University of Sydney

MATH 1901

Differential Calculus (Advanced)

Assignment 2

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1. (a) Using L'Hopital's Rule, we can compute the following limit.

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) &= \lim_{x\to 1}\left(\frac{x-1}{(\ln x)(x-1)}-\frac{\ln x}{(\ln x)(x-1)}\right)\\
&= \lim_{x\to 1}\left(\frac{x-1-\ln x}{(\ln x)(x-1)}\right)\\
&= \lim_{x\to 1}\left(\frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}\left[(\ln x)(x-1)\right]}\right)\\
&= \lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x}\right)\\
&= \lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{1-\frac{1}{x}+\ln x}\right)\\
&= \lim_{x\to 1}\left(\frac{\frac{d}{dx}(1-\frac{1}{x})}{\frac{d}{dx}(1-\frac{1}{x}+\ln x)}\right)\\
&= \lim_{x\to 1}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}}\right)\\
&= \lim_{x\to 1}\left(\frac{\frac{1}{x}}{\frac{1}{x}+1}\right)\\
&= \frac{1}{1+1}\\
\therefore \lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) &= \frac{1}{2}
\end{aligned}$$

(b) To compute the Taylor polynomial of order 5 of the function $f(x) := \frac{e^{x^2}}{x^2}$ about $x = 1$, we must first determine the derivatives, up to and including the fifth derivative, at the point $x = 1$. In order to compute this derivative, we will use the Leibniz formula for the n-th derivative of a product of two functions. The formula to compute these derivatives is as follows:

$$(hg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}h^{(k)}g^{(n-k)}$$

Before calculating the derivatives using the Leibniz formula, we must compute the derivatives of $h = e^{x^2}$ and $g = x^{-2}$. For the function g, the derivatives are as follows.

$$\begin{aligned}
g^{(0)} &= x^{-2}\\
g^{(1)} &= -2x^{-3}\\
g^{(2)} &= 6x^{-4}\\
g^{(3)} &= -24x^{-5}\\
g^{(4)} &= 120x^{-6}\\
g^{(5)} &= -720x^{-7}
\end{aligned}$$

For the function f, the derivatives are less straight forward to calculate.

$$
\begin{aligned}
h^{(0)} &= e^{x^2} \\
\\
h^{(1)} &= \frac{d}{dx}\left[e^{x^2}\right] \\
\therefore h^{(1)} &= 2xe^{x^2} \\
\\
h^{(2)} &= \frac{d}{dx}\left[2xe^{x^2}\right] \\
&= 2e^{x^2} + 4x^2e^{x^2} \\
\therefore h^{(2)} &= 2e^{x^2}\left[2x^2+1\right] \\
\\
h^{(3)} &= \frac{d}{dx}\left[2e^{x^2} + 4x^2e^{x^2}\right] \\
&= 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2} \\
\therefore h^{(3)} &= 4xe^{x^2}\left[2x^2+3\right] \\
\\
h^{(4)} &= \frac{d}{dx}\left[4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2}\right] \\
&= 4e^{x^2} + 8x^2e^{x^2} + 8e^{x^2} + 16x^2e^{x^2} + 24x^2e^{x^2} + 16x^4e^{x^2} \\
&= 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2} \\
\therefore h^{(4)} &= 4e^{x^2}\left[4x^4+12x^2+3\right] \\
\\
h^{(5)} &= \frac{d}{dx}\left[12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}\right] \\
&= 24xe^{x^2} + 96xe^{x^2} + 96x^3e^{x^2} + 64x^3e^{x^2} + 32x^5e^{x^2} \\
&= 120xe^{x^2} + 160x^3e^{x^2} + 32x^5e^{x^2} \\
\therefore h^{(5)} &= 8xe^{x^2}\left[4x^4+20x^2+15\right]
\end{aligned}
$$

In order to calculate the derivatives of $\frac{e^{x^2}}{x^2}$, we will set $h = e^{x^2}$ and $g = x^{-2}$. Using these definitions, we can compute the derivatives at $x = 1$, up to and inclduing, the fifth derivative. Computing the zeroth derivative, in other terms the function itself, at $x = 1$, we get the result:

$$
\begin{aligned}
(hg)^{(0)} &= \frac{e^{x^2}}{x^2} \\
\therefore (hg)^{(0)} &= e
\end{aligned}
$$

Now computing the first derivative of the function using the Leibniz formula, and evaluating at $x = 1$, we get the result:

$$\begin{aligned}
(hg)^{(1)} &= \sum_{k=0}^{1} \binom{1}{k} h^{(k)} g^{(1-k)} \\
&= \binom{1}{0} h^{(0)} g^{(1)} + \binom{1}{1} h^{(1)} g^{(0)} \\
&= \binom{1}{0} \left[(e^{x^2})(-2x^{-3}) \right] + \binom{1}{1} \left[(2xe^{x^2})(x^{-2}) \right] \\
\therefore (hg)^{(1)} &= -2 + 2 \quad \text{Substituting } x = 1 \\
&= 0
\end{aligned}$$

Now computing the second derivative and evaluating at $x = 1$, we get the result:

$$\begin{aligned}
(hg)^{(2)} &= \sum_{k=0}^{2} \binom{2}{k} h^{(k)} g^{(2-k)} \\
&= \binom{2}{0} h^{(0)} g^{(2)} + \binom{2}{1} h^{(1)} g^{(1)} + \binom{2}{2} h^{(2)} g^{(0)} \\
&= \binom{2}{0} \left[(e^{x^2})(6x^{-4}) \right] + \binom{2}{1} \left[(2xe^{x^2})(-2x^{-3}) \right] + \binom{2}{2} \left[(2e^{x^2}(2x^2+1))(x^{-2}) \right] \\
\therefore (hg)^{(2)} &= 6e - 8e + 6e \quad \text{Substituting } x = 1 \\
&= 4e
\end{aligned}$$

Now computing the third derivative at $x = 1$, we get the following result:

$$\begin{aligned}
(hg)^{(3)} &= \sum_{k=0}^{3} \binom{3}{k} h^{(k)} g^{(3-k)} \\
&= \binom{3}{0} h^{(0)} g^{(3)} + \binom{3}{1} h^{(1)} g^{(2)} + \binom{3}{2} h^{(2)} g^{(1)} + \binom{3}{3} h^{(3)} g^{(0)} \\
&= \binom{3}{0} \left[(e^{x^2})(-24x^{-5}) \right] + \binom{3}{1} \left[(2xe^{x^2})(6x^{-4}) \right] \\
&\quad + \binom{3}{2} \left[(2e^{x^2}(2x^2+1))(-2x^{-3}) \right] + \binom{3}{3} \left[(4xe^{x^2}(2x^2+3))(x^{-2}) \right] \\
\therefore (hg)^{(3)} &= -24e + 36e - 36e + 20e \quad \text{Substituting } x = 1 \\
&= -4e
\end{aligned}$$

Now computing the fourth derivative of the function at $x = 1$, we get the result:

$$
\begin{aligned}
(hg)^{(4)} &= \sum_{k=0}^{4} \binom{4}{k} h^{(k)} g^{(4-k)} \\
&= \binom{4}{0} h^{(0)} g^{(4)} + \binom{4}{1} h^{(1)} g^{(3)} + \binom{4}{2} h^{(2)} g^{(2)} + \binom{4}{3} h^{(3)} g^{(1)} + \binom{4}{4} h^{(4)} g^{(0)} \\
&= \binom{4}{0} \left[(e^{x^2})(120x^{-6})\right] + \binom{4}{1} \left[(2xe^{x^2})(-24x^{-5})\right] \\
&\quad + \binom{4}{2} \left[(2e^{x^2}(2x^2+1))(6x^{-4})\right] + \binom{4}{3} \left[(4xe^{x^2}(2x^2+3))(-2x^{-3})\right] \\
&\quad + \binom{4}{4} \left[(4e^{x^2}(4x^4+12x^2+3))(x^{-2})\right] \\
\therefore (hg)^{(4)} &= 120e - 192e + 216e - 160e + 76e \qquad \text{Substituting } x = 1 \\
&= 60e
\end{aligned}
$$

Now computing the fifth derivative of the function at $x = 1$, we get the result that follows:

$$
\begin{aligned}
(hg)^{(5)} &= \sum_{k=0}^{5} \binom{5}{k} h^{(k)} g^{(5-k)} \\
&= \binom{5}{0} h^{(0)} g^{(5)} + \binom{5}{1} h^{(1)} g^{(4)} + \binom{5}{2} h^{(2)} g^{(3)} + \binom{5}{3} h^{(3)} g^{(2)} + \binom{5}{4} h^{(4)} g^{(1)} + \binom{5}{5} h^{(5)} g^{(0)} \\
&= \binom{5}{0} \left[(e^{x^2})(-720x^{-7})\right] + \binom{5}{1} \left[(2xe^{x^2})(120x^{-6})\right] \\
&\quad + \binom{5}{2} \left[(2e^{x^2}(2x^2+1))(-24x^{-5})\right] + \binom{5}{3} \left[(4xe^{x^2}(2x^2+3))(6x^{-4})\right] \\
&\quad + \binom{5}{4} \left[(4e^{x^2}(4x^4+12x^2+3))(-2x^{-3})\right] + \binom{5}{5} \left[(8xe^{x^2}(4x^4+20x^2+15))(x^{-2})\right] \\
\therefore (hg)^{(5)} &= -720e + 1200e - 1440e + 1200e - 760e + 312e \qquad \text{Substituting } x = 1 \\
&= -208e
\end{aligned}
$$

Using the results that we have just calculated, we will amalgamate the values of each derivative for the function $f(x) = \frac{e^{x^2}}{x^2}$ at $x = 1$.

$$
\begin{aligned}
f^{(0)}(1) &= e \\
f^{(1)}(1) &= 0 \\
f^{(2)}(1) &= 4e \\
f^{(3)}(1) &= -4e \\
f^{(4)}(1) &= 60e \\
f^{(5)}(1) &= -208e
\end{aligned}
$$

Now we will examine the general form for the Taylor expansion about some arbitrary point, x_0, of order 5.

$$
\begin{aligned}
T_5(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x - x_0)^4 \\
+ \frac{f^{(5)}(x_0)}{5!}(x - x_0)^5
\end{aligned}
$$

Using the values for the derivatives of $f(x)$ about the point $x = 1$, we can calculate the Taylor expansion of order 5 for the function $f(x) = \frac{e^{x^2}}{x^2}$.

$$
\begin{aligned}
T_5(x) &= f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x - x_0)^4 \\
&\quad + \frac{f^{(5)}(x_0)}{5!}(x - x_0)^5 \\
&= f(1) + f^{(1)}(1)(x - 1) + \frac{f^{(2)}(1)}{2!}(x - 1)^2 + \frac{f^{(3)}(x1)}{3!}(x - 1)^3 + \frac{f^{(4)}(1)}{4!}(x - 1)^4 \\
&\quad + \frac{f^{(5)}(1)}{5!}(x - 1)^5 \\
&= e + 0 + \frac{4e}{2}(x - 1)^2 + \frac{-4e}{6}(x - 1)^3 + \frac{60e}{24}(x - 1)^4 + \frac{-208e}{120}(x - 1)^5 \\
\therefore T_5(x) &= e + 2e(x - 1)^2 - \frac{2e}{3}(x - 1)^3 + \frac{5e}{2}(x - 1)^4 - \frac{26e}{15}(x - 1)^5
\end{aligned}
$$

2. (a) The Mean Value Theorem states that for some function $f : [a, b] \to \mathbb{R}$ be continuous and $f : (a, b) \to \mathbb{R}$ be differentiable, there exists $c \in (a, b)$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$. By considering cases for the given inequality, $\sqrt{1 + x} \leq 1 + \frac{x}{2}$ for $x \in (-1, \infty)$, we can use the Mean Value Theorem to prove this inequality holds $\forall x \in (-1, \infty)$.

Considering the first case, where $x \in (-1, 0)$, we define $f(t) := \sqrt{1 + t}$ for $t \in [x, 0]$. By the Mean Value Theorem, there exists $c \in (x, 0)$ such that:

$$
\begin{aligned}
\frac{f(0) - f(x)}{0 - x} &= f'(c) \\
\therefore \frac{1 - \sqrt{1 + x}}{-x} &= \frac{1}{2\sqrt{1 + c}} \\
\therefore 1 - \sqrt{1 + x} &= \frac{-x}{2\sqrt{1 + c}}
\end{aligned}
$$

As $c \in (x, 0)$ for $x \in (-1, 0)$, it follows that:

$$
0 < \sqrt{1 + c} < 1 \implies \frac{1}{2\sqrt{1 + c}} > \frac{1}{2}
$$

Now, as for $x \in (-1, 0)$, $x < 0$, $\therefore -x > 0$. Thus it follows that:

$$
\begin{aligned}
\frac{-x}{2\sqrt{1 + c}} &> \frac{-x}{2} \\
\therefore 1 - \sqrt{1 + x} &> \frac{-x}{2} \\
\therefore \sqrt{1 + x} &< 1 + \frac{x}{2} \qquad \forall x \in (-1, 0)
\end{aligned}
$$

Now, considering the second case, where $x \in (0, \infty)$, we define $f(t) := \sqrt{1 + t}$ for $t \in [0, x]$. By the Mean Value Theorem, there exists $c \in (0, x)$ such that:

$$\begin{aligned}
\frac{f(x)-f(0)}{x-0} &= f'(c)\\
\therefore \frac{\sqrt{1+x}-1}{x} &= \frac{1}{2\sqrt{1+c}}\\
\therefore \sqrt{1+x}-1 &= \frac{x}{2\sqrt{1+c}}
\end{aligned}$$

As $c \in (0,x)$ for $x \in (0,\infty)$, it follows that:

$$\sqrt{1+c} > 1 \implies \frac{1}{2\sqrt{1+c}} < \frac{1}{2}$$

Now, as for $x \in (0,\infty)$, $x > 0$. Thus it follows that:

$$\begin{aligned}
\frac{x}{2\sqrt{1+c}} &< \frac{x}{2}\\
\therefore \sqrt{1+x}-1 &< \frac{x}{2}\\
\therefore \sqrt{1+x} &< 1+\frac{x}{2} \qquad \forall x \in (0,\infty)
\end{aligned}$$

Now considering the third and final case, where $x = 0$, we get the following results.

$$\begin{aligned}
x = 0 &\implies \begin{cases} \sqrt{1+x} = 1\\ 1+\frac{x}{2} = 1\end{cases}\\
\therefore \sqrt{1+x} &= 1+\frac{x}{2} \qquad \text{for } x = 0
\end{aligned}$$

Thus combining the three cases, in the order Case 1, Case 3, Case 2, we get the following results.

$$\begin{aligned}
\sqrt{1+x} &< 1+\frac{x}{2} \qquad \forall x \in (-1,0)\\
\sqrt{1+x} &= 1+\frac{x}{2} \qquad \forall x \in [0,0]\\
\sqrt{1+x} &< 1+\frac{x}{2} \qquad \forall x \in (0,\infty)
\end{aligned}$$

Thus it is clear that the inequality holds for the union of the continuous intervals $(-1,0) \cup [0,0] \cup (0,\infty)$. As a result, the inequality, $\sqrt{1+x} \le 1+\frac{x}{2}$ holds $\forall x \in (-1,\infty)$, with equality occuring at $x = 0$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Fix some $x_0 \in \mathbb{R}$. Caratheodory's characterisation for differentiability at x_0 asserts that there exists a function $m_{x_0} : \mathbb{R} \to \mathbb{R}$ that is continuous at x_0, such that

$$f(x) = f(x_0) + m_{x_0}(x)(x - x_0)$$

for all $x \in \mathbb{R}$. In this characterisation, $f'(x_0) = m_{x_0}(x_0)$. This characterisation defines the function $f(x)$ to be differentiable at the point x_0. As a result, $f'(x_0) = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$. In other words, $m_{x_0}(x_0) = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.

For the following proof, assume f is bijective with inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$, with $f'(x_0) \neq 0$, and that the inverse is continuous. We are first required to prove that f^{-1} is differentiable at $y_0 \coloneqq f(x_0)$. Examining Caratheodory's characterisation at the point $y_0 \coloneqq f(x_0)$, we get the following result.

$$\begin{aligned}
f(x) &= f(x_0) + m_{x_0}(x)(x - x_0) \\
\therefore m_{x_0}(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{f^{-1}(f(x)) - f^{-1}(f(x_0))} \qquad \text{as } f(x) \text{ is bijective and has an inverse} \\
&= \lim_{y \to y_0} \frac{y - y_0}{f^{-1}(y) - f^{-1}(y_0)} \\
\therefore \frac{1}{m_{x_0}(x_0)} &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \qquad \text{as } f'(x_0) \neq 0 \text{ and thus } m_{x_0}(x_0) \neq 0
\end{aligned}$$

Thus it is clear that the inverse function is defined and differentiable at the point $y_0 \coloneqq f(x_0)$. Defining $m_{y_0}(y_0) \coloneqq \frac{1}{m_{x_0}(x_0)}$, we are able to write the inverse function, $f^{-1}(x)$, in the form of Caratheodory's characterisation.

$$\therefore f^{-1}(y) = f^{-1}(y_0) + m_{y_0}(y)(y - y_0)$$

Thus it is clear that at the point $y_0 \coloneqq f(x_0)$, $f^{-1}(x)$ is differentiable, as it can be written in the form of Caratheodory's characterisation. In order for Caratheodory's characterisation to be valid for the inverse function, we assume that $f^{-1}(x)$ is continuous. Furthermore, $f'(x_0) \neq 0$, and as $f'(x_0) = m_{x_0}(x_0)$, $\therefore m_{x_0}(x_0) \neq 0$, and thus $\frac{1}{m_{x_0}(x_0)} = m_{y_0}(y_0)$ exists, and is defined. We are now required to prove $(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$, and so examining the relationship between Caratheodory's characterisation and the derivative of the function, we get the following results.

$$\begin{aligned}
f'(x_0) &= m_{x_0}(x_0) \\
\therefore \frac{1}{m_{x_0}(x_0)} &= m_{y_0}(y_0) \\
\therefore \frac{1}{f'(x_0)} &= m_{y_0}(y_0) \\
(f^{-1})'(y_0) &= m_{y_0}(y_0) \\
\therefore (f^{-1})'(y_0) &= \frac{1}{f'(x_0)} \\
\therefore (f^{-1})'(y_0) &= \frac{1}{f'(f^{-1}(y_0))}
\end{aligned}$$