The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 9

MATH1903: Integral Calculus and Modelling (Advanced) Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/JM/MATH1903/
Lecturers: Daniel Daners and James Parkinson

Material covered

(1) Solution of first order differential equations by separation of variables

(2) Particular solutions of differential equations

(3) Applications of differential equations in various contexts

Outcomes

After completing this tutorial you should

(1) confident in solving separable first order differential equations with or without initial conditions

(2) be able to deal with differential equations arising from a variety of models and be able to interpret the solutions.

Questions to do before the tutorial

1. Find the general solutions of the following differential equations.

(a) $(1+x^2)\dfrac{dy}{dx}+xy=0$, (b) $x\dfrac{dy}{dx}=y^2-1$.

Questions to complete during the tutorial

2. Find the general solution of the following differential equations.

(a) $(x^2y^2+x^2+y^2+1)\dfrac{dy}{dx}=xy+x$, (b) $ye^x\dfrac{dy}{dx}=y^2+y-2$.

3. Find the particular solutions of the following differential equations.

(a) $\dfrac{dy}{dx}=xe^{y-x^2}$, $y(0)=0$, (b) $\dfrac{dy}{dx}=\dfrac{1+y^2}{1+x^2}$, $y(0)=a$, a a constant.

4. Consider the differential equation $y'=xy^2$.

(a) Sketch the direction field for the given differential equation for $-3\le x,y\le 3$.

(b) Solve the differential equation with the initial condition $y(1)=-2$.

(c) The equilibrium solution $y=0$ is stable if any solution starting near zero stays near zero for all $x>0$. Is the zero solution $y=0$ for the given differential equation stable? Briefly justify your answer.

5. Einstein's Theory of Relativity predicts the existence of black holes: regions in space from which nothing can escape, due to strong gravitational forces. The theory predicts that black holes will be formed when large stars collapse.

However, Einstein's theory did not take into account quantum mechanical effects. In 1975, Stephen Hawking used quantum theory to show that a black hole should glow slightly; that is, it should radiate energy and particles in the same way that a heated object does. Assuming that nothing else falls into the black hole, this causes its mass M to decrease at the rate governed by the differential equation,

$$\frac{dM}{dt} = -\frac{\alpha}{M^2},$$

where t denotes time and α is a constant whose value is not yet known precisely.

(a) Find the general solution $M(t)$ of this differential equation.

(b) Find the particular solution which satisfies the condition that the mass is M_0 when $t = 0$.

(c) How long does it take for a black hole which initially has mass M_0 to lose half its mass? How long does it take for it to evaporate completely?

Extra questions for further practice

6. Find the general solutions of

(a) $\dfrac{dy}{dx} = \dfrac{x + \sin x}{3y^2}$, (b) $\dfrac{dx}{dt} = 1 + t - x - tx$, (c) $\dfrac{dy}{dx} = \dfrac{\ln x}{xy + xy^3}$.

7. Find particular solutions satisfying the given conditions.

(a) $\dfrac{dy}{dx} = \dfrac{1+x}{xy}$ $(x > 0)$, $y(1) = -4$; (b) $\dfrac{dy}{dt} = \dfrac{ty + 3t}{t^2 + 1}$, $y(2) = 2$.

8. Find a function $g(x)$ such that $g'(x) = g(x)\big(1 + g(x)\big)$ and $g(0) = 1$.

9. A molecule of substance A can combine with a molecule of substance B to form a molecule of substance X, in a reaction which is denoted $A + B \to X$. According to the Law of Mass Action, the rate of formation of X is proportional to the product of the amounts of A and B present. A test-tube initially contains amounts a and b of substances A and B, respectively, (measured in billions of molecules), but none of substance X.

(a) Let $x(t)$ denote the amount of substance X (that is, the number of billions of X molecules) produced within the first t seconds. Write down a differential equation for $x(t)$.

(b) Assuming that $a \neq b$, solve this equation to obtain an expression for $x(t)$.

(c) Suppose that initially there are two molecules of B for every molecule of A, and that after 10 seconds there are six molecules of B for every molecule of A. What is the ratio after 30 seconds?

(d) The experiment is repeated, but with the initial amount of substance B halved so as to equal the initial amount a of substance A. (As before, substance X is absent initially.) What fraction of A molecules remain after 30 seconds?