Maps of the Earth: An Introduction

Notes for MATH1906 — Semester 1, 2017

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1 Introduction

We start this exposition on map projections by listing the properties of the ideal map, taken loosely from Johann Heinrich Lambert's seminal 1772 treatise [3] on map making:

> *One readily gives properties of the perfect map.*
>
> 1. *It should not distort the shape of the countries.*
> 2. *The relative size of the countries should not be changed.*
> 3. *The distances between any two places have the correct proportions.*
> 4. *Every straight line on the Earth should be a straight line on the map.*
> 5. *The longitude and latitude of every place should be easy to read from the map.*
>
> *That means that a map should be such that countries, continents or even the whole world appear in the same way as a plan of a house, farm or a city. The Earth is a sphere, and therefore this is impossible. Hence not all requirements can be satisfied in any particular map, but only some of them.*

Lambert then goes on and describes his approach to map making: Choose the projection surface and some properties the map should have. Then derive and solve differential equations for the coordinates. Given the various properties of a map we desire there are many different constructions. This approach to map making has not changed until now. The aim of this exposition is explain some of the map constructions. Some material, particularly Section 4, is taken from [2].

After introducing some preliminary material we first discuss some classes of angle preserving maps. The first one is also one of the oldest, namely the stereographic projection going back to the antiquities. The second one is the Mercator projection going back to Gerardus Mercator 1569. Identifying the plane with the complex plane (or Argand diagram), we show that the complex exponential function transforms the Mercator projection into the stereographic projection. Modification of that idea then lead us to the Lambert conic conformal maps, and other conformal maps.

The second class of maps we discuss are area preserving maps. These are maps with the property that for every region U on the sphere, the ratio of the surface area of U and the surface area of the image $g(U)$ on the map is the same. More precisely,

$$\frac{\text{surface area of } U}{\text{surface area of } g(U)} = \text{constant}$$

for every region U on the sphere.

2 Preliminary Material

In this section we introduce some basic facts and terminology about coordinates systems and transformations. First we introduce a coordinate system on the sphere. Each location on the sphere is given uniquely by its longitude and latitude as done in geographic maps. Let $\mathbb{S}$ denote a sphere in $\mathbb{R}^3$ of radius r centred at the origin, that is,

$$\mathbb{S} = \{(x, y, z) : \; x^2 + y^2 + z^2 = r^2\}.$$

The sphere is a two dimensional surface, so we need two parameters to describe it.

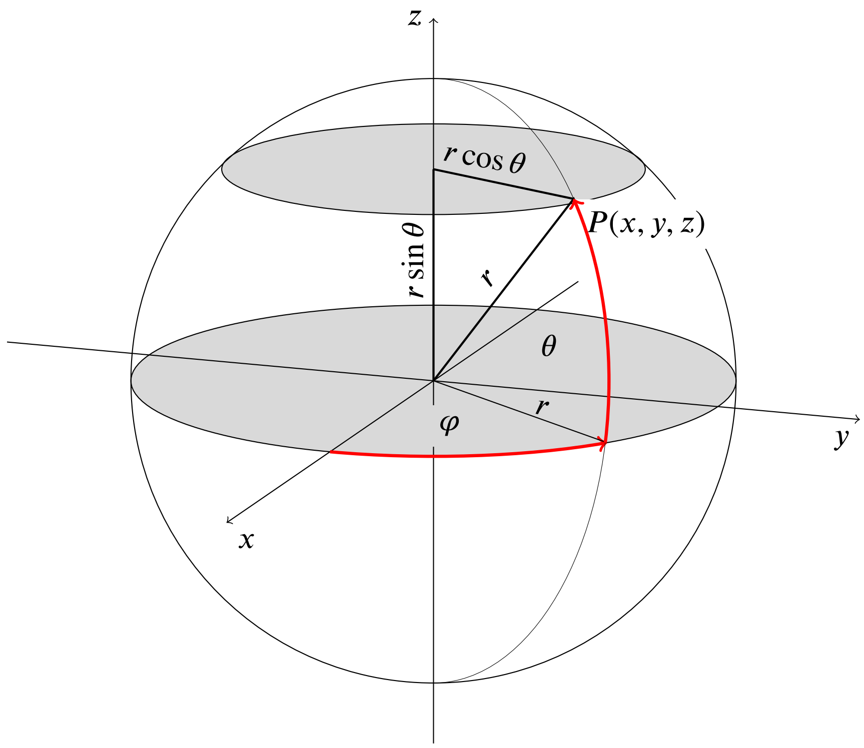

Figure 2.1: Longitude, latitude and spherical coordinates.

Each point $P(x, y, z)$ on the sphere is uniquely determined by an angle $\varphi \in (-\pi, \pi]$ in the xy-plane from the positive x-axis, and an angle $\theta \in [-\pi/2, \pi/2]$ along a great circle through the poles $(0, 0, \pm r)$ as shown in Figure 2.1. We call φ the *longitude* and θ the *latitude* of P. We can express the coordinates (x, y, z) of P by means of these angles. The point P lies on a circle given by the intersection of the plane $z = r\sin\theta$. The radius of that circle is $r\cos\theta$; see Figure 2.1. Hence we can express the x and y coordinates in polar coordinates using that radius. The coordinates of P therefore can be written as

$$\begin{aligned} x &= r\cos\theta\cos\varphi, \\ y &= r\cos\theta\sin\varphi, \\ z &= r\sin\theta. \end{aligned} \tag{2.1}$$

A map is a representation of the sphere, or part thereof, on the plane. The coordinates of the image point $Q(u, v)$ of $P(x, y, z)$ are functions of (x, y, z) or (φ, θ). To describe a map we need to find that function. This may involve geometric constructions, deriving a differential equation. The differential equations usually arise from conditions we impose on the map such as preserving angles or area, and how we want the meridians and parallels to look like on the map.

An important class of maps are angle preserving maps. We also call such maps *conformal* maps. This means that the images any two smooth curves intersect at the same angle as the original curves. The angle of intersection of two curves is defined to be the angle of intersection of their tangents at the point of intersection; see Figure 2.2. The lines ℓ_1 and ℓ_2 are the tangents to the curves γ_1 and γ_2 at the point of intersection. In particular

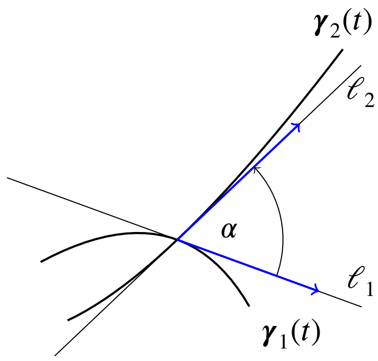

Figure 2.2: Angle of intersection of two curves

all curves intersecting at the same point having the same tangents have the same angle of intersection. We often use this fact to show that a given map is conformal. It allows to replace any curve with a simpler one having the same tangent. A curve on a sphere is the image of a function $\gamma(t) = (x(t), y(t), z(t))$ defined on some interval $J = (a, b)$ with image on the sphere. A curve on the plane only has two component functions $\gamma(t) = (x(t), y(t))$, $t \in J$. Smooth means that γ is differentiable. The derivative by definition is the limit

$$\gamma'(t) := \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

The derivative $\gamma'(t)$ is a vector tangent to the curve at $\gamma(t)$. It is obtained by componentwise differentiation of $\gamma(t)$. If it is a curve in the plane for instance this means that $\gamma'(t) = (x'(t), y'(t))$.

Having one conformal map on the plane we can get other conformal maps by applying suitable transformations. For that purpose it is useful to consider maps and curves on $\mathbb{R}^2$ as maps or curves on $\mathbb{C}$. We identify points in $\mathbb{R}^2$ by the corresponding point on the complex plane, that is, the Argand diagram. A curve $(x(t), y(t))$ is then given by

$$\gamma(t) = x(t) + iy(t) \in \mathbb{C}.$$

Differentiation with respect to $t \in \mathbb{R}$ is as before, namely

$$\gamma'(t) = x'(t) + iy'(t).$$

The advantage of working with transformations on the complex plane is that differentiable functions $f : \mathbb{C} \to \mathbb{C}$ are conformal. Differentiation of a function of a complex variable is defined in exactly the same way as for a function of a real variable, namely

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

It turns out that all rules of differentiation such as the product rule, the quotient rule and the chain rule apply in exactly the same way as for functions of a real variable. Most familiar functions like polynomials and rational functions can be differentiated the same way as their real counterparts. We will also define the complex exponential function e^z, and it turns out that its derivative is e^z.

We now show that differentiable function on the complex plane are conformal. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is differentiable and $\gamma_1(t)$ and $\gamma_2(t)$ are smooth curves that intersect at $z_0 = \gamma_1(0) = \gamma_2(0)$. By the chain rule

$$\frac{d}{dt} f(\gamma_i(t))|_{t=0} = f'(\gamma_i(0))\gamma_i'(0) = f'(z_0)\gamma_i'(0) \tag{2.2}$$

for $i = 1, 2$. The direction of the tangent to $\gamma_1(t)$ and $\gamma_2(t)$ at z_0 is given by $\gamma_1'(0)$ and $\gamma_2(0)$, respectively. Equation (2.2) shows that the direction of the tangents to the images $f(\gamma_1(t))$ and $f(\gamma_2(t)$ at $f(z_0)$ are given by $f'(z_0)\gamma_1'(t)$ and $f'(z_0)\gamma_2'(t)$, respectively. Recall that multiplication by a complex number means a rotation and dilation. In particular, $\gamma_1'(0)$ and $\gamma_2'(0)$ get rotated by the same angle given by the multiplication by the complex number $f'(z_0)$. The situation is illustrated in Figure 2.3.

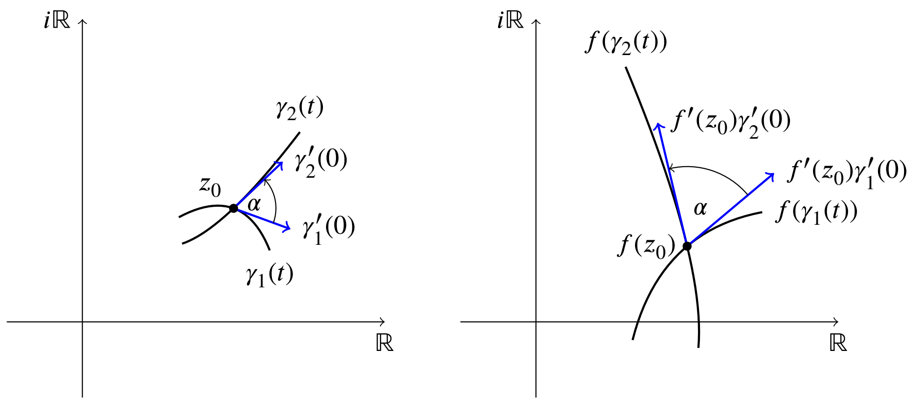

Figure 2.3: Differentiable functions on $\mathbb{C}$ define conformal maps

3 The Stereographic Projection

The stereographic projection is one of the oldest known map projections, and has been used extensively for navigation, and for determining calendar dates in the ancient mulim world. Unlike the Mercator projection, the stereographic projection is a *geometric projection*, and hence geometric arguments can be used to prove many of its properties. The stereographic projection also plays an important role in complex analysis as a model of the complex plane containing the point at infinity, namely the *Riemann sphere*; see for instance [1].

The stereographic projection of a point P on the unit sphere from the North Pole N onto the plane containing the equator is the point Q where the line NP meets the plane as shown in Figure 3.1. We compute the stereographic projection, and show that it is

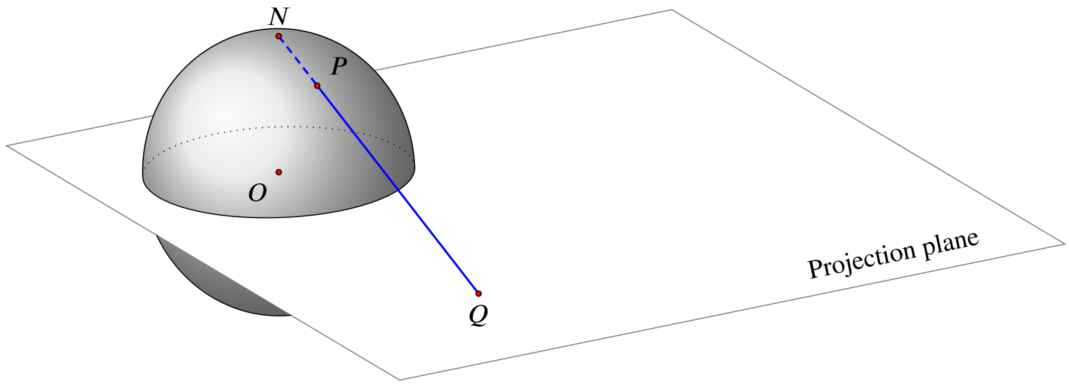

Figure 3.1: Stereographic projection from the North Pole.

conformal (angle preserving).

To compute the stereographic projection $Q(u, v)$ of a point $P(x, y, z)$ is a point on the sphere from the North Pole we need to determine the distance $OQ = \sqrt{u^2 + v^2}$. We look at a cross section through the North Pole, the center of the sphere and the point P as shown in Figrue 3.2. The point Z is the projection of P onto the z-axis parallel to the xy-plane. Looking at the figure we see that $\triangle NOQ$ and $\triangle NZP$ are similar and hence

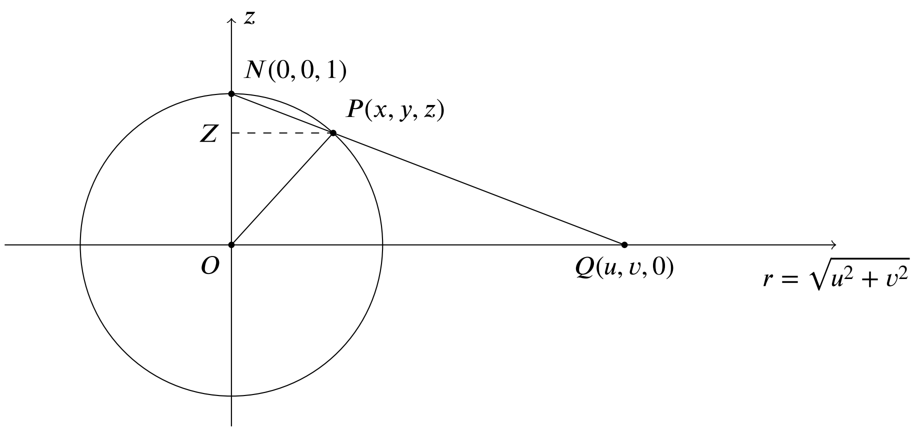

Figure 3.2: Cross section showing the sphere and the points P and Q.

$$\frac{OQ}{ON} = \frac{ZP}{ZN} \tag{3.1}$$

Since P is on a sphere of radius one we have $ON = 1$, $ZN = 1 - z$, $ZP = \sqrt{x^2 + y^2}$ and $OQ = \sqrt{u^2 + v^2}$. Hence (3.1) becomes

$$\sqrt{u^2 + v^2} = \frac{\sqrt{x^2 + y^2}}{1 - z},$$

and so

$$u = \frac{\sqrt{u^2 + v^2}}{\sqrt{x^2 + y^2}} x = \frac{x}{1 - z} \qquad \text{and} \qquad v = \frac{\sqrt{u^2 + v^2}}{\sqrt{x^2 + y^2}} y = \frac{y}{1 - z}.$$

As $x^2 + y^2 + z^2 = 1$ we also have

$$r^2 = u^2 + v^2 = \frac{x^2 + y^2}{(1 - z)^2} = \frac{1 - z^2}{(1 - z)^2} = \frac{1 + z}{1 - z}. \tag{3.2}$$

If we express u, v, z in terms of spherical coordinates (2.1) we get

$$r = \sqrt{\frac{1 + z}{1 - z}} = \sqrt{\frac{(1 + z)^2}{1 - z^2}} = \frac{1 + \sin\theta}{\cos\theta} = \sec\theta + \tan\theta.$$

Note that the stereographic projection of $P(x, y, z)$ from the South Pole is the same as the stereographic projection of $\tilde{P}(x, y, -z)$ from the North Pole. Hence to get the formula for the projection from the South Pole we just need to replace z by $-z$ in the above formulae. In particular, we see from (3.2) that we have to replace r by $1/r$ for the projection from the South Pole. Hence we have proved the first part of the following theorem.

Theorem 3.1 (*Stereographic projection*)

The stereographic projection of a point $P(x, y, z)$ on the unit sphere from the North Pole onto the plane through the equator is given by

$$u = \frac{x}{1 - z}, \qquad v = \frac{y}{1 - z}, \qquad r^2 = u^2 + v^2 = \frac{1 + z}{1 - z}. \tag{3.3}$$

Expressed in terms of longitude φ and latitude θ we have

$$u = r\cos\varphi, \qquad v = r\sin\varphi, \qquad r = \sec\theta + \tan\theta. \tag{3.4}$$

The stereographic projection from the South Pole is given by

$$u = \frac{x}{1 + z}, \qquad v = \frac{y}{1 + z}, \qquad r^2 = u^2 + v^2 = \frac{1 - z}{1 + z}. \tag{3.5}$$

Expressed in terms of φ and θ we have

$$u = r\cos\varphi, \qquad v = r\sin\varphi, \qquad r = \frac{1}{\sec\theta + \tan\theta}. \tag{3.6}$$

Moreover, the stereographic projection is conformal, that is, it is angle preserving.

Figure 3.3 shows the sterographic projection of the Earth from the South Pole. It is turned so that the null meridian is the negative y-axis.

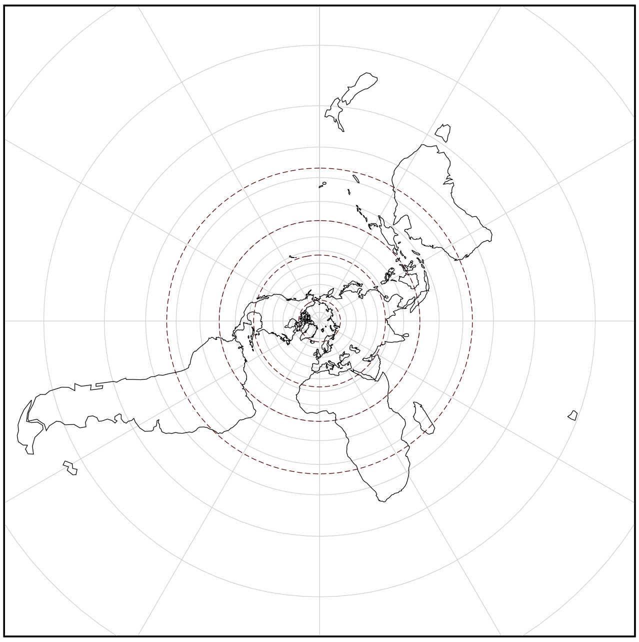

Figure 3.3: Stereographic projection of the Earth from the South Pole

To show that the stereographic projection is conformal we first look at two special curves. The first is a circle C_1 given by the intersection of the sphere with a plane containing the z-axis. The second curve is the circle C_2 given by the intersection of the sphere with an arbitrary other plane through the North Pole. Because the planes defining C_1 and C_2 contain the North Pole their images under the stereographic projection are the straight lines ℓ_1 and ℓ_2 given by the intersection of the xy-plane with the planes containing the circles. The circles intersect at the North Pole and some other point P on the sphere. Let Q be the stereographic projection of P on the xy-plane. Note that Q is the point of intersection of ℓ_1 and ℓ_2. We now show that the two circles on the sphere and the two lines intersect at the same angle. One possible situation is shown in Figure 3.4.

To do so we look at the tangent plane to the sphere at the point P. The tangent plane at P clearly contains the tangents to the cirles C_1 and C_2 at P. The tangent plane intersects the xy-plane in a line ℓ_3. Let A be the point of intersection of ℓ_1 with ℓ_3, and B the point of intersection of ℓ_2 with ℓ_3. We claim that $\triangle AQB$ in the xy-plane, and $\triangle AQB$ in the tangent plane to P are congruent, and therefore $\angle AQB$ and $\angle APB$ are equal. The side AB is in common for both triangles, and because ℓ_1 passes through the origin and ℓ_3 is defined by a tangent plane to the sphere we have $\angle QAB = \angle PAB = \pi/2$. To show the congruency of the two triangles it is sufficient to show that $AP = AQ$, which we prove by looking at Figure 3.5.

As $ON = OP$ we know that $\triangle NPO$ is isosceles. Hence if we let $\alpha := \angle NOP$, then $\angle ONP = \dfrac{\pi - \alpha}{2}$, and so $\angle NQO = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$. As $\triangle OPA$ is right angled at P we see that $\angle OAP = \alpha$, and $\angle PAQ = \dfrac{\pi}{2} - \alpha$. Hence we deduce that

$$\angle NQO = \angle APQ = \frac{\pi}{4} + \frac{\alpha}{2},$$

that is, $\triangle PAQ$ is isosceles. In particular $AP = AQ$, as claimed. The arguments apply

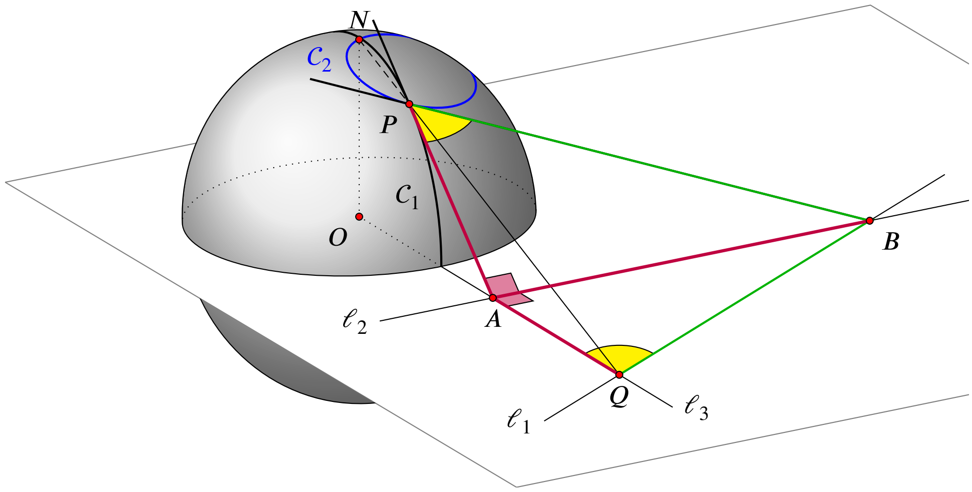

Figure 3.4: Two circles through N intersect at same angle as their projections.

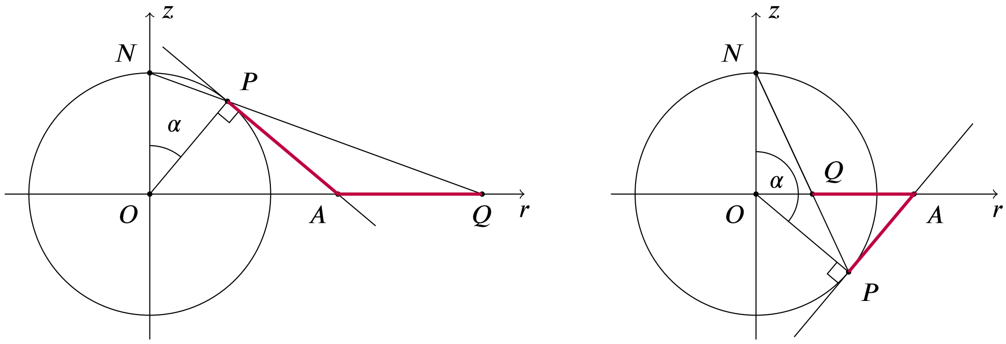

Figure 3.5: The length of AP is equal to that of AQ.

for P in the upper and the lower hemispheres; see also Figure 3.5.

To deal with general curves it is enough to look at a smooth curve on the sphere intersecting the circle C_1 at a point P. Then look at the tangent vector $\boldsymbol{v}$ to the curve at P. Then consider a plane $\mathcal{E}$ passing through N and P parallel to $\boldsymbol{v}$. The intersection of $\mathcal{E}$ with the sphere is a circle C_2. By construction the vector $\boldsymbol{v}$ is also tangent to C_2. The image of C_2 under the stereographic projection is also the projection of the tangent to the curve under the stereographic projection, and coincides with the tangent to the projection of the curve at Q. Hence the situation reduces to the one of two intersecting circles as considered before.

4 The Mercator Projection

The Mercator map is one of the most frequently seen maps. It is very useful for many practical purposes since it is designed so that every path of equal compass bearing on the globe appears as a straight line. In particular, it is angle preserving. The Mercator map is the having the following porperties:

(i) the north–south direction is the vertical direction;

(ii) the east–west direction is the horizontal direction with the length of the equator preserved;

(iii) all paths of equal compass bearing on the sphere are straight lines.

For simplicity we assume the sphere has radius one. The first two conditions imply that the image of the sphere lies in a strip of width 2π. Moreover, the meridians are mapped onto vertical lines and the parallels onto horizontal lines. Hence we only need to determine the spacing of the parallels. On the plane we introduce a rectangular coordinate system with $u = u(\varphi, \theta)$ the horizontal direction and $v = v(\varphi, \theta)$ the vertical direction.

We now consider a line of constant compass bearing on the sphere. Assume that the bearing from due North is α. By (ii) we have $u = \varphi$. Consider a small rectangle at (φ, θ) with $\Delta\varphi$ and $\Delta\theta$ determined by α as shown in Figure 4.1. Because the parallel at latitude

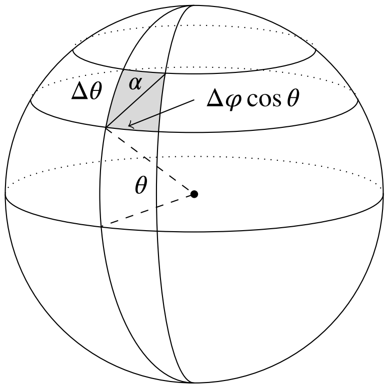

Figure 4.1: Small rectangle on the sphere.

θ has radius $\cos\theta$, the edge along the parallel has approximate length $\Delta\varphi\cos\theta$. The edge parallel to the meridian has length $\Delta\theta$; see Figure 4.1. Hence

$$\cot\alpha \approx \frac{\Delta\theta}{\Delta\varphi\cos\theta}.$$

The image of that path on the map is a straight line with angle α from the v-axis as shown in Figure 4.2. To satisfy (iii) we require that

$$\cot\alpha \approx \frac{\Delta v}{\Delta u} = \frac{\Delta v}{\Delta\varphi}.$$

Equating the two expressions for $\cot\alpha$ we get

$$\frac{\Delta\theta}{\cos\theta} = \Delta\theta\sec\theta \approx \Delta v.$$

If we let $\Delta\theta$ tend to zero we get

$$\frac{dv}{d\theta} = \sec\theta.$$

Integrating we get

$$v(\theta) = \log(\tan\theta + \sec\theta) + C$$

for some constant C. Since we require that $v(0) = 0$ we get $C = 0$.

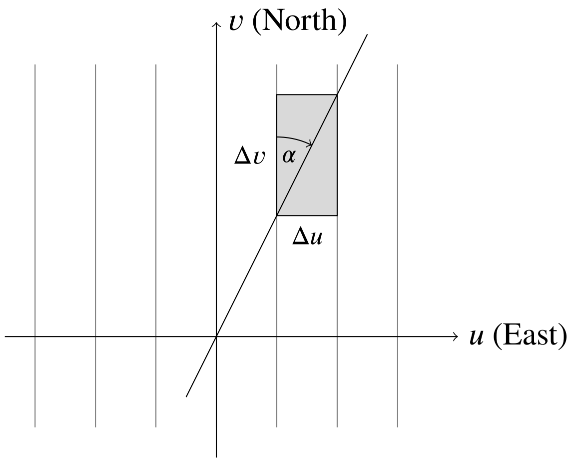

Figure 4.2: Grid for the Mercator projection.

Theorem 4.1 (***Mercator projection***)

The unique map with the properties (i)–(iii) *is given by*

$$\begin{aligned} u(\varphi, \theta) &= \varphi, \\ v(\varphi, \theta) &= \log(\tan\theta + \sec\theta), \end{aligned} \tag{4.1}$$

where φ is longitude and θ is latitude of a point on the sphere. Moreover, the map is a conformal.

An alternative construction is to make sure that the north–south distortion of length is the same as the east–west distortion. See for instance [5] or the comments in [4] for that approach.

Looking at (4.1) we see that $v(\theta) \to \pm\infty$ as $\theta \to \pm\pi/2$. The North and South Poles are at ∞ and $-\infty$, respectively. The sphere without the poles is mapped onto the infinite strip $(-\pi, \pi] \times \mathbb{R}$. This means that there is a huge area distortion in the polar regions. The map is shown in Figure 4.3, where the distortion is obvious by comparing the surface areas of Greenland or Antarctica to Australia for instance.

5 Mercator projection, stereographic projection and the complex exponential function

Apart from being conformal there does not seem to be a close connection between the Mercator and stereographic projections. However, we will show that applying the complex exponential function to the Mercator map, we get the stereographic projection. The idea is to identify points on the plane by the corresponding complex number on the complex plane, that is, the Argand diagram. If (x, y) is a point on the plane $\mathbb{R}^2$, then $z = x + iy$ is the corresponding point on the complex plane $\mathbb{C}$. One possible way to define the complex exponential function is

$$e^z := e^{x+iy} := e^x(\cos y + i \sin y). \tag{5.1}$$

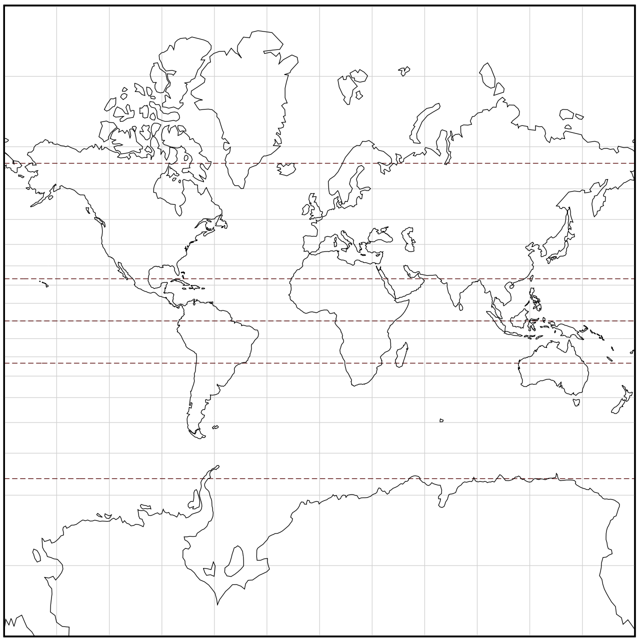

Figure 4.3: The Mercator projection

The reason for using the exponetial notation is that it shares many properties with the real exponential function.

Proposition 5.1

The complex exponential function has the following properties:

(i) $e^{z+w} = e^z e^w$ *for all* $z, w \in \mathbb{C}$;

(ii) *The map* $t \mapsto e^{it}$, $t \in \mathbb{R}$, *is* 2π*-periodic, and* $|e^{it}| = 1$ *for all* $t \in \mathbb{R}$.

(iii) *The line horizontal line* $\operatorname{Im} z = y$ *is mapped onto the ray* re^{iy}, $r > 0$. *The vertical line* $\operatorname{Re} z = x$ *is mapped onto the circle of radius* e^x *centred at the origin; see Figure 5.1.*

(iv) *The map*

$$\{z \in \mathbb{C} : \, -\pi < \operatorname{Im} z \leq \pi\} \to \mathbb{C} \setminus \{0\}, \quad z \mapsto e^z$$

is a bijection. The inverse function is the principal logarithm.

Proof: (i) Let $z = x+iy$ and $w = u+iv$ be complex numbers. Using the familiar trigonometric identities for the sum of two angles and the definition (5.1), we get

$$\begin{aligned} e^z e^w &= e^x e^u (\cos y + i \sin y)(\cos v + i \sin v) \\ &= e^{x+u}\big(\cos y \cos v - \sin y \sin v + i(\sin y \cos v + \cos y \sin v)\big) \\ &= e^{x+u}\big(\cos(y+v) + i \sin(y+v)\big) = e^{x+u+i(y+v)} = e^{z+w} \end{aligned}$$

(ii) The number $e^{it} = \cos t + i \sin t$ for $t \in \mathbb{R}$ represents the point $(\cos t, \sin t)$ on the unit circle centred at the origin. In particular $|e^{it}| = \sqrt{\cos^2 t + \sin^2 t} = 1$. As t varies over a half open interval of length 2π we reach each point on the circle exactly once.

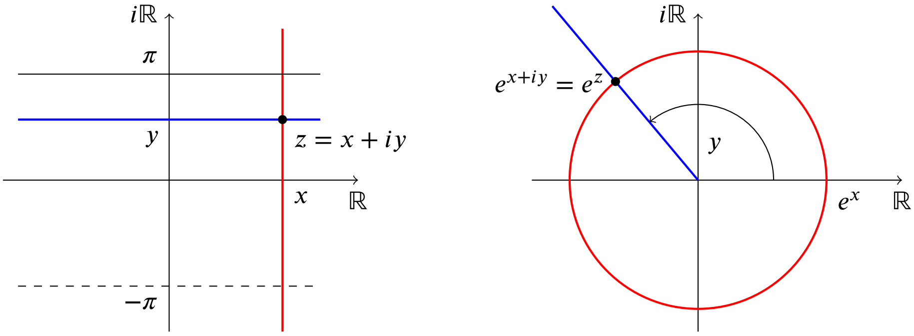

Figure 5.1: Mapping properties of the exponential function.

(iii) By definition of the exponential function (5.1) we see that if y is fixed, then $e^{x+iy} = e^x(\cos y + i \sin y)$ is a point having fixed polar angle y and distance e^x from the origin. Hence all points lie on a ray with polar angle y as seen in Figure 5.1. Similarly, if x is fixed, then $e^{x+iy} = e^x(\cos y + i \sin y)$ lies on a circle of radius e^x as y varies.

(iv) From the discussion in (iii) and Figure 5.1 the mapping is bijective if we restrict the polar angle to the interval $(-\pi, \pi]$, or any other half open interval of length 2π. $\square$

We now return to the Mercator and stereographic projections. The Mercator map covers a strip of width 2π. We can orient it on the complex plane so that north points into the direction of the negative real axis as shown in the left diagram in Figure 5.2. Using the equation (4.1) for the Mercator projection we can represent a point of longitude and latitude φ and θ by the complex number

$$z(\varphi, \theta) = -\log(\sec\theta + \tan\theta) + i\varphi \tag{5.2}$$

on an Argand diagram. Applying the complex exponential function we get

$$e^{z(\varphi,\theta)} = e^{-\log(\sec\theta+\tan\theta)}(\cos\varphi + i\sin\varphi) = \frac{1}{\sec\theta + \tan\theta}(\cos\varphi + i\sin\varphi). \tag{5.3}$$

The last expression exactly corresponds to the coordinates (u, v) for the stereographic projection from the South Pole given in (3.6). The situation is illustrated in Figure 5.2.

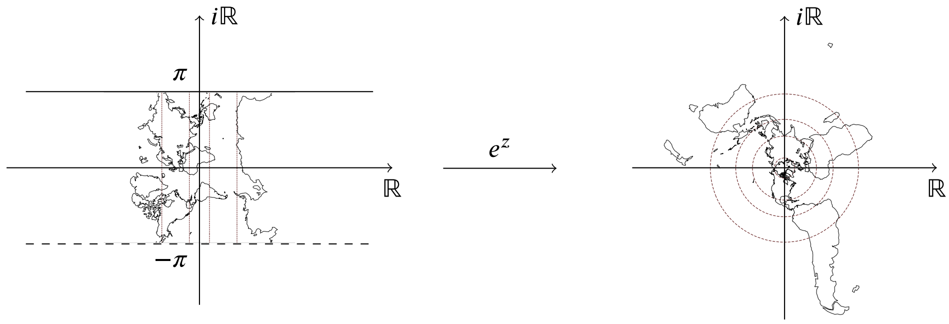

Figure 5.2: Complex exponential maps Mercator to Stereographic projection

If we use the map $z \mapsto e^{-z}$ we get

$$e^{-z(\varphi,\theta)} = e^{\log(\sec\theta+\tan\theta)}(\cos(-\varphi) + i\sin(-\varphi)) = (\sec\theta + \tan\theta)(\cos\varphi - i\sin\varphi),$$

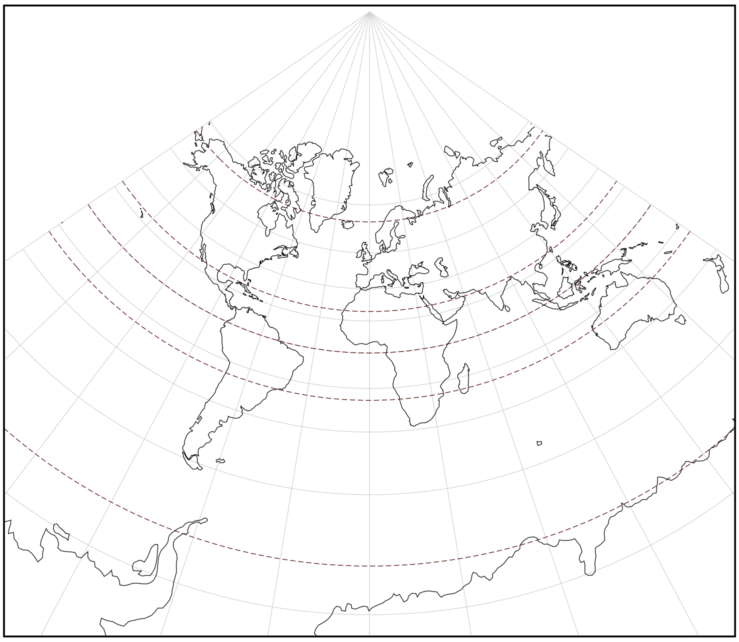

Figure 6.1: Lambert conic conformal map of the Earth

which corresponds to the stereographic projection from the North Pole. The minus sign in the last expression ensures that the map is correctly oriented with east going counter clockwise from the positive real axis.

Combining the two cases considered above we see that the reflection of the circle given by $w \mapsto 1/w$ maps the stereographic projection from the South Pole onto that from the North Pole and vice versa.

6 Lambert's Conic Conformal Projection

In this section we modify the idea used in the previous section. We used that the exponential function e^z maps an interval $\operatorname{Re} z = x$ fixed and $-\pi < \operatorname{Im} z \leq \pi$ onto a full circle. If we multyply z by some constant $\alpha \in [-1, 1]$, then such an interval is mapped onto an arc. Hence the Mercator map as discussed in the previous section is not mapped onto the whole space, but only onto some cone as shown in Figure 6.1. More precisely, we want to look at the map

$$z \mapsto \beta e^{\alpha z} \tag{6.1}$$

with $\alpha \in [-1, 1]$ non-zero and $\beta > 0$. The constant α determines the opening angle of the cone, and β is a scaling factor which allows us to adjust the size of the map. We use the fact that differentiable functions on the complex plane are angle preserving as shown in Section 2. Given that the Mercator map is conformal we then get another conformal map by applying (6.1) to the Mercator map. The purpose of conic map is to create a more accurate map near one or possibly two parallels. From these conditions we compute the constants α and β.

We now derive conditions which guarantee that the length of the parallel of latitude θ_0 is preserved. If the parallel is the equator, then this is achieved by the stereographic projection, and also by the Mercator projection.

Definition 6.1. A parallel whose length is preserved on the map is called a *standard parallel*.

On the Mercator map the parallel of latitude θ_0 is given by (5.2) with $\theta = \theta_0$ and $\varphi \in (-\pi, \pi]$. Its image on the map is an arc of angle $2\pi\alpha$ and radius

$$|\beta e^{\lambda z(\varphi,\theta_0)}| = \beta e^{-\alpha \log(\sec\theta_0+\tan\theta_0)} = \beta(\sec\theta_0 + \tan\theta_0)^{-\alpha}$$

The length of parallel on the sphere is $2\pi\cos\theta_0$ because it is a circle of radius $\cos\theta_0$ as seen from Figure 2.1. Hence, for the parallel of latitude θ_0 to be a standard parallel we need to make sure that the arc representing it on the map has the same length, that is,

$$2\pi\cos\theta = 2\pi|\alpha|\beta(\sec\theta_0 + \tan\theta_0)^{-\alpha}$$

or equivalently

$$\cos\theta_0 = |\alpha|\beta(\sec\theta_0 + \tan\theta_0)^{-\alpha} \tag{6.2}$$

We can play with both parameters, α and β, to construct a map with the properties we want.

In the first example we want to construct a map that little distortion near a given parallel. We imagine that the map is projected onto a cone to the sphere tangent to the sphere at the parallel θ_0 as shown in Figure 6.2. We can see from Figure 6.2 that the radius of the arc

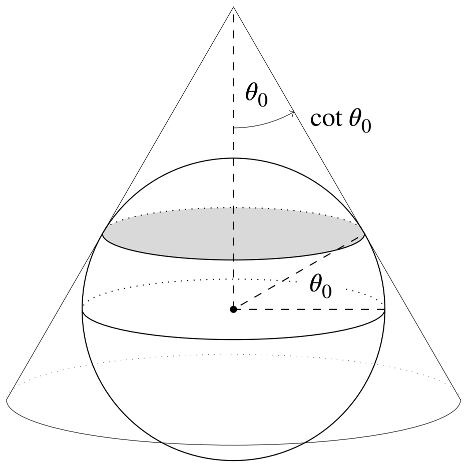

Figure 6.2: Cone tangent to the sphere.

representing the standard parallel is $\cot\theta_0$. Hence we require that $2\pi\cos\theta_0 = 2\pi\cot\theta_0$, so

$$\alpha = \sin\theta_0.$$

From (6.2) we deduce that

$$\beta = |\cot\theta_0|(\sec\theta_0 + \tan\theta_0)^{\sin\theta_0}.$$

Hence the map is given by

$$\beta e^{\alpha z(\varphi,\theta)} = \beta(\sec\theta + \tan\theta)^{-\sin\theta_0}.$$

Theorem 6.2 (***Lambert conic conformal projection***)

The conic conformal map with standard parallel at latitude θ_0 is given by

$$u(\varphi,\theta) = R(\varphi,\theta)\cos(\varphi\sin\theta_0),$$
$$u(\varphi,\theta) = R(\varphi,\theta)\sin(\varphi\sin\theta_0),$$

where

$$R(\varphi,\theta) = |\cot\theta_0|\Big(\frac{\sec\theta_0+\tan\theta_0}{\sec\theta+\tan\theta}\Big)^{\sin\theta_0}.$$

We can then rotate and translate the map to give it the right orientation as done in Figure 6.1.

As a second example we construct a map with two standard parallels. Such maps are good to represent medium sized regions such as Australia, China, Europe, or the USA. Such maps are conformal and the area distortion is minimal. As an example see Figure 9.3 with standard parallels at 40°N and 60°N.

Assume now that the standard parallels have latitude θ_0 and θ_1, respectively. According to (6.2) we require that α and β are such that both equations

$$\cos\theta_0 = |\alpha|\beta(\sec\theta_0+\tan\theta_0)^{-\alpha}$$
$$\cos\theta_1 = |\alpha|\beta(\sec\theta_1+\tan\theta_1)^{-\alpha}$$

are satisfied. Dividing the two equations we get

$$\frac{\cos\theta_0}{\cos\theta_1} = \frac{\sec\theta_1}{\sec\theta_0} = \Big(\frac{\sec\theta_0+\tan\theta_0}{\sec\theta_1+\tan\theta_1}\Big)^{-\alpha}$$

and hence

$$\alpha = \frac{\log(\frac{\sec\theta_0}{\sec\theta_1})}{\log(\frac{\sec\theta_0+\tan\theta_0}{\sec\theta_1+\tan\theta_1})}, \qquad \beta = \frac{\cos\theta_0}{|\alpha|}(\sec\theta_0+\tan\theta_0)^{\alpha}.$$

The conic conformal maps discussed here can also be derived without using complex numbers; see [2].

7 Other conformal maps

Lambert generalised the idea of conic conformal maps. Rather than assuming the meridians are straight lines meeting at one of the poles, and the are parallels circles centred at one of the poles, he assumed that also the meridians are circles that meet at both poles. To obtain such a map we can again use that differentiable maps of a complex variables are conformal maps. To construct the conic conformal maps we used the Mercator map and mapped one of the poles at infinity to zero. Here we use a similar idea, but map one pole to 1 and the other one to -1. This time we start with the stereographic projection. First we shift the North Pole at the center to 1 by using the translation $w \mapsto w-1$. Then we move the other pole at infinity to -1 by dividing by $w+1$. Hence we use the transformation

$$w \mapsto \frac{w-1}{w+1}. \tag{7.1}$$

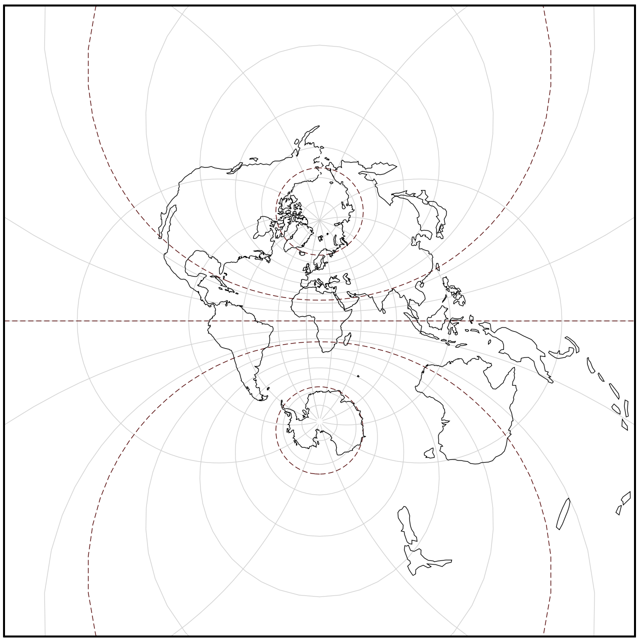

Figure 7.1: Conformal map with meridians intersecting at two given points.

After a rotation to orient the map so that North is up we get the map in Figure 7.1. The map shows massive area distortion away from the central meridian. It is suitable to map an area of the globe over a small range of longitudes and a large range of latitudes. An example of such regions are North and South America, or Europe and Africa; see Figure 7.2. These maps are discussed already by Lambert 1772 in [3].

The transformation (7.1) is a special case of a *fractional linear transformation* or a *Möbius transformation* of the complex plane. One can show that the particular transformation (7.1) can be obtained by rotating the sphere about the y-axis so that the poles are at $(\pm 1, 0, 0)$, and then applying a stereographic projection.

8 Rectangular equal area maps

Consider a cylindrical projection with a cylinder not tangent to the sphere, but passing through the parallels at $\pm\theta_0$. A cross section of the situation is shown in Figure 8.1. The width of the map is $2\pi\cos\theta_0$. To make sure that the map is area preserving we need to equate the surface area on the sphere between latitudes θ and $\theta + \Delta\theta$ to the surface area of a strip on the map between v and $v + \Delta v$. Hence we require

$$2\pi\Delta\theta\cos\theta \approx 2\pi\cos\theta_0\Delta v,$$

where we used that the radius of the parallel of latitude θ is $\cos\theta$ as seen from Figure 2.1. If we divide by $\Delta\theta$ and let $\Delta\theta$ to zero we get

$$\frac{dv}{d\theta} = \frac{\cos\theta}{\cos\theta_0}$$

Taking into account that $v(0) = 0$ we get

$$v(\theta) = \frac{\sin\theta}{\cos\theta_0}$$

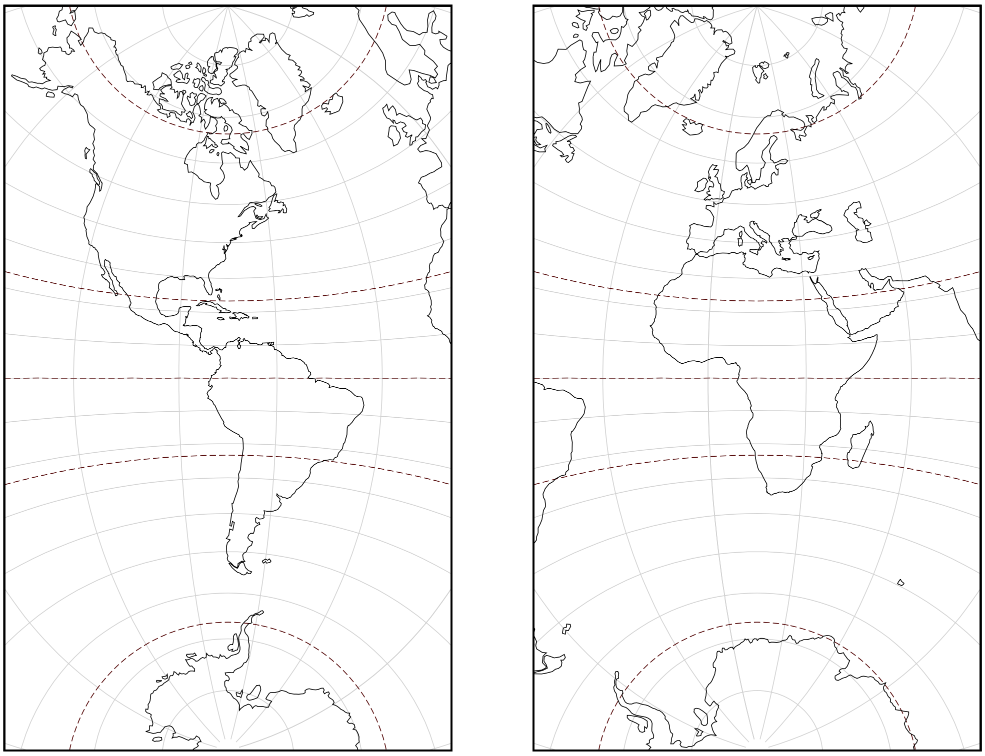

Figure 7.2: Conformal maps of the Americas, and of Europe and Africa.

Hence we get a family of area preserving maps with rectangular coordinate system.

Theorem 8.1 (*Lambert equal area projection*)

Let $\theta_0 \in (0, \pi/2)$. *Then the map given by*

$$u(\varphi, \theta) = \varphi \cos \theta_0, \qquad v(\varphi, \theta) = \frac{\sin \theta}{\cos \theta_0}$$

defines an area preserving map with standard parallels at $\pm\theta_0$.

There are some named special cases:

- Lambert equal area map (1772) with the equator being the standard parallel;
- Behrmann equal area map (Walter Behrmann 1910) with standard parallels at $\pm 30°$.
- Gall-Peters equal area map (James Gall 1855, "reinvented" and made popular by Arno Peters 1983) with standard parallels at 45°.
- Hobo-Dyer equal area map (2002) with standard parallels at $\pm 37.5°$ and South up; see Figure 8.2.

9 Lambert's Conic Equal Area Map

As in case of the conformal maps we want to generalise the rectangular maps, and look at a conical map. This time we cannot use differentiable functions in the complex plane as

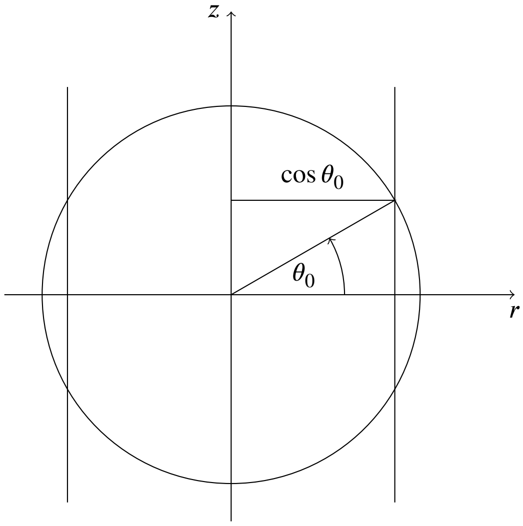

Figure 8.1: Cylindrical projection with standard parallels at latitude $\pm\theta_0$.

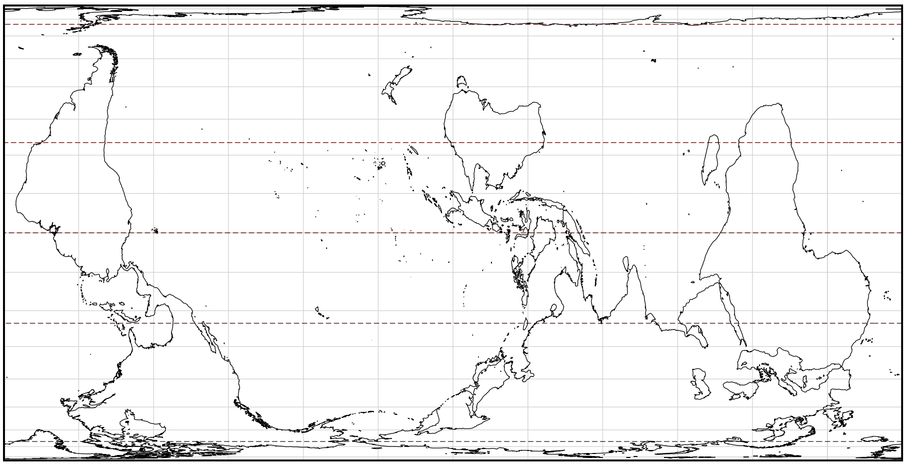

Figure 8.2: Hobo-Dyer equal area map.

these are automatically conformal, which makes it impossible to be area preserving. We therefore look at a general conical projection

$$u = \rho(\theta)\cos(\alpha\varphi), \qquad v = -\rho(\theta)\sin(\alpha\varphi),$$

where φ and θ are longitude and latitude, ρ is the radius of the circles representing the parallels and α determines the opening angle of the cone as shown in Figure 9.1. Assume that the meridians are equally spaced on the map. The aim is to determine the spacing of the parallels so that the map becomes area preserving. To derive a differential equation for $\rho(\theta)$ we consider a small rectangle on the sphere at (φ, θ) with side lengths $\Delta\varphi$ and $\Delta\theta$. The parallel at latitude θ has length $2\pi\cos\theta$. Hence a small rectangle on the sphere of side lengths $\Delta\varphi$ and $\Delta\theta$ has area approximately equal to $\Delta\varphi\Delta\theta\cos\theta$; see also Figure 4.1. On the map that rectangle has approximate area $\rho t\Delta\varphi\Delta\rho$. Hence we require that

$$\Delta\varphi\Delta\theta\cos\theta \approx \alpha\rho\Delta\varphi\Delta\rho$$

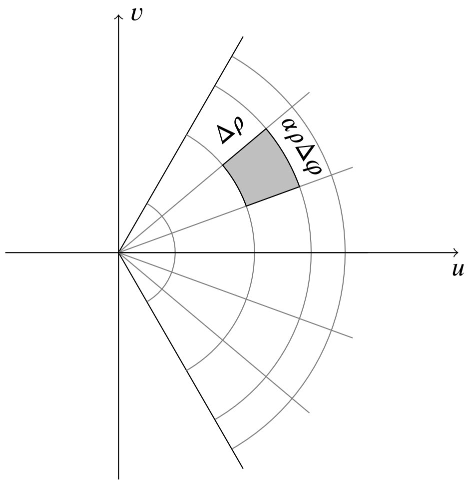

Figure 9.1: Cone showing the image of a small rectangle on the sphere.

as seen in the diagram above. Dividing by $\Delta\varphi\Delta\theta$ and taking into account that ρ is decreasing as θ is increasing we get

$$\cos\theta \approx -\frac{\Delta\rho}{\Delta\theta}\alpha\rho$$

Letting $\Delta\theta$ go to zero we get the differential equation

$$\cos\theta = -\alpha\frac{d\rho}{d\theta}\rho = -\frac{\alpha}{2}\frac{d(\rho^2)}{d\theta} \tag{9.1}$$

Integrating the above differential equation we get

$$\frac{1}{2}\int_{\theta_0}^{\theta}\frac{d\rho^2}{d\theta}\,d\theta = -\frac{1}{\alpha}\int_{\theta_0}^{\theta}\cos\theta\,d\theta$$

and hence

$$\rho^2(\theta) - \rho^2(\theta_0) = -\frac{2}{\alpha}(\sin\theta - \sin\theta_0). \tag{9.2}$$

We have the freedom of choosing the parameters $\rho_0 = \rho(\theta_0)$ and the opening angle given by α. We require the parallels of latitude θ_0 and θ_1 to preserve length. That means $2\pi\alpha\rho_i = 2\pi\cos\theta_i$ and hence

$$\rho_i = \frac{\cos\theta_i}{\alpha} \tag{9.3}$$

for $i = 0, 1$. Substituting into (9.2) we get

$$\frac{\cos^2\theta_1}{\alpha^2} - \frac{\cos^2\theta_0}{\alpha^2} = -\frac{2}{\alpha}(\sin\theta_1 - \sin\theta_0).$$

Therefore

$$\alpha = -\frac{\cos^2\theta_1 - \cos^2\theta_0}{2(\sin\theta_1 - \sin\theta_0)} = \frac{\sin^2\theta_1 - \sin^2\theta_0}{2(\sin\theta_1 - \sin\theta_0)} = \frac{\sin\theta_0 + \sin\theta_1}{2}.$$

Using (9.2) and (9.3) we get the following theorem.

Theorem 9.1 (***Lambert conic equal area projection***)

Let $\theta_0, \theta_1 \in (-\pi/2, \pi/2)$ be such that $\theta_1 \neq -\theta_0$. Then the area preserving conic map with standard parallels at latitude θ_0 and θ_1 is given by

$$u(\varphi, \theta) = R(\theta)\cos(\alpha\varphi), \qquad v(\varphi, \theta) = -R(\theta)\sin(\alpha\varphi),$$

where

$$\alpha = \frac{\sin\theta_0 + \sin\theta_1}{2} \quad \textit{and} \quad R(\theta) = \operatorname{sign}(\alpha)\sqrt{\frac{\cos^2\theta_0}{\alpha^2} + \frac{2}{\alpha}(\sin\theta_0 - \sin\theta)}.$$

The sign of α is there to make sure that the map is correct if the standard parallels are such that the meridians converge towards the South Pole.

As a special case we can have a cone tangent to the sphere at the parallel of latitude θ_0 as shown in Figure 6.2, we just set $\theta_0 = \theta_1$ in the above theorem. A map centred at 150°E and standard parallel of latitude 25°S is shown in Figure 9.2.

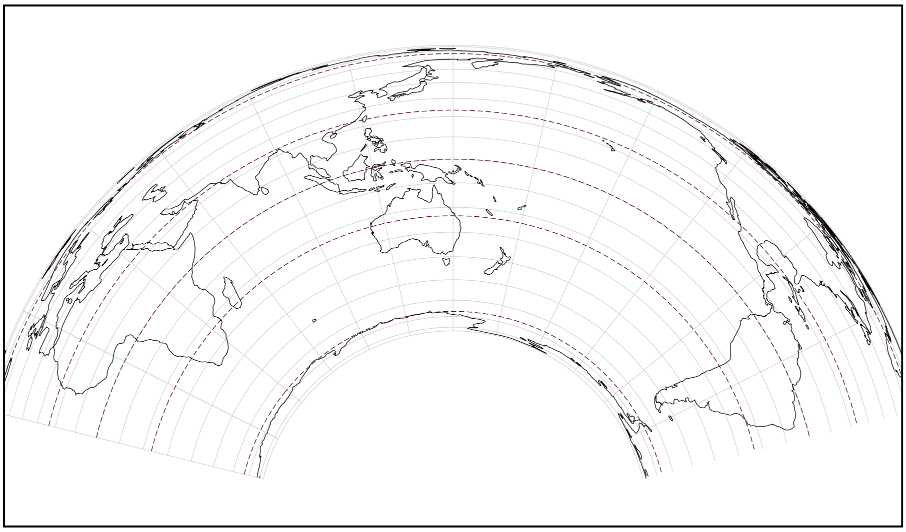

Figure 9.2: Conic area preserving map centred in Australia with standard parallel at 25°S.

As in case of the conic conformal map, we can represent medium sized regions with minimal distortion. Figure 9.3 shows an area preserving map and a conformal map of Europe with standard parallels at 40°N and 60°N.

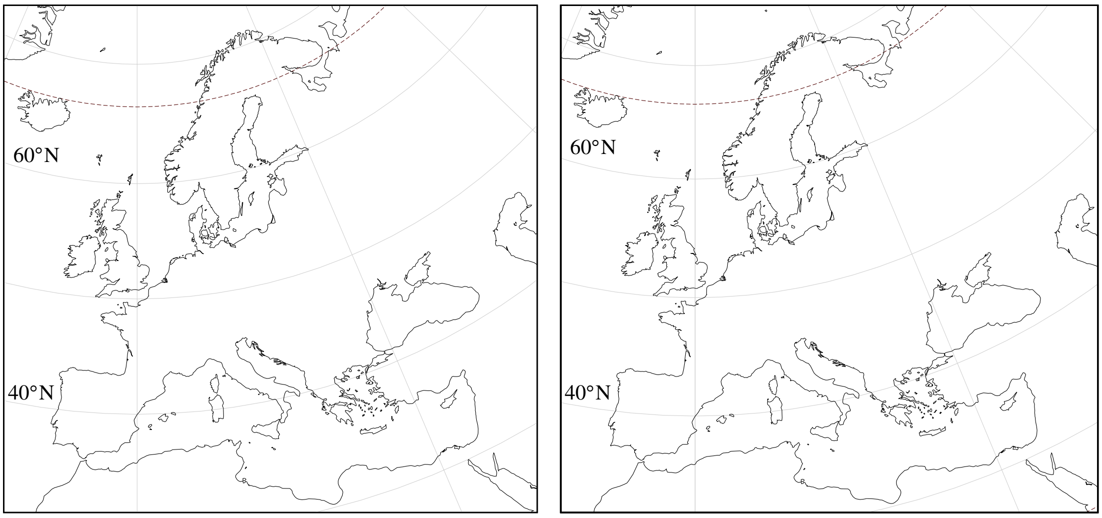

Figure 9.3: Conformal map (left) and area preserving map (right) of Europe with standard parallels at 40°N and 60°N.

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