THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 5

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Intuitive concept of a limit.
- ☐ Formal definition of limits in terms of ε and δ.
- ☐ The limit laws.
- ☐ The squeeze law.

Outcomes

After completing this tutorial you should

- ☐ understand the intuitive notion of limits;
- ☐ be able to work with the ε-δ definition of a limit in concrete and theoretical contexts;
- ☐ be able to prove the limit laws;
- ☐ calculate complicated limits using limit laws and the squeeze law;
- ☐ use the squeeze law to prove theoretical results;

Summary of essential material

Definition of a limit. We say that $\lim_{x\to a} f(x) = \ell$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon.$$

Note that

$$\lim_{x\to a} f(x) = \ell \qquad \Longleftrightarrow \qquad \lim_{x\to a} |f(x) - \ell| = 0.$$

We often use the latter in conjuction with the squeeze law. We can also consider limits from the right: We say that $\lim_{x\to a+} f(x) = \ell$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < x - a < \delta \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon.$$

Similarly we consider limits from the left: replace $x \to a+$ by $x \to a-$ and $0 < x - a < \delta$ by $0 < a - x < \delta$.

- A limit exists if and only if right and left hand limits exist and are equal.

Note on computing limits. The ε-δ definition of a limit cannot be used to compute a limit! It can be used to prove some number is the limit. If you do that you must make sure that the argument you provide works for *every* choice of $\varepsilon > 0$! The *limit laws* and the *squeeze law*, in conjunction with a number of elementary limits, are the main tools to compute limits!

Limit Laws. If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$, then

(1) $\lim_{x\to a}(kf(x)) = k\ell$ for all $k \in \mathbb{R}$.

(2) $\lim_{x\to a}(f(x) + g(x)) = \ell + m$.

(3) $\lim_{x\to a}(f(x)g(x)) = \ell m$.

(4) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{m}$ provided $m \neq 0$.

Squeeze Law. Suppose that

$$f(x) \le g(x) \le h(x) \quad \text{for all } x \text{ near } a \text{ (but not necessarily at } x = a).$$

If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = \ell$ then $\lim_{x\to a} g(x) = \ell$.

Questions to complete during the tutorial

Questions marked by * are harder questions.

1. Calculate the following limits using one or more of the limit laws and squeeze law. It is very tedious to write this down, please at least explain your group exactly how to apply the laws.

(a) $\displaystyle\lim_{x\to 3} \frac{x^2+3x+2}{4x^2-x+1}$

(b) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x-1}$

(c) $\displaystyle\lim_{x\to 1} \frac{x^2-3x+2}{x^3-1}$

(d) $\displaystyle\lim_{x\to 0} \frac{\sqrt{3+2x}-\sqrt{3}}{x}$

2. Use the Squeeze Law to calculate the limit $\displaystyle\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

3. Sketch the function with formula

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 2x+1 & \text{if } x > 0. \end{cases}$$

Find suitable values of δ such that whenever $0 < |x| < \delta$, we have:

(a) $|f(x) - 1| < 0.01$ (b) $|f(x) - 1| < 0.001$ (c) $|f(x) - 1| < \varepsilon$.

4. (a) Let $a > 0$. Use the squeeze law to show that $\displaystyle\lim_{x\to a} \sqrt{x} = \sqrt{a}$.

Hint: Rewrite $|\sqrt{x} - \sqrt{a}|$ so that $|x - a|$ appears in the expression.

(b) Use the ε-δ definition of limits to show that $\lim_{x\to 0+} \sqrt{x} = 0$.

5. The function f is defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

Use the squeeze law to show that $\lim_{x\to 0} f(x) = 0$. Also write down an ε-δ proof.

6. Suppose that $\displaystyle\lim_{x\to a} f(x) = \ell$ and $\displaystyle\lim_{x\to a} g(x) = m$. Use the ε-δ definition of limits to prove the following limit laws:

(a) $\displaystyle\lim_{x\to a} (kf(x)) = k\ell$, $k \in \mathbb{R}$ is a constant.

(b) $\displaystyle\lim_{x\to a} (f(x) + g(x)) = \ell + m$.

7. (a) Assume that $\displaystyle\lim_{x\to a} f(x) = \ell$. Let M_1, M_2 such that $M_1 < \ell < M_2$. Use the ε-δ definition of a limit to show that there exists a $\delta > 0$ such that $M_1 < f(x) < M_2$ for all $x \in (a-\delta, a+\delta)$, $x \neq a$.

*(b) If $\displaystyle\lim_{x\to a} f(x) = \ell$ and $\displaystyle\lim_{x\to a} g(x) = m$, show that $\displaystyle\lim_{x\to a}\big(f(x)g(x)\big) = \ell m$ (limit law of multiplication).

Hint: Write $f(x)g(x) - \ell m = f(x)\big(g(x) - m\big) + m\big(f(x) - \ell\big)$ and use the ε-δ definition of limits.

8. Prove or disprove the following statements:

(a) If $\lim_{x\to 0} f(x)^2$ exists then $\lim_{x\to 0} f(x)$ exists.

(b) If $\lim_{x\to 0} f(x^2)$ exists then $\lim_{x\to 0} f(x)$ exists.

*(c) If $\displaystyle\lim_{x\to a} g(x) = m$ and $\displaystyle\lim_{y\to m} f(y) = \ell$ then $\displaystyle\lim_{x\to a} f(g(x)) = \ell$. Here f and g both have domain $\mathbb{R}$.

Extra questions for further practice

9. Prove that $\lim_{x\to a} f(x) = \ell$ if and only if $\lim_{x\to a} |f(x) - \ell| = 0$.

10. (a) For $x, y \in \mathbb{R}$ we obviously have $|x| = \big|(x-y)+y\big|$ and $|y| = \big|(y-x)+x\big|$. Use the triangle inequality to show that $\big||x| - |y|\big| \le |x-y|$. This is called the *reversed triangle inequalty*.

(b) Hence, use the squeeze law to show $\lim_{x\to a} f(x) = \ell$ implies that $\lim_{x\to a} |f(x)| = |\ell|$.

(c) Is the converse of the statement in part (b) true? Give a proof or a counter example.

11. Prove the following results using the ϵ, δ definition:

(a) $\lim_{x\to 4} f(x) = -3$, where $f(x) = \begin{cases} 5-2x & \text{if } x \neq 4, \\ 100 & \text{if } x = 4. \end{cases}$

(b) $\lim_{x\to 0} g(x) = 0$, where $g(x) = \begin{cases} 3x & \text{if } x \text{ is rational}, \\ 7x & \text{if } x \text{ is irrational}. \end{cases}$

12. Sketch the graph of the function:

$$f(x) = \begin{cases} 0 & x < 0, \\ 1 & x = 0, \\ x+2 & x > 0. \end{cases}$$

Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ (no need for formal proofs). Does $\lim_{x\to 0} f(x)$ exist?

13. Using the limit laws, show that the limit of a function exists as $x \to a$, then the limit is unique. That is, prove that if $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} f(x) = m$, then $\ell = m$.
Hint: Write $\ell - m = (f(x) - m) - (f(x) - \ell)$.

Challenge questions (optional)

14. Prove that $\lim_{x\to 0} f(x)$ does not exist, where $f(x) = \begin{cases} 0 & \text{if } x \text{ rational}, \\ 1 & \text{if } x \text{ is irrational}. \end{cases}$

***15.** Prove the quotient limit law: If $\lim_{x\to a} g(x) = m$ and $m \neq 0$, then $\lim_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{m}$.