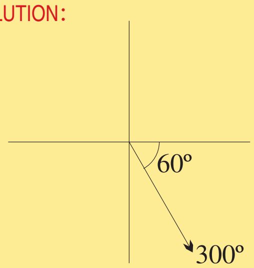
13

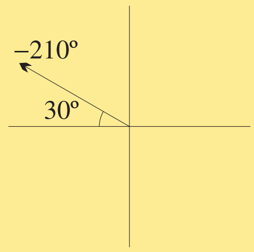
TRIGONOMETRIC FUNCTIONS AT ANY ANGLE: Draw a quadrant diagram, then:
1. Place the ray in the correct quadrant, and use the ASTC rule to work out the sign of the answer.
2. Find the related angle, and work out the value of the trigonometric function at the related angle.

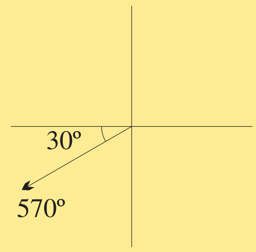
WORKED EXERCISE: Find the exact values of:

(a) $\tan 300°$ (b) $\sin(-210°)$ (c) $\cos 570°$

SOLUTION:







(a) $300°$ is in quadrant 4, the related angle is $60°$, so $\tan 300° = -\tan 60°$
$= -\sqrt{3}\,.$

(b) $-210°$ is in quadrant 2, the related angle is $30°$, so $\sin(-210°) = +\sin 30°$
$= \tfrac{1}{2}.$

(c) $570°$ is in quadrant 3, the related angle is $30°$, so $\cos 570° = -\cos 30°$
$= -\dfrac{\sqrt{3}}{2}\,.$

NOTE: The calculator will give approximate values of the trigonometric functions without any need to find the related angle. But it will *not* give exact values when these values involve surds, and all calculators eventually cut out or become inaccurate for large angles.

General Angles With Pronumerals: This quadrant-diagram method can be used to generate formulae for expressions such as $\sin(180° + A)$ or $\cot(360° - A)$. The trick is to deal with A on the quadrant diagram *as if it were acute*.

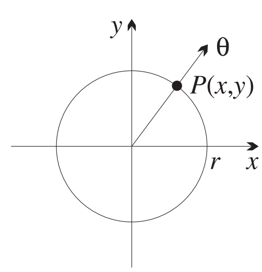
14

SOME FORMULAE WITH GENERAL ANGLES:

$\sin(180° - A) = \sin A$ $\quad$ $\sin(180° + A) = -\sin A$ $\quad$ $\sin(360° - A) = -\sin A$
$\cos(180° - A) = -\cos A$ $\quad$ $\cos(180° + A) = -\cos A$ $\quad$ $\cos(360° - A) = \cos A$
$\tan(180° - A) = -\tan A$ $\quad$ $\tan(180° + A) = \tan A$ $\quad$ $\tan(360° - A) = -\tan A$

Some people prefer to learn this list of identities to evaluate trigonometric functions, but this seems unnecessary when the quadrant-diagram method is so clear.

Specifying a Point in Terms of r and θ: If the definitions of $\sin\theta$ and $\cos\theta$ are rewritten with x and y as the subject:



15

RECOVERING THE COORDINATES OF A POINT:

$x = r\cos\theta$
$y = r\sin\theta$

This means that if a point P is specified in terms of its distance OP from the origin and the angle of the ray OP, then the x and y coordinates of P can be recovered by means of these formulae.

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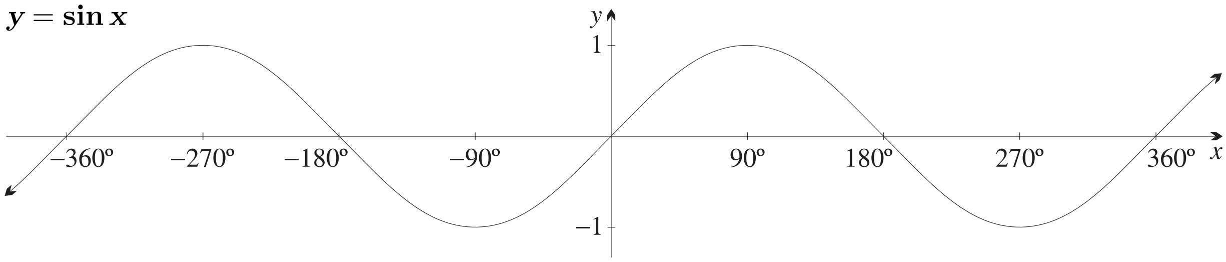
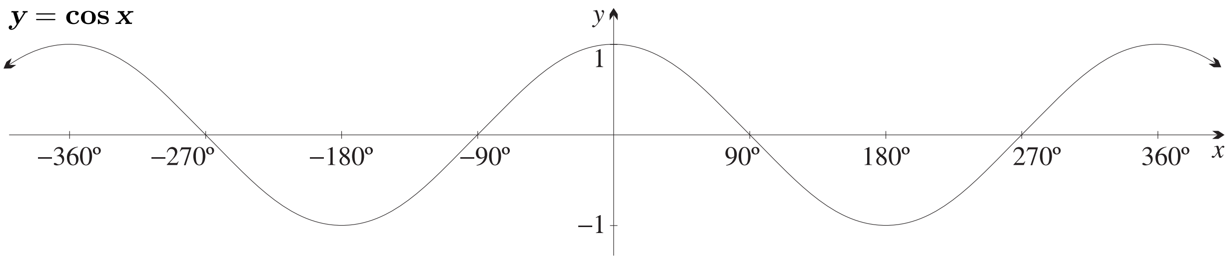
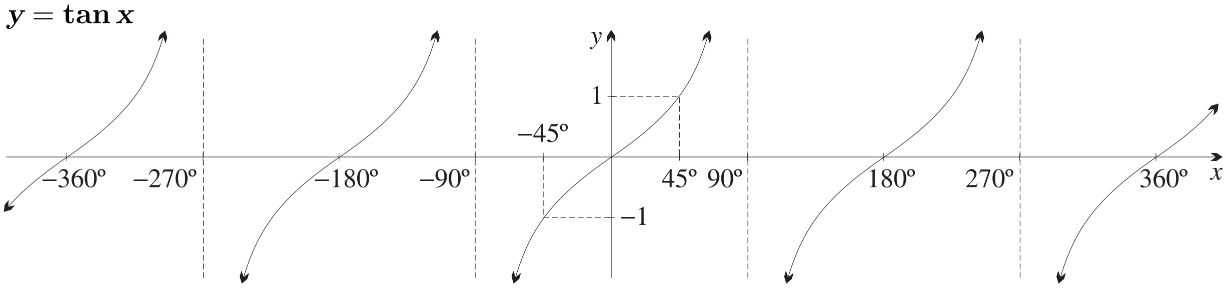
$y = \sin x$



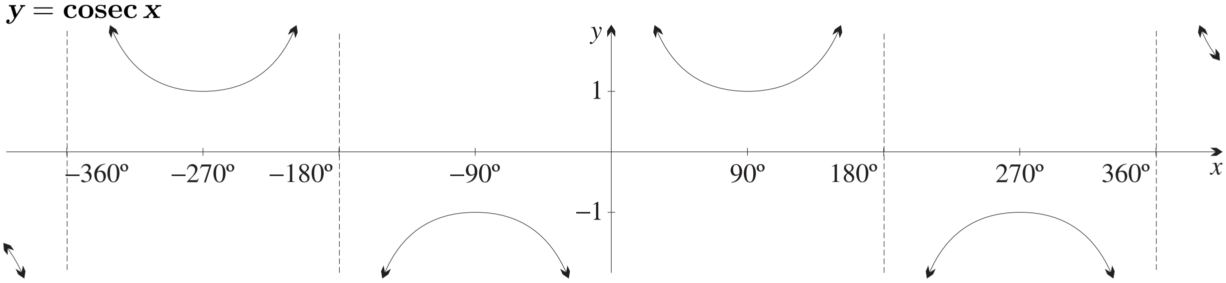
$y = \cos x$



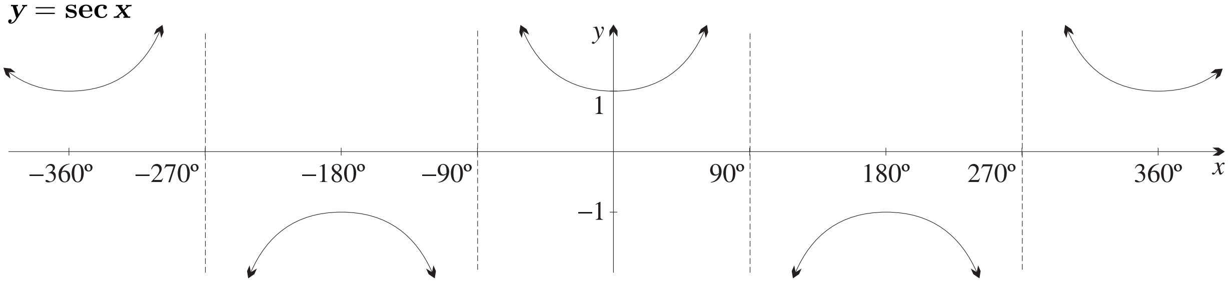
$y = \tan x$



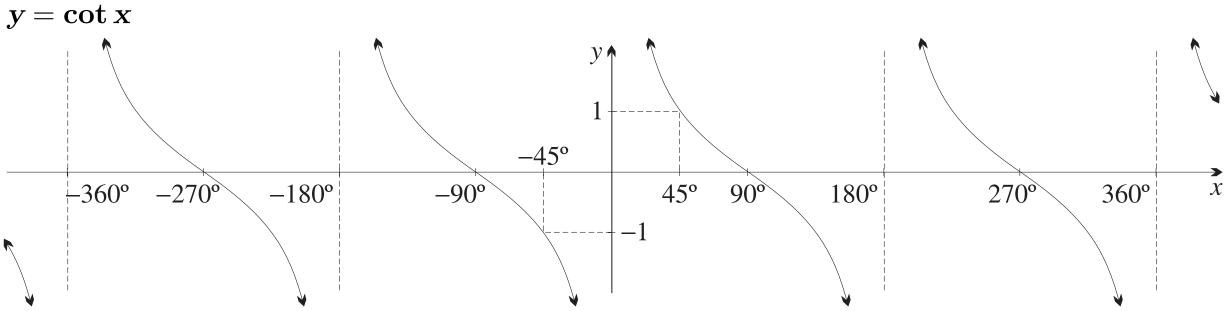
$y = \operatorname{cosec} x$



$y = \sec x$



$y = \cot x$



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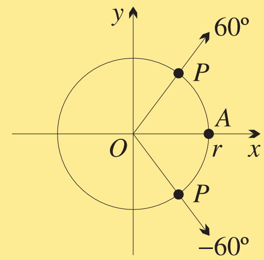
WORKED EXERCISE: The circle $x^2 + y^2 = 36$ meets the positive direction of the x-axis at A. Find the coordinates of the points P on the circle such that $\angle AOP = 60°$.

SOLUTION: The circle has radius 6, so $r = 6$,
and the ray OP has angle $60°$ or $-60°$,
so the coordinates (x, y) of P are

$$\begin{aligned} x &= 6\cos 60° &\quad\text{or}\quad& x = 6\cos(-60°) \\ &= 3 && \quad= 3 \\ y &= 6\sin 60° &\quad\text{or}\quad& y = 6\sin(-60°) \\ &= 3\sqrt{3}, && \quad= -3\sqrt{3}. \end{aligned}$$

So $P = (3, 3\sqrt{3})$ or $P = (3, -3\sqrt{3})$.



The Graphs of the Six Trigonometric Functions: In the diagrams on the previous page, the six trigonometric functions have been drawn over the extended range $-450° \le x \le 450°$ so that it becomes clear how the graphs are built up by infinite repetition of a simple element.

The sine and cosine graphs are waves. It turns out that these are the basic wave shapes, because any wave pattern, no matter how complicated, can always be reduced to a combination of various types of sine and cosine waves. Later in the course, these six graphs will become fundamental to our work in trigonometry. Their distinctive shapes and symmetries should be studied carefully and remembered. (A question in the following exercise discusses these things.)

Exercise 4D

1. Use the ASTC rule to determine the sign (+ or −) of each of these trigonometric ratios:

(a) $\sin 20°$ (b) $\sec 50°$ (c) $\cos 100°$ (d) $\tan 290°$
(e) $\cot 140°$ (f) $\sin 310°$ (g) $\operatorname{cosec} 200°$ (h) $\cos 320°$
(i) $\sin 400°$ (j) $\sec(-30°)$ (k) $\tan(-130°)$ (l) $\cos 500°$
(m) $\cot 600°$ (n) $\operatorname{cosec} 700°$ (o) $\tan(-400°)$ (p) $\sec(-330°)$

2. Find the related angle for each of the following:

(a) $36°$ (b) $150°$ (c) $310°$ (d) $200°$ (e) $-60°$
(f) $-150°$ (g) $-300°$ (h) $430°$ (i) $-500°$ (j) $600°$

3. Write each trigonometric ratio as the ratio of an acute angle with the correct sign attached:

(a) $\tan 130°$ (b) $\cos 310°$ (c) $\sin 220°$
(d) $\cot 260°$ (e) $\sec 170°$ (f) $\operatorname{cosec} 320°$
(g) $\cos(-175)°$ (h) $\operatorname{cosec}(-235°)$ (i) $\tan 500°$
(j) $\sin(-455°)$ (k) $\sec 1000°$ (l) $\cot 2000°$

4. Use the trigonometric graphs to find the values (if they exist) of these trigonometric ratios of boundary angles:

(a) $\sin 90°$ (b) $\cos 180°$ (c) $\cos 270°$
(d) $\tan 360°$ (e) $\tan 90°$ (f) $\sec 360°$
(g) $\operatorname{cosec} 270°$ (h) $\cot 270°$ (i) $\operatorname{cosec}(-270°)$
(j) $\cot 450°$ (k) $\cot 540°$ (l) $\operatorname{cosec} 180°$

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5. Find the exact value of:

(a) $\cos 135^\circ$
(b) $\sin 120^\circ$
(c) $\tan 225^\circ$
(d) $\cos 330^\circ$
(e) $\cot 210^\circ$
(f) $\sec 150^\circ$
(g) $\operatorname{cosec} 330^\circ$
(h) $\sin 405^\circ$
(i) $\sec 480^\circ$
(j) $\cot 660^\circ$
(k) $\operatorname{cosec}(-60^\circ)$
(l) $\cos(-135^\circ)$
(m) $\sin(-210^\circ)$
(n) $\tan 1500^\circ$
(o) $\operatorname{cosec}(-135^\circ)$
(p) $\sec(-150^\circ)$

6. Given that $\sin 25^\circ \doteqdot 0{\cdot}42$ and $\cos 25^\circ \doteqdot 0{\cdot}91$, write down approximate values, without using a calculator, for:

(a) $\sin 155^\circ$
(b) $\cos 205^\circ$
(c) $\cos 335^\circ$
(d) $\sin 335^\circ$
(e) $\sin 205^\circ - \cos 155^\circ$
(f) $\cos 385^\circ - \sin 515^\circ$

7. Given that $\tan 35^\circ \doteqdot 0{\cdot}70$ and $\sec 35^\circ \doteqdot 1{\cdot}22$, write down approximate values for:

(a) $\tan 145^\circ$
(b) $\sec 215^\circ$
(c) $\tan 325^\circ$
(d) $\tan 215^\circ + \sec 145^\circ$
(e) $\sec 325^\circ + \tan 395^\circ$
(f) $\sec(-145)^\circ - \tan(-215)^\circ$

DEVELOPMENT

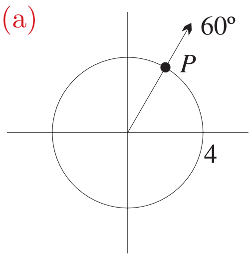
8. Find the value of:

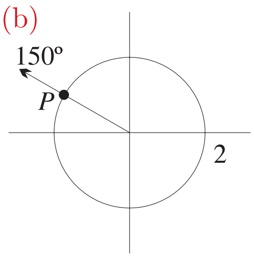
(a) $\sin 240^\circ \cos 150^\circ - \sin 150^\circ \cos 240^\circ$
(b) $3\tan 210^\circ \sec 210^\circ - \sin 330^\circ \cot 135^\circ - \cos 150^\circ \operatorname{cosec} 240^\circ$
(c) $\sin^2 120^\circ \operatorname{cosec} 270^\circ - \cos^2 315^\circ \sec 180^\circ - \tan^2 225^\circ \cot 315^\circ$

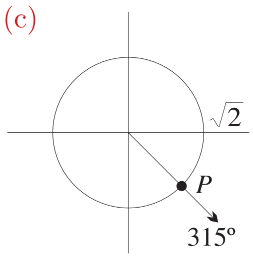
9. Prove:

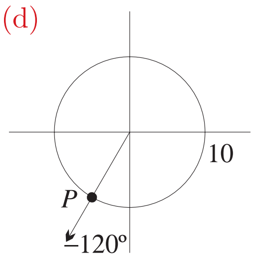
(a) $\sin 330^\circ \cos 150^\circ - \cos 390^\circ \sin 390^\circ = 0$
(b) $\sin 420^\circ \cos 405^\circ + \cos 420^\circ \sin 405^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$
(c) $\dfrac{\sin 135^\circ - \cos 120^\circ}{\sin 135^\circ + \cos 120^\circ} = 3 + 2\sqrt{2}$
(d) $(\sin 150^\circ + \cos 270^\circ + \tan 315^\circ)^2 = \sin^2 135^\circ \cos^2 225^\circ$
(e) $\dfrac{\sin 120^\circ}{\tan 300^\circ} - \dfrac{\cos 240^\circ}{\cot 315^\circ} = \tan^2 240^\circ - \operatorname{cosec}^2 330^\circ$

10. Find the coordinates of the point P in each of the following diagrams:

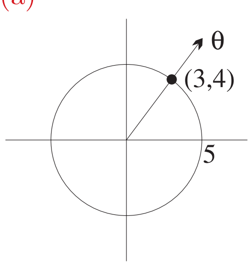
(a)


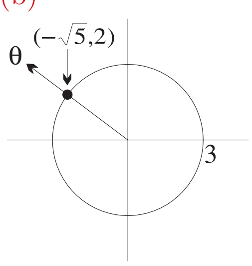
(b)


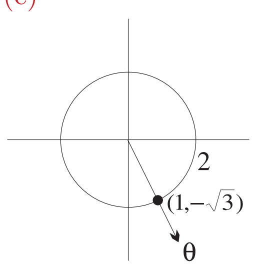
(c)


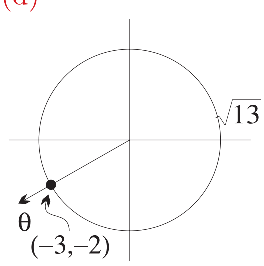
(d)


11. Find the angle θ, correct to the nearest minute where necessary, given that $0^\circ < \theta < 360^\circ$:

(a)


(b)


(c)


(d)


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12. Show that the following relationships are satisfied by the given values:

(a) $\sin 2\theta = 2\sin\theta\cos\theta$, when $\theta = 150°$.

(b) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, when $\theta = 135°$.

(c) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, when $\theta = 225°$.

(d) $\sin(A+B) = \sin A\cos B + \cos A\sin B$, when $A = 300°$ and $B = 240°$.

(e) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$, when $A = 330°$ and $B = 210°$.

13. Write as a trigonometric ratio of A , with the correct sign attached:

(a) $\sin(-A)$	(d) $\sec(-A)$	(g) $\cos(180° - A)$	(j) $\operatorname{cosec}(360° - A)$
(b) $\cos(-A)$	(e) $\sin(180° - A)$	(h) $\tan(180° + A)$	(k) $\cot(180° - A)$
(c) $\tan(-A)$	(f) $\sin(360° - A)$	(i) $\sec(180° + A)$	(l) $\sec(360° - A)$

14. Examine the graphs of the six trigonometric functions on page 124, then answer these questions.

(a) What are the ranges of the six functions?

(b) What is the *period* of each function, that is, how far does one move on the horizontal axis before the graph repeats itself? How is this period related to the identities

$$\sin(\theta + 360°) = \sin\theta, \qquad \sec(\theta + 360°) = \sec\theta, \qquad \tan(\theta + 180°) = \tan\theta?$$

(c) Which functions are even and which are odd?

(d) More generally, about what points do the graphs have point symmetry? (That is, about what points are they unchanged by a rotation of 180°?)

(e) What are the axes of symmetry of the graphs?

EXTENSION

15. Write as a trigonometric ratio of θ with the correct sign attached:

(a) $\sin(90° + \theta)$	(c) $\cos(90° + \theta)$	(e) $\cot(90° + \theta)$
(b) $\sin(90° - \theta)$	(d) $\sin(270° - \theta)$	(f) $\sec(270° - \theta)$

16. Simplify:

(a) $\cos(180° - \alpha)\sec\alpha$	(c) $\sin(90° - \alpha)\sec(90° + \alpha)$
(b) $\sec\alpha\sin(180° - \alpha)$	(d) $\cot(180° + \alpha)\cos(270° - \alpha)$

17. Show that:

(a) $\tan(90° - A)\sec(180° + A)\cos(90° + A) = 1$

(b) $\tan(180° - A)\sin(270° + A)\operatorname{cosec}(360° - A) = -1$

4 E Given One Trigonometric Function, Find Another

When the exact value of one trigonometric function is known for an angle, the exact value of the other trigonometric functions can easily be found using the circle diagram and Pythagoras' theorem.

16 GIVEN ONE TRIGONOMETRIC FUNCTION, FIND ANOTHER: Draw a circle diagram, and use Pythagoras' theorem to find whichever of x, y and r is missing.

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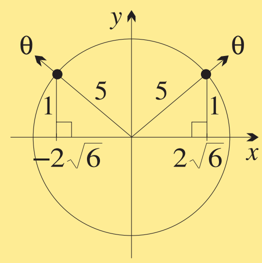
Worked Exercise:

(a) Given that $\sin\theta = \frac{1}{5}$, find $\cos\theta$.

(b) Repeat if it is also known that $\tan\theta$ is negative.

Solution:



(a) First, the angle must be in quadrant 1 or 2.

Since $\sin\theta = \dfrac{y}{r} = \dfrac{1}{5}$, we can take $y = 1$ and $r = 5$,

so by Pythagoras' theorem, $x = \sqrt{24}$ or $-\sqrt{24}$

$= 2\sqrt{6}$ or $-2\sqrt{6}$,

so $\cos\theta = \dfrac{2\sqrt{6}}{5}$ or $-\dfrac{2\sqrt{6}}{5}$.

(b) Since $\tan\theta$ is negative, θ must be in quadrant 2,

so $\cos\theta = -\frac{2}{5}\sqrt{6}$.

Exercise 4E

1. (a) Given that $\cos\theta = \frac{3}{5}$ and θ is acute, find $\sin\theta$ and $\tan\theta$.
(b) Given that $\tan\theta = -\frac{5}{12}$ and θ is obtuse, find $\sin\theta$ and $\sec\theta$.

2. (a) Given that $\sin\alpha = \frac{8}{17}$, find the possible values of $\cos\alpha$ and $\cot\alpha$.
(b) Given that $\cos x = -\frac{3}{4}$ and $90° < x < 180°$, find $\tan x$ and $\operatorname{cosec} x$.

3. (a) Given that $\cot\beta = \frac{3}{2}$ and $\sin\beta < 0$, find $\cos\beta$.
(b) If $\operatorname{cosec}\alpha = -\frac{5}{2}$ and $\cos\alpha > 0$, find $\cot\alpha$.
(c) If $\tan\theta = 2$, find the possible values of $\operatorname{cosec}\theta$.
(d) Suppose that $\sin A = 1$. Find $\sec A$.

4. (a) Given that $\sec P = -3$ and $180° < P < 360°$, find $\operatorname{cosec} P$.
(b) If $\cos\theta = -1$, find $\tan\theta$.
(c) Suppose that $\cos\alpha = \frac{2}{3}$. Find the possible values of $\sin\alpha$ and $\cot\alpha$.
(d) Given that $\cot x = -\frac{3}{5}$, find the possible values of $\operatorname{cosec} x$ and $\sec x$.

DEVELOPMENT

5. Given that $\sin\theta = \dfrac{p}{q}$, with θ obtuse and p and q both positive, find expressions for $\cos\theta$ and $\tan\theta$.

6. If $\tan\alpha = k$, where $k > 0$, find the possible values of $\sin\alpha$ and $\sec\alpha$.

7. (a) Prove the algebraic identity $(1 - t^2)^2 + 4t^2 = (1 + t^2)^2$.
(b) If $\cos x = \dfrac{1 - t^2}{1 + t^2}$ and x is acute, find expressions for $\sin x$ and $\tan x$.

EXTENSION

8. If $\sin\theta = k$ and θ is obtuse, find an expression for $\tan(\theta + 90°)$.

9. If $\sec\theta = a + \dfrac{1}{4a}$, prove that $\sec\theta + \tan\theta = 2a$ or $\dfrac{1}{2a}$.

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4 F Trigonometric Identities and Elimination

Working with the trigonometric functions requires knowledge of a number of formulae called *trigonometric identities*, which relate trigonometric functions to each other. This section introduces eleven of these in four groups: the three *reciprocal identities*, the two *ratio identities*, the three *Pythagorean identities*, and the three *identities concerning complementary angles*.

The Three Reciprocal Identities: It follows immediately from the definitions of the trigonometric functions in terms of x, y and r that:

17

THE RECIPROCAL IDENTITIES: For all angles θ:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad (\text{provided } \sin\theta \neq 0)$$

$$\sec\theta = \frac{1}{\cos\theta} \quad (\text{provided } \cos\theta \neq 0)$$

$$\cot\theta = \frac{1}{\tan\theta} \quad (\text{provided } \tan\theta \neq 0 \text{ and } \cot\theta \neq 0)$$

NOTE: The last identity needs attention. One cannot use the calculator to find $\cot 90^\circ$ or $\cot 270^\circ$ by first finding $\tan 90^\circ$ or $\tan 270^\circ$, because both of these are undefined. We already know, however, that $\cot 90^\circ = \cot 270^\circ = 0$.

The Two Ratio Identities: Again using the definitions of the trigonometric functions:

18

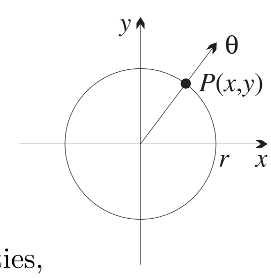
THE RATIO IDENTITIES: For any angle θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad (\text{provided } \cos\theta \neq 0)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (\text{provided } \sin\theta \neq 0)$$

The Three Pythagorean Identities: Since the point $P(x, y)$ lies on the circle with centre O and radius r, its coordinates satisfy

$$x^2 + y^2 = r^2.$$



Dividing through by r^2 gives $\qquad \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1,$

then by the definitions, $\qquad \sin^2\theta + \cos^2\theta = 1.$

Dividing through by $\cos^2\theta$ and using the ratio and reciprocal identities,

$$\tan^2\theta + 1 = \sec^2\theta, \quad \text{provided } \cos\theta \neq 0.$$

Dividing through instead by $\sin^2\theta$, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$, provided $\sin\theta \neq 0$.

These identities are called the *Pythagorean identities* because they rely on the circle equation $x^2 + y^2 = r^2$, which is really just a restatement of Pythagoras' theorem.

19

THE PYTHAGOREAN IDENTITIES: For any angle θ:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta \quad (\text{provided } \cos\theta \neq 0)$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta \quad (\text{provided } \sin\theta \neq 0)$$

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The Three Identities for Complementary Angles: These identities relate the values of the trigonometric functions at any angle θ to their values at the complementary angle $90^\circ - \theta$.

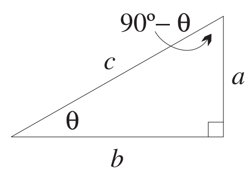
20 THE COMPLEMENTARY IDENTITIES: For any angle θ:

$$\cos(90^\circ - \theta) = \sin\theta$$
$$\cot(90^\circ - \theta) = \tan\theta \quad \text{(provided } \tan\theta \text{ is defined)}$$
$$\operatorname{cosec}(90^\circ - \theta) = \sec\theta \quad \text{(provided } \sec\theta \text{ is defined)}$$

PROOF:

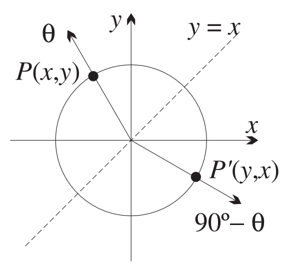
A. [Acute angles] The triangle on the right shows that when θ is acute, viewing the right triangle from $90^\circ - \theta$ instead of from θ exchanges the opposite side and the adjacent side, so:

$$\cos(90^\circ - \theta) = \frac{a}{c} = \sin\theta,$$
$$\cot(90^\circ - \theta) = \frac{a}{b} = \tan\theta,$$
$$\operatorname{cosec}(90^\circ - \theta) = \frac{c}{b} = \sec\theta.$$



B. [General angles] For general angles, we take the full circle diagram, and reflect it in the diagonal line $y = x$. Let P' be the image of P under this reflection.

1. The image OP' of the ray OP corresponds with the angle $90^\circ - \theta$.
2. The image P' of $P(x, y)$ has coordinates $P'(y, x)$. We have seen before that reflection in the line $y = x$ reverses the coordinates of each point. So x and y are interchanged in passing from P to P'.



Applying the definitions of the trigonometric functions to the angle $90^\circ - \theta$:

$$\cos(90^\circ - \theta) = \frac{y}{r} = \sin\theta,$$
$$\cot(90^\circ - \theta) = \frac{y}{x} = \tan\theta, \text{ provided } x \neq 0,$$
$$\operatorname{cosec}(90^\circ - \theta) = \frac{r}{x} = \sec\theta, \text{ provided } x \neq 0.$$

Cosine, Cosecant and Cotangent: The complementary identities are the origin of the 'co-' prefix of cosine, cosecant and cotangent — the prefix is an abbreviation of the prefix 'com-' of complementary angle. The various identities can be easily remembered as:

21 CO-FUNCTIONS: The co-function of a complement is the function of an angle.
The co-function of an angle is the function of the complement.

Proving Identities: An *identity* is a statement that needs to be proven true for all values of θ for which both sides are defined. It is quite different from an equation, which needs to be solved and to have its solutions listed.

22 PROVING TRIGONOMETRIC IDENTITIES: Work separately on the LHS and the RHS until they are the same.

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WORKED EXERCISE: Prove that $\sin A \sec A = \tan A$. NOTE: The necessary restriction to angles for which $\sec A$ and $\tan A$ are defined is implied by the statement.

SOLUTION:
$$\begin{aligned}\text{LHS} &= \sin A \times \frac{1}{\cos A} \quad \text{(reciprocal identity)}\\ &= \tan A \quad \text{(ratio identity)}\\ &= \text{RHS}\end{aligned}$$

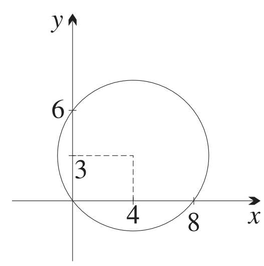
WORKED EXERCISE: Prove that $\dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta} = \sec^2\theta\,\mathrm{cosec}^2\,\theta$.

PROOF:
$$\begin{aligned}\text{LHS} &= \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}\\ &= \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta} \quad \text{(common denominator)}\\ &= \frac{1}{\sin^2\theta\cos^2\theta} \quad \text{(Pythagorean identity)}\\ &= \sec^2\theta\,\mathrm{cosec}^2\,\theta \quad \text{(reciprocal identities)}\\ &= \text{RHS}\end{aligned}$$

Elimination: If x and y are given as functions of θ, then using the techniques of simultaneous equations, the θ can often be eliminated to give a relation (rarely a function) between x and y.

WORKED EXERCISE: Eliminate θ from the following pair, and describe the graph of the relation:

$$x = 4 + 5\cos\theta$$
$$y = 3 - 5\sin\theta$$



SOLUTION: From the first equation, $5\cos\theta = x - 4$,
and from the second equation, $5\sin\theta = 3 - y$.
Squaring and adding, $25\cos^2\theta + 25\sin^2\theta = (x-4)^2 + (3-y)^2$
and since $\cos^2\theta + \sin^2\theta = 1$, $(x-4)^2 + (y-3)^2 = 25$,
which is a circle of radius 5 and centre $(4, 3)$.

Exercise 4F

1. Use your calculator to verify that:
(a) $\sin 16° = \cos 74°$ (c) $\sec 7° = \mathrm{cosec}\, 83°$ (e) $1 + \tan^2 55° = \sec^2 55°$
(b) $\tan 63° = \cot 27°$ (d) $\sin^2 23° + \cos^2 23° = 1$ (f) $\mathrm{cosec}^2\, 32° - 1 = \cot^2 32°$

2. Simplify: (a) $\dfrac{1}{\sin\theta}$ (b) $\dfrac{1}{\tan\alpha}$ (c) $\dfrac{\sin\beta}{\cos\beta}$ (d) $\dfrac{\cos\phi}{\sin\phi}$

3. Simplify: (a) $\sin\alpha\,\mathrm{cosec}\,\alpha$ (b) $\cot\beta\tan\beta$ (c) $\cos\theta\sec\theta$

4. Prove: (a) $\tan\theta\cos\theta = \sin\theta$ (b) $\cot\alpha\sin\alpha = \cos\alpha$ (c) $\sin\beta\sec\beta = \tan\beta$

5. Prove: (a) $\cos A\,\mathrm{cosec}\, A = \cot A$ (b) $\mathrm{cosec}\, x\cos x\tan x = 1$ (c) $\sin y\cot y\sec y = 1$

6. Simplify: (a) $\dfrac{\cos\alpha}{\sec\alpha}$ (b) $\dfrac{\sin\alpha}{\mathrm{cosec}\,\alpha}$ (c) $\dfrac{\tan A}{\sec A}$ (d) $\dfrac{\cot A}{\mathrm{cosec}\, A}$

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7. Simplify: (a) $\dfrac{1}{\sec^2\theta}$ (b) $\sin^2\alpha\,\text{cosec}^2\,\alpha$ (c) $\dfrac{\sin^2\beta}{\cos^2\beta}$ (d) $\dfrac{\cos^2 A}{\sin^2 A}$

8. Simplify: (a) $\sin(90^\circ-\theta)$ (b) $\sec(90^\circ-\alpha)$ (c) $\dfrac{1}{\cot(90^\circ-\beta)}$ (d) $\dfrac{\cos(90^\circ-\phi)}{\sin(90^\circ-\phi)}$

9. Simplify: (a) $\sin^2\alpha+\cos^2\alpha$ (b) $1-\cos^2\beta$ (c) $1+\tan^2\phi$ (d) $\sec^2 x-\tan^2 x$

10. Simplify: (a) $1-\sin^2\beta$ (b) $1+\cot^2\phi$ (c) $\text{cosec}^2\,A-1$ (d) $\cot^2\theta-\text{cosec}^2\,\theta$

DEVELOPMENT

11. Prove the identities:

(a) $(1-\sin\theta)(1+\sin\theta)=\cos^2\theta$

(b) $(1+\tan^2\alpha)\cos^2\alpha=1$

(c) $(\sin A+\cos A)^2=1+2\sin A\cos A$

(d) $\cos^2 x-\sin^2 x=1-2\sin^2 x$

(e) $\tan^2\phi\cos^2\phi+\cot^2\phi\sin^2\phi=1$

(f) $3\cos^2\theta-2=1-3\sin^2\theta$

(g) $2\tan^2 A-1=2\sec^2 A-3$

(h) $1-\tan^2\alpha+\sec^2\alpha=2$

(i) $\cos^4 x+\cos^2 x\sin^2 x=\cos^2 x$

(j) $\cot\theta(\sec^2\theta-1)=\tan\theta$

12. Prove the identities:

(a) $\sin\theta\cos\theta\,\text{cosec}^2\,\theta=\cot\theta$

(b) $(\cos\phi+\cot\phi)\sec\phi=1+\text{cosec}\,\phi$

(c) $\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\cos\alpha}{\sin\alpha}=\sec\alpha\,\text{cosec}\,\alpha$

(d) $\dfrac{1+\tan^2 x}{1+\cot^2 x}=\tan^2 x$

(e) $\sin^4 A-\cos^4 A=\sin^2 A-\cos^2 A$

(f) $\dfrac{1}{1+\sin\theta}+\dfrac{1}{1-\sin\theta}=2\sec^2\theta$

(g) $\sin\beta+\cot\beta\cos\beta=\text{cosec}\,\beta$

(h) $\dfrac{1}{\sec\phi-\tan\phi}-\dfrac{1}{\sec\phi+\tan\phi}=2\tan\phi$

(i) $\dfrac{1+\cot x}{1+\tan x}=\cot x$

(j) $\dfrac{\cos\alpha}{1+\sin\alpha}=\sec\alpha(1-\sin\alpha)$

13. (a) If $x=a\cos\alpha$ and $y=a\sin\alpha$, show that $x^2+y^2=a^2$.

(b) If $x=a\sec\theta$ and $y=b\tan\theta$, show that $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

(c) If $x=r\cos\theta\sin\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\phi$, show that $x^2+y^2+z^2=r^2$.

(d) If $x=a\cos\theta-b\sin\theta$ and $y=a\sin\theta+b\cos\theta$, show that $x^2+y^2=a^2+b^2$.

14. Eliminate θ from each pair of equations:

(a) $x=a\cos\theta$ and $y=b\sin\theta$

(b) $x=a\tan\theta$ and $y=b\sec\theta$

(c) $x=2+\cos\theta$ and $y=1+\sin\theta$

(d) $x=\sin\theta+\cos\theta$ and $y=\sin\theta-\cos\theta$

15. Prove that each expression is independent of θ:

(a) $\dfrac{\cos^2\theta}{1+\sin\theta}+\dfrac{\cos^2\theta}{1-\sin\theta}$

(b) $\tan\theta(1-\cot^2\theta)+\cot\theta(1-\tan^2\theta)$

(c) $\dfrac{\tan\theta+\cot\theta}{\sec\theta\,\text{cosec}\,\theta}$

(d) $\dfrac{\tan\theta+1}{\sec\theta}-\dfrac{\cot\theta+1}{\text{cosec}\,\theta}$

16. Prove the identities:

(a) $\dfrac{2\cos^3\theta-\cos\theta}{\sin\theta\cos^2\theta-\sin^3\theta}=\cot\theta$

(b) $\sec y+\tan y+\cot y=\dfrac{1+\sin y}{\sin y\cos y}$

(c) $\dfrac{\cos A-\tan A\sin A}{\cos A+\tan A\sin A}=1-2\sin^2 A$

(d) $(\sin\phi+\cos\phi)(\sec\phi+\text{cosec}\,\phi)=2+\tan\phi+\cot\phi$

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(e) $\dfrac{1}{1+\tan^2 x} - \dfrac{1}{1+\sec^2 x} = \dfrac{\cos^4 x}{1+\cos^2 x}$ (f) $\dfrac{\cos\theta}{1-\tan\theta} + \dfrac{\sin\theta}{1-\cot\theta} = \sin\theta + \cos\theta$

(g) $(\tan\alpha + \cot\alpha - 1)(\sin\alpha + \cos\alpha) = \dfrac{\sec\alpha}{\operatorname{cosec}^2\alpha} + \dfrac{\operatorname{cosec}\alpha}{\sec^2\alpha}$

(h) $\dfrac{1}{\sec\theta + \tan\theta} = \sec\theta - \tan\theta = \dfrac{\cos\theta}{1+\sin\theta}$

(i) $\dfrac{1}{\cot\theta - \cos\theta} = \dfrac{\tan\theta}{1-\sin\theta} = \dfrac{\sin\theta + \sin^2\theta}{\cos^3\theta}$

(j) $\sin^2 x(1 + n\cot^2 x) + \cos^2 x(1 + n\tan^2 x) = n + 1$
$= \sin^2 x(n + \cot^2 x) + \cos^2 x(n + \tan^2 x)$

(k) $\dfrac{(\sin^2\alpha - \cos^2\alpha)(1 - \sin\alpha\cos\alpha)}{\cos\alpha(\sec\alpha - \operatorname{cosec}\alpha)(\sin^3\alpha + \cos^3\alpha)} = \sin\alpha$

(l) $\dfrac{1 + \operatorname{cosec}^2 A\tan^2 C}{1 + \operatorname{cosec}^2 B\tan^2 C} = \dfrac{1 + \cot^2 A\sin^2 C}{1 + \cot^2 B\sin^2 C}$

EXTENSION

17. Eliminate θ from each pair of equations:
(a) $x = \operatorname{cosec}^2\theta + 2\cot^2\theta$ and $y = 2\operatorname{cosec}^2\theta + \cot^2\theta$
(b) $x = \sin\theta - 3\cos\theta$ and $y = \sin\theta + 2\cos\theta$
(c) $x = \sin\theta + \cos\theta$ and $y = \tan\theta + \cot\theta$ [HINT: Find x^2y.]

18. (a) If $\dfrac{a}{\sin A} = \dfrac{b}{\cos A}$, show that $\sin A\cos A = \dfrac{ab}{a^2+b^2}$.
(b) If $\dfrac{a+b}{\operatorname{cosec} x} = \dfrac{a-b}{\cot x}$, show that $\operatorname{cosec} x\cot x = \dfrac{a^2-b^2}{4ab}$.
(c) If $\tan\theta + \sin\theta = x$ and $\tan\theta - \sin\theta = y$, prove that $x^4 + y^4 = 2xy(8 + xy)$.

4 G Trigonometric Equations

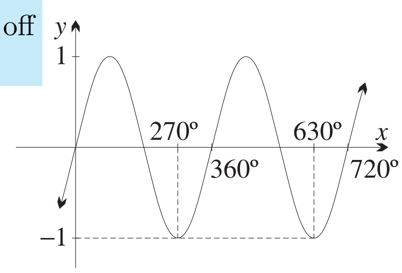
This piece of work is absolutely vital, because so many problems in later work end up with a trigonometric equation that has to be solved. There are many small details and qualifications in the methods, and the subject needs a great deal of careful study.

Pay Attention to the Domain: To begin with a warning, before any other details:

23 THE DOMAIN: Always pay attention to the domain in which the angle can lie.

Equations Involving Boundary Angles: The usual quadrants-and-related-angle method described below doesn't apply to boundary angles, which do not lie in any quadrant.

24 THE BOUNDARY ANGLES: If a trigonometric equation involves boundary angles, read the solutions off a sketch of the graph.



WORKED EXERCISE: Solve $\sin x = -1$, for $0° \le x \le 720°$.

SOLUTION: The graph of $y = \sin x$ is drawn on the right.
Reading from this graph, $x = 270°$ or $630°$.

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The Standard Method — Quadrants and Related Angle: Nearly all our trigonometric equations will eventually come down to something like

$$\sin x = -\tfrac{1}{2}, \quad \text{where } -180^\circ \le x \le 360^\circ.$$

As long as the angle is not a boundary angle, the method is:

25 THE QUADRANTS-AND-RELATED-ANGLE METHOD:

1. Draw a quadrant diagram, then draw a ray in each quadrant that the angle could be in.
2. Find the related angle (only work with positive numbers here):
   (a) using special angles, or
   (b) using the calculator to find an approximation.
3. Mark the angles on the ends of the rays, taking account of any restrictions on x, and write a conclusion.

WORKED EXERCISE: Solve each equation. Give the solution exactly if possible, or else to the nearest degree:

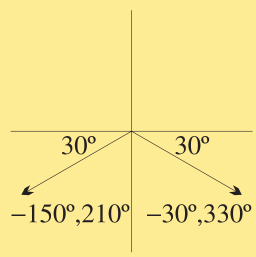
(a) $\sin x = -\tfrac{1}{2}, \quad -180^\circ \le x \le 360^\circ$ (b) $\tan x = -3, \quad 0^\circ \le x \le 360^\circ$

SOLUTION:

(a) $\sin x = -\tfrac{1}{2}$, where $-180^\circ \le x \le 360^\circ$

$x = -150^\circ,\ -30^\circ,\ 210^\circ$ or 330°

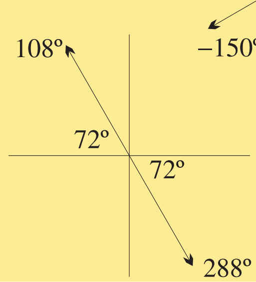
(Since $\sin x$ is negative, x is in quadrants 3 or 4, the sine of the related angle is $+\tfrac{1}{2}$, so the related angle is 30°.)



(b) $\tan x = -3$, where $0^\circ \le x \le 360^\circ$

$x \doteqdot 108^\circ$ or 288°

(Since $\tan x$ is negative, x is in quadrants 2 or 4, the tangent of the related angle is $+3$, so the related angle is about 72°.)



NOTE: When using the calculator, *never* enter a negative number and take an inverse trigonometric function of it. In the example above, the calculator was used to find the *acute* angle whose tan was 3, that is, $71^\circ 34'$. The positive number 3 was entered, not -3.

The Three Reciprocal Functions: Because they are unfamiliar, and also because the calculator doesn't have specific keys for them:

26 THE RECIPROCAL FUNCTIONS: Try to change any of the three reciprocal functions secant, cosecant and cotangent to the three more common functions by taking reciprocals.

WORKED EXERCISE: Suppose we are given that $\operatorname{cosec} x = -2$.
Taking reciprocals of both sides gives $\sin x = -\tfrac{1}{2}$,
which was solved in the previous worked example.

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Equations with Compound Angles: These can cause trouble. Equations like

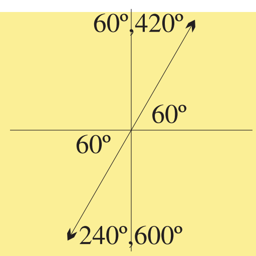
$$\tan 2x = \sqrt{3}, \text{ where } 0° \le x \le 360°, \qquad \text{or}$$
$$\sin(x - 250°) = \frac{\sqrt{3}}{2}, \text{ where } 0° \le x \le 360°,$$

are really trigonometric equations in the compound angles $2x$ and $(x - 250°)$ respectively. The secret lies in solving for the *compound angle*, and in *calculating first the domain for that compound angle*.

27 EQUATIONS WITH COMPOUND ANGLES:
1. Let u be the compound angle.
2. Find the restrictions on u from the given restrictions on x.
3. Solve the trigonometric equation for u.
4. Hence solve for x.

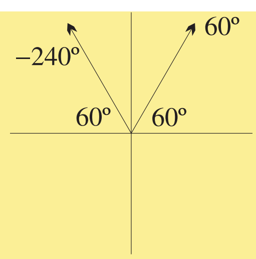
WORKED EXERCISE: Solve $\tan 2x = \sqrt{3}$, where $0° \le x \le 360°$.

SOLUTION: Let $u = 2x$.
Then $\tan u = \sqrt{3}$, where $0° \le u \le 720°$,
(the restriction on u is the key step here),
so from the diagram, $u = 60°, 240°, 420° \text{ or } 600°$.
Since $x = \frac{1}{2}u$, $x = 30°, 120°, 210° \text{ or } 300°$.



WORKED EXERCISE: Solve $\sin(x - 250°) = \frac{\sqrt{3}}{2}$, where $0° \le x \le 360°$.

SOLUTION: Let $u = x - 250°$.
Then $\sin u = \frac{\sqrt{3}}{2}$, where $-250° \le u \le 110°$,
(again, the restriction on u is the key step here),
so from the diagram, $u = -240° \text{ or } 60°$.
Since $x = u + 250°$, $x = 10° \text{ or } 310°$.



Equations Requiring Algebraic Substitutions: If there are powers or reciprocals of the trigonometric function present, as in

$$5\sin^2 x = \sin x, \text{ for } 0° \le x \le 360°, \qquad \text{or}$$
$$\frac{4}{\cos x} - \cos x = 0, \text{ for } -180° \le x \le 180°,$$

but still only the one trigonometric function, then it is probably better to make a substitution so that the algebra can be done without interference by the trigonometric notation.

28 ALGEBRAIC SUBSTITUTION:
1. Substitute u to obtain a purely algebraic equation.
2. Solve the algebraic equation — it may have more than one solution.
3. Solve each of the resulting trigonometric equations.

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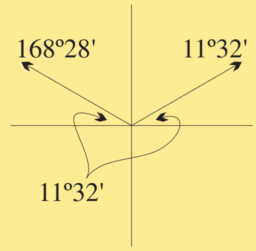
WORKED EXERCISE: Solve $5\sin^2 x = \sin x$, for $0° \le x \le 360°$. Give the exact value of the solutions if possible, otherwise approximate to the nearest minute.

SOLUTION: Let $u = \sin x$.



Then $5u^2 = u$

$\boxed{-u}$ $5u^2 - u = 0$

$u(5u-1) = 0$

$u = 0$ or $u = \frac{1}{5}$,

so $\sin x = 0$ or $\sin x = \frac{1}{5}$.

Using the graph of $y = \sin x$ to solve $\sin x = 0$
and the quadrant-diagram method to solve $\sin x = \frac{1}{5}$,

$x = 0°,\ 180°$ or $360°$, or $x \doteqdot 11°32'$ or $168°28'$.

WORKED EXERCISE: Solve $\dfrac{4}{\cos x} - \cos x = 0$, for $-180° \le x \le 180°$.

SOLUTION: Let $u = \cos x$.

Then $\dfrac{4}{u} - u = 0$

$\boxed{\times u}$ $4 - u^2 = 0$

$u = 2$ or $u = -2$,

so $\cos x = 2$ or $\cos x = -2$.

Neither equation has a solution, because $\cos x$ lies between -1 and 1,
so there are no solutions.

Equations with More than One Trigonometric Function:

Often a trigonometric equation will involve more than one trigonometric function, as, for example,

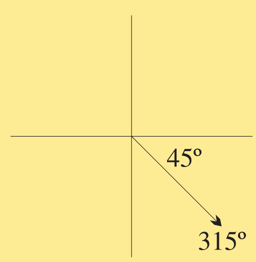
$$\sec^2 x + \tan x = 1, \text{ where } 180° \le x \le 360°.$$

29 EQUATIONS WITH MORE THAN ONE TRIGONOMETRIC FUNCTION: Usually use trigonometric identities to produce an equation in only one trigonometric function, then proceed by substitution as before.
If all else fails, reduce everything to sines and cosines, and hope for the best!

WORKED EXERCISE: Solve $\sec^2 x + \tan x = 1$, where $180° \le x \le 360°$ (as above).

SOLUTION: Recognizing that $\sec^2 x = 1 + \tan^2 x$, the equation becomes

$1 + \tan^2 x + \tan x = 1$, where $180° \le x \le 360°$



$\boxed{-1}$ $\tan^2 x + \tan x = 0$,

$\tan x(\tan x + 1) = 0$,

so $\tan x = 0$ or $\tan x = -1$.

Using the graph of $y = \tan x$ to solve $\tan x = 0$,
and the quadrant-diagram method to solve $\tan x = -1$,

$x = 180°,\ 360°$ or $315°$.

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Homogeneous Equations: One special sort of equation which occurs quite often is called *homogeneous* in $\sin x$ and $\cos x$ because the sum of the indices of $\sin x$ and $\cos x$ in each term is the same. For example, the following equation is homogeneous of degree 2 in $\sin x$ and $\cos x$:

$$\sin^2 x - 3\sin x\cos x + 2\cos^2 x = 0, \quad \text{for } 0^\circ \le x \le 180^\circ.$$

30 HOMOGENEOUS EQUATIONS: To solve a homogeneous equation in $\sin x$ and $\cos x$, divide through by a power of $\cos x$ to produce an equation in $\tan x$.

WORKED EXERCISE: Continuing with the example above,

$\boxed{\div \cos^2 x}$ $\tan^2 x - 3\tan x + 2 = 0.$

Let $u = \tan x$,

then

$$\begin{aligned} u^2 - 3u + 2 &= 0 \\ (u-2)(u-1) &= 0 \\ u = 2 \ \text{ or } \ u &= 1 \\ \tan x = 2 \ \text{ or } \ \tan x &= 1. \end{aligned}$$

So $x \doteqdot 63^\circ 26'$ or $x = 45^\circ$.

Exercise 4G

1. Solve each of these equations for $0^\circ \le \theta \le 360^\circ$ (each related angle is 30°, 45° or 60°):

(a) $\sin\theta = \dfrac{\sqrt{3}}{2}$
(b) $\tan\theta = 1$
(c) $\cos\theta = -\dfrac{1}{\sqrt{2}}$
(d) $\tan\theta = -\sqrt{3}$
(e) $\operatorname{cosec}\theta = -2$
(f) $\sec\theta = -\dfrac{2}{\sqrt{3}}$

2. Solve each of these equations for $0^\circ \le \theta \le 360^\circ$ (the trigonometric graphs are helpful here):

(a) $\sin\theta = 1$
(b) $\cos\theta = -1$
(c) $\cos\theta = 0$
(d) $\sec\theta = 1$
(e) $\tan\theta = 0$
(f) $\cot\theta = 0$

3. Solve each of these equations for $0^\circ \le x \le 360^\circ$. Use your calculator to find the related angle in each case, and give solutions correct to the nearest degree.

(a) $\cos x = \frac{3}{7}$
(b) $\sin x = 0{\cdot}1234$
(c) $\tan x = -7$
(d) $\cot x = -0{\cdot}45$
(e) $\operatorname{cosec} x = -\frac{3}{2}$
(f) $\sec x = 6$

4. Solve each of these equations for α in the given domain. Give solutions correct to the nearest minute where necessary:

(a) $\sin\alpha = 0{\cdot}1$, $0^\circ \le \alpha \le 360^\circ$
(b) $\cos\alpha = -0{\cdot}1$, $0^\circ \le \alpha \le 360^\circ$
(c) $\tan\alpha = -1$, $-180^\circ \le \alpha \le 180^\circ$
(d) $\operatorname{cosec}\alpha = -1$, $0^\circ \le \alpha \le 360^\circ$
(e) $\sin\alpha = 3$, $0^\circ \le \alpha \le 360^\circ$
(f) $\sec\alpha = \sqrt{2}$, $0^\circ \le \alpha \le 360^\circ$
(g) $\cos\alpha = 0$, $-180^\circ \le \alpha \le 180^\circ$
(h) $\cot\alpha = \frac{1}{2}$, α reflex
(i) $\sqrt{3}\tan\alpha + 1 = 0$, α obtuse
(j) $\operatorname{cosec}\alpha + 2 = 0$, α reflex
(k) $2\cos\alpha - 1 = 0$, $0^\circ \le \alpha \le 360^\circ$
(l) $\cot\alpha = 3$, $0^\circ \le \alpha \le 360^\circ$
(m) $\tan\alpha = 0$, $-360^\circ \le \alpha \le 360^\circ$
(n) $\tan\alpha = -0{\cdot}3$, $-180^\circ \le \alpha \le 180^\circ$
(o) $\sin\alpha = -0{\cdot}7$, $0^\circ \le \alpha \le 720^\circ$
(p) $\tan\alpha = 1 - \sqrt{2}$, $0^\circ \le \alpha \le 360^\circ$

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DEVELOPMENT

5. Solve for $0^\circ \le \theta \le 360^\circ$, giving solutions correct to the nearest degree where necessary:
(a) $\cos^2\theta = 1$ (b) $\sec^2\theta = \frac{4}{3}$ (c) $\tan^2\theta = 9$ (d) $\operatorname{cosec}^2\theta = 2$

6. Solve for $0^\circ \le x \le 360^\circ$ (let u be the compound angle):
(a) $\sin 2x = \frac{1}{2}$ (b) $\cos 2x = -\frac{1}{\sqrt{2}}$ (c) $\tan 3x = \sqrt{3}$ (d) $\sec 3x = 0$

7. Solve for $0^\circ \le \alpha \le 360^\circ$ (let u be the compound angle):
(a) $\tan(\alpha - 45^\circ) = \frac{1}{\sqrt{3}}$
(b) $\sin(\alpha + 30^\circ) = -\frac{\sqrt{3}}{2}$
(c) $\cot(\alpha + 60^\circ) = 1$
(d) $\operatorname{cosec}(\alpha - 75^\circ) = -2$

8. Solve for $0^\circ \le \theta \le 360^\circ$:
(a) $\sin\theta = \cos\theta$
(b) $\sqrt{3}\sin\theta + \cos\theta = 0$
(c) $4\sin\theta = 3\operatorname{cosec}\theta$
(d) $\sec\theta - 2\cos\theta = 0$

9. Solve for $0^\circ \le \theta \le 360^\circ$, giving solutions correct to the nearest minute where necessary:
(a) $\cos^2\theta - \cos\theta = 0$
(b) $\cot^2\theta = \sqrt{3}\cot\theta$
(c) $2\sin\theta\cos\theta = \sin\theta$
(d) $\tan^2\theta - \tan\theta - 2 = 0$
(e) $2\sin^2\theta - \sin\theta = 1$
(f) $\sec^2\theta + 2\sec\theta = 8$
(g) $3\cos^2\theta + 5\cos\theta = 2$
(h) $4\operatorname{cosec}^2\theta - 4\operatorname{cosec}\theta - 15 = 0$
(i) $4\sin^3\theta = 3\sin\theta$

10. Solve for $0^\circ \le x \le 360^\circ$, giving solutions correct to the nearest minute where necessary:
(a) $2\sin^2 x + \cos x = 2$
(b) $\sec^2 x - 2\tan x - 4 = 0$
(c) $8\cos^2 x = 2\sin x + 7$
(d) $6\tan^2 x = 5\sec x$
(e) $6\operatorname{cosec}^2 x = \cot x + 8$

11. Solve for $0^\circ \le \alpha \le 360^\circ$, giving solutions correct to the nearest minute where necessary:
(a) $3\sin\alpha = \operatorname{cosec}\alpha + 2$
(b) $3\tan\alpha - 2\cot\alpha = 5$

12. Solve for $0^\circ \le A \le 360^\circ$, giving solutions correct to the nearest minute where necessary:
(a) $\cot A + 4\tan A = 4\operatorname{cosec} A$
(b) $3(\tan A + \sec A) = 2\cot A$

13. Solve for $0^\circ \le x \le 360^\circ$, giving solutions correct to the nearest minute where necessary:
(a) $\cos x\tan x + \tan x = \cos x + 1$
(b) $6\sin x\cos x + 3\sin x = 2\cos x + 1$

14. Solve each of these homogeneous equations for $0^\circ \le x \le 360^\circ$ by dividing both sides by a suitable power of $\cos x$. Give solutions to the nearest minute where necessary.
(a) $\sin x = 3\cos x$
(b) $\sin^2 x - 2\sin x\cos x - 8\cos^2 x = 0$
(c) $5\sin^2 x + 8\sin x\cos x = 4\cos^2 x$
(d) $\sin^3 x + 2\sin^2 x\cos x + \sin x\cos^2 x = 0$

EXTENSION

15. Solve for $0^\circ \le \theta \le 360^\circ$, giving solutions correct to the nearest minute where necessary:
(a) $4\cos^2\theta + 2\sin\theta = 3$
(b) $5\sec^2\theta + 7\tan\theta = 7$
(c) $\cos^2\theta - 8\sin\theta\cos\theta + 3 = 0$
(d) $5\sin^2\theta - 4\sin\theta\cos\theta + 3\cos^2\theta = 2$
(e) $8\cos^4\theta - 10\cos^2\theta + 3 = 0$
(f) $\sqrt{6}\cos\theta + \sqrt{2}\sin\theta + \sqrt{3}\cot\theta + 1 = 0$
(g) $20\cot\theta + 15\cot\theta\operatorname{cosec}\theta - 4\operatorname{cosec}\theta = 3(1 + \cot^2\theta)$
(h) $\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} + \cos\theta = 0$
(i) $(\sqrt{3} + 1)\cos^2\theta - 1 = (\sqrt{3} - 1)\sin\theta\cos\theta$
(j) $\dfrac{1 + 2\sin^2\theta}{\cos^2\theta} + 4\tan\theta = 0$

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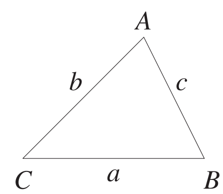
4 H The Sine Rule and the Area Formula

The sine rule, the area rule and the cosine rule belong both to trigonometry and to geometry. On the one hand, they extend the elementary trigonometry of Section 4A to non-right-angled triangles. On the other hand, they generalise Pythagoras' theorem, the isosceles triangle theorem, and some results about altitudes of triangles. They are also closely related to the four standard congruence tests, and the sine rule can be restated as a theorem about the diameter of the circumcircle of a triangle. These last three sections review the rules and their applications. Their proofs should now be given more attention, particularly because they involve connections between trigonometry and Euclidean geometry.

Statement of the Sine Rule: We will often use the convention that each side of a triangle is given the lower-case letter of the opposite vertex, as in the diagram on the right. Using that convention, here are the verbal and symbolic statements of the sine rule.

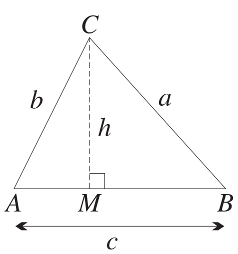


31 THEOREM — THE SINE RULE: In any triangle, the ratio of each side to the sine of the opposite angle is constant. That is, in any triangle $\triangle ABC$,

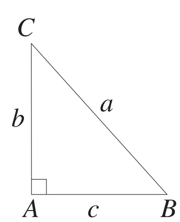
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\,.$$

Proving the Sine Rule by Constructing an Altitude: So far we can only handle right triangles, so any proof of the sine rule must involve a construction with a right angle. The obvious approach is to construct an *altitude*, which is the perpendicular from one vertex to the opposite side.

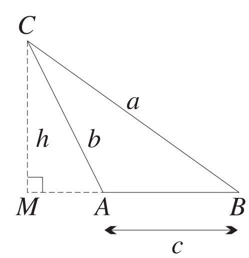
GIVEN: Let ABC be any triangle. There are three cases, depending on whether $\angle A$ is an acute angle, a right angle, or an obtuse angle.



CASE 1: $\angle A$ is acute



CASE 2: $\angle A = 90^\circ$



CASE 3: $\angle A$ is obtuse

AIM: To prove that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

In case 2, $\sin A = \sin 90^\circ = 1$, and $\sin B = \dfrac{b}{a}$, so the result is clear.

CONSTRUCTION: In the remaining cases 1 and 3, construct the altitude from C, meeting AB, produced if necessary, at M. Let h be the length of CM.

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PROOF:

CASE 1 — Suppose that $\angle A$ is acute.

In the triangle ACM, $\dfrac{h}{b} = \sin A$

$\boxed{\times b}$ $\qquad h = b\sin A.$

In the triangle BCM, $\dfrac{h}{a} = \sin B$

$\boxed{\times a}$ $\qquad h = a\sin B.$

Equating these, $b\sin A = a\sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

CASE 3 — Suppose that $\angle A$ is obtuse.

In the triangle ACM, $\dfrac{h}{b} = \sin(180^\circ - A)$,

and since $\sin(180^\circ - A) = \sin A$,

$\boxed{\times b}$ $\qquad h = b\sin A.$

In the triangle BCM, $\dfrac{h}{a} = \sin B$

$\boxed{\times a}$ $\qquad h = a\sin B.$

Equating these, $b\sin A = a\sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

The Area Formula: The well-known formula area $= \frac{1}{2} \times \text{base} \times \text{height}$ can be generalised to a formula involving two sides and the included angle.

32 THEOREM — THE AREA FORMULA: The area of a triangle is half the product of any two sides and the sine of the included angle. That is,

$$\text{area } \triangle ABC = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B = \tfrac{1}{2}ab\sin C.$$

PROOF: We use the same diagrams as in the proof of the sine rule.

In case 2, $\angle A = 90^\circ$ and $\sin A = 1$, so area $= \frac{1}{2}bc = \frac{1}{2}bc\sin A$, as required.

Otherwise, area $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times AB \times h$

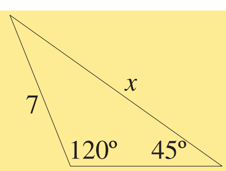
$= \frac{1}{2} \times c \times b\sin A$, since we proved before that $h = b\sin A$.

Using the Sine Rule to Find a Side — The AAS Congruence Situation: For the sine rule to be applied to the problem of finding a side, one side and two angles must be known. This is the situation described by the AAS congruence test, so only one triangle will be possible. The sine rule should be learned in verbal form because the triangle being solved could have any names, or could be unnamed.

33 USING THE SINE RULE TO FIND A SIDE: In the AAS congruence situation:

$$\frac{\text{unknown side}}{\text{sine of opposite angle}} = \frac{\text{known side}}{\text{sine of opposite angle}}.$$

WORKED EXERCISE: Find x in the given triangle.



SOLUTION:

$$\frac{x}{\sin 120^\circ} = \frac{7}{\sin 45^\circ}$$

$\boxed{\times \sin 120^\circ}$

$$x = \frac{7\sin 120^\circ}{\sin 45^\circ}$$

$$= 7 \times \frac{\sqrt{3}}{2} \times \sqrt{2}$$

$$= \tfrac{7}{2}\sqrt{6}$$

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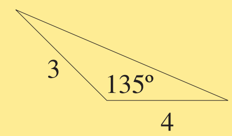
Using the Area Formula — The SAS Congruence Situation: The area formula requires the SAS situation where two sides and the included angle are known.

34 USING THE AREA FORMULA: In the SAS congruence situation:

$$\text{area} = (\text{half the product of two sides}) \times (\text{sine of the included angle})\,.$$

WORKED EXERCISE: Find the area of the given triangle.

SOLUTION:
$$\begin{aligned}\text{Area} &= \frac{1}{2} \times 3 \times 4 \times \sin 135^\circ \\ &= 6 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= 3\sqrt{2}\ \text{square units.}\end{aligned}$$



Using the Sine Rule to Find an Angle — The Ambiguous ASS Situation: It is well known that the SAS congruence test requires that the angle be included between the two sides. When two sides and a non-included angle are known, the situation is normally referred to as 'the spurious ASS test' or 'the ambiguous ASS test', because in many such situations the resulting triangle is not quite determined up to congruence, and two triangles may be possible.

When the sine rule is applied in the ASS situation, there is only one answer for the sine of an angle. Angles in triangles, however, can be acute or obtuse, and the sines of both acute and obtuse angles are positive, so there may be two possible solutions for the angle itself.

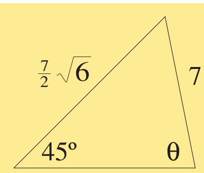
35 USING THE SINE RULE TO FIND AN ANGLE: If two sides and a non-included angle of the triangle are known, corresponding to the ambiguous ASS situation, then:

$$\frac{\text{sine of unknown angle}}{\text{opposite side}} = \frac{\text{sine of known angle}}{\text{opposite side}}\,.$$

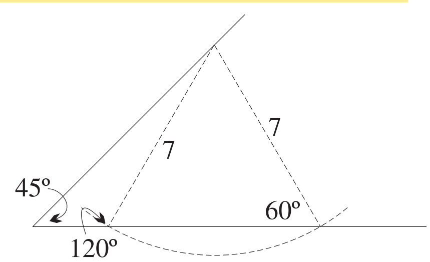
Always check the angle sum to see whether both answers are possible.

WORKED EXERCISE: Find θ in the given triangle.

SOLUTION:
$$\begin{aligned}\frac{\sin\theta}{\frac{7}{2}\sqrt{6}} &= \frac{\sin 45^\circ}{7} \\ \sin\theta &= \tfrac{7}{2}\sqrt{6} \times \frac{1}{7\sqrt{2}} \\ \sin\theta &= \tfrac{1}{2}\sqrt{3}\,,\end{aligned}$$
so $\theta = 60^\circ$ or 120°.



NOTE: There are *two* angles whose sine is $\dfrac{\sqrt{3}}{2}$, one acute and one obtuse. Moreover, $120^\circ + 45^\circ = 165^\circ$, leaving just 15° for the third angle in the obtuse case, so it all seems to work. Opposite is the ruler and compasses construction of the triangle, showing how two different triangles can be produced from the same given ASS measurements.



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In many examples, however, the obtuse angle solution can be excluded by using the fact that the angle sum of the triangle cannot exceed 180°. In particular:

1. In a right triangle, both the other angles must be acute, giving rise to the well-known RHS congruence test.
2. If one angle is obtuse, then both other angles are acute. Hence there is a valid 'OSS' congruence test, which applies to the situation where two sides and a non-included obtuse angle are known.

WORKED EXERCISE: Find θ in the given triangle.

SOLUTION:

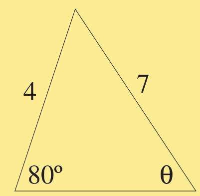
$$\frac{\sin\theta}{4} = \frac{\sin 80^\circ}{7}$$

$$\sin\theta = \frac{4\sin 80^\circ}{7}$$

$$\theta \doteqdot 34^\circ 15' \text{ or } 145^\circ 45'$$

But $\theta \doteqdot 145^\circ 45'$ is impossible, because the angle sum would then exceed 180°,

so $\theta \doteqdot 34^\circ 15'$.



The Sine Rule and the Circumcircle: Now that the three ratios $\dfrac{a}{\sin A}$, $\dfrac{b}{\sin B}$ and $\dfrac{c}{\sin C}$ have been proven all to be equal, we obviously should be asking *what are they equal to*?

First, $\sin A$, $\sin B$ and $\sin C$ are all pure numbers, so the ratios $\dfrac{a}{\sin A}$, $\dfrac{b}{\sin B}$ and $\dfrac{c}{\sin C}$, being lengths over numbers, must all be lengths. Secondly, the sine function cannot exceed 1, so each ratio $\dfrac{a}{\sin A}$, $\dfrac{b}{\sin B}$ and $\dfrac{c}{\sin C}$ is a length greater than or equal to each of the sides.

The following theorem shows that the common value of these three ratios is the diameter of the *circumcircle*, which is the circle passing through all three vertices. This provides an alternative and far more enlightening proof of the sine rule, clearly illustrating connections between trigonometry and the geometry of circles.

36 THEOREM — THE SINE RULE AND THE CIRCUMCIRCLE: In any triangle, the ratio of each side to the sine of the opposite angle is constant, and this constant is equal to the diameter of the circumcircle of the triangle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of the circumcircle}\,.$$

GIVEN: Let O be the centre of the circumcircle of $\triangle ABC$. Let d be the diameter of the circumcircle, and let $\angle A = \alpha$. There are three cases, according as to whether α is acute, obtuse, or a right angle.

AIM: To prove that $\dfrac{a}{\sin\alpha} = d$.

In case 2, $a = d$, and also $\sin\alpha = \sin 90^\circ = 1$ (angle in a semicircle).

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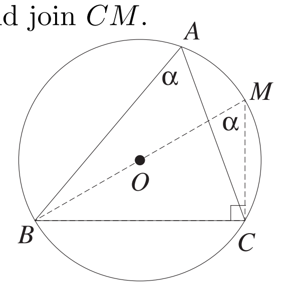
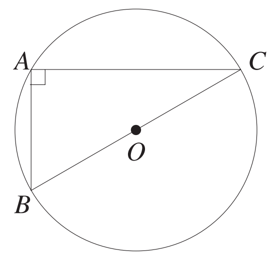
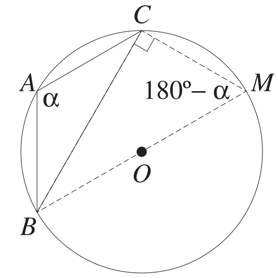
CONSTRUCTION: In the remaining cases 1 and 3, construct the diameter BOM, and join CM.



CASE 1: α is acute



CASE 2: $\alpha = 90^\circ$

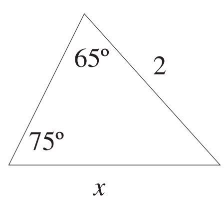


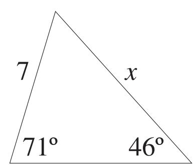
CASE 3: α is obtuse

PROOF: In case 1, $\angle M = \alpha$ (angles on the same arc BC),
and in case 3, $\angle M = 180^\circ - \alpha$ (cyclic quadrilateral $BMCA$).
In both cases, $\sin \angle M = \sin \alpha$, since $\sin(180^\circ - \alpha) = \sin \alpha$.
Also $\angle BCM = 90^\circ$ (angle in a semicircle),
so in $\triangle BCM$, $\dfrac{a}{d} = \sin \alpha$, so that $\dfrac{a}{\sin \alpha} = d$, as required.

Exercise 4H

1. Find x, correct to one decimal place, in each triangle:

(a)


(b)


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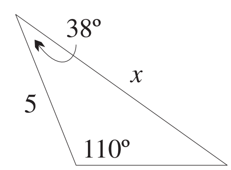
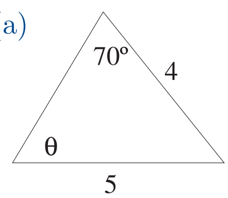
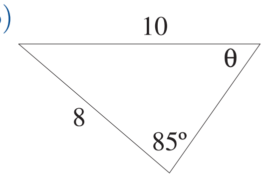
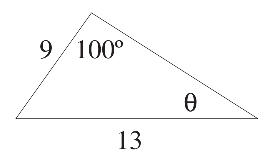


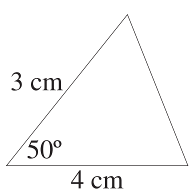
2. Find θ, correct to the nearest degree, in each triangle:

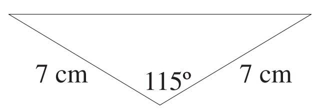
(a)


(b)


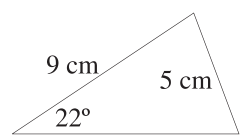
(c)


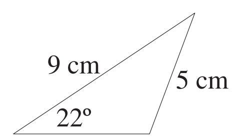
3. Find the area of each triangle, correct to the nearest square centimetre:

(a)


(b)


4. There are two triangles that have sides 9 cm and 5 cm, and in which the angle opposite the 5 cm side is 22°. Find, in each case, the size of the angle opposite the 9 cm side (answer correct to the nearest minute).





5. Sketch $\triangle ABC$ in which $a = 2{\cdot}8$ cm, $b = 2{\cdot}7$ cm and $A = 52^\circ 21'$.
(a) Find B, holding the answer in memory, but writing it correct to the nearest minute.
(b) Hence find C, correct to the nearest minute, but hold the answer on the screen.
(c) Hence find the area of $\triangle ABC$ in cm^2, correct to two decimal places.

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6. Sketch $\triangle PQR$ in which $p = 7$ units, $q = 15$ units and $\angle P = 25°50'$.

(a) Find the two possible sizes of $\angle Q$, correct to the nearest minute.

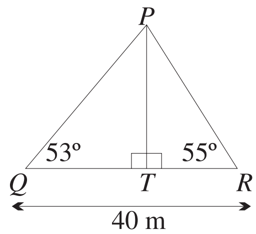
(b) For each of the possible sizes of $\angle Q$, find r, correct to one decimal place.

7. Find all the unknown sides (to one decimal place) and angles (to the nearest minute) of $\triangle ABC$ if $A = 40°$, $a = 7{\cdot}6$ and $b = 10{\cdot}5$.

8. In $\triangle PQR$, $\angle Q = 53°$, $\angle R = 55°$ and $QR = 40$ metres. T is the point on QR such that $PT \perp QR$.

(a) By using the sine rule in the triangle PQR, show that $PQ = \dfrac{40 \sin 55°}{\sin 72°}$.

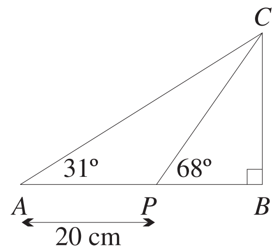
(b) Hence use $\triangle PQT$ to find PT, correct to the nearest metre.



9. In $\triangle ABC$, $\angle B = 90°$ and $\angle A = 31°$. P is a point on AB such that $AP = 20$ cm and $\angle CPB = 68°$.

(a) Show that $PC = \dfrac{20 \sin 31°}{\sin 37°}$.

(b) Hence find PB, correct to the nearest centimetre.

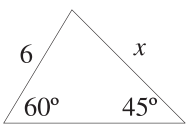


DEVELOPMENT

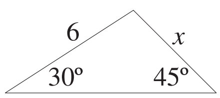
10. In $\triangle ABC$, $\sin A = \frac{1}{4}$, $\sin B = \frac{2}{3}$ and $a = 12$. Find the value of b.

11. Find the exact value of x in each diagram:

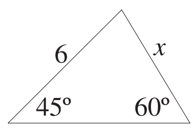
(a)



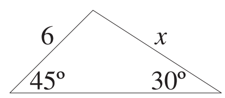
(b)



(c)



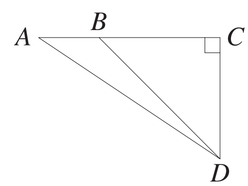
(d)



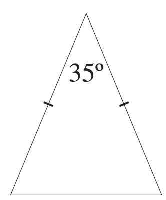
12. The points A, B and C lie on a horizontal line and D lies directly below C. The angles of depression of D from A and B are 34° and 62° respectively, and $AB = 75{\cdot}4$ metres.

(a) Show that $CD = \dfrac{75{\cdot}4 \sin 34° \sin 62°}{\sin 28°}$.

(b) Hence find the height of C above D in metres, correct to one decimal place.



13. The vertical angle of an isosceles triangle is 35°, and its area is $35\,\text{cm}^2$. Find the length of the equal sides, correct to the nearest millimetre.



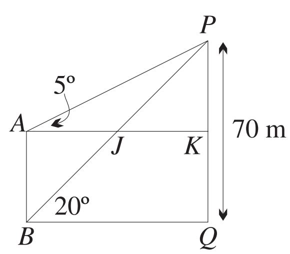
14. Two towers AB and PQ stand on level ground. The angles of elevation of the top of the taller tower from the top and bottom of the shorter tower are 5° and 20° respectively. The height of the taller tower is 70 metres.

(a) Explain why $\angle APJ = 15°$.

(b) Show that $AB = \dfrac{BP \sin 15°}{\sin 95°}$.

(c) Show that $BP = \dfrac{70}{\sin 20°}$.

(d) Hence find the height of the shorter tower, correct to the nearest metre.



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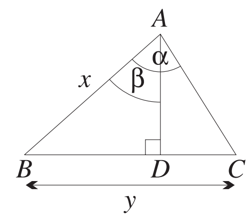
15. In the diagram opposite, $\angle BAC = \alpha$, $\angle BAD = \beta$, $BC = y$ and $AB = x$.

(a) Show that $y = \dfrac{x \sin \alpha}{\sin C}$.

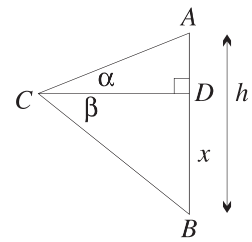
(b) Hence show that $y = \dfrac{x \sin \alpha}{\cos(\alpha - \beta)}$.



16. In the diagram opposite, $\angle ACD = \alpha$, $\angle BCD = \beta$, $AB = h$ and $BD = x$.

(a) Show that $BC = \dfrac{h \cos \alpha}{\sin(\alpha + \beta)}$.

(b) Hence show that $x = \dfrac{h \sin \beta \cos \alpha}{\sin(\alpha + \beta)}$.

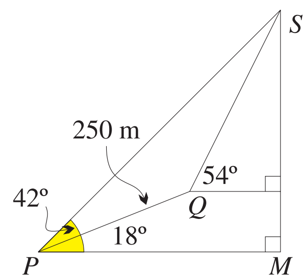


17. The summit S of a mountain is observed from two points P and Q 250 metres apart. PQ is inclined at 18° to the horizontal and the respective angles of elevation of S from P and Q are 42° and 54°.

(a) Explain why $\angle PSQ = 12°$ and $\angle PQS = 144°$.

(b) Show that $SP = \dfrac{250 \sin 144°}{\sin 12°}$.

(c) Hence find the vertical height SM, correct to the nearest metre.

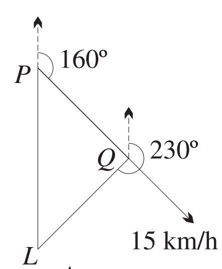


18. A ship is sailing at 15 km/h on a bearing of 160°. At 9:00 am it is at P, and lighthouse L is due south. At 9:40 am it is at Q, and the lighthouse is on a bearing of 230°.

(a) Show that $\angle PQL = 110°$.

(b) Find the distance PL, correct to the nearest km.

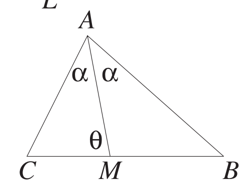
(c) Find the time, to the nearest minute, at which the lighthouse will be due west of the ship.



19. In a triangle ABC, the bisector of angle A meets the opposite side BC at M. Let $\alpha = \angle CAM = \angle BAM$, and let $\theta = \angle CMA$.

(a) Explain why $\sin \angle BMA = \sin \theta$.

(b) Hence show that $AC : AB = MC : MB$.

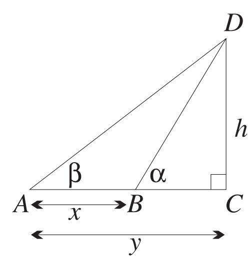


20. (a) Show that $h = \dfrac{x \sin \alpha \sin \beta}{\sin(\alpha - \beta)}$ in the diagram opposite.

(b) Use the fact that $\tan \alpha = \dfrac{h}{y - x}$ and $\tan \beta = \dfrac{h}{y}$ to show that $h = \dfrac{x \tan \alpha \tan \beta}{\tan \alpha - \tan \beta}$.

(c) Combine the expressions in parts (a) and (b) to show that $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$.

(d) Hence find the exact value of $\sin 15°$.



21. Suppose that the sine rule is being used in an ASS situation to find an angle θ in a triangle, and that $\sin \theta$ has been found. Explain why there is only one solution for θ if and only if $\theta = 90°$ or the related angle of θ is less than the known angle.

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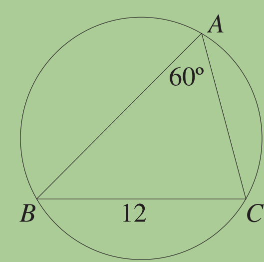
EXTENSION

22. Two ships P and Q are observed to be NW and NE respectively of a port A. From a second port B, which is 1 km due east of A, the ships P and Q are observed to be WNW and NNE respectively. Show that the two ships are approximately 2·61 km apart.

23. [The circumcircle] In one of the proofs of the sine rule, we saw that $\dfrac{a}{\sin A}$, $\dfrac{b}{\sin B}$ and $\dfrac{c}{\sin C}$ are each equal to the diameter of the circumcircle of $\triangle ABC$.

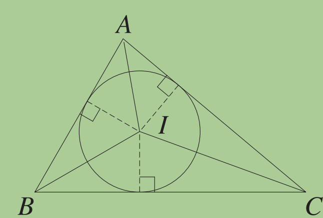


(a) In $\triangle ABC$, $\angle A = 60°$ and $BC = 12$. Find the diameter D_C of the circumcircle.

(b) The triangle $\triangle ABC$ in part (a) is not determined up to congruence. Why does the diameter of the circumcircle nevertheless remain constant as the triangle varies?

(c) A triangle $\triangle PQR$ with $\angle RPQ = 150°$ is inscribed inside a circle of diameter D_C. Find the ratio $D_C : RQ$.

24. [The circumcircle and the incircle]



(a) Let Δ be the area of $\triangle ABC$, and let D_C diameter of its circumcircle. Show that $D_C = \dfrac{abc}{2\Delta}$.

(b) The *incircle* of triangle $\triangle ABC$ is the circle drawn inside the triangle and tangent to each side, as shown in the diagram. Let I, the *incentre*, be the centre of the incircle, and let D_I be its diameter. Find the area of each of the three triangles $\triangle AIB$, $\triangle BIC$ and $\triangle CIA$. Hence show that $D_I = \dfrac{2\Delta}{s}$, where $s = \frac{1}{2}(a+b+c)$ is the *semiperimeter* of the triangle.

(c) Hence find the ratio $D_C : D_I$ and the product $D_C D_I$.

(d) Find also $\dfrac{\text{area of triangle}}{\text{area of circumcircle}}$ and $\dfrac{\text{area of triangle}}{\text{area of incircle}}$.

4 I The Cosine Rule

The cosine rule is a generalisation of Pythagoras' theorem to non-right-angled triangles, because it gives a formula for the square of any side in terms of the squares of the other two sides and the cosine of the opposite angle. The proof is based on Pythagoras' theorem, and again begins with the construction of an altitude.

37 THEOREM — THE COSINE RULE: The square of any side of a triangle equals the sum of the squares of the other two sides minus twice the product of those sides and the cosine of their included angle:

$$a^2 = b^2 + c^2 - 2bc\cos A\,.$$

GIVEN: Let ABC be any triangle. Again, there are three cases, according as to whether α is acute, obtuse, or a right angle.

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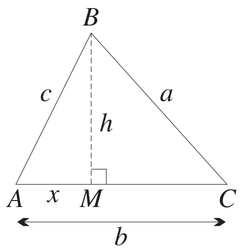
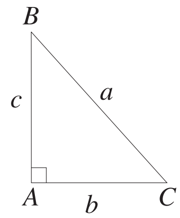
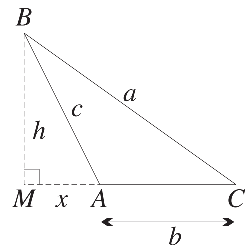


CASE 1: $\angle A$ is acute



CASE 2: $\angle A = 90^\circ$



CASE 3: $\angle A$ is obtuse

AIM: To prove that $a^2 = b^2 + c^2 - 2bc\cos A$.
In case 2, $\cos A = 0$, and this is just Pythagoras' theorem.

CONSTRUCTION: In the remaining cases 1 and 3, construct the altitude from B, meeting AC, produced if necessary, at M. Let $BM = h$ and $AM = x$.

PROOF:

CASE 1 — Suppose that $\angle A$ is acute.
By Pythagoras' theorem in $\triangle BMC$,
$$a^2 = h^2 + (b - x)^2.$$
By Pythagoras' theorem in $\triangle BMA$,
$$h^2 = c^2 - x^2,$$
$$\begin{aligned} \text{so } a^2 &= c^2 - x^2 + (b - x)^2 \\ &= c^2 - x^2 + b^2 - 2bx + x^2 \\ &= b^2 + c^2 - 2bx. \qquad (*) \end{aligned}$$
Using trigonometry in $\triangle ABM$,
$$x = c\cos A.$$
So $a^2 = b^2 + c^2 - 2bc\cos A$.

CASE 3 — Suppose that $\angle A$ is obtuse.
By Pythagoras' theorem in $\triangle BMC$,
$$a^2 = h^2 + (b + x)^2.$$
By Pythagoras' theorem in $\triangle BMA$,
$$h^2 = c^2 - x^2,$$
$$\begin{aligned} \text{so } a^2 &= c^2 - x^2 + (b + x)^2 \\ &= c^2 - x^2 + b^2 + 2bx + x^2 \\ &= b^2 + c^2 + 2bx. \qquad (*) \end{aligned}$$
Using trigonometry in $\triangle ABM$,
$$\begin{aligned} x &= c\cos(180^\circ - A) \\ &= -c\cos A. \end{aligned}$$
So $a^2 = b^2 + c^2 - 2bc\cos A$.

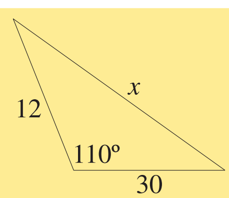
NOTE: The identity $\cos(180^\circ - A) = -\cos A$ is the key step in Case 3 of the proof. The cosine rule appears in Euclid's geometry book, but without any mention of the cosine ratio — the form given there is approximately the two statements in the proof marked with $(*)$.

Using the Cosine Rule to Find a Side — The SAS Situation: For the cosine rule to be applied to find a side, two sides and the included angle must be known, which is the SAS congruence situation.

38 USING THE COSINE RULE TO FIND A SIDE: In the SAS congruence situation:

square of any side = (sum of squares of other two sides)
− (twice the product of those sides) × (cosine of their included angle).

WORKED EXERCISE: Find x in the given triangle.



SOLUTION:
$$\begin{aligned} x^2 &= 12^2 + 30^2 - 2 \times 12 \times 30 \times \cos 110^\circ \\ &= 144 + 900 - 720\cos 110^\circ \\ &= 1044 + 720\cos 70^\circ \end{aligned}$$
So $x \doteqdot 35{\cdot}92$.

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Using the Cosine Rule to Find an Angle — the SSS Situation: Solving the cosine rule above for $\cos A$ gives:

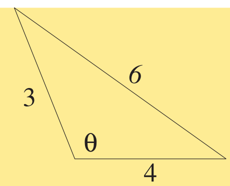
39 THE COSINE RULE WITH $\cos A$ AS SUBJECT: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Application of the cosine rule to find an angle requires that all three sides are known, which is the SSS congruence test. To use the rule, one can either substitute each time into the usual form of the cosine rule, or remember it verbally.

40 USING THE COSINE RULE TO FIND AN ANGLE: In the SSS congruence situation:

$\cos\theta =$ (sum of squares of two including sides) − (square of opposite side)
all divided by (twice the product of the including sides).

WORKED EXERCISE: Find θ in the given triangle.



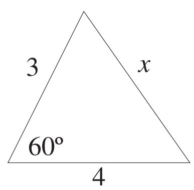
SOLUTION: $\cos\theta = \dfrac{3^2 + 4^2 - 6^2}{2 \times 3 \times 4}$

$= \dfrac{-11}{24}$

so $\theta \doteqdot 117^\circ 17'$.

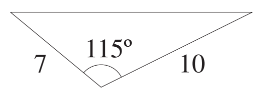
Exercise 4I

1. (a)



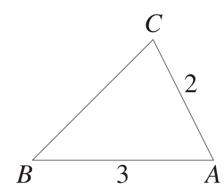
Find the length x as a surd.

(b)



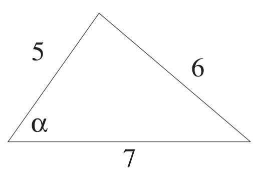
Find the unknown side to two decimal places.

(c)



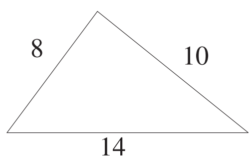
If $\cos A = \frac{1}{4}$, find the exact value of a.

2. (a)



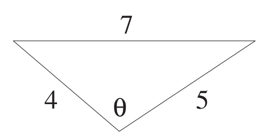
Find the angle α correct to the nearest degree.

(b)



Find the largest angle of the given triangle, to the nearest minute.

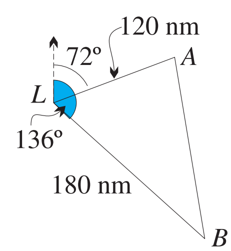
(c)



Find the value of $\cos\theta$.

3. P, Q and R are landmarks. It is known that R is 8·7 km from P and 9·3 km from Q, and that $\angle PRQ = 79^\circ 32'$. Find, in kilometres correct to one decimal place, the distance between P and Q.

4. Ship A is 120 nautical miles from a lighthouse L on a bearing of 72°, while ship B is 180 nautical miles from L on a bearing of 136°. Calculate the distance between the two ships, correct to the nearest nautical mile.



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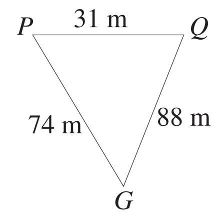
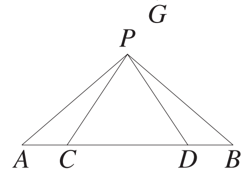
5. A golfer at G wishes to hit a shot between two trees P and Q. The trees are 31 metres apart, and the golfer is 74 metres from P and 88 metres from Q. Find the angle within which the golfer must play the shot (answer to the nearest degree).



6. Roof trusses AP, BP, CP and DP are nailed to a horizontal beam AB, as shown in the diagram opposite. Given that $AP = BP = 7{\cdot}2$ metres, $CP = DP = 5{\cdot}5$ metres, $AB = 10{\cdot}6$ metres and $CD = 7{\cdot}4$ metres, find, correct to the nearest minute:

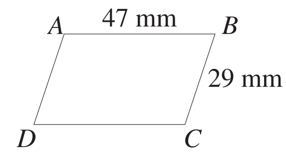
(a) $\angle APB$ (b) $\angle CPD$



7. A parallelogram $ABCD$ has sides $AB = DC = 47$ mm and $AD = BC = 29$ mm. The longer diagonal BD is 60 mm.

(a) Use the cosine rule to find the size of $\angle BCD$.

(b) Use cointerior angles on parallel lines to find the size of $\angle ABC$ (give each answer correct to the nearest minute).



DEVELOPMENT

8. In $\triangle ABC$, $a = 31$ units, $b = 24$ units and $\cos C = \frac{59}{62}$. Show that:

(a) $c = 11$ units (b) $A = 120°$

9. The sides of a triangle are in the ratio 5 : 16 : 19. Find the smallest angle of the triangle, correct to the nearest minute.

10. In $\triangle PQR$, $p = 5\sqrt{3}$ cm, $q = 11$ cm and $R = 150°$. Find: (a) r (b) $\cos P$

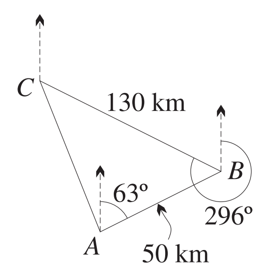
11. In $\triangle ABC$, $a = 4$ cm, $b = 5$ cm and $c = 6$ cm. Find $\cos A$, $\cos B$ and $\cos C$, and hence show that $6\cos A \cos C = \cos B$.

12. A ship sails 50 km from port A to port B on a bearing of 63°, then sails 130 km from port B to port C on a bearing of 296°.

(a) Show that $\angle ABC = 53°$.

(b) Find, to the nearest km, the distance of port A from port C.

(c) Use the cosine rule to find $\angle ACB$, and hence find the bearing of port A from port C, correct to the nearest degree.

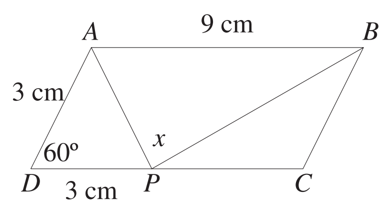


13. $ABCD$ is a parallelogram in which $AB = 9$ cm, $AD = 3$ cm and $\angle ADC = 60°$. The point P is the point on DC such that $DP = 3$ cm.

(a) Explain why $\triangle ADP$ is equilateral and hence find AP.

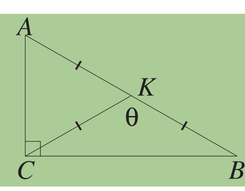
(b) Use the cosine rule in $\triangle BCP$ to find the exact length of BP.

(c) Let $\angle APB = x$. Show that $\cos x = -\frac{1}{14}\sqrt{7}$.



EXTENSION

14. $\triangle ABC$ is right-angled at C, and K is the midpoint of AB. Also, CK has the same length as AK and BK. Prove that $\cos\theta = \dfrac{b^2 - a^2}{b^2 + a^2}$, where $\theta = \angle BKC$.



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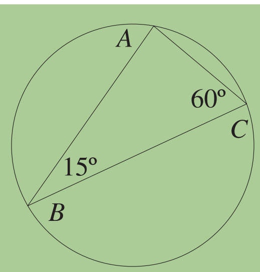
15. (a)

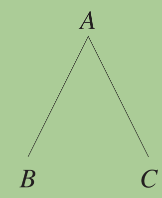


Two of the angles of a triangle ABC are 15° and 60° and the triangle is inscribed in a circle of radius 6 units.

(i) Show that $AC^2 = 36(2 - \sqrt{3}\,)$.

(ii) Find the exact area of the triangle.

(b)



BA and CA are identical rods hinged at A. When $BC = 5\,\text{cm}$, $\angle BAC = 45^\circ$ and when $BC = 6\,\text{cm}$, $\angle BAC = \alpha$. Show that $\cos\alpha = \dfrac{18\sqrt{2} - 11}{25}$.

16. [Heron's formula for the area of a triangle in terms of the side lengths]

(a) By repeated application of factoring by the difference of squares, prove the identity

$$(2ab)^2 - (a^2 + b^2 - c^2)^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c).$$

(b) Let $\triangle ABC$ be any triangle, and let $s = \frac{1}{2}(a+b+c)$ be the *semiperimeter*. Prove that

$$(a + b + c)(a + b - c)(a - b + c)(-a + b + c) = 16s(s - a)(s - b)(s - c).$$

(c) Write down the formula for $\cos C$ in terms of the sides a, b and c, then use (a) and (b) and the Pythagorean identities to prove that $\sin C = \dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{ab}$.

(d) Hence show that the area Δ of the triangle is $\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,$.

17. [The circumcircle and the incircle] In the previous Exercise 4H, formulae were developed for the diameters D_C and D_I of the circumcircle and incircle respectively of a triangle $\triangle ABC$.

(a) Use these formulae, and the methods of the previous question, to find formulae for the diameters of these circles in terms of the side lengths of the triangle.

(b) Show that area of circumcircle : area of incircle $= a^2b^2c^2 : 16(s-a)^2(s-b)^2(s-c)^2$.

4 J Problems Involving General Triangles

A triangle has three lengths and three angles, and most triangle problems involve using three of these six measurements to calculate some of the others. The key to deciding which formula to use is to see which congruence situation applies.

Trigonometry and the Congruence Tests: The four standard congruence tests — RHS, AAS, SAS and SSS — can also be regarded as theorems about constructing triangles from given data. If you know three measurements including one length, then apart from the ambiguous ASS test, there is only one possible triangle with these three measurements, and you can construct it up to congruence.

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41 THE SINE, COSINE AND AREA RULES AND THE STANDARD CONGRUENCE TESTS:
In a right triangle, use simple trigonometry and Pythagoras' theorem. Otherwise:
AAS: Use the sine rule to find each of the other two sides.
ASS: Use the sine rule to find the unknown angle opposite a known side (possibly with two solutions).
SAS: Use the cosine rule to find the third side, and use the area rule to find the area.
SSS: Use the cosine rule to find any angle.

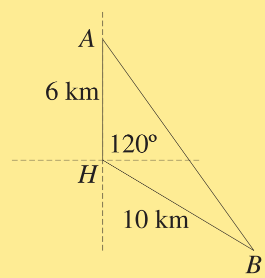
Problems Requiring Two Steps: Various situations with non-right-angled triangles require two steps for their solution, for example, finding the other two angles in an SAS situation, or finding the area given AAS, ASS or SSS situations.

WORKED EXERCISE: A boat sails 6 km due north from the harbour H to A, and a second boat sails 10 km from H to B on a bearing of 120°. What is the bearing of B from A, correct to the nearest minute?

SOLUTION: First, using the cosine rule to find AB,

$$\begin{aligned} AB^2 &= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 120^\circ \\ &= 36 + 100 - 120 \times (-\tfrac{1}{2}) \\ &= 196, \end{aligned}$$

so $\quad AB = 14\,\text{km}$.



Secondly, using the cosine rule to find $\angle A$,

$$\begin{aligned} \cos A &= \frac{6^2 + 14^2 - 10^2}{2 \times 6 \times 14} \\ &= \tfrac{11}{14}, \end{aligned}$$

so $\quad A \doteqdot 38^\circ 13'$, and the bearing of B from A is about $141^\circ 47'$.

Finding the Third Side in the Ambiguous ASS Situation: The cosine rule in the form $a^2 = b^2 + c^2 - 2bc\cos A$ can also be rewritten as a quadratic in c:

$$c^2 - 2bc\cos A + (b^2 - a^2) = 0.$$

This allows the third side to be found in one step in the ambiguous ASS situation when two sides and a non-included angle are given. For there to be two solutions, the quadratic must have two positive solutions.

WORKED EXERCISE: A tree trunk grows at an angle of 30° to the ground, and a 4 metre rod hangs from a point P that is 6 metres along the trunk. Find (to the nearest centimetre) the maximum and minimum distances of the other end E of the rod from the base B of the tree when E is resting on the ground.

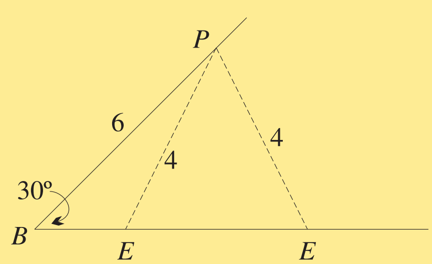
SOLUTION: We rearrange the cosine rule as a quadratic in $p = BE$:

$$4^2 = 6^2 + p^2 - 2 \times 6 \times p \times \cos 30^\circ$$

$$p^2 - 6p\sqrt{3} + 20 = 0.$$

Using the quadratic formula,

$$\begin{aligned} b^2 - 4ac &= 108 - 80 \\ &= 4 \times 7 \\ p &= 3\sqrt{3} + \sqrt{7} \quad \text{or} \quad 3\sqrt{3} - \sqrt{7} \\ &\doteqdot 7{\cdot}84 \text{ metres} \quad \text{or} \quad 2{\cdot}55 \text{ metres}. \end{aligned}$$



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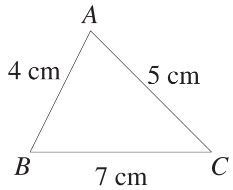
Exercise 4J

1. (a)

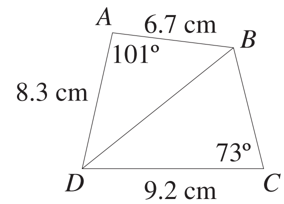


In $\triangle ABC$, $AB = 4$ cm, $BC = 7$ cm and $CA = 5$ cm.

(i) Find $\angle ABC$, correct to 1 minute.

(ii) Hence calculate the area of $\triangle ABC$, correct to $0{\cdot}1\ \text{cm}^2$.

(b)

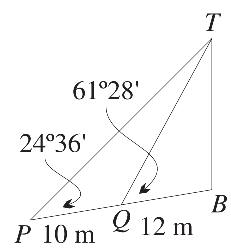


In the diagram above, $AB = 6{\cdot}7$ cm, $AD = 8{\cdot}3$ cm and $DC = 9{\cdot}2$ cm. Also, $\angle A = 101°$ and $\angle C = 73°$.

(i) Find the diagonal BD, correct to the nearest millimetre.

(ii) Hence find $\angle CBD$, correct to the nearest degree.

2. In the diagram opposite, TB is a vertical flagpole at the top of an inclined path PQB.

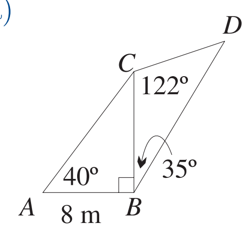


(a) Show that $TQ = \dfrac{10 \sin 24°36'}{\sin 36°52'}$.

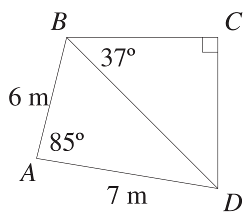
(b) Hence find the height of the flagpole in metres, correct to two decimal places.

3. In each diagram, find CD correct to the nearest centimetre:

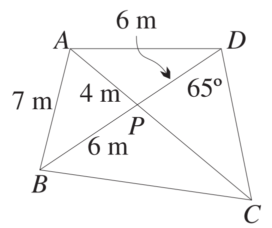
(a)



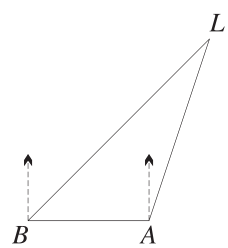
(b)



(c)



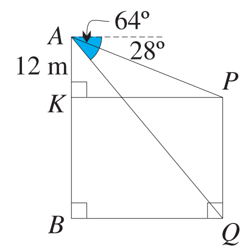
4. A ship at A is 10 nautical miles from a lighthouse L which is on a bearing of N25°E. The ship then sails due west to B, from which the bearing of the lighthouse is N55°E.



(a) Show that $\angle ALB = 30°$.

(b) Using the sine rule, show that $AB = 5 \operatorname{cosec} 35°$, and hence find the distance sailed by the ship from A to B. Give your answer in nautical miles, correct to one decimal place.

5. Two towers AB and PQ stand on level ground. Tower AB is 12 metres taller than tower PQ. From A, the angles of depression of P and Q are 28° and 64° respectively.



(a) Use $\triangle AKP$ to show that $KP = BQ = 12 \tan 62°$.

(b) Use $\triangle ABQ$ to show that $AB = 12 \tan 62° \tan 64°$.

(c) Hence find the height of the shorter tower, correct to the nearest metre.

(d) Solve the problem again by finding AP using $\triangle AKP$ and then using the sine rule in $\triangle APQ$.

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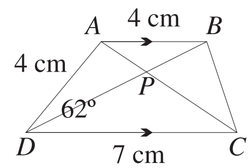
6. In the diagram opposite, $ABCD$ is a trapezium in which $AB \parallel DC$. The diagonals AC and BD meet at P. Also, $AB = AD = 4\,\text{cm}$, $DC = 7\,\text{cm}$ and $\angle ADC = 62°$.



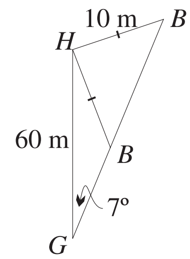
(a) Find $\angle ACD$, correct to the nearest minute. [HINT: Find AC first.]

(b) Explain why $\angle PDC = \frac{1}{2}\angle ADC$.

(c) Hence find, to the nearest minute, the acute angle between the diagonals of the trapezium.

DEVELOPMENT

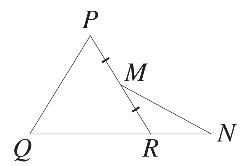
7. With his approach shot to the hole H, a golfer at G landed his ball B 10 metres from H. The direction of the shot was $7°$ away from the direct line between G and H.



(a) Find, correct to the nearest minute, the two possible sizes of $\angle GBH$.

(b) Hence find the two possible distances the ball has travelled (answer in metres to one decimal place).

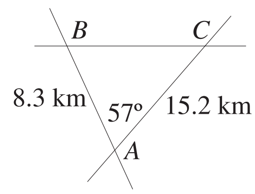
8. PQR is an equilateral triangle with side length 3 cm. M is the midpoint of PR and N is the point in QR produced such that $RN = 2\,\text{cm}$.



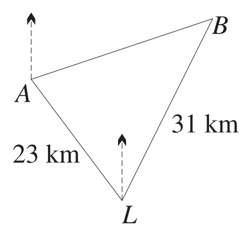
(a) Find MN.

(b) Hence calculate $\angle QNM$, correct to the nearest minute.

9. AB, BC and CA are straight roads. AB and AC intersect at $57°$. $AB = 8{\cdot}3\,\text{km}$ and $AC = 15{\cdot}2\,\text{km}$. Two cars P_1 and P_2 leave A at the same instant. P_1 travels along AB and then BC at 80 km/h while P_2 travels along AC at 50 km/h. Which car reaches C first, and by how many minutes does it do so (answer to one decimal place)?



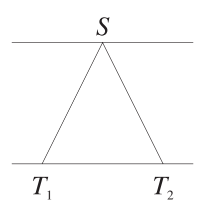
10. Town A is 23 km from landmark L in the direction N56°W, and town B is 31 km from L in the direction N46°E.



(a) Find how far town B is from town A (answer to the nearest km).

(b) Find the bearing of town B from town A (answer to the nearest degree).

11. Two trees T_1 and T_2 on one bank of a river are 86 metres apart. A sign S on the opposite bank is between the trees and the angles ST_1T_2 and ST_2T_1 are $53°30'$ and $60°45'$ respectively.

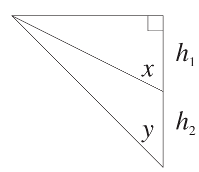


(a) Find ST_1.

(b) Hence find the width of the river, correct to the nearest metre.

12. In the given diagram, prove that

$$h_1 = \frac{h_2 \cos x \sin y}{\sin(x-y)}\,.$$



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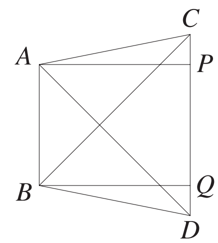
13. AB and CD are vertical lines, while AP and BQ are horizontal lines. From A and B, the angles of elevation of C are $37°$ and $48°$ respectively. From A, the angle of depression of D is $52°$. Let $AP = BQ = x$.

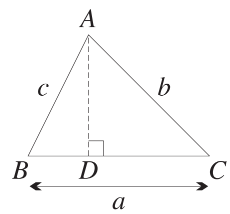


(a) Show that $CP = x \tan 37°$, and write down similar expressions for CQ and PD.

(b) Let α be the angle of elevation of B from D. Explain why $x \tan 48° - x \tan 37° = x \tan 52° - x \tan \alpha$.

(c) Hence find α, correct to the nearest minute.

14. ABC is a triangle and D is the point on BC such that $AD \perp BC$.



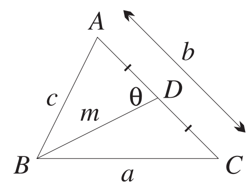
(a) Show that $BD = c \cos B$, and write down a similar expression for DC.

(b) Hence show that $a = b \cos C + c \cos B$.

(c) Show that $b \sin C = c \sin B$.

(d) Use (b) and (c) to show that $\dfrac{a - b\cos C}{b \sin C} = \cot B$.

15. ABC is a triangle and D is the midpoint of AC. $BD = m$ and $\angle ADB = \theta$.



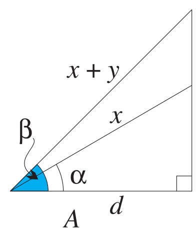
(a) Simplify $\cos(180° - \theta)$.

(b) Show that $\cos\theta = \dfrac{4m^2 + b^2 - 4c^2}{4mb}$, and write down a similar expression for $\cos(180° - \theta)$.

(c) Hence show that $a^2 + c^2 = 2m^2 + \frac{1}{2}b^2$.

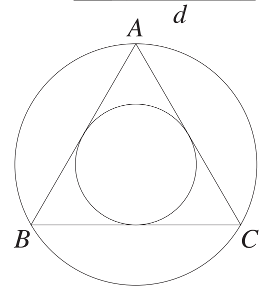
16. The sides of a triangle are $n^2 + n + 1$, $2n + 1$ and $n^2 - 1$, where $n > 1$. Find the largest angle of the triangle.

17. A ladder of length x cm is inclined at an angle α to the ground. The foot of the ladder is fixed. If the ladder were y cm longer, the inclination to the horizontal would be β. Show that the distance from the foot of the ladder to the wall is given by $\dfrac{y \cos\alpha \cos\beta}{\cos\alpha - \cos\beta}$ cm.



18. In $\triangle ABC$, $a \cos A = b \cos B$. Prove, using the cosine rule, that the triangle is either isosceles or right-angled.

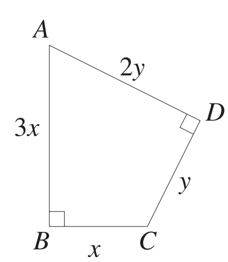
19. The diagram opposite shows an equilateral triangle ABC whose sides are $2x$ units long.



(a) Show that the inscribed circle tangent to all three sides has area $\frac{1}{3}\pi x^2$.

(b) Show that the circumscribed circle passing through all three vertices has four times the area of the inscribed circle.

20. In the quadrilateral $ABCD$ sketched opposite, $\angle B$ and $\angle D$ are right angles, $AB = 3BC = 3x$ and $AD = 2DC = 2y$. Use Pythagoras' theorem and the cosine rule to show that $\angle A = 45°$.



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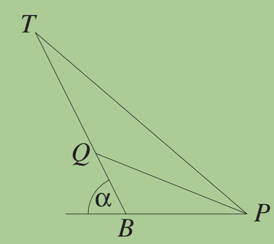
EXTENSION

21. From the top T of a hill BT inclined at α to the horizontal, the angle of depression of a point P on the plane below is $12°$. From Q which is three-quarters of the way down the hill, the angle of depression of P is $6°$.

(a) Using the sine rule, show that $\sin\alpha = 3\sin(\alpha - 12°)$.

(b) Apply the formula $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ to the result in (a) to find α to the nearest minute.



22. [Brahmagupta's formula for the area of a cyclic quadrilateral in terms of its sides]

(a) Use repeated application of the difference of squares to prove the identity

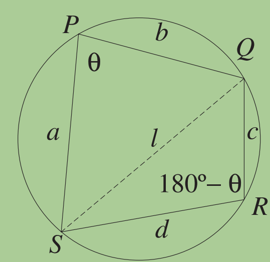
$$4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2 = (a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d).$$

(b) Let $s = \frac{1}{2}(a+b+c+d)$ be the *semiperimeter* of the cyclic quadrilateral $PQRS$ in the diagram below. Prove that

$$(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d) = 16(s-a)(s-b)(s-c)(s-d).$$

(c) Let $\angle P = \theta$, then by circle geometry the opposite angle is $\angle R = 180° - \theta$. By equating expressions for the square of the diagonal $\ell = SQ$, prove that

$$\cos\theta = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$



(d) Hence show that

$$\sin\theta = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}.$$

(e) Hence show that the area of the cyclic quadrilateral is

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}\,.$$

(f) How can Heron's formula be generated as a special case of Brahmagupta's formula?

23. [Diagonals and diameter of a cyclic quadrilateral] Using the results established in the previous question, and with the same notation, let $m = PR$ be the other diagonal, let $\phi = \angle PQR$, and let D_C be the diameter of the circumcircle. Prove further that:

(i) $\ell^2 = \dfrac{(ad+bc)(ac+bd)}{ab+cd}$

(ii) $\dfrac{\ell}{m} = \dfrac{ad+bc}{ab+cd} = \dfrac{\sin\theta}{\sin\phi}$

(iii) $\ell m = ac+bd$

(iv) $D_C = \dfrac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}$

(v) $\dfrac{\text{area of quadrilateral}}{\text{area of circle}} = \dfrac{16(s-a)^{\frac{3}{2}}(s-b)^{\frac{3}{2}}(s-c)^{\frac{3}{2}}(s-d)^{\frac{3}{2}}}{\pi(ab+cd)(ac+bd)(ad+bc)}$

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