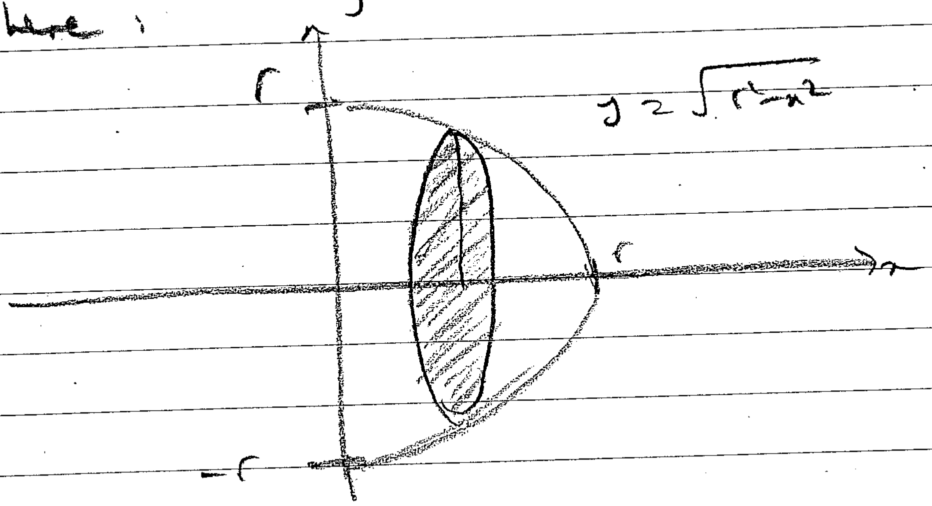
(A)

MATH1903 Lecture 5 Thurs 17/8/2017

Applications of Riemann sums continued

Recall using the disc method to find the volume of $\frac{1}{2}$ sphere:



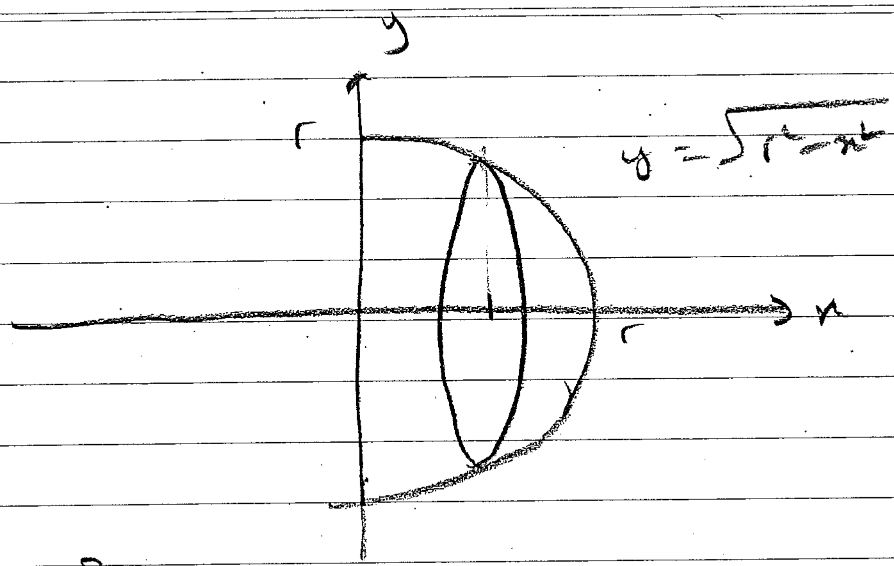
$$V = \int_0^r \pi y^2 \, dx = \pi \int_0^r (r^2 - x^2) \, dx = \dots = \frac{2\pi r^3}{3}.$$

$$\boxed{\text{Volume of full sphere} = \frac{4\pi r^3}{3}}$$

Instead of integrating the circular cross-sectional area of the circles (to get volume), we could integrate the perimeter of the circles to surely get surface area?

layering them side by side "infinitely often" should produce area??

(B)



$$A \overset{?}{=} \int_0^r (\text{perimeter at } x)\, dx$$

$$= \int_0^r 2\pi y \, dx$$

$$= 2\pi \int_0^r \sqrt{r^2 - x^2} \, dx$$

Put $x = r\sin\theta$

so

$\frac{dx}{d\theta} = r\cos\theta$,

$dx = r\cos\theta \, d\theta$

$$= 2\pi \int_0^{\pi/2} r\cos\theta \, r\cos\theta \, d\theta$$

$$= 2\pi r^2 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \pi r^2 \int_0^{\pi/2} 1 + \cos 2\theta \, d\theta$$

$$= \pi r^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \pi r^2 \left(\pi/2 - 0 \right)$$

$$= \frac{\pi^2 r^2}{2}$$

Hence

$$\boxed{\text{area of full sphere} \overset{?}{=} \pi^2 r^2}$$

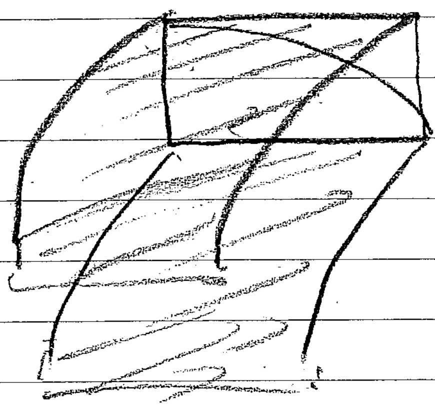
!! But

$$\pi^2 r^2 < \underbrace{4\pi r^2}$$ $(\pi < 4)$

well-known formula for surface area of sphere $= \frac{dV}{dr}$ where $V = \frac{4}{3}\pi r^3$

Contradiction !! ? What went wrong?

Error : thin cylinders underestimate the true surface area

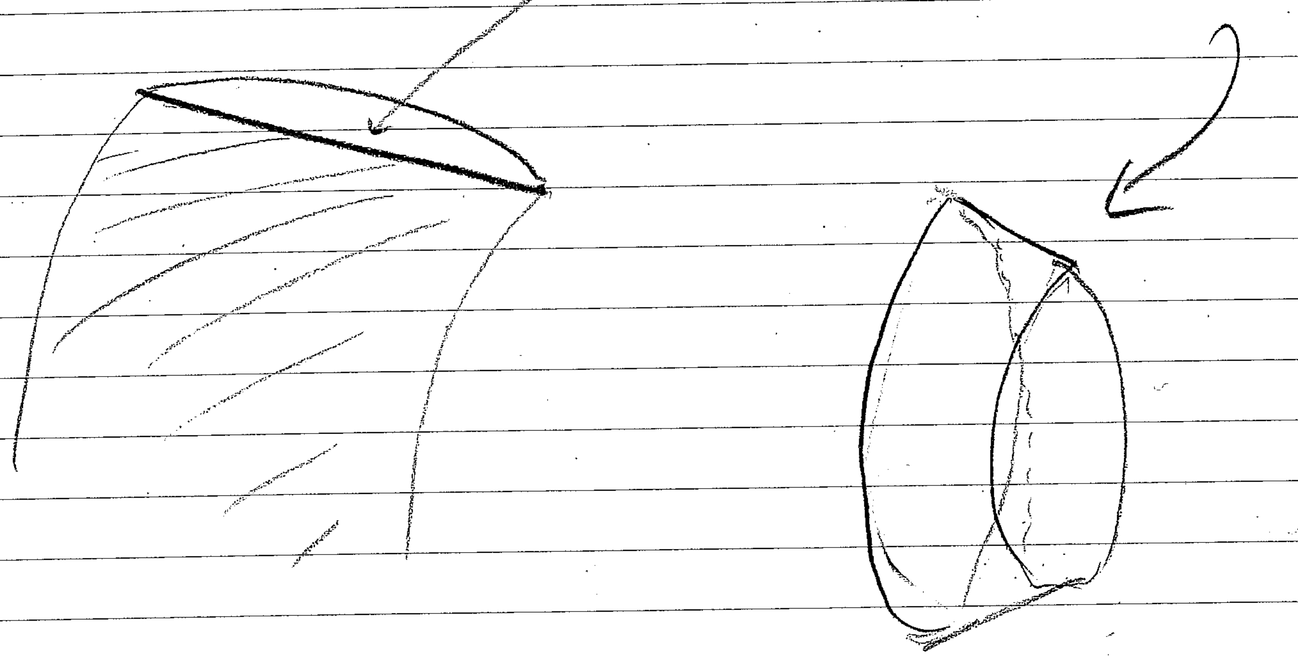


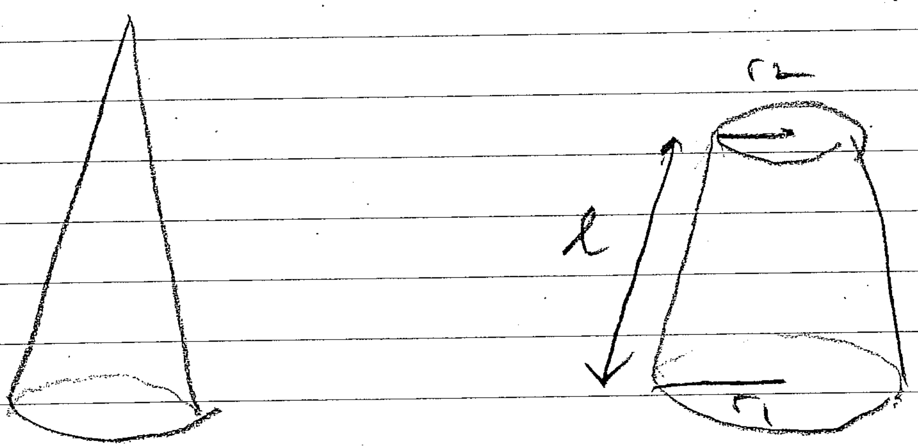
(like trying to approximate the hypotenuse of a triangle using one of the sides)

(set containments of surfaces is also violated, so measure theory cannot be applied immediately)

D

To fix error we use the secant and rotate to get a frustrum





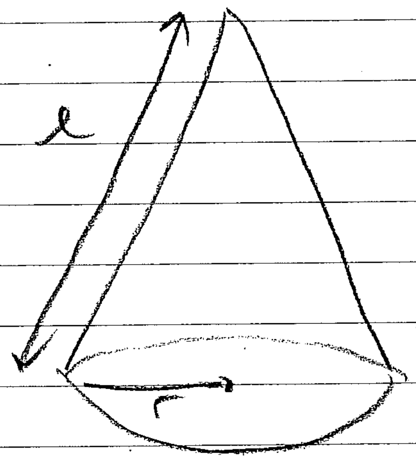
cone

$$\boxed{\text{surface area} = \pi l (r_1 + r_2)}$$

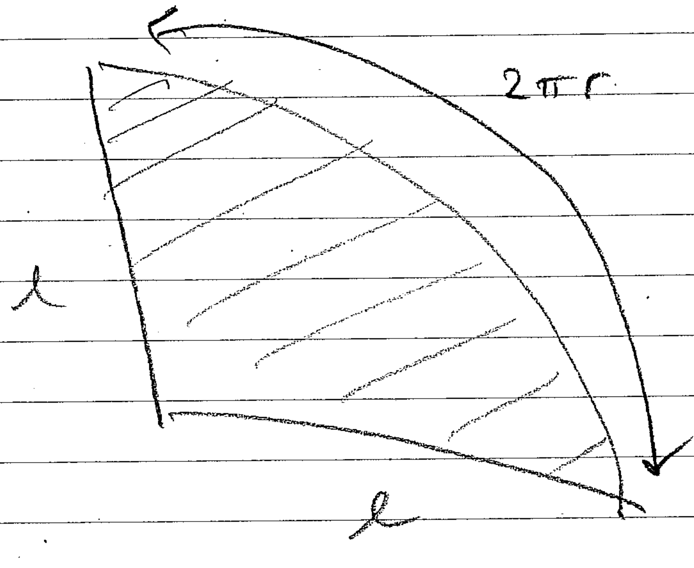
Exercise: deduce this formula from the special case of the surface area of a cone

E

<u>Surface area of a cone:</u>



cut open and flatten out:

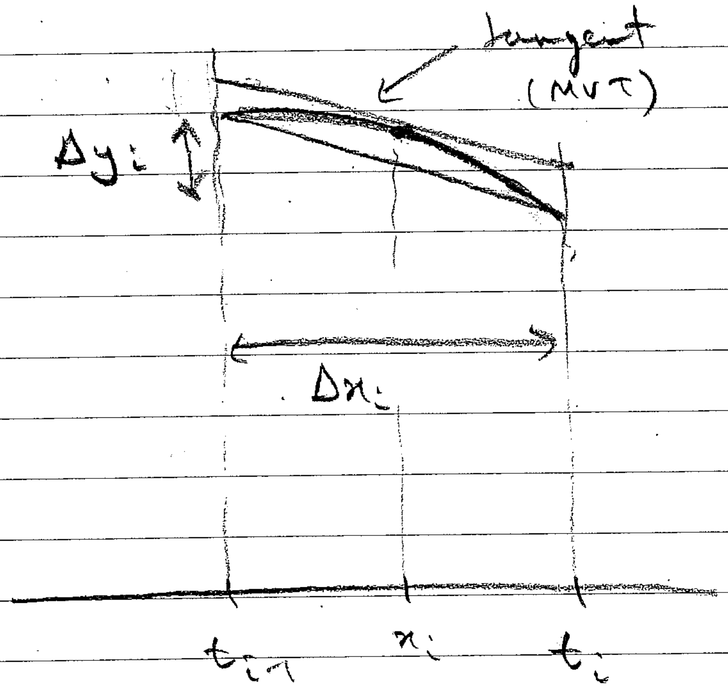


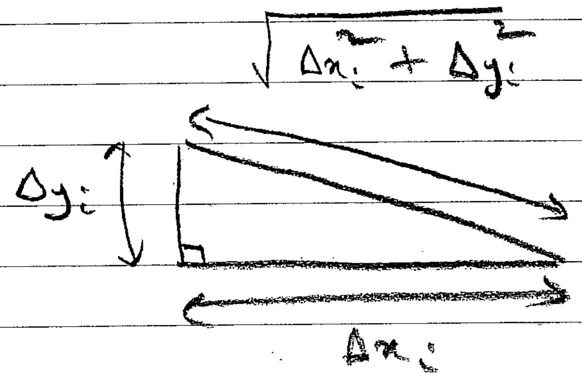
$$\text{Area of cone} = \frac{2\pi r}{2\pi l}\left(\pi l^2\right)$$

$$= \pi r l$$

Ⓕ

Returning to our rotated curve





Choose x_i such that

$$\boxed{\Delta y_i = f'(x_i)\,\Delta x_i} \qquad \text{(MVT)}$$

Surface area of rotated curve

$\approx$ surface area of frustrum

$$= \pi \sqrt{\Delta x_i^2 + \Delta y_i^2}\,\big(f(t_{i-1}) + f(t_i)\big)$$

$$= \pi\,\big(f(t_{i-1}) + f(t_i)\big)\left(\sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}}\right)\Delta x_i$$

Hence, putting together all the frustrums :

surface area of revolution

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \pi \left(f(x_{i-1}) + f(x_i) \right) \sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}} \; \Delta x_i$$

$$= \int_a^b \pi \left(f(x) + f(x) \right) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx$$

$$= \int_a^b \underbrace{2\pi y}_{\text{perimeter}} \; \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}_{\text{correction factor}} \; dx$$

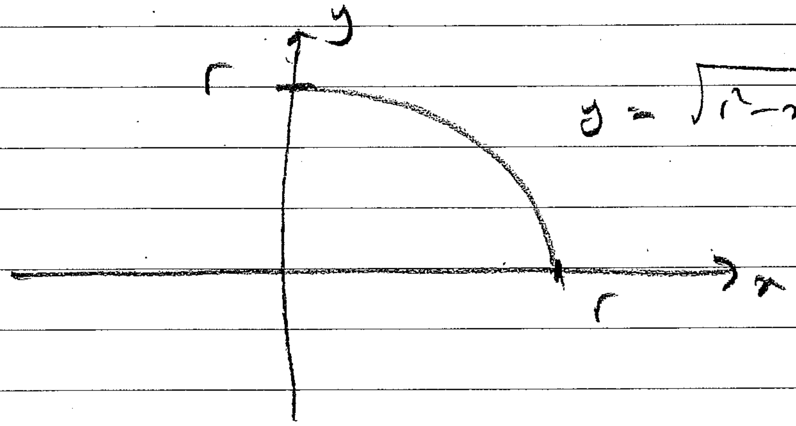
$$= \int_a^b 2\pi y \; ds$$

where $\boxed{ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx}$

is the differential of <u>arc length</u>

pp 64–69 ← length of a curve

Corrected example (surface area of a sphere):



surface area of half sphere

$$= \int_0^r 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = (r^2 - x^2)^{1/2},$$

so $y' = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x)$

$$= \frac{-x}{\sqrt{r^2 - x^2}}$$

so $(y')^2 = \frac{x^2}{r^2 - x^2}$

$$= 2\pi \int_0^r y \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= 2\pi \int_0^r y \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}}\, dx$$

$$= 2\pi \int_0^r y \frac{\sqrt{r^2}}{\sqrt{r^2 - x^2}}\, dx$$

$$= 2\pi \int_0^r \cancel{y} \frac{r}{\cancel{y}}\, dx$$

$$= 2\pi r \int_0^r dx = 2\pi r [x]_0^r = 2\pi r^2$$

so $\boxed{\text{surface area of sphere} = 4\pi r^2}$