

UNIVERSITY OF SYDNEY

MATH 1901

DIFFERENTIAL CALCULUS (ADVANCED)

Assignment 2

Author: Keegan Gyoery
SID: 470413467

Tutor: Daniel Daners
Tutorial: Carslaw Tutorial Room
359, 10am

May 17, 2017

1. (a) Using L'Hopital's Rule, we can compute the following limit.

$$\begin{aligned}
 \lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) &= \lim_{x \rightarrow 1} \left(\frac{x-1}{(\ln x)(x-1)} - \frac{\ln x}{(\ln x)(x-1)} \right) \\
 &= \lim_{x \rightarrow 1} \left(\frac{x-1-\ln x}{(\ln x)(x-1)} \right) \\
 &= \lim_{x \rightarrow 1} \left(\frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}[(\ln x)(x-1)]} \right) \\
 &= \lim_{x \rightarrow 1} \left(\frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x} \right) \\
 &= \lim_{x \rightarrow 1} \left(\frac{1-\frac{1}{x}}{1-\frac{1}{x}+\ln x} \right) \\
 &= \lim_{x \rightarrow 1} \left(\frac{\frac{d}{dx}(1-\frac{1}{x})}{\frac{d}{dx}(1-\frac{1}{x}+\ln x)} \right) \\
 &= \lim_{x \rightarrow 1} \left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}} \right) \\
 &= \lim_{x \rightarrow 1} \left(\frac{\frac{1}{x}}{\frac{1}{x}+1} \right) \\
 &= \frac{1}{1+1} \\
 \therefore \lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) &= \frac{1}{2}
 \end{aligned}$$

- (b) To compute the Taylor polynomial of order 5 of the function $f(x) := \frac{e^{x^2}}{x^2}$ about $x = 1$, we must first determine the derivatives, up to and including the fifth derivative, at the point $x = 1$. In order to compute this derivative, we will use the Leibniz formula for the n -th derivative of a product of two functions. The formula to compute these derivatives is as follows:

$$(hg)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}$$

Before calculating the derivatives using the Leibniz formula, we must compute the derivatives of $h = e^{x^2}$ and $g = x^{-2}$. For the function g , the derivatives are as follows.

$$\begin{aligned}
 g^{(0)} &= x^{-2} \\
 g^{(1)} &= -2x^{-3} \\
 g^{(2)} &= 6x^{-4} \\
 g^{(3)} &= -24x^{-5} \\
 g^{(4)} &= 120x^{-6} \\
 g^{(5)} &= -720x^{-7}
 \end{aligned}$$

For the function f , the derivatives are less straight forward to calculate.

$$h^{(0)} = e^{x^2}$$

$$\begin{aligned} h^{(1)} &= \frac{d}{dx} [e^{x^2}] \\ \therefore h^{(1)} &= 2xe^{x^2} \end{aligned}$$

$$\begin{aligned} h^{(2)} &= \frac{d}{dx} [2xe^{x^2}] \\ &= 2e^{x^2} + 4x^2e^{x^2} \\ \therefore h^{(2)} &= 2e^{x^2} [2x^2 + 1] \end{aligned}$$

$$\begin{aligned} h^{(3)} &= \frac{d}{dx} [2e^{x^2} + 4x^2e^{x^2}] \\ &= 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2} \\ \therefore h^{(3)} &= 4xe^{x^2} [2x^2 + 3] \end{aligned}$$

$$\begin{aligned} h^{(4)} &= \frac{d}{dx} [4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2}] \\ &= 4e^{x^2} + 8x^2e^{x^2} + 8e^{x^2} + 16x^2e^{x^2} + 24x^2e^{x^2} + 16x^4e^{x^2} \\ &= 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2} \\ \therefore h^{(4)} &= 4e^{x^2} [4x^4 + 12x^2 + 3] \end{aligned}$$

$$\begin{aligned} h^{(5)} &= \frac{d}{dx} [12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}] \\ &= 24xe^{x^2} + 96xe^{x^2} + 96x^3e^{x^2} + 64x^3e^{x^2} + 32x^5e^{x^2} \\ &= 120xe^{x^2} + 160x^3e^{x^2} + 32x^5e^{x^2} \\ \therefore h^{(5)} &= 8xe^{x^2} [4x^4 + 20x^2 + 15] \end{aligned}$$

In order to calculate the derivatives of $\frac{e^{x^2}}{x^2}$, we will set $f = e^{x^2}$ and $g = x^{-2}$. Using these definitions, we can compute the derivatives at $x = 1$, up to and including, the fifth derivative. Computing the zeroth derivative, in other terms the function itself, at $x = 1$, we get the result:

$$\begin{aligned} (hg)^{(0)} &= \frac{e^{x^2}}{x^2} \\ \therefore (hg)^{(0)} &= e \end{aligned}$$

Now computing the first derivative of the function using the Leibniz formula, and evaluating at $x = 1$, we get the result:

$$\begin{aligned}
 (hg)^{(1)} &= \sum_{k=0}^1 \binom{1}{k} h^{(k)} g^{(1-k)} \\
 &= \binom{1}{0} h^{(0)} g^{(1)} + \binom{1}{1} h^{(1)} g^{(0)} \\
 &= \binom{1}{0} [(e^{x^2})(-2x^{-3})] + \binom{1}{1} [(2xe^{x^2})(x^{-2})] \\
 \therefore (hg)^{(1)} &= -2 + 2 \quad \text{Substituting } x = 1 \\
 &= 0
 \end{aligned}$$

Now computing the second derivative and evaluating at $x = 1$, we get the result:

$$\begin{aligned}
 (hg)^{(2)} &= \sum_{k=0}^2 \binom{2}{k} h^{(k)} g^{(2-k)} \\
 &= \binom{2}{0} h^{(0)} g^{(2)} + \binom{2}{1} h^{(1)} g^{(1)} + \binom{2}{2} h^{(2)} g^{(0)} \\
 &= \binom{2}{0} [(e^{x^2})(6x^{-4})] + \binom{2}{1} [(2xe^{x^2})(-2x^{-3})] + \binom{2}{2} [(2e^{x^2}(2x^2 + 1))(x^{-2})] \\
 \therefore (hg)^{(2)} &= 6e - 8e + 6e \quad \text{Substituting } x = 1 \\
 &= 4e
 \end{aligned}$$

Now computing the third derivative at $x = 1$, we get the following result:

$$\begin{aligned}
 (hg)^{(3)} &= \sum_{k=0}^3 \binom{3}{k} h^{(k)} g^{(3-k)} \\
 &= \binom{3}{0} h^{(0)} g^{(3)} + \binom{3}{1} h^{(1)} g^{(2)} + \binom{3}{2} h^{(2)} g^{(1)} + \binom{3}{3} h^{(3)} g^{(0)} \\
 &= \binom{3}{0} [(e^{x^2})(-24x^{-5})] + \binom{3}{1} [(2xe^{x^2})(6x^{-4})] \\
 &\quad + \binom{3}{2} [(2e^{x^2}(2x^2 + 1))(-2x^{-3})] + \binom{3}{3} [(4xe^{x^2}(2x^2 + 3))(x^{-2})] \\
 \therefore (hg)^{(3)} &= -24e + 36e - 36e + 20e \quad \text{Substituting } x = 1 \\
 &= -4e
 \end{aligned}$$

Now computing the fourth derivative of the function at $x = 1$, we get the result:

$$\begin{aligned}
 (hg)^{(4)} &= \sum_{k=0}^4 \binom{4}{k} h^{(k)} g^{(4-k)} \\
 &= \binom{4}{0} h^{(0)} g^{(4)} + \binom{4}{1} h^{(1)} g^{(3)} + \binom{4}{2} h^{(2)} g^{(2)} + \binom{4}{3} h^{(3)} g^{(1)} + \binom{4}{4} h^{(4)} g^{(0)} \\
 &= \binom{4}{0} [(e^{x^2})(120x^{-6})] + \binom{4}{1} [(2xe^{x^2})(-24x^{-5})] \\
 &\quad + \binom{4}{2} [(2e^{x^2}(2x^2 + 1))(6x^{-4})] + \binom{4}{3} [(4xe^{x^2}(2x^2 + 3))(-2x^{-3})] \\
 &\quad + \binom{4}{4} [(4e^{x^2}(4x^4 + 12x^2 + 3))(x^{-2})] \\
 \therefore (hg)^{(4)} &= 120e - 192e + 216e - 160e + 76e \quad \text{Substituting } x = 1 \\
 &= 60e
 \end{aligned}$$

Now computing the fifth derivative of the function at $x = 1$, we get the result that follows:

$$\begin{aligned}
 (hg)^{(5)} &= \sum_{k=0}^5 \binom{5}{k} h^{(k)} g^{(5-k)} \\
 &= \binom{5}{0} h^{(0)} g^{(5)} + \binom{5}{1} h^{(1)} g^{(4)} + \binom{5}{2} h^{(2)} g^{(3)} + \binom{5}{3} h^{(3)} g^{(2)} + \binom{5}{4} h^{(4)} g^{(1)} + \binom{5}{5} h^{(5)} g^{(0)} \\
 &= \binom{5}{0} [(e^{x^2})(-720x^{-7})] + \binom{5}{1} [(2xe^{x^2})(120x^{-6})] \\
 &\quad + \binom{5}{2} [(2e^{x^2}(2x^2 + 1))(-24x^{-5})] + \binom{5}{3} [(4xe^{x^2}(2x^2 + 3))(6x^{-4})] \\
 &\quad + \binom{5}{4} [(4e^{x^2}(4x^4 + 12x^2 + 3))(-2x^{-3})] + \binom{5}{5} [(8xe^{x^2}(4x^4 + 20x^2 + 15))(x^{-2})] \\
 \therefore (hg)^{(5)} &= -720e + 1200e - 1440e + 1200e - 760e + 312e \quad \text{Substituting } x = 1 \\
 &= -208e
 \end{aligned}$$

Using the results that we have just calculated, we will amalgamate the values of each derivative for the function $f(x) = \frac{e^{x^2}}{x^2}$ at $x = 1$.

$$\begin{aligned}
 f^{(0)}(1) &= e \\
 f^{(1)}(1) &= 0 \\
 f^{(2)}(1) &= 4e \\
 f^{(3)}(1) &= -4e \\
 f^{(4)}(1) &= 60e \\
 f^{(5)}(1) &= -208e
 \end{aligned}$$

Now we will examine the general form for the Taylor expansion about some arbitrary point, x_0 , of order 5.

$$\begin{aligned}
 T_5(x) &= f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x - x_0)^4 \\
 &\quad + \frac{f^{(5)}(x_0)}{5!}(x - x_0)^5
 \end{aligned}$$

Using the values for the derivatives of $f(x)$ about the point $x = 1$, we can calculate the Taylor expansion of order 5 for the function $f(x) = \frac{e^{x^2}}{x^2}$.

$$\begin{aligned}
T_5(x) &= f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x - x_0)^4 \\
&\quad + \frac{f^{(5)}(x_0)}{5!}(x - x_0)^5 \\
&= f(1) + f^{(1)}(1)(x - 1) + \frac{f^{(2)}(1)}{2!}(x - 1)^2 + \frac{f^{(3)}(1)}{3!}(x - 1)^3 + \frac{f^{(4)}(1)}{4!}(x - 1)^4 \\
&\quad + \frac{f^{(5)}(1)}{5!}(x - 1)^5 \\
&= e + 0 + \frac{4e}{2}(x - 1)^2 + \frac{-4e}{6}(x - 1)^3 + \frac{60e}{24}(x - 1)^4 + \frac{-208e}{120}(x - 1)^5 \\
\therefore T_5(x) &= e + 2e(x - 1)^2 - \frac{2e}{3}(x - 1)^3 + \frac{5e}{2}(x - 1)^4 - \frac{26e}{15}(x - 1)^5
\end{aligned}$$

2. (a) The Mean Value Theorem states that for some function $f : [a, b] \rightarrow \mathbb{R}$ be continuous and $f : (a, b) \rightarrow \mathbb{R}$ be differentiable, there exists $c \in (a, b)$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$. By considering cases for the given inequality, $\sqrt{1+x} \leq 1 + \frac{x}{2}$ for $x \in (-1, \infty)$, we can use the Mean Value Theorem to prove this inequality holds $\forall x \in (-1, \infty)$.

Considering the first case, where $x \in (-1, 0)$, we define $f(t) := \sqrt{1+t}$ for $t \in [x, 0]$. By the Mean Value Theorem, there exists $c \in (x, 0)$ such that:

$$\begin{aligned}
\frac{f(0) - f(x)}{0 - x} &= f'(c) \\
\therefore \frac{1 - \sqrt{1+x}}{-x} &= \frac{1}{2\sqrt{1+c}} \\
\therefore 1 - \sqrt{1+x} &= \frac{-x}{2\sqrt{1+c}}
\end{aligned}$$

As $c \in (x, 0)$ for $x \in (-1, 0)$, it follows that:

$$0 < \sqrt{1+c} < 1 \implies \frac{1}{2\sqrt{1+c}} > \frac{1}{2}$$

Now, as for $x \in (-1, 0)$, $x < 0$, $\therefore -x > 0$. Thus it follows that:

$$\begin{aligned}
\frac{-x}{2\sqrt{1+c}} &> \frac{-x}{2} \\
\therefore 1 - \sqrt{1+x} &> \frac{-x}{2} \\
\therefore \sqrt{1+x} &< 1 + \frac{x}{2} \quad \forall x \in (-1, 0)
\end{aligned}$$

Now, considering the second case, where $x \in (0, \infty)$, we define $f(t) := \sqrt{1+t}$ for $t \in [0, x]$. By the Mean Value Theorem, there exists $c \in (0, x)$ such that:

$$\begin{aligned}
\frac{f(x) - f(0)}{x - 0} &= f'(c) \\
\therefore \frac{\sqrt{1+x} - 1}{x} &= \frac{1}{2\sqrt{1+c}} \\
\therefore \sqrt{1+x} - 1 &= \frac{x}{2\sqrt{1+c}}
\end{aligned}$$

As $c \in (0, x)$ for $x \in (0, \infty)$, it follows that:

$$\sqrt{1+c} > 1 \implies \frac{1}{2\sqrt{1+c}} < \frac{1}{2}$$

Now, as for $x \in (0, \infty)$, $x > 0$. Thus it follows that:

$$\begin{aligned}
\frac{x}{2\sqrt{1+c}} &< \frac{x}{2} \\
\therefore \sqrt{1+x} - 1 &< \frac{x}{2} \\
\therefore \sqrt{1+x} &< 1 + \frac{x}{2} \quad \forall x \in (0, \infty)
\end{aligned}$$

Now considering the third and final case, where $x = 0$, we get the following results.

$$\begin{aligned}
x = 0 &\implies \begin{cases} \sqrt{1+x} = 1 \\ 1 + \frac{x}{2} = 1 \end{cases} \\
\therefore \sqrt{1+x} &= 1 + \frac{x}{2} \quad \text{for } x = 0
\end{aligned}$$

Thus combining the three cases, we get the final result:

$$\sqrt{1+x} \leq 1 + \frac{x}{2} \quad \forall x \in (-1, \infty)$$

- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. Fix some $x_0 \in \mathbb{R}$. Caratheodory's characterisation for differentiability at x_0 asserts that there exists a function $m_{x_0} : \mathbb{R} \rightarrow \mathbb{R}$ that is continuous at x_0 , such that

$$f(x) = f(x_0) + m_{x_0}(x)(x - x_0)$$

for all $x \in \mathbb{R}$. In this characterisation, $f'(x_0) = m_{x_0}(x_0)$. This characterisation defines the function $f(x)$ to be differentiable at the point x_0 . As a result, $f'(x) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$. In other words, $m_{x_0}(x) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$. Thus $f'(x_0) = m_{x_0}(x_0)$.

For the following proof, assume f is bijective with inverse $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$, with $f'(x_0) \neq 0$, and that the inverse is continuous. We are first required to prove that f^{-1} is differentiable at $y_0 := f(x_0)$. Examining Caratheodory's characterisation at the point $y_0 := f(x_0)$, we get the following result.

$$\begin{aligned}
f(x) &= f(x_0) + m_{x_0}(x)(x - x_0) \\
\therefore m_{x_0}(x) &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{f^{-1}(f(x)) - f^{-1}(f(x_0))} \quad \text{as } f(x) \text{ is bijective and has an inverse} \\
&= \lim_{y \rightarrow y_0} \frac{y - y_0}{f^{-1}(y) - f^{-1}(y_0)} \\
\therefore \frac{1}{m_{x_0}(x)} &= \lim_{y \rightarrow y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \quad \text{as } f'(x_0) \neq 0 \text{ and thus } m_{x_0}(x) \neq 0
\end{aligned}$$

Thus it is clear that the inverse function is defined and differentiable at the point $y_0 := f(x_0)$. Defining $m_{y_0}(y) := \frac{1}{m_{x_0}(x)}$, we are able to write the inverse function, $f^{-1}(x)$, in the form of Caratheodory's characterisation.

$$\therefore f^{-1}(y) = f^{-1}(y_0) + m_{y_0}(y)(y - y_0)$$

Thus it is clear that at the point $y_0 := f(x_0)$, $f^{-1}(x)$ is differentiable, as it can be written in the form of Caratheodory's characterisation. In order for Caratheodory's characterisation to be valid for the inverse function, we assume that $f^{-1}(x)$ is continuous. Furthermore, $f'(x_0) \neq 0$, and as $f'(x_0) = m_{x_0}(x)$, $\therefore m_{x_0}(x) \neq 0$, and thus $\frac{1}{m_{x_0}(x)} = m_{y_0}(y)$ exists, and is defined. We are now required to prove $(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$, and so examining the relationship between Caratheodory's characterisation and the derivative of the function, we get the following results.

$$\begin{aligned}
f'(x_0) &= m_{x_0}(x) \\
\therefore \frac{1}{m_{x_0}(x)} &= m_{y_0}(y) \\
\therefore \frac{1}{f'(x_0)} &= m_{y_0}(y) \\
(f^{-1})'(y_0) &= m_{y_0}(y) \\
\therefore (f^{-1})'(y_0) &= \frac{1}{f'(x_0)} \\
\therefore (f^{-1})'(y_0) &= \frac{1}{f'(f^{-1}(y_0))}
\end{aligned}$$