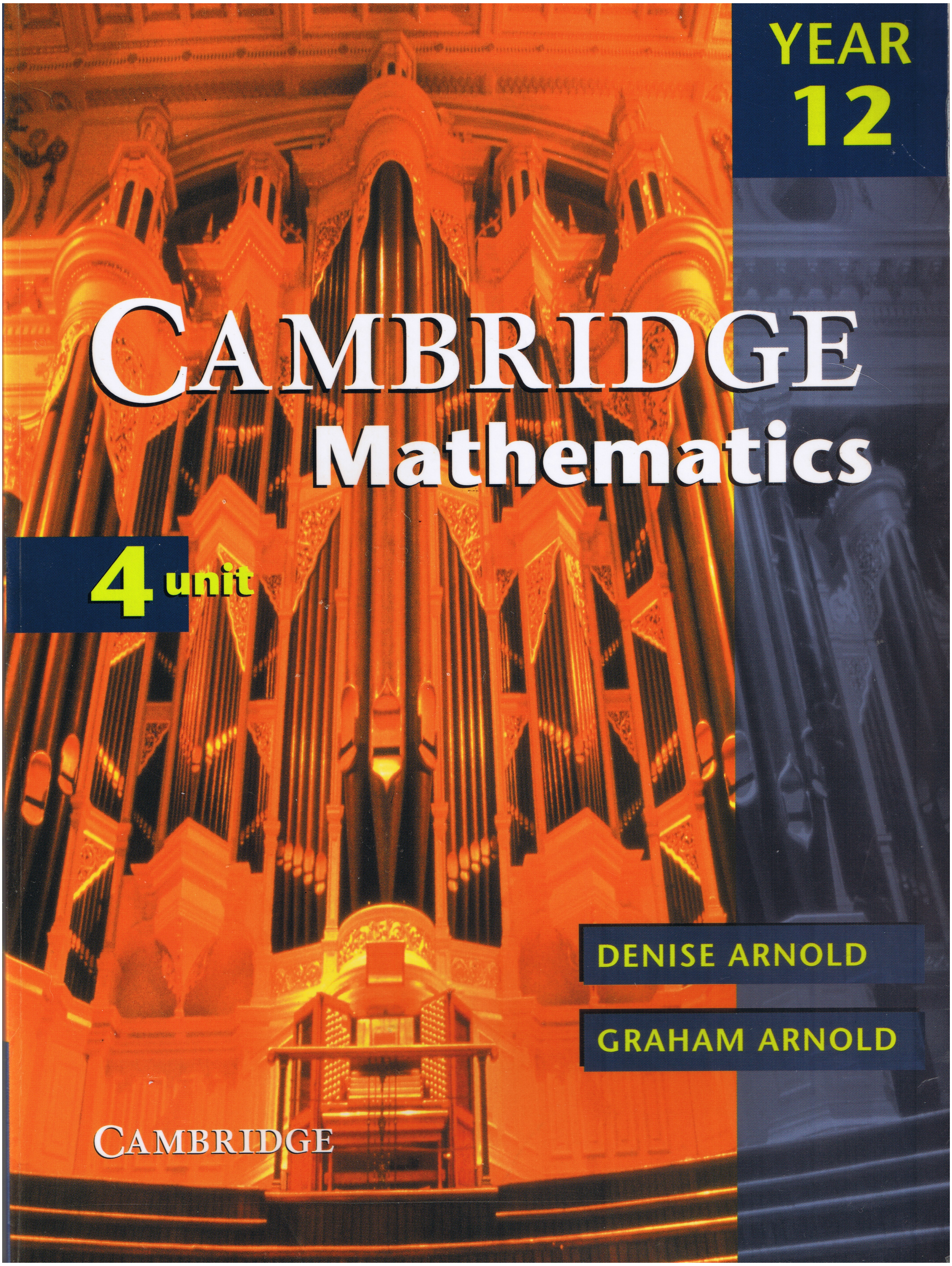

YEAR 12

CAMBRIDGE Mathematics

4 unit

DENISE ARNOLD

GRAHAM ARNOLD

CAMBRIDGE

YEAR
12

CAMBRIDGE Mathematics 4 unit

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CAMBRIDGE
UNIVERSITY PRESS

Dedication
To Mark, Clare, Paul and Helen

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011–4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published by Hodder Education 1993
Reprinted by Cambridge University Press 2000

Printed in Australia by McPhersons Printing Group

Typeface Times Roman 12/14pt *System* QuarkXpress 4.04® [ ]

National Library of Australia Cataloguing in Publication data

Arnold, Denise
Cambridge 4 unit mathematics year 12
ISBN 0 521 00547 7
1.Mathematics 2. Mathematics – Problems, exercises, etc.
I. Arnold, Graham (Graham Colin). II. Title III. Title:
Cambridge four unit mathematics
510

ISBN 0 521 00547 7

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Preface

This textbook has been written to follow the syllabus for the NSW Higher School Certificate course '4 Unit Mathematics'. It is assumed that the reader is familiar with the content of the corresponding 2 Unit and 3 Unit Mathematics syllabuses.

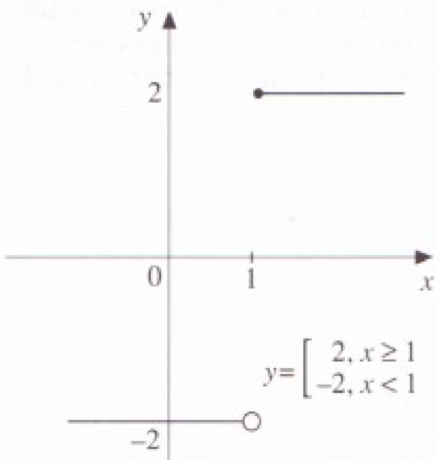
1 Graphs

In this chapter it is assumed that students are familiar with the graphs of basic curves, including the graphs of polynomial functions, the rectangular hyperbola $xy = k$, the circle, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. It is also expected that students can use calculus to find turning points and points of inflection on curves and understand the concepts of limiting values and asymptotes to curves.

1.1 Critical points on curves

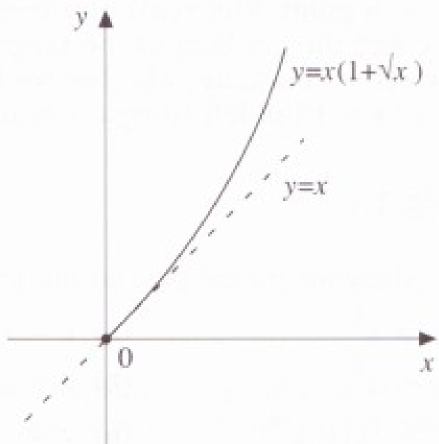
A critical point on the curve $y = f(x)$ is one at which the derivative $f'(x)$ is not defined. Clearly the derivative is not defined at any point of discontinuity on the curve, nor at an endpoint of a finite domain, given the definition of $f'(x)$ as a two-sided limit.

Figure 1.1



There is a discontinuity at (1,2) and $\frac{dy}{dx}$ is not defined at (1,2).

Figure 1.2



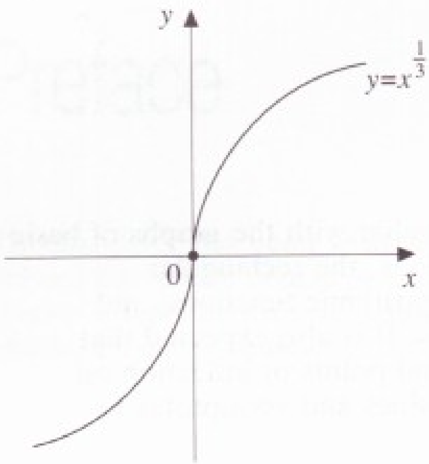
Domain $\{x : x \geqslant 0\}$

$\frac{dy}{dx} = 1 + \frac{3}{2}\sqrt{x},\ x > 0.$

$\frac{dy}{dx} \to 1^+$ as $x \to 0^+$,

$\therefore y = x$ is a tangent line at the critical point (0,0).

Figure 1.3



$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

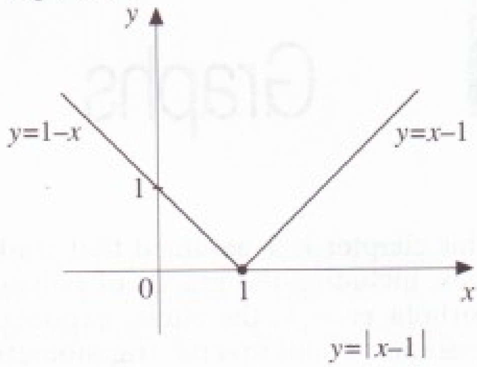
$\frac{dy}{dx}$ is not defined at $x = 0$,

$\therefore$ (0,0) is a critical point.

$\frac{dy}{dx} \to \infty$ as $x \to 0$,

$\therefore$ the tangent line at (0,0) is vertical.

Figure 1.4



$\frac{dy}{dx} = \begin{bmatrix} 1, & x > 1 \\ -1, & x < 1 \end{bmatrix}$

$\frac{dy}{dx} \to 1$ as $x \to 1^+$

$\frac{dy}{dx} \to -1$ as $x \to 1^-$,

$\therefore$ $\frac{dy}{dx}$ is not defined at $x = 1$, and

(1,0) is a critical point.

These examples illustrate four different types of critical point. For $\frac{dy}{dx}$ to be defined at a point $A(a, f(a))$ on the curve, the function f must be continuous at $x = a$ and the gradient of the tangent to the curve must have a limiting value which is the same, whether we trace the curve from right to left towards $A(x \to a^+)$ or from left to right towards $A(x \to a^-)$.

Exercise 1.1

Sketch (showing critical points) the graphs of

1 **(a)** $y = x^{\frac{1}{2}}$ **(b)** $y = x^{\frac{1}{2}} - 2$

2 **(a)** $y = x^{\frac{1}{3}}$ **(b)** $y = x^{\frac{1}{3}} + 2$

3 **(a)** $y = |x + 1|$ **(b)** $y = |x| + 1$

4 **(a)** $y = x\,(3 + \sqrt{x})$ **(b)** $y = x\,(3 - \sqrt{x})$

5 **(a)** $y = x + |x|$ **(b)** $y = x^2 - |x|$

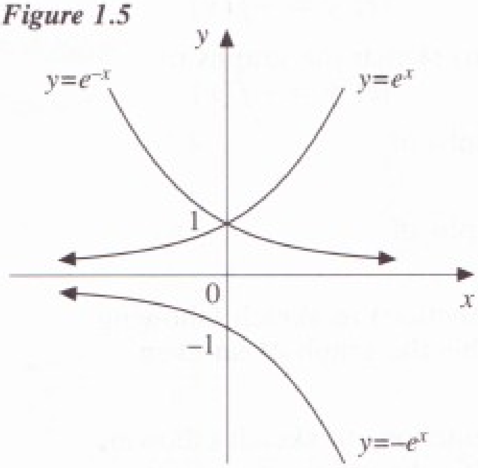
Note in **(a)** that $y = x + x = 2x$ if $x \geqslant 0$
$y = x - x = 0$ if $x < 0$

6 **(a)** $y = |x| + |x - 2|$ **(b)** $y = |x| - |x - 2|$

Note in **(a)** that $y = x + (x - 2) = 2x - 2,$ if $x \geqslant 2$
$y = x - (x - 2) = 2,$ if $0 \leqslant x < 2$
$y = -x - (x - 2) = 2 - 2x,$ if $x < 0.$

1.2 Reflecting graphs in the coordinate axes

Figure 1.5



The transformation $x \to -x$ is a reflection in the y-axis. Hence the graph $y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis. Similarly, as $y \to -y$ is a reflection in the x-axis, the graph $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis. This is illustrated for the function $f(x) = e^x$ in figure 1.5.

This reflection property can be used to sketch the graph $y = |f(x)|$ from the graph of $y = f(x)$.

$$|f(x)| = \begin{cases} f(x), & \{x : f(x) \geqslant 0\} \\ -f(x), & \{x : f(x) < 0\} \end{cases}$$

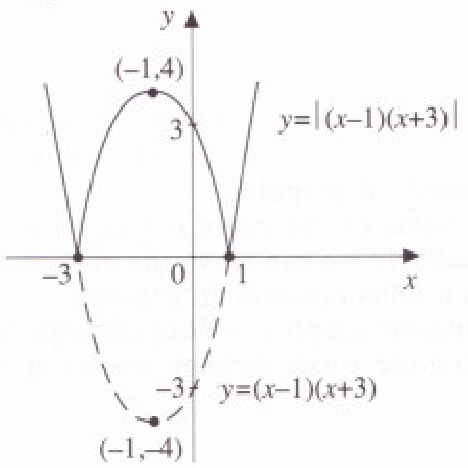
Hence those sections of $y = f(x)$ which lie below the x-axis are reflected in the x-axis.

Example 1

Sketch the graph of $y = |(x - 1)(x + 3)|$.

Solution

Figure 1.6



Note that this graph has critical points at $(-3,0)$ and $(1,0)$, since the limiting tangents have different gradients depending on whether we approach these points by tracing the curve from the left or the right.

Exercise 1.2

1 For the function $f(x) = x^2$ (an even function) sketch the graphs of
(a) $y = f(x)$ **(b)** $y = f(-x)$ **(c)** $y = -f(x)$

2 For the function $f(x) = x^3$ (an odd function) sketch the graphs of
(a) $y = f(x)$ **(b)** $y = f(-x)$ **(c)** $y = -f(x)$

3 Use the graph of $y = \ln x$ to sketch the graphs of
(a) $y = \ln(-x)$ **(b)** $y = -\ln x$

4 Use the graph of $y = \ln x$ to sketch the graphs of
(a) $y = |\ln x|$ **(b)** $y = \ln |x|$

5 Use the graph of $f(x) = 4 - x^2$ (an even function) to sketch (showing critical points) the graph of $y = |f(x)|$. Is this the graph of an even function?

6 Use the graph of $f(x) = x^3 - 3x$ (an odd function) to sketch (showing critical points) the graph of $y = |f(x)|$. Is this the graph of an even function?

1.3 Translation of graphs: addition and subtraction of ordinates

Figure 1.7

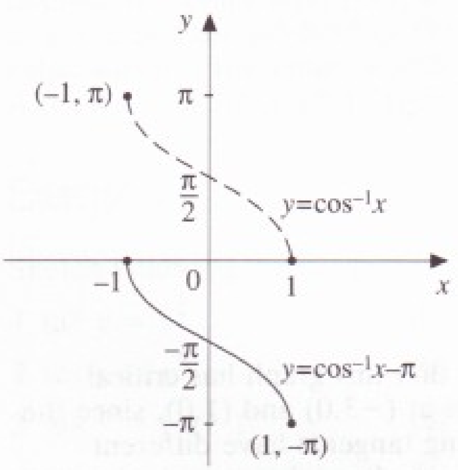
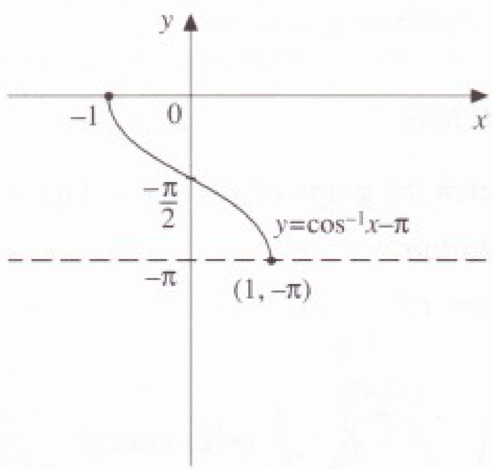


Figure 1.8



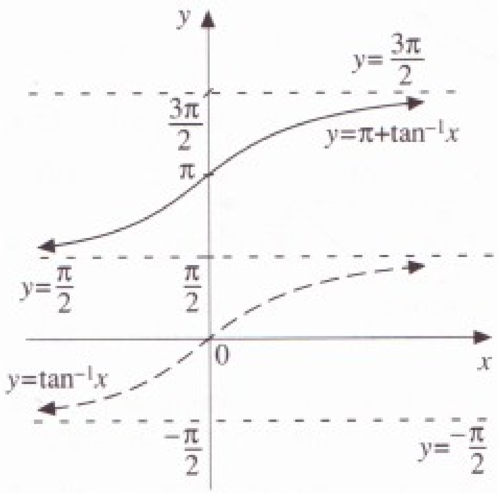
These diagrams illustrate two equivalent methods of graphing $y = \cos^{-1} x - \pi$. We can subtract π from each y-value on the curve $y = \cos^{-1} x$, translating the curve π units downward. Alternatively we can leave the curve $y = \cos^{-1} x$ in position and translate the x-axis π units upward. In general, the graph $y = f(x) \pm c$ is obtained by translating the graph $y = f(x)$ through c units up or down, or equivalently by translating the x-axis through c units in the opposite direction.

Example 2

Sketch $y = \pi + \tan^{-1}x$ and state the domain and range of the function.

Solution

Figure 1.9

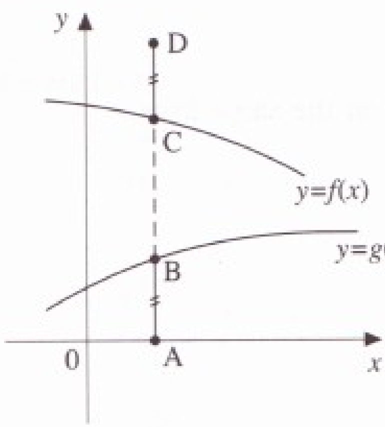


Domain $\{\text{all real } x\}$

Range $\left\{y: \frac{\pi}{2} < y < \frac{3\pi}{2}\right\}$

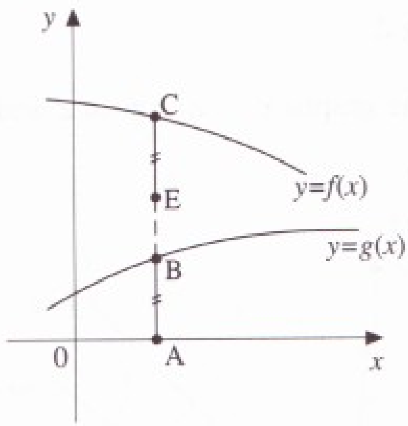
The following diagrams illustrate the procedures of addition and subtraction of ordinates to graph $y = f(x) \pm g(x)$.

Figure 1.10



AC + AB = AD
D is a point on $y = f(x) + g(x)$.

Figure 1.11



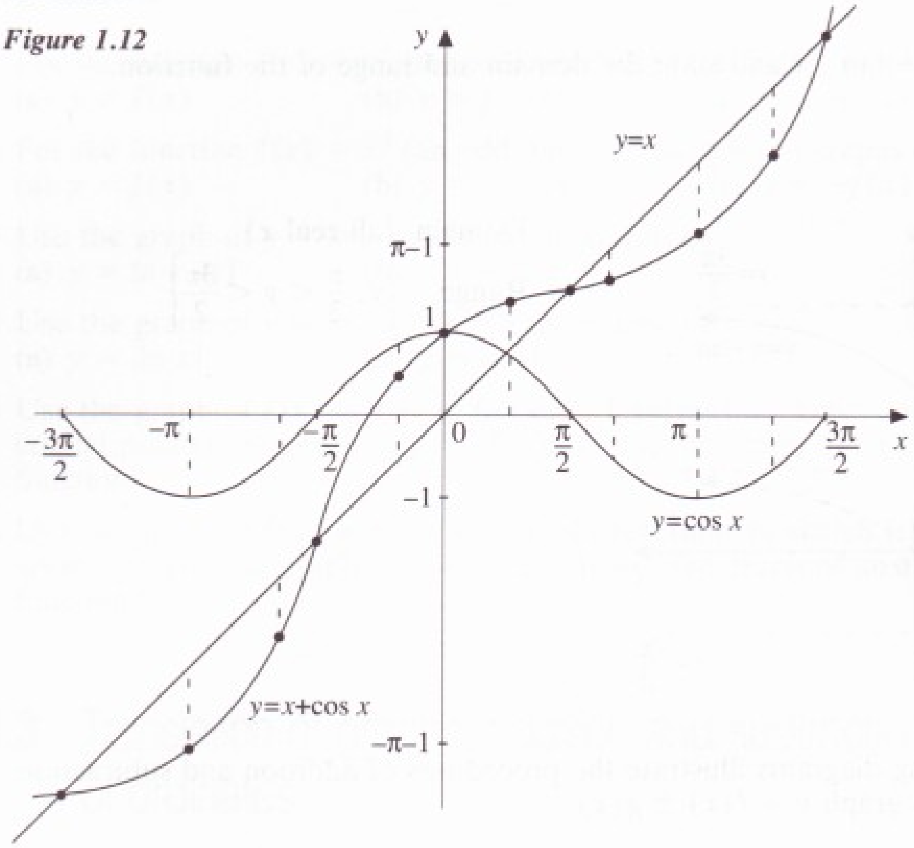
AC − AB = AE
E is a point on $y = f(x) - g(x)$.

Example 3

Sketch the graphs $y = x$, $y = \cos x$ on the same axes. Hence sketch $y = x + \cos x$.

Solution

Figure 1.12

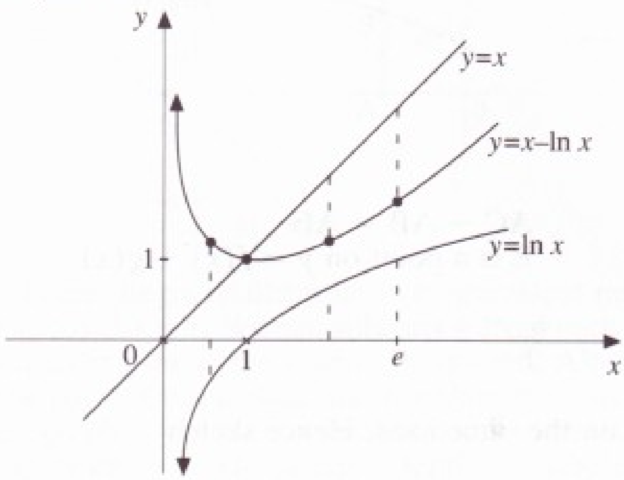


Example 4

Sketch the graphs $y = x$, $y = \ln x$ and $y = x - \ln x$ on the same axes.

Solution

Figure 1.13



Exercise 1.3

1 Use the graph of $y = \sin^{-1} x$ to sketch the graphs of

(a) $y = \sin^{-1} x + \frac{\pi}{2}$ (b) $y = \sin^{-1}(x + 1)$

2 Use the graph of $y = \cos x$ to sketch the graphs of

(a) $y = \cos x - 1$ (b) $y = \cos\left(x - \frac{\pi}{2}\right)$

3 Use the graphs of $y = x$ and $y = e^{-x}$ to sketch the graphs of

(a) $y = x + e^{-x}$ (b) $y = x - e^{-x}$

4 Use the graphs of $y = \ln x$ and $y = \frac{1}{x}$ to sketch the graphs of

(a) $y = \ln x + \frac{1}{x}$ (b) $y = \ln x - \frac{1}{x}$

5 Use the graphs of $y = x$ and $y = \sin x$ (both odd functions) to sketch the graph of $y = x + \sin x$. Is this the graph of an odd function?

6 The functions $g(x)$ and $h(x)$ are both even functions. Show that the function $f(x) = g(x) + h(x)$ is also an even function.

1.4 Multiplication of ordinates

Figure 1.14

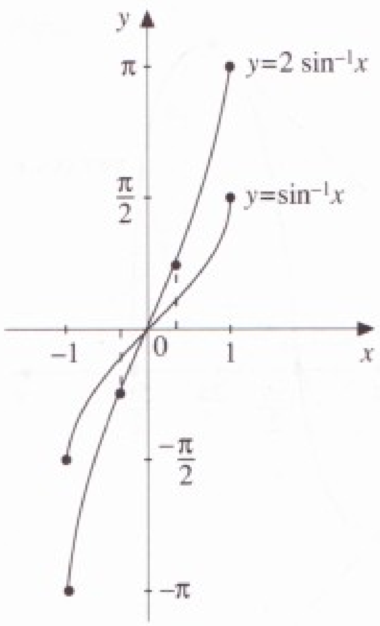
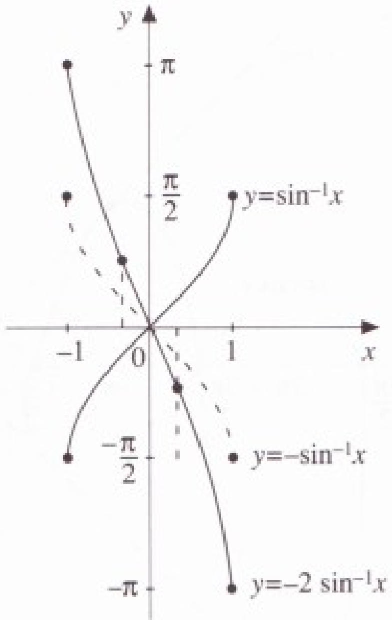


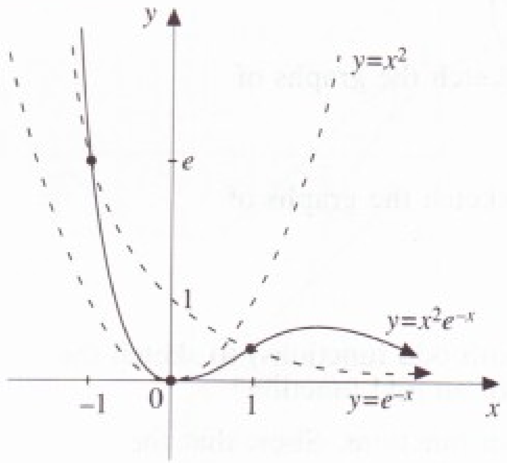
Figure 1.15



As figures 1.14 and 1.15 illustrate, for $f(x) = \sin^{-1} x$, the graph $y = cf(x)$, $c > 0$, is obtained by enlarging $y = f(x)$ by a factor c in the direction parallel to the y-axis, while for $c < 0$, $y = cf(x)$ is a reflection in the x-axis followed by an enlargement parallel to the y-axis.

Examination of the features of $y = f(x)$ and $y = g(x)$ can help you construct the graph $y = f(x)g(x)$. Consider the graph $y = x^2e^{-x}$.

Figure 1.16



Features

- $y = 0$ when $x = 0$.
- $y = x^2e^{-x}$ lies below $y = e^{-x}$ only for $-1 < x < 1$, these graphs intersecting at $x = \pm 1$.
- As $x \to \infty$, $e^{-x} \to 0$ more quickly than any power of $\frac{1}{x}$ and hence $x^2e^{-x} \to 0$.

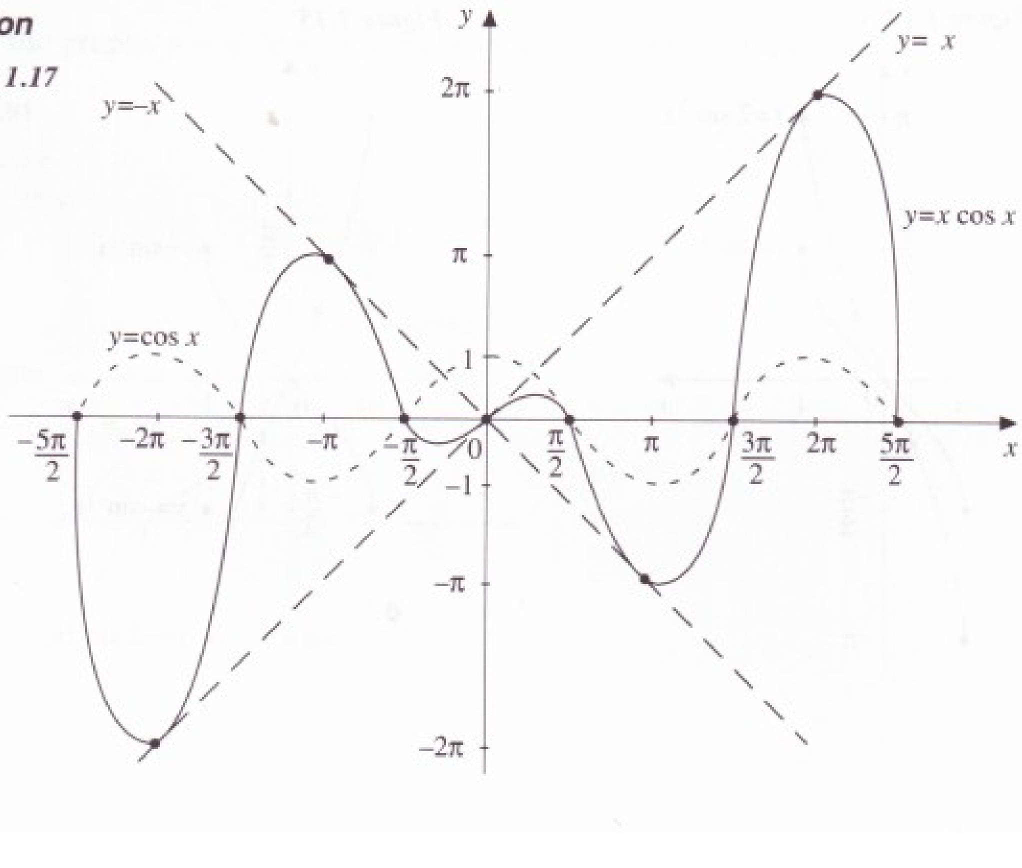
The exact positions of the turning points and points of inflection can be found by using calculus.

Example 5

Sketch the graphs $y = x$, $y = \cos x$ on the same axes. Hence sketch $y = x \cos x$.

Solution

Figure 1.17



We note that $y = x \cos x$ is an odd function and hence its graph has point symmetry about the origin. For $x \geqslant 0$, $-x \leqslant x \cos x \leqslant x$ and hence the graph $y = x \cos x$ lies between the lines $y = \pm x$, touching these lines when $\cos x = \pm 1$.

Exercise 1.4

1 Use the graph of $y = \cos^{-1} x$ to sketch the graphs of

(a) $y = \frac{1}{2}\cos^{-1} x$ **(b)** $y = \cos^{-1}\left(\frac{x}{2}\right)$

2 Use the graph of $y = \sin x$ to sketch the graphs of
(a) $y = 2 \sin x$ **(b)** $y = \sin 2x$

3 Use the graphs of $y = x$ and $y = e^{-x}$ to sketch the graph of $y = xe^{-x}$.

4 Use the graphs of $y = \ln x$ and $y = \frac{1}{x}$ to sketch the graph of $y = \frac{\ln x}{x}$.

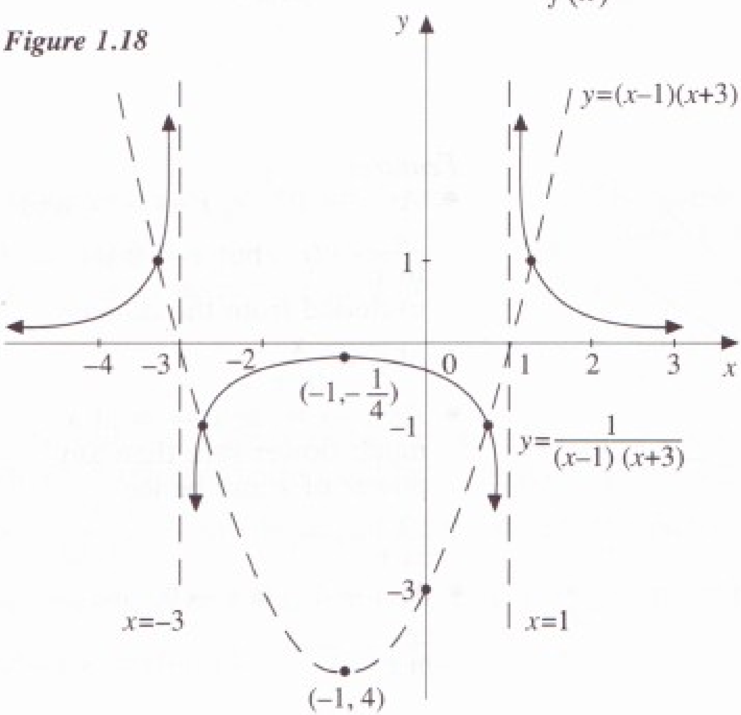
5 Use the graphs of $y = x$ and $y = \sin x$ (both odd functions) to sketch the graph of $y = x \sin x$. Is this the graph of an even function?

6 The functions $g(x)$ and $h(x)$ are both even functions. Show that the function $f(x) = g(x)h(x)$ is also an even function.

1.5 Reciprocal functions and division of ordinates

Consider the graphs $y = f(x)$ and $y = \frac{1}{f(x)}$, where $f(x) = (x - 1)(x + 3)$.

Figure 1.18



Features of figure 1.18

- $f(x)$, $\dfrac{1}{f(x)}$ have the same sign.
- $y = f(x)$, $y = \dfrac{1}{f(x)}$ intersect where $f(x) = \pm 1$.
- x-intercepts of $y = f(x)$ correspond to vertical asymptotes of $y = \dfrac{1}{f(x)}$.
- As $f(x) \to \infty$, $\dfrac{1}{f(x)} \to 0$.
- Minimum turning points of $y = f(x)$ correspond to maximum turning points of $y = \dfrac{1}{f(x)}$, and conversely.

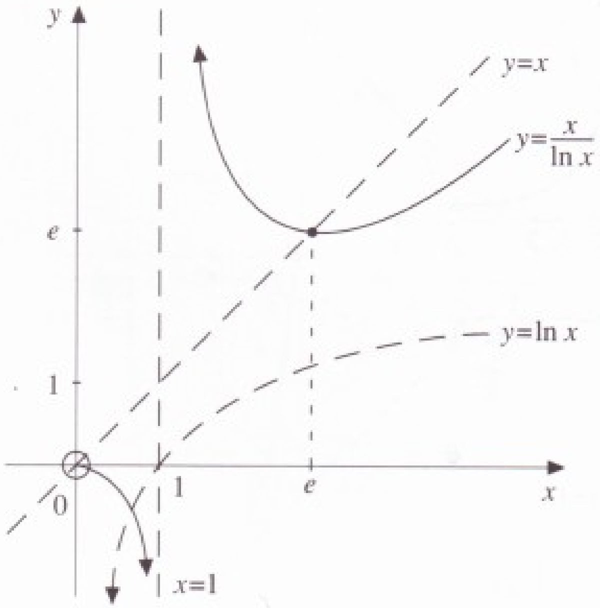
The same general principles apply in graphing $y = \dfrac{1}{f(x)}$ for any function f. Note that $y = \dfrac{1}{f(x)} \Rightarrow \dfrac{dy}{dx} = \dfrac{-f'(x)}{[f(x)]^2}$. Hence $\dfrac{1}{f(x)}$ decreases as $f(x)$ increases, and conversely, and the stationary points of $y = \dfrac{1}{f(x)}$ have the same x-coordinates as the stationary points of $y = f(x)$ for which $f(x) \neq 0$. Graphs of the form $y = \dfrac{f(x)}{g(x)}$ are constructed by considering the features of $y = f(x)$ and $y = g(x)$.

Example 6

Sketch the graphs $y = x$ and $y = \ln x$. Hence sketch $y = \dfrac{x}{\ln x}$.

Solution

Figure 1.19



Features

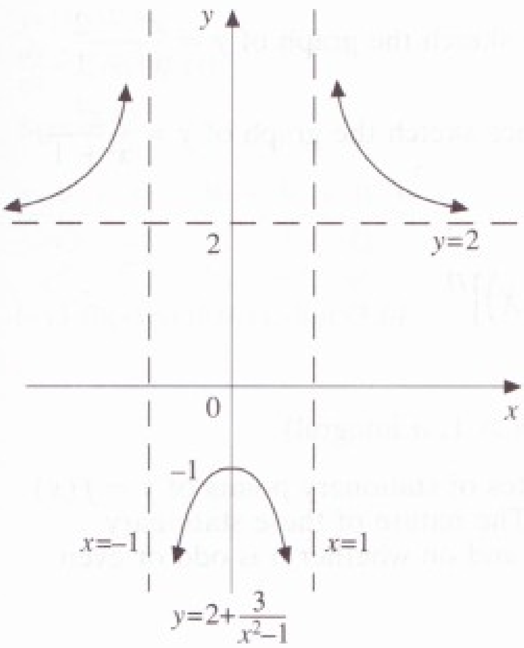
- As $x \to 0^+$, $\ln x \to -\infty$ and $\dfrac{x}{\ln x} \to 0^-$, but $x = 0$ is excluded from the domain of $y = \dfrac{x}{\ln x}$.
- As $x \to \infty$, $\ln x \to \infty$ at a much slower rate than any power of x and hence $\dfrac{x}{\ln x} \to \infty$.
- As $x \to 1^-$, $\ln x \to 0^-$ and $\dfrac{x}{\ln x} \to -\infty$.

Functions of the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, are called rational functions. Whenever the degree of $P(x)$ is equal to that of $Q(x)$, or exceeds that of $Q(x)$ by one, it is often easier to graph $y = \dfrac{P(x)}{Q(x)}$ after using the division transformation for polynomials.

Degree $P(x)$ = degree $Q(x)$

$$y = \frac{2x^2 + 1}{x^2 - 1} = 2 + \frac{3}{x^2 - 1}$$

Figure 1.20



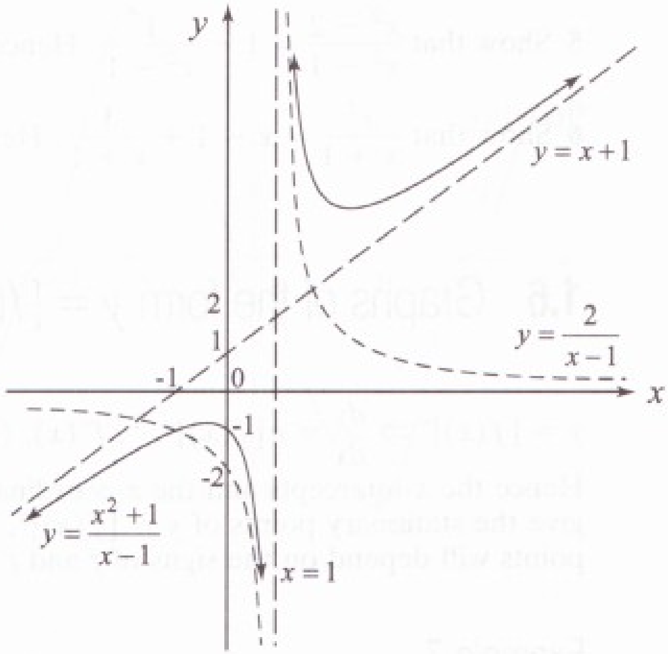
The graph $y = \dfrac{3}{x^2 - 1}$ has been translated two units upward.
$y = 2$ is an asymptote as $x \to \infty$.

Degree $P(x)$ = degree $Q(x) + 1$

$$y = \frac{x^2 + 1}{x - 1}$$

$$y = (x + 1) + \frac{2}{x - 1}$$

Figure 1.21



The graph has been constructed by addition of the ordinates of $y = x + 1$ and $y = \dfrac{2}{x - 1}$.
$y = x + 1$ is an asymptote as $x \to \infty$.

In general, if $P(x)$ and $Q(x)$ have the same degree, the quotient in the division transformation will be a constant, giving a horizontal asymptote of $y = \dfrac{P(x)}{Q(x)}$. If the degree of $P(x)$ exceeds the degree of $Q(x)$ by one, the quotient will be a linear function $ax + b$, $a \neq 0$, and $y = \dfrac{P(x)}{Q(x)}$ will have an oblique asymptote $y = ax + b$ as $x \to \infty$.

Exercise 1.5

1 Use the graph of $f(x) = 4 - x^2$ (an even function) to sketch the graph of $y = \dfrac{1}{f(x)}$. Is this the graph of an even function?

2 Use the graph of $f(x) = x^3 - 3x$ (an odd function) to sketch the graph of $y = \dfrac{1}{f(x)}$. Is this the graph of an odd function?

3 Use the graphs of $y = x$ and $y = e^x$ to sketch the graph of $y = \dfrac{x}{e^x}$.

4 Use the graphs of $y = \ln x$ and $y = x$ to sketch the graph of $y = \dfrac{\ln x}{x}$.

5 Show that $\dfrac{x^2 - 2}{x^2 - 1} = 1 - \dfrac{1}{x^2 - 1}$. Hence sketch the graph of $y = \dfrac{x^2 - 2}{x^2 - 1}$.

6 Show that $\dfrac{x^2}{x + 1} = x - 1 + \dfrac{1}{x + 1}$. Hence sketch the graph of $y = \dfrac{x^2}{x + 1}$.

1.6 Graphs of the form $y = [f(x)]^n$

$y = [f(x)]^n \Rightarrow \dfrac{dy}{dx} = n[f(x)]^{n-1} \cdot f'(x)$, ($n > 1$, n integral).

Hence the x-intercepts and the x-coordinates of stationary points of $y = f(x)$ give the stationary points of $y = [f(x)]^n$. The nature of these stationary points will depend on the signs of f and f' and on whether n is odd or even.

Example 7

Sketch $y = f(x)$, $y = [f(x)]^2$ and $y = [f(x)]^3$ when $f(x) = x(x^2 - 3)$.

Solution

Figure 1.22

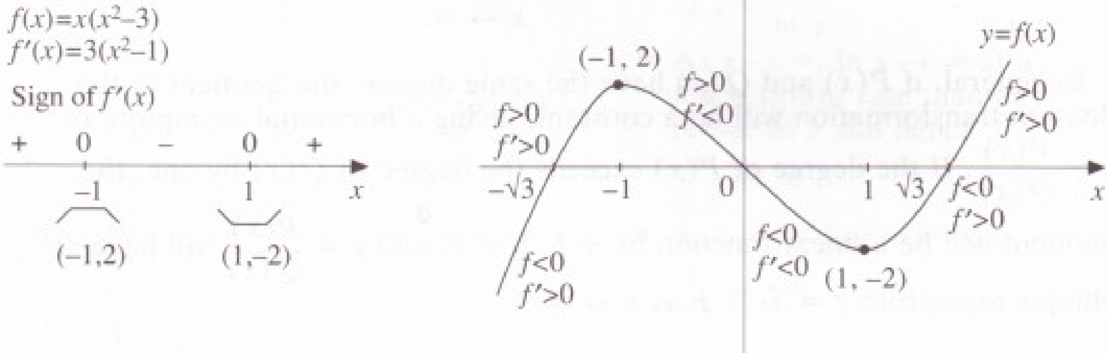


Figure 1.23

$y = [f(x)]^2$

$\dfrac{dy}{dx} = 2f(x)f'(x)$

Sign of $\dfrac{dy}{dx}$

$-\ 0\ +\ 0\ -\ 0\ +\ 0\ -\ 0\ +$

$-\sqrt{3}\quad -1\quad 0\quad 1\quad \sqrt{3}\quad x$

$(-\sqrt{3},0)(-1,4)\ (0,0)\ \ (1,4)\ (\sqrt{3},0)$

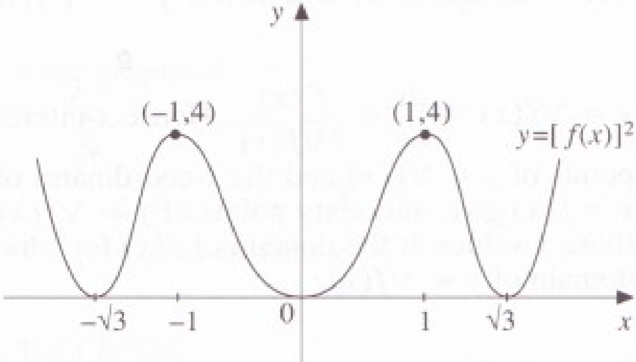


Figure 1.24

$y = [f(x)]^3$

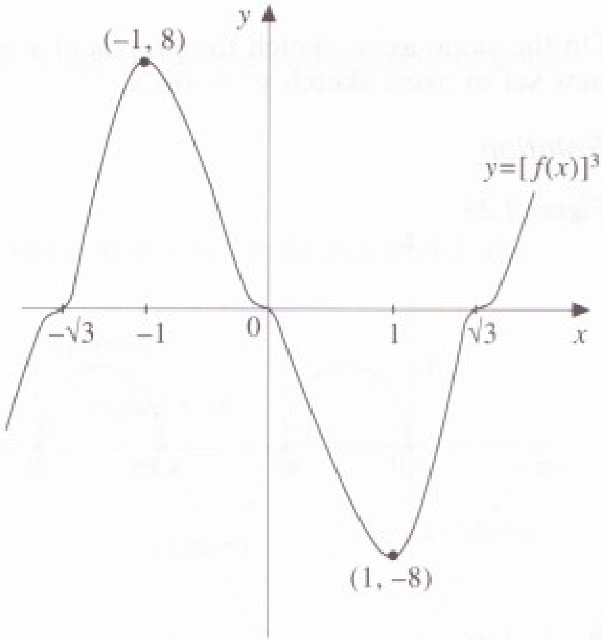
$\dfrac{dy}{dx} = 3[f(x)]^2 f'(x)$

Sign of $\dfrac{dy}{dx}$

$+\ 0\ +\ 0\ -\ 0\ -\ 0\ +\ 0\ +$

$-\sqrt{3}\quad -1\quad 0\quad 1\quad \sqrt{3}$

$(-\sqrt{3},0)(-1,8)\ (0,0)\ (1,-8\ )(\sqrt{3},0)$



Exercise 1.6

Use the graph of $y = f(x)$ to sketch the graphs of

(a) $y = [f(x)]^2$ **(b)** $y = [f(x)]^3$

for each of the following functions

1 $f(x) = x^2 - 1$

2 $f(x) = 3x - \dfrac{x^3}{4}$

3 $f(x) = \cos x$

4 $f(x) = 4 \sin x$

1.7 Graphs of the form $y = \sqrt{f(x)}$

$y = \sqrt{f(x)} \Rightarrow \dfrac{dy}{dx} = \dfrac{f'(x)}{2\sqrt{f(x)}}$. Hence x-intercepts of $y = f(x)$ give critical points of $y = \sqrt{f(x)}$ and the x-coordinates of stationary points of $y = f(x)$ give stationary points of $y = \sqrt{f(x)}$, provided $f(x) \neq 0$. Clearly only those x-values in the domain of $f(x)$ for which $f(x) \geqslant 0$ are included in the domain of $y = \sqrt{f(x)}$.

Example 8

On the same axes, sketch the graphs of $y = \sin x$ and $y = \sqrt{(\sin x)}$. On a new set of axes, sketch $y^2 = \sin x$.

Solution

Figure 1.25

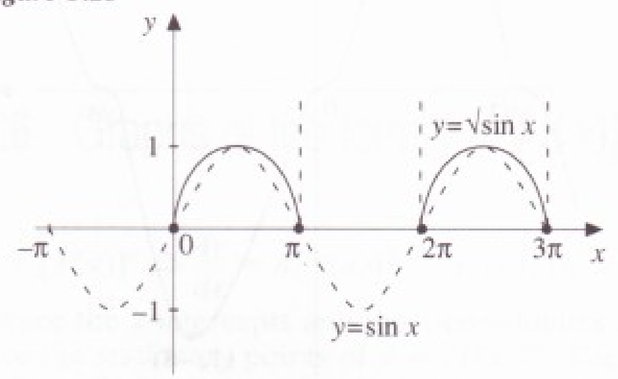
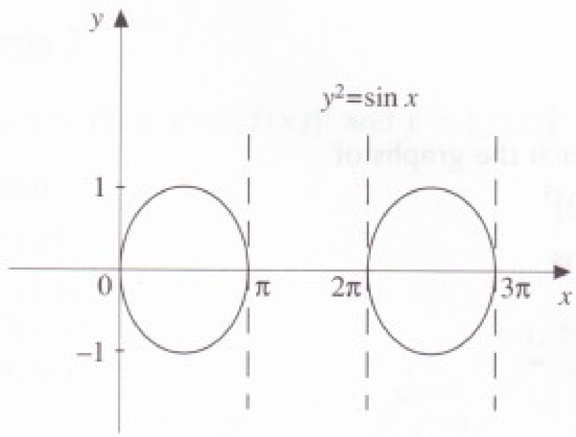


Figure 1.26



$y = \sqrt{(\sin x)}$ has vertical tangent lines at the critical points $(n\pi, 0)$, n integral. Also $0 < \sin x < 1 \Rightarrow \sqrt{(\sin x)} > \sin x$.

Exercise 1.7

Use the graph of $y = f(x)$ to sketch the graphs of
(a) $y = \sqrt{f(x)}$ **(b)** $y^2 = f(x)$
for each of the following functions

1 $f(x) = x^2 - 1$ **2** $f(x) = 3x - \dfrac{x^3}{4}$
3 $f(x) = \cos x$ **4** $f(x) = 4 \sin x$

1.8 Graphs of composite functions

Knowledge of basic curves can often be applied to the construction of graphs of composite functions.

Example 9

Sketch the graphs $y = \ln u$ and $u = \cos x$, $0 \leqslant x \leqslant 2\pi$. Hence sketch the graph $y = \ln(\cos x)$, $0 \leqslant x \leqslant 2\pi$.

Solution

Figure 1.27

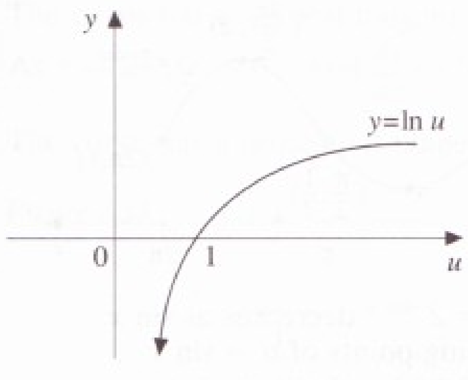


Figure 1.28

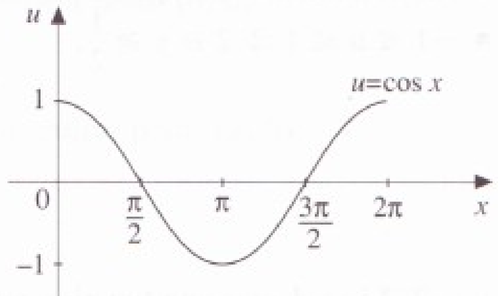
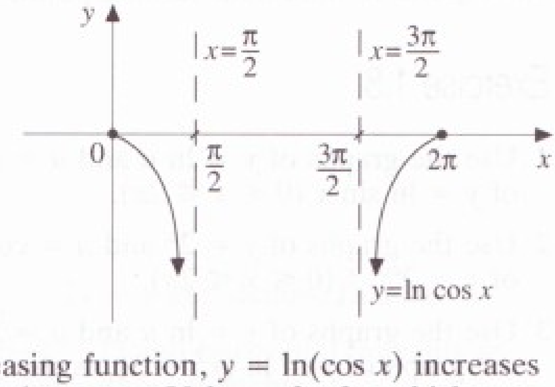


Figure 1.29



Features of figure 1.27
- Vertical asymptote at $u = 0$.
- $y = 0$ when $u = 1$.
- Monotonic increasing throughout domain $\{u: u > 0\}$.

Note that because $y = \ln u$ is an increasing function, $y = \ln(\cos x)$ increases as $\cos x$ increases and decreases as $\cos x$ decreases. Values of x for which $\cos x \leqslant 0$ are not in the domain of $y = \ln(\cos x)$.

Example 10

Sketch the graphs $y = 2^{-u}$ and $u = \sin x$, $0 \leqslant x \leqslant 2\pi$. Hence sketch the graph $y = 2^{-\sin x}$, $0 \leqslant x \leqslant 2\pi$, showing the coordinates of any stationary points.

Solution

Figure 1.30

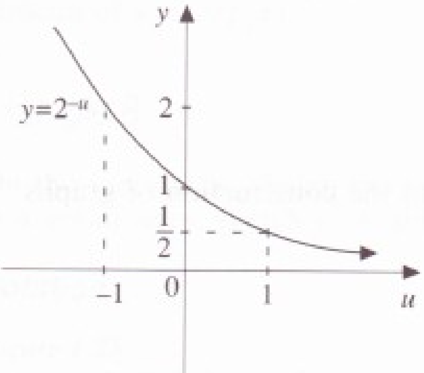


Figure 1.31

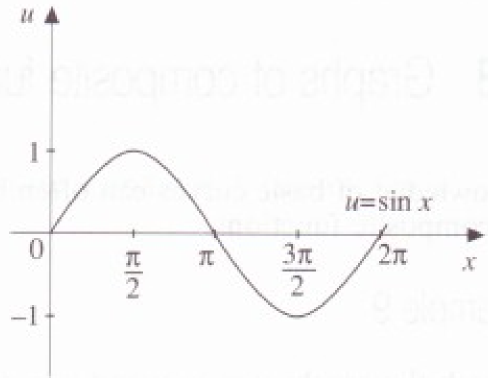
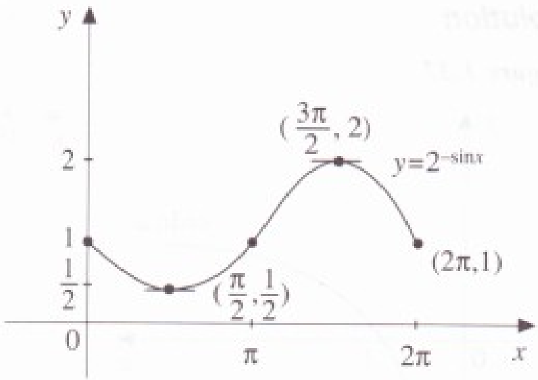


Figure 1.32



Features of figure 1.30

- When $u = 0$, $y = 1$.
- Monotonic decreasing.
- $-1 \leqslant u \leqslant 1 \Rightarrow 2 \geqslant y \geqslant \frac{1}{2}$.

$y = 2^{-u}$ is a decreasing function and hence $y = 2^{-\sin x}$ decreases as $\sin x$ increases, and conversely. Thus maximum turning points of $u = \sin x$ correspond to minimum turning points of $y = 2^{-\sin x}$ and vice versa.

Exercise 1.8

1 Use the graphs of $y = \ln u$ and $u = \sin x$ $(0 \leqslant x \leqslant 2\pi)$ to sketch the graph of $y = \ln \sin x$ $(0 \leqslant x \leqslant 2\pi)$.

2 Use the graphs of $y = 2^u$ and $u = \cos x$ $(0 \leqslant x \leqslant 2\pi)$ to sketch the graph of $y = 2^{\cos x}$ $(0 \leqslant x \leqslant 2\pi)$.

3 Use the graphs of $y = \ln u$ and $u = x^2 - 3$ (an even function) to sketch the graph of $y = \ln(x^2 - 3)$. Is this the graph of an even function?

4 Use the graphs of $y = 2^{-u}$ and $u = |x|$ (an even function) to sketch the graph of $y = 2^{-|x|}$. Is this the graph of an even function?

5 The functions $g(x)$ and $h(x)$ are such that $h(x)$ is an even function. Show that the function $f(x) = g[h(x)]$ is also an even function.

1.9 Implicit differentiation and curve sketching

Example 11

Sketch the graph $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

Solution

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$$

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0 \qquad \left\{\begin{array}{l}\text{Take the derivative of both sides with}\\ \text{respect to } x\text{, remembering that } y \text{ is a}\\ \text{function of } x\text{, and using the chain rule.}\end{array}\right.$$

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{2}}$$

The function has domain $\{x: 0 \leqslant x \leqslant a\}$ and range $\{y: 0 \leqslant y \leqslant a\}$.

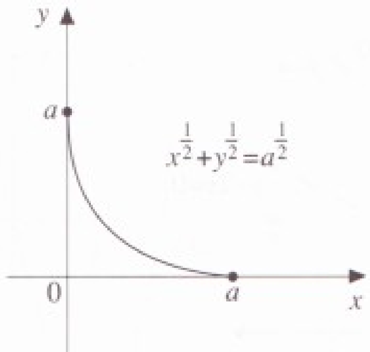
As $x \to 0^+$, $y \to a^-$, and $\frac{dy}{dx} \to -\infty$.

The curve has a vertical tangent line at the critical point $(0, a)$.

As $x \to a^-$, $y \to 0^+$, and $\frac{dy}{dx} \to 0^-$.

The curve has a horizontal tangent line at the critical point $(a, 0)$.

Figure 1.33



Example 12

$x^2 + y^2 = xy + 3$. Show that $(x - 2y)\frac{dy}{dx} = 2x - y$ and deduce that the curve has vertical tangents at $(2, 1)$ and $(-2, -1)$ and horizontal tangents at $(1, 2)$ and $(-1, -2)$. Sketch the curve, showing these tangents.

Solution

$$x^2 + y^2 = xy + 3$$

$$2x + 2y\frac{dy}{dx} = y + x\frac{dy}{dx} \qquad \left\{\begin{array}{l}\text{Take the derivative of both sides with}\\ \text{respect to } x\text{. Consider } y \text{ as a function}\\ \text{of } x \text{ and use the chain and product rules.}\end{array}\right.$$

$$(x - 2y)\frac{dy}{dx} = 2x - y$$

Substitution of $x = 2y$ in the equation of the curve gives

$$x - 2y = 0 \Rightarrow \quad 4y^2 + y^2 = 2y^2 + 3$$
$$3(y^2 - 1) = 0$$
$$\begin{cases} y = 1 \\ x = 2 \end{cases} \text{ or } \begin{cases} y = -1 \\ x = -2 \end{cases}$$

In either case, $2x - y \neq 0$. Hence as $x - 2y \to 0$, $\frac{dy}{dx} \to \infty$ and the curve has vertical tangents at $(2, 1)$ and $(-2, -1)$.

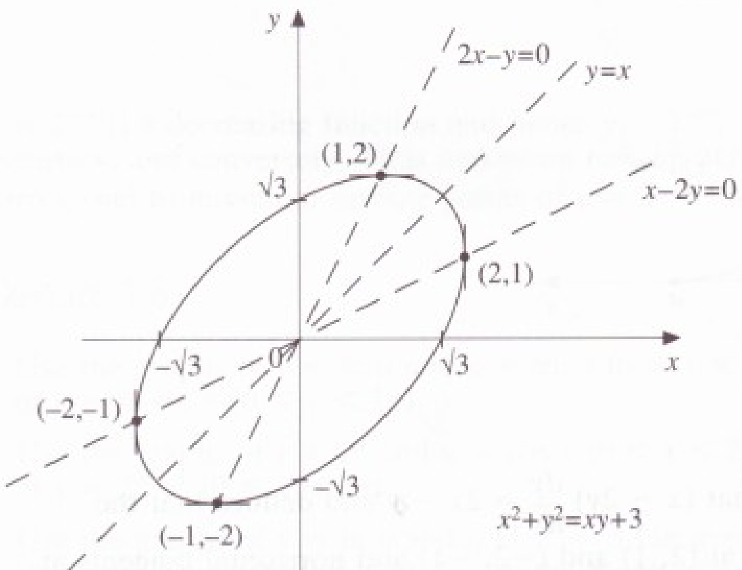
Similarly, $\quad 2x - y = 0 \quad \Rightarrow 3(x^2 - 1) = 0$

$$\begin{cases} x = 1 \\ y = 2 \end{cases} \text{ or } \begin{cases} x = -1 \\ y = -2 \end{cases}$$

In either case, $x - 2y \neq 0$. Hence $2x - y = 0 \Rightarrow \frac{dy}{dx} = 0$ and the curve has horizontal tangents at $(1, 2)$ and $(-1, -2)$.

Clearly the curve is symmetric about $y = x$, since the transformation $y \leftrightarrow x$ leaves the Cartesian equation of the curve unchanged.

Figure 1.34



Exercise 1.9

1 If $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 1$, show that $\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^{\frac{1}{2}}$. Sketch (showing critical points) the graph of $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 1$.

2 If $x^2 + y^2 + xy = 3$, show that $\dfrac{dy}{dx} = -\left(\dfrac{2x + y}{x + 2y}\right)$. Sketch (showing critical points and stationary points) the graph of $x^2 + y^2 + xy = 3$.

3 Sketch (showing critical points and stationary points) the graph of $x^3 + y^3 = 1$.

4 Sketch (showing critical points and stationary points) the graph of $x^2 + 4y^2 = 4$.

5 Sketch (showing critical points) the graph of $x^2 - 4y^2 = 4$.

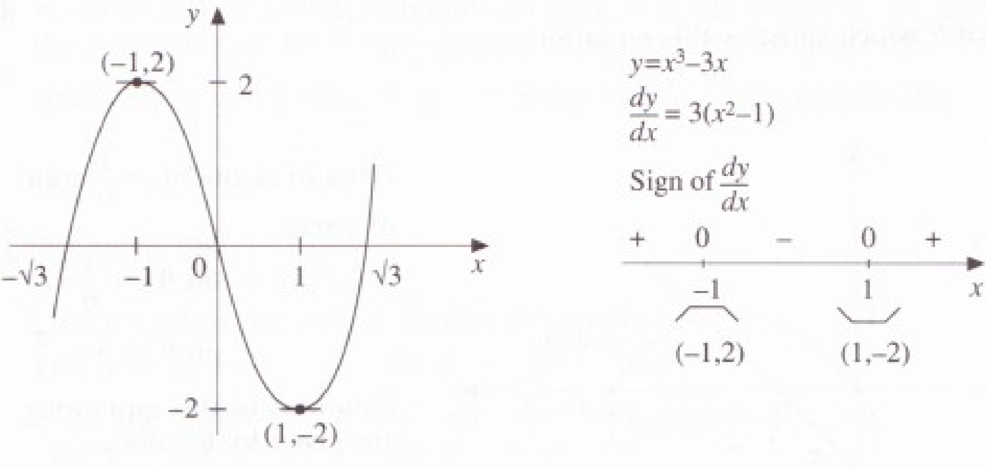
1.10 Using graphs

Example 13

Sketch the graph $y = x(x^2 - 3)$, showing the coordinates of the turning points and the intercepts on the coordinate axes. Deduce the set of values of k for which $x(x^2 - 3) + k = 0$ has exactly one real root.

Solution

Figure 1.35



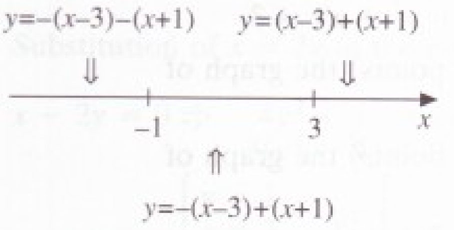
If the graph $y = x(x^2 - 3)$ is translated more than two units either upward or downward, the translated graph $y = x(x^2 - 3) + k$ will cut the x-axis exactly once. Hence $x(x^2 - 3) + k = 0$ will have exactly one real root if $k > 2$ or $k < -2$. Hence the set of values of k is $\{k: |k| > 2\}$.

Example 14

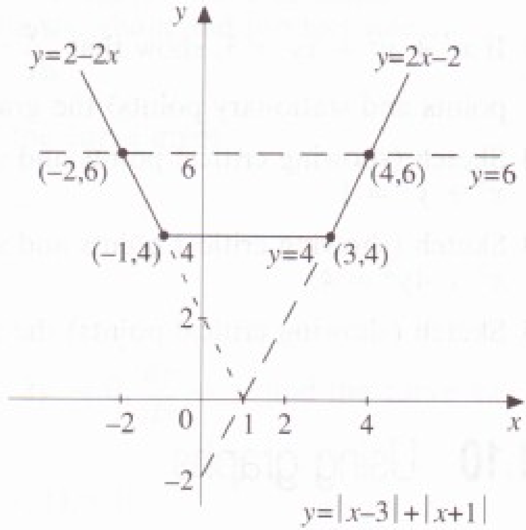
Sketch the graph $y = |x - 3| + |x + 1|$, and hence solve $|x - 3| + |x + 1| > 6$.

Solution

Figure 1.36



$$\therefore y = \begin{cases} 2-2x, & x<-1 \\ 4 \quad, & -1 \le x \le 3 \\ 2x-2, & x>3 \end{cases}$$



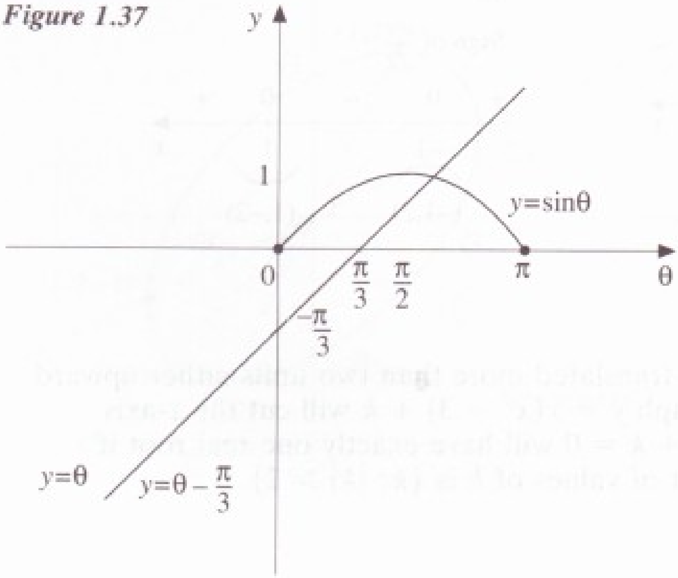
By inspection of the graph, $|x - 3| + |x + 1| > 6$ for $x < -2$ or $x > 4$.

Example 15

A chord subtends an angle θ radians at the centre of a circle. If the area of the minor segment cut off by the chord is one-sixth of the area of the circle, show that $\sin\theta = \theta - \frac{\pi}{3}$. Use a graphical means to show that there is just one value of θ which satisfies this equation.

Solution

Figure 1.37



Area of segment $= \frac{1}{6}$ area of circle,

$$\therefore \frac{1}{2} r^2 (\theta - \sin\theta) = \frac{1}{6}\pi r^2$$

$$\sin\theta = \theta - \frac{\pi}{3}.$$

Solutions to this equation are given by θ-values where $y = \theta - \frac{\pi}{3}$ and $y = \sin\theta$ intersect. Hence the equation has just one solution.

Exercise 1.10

1 Sketch the graph of $y = x^4 - 4x^3$. Use this graph to find the number of real roots of the equation $x^4 - 4x^3 = kx$, where k is a positive real number.

2 Sketch the graph of $y = \dfrac{(x+1)^4}{x^4+1}$. Use this graph to find the set of values of the real number k for which the equation $(x+1)^4 = k\,(x^4+1)$ has two real distinct roots.

3 Sketch the graph of $y = |x| - |x-4|$. Use this graph to solve the inequality $|x| - |x-4| > 2$.

4 Sketch the graph of $y = \cos x$ for $0 \leqslant x \leqslant 2\pi$. Use this graph to solve the inequalities

(a) $\cos x \leqslant \frac{1}{2}$, for $0 \leqslant x \leqslant 2\pi$ **(b)** $|\cos x| \leqslant \frac{1}{2}$, for $0 \leqslant x \leqslant 2\pi$

5 Sketch the graph of $y = \sin 2x$ for $0 \leqslant x \leqslant 2\pi$. Use this graph to solve the inequalities

(a) $\sin 2x \geqslant \frac{1}{2}$, for $0 \leqslant x \leqslant 2\pi$ **(b)** $|\sin 2x| \geqslant \frac{1}{2}$, for $0 \leqslant x \leqslant 2\pi$

6 Sketch the graph of $y = \dfrac{x^2+1}{x^2-1}$. Use this graph to solve the inequality $\dfrac{x^2+1}{x^2-1} < 1$.

7 A chord AB of a circle makes an angle θ with the diameter passing through A. If the area of the minor segment is one-quarter the area of the circle, show that $\sin 2\theta = \frac{\pi}{2} - 2\theta$. Solve this equation graphically.

8 A chord AB of a circle subtends an angle θ at the centre of the circle. If the perimeter of the minor segment is one-half the circumference of the circle, show that $2 \sin \frac{\theta}{2} = \pi - \theta$. Solve this equation graphically.

Diagnostic test 1

Subsection

1 Sketch (showing critical points) the graphs of (1.1)

(a) $y = |x+2|$ **(b)** $y = x\,(2 + \sqrt{x})$

2 Use the graph of $y = \cos^{-1}x$ to sketch the graphs of (1.2)

(a) $y = \cos^{-1}(-x)$ **(b)** $y = -\cos^{-1}x$

3 Use the graph of $y = \sin^{-1}x$ to sketch the graphs of (1.3) (1.4)

(a) $y = \sin^{-1}x - \frac{\pi}{2}$ **(b)** $y = \frac{1}{2}\sin^{-1}x$

4 Use the graphs of $y = x$ and $y = \ln x$ to sketch (1.3) (1.4)
(a) $y = x + \ln x$ **(b)** $y = x \ln x$

5 Use the graphs of $y = x$ and $y = e^x$ to sketch (1.3) (1.5)
(a) $y = x - e^x$ **(b)** $y = \dfrac{e^x}{x}$

6 Use the graph of $y = x\,(x + 2)$ to sketch the graphs of (1.2) (1.5)
(a) $y = |x(x + 2)|$ **(b)** $y = \dfrac{1}{x(x + 2)}$

7 Sketch the graphs of (1.5)
(a) $y = \dfrac{x^2}{x^2 - 1}$ **(b)** $y = \dfrac{x^2 + 4}{x}$

8 Use the graph of $y = 1 - x^2$ to sketch the graphs of (1.6)
(a) $y = (1 - x^2)^2$ **(b)** $y = (1 - x^2)^3$

9 Use the graph of $y = 4 \cos x$ to sketch the graphs of (1.7)
(a) $y = \sqrt{(4 \cos x)}$ **(b)** $y^2 = 4 \cos x$.

10 Sketch the graphs of (1.8)
(a) $y = \ln(1 - x^2)$ **(b)** $y = e^{-x^2}$

11 Sketch (showing critical points and stationary points) the graphs of (1.9)
(a) $x^2 + y^2 = 4$ **(b)** $x^2 - y^2 = 4$

12 **(a)** Sketch the graph of $y = x^3 - 12x$. Use this graph to find the set of values of the real number k for which the equation $x^3 - 12x + k = 0$ has exactly one real root. (1.10)
(b) Sketch the graph of $y = |x| + |x + 2|$. Use this graph to solve the inequality $|x| + |x + 2| > 4$.

Further questions 1

Sketch the graphs of

1 **(a)** $y = x^n$ **(b)** $y = x^{-n}$ **(c)** $y = x^{\frac{1}{n}}$ **(d)** $y = x^{-\frac{1}{n}}$
for $n \geqslant 2$, an even positive integer.

2 **(a)** $y = x^n$ **(b)** $y = x^{-n}$ **(c)** $y = x^{\frac{1}{n}}$ **(d)** $y = x^{-\frac{1}{n}}$
for $n \geqslant 3$, an odd positive integer.

3 **(a)** $|x| + |y| = 1$ **(b)** $|x| - |y| = 1$

4 **(a)** $y = \sin |x|$ **(b)** $y = |\sin x|$

5 **(a)** $y = \dfrac{1}{\cos^{-1} x}$ **(b)** $y = \dfrac{1}{\sin^{-1} x}$

6 **(a)** $y = x + \dfrac{1}{x}$ **(b)** $y = x - \dfrac{1}{x}$

7 **(a)** $y = x + \dfrac{1}{x^2}$ **(b)** $y = x - \dfrac{1}{x^2}$

8 **(a)** $y = x^2 + \frac{1}{x}$ **(b)** $y = x^2 - \frac{1}{x}$

9 **(a)** $y = x^2 + \frac{1}{x^2}$ **(b)** $y = x^2 - \frac{1}{x^2}$

10 **(a)** $y = \frac{1}{x} + \frac{1}{x^2}$ **(b)** $y = \frac{1}{x} - \frac{1}{x^2}$

11 **(a)** $y = x^2 \ln x$ **(b)** $y = \frac{\ln x}{x^2}$

12 **(a)** $y = x^2 e^x$ **(b)** $y = \frac{e^x}{x^2}$

13 **(a)** $y = \frac{x}{x^2 - 1}$ **(b)** $y = \frac{x^2}{x^2 - 1}$

14 **(a)** $y = \frac{1}{2}(e^x - e^{-x})$ **(b)** $y = \frac{1}{2}(e^x + e^{-x})$

(c) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

15 **(a)** $y = \cos x - \sin x$ **(b)** $y = \cos x + \sin x$

(c) $y = \frac{\cos x - \sin x}{\cos x + \sin x}$

16 Sketch the graph of $f(x) = 1 - \frac{9}{x^2} + \frac{18}{x^4}$. Hence find the set of values of the real number k such that the equation $f(x) = k$ has four real distinct roots.

17 Find the gradient of the tangent from the origin to the curve $y = \ln x$. Hence find the set of values of the real number k such that the equation $\ln x = kx$ has two real distinct roots.

18 Find the equation of the tangent to the curve $xy(x + y) + 16 = 0$ at the point on the curve where the gradient is -1.

19 The chord AB of a circle of radius r subtends an angle of 2θ radians at the centre O. The perimeter of the minor segment AB is k times the perimeter of the triangle OAB. Show that $k + (k - 1)\sin\theta = \theta$. Use a graphical method to obtain an estimate of θ in the case when $k = \frac{1}{2}$.

20 A taut belt passes round two circular pulleys of radii 6 cm and 2 cm respectively. The straight portions of the belt are common tangents to the two pulleys and are inclined to each other at an angle of 2θ radians. The total length of the belt is 44 cm. Show that $\pi + \theta + \cot\theta = 5.5$ and hence use a graphical method to obtain an estimate of θ.

2 Complex Numbers

2.1 The arithmetic of complex numbers and the solution of quadratic equations

Why do we need complex numbers?

The number system with which we are familiar consists of the set $\mathbb{R}$ of all real numbers together with the operations $+$ and $\times$. The set $\mathbb{R}$ includes the set $\mathbb{Z}$ of all integers and the set $\mathbb{Q}$ of all rational numbers, where $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Each of $\mathbb{R}$ and $\mathbb{Q}$, together with the operations $+$ and $\times$, is a field. (See appendix 2.) In the real number system we can solve every linear equation $ax + b = 0$. However, the real number system does not enable us to solve every quadratic equation with real coefficients.

Consider the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$. Using the method of completing the square,

$$ax^2 + bx + c = 0$$
$$4a^2x^2 + 4abx = -4ac$$
$$(2ax + b)^2 = b^2 - 4ac, \qquad (1)$$

where $\Delta = b^2 - 4ac$ is known as the discriminant of the quadratic equation.

If $\Delta \geqslant 0$, equation (1) has real roots given by the quadratic formula

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}.$$

If $\Delta < 0$, equation (1) has no real roots since there is no real number which has a negative square.

To solve all quadratic equations with real coefficients we would need to extend the real number system to include numbers with negative squares. Let the number i be defined by $i^2 = -1$. The extended set of numbers would need to include all the real multiples of i of the form $b \times i$, $b \in \mathbb{R}$. The operation $\times$ should be defined to obey the usual number laws in the real number system. Then every real number would have two square roots. For example, -4 could be written $4 \times i^2$. Then -4 has two square roots, $2 \times i$ and $-2 \times i$.

The structure of the complex number system

Consider the set $\mathbb{C}$ of numbers of the form $a + bi$, where a and b are real. The operations $+$ and $\times$ between elements of $\mathbb{C}$ are defined so that the numbers $a + bi$ are formally treated like linear expressions in the pronumeral i, with i^2 then being replaced by -1.

Example 1

Find the sum and product of $2 + 5i$ and $1 + 3i$.

Solution

$(2 + 5i) + (1 + 3i) = 3 + 8i$

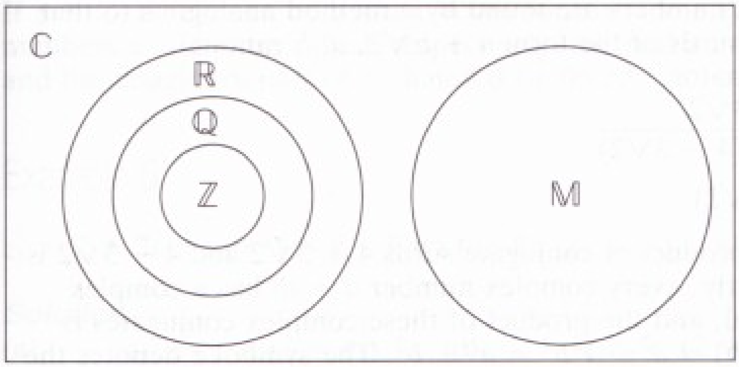
$$\begin{aligned}(2 + 5i)(1 + 3i) &= 2 + 15i^2 + 5i + 6i\\ &= 2 - 15 + 11i\\ &= -13 + 11i\end{aligned}$$

The set of complex numbers $\mathbb{C}$ with the operations $+$ and $\times$ is called the system of complex numbers and is a field which is an extension of the real number field. Numbers $a + 0i$, $a \in \mathbb{R}$ behave exactly like the real numbers with the one-to-one correspondence $a + 0i \leftrightarrow a$. In practice we use the symbol 3 for both the complex number $3 + 0i$ and the real number 3 and we refer to $3 + 0i$ as real. In this sense the set of real numbers $\mathbb{R}$ is a subset of the set of complex numbers $\mathbb{C}$.

Numbers of the form $0 + bi$, $b \in \mathbb{R}$, are called imaginary. For example, $0 + 2i = 2i$ is an imaginary number.

The following Venn diagram shows the relationship between $\mathbb{C}$, $\mathbb{R}$, the set of rational numbers $\mathbb{Q}$, the set of integers $\mathbb{Z}$ and the set of imaginary numbers $\mathbb{M}$.

Figure 2.1



The description 'complex' does not imply 'non-real'. The real numbers 0, 1, $\frac{3}{5}$, $\sqrt{2}$, π are also complex numbers. The numbers $3i$, $1 - 2i$, $\pi + \sqrt{3}\,i$ can be described as 'non-real'.

The definitions of $+$ and $\times$ are consistent with $2 + 3i$ being the same complex number as $2 + 3 \times i$, where 2, 3, and i are regarded as elements of $\mathbb{C}$. The number $1 + (-2)i$ is usually written $1 - 2i$.

Operations $+$, $\times$ on $\mathbb{C}$. Complex conjugates and reciprocals

Addition, subtraction and multiplication of complex numbers follow the same pattern as for surds of the form $a + b\sqrt{2}$, where a and b are rational. Just as $(\sqrt{2})^2$ is replaced by 2, i^2 is replaced by -1.

Addition: $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$
$(a + ib) \quad + (c + id) \quad = (a + c) + i(b + d)$

(Similarly for *subtraction*)
Multiplication: $(a + b\sqrt{2})(c + d\sqrt{2}) = ac + bd(\sqrt{2})^2 + (ad + bc)\sqrt{2}$
$= (ac + 2bd) + (ad + bc)\sqrt{2}$
$(a + ib)(c + id) = ac + i^2bd + i(ad + bc)$
$= (ac - bd) + i(ad + bc)$

Example 2

$z_1 = 2 + 3i$, $z_2 = -1 + 5i$. Find the values of
(a) $z_1 + z_2$ **(b)** $z_1 - z_2$ **(c)** $3z_1$ **(d)** $3iz_1$ **(e)** z_1z_2

Solution

(a) $z_1 + z_2 = 1 + 8i$ **(b)** $z_1 - z_2 = 3 - 2i$ **(c)** $3z_1 = 6 + 9i$
(d) $3iz_1 = 6i + 9i^2 = -9 + 6i$ **(e)** $z_1z_2 = -2 + 15i^2 - 3i + 10i = -17 + 7i$

Note that when we write a general complex number $a + ib$ we write the symbol i before the pronumeral, but when we write a specific complex number $2 + 3i$ we write the i after the 3. It is usual to write the term in i last, for example $-9 + 6i$ rather than $6i - 9$.

Reciprocals of complex numbers are found by a method analogous to that for finding reciprocals of surds of the form $a + b\sqrt{2}$, a, b rational. Rationalising the denominator,

$$\frac{1}{4 + 3\sqrt{2}} = \frac{4 - 3\sqrt{2}}{(4 + 3\sqrt{2})(4 - 3\sqrt{2})}$$
$$= -\frac{1}{2}(4 - 3\sqrt{2}).$$

We use the fact that the product of conjugate surds $4 + 3\sqrt{2}$ and $4 - 3\sqrt{2}$ is a rational number. Similarly, every complex number $a + ib$ has a complex conjugate $a - ib$, a, b real, and the product of these complex conjugates is real since $(a + ib)(a - ib) = a^2 - i^2b^2 = a^2 + b^2$. The symbol $\bar{z}$ denotes the complex conjugate of z. For example, $\overline{3 + 5i} = 3 - 5i$.

Example 3

For each of the following values of z find $\bar{z}$ and $z\bar{z}$.
(a) $2 - 3i$ **(b)** i **(c)** 2

Solution

(a) $\bar{z} = 2 + 3i$, $z\bar{z} = 4 + 9 = 13$
(b) $\bar{z} = -i$, $z\bar{z} = 1$ (since $i = 0 + 1i$)
(c) $\bar{z} = 2$, $z\bar{z} = 4$ (since $2 = 2 + 0i$)

The reciprocal of a non-real complex number is found by the process of realising the denominator, using the fact that the product of complex conjugates is real.

Example 4

(a) Find the reciprocal of **(i)** i **(ii)** $4 + 3i$.
(b) Write $(2 + 3i) \div (1 - 2i)$ in the form $a + ib$.

Solution

(a) **(i)** $i(-i) = 1$

$i, -i$ are reciprocals

(ii) $$\frac{1}{4+3i} = \frac{4-3i}{(4+3i)(4-3i)} = \frac{4-3i}{16+9} = \frac{4}{25} - \frac{3}{25}i$$

(b) $$\frac{2+3i}{1-2i} = \frac{(2+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{(2-6)+(4+3)i}{1+4} = -\frac{4}{5} + \frac{7}{5}i$$

Real and imaginary parts of a complex number

If $z \in \mathbb{C}$, then z can be written in the form $z = x + iy$, $x, y \in \mathbb{R}$. The real numbers x and y are called, respectively, the real part of z, denoted by $Re\ z$, and the imaginary part of z, denoted by $Im\ z$.

Example 5

$z_1 = 2 + 3i$, $z_2 = 1 + i$. Find **(a)** $Re\ (z_1 + z_2)$ **(b)** $Im\ (z_1\bar{z}_2)$.

Solution

(a) $$z_1 + z_2 = (3 + 4i)$$
$$Re\ (z_1 + z_2) = 3$$

(b) $$z_1\bar{z}_2 = (2 + 3i)(1 - i) = 5 + i$$
$$Im\ (z_1\bar{z}_2) = 1$$

Real numbers z have $Im\ z = 0$ and imaginary numbers z have $Re\ z = 0$. For example, $Im\ 2 = 0$, and $Re\ (3i) = 0$. The only number z which has $Re\ z = Im\ z = 0$ is $z = 0$.

Example 6

Find $z \in \mathbb{C}$ such that $Re\ z = 2$ and z^2 is imaginary.

Solution

$Re\ z = 2 \Rightarrow z = 2 + iy$ and $z^2 = (4 - y^2) + i\,(4y)$, $y \in \mathbb{R}$
z^2 imaginary $\Rightarrow 4 - y^2 = 0 \Rightarrow y = \pm 2$,
$\therefore z = 2 + 2i$ or $z = 2 - 2i$.

Equality of complex numbers

If $a + ib = c + id$, $a,b,c,d \in \mathbb{R}$,

then $(a - c) + i(b - d) = 0 \Rightarrow \begin{cases} a - c = Re\, 0 = 0 \\ b - d = Im\, 0 = 0 \end{cases} \Rightarrow \begin{cases} a = c \\ b = d \end{cases}$

Two complex numbers are equal if and only if both their real parts are equal and their imaginary parts are equal.

Unlike real numbers, complex numbers are not ordered. Non-real numbers cannot be compared using one of the symbols $>$ or $<$. In particular, non-real numbers cannot be designated positive or negative.

Example 7

Find real a and b such that $x^3 - x + 2 = (x^2 + 1)\, Q(x) + ax + b$, for all x.

Solution

Substituting $x = i$, $\quad -i - i + 2 = 0 + ai + b \quad$ (since $i^2 = -1$),
$\therefore 2 - 2i = b + ai, \quad a, b \in \mathbb{R}$,
$\therefore b = 2$ and $a = -2$.

Solving quadratic equations with real coefficients

Every negative real number has two complex square roots. For example, $-16 = 16i^2$, $\therefore -16$ has square roots $4i$ and $-4i$.

Example 8

Use the method of completing the square to solve $x^2 + 2x + 3 = 0$.

Solution

$x^2 + 2x + 3 = 0 \Rightarrow (x + 1)^2 = -2 \Rightarrow (x + 1)^2 = 2i^2$,
$\therefore x + 1 = \pm \sqrt{2}i, \quad \therefore x = -1 \pm \sqrt{2}i$.

This procedure enables us to extend the quadratic formula to solve every quadratic equation with real coefficients.

$ax^2 + bx + c = 0,\ a,b,c, \in \mathbb{R}$
$4a^2x^2 + 4abx = -4ac$
$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$
$(2ax + b)^2 = \Delta$, where $\Delta = b^2 - 4ac$ is a real number.

If $\Delta \geqslant 0$, this equation has two real roots given by $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$.

If $\Delta < 0$, then $\Delta = i^2\,|\Delta|$ has two square roots $i\sqrt{|\Delta|}$ and $-i\sqrt{|\Delta|}$.

Hence the given quadratic equation has two non-real roots given by
$x = \dfrac{-b \pm i\sqrt{|\Delta|}}{2a}$.

Example 9

Solve **(a)** $x^2 - 2x + 5 = 0$ **(b)** $2x^2 + \sqrt{3}x + 1 = 0$

Solution

(a) $\Delta = -16 = 16i^2$,

$\therefore x = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

(b) $\Delta = -5 = 5i^2$,

$\therefore x = \dfrac{-\sqrt{3} \pm \sqrt{5}i}{4} = -\dfrac{\sqrt{3}}{4} \pm \dfrac{\sqrt{5}i}{4}$

Note that in each of these examples the roots of the quadratic equation are complex conjugates. This will always be the case when the equation has real coefficients and discriminant $\Delta < 0$. In comparison, a quadratic equation with rational coefficients and $\sqrt{\Delta}$ irrational has two irrational roots which are conjugate surds. For example, $x^2 - 4x + 2 = 0$ has roots $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

Square roots of complex numbers

Example 10

$z^2 = 3 + 4i$. Find z.

Solution

Let $z = x + iy$, $x, y \in \mathbb{R}$.
Then $(x + iy)^2 = 3 + 4i \Rightarrow (x^2 - y^2) + (2xy)i = 3 + 4i$.
Equating real and imaginary parts: $x^2 - y^2 = 3$ and $2xy = 4$
$\therefore x^4 - x^2y^2 = 3x^2$ and $x^2y^2 = 4$

Then $x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2 - 4)(x^2 + 1) = 0$, x real,
$\therefore x = 2, y = 1 \Rightarrow z = 2 + i$, or $x = -2, y = -1 \Rightarrow z = -2 - i$.
Hence $3 + 4i$ has two square roots $2 + i$ and $-(2 + i)$.

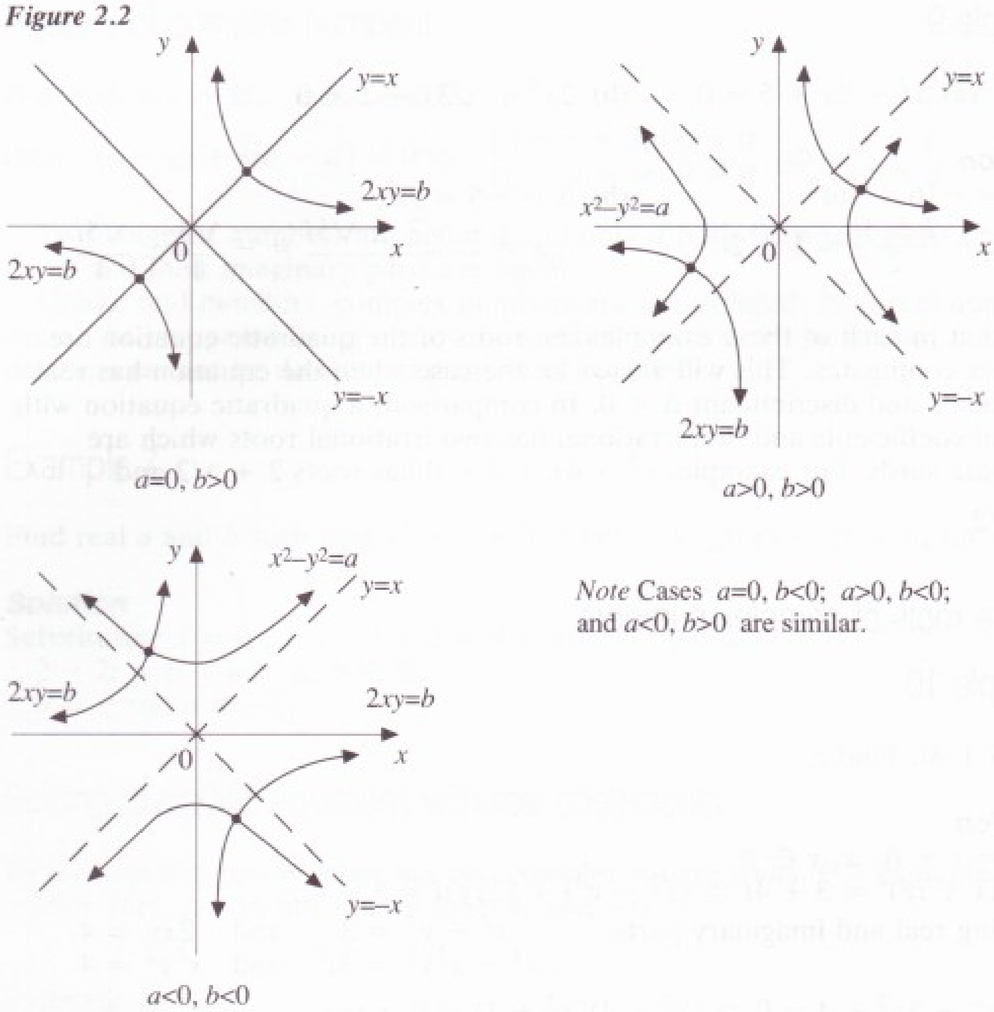
In general, to find the square roots of $a + ib$, $a, b \in \mathbb{R}$, $b \neq 0$ we solve $z^2 = a + ib$, where $z = x + iy$, $x, y \in \mathbb{R}$.
Then $(x + iy)^2 = a + ib \Rightarrow (x^2 - y^2) + (2xy)i = a + ib$.

Equating real and imaginary parts gives $x^2 - y^2 = a$ and $2xy = b$. Solution of these simultaneous equations will give the real and imaginary parts of z. Each such value of z is a square root of $a + ib$.

Consider the graphs $x^2 - y^2 = a$ and $2xy = b$, $b \neq 0$.
$2xy = b$ is a rectangular hyperbola with asymptotes $x = 0$ and $y = 0$.
$x^2 - y^2 = a$, $a \neq 0$, is a rectangular hyperbola with asymptotes $y = \pm x$.
$x^2 - y^2 = 0$ is the pair of lines $y = x$, $y = -x$.

In each case the graphs $x^2 - y^2 = a$ and $2xy = b$, $b \neq 0$, have two points of intersection and the symmetry in these diagrams shows that if (x_1, y_1) is one point of intersection, the other is $(-x_1, -y_1)$. Hence every non-real number has two distinct square roots, z_1 and $-z_1$. We have already seen that all non-zero real numbers have two distinct square roots of this form.

Figure 2.2



Note Cases $a=0, b<0$; $a>0, b<0$; and $a<0, b>0$ are similar.

Solving quadratic equations with complex coefficients

Consider the quadratic equation $ax^2 + bx + c = 0$. We have seen that completing the square gives $(2ax + b)^2 = \Delta$, where $\Delta = b^2 - 4ac$. If at least one of the coefficients a, b, c is non-real, then Δ need not be real. However, every non-zero complex number has two square roots and these sum to zero.

If $\Delta \neq 0$, let $\alpha, -\alpha$ be the square roots of Δ. Then $ax^2 + bx + c = 0$ has two distinct complex roots given by $x = \dfrac{-b \pm \alpha}{2a}$, $\alpha \neq 0$, $\alpha^2 = \Delta$.

If $\Delta = 0$, then $ax^2 + bx + c = 0$ has two equal complex roots, $x = -\dfrac{b}{2a}$.

Example 11

Solve $2x^2 + (1 - i)\,x + (1 - i) = 0$.

Solution

Find Δ: $\Delta = (1 - i)^2 - 8(1 - i) = -8 + 6i$.

Find square roots of Δ: Let $(a + ib)^2 = -8 + 6i$, $a,b \in \mathrm{R}$.

Then $(a^2 - b^2) + (2ab)i = -8 + 6i$.

Equating real and imaginary parts, $a^2 - b^2 = -8$ and $ab = 3$.

$$a^2 - \frac{9}{a^2} = -8 \Rightarrow a^4 + 8a^2 - 9 = 0$$

$$(a^2 + 9)(a^2 - 1) = 0,\ a \text{ real}$$
$$\Rightarrow a = 1,\ b = 3 \text{ or } a = -1,\ b = -3.$$

Hence Δ has square roots $1 + 3i$, $-1 - 3i$.

Use the quadratic formula: $2x^2 + (1 - i)x + (1 - i) = 0$ has solutions

$$x = \frac{-(1 - i) \pm (1 + 3i)}{4},$$

$$\therefore x = i \quad \text{or} \quad x = -\frac{1}{2} - \frac{1}{2}i.$$

Note that when the coefficients of the quadratic equation are not all real, the roots of the equation need not be complex conjugates. However, the relationship between the coefficients and the sum and product of the roots is the same as for quadratic equations with real coefficients. In the above example the sum of the roots is $i + (-\frac{1}{2} - \frac{1}{2}i) = -\frac{1}{2}(1 - i)$,

while the product of the roots is $i(-\frac{1}{2} - \frac{1}{2}i) = \frac{1}{2}(1 - i)$.

Exercise 2.1

1 Copy the Venn diagram in figure 2.1. Write the following complex numbers in the appropriate place on the diagram

$\sqrt{2}$, $3i$, $\frac{3}{5}$, $1 - 2i$, 0, 2.3, π, $\pi + \sqrt{3}i$, i, $-i$, -1.

2 $z_1 = 2 - 3i$, $z_2 = 1 + 4i$. Evaluate

(a) $z_1 + z_2$ **(b)** $z_1 - z_2$ **(c)** $z_1 z_2$ **(d)** ${z_1}^2$

(e) $\dfrac{1}{z_2}$ **(f)** $z_2 \div z_1$ **(g)** ${z_1}^2 - {z_2}^2$ **(h)** ${z_1}^3 - {z_2}^3$

3 $z = -3 + 2i$

(a) Evaluate $\bar{z}$. Verify that $z\bar{z}$ is real.

(b) Use $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}}$ to find $\dfrac{1}{z}$ in the form $a + ib$; $a,b \in \mathbb{R}$.

4 Prove the following results about complex conjugates

(a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ **(b)** $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ **(c)** $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

(d) $\overline{\left(\dfrac{1}{z}\right)} = \dfrac{1}{(\bar{z})}$ **(e)** $\overline{z_1 \div z_2} = \bar{z}_1 \div \bar{z}_2$ **(f)** $\overline{5z} = 5\bar{z}$

5 (a) $a\alpha^2 + b\alpha + c = 0$, where $a, b, c \in \mathbb{R}$ and $\alpha \in \mathbb{C}$. Use the results in question 4 to show that $a(\bar{\alpha})^2 + b\bar{\alpha} + c = 0$.
(b) Deduce that if α is a non-real root of $ax^2 + bx + c = 0$, where a, b, c are real, then $\bar{\alpha}$ is the other root of this quadratic equation.

6 (a) $z \in \mathbb{C}$ such that $Im\ z = 2$ and z^2 is real. Find z.
(b) $z \in \mathbb{C}$ such that $Re\ z = 2Im\ z$, and $z^2 - 4i$ is real. Find z.

7 $z \in C$ such that $\dfrac{z}{z - i}$ is real. Show that z is imaginary.

8 Find the square roots of the following complex numbers
(a) -25 **(b)** $-6i$ **(c)** i **(d)** $-4 + 3i$ **(e)** $-5 - 12i$

9 Solve the following quadratic equations
(a) $x^2 + x + 1 = 0$ **(b)** $2x^2 - 4x + 3 = 0$
(c) $4x^2 - 4(1 + 2i)x - (3 - 4i) = 0$ **(d)** $ix^2 - 2(i + 1)x + 10 = 0$

10 (a) $3 - 2i$ is one root of $x^2 + bx + c = 0$ where b and c are real. Find b and c.
(b) $x^2 + 6x + k = 0$ has one root α where $Im(\alpha) = 2$. If k is real, find both roots of the equation and the value of k.
(c) $1 - 2i$ is one root of $x^2 - (3 + i)x + k = 0$. Find k and the other root of the equation.

2.2 Geometrical representation of a complex number as a point in an Argand diagram

Two complex numbers are equal if and only if both their real parts are equal and their imaginary parts are equal. Hence there is a one-to-one correspondence between the complex number $a + ib$ and the ordered pair (a, b), $a, b \in \mathbb{R}$. This suggests using the point A with Cartesian coordinates (a, b) to represent the complex number $a + ib$.

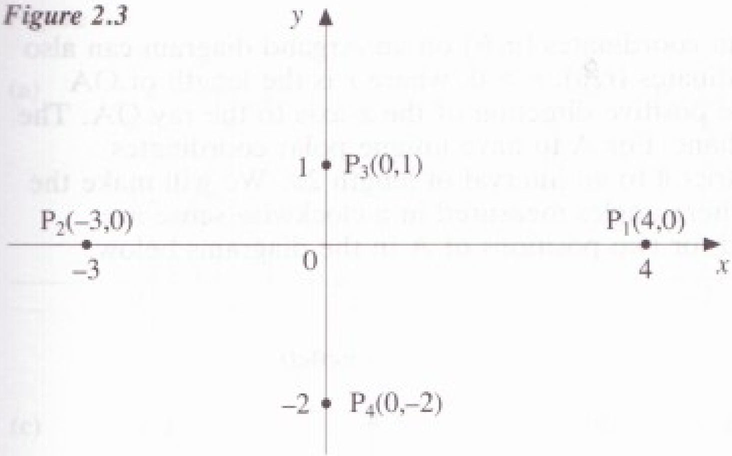
As this idea was first expressed clearly by the French mathematician Jean Robert Argand (1768–1822), his name is given to the diagram which represents a complex number in this way. On an Argand diagram, the real part of a complex number is measured along the x-axis, and the imaginary part along the y-axis. We refer to the x- and y-axes as the real and imaginary axes respectively.

Example 12

On an Argand diagram show the points P_1, P_2, P_3, P_4, representing the complex numbers 4, -3, i, $-2i$ respectively.

Solution

Figure 2.3



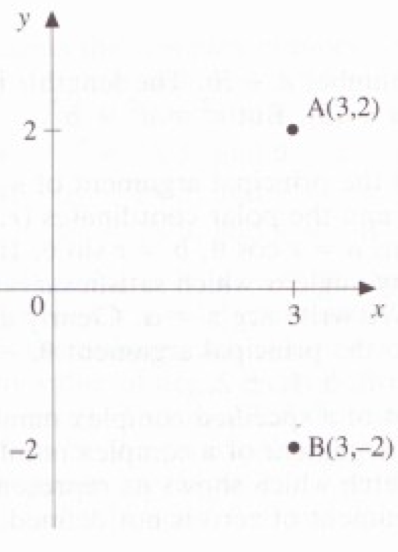
Note that if $z = x + iy$ is real, $y = 0$ and hence the point P representing z lies on the x-axis. Similarly, if $z = x + iy$ is imaginary, $x = 0$ and the point P representing z lies on the y-axis.

Example 13

If $z = 3 + 2i$, express $\bar{z}$ in the form $a + ib$. On an Argand diagram, plot the points A and B representing z and $\bar{z}$ respectively.

Solution

Figure 2.4



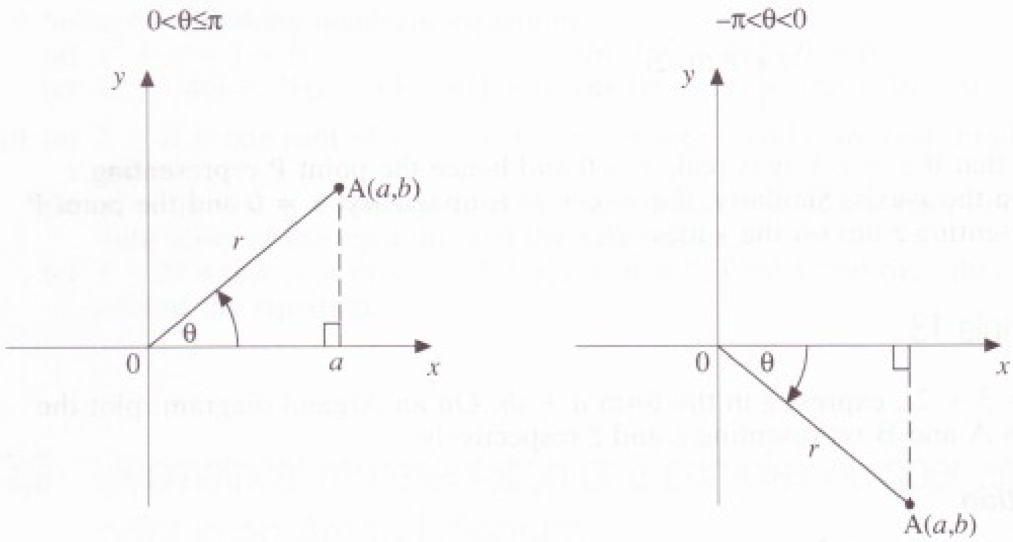
$z = 3 + 2i, \quad \bar{z} = 3 - 2i.$
Note that B is the reflection of A in the real axis.

In general, if $z = x + iy$ is represented by P(x, y) then $\bar{z} = x - iy$ is represented by Q$(x, -y)$, and Q is the reflection of P in the real axis.

Modulus and argument of a complex number

The point A with Cartesian coordinates (a,b) on an Argand diagram can also be specified by polar coordinates (r,θ), $r > 0$, where r is the length of OA and θ is the angle from the positive direction of the x-axis to the ray OA. The angle θ is measured in radians. For A to have unique polar coordinates (r,θ) it is necessary to restrict θ to an interval of length 2π. We will make the restriction $-\pi < \theta \leqslant \pi$, where angles measured in a clockwise sense are negative. This is illustrated for two positions of A in the diagrams below.

Figure 2.5



A(a,b) represents the complex number $a + ib$. The length r is called the modulus of $a + ib$ and is written $|a + ib|$. But $r^2 = a^2 + b^2$,
$\therefore |a + ib| = \sqrt{(a^2 + b^2)}$.
The angle θ, $-\pi < \theta \leqslant \pi$, is called the principal argument of $a + ib$.

The Cartesian coordinates (a,b) and the polar coordinates (r,θ) of the point A are related by the equations $a = r\cos\theta$, $b = r\sin\theta$. If $z = a + ib$ and $r = \sqrt{(a^2 + b^2)}$ we refer to any angle α which satisfies $a = r\cos\alpha$ and $b = r\sin\alpha$, as an argument of z. We write $\arg z = \alpha$. Clearly $\arg z$ has infinitely many values, α, related to the principal argument θ, $-\pi < \theta \leqslant \pi$, by the equation $\alpha = \theta + 2n\pi$, $n = 0, \pm 1, \pm 2,\ldots$

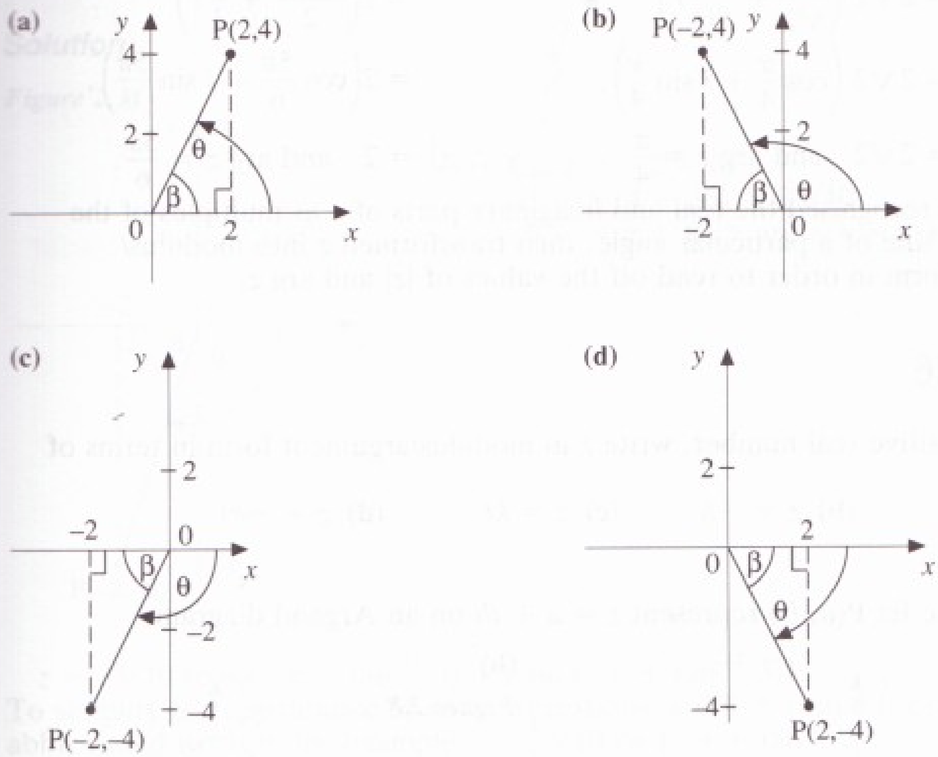
When asked to find the argument of a specified complex number we usually quote the principal argument. The argument of a complex number is most easily obtained by considering a sketch which shows its representation on an Argand diagram. Note that the argument of zero is not defined.

Example 14

Find the modulus and principal argument of
(a) $2 + 4i$ **(b)** $-2 + 4i$ **(c)** $-2 - 4i$ **(d)** $2 - 4i$

Solution

Figure 2.6



P(a, b) represents the complex number $z = a + ib$, where z has modulus r and argument θ. In each case, $r = \sqrt{(4 + 16)} = 2\sqrt{5}$ and $\beta = \tan^{-1} 2$.

(a) $z = 2 + 4i$ $\quad |z| = 2\sqrt{5}$, and $\theta = \beta$ $\quad \Rightarrow \quad \arg z = \tan^{-1} 2$

(b) $z = -2 + 4i$ $\quad |z| = 2\sqrt{5}$, and $\theta = \pi - \beta$ $\quad \Rightarrow \quad \arg z = \pi - \tan^{-1} 2$

(c) $z = -2 - 4i$ $\quad |z| = 2\sqrt{5}$, and $\theta = -\pi + \beta$ $\quad \Rightarrow \quad \arg z = -\pi + \tan^{-1} 2$

(d) $z = 2 - 4i$ $\quad |z| = 2\sqrt{5}$, and $\theta = -\beta$ $\quad \Rightarrow \quad \arg z = -\tan^{-1} 2$

By definition, if $z = x + iy$, $x, y \in \mathbb{R}$, then $|z| = \sqrt{(x^2 + y^2)}$ and $\arg z$ is any value of θ for which $x = |z| \cos \theta$ and $y = |z| \sin \theta$. The principal argument of z is the unique such value of θ in the interval $-\pi < \theta \leqslant \pi$. Hence if θ is any value of $\arg z$, z can be written in the form $z = |z|(\cos \theta + i \sin \theta)$. This is called the modulus/argument form of z. We usually choose θ to be the principal argument of z.

Conversely, suppose $z = k(\cos \theta + i \sin \theta)$, where k is a positive real number. Then $|z| = \sqrt{[k^2(\cos^2 \theta + \sin^2 \theta)]} = k$, and θ is an argument of z.

Example 15

Find the modulus and argument of **(a)** $2 + 2i$ **(b)** $-\sqrt{3} + i$

Solution

(a) $z = 2 + 2i$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$

$\therefore |z| = 2\sqrt{2}$ and $\arg z = \frac{\pi}{4}$.

(b) $z = -\sqrt{3} + i$

$= 2\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right),$

$\therefore |z| = 2$ and $\arg z = \frac{5\pi}{6}$.

Here we recognised the real and imaginary parts of z as multiples of the cosine and sine of a particular angle, then transformed z into modulus/argument form in order to read off the values of $|z|$ and $\arg z$.

Example 16

If k is a positive real number, write z in modulus/argument form in terms of k if

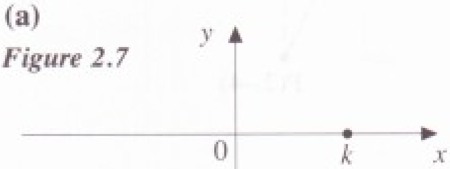
(a) $z = k$ **(b)** $z = -k$ **(c)** $z = ki$ **(d)** $z = -ki$

Solution

In each case let $P(a, b)$ represent $z = a + ib$ on an Argand diagram.

(a)

Figure 2.7



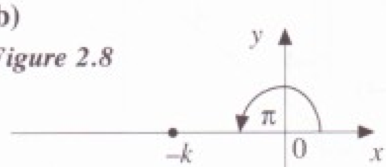
$z = k$

$|z| = k$ and $\arg z = 0$

$z = k(\cos 0 + i\sin 0)$

(b)

Figure 2.8



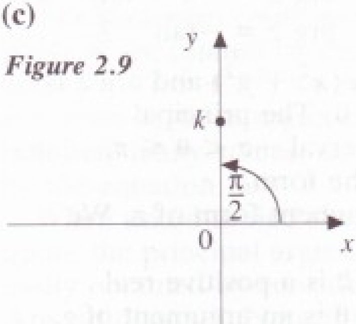
$z = -k$

$|z| = k$ and $\arg z = \pi$

$z = k(\cos\pi + i\sin\pi)$

(c)

Figure 2.9



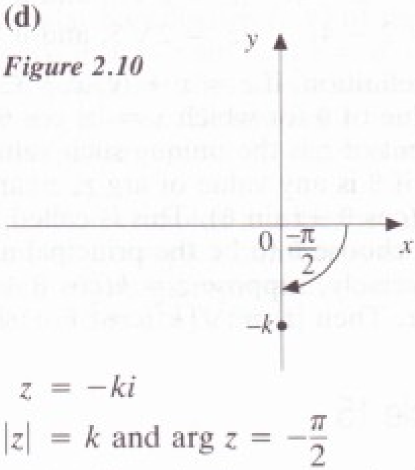
$z = ki$

$|z| = k$ and $\arg z = \frac{\pi}{2}$

$z = k\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

(d)

Figure 2.10



$z = -ki$

$|z| = k$ and $\arg z = -\frac{\pi}{2}$

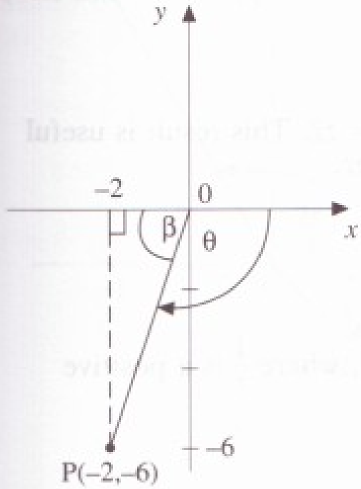
$z = k\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$

Example 17

Write $-2-6i$ in modulus/argument form.

Solution

Figure 2.11



Let $z=-2-6i$
$|z|=\sqrt{(4+36)}=2\sqrt{10}$
$\theta=-\pi+\beta \Rightarrow \arg z=-\pi+\tan^{-1} 3$

$z=2\sqrt{10}\,[\cos(-\pi+\tan^{-1} 3)+i\sin(-\pi+\tan^{-1} 3)]$.
To simplify the appearance of such expressions, $\cos\theta+i\sin\theta$ is often abbreviated to cis θ, for example $z=2\sqrt{10}\operatorname{cis}(-\pi+\tan^{-1} 3)$.

Products and quotients using modulus/argument form

Let $z_1=r_1(\cos\alpha+i\sin\alpha)$, $z_2=r_2(\cos\beta+i\sin\beta)$, where $r_1=|z_1|$ and $r_2=|z_2|$ are positive real numbers and $\arg z_1=\alpha$, $\arg z_2=\beta$.
Then $z_1z_2 = r_1r_2(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)$
$= r_1r_2[(\cos\alpha\cos\beta-\sin\alpha\sin\beta)+i(\sin\alpha\cos\beta+\cos\alpha\sin\beta)]$
$= r_1r_2[\cos(\alpha+\beta)+i\sin(\alpha+\beta)]$,
and r_1r_2 is a positive real number,
$\therefore |z_1z_2|=r_1r_2=|z_1||z_2|$ and $\alpha+\beta$ is one value of $\arg z_1z_2$.

We usually write $\arg z_1z_2=\arg z_1+\arg z_2$ and understand this to mean that one value of $\arg z_1z_2$ is obtained by adding any value of $\arg z_1$ to any value of $\arg z_2$.

Example 18

If $z_1=3\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$ and $z_2=2\left(\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}\right)$, write z_1z_2

(a) in modulus/argument form, and **(b)** in the form $a+ib$.

Solution

(a) $|z_1| = 3,\ |z_2| = 2 \Rightarrow |z_1 z_2| = 6$

$\arg z_1 = \dfrac{2\pi}{3}$ and $\arg z_2 = \dfrac{5\pi}{6} \Rightarrow \arg z_1 z_2 = \dfrac{2\pi}{3} + \dfrac{5\pi}{6} = \dfrac{3\pi}{2}$.

But $\dfrac{3\pi}{2} > \pi$. The principal argument of $z_1 z_2$ is $\dfrac{3\pi}{2} - 2\pi = -\dfrac{\pi}{2}$

$\therefore z_1 z_2 = 6\left[\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right]$.

(b) $z_1 z_2 = 6(0 - i) = 0 - 6i$

If $z = x + iy$, $|z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z\bar{z}$. This result is useful when considering the reciprocal of a complex number.

Let z have modulus r and argument θ. Then $z = r(\cos\theta + i\sin\theta)$ and $\bar{z} = r(\cos\theta - i\sin\theta)$.

But $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{\bar{z}}{|z|^2}, \qquad z \neq 0,$

$\therefore \dfrac{1}{z} = \dfrac{r(\cos\theta - i\sin\theta)}{r^2} = \dfrac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$, where $\dfrac{1}{r}$ is a positive real number.

Thus $\left|\dfrac{1}{z}\right| = \dfrac{1}{r}$ and $-\theta$ is one value of $\arg\left(\dfrac{1}{z}\right)$.

Hence $\quad \left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$ and $\arg\left(\dfrac{1}{z}\right) = -\arg z$

Then

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \Rightarrow \begin{cases} \left|\dfrac{z_1}{z_2}\right| = |z_1| \cdot \left|\dfrac{1}{z_2}\right| = |z_1| \cdot \dfrac{1}{|z_2|} = \dfrac{|z_1|}{|z_2|} \\ \arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 + \arg\left(\dfrac{1}{z_2}\right) = \arg z_1 - \arg z_2. \end{cases}$$

The method of mathematical induction can be used to prove that for any $z \neq 0$, $|z^n| = |z|^n$ and $\arg(z^n) = n\arg z$ for all positive integers n.

This result can be extended to negative integers. Let n be a positive integer, $z \neq 0$. Then

$$\begin{aligned} |z^{-n}| &= \left|\left(\frac{1}{z}\right)^n\right| \\ &= \left|\frac{1}{z}\right|^n \\ &= \left(\frac{1}{|z|}\right)^n \\ &= |z|^{-n} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \arg(z^{-n}) &= \arg\left[\left(\frac{1}{z}\right)^n\right] \\ &= n\arg\left(\frac{1}{z}\right) \\ &= n(-\arg z) \\ &= -n\arg z \end{aligned}$$

Example 19

$z_1 = 1 - i$ and $z_2 = -1 + \sqrt{3}i$.

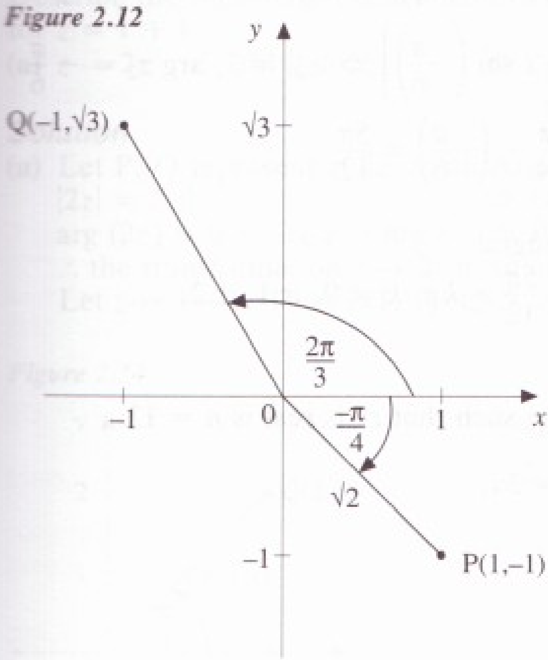
(a) Find the moduli and principal arguments of z_1, z_2, $z_1 z_2$.

(b) Use the given forms of z_1, z_2 to find $z_1 z_2$ in the form $a + ib$. Hence evaluate $\cos\dfrac{5\pi}{12}$ as a surd.

Solution

Let P(1, −1), Q(−1, $\sqrt{3}$) represent z_1, z_2 respectively.

Figure 2.12



(a) $|z_1| = \sqrt{2}$, $|z_2| = 2 \qquad \Rightarrow \quad |z_1 z_2| = 2\sqrt{2}$

$\arg z_1 = -\frac{\pi}{4}$, $\arg z_2 = \frac{2\pi}{3} \qquad \Rightarrow \quad \arg(z_1 z_2) = -\frac{\pi}{4} + \frac{2\pi}{3} = \frac{5\pi}{12}$

(b) $z_1 z_2 = (1 - i)(-1 + \sqrt{3}i) = (\sqrt{3} - 1) + (\sqrt{3} + 1)i$

But $\qquad z_1 z_2 = 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i \sin\frac{5\pi}{12}\right)$

Equating real parts: $\quad 2\sqrt{2}\cos\frac{5\pi}{12} = \sqrt{3} - 1 \Rightarrow \cos\frac{5\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$

Example 20

(a) If $z_1 = 1 + i$, $z_2 = \sqrt{3} - i$, find the moduli and principal arguments of z_1, z_2 and $\frac{z_1}{z_2}$.

(b) If $z = \frac{1 + i}{\sqrt{3} - i}$, find the smallest positive integer n such that z^n is real, and evaluate z^n for this integer n.

Solution

(a) $z_1 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow |z_1| = \sqrt{2},\ \arg z_1 = \frac{\pi}{4},$

$z_2 = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right] \Rightarrow |z_2| = 2,\ \arg z_2 = -\frac{\pi}{6}$

$\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{1}{\sqrt{2}}$ and $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{4} - \left(-\frac{\pi}{6}\right) = \frac{5\pi}{12}$

(b) If z^n is real, then $\arg z^n = k\pi$, k integral.

But $\arg z^n = n \arg z$. Therefore $n \cdot \frac{5\pi}{12} = k\pi,\ k = 0, \pm 1, \pm 2, \ldots,$

$\therefore n = \frac{12}{5}k, \quad k = 0, \pm 1, \pm 2, \ldots$

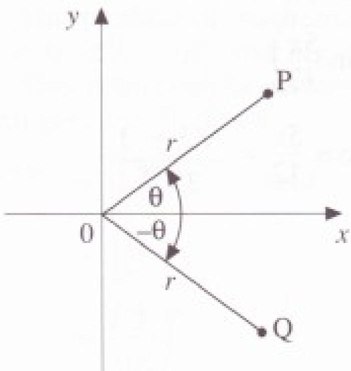
Hence the smallest positive integer n such that z^n is real is $n = 12$.

$|z^{12}| = \left(\frac{1}{\sqrt{2}}\right)^{12} = \frac{1}{64}$ and $\arg(z^{12}) = 5\pi,$

$\therefore z^{12} = -\frac{1}{64}$

Geometrical relationship between points on an Argand diagram

Figure 2.13



P, Q represent z, $\bar{z}$ respectively, where $z = r(\cos\theta + i\sin\theta)$.

P and Q are reflections of each other in the real axis. The transformation $z \to \bar{z}$ acting on all complex numbers z corresponds to a reflection in the real axis on an Argand diagram.

Similarly if we consider the relationship between points P, Q, representing z, cz, we will be able to describe geometrically the transformation $z \to cz$, where $c \in \mathbb{C}$.

Example 21

Describe the following transformations and illustrate on an Argand diagram for $z = 1 + i$.

(a) $z \to 2z$ **(b)** $z \to iz$ **(c)** $z \to -z$ **(d)** $z \to -3z$

Solution

(a) Let P, Q represent z, $2z$ on an Argand diagram.

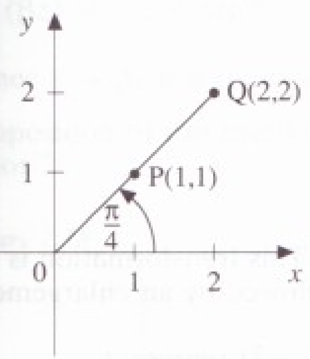
$|2z| = 2|z| \qquad \Rightarrow OQ = 2\,OP$

$\arg(2z) = 0 + \arg z = \arg z \qquad \Rightarrow$ P, Q lie on same ray from O,

$\therefore$ the transformation $z \to 2z$ is an enlargement about O by a factor 2.

Let $z = 1 + i$. Then P(1, 1), Q(2, 2) represent z, $2z$.

Figure 2.14



$\arg(2z) = \arg z = \frac{\pi}{4}$

$|2z| = 2|z| = 2\sqrt{2}$

Note that similarly, $z \to cz$, where c is a positive real number, is an enlargement (or reduction) about O by a factor c.

(b) Let P, Q represent z, iz on an Argand diagram.

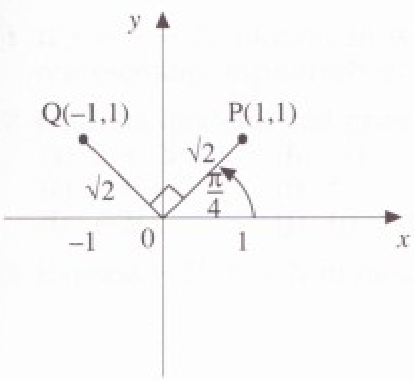
$|iz| = |i||z| = |z| \qquad \Rightarrow OQ = OP$

$\arg(iz) = \frac{\pi}{2} + \arg z \qquad \Rightarrow$ ray OQ makes an angle $\frac{\pi}{2}$ with ray OP.

The transformation $z \to iz$ is a rotation anticlockwise about O through $\frac{\pi}{2}$.

Let $z = 1 + i$. Then P(1, 1), Q(−1, 1) represent z, iz.

Figure 2.15

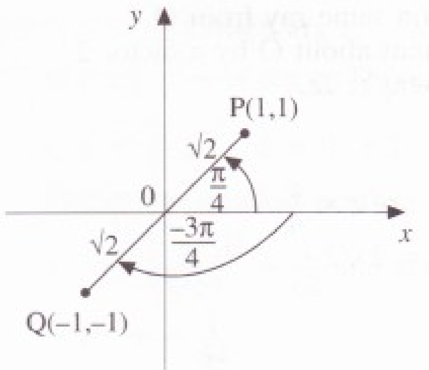


$|iz| = |z| = \sqrt{2}$

$\arg(iz) = \frac{\pi}{2} + \frac{\pi}{4}$

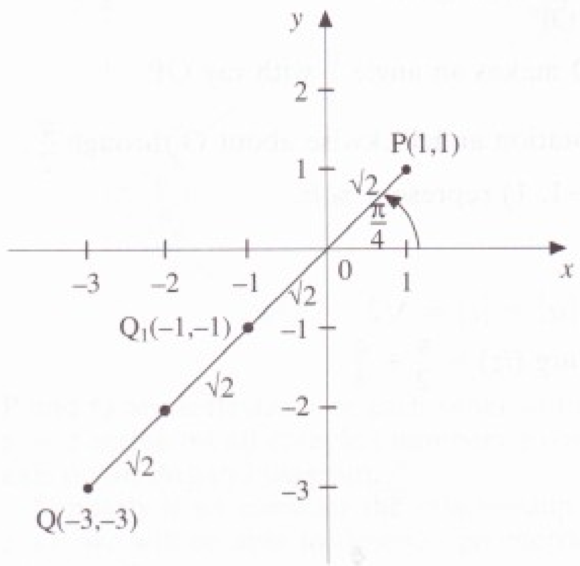
(c) Let P, Q represent z, $-z$ respectively on an Argand diagram.
$-z = i^2 z = i(iz)$. The transformation $z \to iz \to i(iz)$ is the composition of two rotations, each anticlockwise about O through $\frac{\pi}{2}$. Hence the transformation $z \to -z$ is a rotation anticlockwise about O through π, or equivalently a reflection in the point O.
Let $z = 1 + i$. Then P(1, 1), Q(−1, −1) represent z, $-z$.

Figure 2.16



(d) $z \to -3z$ can be expressed as $z \to -z \to 3(-z)$. This transformation is the composition of a reflection in the point O followed by an enlargement about O by a factor 3.
Let $z = 1 + i$. Then P(1, 1), $Q_1(-1, -1)$, Q(−3, −3) represent z, $-z$, $-3z$ respectively.

Figure 2.17



Example 22

Write $\alpha = 1 - \sqrt{3}i$ in modulus argument form. Hence describe the transformation $z \to \alpha z$ and illustrate on an Argand diagram for $z = 1 + i$.

Solution

$$\alpha = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right],$$

$\therefore \alpha = 2\beta$, where $\beta = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$.

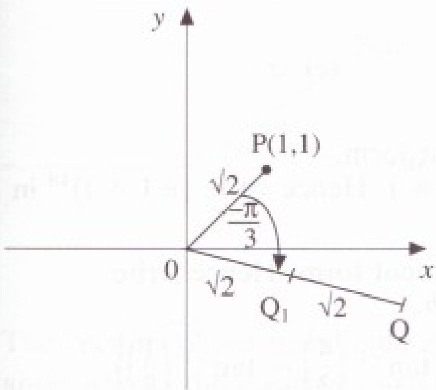
$z \to \alpha z$ can be expressed as $z \to \beta z \to 2\beta z$. Let P, Q_1, Q represent z, βz, $2\beta z$ respectively. Then

$|\beta z| = |\beta||z| = |z| \quad \Rightarrow OQ_1 = OP$

$\arg(\beta z) = -\frac{\pi}{3} + \arg z \quad \Rightarrow$ ray OQ_1 makes an angle $-\frac{\pi}{3}$ with ray OP.

Hence $z \to \beta z$ is a rotation clockwise about O through $\frac{\pi}{3}$ and $z \to \alpha z$ is the composition of this rotation followed by an enlargement about O by a factor 2.

Figure 2.18



$z = 1 + i$, $|z| = \sqrt{2}$ and $\arg z = \frac{\pi}{4}$

$|\beta z| = \sqrt{2}$ and $\arg(\beta z) = -\frac{\pi}{3} + \frac{\pi}{4} = -\frac{\pi}{12}$

$|\alpha z| = 2\sqrt{2}$ and $\arg(\alpha z) = -\frac{\pi}{12}$

Exercise 2.2

1 If $z = 3 - 2i$, plot on an Argand diagram the points P, Q, S, T, V, representing respectively z, $\bar{z}$, iz, $2z$, $-2iz$.

2 Find the modulus and principal arguments of

(a) $2 + 2i$	**(b)** $-1 + \sqrt{3}i$	**(c)** $-1 - \sqrt{3}i$	**(d)** $2 - i$
(e) $-3 + 2i$	**(f)** 5	**(g)** -5	**(h)** i
(i) $-2i$	**(j)** $i(i + 1)$		

3 Express $-2\sqrt{3} + 2i$ in modulus/argument form.

4 Use the properties of modulus and argument of a complex number to deduce that

(a) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$ **(b)** $\overline{\left(\frac{1}{z}\right)} = \frac{1}{(\bar{z})}$ **(c)** $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\overline{z_1})}{(\overline{z_2})}$

5 Use the method of mathematical induction to prove that $|z^n| = |z|^n$ and $\arg(z^n) = n \arg z$ for all positive integers n.

6 $z_1 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. Write down the modulus and principal argument of

(a) ${z_1}^3$ **(b)** $\frac{1}{z_2}$ **(c)** $\frac{{z_1}^3}{z_2}$

7 Write down the moduli and arguments of $-\sqrt{3} + i$ and $4 + 4i$. Hence express in modulus/argument form

(a) $(-\sqrt{3} + i)(4 + 4i)$ **(b)** $\frac{-\sqrt{3} + i}{4 + 4i}$

8 $z = 1 + \sqrt{3}i$. Find the smallest positive integer n for which z^n is real and evaluate z^n for this value of n. Show that there is no integral value of n for which z^n is imaginary.

9 z has modulus r and argument θ. Find in terms of r and θ the modulus and one argument of

(a) z^2 **(b)** $\frac{1}{z}$ **(c)** iz

10 **(a)** Write $(1 + \sqrt{3}i)^{-1}$ in modulus/argument form.
(b) State the modulus and argument of $-1 + i$. Hence write $(-1 + i)^{18}$ in the form $a + ib$.

11 Write $\sqrt{3} + i$ and $\sqrt{3} - i$ in modulus/argument form. Hence write $(\sqrt{3} + i)^{10} + (\sqrt{3} - i)^{10}$ in the form $a + ib$.

12 Find the modulus of $\frac{7 - i}{3 - 4i}$. Evaluate $\tan\left\{\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right)\right\}$.

Hence find the principal argument of $\frac{7 - i}{3 - 4i}$ in terms of π.

13 Describe geometrically the transformation $z \to \alpha z$, where $\alpha = -2 + 2i$. Illustrate on an Argand diagram for $z = 3i$.

14 **(a)** If p is real, and $-2 < p < 2$, show that the roots of the equation $x^2 + px + 1 = 0$ are non-real complex numbers with modulus 1.
(b) Solve the equations $x^2 + x + 1 = 0$ and $x^2 - \sqrt{3}x + 1 = 0$. Plot on an Argand diagram the points A and B representing the solutions of the first equation, and C and D representing the solutions of the second, choosing A and C to lie above the real axis.
(c) Find the angles $A\hat{O}B$, $C\hat{O}D$, $C\hat{O}A$ and $A\hat{C}B$.

15 (a) Obtain in the form $a + ib$ the roots of the equation $x^2 + 2x + 3 = 0$. Find the modulus and argument of each root and represent the roots on an Argand diagram by the points A and B.

(b) Let H and K be the points representing the roots of $x^2 + 2px + q = 0$, where p and q are real and $p^2 < q$. Find the algebraic relation satisfied by p and q in each of the following cases

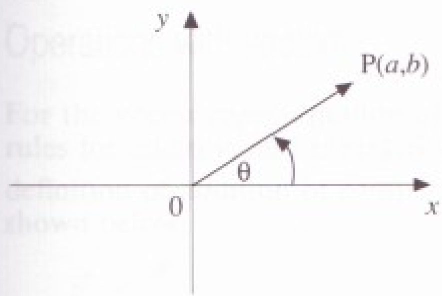
(i) $\mathrm{H\widehat{O}K}$ is a right angle.

(ii) A, B, H and K are equidistant from O.

2.3 Geometrical representation of a complex number as a vector

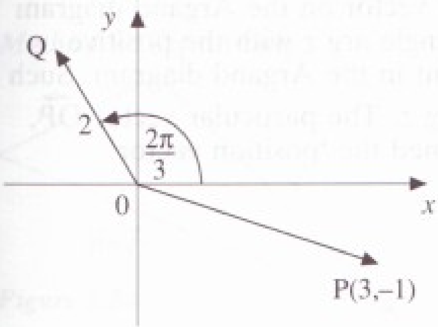
$z = a + ib$ can be represented by the point $P(a, b)$ on an Argand diagram. Alternatively, z could be represented by the vector $\overrightarrow{OP}$, denoted by an arrow with tail at O and tip at P.

Figure 2.19



The vector $\overrightarrow{OP}$ has magnitude equal to its length, and direction given by the angle θ from the positive x-axis to the vector. Hence the magnitude of $\overrightarrow{OP}$ is $|z|$ and the direction of $\overrightarrow{OP}$ is determined by the principal argument of z. For example:

Figure 2.20

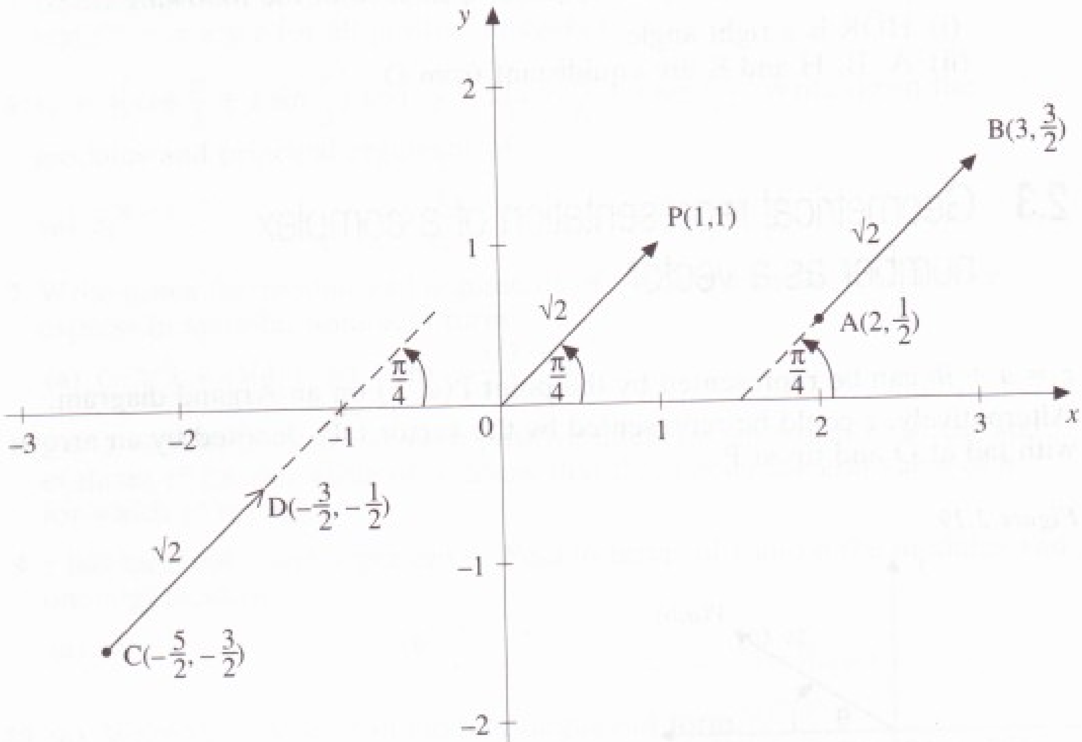


$\overrightarrow{OP}$ represent $3 - i$

$\overrightarrow{OQ}$ represents $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

If we do not insist that the tail of the vector is at the origin, then there are infinitely many vectors which could represent z. Consider $z = 1 + i$ and let $\overrightarrow{OP}$ represent z.

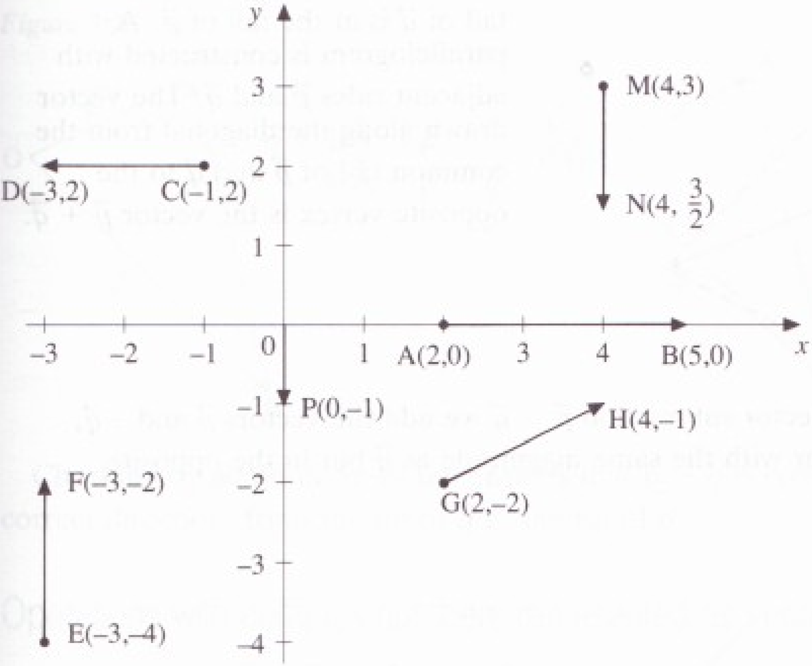
Figure 2.21



The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ each have magnitude $\sqrt{2} = |z|$, and make an angle $\frac{\pi}{4} = \arg z$ with the positive x-axis. Hence $\overrightarrow{AB}$ and $\overrightarrow{CD}$ also represent z. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are translations of $\overrightarrow{OP}$, that is $\overrightarrow{OP}$ moved parallel to itself to a new position in the plane. $z = 1 + i$ is represented by any member of the set of translations of the vector $\overrightarrow{OP}$.

In general, $z = a + ib$ is represented by any vector on the Argand diagram which has magnitude $|z|$ and which makes an angle $\arg z$ with the positive x-axis. The tail of the vector can be at any point in the Argand diagram. Such a vector is termed the 'free vector' representing z. The particular vector $\overrightarrow{OP}$, with tail at the origin and tip at $P(a, b)$, is termed the 'position vector' representing z. For example:

Figure 2.22



Vector	Number
$\overrightarrow{OP}$	$-i$
$\overrightarrow{AB}$	3
$\overrightarrow{MN}$	$-\frac{3}{2}i$
$\overrightarrow{CD}$	-2
$\overrightarrow{EF}$	$2i$
$\overrightarrow{GH}$	$2+i$

Operations with vectors

For the vector representation of complex numbers to be useful, the usual rules for addition and subtraction of vectors should be consistent with the definition of addition of complex numbers. Consider the free vectors $\vec{p}$ and $\vec{q}$ shown below:

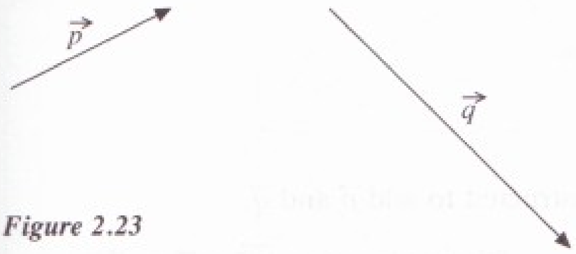


Figure 2.23

There are two equivalent ways of forming the vector sum $\vec{p}+\vec{q}$.

Method 1 (Tip-to-tail)

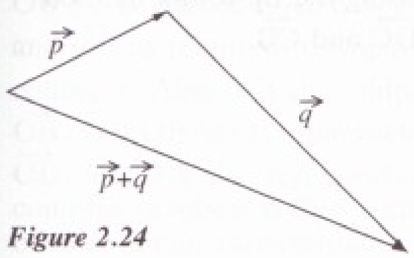


Figure 2.24

The vector $\vec{q}$ is translated (moved parallel to itself) until the tail of $\vec{q}$ is at the tip of $\vec{p}$. A third vector is constructed from the tail of $\vec{p}$ to the tip of $\vec{q}$. This vector is the sum $\vec{p}+\vec{q}$.

Method 2 (Parallelogram)

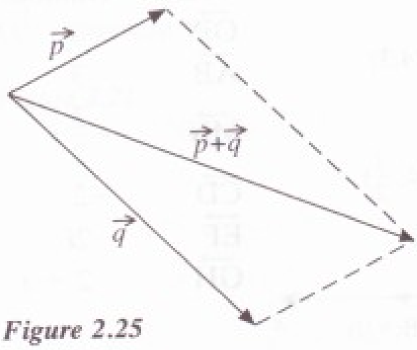
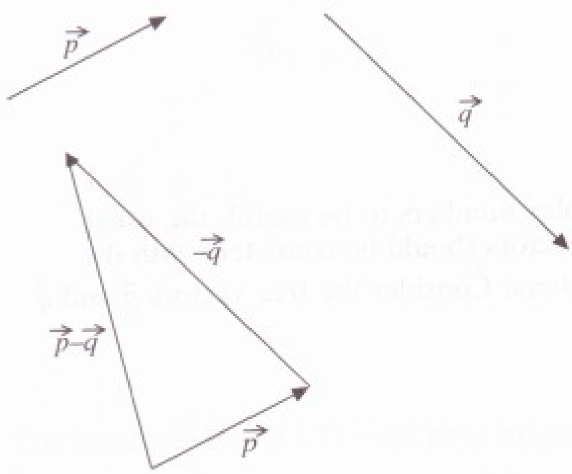


Figure 2.25

The vector $\vec{q}$ is translated until the tail of $\vec{q}$ is at the tail of $\vec{p}$. A parallelogram is constructed with adjacent sides $\vec{p}$ and $\vec{q}$. The vector drawn along the diagonal from the common tail of $\vec{p}$ and $\vec{q}$ to the opposite vertex is the vector $\vec{p} + \vec{q}$.

To perform the vector subtraction $\vec{p} - \vec{q}$ we add the vectors $\vec{p}$ and $-\vec{q}$, where $-\vec{q}$ is a vector with the same magnitude as $\vec{q}$ but in the opposite direction.

Figure 2.26



Consider again the parallelogram constructed to add $\vec{p}$ and $\vec{q}$.

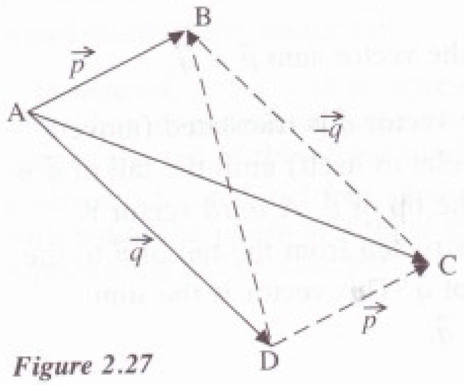
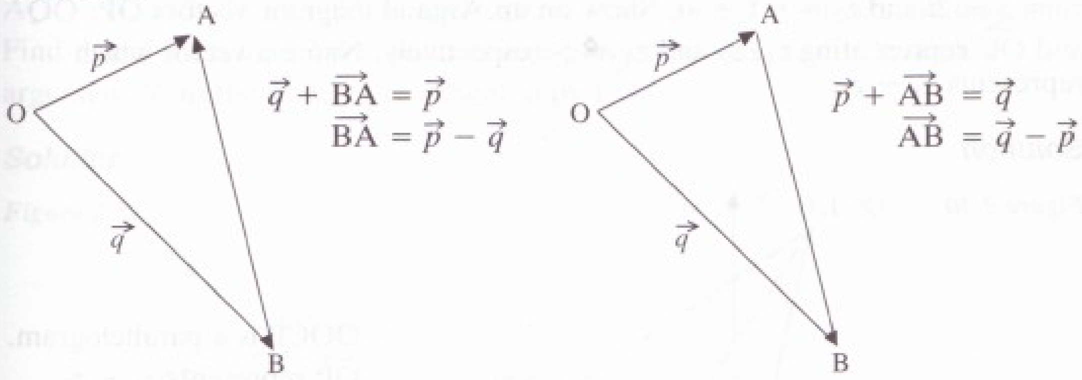


Figure 2.27

Diagonal vector $\overrightarrow{AC}$ is $\vec{p} + \vec{q}$, using the parallelogram method to add $\overrightarrow{AD}$ and $\overrightarrow{AB}$. Diagonal vector $\overrightarrow{DB}$ is $\vec{p} - \vec{q}$, using the tip-to-tail method to add $\overrightarrow{DC}$ and $\overrightarrow{CB}$.

Compare the diagrams below:

Figure 2.28

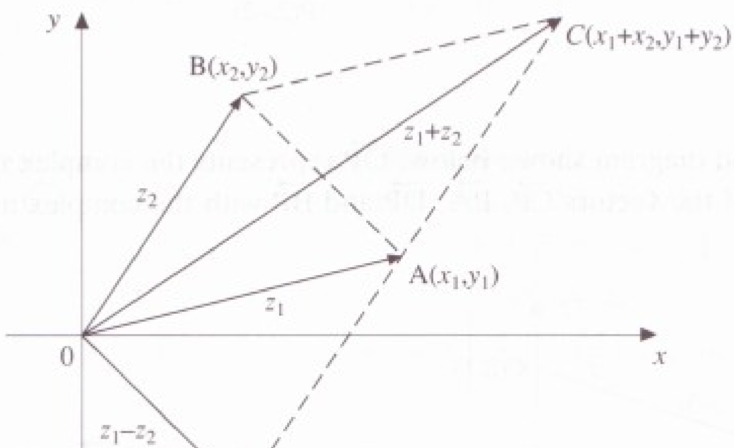


Checking by addition tip-to-tail ensures that $\vec{p} - \vec{q}$ is constructed in the correct direction, from the tip of $\vec{q}$ to the tip of $\vec{p}$.

Operations with complex numbers represented as vectors

Let the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ represent $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_1 + z_2$ and $z_1 - z_2$ respectively.

Figure 2.29



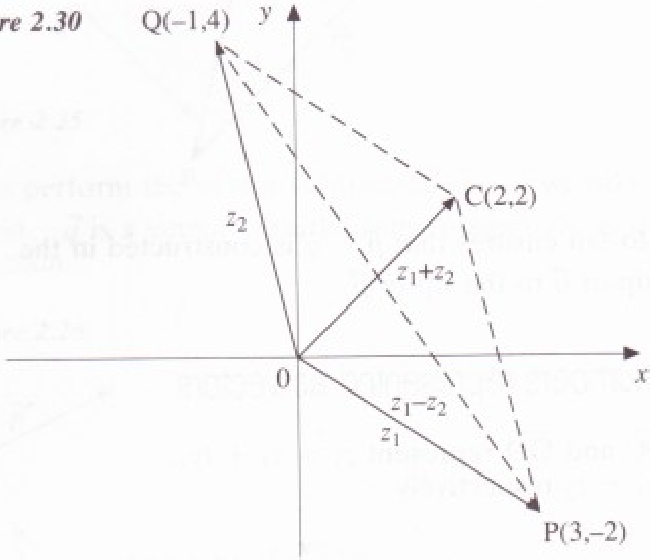
Consider OACB. Comparison of gradients gives OA ∥ BC and OB ∥ AC. Hence OACB is a parallelogram and $\overrightarrow{OC}$ is the vector sum of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as required. Comparison of gradients shows that D, A and C are collinear. Also A is the midpoint of DC. Hence $\overrightarrow{DA}$ is equal and parallel to $\overrightarrow{OB}$, and OBAD is a parallelogram. Therefore $\overrightarrow{BA}$ is equal and parallel to $\overrightarrow{OD}$, and $\overrightarrow{BA}$ also represents $z_1 - z_2$ as required. Addition and subtraction of complex numbers is thus consistent with the rules for addition and subtraction of their vector representations.

Example 23

$z_1 = 3 - 2i$ and $z_2 = -1 + 4i$. Show on an Argand diagram vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OC}$ representing z_1, z_2 and $z_1 + z_2$ respectively. Name a vector which represents $z_1 - z_2$.

Solution

Figure 2.30



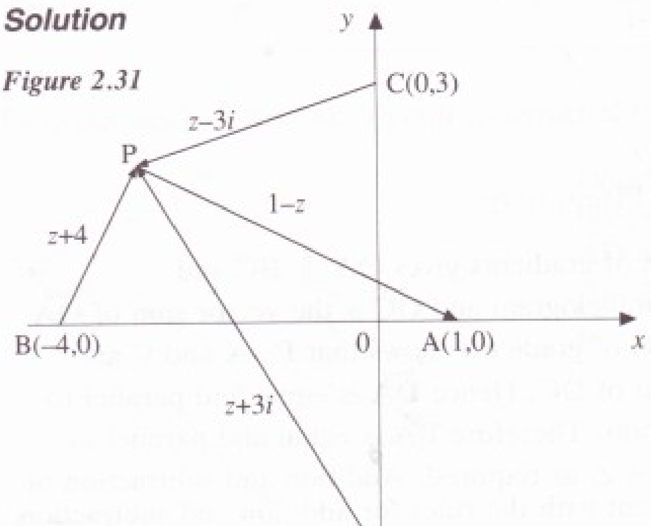
OQCP is a parallelogram. $\overrightarrow{QP}$ represents $z_1 - z_2$.

Example 24

In the Argand diagram shown below, $\overrightarrow{OP}$ represents the complex number z. Label each of the vectors $\overrightarrow{CP}$, $\overrightarrow{PA}$, $\overrightarrow{DP}$ and $\overrightarrow{BP}$ with the complex number it represents.

Solution

Figure 2.31



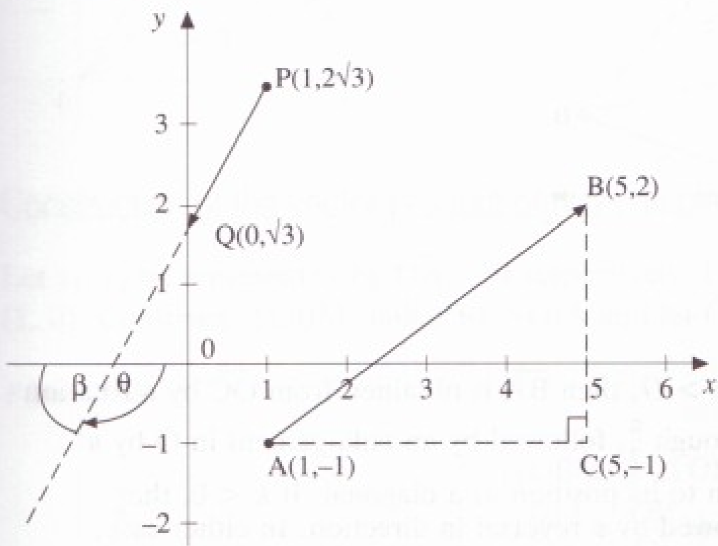
Note that $\overrightarrow{BP}$ represents $z - (-4) = z + 4$ and $\overrightarrow{DP}$ represents $z - (-3i) = z + 3i$.

Example 25

A$(1, -1)$, B$(5, 2)$, P$(1, 2\sqrt{3})$ and Q$(0, \sqrt{3})$ are points in an Argand diagram. Find in the form $a + ib$ the number represented by $\overrightarrow{AB}$, and in modulus/argument form the number represented by $\overrightarrow{PQ}$.

Solution

Figure 2.32



Let $\overrightarrow{AB}$ represent z_1.

$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$

$z_1 = 4 + 3i$

Let PQ represent z_2.

$\tan \beta = |\text{grad PQ}| = \sqrt{3}$,

$\therefore \theta = -\pi + \beta = -\pi + \frac{\pi}{3} = \frac{-2\pi}{3}$.

$PQ = 2$,

$\therefore z_2 = 2\left[\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right)\right]$.

We have seen that the transformation $z \to iz$ corresponds to a rotation anticlockwise about O through an angle $\frac{\pi}{2}$ in the Argand diagram. In general, if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent z and iz respectively, then the angle from $\overrightarrow{AB}$ to $\overrightarrow{CD}$ is $\frac{\pi}{2}$.

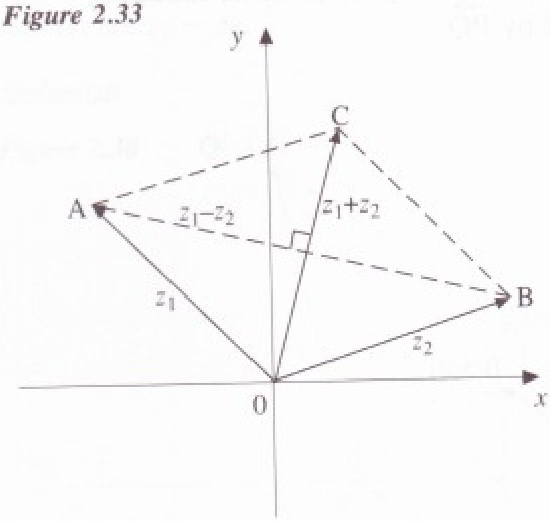
Example 26

$\frac{z_1 - z_2}{z_1 + z_2} = ki$, $k \in \mathbb{R}$. Show that $|z_1| = |z_2|$.

Solution

Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2. Construct the parallelogram OACB. Then $\overrightarrow{OC}$, $\overrightarrow{BA}$ represent $z_1 + z_2$, $z_1 - z_2$ respectively.

Figure 2.33



$z_1 - z_2 = ki(z_1 + z_2)$. If $k > O$, then $\overrightarrow{BA}$ is obtained from $\overrightarrow{OC}$ by a rotation anticlockwise about O through $\frac{\pi}{2}$, followed by an enlargement in O by a factor k, then a translation to its position as a diagonal. If $k < 0$, the enlargement by $|k|$ is followed by a reversal in direction. In either case, diagonals OC and AB of parallelogram OACB meet at right angles and OACB is a rhombus. Hence OA = OB and $|z_1| = |z_2|$.

Example 27

For any complex numbers z_1, z_2 show that
$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$. Interpret this result geometrically.

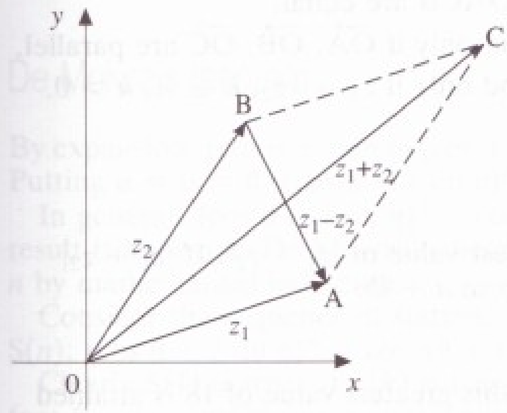
Solution

$$\begin{aligned}|z_1 + z_2|^2 + |z_1 - z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)} \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= 2(z_1\bar{z}_1 + z_2\bar{z}_2) \\ &= 2(|z_1|^2 + |z_2|^2)\end{aligned}$$

Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2 on an Argand diagram and complete the parallelogram OACB. Then $\overrightarrow{OC}$, $\overrightarrow{BA}$ represent $z_1 + z_2$, $z_1 - z_2$ respectively.

$$\begin{aligned}|z_1 + z_2|^2 + |z_1 - z_2|^2 &= 2(|z_1|^2 + |z_2|^2) \\ OC^2 + BA^2 &= 2(OA^2 + OB^2) \\ &= OA^2 + BC^2 + OB^2 + AC^2\end{aligned}$$

Figure 2.34

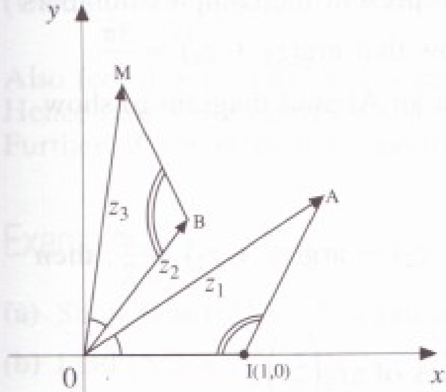


Hence the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

Construction of the vector product of two complex numbers

Let z_1, z_2 be represented by $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively. Let I have coordinates (1, 0). Construct $\triangle$OBM similar to $\triangle$OIA and let $\overrightarrow{OM}$ represent z_3.

Figure 2.35



$\triangle OBM \,|||\, \triangle OIA$,

$\therefore \dfrac{OM}{OA} = \dfrac{OB}{OI} \Rightarrow \dfrac{|z_3|}{|z_1|} = \dfrac{|z_2|}{1}$

$M\widehat{O}B = A\widehat{O}I \Rightarrow \arg z_3 - \arg z_2 = \arg z_1$,

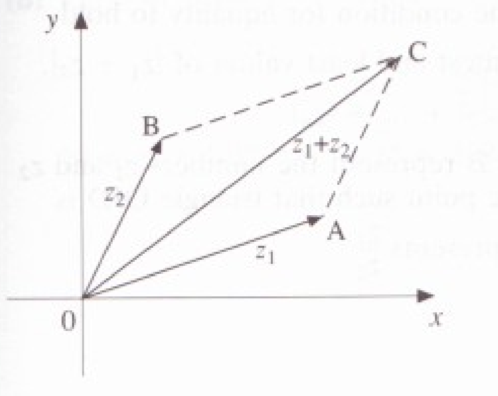
$\therefore |z_3| = |z_1|\,|z_2| = |z_1 z_2|$ and

$\arg z_3 = \arg z_1 + \arg z_2 = \arg(z_1 z_2)$,

$\therefore z_3 = z_1 z_2$ and $\overrightarrow{OM}$ represents the product of z_1 and z_2.

The triangle inequality

Figure 2.36



$\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2. OACB is a parallelogram and $\overrightarrow{OC}$ represents $z_1 + z_2$.

OC $\leqslant$ OA + AC, with equality if and only if O, A and C are collinear.
OC $\leqslant$ OA + OB, since opposite sides of OACB are equal,
$\therefore |z_1 + z_2| \leqslant |z_1| + |z_2|$, with equality if and only if $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are parallel,
$\therefore |z_1 + z_2| \leqslant |z_1| + |z_2|$, with equality if and only if $z_1 = kz_2$, $k \in \mathbb{R}$, $k > 0$.

Example 28

If $z_1 = 3 + 4i$ and $|z_2| = 13$, find the greatest value of $|z_1 + z_2|$. If $|z_1 + z_2|$ takes its greatest value, express z_2 in the form $a + ib$.

Solution

$|z_1 + z_2| \leqslant |z_1| + |z_2| = 5 + 13 = 18$ and this greatest value of 18 is attained when $z_2 = kz_1$ for some positive real k.
But $|z_2| = 13$ and $z_2 = kz_1 \Rightarrow 13 = 5k$.
$z_2 = \dfrac{13}{5}(3 + 4i) = \dfrac{39}{5} + \dfrac{52}{5}i$ when $|z_1 + z_2| = 18$.

Exercise 2.3

1 On an Argand diagram the points A and B represent the complex numbers $z_1 = i$ and $z_2 = \dfrac{1}{\sqrt{2}}(1 + i)$ respectively. Show that $\arg(z_1 + z_2) = \dfrac{3\pi}{8}$.

2 Use the vector representation of z_1 and z_2 on an Argand diagram to show that

(a) If $|z_1| = |z_2|$, then $\dfrac{z_1 + z_2}{z_1 - z_2}$ is imaginary.

(b) If $0 < \arg z_2 < \arg z_1 < \dfrac{\pi}{2}$, and $\arg(z_1 - z_2) - \arg(z_1 + z_2) = \dfrac{\pi}{2}$, then $|z_1| = |z_2|$.

3 If $|z_1 + z_2| = |z_1 - z_2|$, find the possible values of $\arg\left(\dfrac{z_1}{z_2}\right)$.

4 On an Argand diagram the points P and Q represent z and $z + iz$ respectively. Show that OPQ is a right-angled triangle.

5 On an Argand diagram the points P and Q represent the numbers z_1 and z_2 respectively. OPQ is an equilateral triangle. Show that $z_1^2 + z_2^2 = z_1 z_2$.

6 Show that $||z_1| - |z_2|| \leqslant |z_1 + z_2|$. State the condition for equality to hold.

7 If $z_1 = 24 + 7i$ and $|z_2| = 6$, find the greatest and least values of $|z_1 + z_2|$.

8 Show that $|z_1 + z_2 + \ldots + z_n| \leqslant |z_1| + |z_2| + \ldots + |z_n|$.

9 On an Argand diagram the points A and B represent the numbers z_1 and z_2 respectively. I is the point (1, 0). D is the point such that triangle OID is similar to triangle OBA. Show that D represents $\dfrac{z_1}{z_2}$.

2.4 Powers and roots of complex numbers

De Moivre's theorem

By expansion, $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$. (1)
Putting $\alpha = \beta = \theta$, $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$.

In general, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all integers n. This result is known as De Moivre's theorem. It can be proved for positive integers n by mathematical induction.

Consider the sequence of statements
S(n): $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $n = 1,2,3 \ldots$

Clearly S(1) is true. If S(k) is true, then considering S($k+1$)

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\ &= \cos(k+1)\theta + i\sin(k+1)\theta, \text{ using equation (1).}\end{aligned}$$

S(1) is true, and for all positive integers k, if S(k) is true then S($k+1$) is true. Hence by induction, S(n) is true for all positive integers n.

$$\begin{aligned}\text{Now } (\cos\theta + i\sin\theta)^{-n} &= [(\cos\theta + i\sin\theta)^n]^{-1}\\ &= [\cos n\theta + i\sin n\theta]^{-1}, \ n = 1,2,3 \ldots\\ &= \cos(-n\theta) + i\sin(-n\theta).\end{aligned}$$

(Using $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$, where $z = \cos n\theta + i\sin n\theta$.)

Also $(\cos\theta + i\sin\theta)^0 = 1 = \cos 0 + i\sin 0$.
Hence $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all integers n.
Further, if $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Example 29

(a) Show that $(\bar{z})^n = \overline{z^n}$, n integral.

(b) Find $(\sqrt{3} + i)^8 + (\sqrt{3} - i)^8$ in the form $a + ib$.

Solution

(a) Let $z = r(\cos\theta + i\sin\theta)$

$(\bar{z})^n = r^n[\cos(-\theta) + i\sin(-\theta)]^n = r^n[\cos(-n\theta) + i\sin(-n\theta)] = \overline{z^n}$.

(b) Let $z = \sqrt{3} + i$. Then $z = 2\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$.

$$\therefore z^8 = 2^8\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$

Now $z^8 + (\bar{z})^8 = z^8 + \overline{z^8} = 2\,Re(z^8) = 2^9\cos\dfrac{4\pi}{3} = -256$.

Example 30

(a) By expressing $\cos 4\theta$, $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$, show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(b) Hence solve the equation $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$.

Solution

(a) $\cos 4\theta + i\sin 4\theta$

$= (\cos\theta + i\sin\theta)^4$ (using De Moivre's theorem)

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$.

Equating real and imaginary parts:

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$, and

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$,

$$\therefore \tan 4\theta = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

(b) Let $t = \tan\theta$. Then $\tan 4\theta = 1 \Leftrightarrow \dfrac{4t - 4t^3}{1 - 6t^2 + t^4} = 1$,

$$\therefore \tan 4\theta = 1 \Leftrightarrow t^4 + 4t^3 - 6t^2 - 4t + 1 = 0.$$

$\tan 4\theta = 1$ has solutions $4\theta = \dfrac{\pi}{4} + n\pi$, n integral,

$$\theta = \left(\frac{4n+1}{16}\right)\pi,\ n \text{ integral}$$

$n = 0, \pm 1, -2$ give four distinct values of $t = \tan\theta$. Any other integer n will repeat one of these values of t,

$\therefore t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$ has roots $\tan\dfrac{\pi}{16}$, $-\tan\dfrac{3\pi}{16}$, $\tan\dfrac{5\pi}{16}$, $-\tan\dfrac{7\pi}{16}$.

De Moivre's theorem also enables us to express powers of $\cos\theta$ and $\sin\theta$ in terms of the cosine and sine ratios of multiples of θ.

Let $z = \cos\theta + i\sin\theta$. Then $z^n = \cos n\theta + i\sin n\theta$ and $z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$, for all positive integers n. Hence

- $z^n + z^{-n} = 2\cos n\theta$, with special case $z + z^{-1} = 2\cos\theta$
- $z^n - z^{-n} = 2i\sin n\theta$, with special case $z - z^{-1} = 2i\sin\theta$.

Example 31

Show that $\sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$.

Solution

Let $z = \cos\theta + i\sin\theta$. Then $2i\sin\theta = z - z^{-1}$ and $8i^3\sin^3\theta = (z - z^{-1})^3$.

But $(z - z^{-1})^3 = z^3 - 3z + 3z^{-1} - z^{-3}$

$= (z^3 - z^{-3}) - 3(z - z^{-1})$.

Hence $-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

and $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.

Example 32

Show that $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$. Hence use the substitution $x = 2\sin\theta$ to evaluate $\int_0^2 (4 - x^2)^{\frac{3}{2}}\,dx$.

Solution

Let $z = \cos\theta + i\sin\theta$. Then $2\cos\theta = z + z^{-1}$ and $16\cos^4\theta = (z + z^{-1})^4$.

But $(z + z^{-1})^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$

$= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$.

Hence $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$.

Let $I = \int_0^2 (4 - x^2)^{\frac{3}{2}}\,dx$

Substitute $x = 2\sin\theta$, $\frac{-\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$

$dx = 2\cos\theta\,d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 2 \Rightarrow \theta = \frac{\pi}{2}$

$(4 - x^2)^{\frac{3}{2}} = 8\cos^3\theta$

$$I = \int_0^{\frac{\pi}{2}} 8\cos^3\theta . 2\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 16\cos^4\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} (2\cos 4\theta + 8\cos 2\theta + 6)\,d\theta$$

$$= \left[\frac{1}{2}\sin 4\theta + 4\sin 2\theta + 6\theta\right]_0^{\frac{\pi}{2}}$$

$$= 3\pi$$

Using De Moivre's theorem to find roots of complex numbers

Consider the complex nth roots of unity, where n is a positive integer.

$z^n = 1 \Rightarrow |z| = 1$. Hence the nth roots of unity have modulus 1 and their representations P on an Argand diagram lie on the unit circle with centre the origin. Let $z = \cos\theta + i\sin\theta$ satisfy $z^n = 1$. Using De Moivre's theorem,

$\cos n\theta + i\sin n\theta = 1 + 0i$

$\therefore \cos n\theta = 1$ and $\sin n\theta = 0$

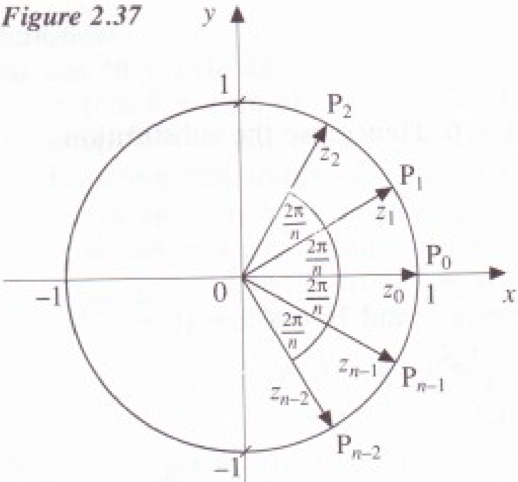
$\therefore n\theta = 2\,k\pi$

$\therefore \theta = \frac{2\pi}{n}k, \quad k$ integral.

Taking $\theta = \frac{2\pi}{n} k$, $\quad k = 0, 1, 2, \ldots, n-1$ gives n distinct numbers z with argument $\frac{2\pi}{n} k$.

Any other integer values of k repeat the values of $\cos\theta$ and $\sin\theta$ and hence the values of z. Let the n complex nth roots of unity be $z_0, z_1, \ldots, z_{n-1}$ with representations $P_0, P_1, \ldots, P_{n-1}$ in the Argand diagram.

Figure 2.37



$P_0, \ldots, P_{n-1}$ are equally spaced around the unit circle, with angular spacing $\frac{2\pi}{n}$.

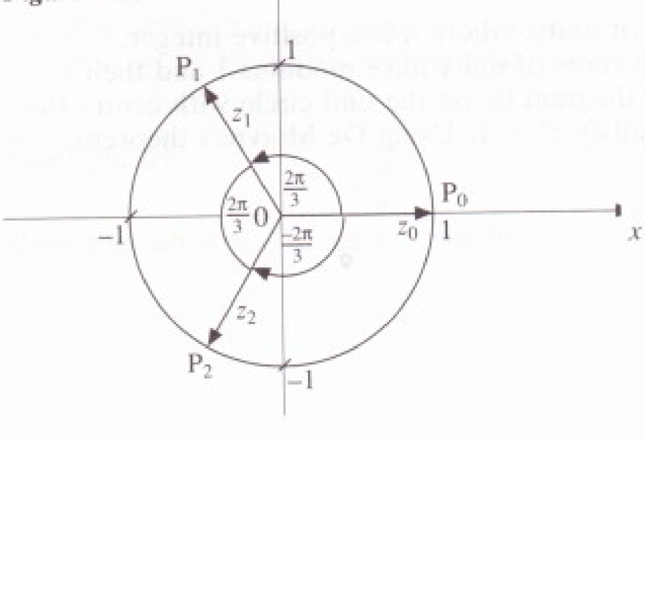
Considering the arguments of z_k and z_{n-k}, $k = 1, 2, \ldots, n-1$, we can deduce that the non-real roots of unity occur in complex conjugate pairs. Further, by noting that $\arg z_k = k \arg z_1$ we can deduce that $z_k = {z_1}^k$ and hence that all the nth roots of unity are powers of the particular nth root with argument $\frac{2\pi}{n}$.

Example 33

Show on an Argand diagram P_0, P_1, P_2 representing the complex cube roots of unity z_0, z_1, z_2. Find the non-real cube roots in the form $a + ib$.

Solution

Figure 2.38



$$z_1 = \text{cis}\,\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$$

$$z_2 = \text{cis}\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$$

Noting that $\arg(z_2{}^2) = -\frac{4\pi}{3}$, we can deduce that P_1 represents $z_2{}^2$ and hence $z_1 = z_2{}^2$. Similarly $z_2 = z_1{}^2$. Hence the cube roots of unity can be written as 1, ω, ω^2, where ω is either of the two non-real roots.

Example 34

If ω is a non-real cube root of unity, show that $1 + \omega + \omega^2 = 0$, and deduce that $(1 + \omega)^3(1 + 2\omega + 2\omega^2) = 1$.

Solution

The cube roots of unity satisfy $x^3 - 1 = 0$. But $x^3 - 1 = (x - 1)(x^2 + x + 1)$. Hence $\omega \neq 1 \Rightarrow \omega^2 + \omega + 1 = 0$.

Then $(1 + \omega)^3 = (-\omega^2)^3 = -(\omega^3)^2 = -1$, (since $\omega^3 = 1$)
and $1 + 2\omega + 2\omega^2 = 2(1 + \omega + \omega^2) - 1 = -1$, (since $1 + \omega + \omega^2 = 0$)
$\therefore (1 + \omega)^3(1 + 2\omega + 2\omega^2) = 1$.

A similar technique enables us to find the complex nth roots of any complex number. Let $z = \cos\theta + i\sin\theta$ satisfy $z^n = \cos\alpha + i\sin\alpha$.

Then $\cos n\theta + i\sin n\theta = \cos\alpha + i\sin\alpha$ (by De Moivre's theorem),

$\therefore n\theta = \alpha + 2\pi k, \quad \theta = \frac{\alpha}{n} + \frac{2\pi}{n}k$, k integral.

The complex nth roots of $\cos\alpha + i\sin\alpha$ are equally spaced around the unit circle with angular spacing $\frac{2\pi}{n}$, one such root having argument $\frac{\alpha}{n}$.

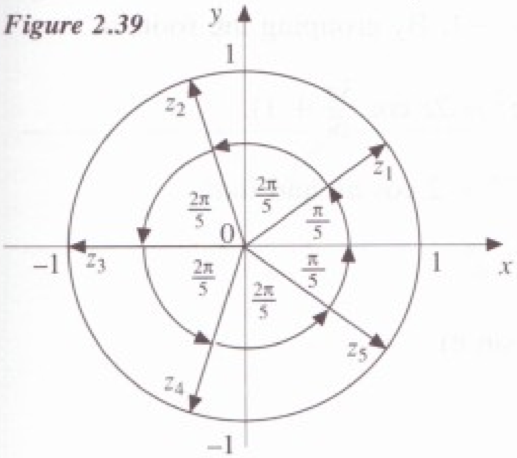
Example 35

Find the complex 5th roots of -1.

Solution

$|-1| = 1$ and $\arg(-1) = \pi$. Hence the complex 5th roots of -1 all have modulus 1 and by De Moivre's theorem one complex 5th root of -1 has argument $\frac{\pi}{5}$, the others being equally spaced around the unit circle in the Argand diagram by an angle $\frac{2\pi}{5}$.

Figure 2.39



By inspection, the complex 5th roots of -1 are $\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$, and -1.

De Moivre's theorem provides another method for finding the square roots of a complex number.

Example 36

Find the square roots of $z = 2\sqrt{2}(-1 + i)$.

Solution

$$z = 4\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, i\right) = 4\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$$

$\therefore\ |z| = 4$ and $\arg z = \dfrac{3\pi}{4}$.

By De Moivre's theorem, one square root of z has modulus 2 and argument $\dfrac{3\pi}{8}$. Hence the two square roots of z are $\pm\, 2\left(\cos\dfrac{3\pi}{8} + i\sin\dfrac{3\pi}{8}\right)$.

Exercise 2.4

1 Express $1 + i$ and $1 - i$ in modulus/argument form. Use De Moivre's theorem to evaluate $(1 + i)^{20} + (1 - i)^{20}$.

2 Express $-1 + \sqrt{3}i$ in modulus/argument form. Use De Moivre's theorem to show that $(-1 + \sqrt{3}i)^n + (-1 - \sqrt{3}i)^n = 2^{n+1}\cos\left(\dfrac{2n\pi}{3}\right)$, n a positive integer. Evaluate this expression in each of the cases $n = 3m$, $n = 3m \pm 1$ where m is a positive integer.

3 Use De Moivre's theorem to solve the equation $z^5 = 1$. Show that the points representing the five roots of this equation on an Argand diagram form the vertices of a regular pentagon of area $\dfrac{5}{2}\sin\dfrac{2\pi}{5}$ and perimeter $10\sin\dfrac{\pi}{5}$.

4 Use De Moivre's theorem to solve $z^5 = -1$. By grouping the roots in complex conjugate pairs, show that

$$z^5 + 1 = (z + 1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right).$$

5 If $z = \cos\theta + i\sin\theta$, show that $z^n - z^{-n} = 2\cos n\theta$ and that $z^n - z^{-n} = 2i\sin n\theta$. Hence show that

(a) $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

(b) $\sin^5\theta = \dfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

6 Factorise $z^3 - 1$. If z is one of the three cube roots of unity, find the two possible values of $z^2 + z + 1$.

7 1, ω, and ω^2 are the three cube roots of unity. State the values of ω^3 and $1 + \omega + \omega^2$. Hence simplify each of the expressions $(1 + 3\omega + \omega^2)^2$ and $(1 + \omega + 3\omega^2)^2$ and show that their sum is -4 and their product is 16.

8 Use De Moivre's theorem to find in modulus/argument form

(a) the square roots of $\sqrt{3} + i$ **(b)** the cube roots of $-2 - 2i$

2.5 Curves and regions in the Argand diagram

Let P be the representation on an Argand diagram of a complex number z which satisfies the equation $Re\ z = 3$. Let $z = x + iy$. Then $Re\ z = 3 \Rightarrow x = 3 \Rightarrow$ P lies on the line $x = 3$.

Conversely, any point on the line $x = 3$ is the representation of a complex number z which satisfies $Re\ z = 3$. Hence, on an Argand diagram, the locus of z satisfying $Re\ z = 3$ is the straight line with Cartesian equation $x = 3$. We say that the equation $Re\ z = 3$ defines the line $x = 3$ in the Argand diagram.

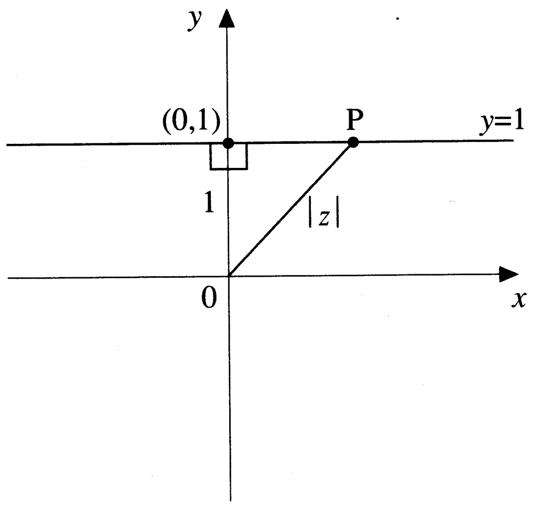
Example 37

Sketch the curve in the Argand diagram defined by the equation $Im(z - 1 + 3i) = 4$. Find the minimum value of $|z|$ subject to this condition and state the value of z for which this minimum is attained.

Solution

Let $z = x + iy$. Then $z - 1 + 3i = (x - 1) + i(y + 3)$,
$\therefore Im(z - 1 + 3i) = 4 \Rightarrow y + 3 = 4 \Rightarrow y = 1$.
The curve has Cartesian equation $y = 1$.

Figure 2.40



If P represents z, OP $= |z|$.
OP takes a minimum value of 1 when P has coordinates $(0, 1)$.
$|z|$ has a minimum value of 1 when $z = i$.

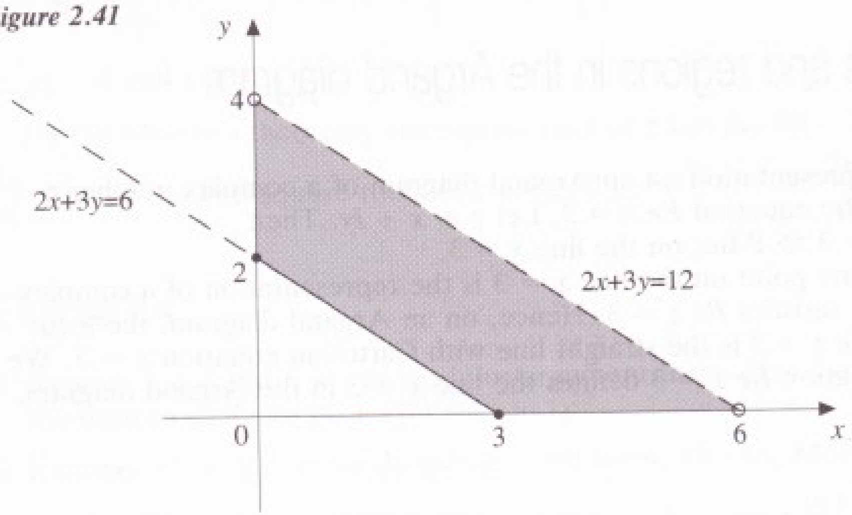
Example 38

Sketch the region in the Argand diagram defined by
$6 \leqslant Re[(2-3i)z] < 12$ and $Re\, z.Im\, z \geqslant 0$.

Solution

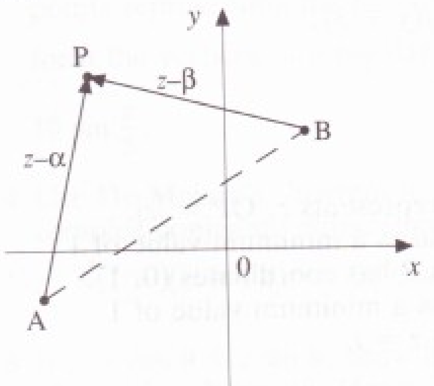
Let $z = x + iy$. Then $(2-3i)(x+iy) = (2x+3y) + i(2y-3x)$.
$6 \leqslant Re[(2-3i)z] < 12 \Rightarrow 6 \leqslant 2x + 3y < 12$,
and $Re\, z.Im\, z \geqslant 0 \Rightarrow xy \geqslant 0$.

Figure 2.41



Consider the equation $|z-\alpha| = |z-\beta|$, where α and β are fixed complex numbers represented by A and B respectively in an Argand diagram.

Figure 2.42



If P represents z, then $\overrightarrow{AP}$, $\overrightarrow{BP}$ represent $z-\alpha$, $z-\beta$ respectively.

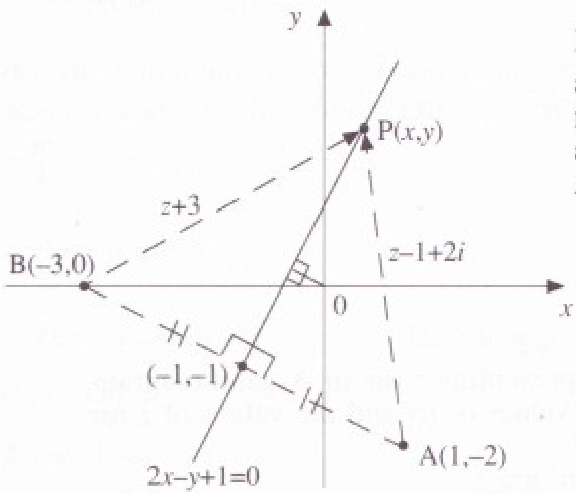
If z satisfies $|z-\alpha| = |z-\beta|$, then $\triangle APB$ is isosceles with AP = BP, and P lies on the perpendicular bisector of AB. Conversely, if P lies on the perpendicular bisector of AB, then AP = BP and $|z-\alpha| = |z-\beta|$. Hence the locus of the point P representing z is the perpendicular bisector of AB.

Example 39

z satisfies $|z - 1 + 2i| = |z + 3|$. Sketch the locus of the point P representing z in the Argand diagram and find its Cartesian equation. Find the minimum value of $|z|$.

Solution

Figure 2.43



Let P, A and B represent z, $1 - 2i$ and -3 respectively. Then $\overrightarrow{AP}$, $\overrightarrow{BP}$ represent $z - (1 - 2i)$, $z - (-3)$, and $|z - 1 + 2i| = |z + 3| \Rightarrow$ AP = BP

The locus of P is the perpendicular bisector of AB. Since AB has midpoint $(-1, -1)$ and gradient $-\frac{1}{2}$, the locus of P passes through $(-1, -1)$ with gradient 2 and has Cartesian equation $2x - y + 1 = 0$. Now OP $= |z|$. Hence the minimum value of $|z|$ is the perpendicular distance from $(0, 0)$ to the locus of P. Therefore the minimum value of $|z|$ is $\frac{1}{\sqrt{5}}$.

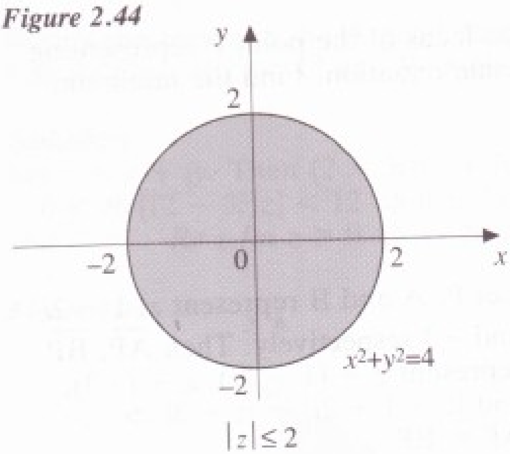
The examples we have considered illustrate the two techniques used to find a curve in the Argand diagram determined by an equation in z. We can use the representation $z = x + iy$ to determine algebraically the Cartesian equation (or inequation) of the locus of P representing z, sketch the graph of the corresponding curve (or region), and hence describe the locus geometrically. Alternatively, we can use the vector representation of a complex number to determine the locus of P geometrically, then sketch the locus and deduce its Cartesian equation. The vector approach enables us to use known geometrical results and is often more efficient.

Example 40

Sketch the region in the Argand diagram determined by $|z| \leqslant 2$.

Solution

Figure 2.44



If $\overrightarrow{OP}$ represents z, then $OP = |z|$.
$|z| \leqslant 2 \Rightarrow$ P lies on or inside the circle, centre $(0, 0)$ and radius 2.

Example 41

z satisfies the equation $|z - 2 - 2i| = \sqrt{2}$.

(a) Sketch the locus of the point P representing z on an Argand diagram.

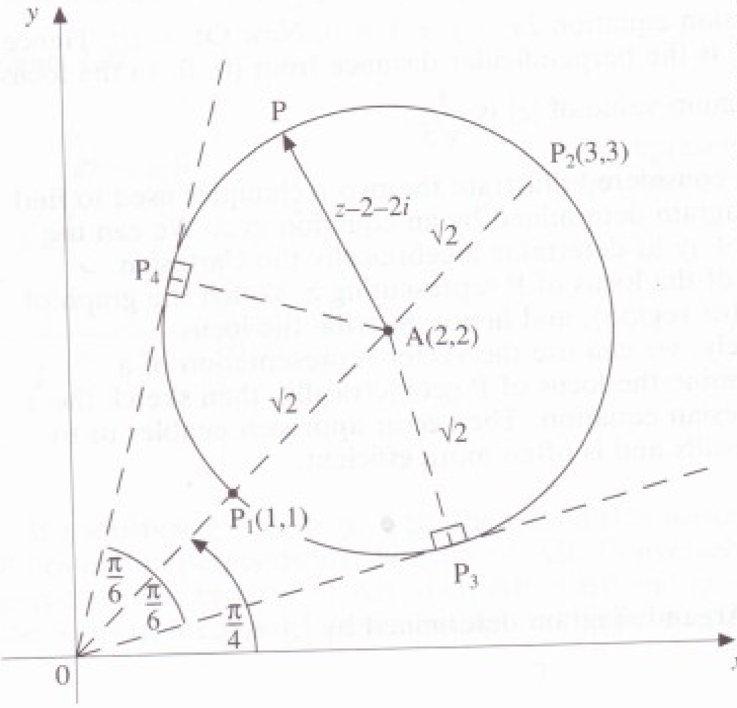
(b) Find the maximum and minimum values of $|z|$ and the values of z for which these extremes are attained.

(c) Find the range of possible values of arg z.

Solution

(a)

Figure 2.45



Let A represent $2 + 2i$.
Then $\overrightarrow{AP}$ represents $z - (2 + 2i)$ and $|z - 2 - 2i| = \sqrt{2}$
$\Rightarrow AP = \sqrt{2}$,
$\therefore$ P lies on the circle centre A(2,2) and radius $\sqrt{2}$.

(b) Let the circle diameter which lies on the line OA be P_1P_2 as shown. Then P_1, P_2 have coordinates (1, 1), (3, 3) respectively. Now the maximum value of $|z|$ is $OP_2 = 3\sqrt{2}$ when $z = 3 + 3i$ and the minimum value of $|z|$ is $OP_1 = \sqrt{2}$ when $z = 1 + i$.

(c) Let tangents from (0, 0) meet the circle in P_3, P_4 as shown. Now

$$\sin A\widehat{O}P_3 = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

$$\therefore A\widehat{O}P_3 = A\widehat{O}P_4 = \frac{\pi}{6}$$

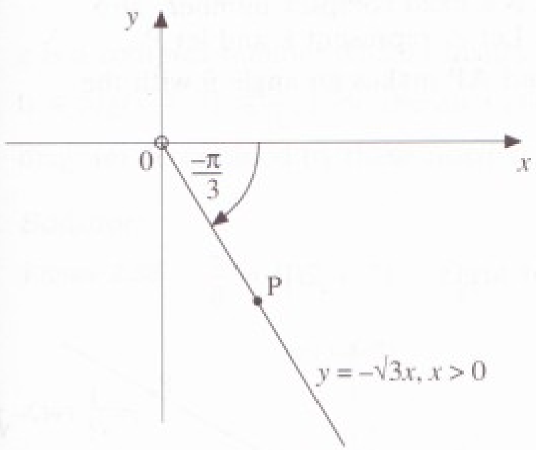
But the minimum and maximum values of arg z are the angles from the positive x-axis to the vectors $\overrightarrow{OP_3}$ and $\overrightarrow{OP_4}$ respectively,

$$\therefore \frac{\pi}{4} - \frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{4} + \frac{\pi}{6},$$

$$\therefore \frac{\pi}{12} \leqslant \arg z \leqslant \frac{5\pi}{12}.$$

The equation $\arg z = -\frac{\pi}{3}$ determines a ray in the Argand diagram. Let $\overrightarrow{OP}$ represent z.

Figure 2.46



The gradient of $\overrightarrow{OP}$ is $\tan\left(-\frac{\pi}{3}\right)$
$= -\sqrt{3}$.
The locus of P is the ray
$y = -\sqrt{3}x$, $x > 0$.

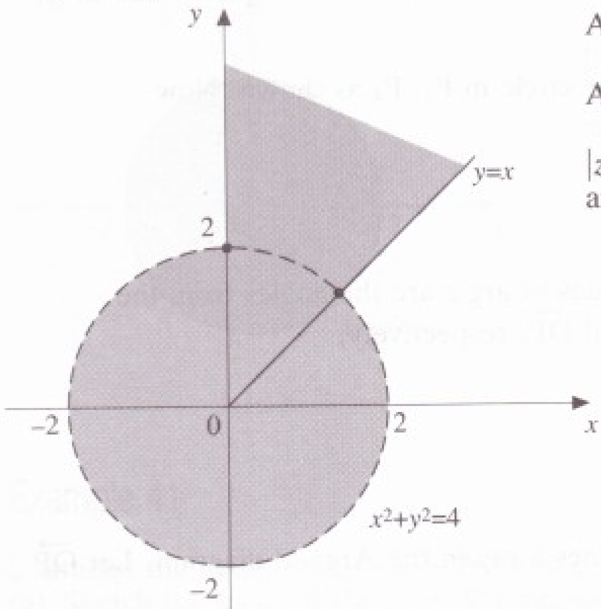
Note that $\arg z = -\frac{\pi}{3} \Rightarrow z \neq 0$, hence we must exclude (0, 0) from the locus of P.

Example 42

Shade the region in the Argand diagram defined by the inequalities
$\frac{\pi}{4} \leqslant \arg z \leqslant \frac{\pi}{2}$ or $|z| < 2$.

Solution

Figure 2.47



Arg $z = \frac{\pi}{2}$ is the positive y-axis.

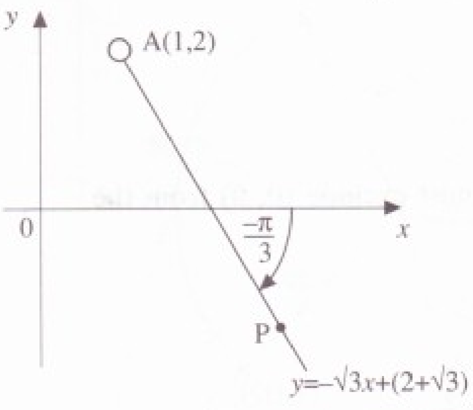
Arg $z = \frac{\pi}{4}$ is the ray $y = x$, $x > 0$.

$|z| = 2$ is the circle, centre (0, 0) and radius 2.

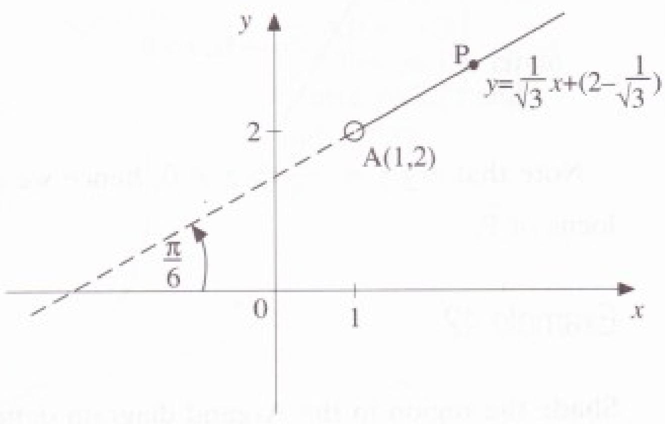
The equation $\arg(z - \lambda) = \theta$, where λ is a fixed complex number, also determines a ray in the Argand diagram. Let A represent λ and let P represent z. Then $\overrightarrow{AP}$ represents $z - \lambda$ and $\overrightarrow{AP}$ makes an angle θ with the direction of the positive x-axis.
For example

Figure 2.48

(a) $\arg(z - [1 + 2i]) = -\frac{\pi}{3}$



(b) $\arg(z - [1 + 2i]) = \frac{\pi}{6}$



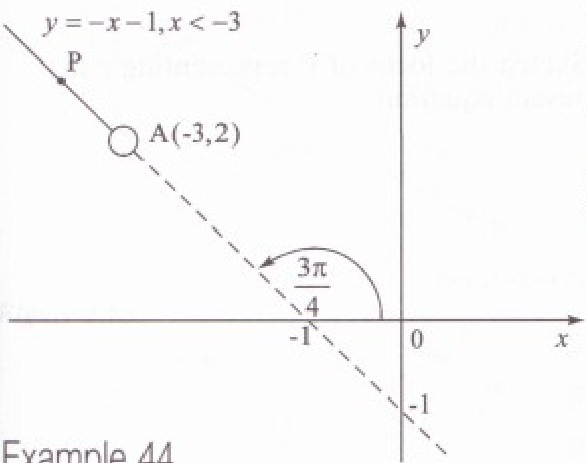
Note that A is excluded from the locus in each case, since $z \neq 1 + 2i$ as arg 0 is undefined.

Example 43

$\operatorname{Arg}(z + 3 - 2i) = \frac{3\pi}{4}$. Sketch the locus of the point P representing z in the Argand diagram and write down its Cartesian equation.

Solution

Figure 2.49



Let A$(-3, 2)$ represent $-3 + 2i$.
Then $\overrightarrow{AP}$ represents $z - (-3 + 2i)$.
AP has gradient $\tan\left(\frac{3\pi}{4}\right) = -1$.

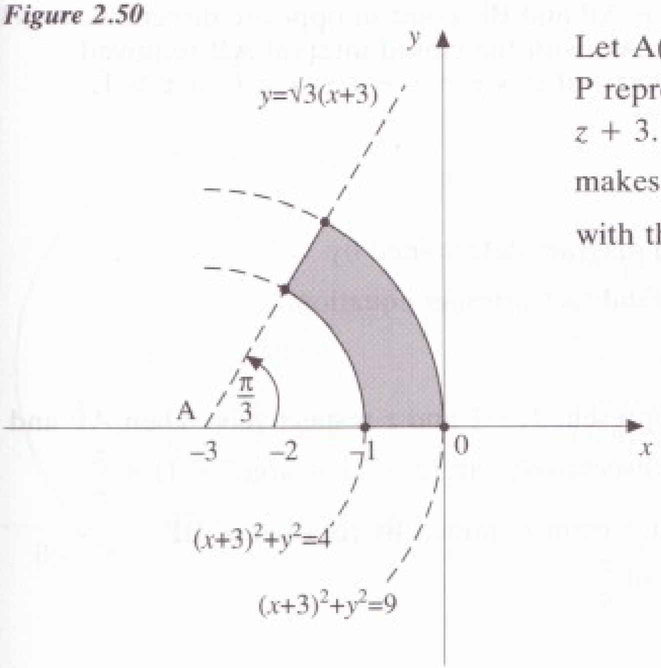
Hence the locus of P has Cartesian equation $y = -x - 1$, $x < -3$.

Example 44

z is a complex number which satisfies $2 \leqslant |z + 3| \leqslant 3$ and $0 \leqslant \arg(z + 3) \leqslant \frac{\pi}{3}$. Find the area and perimeter of the region in the Argand diagram determined by these restrictions on z.

Solution

Figure 2.50



Let A$(-3, 0)$ represent -3. Then, if P represents z, $\overrightarrow{AP}$ represents $z + 3$. Hence $2 \leqslant AP \leqslant 3$ and $\overrightarrow{AP}$ makes an angle between 0 and $\frac{\pi}{3}$ with the positive x-axis.

Area is $\frac{1}{6} \cdot \pi(3^2 - 2^2) = \frac{5\pi}{6}$ sq. units and perimeter is $\frac{1}{6}.2\pi(3 + 2) + 2$
$= 2 + \frac{5\pi}{3}$ units.

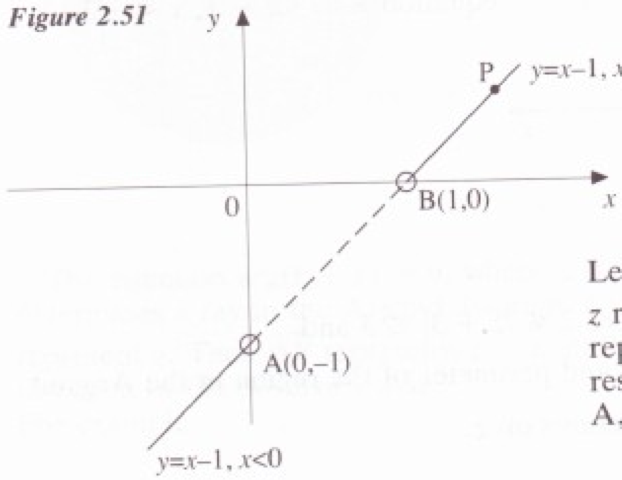
If z satisfies $\arg(z - \alpha) = \arg(z - \beta)$, where α and β are fixed complex numbers, then the vectors representing $z - \alpha$ and $z - \beta$ are parallel.

Example 45

z satisfies $\arg(z + i) = \arg(z - 1)$. Sketch the locus of P representing z in the Argand diagram and find its Cartesian equation.

Solution

Figure 2.51



Let A, B and P represent $-i$, 1 and z respectively. Then $\overrightarrow{AP}$, $\overrightarrow{BP}$ representing $z + i$, $z - 1$ respectively are parallel. Hence A, B and P are collinear.

If P lies in the interval AB, then $\overrightarrow{AP}$ and $\overrightarrow{BP}$ point in opposite directions. Hence the locus of P is the line AB with the closed interval AB removed. The Cartesian equation of the locus of P is $y = x - 1, \quad x < 0$ or $x > 1$.

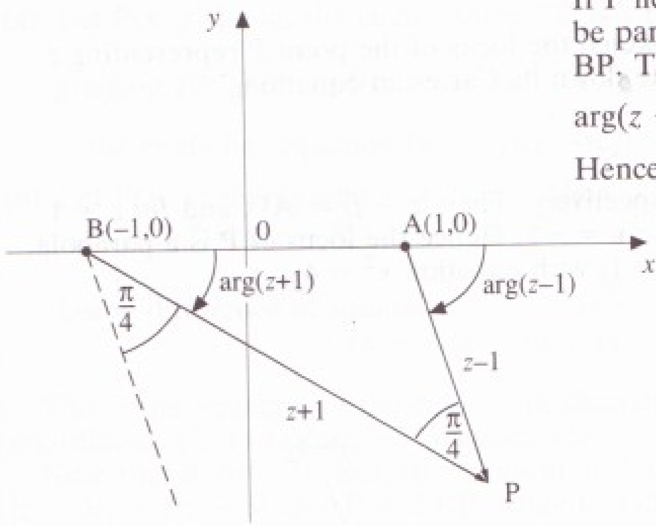
Example 46

Sketch the curve in the Argand diagram determined by
$\arg(z - 1) = \arg(z + 1) + \frac{\pi}{4}$. Find its Cartesian equation.

Solution

Let A(1, 0), B(−1, 0) and P represent 1, −1 and z respectively. Then $\overrightarrow{AP}$ and $\overrightarrow{BP}$ represent $z - 1$ and $z + 1$ respectively. $\arg(z - 1) = \arg(z + 1) + \frac{\pi}{4}$ requires $\overrightarrow{AP}$ to be parallel to the vector obtained by rotation of $\overrightarrow{BP}$ anticlockwise through an angle of $\frac{\pi}{4}$.

Figure 2.52

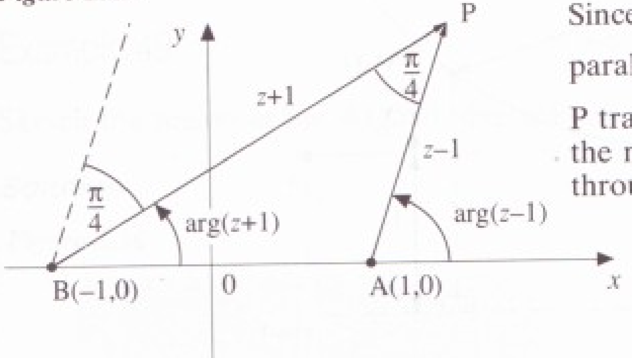


If P lies below the x-axis, AP must be parallel to a clockwise rotation of BP. This diagram shows

$\arg(z - 1) = \arg(z + 1) - \frac{\pi}{4}$.

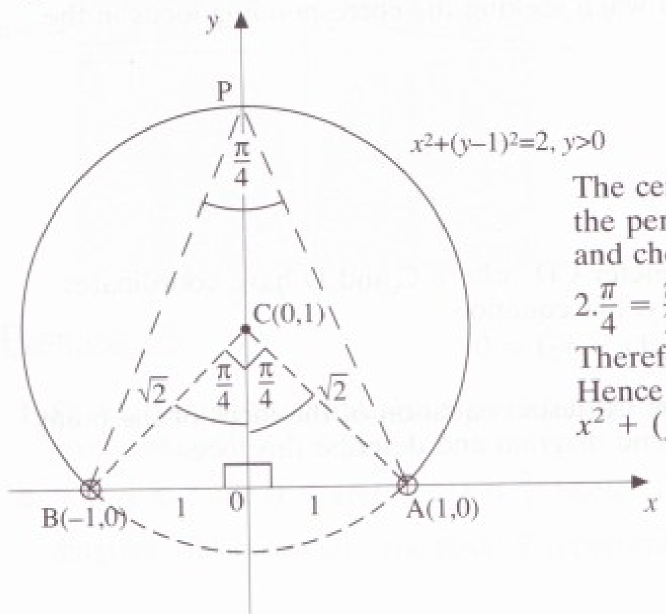
Hence P must lie above the x-axis.

Figure 2.53



Since alternate angles between parallel lines are equal, $B\widehat{P}A = \frac{\pi}{4}$ as P traces its locus. Hence P lies on the major arc AB of a circle through A and B.

Figure 2.54



The centre C of this circle lies on the perpendicular bisector of AB, and chord AB subtends an angle $2.\frac{\pi}{4} = \frac{\pi}{2}$ at C.

Therefore $OC = 1$ and $AC = \sqrt{2}$. Hence the locus of P has equation $x^2 + (y - 1)^2 = 2,\ y > 0$.

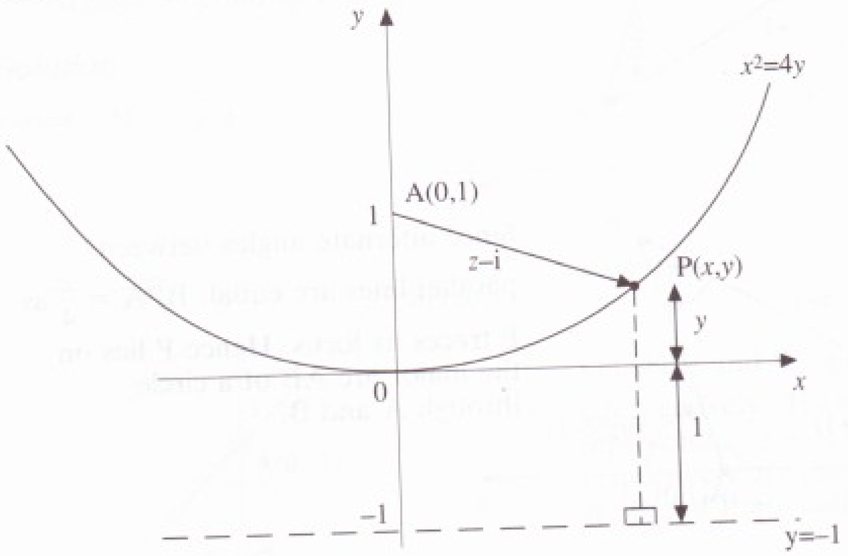
Example 47

z satisfies $|z - i| = Im\, z + 1$. Sketch the locus of the point P representing z in the Argand diagram and write down its Cartesian equation.

Solution

Let A(0, 1), P represent i, z respectively. Then $|z - i| = AP$, and $Im\, z + 1$ is the distance from P to the line $y = -1$. Hence the locus of P is a parabola, focus A(0, 1) and directrix $y = -1$, with equation $x^2 = 4y$.

Figure 2.55



There are some equations in z for which algebraic methods are easier than using the vector representation when seeking the corresponding locus in the Argand diagram.

Example 48

(a) Show that a circle with diameter CD, where C and D have coordinates (x_1, y_1), (x_2, y_2) respectively, has equation $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

(b) $|z - 3i| = 2|z - 3|$. Find the Cartesian equation of the locus of the point P representing z in the Argand diagram and describe this locus geometrically.

Solution

(a) Let P(x, y) lie on the circle. Since CD is a diameter, $C\widehat{P}D = \frac{\pi}{2}$. Hence gradient PC.gradient PD $= -1 \Rightarrow \frac{(y - y_1)}{(x - x_1)} \cdot \frac{(y - y_2)}{(x - x_2)} = -1$,
$\therefore$ the circle has equation $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

(b) Let P(x, y) represent $z = x + iy$. Then

$$|z - 3i|^2 = 4|z - 3|^2 \quad \Rightarrow \quad x^2 + (y - 3)^2 = 4(x - 3)^2 + 4y^2$$
$$(y - 3)^2 - 4y^2 = 4(x - 3)^2 - x^2.$$

Using difference of squares: $(3y - 3)(-y - 3) = (3x - 6)(x - 6)$,
$$\therefore (x - 2)(x - 6) + (y - 1)(y + 3) = 0.$$

This is the equation of the circle with diameter CD, where C and D have coordinates $(2, 1)$ and $(6, -3)$ respectively.

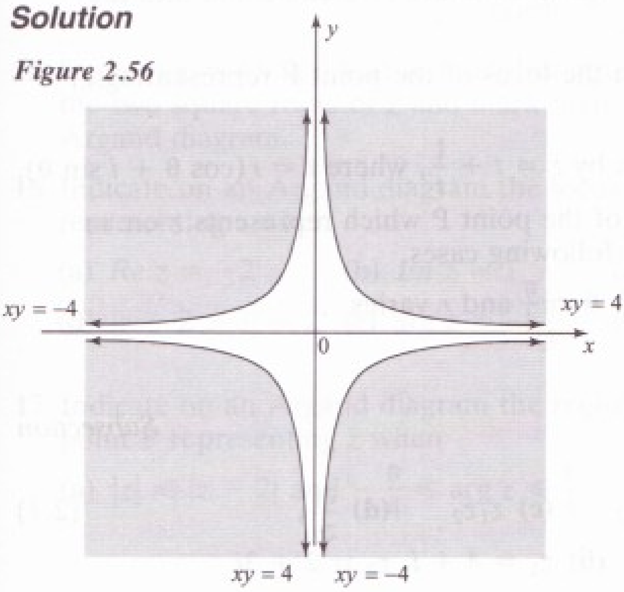
Note that if A$(0, 3)$, B$(3, 0)$ represent $3i$, 3 respectively, then $|z - 3i| = 2|z - 3| \Rightarrow$ AP = 2 BP, while C and D are the points dividing the interval AB internally and externally respectively in the ratio 2:1.

Example 49

Sketch the region in the Argand diagram defined by $|z^2 - (\bar{z})^2| \geqslant 16$.

Solution

Figure 2.56



Let $z = x + iy$.
$$z^2 - (\bar{z})^2 = (z - \bar{z})(z + \bar{z}) = (2iy).(2x),$$
$$\therefore |z^2 - (\bar{z})^2| = 4|xy|,$$
$$\therefore |xy| \geqslant 4.$$

Exercise 2.5

1 Sketch on an Argand diagram the locus of the point P representing z, given that $|z|^2 = z + \bar{z} + 1$.

2 $|z + i| \leqslant 2$ and $0 \leqslant \arg(z + 1) \leqslant \frac{\pi}{4}$. Sketch the region in an Argand diagram which contains the point P representing z.

3 $|z - 1| \leqslant |z - i|$ and $|z - 2 - 2i| \leqslant 1$. Sketch the region in the Argand diagram which contains the point P representing z. If P describes the boundary of this region, find the value of z when $\arg(z - 1) = \frac{\pi}{4}$.

4 $\operatorname{Arg}(z + 3) = \frac{\pi}{3}$. Sketch the locus of the point P representing z on an Argand diagram. Find the modulus and argument of z when $|z|$ takes its least value. Hence find, in the form $a + ib$, the value of z for which $|z|$ is a minimum.

5 $|z - 1| = 1$. Sketch the locus of the point P representing z on an Argand diagram. Hence deduce that $\arg(z - 1) = \arg(z^2)$.

6 $z = x + iy$ is such that $\frac{z - i}{z + 1}$ is purely imaginary. Find the equation of the locus of the point P representing z and show this locus on an Argand diagram.

7 $Re\left(z - \frac{1}{z}\right) = 0$. Find the equation of the locus of the point P representing z on an Argand diagram and sketch this locus.

8 If $\arg(z - 2) = \arg(z + 2) + \frac{\pi}{3}$, show that the locus of the point P representing z on an Argand diagram is an arc of a circle and find the centre and radius of this circle.

9 Indicate on an Argand diagram the locus of the point P representing z, given that $|z^2 - \overline{z}^2| = 4$.

10 The complex number z is given by $z = t + \frac{1}{t}$, where $t = r(\cos\theta + i\sin\theta)$. Find the equation of the locus of the point P which represents z on an Argand diagram in each of the following cases

(a) $r = 2$ and θ varies **(b)** $\theta = \frac{\pi}{4}$ and r varies

Diagnostic test 2

Subsection

1 Find **(a)** $z_1 + z_2$ **(b)** $z_1 - z_2$ **(c)** $z_1 z_2$ **(d)** $\frac{z_1}{z_2}$, (2.1)
when **(i)** $z_1 = 2 + i$, $z_2 = i$, **(ii)** $z_1 = 4 + i$, $z_2 = 2 + 3i$

2 Find **(a)** $Re\, z$ **(b)** $Im\, z$ **(c)** $\overline{z}$, (2.1)
when **(i)** $z = 3$ **(ii)** $z = 4i$ **(iii)** $z = 3 + 4i$

3 Find real x and y, such that $(x + iy)^2 = 3 + 4i$ (2.1)

4 Solve **(a)** $x^2 + 2x + 2 = 0$ **(b)** $x^2 + (2 - i)x - 2i = 0$ (2.1)

5 Find $|z|$ and $\arg z$ when (2.2)

(a) $z = 2$ **(b)** $z = 2i$ **(c)** $z = 1 + \sqrt{3}i$ **(d)** $z = -\sqrt{3} - i$

6 Express in modulus/argument form **(a)** $-1 + i$ **(b)** $1 - i$ (2.2)

7 Write z in the form $a + ib$ when (2.2)

(a) $|z| = 4$ and $\arg z = \frac{2\pi}{3}$ **(b)** $|z| = 2$ and $\arg z = -\frac{\pi}{6}$

8 $z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $z_2 = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$. Find (2.2)

(a) $|z_1 z_2|$ and $\arg(z_1 z_2)$ **(b)** $\left|\frac{z_1}{z_2}\right|$ and $\arg\left(\frac{z_1}{z_2}\right)$

9 $z = 1 + i$. Find $|z^{10}|$ and $\arg(z^{10})$. (2.2)

10 $z = 1 + i$. Mark on an Argand diagram the points representing (2.2)
(a) z **(b)** $\bar{z}$ **(c)** iz **(d)** $z + 1$ **(e)** $z - 2i$

11 Show geometrically how to construct the vectors representing (2.3)
(a) $z_1 + z_2$ **(b)** $z_1 - z_2$ **(c)** $z_2 - z_1$,
when **(i)** $z_1 = 2$, $z_2 = i$ **(ii)** $z_1 = 4 + 2i$, $z_2 = 1 + 3i$

12 Express $(\cos\theta + i\sin\theta)^4$ in modulus/argument form. (2.4)

13 Express $\cos 2\theta - i\sin 2\theta$ in the form $(\cos\theta + i\sin\theta)^n$. (2.4)

14 Use De Moivre's theorem with $n = 2$ to show that (2.4)
$\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.
Hence show that $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$.

15 Express $z = 4\sqrt{2}(1 + i)$ in modulus/argument form. Hence find the two square roots of z and mark their representations on an Argand diagram. (2.4)

16 Indicate on an Argand diagram the locus of the point P representing z when (2.5)

(a) $Re\, z = -2$ **(b)** $Im\, z = 1$ **(c)** $|z| = 2$
(d) $|z - 2 - i| = 2$ **(e)** $\arg z = -\frac{\pi}{3}$ **(f)** $\arg(z + i) = \frac{3\pi}{4}$

17 Indicate on an Argand diagram the region which contains the point P representing z when (2.5)

(a) $|z| \leqslant |z - 2|$ and $-\frac{\pi}{4} \leqslant \arg z \leqslant \frac{\pi}{4}$ **(b)** $|z| \leqslant 1$ or $0 \leqslant \arg z \leqslant \frac{\pi}{2}$

Further questions 2

1 Express $(3 + 2i)(5 + 4i)$ and $(3 - 2i)(5 - 4i)$ in the form $a + ib$. Hence find the prime factors of $7^2 + 22^2$.

2 Complex numbers $z_1 = \frac{a}{1 + i}$ and $z_2 = \frac{b}{1 + 2i}$, where a and b are real, are such that $z_1 + z_2 = 1$. Find a and b.

3 $1 + i$ is a root of the equation $x^2 + (a + 2i)x + (5 + ib) = 0$, where a and b are real. Find the values of a and b.

4 $1 - 2i$ is one root of the equation $x^2 + (1 + i)x + k = 0$. Find the other root and the value of k.

5 a and b are real numbers such that the sum of the squares of the roots of the equation $x^2 + (a + ib)x + 3i = 0$ is 8. Find all possible pairs of values a, b.

6 Solve $x^2 - 4x + (1 - 4i) = 0$.

7 Find the modulus and argument of each of the complex numbers $z_1 = 2i$ and $z_2 = 1 + \sqrt{3}i$. Mark on an Argand diagram the points P, Q, R and S representing z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ respectively. Deduce the exact values of $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$.

8 On an Argand diagram, the points A, B, C and D represent z_1, z_2, z_3 and z_4 respectively. Show that if $z_1 - z_2 + z_3 - z_4 = 0$, then ABCD is a parallelogram, and if also $z_1 + iz_2 - z_3 - iz_4 = 0$, then ABCD is a square.

9 If $|z| = r$ and $\arg z = \theta$, show that $\dfrac{z}{z^2 + r^2}$ is real and give its value.

10 1, ω and ω^2 are the cube roots of unity. State the values of ω^3 and $1 + \omega + \omega^2$. Hence show that $(1 + \omega^2)^{12} = 1$ and $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)(1 - \omega^7)(1 - \omega^8) = 27$.

11 1, ω and ω^2 are the three cube roots of unity. Show that if the equations $z^3 - 1 = 0$ and $pz^5 + qz + r = 0$ have a common root, then $(p + q + r)(p\omega^5 + q\omega + r)(p\omega^{10} + q\omega^2 + \mathrm{r}) = 0$.

12 Show that the roots of $z^6 + z^3 + 1 = 0$ are among the roots of $z^9 - 1 = 0$. Hence find the roots of $z^6 + z^3 + 1 = 0$ in modulus/argument form.

13 Indicate on an Argand diagram the region defined by the pair of inequalities $|z| \leqslant 6$ and $|z - 5| \leqslant 5$. Write down the range of values of arg z for such z. Find the values of z for which both $|z| = 6$ and $|z - 5| = 5$.

14 The point P represents the complex number z on an Argand diagram. Describe the locus of P in each of the following cases

(a) $|z| = |z - 2|$ **(b)** $\arg(z - 2) = \arg(z + 2) + \dfrac{\pi}{2}$

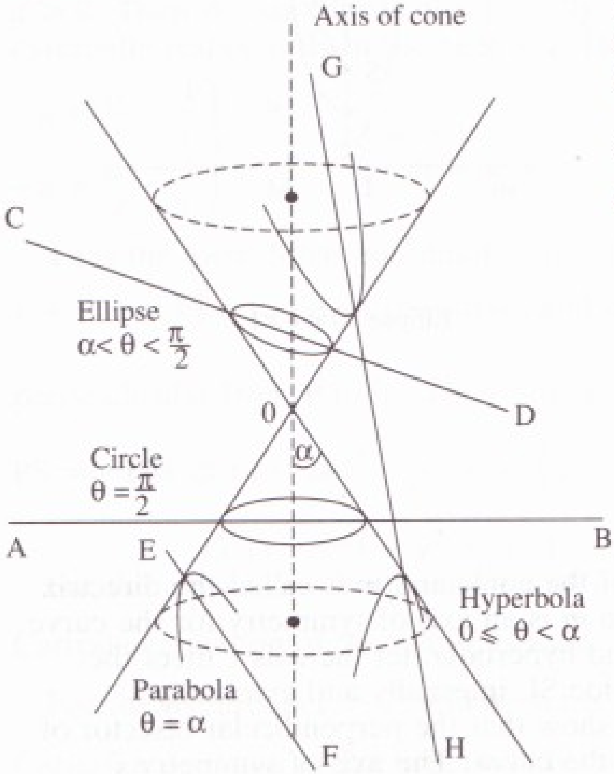
Find the complex number z which satisfies both of these equations.

3 Conics

3.1 Focus/directrix definitions and Cartesian equations

The curve formed when a plane cuts a double cone is called a conic section. The nature of the curve is determined by the relation between the semi-vertical angle α of the cone and the angle θ between the plane and the axis of the cone, as shown in Figure 3.1.

Figure 3.1



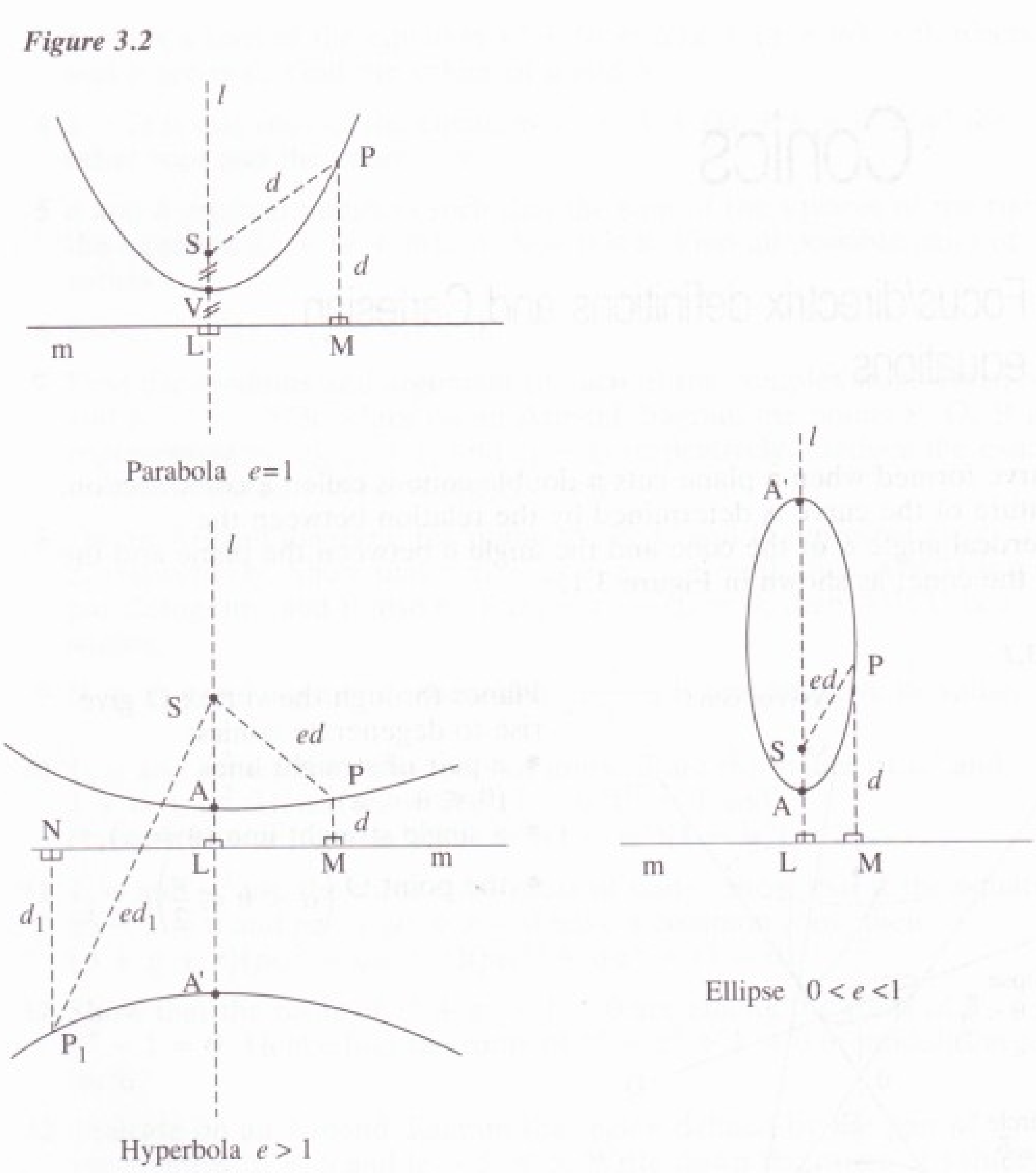
Planes through the vertex O give rise to degenerate conics:

- a pair of straight lines ($0 \leqslant \theta < \alpha$),
- a single straight line ($\theta = \alpha$),
- the point O $\left(\alpha < \theta \leqslant \frac{\pi}{2}\right)$.

The parabola has been studied in detail in the 3 Unit course. The ellipse and hyperbola have similar focus/directrix definitions, from which we can derive their Cartesian equations.

Let S be a fixed point and m a fixed line in a plane. Define the locus of a point P in the plane by the condition that the ratio of the distances from P to S and from P to m is e:1 for some constant $e > 0$. The shape of the curve is determined by the value of e, which is called the eccentricity of the curve.

Figure 3.2



Parabola $e=1$

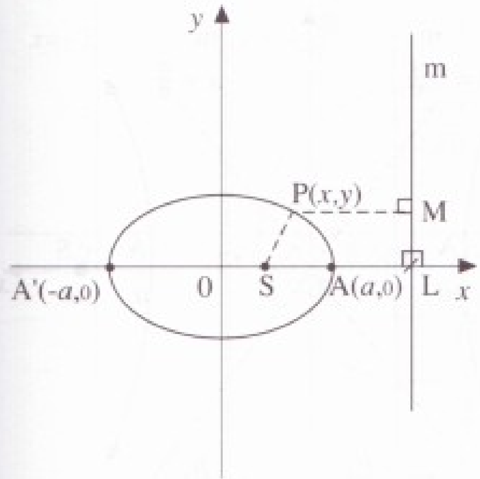
Hyperbola $e>1$

Ellipse $0<e<1$

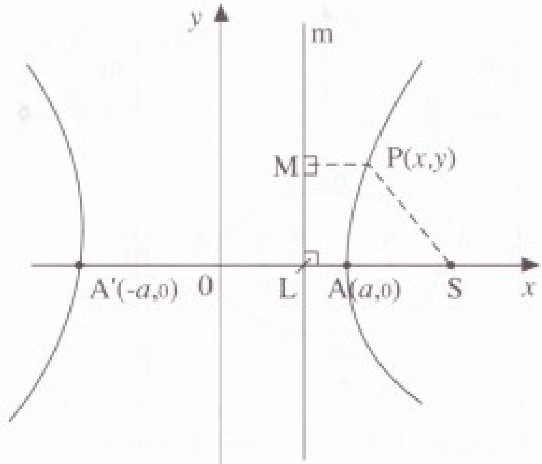
In each case S is called the focus of the conic and m is called the directrix. The line ℓ through S perpendicular to m is an axis of symmetry for the curve. Let ℓ meet m in L. For the ellipse and hyperbola, let the axis ℓ meet the curve in points A, A′ which then divide SL internally and externally respectively in the ratio e:1. We will show that the perpendicular bisector of AA′ is a second axis of symmetry of the curve. The axis of symmetry ℓ perpendicular to the directrix m is called the major axis of the ellipse or hyperbola, while the axis of symmetry parallel to the directrix is called the minor axis.

To obtain the standard form of the Cartesian equation of the ellipse or hyperbola, ℓ is chosen to be the x-axis while the perpendicular bisector of AA′ is chosen to be the y-axis. The origin O is now the midpoint of AA′.

Figure 3.3



Ellipse



Hyperbola

Let A, S, L have coordinates $(a, 0)$, $(k, 0)$, $(c, 0)$ respectively, where $a > 0$. Then A′ has coordinates $(-a, 0)$. A, A′ divide SL internally and externally respectively in the ratio e:1. Hence

$$\left.\begin{aligned} a &= \frac{ec + k}{e + 1} \\ -a &= \frac{ec - k}{e - 1} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} ae + a &= ec + k \text{ (i)} \\ -ae + a &= ec - k \text{ (ii)} \end{aligned}\right] \Rightarrow \begin{aligned} c &= \frac{a}{e} \quad [\text{(i)} + \text{(ii)}] \\ k &= ae \quad [\text{(i)} - \text{(ii)}] \end{aligned}$$

Thus the focus S has coordinates $(ae, 0)$ and the directrix m has equation $x = \frac{a}{e}$. Let $P(x, y)$ lie on the curve, and let M be the foot of the perpendicular from P to m. Then M has coordinates $\left(\frac{a}{e}, y\right)$.

$$PS = e.PM \Rightarrow (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$$

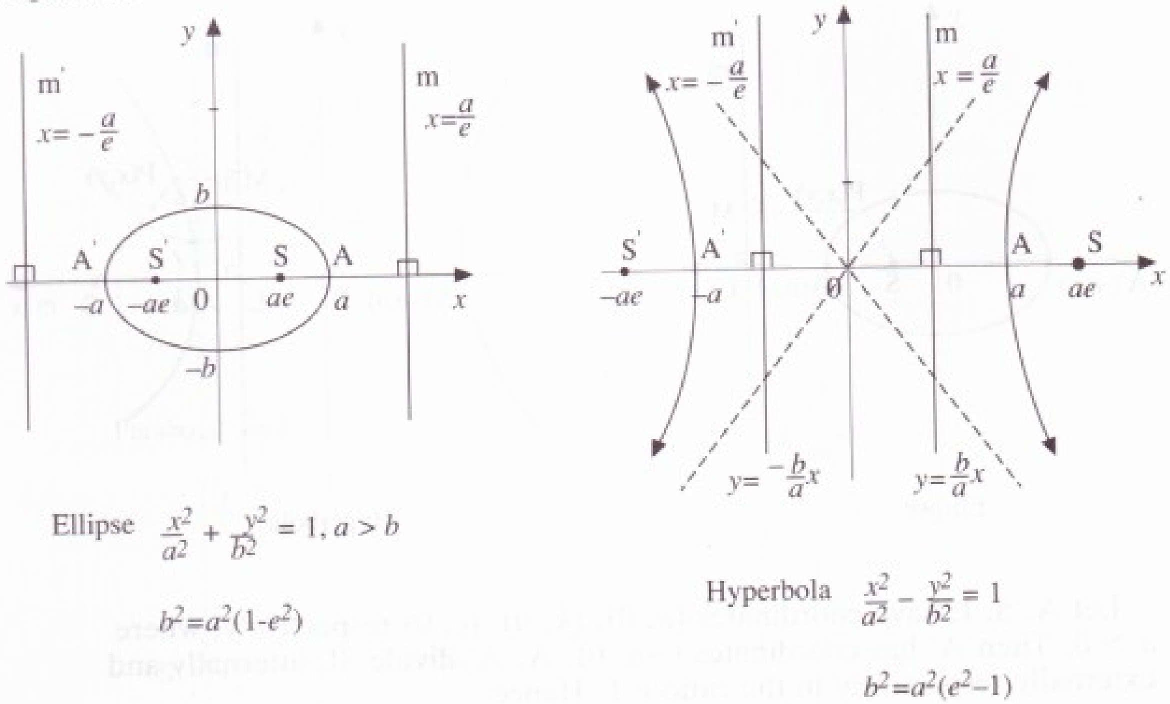
$$x^2(1 - e^2) + y^2 = a^2(1 - e^2).$$

For the ellipse, $0 < e < 1$ and we let $b^2 = a^2(1 - e^2)$, where $b > 0$. The Cartesian equation of the ellipse is then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

For the hyperbola $e > 1$, and we let $b^2 = a^2(e^2 - 1)$, where $b > 0$. The Cartesian equation of the hyperbola is then $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Clearly, in either case, if $P(x, y)$ lies on the curve, then $P_1(-x, y)$ also lies on the curve. Hence the y-axis is an axis of symmetry for the curve. Let S′, m' be the reflections of S, m respectively in the y-axis.

Figure 3.4



If S′, m' had been used as the fixed point and fixed line in the locus definition of P, we would have obtained the same curve. The ellipse and hyperbola thus have two foci S(ae, 0) and S′($-ae$, 0), and two directrices m:$x = \frac{a}{e}$ and m′: $x = -\frac{a}{e}$. The length of the major axis AA′ is $2a$, while OA, the semi-major axis, has length a. Similarly the lengths of the minor axis and semi-minor axis are $2b$ and b respectively, even in the case of the hyperbola where this fixed length $2b$ is not a geometrical feature of the curve as it is for the ellipse.

The hyperbola has two oblique asymptotes.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \left|\frac{x}{a} - \frac{y}{b}\right| . \left|\frac{x}{a} + \frac{y}{b}\right| = 1.$$

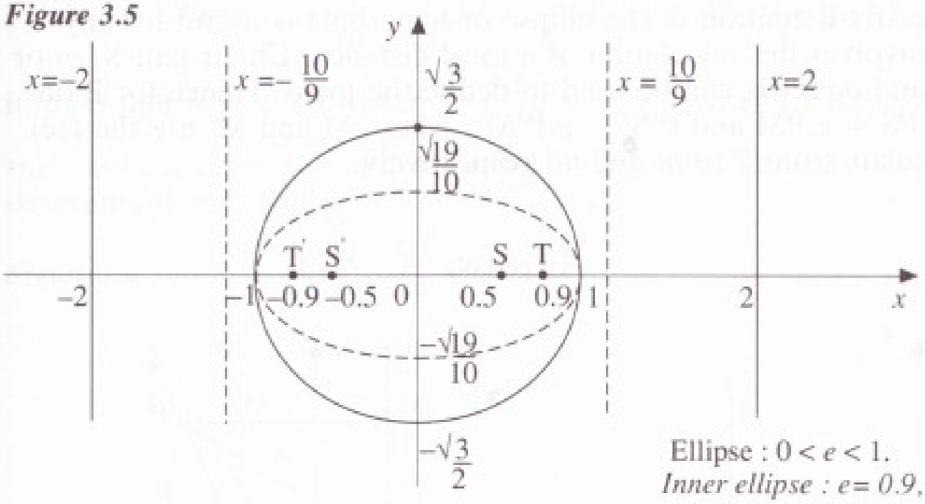
$$\therefore \left|\frac{b}{a}x - y\right| = \frac{b}{\left|\frac{x}{a} + \frac{y}{b}\right|} \to 0 \text{ as } \begin{cases} x \to +\infty \text{ and } y \to +\infty \text{ or} \\ x \to -\infty \text{ and } y \to -\infty \end{cases}$$

$$\therefore \left|\frac{b}{a}x + y\right| = \frac{b}{\left|\frac{x}{a} + \frac{y}{b}\right|} \to 0 \text{ as } \begin{cases} x \to +\infty \text{ and } y \to -\infty \text{ or} \\ x \to -\infty \text{ and } y \to +\infty \end{cases}$$

Hence $y = \pm \frac{b}{a}x$ are asymptotes to the hyperbola, as shown in Figure 3.4.

For both the ellipse and the hyperbola, changing the value of the eccentricity e changes the shape of the curve.

Figure 3.5

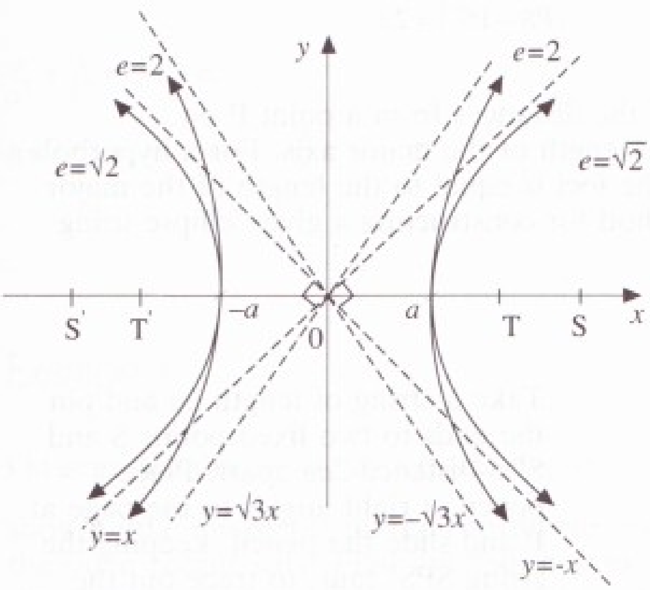


Ellipse : $0 < e < 1$.
Inner ellipse : e= 0.9, a= 1. Foci at T,T′
Outer ellipse : e= 0.5, a= 1. Foci at S,S′

$$\frac{b^2}{a^2} = 1 - e^2, \quad a > b > 0.$$

As e decreases, $\frac{b}{a}$ increases and the ellipse becomes less elongated. The foci move in towards the centre and the directrices move out from the centre. In the limit as $e \to 0$, $\frac{b}{a} \to 1$ and the ellipse becomes a circle (S, S′ coincident with O, and directrices at ∞).

Figure 3.6



Hyperbola , $e>1$.
Hyperbola with $e = 2$ has foci S,S′.
Hyperbola with $e = \sqrt{2}$ has foci T,T′.

$$\frac{b^2}{a^2} = e^2 - 1, \quad a > 0, b > 0.$$

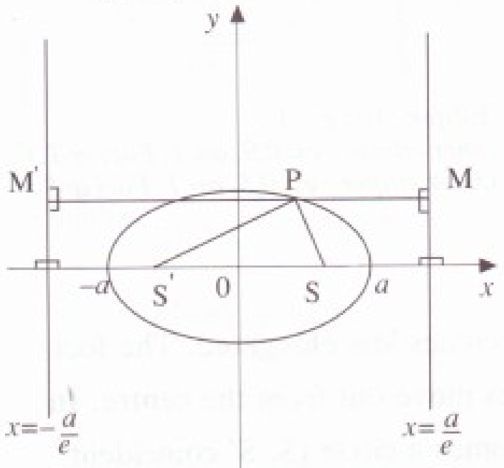
As e increases, $\frac{b}{a}$ increases.

The asymptotes $y = \pm \frac{b}{a} x$ become steeper and the curvature of the hyperbola becomes less pronounced. The hyperbola with $e = \sqrt{2}$ has asymptotes $y = \pm x$ which meet at right angles, and is called a rectangular hyperbola.

The focus/directrix definition of the ellipse or hyperbola is useful for any problem which involves the calculation of a focal distance. Either pair S, m or S$'$, m' of focus and directrix can be used to define the locus. Hence for P on the curve, both PS $= e$.PM and P$'$S$' = e$.P$'$M$'$, where M and M$'$ are the feet of the perpendiculars from P to m and m' respectively.

Figure 3.7

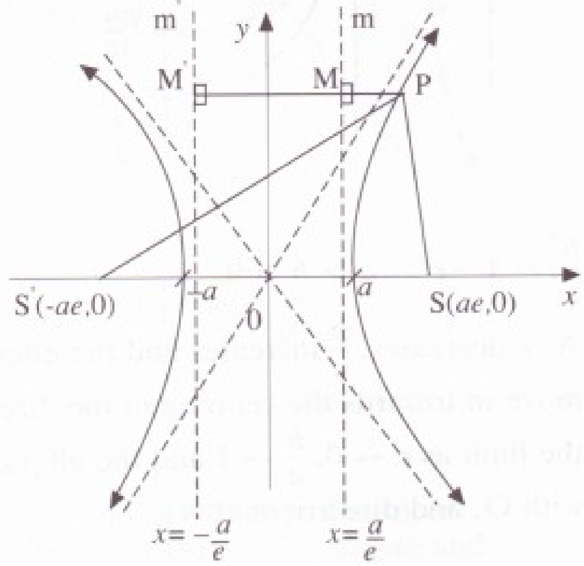
Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$PS + PS' = e\,(PM + PM') = e\,MM'$,

$\therefore\ PS + PS' = 2a.$

Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

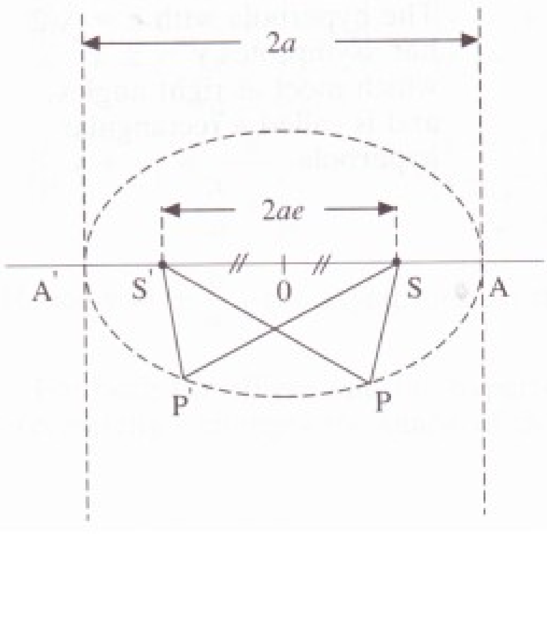


$|PS - PS'| = e\,|PM - PM'| = e\,MM'$.

$|PS - PS'| = 2a.$

Hence for an ellipse, the sum of the distances from a point P on the curve to the foci is equal to the length of the major axis. For a hyperbola, the difference of the distances to the foci is equal to the length of the major axis. This property provides a method for constructing a given ellipse using two pins and a length of string.

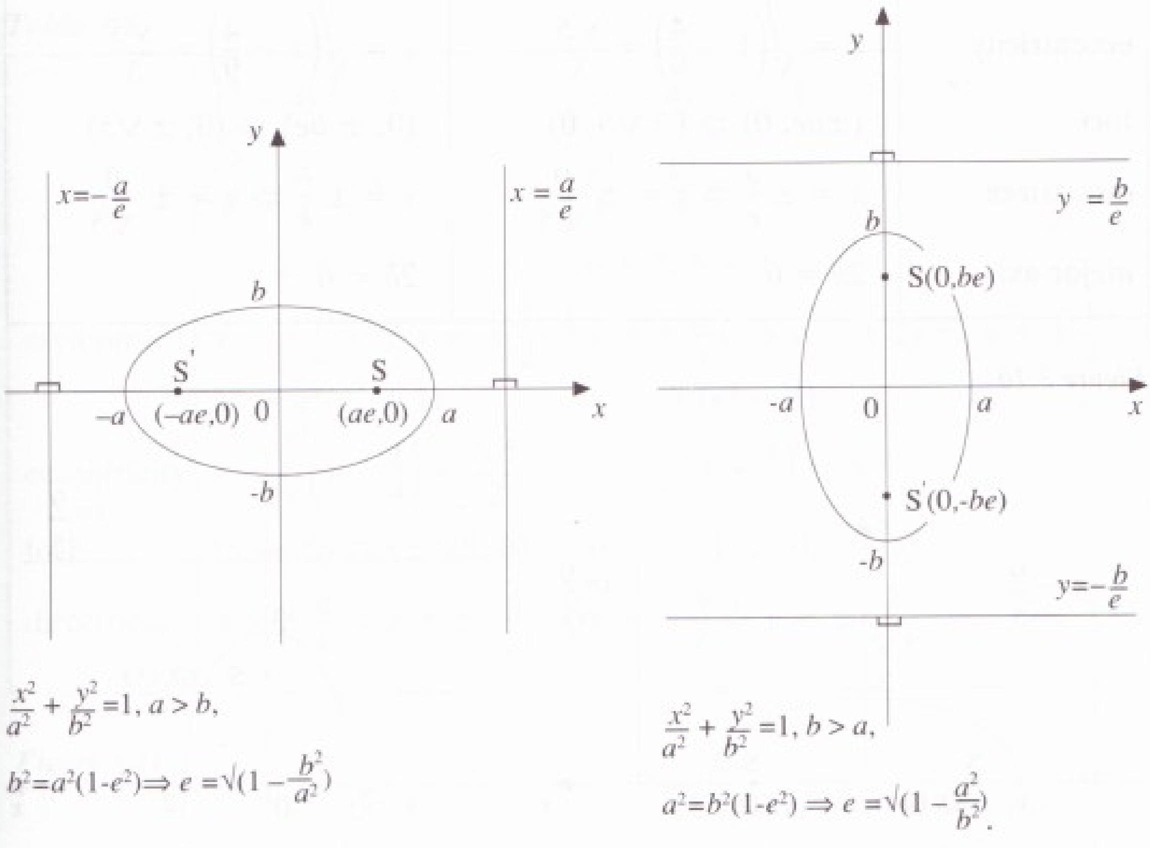
Figure 3.8



Take a string of length $2a$ and pin the ends to two fixed points S and S$'$ a distance $2ae$ apart. Place a pencil at right angles to the page at P and slide the pencil, keeping the string SPS$'$ taut, to trace out the ellipse.

For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $b^2 = a^2(1 - e^2)$, where a and b are positive, implies that $b < a$. However $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $b > a$, also defines an ellipse. In this case the roles of x and y are interchanged, as are the roles of a and b in determining e, S and m.

Figure 3.9



Example 1

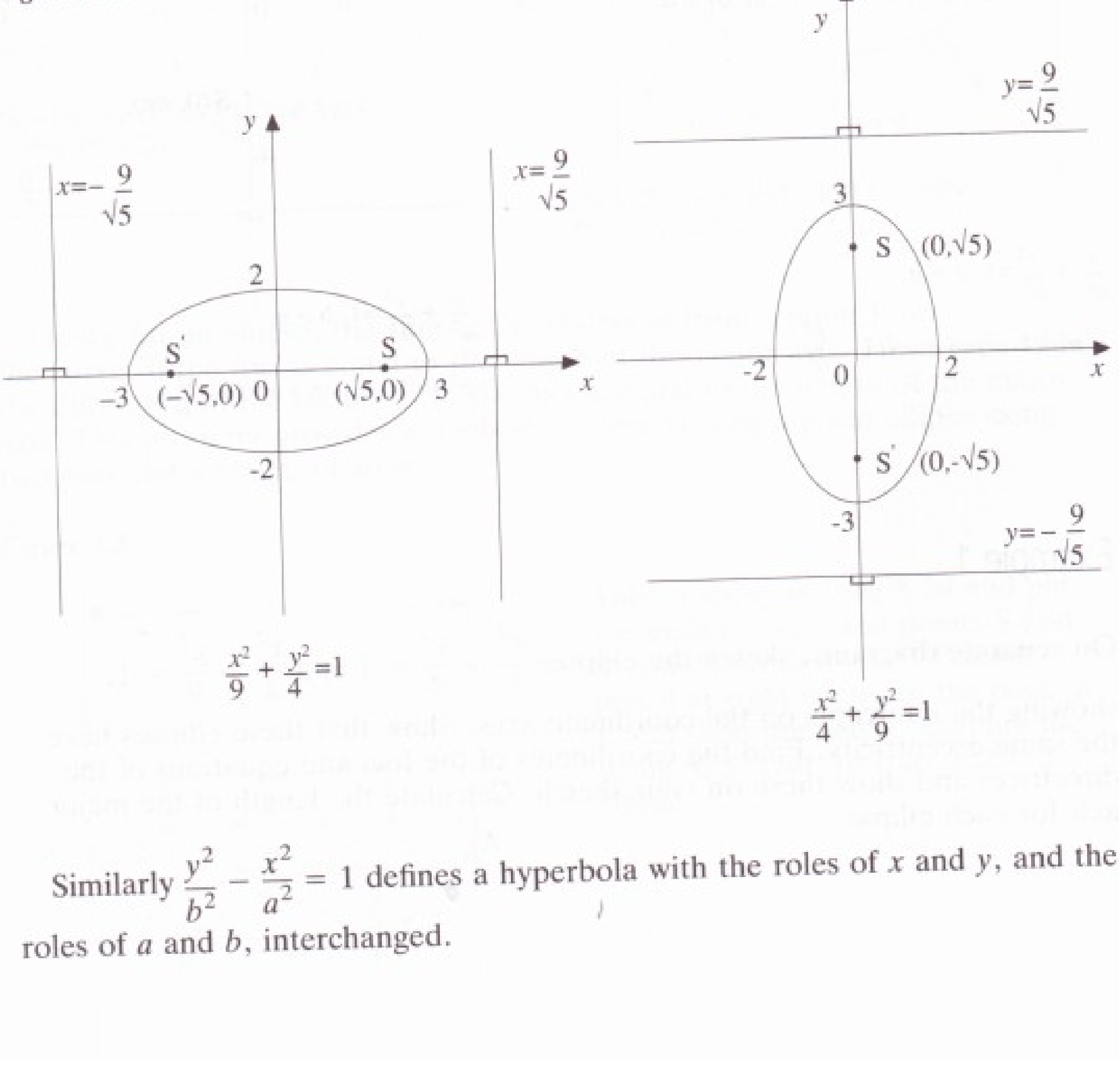
On separate diagrams, sketch the ellipses $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, showing the intercepts on the coordinate axes. Show that these ellipses have the same eccentricity. Find the coordinates of the foci and equations of the directrices and show these on your sketch. Calculate the length of the major axis for each ellipse.

Solution

Table 3.1

	$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$	$\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$
	$a = 3,\ b = 2 \Rightarrow b < a.$ $b^2 = a^2(1 - e^2)$	$a = 2,\ b = 3 \Rightarrow b > a.$ $a^2 = b^2(1 - e^2)$
eccentricity	$e = \sqrt{\left(1 - \frac{4}{9}\right)} = \dfrac{\sqrt{5}}{3}$	$e = \sqrt{\left(1 - \frac{4}{9}\right)} = \dfrac{\sqrt{5}}{3}$
foci	$(\pm ae, 0) \Rightarrow (\pm\sqrt{5}, 0)$	$(0, \pm\ be) \Rightarrow (0, \pm\sqrt{5})$
directrices	$x = \pm\dfrac{a}{e} \Rightarrow x = \pm\dfrac{9}{\sqrt{5}}$	$y = \pm\dfrac{b}{e} \Rightarrow y = \pm\dfrac{9}{\sqrt{5}}$
major axis	$2a = 6$	$2b = 6$

Figure 3.10



Similarly $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$ defines a hyperbola with the roles of x and y, and the roles of a and b, interchanged.

Example 2

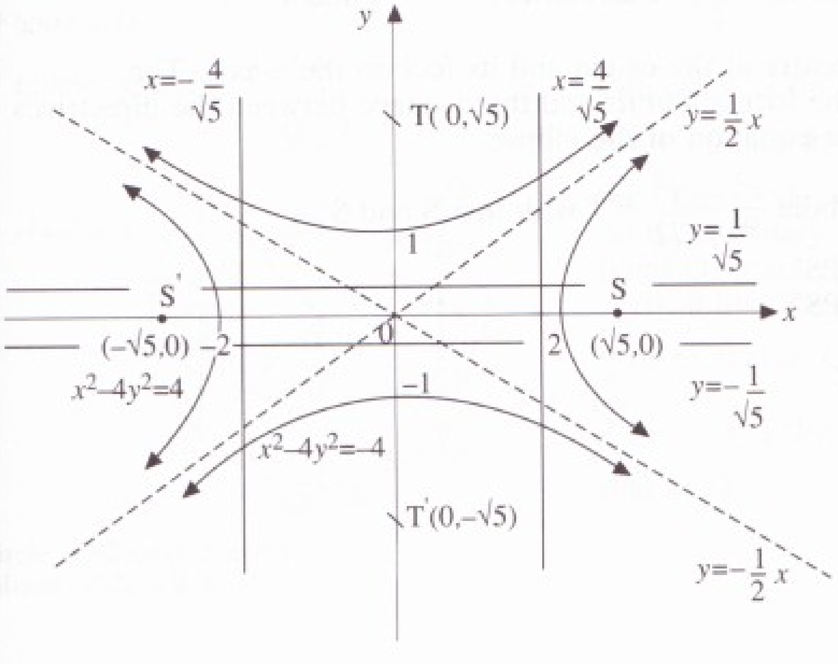
On the same diagram, sketch the hyperbolas $x^2 - 4y^2 = 4$ and $x^2 - 4y^2 = -4$, showing that they have common asymptotes. Calculate the eccentricity of each hyperbola. Find the coordinates of the foci and the equations of the directrices and show these on your sketch.

Solution

Table 3.2

	$x^2 - 4y^2 = 4$	$x^2 - 4y^2 = -4$
	$\frac{x^2}{4} - \frac{y^2}{1} = 1.$	$\frac{y^2}{1} - \frac{x^2}{4} = 1.$
	$a = 2, b = 1.$	$a = 2, b = 1.$
asymptotes	$y = \pm \frac{b}{a} x \Rightarrow y = \pm \frac{1}{2}x$	$x = \pm \frac{a}{b} y \Rightarrow x = \pm 2y \Rightarrow y = \pm \frac{1}{2}x.$
	$b^2 = a^2(e^2 - 1)$	$a^2 = b^2(e^2 - 1)$
eccentricity	$e = \sqrt{\left(1 + \frac{1}{4}\right)} = \frac{\sqrt{5}}{2}$	$e = \sqrt{\left(1 + \frac{4}{1}\right)} = \sqrt{5}$
foci	$(\pm ae, 0) \Rightarrow (\pm\sqrt{5}, 0)$	$(0, \pm be) \Rightarrow (0, \pm\sqrt{5})$
directrices	$x = \pm \frac{a}{e} \Rightarrow x = \pm \frac{4}{\sqrt{5}}$	$y = \pm \frac{b}{e} \Rightarrow y = \pm \frac{1}{\sqrt{5}}$

Figure 3.11



Exercise 3.1

1 Find the locus of a variable point $P(x, y)$ which moves so that
(a) its distance from $(1, 0)$ is one-third its distance from $x = 9$
(b) its distance from $(0, 1)$ is one-half its distance from $y = 4$
(c) its distance from $(9, 0)$ is three times its distance from $x = 1$
(d) its distance from $(0, 4)$ is two times its distance from $y = 1$

2 For each of the ellipses
(a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$, **(b)** $\frac{x^2}{16} + \frac{y^2}{25} = 1$, **(c)** $\frac{x^2}{3} + \frac{y^2}{2} = 1$, **(d)** $x^2 + 2y^2 = 4$, find
(i) the eccentricity, **(ii)** the coordinates of the foci, **(iii)** the equations of the directrices. Sketch each ellipse.

3 For each of the hyperbolas
(a) $\frac{x^2}{9} - \frac{y^2}{16} = 1$, **(b)** $\frac{y^2}{16} - \frac{x^2}{9} = 1$, **(c)** $\frac{x^2}{2} - \frac{y^2}{4} = 1$, **(d)** $x^2 - y^2 = 4$, find
(i) the eccentricity, **(ii)** the coordinates of the foci, **(iii)** the equations of the directrices, **(iv)** the equations of the asymptotes. Sketch each hyperbola.

4 Find the equation of the ellipse
(a) which has eccentricity $\frac{4}{5}$ and foci $(-4, 0)$ and $(4, 0)$
(b) which has eccentricity $\frac{2}{3}$ and directrices $x = -9$ and $x = 9$

5 Find the equation of the hyperbola
(a) which has eccentricity $\frac{5}{4}$ and foci $(-5, 0)$ and $(5, 0)$
(b) which has eccentricity $\frac{3}{2}$ and directrices $x = -4$ and $x = 4$

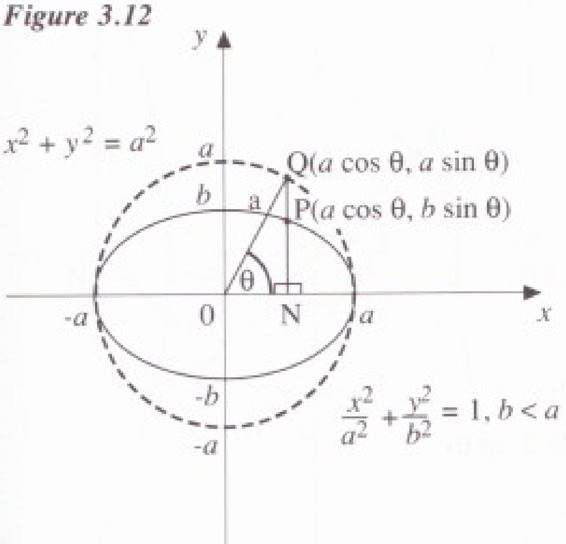
6 An ellipse has its centre at the origin and its foci on the x-axis. The distance between the foci is 4 units and the distance between the directrices is 16 units. Find the equation of the ellipse.

7 P lies on the hyperbola $\frac{x^2}{9} - \frac{y^2}{72} = 1$ with foci S and S′.
(a) If PS = 2 find PS′.
(b) If PS = 8 find PS′.

3.2 Parametric equations for the ellipse and hyperbola

Ellipse

Figure 3.12

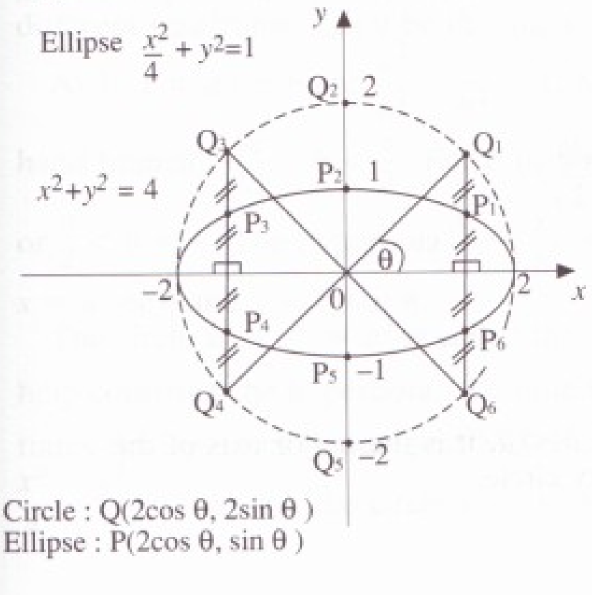


Let N be the foot of a perpendicular from a point P on the ellipse to the x-axis. Produce NP to a point Q on the circle $x^2 + y^2 = a^2$. Let OQ make an angle θ with the positive x-axis, $-\pi < \theta \leqslant \pi$. Then Q has coordinates $(a\cos\theta, a\sin\theta)$.

At P, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x = a\cos\theta$, hence $y = b\sin\theta$.

Hence the ellipse with Cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has parametric equations $x = a\cos\theta$ and $y = b\sin\theta$, $-\pi < \theta \leqslant \pi$. The circle $x^2 + y^2 = a^2$ is called the auxiliary circle and it can be used to help construct the ellipse. For example, to construct $\frac{x^2}{4} + y^2 = 1$ we use the auxiliary circle $x^2 + y^2 = 4$.

Figure 3.13



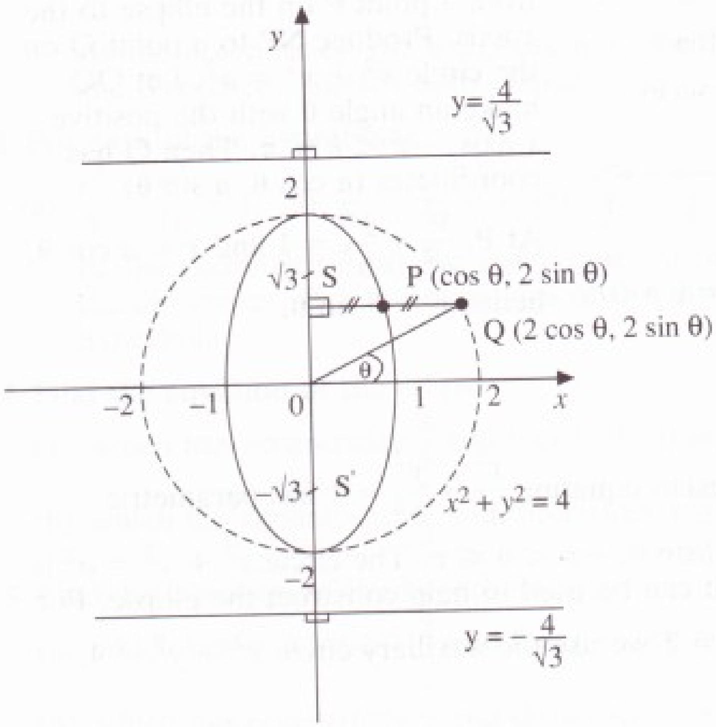
$Q(a\cos\theta, a\sin\theta)$ lies on the circle while $P(a\cos\theta, \frac{b}{a}.a\sin\theta)$ lies on the ellipse. Mark several points Q on the auxiliary circle. For each point Q, mark the point P on a vertical line through Q with ordinate $\frac{b}{a}$ times that of Q. For $\frac{x^2}{4} + y^2 = 1$ the ordinate of P is half that of Q.

Example 3

Find the Cartesian equation of the ellipse with parametric equations $x = \cos\theta$ and $y = 2\sin\theta$. Sketch the ellipse showing its foci and directrices. State the length of the semi-major axis. Show the auxiliary circle on your sketch and state its equation.

Solution

Figure 3.14



$x^2 + \dfrac{y^2}{4} = \cos^2\theta + \sin^2\theta = 1$

$a = 1,\ b = 2 \Rightarrow b > a.$

$a^2 = b^2(1 - e^2) \Rightarrow e = \sqrt{\left(1 - \frac{1}{4}\right)} = \dfrac{\sqrt{3}}{2}.$

Foci $\quad (0, \pm\, be) \Rightarrow (0, \pm\, \sqrt{3}).$

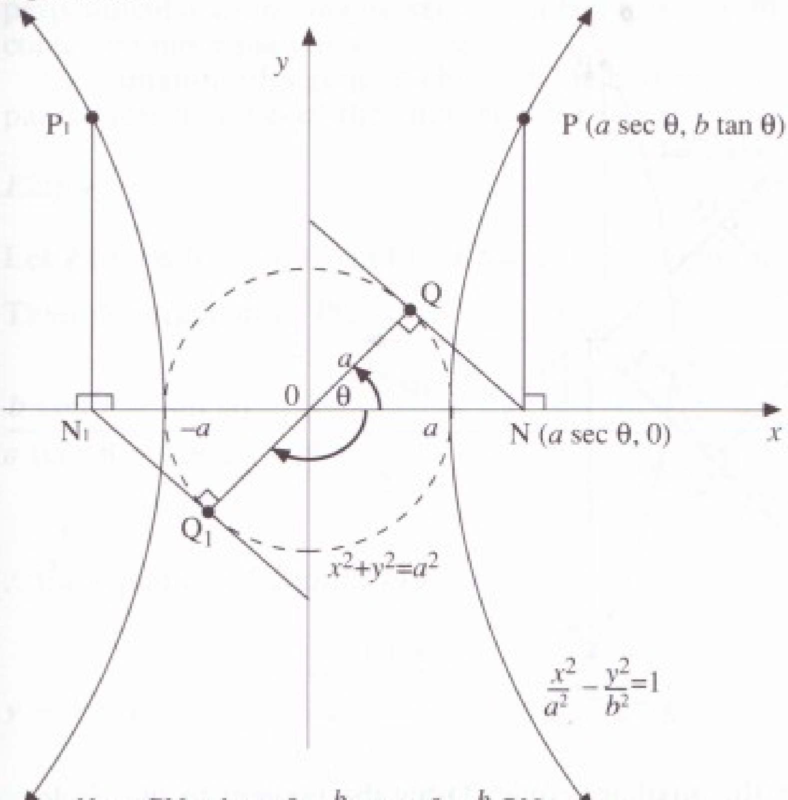
Directrices $y = \pm\,\dfrac{b}{e} \Rightarrow y = \pm\,\dfrac{4}{\sqrt{3}}.$

Length of semi-major axis is 2.

Note that in both the cases, $b > a$ and $b < a$, it is the major axis of the ellipse that is the diameter of the auxiliary circle.

Hyperbola

Figure 3.15



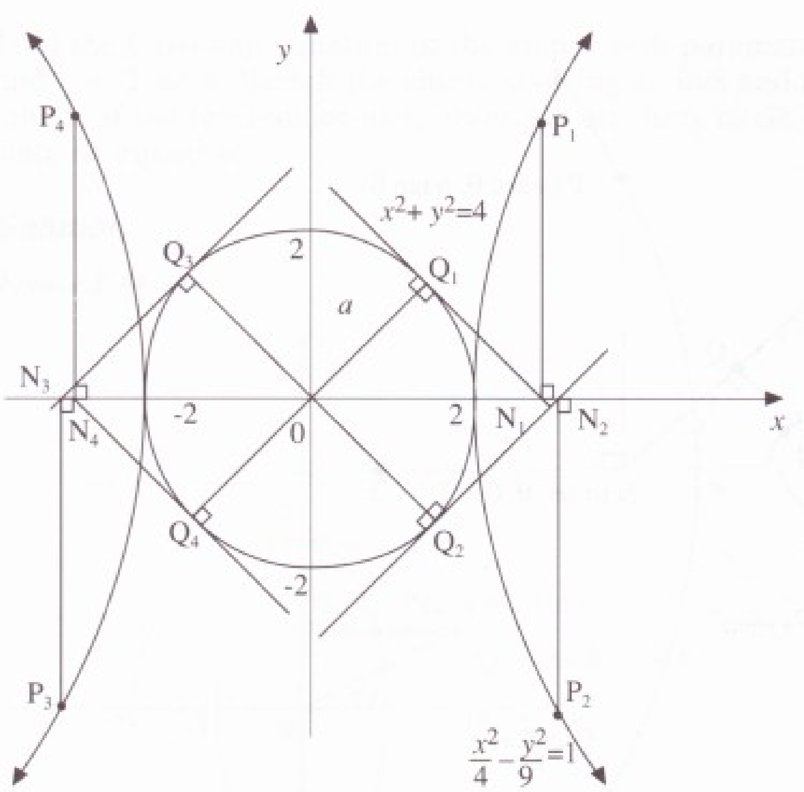
Note: PN $= b \tan\theta = \frac{b}{a}.\, a\tan\theta = \frac{b}{a}$ QN

N is the foot of the perpendicular from a point P on the hyperbola to the x-axis. From N, a tangent is drawn to the circle $x^2 + y^2 = a^2$, meeting the circle in Q. If P is on the right-hand branch, choose Q in the same quadrant as P. If P is on the left-hand branch, choose Q so that P and Q are in different quadrants. Let θ be the angle OQ makes with the positive x-axis.

At P, $x = a \sec\theta$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, hence $y = b\tan\theta$. For P on the right-hand branch, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. For P on the left-hand branch, $-\pi < \theta < -\frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leqslant \pi$. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has parametric equations $x = a\sec\theta$ and $y = b\tan\theta$.

The circle $x^2 + y^2 = a^2$ is called the auxiliary circle and it can be used to help construct the hyperbola. We note that the length of the ordinate at P is $\frac{b}{a}$ times the length of the tangent NQ. For example, to construct the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ we use the circle $x^2 + y^2 = 4$.

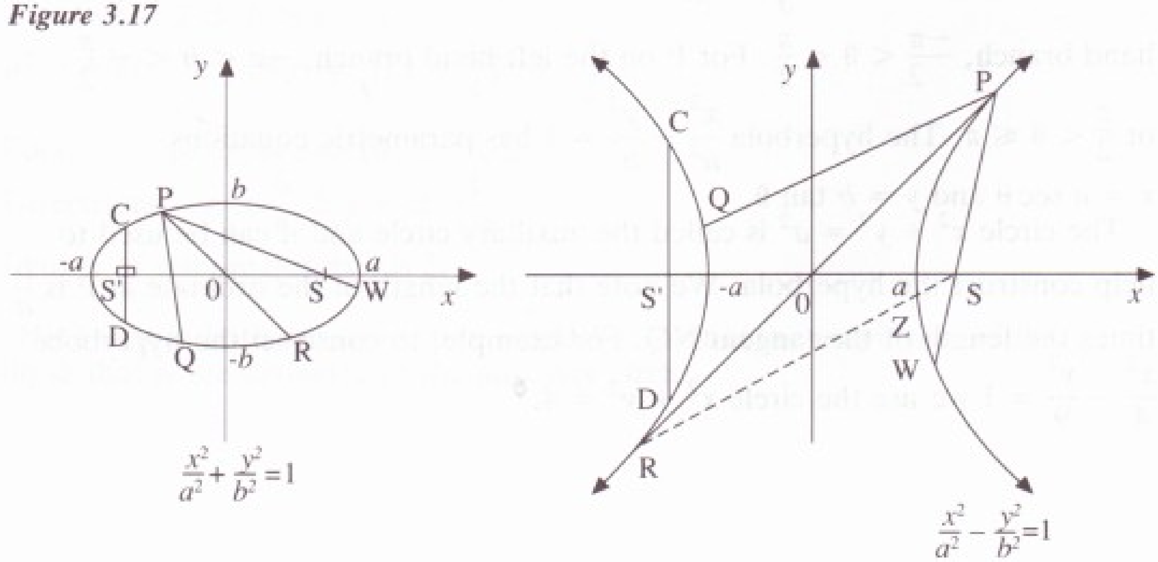
Figure 3.16



Mark several points Q on the auxiliary circle. Draw the tangent to the circle at Q, meeting the x-axis in N. Mark P on a vertical line through N so that $NP = \frac{3}{2}NQ$. Choose P in the correct quadrant given the position of Q: same quadrant as Q on the right-hand branch; different quadrant from Q on the left-hand branch.

Equation of a chord on an ellipse or hyperbola

Figure 3.17



PQ, PR, PW are chords of the conic. Chord PR through the centre is a diameter, while PW through the focus S is a focal chord. (Note that RZ is also a focal chord of the hyperbola.) CD is a focal chord which is perpendicular to the major axis. Such a chord is termed a latus rectum of the conic and has equation $x = \pm ae$.

The equation of a general chord PQ is best expressed in terms of the parameters θ and ϕ of the endpoints P and Q.

Ellipse

Let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Then the gradient of PQ is

$$\frac{b\,(\sin\theta - \sin\phi)}{a\,(\cos\theta - \cos\phi)} = \frac{b}{a}\cdot\frac{2\sin\left(\dfrac{\theta-\phi}{2}\right)\cos\left(\dfrac{\theta+\phi}{2}\right)}{-2\sin\left(\dfrac{\theta-\phi}{2}\right)\sin\left(\dfrac{\theta+\phi}{2}\right)} = -\frac{b}{a}\cdot\frac{\cos\left(\dfrac{\theta+\phi}{2}\right)}{\sin\left(\dfrac{\theta+\phi}{2}\right)}$$

$\therefore$ the equation of chord PQ is

$$y - b\sin\theta = -\frac{b}{a}\cdot\frac{\cos\left(\dfrac{\theta+\phi}{2}\right)}{\sin\left(\dfrac{\theta+\phi}{2}\right)}(x - a\cos\theta)$$

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\theta\cos\left(\frac{\theta+\phi}{2}\right) + \sin\theta\sin\left(\frac{\theta+\phi}{2}\right)$$

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right).$$

In the special case where PQ is a diameter, (0, 0) lies on the chord and hence $|\theta - \phi| = \pi$.

Hyperbola

Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ lie on $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Then the gradient of PQ is

$$\frac{b\,(\tan\theta - \tan\phi)}{a\,(\sec\theta - \sec\phi)} = \frac{b}{a}\,\frac{\sin(\theta-\phi)}{(\cos\phi - \cos\theta)} = \frac{b}{a}\cdot\frac{2\sin\left(\dfrac{\theta-\phi}{2}\right)\cos\left(\dfrac{\theta-\phi}{2}\right)}{2\sin\left(\dfrac{\theta-\phi}{2}\right)\sin\left(\dfrac{\theta+\phi}{2}\right)}.$$

$\therefore$ the equation of PQ is

$$y - b\tan\theta = \frac{b\cos\left(\frac{\theta-\phi}{2}\right)}{a\sin\left(\frac{\theta+\phi}{2}\right)}(x - a\sec\theta)$$

$$\begin{aligned}\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) &= \cos\left(\frac{\theta-\phi}{2}\right)\sec\theta - \sin\left(\frac{\theta+\phi}{2}\right)\tan\theta \\ &= \sec\theta\left\{\cos\left(\theta - \frac{\theta+\phi}{2}\right) - \sin\theta\sin\left(\frac{\theta+\phi}{2}\right)\right\} \\ &= \sec\theta\cos\theta\cos\left(\frac{\theta+\phi}{2}\right).\end{aligned}$$

Hence the equation of PQ is $\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$.

In the special case where PQ is a diameter, $(0, 0)$ lies on the chord and hence $|\theta + \phi| = \pi$.

Example 4

(a) Given that the equation of the chord PQ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$, where P, Q have parameters θ, ϕ, show that if PQ is a focal chord then $\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right)$ takes one of the values $-\left(\frac{1-e}{1+e}\right)$ or $-\left(\frac{1+e}{1-e}\right)$.

(b) PQ is a focal chord of $\frac{x^2}{4} + \frac{y^2}{3} = 1$, where P has coordinates $\left(1, \frac{3}{2}\right)$. Find the coordinates of Q.

Solution

(a) If PQ is a focal chord through $S(ae, 0)$, then $e\cos\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$.

Expanding both cosines gives

$(e-1)\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi}{2}\right) = (e+1)\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi}{2}\right)$.

Hence $\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right) = -\left(\frac{1+e}{1-e}\right)$.

Similarly, if PQ is a focal chord through $S'(-ae, 0)$, then replacing e by $-e$,

$\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right) = -\left(\frac{1-e}{1+e}\right)$.

(b) $\frac{x^2}{4} + \frac{y^2}{3} = 1 \Rightarrow a = 2$ and $b = \sqrt{3}$, $\therefore P\left(1, \frac{3}{2}\right) \equiv P\left(2\cos\frac{\pi}{3}, \sqrt{3}\sin\frac{\pi}{3}\right)$.

Also $a > b \Rightarrow b^2 = a^2(1 - e^2)$ $\therefore e = \sqrt{\left(1 - \frac{3}{4}\right)} = \frac{1}{2}$.

Let Q have parameter ϕ. P has parameter $\frac{\pi}{3}$. Hence

$\tan\left(\frac{\phi}{2}\right)\tan\frac{\pi}{6} = -\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)$, or $\tan\left(\frac{\phi}{2}\right)\tan\frac{\pi}{6} = -\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$,

$\therefore \tan\left(\frac{\phi}{2}\right) = -\frac{1}{\sqrt{3}}$,

$$\cos\phi = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{1}{2},$$

$$\text{and } \sin\phi = \frac{\frac{-2}{\sqrt{3}}}{1+\frac{1}{3}} = -\frac{\sqrt{3}}{2}.$$

$\tan\left(\frac{\phi}{2}\right) = -3\sqrt{3}$,

$$\cos\phi = \frac{1-27}{1+27} = -\frac{13}{14},$$

$$\text{and } \sin\phi = \frac{-6\sqrt{3}}{1+27} = -\frac{3\sqrt{3}}{14}.$$

Q has coordinates $(2\cos\phi, \sqrt{3}\sin\phi) \Rightarrow Q\left(1, \frac{-3}{2}\right)$ or $Q\left(\frac{-13}{7}, \frac{-9}{14}\right)$.

Exercise 3.2

1 Find the parametric equations of

(a) The ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(b) The ellipse $x^2 + 4y^2 = 4$

(c) The hyperbola $\frac{x^2}{16} - \frac{y^2}{25} = 1$

(d) The hyperbola $x^2 - y^2 = 4$

2 Find the Cartesian equations of

(a) The ellipse $x = 3\cos\theta$, $y = 2\sin\theta$

(b) The ellipse $x = 5\cos\theta$, $y = 4\sin\theta$

(c) The hyperbola $x = 3\sec\theta$, $y = 4\tan\theta$

(d) The hyperbola $x = 2\sec\theta$, $y = 5\tan\theta$

3 $P(a\cos\theta, b\sin\theta)$ and $Q[a\cos(\pi + \theta), b\sin(\pi + \theta)]$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that PQ passes through (0, 0).

4 P(a sec θ, b tan θ) and Q[a sec (π − θ), b tan (π − θ)] lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Show that PQ passes through (0, 0).

5 P(a cos θ, b sin θ) and Q[a cos(−θ), b sin(−θ)] are the extremities of the latus rectum $x = ae$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(a) Show that cos θ = e **(b)** Show that PQ has length $2\frac{b^2}{a}$

6 P(a sec θ, b tan θ) lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
with foci S(ae, 0) and S′($-ae$, 0).
(a) Show that PS = $a(e \sec\theta - 1)$ and PS′ = $a(e \sec\theta + 1)$
(b) Deduce that |PS − PS′| = $2a$

7 P(a cos θ, b sin θ) and Q(a cos ϕ, b sin ϕ) lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(a) If PQ subtends a right angle at (0, 0), show that $\tan\theta \tan\phi = -\frac{a^2}{b^2}$

(b) If PQ subtends a right angle at (a, 0) show that $\tan\frac{\theta}{2} \tan\frac{\phi}{2} = -\frac{b^2}{a^2}$

8 P(a sec θ, b tan θ) and Q(a sec ϕ, b tan ϕ) lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(a) If PQ subtends a right angle at (0, 0), show that $\sin\theta \sin\phi = -\frac{a^2}{b^2}$

(b) If PQ subtends a right angle at (a, 0), show that $\tan\frac{\theta}{2} \tan\frac{\phi}{2} = -\frac{b^2}{a^2}$

9 **(a)** P(a sec θ, b tan θ) and Q(a sec ϕ, b tan ϕ) lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Use the result that the chord PQ has equation $\frac{x}{a}\cos\left(\frac{\theta - \phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta + \phi}{2}\right)$ to show that if PQ is a focal chord, then $\tan\frac{\theta}{2} \tan\frac{\phi}{2}$ takes one of the values $\frac{1-e}{1+e}$ or $\frac{1+e}{1-e}$.

(b) P($2\sqrt{3}$, $3\sqrt{3}$) is one extremity of a focal chord on the hyperbola $\frac{x^2}{3} - \frac{y^2}{9} = 1$. Find the coordinates of the other extremity Q.

3.3 Tangents and normals to the ellipse and hyperbola

Equations of tangents and normals to the ellipse

Cartesian form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

By implicit differentiation

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0,$$

$\therefore$ the gradient of the tangent at $P(x_1, y_1)$ is $\dfrac{-b^2}{a^2}\dfrac{x_1}{y_1}$

$\therefore$ the equation of the tangent at P is

$$y - y_1 = -\frac{b^2}{a^2}\frac{x_1}{y_1}(x - x_1)$$

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$\therefore$ $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ is the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at $P(x_1, y_1)$.

Also the gradient of the normal at P is $\dfrac{a^2}{b^2}\dfrac{y_1}{x_1}$,

$\therefore$ the equation of the normal at P is

$$y - y_1 = \frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1).$$

$\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y} = a^2 - b^2$ is the normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at $P(x_1, y_1)$.

Parametric form

$$x = a\cos\theta \Rightarrow \frac{dx}{d\theta} = -a\sin\theta$$

$$y = b\sin\theta \Rightarrow \frac{dy}{d\theta} = b\cos\theta$$

$$\therefore \frac{dy}{dx} = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$$

$\therefore$ the gradient of the tangent at $P(a\cos\phi, b\sin\phi)$ is $-\dfrac{b}{a}\dfrac{\cos\phi}{\sin\phi}$,

$\therefore$ the equation of the tangent at P is

$$y - b\sin\phi = -\frac{b}{a}\frac{\cos\phi}{\sin\phi}(x - a\cos\phi)$$

$$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = \cos^2\phi + \sin^2\phi = 1$$

$\therefore$ $\dfrac{x\cos\phi}{a} + \dfrac{y\sin\phi}{b} = 1$ is the tangent to the ellipse $x = a\cos\theta$, $y = b\sin\theta$ at $P(a\cos\phi, b\sin\phi)$.

Also the gradient of the normal at P is $\dfrac{a\sin\phi}{b\cos\phi}$,

$\therefore$ the equation of the normal at P is

$$y - b\sin\phi = \frac{a\sin\phi}{b\cos\phi}(x - a\cos\phi).$$

$\dfrac{ax}{\cos\phi} - \dfrac{by}{\sin\phi} = a^2 - b^2$ is the normal to the ellipse $x = a\cos\theta$, $y = b\sin\theta$ at $P(a\cos\phi, b\sin\phi)$.

Equations of tangents and normals to the hyperbola

Cartesian form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

By implicit differentiation

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$\therefore$ the gradient of the tangent at $P(x_1, y_1)$ is $\frac{b^2}{a^2}\frac{x_1}{y_1}$

$\therefore$ the equation of the tangent at P is

$$y - y_1 = \frac{b^2}{a^2}\frac{x_1}{y_1}(x - x_1)$$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^{\,2}}{a^2} - \frac{y_1^{\,2}}{b^2} = 1$$

> $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ is the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$.

Also the gradient of the normal at P is $\frac{-a^2}{b^2}\frac{y_1}{x_1}$,

$\therefore$ the equation of the normal at P is

$$y - y_1 = -\frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1).$$

> $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$ is the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$.

Parametric form

$$x = a\sec\theta \Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$$y = b\tan\theta \Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{b}{a}\frac{\sec\theta}{\tan\theta}$$

$\therefore$ the gradient of the tangent at $P(a\sec\phi, b\tan\phi)$ is $\frac{b}{a}\frac{\sec\phi}{\tan\phi}$,

$\therefore$ the equation of the tangent at P is

$$y - b\tan\phi = \frac{b\sec\phi}{a\tan\phi}(x - a\sec\phi),$$

$$\frac{x\sec\phi}{a} - \frac{y\tan\phi}{b} = \sec^2\phi - \tan^2\phi = 1$$

> $\frac{x\sec\phi}{a} - \frac{y\tan\phi}{b} = 1$ is the tangent to the hyperbola $x = a\sec\phi$, $y = b\tan\phi$ at $P(a\sec\phi, b\tan\phi)$.

Also the gradient of the normal at P is $\frac{-a\tan\phi}{b\sec\phi}$

$\therefore$ the equation of the normal at P is

$$y - b\tan\phi = -\frac{a\tan\phi}{b\sec\phi}(x - a\sec\phi).$$

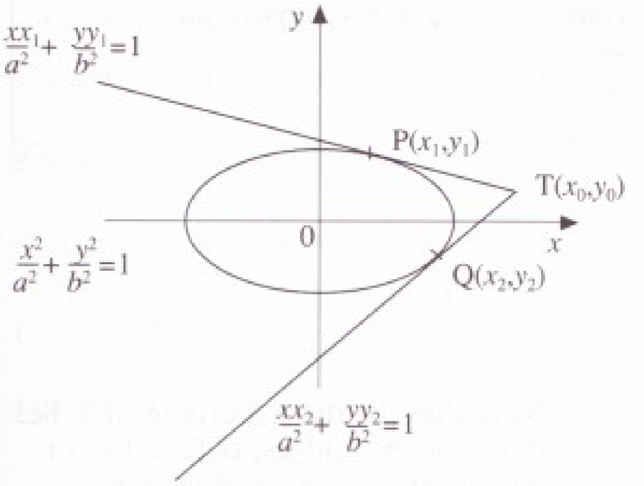
> $\frac{ax}{\sec\phi} + \frac{by}{\tan\phi} = a^2 + b^2$ is the normal to the hyperbola $x = a\sec\theta$, $y = b\tan\theta$ at $P(a\sec\phi, b\tan\phi)$.

Each of the parametric and Cartesian forms can be obtained from the other by using the substitutions $x_1 = a\cos\phi$, $y_1 = b\sin\phi$ for the ellipse, and $x_1 = a\sec\phi$, $y_1 = b\tan\phi$ for the hyperbola.

Chord of contact for the ellipse and hyperbola

From any point T external to the ellipse or hyperbola, two tangents can be drawn to the curve, meeting the conic in P and Q. PQ is called the chord of contact of tangents from T.

Figure 3.18



For the ellipse in Figure 3.18, T(x_0, y_0) lies on both tangents, TP and TQ.

$\therefore \dfrac{x_0x_1}{a^2} + \dfrac{y_0y_1}{b^2} = 1$ and

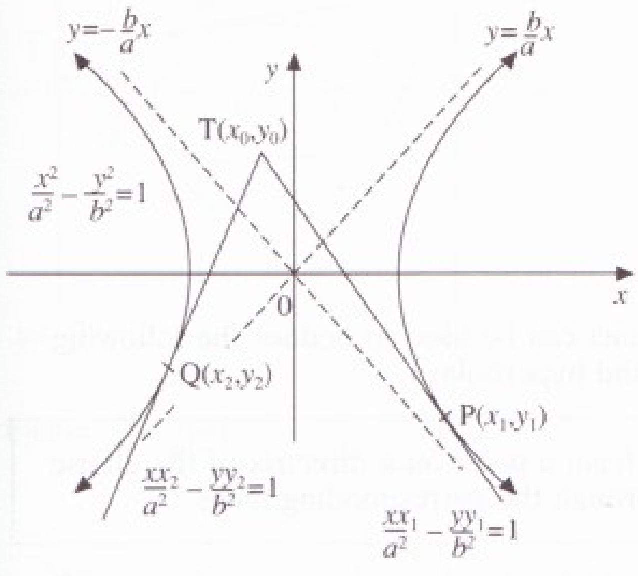
$\dfrac{x_0x_2}{a^2} + \dfrac{y_0y_2}{b^2} = 1.$

Hence both P(x_1, y_1) and Q(x_2, y_2) satisfy

$\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} = 1.$

But this is the equation of a straight line and is thus the equation of the chord PQ.

Figure 3.19

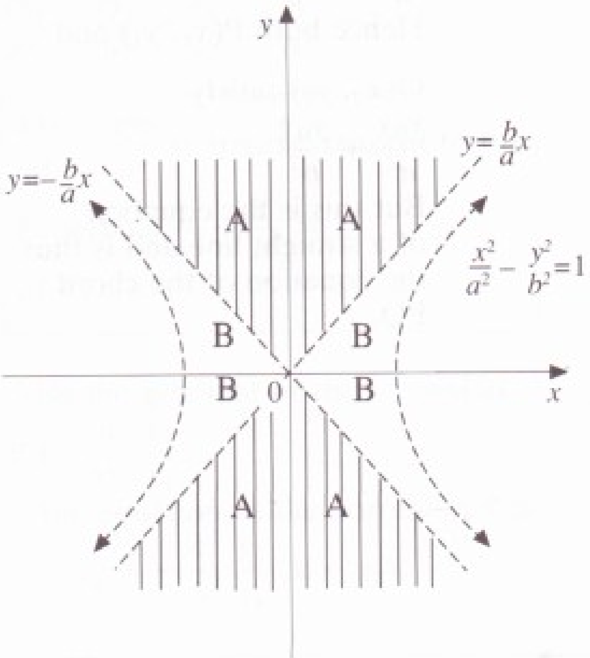


Similarly $\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = 1$ is the equation of the chord PQ of the hyperbola in Figure 3.19.

The chord of contact of tangents from $T(x_0, y_0)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has equation $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$.

The chord of contact of tangents from $T(x_0, y_0)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has equation $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$.

Figure 3.20



Note that for the hyperbola, if T lies in region A, endpoints P and Q of the chord of contact of tangents from T lie on different branches of the hyperbola. If T lies in region B, then P and Q lie on the same branch.

The equation of the chord of contact can be used to deduce the following geometric properties of the ellipse and hyperbola:

The chord of contact of tangents from a point on a directrix of the ellipse (or hyperbola) is a focal chord through the corresponding focus,

and conversely:

The tangents at the endpoints of a focal chord of the ellipse (or hyperbola) meet on the corresponding directrix.

Proof

Consider the ellipse in Figure 3.18. If $T(x_0, y_0)$ lies on the directrix $x = \frac{a}{e}$, then $x_0 = \frac{a}{e}$ and the chord of contact PQ has equation $\frac{x}{ae} + \frac{yy_0}{b^2} = 1$. But $S(ae, 0)$ satisfies this equation and hence PQ is a focal chord through S. Similarly, if T lies on $x = -\frac{a}{e}$, then PQ is a focal chord through $S'(-ae, 0)$.

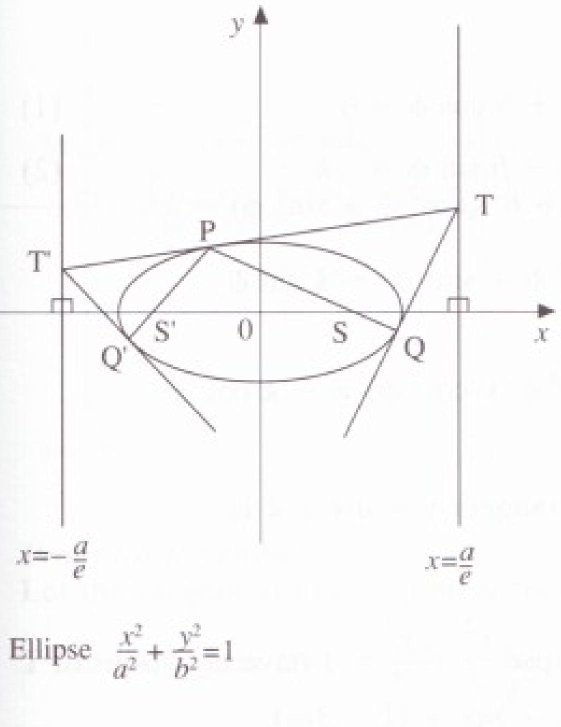
Conversely, if PQ is a focal chord with tangents at P and Q meeting in $T(x_0, y_0)$, then PQ has equation $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$. Hence

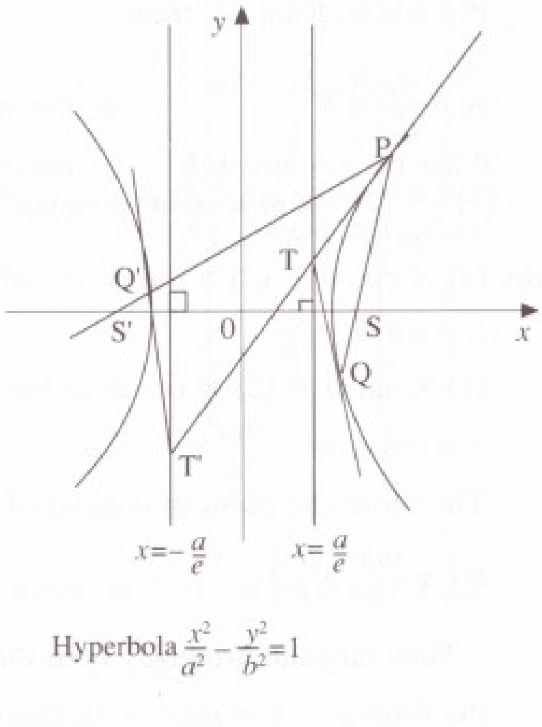
if $S(ae, 0)$ lies on PQ, then $x_0 = \frac{a}{e}$ and T lies on the directrix $x = \frac{a}{e}$;

if $S(-ae, 0)$ lies on PQ, then $x_0 = -\frac{a}{e}$ and T lies on the directrix $x = -\frac{a}{e}$.

The proof for the hyperbola is similar. These results are illustrated in Figure 3.21.

Figure 3.21





While problems involving the chord of contact of tangents from a point are most easily investigated using Cartesian methods, many other problems can be solved by using either a Cartesian or a parametric approach.

Example 5

(a) Show that if $y = mx + k$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $m^2a^2 + b^2 = k^2$.

(b) Find the equations and points of contact of tangents from (3, 1) to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Solution

The standard Cartesian method solves $y = mx + k$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

simultaneously by substitution for y, forming a quadratic equation in x. If the line is a tangent to the ellipse, the discriminant of this quadratic equation is zero, giving $m^2a^2 + b^2 = k^2$. However, parametric methods provide an easier solution.

(a) The ellipse has parametric equations $x = a\cos\theta$ and $y = b\sin\theta$. Hence $\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$. If $y = mx + k$ is a tangent to the ellipse at P$(a\cos\phi, b\sin\phi)$, then

$m = \frac{dy}{dx}$ at P $\quad \Rightarrow ma\sin\phi + b\cos\phi = 0 \qquad (1)$

P lies on $y = mx + k \quad \Rightarrow ma\cos\phi - b\sin\phi = -k \qquad (2)$

$(1)^2 + (2)^2 \Rightarrow m^2a^2(\sin^2\phi + \cos^2\phi) + b^2(\cos^2\phi + \sin^2\phi) = k^2$

$\therefore m^2a^2 + b^2 = k^2$.

(b) $(1) \times \cos\phi - (2) \times \sin\phi \Rightarrow \quad b(\cos^2\phi + \sin^2\phi) = k\sin\phi$

$\therefore b\sin\phi = \frac{b^2}{k}$

$(1) \times \sin\phi + (2) \times \cos\phi \Rightarrow ma(\sin^2\phi + \cos^2\phi) = -k\cos\phi$

$\therefore a\cos\phi = -\frac{ma^2}{k}$.

Therefore the point of contact of the tangent $y = mx + k$ is P$\left(-\frac{ma^2}{k}, \frac{b^2}{k}\right)$.

Now tangents from (3, 1) to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ have equations of the form $y - 1 = m(x - 3)$, that is, $y = mx + (1 - 3m)$.

$$\begin{aligned} \text{Hence } m^2a^2 + b^2 = k^2 \Rightarrow 4m^2 + 9 &= (1 - 3m)^2 \\ 5m^2 - 6m - 8 &= 0 \\ (5m + 4)(m - 2) &= 0 \end{aligned}$$

$\therefore m = -\frac{4}{5}$, $k = 1 - 3m = \frac{17}{5}$ and P$\left(\frac{-ma^2}{k}, \frac{b^2}{k}\right) \equiv$ P$\left(\frac{-16}{17}, \frac{45}{17}\right)$,

or $m = 2$, $k = 1 - 3m = -5$ and P$\left(\frac{-ma^2}{k}, \frac{b^2}{k}\right) \equiv$ P$\left(\frac{8}{5}, \frac{-9}{5}\right)$, Hence the tangents from (3, 1) to $\frac{x^2}{4} + \frac{y^2}{9} = 1$ are

$y = -\frac{4}{5}x + \frac{17}{5}$, with point of contact P $\left(\frac{-16}{17}, \frac{45}{17}\right)$ and

$y = 2x - 5$, with point of contact P $\left(\frac{8}{5}, -\frac{9}{5}\right)$.

Further geometric properties of the ellipse and hyperbola

> The segment of the tangent to an ellipse or hyperbola between the point of contact and the directrix subtends a right angle at the corresponding focus.

Figure 3.22

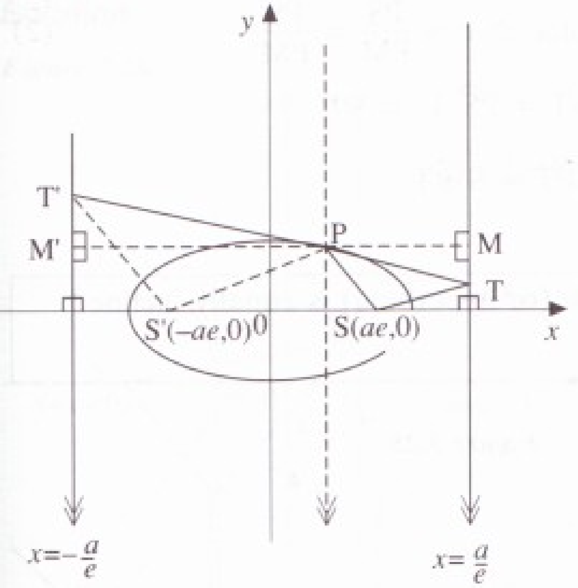
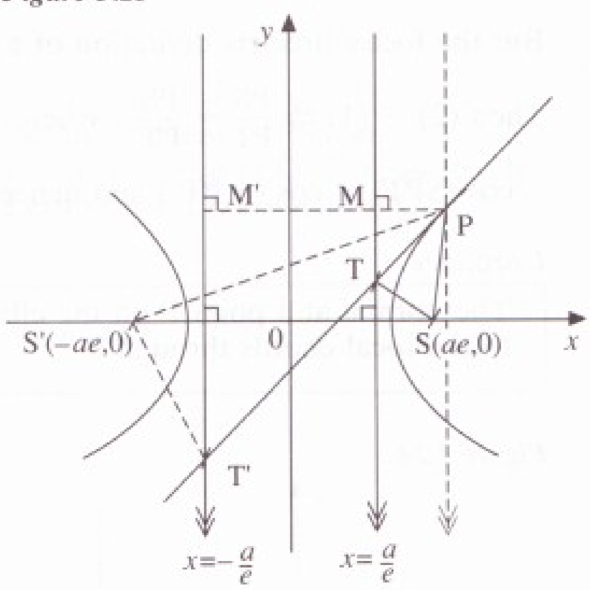


Figure 3.23



Proof for the ellipse

Let the tangent at $P(x_1, y_1)$ meet the directrices at T, T′ as in Figure 3.22.

T lies on tangent PT $\Rightarrow \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ (1)

T lies on directrix m $\Rightarrow x = \dfrac{a}{e}$ (2)

$\therefore$ T has coordinates $\left(\dfrac{a}{e}, \dfrac{b^2(ae - x_1)}{aey_1}\right)$, and

$$\text{gradient PS.gradient ST} = \frac{y_1}{x_1 - ae} \cdot \frac{b^2(ae - x_1)}{aey_1\left(\dfrac{a}{e} - ae\right)} = -\frac{b^2}{a^2(1 - e^2)}.$$

Hence $b^2 = a^2(1 - e^2) \Rightarrow$ gradient PS.gradient ST $= -1$ $\therefore$ PS $\perp$ ST

Similarly, replacing e by $-e$, PS$'$ $\perp$ S$'$T$'$. Hence $P\widehat{S}T = P\widehat{S'}T' = 90°$.

The proof for the hyperbola is similar, with b^2 replaced by $-b^2$ and $b^2 = a^2(e^2 - 1)$ used to simplify the product of the gradients of PS and ST. These results can be used to prove the reflection property for the ellipse or hyperbola.

> The tangent at a point P on the ellipse (or hyperbola) is equally inclined to the focal chords through P.

Proof (for either ellipse or hyperbola)
Let the feet of the perpendiculars from P to the directrices m, m' be M, M$'$ respectively, as in Figures 3.22 and 3.23. Construct the line through P parallel to m and m'. Then the intercepts made on the transversals TT$'$ and MM$'$ by this set of three parallel lines are in proportion,

$$\therefore \frac{PT}{PT'} = \frac{PM}{PM'} \text{ and hence } \frac{PT}{PM} = \frac{PT'}{PM'} \tag{1}$$

But the focus/directrix definition of a conic $\Rightarrow e = \dfrac{PS}{PM} = \dfrac{PS'}{PM'}$ (2)

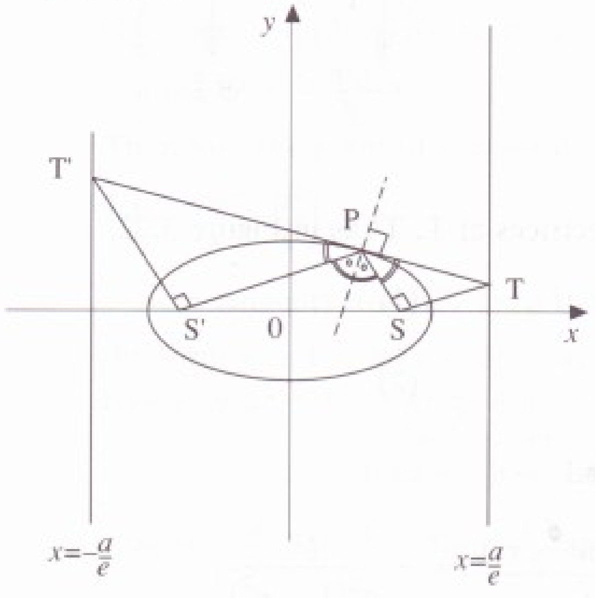
Then (2) $\div$ (1) $\Rightarrow \dfrac{PS}{PT} = \dfrac{PS'}{PT'}$, where $P\widehat{S}T = P\widehat{S'}T' = 90°$

$\therefore \cos(S\widehat{P}T) = \cos(S'\widehat{P}T')$ and hence $S\widehat{P}T = S'\widehat{P}T'$.

Corollary

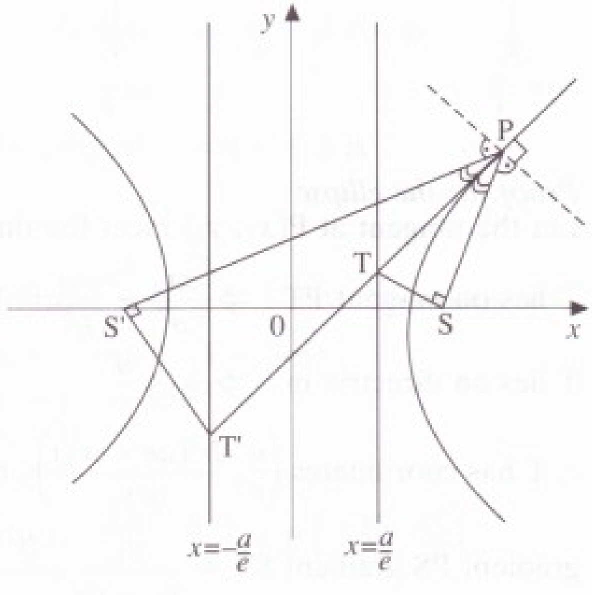
> The normal at a point P on the ellipse (or hyperbola) is equally inclined to the focal chords through P.

Figure 3.24



Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Figure 3.25



Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

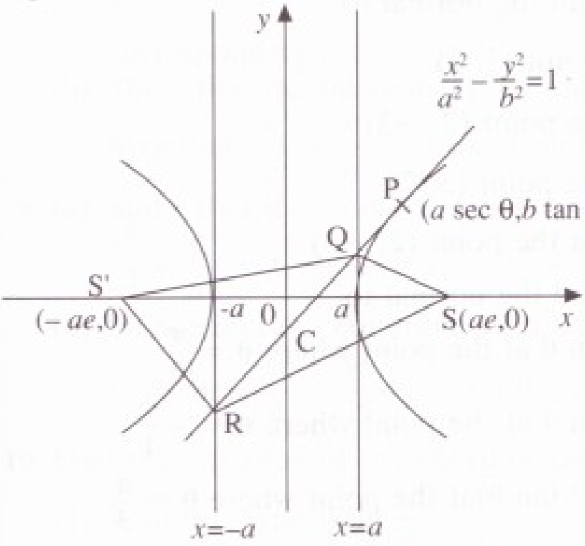
These geometric properties are illustrated in Figures 3.24 and 3.25. We can further deduce that the sets of points P, S, T, M and P, S′, T′, M′, are each concyclic, with PT and PT′ respectively being the diameters of the circles and the normal at P being a common tangent.

Example 6

P(a sec θ, b tan θ) lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at P meets the tangents at the ends of the major axis at Q and R. Show that QR subtends a right angle at either focus. Deduce that if P is the point $\left(5, \frac{4}{3}\right)$ on the hyperbola $\frac{x^2}{9} - y^2 = 1$, with foci S and S′, then Q, R, S, S′ are concyclic, and find the equation of the circle through these points.

Solution

Figure 3.26



Let the tangent at P meet $x = a$, $x = -a$ in Q, R respectively. Let QR meet the y-axis in C.

Tangent PR has equation $\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$. Hence Q has coordinates $\left[a, \frac{b}{\tan \theta}(\sec \theta - 1)\right]$ and R has coordinates $\left[-a, -\frac{b}{\tan \theta}(\sec \theta + 1)\right]$.

$$\text{Gradient } QS.\text{gradient RS} = \frac{b(\sec \theta - 1)}{a \tan \theta \,(1 - e)} \cdot \frac{-b(\sec \theta + 1)}{-a \tan \theta \,(1 + e)}$$
$$= \frac{b^2}{a^2 (1 - e^2)} \cdot \frac{(\sec^2 \theta - 1)}{\tan^2 \theta}.$$

Then $b^2 = a^2(e^2 - 1) \Rightarrow$ gradient QS . gradient RS $= -1$ $\therefore$ QS $\perp$ RS. Similarly, replacing e by $-e$, QS$'$ $\perp$ RS$'$. Hence QR subtends angles of 90° at each of S and S$'$, and Q, S, R, S$'$ are concylic, with QR the diameter of the circle through the points. The y-axis is the perpendicular bisector of the chord SS$'$, hence the centre of this circle is the point C where the diameter QR meets the y-axis.

If $P\left(5, \frac{4}{3}\right)$ lies on the hyperbola $\frac{x^2}{9} - y^2 = 1$, then QR has equation $\frac{5x}{9} - \frac{4y}{3} = 1$ and meets the y-axis in $C\left(0, -\frac{3}{4}\right)$. Also $b^2 = a^2(e^2 - 1)$ gives $e^2 = \frac{10}{9}$, and S has coordinates $(\sqrt{10}, 0)$. Hence $CS^2 = \frac{169}{16}$ and the circle through Q, R, S, S$'$ has equation $x^2 + \left(y + \frac{3}{4}\right)^2 = \frac{169}{16}$.

Exercise 3.3

1 Find the equations of the tangent and the normal to

(a) the ellipse $\frac{x^2}{15} + \frac{y^2}{10} = 1$ at the point $(3, 2)$

(b) the ellipse $3x^2 + 4y^2 = 48$ at the point $(2, -3)$

(c) the hyperbola $\frac{x^2}{6} - \frac{y^2}{8} = 1$ at the point $(3, 2)$

(d) the hyperbola $9x^2 - 2y^2 = 18$ at the point $(2, -3)$

2 Find the equations of the tangent and the normal to

(a) the ellipse $x = 6\cos\theta$, $y = 2\sin\theta$ at the point where $\theta = \frac{\pi}{6}$

(b) the ellipse $x = 4\cos\theta$, $y = 2\sin\theta$ at the point where $\theta = -\frac{\pi}{4}$

(c) the hyperbola $x = 2\sec\theta$, $y = 3\tan\theta$ at the point where $\theta = \frac{\pi}{3}$

(d) the hyperbola $x = 2\sec\theta$, $y = 4\tan\theta$ at the point where $\theta = -\frac{\pi}{4}$

3 Find the equation of the chord of contact of tangents from the point

(a) $(5, 4)$ to the ellipse $\frac{x^2}{15} + \frac{y^2}{10} = 1$

(b) $(6, 4)$ to the ellipse $3x^2 + 4y^2 = 48$

(c) $(1, 2)$ to the hyperbola $\frac{x^2}{6} - \frac{y^2}{8} = 1$

(d) $(1, 2)$ to the hyperbola $9x^2 - 2y^2 = 18$

4 P($a\cos\theta$, $b\sin\theta$) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The normal at P cuts the x-axis at X and the y-axis at Y. Show that $\frac{\text{PX}}{\text{PY}} = \frac{b^2}{a^2}$.

5 P($a\sec\theta$, $b\tan\theta$) lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that $\frac{\text{PX}}{\text{PY}} = \sin^2\theta$ and deduce that if P is an extremity of a latus rectum, then $\frac{\text{PX}}{\text{PY}} = \frac{e^2 - 1}{e^2}$.

6 S is a focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at $(a, 0)$ meets an asymptote at the point T. Show that OT = OS.

7 P($a\sec\theta$, $b\tan\theta$) lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at P meets the asymptotes at the points M and N. Show that PM = PN.

8 **(a)** Show that the chord of contact of the tangents from a point on a directrix of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a focal chord through the corresponding focus.

(b) Show that the tangents at the endpoints of a focal chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet on the corresponding directrix.

9 **(a)** Show that if $y = mx + k$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $m^2a^2 - b^2 = k^2$

(b) Hence find the equations of the tangents from the point (1, 3) to the hyperbola $\frac{x^2}{4} - \frac{y^2}{15} = 1$ and the coordinates of their points of contact.

10 Find the equation of the chord of contact of the tangents from the point (1, 3) to the hyperbola $\frac{x^2}{4} - \frac{y^2}{15} = 1$. Hence find the coordinates of their points of contact and the equations of these tangents.

11 P($a\cos\theta$, $b\sin\theta$) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The tangent at P meets the tangents at the ends of the major axis at Q and R. Show that QR subtends a right angle at either focus. Deduce that if P is the point $(1, \frac{2}{3}\sqrt{2})$ on the ellipse $\frac{x^2}{9} + y^2 = 1$ with foci S and S′, then Q, R, S, S′ are concyclic, and find the equation of the circle through these points.

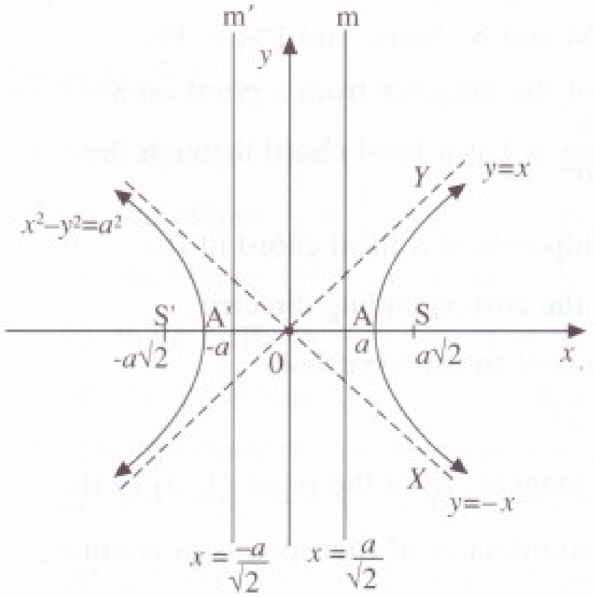
3.4 The rectangular hyperbola

A hyperbola is called rectangular if its asymptotes meet at right angles. The asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ have equations $y = \pm \frac{b}{a}x$. Hence asymptotes perpendicular $\Rightarrow \left(\frac{b}{a}\right).\left(-\frac{b}{a}\right) = -1 \Rightarrow a = b$.

Then $b^2 = a^2(e^2 - 1) \Rightarrow e = \sqrt{\left(1 + \frac{b^2}{a^2}\right)} \therefore e = \sqrt{2}$.

Hence a rectangular hyperbola, with major axis along the x-axis, has equation $x^2 - y^2 = a^2$, eccentricity $e = \sqrt{2}$ and asymptotes $y = \pm x$.

Figure 3.27



If we rotate the coordinate axes clockwise through $\frac{\pi}{4}$, the new X- and Y-axes become the asymptotes of the hyperbola. The original x- and y-axes have equations $Y = X$ and $Y = -X$ respectively, relative to the new axes.

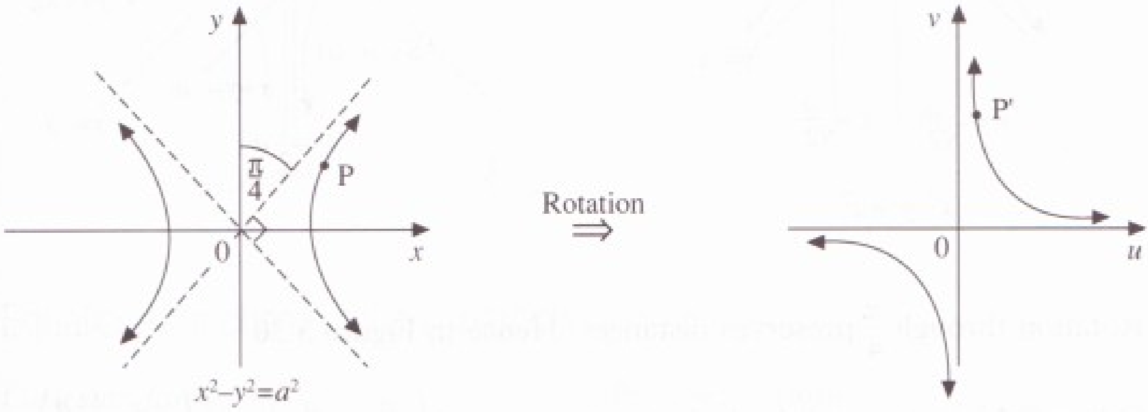
We recognise that relative to the new $X-Y$ axes, the hyperbola has equation $XY = c^2$ for some constant c, $c > 0$. Then at A and A′ $Y = X$ and $XY = c^2 \Rightarrow$ A, A′ have coordinates (c, c), $(-c, -c)$.

Then $\text{OA} = c\sqrt{2}$ and $\text{OA} = a \Rightarrow c^2 = \frac{1}{2}a^2$.

Hence the hyperbola has equation $XY = \frac{1}{2}a^2$ relative to the new $X-Y$ axes.

We can prove this result more rigorously by recalling some results obtained in Section 2.2 of Chapter 2 on complex numbers. We can consider points P on the hyperbola as points in an Argand diagram, representing complex numbers $z = x + iy$, where $x^2 - y^2 = a^2$. Rotation of the axes clockwise through $\frac{\pi}{4}$ is equivalent to the rotation of points in the plane (and hence the hyperbola) anticlockwise through $\frac{\pi}{4}$. This increases the argument of z by $\frac{\pi}{4}$ and is equivalent to multiplication of z by $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$. Hence if P becomes P′ after the rotation, P′ represents the complex number $\frac{1}{\sqrt{2}}(1+i)z$.

Figure 3.28



P(x, y) represents $z = x + iy$, and P′(u, v) represents $\frac{1}{\sqrt{2}}(1+i)z$.

$$u + iv = \frac{1}{\sqrt{2}}(1+i)(x+iy) = \left(\frac{x-y}{\sqrt{2}}\right) + i\left(\frac{x+y}{\sqrt{2}}\right).$$

Then $\left.\begin{aligned} u &= \frac{x-y}{\sqrt{2}} \\ v &= \frac{x+y}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \qquad \begin{aligned} uv &= \frac{x^2-y^2}{2} \\ &= \frac{1}{2}a^2 \quad (\text{P on } x^2 - y^2 = a^2). \end{aligned}$

P′ lies on the curve $uv = \frac{1}{2}a^2$.

Hence the rectangular hyperbola with major axis of length $2a$ has equation $x^2 - y^2 = a^2$, if we choose the x-axis along the major axis with the origin at the point of intersection of the asymptotes (Figure 3.29), and equation $xy = \frac{1}{2}a^2$, if we choose the asymptotes as the x- and y-axes with the curve in the first and third quadrants (Figure 3.30).

Figure 3.29

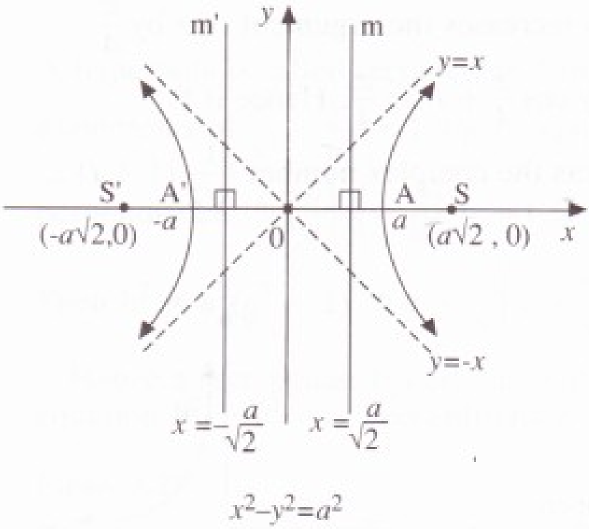
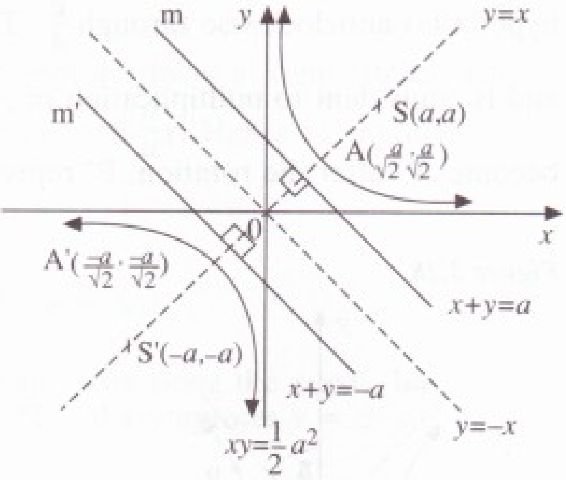


Figure 3.30



Rotation through $\frac{\pi}{4}$ preserves distances. Hence in Figure 3.30

$OA = OA' = a$ and A, A′ lie on $y = x \Rightarrow A\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right), A'\left(\frac{-a}{\sqrt{2}}, \frac{-a}{\sqrt{2}}\right)$

$OS = OS' = a\sqrt{2}$ and S, S′ lie on $y = x \Rightarrow S(a, a), S'(-a, -a)$.
Directrices m, m' parallel to $y = -x \Rightarrow$ each has equation of the form $x + y + k = 0$, k constant.

Distance from 0 to m, m' is $\frac{a}{\sqrt{2}} \Rightarrow \frac{|0 + 0 + k|}{\sqrt{2}} = \frac{a}{\sqrt{2}} \Rightarrow k = \pm a$,

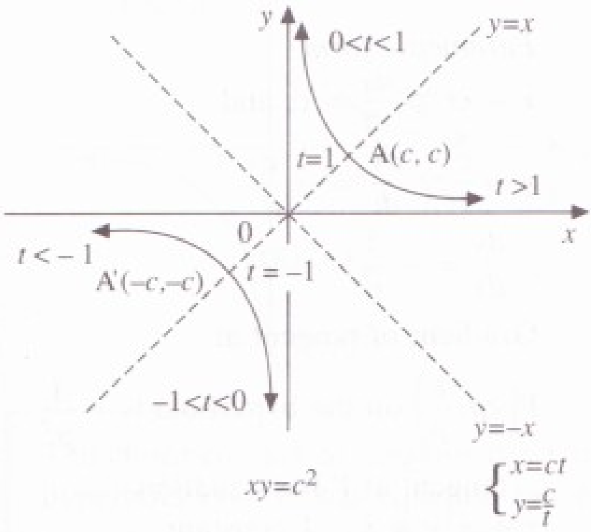
$\therefore$ directrices have equations $x + y = \pm a$.

The hyperbola $xy = c^2$ thus has eccentricity $e = \sqrt{2}$, foci $S(c\sqrt{2}, c\sqrt{2})$ and $S'(-c\sqrt{2}, -c\sqrt{2})$, directrices $x + y = \pm c\sqrt{2}$ and endpoints of the major axis $A(c, c)$ and $A'(-c, -c)$.

Parametric equations of the rectangular hyperbola

The standard parametric form of the rectangular hyperbola $xy = c^2$ is $x = ct$ and $y = \frac{c}{t}$, where the value of the parameter t depends on the position of $P\left(ct, \frac{c}{t}\right)$ on the curve as shown in Figure 3.31.

Figure 3.31



Equation of a chord of the hyperbola $xy = c^2$

Cartesian form

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ lie on $xy = c^2$.
The gradient of PQ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{x_2 - x_1}\left(\frac{c^2}{x_2} - \frac{c^2}{x_1}\right)$$
$$= \frac{-c^2}{x_1 x_2}$$

$\therefore$ chord PQ has equation
$c^2 x + x_1 x_2\, y = k$, $\quad k$ constant.

P lies on PQ and $x_1 y_1 = c^2$
$\therefore k = c^2(x_1 + x_2)$

> The chord from $P(x_1, y_1)$ to $Q(x_2, y_2)$ on $xy = c^2$ has equation
> $c^2 x + x_1 x_2 y = c^2(x_1 + x_2)$.

Parametric form

Let $P\left(cp, \frac{c}{p}\right)$, $Q\left(cq, \frac{c}{q}\right)$ lie on $xy = c^2$. The gradient of PQ is

$$\frac{c\left(\frac{1}{q} - \frac{1}{p}\right)}{c(q - p)} = -\frac{1}{pq}$$

$\therefore$ chord PQ has equation
$x + pqy = k$, $\quad k$ constant.

$P\left(cp, \frac{c}{p}\right)$ lies on PQ
$\therefore k = c(p + q)$

> The chord from $P\left(cp, \frac{c}{p}\right)$ to $Q\left(cq, \frac{c}{q}\right)$ on $xy = c^2$ has equation
> $x + pqy = c(p + q)$.

Equations of tangents and normals to the hyperbola $xy = c^2$

Cartesian form

By implicit differentiation,

$$xy = c^2 \Rightarrow y + x\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}$$

Gradient of tangent at $P(x_1, y_1)$ on the hyperbola is $-\frac{y_1}{x_1}$

$\therefore$ tangent at P has equation

$y_1 x + x_1 y = k$, k constant.

But P lies on tangent

$\therefore k = 2x_1 y_1 = 2c^2$.

> The tangent to $xy = c^2$ at $P(x_1, y_1)$ has equation $xy_1 + yx_1 = 2c^2$.

Also the normal at $P(x_1, y_1)$ has gradient $\frac{x_1}{y_1}$ and equation

$$y - y_1 = \frac{x_1}{y_1}(x - x_1).$$

The normal to $xy = c^2$ at $P(x_1, y_1)$ has equation $xx_1 - yy_1 = x_1^2 - y_1^2$.

> The normal to $xy = c^2$ at $P(x_1, y_1)$ has equation $xx_1 - yy_1 = x_1^2 - y_1^2$.

Parametric form

$$x = ct \Rightarrow \frac{dx}{dt} = c, \text{ and}$$

$$y = \frac{c}{t} \Rightarrow \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{t^2}.$$

Gradient of tangent at $P\left(cp, \frac{c}{p}\right)$ on the hyperbola is $-\frac{1}{p^2}$

$\therefore$ tangent at P has equation

$x + p^2 y = k$, k constant.

But P lies on tangent

$\therefore k = 2cp$.

> The tangent to $xy = c^2$ at $P\left(cp, \frac{c}{p}\right)$ has equation $x + p^2 y = 2cp$.

Also the normal at $P\left(cp, \frac{c}{p}\right)$ has gradient p^2 and equation

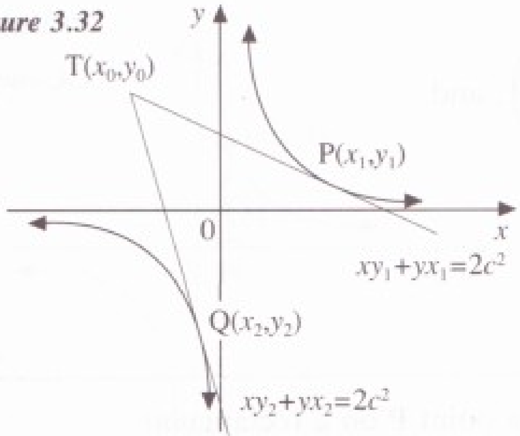
$$y - \frac{c}{p} = p^2(x - cp).$$

> The normal to $xy = c^2$ at $P\left(cp, \frac{c}{p}\right)$ has equation $px - \frac{1}{p}y = c\left(p^2 - \frac{1}{p^2}\right)$.

Equation of the chord of contact of tangents from an external point

Let tangents from an external point $T(x_0, y_0)$ meet the hyperbola $xy = c^2$ in $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Figure 3.32



T lies on tangents PT and QT,
$\therefore x_0 y_1 + y_0 x_1 = 2c^2$ and
$x_0 y_2 + y_0 x_2 = 2c^2$.
Hence $P(x_1, y_1)$ and $Q(x_2, y_2)$ both satisfy $x_0 y + y_0 x = 2c^2$, and hence this linear equation must be the equation of PQ.

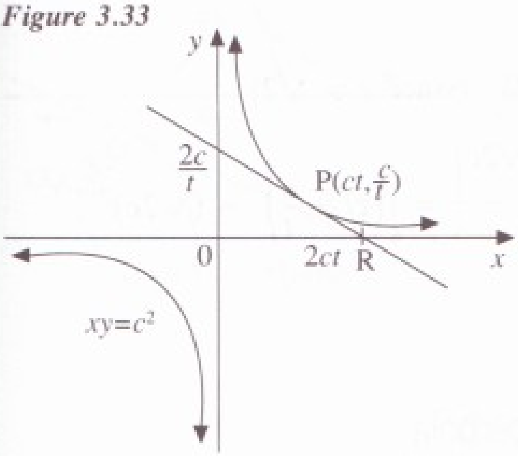
> The chord contact of tangents from an external point $T(x_0, y_0)$ to the hyperbola $xy = c^2$ has equation $xy_0 + yx_0 = 2c^2$.

All the geometric properties proved for the general hyperbola are clearly true for the rectangular hyperbola.

Further geometric properties of the rectangular hyperbola

> The area of the triangle bounded by a tangent and the asymptotes is a constant.

Figure 3.33



Proof

Let the tangent at $P\left(ct, \frac{c}{t}\right)$ on $xy = c^2$ meet the x- and y-axes in R and T respectively. The tangent has equation $x + t^2 y = 2ct$.

At T, $x = 0$, $y = \frac{2c}{t}$,

and at R, $y = 0$, $x = 2ct$,

$\therefore$ area $\Delta OTR = \frac{1}{2}.2ct.\frac{2c}{t} = 2c^2$.

> The length of the intercept cut off a tangent by the asymptotes is twice the distance from the point of intersection of the asymptotes to the point of contact of the tangent.

Proof

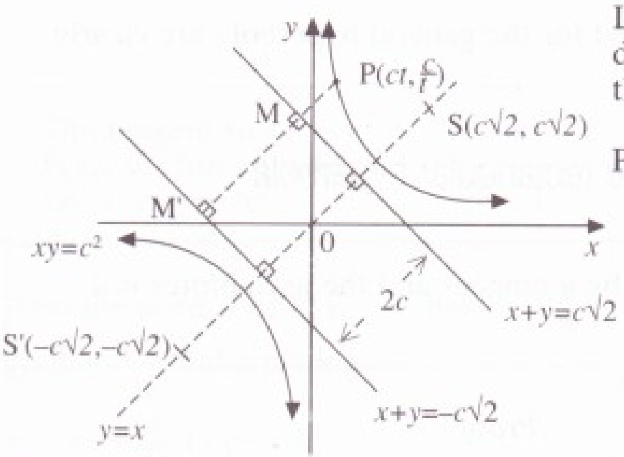
$$\mathrm{TR}^2 = \left(\frac{2c^2}{t}\right) + (2ct)^2 = 4c^2\left(t^2 + \frac{1}{t^2}\right), \text{ and}$$

$$\mathrm{OP}^2 = (ct)^2 + \left(\frac{c}{t}\right)^2 = c^2\left(t^2 + \frac{1}{t^2}\right).$$

$$\mathrm{TR}^2 = 4\mathrm{OP}^2,$$
$$\therefore \mathrm{TR} = 2\mathrm{OP}.$$

> The product of the focal distances of a point P on a rectangular hyperbola is equal to the square of the distance from P to the point of intersection of the asymptotes.

Figure 3.34



Proof

Let $xy = c^2$ have foci S and S′, and directrices m and m'. Let M, M′ be the feet of the perpendiculars from $\mathrm{P}\left(ct, \frac{c}{t}\right)$ to m, m' respectively.

Then $\mathrm{PS.PS'} = e\ \mathrm{PM}.\ e\mathrm{PM'} = 2\ \mathrm{PM.PM'}$ (since $e = \sqrt{2}$),

$$\therefore \mathrm{PS.PS'} = 2.\frac{\left|ct + \frac{c}{t} - \sqrt{2}c\right|}{\sqrt{2}}.\frac{\left|ct + \frac{c}{t} + \sqrt{2}c\right|}{\sqrt{2}} = \left|\left(ct + \frac{c}{t}\right)^2 - (\sqrt{2}c)^2\right|,$$

$$\therefore \mathrm{PS.PS'} = (ct)^2 + \left(\frac{c}{t}\right)^2 = \mathrm{OP}^2.$$

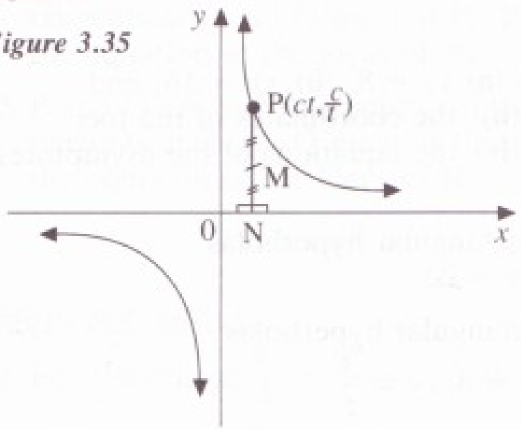
Locus problems and the rectangular hyperbola

Example 7

$\mathrm{P}\left(ct, \frac{c}{t}\right)$ is a point on the rectangular hyperbola $xy = c^2$. N is the foot of the perpendicular from P to the x-axis and M is the midpoint of PN. Find the Cartesian equation of the locus of M as the position of P varies, and describe this locus geometrically.

Solution

Figure 3.35



M has coordinates $\left(ct, \dfrac{c}{2t}\right)$. Hence the locus of M has parametric equations $x = ct$ and $y = \dfrac{c}{2t}$ and Cartesian equation $xy = \dfrac{1}{2}c^2$

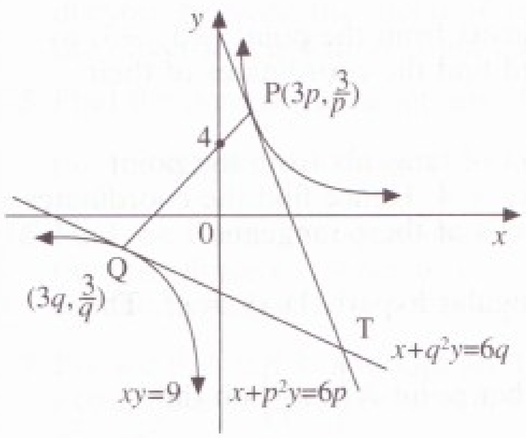
M traces a rectangular hyperbola as the position of P varies.

Example 8

P $\left(3p, \dfrac{3}{p}\right)$ and Q $\left(3q, \dfrac{3}{q}\right)$ are points on different branches of the hyperbola $xy = 9$. Find the coordinates of the point of intersection T of the tangents at P and Q. Find the locus of T if the positions of P and Q vary so that the chord PQ passes through (0, 4).

Solution

Figure 3.36



T lies on the tangents at P and at Q.

At T, $\quad x + p^2y = 6p \qquad (1)$

$\qquad\quad x + q^2y = 6p \qquad (2)$

$(1) - (2) \Rightarrow (p^2 - q^2)y = 6(p - q).$

$p \neq q \therefore y = \dfrac{6}{p + q}$

Then $(1) \Rightarrow x = \dfrac{6pq}{p + q}$

T has coordinates $\left(\dfrac{6pq}{p + q}, \dfrac{6}{p + q}\right)$

If (0, 4) lies on chord PQ, with equation $x + pqy = 3(p + q)$, then $4pq = 3(p + q)$ and T is T $\left(\dfrac{9}{2}, \dfrac{9}{2pq}\right)$.

P, Q on different branches $\Rightarrow pq < 0$,

$\therefore$ locus of T is $x = \dfrac{9}{2}, y < 0$.

In the above example the locus of T could have been found without finding the coordinates of T. Let T have coordinates (x_0, y_0). PQ is the chord of contact of tangents from T to the hyperbola. Hence PQ has equation $xy_0 + yx_0 = 18$. Then (0, 4) lies on PQ $\Rightarrow x_0 = \dfrac{9}{2}$. Considering Figure 3.20, since P and Q lie on different branches, T cannot lie in the first quadrant. Hence the locus of T has equation $x = \dfrac{9}{2}, y < 0$.

Exercise 3.4

1 For each of the rectangular hyperbolas **(a)** $xy = 8$, **(b)** $xy = 16$, find
(i) the eccentricity **(ii)** the coordinates of the foci
(iii) the equations of the directrices **(iv)** the equations of the asymptotes
Sketch each hyperbola.

2 Find the parametric equations of the rectangular hyperbolas
(a) $xy = 4$ **(b)** $xy = 25$

3 Find the Cartesian equations of the rectangular hyperbolas
(a) $x = 4t,\ y = \dfrac{4}{t}$ **(b)** $x = 3t,\ y = \dfrac{3}{t}$

4 Find the equations of the tangent and the normal to the rectangular hyperbolas
(a) $xy = 8$ at the point $(4, 2)$
(b) $xy = 12$ at the point $(-3, -4)$
(c) $x = 2t,\ y = \dfrac{2}{t}$ at the point where $t = 4$
(d) $x = 3t,\ y = \dfrac{3}{t}$ at the point where $t = -1$

5 Find the equation of the chord of contact of tangents from the point
(a) $(2, 1)$ to $xy = 10$ **(b)** $(1, -2)$ to $xy = 6$

6 (a) Show that if $y = mx + k$ is a tangent to the rectangular hyperbola $xy = c^2$, then $k^2 + 4mc^2 = 0$.
(b) Hence find the equations of the tangents from the point $(-1, -3)$ to the rectangular hyperbola $xy = 4$ and find the coordinates of their points of contact.

7 Find the equation of the chord of contact of tangents from the point $(-1, -3)$ to the rectangular hyperbola $xy = 4$. Hence find the coordinates of their points of contact and the equations of these tangents.

8 P$\left(cp, \dfrac{c}{p}\right)$ and Q$\left(cq, \dfrac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. The chord PQ subtends a right angle at another point $R\left(cr, \dfrac{c}{r}\right)$ on the hyperbola. Show that the normal at R is parallel to PQ.

9 P$\left(ct, \dfrac{c}{t}\right)$, where $t \neq 1$, lies on the rectangular hyperbola $xy = c^2$. The tangent and normal at P meet the line $y = x$ at T and N respectively. Show that
(a) OP = PN **(b)** OT.ON $= 4c^2$

10 P$\left(cp, \dfrac{c}{p}\right)$ and Q$\left(cq, \dfrac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. The tangents at P and Q meet at R, and OR cuts PQ at M. Show that M is the midpoint of PQ.

11 P and Q are variable points on the rectangular hyperbola $xy = 9$. The tangents at P and Q meet at R. If PQ passes through the point (6, 2), find the equation of the locus of R.

12 P and Q are variable points on the rectangular hyperbola $xy = c^2$. The tangents at P and Q meet at R. If PQ passes through the point $(a, 0)$, find the equation of the locus of R.

Diagnostic test 3

Subsection

1 For the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$, find (3.1)

(a) the eccentricity **(b)** the coordinates of the foci
(c) the equations of the directrices
Sketch the ellipse.

2 For the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$, find (3.1)

(a) the eccentricity, **(b)** the coordinates of the foci
(c) the equations of the directrices, **(d)** the equations of the asymptotes
Sketch the hyperbola.

3 P lies on the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$ with foci S and S′. If PS = 2, find PS′. (3.1)

4 A hyperbola has centre at the origin and foci on the x-axis. The distance between the foci is 16 units and the distance between the directrices is 4 units. Find the equation of the hyperbola. (3.1)

5 Find the parametric equations of (3.2)

(a) the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **(b)** the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

6 Find the Cartesian equations of (3.2)
(a) the ellipse $x = 4 \cos \theta$, $y = 3 \sin \theta$
(b) the hyperbola $x = 4 \sec \theta$, $y = 5 \tan \theta$

7 $P(a \sec \theta, b \tan \theta)$ and $Q[a \sec(-\theta), b \tan(-\theta)]$ are the extremities of the latus rectum $x = ae$ of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Show that (3.2)

(a) $\sec \theta = e$ **(b)** PQ has length $2\dfrac{b^2}{a}$

8 $P(a \cos \theta, b \sin \theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci $S(ae, 0)$ and $S'(-ae, 0)$. Show that (3.2)
(a) $PS = a(1 - e \cos \theta)$ and $PS' = a(1 + e \cos \theta)$
(b) $PS + PS' = 2a$

9 Find the equations of the tangent and normal to (3.3)

(a) the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ at the point (2, 1)

(b) the ellipse $x = 4 \cos \theta$, $y = 2 \sin \theta$ at the point where $\theta = \frac{\pi}{3}$

(c) the hyperbola $\frac{x^2}{12} - \frac{y^2}{27} = 1$ at the point (4,3)

(d) the hyperbola $x = 3 \sec \theta$, $y = 6 \tan \theta$ at the point where $\theta = \frac{\pi}{6}$

10 Find the equations of the chords of contact of tangents from (3.3)

(a) (4, 3) to the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$

(b) (2, 1) to the hyperbola $\frac{x^2}{12} - \frac{y^2}{27} = 1$

11 P($a \cos \theta$, $b \sin \theta$) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The tangent at P cuts the x-axis at X and the y-axis at Y. (3.3)
Show that $\frac{PX}{PY} = \tan^2 \theta$ and deduce that if P is an extremity of a latus rectum, then $\frac{PX}{PY} = \frac{1 - e^2}{e^2}$.

12 P($a \sec \theta$, $b \tan \theta$) lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The normal at P cuts the x-axis at X and the y-axis at Y. (3.3)
Show that $\frac{PX}{PY} = \frac{b^2}{a^2}$.

13 For the rectangular hyperbola $xy = 18$, find (3.4)

(a) the eccentricity,
(b) the coordinates of the foci,
(c) the equations of the directrices,
(d) the equations of the asymptotes

Sketch the rectangular hyperbola.

14 **(a)** Find the parametric equations of the rectangular hyperbola $xy = 9$. (3.4)

(b) Find the Cartesian equation of the rectangular hyperbola $x = 5t$, $y = \frac{5}{t}$.

15 Find the equations of (3.4)

(a) the tangent and the normal to $xy = 6$ at the point (3, 2)

(b) the tangent and the normal to $x = 4t$, $y = \frac{4}{t}$ at the point where $t = 2$

(c) the chord of contact of tangents from (2, −1) to $xy = 4$

16 $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that (3.4)
(a) PX = PY **(b)** the area of $\triangle$YOX is independent of t

Further questions 3

1 $P(a \cos \theta, b \sin \theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The tangent at P cuts the x-axis at A and the y-axis at B. T and T′ are the feet of the perpendiculars from the foci S and S′ respectively to this tangent. X and Y are the feet of the perpendiculars from P to the x- and y-axis respectively. Show that
(a) $OX.OA = a^2$ **(b)** $OY.OB = b^2$ **(c)** $ST.S'T' = b^2$

2 $P(a \sec \theta, b \tan \theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at P cuts the x-axis at A and the y-axis at B. T and T′ are the feet of the perpendiculars from the foci S and S′ respectively to this tangent. X and Y are the feet of the perpendiculars from P to the x- and y-axis respectively. Show that
(a) $OX.OA = a^2$ **(b)** $OY.OB = b^2$ **(c)** $ST.S'T' = b^2$

3 $P(a \cos \theta, b \sin \theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The normal at P cuts the x-axis at G, and N is the foot of the perpendicular from P to the x-axis. Show that
(a) $OG = e^2\ ON$ **(b)** $SG = e\ SP$, and $S'G = e\ S'P$

4 $P(a \sec \theta, b \tan \theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The normal at P cuts the x-axis at G, and N is the foot of the perpendicular from P to the x-axis. Show that
(a) $OG = e^2\ ON$ **(b)** $SG = e\ SP$ and $S'G = e\ S'P$

5 $P(a \cos \theta, b \sin \theta)$ lies at an extremity of a latus rectum through one focus S of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The tangent at P cuts the y-axis at Q. Show that the normal at P is parallel to QS′, where S′ is the other focus.

6 $P(a \cos \theta, b \sin \theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The tangent at P cuts the tangent at $A(a, 0)$ at R. Show that OR is parallel to A′P, where A′ is the point $(-a, 0)$.

7 Find the equations and the coordinates of the points of contact of the tangents to $x^2 + 2y^2 = 19$ which are parallel to $x + 6y = 5$.

8 Find the equations and the coordinates of the points of contact of the tangents to $2x^2 - 3y^2 = 5$ which are parallel to $8x = 9y$.

9 Find the equations and the coordinates of the points of contact of the tangents to $x^2 - y^2 = 7$ which are parallel to $3y = 4x$.

10 Find the equations and the coordinates of the points of contact of the tangents to $8x^2 + 3y^2 = 35$ from the point $\left(\frac{5}{4}, 5\right)$.

11 Find the equations and the coordinates of the points of contact of the tangents to $x^2 - 9y^2 = 9$ from the point $(3, 2)$.

12 $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the equation of the chord PQ. Hence show that if PQ subtends a right angle at the point $A(a, 0)$ then PQ passes through a fixed point on the x-axis.

13 $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci S and S′. The normal at P meets SS′ at G. Show that $PG^2 = (1 - e^2)\,PS.PS'$.

14 $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent and normal at P cut the y-axis at T and G respectively. Show that the circle on GT as diameter passes through the foci S and S′.

15 $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The line through P perpendicular to the x-axis meets an asymptote at Q and the normal at P meets the x-axis at N. Show that QN is perpendicular to the asymptote.

16 The asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are inclined to each other at an angle α. Show that $\tan\alpha = \frac{2ab}{|a^2 - b^2|}$.

17 $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the rectangular hyperbola $x^2 - y^2 = a^2$ at Q and R. Show that P is the midpoint of QR.

18 $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the hyperbola again at Q. The circle on PQ as diameter meets the hyperbola again at R. Find the coordinates of Q and R.

19 $P(a\sec\theta, a\tan\theta)$ lies on the rectangular hyperbola $x^2 - y^2 = a^2$. A is the point $(a, 0)$. M is the midpoint of AP. Find the equation of the locus of M.

20 $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the hyperbola again at Q. M is the midpoint of PQ. Find the equation of the locus of M.

4 Polynomials

4.1 Factors and zeros of polynomials

A polynomial in x over the field $\mathbb{F}$ (See Appendix 2) has the form

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0, \qquad a_0, \ldots, a_n \in \mathbb{F},$$

for some integer n. The highest power of x with a non-zero coefficient is called the degree of $P(x)$. We write deg P for the degree of $P(x)$. If $a_n \neq 0$, then a_n is called the leading coefficient, while a_nx^n is the leading term. If $a_n = 1$, we describe $P(x)$ as a monic polynomial. We use the symbols $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Q}$ to denote the fields of complex, real and rational numbers respectively, where $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Consider $P(x) = x^4 - 2x^2 - 3$. The coefficients of $P(x)$ are all elements of $\mathbb{Q}$, so we can describe $P(x)$ as a polynomial over $\mathbb{Q}$. $P(x)$ can also be considered as a polynomial over $\mathbb{R}$, or over $\mathbb{C}$. When we consider the factors of $P(x)$, we have to state the field ($\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) from which the coefficients of the polynomial factors may be chosen. The factors of $P(x)$ over $\mathbb{Q}$ are $P(x) = (x^2 - 3)(x^2 + 1)$.

If we allow the factors to have irrational coefficients, then the factors of $P(x)$ over $\mathbb{R}$ are $P(x) = (x - \sqrt{3})(x + \sqrt{3})(x^2 + 1)$.
$P(x)$ can be factored further if we allow non-real coefficients. The factorisation of $P(x)$ over $\mathbb{C}$ is $P(x) = (x - \sqrt{3})(x + \sqrt{3})(x + i)(x - i)$.

The values of x which make $P(x)$ take the value zero are referred to as the zeros of $P(x)$. They are obtained from the linear factors of $P(x)$ over the specified field. They are also the solutions of $P(x) = 0$. We refer to the roots of the polynomial equation $P(x) = 0$, but the zeros of the polynomial function $P(x)$.

Example 1

Find the zeros of $P(x) = x^4 - 2x^2 - 3$
(a) over $\mathbb{Q}$ **(b)** over $\mathbb{R}$ **(c)** over $\mathbb{C}$

Solution

(a) $P(x) = (x^2 - 3)(x^2 + 1)$ and cannot be factored further over $\mathbb{Q}$. We describe these factors as irreducible over $\mathbb{Q}$. $P(x)$ has no linear factors over $\mathbb{Q}$ and $P(x) = 0$ has no solutions in the field of rational numbers. Hence $P(x)$ has no zeros over $\mathbb{Q}$.

(b) Irreducible factors of $P(x)$ over $\mathbb{R}$ are $P(x) = (x - \sqrt{3})(x + \sqrt{3})(x^2 + 1)$. Each linear factor gives rise to a zero of $P(x)$ since $P(\sqrt{3}) = P(-\sqrt{3}) = 0$. Hence the zeros of $P(x)$ over $\mathbb{R}$, are $\sqrt{3}$ and $-\sqrt{3}$.

(c) Factors of $P(x)$ over $\mathbb{C}$ are $P(x) = (x - \sqrt{3})(x + \sqrt{3})(x - i)(x + i)$. Hence the zeros of $P(x)$ over $\mathbb{C}$ are $\pm\sqrt{3}$ and $\pm i$.

The remainder and factor theorems, with which we are familiar for polynomials over $\mathbb{R}$, also apply for polynomials over $\mathbb{C}$. In general:

- When $P(x)$ is divided by $(x - \alpha)$, $\alpha \in \mathbb{C}$, the remainder is $P(\alpha)$.
- $(x - \alpha)$ is a factor of $P(x)$ over $\mathbb{C}$ if $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$.

These results follow from the division transformation. When $P(x)$ is divided by $(x - \alpha)$, the quotient $Q(x)$ is a polynomial over $\mathbb{C}$, while the remainder is a complex number r. Then

$P(x) = (x - \alpha)Q(x) + r \Rightarrow P(\alpha) = r$, and hence $(x - \alpha)$ is a factor of $P(x)$ if $P(\alpha) = 0$.

Example 2

Divide $P(x) = x^2 + x + 3$ by $(x - i)$. Verify that the remainder is $P(i)$.

Solution

$$
\begin{array}{r l}
 & x + (1 + i) \leftarrow \text{quotient} \\
x - i\,\big) & x^2 + x + 3 \\
 & \underline{x^2 - ix} \\
 & (1 + i)\,x + 3 \\
 & \underline{(1 + i)\,x + (1 - i)} \\
 & 2 + i \leftarrow \text{remainder}
\end{array}
$$

$$\therefore \frac{x^2 + x + 3}{x - i} = \{x + (1 + i)\} + \frac{2 + i}{x - i} \quad \text{or}$$

$$x^2 + x + 3 = (x - i)\{x + (1 + i)\} + (2 + i).$$

Also $P(x) = x^2 + x + 3 \Rightarrow P(i) = -1 + i + 3 = 2 + i$.

Multiple zeros of a polynomial

A multiple zero of $P(x)$ is a number α such that $P(x)$ can be expressed in the form $P(x) = (x - \alpha)^2 Q(x)$, for some polynomial $Q(x)$.

If $P(x) = (x - \alpha)^r Q(x)$, where r is a positive integer, and $Q(\alpha) \neq 0$, then $(x - \alpha)^r$ is the greatest power of $(x - \alpha)$ which is a factor of $P(x)$, and we describe α as a zero of multiplicity r of $P(x)$. A zero of multiplicity 1 is called a single zero. We can use the division transformation to derive a procedure for determining the multiplicity of a zero of a polynomial $P(x)$.

> If α is a zero of multiplicity r of $P(x)$, $r > 1$, then α is a zero of multiplicity $r - 1$ of $P'(x)$. Conversely, if $P(x)$ and $P'(x)$ have a common zero α, then α is a multiple zero of $P(x)$.
> Further, if $P(\alpha) = P'(\alpha) = P''(\alpha) = \ldots = P^{(r-1)}(\alpha) = 0$, and $P^{(r)}(\alpha) \neq 0$, then α is a zero of multiplicity r of $P(x)$.

Proof
If $P(x) = (x - \alpha)^r Q(x)$, $Q(x)$ a polynomial such that $Q(\alpha) \neq 0$,
then $P'(x) = r(x - \alpha)^{r-1}Q(x) + (x - \alpha)^r Q'(x) = (x - \alpha)^{r-1} Q_1(x)$,
where $Q_1(x) = rQ(x) + (x - \alpha) Q'(x) \Rightarrow Q_1(\alpha) \neq 0$.
Hence if α is a zero of multiplicity r of $P(x)$, then α is a zero of multiplicity $r - 1$ of $P'(x)$.

Conversely, let $P(x)$ and $P'(x)$ have a common zero α. Then
$P(x) = (x - \alpha)Q(x)$ and $P'(x) = Q(x) + (x - \alpha)Q'(x)$,
for some polynomial $Q(x)$.
But $P'(\alpha) = 0 \Rightarrow Q(\alpha) = 0 \Rightarrow (x - \alpha)$ is a factor of $Q(x)$.
Hence $(x - \alpha)^2$ is a factor of $P(x)$ and α is a multiple zero of $P(x)$.
Using the previous result, the multiplicity of α as a zero of $P(x)$ is one more than the multiplicity of α as a zero of $P'(x)$. Let α be a zero of $P(x)$, such that $P(\alpha) = P'(\alpha) = P''(\alpha) = \ldots = P^{(r-1)}(\alpha) = 0$, and $P^{(r)}(\alpha) \neq 0$.
Then α is a single zero of $P^{(r-1)}(x)$ and a common zero of $P^{(r-1)}(x)$ and $P^{(r-2)}(x)$.
Hence α is a zero of multiplicity 2 of $P^{(r-2)}(x)$.
Continuing in this way, each step backwards along the chain of derivatives increases the multiplicity of the zero α by 1, so that as a zero of $P(x)$, α has multiplicity r.

Example 3

Show that $P(x) = x^4 - 2x^3 + 2x - 1$ has a multiple zero. Find this zero and determine its multiplicity. Factor $P(x)$ over $\mathbb{R}$.

Solution

$$\begin{array}{lll} P(x) = x^4 - 2x^3 + 2x - 1 & \Rightarrow & \left.\begin{array}{l} P(1) = 0 \\ P'(1) = 0 \end{array}\right\} \Rightarrow \text{1 is a multiple zero of } P(x). \\ P'(x) = 4x^3 - 6x^2 + 2 & \Rightarrow & \\ P''(x) = 12x^2 - 12x & \Rightarrow & P''(1) = 0 \\ P^{(3)}(x) = 24x - 12 & \Rightarrow & P^{(3)}(1) \neq 0. \end{array}$$

Hence 1 is a zero of multiplicity 3 of $P(x)$, and $P(x) = (x - 1)^3 (x + k)$ for some constant k, as $P(x)$ is a monic polynomial of degree 4. Then $P(0) = -1 \Rightarrow k = 1$ and $P(x) = (x - 1)^3 (x + 1)$.

Example 4

Show that $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ has no multiple zero for any integer $n > 1$.

Solution

Let $P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$. Then

$$P'(x) = \quad 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!}.$$

$$\therefore P(x) - P'(x) = \frac{x^n}{n!} \qquad (1)$$

Suppose α is a multiple zero of $P(x)$. Then $P(\alpha) = P'(\alpha) = 0$, and $P(\alpha) - P'(\alpha) = 0 \Rightarrow \alpha = 0$, using (1). But $P(0) \neq 0$.
Hence $P(x)$ has no multiple zero.

Factorising polynomials and solving polynomial equations often involve testing for zeros by a process of trial and error, then applying the factor theorem. When $P(x)$ has integer coefficients, the set of possible rational zeros is limited.

Example 5

$P(x)$ is a polynomial with integer coefficients.

(a) Show that if α is an integral zero of $P(x)$, then α is a divisor of the constant term.

(b) Show that if $\beta = \frac{p}{q}$ is a rational zero of $P(x)$, where p and q have no common factor, then p is a divisor of the constant term and q is a divisor of the leading coefficient.

(c) Show that $4x^3 + 2x^2 - 14x + 3$ has exactly one rational zero, and factorise the given polynomial over $\mathbb{Q}$.

Solution

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \qquad a_0, \ldots, a_n$ integral.

(a) $P(\alpha) = \alpha(a_n\alpha^{n-1} + a_{n-1}\alpha^{n-2} + \ldots + a_1) + a_0$

$\therefore P(\alpha) = 0 \Rightarrow a_0 = \alpha(-a_n\,\alpha^{n-1} - a_{n-1}\alpha^{n-2} - \ldots - a_1)$

But α integral $\Rightarrow (-a_n\,\alpha^{n-1} - a_{n-1}\alpha^{n-2} - \ldots - a_1)$ integral.

Hence α is a divisor of a_0.

(b) $P\left(\frac{p}{q}\right) = a_n \frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \ldots + a_1\frac{p}{q} + a_0,$

$$\therefore P\left(\frac{p}{q}\right) = 0 \Rightarrow p(a_n\,p^{n-1} + a_{n-1}\,qp^{n-2} + \ldots + a_1 q^{n-1}) = -a_0\,q^n, \text{ and}$$
$$q(a_{n-1}\,p^{n-1} + a_{n-2}\,qp^{n-2} + \ldots + a_0 q^{n-1}) = -a_n\,p^n$$

But, as p and q are integral, each bracketed expression is an integer.
Also, since p and q have no common factor, neither pair of integers p, q^n nor p^n, q has a common factor.
Hence p is a divisor of a_0 and q is a divisor of a_n.

(c) $P(x) = 4x^3 + 2x^2 - 14x + 3$. All rational zeros of $P(x)$ have the form $\frac{p}{q}$, where p and q are integer divisors of 3 and 4 respectively.
Hence the only possible rational zeros of $P(x)$ are ± 1, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, ± 3, $\pm\frac{3}{2}$, $\pm\frac{3}{4}$. But of these, only $\frac{3}{2}$ satisfies $P(x) = 0$.
Hence $\frac{3}{2}$ is the only rational zero of $P(x)$, and $(2x - 3)$ is a factor of $P(x)$.
By inspection, or by polynomial division,
$4x^3 + 2x^2 - 14x + 3 = (2x - 3)(2x^2 + 4x - 1)$, and these are irreducible factors over $\mathbb{Q}$.

Exercise 4.1

1 Find the zeros of $P(x)$ over **(i)** the rational numbers, **(ii)** the real numbers, **(iii)** the complex numbers, if
(a) $P(x) = x^4 - 5x^2 + 4$ **(b)** $P(x) = x^4 - 3x^2 + 2$
(c) $P(x) = x^4 + 3x^2 - 4$

2 Find the roots of $P(x) = 0$ over **(i)** the rational numbers, **(ii)** the real numbers, **(iii)** the complex numbers, if
(a) $P(x) = x^4 - 5x^2 + 6$ **(b)** $P(x) = x^4 - x^2 - 2$
(c) $P(x) = x^4 + 5x^4 + 4$

3 If $P(x) = x^3 - x^2 + x - 1$
(a) divide $P(x)$ by $x - 1$ and verify that the remainder is $P(1)$
(b) divide $P(x)$ by $x - i$ and verify that the remainder is $P(i)$

4 If $P(x) = x^3 - 3x^2 + 4x - 2$
(a) divide $P(x)$ by $x + 1$ and verify that the remainder is $P(-1)$
(b) divide $P(x)$ by $x + i$ and verify that the remainder is $P(-i)$

5 Express $P(x)$ as a product of irreducible factors over **(i)** $\mathbb{Q}$, **(ii)** $\mathbb{R}$, **(iii)** $\mathbb{C}$, if
(a) $P(x) = x^3 + x^2 - 3x - 3$ **(b)** $P(x) = x^3 - 2x^2 + 4x - 8$

6 Express $P(x)$ as a product of irreducible factors over **(i)** $\mathbb{Q}$, **(ii)** $\mathbb{R}$, **(iii)** $\mathbb{C}$, if
(a) $P(x) = x^4 + 3x^3 - 6x^2 - 6x + 8$
(b) $P(x) = x^4 - x^3 - 5x^2 - x - 6$

7 **(a)** Find $P(x)$, given that $P(x)$ is monic, of degree 3, with 5 as a single zero and -2 as a zero of multiplicity 2.
(b) Find $P(x)$, given that $P(x)$ is monic, of degree 4, with -1 as a single zero and 3 as a zero of multiplicity 3.

8 If $P(x) = x^3 - 3x^2 + 4$, show that $P(x)$ has a multiple zero and find this zero and its multiplicity.

9 If $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$, show that $P(x) = 0$ has a multiple root and find this root and its multiplicity.

10 If $P(x) = 4x^3 + 12x^2 - 15x + 4$ has a double zero, find all the zeros and factorise $P(x)$ over the real numbers.

11 If $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$ has a triple zero, find all the zeros and factor $P(x)$ over the real numbers.

12 If $P(x) = x^3 - 3x^2 - 9x + c$ has a double zero, find c and factorise $P(x)$ over the real numbers.

13 If $P(x) = x^4 + 2x^3 - 12x^2 - 40x + c$ has a triple zero, find c and factorise $P(x)$ over the real numbers.

14 If $ax^3 + bx^2 + d = 0$ has a double root, show that $27a^2d + 4b^3 = 0$.

15 If $ax^3 + cx + d = 0$ has a double root, show that $27ad^2 + 4c^3 = 0$.

16 If $P(x) = 1 - x + \frac{x^2}{2!} - \ldots + (-1)^n \frac{x^n}{n!}$, show that $P(x)$ has no multiple zero for $n \geqslant 2$.

17 Given that $P(x)$ has a rational zero, find this zero and factorise $P(x)$ over the real numbers if
(a) $P(x) = 2x^3 - 3x^2 + 2x - 3$ **(b)** $P(x) = 2x^3 + x^2 - 4x - 2$

18 Given that $P(x) = 4x^4 + 8x^3 + 5x^2 + x - 3$ has two rational zeros, find these zeros and factorise $P(x)$ over the real numbers.

4.2 Factorisation of polynomials and the fundamental theorem of algebra

The division transformation

If $P(x)$ and $D(x)$ are polynomials over a field $\mathbb{F}$, the process of polynomial division of $P(x)$ by $D(x)$ yields a quotient $Q(x)$ and a remainder $R(x)$, which are both polynomials over $\mathbb{F}$, such that $\deg R < \deg D$. Then $P(x) \equiv D(x)Q(x) + R(x)$, identically over $\mathbb{F}$. This identity is said to be a division transformation of $P(x)$.

For example, consider $P(x) = x^4 - x + 2$ and $D(x) = x^2 + 1$.

$$\begin{array}{r|l} & x^2 \quad\; - 1 \\ \hline x^2 + 1 & x^4 - x + 2 \\ & \underline{x^4 + x^2 \qquad\quad} \\ & \quad -x^2 - x + 2 \\ & \quad \underline{-x^2 \qquad - 1} \\ & \qquad\quad -x + 3 \end{array}$$

$$\begin{array}{ccccccc} x^4 - x + 2 \equiv & (x^2 + 1) & & (x^2 - 1) & + & (-x + 3) & (2) \\ \uparrow & \uparrow & & \uparrow & & \uparrow & \\ P(x) \equiv & D(x) & \cdot & Q(x) & + & R(x) & \end{array}$$

$P(x)$ and $D(x)$ are polynomials over $\mathbb{Q}$. Hence the quotient and remainder are also polynomials with rational coefficients, and equation (2) is true for all rational x. However, since $P(x)$ and $D(x)$ can also be considered as polynomials over $\mathbb{C}$, equation (2) is true for all complex numbers x.

Hence there are two important facts to note about the division transformation for polynomials $P(x)$ and $D(x)$:

> The identity $P(x) \equiv D(x).Q(x) + R(x)$ is true for all complex numbers x, where $Q(x)$ and $R(x)$ are polynomials such that deg $R <$ deg D.
> The nature of the coefficients of $Q(x)$ and $R(x)$ is determined by the nature of the coefficients of $P(x)$ and $D(x)$.

Example 6

Find the remainder when $P(x) = x^3 - 2x + 1$ is divided by
(a) $x - i$ **(b)** $x^2 + 1$

Solution

(a) $x - i$ is a linear divisor. Hence we can use the remainder theorem, and the remainder is $P(i) = -i - 2i + 1 = 1 - 3i$.

(b) $P(x) = x^3 - 2x + 1$ and $D(x) = x^2 + 1$ are polynomials over $\mathbb{Q}$.
By the division transformation, $P(x) \equiv (x^2 + 1)Q(x) + R(x)$, where $Q(x)$ and $R(x)$ are polynomials over $\mathbb{Q}$, such that deg $R <$ deg $D = 2$.
Thus $P(x) \equiv (x^2 + 1)Q(x) + ax + b$, a, b rational, (3)
$\therefore P(i) \quad = 0 + ai + b$ (Equation (3) is true for all $x \in \mathbb{C}$)
$\therefore 1 - 3i \quad = ai + b \Rightarrow a = -3$ and $b = 1$ (a, b real).
Hence the remainder $ax + b$ is $-3x + 1$.

The division transformation is useful for avoiding polynomial division, or solving algebraic or theoretical problems involving polynomial division.

Example 7

When $x^4 - kx + 1$ is divided by $x^2 + 1$, the remainder is $3x + 2$. Find k.

Solution

By the division transformation, $x^4 - kx + 1 \equiv (x^2 + 1)\, Q(x) + 3x + 2$.
Substituting $x = i$, $\quad 2 - ki = 0 + 3i + 2 \Rightarrow k = -3$.

The fundamental theorem of algebra

The fundamental theorem of algebra states that every polynomial of degree greater than or equal to one has at least one zero over $\mathbb{C}$. This theorem was first proved by the German mathematician Gauss (1777–1855). Using this theorem we can establish by induction that every polynomial of degree n has exactly n zeros over $\mathbb{C}$ (not necessarily distinct).

Consider the set of statements $\{S(n), n = 1, 2, 3 \ldots\}$ defined by
$S(n)$: Every polynomial of degree n has exactly n zeros over $\mathbb{C}$.

Clearly $S(1)$ is true, as a polynomial of degree 1 has the form
$P(x) = ax + b$, $a \neq 0$, which has exactly one zero, $-\frac{b}{a}$.

If $S(k)$ is true, k a positive integer, consider $S(k + 1)$.
Let $P(x)$ be a polynomial of degree $k + 1$. By the fundamental theorem, there is an $\alpha \in C$ such that $P(\alpha) = 0$.
Then $P(x) \equiv (x - \alpha)\, Q(x)$ for some polynomial $Q(x)$ of degree k.
But if $S(k)$ is true, $Q(x)$ has exactly k zeros over $\mathbb{C}$ and these k zeros of $Q(x)$ are also zeros of $P(x)$.
Then $P(x) \equiv (x - \alpha)Q(x)$ for some polynomial $Q(x)$ of degree k.

Hence if $S(k)$ is true, then $S(k + 1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all positive integers n.

Note that the n zeros of $P(x)$ of degree n over $\mathbb{C}$ need not be distinct and they may be non-real. For example, consider $P(x) = (x - 1)^2\,(x^2 + 1)$. This polynomial has degree 4 but only two real zeros 1 and 1. However, over $\mathbb{C}$, $P(x) = (x - 1)^2(x + i)(x - i)$ with four zeros 1, 1, i and $-i$.

Clearly, since a polynomial $P(x)$ of degree n has exactly n zeros over $\mathbb{C}$, $P(x)$ can be completely factorised into n linear factors over $\mathbb{C}$. If the zeros of $P(x)$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $P(x) = k(x - \alpha_1)\,(x - \alpha_2)\ldots(x - \alpha_n)$, for some $k \in \mathbb{C}$.

Polynomials with real coefficients

Suppose $P(x)$ is a polynomial with real coefficients and a non-real zero α.
Consider the polynomial $D(x) = (x - \alpha)(x - \bar{\alpha})$.
$D(x) = x^2 - (\alpha + \bar{\alpha})\,x + \alpha\bar{\alpha} = x^2 - 2\,\mathrm{Re}(\alpha)x + |\alpha|^2$.
Hence $D(x)$ also has real coefficients and by the division transformation, $P(x) \equiv D(x)Q(x) + R(x)$, where $Q(x)$ and $R(x)$ are polynomials with real coefficients such that $\deg R < \deg D = 2$. Then

$$P(x) \equiv (x - a)(x - \bar{\alpha})Q(x) + cx + d, \qquad c, d \text{ real.}$$

$$P(\alpha) = c\alpha + d \quad \text{and} \quad P(\bar{\alpha}) = c\bar{\alpha} + d = \overline{c\alpha + d} = \overline{P(\alpha)}.$$

$$\therefore P(\alpha) = 0 \Rightarrow P(\bar{\alpha}) = 0.$$

Hence, if $P(x)$ is a polynomial with real coefficients, the non-real zeros of $P(x)$ occur in complex conjugate pairs. If $P(x)$ has odd degree, then $P(x)$ has at least one real zero. The real zeros of $P(x)$ give rise to linear factors over $\mathbb{R}$, while the non-real zeros can be grouped in complex conjugate pairs, giving rise to factors of the form $(x - \alpha)(x - \bar{\alpha})$ over $\mathbb{C}$. But we have seen that $(x - \alpha)(x - \bar{\alpha}) = x^2 - 2\,\mathrm{Re}(\alpha)\,x + |\alpha|^2$ is a quadratic polynomial with real coefficients. Hence every polynomial with real coefficients can be factorised into a product of real linear and real quadratic factors irreducible over $\mathbb{R}$. (Note that the zeros of polynomials with non-real coefficients need not occur in complex conjugate pairs. For example, $x^2 + ix + 2 = (x - i)(x + 2i)$ has zeros i and $-2i$ which are not complex conjugates.)

Example 8

Find all the zeros of $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$ over $\mathbb{C}$, given that $1 + i$ is a zero. Factorise $P(x)$ fully over $\mathbb{R}$.

Solution

$P(x)$ has real coefficients. Hence $P(1 + i) = 0 \Rightarrow P(1 - i) = 0$ and then $[x - (1 + i)][x - (1 - i)] = x^2 - 2x + 2$ is a factor of $P(x)$.

By inspection, $x^4 - x^3 - 2x^2 + 6x - 4 = (x^2 - 2x + 2)(x^2 + x - 2)$

$\therefore P(x) = (x^2 - 2x + 2)(x + 2)(x - 1)$.

This is the factorisation of $P(x)$ into irreducible factors over $\mathbb{R}$, and $P(x)$ has zeros $1 + i$, $1 - i$, -2 and 1 over $\mathbb{C}$.

Example 9

$P(x)$ is a monic polynomial of degree 4 with integer coefficients and constant term 2. $P(x)$ has a zero i, and a rational zero. The sum of the zeros of $P(x)$ is a positive real number. Find $P(x)$ factorised into irreducible factors over $\mathbb{R}$.

Solution

$P(x)$ has real coefficients. Hence $P(i) = 0 \Rightarrow P(-i) = 0$, and then $(x - i)(x + i) = (x^2 + 1)$ is a factor of $P(x)$. The rational zero of $P(x)$ is $\frac{p}{q}$, where q is a divisor of the leading coefficient 1, and p is a divisor of the constant term 2. Hence $P(x)$ has the form $(x^2 + 1)(x - \alpha)(x - \beta)$, where the rational zero α takes one of the values ± 1 or ± 2 (since $P(x)$ is a monic polynomial of degree 4). Given that the constant term is 2, $\alpha\beta = 2$, and hence the zeros of $P(x)$ are i, $-i$, 1 and 2, or i, $-i$, -1 and -2. But the sum of the zeros is positive. Thus $P(x) = (x^2 + 1)(x - 1)(x - 2)$ and these factors are irreducible over $\mathbb{R}$.

Exercise 4.2

1 Find the remainder when $P(x) = x^3 + 2x^2 + 1$ is divided by
(a) $x + i$ **(b)** $x^2 + 1$

2 Find the remainder when $P(x) = x^5 - 3x^4 + 2x - 1$ is divided by
(a) $x - 2i$ **(b)** $x^2 + 4$

3 When $P(x) = x^4 + ax^2 + 2x$ is divided by $x^2 + 1$, the remainder is $2x + 3$. Find the value of a.

4 When $P(x) = x^4 + ax^3 + 3x - 11$ is divided by $x^2 + 4$, the remainder is $x + 5$. Find the value of a.

5 When $P(x) = x^4 + ax^2 + bx + 2$ is divided by $x^2 + 1$, the remainder is $-x + 1$. Find the values of a and b.

6 When $P(x) = x^4 + ax^3 + b$ is divided by $x^2 + 4$, the remainder is $-x + 13$. Find the values of a and b.

7 Sketch the graphs of $x^2 - y^2 = 0$ and $xy = 2$ on the same axes. By observing the number of points of intersection of the graphs, deduce that the equation $z^2 = 4i$ has at least one root (in fact two roots).

8 Show that if the polynomials $P_1(z) = b_n z^n + b_{n-1}z^{n-1} + \ldots + b_0$ and $P_2(z) = c_n z^n + c_{n-1}z^{n-1} + \ldots + c_0$ are equal for more than n values of z, then $b_n = c_n$, $b_{n-1} = c_{n-1}$, $\ldots$, $b_0 = c_0$.

9 Find the zeros of $P(x) = x^4 + x^3 - x^2 + x - 2$ over $\mathbb{C}$, given that i is a zero. Factor $P(x)$ fully over $\mathbb{R}$.

10 Find the zeros of $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$ over $\mathbb{C}$, given that $2 - i$ is a zero. Factor $P(x)$ fully over $\mathbb{R}$.

11 Find the zeros of $P(x) = x^4 - 2x^3 + 6x^2 - 2x + 5$ over $\mathbb{C}$, given that $1 + 2i$ is a zero. Factorise $P(x)$ fully over $\mathbb{R}$.

12 $P(x)$ is a monic polynomial of degree 4 with integer coefficients and constant term 3. One zero is i, another zero is rational and the sum of the zeros is negative. Factorise $P(x)$ fully over $\mathbb{R}$.

13 $P(x)$ is an even monic polynomial of degree 4 with integer coefficients. One zero is $2i$ and the product of the zeros is -8. Factorise $P(x)$ fully over $\mathbb{R}$.

14 $P(x)$ is a monic polynomial of degree 4 with integer coefficients and constant term 4. One zero is $\sqrt{2}$, another zero is rational and the sum of the zeros is positive. Factorise $P(x)$ fully over $\mathbb{R}$.

4.3 The relationship between the roots and coefficients of a polynomial equation

Let $ax^4 + bx^3 + cx^2 + dx + e = 0$ have roots $\alpha, \beta, \gamma, \delta$ over $\mathbb{C}$. Then $ax^4 + bx^3 + cx^2 + dx + e \equiv a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$.

Consider the term in x^2 on the right. This term is obtained by selecting x from two brackets and the constant from the remaining brackets in all possible ways. There are $\binom{4}{2}$ such terms, each of the form $ax^2(-\alpha)(-\beta)$. Equating coefficients of x^2 gives $c = a(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\gamma + \gamma\delta)$. We refer to this bracketed expression as the sum of products of the roots taken two at a time and we will use the notation $\Sigma\alpha\beta$ to denote such a sum. Similarly $\Sigma\alpha$ and $\Sigma\alpha\beta\gamma$ will denote, respectively, the sum of the roots and the sum of products taken three at a time. Then by a process similar to that described above for determining the coefficient of x^2

$$\Sigma\alpha = -\frac{b}{a}, \quad \Sigma\alpha\beta = \frac{c}{a}, \quad \Sigma\alpha\beta\gamma = -\frac{d}{a}, \quad \alpha\beta\gamma\delta = \frac{e}{a}.$$

Note the pattern of alternating signs and the leading coefficient as divisor. Clearly, this pattern extends to polynomials of any degree, giving:

For $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$
the sum of the products of roots taken r at a time $= (-1)^r \dfrac{a_{n-r}}{a_n}$.

Example 10

Expand $P(x) = (x - 1)(x + 2)(x - 3)(x + 1)$.

Solution

$P(x)$ has zeros $1, -1, -2, 3$, and form $x^4 + bx^3 + cx^2 + dx + e$, since $P(x)$ is monic of degree four. If $\alpha, \beta, \gamma, \delta$ denote the zeros of $P(x)$, then

$$\Sigma\alpha = 1 \Rightarrow b = -1 \qquad \Sigma\alpha\beta = -7 \Rightarrow c = +(-7)$$
$$\Sigma\alpha\beta\gamma = -1 \Rightarrow d = -(-1) \qquad \alpha\beta\gamma\delta = 6 \Rightarrow e = +6$$

Hence $P(x) = x^4 - x^3 - 7x^2 + x + 6$.

Example 11

Two of the roots of $2x^4 - 5x^3 + cx^2 + dx - 18 = 0$ are reciprocals, while the remaining roots are opposites of each other. Find the four roots of the equation and evaluate c and d.

Solution

Let the roots be $\alpha, \frac{1}{\alpha}, \beta, -\beta$. Then

Sum of roots is $\frac{5}{2}$

$\therefore \alpha + \frac{1}{\alpha} = \frac{5}{2}$

$2\alpha^2 - 5\alpha + 2 = 0,$

$\therefore (2\alpha - 1)(\alpha - 2) = 0$

$\alpha = \frac{1}{2}, 2$

Product of roots is $-\beta^2$,

$\therefore -\beta^2 = -9$

$\beta = \pm 3$

The roots are $\frac{1}{2}, 2, 3, -3$

$\therefore \Sigma\alpha\beta = -8 \Rightarrow c = -16$ and $\Sigma\alpha\beta\gamma = -\frac{45}{2} \Rightarrow d = 45$

Example 12

$2x^3 - 5x^2 + 4x + 6 = 0$ has roots α, β, γ. Find an equation with roots $2\alpha, 2\beta, 2\gamma$.

Solution

There are two methods to obtain the new equation.
First, using the relations between roots and coefficients, for the new equation:

The sum of roots is $2(\alpha + \beta + \gamma) = 2.\frac{5}{2} = 5$

The sum of products two at a time is $4\Sigma\alpha\beta = 4.\frac{4}{2} = 8$

The products of roots is $8\alpha\beta\gamma = 8.\left(-\frac{6}{2}\right) = -24$

Hence the required equation is $x^3 - 5x^2 + 8x + 24 = 0$.
Alternatively, since α, β, γ satisfy $2x^3 - 5x^2 + 4x + 6 = 0$.

$2\alpha, 2\beta, 2\gamma$ satisfy $2\left(\frac{x}{2}\right)^3 - 5\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) + 6 = 0$

$\therefore$ the required equation is $x^3 - 5x^2 + 8x + 24 = 0$.

The second method is easier and can be used to find equations with roots $k\alpha, k\beta, k\gamma$ $\left(\text{replace } x \text{ by } \frac{x}{k}\right)$; $\alpha + k, \beta + k, \gamma + k$ (replace x by $x - k$); $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ $\left(\text{replace } x \text{ by } \frac{1}{x}\right)$; $\alpha^2, \beta^2, \gamma^2$, (replace x by $x^{\frac{1}{2}}$), where k is any constant.

Example 13

$2x^3 - 5x^2 + 4x + 6 = 0$ has roots α, β, γ. Find equations with roots

(a) $\alpha + 1, \beta + 1, \gamma + 1$ **(b)** $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ **(c)** $\alpha\beta, \beta\gamma, \gamma\alpha$

Solution

(a) $\alpha + 1, \beta + 1, \gamma + 1$ satisfy $2(x-1)^3 - 5(x-1)^2 + 4(x-1) + 6 = 0$

$\therefore$ the required equation is $2x^3 - 11x^2 + 20x - 5 = 0$

(b) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, satisfy $2\left(\frac{1}{x}\right)^3 - 5\left(\frac{1}{x}\right)^2 + 4\left(\frac{1}{x}\right) + 6 = 0$

$\therefore$ the required equation is $2 - 5x + 4x^2 + 6x^3 = 0$

(c) $\alpha\beta, \beta\gamma, \gamma\alpha$ can be rewritten $\frac{\alpha\beta\gamma}{\gamma}, \frac{\alpha\beta\gamma}{\alpha}, \frac{\alpha\beta\gamma}{\beta}$. But $\alpha\beta\gamma = -\frac{6}{2} = -3$. Hence the required equation has roots $-\frac{3}{\alpha}, -\frac{3}{\beta}, -\frac{3}{\gamma}$.

From (b), $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, satisfy $2 - 5x + 4x^2 + 6x^3 = 0$

$\therefore \frac{-3}{\alpha}, \frac{-3}{\beta}, \frac{-3}{\gamma}$ satisfy $2 - 5\left(\frac{-x}{3}\right) + 4\left(\frac{-x}{3}\right)^2 + 6\left(\frac{-x}{3}\right)^3 = 0$

$\therefore$ the required equation is $18 + 15x + 4x^2 - 2x^3 = 0$

Example 14

$x^3 + 4x^2 + 2x - 1 = 0$ has roots α, β, γ.

(a) Find an equation with roots $\alpha^2, \beta^2, \gamma^2$ **(b)** Evaluate $\alpha^3 + \beta^3 + \gamma^3$

Solution

(a) α, β, γ satisfy $x^3 + 4x^2 + 2x - 1 = 0$. Hence $\alpha^2, \beta^2, \gamma^2$ satisfy $(x^{\frac{1}{2}})^3 + 4(x^{\frac{1}{2}})^2 + 2(x^{\frac{1}{2}}) - 1 = 0$.

Rearrangement gives $(x^{\frac{1}{2}})(x + 2) = -(4x - 1)$.

Hence the required equation is $x(x + 2)^2 = (4x - 1)^2$

$$x^3 - 12x^2 + 12x - 1 = 0.$$

(b) $\alpha^3 + 4\alpha^2 + 2\alpha - 1 = 0$
$\beta^3 + 4\beta^2 + 2\beta - 1 = 0$ (since α, β, γ are roots of the given equation).
$\gamma^3 + 4\gamma^2 + 2\gamma - 1 = 0$
$\therefore (\alpha^3 + \beta^3 + \gamma^3) + 4(\alpha^2 + \beta^2 + \gamma^2) + 2(\alpha + \beta + \gamma) - 3 = 0.$
But $\gamma^2 + \beta^2 + \gamma^2$ are roots of $x^3 - 12x^2 + 12x - 1 = 0$ from (a), hence $\alpha^2 + \beta^2 + \gamma^2 = 12$. Then $(\alpha^3 + \beta^2 + \gamma^3) + 48 - 8 - 3 = 0$
$\therefore \alpha^3 + \beta^3 + \gamma^3 = -37.$

Example 15

α, β, γ are roots of $x^3 + qr + r = 0$. Find in terms of q, r the coefficients of an equation with roots $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}, \dfrac{1}{\gamma^2}$

Solution

α, β, γ satisfy $x^3 + qr + r = 0.$

$\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ satisfy $\left(\dfrac{1}{x}\right)^3 + q\left(\dfrac{1}{x}\right) + r = 0$

$1 + qx^2 + rx^3 = 0.$

$\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}, \dfrac{1}{\gamma^2}$ satisfy $1 + q(x^{\frac{1}{2}})^2 + r(x^{\frac{1}{2}})^3 = 0.$

Rearrangement gives $1 + qx = -rx^{\frac{3}{2}}$. Squaring and simplifying, the required equation is $r^2x^3 - q^2x^2 - 2qx - 1 = 0.$

Exercise 4.3

1 Find the monic polynomial of degree 3 with zeros 1, 2 and 3.

2 Find the monic polynomial of degree 4 with zeros -3, -1, 1 and 3.

3 Two of the roots of $3x^3 + ax^2 + 23x - 6 = 0$ are reciprocals. Find the value of a and the three roots.

4 Two of the roots of $x^3 - 3x^2 - 4x + a = 0$ are opposites. Find the value of a and the three roots.

5 The equation $x^4 + px^3 + qx^2 + rx + s = 0$ has two roots which are reciprocals and two other roots which are opposites. Show that $q = 1 + s$ and $r = ps$.

6 The equation $px^3 + qx^2 + rx + s = 0$ has roots $a - c$, a and $a + c$, which are in arithmetic progression. Show that $a = -\dfrac{q}{3p}$ and hence show that $2q^3 - 9pqr + 27p^2s = 0$.

7 Solve the equation $18x^3 + 27x^2 + x - 4 = 0$ given that the roots are in arithmetic progression.

8 The equation $x^3 - 6x^2 + ax + 10 = 0$ has roots that are in arithmetic progression. Find the value of a and solve the equation.

9 The equation $px^3 + qx^2 + rx + s = 0$ has roots ac, a and $\frac{a}{c}$, which are in geometric progression. Show that $a = \sqrt[3]{\left(-\frac{s}{p}\right)}$ and hence show that $pr^3 - qs^3 = 0$.

10 Solve the equation $2x^3 - 13x^2 - 26x + 16 = 0$, given that the roots are in geometric progression.

11 The equation $x^3 - 13x^2 + ax - 27 = 0$ has roots that are in geometric progression. Find the value of a and solve the equation.

12 The equation $x^3 + 3x^2 - 2x - 2 = 0$ has roots α, β and γ. Find the equations with roots
(a) 2α, 2β, and 2γ **(b)** $\alpha - 2$, $\beta - 2$ and $\gamma - 2$
(c) $\frac{1}{\alpha}, \frac{1}{\beta}$, and $\frac{1}{\gamma}$ **(d)** α^2, β^2 and γ^2

13 The equation $x^4 + 4x^3 - 3x^2 - 4x + 2 = 0$ has roots α, β, γ and δ. Find the equations with roots
(a) 2α, 2β, 2γ and 2δ **(b)** $\alpha - 2$, $\beta - 2$, $\gamma - 2$ and $\delta - 2$
(c) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ **(d)** α^2, β^2, γ^2 and δ^2

14 The equation $x^3 + x^2 - 2x - 3 = 0$ has roots α, β and γ. Find the equations with roots
(a) $\alpha^2\beta\gamma$, $\alpha\beta^2\gamma$, and $\alpha\beta\gamma^2$
(b) $2\alpha + \beta + \gamma$, $\alpha + 2\beta + \gamma$ and $\alpha + \beta + 2\gamma$

15 The equation $x^3 + 2x + 1 = 0$ has roots α, β and δ. Find the equations with roots
(a) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ **(b)** α^2, β^2 and γ^2 **(c)** $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$

16 The equation $x^3 + px^2 + r = 0$ has roots α, β and γ. Find the equations with roots
(a) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ **(b)** α^2, β^2 and γ^2 **(c)** $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$

17 The equation $x^3 + x^2 + 2 = 0$ has roots α, β and γ. Evaluate
(a) $\alpha + \beta + \gamma$ **(b)** $\alpha^2 + \beta^2 + \gamma^2$
(c) $\alpha^3 + \beta^3 + \gamma^3$ **(d)** $\alpha^4 + \beta^4 + \gamma^4$

18 The equation $x^3 + qx + r = 0$ has roots α, β and γ. Find expressions for
(a) $\alpha^2 + \beta^2 + \gamma^2$ **(b)** $\alpha^3 + \beta^3 + \gamma^3$ **(c)** $\alpha^5 + \beta^5 + \gamma^5$

4.4 Solutions of polynomial equations

Example 16

Show that i is a zero of $P(x) = x^3 - 5x + 6i$. Hence solve $x^3 - 5x + 6i = 0$.

Solution

$P(i) = -i - 5i + 6i = 0 \Rightarrow (x - i)$ is a factor of $P(x)$.
By inspection, $x^3 - 5x + 6i = (x - i)(x^2 + ix - 6)$
$\therefore P(x) = 0 \Rightarrow \quad x = i \quad \text{or} \quad x^2 + ix - 6 = 0$
Using the quadratic formula, $x = i \quad \text{or} \quad x = \dfrac{-i \pm \sqrt{23}}{2}$

Quartic equations with symmetric coefficients $ax^4 + bx^3 + cx^2 \pm bx + a = 0$ can be converted to quadratic equations in $\left(x \pm \dfrac{1}{x}\right)$.

Example 17

Solve $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$ over $\mathbb{C}$. Factorise $P(x) = x^4 - 4x^3 + 5x^2 - 4x + 1$ over $\mathbb{Q}$ and over $\mathbb{R}$.

Solution

$$P(x) = x^2\left(x^2 - 4x + 5 - \frac{4}{x} + \frac{1}{x^2}\right) = x^2\left\{\left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5\right\}.$$

Using $\left(x + \dfrac{1}{x}\right)^2 = \left(x^2 + \dfrac{1}{x^2}\right) + 2$, $P(x) = x^2\left\{\left(x + \dfrac{1}{x}\right)^2 - 4\left(x + \dfrac{1}{x}\right) + 3\right\}$.

Since 0 is not a zero of $P(x)$, the solutions of $P(x) = 0$ are the solutions of $\left(x + \dfrac{1}{x}\right)^2 - 4\left(x + \dfrac{1}{x}\right) + 3 = 0$. By factorising this quadratic

$$P(x) = x^2\left(x + \frac{1}{x} - 3\right)\left(x + \frac{1}{x} - 1\right) = (x^2 - 3x + 1)(x^2 - x + 1) \qquad (4)$$

$$\therefore P(x) = 0 \Rightarrow \quad x^2 - 3x + 1 = 0 \quad \text{or} \quad x^2 - x + 1 = 0,$$
$$x = \frac{3 \pm \sqrt{5}}{2} \quad \text{or} \quad x = \frac{1 \pm i\sqrt{3}}{2}.$$

From (4), $P(x) = (x^2 - 3x + 1)(x^2 - x + 1)$ over $\mathbb{Q}$ and

$P(x) = \left(x - \dfrac{3 + \sqrt{5}}{2}\right)\left(x - \dfrac{3 - \sqrt{5}}{2}\right)(x^2 - x + 1)$ over $\mathbb{R}$.

Polynomial equations of the form $x^n \pm 1 = 0$ can be solved by finding the complex nth roots of ∓ 1. This technique also enables us to solve equations of the form $x^{n-1} + x^{n-2} + \ldots + x + 1 = 0$ or $x^{n-1} - x^{n-2} + \ldots + (-1)^{n-1} = 0$, since multiplication by one of $(x + 1)$ or $(x - 1)$ will give $x^n \pm 1 = 0$.

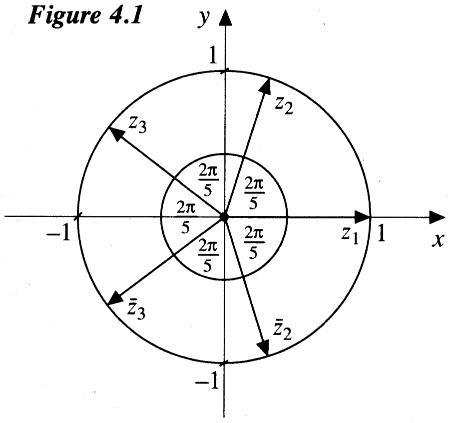
Example 18

Find all the zeros of $P(x) = x^4 + x^3 + x^2 + x + 1$. Hence factorise $P(x)$ into irreducible factors over $\mathbb{R}$. Deduce that $\cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$.

Solution

$(x - 1)(x^4 + x^3 + x^2 + x + 1) = x^5 - 1$. Hence the zeros of $P(x)$ are the solutions of $x^5 = 1$, $x \neq 1$. The complex fifth roots of unity are equally spaced around the unit circle in the Argand diagram, the angular spacing being $\frac{2\pi}{5}$. Clearly $z_1 = 1$ is one fifth root of unity.

Figure 4.1



The non-real fifth roots of unity are:

z_2 and $\bar{z}_2$, where $z_2 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$,

z_3 and $\bar{z}_3$, where $z_3 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}$.

Hence the zeros of $P(x)$ over $\mathbb{C}$ are

$\cos \frac{2\pi}{5} \pm i \sin \frac{2\pi}{5}$,

$\cos \frac{4\pi}{5} \pm i \sin \frac{4\pi}{5}$.

$$\begin{aligned} P(x) &= (x - z_2)(x - \bar{z}_2)(x - z_3)(x - \bar{z}_3) \\ &= \{x^2 - 2 \operatorname{Re}(z_2)x + |z_2|^2\}\{x^2 - 2 \operatorname{Re}(z_3)x + |z_3|^2\}, \end{aligned}$$

$$\therefore x^4 + x^3 + x^2 + x + 1 = \left(x^2 - 2 \cos \frac{2\pi}{5} x + 1\right)\left(x^2 - 2 \cos \frac{4\pi}{5} x + 1\right).$$

These factors of $P(x)$ are irreducible over $\mathbb{R}$. Further, equating coefficients of x,

$$2 \cos \frac{4\pi}{5} + 2 \cos \frac{2\pi}{5} + 1 = 0.$$

Using $\cos 2\theta = 2 \cos^2 \theta - 1$, $\quad 4 \cos^2 \frac{2\pi}{5} + 2 \cos \frac{2\pi}{5} - 1 = 0.$

But $0 < \frac{2\pi}{5} < \frac{\pi}{2} \Rightarrow \cos \frac{2\pi}{5} > 0 \quad \therefore \cos \frac{2\pi}{5} = \frac{-2 + 2\sqrt{5}}{8} = \frac{\sqrt{5} - 1}{4}$.

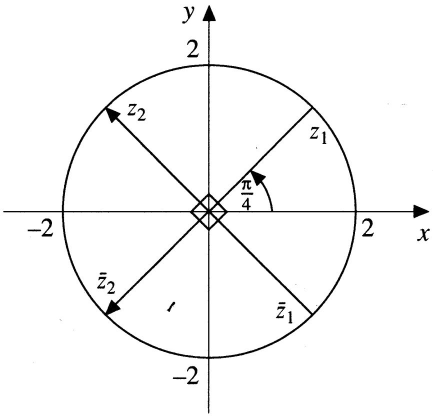
Example 19

Solve $z^5 + 16z = 0$ over $\mathbb{C}$.

Solution

$z^5 + 16z = 0 \Rightarrow z(z^4 + 16) = 0$. Hence $z = 0$, or z is a complex fourth root of -16. Clearly one such root has argument $\frac{\pi}{4}$ and modulus 2, since $\arg(-16) = \pi$ and $|-16| = 2^4$. The other fourth roots of -16 are equally spaced by $\frac{\pi}{2}$ around a circle of radius 2 and centre (0, 0) in the Argand diagram.

Figure 4.2



The fourth roots of -16 are

$2\left(\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}\right)$ and

$2\left(\cos\frac{3\pi}{4} \pm i\sin\frac{3\pi}{4}\right)$.

Hence $z^5 + 16z = 0$ has roots $0, \sqrt{2}\,(1 \pm i), \sqrt{2}\,(-1 \pm i)$.

De Moivre's theorem can also be used to convert some polynomial equations into trigonometric equations. Let $z = \cos\theta + i\sin\theta$.

Then by De Moivre's theorem, $z^n = \cos n\theta + i\sin n\theta, \qquad n = 1, 2, \ldots$

But by the Binomial theorem, $\qquad z^n = \sum_{k=0}^{n}\binom{n}{k} i^k \sin^k\theta\cos^{n-k}\theta$

Equating real and imaginary parts,

$$\cos n\theta = \cos^n\theta - \binom{n}{2}\sin^2\theta\cos^{n-2}\theta + \binom{n}{4}\sin^4\theta\cos^{n-4}\theta - \ldots$$

$$\sin n\theta = \binom{n}{1}\sin\theta\cos^{n-1}\theta - \binom{n}{3}\sin^3\theta\cos^{n-3}\theta + \binom{n}{5}\sin^5\theta\cos^{n-5}\theta - \ldots$$

Example 20

Use De Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Hence solve $8x^3 - 6x - 1 = 0$. Deduce that $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$.

Solution

Using the above method for $n = 3$, we obtain by De Moivre's theorem
$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$.

Let $x = \cos\theta$. Then $\cos 3\theta = \frac{1}{2} \Leftrightarrow 4x^3 - 3x = \frac{1}{2}$. (5)

Hence if θ is a solution of $\cos 3\theta = \frac{1}{2}$, $\cos\theta$ is a root of

$8x^3 - 6x - 1 = 0$ (rearranging (5))

$$\cos 3\theta = \frac{1}{2} \Rightarrow \quad 3\theta = \pm\frac{\pi}{3} + 2n\pi,\ n \text{ integral}$$

$$\theta = (6n \pm 1)\frac{\pi}{9}, \quad n = 0, \pm 1, \pm 2, \ldots$$

These values of θ give exactly three distinct values of $\cos\theta$, namely

$\cos\frac{\pi}{9}$, $\cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9}$, and $\cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$.

Hence the roots of $8x^3 - 6x - 1 = 0$ are $\cos\frac{\pi}{9}$, $-\cos\frac{4\pi}{9}$ and $-\cos\frac{2\pi}{9}$.

But the coefficient of x^2 is zero and hence the sum of the roots is zero, giving
$\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$.

Example 21

Use De Moivre's theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$. Hence show that $x^4 - 10x^2 + 5 = 0$ has roots $\pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$. Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$. By solving $x^4 - 10x^2 + 5 = 0$ another way, find the value of $\tan\frac{\pi}{5}$ as a surd.

Solution

Using De Moivre's theorem and the binomial theorem as before

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5\sin\theta\cos^4\theta - 10\sin^3\theta\cos^2\theta + \sin^5\theta}{\cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\sin^4\theta\cos\theta}$$

$$\therefore \tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

Let $x = \tan\theta$. Then $\tan 5\theta = 0 \Leftrightarrow x(x^4 - 10x^2 + 5) = 0$.
Hence the roots of $x^4 - 10x^2 + 5 = 0$ are the non-zero values of $\tan\theta$, where θ is a solution of $\tan 5\theta = 0$.

But $\tan 5\theta = 0 \Rightarrow 5\theta = n\pi \Rightarrow \theta = n\frac{\pi}{5}$, $n = 0, \pm 1, \pm 2, \ldots$

For these values of θ, there are four distinct non-zero values of $\tan\theta$, namely $\pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$, and hence these are the roots of $x^4 - 10x^2 + 5 = 0$.

However $-\tan\frac{\pi}{5} = \tan\frac{4\pi}{5}$, $-\tan\frac{2\pi}{5} = \tan\frac{3\pi}{5}$ and the product of the roots of the given equation is 5 $\quad\therefore \tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$.

Considering $x^4 - 10x^2 + 5 = 0$ as a quadratic in x^2,

$x^2 = \dfrac{10 \pm 4\sqrt{5}}{2} = 5 \pm 2\sqrt{5}$

But $x^2 = \tan^2\frac{\pi}{5}$ or $x^2 = \tan^2\frac{2\pi}{5}$.

Then $0 < \tan\frac{\pi}{5} < \tan\frac{2\pi}{5} \Rightarrow \tan\frac{\pi}{5} = \sqrt{(5 - 2\sqrt{5})}$.

Exercise 4.4

1 Show that $-i$ is a zero of $P(x) = x^3 + ix^2 - 4x - 4i$. Hence factorise $P(x)$ over $\mathbb{C}$.

2 Show that $2i$ is a zero of $P(x) = x^3 - 2ix^2 - 3x + 6i$. Hence factorise $P(x)$ over $\mathbb{C}$.

3 If $P(x) = 3x^4 + 10x^3 + 6x^2 + 10x + 3$, solve $P(x) = 0$ over $\mathbb{C}$, and factorise $P(x)$ over $\mathbb{R}$.

4 If $P(x) = 2x^4 + 7x^3 + 2x^2 - 7x + 2$, solve $P(x) = 0$ over $\mathbb{C}$, and factorise $P(x)$ over $\mathbb{R}$.

5 Show that the zeros of $P(x) = x^4 + x^2 + 1$ are included in the zeros of $x^6 - 1$. Hence factorise $P(x)$ over $\mathbb{R}$.

6 Show that the zeros of $P(x) = x^4 - x^2 + 1$ are included in the zeros of $x^6 + 1$. Hence factorise $P(x)$ over $\mathbb{R}$.

7 Solve $z^5 - 4z = 0$ over $\mathbb{C}$.

8 Solve $4z^5 + z = 0$ over $\mathbb{C}$.

9 Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. Hence
(a) Solve the equation $8x^4 - 8x^2 + 1 = 0$ and deduce the exact values of $\cos\frac{\pi}{8}$ and $\cos\frac{5\pi}{8}$.
(b) Solve the equation $16x^4 - 16x^2 + 1 = 0$ and deduce the exact values of $\cos\frac{\pi}{12}$ and $\cos\frac{5\pi}{12}$.

10 Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. Hence
(a) Solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ and deduce the exact values of $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$.

(b) Solve the equation $32x^5 - 40x^3 + 10x - 1 = 0$ and deduce that

(i) $\cos\frac{\pi}{15} + \cos\frac{7\pi}{15} + \cos\frac{13\pi}{15} + \cos\frac{19\pi}{15} = -\frac{1}{2}$

(ii) $\cos\frac{\pi}{15}\cos\frac{7\pi}{15}\cos\frac{13\pi}{15}\cos\frac{19\pi}{15} = \frac{1}{16}$

4.5 Partial fractions

The highest common factor of two polynomials $A(x)$, $B(x)$ is the monic polynomial of highest degree which is a factor of both $A(x)$ and $B(x)$. It can be found by repeated use of the division transformation.

Example 22

Find the highest common factor of $A(x) = x^4 - 3x^3 + 2x^2 - 3x - 9$ and $B(x) = x^3 - 4x^2 + 5x - 6$.

Solution

(1) $A(x) \div B(x)$

$$\Rightarrow \underbrace{x^4 - 3x^3 + 2x^2 - 3x - 9}_{A(x)} = \underbrace{(x^3 - 4x^2 + 5x - 6)}_{B(x)}\,.\,\underbrace{(x + 1)}_{Q_1(x)} + \underbrace{(x^2 - 2x - 3)}_{R_1(x)}$$

(2) $B(x) \div R_1(x) \Rightarrow$

$$\underbrace{x^3 - 4x^2 + 5x - 6}_{B(x)} = \underbrace{(x^2 - 2x - 3)}_{R_1(x)}\,.\,\underbrace{(x - 2)}_{Q_2(x)} + \underbrace{4(x - 3)}_{R_2(x)}$$

(3) $R_1(x) \div R_2(x) \Rightarrow$

$$\underbrace{x^2 - 2x - 3}_{R_1(x)} = \underbrace{4(x - 3)}_{R_2(x)}\,.\,\underbrace{\tfrac{1}{4}(x + 1)}_{Q_3(x)} + 0$$

The last non-zero remainder $4(x - 3)$ gives the highest common factor as $(x - 3)$, a monic polynomial. Following the chain of divisions in reverse, $(x - 3)$ is a factor of $R_2(x)$ and $R_1(x)$ and hence of $B(x)$ (from step (2)), then of $A(x)$ (from step (1)). Following the chain of divisions in the direction $(1) \to (2) \to (3)$, we can see that any common factor of $A(x)$ and $B(x)$ is a factor of $(x - 3)$.
Obtaining $R_1(x)$ from step (1) and substituting this expression for $R_1(x)$ in step (2) gives

$$\begin{aligned} R_2(x) &= B(x) - [A(x) - B(x)Q_1(x)]Q_2(x) \\ &= A(x)\,[-Q_2(x)] + B(x)\,[Q_1(x)\,Q_2(x) + 1] \\ \therefore (x - 3) &= A(x)\,[-\tfrac{1}{4}(x - 2)] + B(x)\,[\tfrac{1}{4}(x^2 - x - 1)] \end{aligned}$$

In general, for any polynomials $A(x)$ and $B(x)$ with highest common factor $H(x)$, we can use the method shown above to find unique polynomials $U(x)$ and $V(x)$ such that $H(x) \equiv A(x)U(x) + B(x)V(x)$. This is an identity over $\mathbb{C}$ but the nature of the coefficients of $H(x)$, $U(x)$ and $V(x)$ is determined by that of $A(x)$ and $B(x)$. An important consequence of this result is the decomposition of a rational function into partial fractions.

Consider $\dfrac{P(x)}{A(x)B(x)}$ where $\deg P < \deg A + \deg B$, and $A(x)$ and $B(x)$ have no common factor. Since the highest common factor of $A(x)$ and $B(x)$ is 1, we can find polynomials $U(x)$ and $V(x)$ such that

$$1 \equiv A(x)U(x) + B(x)V(x)$$

Then $\dfrac{P(x)}{A(x)\,B(x)} \equiv \dfrac{P(x)U(x)}{B(x)} + \dfrac{P(x)V(x)}{A(x)}$

But by the division transformation, $P(x)U(x) \equiv B(x)F(x) + L(x)$ and $P(x)V(x) \equiv A(x)G(x) + K(x)$, where $F(x)$, $G(x)$, $L(x)$ and $K(x)$ are polynomials such that $\deg L < \deg B$ and $\deg K < \deg A$. Then

$$\frac{P(x)}{A(x)B(x)} \equiv \{F(x) + G(x)\} + \frac{L(x)}{B(x)} + \frac{K(x)}{A(x)}.$$

But if $F(x) + G(x) \not\equiv 0$, $\deg P \geqslant \deg A + \deg B$ $\therefore F(x) + G(x) \equiv 0$.
Hence we can find unique polynomials $K(x)$ and $L(x)$ with $\deg K < \deg A$ and $\deg L < \deg B$, such that $\dfrac{P(x)}{A(x)B(x)} \equiv \dfrac{K(x)}{A(x)} + \dfrac{L(x)}{B(x)}$. (The uniqueness of these polynomials follows from the nature of the algorithm used to find $U(x)$ and $V(x)$ and of the division transformation.) This is called the decomposition of the rational function $\dfrac{P(x)}{A(x)B(x)}$ into partial fractions. Note that $A(x)$ and $B(x)$ must have no common factor and $\deg P < \deg A + \deg B$.

Consider $\dfrac{P(x)}{Q(x)}$, where $\deg P < \deg Q$, and
$Q(x) \equiv (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$, $\alpha_1, \alpha_2, \ldots, \alpha_n$ distinct.
Let $A(x) = (x - \alpha_1)$ and $B(x) = (x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)$.
Then we can find a constant c_1 and a polynomial $L_1(x)$ with $\deg L_1 < \deg B$, such that

$\dfrac{P(x)}{Q(x)} \equiv \dfrac{c_1}{x - \alpha_1} + \dfrac{L_1(x)}{B(x)}$. Continuing this process (or applying induction on $\deg Q$)

$\dfrac{P(x)}{Q(x)} \equiv \dfrac{c_1}{x - \alpha_1} + \dfrac{c_2}{x - \alpha_2} + \ldots + \dfrac{c_n}{x - \alpha_n}$, $c_1, c_2, \ldots, c_n$, constant.

In practice, $c_1, c_2, \ldots$ are obtained from the identity by substitution or by equating coefficients in the usual way.

Example 23

Express $\dfrac{3x-2}{(x-1)(x-2)}$ as a sum of partial fractions.

Solution

Let $\dfrac{3x-2}{(x-1)(x-2)} \equiv \dfrac{c_1}{x-1} + \dfrac{c_2}{x-2}$.

Then $\quad 3x - 2 \equiv c_1(x-2) + c_2(x-1)$.

Putting $x = 1$ gives $c_1 = -1$, while $x = 2$ gives $c_2 = 4$.

Hence $\dfrac{3x-2}{(x-1)(x-2)} \equiv \dfrac{-1}{x-1} + \dfrac{4}{x-2}$.

An alternative method of obtaining $c_1, \ldots, c_n$ in the identity $\dfrac{P(x)}{Q(x)} \equiv \dfrac{c_1}{x-\alpha_1} + \ldots + \dfrac{c_n}{x-\alpha_n}$, where $Q(x) = (x-\alpha_1)\ldots(x-\alpha_n)$, is to note that $P(x).\dfrac{x-\alpha_1}{Q(x)-Q(\alpha_1)} \equiv c_1 + c_2\dfrac{x-\alpha_1}{x-\alpha_2} + \ldots + c_n\dfrac{x-\alpha_1}{x-\alpha_n}$ since $Q(\alpha_1) = 0$. Taking limits of both sides as $x \to \alpha_1$, $c_1 = \dfrac{P(\alpha_1)}{Q'(\alpha_1)}$.

Similarly, $c_k = \dfrac{P(\alpha_k)}{Q'(\alpha_k)}$, $k = 1, 2, \ldots, n$. In the above example,

$c_1 = \dfrac{3x-2}{\dfrac{d}{dx}(x^2-3x+2)}$ when $x = 1 \Rightarrow c_1 = -1$.

$c_2 = \dfrac{3x-2}{\dfrac{d}{dx}(x^2-3x+2)}$ when $x = 2 \Rightarrow c_2 = 4$.

If the numerator of the rational function has greater degree than the denominator, it is necessary to perform the division transformation before seeking partial fractions.

Example 24

Use partial fractions to evaluate $\displaystyle\int_0^1 \frac{x^3-2x^2-2x-1}{x^2-2x-3}\,dx$.

Solution

By division, or by inspection, $x^3 - 2x^2 - 2x - 1 \equiv (x^2-2x-3)x + (x-1)$

Hence $\qquad \dfrac{x^3-2x^2-2x-1}{x^2-2x-3} = x + \dfrac{x-1}{x^2-2x-3}$

But $x^2 - 2x - 3 = (x-3)(x+1)$

$\therefore \dfrac{x-1}{x^2-2x-3} \equiv \dfrac{c_1}{x+1} + \dfrac{c_2}{x-3}$, c_1, c_2 constant.

Then $x - 1 \equiv c_1(x - 3) + c_2(x + 1)$

Putting $x = -1$ gives $-2 = -4c_1 \Rightarrow c_1 = \frac{1}{2}$,

while $x = 3$ gives $2 = 4c_2 \Rightarrow c_2 = \frac{1}{2}$,

$$\begin{aligned}
\therefore \int_0^1 \frac{x^3 - 2x^2 - 2x - 1}{x^2 - 2x - 3}\,dx &= \int_0^1 x\,dx + \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx + \frac{1}{2}\int_0^1 \frac{1}{x-3}\,dx \\
&= \left[\frac{1}{2}x^2\right]_0^1 + \frac{1}{2}\Big[\ln|x+1|\Big]_0^1 + \frac{1}{2}\Big[\ln|x-3|\Big]_0^1 \\
&= \frac{1}{2}(1 + \ln 2 + \ln 2 - \ln 3) \\
&= \frac{1}{2}\left\{1 + \ln\left(\frac{4}{3}\right)\right\}.
\end{aligned}$$

Partial fraction decompositions can also be found when the denominator of the rational function has an irreducible quadratic factor, but in this case the partial fraction associated with this quadratic factor will have a linear numerator.

Example 25

Express $\dfrac{x^2 + 1}{(x + 2)(x - 1)(x^2 + x + 1)}$ as a sum of partial fractions over $\mathbb{R}$.

Solution

Let $\dfrac{x^2 + 1}{(x + 2)(x - 1)(x^2 + x + 1)} \equiv \dfrac{c_1}{x + 2} + \dfrac{c_2}{x - 1} + \dfrac{ax + b}{x^2 + x + 1}$

Then $x^2 + 1$

$\equiv c_1(x - 1)(x^2 + x + 1) + c_2(x + 2)(x^2 + x + 1) + (ax + b)(x + 2)(x - 1)$

Put $x = 1$: $\quad 2 = 9c_2 \Rightarrow c_2 = \frac{2}{9}$

Put $x = -2$: $\quad 5 = -9c_1 \Rightarrow c_1 = \frac{-5}{9}$

Equate coefficients of x^3: $\quad 0 = c_1 + c_2 + a \Rightarrow a = \frac{1}{3}$

$$\therefore \frac{x^2 + 1}{(x + 2)(x - 1)(x^2 + x + 1)} \equiv -\frac{5}{9(x + 2)} + \frac{2}{9(x - 1)} + \frac{1}{3(x^2 + x + 1)}$$

Example 26

Express $\dfrac{x^3+1}{(x^2+2)(x^2+8)}$ as a sum of partial fractions over $\mathbb{R}$.

Solution

Let $\dfrac{x^3+1}{(x^2+2)(x^2+8)} \equiv \dfrac{ax+b}{x^2+2} + \dfrac{cx+d}{x^2+8}$.

Then $\qquad x^3+1 \equiv (ax+b)(x^2+8) + (cx+d)(x^2+2)$.

$$\left.\begin{array}{ll}\text{Equate coefficients of } x^3\text{:} & 1 = a + c \\ \text{Equate coefficients of } x\text{:} & 0 = 8a + 2c\end{array}\right\} \Rightarrow a = \frac{-1}{3} \text{ and } c = \frac{4}{3}$$

$$\left.\begin{array}{ll}\text{Equate coefficients of } x^2\text{:} & 0 = b + d \\ \text{Equate constant terms:} & 1 = 8b + 2d\end{array}\right\} \Rightarrow b = \frac{1}{6} \text{ and } d = -\frac{1}{6}$$

$$\therefore \frac{x^3+1}{(x^2+2)(x^2+8)} \equiv \frac{-(2x+1)}{6(x^2+2)} + \frac{8x-1}{6(x^2+8)}$$

In the partial fraction decompositions considered so far, the denominator of the rational function has no multiple zeros.

Consider $\dfrac{P(x)}{(x-\alpha)^n}$, where $P(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0$.

We can express $P(x)$ as a polynomial in $(x - \alpha)$

Let $G(x) = P(x+\alpha) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_0$

Then $G(x-\alpha) = P(x) = b_{n-1}(x-\alpha)^{n-1} + b_{n-2}(x-\alpha)^{n-2} + \ldots + b_0$

Now $\dfrac{P(x)}{(x-\alpha)^n} \equiv \dfrac{b_{n-1}}{(x-\alpha)} + \dfrac{b_{n-2}}{(x-\alpha)^2} + \ldots + \dfrac{b_0}{(x-\alpha)^n}$.

Example 27

Write $\dfrac{x+3}{x^4-2x^2+1}$ as a sum of partial fractions.

Solution

$x^4 - 2x^2 + 1 = (x^2-1)^2 = (x-1)^2(x+1)^2$

$$\therefore \frac{x+3}{x^4-2x^2+1} \equiv \frac{c_1}{x-1} + \frac{c_2}{(x-1)^2} + \frac{c_3}{x+1} + \frac{c_4}{(x+1)^2},$$

c_1, c_2, c_3, c_4 constants.

$x + 3 \equiv c_1(x-1)(x+1)^2 + c_2(x+1)^2 + c_3(x+1)(x-1)^2 + c_4(x-1)^2$.

Put $x = 1$: $\quad$ then $c_2 = 1$. $\quad$ Put $x = -1$: $\quad$ then $c_4 = \dfrac{1}{2}$.

$$\left.\begin{array}{ll}\text{Equate coefficients of } x^3\text{:} & 0 = c_1 + c_3 \\ \text{Put } x = 0\text{:} & 3 = -c_1 + c_3 + c_2 + c_4\end{array}\right\} \Rightarrow c_3 = \frac{3}{4} \text{ and } c_1 = \frac{-3}{4}$$

$$\therefore \frac{x+3}{x^4-2x^2+1} \equiv \frac{-3}{4(x-1)} + \frac{1}{(x-1)^2} + \frac{3}{4(x+1)} + \frac{1}{2(x+1)^2}$$

Exercise 4.5

Express as a sum of partial fractions

1 $\dfrac{2x+10}{(x-1)(x+3)}$ 2 $\dfrac{4x+5}{2x^2+5x+3}$ 3 $\dfrac{6}{2x^2-5x+2}$

4 $\dfrac{x^2+x+2}{x(x+1)}$ 5 $\dfrac{2x+4}{(x-2)(x^2+4)}$ 6 $\dfrac{3x^2-3x+2}{(2x-1)(x^2+1)}$

7 $\dfrac{x^3+2x^2+6x+10}{(x+1)(x^2+4)}$ 8 $\dfrac{5-x}{(2x+3)(x^2+1)}$ 9 $\dfrac{x^2+7}{(x^2+1)(x^2+4)}$

10 $\dfrac{3x}{(x^2+1)(x^2+4)}$

Diagnostic test 4

Subsection

1 Find the zeros of $P(x)$ (4.1)
(a) over $\mathbb{Q}$ **(b)** over $\mathbb{R}$ **(c)** over $\mathbb{C}$
if **(i)** $P(x) = x^4 - 4x^2 + 3$ **(ii)** $P(x) = x^4 - 2x^2 - 3$

2 Express $P(x) = x^4 + 4x^3 - 3x^2 + 8x - 10$ as a product of irreducible factors (4.1)
(a) over $\mathbb{Q}$ **(b)** over $\mathbb{R}$ **(c)** over $\mathbb{C}$

3 If $P(x) = 4x^3 + 15x^2 + 12x - 4$ has a double zero, find all the zeros and factorise $P(x)$ fully over the real numbers. (4.1)

4 If $P(x) = 2x^3 - x^2 - 6x + 3$ has a rational zero, find all the zeros and factorise $P(x)$ fully over the real numbers. (4.1)

5 Find the remainder when $P(x) = x^3 + 2x^2 - 1$ is divided by (4.2)
(a) $x - i$ **(b)** $x^2 + 1$

6 When $P(x) = x^4 + ax^2 + bx$ is divided by $x^2 + 1$, the remainder is $x + 2$. Find the values of a and b. (4.2)

7 If $P(x) = x^4 - 2x^3 - x^2 + 6x - 6$ has a zero $1 - i$, find the zeros of $P(x)$ over $\mathbb{C}$, and factorise $P(x)$ fully over $\mathbb{R}$. (4.2)

8 $P(x)$ is an even monic polynomial of degree 4 with integer coefficients. If $\sqrt{2}$ is a zero, and the constant term is 6, factorise $P(x)$ fully over $\mathbb{R}$. (4.2)

9 The equation $x^3 - 3x^2 + ax + 8 = 0$ has roots that are in arithmetic progression. Find the value of a and solve the equation. (4.3)

10 The equation $x^3 + x^2 - 2x - 3 = 0$ has roots α, β and γ. Find the equations with roots (4.3)
(a) 2α, 2β and 2γ **(b)** $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$
(c) $\alpha - 2$, $\beta - 2$ and $\gamma - 2$ **(d)** $\alpha + 2$, $\beta + 2$ and $\gamma + 2$

11 The equation $x^3 + qx + r = 0$ has roots α, β and γ. Find the equations with roots (4.3)

(a) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ **(b)** α^2, β^2 and γ^2

12 The equation $x^3 + 2x + 1 = 0$ has roots α, β and γ. Evaluate (4.3)

(a) $\alpha + \beta + \gamma$ **(b)** $\alpha^2 + \beta^2 + \gamma^2$
(c) $\alpha^3 + \beta^3 + \gamma^3$ **(d)** $\alpha^4 + \beta^4 + \gamma^4$

13 If $P(x) = 3x^4 - 4x^3 - 14x^2 - 4x + 3$, solve the equation $P(x) = 0$ over $\mathbb{C}$ and factor $P(x)$ fully over $\mathbb{R}$. (4.4)

14 Solve the equation $z^5 - 16z = 0$ over $\mathbb{C}$ (4.4)

15 Show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$. Solve the equation $x^3 - 3x^2 - 3x + 1 = 0$ and hence find the exact values of $\tan\frac{\pi}{12}$ and $\tan\frac{5\pi}{12}$. (4.4)

16 Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. Solve the equation $16x^5 - 20x^3 + 5x = 0$ and hence find the exact values of $\cos\frac{\pi}{10}$ and $\cos\frac{3\pi}{10}$. (4.4)

17 Express $\frac{3x - 4}{x^2 - x - 6}$ as a sum of partial fractions. (4.5)

18 Express $\frac{3x^2 - 6x + 10}{(x - 4)(x^2 + 1)}$ as a sum of partial fractions. (4.5)

Further questions 4

1 Show that $\sqrt{7}$ is a root of the equation $2x^3 - 13x - \sqrt{7} = 0$. Hence solve the equation fully over $\mathbb{R}$.

2 If $P(x) = x^6 + x^4 + x^2 + 1$, show that the solutions of the equation $P(x) = 0$ are among the solutions of the equation $x^8 - 1 = 0$. Hence factorise $P(x)$ fully over $\mathbb{R}$.

3 If $P(x) = 5x^4 - 11x^3 + 16x^2 - 11x + 5$, solve $P(x) = 0$ over $\mathbb{C}$ and factorise $P(x)$ fully over $\mathbb{R}$.

4 If $P(x) = x^4 - 8x^3 + 30x^2 - 56x + 49$ has a non-real double zero, solve the equation $P(x) = 0$ over $\mathbb{C}$ and factorise $P(x)$ fully over $\mathbb{R}$.

5 Show that $1 + i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$. Hence solve the equation over $\mathbb{C}$.

6 The equation $x^3 + 3x + 2 = 0$ has roots α, β and γ. Find the equation with roots $\alpha + \frac{1}{\alpha}$, $\beta + \frac{1}{\beta}$ and $\gamma + \frac{1}{\gamma}$.

7 The equation $x^4 - px^3 + qx^2 - pqx + 1 = 0$ has roots α, β, γ and δ. Show that $(\alpha + \beta + \gamma)(\alpha + \beta + \delta)(\alpha + \gamma + \delta)(\beta + \gamma + \delta) = 1$.

8 The equation $x^4 - px^3 + qx^2 - rx + s = 0$ has roots α, β, γ and δ. Show that

(a) If $\alpha\beta = \gamma\delta$, then $r^2 = p^2 s$

(b) If $\alpha + \beta = \gamma + \delta$, then $p^3 - 4pq + 8r = 0$

9 The equation $z^n - 1 = 0$ has roots $1, z_1, z_2, \ldots, z_{n-1}$. Show that $(1 - z_1)(1 - z_2) \ldots (1 - z_{n-1}) = n$.

10 The equation $x^3 + 3px^2 + 3qx + r = 0$, where $p^2 \neq q$, has a double root. Show that $(pq - r)^2 = 4(p^2 - q)(q^2 - pr)$.

11 The equation $x^n + px - q = 0$ has a double root. Show that

$$\left(\frac{p}{n}\right)^n + \left(\frac{q}{n-1}\right)^{n-1} = 0.$$

12 The roots of the equation $\sum_{r=0}^{n} a_r x^{n-r} = 0$, where $a_0 = 1$, are the first n positive integers. Find expressions for a_1 and a_n and show that $a_2 = \frac{1}{24}n(n + 1)(n - 1)(3n + 2)$.

5 Integration

5.1 Using a table of standard integrals

In the following tables, the constant of integration has been omitted.

Table 5.1

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \qquad n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x|$$

$$\int e^{ax}\, dx = \frac{1}{a}e^{ax}, \qquad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \qquad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \qquad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \qquad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \qquad a > 0$$

$$\int \frac{1}{\sqrt{(a^2 - x^2)}} dx = \sin^{-1}\frac{x}{a}, \qquad a > 0$$

$$\int \frac{1}{\sqrt{(x^2 - a^2)}}\, dx = \ln|x + \sqrt{(x^2 - a^2)}|$$

$$\int \frac{1}{\sqrt{(x^2 + a^2)}}\, dx = \ln\{x + \sqrt{(x^2 + a^2)}\}$$

Table 5.2

$$\int \{f(x)\}^n f'(x)\, dx = \frac{1}{n+1}\{f(x)\}^{n+1}, \; n \neq -1$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)|$$

$$\int e^{f(x)} f'(x)\, dx = e^{f(x)}$$

$$\int \cos\{f(x)\} f'(x)\, dx = \sin\{f(x)\}$$

$$\int \sin\{f(x)\} f'(x)\, dx = -\cos\{f(x)\}$$

$$\int \sec^2\{f(x)\}\, f'(x)\, dx = \tan\{f(x)\}$$

$$\int \sec\{f(x)\}\tan\{f(x)\}\, f'(x)\, dx = \sec\{f(x)$$

$$\int \frac{f'(x)}{a^2 + \{f(x)\}^2} dx = \frac{1}{a}\tan^{-1}\left\{\frac{1}{a}f(x)\right\}, \quad a >$$

$$\int \frac{f'(x)}{\sqrt{[a^2 - \{f(x)\}^2]}}\, dx = \sin^{-1}\left\{\frac{1}{a}f(x)\right\}, \; a >$$

$$\int \frac{f'(x)}{\sqrt{[\{f(x)\}^2 - a^2]}}\, dx$$

$$= \ln|f(x) + \sqrt{[\{f(x)\}^2 - a^2]}|$$

$$\int \frac{f'(x)}{\sqrt{[\{f(x)\}^2 + a^2]}}\, dx$$

$$= \ln\{f(x) + \sqrt{([f(x)]^2 + a^2)}\}$$

Table 5.1 lists standard integrals. Each entry in this table provides a pattern of integration which can be applied to a family of more complicated integrals represented by the corresponding entry in Table 5.2. In each case, the integral in the right-hand column can be reduced to the simpler integral on the left by making the substitution $u = f(x)$. For example, consider $\int e^{f(x)} f'(x)\, dx$.

$$u = f(x) \qquad\qquad \int e^{f(x)} f'(x)\, dx = \int e^u\, du$$

$$\frac{du}{dx} = f'(x) \qquad\qquad = e^u + c, \quad \text{c constant}$$

$$du = f'(x)\, dx \qquad\qquad = e^{f(x)} + c$$

(Note that $du = f'(x)\, dx$ is not obtained algebraically from the preceding statement but should be considered as a statement that $f'(x)\, dx$ is replaced by du when performing the substitution $u = f(x)$.) Comparison of the left- and right-hand entries shows that when x is replaced by $f(x)$, the factor $f'(x)$ is required in the integrand for the same pattern of integration to apply. Recognition of these patterns makes the formal process of substitution unnecessary for many quite complicated integrals.

Example 1

Find $\int \sin^4 x \cos x\, dx$.

Solution

We notice that if $f(x) = \sin x$, then $f'(x) = \cos x$, and the given integral follows the pattern $\int \{f(x)\}^4 f'(x)\, dx = \frac{1}{5}\{f(x)\}^5 + c$, by comparison with the standard integral $\int x^4\, dx = \frac{1}{5}x^5 + c$. We write

$$\int \sin^4 x \cos x\, dx = \frac{1}{5}\sin^5 x + c, \ c \text{ constant.}$$

We think of the factor $f'(x) = \cos x$ as being 'used up' in the process of integration, just as $f'(x)\, dx$ formally became du when performing the substitution $u = f(x)$.

Example 2

Evaluate $\int_0^{\frac{\pi}{4}} \frac{e^{\tan x}}{\cos^2 x}\, dx$

Solution

If $f(x) = \tan x$, then $f'(x) = \sec^2 x = \dfrac{1}{\cos^2 x}$, and the given integral follows the pattern $\int e^{f(x)} f'(x)\, dx = e^{f(x)} + c$, by comparison with the standard integral $\int e^x\, dx = e^x + c$.

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{e^{\tan x}}{\cos^2 x}\, dx &= \int_0^{\frac{\pi}{4}} e^{\tan x} \sec^2 x\, dx \\ &= [e^{\tan x}]_0^{\frac{\pi}{4}} \\ &= e - 1\end{aligned}$$

Example 3

Find $\int x \cos (x^2)\, dx$.

Solution

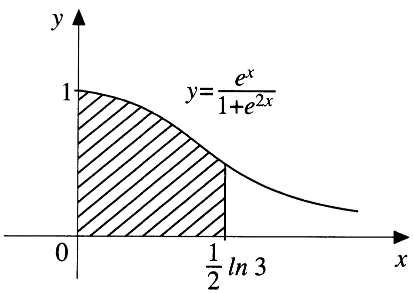
$$\int x \cos (x^2)\, dx = \frac{1}{2}\int \cos (x^2) . 2x\, dx = \frac{1}{2} \sin (x^2) + c, \qquad c \text{ constant.}$$

Example 4

Calculate the area bounded by the curve $y = \dfrac{e^x}{1 + e^{2x}}$, the x-axis, the y-axis and the ordinate $x = \dfrac{1}{2} \ln 3$.

Solution

Figure 5.1



Shaded area is A sq. units, where

$$\begin{aligned}A &= \int_0^{\frac{1}{2}\ln 3} \frac{e^x}{1 + e^{2x}}\, dx \\ &= \int_0^{\frac{1}{2}\ln 3} \frac{e^x}{1 + (e^x)^2}\, dx \\ &= \Big[\tan^{-1}(e^x)\Big]_0^{\frac{1}{2}\ln 3} \\ &= \tan^{-1}(e^{\ln\sqrt{3}}) - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4}\end{aligned}$$

Area is $\frac{\pi}{12}$ sq. units.

In this example we have noticed that if $f(x) = e^x$, then $f'(x) = e^x$ and the required integral follows the pattern $\int \frac{f'(x)}{1 + \{f(x)\}^2}\, dx = \tan^{-1}\{f(x)\} + c$, by comparison with the standard integral.

Example 5

Evaluate $\int_{\frac{3\pi}{4}}^{\pi} \tan x \, dx$

Solution

$$\begin{aligned}
\int_{\frac{3\pi}{4}}^{\pi} \tan x \, dx &= \int_{\frac{3\pi}{4}}^{\pi} \frac{\sin x}{\cos x} dx \\
&= -\int_{\frac{3\pi}{4}}^{\pi} -\frac{\sin x}{\cos x} dx \\
&= -\Big[\ln|\cos x|\Big]_{\frac{3\pi}{4}}^{\pi} \\
&= -\left\{\ln|-1| - \ln\left|-\frac{1}{\sqrt{2}}\right|\right\} \\
&= -\frac{1}{2}\ln 2.
\end{aligned}$$

Using the pattern
$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$
with $f(x) = \cos x$

Note the use of the absolute value signs in this example. The result $\int \frac{1}{x} dx = \ln x + c$ is true only if $x > 0$. For $x < 0$, making the substitution $u = -x$, we have $u > 0$ and

$$u = -x$$
$$\frac{du}{dx} = -1$$
$$du = -dx$$

$$\begin{aligned}
\int \frac{1}{x}\, dx &= \int -\frac{1}{u} . -du \\
&= \int \frac{1}{u} du \\
&= \ln u + c \qquad \text{(since } u > 0\text{)} \\
&= \ln(-x) + c
\end{aligned}$$

But $|x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$, hence $\int \frac{1}{x} dx = \ln|x| + c$ for both $x > 0$ and $x < 0$.

Similarly, when $f(x)$ is not positive throughout its domain,
$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c.$

Example 6

Find $\displaystyle\int \frac{dx}{x \ln x}$

Solution

$$\int \frac{dx}{x \ln x} = \int \frac{\left(\frac{1}{x}\right)}{\ln x} dx$$

$$= \ln|\ln x| + c$$

Using $f(x) = \ln x$ and pattern $\displaystyle\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c.$

Note that the natural domain of the integrand $\dfrac{1}{x \ln x}$ includes the interval $0 < x < 1$, hence $\ln x$ takes both positive and negative values and the absolute value signs are required. Compare this case with the last entry in Table 5.1. $\{x + \sqrt{(x^2 + a^2)}\} > 0$ for all values of x, since $\sqrt{(x^2 + a^2)} > \sqrt{x^2} = |x|$. Hence $\displaystyle\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \ln\{x + \sqrt{(x^2 + a^2)}\} + c$, and the absolute value signs are unnecessary (similarly with the last entry in Table 5.2).

Example 7

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{(2 - \cos^2 x)}} dx$

Solution

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{(2 - \cos^2 x)}} dx$$

$$= -\int_0^{\frac{\pi}{2}} \frac{-\sin x}{\sqrt{[(\sqrt{2})^2 - (\cos x)^2]}} dx$$

$$= -\left[\sin^{-1}\left(\frac{1}{\sqrt{2}} \cos x\right)\right]_0^{\frac{\pi}{2}}$$

$$= -\left\{\sin^{-1} 0 - \sin^{-1} \frac{1}{\sqrt{2}}\right\}$$

$$= \frac{\pi}{4}$$

Using $f(x) = \cos x$, $a = \sqrt{2}$, and the pattern

$$\int \frac{f'(x)}{\sqrt{[a^2 - \{f(x)\}^2]}} dx = \sin^{-1}\left\{\frac{1}{a} f(x)\right\} + c$$

Exercise 5.1

Find

1 $\displaystyle\int \frac{x}{1 + x^2}\, dx$

2 $\displaystyle\int \frac{x}{(1 + x^2)^2}\, dx$

3 $\displaystyle\int e^{\sin x} \cos x\, dx$

4 $\displaystyle\int e^x \sin (e^x)\, dx$

5 $\displaystyle\int x\sqrt{(1 + x^2)}\, dx$

6 $\displaystyle\int \frac{1}{\sqrt{(4 - x^2)}}\, dx$

7 $\displaystyle\int \frac{1}{1 + 4x^2}\, dx$

8 $\displaystyle\int \tan^3 x \sec^2 x\, dx$

9 $\displaystyle\int x \sec^2(x^2)\, dx$

10 $\displaystyle\int \frac{1}{\sqrt{x}} e^{\sqrt{x}}\, dx$

11 $\displaystyle\int \sec^3 x \tan x\, dx$

12 $\displaystyle\int \frac{\sin 2x}{2 + \sin^2 x}\, dx$

Evaluate

13 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \cot x\, dx$

14 $\displaystyle\int_1^e \frac{1}{x} \cos (\ln x)\, dx$

15 $\displaystyle\int_0^2 \frac{1}{4 + x^2}\, dx$

16 $\displaystyle\int_{\sqrt{2}}^3 \frac{1}{\sqrt{(x^2 - 1)}}\, dx$

17 $\displaystyle\int_0^2 \frac{1}{\sqrt{(x^2 + 4)}}\, dx$

18 $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1 - 4x^2)}}\, dx$

19 $\displaystyle\int_0^{\frac{\pi}{6}} \tan 2x \sec 2x\, dx$

20 $\displaystyle\int_0^{\ln 3} \frac{e^x}{1 + e^x}\, dx$

5.2 Reduction to standard form by algebraic rearrangement of the integrand

Some integrals can be reduced to standard form by expressing the integrand as a series of terms in rational powers of x.

Example 8

Find $\displaystyle\int \frac{(\sqrt{x}+1)^2}{\sqrt{x}}\,dx$

Solution

$(\sqrt{x}+1)^2 = x + 2\sqrt{x} + 1$

$\therefore \dfrac{(\sqrt{x}+1)^2}{\sqrt{x}} = \sqrt{x} + 2 + \dfrac{1}{\sqrt{x}}$

$$\begin{aligned}\int \frac{(\sqrt{x}+1)^2}{\sqrt{x}}\,dx &= \int (x^{\frac{1}{2}} + 2 + x^{-\frac{1}{2}})\,dx \\ &= \frac{2}{3}x^{\frac{3}{2}} + 2x + 2x^{\frac{1}{2}} + c \\ &= \frac{2}{3}x\sqrt{x} + 2x + 2\sqrt{x} + c\end{aligned}$$

When the integrand is a rational function of the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, using the division transformation and decomposition into partial fractions reduces the integral to a combination of standard forms. These procedures are investigated in Chapter 4, Sections 4.2 and 4.5.

Case 1
$Q(x)$ is linear and deg $P \geqslant$ deg Q.

Example 9

Find **(a)** $\displaystyle\int \frac{3x-2}{2x+1}\,dx$ **(b)** $\displaystyle\int \frac{2x^2-x+1}{x-2}\,dx$

Solution

(a)
$$\begin{aligned}\frac{3x-2}{2x+1} &= \frac{\frac{3}{2}(2x+1) - \frac{7}{2}}{2x+1} \\ &= \frac{3}{2} - \frac{7}{2}\cdot\frac{1}{2x+1}\end{aligned}$$

$$\begin{aligned}\int \frac{3x-2}{2x+1}\,dx &= \int \left(\frac{3}{2} - \frac{7}{4}\cdot\frac{2}{2x+1}\right)dx \\ &= \frac{3}{2}x - \frac{7}{4}\ln|2x+1| + c\end{aligned}$$

(Note that it is possible to proceed by inspection rather than by formal division.)

(b)
$$\begin{array}{r} 2x + 3 \\ x - 2\overline{\big)\,2x^2 - x + 1} \\ \underline{2x^2 - 4x} \\ 3x + 1 \\ \underline{3x - 6} \\ 7 \end{array}$$

$$\begin{aligned}&\int \frac{2x^2-x+1}{x-2}\,dx \\ &= \int \left(2x + 3 + \frac{7}{x-2}\right)dx \\ &= x^2 + 3x + 7\ln|x-2| + c\end{aligned}$$

Case 2
$Q(x)$ is a product of linear factors and deg $P <$ deg Q.

Example 10

Find $\displaystyle\int \frac{3x - 2}{(x - 1)(x - 2)}\,dx$

Solution

$$\frac{3x - 2}{(x - 1)(x - 2)} \equiv \frac{a}{x - 1} + \frac{b}{x - 2}$$

$$3x - 2 \equiv a(x - 2) + b(x - 1)$$

Put $x = 2$: $\quad b = 4$
Put $x = 1$: $\quad a = -1$

$$\int \frac{3x - 2}{(x - 1)(x - 2)}\,dx$$
$$= \int \left(\frac{-1}{x - 1} + \frac{4}{x - 2}\right) dx$$
$$= -\ln|x - 1| + 4\ln|x - 2| + c$$
$$= \ln\left\{\frac{(x - 2)^4}{|x - 1|}\right\} + c$$

Case 3
$Q(x)$ is a product of linear factors and deg $P \geqslant$ deg Q.
Note: In this case it is necessary to use the division transformation before seeking partial fractions.

Example 11

Evaluate $\displaystyle\int_2^3 \frac{x^2 + 1}{x^2 - x}\,dx$

Solution

By division, $\dfrac{x^2 + 1}{x^2 - x} = 1 + \dfrac{x + 1}{x^2 - x}$

Let $\dfrac{x + 1}{x(x - 1)} \equiv \dfrac{a}{x} + \dfrac{b}{x - 1}$

$$x + 1 \equiv a(x - 1) + bx$$

Put $x = 0$: $\quad a = -1$
Put $x = 1$: $\quad b = 2$

$$\int_2^3 \frac{x^2 + 1}{x^2 - x}\,dx$$
$$= \int_2^3 \left(1 - \frac{1}{x} + \frac{2}{x - 1}\right) dx$$
$$= [x]_2^3 - [\ln|x|]_2^3 + 2[\ln|x - 1|]_2^3$$
$$= 1 - \ln\frac{3}{2} + 2\ln 2$$
$$= 1 + \ln\frac{8}{3}$$

Case 4

$Q(x)$ has an irreducible quadratic factor. (Note that a quadratic has no linear factors with real coefficients if its discriminant is negative. Such a quadratic is termed irreducible over $\mathbb{R}$.)

Example 12

Evaluate $\displaystyle\int_{-1}^{0} \frac{2x+1}{x^2+2x+2}\,dx$

Solution

$x^2 + 2x + 2$ is irreducible ($\Delta < 0$), and completing the square gives $x^2 + 2x + 2 = (x+1)^2 + 1$. Make the substitution $u = x + 1$.

$u = x + 1$
$\dfrac{du}{dx} = 1$
$du = dx$
$x = -1 \Rightarrow u = 0$
$x = 0 \quad \Rightarrow u = 1$

$$\begin{aligned}
\int_{-1}^{0} \frac{2x+1}{x^2+2x+2}\,dx &= \int_{-1}^{0} \frac{2x+1}{(x+1)^2+1}\,dx \\
&= \int_{0}^{1} \frac{2u-1}{u^2+1}\,du \\
&= \int_{0}^{1}\left(\frac{2u}{u^2+1} - \frac{1}{u^2+1}\right)du \\
&= [\ln(u^2+1)]_0^1 - [\tan^{-1} u]_0^1 \\
&= \ln 2 - \frac{\pi}{4}
\end{aligned}$$

Example 13

Evaluate $\displaystyle\int_{-1}^{0} \frac{7x^2-5x+4}{(x-1)(x^2+1)}\,dx$

Solution

Let $\dfrac{7x^2-5x+4}{(x-1)(x^2+1)} \equiv \dfrac{a}{x-1} + \dfrac{bx+c}{x^2+1}$ $\quad a, b, c$ constants.

Then $\quad 7x^2 - 5x + 4 \equiv a(x^2+1) + (bx+c)(x-1)$

Put $x = 1$: $\quad 6 = 2a \Rightarrow a = 3$

Equate coefficients of x^2: $\quad 7 = a + b \Rightarrow b = 4$

Equate constant terms: $\quad 4 = a - c \Rightarrow c = -1$

$$\text{Hence } \int_{-1}^{0} \frac{7x^2 - 5x + 4}{(x-1)(x^2+1)}\,dx = \int_{-1}^{0} \left(\frac{3}{x-1} + \frac{4x-1}{x^2+1}\right) dx$$

$$= \int_{-1}^{0} \left(3\frac{1}{x-1} + 2\frac{2x}{x^2+1} - \frac{1}{x^2+1}\right) dx$$

$$= \left[3\ln|x-1| + 2\ln(x^2+1) - \tan^{-1} x\right]_{-1}^{0}$$

$$= -5\ln 2 - \frac{\pi}{4}$$

Example 14

Find $\displaystyle\int \frac{3x-1}{(x^2+1)(x^2+2)}\,dx$

Solution

Let $\displaystyle\frac{3x-1}{(x^2+1)(x^2+2)} \equiv \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+2}$, a, b, c, d constants.

Then $3x - 1 \equiv (ax+b)(x^2+2) + (cx+d)(x^2+1)$

$$\left.\begin{array}{ll}\text{Equate coefficients of } x^3: & 0 = a + c \\ \text{Equate coefficients of } x: & 3 = 2a + c\end{array}\right\} \Rightarrow a = 3 \text{ and } c = -3$$

$$\left.\begin{array}{ll}\text{Equate coefficients of } x^2: & 0 = b + d \\ \text{Equate constant terms:} & -1 = 2b + d\end{array}\right\} \Rightarrow b = -1 \text{ and } d = 1$$

$$\int \frac{3x-1}{(x^2+1)(x^2+2)}\,dx$$

$$= \int \left(\frac{3x-1}{x^2+1} + \frac{-3x+1}{x^2+2}\right) dx$$

$$= \int \left(\frac{3}{2}\frac{2x}{x^2+1} - \frac{1}{x^2+1} - \frac{3}{2}\frac{2x}{x^2+2} + \frac{1}{x^2+2}\right) dx$$

$$= \frac{3}{2}\ln(x^2+1) - \tan^{-1} x - \frac{3}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$$

Note that in each of examples 12, 13 and 14, the numerator of the integrand has lower degree than the denominator. If this is not the case it is necessary to perform the division transformation before seeking partial fractions, as in Example 11.

Exercise 5.2

Find

1 $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$ {Show first that $x^2+2x+2=1+(x+1)^2$}

2 $\displaystyle\int \frac{1}{\sqrt{(2x-x^2)}}\,dx$ {Show first that $2x-x^2=1-(x-1)^2$}

3 $\displaystyle\int \frac{x-1}{x^2+1}\,dx$

4 $\displaystyle\int \frac{x(2x+1)}{x+1}\,dx$

5 $\displaystyle\int \frac{x+1}{x(2x+1)}\,dx$

6 $\displaystyle\int \frac{x^2}{(x+1)(x+2)}\,dx$

7 $\displaystyle\int \frac{2x+3}{x^2+2x+5}\,dx$

8 $\displaystyle\int \frac{6x-10}{(x+1)(x-3)}\,dx$

9 $\displaystyle\int \frac{4}{x^2-2x-1}\,dx$

10 $\displaystyle\int \frac{4x-x^2}{(x+1)(x^2+4)}\,dx$

11 $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

12 $\displaystyle\int \frac{3}{(x^2+1)(x^2+4)}\,dx$

Evaluate

13 $\displaystyle\int_1^3 \frac{1}{x^2-2x+5}\,dx$ {Show first that $x^2-2x+5=(x-1)^2+4$}

14 $\displaystyle\int_{-1}^0 \frac{1}{\sqrt{(3-2x-x^2)}}\,dx$ {Show first that $3-2x-x^2=4-(x+1)^2$}

15 $\displaystyle\int_0^2 \frac{x+1}{x^2+4}\,dx$

16 $\displaystyle\int_0^1 \frac{7+x-2x^2}{2-x}\,dx$

17 $\displaystyle\int_1^2 \frac{2x-3}{x^2-2x+2}\,dx$

18 $\displaystyle\int_0^3 \frac{x^2+4x+5}{(x+1)(x+3)}\,dx$

19 $\displaystyle\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}\,dx$

20 $\displaystyle\int_0^{\sqrt{3}} \frac{8}{(x^2+1)(x^2+9)}\,dx$

5.3 Using substitution to reduce integrals to standard form

Example 15

Evaluate $\displaystyle\int_4^{12} \frac{1}{(4+x)\sqrt{x}}\,dx$

Solution

The substitution $u = \sqrt{x}$ reduces the integrand to a rational function.

$$u = \sqrt{x}$$
$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$
$$du = \frac{1}{2\sqrt{x}}\,dx$$
$$x = 4 \Rightarrow u = 2$$
$$x = 12 \Rightarrow u = 2\sqrt{3}$$

$$\begin{aligned}\int_4^{12} \frac{1}{(4+x)\sqrt{x}}\,dx &= 2\int_4^{12} \frac{1}{(4+x)}\frac{1}{2\sqrt{x}}\,dx \\ &= 2\int_2^{2\sqrt{3}} \frac{1}{4+u^2}\,du \\ &= 2\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_2^{2\sqrt{3}} \\ &= \tan^{-1}\sqrt{3} - \tan^{-1} 1 \\ &= \frac{\pi}{12}\end{aligned}$$

Example 16

Find $\displaystyle\int \frac{x^3}{\sqrt{(x^2+1)}}\,dx$

Solution

Make the substitution $u = \sqrt{(x^2+1)}$, or equivalently $u^2 = x^2 + 1$, $u > 0$.

$$\begin{aligned}u^2 &= x^2 + 1,\ u > 0 \\ 2u\frac{du}{dx} &= 2x \\ udu &= xdx \\ \sqrt{(x^2+1)} &= |u| = u\end{aligned}$$

$$\begin{aligned}\int \frac{x^3}{\sqrt{(x^2+1)}}\,dx &= \int \frac{x^2}{\sqrt{(x^2+1)}}\,xdx \\ &= \int \frac{u^2-1}{u}\,udu \\ &= \int (u^2-1)\,du \\ &= \frac{1}{3}u^3 - u + c \\ &= \frac{1}{3}(x^2-2)\sqrt{(x^2+1)} + c\end{aligned}$$

Note that using implicit differentiation (Section 1.9, in Chapter 1) on $u^2 = x^2 + 1$ is much easier than deriving $u = \sqrt{(x^2 + 1)}$ to find $\frac{du}{dx}$. When substitutions like $u^2 = x^2 + 1$ are made, it is necessary to restrict the values of u so that each x-value in the domain of the integrand corresponds to exactly one value of u. We have chosen the restriction $u > 0$, and this restriction then applies when substituting for x in the integrand.

Trigonometric substitutions are useful when the integrand involves factors like $\sqrt{(a^2 - x^2)}$ or $\sqrt{(x^2 \pm a^2)}$, since the identities $\cos^2 \theta + \sin^2 \theta = 1$ and $1 + \tan^2 \theta = \sec^2 \theta$ can then be used to simplify such expressions. For example

$x = a \sin \theta \Rightarrow \sqrt{(a^2 - x^2)} = \sqrt{[a^2(1 - \sin^2 \theta)]} = |a \cos \theta|$

Choosing $a > 0$, and restricting θ so that $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$, each value of x in the domain of $\sqrt{(a^2 - x^2)}$ corresponds to exactly one value of θ, and $a \cos \theta > 0 \Rightarrow \sqrt{(a^2 - x^2)} = a \cos \theta$.
Similarly, for $a > 0$

$x = a \tan \theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \Rightarrow \sqrt{(a^2 + x^2)} = a \sec \theta$

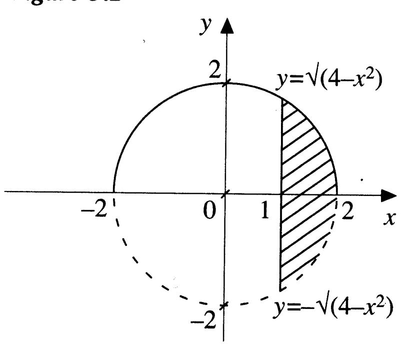
$x = a \sec \theta,\ -\pi < \theta \leqslant -\frac{\pi}{2}$ or $0 \leqslant \theta < \frac{\pi}{2} \Rightarrow \sqrt{(x^2 - a^2)} = a \tan \theta$

Example 17

Find the area of the minor segment of a circle of radius 2 cut off by a chord 1 unit from the centre of the circle.

Solution

Figure 5.2



Let the shaded area be A sq. units.

Then $A = 2\int_1^2 \sqrt{(4 - x^2)}\, dx$

Let $x = 2\sin\theta,\ -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$

$$\frac{dx}{d\theta} = 2\cos\theta$$
$$dx = 2\cos d\theta$$
$$x = 2 \Rightarrow \theta = \frac{\pi}{2}$$
$$x = 1 \Rightarrow \theta = \frac{\pi}{6}$$
$$\sqrt{(4 - x^2)} = \sqrt{[4(1 - \sin^2\theta)]} = 2\cos\theta$$

$$A = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cos\theta . 2\cos\theta\, d\theta$$
$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$$
$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$$
$$= 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= 4\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + 2\left(\sin\pi - \sin\frac{\pi}{3}\right)$$

$\therefore$ area is $\frac{4\pi}{3} - \sqrt{3}$ sq. units

Example 18

Find $\displaystyle\int \frac{dx}{x^2\sqrt{(x^2 - 4)}}$

Solution

Let $x = 2\sec\theta,\ -\pi \leqslant \theta < -\frac{\pi}{2},\ 0 \leqslant \theta < \frac{\pi}{2}$

$$\frac{dx}{d\theta} = 2\sec\theta\tan\theta$$
$$dx = 2\sec\theta\tan\theta\, d\theta$$
$$\sqrt{(x^2 - 4)} = \sqrt{[4(\sec^2\theta - 1)]} = 2\tan\theta$$

$$\int \frac{dx}{x^2\sqrt{(x^2 - 4)}} = \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta . 2\tan\theta}\, d\theta$$
$$= \frac{1}{4}\int \cos\theta\, d\theta$$
$$= \frac{1}{4}\sin\theta + c$$

But $x = 2\sec\theta \Rightarrow \frac{2}{x} = \cos\theta$, $\quad \therefore \sin^2\theta = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$

$$\left.\begin{array}{l} x > 0 \Rightarrow 0 \leqslant \theta < \frac{\pi}{2} \Rightarrow \sin\theta > 0 \\ x < 0 \Rightarrow -\pi \leqslant \theta < -\frac{\pi}{2} \Rightarrow \sin\theta < 0 \end{array}\right\} \Rightarrow \quad \sin\theta = \frac{\sqrt{(x^2 - 4)}}{x}$$

$$\therefore \int \frac{dx}{x^2\sqrt{(x^2 - 4)}} = \frac{\sqrt{(x^2 - 4)}}{4x} + c$$

The standard integral $\displaystyle\int \frac{1}{\sqrt{(x^2 + a^2)}}\, dx = \ln\{x + \sqrt{(x^2 + a^2)}\} + c$ can be established by using the substitution $x = a\tan\theta$, $(a > 0)$.

Let $x = a \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\frac{dx}{d\theta} = a \sec^2 \theta$$

$$dx = a \sec^2 \theta \, d\theta$$

$$\sqrt{(x^2 + a^2)} = \sqrt{[a^2(\tan^2 \theta + 1)]} = a \sec \theta$$

$$\begin{aligned}
\int \frac{1}{\sqrt{(x^2 + a^2)}} dx &= \int \frac{1}{a \sec \theta} . a \sec^2 \theta \, d\theta \\
&= \int \sec \theta \, d\theta \\
&= \int \frac{\sec \theta(\sec \theta + \tan \theta)}{(\sec \theta + \tan \theta)} d\theta \\
&= \int \frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} d\theta \\
&= \ln |\sec \theta + \tan \theta| + c \\
&= \ln \left| \frac{1}{a} (\sqrt{(x^2 + a^2)} + x) \right| + c \\
&= \ln \{x + \sqrt{(x^2 + a^2)}\} + k
\end{aligned}$$

(Note that $k = c - \ln a$ is a constant, and $x + \sqrt{(x^2 + a^2)} > 0$. Note also the procedure for integrating $\sec \theta$. Similarly, by writing

$\operatorname{cosec} \theta = \dfrac{\operatorname{cosec} \theta(\operatorname{cosec} \theta - \cot \theta)}{(\operatorname{cosec} \theta - \cot \theta)} = \dfrac{-\operatorname{cosec} \theta \cot \theta + \operatorname{cosec}^2 \theta}{\operatorname{cosec} \theta - \cot \theta}$, we can

obtain $\int \operatorname{cosec} \theta \, d\theta = \ln |\operatorname{cosec} \theta - \cot \theta|$)

The substitution $t = \tan \frac{\theta}{2}$, and the identities $\sin \theta = \dfrac{2t}{1 + t^2}$, $\cos \theta = \dfrac{1 - t^2}{1 + t^2}$ and $\tan \theta = \dfrac{2t}{1 - t^2}$ can be used to convert an integrand involving trigonometric functions into a rational function of t.

Example 19

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin \theta} d\theta$

Solution

Let $t = \tan \frac{\theta}{2}$, $-\pi < \theta < \pi$

$$\frac{dt}{d\theta} = \frac{1}{2} \sec^2\left(\frac{\theta}{2}\right)$$

$$\frac{dt}{d\theta} = \frac{1}{2}(1 + t^2)$$

$$d\theta = \frac{2}{1 + t^2} dt$$

$\theta = 0 \Rightarrow t = 0$

$\theta = \frac{\pi}{2} \Rightarrow t = 1$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin \theta} d\theta &= \int_0^1 \frac{1}{1 + \left(\frac{2t}{1 + t^2}\right)} . \frac{2}{(1 + t^2)} dt \\
&= \int_0^1 \frac{2}{1 + t^2 + 2t} dt \\
&= 2\int_0^1 \frac{1}{(1 + t)^2} dt \\
&= -2\left[\frac{1}{1 + t}\right]_0^1 \\
&= 1
\end{aligned}$$

Exercise 5.3

Find, using the given substitution

1 $\displaystyle\int \frac{1}{\sqrt{x}\,\sqrt{(1-x)}}\,dx \quad (u = \sqrt{x})$

2 $\displaystyle\int \frac{x}{x^4 - 1}\,dx \quad (u = x^2)$

3 $\displaystyle\int x\sqrt{(x+1)}\,dx \quad (u^2 = x + 1)$

4 $\displaystyle\int x^2\sqrt{(x-1)}\,dx \quad (u^2 = x - 1)$

5 $\displaystyle\int \frac{1}{e^x + 1}\,dx \quad (u = e^x)$

6 $\displaystyle\int \frac{e^x + e^{2x}}{1 + e^{2x}}\,dx \quad (u = e^x)$

7 $\displaystyle\int \frac{\sqrt{x}}{1 + x}\,dx \quad (u = \sqrt{x})$

8 $\displaystyle\int \sqrt{\left(\frac{x}{1-x}\right)}\,dx \quad (x = \sin^2\theta)$

9 $\displaystyle\int \frac{\sqrt{(16 - x^2)}}{x^2}\,dx \quad (x = 4\sin\theta)$

10 $\displaystyle\int \sqrt{\left(\frac{1+x}{1-x}\right)}\,dx \quad (x = \cos 2\theta)$

11 $\displaystyle\int \operatorname{cosec} x\,dx \quad \left(t = \tan\frac{x}{2}\right)$

12 $\displaystyle\int \sec x\,dx \quad \left(t = \tan\frac{x}{2}\right)$

Evaluate, using the given substitution

13 $\displaystyle\int_0^1 \frac{x}{x^4 + 1}\,dx \quad (u = x^2)$

14 $\displaystyle\int_4^9 \frac{1}{(x-1)\sqrt{x}}\,dx \quad (u = \sqrt{x})$

15 $\displaystyle\int_2^6 x\sqrt{(6-x)}\,dx \quad (u^2 = 6 - x)$

16 $\displaystyle\int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{x^2\sqrt{(1+x^2)}}\,dx \quad (x = \tan\theta)$

17 $\displaystyle\int_0^1 \frac{\sqrt{x}}{1+x}\,dx \quad (x = \tan^2\theta)$

18 $\displaystyle\int_0^2 \sqrt{\left(\frac{x}{4-x}\right)}\,dx \quad (x = 4\sin^2\theta)$

19 $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{1 - \sin x}\,dx \quad \left(t = \tan\frac{x}{2}\right)$

20 $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{3 + 5\cos x}\,dx \quad \left(t = \tan\frac{x}{2}\right)$

5.4 Integrals of trigonometric functions

The standard trigonometric identities are useful for finding some trigonometric integrals. In Example 17, the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ was used to find the integral of $\cos^2\theta$. Similarly, the integral of $\sin^2\theta$ is found by using $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$. The integrals of $\tan^2\theta$ and $\cot^2\theta$ are found using the identities $\tan^2\theta = \sec^2\theta - 1$ and $\cot^2\theta = \operatorname{cosec}^2\theta - 1$ respectively.

Example 20

Find $\int \tan^3 \theta \, d\theta$

Solution

$$\begin{aligned}\int \tan^3 \theta \, d\theta &= \int \tan \theta . \tan^2 \theta \, d\theta \\ &= \int \tan \theta \, (\sec^2 \theta - 1) \, d\theta \\ &= \int (\tan \theta \sec^2 \theta - \tan \theta) \, d\theta \\ &= \int \left(\tan \theta \sec^2 \theta + \frac{-\sin \theta}{\cos \theta}\right) d\theta \\ &= \frac{1}{2} \tan^2 \theta + \ln|\cos \theta| + c\end{aligned}$$

Note that the introduction of $\sec^2 \theta$ gave the pattern $\int f(\theta) f'(\theta) \, d\theta = \frac{1}{2}\{f(\theta)\}^2$, where $f(\theta) = \tan \theta$.

Example 21

Find $\int \sin^5 \theta \cos^4 \theta \, d\theta$

Solution

We note that $\frac{d}{d\theta} \cos \theta = -\sin \theta$, and that $\sin^4 \theta = (1 - \cos^2 \theta)^2$. Hence we can rewrite the integrand as a sum of terms of the form $\cos^n \theta(-\sin \theta)$, each of which follows the pattern $\{f(\theta)\}^n f'(\theta)$, with integral $\frac{1}{n+1}\{f(\theta)\}^{n+1}$, as in Example 1.

Let $I = \int \sin^5 \theta \cos^4 \theta \, d\theta = -\int \sin^4 \theta \cos^4 \theta(-\sin \theta) \, d\theta$

Then $\sin^4 \theta = (1 - \cos^2 \theta)^2 = 1 - 2\cos^2 \theta + \cos^4 \theta$ gives

$$I = -\int \cos^4 \theta(-\sin \theta) \, d\theta + 2\int \cos^6 \theta(-\sin \theta) \, d\theta - \int \cos^8 \theta(-\sin \theta) \, d\theta$$

$$= -\frac{1}{5} \cos^5 \theta + \frac{2}{7} \cos^7 \theta - \frac{1}{9} \cos^9 \theta + c$$

All integrals of the form $\int \sin^m \theta \cos^n \theta \, d\theta$, where at least one of m and n is odd, could be found in this way.

When the integrand is a product of trigonometric functions of the form $\sin m\theta \cos n\theta$, $\cos m\theta \cos n\theta$, or $\sin m\theta \sin n\theta$, we convert these products into sums by using the following identities:

$2 \sin p \cos q = \sin (p - q) + \sin (p + q)$

$2 \cos p \cos q = \cos (p - q) + \cos (p + q)$

$2 \sin p \sin q = \cos (p - q) - \cos (p + q)$

Proof is by expansion of right-hand sides.

Example 22

Evaluate $\int_0^{\frac{\pi}{6}} \sin 2\theta \cos 3\theta \, d\theta$

Solution

$2 \sin 2\theta \cos 3\theta = \sin (2\theta - 3\theta) + \sin (2\theta + 3\theta) = -\sin \theta + \sin 5\theta$

$$\begin{aligned}
\therefore \int_0^{\frac{\pi}{6}} \sin 2\theta \cos 3\theta \, d\theta &= \frac{1}{2}\int_0^{\frac{\pi}{6}} (\sin 5\theta - \sin\theta) \, d\theta \\
&= \frac{1}{2}\left[-\frac{1}{5}\cos 5\theta + \cos \theta\right]_0^{\frac{\pi}{6}} \\
&= \frac{1}{2}\left\{-\frac{1}{5}\left(\cos \frac{5\pi}{6} - \cos 0\right) + \left(\cos \frac{\pi}{6} - \cos 0\right)\right\} \\
&= \frac{3\sqrt{3} - 4}{10}
\end{aligned}$$

Exercise 5.4

Find, using the given substitution or otherwise

1 $\int \sin^2 x \cos^3 x \, dx \quad (u = \sin x)$

2 $\int \cos^2 x \sin^5 x \, dx \quad (u = \cos x)$

3 $\int \frac{\cos^3 x}{\sin^2 x} \, dx \quad (u = \sin x)$

4 $\int \frac{\sin^3 x}{\cos^5 x} \, dx \quad (u = \cos x)$

5 $\int \sqrt[3]{(\sin x)} \cos^3 x \, dx \quad (u = \sin x)$

6 $\int \cos 6x \cos 2x \, dx$

7 $\int \sin 6x \sin 2x \, dx$

8 $\int \sin 3x \cos x \, dx$

9 $\int \cos 3x \sin x \, dx$

10 $\int \cos 4x \cos 2x \, dx$

11 $\int \sin 4x \cos 3x \, dx$

12 $\int \cos 5x \sin 2x \, dx$

Evaluate

13 $\int_0^{\frac{\pi}{4}} (\tan^3 x + \tan x) \, dx$

14 $\int_0^{\frac{\pi}{3}} (\sin^3 x - \sin x) \, dx$

15 $\int_0^{\frac{\pi}{2}} \surd(\cos x) \sin^3 x \, dx$

16 $\int_0^{\frac{\pi}{4}} \sin 4x \sin 2x \, dx$

5.5 Integration by parts

Let $F(x)$ be a primitive function of $f(x)$. Using the product rule for differentiation,
$\frac{d}{dx}F(x)g(x) = f(x)g(x) + F(x)g'(x)$. Integrating both sides with respect to x,

$$F(x)g(x) = \int f(x)g(x) \, dx + \int F(x)g'(x) \, dx, \quad \text{therefore}$$

$\int f(x)g(x) \, dx$	$=$	$F(x)g(x)$	$-$	$\int F(x)g'(x) \, dx$
$\int$ Product	$=$	[integrate first leave second]	$-$	$\int$ [integrate first derive second]

This identity enables us to convert an integral which cannot be done directly into a different integral. This procedure is called *integration by parts* and the identity above is perhaps remembered more readily in words than in symbols.

Example 23

(a) Find $\int x \ln x \, dx$

(b) Evaluate $\int_1^2 \ln x \, dx$

Solution

We note that $\frac{d}{dx} \ln x = \frac{1}{x}$. Hence, integration by parts, with $\ln x$ as the second function, removes $\ln x$ from the integrand, leaving powers of x.

(a) $$\begin{aligned}\int x \ln x \, dx &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \frac{1}{x} \, dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c \\ &= \frac{1}{4}x^2 (2 \ln x - 1) + c\end{aligned}$$

(b) $$\begin{aligned}\int_1^2 \ln x \, dx &= \int_1^2 1 . \ln x \, dx \\ &= \Big[x \ln x \Big]_1^2 - \int_1^2 x \frac{1}{x} \, dx \\ &= 2 \ln 2 - \int_1^2 1 \, dx \\ &= 2 \ln 2 - 1\end{aligned}$$

Example 24

Evaluate $\int_0^{\pi} e^x \sin x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi} e^x \sin x \, dx &= \Big[e^x \sin x \Big]_0^{\pi} - \int_0^{\pi} e^x \cos x \, dx \\ &= 0 - \int_0^{\pi} e^x \cos x \, dx \\ &= -\left\{ \Big[e^x \cos x \Big]_0^{\pi} - \int_0^{\pi} e^x (- \sin x) \, dx \right\} \\ &= e^{\pi} + 1 - \int_0^{\pi} e^x \sin x \, dx\end{aligned}$$

$\therefore 2\int_0^{\pi} e^x \sin x \, dx = e^{\pi} + 1.$ Hence $\int_0^{\pi} e^x \sin x \, dx = \frac{1}{2} (e^{\pi} + 1).$

Note that integration by parts can be used to find both definite and indefinite integrals and sometimes must be carried out more than once, as in Example 24, where we found an equation for the given integral, rather than perform a direct integration. Note also the technique of introducing the factor 1 in the integrand to enable integration by parts (Example 23 (b)). Frequently the new integral resulting from integration by parts requires recognition of a pattern of integration, algebraic rearrangement of the new integrand, or substitution, in order to complete the process.

Example 25

(a) Find $\int \sin^{-1} x \, dx$ **(b)** Evaluate $\int_0^1 x \sin^{-1} x \, dx$

Solution

(a) $$\int 1 . \sin^{-1} x \, dx = x \sin^{-1} x - \int x . \frac{1}{\sqrt{(1 - x^2)}} \, dx$$

$$= x \sin^{-1} x + \frac{1}{2} \int (1 - x^2)^{-\frac{1}{2}} (-2x) \, dx$$

$\therefore$ using the pattern $\int \left\{f(x)\right\}^{-\frac{1}{2}} f'(x) \, dx = 2 \left\{f(x)\right\}^{\frac{1}{2}}$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{(1 - x^2)} + c$$

(b) $$\int_0^1 x \sin^{-1} x \, dx = \left[\frac{1}{2} x^2 \sin^{-1} x\right]_0^1 - \int_0^1 \frac{1}{2} x^2 \frac{1}{\sqrt{(1 - x^2)}} \, dx = \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{x^2}{\sqrt{(1 - x^2)}} \, d$$

Let $u^2 = 1 - x^2, \quad u > 0$

$2u \dfrac{du}{dx} = -2x$

$u du = -x dx$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 0$

$$\int_0^1 \frac{x}{\sqrt{(1 - x^2)}} x \, dx = \int_1^0 \frac{\sqrt{(1 - u^2)}}{u} (-u) \, du$$

$$= \int_0^1 \sqrt{(1 - u^2)} \, du$$

$$= \frac{\pi}{4}$$

(Area of $\frac{1}{4}$ of a circle of radius 1.)

Hence $\int_0^1 x \sin^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}$

Recurrence formulae

Repeated application of integration by parts can be used to reduce the power of a function in the integrand stepwise until the integral is known.

Example 26

$I_n = \int x^n e^x \, dx$. Show that $I_n = x^n e^x - nI_{n-1}$, and hence find $\int x^3 e^x \, dx$.

Solution

$$\int x^n e^x \, dx = x^n e^x - \int nx^{n-1} e^x \, dx \qquad \text{(integration by parts)}$$

$$\begin{aligned}
I_n &= x^n e^x - nI_{n-1}{}^* && \text{(recurrence formula)} \\
\text{Hence } I_3 &= x^3 e^x - 3I_2 && (\text{*with } n = 3) \\
&= x^3 e^x - 3\,(x^2 e^x - 2I_1) && (\text{* with } n = 2) \\
&= (x^3 - 3x^2)\, e^x + 6\,(xe^x - 1I_0) && (\text{* with } n = 1) \\
\therefore I_3 &= (x^3 - 3x^2 + 6x)\, e^x - 6I_0
\end{aligned}$$

Then $I_0 = \int e^x \, dx = e^x + c \Rightarrow I_3 = (x^3 - 3x^2 + 6x - 6)\, e^x + c$

$$\therefore \int x^3 e^x \, dx = (x^3 - 3x^2 + 6x - 6)\, e^x + c$$

Using the recurrence (or reduction) formula * is equivalent to repeated application of integration by parts.

Example 27

$I_n = \int_0^1 x^n \sqrt{(1 - x)} \, dx, \quad n = 0, 1, 2, \ldots$ Show that $I_n = \dfrac{2n}{2n + 3} I_{n-1}$ and hence evaluate $\int_0^1 x^3 \sqrt{(1 - x)} \, dx$. Show that $I_n = \dfrac{n!(n + 1)!}{(2n + 3)!} 4^{n+1}$

Solution

$$\int_0^1 x^n \sqrt{(1-x)}\,dx = \left[x^n\left\{-\frac{2}{3}(1-x)^{\frac{3}{2}}\right\}\right]_0^1 + \frac{2}{3}\int_0^1 nx^{n-1}(1-x)\sqrt{(1-x)}\,dx$$

$$= 0 + \frac{2n}{3}\int_0^1 [x^{n-1}\sqrt{(1-x)} - x^n\sqrt{(1-x)}]\,dx$$

$$\therefore 3I_n = 2n(I_{n-1} - I_n)$$

$$\text{Hence } I_n = \frac{2n}{2n+3} I_{n-1}$$

$$\therefore I_3 = \frac{6}{9} I_2 = \frac{6}{9}\frac{4}{7} I_1 = \frac{6}{9}\frac{4}{7}\frac{2}{5} I_0$$

$$\text{But } I_0 = \int_0^1 \sqrt{(1-x)}\,dx = -\frac{2}{3}\left[(1-x)^{\frac{2}{3}}\right]_0^1 = \frac{2}{3}$$

$$\therefore I_3 = \frac{6}{9}\frac{4}{7}\frac{2}{5}\frac{2}{3} \Rightarrow \int x^3 \sqrt{(1-x)}\,dx = \frac{32}{315}$$

$$I_n = \frac{2n}{2n+3} I_{n-1} = \frac{2n}{2n+3}\frac{2n-2}{2n+1} I_{n-2}$$

Continued application of the recurrence formula gives

$$I_n = \frac{2n}{2n+3}\frac{2n-2}{2n+1}\frac{2n-4}{2n-1}\cdots\frac{6}{9}\frac{4}{7}\frac{2}{5} I_0$$

$$= \frac{2^n\, n(n-1)(n-2)\ldots 3.2.1}{(2n+3)(2n+1)(2n-1)\ldots 9.7.5}\frac{2}{3}$$

$$= \frac{2^{n+1}\, n!\,(2n+2)(2n)(2n-2)\ldots 8.6.4.2}{(2n+3)!}$$

$$= \frac{2^{n+1}\, n!\, 2^{n+1}(n+1)n(n-1)\ldots 4.3.2.1}{(2n+3)!}$$

$$= \frac{n!(n+1)!}{(2n+3)!} 4^{n+1}$$

In Example 32 in Chapter 2 we used De Moivre's theorem to convert $\cos^4\theta$ into a sum of terms of the form $\cos n\theta$ and hence found $\int \cos^4\theta\, d\theta$.

Alternatively, such integrals can be found by establishing recurrence formulae using integration by parts.

Example 28

$I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$, $n = 0, 1, \ldots$ Show that $I_n = \frac{n-1}{n} I_{n-2}$ and hence evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x\, dx$ and $\int_0^{\frac{\pi}{2}} \cos^5 x\, dx$

Solution

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^n x dx &= \int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x\, dx \\
&= \Big[\sin x \cos^{n-1} x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \,.\, (n-1) \cos^{n-2} x\, (-\sin x)\, dx \\
&= 0 + (n-1) \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^{n-2} x\, dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} (\cos^{n-2} x - \cos^n x)\, dx \\
\therefore I_n &= (n-1)(I_{n-2} - I_n). \qquad \text{Hence } I_n = \frac{n-1}{n} I_{n-2}
\end{aligned}$$

Therefore $I_4 = \frac{3}{4} I_2 = \frac{3}{4}\frac{1}{2} I_0$, where $I_0 = \int_0^{\frac{\pi}{2}} 1\, dx = \frac{\pi}{2}$

and $I_5 = \frac{4}{5} I_3 = \frac{4}{5}\frac{2}{3} I_1$, where $I_1 = \int_0^{\frac{\pi}{2}} \cos x\, dx = 1$

$\therefore \int_0^{\frac{\pi}{2}} \cos^4 x\, dx = \frac{3}{16}\pi$ and $\int_0^{\frac{\pi}{2}} \cos^5 x\, dx = \frac{8}{15}$

Note that when the recurrence formula relates I_n to I_{n-2} it is necessary to consider separately the cases n even, and n odd, when finding expressions for I_n.

Using the recurrence formula $I_n = \frac{n-1}{n} I_{n-2}$ for $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$,

for n even, $I_n = \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4} \cdots \frac{3}{4}\frac{1}{2} I_0$, where $I_0 = \frac{\pi}{2}$

$$\begin{aligned}
&= \frac{n(n-1)(n-2)(n-3) \ldots 3.2.1}{\{n(n-2)(n-4) \ldots 4.2\}^2} \frac{\pi}{2} \\
&= \frac{n!\, \pi}{\left\{\left(\frac{1}{2} n\right)!\right\}^2 2^{n+1}}
\end{aligned}$$

$$\text{for } n \text{ odd, } I_n = \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{4}{5}\frac{2}{3}I_1, \qquad \text{where } I_1 = 1$$

$$= \frac{\{(n-1)(n-3)(n-5)\ \dots\ 4.2\}^2}{n!}.\ 1$$

$$= \frac{\left\{\left[\frac{1}{2}(n-1)\right]!\right\}^2}{n!}\ 2^{n-1}$$

Exercise 5.5

Find

1 $\int x\, e^x\, dx$

2 $\int x^2 \ln x\, dx$

3 $\int x \sin x\, dx$

4 $\int x^2 \cos x\, dx$

5 $\int x \cos^2 x\, dx$

6 $\int \tan^{-1} x\, dx$

7 $\int x \tan^{-1} x\, dx$

8 $\int e^x \cos x\, dx$

9 $\int x \sec x \tan x\, dx$

10 $\int \sec^3 x\, dx$

Evaluate

11 $\int_0^1 x\, e^{2x}\, dx$

12 $\int_1^e (\ln x)^2\, dx$

13 $\int_0^{\frac{\pi}{2}} x \cos x\, dx$

14 $\int_0^{\frac{\pi}{2}} x \sin x \cos x\, dx$

15 If $I_n = \int \tan^n x\, dx$ for $n \geqslant 0$ show that $I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$ for $n \geqslant 2$

16 If $I_n = \int x (\ln x)^n\, dx$ for $n \geqslant 0$ show that $I_n = \frac{1}{2}x^2 (\ln x)^n - \frac{n}{2} I_{n-1}$ for $n \geqslant 1$

17 If $I_n = \int \sin^n x \, dx$ for $n \geqslant 0$ show that $I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$ for $n \geqslant 2$. Hence show that if $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ for $n \geqslant 0$, then $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geqslant 2$ and deduce that $I_5 . I_6 = \frac{\pi}{12}$

18 If $I_n = \int \sec^n x \, dx$ for $n \geqslant 0$ show that
$I_n = \frac{1}{n-1} \tan x \sec^{n-2} x + \frac{n-2}{n-1} I_{n-2}$ for $n \geqslant 2$

Hence show that if $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$ for $n \geqslant 0$, then
$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ for $n \geqslant 2$ and deduce that $I_6 = \frac{28}{15}$

19 If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$ for $n \geqslant 0$ show that

$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) \, I_{n-2}$ for $n \geqslant 2$. Hence evaluate I_6.

20 If $I_n = \int_0^1 x(1 - x^3)^n \, dx$ for $n \geqslant 0$ show that $I_n = \frac{3n}{3n+2} I_{n-1}$ for $n \geqslant 1$. Hence find an expression for I_n in terms of n for $n \geqslant 0$.

5.6 Further properties of definite integrals

While the indefinite integral $\int f(x) \, dx$ is a function of the variable in the integrand, the definite integral $\int_a^b f(x) \, dx$ is a constant obtained by evaluating $F(b) - F(a)$, where $F'(x) = f(x)$. Clearly the value of a definite integral is independent of the variable of integration. Hence $\int_a^b f(x) \, dx = \int_a^b f(t) \, dt$, whatever the values of a and b.

Example 29

Use the substitution $u = -x$ to evaluate $\displaystyle\int_{-2}^{2} \frac{x^2}{e^x + 1}\,dx$

Solution

$u = -x$

$\dfrac{du}{dx} = -1$

$du = -dx$

$x = -2 \Rightarrow u = 2$

$x = 2 \quad \Rightarrow u = -2$

$$\int_{-2}^{2} \frac{x^2}{e^x + 1}\,dx = \int_{2}^{-2} \frac{u^2}{e^{-u} + 1}(-du)$$

$$= \int_{-2}^{2} \frac{u^2 e^u}{1 + e^u}\,du$$

$$= \int_{-2}^{2} \frac{u^2(e^u + 1 - 1)}{e^u + 1}\,du$$

$$= \int_{-2}^{2} u^2\,du - \int_{-2}^{2} \frac{u^2}{e^u + 1}\,du$$

but $$\int_{-2}^{2} \frac{x^2}{e^x + 1}\,dx = \int_{-2}^{2} \frac{u^2}{e^u + 1}\,du$$

$$\therefore 2\int_{-2}^{2} \frac{x^2}{e^x + 1}\,dx = \int_{-2}^{2} u^2\,du$$

$$= \frac{1}{3}\Big[u^3\Big]_{-2}^{2}$$

$$\therefore \int_{-2}^{2} \frac{x^2}{e^x + 1}\,dx = \frac{1}{6}.\ 16 = \frac{8}{3}$$

Example 30

Show that $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x) + f(-x)\}\,dx$. Hence deduce that

$\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even, and

$\displaystyle\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd.

Evaluate $\displaystyle\int_{-\pi}^{\pi} \sin^5 x \cos^8 x\,dx$

Solution

Using the substitution $u = -x$,
$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-u)\,du$$
$$= \int_{0}^{a} f(-u)\,du$$
$$= \int_{0}^{a} f(-x)\,dx$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{-a}^{0} f(x)\,dx$$
$$= \int_{0}^{a} \{f(x) + f(-x)\}\,dx$$

If $f(x)$ is even, $f(-x) = f(x) \Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

If $f(x)$ is odd, $f(-x) = -f(x) \Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

Let $f(x) = \sin^5 x \cos^8 x$. Then $f(-x) = (-\sin x)^5 \cos^8 x = -f(x)$

Hence $f(x)$ is odd and $\int_{-\pi}^{\pi} \sin^5 x \cos^8 x\,dx = 0$

Example 31

Show that $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$. Hence evaluate $\int_{0}^{\pi} x \sin^2 x\,dx$.

Solution

Let $u = a - x$
$du = -dx$
$x = 0 \Rightarrow u = a$
$x = a \Rightarrow u = 0$

$$\int_{0}^{a} f(x)\,dx = -\int_{a}^{0} f(a-u)\,du$$
$$= \int_{0}^{a} f(a-u)\,du$$
$$= \int_{0}^{a} f(a-x)\,dx$$

Then
$$\int_{0}^{\pi} x \sin^2 x\,dx = \int_{0}^{\pi} (\pi - x) \sin^2 (\pi - x)\,dx$$
$$= \int_{0}^{\pi} (\pi - x) \sin^2 x\,dx$$
$$= \pi \int_{0}^{\pi} \sin^2 x\,dx - \int_{0}^{\pi} x \sin^2 x\,dx$$

$$2\int_0^{\pi} x \sin^2 x \, dx = \frac{\pi}{2}\int_0^{\pi} (1 - \cos 2x) \, dx$$

$$\therefore 2\int_0^{\pi} x \sin^2 x \, dx = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{2}$$

$$\int_0^{\pi} x \sin^2 x \, dx = \frac{\pi^2}{4}$$

Exercise 5.6

1 (a) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(b) Hence show that $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx = \frac{\pi^2}{4}$

2 Use the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(a) to show that $\int_0^{\frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \sin 2x}\,dx = \int_0^{\frac{\pi}{4}} \tan^2 x\,dx$ and hence evaluate the given integral.

(b) to show that $\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos^3 x + \sin^3 x}\,dx$ and hence evaluate the given integral.

3 (a) Show that $\int_{-a}^a f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$

(b) Hence show that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x}{1 + e^x} \sin^2 x\,dx = \frac{\pi}{4}$

4 Use the result $\int_{-a}^a f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$

(a) to show that $\int_{-1}^1 \frac{1}{1 + e^{-x}}\,dx = \int_0^1 1\,dx$ and hence evaluate the given integral.

(b) to show that $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1 + \sin x}\,dx = \int_0^{\frac{\pi}{4}} 2\sec^2 x\,dx$ and hence evaluate the given integral.

Diagnostic test 5

Subsection

Find

1 $\displaystyle\int \frac{\ln x}{x}\,dx$ (5.1)

2 $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$ (5.1)

3 $\displaystyle\int \frac{\cos x}{2 + \sin x}\,dx$ (5.1)

4 $\displaystyle\int_0^2 \frac{x}{\sqrt{(1 + x^2)}}\,dx$ (5.1)

5 $\displaystyle\int \frac{x + 1}{x^2 + 9}\,dx$ (5.2)

6 $\displaystyle\int \frac{x + 7}{(x - 1)(x + 3)}\,dx$ (5.2)

7 $\displaystyle\int \frac{2x^2 - 2x + 1}{(x - 2)(x^2 + 1)}\,dx$ (5.2)

8 $\displaystyle\int_2^4 \frac{(x^2 - 1)^2}{x}\,dx$ (5.2)

9 $\displaystyle\int \frac{1}{(1 + x)\sqrt{x}}\,dx \qquad (u = \sqrt{x})$ (5.3)

10 $\displaystyle\int \frac{x}{\sqrt{(x + 1)}}\,dx \qquad (u^2 = x + 1)$ (5.3)

11 $\displaystyle\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{(x^2 - 1)}}\,dx \qquad (x = \sec \theta)$ (5.3)

12 $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x}\,dx \qquad \left(t = \tan \frac{x}{2}\right)$ (5.3)

13 $\displaystyle\int \sqrt{(\sin x)}\cos^3 x\,dx$ (5.4)

14 $\displaystyle\int \sin 4x \cos x\,dx$ (5.4)

15 $\displaystyle\int x^2 e^x\,dx$ (5.5)

Subsection

16 $\int x \cos 2x \, dx$ (5.5)

17 $\int_0^1 \tan^{-1} x \, dx$ (5.5)

18 If $I_n = \int_1^e (\ln x)^n \, dx$ for $n \geqslant 0$, show that $I_n = e - nI_{n-1}$ (5.5)
for $n \geqslant 1$.
Hence evaluate I_4.

19 Show that $f(x) = x^6 \sin x$ is an odd function. Hence evaluate (5.6)
$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^6 \sin x \, dx$

20 Show that $\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx$. Hence show that (5.6)
$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} \, dx = \frac{\pi}{4}$

Further questions 5

Find

1 $\int \frac{\tan^{-1} x}{1 + x^2} \, dx$ **2** $\int \frac{1}{x^2 + 4x + 3} \, dx$ **3** $\int \frac{1}{1 + e^{-x}} \, dx$

4 $\int \frac{1}{x^2 + 4x + 5} \, dx$ **5** $\int \ln (x^2 - 1) \, dx$ **6** $\int \frac{1 + \sin x}{\cos^2 x} \, dx$

7 $\int \sin 5x \cos 3x \, dx$ **8** $\int \frac{1}{3 + 2x - x^2} \, dx$ **9** $\int \frac{1}{e^x + e^{-x}} \, dx$

10 $\int \ln (x^2 + 1) \, dx$ **11** $\int (\tan x + \cot x) \, dx$ **12** $\int \frac{1}{\sqrt{(3 + 2x - x^2)}} \, dx$

13 $\int \sin 4x \sin 2x \, dx$ **14** $\int \frac{x^2}{x^2 - 1} \, dx$ **15** $\int \sqrt{(4 - x^2)} \, dx$

16 $\int \frac{1}{x} \sec^2(\ln x) \, dx$ **17** $\int \frac{1}{1 - \cos x} \, dx$ **18** $\int \frac{2x + 1}{x^2 + 2x + 2} \, dx$

19 $\int \sin^3 x \cos^2 x \, dx$ **20** $\int \sqrt{(16 + x^2)} \, dx$ **21** $\int \frac{e^{\sin^{-1} x}}{\sqrt{(1 - x^2)}} \, dx$

22 $\int \frac{3x^2 - 6x + 1}{(x - 3)(x^2 + 1)} \, dx$ **23** $\int \frac{1}{x^2 \sqrt{(x - 1)}} \, dx$ **24** $\int \frac{2x^2 - x + 20}{(x - 2)(x^2 + 9)} \, dx$

25 $\int \frac{e^x}{\sqrt{(1 - e^{2x})}} \, dx$ **26** $\int \frac{12}{(x^2 + 4)(x^2 + 16)} \, dx$ **27** $\int \frac{\sin^3 x}{\cos^2 x} \, dx$

Evaluate

28 $\int_0^{\frac{\pi}{4}} \frac{1 - \tan x}{1 + \tan x} \, dx$ **29** $\int_0^1 \frac{x + 1}{x^2 + 1} \, dx$ **30** $\int_0^{\frac{\pi}{2}} \sqrt{(1 + \sin 2x)} \, dx$

31 $\int_0^2 \frac{5x^2 + 4x - 20}{(x + 2)(x^2 + 4)} \, dx$ **32** $\int_0^{\frac{\pi}{2}} \frac{1}{3 \cos x + 4 \sin x + 5} \, dx$

33 $\int_0^a \frac{3x^2 - ax}{(x - 2a)(x^2 + a^2)} \, dx$

34 If n is a positive integer, find in terms of n the three possible values of

$$\int_{\frac{\pi}{2}}^{\pi} \cos nx \, dx$$

35 Show that $\cos mx \cos nx = \frac{1}{2}\{\cos (m + n)x + \cos (m - n)x\}$. Hence evaluate $\int_0^{2\pi} \cos mx \cos nx \, dx$ in the cases **(a)** $m \neq n$ **(b)** $m = n$

36 Show that $\frac{1}{9 - 8 \sin^2 x} = \frac{\sec^2 x}{9 + \tan^2 x}$. Hence use the substitution $u = \tan x$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{9 - 8 \sin^2 x} \, dx$

37 Show that $\frac{1}{9 - 10 \sin^2 x} = \frac{\sec^2 x}{9 - \tan^2 x}$. Hence use the substitution $u = \tan x$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{9 - 10 \sin^2 x} \, dx$

38 Use the substitution $x = 5 \sin^2 \theta + \cos^2 \theta$ to evaluate $\int_2^4 \sqrt{\left(\frac{5 - x}{x - 1}\right)} \, dx$

39 Use the substitution $x = 5 \sin^2 \theta + \cos^2 \theta$ to evaluate

$$\int_2^3 \frac{1}{2\sqrt{[(x-1)(5-x)]}}\, dx$$

40 Find $\displaystyle\int \frac{\ln x}{x^n}\, dx$ in the cases **(a)** $n = 0$ **(b)** $n = 1$ **(c)** $n \neq 0$ or 1.

41 Find $\displaystyle\int \frac{x^3}{(x^2+1)^3}\, dx$

(a) using the substitution $u = x^2 + 1$ **(b)** using the substitution $x = \tan \theta$.
Show that the answers agree.

42 Show that $\displaystyle\int \sqrt{(x^2 - a^2)}\, dx = \frac{1}{2} x \sqrt{(x^2 - a^2)} - \frac{1}{2} a^2 \ln|\, x + \sqrt{(x^2 - a^2)}|$

(a) using the substitution $x = a \sec \theta$ **(b)** using integration by parts.

43 If $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\,\theta}{\sin \theta}\, d\theta$, show that $I_n - I_{n-1} = 0$ for $n \geqslant 1$. Hence find the value of I_n for $n \geqslant 0$.

44 If $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)\,\theta}{\cos \theta}\, d\theta$, show that $I_n + I_{n-1} = 0$ for $n \geqslant 1$. Hence find the value of I_n for $n \geqslant 0$.

45 If $I_n = \displaystyle\int_0^{a} (a^2 - x^2)^n\, dx$ for $n \geqslant 0$, show that $I_n = \dfrac{2a^2 n}{2n+1} I_{n-1}$ for $n \geqslant 1$.

46 If $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x\, dx$ for $n \geqslant 0$, show that $I_n = \dfrac{n-1}{n+2} I_{n-2}$ for $n \geqslant 2$. Hence show that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x\, dx = \frac{\pi}{32}$

47 If $I_n = \displaystyle\int_0^{\frac{\pi}{2}} x^n \sin x\, dx$ for $n \geqslant 0$, show that

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$$

for $n \geqslant 2$. Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x^4 \sin x\, dx$.

48 If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ for $n \geqslant 0$, show that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geqslant 2$.

Hence find I_{10} and use a suitable change of variable to find $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^{10} 2\theta \, d\theta$

49 Show that $\int_0^{\pi} \left(x - \frac{\pi}{2}\right)^6 \cos^3 x \, dx = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^6 \sin^3 x \, dx = 0$

50 Show that $\int_0^a f(x) \, dx = \int_0^{\frac{a}{2}} \{f(x) + f(a - x)\} \, dx$

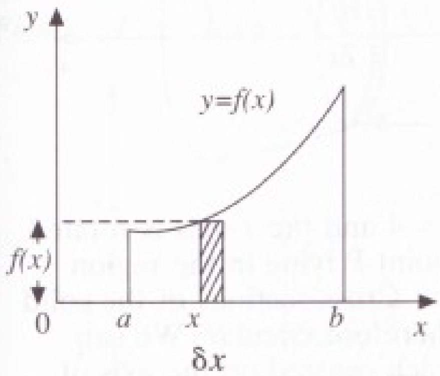
Hence show that $\int_0^{\pi} x \sin^6 x \, dx = \pi \int_0^{\frac{\pi}{2}} \sin^6 x \, dx = \frac{5\pi^2}{32}$

6 Volumes

6.1 Volumes of solids of revolution by slicing

Definite integrals as limiting sums

Figure 6.1



δx denotes a small change in horizontal position, or a small horizontal distance.

As shown in figure 6.1, the area A bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$, $x = b$ can be approximated by summing the areas of thin rectangular strips of width δx, where the first strip has leading edge at a and the last at $b - \delta x$. The area of a typical strip is $f(x)\delta x$, and we write

$A \doteqdot \sum_{x=a}^{b} f(x)\delta x$ to denote the sum of the areas of the strips. Clearly the approximation gets better as we take more and thinner strips, and A is the limiting value of this sum as $\delta x \to 0$.

Hence $A = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\delta x$. But using the fundamental theorem of calculus

$$A = \int_a^b f(x)dx \quad \therefore \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\delta x = \int_a^b f(x)dx$$

Any limiting sum of this form can hence be evaluated as an appropriate definite integral. This concept enables us to find the volume of any solid figure which has parallel cross-sections of similar shape.

Volumes of solids of revolution

A solid of revolution is formed when a bounded region of the Cartesian plane is rotated about a fixed line in the plane called the axis of rotation.

Figure 6.2

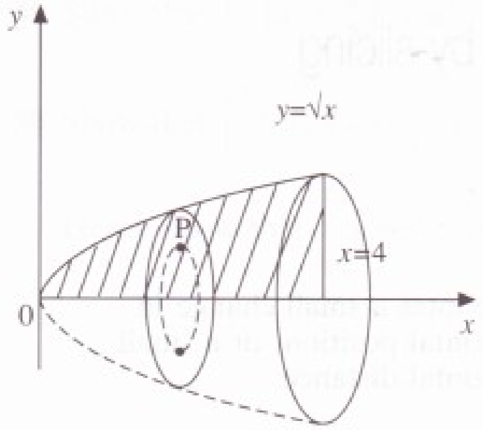
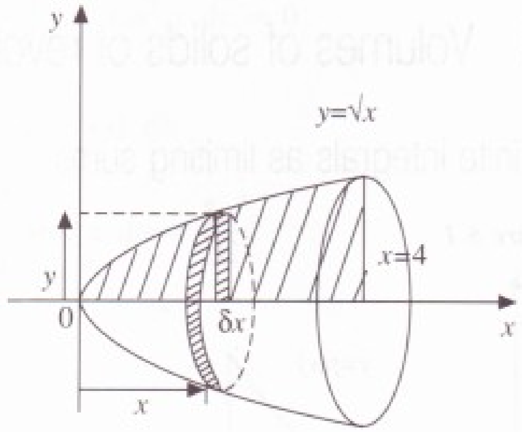


Figure 6.3



In figure 6.2, the region bounded by $y = \sqrt{x}$, $x = 4$ and the x-axis is rotated around the x-axis, forming a paraboloid. Every point P lying in the region sweeps out a circle centred on the axis of rotation. Cross-sections of the solid taken at right angles to the axis of rotation are therefore circular. We can think of the solid as being a stack of thin discs, each centred on the axis of rotation, with the radius of the disc varying according to its position in the stack. A typical disc is illustrated in figure 6.3. This disc of width δx is swept out by a rectangular strip of width δx, perpendicular to the x-axis and with leading edge x units from the origin. The volume of the disc is $\pi y^2 \delta x$.

Just as the shaded area is approximated by the sum of the rectangular strips, so is the volume of the solid approximated by the sum of the volumes of the discs swept out by the strips. If we denote the volume of the solid by V, and that of the typical disc by δV, then, since $y = \sqrt{x}$,

$$\delta V = \pi(\sqrt{x})^2\, \delta x = \pi x \delta x \quad \text{and} \quad V \doteqdot \sum_{x=0}^{4} \pi x \delta x$$

This approximation improves as we take more and thinner strips, hence V is the limiting sum as $\delta x \to 0$.

$$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{4} \pi x \delta x = \int_0^4 \pi x \, dx$$

Evaluating this definite integral, the volume of the paraboloid is 8π cubic units.

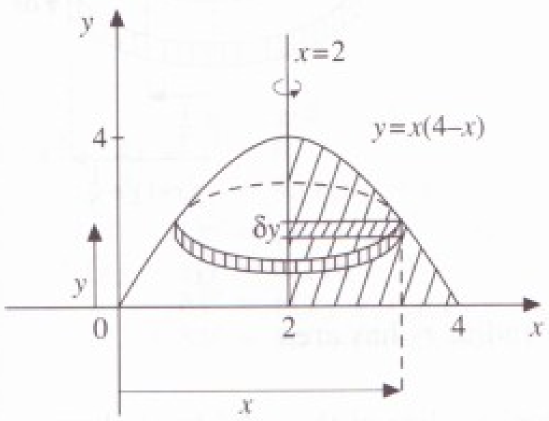
The axis of the solid of revolution need not be either of the coordinate axes. To obtain the slices, identify the area which is rotated through 360° about the axis to sweep out the solid, and divide this area into thin rectangular strips perpendicular to the axis of rotation. A typical thin slice of the solid, taken at right angles to the axis of rotation, is the disc swept out by the rotation of a typical rectangular strip.

Example 1

The region bounded by the curve $y = x(4 - x)$ and the x-axis is rotated through 180° about the line $x = 2$. Find the volume of the solid of revolution.

Solution

Figure 6.4



Note that the parabola is symmetrical about $x = 2$, and rotation of the given region through 180° is equivalent to rotation of the shaded region through 360°. A typical thin slice taken perpendicular to the axis of rotation is a disc of thickness δy and radius $(x - 2)$, with volume δV.

$$\begin{aligned}
\delta V &= \pi(x-2)^2\,\delta y \\
y &= x(4-x) \\
x^2 - 4x &= -y \\
(x-2)^2 &= 4 - y \\
\therefore\quad \delta V &= \pi(4-y)\delta y
\end{aligned}$$

$$\begin{aligned}
\therefore\ V &= \lim_{\delta y \to 0} \sum_{y=0}^{4} \pi(4-y)\delta y \\
&= \pi \int_0^4 (4-y)\,dy \\
&= \pi\left[4y - \frac{1}{2}y^2\right]_0^4 \\
&= 8\pi
\end{aligned}$$

$\therefore$ the volume of the solid is 8π cubic units.

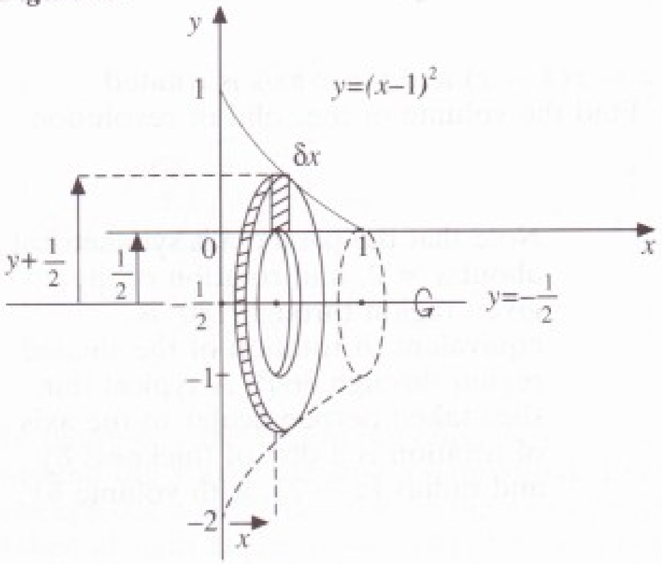
When some solids of revolution are sliced at right angles to the axis of rotation, the cross-section is an annulus rather than a circle.

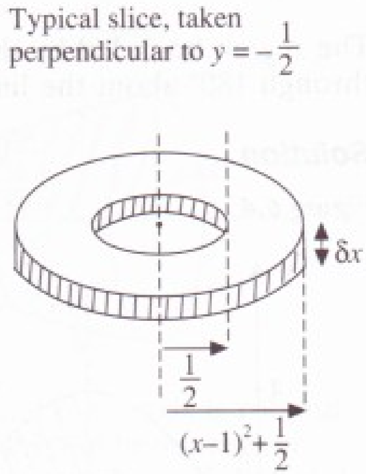
Example 2

The region bounded by the curve $y = (x - 1)^2$ and the x- and y-axes is rotated about the line $y = -\frac{1}{2}$. Find the volume of the solid of revolution.

Solution

Figure 6.5





An annulus with inner radius r_1 and outer radius r_2 has area $\pi(r_2{}^2 - r_1{}^2) = \pi(r_2 + r_1)(r_2 - r_1)$.

Putting $r_1 = \frac{1}{2}$, $r_2 = (x - 1)^2 + \frac{1}{2}$, the typical slice of the solid has volume

$$\delta V = \pi\{(x - 1)^2 + 1\}(x - 1)^2\delta x = \pi\{(x - 1)^4 + (x - 1)^2\}\delta x$$

$$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{1} \pi\{(x - 1)^4 + (x - 1)^2\}\delta x$$

$$= \pi\int_0^1 \{(x - 1)^4 + (x - 1)^2\}\, dx$$

$$= \pi\left[\frac{1}{5}(x - 1)^5 + \frac{1}{3}(x - 1)^3\right]_0^1$$

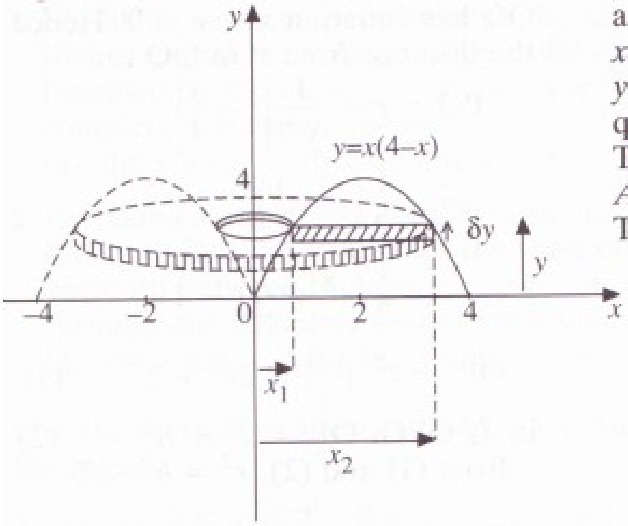
$$\therefore V = \pi\left\{\frac{1}{5} + \frac{1}{3}\right\}.$$ The volume is $\frac{8}{15}\pi$ cubic units.

Example 3

The region bounded by the curve $y = x(4 - x)$ and the x-axis is rotated about the y-axis. Find the volume of the solid of revolution by taking slices perpendicular to the y-axis.

Solution

Figure 6.6



The cross-section of each slice is an annulus with radii x_1, x_2, where $x_2 > x_1$ and x_1, x_2 are the roots of $y = x(4 - x)$ considered as a quadratic equation in x.
The annulus has area
$A = \pi(x_2 + x_1)(x_2 - x_1)$.
The slice has volume $\delta V = A\delta y$.

$$\begin{aligned}
y &= x(4 - x) \\
x^2 - 4x + y &= 0 \\
\therefore x_2 + x_1 &= 4,\ x_1x_2 = y \\
(x_2 - x_1)^2 &= (x_2 + x_1)^2 - 4x_1x_2 \\
\therefore (x_2 - x_1) &= \sqrt{(16 - 4y)} \\
\delta V &= \pi(x_2 + x_1)(x_2 - x_1)\delta y \\
\therefore \delta V &= 8\pi\sqrt{(4 - y)}\,\delta y
\end{aligned}$$

$$\begin{aligned}
\therefore V &= \lim_{\delta y \to 0} \sum_{y=0}^{4} 8\pi\sqrt{(4 - y)}\,\delta y \\
&= 8\pi\int_0^4 \sqrt{(4 - y)}\,dy \\
&= 8\pi\,\frac{2}{3}\left[-(4 - y)^{\frac{3}{2}}\right]_0^4 \\
&= \frac{16\pi}{3}\,8 \\
&= \frac{128\pi}{3}
\end{aligned}$$

$\therefore$ the volume of the solid is $\dfrac{128\pi}{3}$ cubic units.

The axis of rotation need not be parallel to either of the coordinate axes.

Example 4

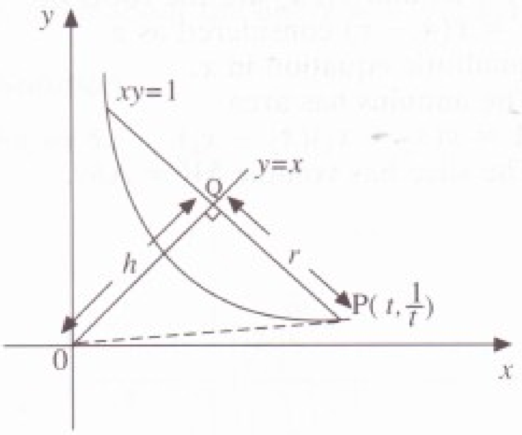
(a) $P\left(t, \dfrac{1}{t}\right)$ lies on the hyperbola $xy = 1$. Q is the foot of the perpendicular from P to the line $y = x$. If $PQ = r$ and $OQ = h$, show that $r^2 = h^2 - 2$.

(b) The region bounded by the hyperbola $xy = 1$ and the line $x + y = 4$ is rotated through 180° about the line $y = x$. Find the volume of the solid of revolution by slicing perpendicular to the axis of rotation.

Solution

(a)

Figure 6.7



OQ has equation $x - y = 0$. Hence the distance from P to OQ is

$$PQ = r = \frac{1}{\sqrt{2}}\left|t - \frac{1}{t}\right|$$

$$r^2 = \frac{1}{2}\left(t - \frac{1}{t}\right)^2$$

$$= \frac{1}{2}\left(t^2 + \frac{1}{t^2}\right) - 1$$

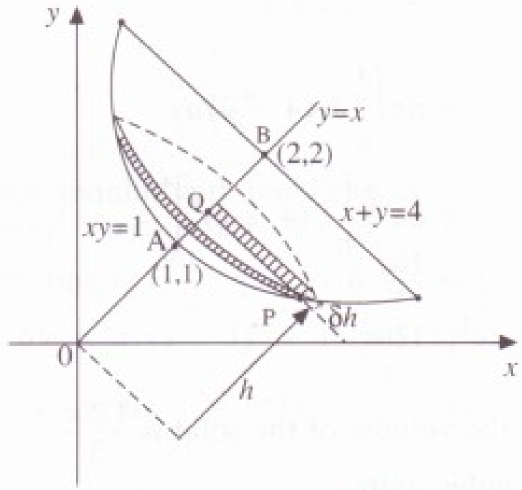
$$\therefore OP^2 = t^2 + \frac{1}{t^2} = 2r^2 + 2 \qquad (1)$$

In $\triangle$ OPQ, $OP^2 = r^2 + h^2 \qquad (2)$

$\therefore$ from (1) and (2), $r^2 = h^2 - 2$

(b)

Figure 6.8



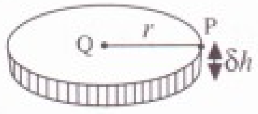
The slice shown is a disc of thickness δh and radius r, with h the distance from 0 to the centre of the disc. From (a),

$r^2 = h^2 - 2$.

$\delta V = \pi r^2 \delta h = \pi(h^2 - 2)\,\delta h$

At A(1,1), $h = OA = \sqrt{2}$

At B(2,2), $h = OB = 2\sqrt{2}$



$$\text{Hence} \qquad V = \lim_{\delta h \to 0} \sum_{h=\sqrt{2}}^{2\sqrt{2}} \pi(h^2 - 2)\delta h$$

$$= \pi\int_{\sqrt{2}}^{2\sqrt{2}} (h^2 - 2)dh$$

$$= \pi\left[\frac{1}{3}h^3 - 2h\right]_{\sqrt{2}}^{2\sqrt{2}}$$

$$= \pi\left(\frac{14}{3}\sqrt{2} - 2\sqrt{2}\right)$$

The volume of the solid is $\frac{8}{3}\pi\sqrt{2}$ cubic units.

Exercise 6.1

1 By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region bounded by the curve $y = x^2$, the x-axis and the line $x = 1$ through one complete revolution about
(a) the x-axis **(b)** the line $x = 1$ **(c)** the y-axis **(d)** the line $y = 1$

2 By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region enclosed between the curve $y = 4 - x^2$ and the lines $x = 2$ and $y = 4$ through one complete revolution about
(a) the x-axis **(b)** the y-axis **(c)** the line $x = 2$ **(d)** the line $y = 4$

Use the method of slicing to find the volume of the solid obtained by rotating the region

3 $\{(x, y) : 0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant 2x - x^2\}$ about the y-axis

4 $\{(x, y) : 0 \leqslant x \leqslant \sqrt{6}, \quad 0 \leqslant y \leqslant 6x^2 - x^4\}$ about the y-axis

5 $\{(x, y) : 0 \leqslant x \leqslant \frac{\pi}{2}, \quad 0 \leqslant y \leqslant \sin x\}$ about the x-axis

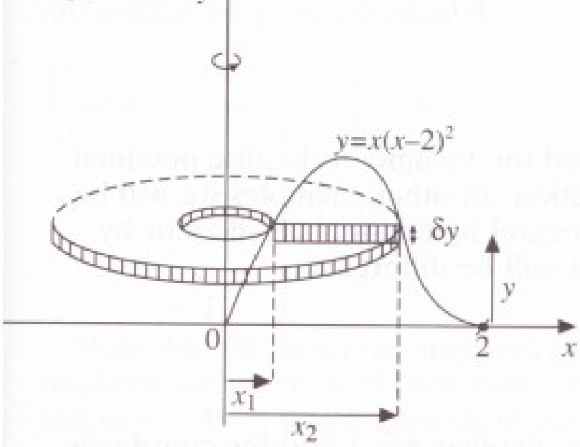
6 $\{(x, y) : 0 \leqslant x \leqslant \frac{\pi}{2}, \quad 0 \leqslant y \leqslant \cos x\}$ about the line $y = 1$

7 enclosed within the circle $(x - 1)^2 + y^2 = 1$ about the y-axis

8 enclosed within the ellipse $(x - 1)^2 + \dfrac{y^2}{4} = 1$ about the y-axis

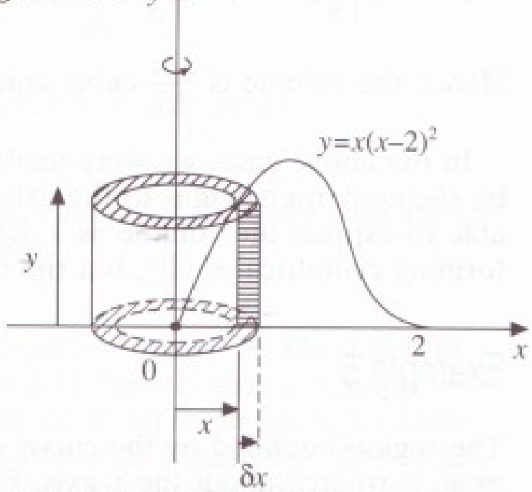
6.2 Volumes of solids of revolution by cylindrical shells

Figure 6.9



Slices formed by strips taken perpendicular to the axis of rotation.

Figure 6.10



Cylindrical shells formed by strips taken parallel to the axis of rotation.

Consider the region bounded by the curve $y = x(x-2)^2$ and the x-axis. When such a region is rotated about the y-axis, we have seen in section 6.1 that thin rectangular strips taken perpendicular to the axis of rotation sweep out slices which are circular or annular in cross-section, as in figure 6.9. Thin rectangular strips taken parallel to the axis of rotation sweep out thin cylindrical shells as in figure 6.10. The volume of the solid of revolution can be found either by stacking the slices in figure 6.9 one on top of the other, and taking the limiting sum of their volumes, or by stacking the cylindrical shells in figure 6.10 one inside the other and taking the limiting sum of their volumes.

To find the volume of the solid by the slice method of figure 6.9, we would need to find $x_2 + x_1$, $x_2 - x_1$, where x_2 and x_1 are the smallest two of the three roots of $y = x(x-2)^2$, considered as a cubic equation in x, as well as finding the maximum value of $x(x-2)^2$ for $0 \leqslant x \leqslant 2$.

We encounter no such problems in finding the volume of the typical cylindrical shell in figure 6.10. This shell has inner radius x, outer radius $x + \delta x$, and height $y = x(x-2)^2$. If the volume of the shell is δV, then

$$\begin{aligned}\delta V &= \pi x(x-2)^2\,\{(x+\delta x)^2 - x^2\}\\ &= \pi x(x-2)^2\,\{2x + \delta x\}\,\delta x \qquad \text{(difference of squares)}\end{aligned}$$

Terms in $(\delta x)^2$ are negligible compared with terms in δx as $\delta x \to 0$. Hence we ignore such terms (referred to as second order terms), and write

$\delta V = 2\pi x^2(x-2)^2\delta x$.

The volume of the solid is now obtained by calculating the limiting sum

$$\begin{aligned}V &= \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi x^2(x-2)^2\,\delta x\\ &= 2\pi\int_0^2 x^2(x-2)^2\,dx\\ &= 2\pi\int_0^2 (x^4 - 4x^3 + 4x^2)\,dx\end{aligned}$$

$$\therefore\ V = 2\pi\left[\frac{1}{5}x^5 - x^4 + \frac{4}{3}x^3\right]_0^2 = 2\pi\left(\frac{32}{5} - 16 + \frac{32}{3}\right)$$

Hence the volume is $\dfrac{32\pi}{15}$ cubic units.

In the above case, we were unable to find the volume of the disc obtained by slicing perpendicular to the axis of rotation. In other examples we will be able to express the volume as a definite integral by taking such slices or by forming cylindrical shells, but the integrals will be different.

Example 5

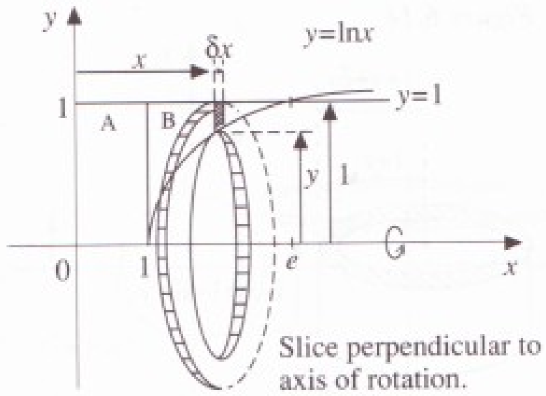
The region bounded by the curve $y = \ln x$, the line $y = 1$ and the coordinate axes, is rotated about the x-axis. Express the volume of the solid of revolution as a definite integral

(a) by slicing perpendicular to the axis of revolution
(b) by forming cylindrical shells
Find the volume of the solid.

Solution

(a)

Figure 6.11



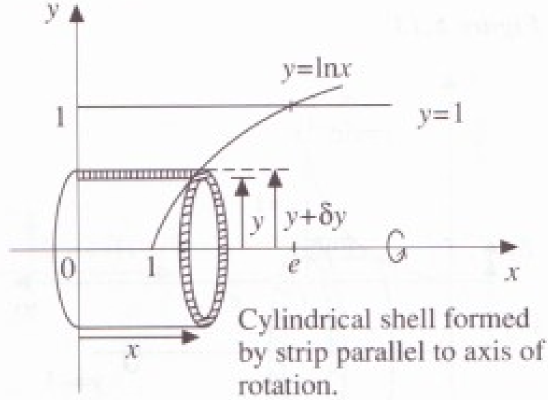
Rotating area A about the x-axis gives a cylinder of volume π. The volume formed by area B gives annular slices of thickness δx and radii $\ln x$, 1. This slice has volume
$\delta V = \pi\{1 - (\ln x)^2\}\delta x$

$$\therefore\ V = \pi + \lim_{\delta x \to 0} \sum_{x=1}^{e} \pi\{1 - (\ln x)^2\}\delta x$$

$$= \pi + \pi \int_1^e \{1 - (\ln x)^2\}\, dx$$

(b)

Figure 6.12



The typical cylindrical shell has radii y, $y + \delta y$, and height $x = e^y$. This shell has volume

$$\delta V = \pi e^y\{(y + \delta y)^2 - y^2\}$$
$$= \pi e^y\{2y + \delta y\}\delta y$$
$$= 2\pi y e^y \delta y \quad \text{(ignoring } (\delta y)^2\text{)}$$

$$\therefore\ V = \lim_{\delta y \to 0} \sum_{y=0}^{1} 2\pi y e^y \delta y$$

$$= 2\pi \int_0^1 y e^y\, dy$$

The integral in **(b)** is easier to calculate, and we use the technique of integration by parts (Section 5.5)

$$\int_0^1 y e^y dy = \Big[y e^y \Big]_0^1 - \int_0^1 e^y dy$$
$$= e - \Big[e^y \Big]_0^1$$
$$= e - (e - 1)$$
$$= 1$$

Hence $V = 2\pi \int_0^1 y e^y dy$
$= 2\pi$
The volume of the solid is 2π cubic units.

Note that the functions involved in the integrals obtained by the different methods are inverses of each other. In figure 6.11 the slice has a variable radius $y = \ln x$, which appears in the integral for V, while in figure 6.12 the cylindrical shell has a variable height $x = e^y$. Since the function e^y gives rise to simpler integrals than $\ln x$, we could have anticipated that the method of cylindrical shells would be easier.

Example 6

The region bounded by the curve $y = \sin^{-1} x$, the x-axis and the ordinate $x = 1$ is rotated about the line $y = -1$. By considering the solid of revolution as a sum of cylindrical shells, find its volume.

Solution

Figure 6.13

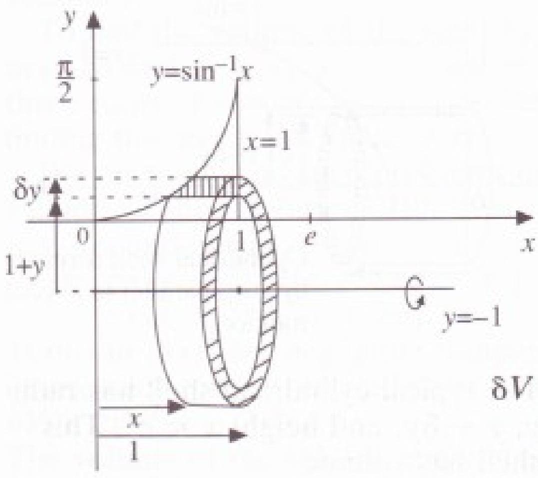
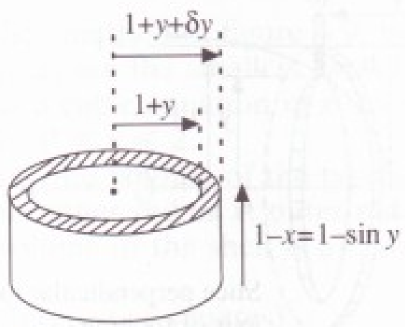


Figure 6.14



$$\begin{aligned}\delta V &= \pi(1 - \sin y)\{(1 + y + \delta y)^2 - (1 + y)^2\}\\ &= \pi(1 - \sin y)\{2(1 + y) + \delta y\}\delta y\\ &= 2\pi(1 + y)(1 - \sin y)\delta y\end{aligned}$$

(Ignoring terms in $(\delta y)^2$)

Hence $V = \lim_{\delta y \to 0} \sum_{y=0}^{\frac{\pi}{2}} 2\pi(1 + y)(1 - \sin y)\,\delta y$

$V = 2\pi\int_0^{\frac{\pi}{2}} (1 + y)(1 - \sin y)dy.$ Integration by parts gives

$$\begin{aligned}V &= 2\pi\{[(1 + y)(y + \cos y)]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} (y + \cos y)dy\}\\ &= 2\pi\left\{\left(1 + \frac{\pi}{2}\right)\frac{\pi}{2} - 1 - \left[\frac{1}{2}y^2 + \sin y\right]_0^{\frac{\pi}{2}}\right\}\\ &= 2\pi\left\{\frac{\pi}{2} + \frac{\pi^2}{4} - 1 - \left(\frac{\pi^2}{8} + 1\right)\right\}\end{aligned}$$

Hence the solid has volume $\pi\left(\frac{\pi^2}{4} + \pi - 4\right)$ cubic units.

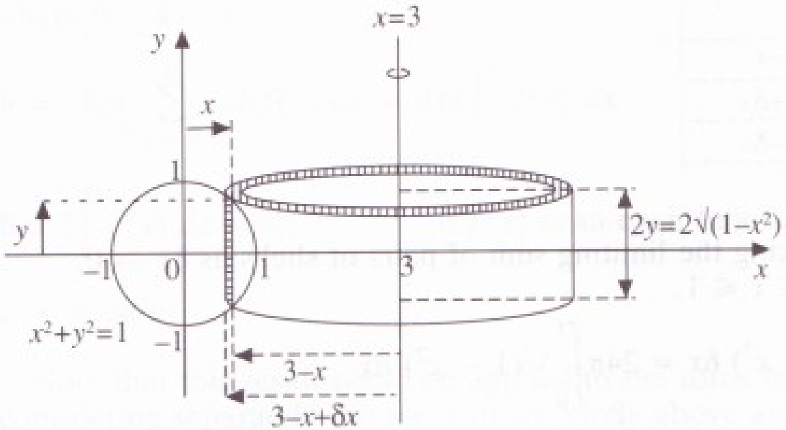
Note that to form cylindrical shells we rotate strips taken parallel to the axis of rotation. In more complicated cases, it is advisable to extract the shell as in figure 6.14, carefully identifying its radii and its height. The thickness of the strip being δy alerts us to the fact that V will be expressed as an integral with respect to y, hence the radii and height of the shell are expressed in terms of y.

Example 7

Use the method of cylindrical shells to find the volume of a torus (doughnut) with inner radius 2 and outer radius 4.

Solution

Figure 6.15



The torus can be formed by rotating the circle $x^2 + y^2 = 1$ about the line $x = 3$. Strips parallel to the axis of rotation sweep out cylindrical shells as in figure 6.15. The volume of a typical shell is

$$\delta V = \pi . 2\sqrt{(1-x^2)}\{(3-x+\delta x)^2 - (3-x)^2\}$$
$$= 2\pi\sqrt{(1-x^2)}\{2(3-x)+\delta x\}\delta x$$
$$\therefore \delta V = 4\pi(3-x)\sqrt{(1-x^2)}\delta x \quad \text{(ignoring terms in } (\delta x)^2\text{)}$$

Hence $V = \lim_{\delta x \to 0} \sum_{x=-1}^{1} 4\pi(3-x)\sqrt{(1-x^2)}\delta x$

$$V = 4\pi\int_{-1}^{1} (3-x)\sqrt{(1-x^2)}\,dx$$

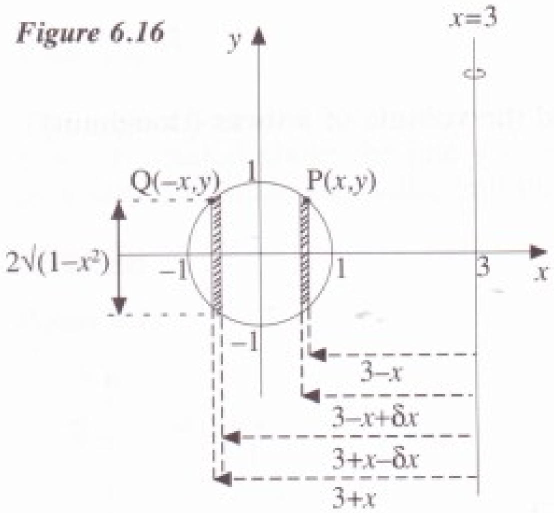
$$= 12\pi\int_{-1}^{1} \sqrt{(1-x^2)}\,dx - 4\pi\int_{-1}^{1} x\sqrt{(1-x^2)}\,dx$$

$$\therefore V = 6\pi^2 - 0 = 6\pi^2$$

(since $\int_{-1}^{1} \sqrt{(1-x^2)}\,dx$ is the area of a semicircle of radius 1, while $x\sqrt{(1-x^2)}$ is an odd function (see section 5.6)).

Alternatively, we could use the symmetry of $x^2 + y^2 = 1$ about the y-axis and consider the cylindrical shells in pairs formed by strips on either side of the y-axis.

Figure 6.16



The volume of the shell swept out by the strip at P is

$\delta V_P = 4\pi(3 - x)\sqrt{(1 - x^2)}\delta x$

The volume of the strip swept out by the strip at Q is

$\delta V_Q = 4\pi(3 + x)\sqrt{(1 - x^2)}\ \delta x$

$\delta V_P + \delta V_Q = 24\pi\sqrt{(1 - x^2)}\ \delta x$

V is now obtained by taking the limiting sum of pairs of shells as $\delta x \to 0$, where x takes values $0 \leqslant x \leqslant 1$.

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 24\pi\sqrt{(1 - x^2)}\ \delta x = 24\pi\int_0^1 \sqrt{(1 - x^2)}\ dx$$

But $\displaystyle\int_0^1 \sqrt{(1 - x^2)}\ dx = \frac{\pi}{4}$ $\quad$ ($\frac{1}{4}$ the area of a circle of radius 1)

Hence the volume of the torus is $6\pi^2$ cubic units.

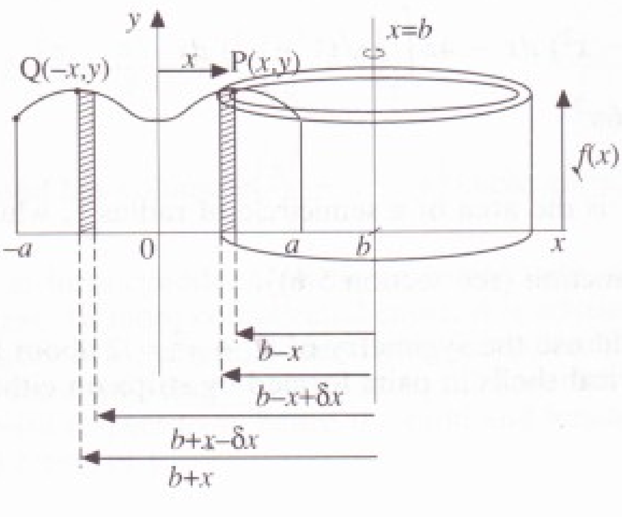
This process of pairing cylindrical shells can be applied whenever the region being rotated has an axis of symmetry parallel to the axis of rotation.

Example 8

$f(x)$ is an even function such that $f(x) \geqslant 0$ for $-a \leqslant x \leqslant a$. The region bounded by $y = f(x)$, the x-axis and the ordinates $x = -a$ and $x = a$ has area A. This region is rotated about the line $x = b$, where $b > a > 0$. Show that the volume of the solid of rotation is $2\pi bA$.

Solution

Figure 6.17



Since $f(x)$ is an even function, the strips taken parallel to the axis of rotation at P and Q each give rise to cylindrical shells of height $f(x)$, the volumes of these shells being respectively δV_P and δV_Q, where

$\delta V_P = \pi\{(b - x + \delta x)^2 - (b - x)^2\} f(x)$.

Ignoring terms in $(\delta x)^2$, $\quad \delta V_P = 2\pi(b - x) f(x)\,\delta x$

Similarly $\quad \delta V_Q = 2\pi(b + x) f(x)\,\delta x$

$\therefore \delta V_P + \delta V_Q = 4\pi b f(x)\,\delta x$

Taking the limiting sum of the volumes of these pairs of shells as $\delta x \to 0$, where $0 < x < a$

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{a} 4\pi b f(x)\,\delta x = 4\pi b \int_0^a f(x)\,dx$$

But $2\int_0^a f(x)\,dx = A$, $\quad$ as $f(x)$ is an even function

$\therefore V = 2\pi b A$

Note that this result could be applied to the torus in example 7 by considering separately the areas of the circle above and below the x-axis, giving $V = 2\pi.3.A$, where $A = \pi$ is the area of the unit circle rotated to form the torus. If we think of making a radial cut on the torus, then opening it out to form a cylinder of the same cross-section, the torus and the cylinder will have the same volume.

Exercise 6.2

1 By taking strips parallel to the axis of rotation, use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curve $y = x^2$, the x-axis and the line $x = 1$ through one complete revolution about
(a) the x-axis **(b)** the line $x = 1$ **(c)** the y-axis **(d)** the line $y = 1$

2 By taking strips parallel to the axis of rotation, use the method of cylindrical shells to find the volume of the solid obtained by rotating the region enclosed between the curve $y = 4 - x^2$ and the lines $x = 2$ and $y = 4$ through one complete revolution about
(a) the x-axis **(b)** the y-axis **(c)** the line $x = 2$ **(d)** the line $y = 4$

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region

3 $\{(x, y) : 0 \leqslant x \leqslant 2,\ \ 0 \leqslant y \leqslant 2x - x^2\}$ about the y-axis

4 $\{(x, y) : 0 \leqslant x \leqslant \sqrt{6},\ \ 0 \leqslant y \leqslant 6x^2 - x^4\}$ about the y-axis

5 $\{(x, y) : 0 \leqslant x \leqslant \frac{\pi}{2},\ \ 0 \leqslant y \leqslant \sin x\}$ about the y-axis

6 $\{(x, y) : 0 \leqslant x \leqslant \frac{\pi}{2},\ \ 0 \leqslant y \leqslant \cos x\}$ about the line $x = \frac{\pi}{2}$

7 enclosed within the circle $(x-1)^2 + y^2 = 1$ about the y-axis

8 enclosed within the ellipse $(x-1)^2 + \frac{y^2}{4} = 1$ about the y-axis

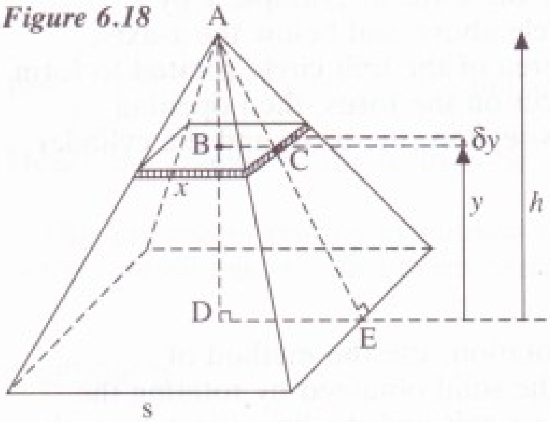
6.3 Volumes of solids with parallel cross-sections of similar shapes

Example 9

Show that the volume of a right square pyramid of height h on a base of side s is given by $V = \frac{1}{3}s^2h$.

Solution

Figure 6.18



Slicing the pyramid parallel to its base gives square slices of side x and width δy, where y is the height of the slice above the base. E, C are midpoints of the sides of the base and slice respectively.

$$\Delta ABC \parallel\!\!| \Delta ADE \Rightarrow \frac{BC}{DE} = \frac{AB}{AD} \Rightarrow \frac{x}{s} = \frac{h-y}{h}$$

Let the volume of the slice be δV. Then $\delta V = x^2 \delta y = \frac{s^2}{h^2}(h-y)^2\,\delta y$.

The volume of the pyramid is the limiting sum of the volumes of the slices as $\delta y \to 0$, where $0 \leqslant y \leqslant h$

$$\therefore V = \lim_{\delta y \to 0} \sum_{y=0}^{h} \frac{s^2}{h^2}(h-y)^2 \delta y$$

$$= \frac{s^2}{h^2}\int_0^h (h-y)^2\,dy$$

$$\therefore V = \frac{s^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_0^h = \frac{1}{3}s^2h$$

Example 10

The base of a solid is the segment of the parabola $x^2 = 4y$ cut off by the line $y = 2$. Cross-sections taken perpendicular to the axis of the parabola are right-angled isosceles triangles with hypotenuse in the base of the solid. Find the volume of the solid.

Solution

Figure 6.19

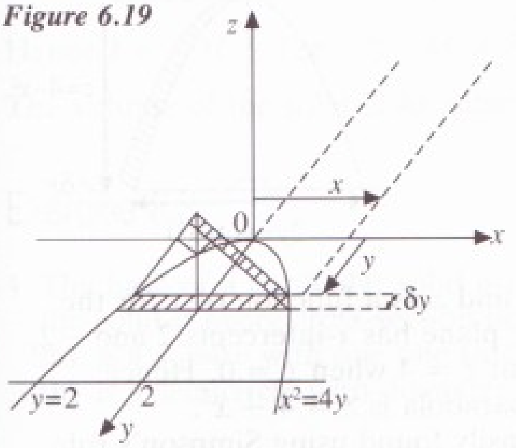
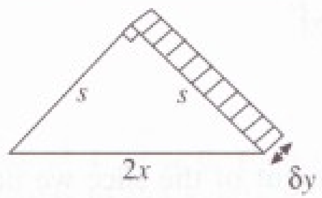


Figure 6.20

Slice with area of cross-section A.



$2s^2 = 4x^2$ (Pythagoras)

$A = \frac{1}{2}s^2 = x^2 = 4y$

If the slice has volume δV, then $\delta V = A\ \delta y$

$\therefore\ \delta \text{V} = 4y\ \delta y$ and $V = \lim\limits_{\delta y \to 0} \sum\limits_{y=0}^{2} 4y\delta y$

$\therefore\ V = 4\int_0^2 y\ dy = 2[y^2]_0^2$. The volume of the solid is 8 cubic units.

Note that it is easier to visualise the solid and its slices if we use a diagram with three mutually perpendicular axes, with the x- and y-axes in the horizontal plane and the z-axis at right angles to this plane. Note the usual convention for the relative orientation of the x- and y-axes. The triangular cross-section in the description of the solid determines the set of parallel slices we use to express its volume as a limiting sum.

Example 11

A solid has an elliptical base with semi-axes 2 and 1. Cross-sections perpendicular to the major axis of the ellipse are parabolic segments with axis passing through the major axis of the ellipse. The height of each such segment is determined by a bounding parabola of height 4, as shown in figure 6.21. Find the volume of the solid.

Solution

Figure 6.21

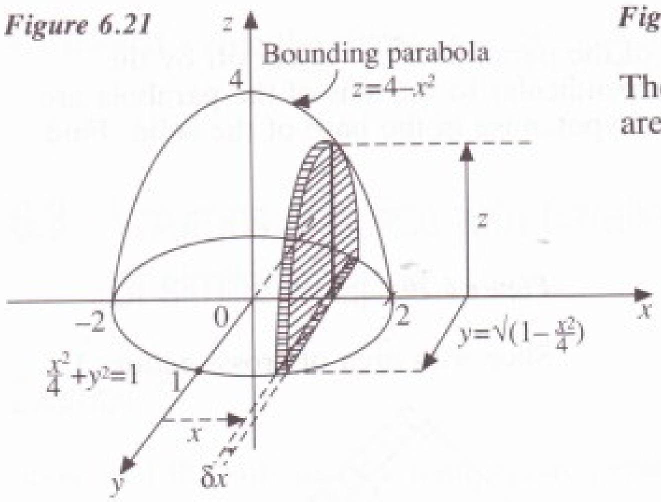
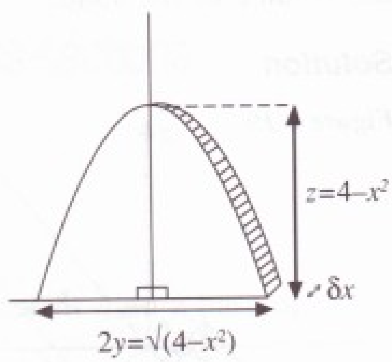


Figure 6.22

The slice is a parabolic segment with area of cross-section A, thickness δx.



To find the height of the slice we need to find z as a function of x for the bounding parabola. This parabola in the x–z plane has x-intercepts 2 and -2, and hence equation $z = k(x - 2)(x + 2)$. But $z = 4$ when $x = 0$. Hence $k = -1$, and the equation of the bounding parabola is $z = 4 - x^2$.

The area of a parabolic segment is most easily found using Simpson's rule, since this rule for approximating definite integrals gives the exact value for the area under a parabola. Hence, considering figure 6.22 and using Simpson's rule with three function values

$$A = \frac{1}{6}\sqrt{(4 - x^2)}\{0 + 4(4 - x^2) + 0\} = \frac{2}{3}(4 - x^2)^{\frac{3}{2}}$$

Hence the volume of the slice is $\quad \delta V = \frac{2}{3}(4 - x^2)^{\frac{3}{2}}\ \delta x$

The volume of the solid is $\quad V = \lim_{\delta x \to 0} \sum_{x=-2}^{2} \frac{2}{3}(4 - x^2)^{\frac{3}{2}}\ \delta x$

$$= \frac{2}{3}\int_{-2}^{2}(4 - x^2)^{\frac{3}{2}}\ \delta x$$

But $(4 - x^2)^{\frac{3}{2}}$ is even $\qquad \therefore V = \frac{4}{3}\int_{0}^{2}(4 - x^2)^{\frac{3}{2}}\ dx$

This integral could be evaluated by using the substitution $x = 2 \sin \theta$. However, integration by parts provides an easier method.

Let $I = \int_0^2 (4 - x^2)^{\frac{3}{2}} dx.$

Then $I = \int_0^2 1.(4 - x^2)^{\frac{3}{2}}dx = [x(4 - x^2)^{\frac{3}{2}}]_0^2 - \int_0^2 \{-3x^2(4 - x^2)^{\frac{1}{2}}dx$

$$= 0 - 3\int_0^2 (4 - x^2 - 4)(4 - x^2)^{\frac{1}{2}}dx$$

$$= -3\int_0^2 (4 - x^2)^{\frac{3}{2}}\,dx + 12\int_0^2 (4 - x^2)^{\frac{1}{2}}dx$$

But $\int_0^2 (4 - x^2)^{\frac{1}{2}}dx$ is $\frac{1}{4}$ of the area of a circle of radius 2.

Hence $I = -3I + 12\pi \Rightarrow 4I = 12\pi \quad \therefore V = \frac{4}{3}I = 4\pi.$

The volume of the solid is 4π cubic units.

Exercise 6.3

1 The base of a particular solid is the circle $x^2 + y^2 = 9$. Find the volume of the solid if every cross-section perpendicular to the x-axis
(a) is a square with one one side in the base of the solid
(b) is a semicircle with diameter in the base of the solid

2 The base of a particular solid is the ellipse $\frac{x^2}{4} + y^2 = 1$. Find the volume of the solid if every cross-section perpendicular to the major axis
(a) is an equilateral triangle with one side in the base of the solid
(b) is an isosceles right-angled triangle with hypotenuse in the base of the solid

3 A right circular cone has height h and base radius r. By considering cross-sections parallel to the base of the cone, show that its volume is given by $V = \frac{1}{3}\pi r^2 h$.

4 A hemisphere has radius r. By considering cross-sections parallel to the base of the hemisphere, show that its volume is given by $V = \frac{2}{3}\pi r^3$.

5 The base of a particular solid is the region bounded by the parabola $y^2 = 4x$ between its vertex $(0, 0)$ and its latus rectum. Find the volume of the solid if every cross-section perpendicular to the x-axis
(a) is a square with one side in the base of the solid
(b) is an equilateral triangle with one side in the base of the solid

6 The base of a particular solid is the region bounded by the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ between its vertex $(2, 0)$ and the corresponding latus rectum. Find the volume of the solid if every cross-section perpendicular to the major axis
(a) is a semicircle with diameter in the base of the solid
(b) is an isosceles right-angled triangle with hypotenuse in the base of the solid

7 The base of a particular solid is $x^2 + y^2 = 4$. Find the volume of the solid if every cross-section perpendicular to the x-axis is a parabolic segment with axis of symmetry passing through the x-axis and height the length of its base.

8 (a) Show that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

(b) The base of a particular solid is the circle $x^2 + y^2 = 4$. Find the volume of the solid if every cross-section perpendicular to the x-axis is a semi-ellipse with minor axis in the base of the solid and semi-major axis equal to its minor axis.

Diagnostic test 6

Subsection

1 Consider the region $\{(x, y) : 0 \leqslant x \leqslant 1, \sqrt{x} \leqslant y \leqslant 1\}$. (6.1)

(a) Use slices perpendicular to the line $y = 1$ to find the volume of the solid formed when this region is rotated about the line $y = 1$.

(b) Use slices perpendicular to the x-axis to find the volume of the solid formed when this region is rotated about the x-axis.

2 The region $\{(x, y) : 0 \leqslant x \leqslant \sqrt{2},\ \ 0 \leqslant y \leqslant 2x^2 - x^4\}$ is rotated about the y-axis. Use slices perpendicular to the y-axis to find the volume of the solid formed. (6.1)

3 Consider the region $\{(x, y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant \sqrt{x}\}$. (6.2)

(a) Use cylindrical shells with strips parallel to the y-axis to find the volume of the solid formed when this region is rotated about the y-axis.

(b) Use cylindrical shells with strips parallel to the line $x = 1$ to find the volume of the solid formed when this region is rotated about the line $x = 1$.

4 The region $\{(x, y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x - x^2\}$ is rotated about the x-axis. Use cylindrical shells with strips parallel to the x-axis to find the volume of the solid formed. (6.2)

5 The base of a particular solid is the circle $x^2 + y^2 = 9$. Find the volume of the solid if every cross-section perpendicular to the x-axis is an isosceles right-angled triangle with hypotenuse in the base of the solid. (6.3)

6 The base of a particular solid is the ellipse $\frac{x^2}{4} + y^2 = 1$. Find the volume of the solid if every cross-section perpendicular to the major axis is a square with one side in the base of the solid. (6.3)

Further questions 6

Use **(i)** the method of slicing, and **(ii)** cylindrical shells to find the volume obtained by rotating the region

1 $\{(x, y) : 1 \leqslant x \leqslant e, 0 \leqslant y \leqslant \ln x\}$ about **(a)** the x-axis **(b)** the y-axis

2 $\{(x, y) : 0 \leqslant x \leqslant 1, e^x \leqslant y \leqslant e\}$ about **(a)** the line $y = e$ **(b)** the line $x = 1$

3 $\{(x, y) : 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant \dfrac{1}{1 + x^2}\}$ about the y-axis

4 $\{(x, y) : 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant e^{-x^2}\}$ about the y-axis.

5 $\{(x, y) : 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant \tan^{-1}x\}$ about the y-axis

6 enclosed by the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coordinate axes about the y-axis

7 enclosed by the parabola $y^2 = 4ax$ between its vertex $(0, 0)$ and its latus rectum about its latus rectum

8 enclosed by the circle $(x - b)^2 + y^2 = a^2$ (where $b > a$) about the y-axis.

9 The base of a particular solid is the circle $x^2 + y^2 = a^2$. Find the volume of the solid if every cross-section perpendicular to the x-axis
(a) is a square with one side in the base of the solid
(b) is a semicircle with diameter in the base of the solid

10 The base of a particular solid is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Find the volume of the solid if every cross-section perpendicular to the x-axis
(a) is an equilateral triangle with one side in the base of the solid
(b) is an isosceles right-angled triangle with hypotenuse in the base of the solid

7 Mechanics

7.1 Motion of a particle in one dimension

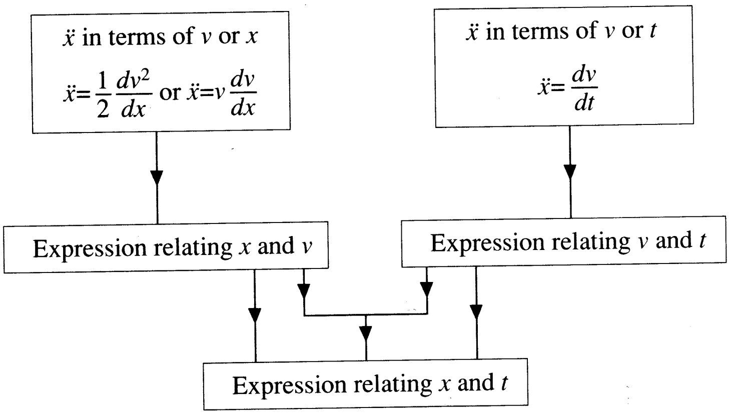
When investigating the motion of a particle in one dimension, we consider the particle to be moving along a number line. We choose the positive direction, and the position of the origin O, which is then the fixed point to which we refer the position x of the particle at time t. The instantaneous velocity v of the particle is its rate of change of position $\frac{dx}{dt}$, while the instantaneous acceleration is its rate of change of velocity $\frac{dv}{dt} = \frac{d^2x}{dt^2}$.
We write $\dot{x}$, $\ddot{x}$ for $\frac{dx}{dt}, \frac{d^2x}{dt^2}$ respectively.
The mathematical representation of the motion of the particle usually comprises
- the equation of motion: $\ddot{x}$ expressed as a function of t, v, or x
- a set of initial conditions: values of x and v when $t = 0$

By integration, using the initial conditions to evaluate constants, we can obtain expressions relating x, v and t. These expressions can then be used to describe the motion, keeping in mind the following results:
- the sign of v indicates the direction of motion
- $v = 0$ when the particle is instantaneously at rest
- $v = 0$ and changes sign when the particle changes direction of motion
- v and $\ddot{x}$ have the same sign when the particle is speeding up, and different signs when it is slowing down.

Position, velocity and acceleration are vector quantities as they have both magnitude and direction. If the vector points towards the positive end of the number line, the corresponding mathematical representation x, v or $\ddot{x}$ has a positive sign. The last result in the above list states that the particle is speeding up if the acceleration vector acts in the same direction as the current direction of motion.

When the equation of motion gives $\ddot{x}$ as a function of v or x, the results $\ddot{x} = \frac{1}{2}\frac{dv^2}{dx}$ or $\ddot{x} = v\frac{dv}{dx}$ can be used.

Figure 7.1



Example 1

Initially a particle is observed to be travelling in a straight line with speed V ms^{-1}. Throughout the subsequent motion, the particle is slowing down at a rate proportional to its speed. Find expressions for the velocity v of the particle when it is x metres from its initial position, and the velocity of the particle and its displacement from its initial position after t seconds. Show that this displacement has a limiting value as $t \to \infty$.

Solution

Origin and positive direction: Choose initial position as origin, and initial direction of motion as positive.

Equation of motion: $\ddot{x} = -kv$, $k > 0$ constant

Initial conditions: $t = 0 \Rightarrow x = 0$ and $v = V$

Relation between x and v

$$\ddot{x} = -kv$$
$$v\frac{dv}{dx} = -kv$$
$$\frac{dv}{dx} = -k$$
$$v = -kx + c, \quad c \text{ constant}$$
$$x = 0,\ v = V \Rightarrow c = V$$
$$\therefore v = V - kx \qquad (1)$$

Relation between v and t

$$\ddot{x} = -kv$$
$$\frac{dv}{dt} = -kv$$
$$-k\frac{dt}{dv} = \frac{1}{v}$$
$$-kt = \ln(Av), \quad A \text{ constant} \qquad (^{**})$$
$$t = 0,\ v = V \Rightarrow AV = 1$$
$$\therefore -kt = \ln\left(\frac{v}{V}\right), \quad v = Ve^{-kt} \qquad (2)$$

From (1) and (2), $\quad V - kx = Ve^{-kt}, \quad \therefore x = \frac{V}{k}(1 - e^{-kt}) \qquad (3)$

From (2), $\quad 0 < v \leqslant V$ and $v \to 0^+$ as $t \to \infty$

From (3), $\quad 0 \leqslant x < \frac{V}{k}$ and $x \to \left(\frac{V}{k}\right)^-$ as $t \to \infty$

Note that when we integrate $\frac{1}{v}$ with respect to $v(**)$, it is more convenient to incorporate the constant of integration in the argument of the logarithm function. $\ln Av = \ln v + \ln A = \ln v + c$, where $c = \ln A$.
Hence $\ln Av$, A constant, and $\ln v + c$, c constant, are equivalent expressions for the indefinite integral.

Newton's laws of motion

When a particle travels at a speed much less than the speed of light, the motion of the particle follows the principles of Newtonian mechanics proposed by Sir Isaac Newton in 1687. The first two laws of Newton can be stated as follows:

1 If the resultant force on a particle is zero, the particle remains at rest, or continues to move in a straight line with constant velocity.
2 If the resultant force on a particle is not zero, the observed acceleration is related to the resultant force by the vector equation $\vec{F} = m\vec{a}$. If m and a are measured in kg and ms^{-2} respectively, F is measured in newtons (N).

If we can identify all the physical forces acting on a particle, we can find the vector sum of all these forces to obtain the resultant force and derive the equation of motion of the particle using Newton's laws. Conversely, if we can deduce the acceleration of the particle by observing its motion, we can find the resultant force from Newton's laws and make deductions about the physical forces on the particle.
When the motion of the particle is restricted to one dimension, Newton's second law can be written $F = m\ddot{x}$.

Example 2

A particle of mass m moves in a straight line subject to a resistance force of magnitude $mk(v + v^3)$, where v is the velocity of the particle. Initially the particle has velocity V. If x is the displacement of the particle from its initial position at time t, derive expressions for x in terms of v, and for v^2 in terms of t. Deduce the limiting values of v and x as $t \to \infty$.

Solution

Origin and positive direction:	initial position and direction of motion
Equation of motion:	$F = -mk(v + v^3) \Rightarrow \ddot{x} = -k(v + v^3)$
Initial conditions:	$t = 0 \Rightarrow x = 0$ and $v = V$

Expression relating x and v

$$\ddot{x} = -k(v + v^3)$$
$$v\frac{dv}{dx} = -kv(1 + v^2)$$
$$-k\frac{dx}{dv} = \frac{1}{1 + v^2}$$
$$-kx = \tan^{-1} v + c, \quad c \text{ constant}$$
$$x = 0,\ v = V \Rightarrow c = -\tan^{-1} V$$
$$kx = \tan^{-1} V - \tan^{-1} v$$
$$x = \frac{1}{k}(\tan^{-1} V - \tan^{-1} v) \quad (1)$$

Expression relating v and t

$$\ddot{x} = -k(v + v^3)$$
$$\frac{dv}{dt} = -kv(1 + v^2)$$
$$-k\frac{dt}{dv} = \frac{1}{v(1 + v^2)}$$
$$= \frac{1}{v} - \frac{v}{1 + v^2} \quad \text{(partial fractions)}$$
$$-kt = \ln Av - \frac{1}{2}\ln(1 + v^2)$$
$$-2kt = \ln\left(A^2\frac{v^2}{1 + v^2}\right), \quad A \text{ constant.}$$
$$t = 0,\ v = V \Rightarrow A^2 = \frac{1 + V^2}{V^2}$$
$$\therefore e^{-2kt} = \frac{1 + V^2}{V^2}\,\frac{v^2}{1 + v^2}$$
$$\frac{v^2}{1 + v^2} = \frac{V^2 e^{-2kt}}{1 + V^2}$$
$$v^2 = \frac{V^2 e^{-2kt}}{1 + V^2(1 - e^{-2kt})} \quad (2)$$

From (2), $\quad v^2 \to 0$ as $t \to \infty \quad \therefore v \to 0^+$ as $t \to \infty$

Then from (1), $\quad x \to \left(\frac{1}{k}\tan^{-1} V\right)^-$ as $t \to \infty$

Simple harmonic motion

If a particle moves is a straight line subject to a force directed towards a fixed point O, with magnitude proportional to the distance of the particle from O, the ensuing motion is oscillatory with O the centre of oscillation, and is termed *simple harmonic*. The force acting on the particle is described as a restoring force. If we choose O as the origin, the restoring force is given by $F = -mn^2x$, where m is the mass of the particle and n is a positive constant. Hence, using Newton's second law, the equation of motion is $\ddot{x} = -n^2x$.

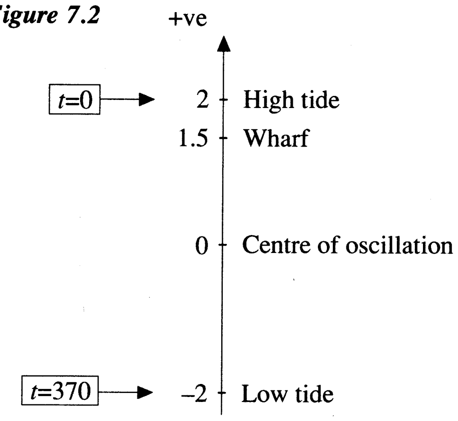
This equation has solution $x = A\cos(nt + \alpha)$, where A and α are constants determined by the initial conditions. This can be verified by differentiation. The maximum displacement A is called the amplitude of the oscillation, while the time taken for one complete oscillation T is called the period, where $T = \frac{2\pi}{n}$. Differentiation gives $v = -nA\sin(nt + \alpha)$, and hence $v^2 = n^2(A^2 - x^2)$.

Example 3

A man sat on a wharf fishing from high tide at 7.00 am until low tide at 1.10 pm. At 7.00 am he noticed that the top of a buoy was 0.5 m above the level of the wharf, while at 1.10 pm it was 3.5 m below the level of the wharf. If the motion of the tide is assumed to be simple harmonic, at what time was the top of the buoy level with the wharf? What is the maximum speed of the buoy?

Solution

Figure 7.2



Period is $2 \times 370 = 740$ minutes

Amplitude is $\frac{1}{2}(0.5 + 3.5) = 2$ m

Motion is simple harmonic $\Rightarrow \ddot{x} = -n^2 x, \quad n = \dfrac{2\pi}{740}$

This equation has solution $x = 2\cos(nt + \alpha), \quad 0 \leqslant \alpha < 2\pi$

Initial conditions: $\quad t = 0, x = 2 \;\Rightarrow\; \cos\alpha = 1 \;\Rightarrow\; \alpha = 0$

$$\therefore x = 2\cos\left(\frac{\pi}{370}t\right) \quad \text{and} \quad v = -\frac{\pi}{185}\sin\left(\frac{\pi}{370}t\right)$$

Bouy level with wharf $\;\Rightarrow\; x = 1.5 \qquad \therefore \cos\left(\dfrac{\pi}{370}t\right) = 0.75$

The first such t is $\; t = \dfrac{370}{\pi}\cos^{-1} 0.75 = 85.$

Hence the top of the buoy is level with the wharf at approximately 8.25 am.

$v = -\dfrac{\pi}{185}\sin\left(\dfrac{\pi}{370}t\right) \;\Rightarrow\;$ maximum v is $\dfrac{\pi}{185} \doteqdot 0.016\,98$ m/min

Hence the maximum speed of the buoy is 2.8×10^{-4} ms^{-1}

Exercise 7.1

1 A particle moves in a straight line with acceleration which is inversely proportional to t^3, where t is the time. The particle has a velocity of 3 ms^{-1} when $t = 1$ and its velocity approaches a limiting value of 5 ms^{-1}. Find an expression for its velocity at time t.

2 A particle moves in a straight line with retardation which increases uniformly with the distance moved. Initially the retardation is 5 ms^{-2} and when the particle has moved a distance of 12 m the retardation is 11 ms^{-2}. Find the distance moved by the particle in coming to rest if the initial velocity is 20 ms^{-1}.

3 A particle moves in a straight line away from a fixed point O in the line, such that when its distance from O is x its speed v is given by $v = \frac{k}{x}$, for some constant k.
(a) Show that the particle has a retardation which is inversely proportional to x^3.
(b) If A, B, C and D are points in that order on the straight line, such that the distances AB, BC and CD are all equal, show that the times taken to travel these successive distances increase in arithmetic progression.

4 A particle moves in a straight line away from a fixed point O in the line, such that at time t its displacement from O is x and its velocity is v. At time $t = 0$, $x = 0$ and $v = V$. Subsequently the particle is slowing down at a rate proportional to the square of its speed. Find expressions for
(a) the velocity v in terms of the displacement x
(b) the velocity v and the displacement x in terms of the time t

5 A particle moves in a straight line away from a fixed point O in the line, such that at time t its displacement from O is x and its velocity v is given by $\frac{1}{v} = A + Bt$, for some positive constants A and B.
(a) Show that the retardation of the particle is proportional to the square of the speed.
(b) If the retardation is 1 ms^{-2} and the velocity is 80 ms^{-1} when $t = 0$, find the values of A and B. Express x in terms of t, and v in terms of x.

6 A particle of mass m moves in a horizontal straight line. The only force acting on the particle is a resistance of magnitude mkv^3 where v is its speed and k is a positive constant. At time t the distance from a fixed point on the line is x. When $t = 0$, $x = 0$ and $v = u$. Show that

$$v = \frac{u}{\sqrt{(1 + 2ku^2t)}} = \frac{u}{1 + kux}\cdot$$

7 A particle of mass m moves in a horizontal straight line. The particle is resisted by a constant force mk and a variable force mv^2, where k is a positive constant and v is the speed. When $t = 0$, $v = u$. Find the distance travelled and the time taken as the particle is brought to rest.

8 A particle of mass m moves in a horizontal straight line. The only force acting on the particle is a resistance of magnitude $mk(v^2 + c^2)$, where k and c are positive constants and v is the speed. If $v = 2c$ when $t = 0$, find
(a) the distance travelled and the time taken for the speed to be halved
(b) the additional distance travelled and the additional time taken for the particle to come to rest

9 The force of attraction experienced by a particle of mass m at a distance $x(> r)$ from the centre O of the earth towards O is $\frac{mgr^2}{x^2}$, where r is the radius of the earth. A particle of mass m starts from the surface with speed u directly away from O.
(a) Find the subsequent speed when the particle is distance x from O.
(b) Deduce that the particle will escape from the attraction of the earth if $u^2 > 2gr$.

In questions 10 and 11, assume that small oscillations of a simple pendulum of length l are simple harmonic with period $2\pi\sqrt{\left(\frac{l}{g}\right)}$.

10 At ground level, where $g = 9.81 \text{ ms}^{-2}$, a simple pendulum beats exact seconds (each half-oscillation takes one second). If it is taken up a mountain to a place where $g = 9.80 \text{ ms}^{-2}$, find by how many seconds per day it will be wrong.

11 At ground level, where $g = 9.81 \text{ ms}^{-2}$, a simple pendulum beats exact seconds. If it is taken up a mountain to a place where it loses 30 seconds per day, find the value of g at the new location.

In questions 12 and 13, assume that tidal motion is simple harmonic.

12 On a certain day, the depth of water in a harbour at high tide at 5 am is 9 m. At the following low tide at 11.20 am the depth is 3 m. Find the latest time before noon that a ship can enter the harbour if a minimum depth of 7.5 m of water is required.

13 On a certain day, low water for a harbour occurs at 3.30 am and high water at 9.45 am, the corresponding depths of water being 5 m and 15 m. Find
(a) between what times during the morning a ship drawing 12.5 m of water can safely enter the harbour
(b) the rate at which the level of water is rising or falling when the depth of water is 13 m

7.2 Motion of a particle moving in a straight line in a resisting medium under gravity

Before applying Newton's laws we need to find the resultant force on the particle if more than one physical force is acting. When a particle moves in a vertical line in a resisting medium, the resistance force of magnitude R and the gravitational force of magnitude mg act in the line of motion, where m is the mass of the particle and g is the acceleration due to gravity at the surface of the earth.

Upward motion	Downward motion
$R \downarrow \downarrow mg \qquad F = mg + R\downarrow$	$R\uparrow \downarrow mg \qquad F = mg - R\downarrow$
Choose projection point as origin,	Choose highest point as origin,
$\uparrow$ as positive direction	$\downarrow$ as positive direction
$\therefore F = -(mg + R) \qquad$ (since $F \downarrow$)	$\therefore F = mg - R \qquad$ (since $F \downarrow$)
$\ddot{x} = -\left(g + \dfrac{R}{m}\right)$	$\ddot{x} = g - \dfrac{R}{m}$

In any problem involving both upward and downward motion, the upward and downward sections should be investigated separately, choosing a new origin and positive direction in each case as indicated above. The resistance force is caused by collisions with particles in the medium. The greater the speed of the particle under investigation, the more frequent are these collisions and hence the greater the resistance. From the equation of motion for downward motion, a particle dropped from rest will fall with increasing velocity until $R = mg$. From this point the particle would fall with constant velocity, termed the terminal velocity of the particle, since $F = 0$. Hence, during the downward motion

$$\boxed{\text{The terminal velocity is the limiting value of } v \text{ as } \ddot{x} \to 0.}$$

Example 4

A particle of mass m is projected vertically upward in a medium where the resistance to motion has magnitude $R = mkv$.

(a) Find the maximum height, and the time taken to reach this maximum height, in terms of the velocity of projection V.

(b) If the speed of projection is twice the terminal velocity

(i) Show that the velocity of the particle on return to its projection point is given by $\dfrac{kv}{g} + 2 - \ln 3 + \ln\left(1 - \dfrac{kv}{g}\right) = 0$. Use Newton's method to show that an approximate solution to this equation is $\dfrac{kv}{g} \doteqdot 0.82$, and deduce the percentage of its terminal velocity that the particle has acquired on return to its projection point.

(ii) Find the ratio of the time taken to reach maximum height to the time to fall from maximum height to the point of projection.

Solution

(a) *Upward motion*

Origin is point of projection

$\uparrow$ is positive direction

Initial conditions: $t = 0$, $x = 0$, $v = V$

$$\boxed{\begin{array}{l}\downarrow R = mkv \\ \downarrow mg\end{array}} \Rightarrow F = -(mg + mkv)$$

Equation of motion $\ddot{x} = -(g + kv)$

Expression relating x and v.

$$\ddot{x} = -(g + kv)$$

$$v\frac{dv}{dx} = -(g + kv)$$

$$\frac{dx}{dv} = \frac{-v}{g + kv}$$

$$k\frac{dx}{dv} = -1 + \frac{g}{g + kv}$$

$$\frac{k^2}{g}\frac{dx}{dv} = -\frac{k}{g} + \frac{k}{g + kv}$$

$$\therefore \frac{k^2}{g}x = -\frac{k}{g}v + \ln(g + kv) + c$$

$$0 = -\frac{k}{g}V + \ln(g + kV) + c$$

(c constant, initially $x = 0$, $v = V$)

$$\frac{k^2}{g}x = \frac{k}{g}(V - v) - \ln\left\{\frac{g + kV}{g + kv}\right\} \quad (1)$$

Expression relating v and t.

$$\ddot{x} = -(g + kv)$$

$$\frac{dv}{dt} = -(g + kv)$$

$$\frac{dt}{dv} = -\frac{1}{g + kv}$$

$$-k\frac{dt}{dv} = \frac{k}{g + kv}$$

$-kt = \ln\{A(g + kv)\}$, A constant

$t = 0$, $v = V \Rightarrow A(g + kV) = 1$

$$\therefore kt = \ln\left\{\frac{g + kV}{g + kv}\right\} \quad (2)$$

At maximum height h, $v = 0$. Let T be the time to reach maximum height.

$$(1) \Rightarrow \quad \frac{k^2}{g}h = \frac{k}{g}V - \ln\left(1 + \frac{k}{g}V\right) \quad \text{and } (2) \Rightarrow \quad kT = \ln\left(1 + \frac{k}{g}V\right)$$

$$\therefore h = \frac{g}{k^2}\left\{\frac{k}{g}V - \ln\left(1 + \frac{k}{g}V\right)\right\} \quad \text{and} \quad T = \frac{1}{k}\ln\left(1 + \frac{k}{g}V\right) \quad (3)$$

(b) *Downward motion*

Origin is at maximum height

$\downarrow$ positive direction

Initial conditions: $t = 0$, $x = 0$, $v = 0$

Terminal velocity: As $\ddot{x} \to 0$, $v \to \left(\frac{g}{k}\right)$

$\boxed{mg \downarrow \ \uparrow R = mkv} \Rightarrow F = mg - mkv$

Equation of motion $\ddot{x} = g - kv$.

Expression relating x and v.

$$\ddot{x} = g - kv$$

$$v\frac{dv}{dx} = g - kv$$

$$-k\frac{dx}{dv} = \frac{-kv}{g - kv}$$

$$-k\frac{dx}{dv} = 1 - \frac{g}{g - kv}$$

$$-\frac{k^2}{g}\frac{dx}{dv} = \frac{k}{g} + \frac{-k}{g - kv}$$

Expression relating v and t.

$$\ddot{x} = g - kv$$

$$\frac{dv}{dt} = g - kv$$

$$-k\frac{dt}{dv} = \frac{-k}{g - kv}$$

$-kt = \ln\{A(g - kv)\}$, A constant

$t = 0$, $v = 0 \Rightarrow Ag = 1$

$$kt = -\ln\left(1 - \frac{k}{g}v\right) \quad (5)$$

$$\therefore -\frac{k^2}{g}x = \frac{k}{g}v + \ln(g - kv) + c$$

$x = 0,\ v = 0 \Rightarrow c = -\ln g$

$$\therefore -\frac{k^2}{g}x = \frac{k}{g}v + \ln\left(1 - \frac{k}{g}v\right) \qquad (4)$$

(i) If $V = \dfrac{2g}{k}$, $h = \dfrac{g}{k^2}(2 - \ln 3)$ from (3)

$\therefore$ when the particle returns to its projection point, $x = h = \dfrac{g}{k^2}(2 - \ln 3)$

$$\therefore \frac{k}{g}v + (2 - \ln 3) + \ln\left(1 - \frac{k}{g}v\right) = 0 \quad \text{from (4)}$$

Let $\lambda = \dfrac{k}{g}v$ and $f(\lambda) = \lambda + \ln(1 - \lambda) + 2 - \ln 3$

Then $f'(\lambda) = 1 - \dfrac{1}{1 - \lambda}$

Using Newton's method with first approximation $\lambda_1 = 0.82$, the second approximation is $\lambda_2 = 0.82 - \dfrac{f(0.82)}{f'(0.82)} = 0.82$ (2 decimal places)

Hence $\dfrac{kv}{g} \doteqdot 0.82 \Rightarrow v \doteqdot 0.82\left(\dfrac{g}{k}\right)$, and the particle has acquired 82% of terminal velocity on return to its projection point.

(ii) Using (5), the time taken to fall from the maximum height to the projection point is given by $kt \doteqdot -\ln(1 - 0.82) = \ln\left(\dfrac{100}{18}\right)$

Using (3), $V = \dfrac{2g}{k} \Rightarrow kT = \ln 3$

$$\text{Hence } \frac{\text{time to maximum height}}{\text{time to fall to projection point}} \doteqdot \frac{\ln 3}{\ln\left(\frac{100}{18}\right)} \doteqdot 0.64$$

Note that during the upward or downward motion we could find x in terms of t by integrating with respect to t the expression for v in terms of t. For example, during the downward motion

$$(5) \Rightarrow \quad kt = -\ln\left(1 - \frac{k}{g}v\right) \quad \Rightarrow \quad \frac{k}{g}v = 1 - e^{-kt}$$

$$\frac{k}{g}\frac{dx}{dt} = 1 - e^{-kt} \quad \Rightarrow \quad \frac{k}{g}x = t + \frac{1}{k}e^{-kt} + c, \quad c \text{ constant}$$

But $t = 0,\ x = 0 \ \Rightarrow\ c = -\dfrac{1}{k} \qquad \therefore x = \dfrac{g}{k}t - \dfrac{g}{k^2}(1 - e^{-kt})$

$$\text{Hence } \quad x = \frac{g}{k^2}\{kt - (1 - e^{-kt})\}$$

Example 5

A particle is projected vertically upward in a medium where the resistance is proportional to the square of the velocity. The velocity of projection is V, the mass of the particle is m and the initial resistance is mkV^2.

(a) Find in terms of V and k the maximum height attained and the time to reach this maximum height.

(b) If the speed of projection is half the terminal velocity, find

(i) the ratio of the velocity on return to the projection point to the velocity of projection

(ii) the ratio of the time taken to fall from the maximum height to the point of projection, to the time taken to attain maximum height

(iii) when and where the particle attains 80% of its terminal velocity

Solution

(a) *Upward motion*

Origin at point of projection

$\uparrow$ positive direction

Initial conditions: $t = 0$, $x = 0$, $v = V$

$$\boxed{\begin{array}{l}\downarrow R = mkv^2 \\ \downarrow mg\end{array}} \Rightarrow F = -(mg + mkv^2)$$

Equation of motion $\ddot{x} = -(g + kv^2)$

Expression relating x and v.

$$\ddot{x} = -(g + kv^2)$$

$$\frac{1}{2}\frac{dv^2}{dx} = -(g + kv^2)$$

$$-2k\frac{dx}{dv^2} = \frac{k}{g + kv^2}$$

$$-2kx = \ln\{A(g + kv^2)\},$$

A constant

$$x = 0,\ v = V \Rightarrow A(g + kV^2) = 1$$

$$-2kx = \ln\left(\frac{g + kv^2}{g + kV^2}\right)$$

$$x = \frac{1}{2k}\ln\left(\frac{g + kV^2}{g + kv^2}\right) \qquad (1)$$

Expression relating v and t.

$$\ddot{x} = -(g + kv^2)$$

$$\frac{dv}{dt} = -(g + kv^2)$$

$$-\frac{dt}{dv} = \frac{1}{g + kv^2}$$

$$-\sqrt{(gk)}\,\frac{dt}{dv} = \frac{\sqrt{(gk)}}{g + kv^2}$$

$$-\sqrt{(gk)}t = \tan^{-1}\sqrt{\left(\frac{k}{g}\right)}v + c,$$

c constant

$$t = 0,\ v = V \Rightarrow c = -\tan^{-1}\sqrt{\left(\frac{k}{g}\right)}V$$

$$\sqrt{(gk)}t = \tan^{-1}\sqrt{\left(\frac{k}{g}\right)}V - \tan^{-1}\sqrt{\left(\frac{k}{g}\right)}v \qquad (2)$$

At maximum height, $v = 0$. Let the maximum height be h and the time taken to maximum height be T. From (1) and (2)

$$h = \frac{1}{2k}\ln\left(1 + \frac{k}{g}V^2\right) \quad \text{and} \quad T = \frac{1}{\sqrt{(gk)}}\tan^{-1}\sqrt{\left(\frac{k}{g}\right)}V \qquad (3)$$

(b) *Downward motion*

Origin at maximum height

$\downarrow$ positive direction

Initial conditions: $t = 0$, $x = 0$, $v = 0$

$\boxed{mg \downarrow \uparrow R = mkv^2} \Rightarrow F = mg - mkv^2$

Equation of motion $\ddot{x} = g - kv^2$

Terminal velocity: as $\ddot{x} \to 0$, $v \to \sqrt{\left(\frac{g}{k}\right)}$

Expression relating x and v.

$$\ddot{x} = g - kv^2$$

$$\frac{1}{2}\frac{dv^2}{dx} = g - kv^2$$

$$-2k\frac{dx}{dv^2} = \frac{-k}{g - kv^2}$$

$$-2kx = \ln\{B(g - kv^2)\},$$

B constant

$x = 0, v = 0 \Rightarrow Bg = 1$

$$-2kx = \ln\left(1 - \frac{k}{g}v^2\right) \qquad (4)$$

Expression relating v and t.

$$\ddot{x} = g - kv^2$$

$$\frac{dv}{dt} = g - kv^2$$

$$\sqrt{k}\frac{dt}{dv} = \frac{\sqrt{k}}{g - kv^2}$$

$$= \frac{1}{2\sqrt{g}}\left\{\frac{\sqrt{k}}{\sqrt{g} - v\sqrt{k}} + \frac{\sqrt{k}}{\sqrt{g} + v\sqrt{k}}\right\}$$

$$2\sqrt{(gk)}\frac{dt}{dv} = \frac{\sqrt{k}}{\sqrt{g} - v\sqrt{k}} + \frac{\sqrt{k}}{\sqrt{g} + v\sqrt{k}}$$

$$2\sqrt{(gk)}t = \ln\left\{\frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}}C\right\},$$

C constant

$t = 0, v = 0 \Rightarrow C = 1$

$$2\sqrt{(gk)}t = \ln\left\{\frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}}\right\} \qquad (5)$$

(i) If $V = \frac{1}{2}\sqrt{\left(\frac{g}{k}\right)}$ when the particle returns to its point of projection, $x = h$

$$\therefore 2kx = 2kh = \ln\left(\frac{5}{4}\right) \Rightarrow \ln\left(1 - \frac{k}{g}v^2\right) = -\ln\left(\frac{5}{4}\right) \quad \text{from (3) and (4).}$$

$$\frac{k}{g}v^2 = \frac{1}{5}$$

$$\therefore \frac{v}{V} = \frac{2}{\sqrt{5}}$$

(ii) If $V = \frac{1}{2}\sqrt{\left(\frac{g}{k}\right)}$, $\quad \sqrt{(gk)}\, T = \tan^{-1}\left(\frac{1}{2}\right)$, $\quad$ from (3)

and on return to the projection point

$$v = \frac{1}{\sqrt{5}}\sqrt{\left(\frac{g}{k}\right)} \Rightarrow 2\sqrt{(gk)}t = \ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right) \qquad \text{from (5)}$$

$$\therefore \frac{t}{T} = \frac{1}{2}\ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)\left\{\tan^{-1}\left(\frac{1}{2}\right)\right\}^{-1} \doteqdot 1.5$$

(iii) $v = 0.8\sqrt{\left(\frac{g}{k}\right)} \Rightarrow -2kx = \ln(0.36)$ and $2\sqrt{(gk)}t = \ln 9$ from (4) and (5).

The particle achieves 80% of its terminal velocity at a distance

$$x - h = \frac{1}{2k}\left\{-\ln(0.36) - \ln\left(\frac{5}{4}\right)\right\} = \frac{1}{2k}\ln\left(\frac{20}{9}\right) \text{ below the}$$

point and at time $t + T = \frac{1}{\sqrt{(gk)}}\left\{\ln 3 + \tan^{-1}\left(\frac{1}{2}\right)\right\}$

after projection.

Exercise 7.2

1 A particle is moving vertically downward in a medium which exerts a resistance to the motion which is proportional to the speed of the particle. The particle is released from rest at O, and at time t its position is at a distance x below O and its speed is v. If the terminal velocity is V, show that $gx + Vv = Vgt$.

2 A particle is moving vertically downward in a medium which exerts a resistance to the motion which is proportional to the square of the speed of the particle. It is released from rest at O and its terminal velocity is V. Find the distance it has fallen below O and the time taken when its velocity is one-half of its terminal velocity.

3 A parachutist of mass m falls freely until his parachute opens. When it is open he experiences an upward resistance mkv, where v is his speed and k is a positive constant. The parachutist falls freely for a time $\frac{1}{2k}$ and then opens his parachute. Find the total distance he has fallen when his speed is $\frac{3g}{4k}$.

4 A particle of mass m falls from rest under gravity and the resistance to its motion is mkv^2, where v is its speed and k is a positive constant.

(a) Show that $v^2 = \frac{g}{k}(1 - e^{-2kx})$, where x is the distance fallen.

(b) As the distance it has fallen increases from d_1 to $2d_1$, the speed increases from v_1 to $\frac{5}{4}v_1$. Express the greatest possible speed of the particle in terms of v_1.

5 A particle of mass m is projected vertically upward under gravity with speed u in a medium in which the resistance is mk times the speed, where k is a positive constant.

(a) If the particle reaches its greatest height H in time T, show that $u = gT + kH$.

(b) If the particle returns to its original position with speed w after a further time T′, show that $w = gT' - kH$.

6 A particle of mass m is projected vertically upward under gravity in a medium in which the resistance is mk times the square of the speed, where k is a positive constant. If its speed of projection is equal to the terminal velocity V in the medium, show that when it returns to the point of projection its speed is $\dfrac{V}{\sqrt{2}}$.

7.3 Motion of a particle in two dimensions

Projectile motion

When investigating the motion of a particle in two dimensions, we consider the particle to be moving on a number plane. We choose the origin and the positive directions of the x- and y-axes, then consider separately the horizontal and vertical components of the motion. The resultant force F on the particle is the vector sum of all the physical forces acting on it. Newton's second law can be expressed as the vector equation $\vec{F} = m\vec{a}$, or equivalently as the pair of component equations $F_H = ma_H$, $F_V = ma_V$, where F_H and F_V are the horizontal and vertical components of $\vec{F}$ respectively. The simplest two-dimensional motion occurs when $\vec{F}$ is constant in both magnitude and direction, as for projectile motion in a non-resistive medium, where $\vec{F}$ has magnitude mg and is directed vertically downward throughout the motion.

Example 6

A particle is projected upward over horizontal ground with velocity U inclined at an angle θ to the horizontal. A second particle is projected simultaneously from the same point in the same direction with velocity $V > U$.

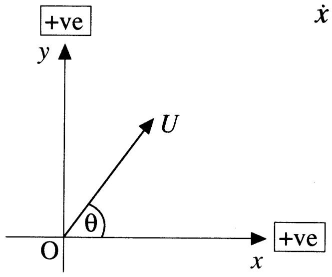
(a) Show that throughout the motion, the line joining the positions of the particles makes a constant angle with the horizontal.

(b) Given that the range and time of flight of the first particle are respectively $\dfrac{1}{g}U^2 \sin 2\theta$ and $\dfrac{2}{g}U \sin \theta$, find the position of the faster particle and the direction of its velocity vector when the slower particle hits the ground. If this velocity vector is directed upward at an angle $\beta = \dfrac{1}{2}\theta$ to the horizontal, find $\dfrac{U}{V}$ in terms of θ and deduce that $\dfrac{1}{4} < \dfrac{U}{V} < \dfrac{1}{2}$.

Solution

Axes and origin:

Figure 7.3



O is the point of projection

Initial conditions:
when $t = 0$
$x = y = 0$
$\dot{x} = U\cos\theta$, $\dot{y} = U\sin\theta$

Equations of motion:
$\ddot{x} = 0$
$\ddot{y} = -g$

Horizontal component
$\ddot{x} = 0$
$\dot{x} = U\cos\theta$
$x = Ut\cos\theta$

Vertical component
$\ddot{y} = -g$
$\dot{y} = -gt + U\sin\theta$
$y = -\frac{1}{2}gt^2 + Ut\sin\theta$

Hence after t seconds, the two particles are at positions

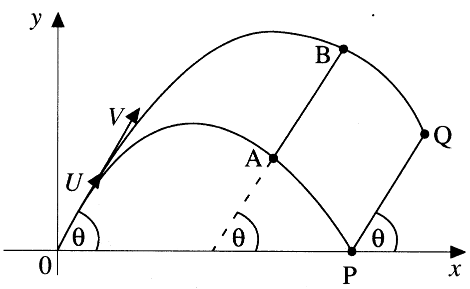
$A\left(Ut\cos\theta,\ -\frac{1}{2}gt^2 + Ut\sin\theta\right)$ (slower particle)

$B\left(Vt\cos\theta,\ -\frac{1}{2}gt^2 + Vt\sin\theta\right)$ (faster particle)

(a) Gradient AB $= \dfrac{(V-U)\,t\sin\theta}{(V-U)\,t\cos\theta} = \tan\theta$. Hence AB makes an angle θ with the horizontal throughout the motion.

(b) $AB^2 = (V-U)^2\,t^2\cos^2\theta + (V-U)^2\,t^2\sin^2\theta$
$AB = (V-U)\,t$ throughout the motion

Figure 7.4



The slower particle hits the ground at P when the faster particle is at Q and $t = \frac{2}{g}U\sin\theta$. Hence QP makes an angle θ with the horizontal and $QP = (V-U)\,\frac{2}{g}U\sin\theta$

At Q, $x = \mathrm{OP} + \mathrm{QP}\cos\theta = \frac{1}{g}U^2\sin 2\theta + \frac{1}{g}(V - U)\,U\sin 2\theta$

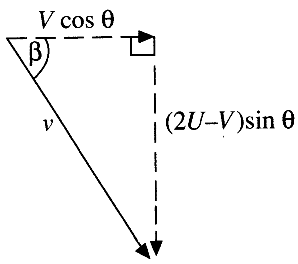
$$y = \mathrm{QP}\sin\theta = \frac{2}{g}U(V - U)\sin^2\theta$$

$\therefore$ Q has coordinates $\left(\frac{1}{g}UV\sin 2\theta,\ \frac{2}{g}U(V - U)\sin^2\theta\right)$

The components of the velocity vector of the faster particle at Q, when $t = \frac{2}{g}U\sin\theta$ are $\dot{x} = V\cos\theta$, $\dot{y} = (V - 2U)\sin\theta$

Figure 7.5

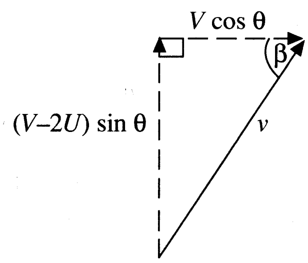
Case 1 $V<2U$



$\tan\beta = (2\frac{U}{V} - 1)\tan\theta$

Figure 7.6

Case 2 $V>2U$



$\tan\beta = (1-2\frac{U}{V})\tan\theta$

Note that if $V = 2U$, then $\vec{v}$ is horizontal, and the faster particle is at its greatest height.

If $\beta = \frac{1}{2}\theta$ and $\vec{v}$ is upward, then $V > 2U$ and $\tan\left(\frac{1}{2}\theta\right) = \left(1 - 2\frac{U}{V}\right)\tan\theta$

Let $\lambda = \tan\left(\frac{1}{2}\theta\right)$.

Then $\frac{U}{V} < \frac{1}{2}$ and $\lambda = \left(1 - 2\frac{U}{V}\right)\frac{2\lambda}{1 - \lambda^2}$

$$\therefore 1 - \lambda^2 = \left(2 - 4\frac{U}{V}\right)$$

$$\lambda^2 = 4\frac{U}{V} - 1$$

$$\frac{U}{V} = \frac{1}{4}(\lambda^2 + 1)$$

$$\therefore \frac{U}{V} = \frac{1}{4}\sec^2\left(\frac{1}{2}\theta\right) \quad \text{and} \quad \frac{1}{4} < \frac{U}{V} < \frac{1}{2}$$

Exercise 7.3

1 A particle is projected with speed V and angle of elevation α from a point O on level ground. The horizontal range is R where $R < \frac{V^2}{g}$.

(a) Show that there are two possible angles of projection for a given R and that if α_1 and α_2 are these two angles, then $\alpha_1 + \alpha_2 = \frac{\pi}{2}$.

(b) Show that if t_1 and t_2 are the respective times of flight, and h_1 and h_2 are the greatest heights reached, then $R = \frac{1}{2} g t_1 t_2 = 4\sqrt{(h_1 h_2)}$.

2 A particle is projected with speed V and angle of elevation α from a point O on the edge of a cliff of height h. When the particle hits the ground its path makes an angle $\tan^{-1}(2 \tan \alpha)$ with the horizontal. Find

(a) the distance from the foot of the cliff to the point where it lands

(b) its speed of projection

3 A particle is projected from a point O at time $t = 0$ with speed V and angle of elevation α. It moves under gravity and reaches its horizontal range R at time $t = T$.

(a) If the direction of motion of the particle makes an angle β with the horizontal when $t = \frac{1}{4} T$, show that $\tan \beta = \frac{1}{2} \tan \alpha$.

(b) If the line OP at a certain point P on its trajectory makes an angle γ below the horizontal, such that $\tan \gamma = \frac{1}{3} \tan \alpha$, show that $t = \frac{4}{3} T$.

4 A particle is projected from a point O with speed V and angle of elevation α. At a certain point P on its trajectory, the direction of motion of the particle and the line OP are inclined (in opposite senses) at equal angles β to the horizontal. Show that

(a) the time taken to reach P from O is $\frac{4V \sin \alpha}{3g}$

(b) $3 \tan \beta = \tan \alpha$

5 A and B are two points on level ground, 40 m apart. Simultaneously a particle is projected from A towards B and another particle is projected from B towards A, each with speed 20 ms^{-1} at an angle of elevation of 45°. Given that the two particles collide, find the time and the height above AB at which this occurs.

6 A and B are two points on level ground a distance X apart. A particle is projected from A towards B with speed V_1 and angle of elevation θ_1. Simultaneously another particle is projected from B towards A with speed V_2 and angle of elevation θ_2.

(a) If the two particles are to collide, show that $V_1 \sin \theta_1 = V_2 \sin \theta_2$ and find a second condition which must also be satisfied.

(b) If $V_1 = 45$, $V_2 = 60$, $\tan\theta_1 = \frac{4}{3}$, $\tan\theta_2 = \frac{3}{4}$ and $X = 150$, show that the two particles do collide and find the time and the height above AB at which this occurs.

7 A particle is projected from a point O with speed 40 ms^{-1} at an angle of elevation α, where $\tan\alpha = \frac{3}{4}$. Two seconds later, a second particle is projected from O and it collides with the first particle one second after leaving O. Find the initial velocity of the second particle.

8 A particle is projected from a point O with speed $\frac{150}{7}$ ms^{-1} at an angle of elevation α, where $\tan\alpha = \frac{4}{3}$. One second later, another particle is projected from O with speed $\frac{225}{7}$ ms^{-1} at an angle of elevation β, where $\tan\beta = \frac{3}{4}$ and in the same vertical plane through O as the first particle. Show that the two particles collide, and find when this occurs. (Take $g = 10$ ms^{-2}.)

9 Two particles A and B are projected simultaneously under gravity, A from a point O on horizontal ground and B from a point 45 m vertically above O. B is projected horizontally with speed 30 ms^{-1}. The particles hit the ground simultaneously at the same point. Taking $g = 10$ ms^{-2}, find
(a) the time taken for B to reach the ground and the horizontal distance it has then travelled
(b) the magnitude and direction of the velocity with which A is projected

10 O is a point on horizontal ground. D is a point a distance d vertically above O. A particle is projected from O with speed U at an angle of elevation α. Simultaneously a second particle is projected horizontally from D with speed V on the same side of OD as A and in the same vertical plane through O as the first particle.
(a) If the two particles are to collide, show that $V = U\cos\alpha$ and find a second condition which must also be satisfied.
(b) If $U = 51$, $V = 45$, $d = 60$ and $\tan\alpha = \frac{8}{15}$, show that the two particles do collide and find the time and the height above O at which this occurs. (Take $g = 10$ ms^{-2}.)

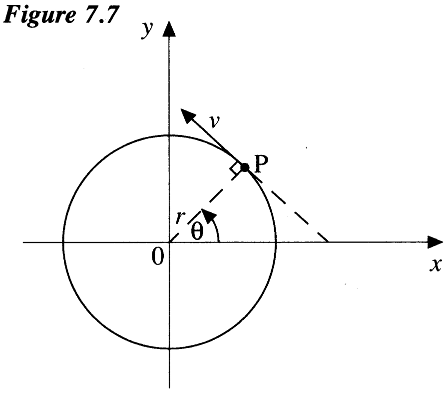
11 A projectile is fired from a point O on level ground with speed 13 ms^{-1} at an angle of elevation α, where $\tan\alpha = \frac{12}{5}$. The projectile just clears the top of a wall in its path and then reaches a maximum height of twice the height of the wall. At the instant of projection, a target is fired horizontally from the top of the wall and continues to move horizontally with constant speed u in the plane of the path of the projectile away from O. Find
(a) the distance of the base of the wall from O
(b) the value of u, given that the projectile hits the target

12 (a) A projectile is fired with speed V at an angle of elevation α from a point O and hits a stationary target at a distance d from O on the same level. Find the value of V.

(b) On another occasion the projectile is fired from O with the same speed and angle of projection as before. At the instant of projection, the target is fired from its original position with speed u and angle of elevation β in the plane of the path of the projectile and away from O. Given that the projectile hits the target, find the time at which this occurs.

7.4 Circular motion in a horizontal plane

Figure 7.7



θ in radians

P moves around a circle of radius r, centre O. The position vector $\overrightarrow{OP}$ makes an angle θ with the positive x-axis. The angular velocity of P is $\omega = \dfrac{d\theta}{dt} = \dot{\theta}$, while the angular acceleration of P is $\alpha = \dot{\omega}$.

The linear velocity of P is a vector $\vec{v}$ in the direction of motion and is directed along the tangent at P. If ℓ is the arc length from the x-axis to P, then $v = \dfrac{d\ell}{dt} = \dfrac{d}{dt}(r\theta) = r\dfrac{d\theta}{dt} = r\omega$. The linear velocity $\vec{v}$, measured in ms^{-1}, gives the speed at which P moves along the circumference, while the angular velocity ω, measured in radians s^{-1}, is related to the number of revolutions per second.

Since the direction of the velocity vector $\vec{v}$ is changing throughout the motion, P has a linear acceleration $\vec{a}$, even if its speed is constant. $\vec{a}$ can be considered as the vector sum of components a_T directed along the tangent at P and a_R directed along the radius. Since a_T is in the same line as $\vec{v}$, a_T changes the speed of the particle as it travels around the circle, while a_R at right angles to $\vec{v}$ changes its direction. If P travels around the circle with constant speed we can deduce that $a_T = 0$, and $\vec{a}$ is directed along the radius and, as it turns the velocity vector inward, it must be directed towards O. In general, $a_T = \dfrac{d^2\ell}{dt^2} = r\dot{\omega}$, hence $a_T = r\alpha$, where α is the angular acceleration.

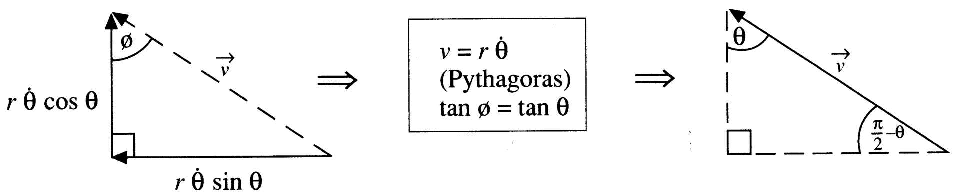
To justify these results mathematically, and to find the magnitude of the radial component a_R of the linear acceleration, we need to investigate the linear velocity $\vec{v}$ as the vector sum of $\dot{x}$ and $\dot{y}$, and the linear acceleration $\vec{a}$ as the vector sum of $\ddot{x}$ and $\ddot{y}$.

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

$\dot{x} = -r\dot{\theta}\sin\theta$ $\qquad$ $\dot{y} = r\dot{\theta}\cos\theta$

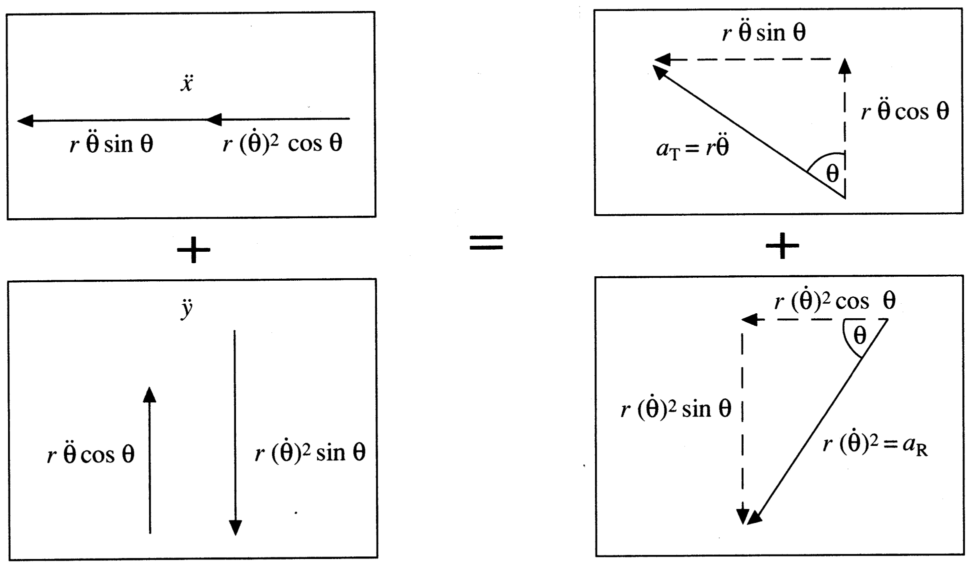
$\ddot{x} = -r\ddot{\theta}\sin\theta - r(\dot{\theta})^2\cos\theta$ $\qquad$ $\ddot{y} = r\ddot{\theta}\cos\theta - r(\dot{\theta})^2\sin\theta$

Figure 7.8



Hence $v = r\omega$ directed along the tangent at P (in the direction of travel).

Figure 7.9



Note that the vectors comprising $\ddot{x}$ have directions according to the signs of the corresponding expressions in r and θ and similarly for $\ddot{y}$.

To obtain the vector sum $\vec{a}$ of $\ddot{x}$ and $\ddot{y}$, we can add these four vectors in any order. Hence $\vec{a}$ is the vector sum of $a_T = r\ddot{\theta}$, directed along the tangent, and $a_R = r(\dot{\theta})^2$, directed along the radius towards O.

We could have obtained these results more easily by considering P as a point in an Argand diagram representing $z = x + iy$. Then $\dot{x} + i\dot{y}$ and $\ddot{x} + i\ddot{y}$ are complex number representations of the vector sums of $\dot{x}$ and $\dot{y}$, and of $\ddot{x}$ and $\ddot{y}$, respectively.
Then $\dot{x} + i\dot{y} = -r\dot{\theta} \sin \theta + i(r\dot{\theta} \cos \theta) = \dot{\theta}\{ir(\cos \theta + i \sin \theta)\}$, which corresponds to a rotation of OP anticlockwise by $\frac{\pi}{2}$, followed by an enlargement by $\dot{\theta}$ (Section 2.2). Hence $v = r\dot{\theta}$ along the tangent.

$$\begin{aligned}\ddot{x} + i\ddot{y} &= -r\ddot{\theta} \sin \theta - r(\dot{\theta})^2 \cos \theta + i\{r\ddot{\theta} \cos \theta - r(\dot{\theta})^2 \sin \}\\ &= \ddot{\theta}\{ir(\cos \theta + i \sin \theta)\} + (\dot{\theta})^2 \{- r(\cos \theta + i \sin \theta)\}\end{aligned}$$

The first expression represents the vector $\overrightarrow{OP}$ rotated by $\frac{\pi}{2}$, then enlarged by $\ddot{\theta}$, while the second represents the vector opposite $\overrightarrow{OP}$ enlarged by $(\dot{\theta})^2$. Hence $\vec{a}$ is composed of a tangential component of magnitude $r\ddot{\theta}$ and a radial component $r(\dot{\theta})^2$. Representing vectors as complex numbers is a useful means of converting vector problems into algebraic form.

Using Newton's second law, we can deduce that if a particle P of mass m is observed to perform circular motion with angular velocity ω and angular acceleration α, the resultant force $\vec{F}$ on P is composed of a component $mr\alpha$ directed along the tangent and a component $mr\omega^2$ directed towards the centre of the circle.

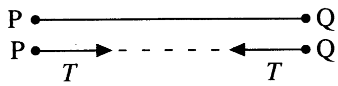
> If P is moving in a circle of radius r with constant speed v and constant angular velocity ω, the motion is described as *uniform circular motion* and the following results apply:
> - Linear velocity is $v = r\omega$, directed along the tangent.
> - Linear acceleration is $a = r\omega^2$, directed towards the centre.
> - The vector sum of all the physical forces on P (the resultant force) is $F = mr\omega^2$ (or equivalently $F = \frac{mv^2}{r}$) directed towards the centre.
> - If f is the number of revolutions per second (frequency) and T is the time taken for one revolution (period), $\omega = 2\pi f = \frac{2\pi}{T}$.

In typical problems involving a particle P observed to perform uniform circular motion, the physical forces on P are gravity, tension (when P is connected by a string or rod to a fixed point or another particle) or reaction forces (when P is in contact with a surface).

Tension forces

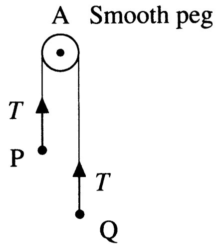
When particles are connected by a taut string, the string exerts a tension force on each particle.

Figure 7.10



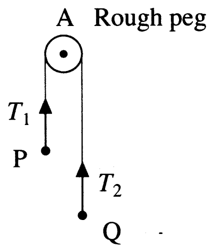
Equal and opposite tension forces on P and Q.

Figure 7.11



Equal tension forces on P and Q, and the particles exert a force $2T$ downward on the peg.

Figure 7.12

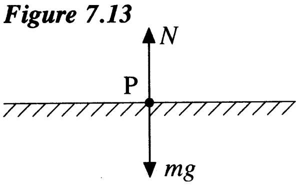


Unequal tension forces, as if there were two strings, or a string tied at A.

Reaction forces

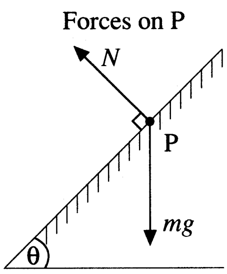
A body P in contact with a surface exerts a force on the surface and the surface exerts an equal and opposite force on P (Newton's third law). These forces are termed an *action-reaction pair* and the force exerted on P by the surface is called a *reaction force*. The reaction force acts at right angles to the surface.

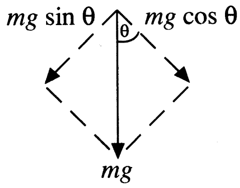
Figure 7.13



P, with mass m, is at rest or moving with constant velocity on a horizontal surface. The forces on P are the reaction force $\vec{N}$ and the gravitational force mg. The acceleration of P is zero, hence the vector sum of these forces is zero, and $N = mg$.

Figure 7.14



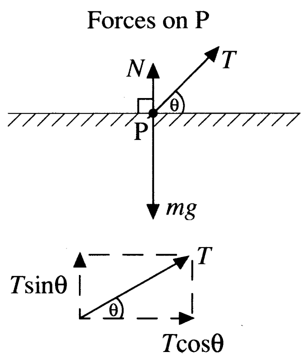


Resolution of mg into components.

P of mass m is sliding down a smooth slope inclined at an angle θ to the horizontal. The acceleration of P is directed down the slope, hence the vector sum of the reaction force $\vec{N}$ and the gravitational force mg is $mg \sin \theta$ directed down the slope.

Since we have deduced that the resultant force is down the slope, we resolve the forces on P into components down the slope and at right angles to the slope. $\vec{N}$ has no component down the slope and we think of $\vec{N}$ as being balanced by the component of mg acting opposite to $\vec{N}$ at right angles to the slope. The remaining component of mg is directed down the slope and must be the magnitude of the resultant force. We write $N = mg \cos \theta$, $ma = mg \sin \theta$, where a is the observed acceleration of P.

Figure 7.15



Resolution of T into components

P of mass m moves in a straight line on a smooth horizontal table, pulled along by a string making an angle θ with the table. P has acceleration a directed horizontally to the right, hence we choose to resolve into forces horizontally and vertically.

The resultant force has zero vertical component $\Rightarrow$ $N + T \sin \theta = mg$
The resultant force has horizontal component ma $\Rightarrow$ $T \cos \theta = ma$

These examples illustrate the procedure for analysing the motion of a particle P acted on by several forces.

- Identify all the physical forces acting on P, and show these on a diagram. Dimensions and forces should not be shown on the same diagram.
- From the observed acceleration of P, deduce the direction and magnitude of the resultant force on P.
- Choose two perpendicular directions in which to resolve the forces on P. (Usually one is the direction of the resultant.)
- Use your deductions about the resultant force to write a relation between the forces for each direction.

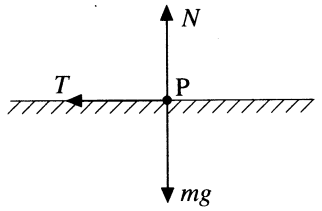
Example 7

A particle P of mass m moves in a horizontal circle in contact with a smooth table. The particle is joined by a taut string of length ℓ to a fixed point O on the table. Find the tension T in the string if the particle has constant speed v. The string can just support a mass of $3m$ without breaking if the mass hangs vertically. Find the maximum linear velocity of the particle around the circle without the string breaking.

Solution

Figure 7.16

Forces on P



Observed acceleration is $\ell\omega^2 = \dfrac{v^2}{\ell}$ towards O. Hence the vector sum of forces on P is $\dfrac{mv^2}{\ell}$ towards O.

The resultant has vertical component zero $\Rightarrow$ $N = mg$

The resultant has horizontal component $\dfrac{mv^2}{\ell}$ $\Rightarrow$ $T = \dfrac{mv^2}{\ell}$

The string breaks if $T > 3mg$. Hence $\dfrac{mv^2}{\ell} \leqslant 3mg \;\Rightarrow\; v \leqslant \sqrt{(3g\ell)}$

The maximum speed without the string breaking is $\sqrt{(3g\ell)}$

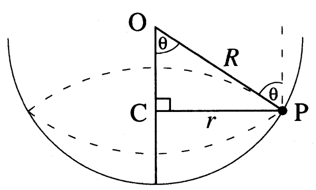
Example 8

A particle P of mass m travels with constant speed in a horizontal circle around the inside of a smooth hemispherical bowl of radius R, centre O. C is the centre of the circle of motion and OP makes an angle θ with OC. Show that if the angular velocity of P is ω, then $\omega^2 = \dfrac{g}{R\cos\theta}$, and find the reaction force in terms of ω. Describe what would happen if ω were increased.

Solution

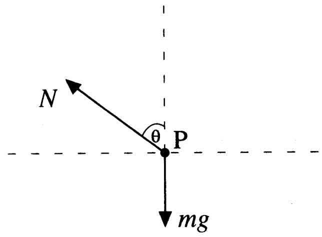
Figure 7.17

Dimension diagram



$r = R\sin\theta$

Forces on P



Note: N is normal to the surface at P, hence directed towards O.

P performs uniform circular motion about C, hence the resultant force is directed towards C.

The resultant has vertical component zero $\Rightarrow$ $N\cos\theta = mg$

The resultant has horizontal component $mr\omega^2$ $\Rightarrow$ $N\sin\theta = m(R\sin\theta)\,\omega^2$

Hence $N = mR\omega^2$ and $\cos\theta = \dfrac{g}{R\omega^2} \;\Rightarrow\; \omega^2 = \dfrac{g}{R\cos\theta}$

As ω increases, $\cos \theta$ decreases and hence θ increases. The particle moves up the side the bowl, so the plane of its circle of motion is closer to O. The reaction force $\vec{N}$ exerted by the surface on the bowl increases in magnitude.

Note that when a surface is described as smooth, no friction forces act on the particle.

Exercise 7.4

1 A partical of mass m kg is travelling at constant speed v ms^{-1} round a circle of radius r m.
(a) If $v = 8$ and $r = 2$, find the magnitude of the linear acceleration.
(b) If $v = 3$, $r = 6$, and the force acting towards the centre of the circle is of constant magnitude 6 N, find the value of m.

2 A particle of mass 0.25 kg is attached to one end of a light inextensible string of length 0.5 m. The other end is fixed to a point A on a smooth horizontal table. The particle is set in motion in a circular path.
(a) If the speed of the particle is 8 ms^{-1}, find the tension in the string and the reaction with the table.
(b) If the string breaks when the tension in it exceeds 50 N, find the greatest angular velocity at which the particle can travel.

3 A mass of 2 kg is revolving at the end of a string 2 m long on a smooth horizontal table with uniform angular speed of 1 revolution per second.
(a) Find the tension in the string.
(b) If the string would break under a tension equal to the weight of 20 kg, find the greatest possible speed of the mass.

4 A mass of 1 kg is fastened by a string of length 1 m to a point 0.5 m above a smooth horizontal table and is describing a circle on the table with uniform angular speed of 1 revolution in 2 seconds. Find the force exerted on the table and the tension in the string.

5 A particle moves with constant angular velocity ω in a horizontal circle of radius r on the inside of a fixed smooth hemispherical bowl of internal radius $2r$. Show that $\omega^2 = \dfrac{g}{r\sqrt{3}}$.

6 One end of a light inextensible string of length ℓ is attached to a fixed point O which is at a height $\frac{1}{3}\ell$ above a smooth horizontal table. A particle of mass m is attached to the other end of the string and rests on the table with the string taut. The particle is set in motion so that it moves in a circle on the table with constant speed v.
(a) Find the tension in the string and the reaction exerted on the particle by the table.
(b) Show that $v^2 \leqslant \dfrac{8g\ell}{3}$.

7 A particle P of mass 0.2 kg moving on a smooth horizontal table with constant speed v ms^{-1} describes a circle with centre O such that OP $= r$ m. The particle is subject to two fores, one towards O with magnitude $8v$ N and one away from O with magnitude $\frac{k}{r^2}$ N, where k is a positive constant.

(a) Given that $k = 75$ and $r = 1$, find the possible values of v.

(b) If the period of revolution is $\frac{\pi}{5}$ when $v = 20$, find the values of r and k.

(c) If $r = 1$, find the set of possible values of k.

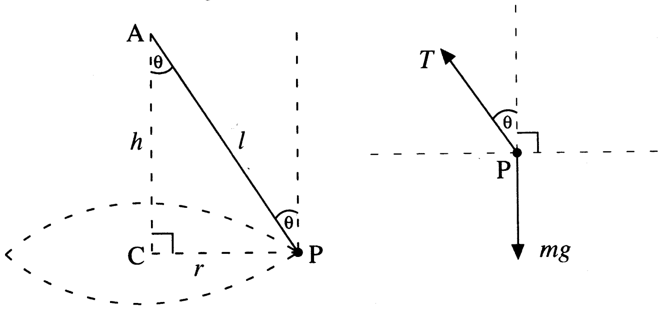
7.5 The conical pendulum

Example 9

A bob P of mass m is suspended from a fixed point A by a light inextensible string of length ℓ. P is observed to perform uniform circular motion with angular velocity ω in a plane below A. Show that the depth of the bob below the suspension point is independent of the length of the string. Describe what happens if the rotational speed of the bob is increased.

Solution

Figure 7.18 Dimension diagram Forces on P



$r = l \sin\theta$ $h = l \cos\theta$

The resultant force is $mr\omega^2$ towards the centre of the circle.
The resultant has a vertical component zero $\Rightarrow$ $T\cos\theta = mg$.
The resultant has a horizontal component $mr\omega^2$ $\Rightarrow$ $T\sin\theta = m(\ell\sin\theta)\omega^2$.

$$\left.\begin{aligned} T\cos\theta &= mg \quad (1)\\ T &= m\ell\omega^2 \quad (2)\end{aligned}\right\} \Rightarrow \quad \cos\theta = \frac{g}{\ell\omega^2} \qquad (1) \div (2)$$

$$\therefore h = \frac{g}{\omega^2}$$

Hence h is independent of ℓ and as ω increases, h decreases and the bob rises.

The system of string and bob described in this example is called a conical pendulum.

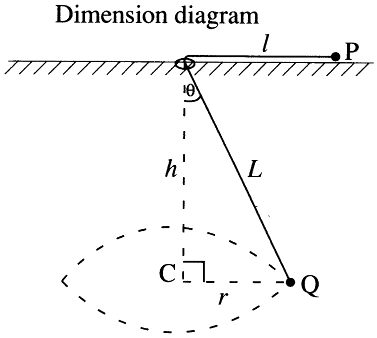
Example 10

A string passing through a smooth hole in a smooth horizontal table connects a particle P of mass m on the table to a particle Q of mass M suspended below the table. P and Q are both performing uniform circular motion with angular velocity ω, where Q moves in a horizontal circle at a depth h below the table. The lengths of string above and below the table are ℓ and L respectively. Show that h depends only on ω, and that $\frac{\ell}{L} = \frac{M}{m}$.

Solution

Figure 7.19

Dimension diagram



$r = L\sin\theta$, $h = L\cos\theta$

The resultant force on P is horizontal, with magnitude $m\ell\omega^2$.
The resultant force on Q is horizontal, with magnitude $Mr\omega^2 = M(L\sin\theta)\,\omega^2$.

Figure 7.20

Forces on P

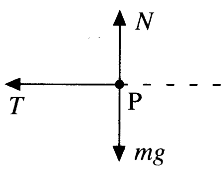
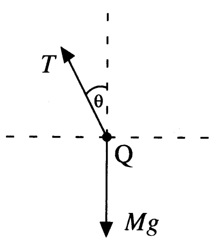


Figure 7.21

Forces on Q



$N = mg$ (1)
$T = m\ell\omega^2$ (2)

$T\cos\theta = Mg$ (3)
$T\sin\theta = ML\sin\theta\,\omega^2.$ (4)

From (2) amd (4), $m\ell = ML \;\Rightarrow\; \frac{\ell}{L} = \frac{M}{m}$

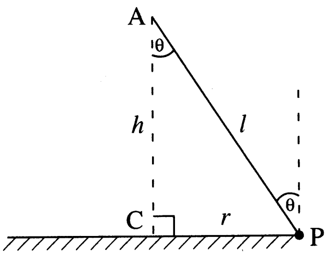
(3) ÷ (4) $\Rightarrow$ $\cos\theta = \frac{g}{L\omega^2}$ $\quad\therefore h = \frac{g}{\omega^2}$. Hence h depends only on ω.

Example 11

A particle P of mass m is connected to a fixed point A by a string of length ℓ. P moves in a circle with constant angular velocity ω on a smooth horizontal table, distance h below A. Show that $\omega < \sqrt{\left(\frac{g}{h}\right)}$. Describe what happens as ω increases to this critical value.

Solution

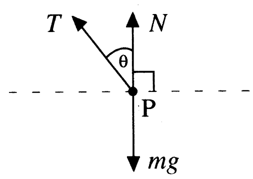
Figure 7.22



$r = l\sin\theta$ $h = l\cos\theta$

Figure 7.23

Forces on P



The resultant force on P is horizontal towards C, of magnitude $mr\omega^2$.

$N + T\cos\theta = mg$ (1) (Vertical components sum to 0.)

$T\sin\theta = m\ell\sin\theta\,\omega^2$

$\Rightarrow T = m\ell\omega^2$ (2) $(r = \ell\sin\theta)$

Substitute for T in (1) $N = m(g - \ell\cos\theta\,\omega^2)$

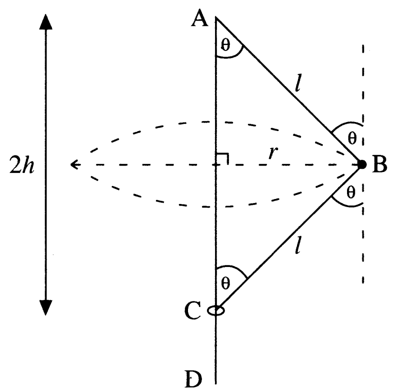
But $h = \ell\cos\theta$ $\therefore N = m(g - h\omega^2)$

While P stays in contact with the table, $N > 0$ and hence $\omega < \sqrt{\left(\frac{g}{h}\right)}$.

As ω increases, T increases and N decreases. When $\omega = \sqrt{\left(\frac{g}{h}\right)}$, the particle loses contact the table. For $\omega > \sqrt{\left(\frac{g}{h}\right)}$, the forces on P are is in figure 7.23 without the reaction force $\vec{N}$. Hence the system behaves like a conical pendulum above the table, with $\omega = \sqrt{\left(\frac{g}{x}\right)}$, where $x < h$ is the depth of the circle below A.

Example 12

Figure 7.24



$r = l \sin \theta \qquad h = l \cos \theta$

AD is a fixed smooth rod, AB and BC are light rods smoothly jointed at B and at A. The mass m at B moves in a horizontal circle about rod AD with constant angular velocity ω. C is a smooth collar of mass M which can slide freely on the rod AD.

Find in terms of ω, m, M and ℓ, the tensions in each of the rods AB and BC, and the depth of C below A, when the collar C is in equilibrium.

Solution

Figure 7.25

Forces on B

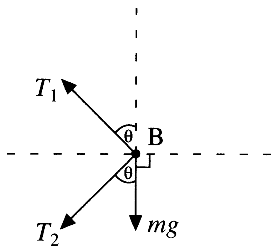
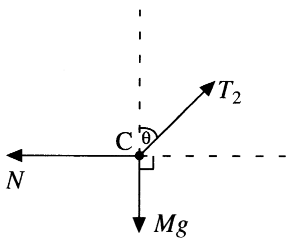


Figure 7.26

Forces on C



The resultant force on B is $mr\omega^2$ horizontally to the left.
The sum of vertical components is 0
$\therefore\ T_1 \cos \theta = T_2 \cos \theta + mg$
The sum of horizontal components is $mr\omega^2$,
$\therefore\ T_1 \sin \theta + T_2 \sin \theta = mr\omega^2$

$(T_1 - T_2) \cos \theta = mg \qquad (1)$

$(T_1 + T_2) \sin \theta = m\ell \sin \theta\, \omega^2 \qquad (2)$

The collar exerts a force to the right on the rod AD, hence the rod AD exerts an equal and opposite force $\vec{N}$ on the collar. The rod exerts the same tension T_2 on B and C. C is in equilibrium, hence the resultant force on C is zero.

$T_2 \cos \theta = Mg \qquad (3)$

$T_2 \sin \theta = N \qquad (4)$

(1) + (3) $\Rightarrow$ $T_1 \cos \theta = (M + m)g \qquad (5)$

(1) + $\cot \theta \times$ (2) $\Rightarrow$ $2T_1 \cos \theta = m(g + \ell\omega^2 \cos \theta) \qquad (6)$

From (5) and (6) $\quad 2(M + m)g = m(g + \ell\omega^2 \cos \theta)$

$$\therefore \cos\theta = \frac{g}{\ell\omega^2}\left\{2\frac{M}{m}+1\right\} \Rightarrow \qquad T_1 = \frac{(M+m)\,\ell\omega^2}{\left(2\frac{M}{m}+1\right)} \qquad \text{from (5)}$$

$$\text{Then } T_2 = \frac{M\ell\omega^2}{\left(2\frac{M}{m}+1\right)} \qquad \text{from (3)}$$

$$\text{Also } h = \ell\cos\theta \Rightarrow 2h = \frac{2g}{\omega^2}\left\{2\frac{M}{m}+1\right\}$$

As ω increases, $2h$ decreases and the collar slides up the rod AD. The tension in each the rods AB and BC increases. Considering (4), the collar C exerts a greater force on the rod AD (equal and opposite to $\vec{N}$, with $\sin\theta$ increasing as ω increases).

The system in figure 7.24 is an example of a governor. Such a device was used to control the speed of a steam engine by using C to open or close a valve according to its position on the rod AD, in response to the angular velocity ω of the mass at B. In practice, B was counter-balanced by a second mass m diametrically opposite B and similarly connected to A and C.

Exercise 7.5

1 An inextensible string of length 2 m is fixed at one end A and carries at its other end B a particle of mass 6 kg which is rotating in a horizontal circle whose centre is 1 m vertically below A. Find the tension in the string and the angular velocity of the particle.

2 Two light inextensible strings AB and BC each of length ℓ are attached to a particle of mass m at B. The other ends A and C are fixed to two points in a vertical line such that A is a distance ℓ above C. The particle describes a horizontal circle with constant angular velocity ω. Find
(a) the tensions in the strings
(b) the least value of ω in order that both strings are taut

3 A light inextensible string of length 3 ℓ is threaded through a smooth ring and carries a particle at each end. One particle A, of mass m, is at rest at a distance ℓ below the ring while the other particle B, of mass M, is rotating in a horizontal circle whose centre is A. Find
(a) m in terms of M
(b) the angular velocity of B

4 The base of a hollow cone of semi-vertical angle 30° is fixed to a horizontal table. Two particles each of mass m are connected by a light inextensible string which passes through a small smooth hole in the vertex C of the cone. One particle A hangs at rest inside the cone while the other particle B moves on the outer smooth surface of the cone in a horizontal circle with centre A. Find
(a) the tension in the strings and the normal reaction of the cone on B
(b) the angular velocity of B

5 Two particles A and B of masses m and M respectively are attached to the ends of a light inextensible string which passes over a smooth hook at O which is free to rotate. The particle A hangs at rest vertically below O while the particle B moves in a horizontal circle with constant speed v. Find expressions for
(a) the length OB and the angle AOB
(b) the radius of the circle in which B moves

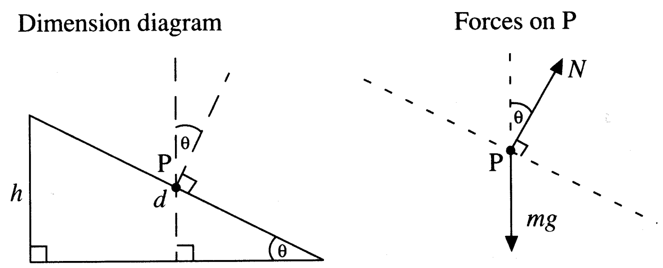
6 A particle A of mass $2m$ is attached by a light inextensible string of length ℓ to a fixed point O and is also attached by another light inextensible string of the same length to a small ring B of mass $3m$ which can slide on a fixed smooth vertical wire passing through O. The particle A describes a horizontal circle with OA inclined at an angle $\frac{\pi}{3}$ with the downward vertical. Find
(a) the tensions in the strings
(b) the angular velocity of A

7 Two light rigid rods AB and BC, each of length 2 m, are smoothly jointed at B and the rod AB is smoothly jointed at A to a fixed smooth vertical rod. The joint at B has a particle of mass 2 kg attached. A small ring of mass 1 kg is smoothly jointed to BC at C and can slide on the vertical rod below A. The ring rests on a smooth horizontal ledge fixed to the vertical rod at a distance $2\sqrt{3}$ m below A. The system rotates about the vertical rod with constant angular velocity ω.
(a) Find the forces in the rods and the force exerted on the ring by the ledge.
(b) What happens to the system when $\omega^2 > \frac{2g}{\sqrt{3}}$?

7.6 Motion around a circular banked track

A body P of mass m travels with constant speed v in a horizontal circular arc with radius of curvature R on a surface inclined at an angle θ to the horizontal. The difference in height across the width d of the road is h.

Figure 7.27



If there is no tendency for the body to slip sideways, no friction force operates and the force exerted by the surface on the body is the reaction force $\vec{N}$ at right angles to the surface.
Observed motion is uniform circular, hence the resultant force on P is horizontal towards the centre of the circle with magnitude $\frac{mv^2}{R}$.

The vertical components sum to 0 $\Rightarrow$ $N \cos \theta = mg$ (1)

The horizontal components sum to $\frac{mv^2}{R}$ $\Rightarrow$ $N \sin \theta = \frac{mv^2}{R}$ (2)

(2) ÷ (1) $\Rightarrow$ $\tan \theta = \frac{v^2}{Rg}$ (3)

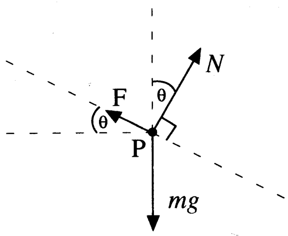
Hence for a fixed angle of banking θ, the correct speed to negotiate the curve so that no lateral force on the wheels is required is $v_0 = \sqrt{(Rg \tan \theta)}$.

If θ is small, $\sin \theta \doteqdot \theta \doteqdot \tan \theta$, and $\frac{h}{d} \doteqdot \tan \theta$

Using (3) $h \doteqdot \frac{v_0^2 d}{Rg}$ (θ small)

If P travels at a speed $V \neq \sqrt{(Rg \tan \theta)}$ around the curve, it relies on friction between P and the road to negotiate the curve.

Figure 7.28



Let $\vec{F}$ be the friction force exerted on P by the road. P travels in the horizontal circular arc with constant speed V.

The vertical components sum to 0 $\Rightarrow$ $N \cos \theta + F \sin \theta = mg$ (4)

The horizontal components sum to $\frac{mV^2}{R}$ $\Rightarrow$ $N \sin \theta - F \cos \theta = \frac{mV^2}{R}$ (5)

$$(4) \times \sin \theta - (5) \times \cos \theta \Rightarrow F(\sin^2 \theta + \cos^2 \theta) = m\left(g \sin \theta - \frac{V^2}{R} \cos \theta\right)$$

$$\therefore F = m \cos \theta \left(g \tan \theta - \frac{V^2}{R}\right), \quad \text{where } \tan \theta = \frac{v_0^2}{Rg} \quad \text{from (3)}$$

$$F = \frac{mg}{\sqrt{(v_0^4 + R^2g^2)}}(v_0^2 - V^2), \quad \left(\text{using } \cos \theta = \frac{Rg}{\sqrt{(v_0^4 + R^2g^2)}}\right).$$

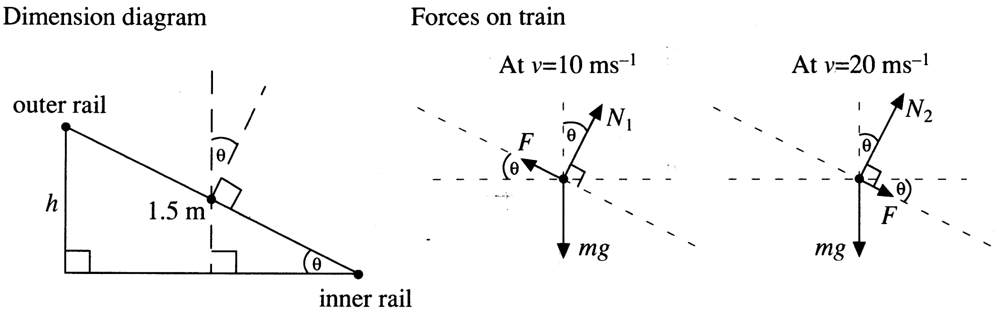
Hence, if $V < v_0$, P tends to slide down the slope and the friction force $\vec{F}$ is in the direction opposing the slide, as in figure 7.28. If $V > v_0$, $F < 0$ and the friction force is in the opposite direction to that indicated in figure 7.28. In this case, P tends to slide up the slope, the friction force again opposing motion.

Example 13

A railway line around a circular arc of radius 800 m is banked by raising the outer rail h m above the inner rail, where the distance between the rails is 1.5 m. When the train travels around the curve at 10 ms^{-1}, the lateral thrust on the inner rail is equal to the lateral thrust on the outer rail when the speed is 20 ms^{-1}. Calculate the value of h and the speed of the train when no lateral thrust is exerted on the rails. (Take $g = 9.8$ ms^{-2}.)

Solution

Figure 7.29



Note that at $v = 10$, the inner rail exerts a lateral thrust $\vec{F}$ up the slope (equal and opposite to the lateral thrust the train exerts on the inner rail by Newton's third law), while at $v = 20$, the lateral thrust $\vec{F}$ (given equal in magnitude) exerted by the outer rail acts down the slope. The normal reaction forces exerted by the rails on the train are shown separately as $\vec{N_1}$ and $\vec{N_2}$.
In each case, the resultant force is directed horizontally to the centre of the circle with magnitude $\dfrac{mv^2}{800}$, so the vertical components sum to 0, while the horizontal components sum to $\dfrac{mv^2}{800}$.

$N_1 \cos\theta + F\sin\theta = mg \quad (1) \qquad N_2\cos\theta - F\sin\theta = mg \quad (3)$

$N_1 \sin\theta - F\cos\theta = \dfrac{100m}{800} \quad (2) \qquad N_2\sin\theta + F\cos\theta = \dfrac{400m}{800} \quad (4)$

$$(1)\times\sin\theta - (2)\times\cos\theta \;\Rightarrow\; F = m\left(g\sin\theta - \frac{1}{8}\cos\theta\right)$$

$$(3)\times\sin\theta - (4)\times\cos\theta \;\Rightarrow\; F = m\left(\frac{1}{2}\cos\theta - g\sin\theta\right)$$

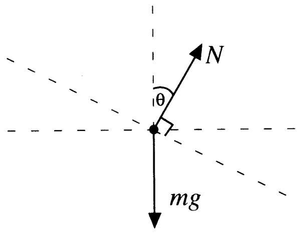
Eliminating F, $\quad 2g \sin \theta = \frac{5}{8} \cos \theta \qquad \therefore \tan \theta = \frac{5}{16g}$

But $\sin \theta \doteqdot \theta \doteqdot \tan \theta$, for θ small, $\qquad \therefore h = 1.5 \sin \theta \doteqdot 1.5 \tan \theta$

Hence $h \doteqdot 1.5 \times \frac{5}{16g} \doteqdot 0.048$

If there is no lateral sideways force, $F = 0$.

Figure 7.30



$$N \cos \theta = mg \qquad \text{(vertical component 0)}$$

$$N \sin \theta = \frac{mv^2}{800} \qquad \left(\text{horizontal component } \frac{mv^2}{800}\right)$$

$$\tan \theta = \frac{v^2}{800g}$$

$$\text{Then } \frac{5}{16g} = \frac{v^2}{800g} \Rightarrow v^2 = 250$$

Hence no lateral thrust is exerted on the rails when $v \doteqdot 16 \text{ ms}^{-1}$.

In this example, the lateral thrust exerted by the rails on the train replaces the friction force which the road exerts on the body as described in the previous discussion. Like the friction force, this lateral thrust opposes the tendency of the body to move down the slope if $v < \sqrt{(Rg \tan \theta)}$, or up the slope if $v > \sqrt{(Rg \tan \theta)}$. It is a reaction to the lateral force the train exerts on the rail, while the normal reaction $\vec{N}$ is a reaction to the force the train exerts at right angles to the rail.

Exercise 7.6 (Take $g = 9.80 \text{ ms}^{-2}$.)

1 A car has no tendency to slip when travelling at a speed of v ms^{-1} round a section of track of radius 100 m which is banked at an angle of 12°. Find the speed of the car.

2 A car has no tendency to slip when travelling at a speed of 30 ms^{-1} round a section of track of radius 200 m which is banked at an angle θ°. Find the angle of banking of the track.

3 At what speed should a car travel round a bend of radius 80 m which is banked at an angle of 10°?

4 At what angle should an aircraft be banked when flying at a speed of 100 ms^{-1} in a horizontal circle of radius 4 km?

5 A bend on a racetrack is designed with variable banking so that cars on the inside can corner at 80 kmh^{-1} and those on the outside can corner at 160 kmh^{-1}, in both cases without any tendency to slip. If the inner radius is 200 m and the outer radius is 220 m, find the difference between the angles of banking at the inside and the outside of the track.

6 A railway line is taken round a circular bend of radius 1000 m. The distance between the rails is 1.5 m. At what height above the inner rail should the outer rail be raised in order to eliminate lateral thrust for an engine travelling at a speed of 40 kmh^{-1} round the bend?

7 A railway line is taken round a circular bend of radius 1000 m. The distance between the rails is 1.5 m and the line is banked by raising the outer rail a height h above the inner rail. For an engine travelling around the bend, the sideways thrust on the inner rail at 10 ms^{-1} is equal to the sideways thrust on the outer rail at 20 ms^{-1}. Find the value of h.

Diagnostic test 7

Subsection

1 A particle moves in a straight line with retardation $\frac{1}{3v^2}$ where v is its velocity at time t. Initially the particle is at a fixed point O on the line with velocity $u > 0$. Find expressions in terms of u for the time taken and the distance travelled for the particle to come to rest. (7.1)

2 A particle moves in a straight line away from a fixed point O in the line such that at time t its displacement from O is x and its velocity is v. At time $t = 0$, $x = 0$ and $v = V$. Subsequently the particle is slowing down at a rate equal to kv^3, where k is a positive constant. Show that (7.1)

(a) $kx = \frac{1}{v} - \frac{1}{V}$ **(b)** $t = \frac{x}{V} + \frac{1}{2}kx^2$

3 A particle of mass m moves in a horizontal straight line. The particle is resisted by a constant force $2m$ and a variable force mv, where v is the speed. When $t = 0$, $v = 4$. Find the distance travelled and the time taken for the particle to come to rest. (7.1)

4 A particle of mass m moves in a horizontal straight line away from a fixed point O in the line. The particle is resisted by a force $mkv^{\frac{3}{2}}$, where k is a positive constant and v is the speed. When $t = 0$, $v = u > 0$. Show that the particle is never brought to rest and that its distance from O is at most $\frac{2}{k}\sqrt{u}$. (7.1)

5 Find the length of the pendulum of a clock which beats seconds at a place where $g = 9.812\ \text{ms}^{-2}$. The clock is moved to a place where $g = 9.921\ \text{ms}^{-2}$. Find (7.1)

(a) by how much it will gain or lose during one day

(b) to what length the pendulum should be altered if the clock is to register correctly

6 The depth of water in a harbour is 7.2 m at low water and 13.6 m at high water. On Monday, low water is at 2.05 pm and high water at 8.20 pm. The captain of a ship drawing 12.3 m of water wants to leave harbour as early on Monday afternoon as he can. Find (7.1)
(a) between what times he can leave on Monday
(b) his earliest leaving time on Wednesday if he fails to leave on Monday or Tuesday

7 A particle of mass 0.5 kg is released from rest and moves vertically downward under gravity in a medium which exerts a resistance to the motion of $\frac{1}{10}v^2$. At time t after release it has fallen a distance x and has velocity v. Taking $g = 10 \text{ ms}^{-2}$, show that $v^2 = 50(1 - e^{-0.4x})$ and $\ddot{x} = 10e^{-0.4x}$. (7.2)

8 A particle is moving vertically downward under gravity in a medium which exerts a resistance to the motion which is proportional to the speed of the particle. It is released from rest at O, and its terminal velocity is V. Find the distance it has fallen below O and the time taken when its velocity is one half of its terminal velocity. (7.2)

9 A and B are two points on level ground 110 m apart. A particle is projected from A towards B with speed 60 ms^{-1} at an angle of elevation of 30°. At the same instant another particle is projected form B towards A with speed 50 ms^{-1}. Given that the two particles collide, find (7.3)
(a) the angle of projection of the second particle
(b) the time of collision

10 A particle is projected from a point O with speed 20 ms^{-1} at an angle of elevation α. T seconds later another particle is projected from O with the same speed but at an angle of elevation β, where $\beta < \alpha$. The two particles collide at a point 24 m horizontally from O and 12 m vertically above O. Find the value of T. (Take $g = 10 \text{ ms}^{-2}$.) (7.3)

11 A particle is projected from a point O with speed v at an angle of elevation α. After a time t, where $t < \dfrac{v \sin \alpha}{g}$, the angle of elevation of the particle from O is θ and the angle which the direction of the velocity makes with the horizontal is ϕ. Show that $2 \tan \theta = \tan \phi + \tan \alpha$. (7.3)

12 Two particles are projected simultaneously from a point O with speeds U and V and angles of elevation α and β respectively. Show that at any time t during their flight, the line joining them is inclined at an angle θ to the horizontal, where $\tan \theta = \dfrac{U \sin \alpha - V \sin \beta}{U \cos \alpha - V \cos \beta}$. (7.3)

13 A particle of mass 0.5 kg is attached to one end of a light inextensible string of length 2 m. The other end is fixed to a point A on a smooth horizontal table. The particle is set in motion in a circular path. (7.4)

(a) If the speed of the particle is 12 ms^{-1}, find the tension in the string.

(b) If the string breaks when the tension in it exceeds 64 N, find the greatest speed at which the particle can travel.

14 A particle of mass 0.1 kg moving on a smooth horizontal table with constant speed v ms^{-1} describes a circle with centre O and radius r m. The particle is attracted towards O by a force of magnitude $4v$ N and repelled from O by a force of magnitude $\frac{k}{r}$ N where k is a constant. (7.4)

(a) Given that $v = 40$ and the time of one revolution is $\frac{\pi}{10}$ seconds, find the values of r and k.

(b) Given that $k = 30$ and $r = 1$, find the possible values of v.

(c) If $r = 1$, find the set of possible values of k.

15 A light inextensible string of length 5ℓ has one end fixed at a point A and the other end fixed at a point B which is at a distance 4ℓ vertically below A. A particle P of mass m is fastened to the midpoint of the string and moves with speed v, with the parts AP and BP of the string both taut, in a horizontal circular path whose centre is the midpoint of AB. (7.5)

(a) Find the tensions in the two parts of the string.

(b) Show that the motion described can take place only if $8v^2 \geqslant 9g\ell$.

16 Two rigid light rods AB and BC, each of length 0.5 m, are smoothly jointed at B and the rod AB is smoothly jointed at A to a fixed smooth vertical rod. The joint at B has a particle of mass 2 kg attached. A small ring of mass 1 kg is smoothly jointed to BC at C and can slide on the vertical rod below A. The ring rests on a smooth horizontal ledge at a distance $\frac{\sqrt{3}}{2}$ m below A. The system rotates about the vertical rod with constant angular velocity 6 radians per second. Find (7.5)

(a) the forces in the rods AB and BC

(b) the force exerted by the ledge on the ring

17 A car describes a horizontal circle of radius 100 m at a speed of 60 kmh^{-1} on a track which is banked at an angle α. Taking $g = 10$ ms^{-2} show that if $\tan\alpha = \frac{5}{18}$, the car has no tendency to slip. (7.6)

18 A railway line has been constructed around a circular curve of radius 600 m. The distance between the rails is 1.5 m and the outside rail is 0.1 m above the inside rail. Find the speed which eliminates a sideways thrust on the wheels for a train on this curve. (7.6)

Further questions 7

1 A particle moves in a straight line away from a fixed point O in the line such that at time t its displacement from O is x and its velocity is v. At time $t = 0$, $x = 0$ and $v = 1$. Subsequently the particle experiences a retardation of magnitude e^v.
(a) Find the time t_1 for the particle to slow to half its initial speed and the further time t_2 for the particle to come to rest. Deduce that $\frac{t_2}{t_1} = e^{\frac{1}{2}}$.
(b) Find the distance travelled by the particle in coming to rest.

2 A particle of mass m moves in a straight line away from a fixed point O in the line such that at time t its displacement from O is x and its velocity is v. At time $t = 0$, $x = 1$ and $v = 0$. Subsequently the only force acting on the particle is one of magnitude $m\frac{k}{x^2}$, where k is a positive constant, in a direction away from O.
(a) Show that v cannot exceed $\sqrt{(2k)}$.
(b) Find the time taken by the particle to move from $x = 2$ to $x = 4$.

3 A particle moves in a straight line with simple harmonic motion. At distances x_1 and x_2 from the centre of the motion its speeds are v_1 and v_2 respectively. Show that
(a) its amplitude is $\sqrt{\left(\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}\right)}$
(b) its period is $2\pi\sqrt{\left(\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}\right)}$

4 A particle P moves with uniform angular speed ω in a circle of radius r. O is the centre of the circle, AB is a diameter and at time t, $P\hat{O}B = \theta$. N is the foot of the perpendicular from P to AB. Show that as P moves in the circle, N moves in the diameter AB with simple harmonic motion.

5 A particle of mass 1 kg is projected vertically upward under gravity with speed $2c$ in a medium in which the resistance to motion is $\frac{g}{c^2}$ times the square of the speed, where c is a positive constant.
(a) Find the time of ascent of the particle.
(b) Find the speed with which the particle returns to its starting point.

6 A particle of mass m is projected vertically upward under gravity with speed nV in a medium in which the resistance to motion is mk times the square of the speed of the particle, where k and n are positive constants and V is the terminal velocity of the particle in this medium.
(a) Find the time taken by the particle to return to its starting point.
(b) Find the speed with which the particle returns to its starting point.

7 A particle is projected from a point O on level ground with speed V at an angle of elevation α. The particle just clears a wall of height h at a distance d from O.

(a) Show that if the angle of projection is fixed, the particle hits the ground at a distance $\dfrac{dh}{d \tan \alpha - h}$ beyond the wall.

(b) Show that if the speed of projection is fixed, the particle hits the ground at a distance c beyond the wall, where $g\{d^2c^2 + h^2(c + d)^2\} = 2\,dhV^2c$.

8 A particle is projected with speed V at an angle of elevation α from a point A on the edge of a cliff of height h. Simultaneously another particle is projected with speed $2V$ at an angle of elevation β from a point B, distance d from the foot of the cliff, the trajectories of the two particles being in the same vertical plane. If the two particles collide, show that $2 \sin(\beta + \gamma) = \sin(\alpha + \gamma)$, where $\gamma = \tan^{-1} \dfrac{h}{d}$.

9 A particle is projected from a point O up a plane inclined at an angle α above the horizontal. The speed of projection is V and the angle of elevation θ, where $\theta > \alpha$. Find

(a) the range of the particle on the inclined plane

(b) the maximum range up the plane

10 A particle is projected with speed V from the top of a cliff of height h above sea level. Find the greatest horizontal distance the particle can cover before landing in the sea.

11 A small ring C can move freely on a light inextensible string. The two ends of the string are attached to points A and B, where A is vertically above B and at a distance c from it. When the ring C is describing a horizontal circle with constant angular velocity ω, the distances of C from A and B are b and a respectively. Show that $2gc(a + b) = \omega^2(a - b)\{c^2 - (a + b)^2\}$.

12 A particle hangs by a light inextensible string of length ℓ from a fixed point O. A second particle of the same mass hangs from the first particle by a second string of the same length. The whole system moves with constant angular velocity ω about the vertical through O, the upper and lower strings making angles α and β respectively with the vertical. Show that

(a) $\tan \alpha = \dfrac{\ell w^2}{g}\left(\sin \alpha + \dfrac{1}{2} \sin \beta\right)$ **(b)** $\tan \beta = \dfrac{\ell w^2}{g}(\sin \alpha + \sin \beta)$

8 Harder 3 Unit Topics

8.1 Inequalities between real numbers

The fact that the square of a real number is non-negative gives rise to many inequalities involving real numbers. There are two basic inequalities which are often useful in establishing more complicated relationships:

$$a^2 + b^2 = (a - b)^2 + 2ab \Rightarrow a^2 + b^2 \geqslant 2ab \qquad (1)$$

$$(a + b)^2 = (a - b)^2 + 4ab \Rightarrow (a + b)^2 \geqslant 4ab \qquad (2)$$

In each case, equality holds if and only if (abbreviated iff) $a = b$

Example 1

Show that if a, b, c and d are positive, then

(a) $a^4 + b^4 + c^4 + d^4 \geqslant 4abcd$ **(b)** $a^2 + b^2 + c^2 \geqslant ab + bc + ca$

Deduce that if $a + b + c = 1$, then $ab + bc + ca \leqslant \frac{1}{3}$. State the condition for equality to hold.

Solution

(a) Using (1) for a^2, b^2 $\quad a^4 + b^4 \geqslant 2a^2b^2$ (equality iff $a^2 = b^2$)

Similarly $\quad c^4 + d^4 \geqslant 2c^2d^2$ (equality iff $c^2 = d^2$)

By addition $\quad a^4 + b^4 + c^4 + d^4 \geqslant 2(a^2b^2 + c^2d^2)$

Using (1) for ab, cd $\quad a^2b^2 + c^2d^2 \geqslant 2abcd$ (equality iff $ab = cd$)

Hence $\quad a^4 + b^4 + c^4 + d^4 \geqslant 4abcd$ (equality iff $a = b = c = d$)

(b) We cannot pair off the terms as we did in (a), but the symmetry of the result suggests that we consider separately $a^2 + b^2$, $b^2 + c^2$, $c^2 + a^2$ and then add the results. Using (1) in each case:

$$(a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2) \geqslant 2ab + 2bc + 2ca$$

Hence $a^2 + b^2 + c^2 \geqslant ab + bc + ca$ (equality iff $a = b = c$)

Then $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\geqslant 3(ab + bc + ca)$ (equality iff $a = b = c$)

$\therefore a + b + c = 1 \Rightarrow ab + bc + ca \leqslant \frac{1}{3}$ (equality iff $a = b = c = \frac{1}{3}$)

This example illustrates two techniques for establishing inequalities from simpler relationships: substitution of an expression for a single variable (e.g. $a \to a^2$, $b \to b^2$ in (1)), and addition of inequalities, either by pairing variables as in (a), or by pairing symmetrically as in (b) when the number of variables is odd.

Example 2

(a) Show that if $a > 0$, $a + \frac{1}{a} \geqslant 2$

(b) Decude that if a, b and c are positive, then $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geqslant 4$ and

$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geqslant 9$

(c) Hence show that $\frac{9}{a + b + c} \leqslant \frac{2}{b + c} + \frac{2}{c + a} + \frac{2}{a + b} \leqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, and state the conditions under which equality holds.

Solution

(a) Using (2), $\left(a + \frac{1}{a}\right)^2 = \left(a - \frac{1}{a}\right)^2 + 4a.\frac{1}{a} \Rightarrow \left(a + \frac{1}{a}\right)^2 \geqslant 4$,

with equality iff $a = \frac{1}{a}$

$\therefore a > 0 \Rightarrow \left(a + \frac{1}{a}\right) \geqslant 2$ (equality iff $a = 1$)

(b) $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) = 1 + 1 + \left(\frac{a}{b} + \frac{b}{a}\right) \geqslant 2 + 2$ $\left(\text{using (a), } a \to \frac{a}{b}\right)$

$\therefore (a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geqslant 4$ (equality iff $\frac{a}{b} = 1$, i.e. $a = b$)

$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 1 + 1 + 1 + \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right)$

$\geqslant 3 + 2 + 2 + 2$ (using (a) as above)

$\therefore (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geqslant 9$ (equality iff $a = b = c$)

(c) Examination of the right-hand inequality suggests use of the first of the inequalities in (b) for symmetrical pairs a and b, b and c, c and a, and then adding.

$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geqslant 4 \Rightarrow \frac{4}{a + b} \leqslant \frac{1}{a} + \frac{1}{b}$ (equality iff $a = b$)

Similarly $\frac{4}{b + c} \leqslant \frac{1}{b} + \frac{1}{c}$ (equality iff $b = c$)

and $\frac{4}{c + a} \leqslant \frac{1}{c} + \frac{1}{a}$ (equality iff $c = a$)

By addition $\dfrac{4}{a+b}+\dfrac{4}{b+c}+\dfrac{4}{c+a}\leqslant 2\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$

$$\therefore \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}\leqslant \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \text{ (equality iff } a=b=c)$$

Examination of the left-hand inequality suggests that we use the second of the two inequalities in (b). However, the substitution $a \to a+b,\ b \to b+c,\ c \to c+a$, will be necessary to obtain the factor $\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}$.

$$\{(a+b)+(b+c)+(c+a)\}\left\{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right\}\leqslant 9$$

with equality iff $a+b=b+c=c+a$ from (b)

$$\therefore \frac{9}{a+b+c}\leqslant \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \qquad \text{(equality iff } a=b=c)$$

$$\therefore \frac{9}{a+b+c}\leqslant \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}\leqslant \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

(equality iff $a=b=c$)

When the inequality to be established does not follow a recognised pattern, rearrangement to compare a variable expression with zero is a useful technique.

Example 3

Show that $(b+c-a)(c+a-b)\leqslant c^2$. Hence show that if a, b and c are the sides of a triangle, then $(b+c-a)(c+a-b)(a+b-c)\leqslant abc$. What is the nature of the triangle if equality holds?

Solution

$c^2-(b+c-a)(c+a-b)=a^2+b^2-2ab=(a-b)^2\geqslant 0$

$\therefore (b+c-a)(c+a-b)\leqslant c^2$ (equality iff $a=b$)

Clearly the role of c in this inequality is different from that of a and b. However the final inequality is symmetrical in a, b and c. This suggests combination of this inequality with two similar inequalities.

$$\begin{array}{lll} & (b+c-a)(c+a-b)\leqslant c^2 & \text{(equality iff } a=b) \\ a\to b,\ b\to c,\ c\to a & (c+a-b)(a+b-c)\leqslant a^2 & \text{(equality iff } b=c) \\ & (a+b-c)(b+c-a)\leqslant b^2 & \text{(equality iff } c=a) \end{array}$$

Since a, b and c are the sides of a triangle, all the expressions involved are positive. Hence multiplication gives

$(b+c-a)^2(c+a-b)^2(a+b-c)^2\leqslant c^2a^2b^2$

$\therefore (b+c-a)(c+a-b)(a+b-c)\leqslant abc$, with equality iff $a=b=c$, and the triangle is equilateral.

Note that the significance of expressions or variables being positive is that multiplication and division preserve the direction of the inequality, and square roots can be taken without introducing absolute values, since strictly

$a^2<b^2 \Leftrightarrow |a|<|b|$.

The triangle inequalities

Inequalities involving absolute values are often related to the triangle inequalities $|a + b| \leqslant |a| + |b|$ and $|a - b| \geqslant ||a| - |b||$. (The reason for the term *triangle inequalities* is clear when these results are considered for complex numbers – see figure 2.36 in Section 2.3.)

For real numbers, the result can be proved algebraically using $|x| = \sqrt{x^2}$.

$$2\,|a||b| = 2|ab| = \begin{bmatrix} 2ab, & ab \geqslant 0 \\ -2ab, & ab \leqslant 0 \end{bmatrix}$$

$$\begin{aligned} |a|^2 + |b|^2 - 2|a|\,|b| &\leqslant a^2 + b^2 - 2ab \leqslant |a|^2 + |b|^2 + 2|a||b| \\ (|a| - |b|)^2 &\leqslant \quad (a - b)^2 \quad \leqslant \quad (|a| + |b|)^2 \\ ||a| - |b|| &\leqslant \quad |a - b| \quad \leqslant \quad |a| + |b| \end{aligned}$$

The left-hand inequality is true with equality iff $ab \geqslant 0$, while equality holds on the right iff $ab \leqslant 0$. Replacing b by $-b$ in the right-hand inequality, we obtain $|a + b| < |a| + |b|$, with equality iff $(-ab) \leqslant 0$. Hence we have proved the quoted triangle inequalities and established that equality holds in both cases if and only if $ab \geqslant 0$.

Example 4

Show

(a) $|a + b + c| \leqslant |a| + |b| + |c|$

(b) $|a - b| \geqslant ||a - c| - |b - c||$

Solution

(a) $|a + b + c| = |(a + b) + c| \leqslant |a + b| + |c| \leqslant |a| + |b| + |c|$

(b) $|a - b| = |(a - c) - (b - c)| \geqslant ||a - c| - |b - c||$

Example 5

Show that $|\sin(\alpha + \beta)| \leqslant |\sin \alpha| + |\sin \beta|$. Under what conditions will equality hold?

Solution

$$\begin{aligned} |\sin(\alpha + \beta)| &= |\sin\alpha\cos\beta + \cos\alpha\sin\beta| \\ &\leqslant |\sin\alpha\cos\beta| + |\cos\alpha\sin\beta| \\ &= |\sin\alpha||\cos\beta| + |\cos\alpha||\sin\beta| \\ &\leqslant |\sin\alpha| + |\sin\beta| \qquad (\text{since } |\cos\theta| \leqslant 1) \qquad (3) \end{aligned}$$

Clearly equality holds if one of α and β is a multiple of π. If neither α nor β is a multiple of π then both $|\cos \alpha|$ and $|\cos \beta|$ are less than one, and equality cannot hold in (3). Hence equality holds if and only if one of α and β is a multiple of π.

Application of calculus to inequalities

Example 6

Show that $x \geqslant \ln(1 + x)$ for all $x > -1$. State when equality holds.

Solution

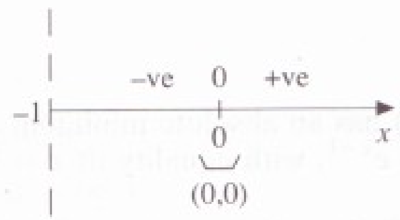
Let $f(x) = x - \ln(1 + x)$

$$f'(x) = 1 - \frac{1}{1 + x}$$

$$f'(x) = 0 \Leftrightarrow x = 0$$

Figure 8.1

Sign of $f'(x)$



$\therefore f(x)$ has an absolute minimum of 0 when $x = 0$.
Hence, for $x > -1$, $x \geqslant \ln(1 + x)$, with equality iff $x = 0$.

Example 7

Show that $\cos x \geqslant 1 - \frac{1}{2}x^2$ for all x.

Solution

Let $f(x) = \cos x - \left(1 - \frac{1}{2}x^2\right)$. Then $f(x)$ is an even function.

$$f'(x) = x - \sin x$$

$$f''(x) = 1 - \cos x \geqslant 0 \quad \text{for all } x$$

$f'(0) = 0$ and $f'(x)$ is a non-decreasing function $\Rightarrow f'(x) \geqslant 0$ for all $x \geqslant 0$
$\therefore f(x)$ is a non-decreasing function for $x > 0$
$\therefore f(0) = 0$ and $f(x)$ is even $\Rightarrow f(x) \geqslant 0$ for all x
$\therefore \cos x \geqslant 1 - \frac{1}{2}x^2$ for all x

Example 8

(a) Show that $x \leqslant e^{x-1}$ for all x

(b) Let $a_1, a_2 \ldots, a_n$ be positive numbers with arithmetic mean A. Show that

$$\frac{a_1, a_2, \ldots a_n}{A^n} \leqslant 1.$$

(c) Deduce that the geometric mean of n positive numbers cannot exceed their arithmetic mean. Under what conditions will their geometric and arithmetic means be equal?

Solution

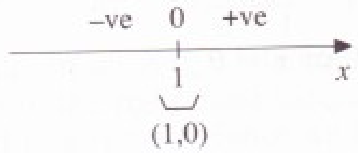
(a) Let $f(x) = e^{x-1} - x$

$f'(x) = e^{x-1} - 1$

$f'(x) = 0$ when $x = 1$

Figure 8.2

Sign of $f'(x)$



$\therefore f(x)$ has an absolute minimum of 0 when $x = 1$.

$\therefore x \leqslant e^{x-1}$, with equality iff $x = 1$.

(b) $\dfrac{a_k}{A} \leqslant e^{\frac{a_k}{A} - 1} \Rightarrow \dfrac{a_1}{A}\dfrac{a_2}{A}\ldots\dfrac{a_n}{A} \leqslant e^{\frac{a_1}{A} - 1 + \frac{a_2}{A} - 1 + \ldots + \frac{a_n}{A} - 1}$

But $\dfrac{a_1}{A} - 1 + \dfrac{a_2}{A} - 1 + \ldots + \dfrac{a_n}{A} - 1 = (a_1 + a_2 + \ldots + a_n)\dfrac{1}{A} - n = 0$

$\therefore \dfrac{a_1\, a_2 \ldots a_n}{A^n} \leqslant 1 \qquad \left(\text{equality iff } \dfrac{a_1}{A} = \dfrac{a_2}{A} = \ldots = \dfrac{a_n}{A} = 1\right)$

(c) $a_1\, a_2\, a_3 \ldots a_n \leqslant A^n \Rightarrow (a_1\, a_2\, a_3 \ldots a_n)^{\frac{1}{n}} \leqslant A$

with equality iff $a_1 = a_2 = \ldots = a_n$

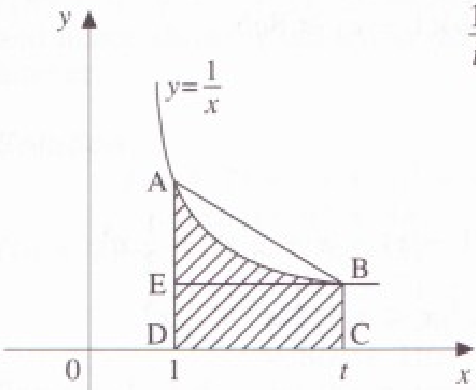
Example 9

Use $\ln t = \displaystyle\int_1^t \frac{1}{x}\, dx$ for $t > 1$ to deduce that

$$1 - \frac{1}{t} \leqslant \ln t \leqslant \frac{1}{2}\left(t - \frac{1}{t}\right) \qquad \text{for } t \geqslant 1$$

Solution

Figure 8.3



Area EBCD < shaded area < area ABCD

$$\frac{1}{t}(t-1) < \int_1^t \frac{1}{x}\,dx < \frac{1}{2}\left(1+\frac{1}{t}\right)(t-1)$$

$$1-\frac{1}{t} < \ln t < \frac{1}{2}\left(t-\frac{1}{t}\right), \quad t > 1$$

But equalities hold when $t = 1$

$$\therefore 1-\frac{1}{t} \leqslant \ln t \leqslant \frac{1}{2}\left(t-\frac{1}{t}\right), \quad t \geqslant 1$$

Exercise 8.1

1 (a) If $0 < a < b$ show that $a < \dfrac{a+b}{2} < b$

(b) If $0 < a < b$ show that $a < \sqrt{(ab)} < b$

2 (a) If $a > 0$, $b > 0$ show that $\dfrac{a+b}{2} \geqslant \sqrt{(ab)}$

(b) If $a > 0$, $b > 0$ show that $\dfrac{a+b}{2} \leqslant \sqrt{\left(\dfrac{a^2+b^2}{2}\right)}$

3 Show that $(a^2-b^2)(c^2-d^2) \leqslant (ac-bd)^2$
Deduce that $(a^2-b^2)(a^4-b^4) \leqslant (a^3-b^3)^2$

4 If $a > 0$, $b > 0$ and $c > 0$, show that $(a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right) \geqslant 9$
Deduce that $(a+b+c)(ab+bc+ca) \geqslant 9abc$ and hence deduce that $a^2b+b^2c+c^2a+ab^2+bc^2+ca^2 \geqslant 6abc$.

5 (a) Show that
$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

(b) If $a > 0$, $b > 0$ and $c > 0$, show that $a^2+b^2+c^2 \geqslant ab+bc+ca$.
Deduce that $a^3+b^3+c^3 \geqslant 3abc$ and hence deduce that
$\dfrac{a+b+c}{3} \geqslant \sqrt[3]{(abc)}$ and $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a} \geqslant 3$

6 (a) If $a > 0$, $b > 0$ and $c > 0$ show that $(a+b)(b+c)(c+a) \geqslant 8abc$

(b) If $a > 0$, $b > 0$, $c > 0$ and $d > 0$ show that
$(b+c+d)(a+c+d)(a+b+d)(a+b+c) \geqslant 81abcd$

7 If $a > 0$, $b > 0$ and $a+b=t$, show that

(a) $\dfrac{1}{a}+\dfrac{1}{b} \geqslant \dfrac{4}{t}$ **(b)** $\dfrac{1}{a^2}+\dfrac{1}{b^2} \geqslant \dfrac{8}{t^2}$

8 If $a > 0$, $b > 0$, $c > 0$ and $a + b + c = 1$ show that

(a) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geqslant 9$ and $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geqslant 27$

(b) $ab + bc + ca \geqslant 9abc$ and $(1 - a)(1 - b)(1 - c) \geqslant 8abc$

9 **(a)** Show that for $x > 0$ $\quad e^x > 1 + x$

(b) Show that for $x > 0$ $\quad x > \dfrac{3 \sin x}{2 + \cos x}$

10 **(a)** Show that for $x > 0$ $\quad x - \frac{1}{2}x^2 < \ln(1 + x) < x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

(b) Show that for $x > 0$ $\quad x - \frac{1}{3}x^3 < \tan^{-1} x < x - \frac{1}{3}x^3 + \frac{1}{3}x^5$

11 If $0 < t < 1$ show that $\frac{1}{2} < \dfrac{1}{1 + t} < 1$. By integrating between 0 and u, deduce that for $0 < u < 1$, $\frac{u}{2} < \ln(1 + u) < u$.

12 If $t > 0$ show that $\dfrac{1}{(1 + t)^2} < \dfrac{1}{1 + t} < 1$. By integrating between 0 and u, deduce that for $u > 0$, $\dfrac{u}{1 + u} < \ln(1 + u) < u$.

8.2 Mathematical induction

Mathematical induction is a means of successively establishing the truth of each member of a sequence of statements S(1), S(2), ... There are two patterns of proof by induction:
- establish the truth of S(1)
- show that for all positive integers k, S(k) true implies S ($k + 1$) true
- deduce that S(n) is true for all positive integers n

Or
- establish the truth of S(1) and the next few statements as required
- show that for all positive integers k, S(1), S(2), ... S(k) all true implies S($k + 1$) true
- deduce that S(n) is true for all positive integers n

This method of proof is useful for verifying a formula for the nth term in a sequence, for establishing a sequence of inequalities, or for investigating divisibility.

Example 10

$f(n) = n^4 + 4n^2 + 11$. Show that $f(n+2) - f(n) = 8(n+1)(n^2+2n+4)$, and hence show by induction that $f(n)$ is divisible by 16 if n is an odd positive integer.

Solution

$$\begin{aligned}
f(n+2) &= (n+2)^4 + 4(n+2)^2 + 11 \\
f(n) &= n^4 + 4n^2 + 11 \\
f(n+2) - f(n) &= \{(n+2)^4 - n^4\} + 4\{(n+2)^2 - n^2\} \\
&= \{(n+2)^2 - n^2\}\{(n+2)^2 + n^2 + 4\} \\
&= 2(2n+2)(2n^2+4n+8) \\
&= 8(n+1)(n^2+2n+4)
\end{aligned}$$

For $n = 1, 3, 5\ldots$ let the statement $S(n)$ be defined by
$S(n)$: $f(n)$ is divisible by 16.
Consider $S(1)$ $\quad f(1) = 1 + 4 + 11 = 16, \quad \therefore S(1)$ is true.
Let k be a positive odd integer. If $S(k)$ is true, then $f(k) = 16M$ for some integer M.
Consider $S(k+2)$ $\quad$ ($k+2$ is the next odd integer)

$$\begin{aligned}
f(k+2) - f(k) &= 8(k+1)(k^2+2k+4) \text{ and } k \text{ odd} \Rightarrow (k+1) \text{ even} \\
\therefore f(k+2) - f(k) &= 8.2J(k^2+2k+4), \qquad \text{for some integer } J \\
f(k+2) &= 16J(k^2+2k+4) + f(k) \\
&= 16\{J(k^2+2k+4) + M\}, \text{ if } S(k) \text{ is true.}
\end{aligned}$$

Now k, J, M integral $\Rightarrow J(k^2+2k+4) + M$ integral.
Hence for all odd positive integers k, $S(k)$ true implies $S(k+2)$ true.
But $S(1)$ is true. Hence by induction, $S(n)$ is true for all odd positive integers n $\therefore f(n)$ is divisible by 16 for all odd positive integers n.

The technique of finding (in factorised form) an expression for the difference of two successive terms in the sequence, before commencing the induction process, is useful for statements about divisibility, since the second step in the induction process is then more straightforward.

Example 11

For positive a and b, show by induction that $\left(\dfrac{a+b}{2}\right)^n \leqslant \dfrac{a^n+b^n}{2}$ for all positive integers n.

Solution

Define the statement $S(n)$: $\quad \left(\dfrac{a+b}{2}\right)^n \leqslant \dfrac{a^n+b^n}{2}$, $n = 1, 2, \ldots$

Clearly $S(1)$ is true. If $S(k)$ is true, then $\left(\dfrac{a+b}{2}\right)^k \leqslant \dfrac{a^k+b^k}{2}$

Consider S$(k+1)$, equivalent to $\dfrac{a^{k+1}+b^{k+1}}{2}-\left(\dfrac{a+b}{2}\right)^{k+1}\geqslant 0$

$$\begin{aligned}
\frac{a^{k+1}+b^{k+1}}{2}-\left(\frac{a+b}{2}\right)^{k+1} &= \frac{a^{k+1}+b^{k+1}}{2}-\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}\right)^{k} \\
&\geqslant \frac{a^{k+1}+b^{k+1}}{2}-\left(\frac{a+b}{2}\right)\left(\frac{a^k+b^k}{2}\right), \\
&\qquad \text{if S}(k) \text{ is true} \\
&= \frac{a^{k+1}+b^{k+1}}{2}-\frac{a^{k+1}+b^{k+1}}{4}-\frac{ab^k+ba^k}{4} \\
&= \frac{1}{4}\left(a^{k+1}+b^{k+1}-ab^k-ba^k\right) \\
&= \frac{1}{4}(a-b)(a^k-b^k)
\end{aligned}$$

Now the bracketed expressions on the right are either both positive, or both negative, and their product is therefore positive,

$\therefore \dfrac{a^{k+1}+b^{k+1}}{2}-\left(\dfrac{a+b}{2}\right)^{k+1}\geqslant 0,\qquad$ if S(k) is true.

Hence for all positive integers k, S(k) true implies S$(k+1)$ true. But S(1) is true. Hence S(n) is true for all positive integers n.

$\therefore \left(\dfrac{a+b}{2}\right)^{n}\leqslant \dfrac{a^n+b^n}{2}$ for all positive integers n.

Example 12

Show by induction that if $f(x)=\sin ax$, then $f^{(n)}(x)=a^n \sin\left(ax+n\dfrac{\pi}{2}\right)$ for all positive integers n.

Solution

Define the statement S(n): $f^{(n)}(x)=a^n\sin\left(ax+n\dfrac{\pi}{2}\right)$, $n=1,2,\ldots$

Consider S(1) $\qquad f'(x)=a\cos ax=a\sin\left(ax+\dfrac{\pi}{2}\right)\qquad \therefore$ S(1) is true

If S(k) is true, then $f^{(k)}(x)=a^k\sin\left(ax+k\dfrac{\pi}{2}\right)$.

Consider S$(k+1)$.

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\, f^{(k)}(x) \\ &= \frac{d}{dx}\, a^k \sin\left(ax + k\,\frac{\pi}{2}\right), \qquad \text{if S}(k)\text{ is true} \\ &= a^k . a \cos\left(ax + k\,\frac{\pi}{2}\right) \end{aligned}$$

But $\cos\left(ax + k\,\frac{\pi}{2}\right) = \sin\left(ax + k\,\frac{\pi}{2} + \frac{\pi}{2}\right)$

$\therefore\ f^{(k+1)}(x) = a^{k+1} \sin\left(ax + [k+1]\,\frac{\pi}{2}\right)$, if S($k$) is true.

Hence for all positive integers k, S(k) true implies S($k + 1$) true. But S(1) is true, therefore by induction, S(n) is true for all positive integers n.

$\therefore\ f^{(n)}(x) = a^n \sin\left(ax + n\,\frac{\pi}{2}\right)$, for all positive integers n.

Example 13

A sequence of terms u_n, $n = 1, 2, 3, \ldots$ is defined by the recurrence relation $u_n = 4u_{n-1} - 5u_{n-2} + 2u_{n-3}$, $n = 4, 5, 6, \ldots$, together with the initial conditions $u_1 = 3$, $u_2 = 1$, $u_3 = 0$. Show by induction that $u_n = 2^{n-1} - 3n + 5$, $n = 1, 2, 3, \ldots$

Solution

Define the statement S(n): $\quad u_n = 2^{n-1} - 3n + 5,\ n = 1, 2, 3, \ldots$
Consider S(1) $\qquad 2^0 - 3 + 5 = 3 = u_1 \Rightarrow$ S(1) true
Consider S(2) $\qquad 2^1 - 6 + 5 = 1 = u_2 \Rightarrow$ S(2) true
Consider S(3) $\qquad 2^2 - 9 + 5 = 0 = u_3 \Rightarrow$ S(3) true
Let k be a positive integer, $k \geqslant 3$. If S(n) is true for all integers $n \leqslant k$, then $u_n = 2^{n-1} - 3n + 5$, for $n = 1, 2, 3, \ldots, k$.
Consider S($k + 1$) $\qquad u_{k+1} = 4u_k - 5u_{k-1} + 2u_{k-2}$ (since $k + 1 \geqslant 4$)

$$\begin{aligned} \therefore u_{k+1} &= 4\{2^{k-1} - 3k + 5\} - 5\,\{2^{k-2} - 3(k-1) + 5\} + 2\,\{2^{k-3} - 3(k-2) + 5\} \\ &= 2^{k-2}(8 - 5 + 1) - 3k(4 - 5 + 2) + (20 - 40 + 22) \\ &= 2^k - 3(k+1) + 5, \qquad \text{if S}(n)\text{ is true, } n = 1, 2, \ldots, k \end{aligned}$$

For $k = 3, 4, \ldots$, S(n) true for all positive integers $n \leqslant k$ implies S($k + 1$) is true. But S(1), S(2), S(3) are true. Hence by induction, S(n) is true for all positive integers n.
$\therefore u_n = 2^{n-1} - 3n + 5,\ n = 1, 2, 3, \ldots$

Note that the recurrence relation is only true for $n \geqslant 4$, hence its use in step two of the induction process requires $k + 1 \geqslant 4$ and the truth of the previous three statements is then required to deduce the truth of S($k + 1$). Hence we must begin by establishing the truth of the first three statements and use the second pattern for a proof by induction.

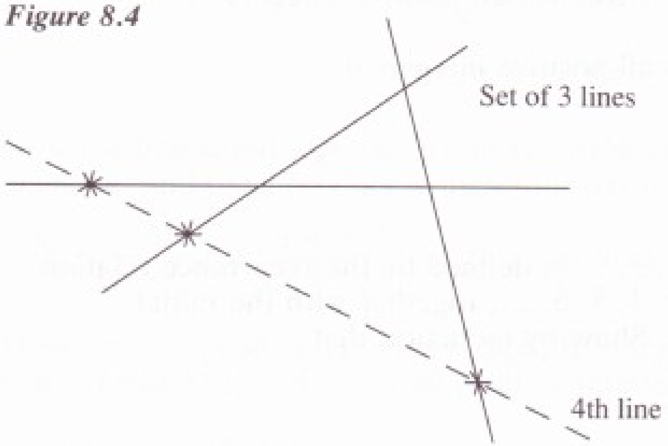
Example 14

Show by induction that n lines, no two of which are coincident or parallel, and no three of which are concurrent, divide the plane into $\frac{1}{2}(n^2 + n + 2)$ regions, for all positive integers n.

Solution

The second step of the induction process requires that we investigate what happens to the regions of the plane associated with k such lines when one extra line is drawn. If we let u_n be the number of regions formed by n such lines, we are seeking a recurrence relation between u_{n+1} and u_n.

Figure 8.4



The $(n + 1)$th line intersects each of the other lines. Hence there are n distinct intersection points along the additional line, dividing this line into $n + 1$ segments or rays, each being a new boundary partitioning one existing region. Hence $n + 1$ new regions have been added.

$\therefore u_{n+1} = u_n + (n + 1), \qquad n = 1, 2, 3, \ldots$

Since this recurrence relation involves u_{n+1} in terms of u_n only, the simpler version of the induction process will suffice.

Define the statement S(n): $\quad u_n = \frac{1}{2}(n^2 + n + 2), \; n = 1, 2, 3, \ldots$

Clearly S(1) is true, since one line divides the plane into two regions.

If S(k) is true, then $u_k = \frac{1}{2}(k^2 + k + 2)$. Consider S($k$ + 1).

$$\begin{aligned} u_{k+1} &= u_k + (k + 1) \\ &= \frac{1}{2}(k^2 + k + 2) + (k + 1), \qquad \text{if S}(k) \text{ is true} \\ &= \frac{1}{2}(k^2 + 3k + 4) \end{aligned}$$

$$\therefore u_{k+1} = \frac{1}{2}\{(k + 1)^2 + (k + 1) + 2\}, \qquad \text{if S}(k) \text{ is true}$$

$\therefore$ for all positive integers k, S(k) true implies S(k + 1) true. But S(1) is true, hence S(n) is true for all positive integers n.

$\therefore$ n such lines divide the plane into $\frac{1}{2}(n^2 + n + 2)$ regions.

In the last two examples we have used a recurrence relation in the induction process to infer the truth of $S(k+1)$ from the truth of $S(k)$ (or of $S(k)$, $S(k-1)$, ..., $S(1)$). Conversely, this step in the induction process can determine recurrence relations for unspecified numbers in the induction statements.

Example 15

Show by induction that for each positive integer n there are unique integers p_n and q_n such that $(\sqrt{3}-1)^n = p_n + q_n\sqrt{3}$. Show that $p_n^2 - 3q_n^2 = (-2)^n$ and deduce that $(\sqrt{3}+1)^n = (-1)^n p_n + (-1)^{n+1} q_n\sqrt{3}$.
What can you deduce about the signs of p_n and q_n?

Solution

Define the statement $S(n)$: There are unique integers p_n and q_n such that $(\sqrt{3}-1)^n = p_n + q_n\sqrt{3}$, $n = 1, 2, 3, \ldots$

Clearly $S(1)$ is true with $p_1 = -1$ and $q_1 = 1$. (4)

If $S(k)$ is true, then $(\sqrt{3}-1)^k = p_k + q_k\sqrt{3}$ for unique integers p_k, q_k

Consider $S(k+1)$ $\quad (\sqrt{3}-1)^{k+1} = (\sqrt{3}-1)(\sqrt{3}-1)^k$

If $S(k)$ is true, $\quad (\sqrt{3}-1)^{k+1} = (\sqrt{3}-1)(p_k + q_k\sqrt{3})$

$$= (3q_k - p_k) + (p_k - q_k)\sqrt{3},$$

where $p_{k+1} = 3q_k - p_k$ and $q_{k+1} = p_k - q_k$ are integers uniquely determined by integers p_k and q_k. (5)

$\therefore$ for all positive integers k, $S(k)$ true implies $S(k+1)$ true.

But $S(1)$ is true.

$\therefore$ $S(n)$ is true for all positive integers n.

$\therefore$ $(\sqrt{3}-1)^n = p_n + q_n\sqrt{3}$, for unique integers p_n, q_n, $n = 1, 2, \ldots$

From (4) and (5), p_n and q_n are determined by the recurrence relations

$p_n = 3q_{n-1} - p_{n-1}$, $q_n = p_{n-1} - q_{n-1}$ and $p_1 = -1$, $q_1 = 1$

Hence $\quad p_n^2 - 3q_n^2 = (3q_{n-1} - p_{n-1})^2 - 3(p_{n-1} - q_{n-1})^2$

$$= -2(p_{n-1}^2 - 3q_{n-1}^2)$$

$$\therefore p_n^2 - 3q_n^2 = -2(p_{n-1}^2 - 3q_{n-1}^2) = (-2)^2(p_{n-2}^2 - 3q_{n-2}^2) = \ldots$$

$$\therefore p_n^2 - 3q_n^2 = (-2)^{n-1}(p_1^2 - 3q_1^2) = (-2)^n$$

Now $(\sqrt{3}-1)^n(\sqrt{3}+1)^n = \{(\sqrt{3}-1)(\sqrt{3}+1)\}^n = 2^n$

$\therefore (p_n + q_n\sqrt{3})(p_n - q_n\sqrt{3}) = p_n - 3q_n^2 = (-1)^n (\sqrt{3}-1)^n(\sqrt{3}+1)^n$

But $p_n + q_n\sqrt{3} = (\sqrt{3}-1)^n \quad \therefore (\sqrt{3}+1)^n = (-1)^n(p_n - q_n\sqrt{3})$

$\therefore (\sqrt{3}+1)^n = (-1)^n p_n + (-1)^{n+1} q_n\sqrt{3}$

But from the binomial expansion of $(\sqrt{3}+1)^n$ it is clear that $(-1)^n p_n$ and $(-1)^{n+1} q_n$ are positive. Hence for n even, $p_n > 0$ and $q_n < 0$, while for n odd, $p_n < 0$ and $q_n > 0$.

Exercise 8.2

1 Show that for $n \geqslant 1$, $2.1! + 5.2! + 10.3! + \ldots + (n^2 + 1)n! = n(n + 1)!$

2 Show that for $n \geqslant 1$, $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \ldots + \dfrac{n}{(n + 1)!} = 1 - \dfrac{1}{(n + 1)!}$

3 If $u_1 = 7$ and $u_n = 2u_{n-1} - 1$ for $n \geqslant 2$, show that $u_n = 3.2^n + 1$ for $n \geqslant 1$.

4 If $u_1 = 5$, $u_2 = 11$ and $u_n = 4u_{n-1} - 3u_{n-2}$ for $n \geqslant 3$, show that $u_n = 2 + 3^n$ for $n \geqslant 1$.

5 If $u_1 = 8$, $u_2 = 20$ and $u_n = 4u_{n-1} - 4u_{n-2}$ for $n \geqslant 3$, show that $u_n = (n + 3)2^n$ for $n \geqslant 1$.

6 If $u_1 = 7$, $u_2 = 29$ and $u_n = 7u_{n-1} - 10u_{n-2}$ for $n \geqslant 3$, show that $u_n = 2^n + 5^n$ for $n \geqslant 1$.

7 If $u_1 = 5$ and $u_n = 3u_{n-1} + 2$ for $n \geqslant 2$, show that $u_n = 2.3^n - 1$ for $n \geqslant 1$.

8 If $u_1 = 2$, $u_2 = 22$ and $u_n = 6u_{n-1} - 5u_{n-2}$ for $n \geqslant 3$, show that $u_n = 5^n - 3$ for $n \geqslant 1$.

9 If $u_1 = 0$, $u_2 = 9$ and $u_n = 6u_{n-1} - 9u_{n-2}$ for $n \geqslant 3$, show that $u_n = (n - 1)3^n$ for $n \geqslant 1$.

10 If $u_1 = 2$, $u_2 = 16$ and $u_n = 8u_{n-1} - 15u_{n-2}$ for $n \geqslant 3$, show that $u_n = 5^n - 3^n$ for $n \geqslant 1$.

11 If $u_n = 9^{n+1} - 8n - 9$, show that $u_{n+1} = 9u_n + 64n + 64$, and hence show that u_n is divisible by 64 for $n \geqslant 1$.

12 If $u_n = 5^{2n} + 3n - 1$, show that u_n is divisible by 9 for $n \geqslant 1$.

13 If $u_n = 2^{n+2} + 3^{2n+1}$, show that $u_{n+1} = 2u_n + 7.3^{2n+1}$, and hence show that u_n is divisible by 7 for $n \geqslant 1$.

14 If $u_n = 3^{4n+2} + 2.4^{3n+1}$, show that u_n is divisible by 17 for $n \geqslant 1$.

15 Show that $7^n + 11^n$ is divisible by 9 for odd $n \geqslant 1$.

16 Show that $3^n + 7^n$ is divisible by 10 for odd $n \geqslant 1$.

17 If $u_n = 3^n - 2n - 1$, show that $u_{n+1} = 3u_n + 4n$, and hence show that $u_n > 0$ for $n \geqslant 2$.

18 If $u_n = 5^n - 4n - 1$, show that $u_n > 0$ for $n \geqslant 2$.

19 If $u_1 = 1$ and $u_n = \sqrt{(2u_{n-1})}$ for $n \geqslant 2$
(a) show that $u_n < 2$ for $n \geqslant 1$
(b) deduce that $u_{n+1} > u_n$ for $n \geqslant 1$

20 If $u_1 = 1$ and $u_n = \sqrt{(3 + 2u_{n-1})}$ for $n \geqslant 2$
(a) show that $u_n < 3$ for $n \geqslant 1$
(b) deduce that $u_{n+1} > u_n$ for $n \geqslant 1$

8.3 Geometry of the circle

Exercise 8.3

1 (The three medians of a triangle are concurrent.)
ABC is a triangle. E and F are the midpoints of CA and AB respectively. BE and CF intersect at G. AG produced cuts BC at D. H is the point on AGD produced, such that AG = GH. Show that
(a) GBHC is a parallelogram
(b) BD = DC

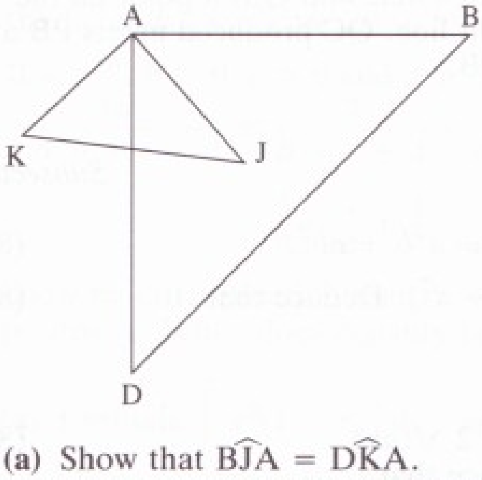
2 (The internal bisectors of the three angles of a triangle are concurrent.)
ABC is a triangle. The internal bisectors of $\widehat{B}$ and $\widehat{C}$ meet at D. DP, DQ and DR are the perpendiculars from D to BC, CA and AB respectively. Show that
(a) DQ = DR
(b) AD is the internal bisector of $\widehat{A}$

3 AB is a chord of a circle. X is a point on AB produced. XT is a tangent from X to the circle.
(a) Show that Δ XAT ||| Δ XTB.
(b) Deduce that $XT^2 = XA.XB$.

4 AB and CD are chords of a circle. AB produced and CD produced meet at X.
(a) Show that Δ XAC ||| Δ XDB.
(b) Deduce that XA.XB = XC.XD.

5 In figure 8.5, ABD and AJK are two isosceles triangles, right angled at A.

Figure 8.5



(a) Show that $B\widehat{J}A = D\widehat{K}A$.
(b) BJ is produced to meet DK at X. Show that BX $\perp$ DK.
(c) The square ABCD is completed. Show that $B\widehat{X}C = 45°$.

6 Two circles with centres O and P and radii r and s (where $r < s$) respectively touch externally at T. ABC and ADE are common tangents to the two circles.

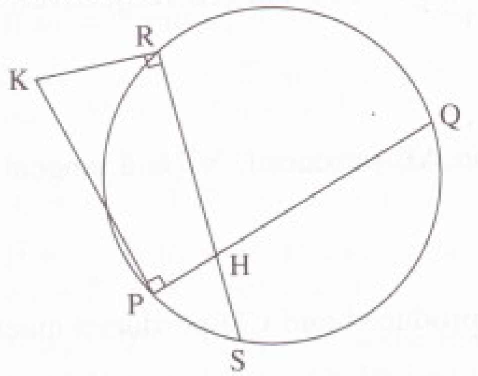
(a) Show that A, O, T and P are collinear.

(b) Show that $AO = \dfrac{r(r+s)}{s-r}$

7 In ΔABC, $AB = AC$. The bisector of $A\hat{B}C$ meets AC at K. The circle through A, B and K cuts BC at D. Show that $AK = CD$.

8 In ΔABC, P and Q are points on the sides CA and AB respectively, such that $B\hat{P}C = C\hat{Q}B$. BP and CQ intersect at K. X and Y are points on CA and AB respectively, such that AYKX is a parallelogram. Show that $AX.XC = AY.YB$.

Figure 8.6



9 In figure 8.6, PQ and RS are two chords of a circle. PQ and RS intersect at H. K is a point such $K\hat{P}Q$ and $K\hat{R}S$ are right angles. Show that KH produced is perpendicular to QS.

10 Two circles intersect at A and B. The centre C of the first circle lies on the second circle. P is a point on the first circle and Q is a point on the second circle such that PAQ is a straight line. QC produced meets PB at X. Show that QX is perpendicular to PB.

Diagnostic test 8

Subsection

1 If $a > 0$ and $b > 0$, show that $a^3 + b^3 \geqslant a^2b + ab^2$. (8.1)

2 Show that $(ac + bd)^2 \leqslant (a^2 + b^2)(c^2 + d^2)$. Deduce that (8.1)

(a) $(a + b)^2 \leqslant 2(a^2 + b^2)$

(b) $(a^3 + b^3)^2 \leqslant (a^2 + b^2)(a^4 + b^4)$

3 If $a > 0$ and $b > 0$, show that $a + b \geqslant 2\sqrt{(ab)}$ (8.1)

If $a > 0$, $b > 0$, $c > 0$ and $d > 0$, deduce that

(a) $a + b + c + d \geqslant 4\sqrt[4]{(abcd)}$

(b) $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d} + \dfrac{d}{a} \geqslant 4$

4 If $0 < t < 1$, show that $\frac{1}{2} < \frac{1}{1 + t^2} < 1$. By integrating between 0 and u, deduce that for $0 < u < 1$, $\frac{u}{2} < \tan^{-1} u < u$. (8.1)

5 Show that $1.1! + 2.2! + 3.3! + \ldots + n.n! = (n + 1)! - 1$, for $n \geqslant 1$. (8.2)

6 If $u_1 = 1$, $u_2 = 5$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for $n \geqslant 3$, show that $u_n = 3^n - 2^n$ for $n \geqslant 1$. (8.2)

7 If $u_n = 5^n + 12n - 1$, show that $u_{n+1} = 5u_n - 48n + 16$ and hence show that u_n is divisible by 16 for $n \geqslant 1$. (8.2)

8 If $u_1 = 1$ and $u_n = \sqrt{(2 + u_{n-1})}$ for $n \geqslant 2$ (8.2)
(a) show that $u_n < 2$ for $n \geqslant 1$
(b) deduce that $u_{n+1} > u_n$ for $n \geqslant 1$

9 ABC is an acute-angled triangle. The altitudes BE and CF intersect at G. AG produced cuts BC at D. (8.3)
(a) Explain why AFGE and CEFB are cyclic quadrilaterals.
(b) Show that $F\hat{G}A = F\hat{B}D$. Deduce that AD is perpendicular to BC.

10 ABC is an acute-angled triangle. The altitudes AD and BE intersect at G. AD produced cuts the circle through A, B and C at H. Show that GD = DH. (8.3)

Further questions 8

1 Show that $(\ell x + my + nz)^2 \leqslant (\ell^2 + m^2 + n^2)(x^2 + y^2 + z^2)$. Deduce that
(a) $(a + b + c)^2 \leqslant 3(a^2 + b^2 + c^2)$
(b) $(a^3 + b^3 + c^3)^2 \leqslant (a^2 + b^2 + c^2)(a^4 + b^4 + c^4)$

2 If $a > 0$, $b > 0$, $c > 0$ and $d > 0$, show that

$$\frac{16}{a+b+c+d} \leqslant \frac{3}{b+c+d} + \frac{3}{a+c+d} + \frac{3}{a+b+d} + \frac{3}{a+b+c} \leqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

3 Show by differentiation that $xy \leqslant e^{x-1} + y \ln y$ for all real x and all positive y. When does equality hold?

4 **(a)** Evaluate $\int_0^1 x^2(1-x)^2\,dx$ and $\int_0^1 \frac{x^2(1-x)^2}{x+2}\,dx$

(b) Show that $\frac{1}{3}\int_0^1 x^2(1-x)^2\,dx < \int_0^1 \frac{x^2(1-x)^2}{x+2}\,dx < \frac{1}{2}\int_0^1 x^2(1-x)^2\,dx$,

and hence deduce that $\frac{2627}{6480} < \ln\frac{3}{2} < \frac{2628}{6480}$

5 **(a)** Evaluate $\displaystyle\int_0^1 x^4(1-x)^4\,dx$ and $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$

(b) Show that $\displaystyle\frac{1}{2}\int_0^1 x^4(1-x)^4\,dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx < \int_0^1 x^4(1-x)^4\,dx,$

and hence deduce that $\dfrac{22}{7} - \dfrac{1}{630} < \pi < \dfrac{22}{7} - \dfrac{1}{1260}$

6 If $0 < x < \dfrac{\pi}{2}$, show that $\dfrac{2x}{\pi} < \sin x < x$. Deduce that

$$1 - e^{-\pi/2} < \int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < (e-1)\,\frac{\pi}{2e}$$

7 Show that for $n \geqslant 1$,

$$1 \ln \frac{2}{1} + 2 \ln \frac{3}{2} + \ldots + n \ln \left(\frac{n+1}{n}\right) = \ln \left(\frac{(n+1)^n}{n!}\right)$$

8 Show that for $n \geqslant 1$,

$$1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \ldots + \frac{x(x+1)\ldots(x+n-1)}{n!}$$
$$= \frac{(x+1)(x+2)\ldots(x+n)}{n!}$$

9 Show that $(35)^n + 3.7^n + 2.5^n + 6$ is divisible by 12 for $n \geqslant 1$.

10 **(a)** Show that $(1+x)^n - 1$ is divisible by x for $n \geqslant 1$.
(b) Show that $(1+x)^n - nx - 1$ is divisible by x^2 for $n \geqslant 2$.

11 **(a)** Using the product rule for differentiation, show that for $n \geqslant 1$,

$$\frac{d}{dx}x^n = nx^{n-1}$$

(b) Using integration by parts, show that for $n \geqslant 1$, $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

12 **(a)** Show that for $n \geqslant 1$, $\dfrac{d^n}{dx^n}\ln(1+x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

(b) Show that for $n \geqslant 1$, $\dfrac{d^n}{dx^n}\ln(1-x) = \dfrac{(-1)\quad(n-1)!}{(1-x)^n}$

13 Show that a convex polygon with $n \geqslant 4$ sides has $\dfrac{n(n-3)}{2}$ diagonals.

14 Show that $n \geqslant 2$ lines, no two of which are parallel and no three of which are concurrent, have $\dfrac{n(n-1)}{2}$ points of intersection.

15 If $u_1 = 1$ and $u_n = \dfrac{2u_{n-1}^{\ 3} + 27}{3u_{n-1}^{\ 2}}$ for $n \geqslant 2$, show that

(a) $u_n > 3$ for $n \geqslant 2$

(b) $u_{n+1} < u_n$ for $n \geqslant 2$

16 If $u_1 = 1$ and $u_n = \dfrac{1}{2}\left(u_{n-1} + \dfrac{3}{u_{n-1}}\right)$ for $n \geqslant 2$, show that

(a) $u_n^{\ 2} > 3$ for $n \geqslant 2$

(b) $u_{n+1} < u_n$ for $n \geqslant 2$

17 Show that for $n \geqslant 1$, $n! \geqslant 2^{n-1}$. Deduce that

$$\frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \ldots + \frac{1}{(n!)^2} \leqslant \frac{4}{3}\left(1 - \frac{1}{4^n}\right)$$

18 If $u_1 = 1$, $u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n \geqslant 3$, show that

$$u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \text{ for } n \geqslant 1$$

19 ABCD is a quadrilateral such that $A\hat{B}D = D\hat{B}C = C\hat{D}A = 45°$. Q is the point on BD such that CQ bisects $B\hat{C}A$. Show that AQCD is a cyclic quadrilateral.

20 ABC is a triangle. The bisector of $C\hat{A}B$ cuts BC at D. K is the point on CB produced such that BK = AC. AB produced cuts the circle through A, K and D at P. Show that BP = DC.

Appendix 1
Specimen Papers 1–6

Specimen Paper 1

1 (a) **(i)** Sketch the graph of $f(x) = x \ln x$

(ii) Use your graph to sketch the graph of $h(x) = \dfrac{1}{x \ln x}$

(b) **(i)** Sketch the graph of the function $g(x) = x \ln x$ for $x \geqslant \dfrac{1}{e}$, showing clearly the coordinates of its endpoint and its points of intersection with the x-axis and the line $y = x$

(ii) On the same axes, sketch the graph of the inverse function $g^{-1}(x)$, showing clearly the coordinates of its endpoint and its points of intersection with the y-axis and the line $y = x$

(iii) Evaluate $\displaystyle\int_1^e x \ln x \, dx$. Hence find the area bounded by the x-axis between $x = 0$ and $x = 1$, the y-axis between $y = 0$ and $y = 1$, and the graphs of $y = g(x)$ and $y = g^{-1}(x)$

2 (a) Use the substitution $u = e^x$ to find $\displaystyle\int \frac{e^{2x}}{1 + e^x} \, dx$

(b) Find $\displaystyle\int \frac{\sin^3 x}{\cos^4 x} \, dx$

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4 + 5 \sin x} \, dx$

(d) Evaluate $\displaystyle\int_0^1 \frac{\sin^{-1} x}{\sqrt{(1 + x)}} \, dx$

3 (a) Show that the gradient of the tangent to the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{7} = 1$ at the extremity in the first quadrant of its latus rectum is equal to the eccentricity of the hyperbola.

(b) P$(a \cos \theta, b \sin \theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a > b > 0$. The tangent and the normal at P cut the y-axis at A and B respectively, and S is a focus of the ellipse.

(i) Show that $\mathrm{A\hat{S}B} = 90°$

(ii) Hence show that A, P, S and B are concyclic and state the location of the centre of the circle through A, P, S and B.

4 (a) **(i)** Find real numbers a and b such that $(a + ib)^2 = -3 + 4i$
(ii) Hence solve the equation $z^2 - 3z + (3 - i) = 0$

(b) **(i)** Express each of $1 + \sqrt{3}i$ and $1 - \sqrt{3}i$ in modulus argument form.
(ii) Hence simplify $(1 + \sqrt{3}i)^{10} + (1 - \sqrt{3}i)^{10}$

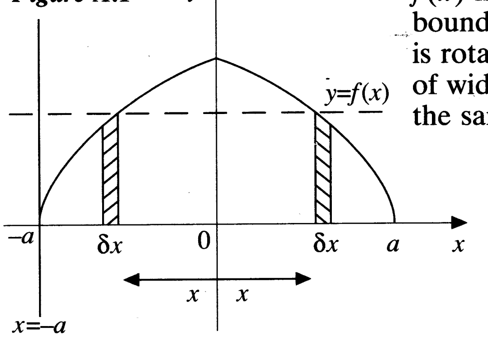
(c) **(i)** By considering the points of intersection of the curve $y = x^3 - x + 2$ and the line $y = mx$, show that there is only one tangent to the curve which passes through the origin.
(ii) Find the equation of this tangent and its point of contact with the curve.

5 (a) On separate Argand diagrams, shade in the regions containing all points representing complex numbers z such that

(i) $|z| \leqslant 1$ and $0 \leqslant \arg z \leqslant \frac{\pi}{4}$

(ii) $|z| \leqslant 1$ or $0 \leqslant \arg z \leqslant \frac{\pi}{4}$

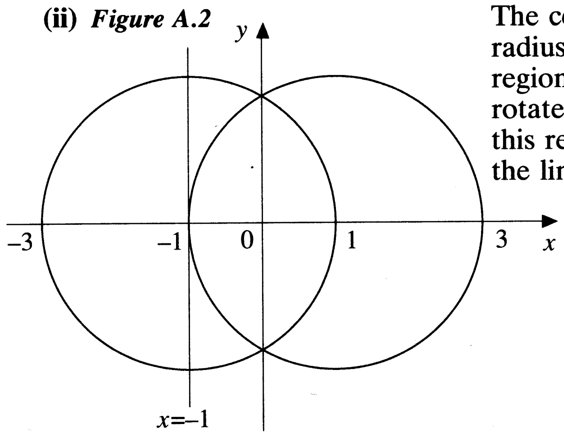
(b) **(i)** ***Figure A.1***



$f(x)$ is an even function. The area bounded by $y = f(x)$ and the x-axis is rotated about $x = -a$. The strips of width δx form cylindrical shells of the same height.

Show that the volume of the solid is given by $V = 4\pi a \int_0^a f(x)\, dx$

(ii) ***Figure A.2***



The centres of two circles, each of radius 2 cm, are 2 cm apart. The region common to the two circles is rotated about one of the tangents to this region which is perpendicular to the line joining the centres.

Show that the volume of the solid formed is given by

$V = 8\pi \int_0^1 \sqrt{[4 - (x + 1)^2]}\, dx$, and hence find this volume.

6 The ends of a light string are fixed to two points A and B in the same vertical plane with A above B, and the string passes through a small smooth ring of mass m. The ring is fastened to the string at a point P and when the string is taut, $A\hat{P}B = 90°$, $B\hat{A}P = \theta$ and the distance of P from AB is r.
The ring revolves in a horizontal circle with constant angular velocity ω and with the string taut.
 (i) Find the tensions T_1 and T_2 in the parts AP and PB of the string. Hence, given that $AB = 5\ell$ and $AP = 4\ell$, show that $16\ell\omega^2 > 5g$.
 (ii) If the ring is free to move on the string, instead of being fastened, show that if it remains in the same position on the string as before, revolving in a horizontal circle with constant angular velocity Ω, then Ω satisfies the equation $12\ell\,\Omega^2 = 35g$.

7 (a) (i) Show that the remainder when the polynomial $P(x)$ is divided by $(x - a)^2$ is $P'(a)x + P(a) - aP'(a)$.
 (ii) Find the value of k for which $x - 1$ is a factor of the polynomial $P(x) = x^{11} - 3x^6 + kx^4 + x^2$.
 For this value of k, find the remainder on dividing $P(x)$ by $(x - 1)^2$.
 (b) Owing to the tides, the height of water in an estuary may be assumed to rise and fall with time in simple harmonic motion. At a certain place there is a danger of flooding when the height of water is above 1.25 m. One day the high tide had a height of 1.5 m at 1.00 am and the following low tide had a height of 0.5 m at 7.30 am. Assuming that the following high tide also had a height of 1.5 m, find the times that day when there was a danger of flooding.

8 (a) If $u_1 = 5$, $u_2 = 13$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for $n \geqslant 3$, show that $u_n = 2^n + 3^n$ for $n \geqslant 1$.
 (b) (i) If $a > 0$, $b > 0$, show that $a + b \geqslant 2\sqrt{(ab)}$
 (ii) Hence show that
 If $a > 0$, $b > 0$ and $c > 0$, then $(a + b)(b + c)(c + a) \geqslant 8abc$
 If $a > 0$, $b > 0$, $c > 0$ and $d > 0$, then $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geqslant 4$

Specimen Paper 2

1 (a) (i) Find the domain and the range of the function $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$. Sketch the graph of the function.
 (ii) $P(x_1, y_1)$ lies on the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$. The tangent at P meets the x-axis and the y-axis at Q and R respectively. Show that OQ + OR is independent of the position of P.

(b) **(i)** Find the gradient of the tangent to the curve $y = e^x$ which passes through the origin.
(ii) Hence find the values of the real number k for which the equation $e^x = kx$ has exactly two real solutions.

2 **(a)** Express $3 + 2x - x^2$ in the form $b^2 - (x - a)^2$. Hence evaluate

$$\int_1^3 \sqrt{(3 + 2x - x^2)}\, dx$$

(b) Find $\int x \sec^2 x\, dx$

(c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x + \sin x}\, dx$. Hence use the substitution $u = \frac{\pi}{2} - x$

to evaluate $\int_0^{\frac{\pi}{2}} \frac{x}{1 + \cos x + \sin x}\, dx$

3 **(a)** Show that the equation $\frac{x^2}{29 - k} + \frac{y^2}{4 - k} = 1$, where k is a real number, represents
(i) an ellipse if $k < 4$
(ii) a hyperbola if $4 < k < 29$
Show that the foci of each ellipse in **(i)** and each hyperbola in **(ii)** are independent of the value of k.

(b) $P\left(ct, \frac{c}{t}\right)$, where $t \neq 1$, $t \neq -1$, lies on the rectangular hyperbola $xy = c^2$.

The tangent at P meets the x-axis and the y-axis at Q and R respectively. The normal at P meets the lines $y = x$ and $y = -x$ at S and T respectively. Show that QSRT is a rhombus.

4 **(a)** **(i)** Express the complex numbers $z_1 = 2i$ and $z_2 = -1 + \sqrt{3}i$ in modulus argument form. On an Argand diagram, plot the points P and Q which represent z_1 and z_2 respectively.
(ii) On the same diagram, construct the vectors which represent the complex numbers $z_1 + z_2$ and $z_1 - z_2$ respectively. Deduce the exact values of $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$.

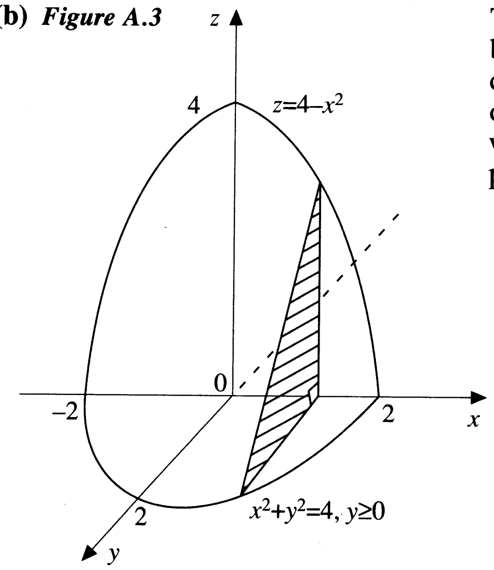
(b) If $\arg(z - 2) - \arg(z + 2) = \frac{\pi}{4}$, show that the locus of the point P representing the complex number z is an arc of a circle, and find the centre and the radius of this circle.

(c) The polynomial $P(x) = x^3 + ax^2 - x - 2$, where a is a constant, has three real zeros, one of which is twice another. Find the value of a and factorise $P(x)$ over the real numbers.

5 **(a)** **(i)** If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$

(ii) Hence show that $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

(b) ***Figure A.3***



The solid shown has a semicircular base of radius 2 units. Vertical cross-sections perpendicular to the diameter are right-angled triangles whose height is bounded by the parabola $z = 4 - x^2$.

By slicing at right angles to the x-axis, show that the volume of the solid is given by $V = \displaystyle\int_0^2 (4 - x^2)^{\frac{3}{2}}\,dx$, and hence calculate this volume.

6 Two particles are connected by a light inextensible string which passes through a small hole with smooth edges in a smooth horizontal table. One particle of mass m travels on the table with constant angular velocity ω. Another particle of mass M travels in a circle with constant angular velocity Ω on a smooth horizontal floor, distance x below the table. The lengths of string on the table and below the table are ℓ and L respectively and the length L makes an angle θ with the vertical.

(a) **(i)** If the floor exerts a force N on the lower particle, show that $N = M(g - x\Omega^2)$. Find the maximum possible value of Ω for the motion to continue as described. What happens if Ω exceeds this value?

(ii) By considering the tension force in the string, show that $\dfrac{L}{\ell} = \dfrac{m}{M}\left(\dfrac{\omega}{\Omega}\right)^2$. If the lower particle exerts zero force on the floor, show that the tension force in the string is given by $T = \dfrac{MgL}{x}$.

(b) The table is 80 cm high and the string is 1.5 m long, while the masses on the table and on the floor are 0.4 kg and 0.2 kg respectively. The particles are observed to have the same angular velocity. If the lower particle exerts zero force on the floor, find

(i) the tension in the string

(ii) the speed of the particle on the table if the string were to break

7 (a) Show that the remainder when the polynomial $P(x)$ is divided by $x^2 - a^2$ is

$$\frac{1}{2a}\{P(a) - P(-a)\}x + \frac{1}{2}\{P(a) + P(-a)\}$$

Find the remainder when $P(x) = x^n - a^n$ is divided by $x^2 - a^2$ in each of the cases **(i)** n even **(ii)** n odd

(b) A particle P is projected from a point O and inclined at an angle of 45° above the horizontal. The particle describes a parabola under gravity. Coordinate axes are taken horizontally and vertically through O. The particle just clears the tops of two vertical poles a distance 40 m apart and each 15 m above the point of projection. Find the horizontal range of the projectile.

8 (a) Show that $7^n + 15^n$ is divisible by 11 for all odd $n \geqslant 1$

(b) **(i)** If $a > 0$ and $b > 0$, show that $a + b \geqslant 2\sqrt{(ab)}$

(ii) If also $a + b = 1$, show that $\frac{1}{a} + \frac{1}{b} \geqslant 4$ and $\frac{1}{a^2} + \frac{1}{b^2} \geqslant 8$

Specimen Paper 3

1 **(i)** Sketch the graph of $y = \dfrac{3x}{x^2 - 1}$

(ii) Solve the equation $\dfrac{3x}{x^2 - 1} = 2$

Use your graph to solve the inequality $\dfrac{3x}{x^2 - 1} > 2$

(iii) Find the gradient of the curve $y = \dfrac{3x}{x^2 - 1}$ at the origin. Use your graph to find the values of the negative real number k for which the equation $\dfrac{3x}{x^2 - 1} = kx$ has exactly one real solution.

(iv) Find the area of the region bounded by the curve $y = \dfrac{3x}{x^2 - 1}$, the x-axis and the line $x = \dfrac{1}{2}$

2 (a) Find $\displaystyle\int \frac{x + 3}{(x + 1)(x^2 + 1)}\,dx$

(b) Use the substitution $x = 4\sin^2\theta$ to evaluate $\displaystyle\int_0^2 \sqrt{[x(4 - x)]}\,dx$

(c) **(i)** If $\mathrm{I}_n = \displaystyle\int_0^1 x^n e^x\,dx$ for $n \geqslant 0$ show that $\mathrm{I}_n = e - n\mathrm{I}_{n-1}$ for $n \geqslant 1$

(ii) Find the value of I_4 and hence evaluate $\displaystyle\int_0^{\frac{1}{2}} x^4 e^{2x}\,dx$

3 (a) P$\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the normal at P cuts the hyperbola again at the point Q with coordinates $\left(-\frac{c}{t^3}, -ct^3\right)$. Hence find the coordinates of the point R where the normal at Q cuts the hyperbola again.

(ii) The normal at P meets the x-axis at A and the tangent at P meets the y-axis at B. M is the midpoint of AB. Find the equation of the locus of M as P moves on the hyperbola.

4 (a) The complex number z and its conjugate $\bar{z}$ satisfy the equation $z\bar{z} + 2iz = 12 + 6i$. Find the possible values of z.

(b) Find in modulus argument from the five roots of $z^5 = -1$.

(i) Show that when these five roots are plotted on an Argand diagram they form the vertices of a regular pentagon of area $\frac{5}{2} \sin \frac{2\pi}{5}$.

(ii) By combining appropriate pairs of roots show that

$$z^4 - z^3 + z^2 - z + 1 = \left(z^2 - 2z \cos \frac{\pi}{5} + 1\right)\left(z^2 - 2z \cos \frac{3\pi}{5} + 1\right).$$

Deduce that $\cos \frac{\pi}{5}$ and $\cos \frac{3\pi}{5}$ are the roots of the equation $4x^2 + 2x - 1 = 0$, and hence find their values.

5 (a) Find integers m and n such that $(x + 1)^2$ is a factor of the polynomial $P(x) = x^5 + 2x^2 + mx + n$

(b) The equation $x^3 - px - q = 0$ has roots α, β and γ

(i) Show that $\alpha^2 + \beta^2 + \gamma^2 = 2p$

(ii) Express $\alpha^3 + \beta^3 + \gamma^3$ and $\alpha^5 + \beta^5 + \gamma^5$ in terms of p and q

(c) The base of a particular solid is the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Find the volume of the solid if every cross-section perpendicular to the major axis of the ellipse is an equilateral triangle with one side in the base of the solid.

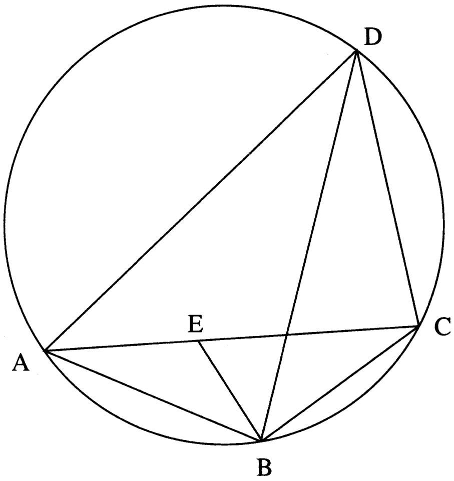
6 (a) A railway line has been constructed around a circular curve of radius 500 m. The distance between the rails is 1.5 m and the outside rail is 0.1 m above the inside rail. Find the speed that eliminates a sideways force on the wheels for a train on this curve. (Take $g = 9.8\ \text{ms}^{-2}$.)

(b) A particle of mass m is set in motion with speed u. Subsequently the only force acting upon the particle directly opposes its motion and is of magnitude $mk(1 + v^2)$ where k is a constant and v is its speed at time t.

(i) Show that the particle is brought to rest after a time $\frac{1}{k} \tan^{-1} u$

(ii) Find an expression for the distance travelled by the particle in this time.

7 (a) *Figure A.4*



In figure A.4, ABCD is a cyclic quadrilateral. E is the point on AC such that $A\hat{B}E = D\hat{B}C$.

(i) Show that $\triangle ABE \;|||\; \triangle DBC$ and $\triangle ABD \;|||\; \triangle EBC$

(ii) Hence show that AB . DC + AD . BC = AC . DB

(b) ABC is an equilateral triangle inscribed in a circle. P is a point on the minor arc AB of the circle. Show that PC = PA + PB.

8 (a) A tripod is made of three rods OA, OB and OC, each 12 cm long and hinged at O. The ends A, B and C rest on a horizontal table such that each of the angles AOB, BOC and COA is 60°. Calculate

(i) the height of O above the table

(ii) the tangent of the angle that OA makes with the table.

(b) A fair die is thrown six times. Find the probabilities that the six scores obtained will

(i) be 1, 2, 3, 4, 5, 6 in some order

(ii) have a product which is an even number

(iii) consist of exactly two 6's and four odd numbers

(iv) be such that a 6 occurs only on the last throw and exactly three of the first five throws result in odd numbers.

Specimen Paper 4

1 (a) (i) If $P(x) = x^3 - 6x^2 + 9x + c$ for some real number c, find the values of x for which $P'(x) = 0$. Hence find the values of c for which the equation $P(x) = 0$ has a repeated root.

(ii) Sketch the graphs of $y = P(x)$ for these values of c. Hence find the set of values of c for which the equation $P(x) = 0$ has only one real root.

(b) (i) Find the domain and the range of the function $f(x) = \cos^{-1}(e^x)$.

(ii) Sketch the graph of $y = \cos^{-1}(e^x)$.

2 (a) Express $x^2 + 2x + 5$ in the form $(x + a)^2 + b^2$. Hence evaluate

$$\int_{-1}^{1} \frac{1}{x^2 + 2x + 5}\, dx$$

(b) Find $\displaystyle\int \frac{\ln x}{\sqrt{x}}\, dx$

(c) (i) Show that $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{x(\pi - 2x)}\, dx = \frac{2}{\pi} \ln 2$

(ii) Hence use the substitution $u = \dfrac{\pi}{2} - x$ to evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi - 2x)}\, dx$

3 (a) Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ meet at right angles. Find the equation of the circle through the points of intersection of these two curves.

(b) $P(a \sec \theta, b \tan \theta)$ lies on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > b > 0$. The tangent at P passes through a focus of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Show that it is parallel to one of the lines $y = x$ and $y = -x$ and that its point of contact with the hyperbola lies on a directrix of the ellipse.

4 (a) Express $z_1 = \dfrac{7 + 4i}{3 - 2i}$ in the form $a + ib$, where a and b are real. On an Argand diagram, sketch the locus of the point representing the complex number z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of $|z|$ subject to this condition.

(b) The complex number $z = x + iy$, where x and y are real, is such that $|z - i| = \text{Im } z$. Show that the locus of the point representing z has equation $y = \dfrac{1}{2}(x^2 + 1)$. Find the gradients of the tangents to this curve which pass through the origin, and hence find the set of possible values of $\arg z$.

(c) Given that $a + b + c = -3$, $a^2 + b^2 + c^2 = 29$ and $abc = -6$, form the monic cubic equation whose roots are a, b and c. Hence find the values of a, b and c.

5 (a) Expand the complex number $z = (1 + ic)^6$ in powers of c and hence find the five real values of c for which z is real.

(b) Two of the zeros of the polynomial $P(x) = x^4 + bx^3 + cx^2 + dx + e$, where b, c, d and e are real, are $2 + i$ and $1 - 3i$. Find the other two zeros and hence find the values of b and e.

(c) On the same axes, sketch the graphs of the functions $y = \frac{1}{2}(e^x + e^{-x})$ and $y = \frac{1}{2}(e^x - e^{-x})$. The region between the two curves bounded by the y-axis and the line $x = 1$ is rotated about the y-axis. Use cylindrical shells to show that the volume of the solid generated is given by

$V = 2\pi\int_0^1 xe^{-x}\,dx$, and hence calculate this volume.

6 Air resistance to the motion of a particle of mass m has magnitude mkv^2, where v is the speed of the particle and k is a constant.

(i) The particle is projected vertically upward under gravity with initial speed v_0. Write down the equation of motion of the particle during its upward motion and hence show that the greatest height reached is

$$\frac{1}{2k}\ln\left(\frac{g + kv_0^2}{g}\right)$$

(ii) The particle falls from its greatest height. Write down the equation of motion of the particle during its downward motion and hence find its terminal velocity. If the particle returns to its point of projection with speed v_1, show that $(g + kv_0^2)(g - kv_1^2) = g^2$

7 (a) **(i)** Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ for all real θ.

(ii) Hence find in surd form the values of $\cot\frac{\pi}{8}$ and $\cot\frac{\pi}{12}$, and show that $\operatorname{cosec}\frac{2\pi}{15} + \operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} = 0$.

(b) The vertices of a quadrilateral ABCD lie on a circle of radius r. The angles subtended at the centre of the circle by the sides AB, BC, CD and DA respectively are in arithmetic sequence with first term α and common difference β.

(i) Show that $2\alpha + 3\beta = \pi$ and interpret this result geometrically.

(ii) Show that the area of the quadrilateral is $2r^2\cos\beta\cos\left(\frac{\beta}{2}\right)$.

8 (a) If $u_1 = 1$ and $u_n = \sqrt{(3u_{n-1})}$ for $n \geqslant 2$, show that

(i) $u_n < 3$ for $n \geqslant 1$

(ii) $u_{n+1} > u_n$ for $n \geqslant 1$

(b) Show that $a^2 + b^2 + c^2 \geqslant ab + bc + ca$.
Hence show that $a^4 + b^4 + c^4 \geqslant a^2b^2 + b^2c^2 + c^2a^2 \geqslant abc\,(a + b + c)$.

Specimen Paper 5

1 **(a)** Sketch the graph of $y = 3x^4 - 4x^3 - 12x^2$, showing clearly the coordinates of the turning points. Use your graph to find the sets of values of k for which the equation $3x^4 - 4x^3 - 12x^2 - k = 0$ has
(i) no real roots
(ii) just two distinct real roots
(iii) four distinct real roots

(b) **(i)** Find the value of k for which 1 is a root of the equation $3x^4 - 4x^3 - 12x^2 - k = 0$. For this value of k, find the two consecutive integers between which the other real root α lies.
(ii) By considering the product of the four roots, express the modulus of the non-real complex roots in terms of α and so determine an interval in which the modulus must lie.
By considering the sum of the four roots, express the real part of the non-real complex roots in terms of α and so determine an interval in which the real part must lie.

2 **(a)** Find $\displaystyle\int \frac{x^2}{(x-1)(x-2)}\,dx$

(b) Find **(i)** $\displaystyle\int \cos^3 x\,dx$ **(ii)** $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{(1-x^2)}}\,dx$

(c) Show that $\displaystyle\int_0^1 \frac{\sqrt{x}}{1+x}\,dx = 2 - \frac{\pi}{2}$, and hence evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\ln(1+x)dx$

3 **(a)** Show that the chord of contact of the tangents from the point $P_0(x_0, y_0)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has equation $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$.

(b) Write down the equation of the chord of contact of the tangents from the point $(4, -1)$ to the ellipse $x^2 + 2y^2 = 6$. Hence find the coordinates of the points of contact and the equations of these tangents.

4 **(a)** If $z_1 = 4\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$ and $z_2 = 2\left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right)$, find the modulus and argument of each of the following
(i) z_1^3 **(ii)** $\dfrac{1}{z_2}$ **(iii)** $\dfrac{z_1^3}{z_2}$

(b) If z is any complex number such that $|z| = 1$, show using an Argand diagram, or otherwise, that
(i) $1 \leqslant |z+2| \leqslant 3$ **(ii)** $-\dfrac{\pi}{6} \leqslant \arg(z+2) \leqslant \dfrac{\pi}{6}$

(c) Find the values of the real numbers p and q if $x^2 + 1$ is a factor of the polynomial $P(x) = x^4 + px^3 + 2x + q$. Hence factorise $P(x)$ over $\mathbb{R}$ and over $\mathbb{C}$.

5 (a) Express $(6 + 5i)(7 + 2i)$ in the form $a + ib$, where a and b are real, and write down $(6 - 5i)(7 - 2i)$ in a similar form. Hence find the prime factors of $32^2 + 47^2$.

(b) The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ. Find the monic equations with roots

(i) $-\alpha$, $-\beta$ and $-\gamma$ **(ii)** $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$ **(iii)** α^2, β^2 and γ^2

(c) The region $\{(x, y) : 0 \leqslant x \leqslant 2 \text{ and } 0 \leqslant y \leqslant 2x - x^2\}$ is rotated about the y-axis. Use the method of slicing to find the volume of the solid formed.

6 A particle moves under gravity in a medium in which the resistance to its motion per unit mass is k times its speed, where k is a constant.

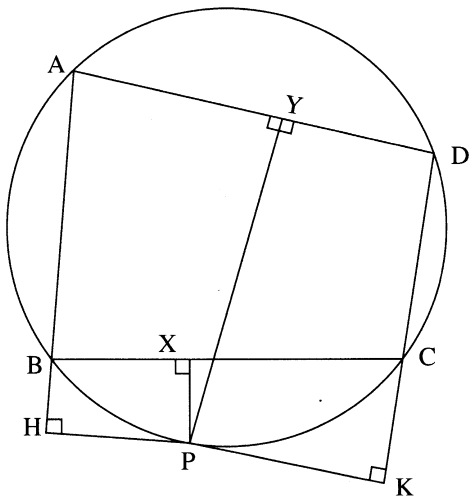
(i) If the particle falls vertically from rest, show that its terminal velocity is given by $V = \frac{g}{k}$.

(ii) If the particle is projected vertically upward with speed V, show that after time t its speed v and height x are given by

$v = V(2e^{-kt} - 1)$ and $x = \frac{1}{k}V(2 - 2e^{-kt} - kt)$.

Hence show that the greatest height H that the particle can reach is given by $H = \frac{1}{k}V(1 - \ln 2)$.

7 ***Figure A.5***



ABCD is a cyclic quadrilateral. P is a point on the circle through A, B, C and D.
PH, PX, PK and PY are the perpendiculars from P to AB produced, BC, DC produced and DA, respectively.

(i) Show that $\triangle XPK \;|||\; \triangle HPY$

(ii) Hence show that $PX.PY = PH.PK$ and $\dfrac{PX.PK}{PH.PY} = \dfrac{(XK)^2}{(HY)^2}$

8 (a) A vertical pole of height 2 m, with base at the point O, stands on the west side of a canal with straight parallel sides running from north to south. Two points A and B both lie on the east side of the canal, A to the north and B to the south of the pole, such that the angle AOB is 150°. The angles of elevation of the top P of the pole are 45° from A and 30° from B respectively. Find

(i) the distance AB

(ii) the width of the canal

(b) If six lines are drawn in a plane, no two of which are parallel and no three of which are concurrent, show that there are 15 points of intersection.

(i) If three of these points are chosen at random, find the probability that they all lie on one of the given lines.

(ii) If four of these points are chosen at random, find the probability that they do not all lie on one of the given lines.

Specimen Paper 6

1 (a) For the curve $x^2y^2 - x^2 + y^2 = 0$

(i) Show that $|y| \leqslant 1$ and $|y| \leqslant |x|$

(ii) Find the equations of the asymptotes and the equations of the tangents at the origin. Hence sketch the curve.

(b) (i) Sketch the graph of $y = x^3 - 3px + q$, where $p > 0$ and q are real, showing clearly the coordinates of the turning points.

(ii) Hence show that the roots of the equation $x^3 - 3px + q = 0$ are all real if and only if $q^2 \leqslant 4p^3$.

2 (a) Find $\displaystyle\int \frac{4}{x^2 + 2x - 1}\,dx$

(b) If $a > 0$, use the substitution $u = \dfrac{1}{x}$ to evaluate $\displaystyle\int_{\frac{1}{a}}^{a} \frac{\ln x}{x^2 + 1}\,dx$

(c) If $\mathrm{I}_n = \displaystyle\int_0^1 (1 - x^2)^n\,dx$ for $n \geqslant 0$, show that $\mathrm{I}_n = \dfrac{2n}{2n + 1}\mathrm{I}_{n-1}$ for $n \geqslant 1$.

Hence, or otherwise, show that $\mathrm{I}_n = \dfrac{2^{2n}(n!)^2}{(2n + 1)!}$ for $n \geqslant 1$.

3 The point P$(a \sec\theta, b\tan\theta)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is joined to the vertices A$(a, 0)$ and A$'(-a, 0)$. The lines AP and A$'$P meet the asymptote $y = \dfrac{bx}{a}$ at Q and R respectively.

(i) Find the coordinates of Q and R.

(ii) Hence find the length QR, showing that it is independent of θ, and show that the area of triangle PQR is $\frac{1}{2}\,|ab(\sec\theta - \tan\theta)|$ square units

4 (a) The equation $x^3 + px + q = 0$ has roots α, β and γ. Find the monic cubic equation with roots α^2, β^2 and γ^2.

(b) Use De Moivre's theorem to show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(i) Find the general solution of $\tan 4\theta = 1$

(ii) Hence find the roots of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ in trigonometric form and show that

$$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$$

5 (a) **(i)** Express the complex number $z = -1 + \sqrt{3}i$ in modulus argument form.

(ii) Indicate on an Argand diagram the points P, Q, R and S representing the complex numbers z, $\bar{z}$, z^2 and $\frac{1}{z}$ respectively.

(b) $1 - 2i$ is a root of the equation $z^2 + (2 + i)z + k = 0$. Find

(i) the other root

(ii) the value of k

(c) The region $\{(x, y) : 0 \leqslant x \leqslant 2 \text{ and } 0 \leqslant y \leqslant 4x^2 - x^4\}$ is rotated about the y-axis. Use the method of slicing to find the volume of the solid formed.

6 A projectile is fired vertically upward from the surface of the earth with speed V. The acceleration due to gravity is $\dfrac{gR^2}{x^2}$ where R is the radius of the earth and x is the distance from the centre of the earth.

(i) Neglecting air resistance, show that if $V = \sqrt{(gR)}$, then the speed v of the projectile at distance x from the centre of the earth is given by

$$v = \sqrt{(gR)}\sqrt{\left(\frac{2R - x}{x}\right)}$$

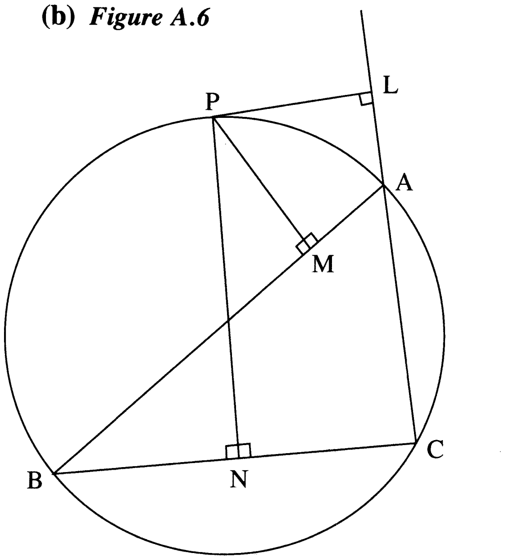
(ii) Hence show that the projectile reaches a height R above the surface of the earth, and find the time taken to reach this height.

7 (a) ADB is a straight line with AD $= a$ and DB $= b$. A circle is drawn on AB as diameter. DC is drawn perpendicular to AB to meet this circle at C.

(i) Show that $\triangle$ADC $|||$ $\triangle$CDB, and hence show that DC $= \sqrt{(ab)}$

(ii) Deduce geometrically that if $a > 0$ and $b > 0$, then $\sqrt{(ab)} \leqslant \dfrac{a + b}{2}$

(b) *Figure A.6*



ABC is a triangle inscribed in a circle. P is a point on the minor arc AB. L, M and N are the feet of the perpendiculars from P to CA produced, AB, and BC respectively.

Show that L, M and N are collinear.

8 (a) If $a > 0$, show that $a^2 + \dfrac{1}{a^2} \geqslant a + \dfrac{1}{a} \geqslant 2$

(b) The equation $x^2 - x + 1 = 0$ has roots α and β, and $A_n = \alpha^n + \beta^n$ for $n \geqslant 1$

(i) Without solving the equation, show that $A_1 = 1$, $A_2 = -1$ and $A_n = A_{n-1} - A_{n-2}$ for $n \geqslant 3$

(ii) Hence use induction to show that $A_n = 2 \cos \dfrac{n\pi}{3}$ for $n \geqslant 1$

Appendix 2
Number Fields

A number system comprising a set of numbers $\mathbb{F}$ and operations $\times$ and $+$ is called a *field* if the following laws are obeyed:

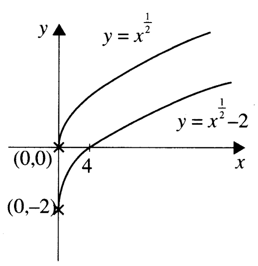
Law of closure:	$a, b \in \mathbb{F} \Rightarrow a\times b \in \mathbb{F}$	and $a+b \in \mathbb{F}$
Associative law:	$a\times(b\times c) = (a\times b)\times c$	and $a+(b+c) = (a+b)+c$
Commutative law:	$a\times b = b\times a$	and $a+b = b+a$
Existence of identity:	$1\times a = a\times 1 = a$	and $0+a = a+0 = a$
Existence of inverse:	Every non-zero number a has a multiplicative inverse a^{-1} such that $a\times a^{-1} = a^{-1}\times a = 1$. Every number a has an additive inverse $-a$ such that $a+(-a) = (-a)+a = 0$.	
Distributive law:	$a\times(b+c) = a\times b+a\times c$	

Answers

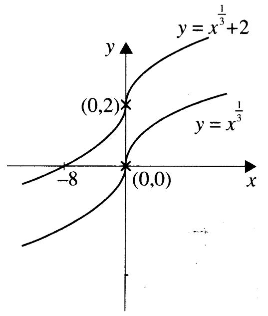
1 Graphs

Exercise 1.1

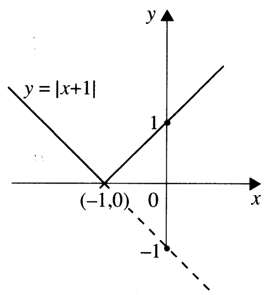
1



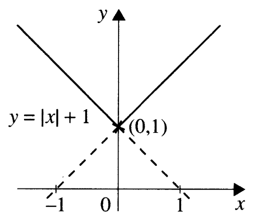
2



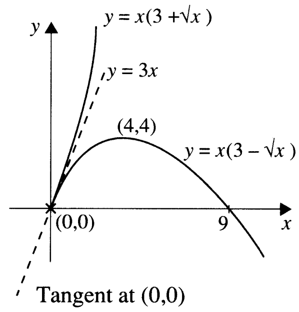
3(a)



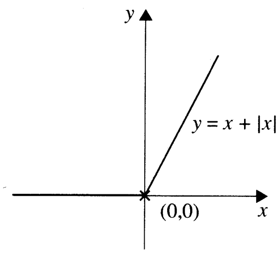
3(b)



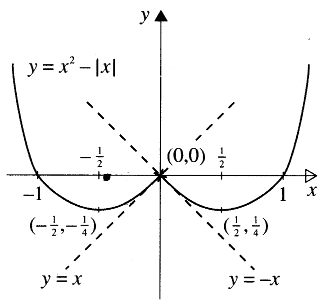
4



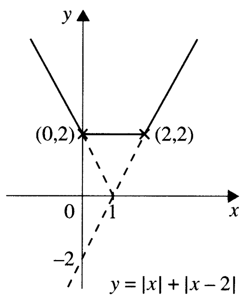
5(a)



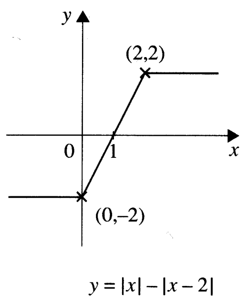
5(b)



6(a)

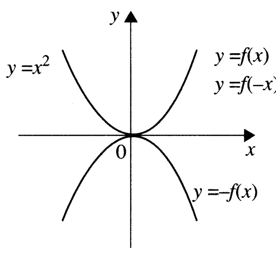


6(b)

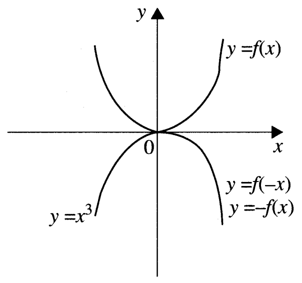


Exercise 1.2

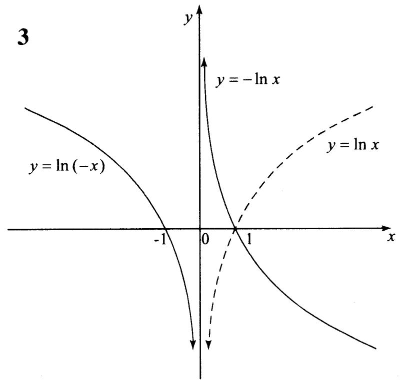
1



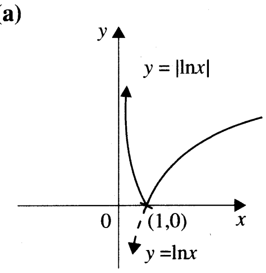
2



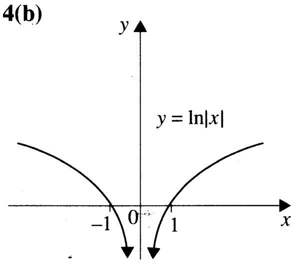
3



4(a)



4(b)



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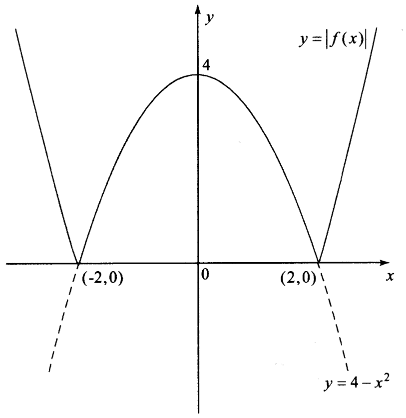
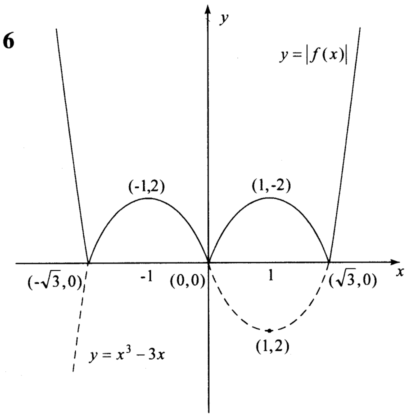
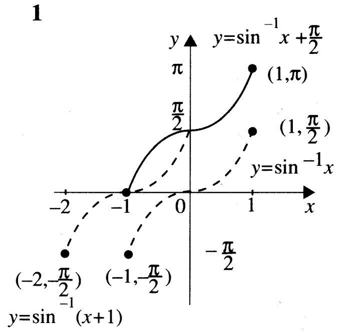


6

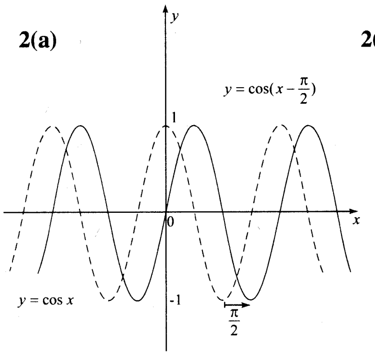


Exercise 1.3

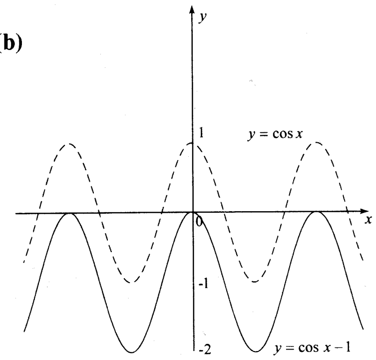
1



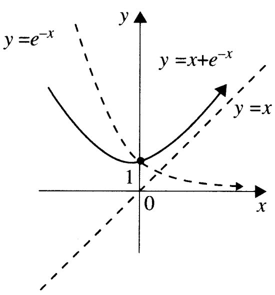
2(a)



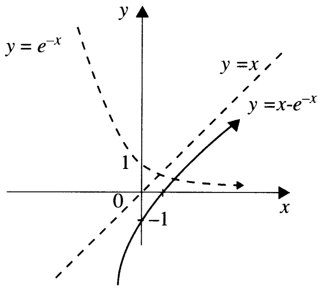
2(b)



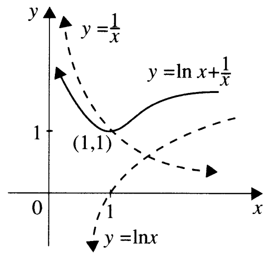
3(a)



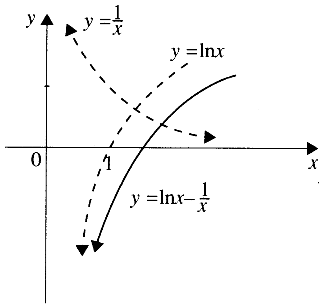
3(b)



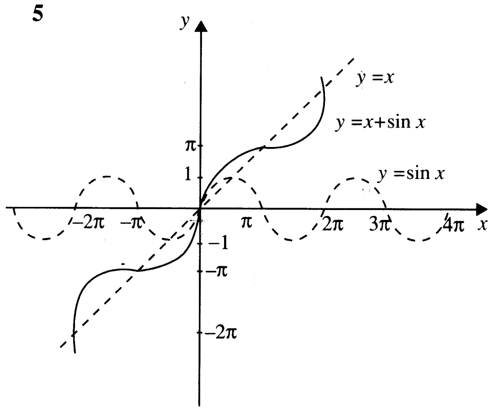
4(a)



4(b)

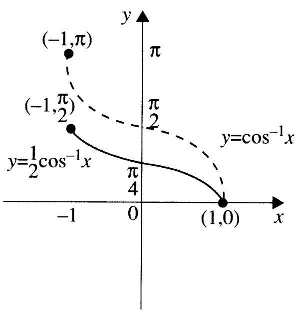


5

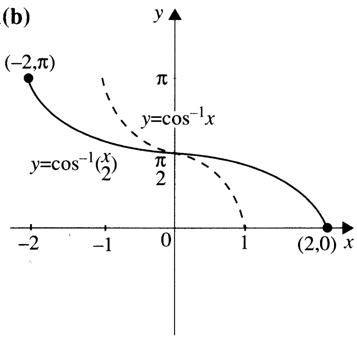


Exercise 1.4

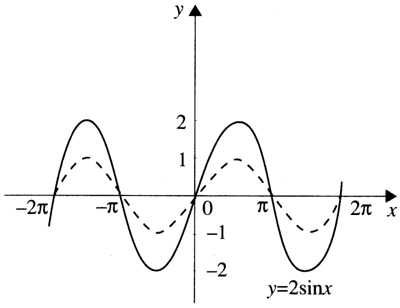
1(a)



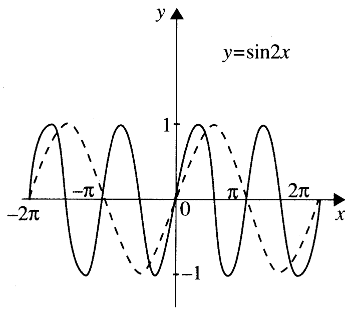
1(b)



2(a)



2(b)



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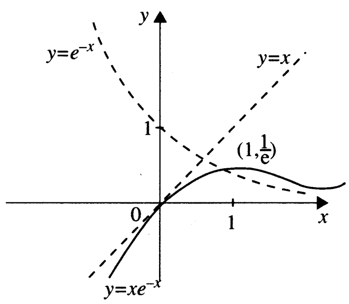
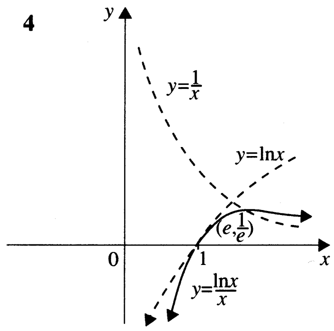
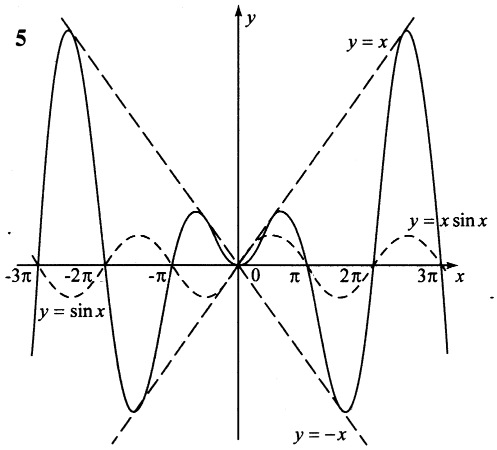


4

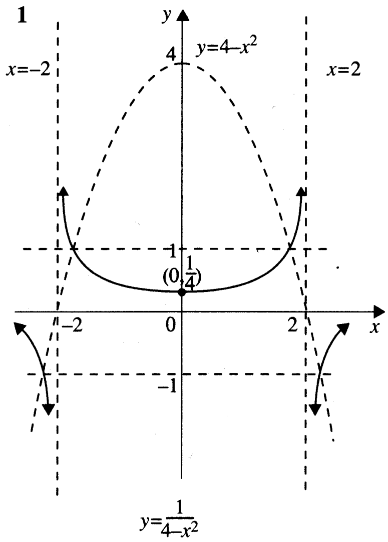


5

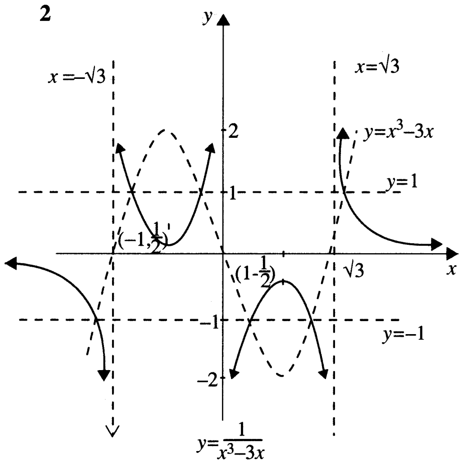


Exercise 1.5

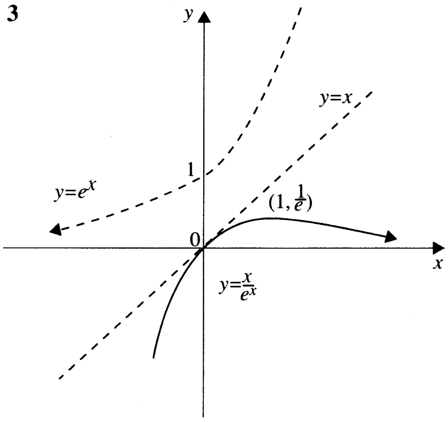
1



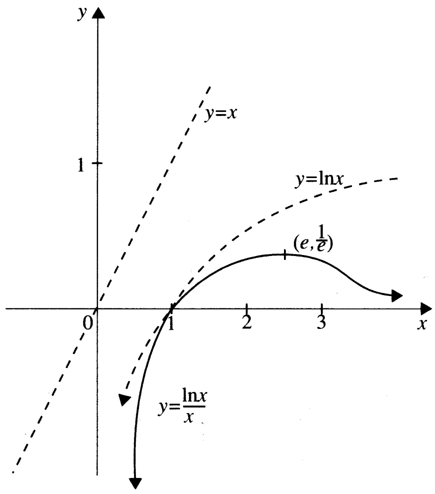
2



3



4



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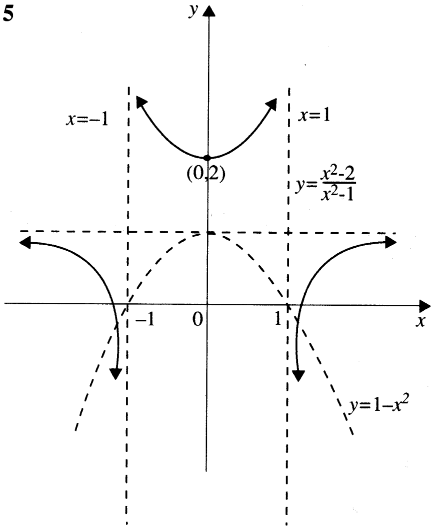
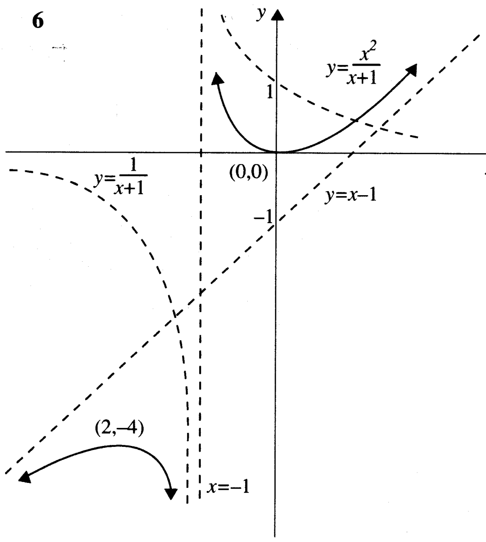
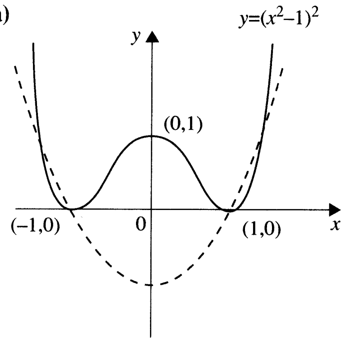


6

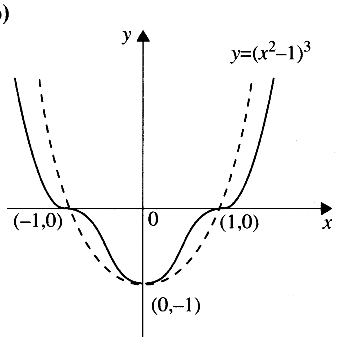


Exercise 1.6

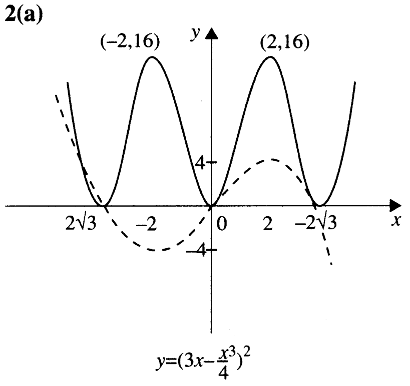
1(a)



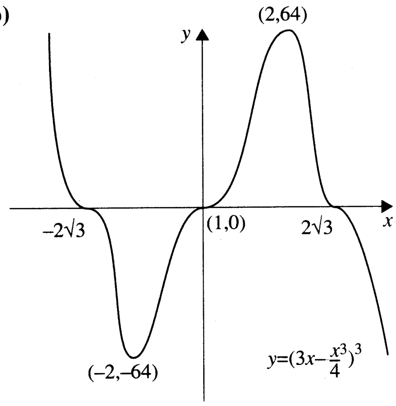
1(b)



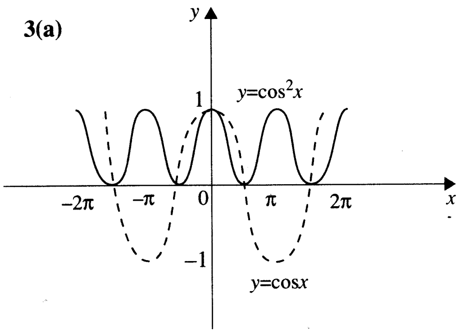
2(a)



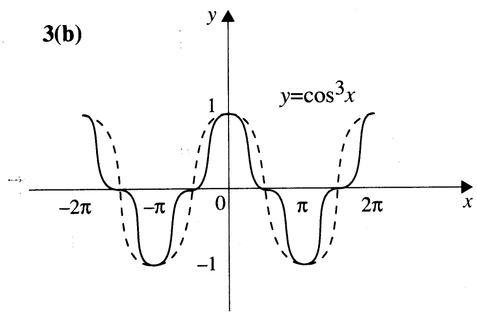
2(b)



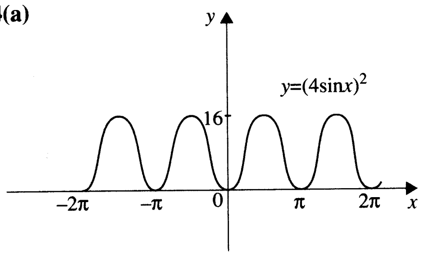
3(a)



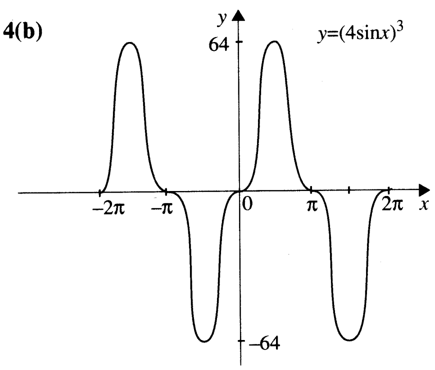
3(b)



4(a)

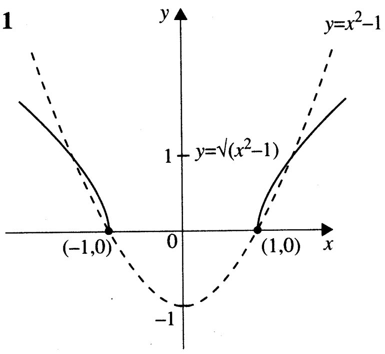


4(b)

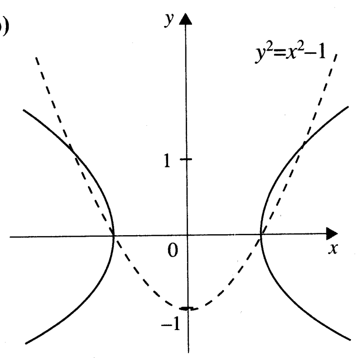


Exercise 1.7

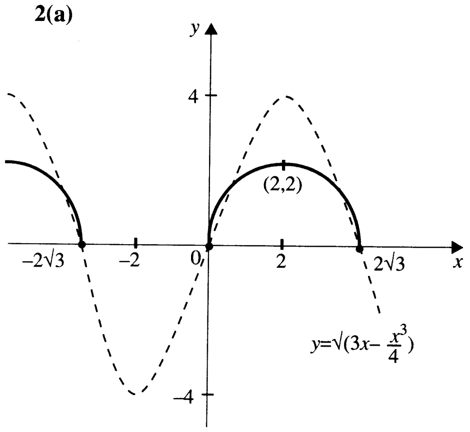
1



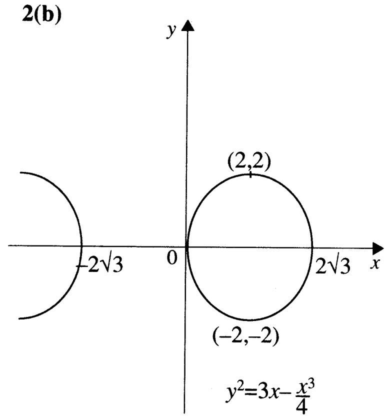
1(b)



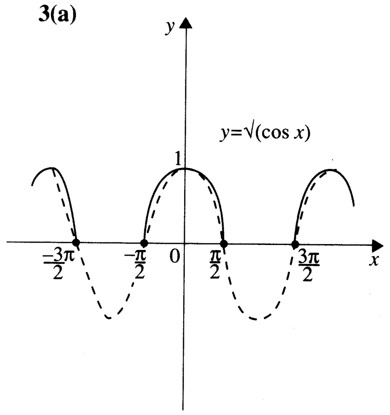
2(a)



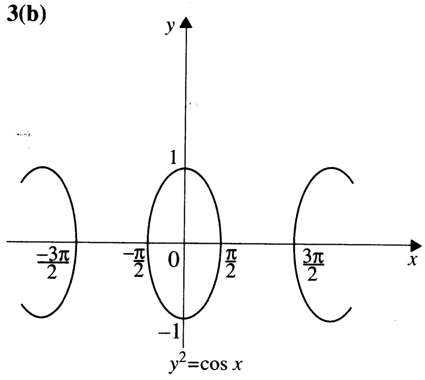
2(b)



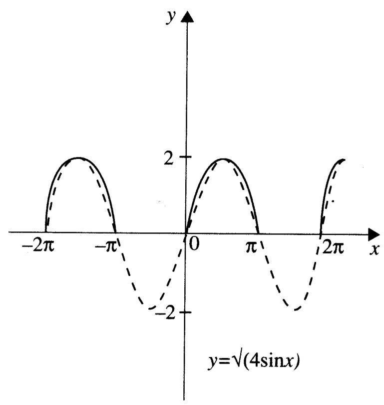
3(a)



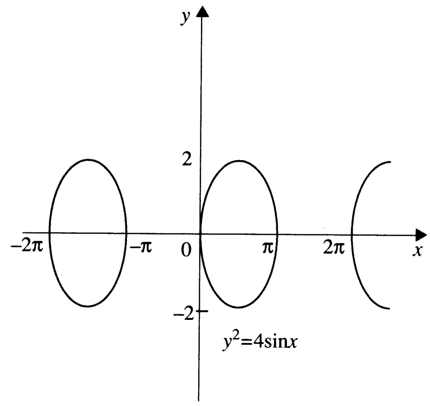
3(b)



4(a)

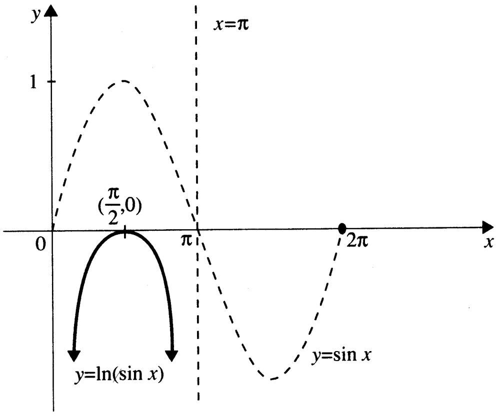


4(b)

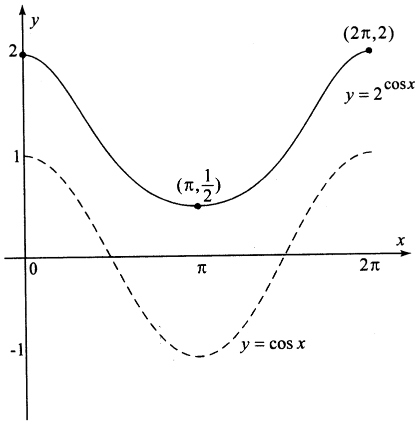


Exercise 1.8

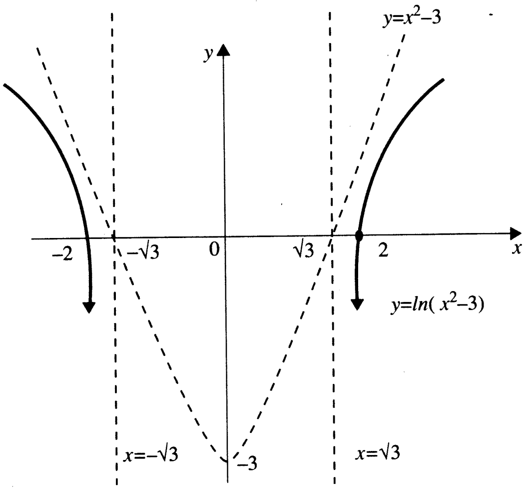
1



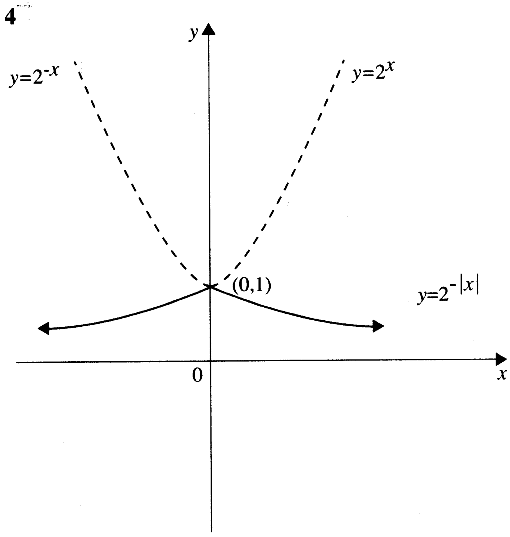
2



3

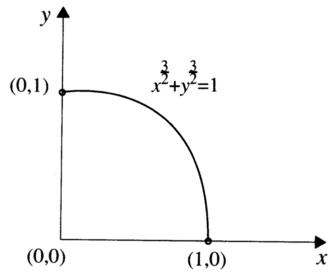


4

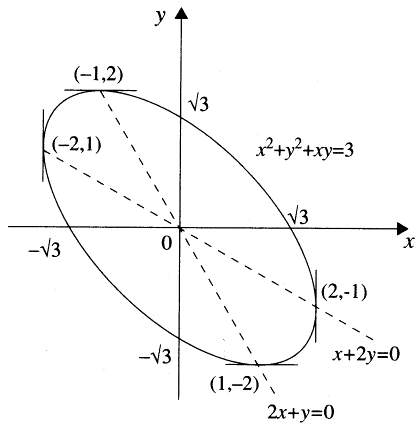


Exercise 1.9

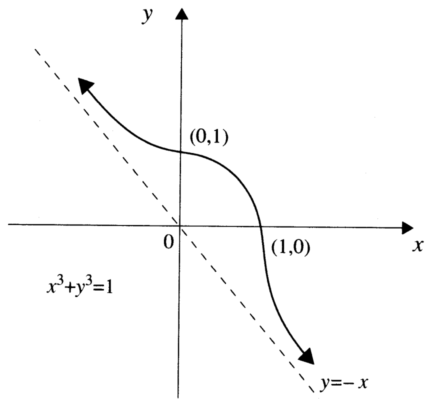
1



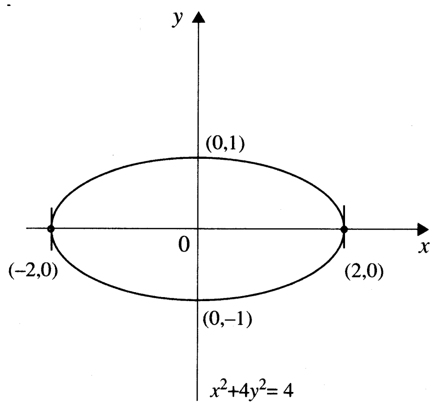
2



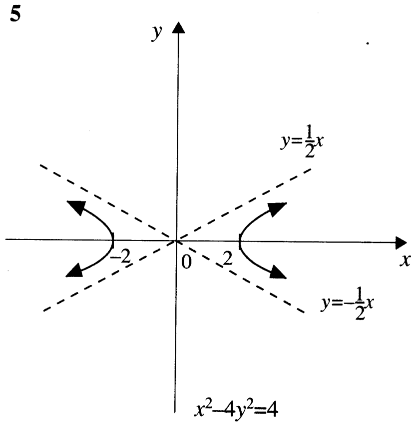
3



4

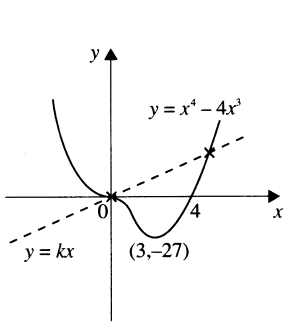


5



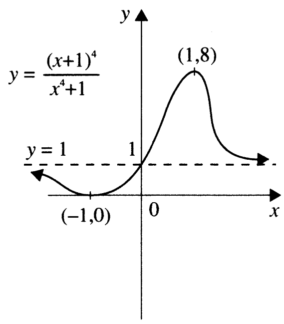
Exercise 1.10

1



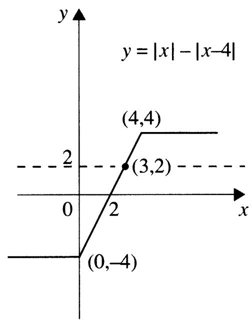
2 real roots

2



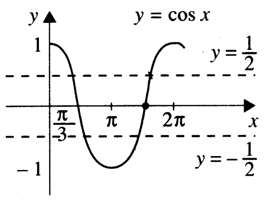
$\{k: 0 < k < 8, k \neq 1\}$

3



$x > 3$

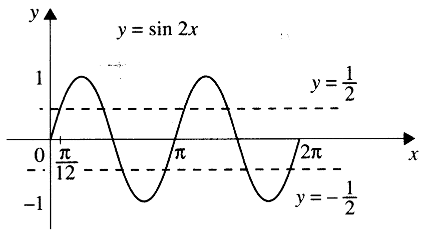
4



(a) $\frac{\pi}{3} \leq x \leq \frac{5\pi}{3}$

(b) $\frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$, $\frac{4\pi}{3} \leq x \leq \frac{5\pi}{3}$

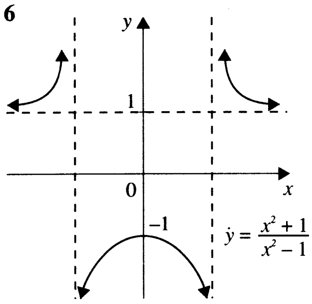
5



(a) $\frac{\pi}{12} \leq x \leq \frac{5\pi}{12}$, $\frac{13\pi}{12} \leq x \leq \frac{17\pi}{12}$

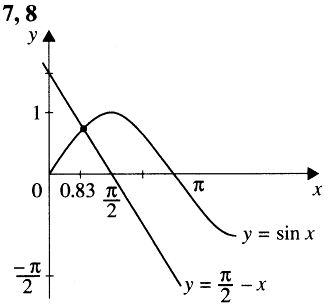
(b) $\frac{\pi}{12} \leq x \leq \frac{5\pi}{12}$, $\frac{13\pi}{12} \leq x \leq \frac{17\pi}{12}$, $\frac{7\pi}{12} \leq x \leq \frac{11\pi}{12}$, $\frac{19\pi}{12} \leq x \leq \frac{23\pi}{12}$

6



$-1 < x < 1$

7, 8



(7) $2\theta \doteqdot 0.83$

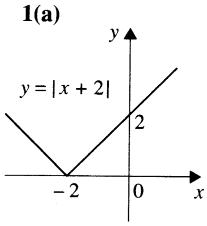
$\theta \doteqdot 0.4$

(8) $\frac{\theta}{2} \doteqdot 0.83$

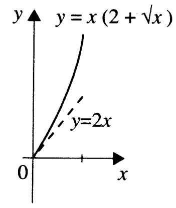
$\theta \doteqdot 1.7$

Diagnostic Test 1

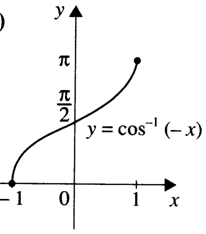
1(a)



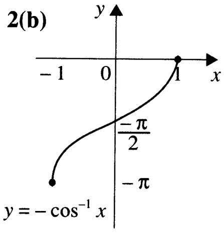
1(b)



2(a)



2(b)



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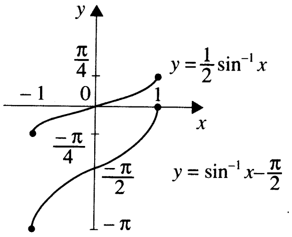

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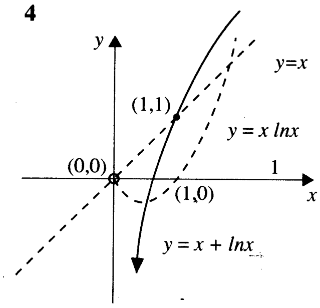

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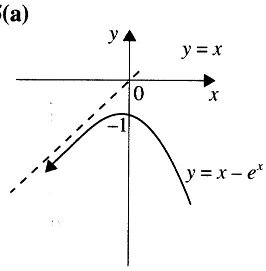

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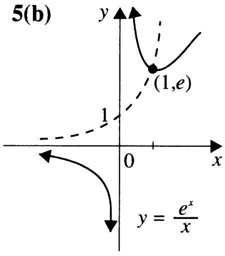

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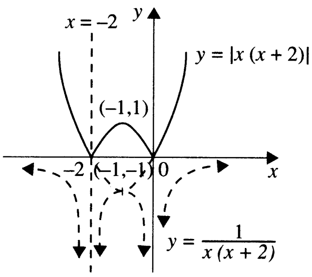
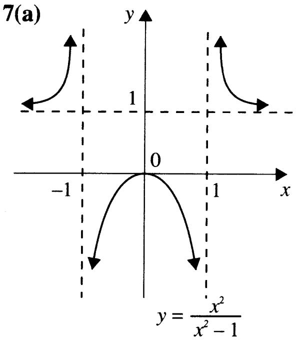
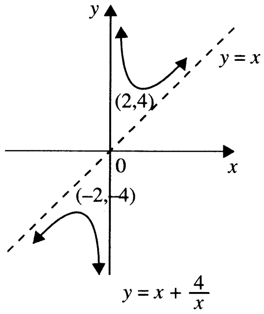


7(a)



7(b)



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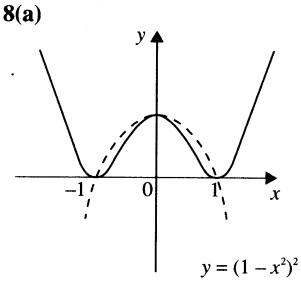
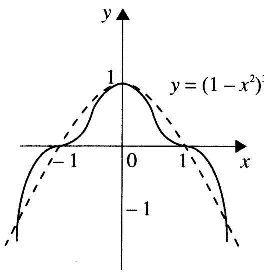
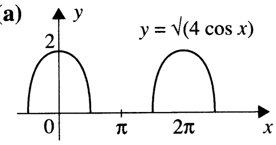


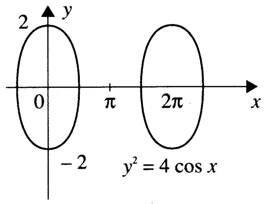
8(b)



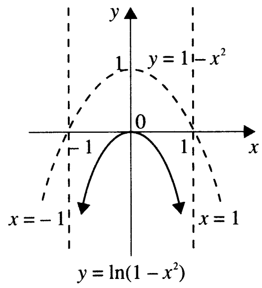
9(a)



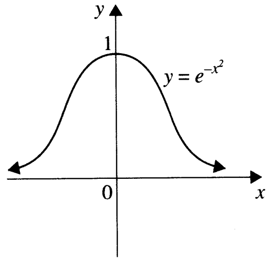
9(b)



10(a)



10(b)



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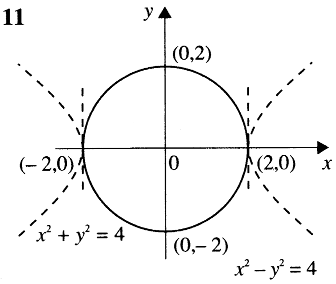
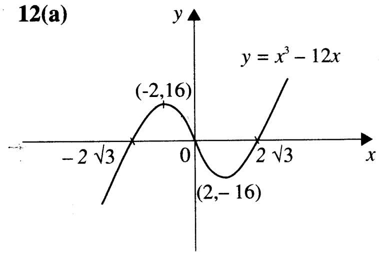
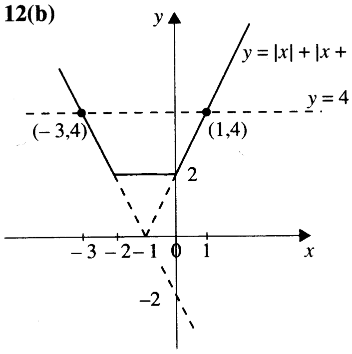


12(a)



$\{k : k > 16 \text{ or } k < -16\}$

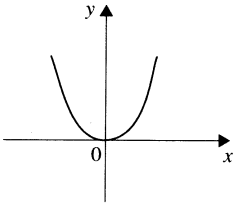
12(b)



$x < -3$ or $x > 1$

Further questions 1

1(a)



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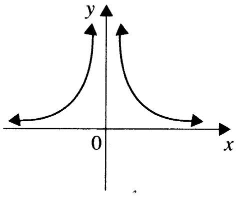
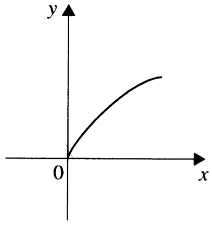
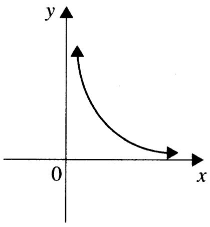


1(c)



1(d)



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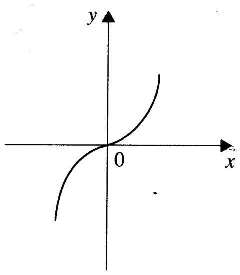

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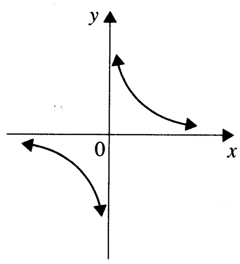

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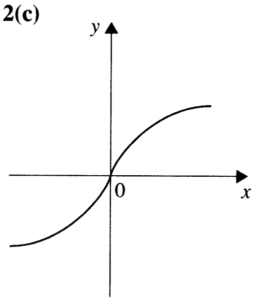
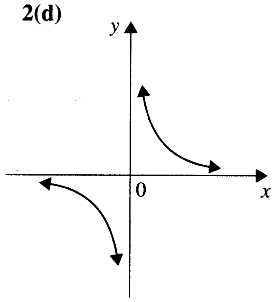


2(d)



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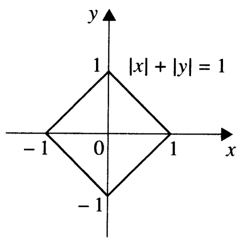
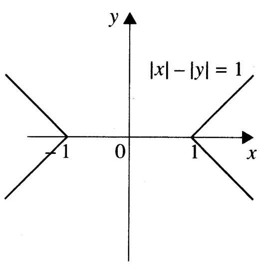


3(b)



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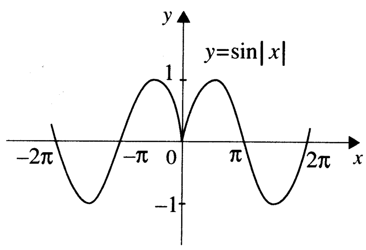
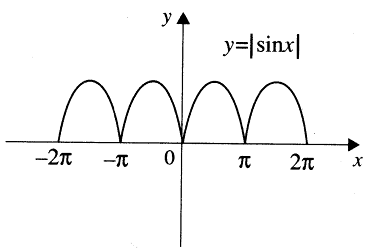
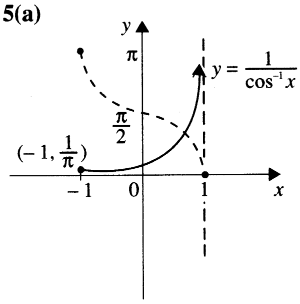


4(b)



5(a)



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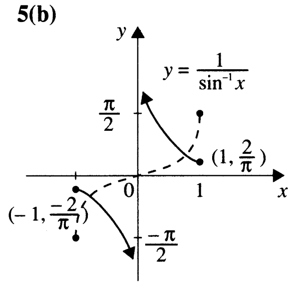

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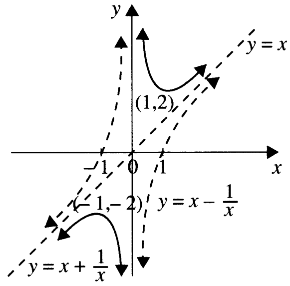

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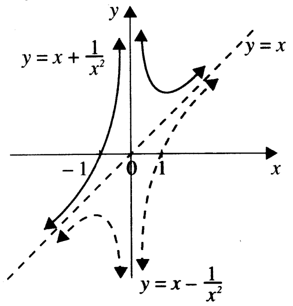

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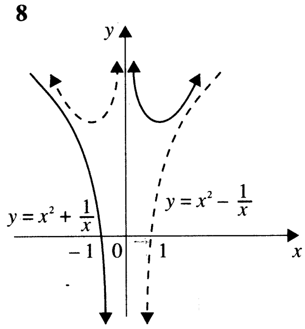

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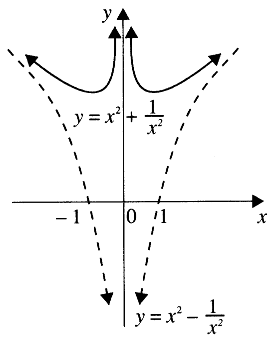

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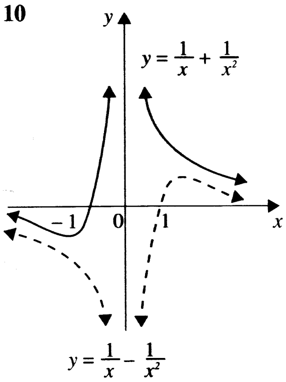
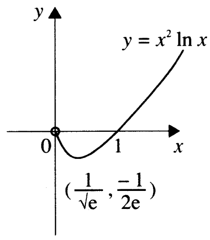
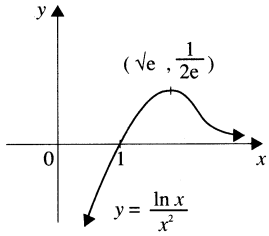


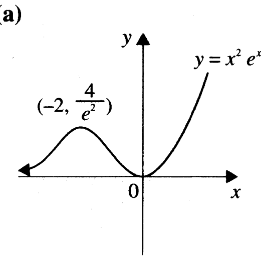
11(a)



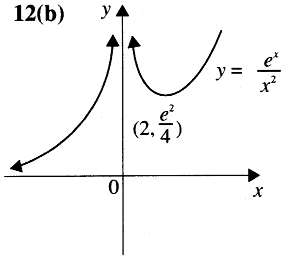
11(b)



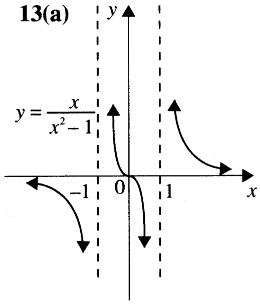
12(a)



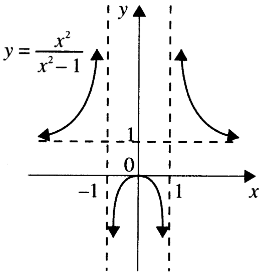
12(b)



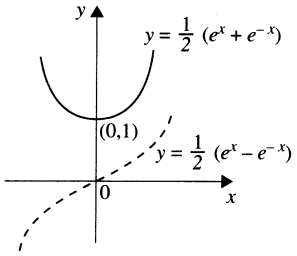
13(a)



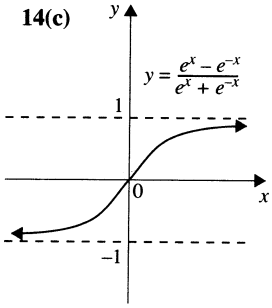
13(b)



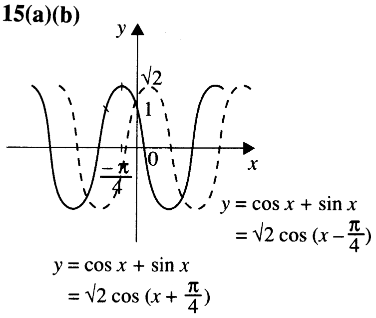
14(a)(b)



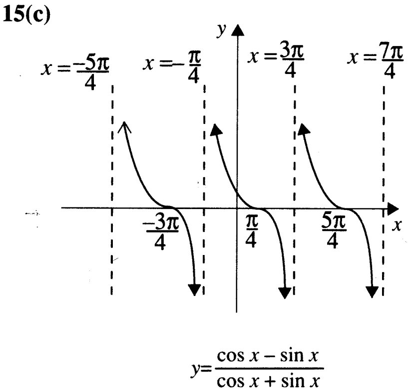
14(c)



15(a)(b)



15(c)



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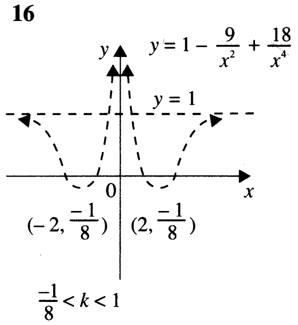

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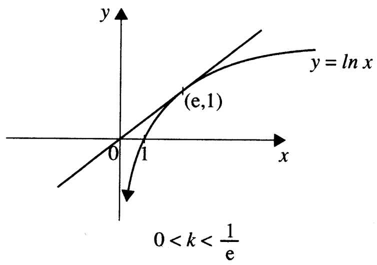
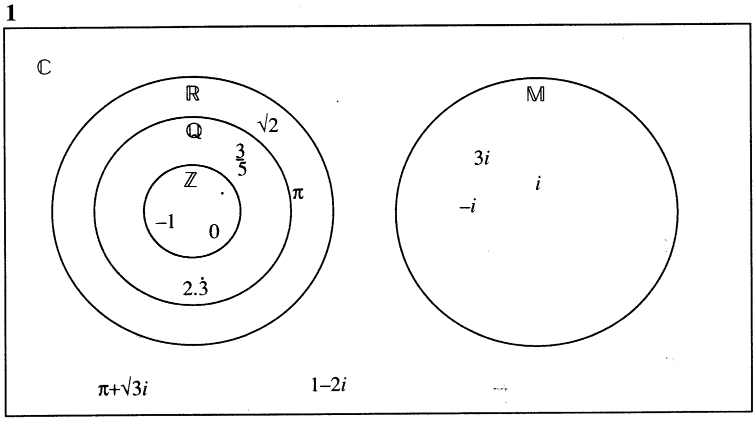


18 $x + y + 4 = 0$ **19** 0.34 **20** 0.50

2 Complex Numbers

Exercise 2.1

1



2 $3+i,\ 1-7i,\ 14+5i,\ -5-12i,\ \frac{1}{17}-\frac{4}{17}i,\ \frac{-10}{13}+\frac{11}{13}i,\ 10-20i,\ 1+43i$

3 $-3-2i,\ 13;\ \frac{-3}{13}-\frac{2}{13}i$ **6** $2i,\ \pm(2+i)$

8 $\pm 5i,\ \pm\sqrt{3}(1-i),\ \pm\frac{1}{\sqrt{2}}(1+i),\ \pm\frac{1}{\sqrt{2}}(1+3i),\ \pm(2-3i)$

9 $x=-\frac{1}{2}(1\pm\sqrt{3}i);\ x=1\pm\frac{1}{\sqrt{2}}i;\ x=\frac{1}{2}+i,\ x=-1-3i$ or $x=3+i$

10 $b=-6,\ c=13;\ -3\pm 2i,\ k=13;\ k=8-i,\ 2+3i$

Exercise 2.2

2 $2\sqrt{2},\frac{\pi}{4};\ 2,\frac{2\pi}{3};\ 2,\frac{-2\pi}{3};\ \sqrt{5},\ -\tan^{-1}\frac{1}{2};\ \sqrt{13},\ \pi-\tan^{-1}\frac{2}{3};\ 5,0;\ 5,\pi;\ 1,\frac{\pi}{2};\ 2,\frac{-\pi}{2};\ \sqrt{2},\frac{3\pi}{4}$

3 $4\ \text{cis}\ \frac{5\pi}{6}$ **6** $64,\pi;\ \frac{1}{2},-\frac{\pi}{6};\ 32,\frac{5\pi}{6}$ **7** $2,\frac{5\pi}{6};\ 4\sqrt{2},\frac{\pi}{4};\ 8\sqrt{2}\ \text{cis}\left(\frac{-11\pi}{12}\right);\ \frac{1}{2\sqrt{2}}\ \text{cis}\ \frac{7\pi}{12}$

8 $n=3,\ -8$ **10** $\frac{1}{2}\ \text{cis}\left(\frac{-\pi}{3}\right);\ \sqrt{2},\frac{3\pi}{4};\ -512i$ **11** $2\ \text{cis}\ \frac{\pi}{6},\ 2\ \text{cis}\left(\frac{-\pi}{6}\right);\ 1024$ **12** $\sqrt{2};\ 1,\frac{\pi}{4}$

13 Rotation anticlockwise by $\frac{3\pi}{4}$, enlargement by $2\sqrt{2}$; $3i\rightarrow -6(1+i)$

14 $x=-\frac{1}{2}(1\pm\sqrt{3}i),\ x=\frac{1}{2}(\sqrt{3}\pm i);\ \frac{2\pi}{3},\frac{\pi}{3},\frac{\pi}{2},\frac{\pi}{3}$

15 $x=-1\pm\sqrt{2}i,\ \sqrt{3}\text{cis}\,[\pm(\pi-\tan^{-1}\sqrt{2})];\ 2p^2-q=0,\ q=3$

Exercise 2.3

3 $\pm\frac{\pi}{2}$ **7** 31, 19

Exercise 2.4

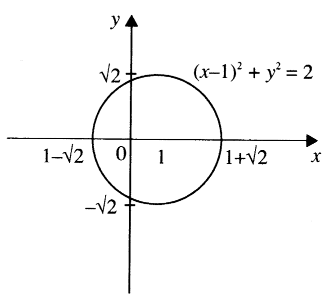
1 $\sqrt{2}\ \text{cis}\left(\pm\frac{\pi}{4}\right)$, -2048 **2** $2\ \text{cis}\left(\frac{2\pi}{3}\right)$; 2^{n+1}, -2^n **3** $z=1$, $\text{cis}\left(\pm\frac{2\pi}{5}\right)$, $\text{cis}\left(\pm\frac{4\pi}{5}\right)$

4 $z=-1$, $\text{cis}\left(\pm\frac{\pi}{5}\right)$, $\text{cis}\left(\pm\frac{3\pi}{5}\right)$

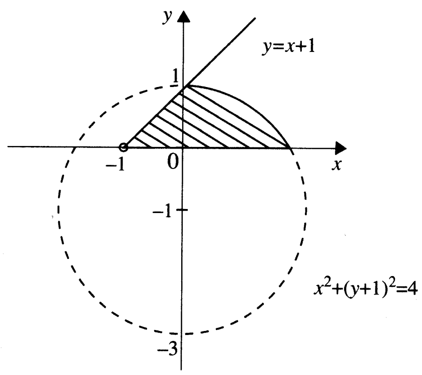
6 0,3 **8** $\pm\sqrt{2}\ \text{cis}\left(\frac{\pi}{12}\right)$; $\sqrt{2}\ \text{cis}\left(\frac{-\pi}{4}\right)$, $\sqrt{2}\ \text{cis}\left(\frac{5\pi}{12}\right)$, $\sqrt{2}\ \text{cis}\left(\frac{-11\pi}{12}\right)$

Exercise 2.5

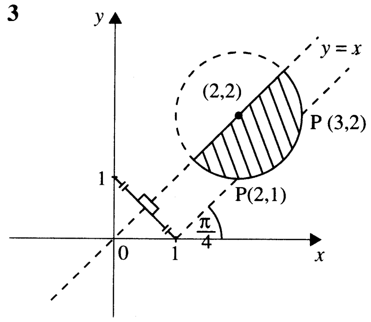
1



2

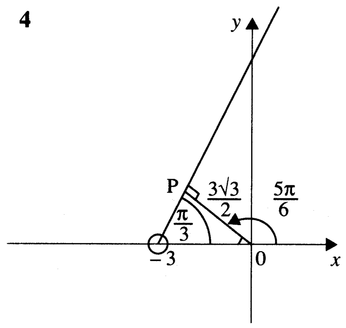


3



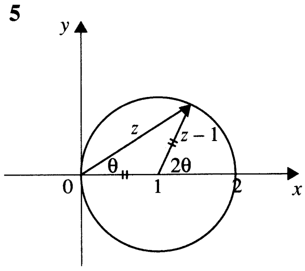
$\arg(z-1)=\frac{\pi}{4}$ for $z=2+i$
or $z=3+2i$

4



$z=\frac{3}{4}(-3+\sqrt{3}\,i)$

5



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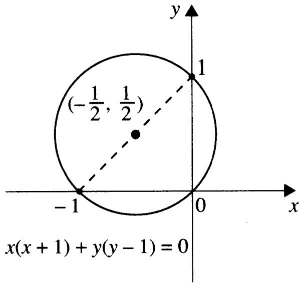

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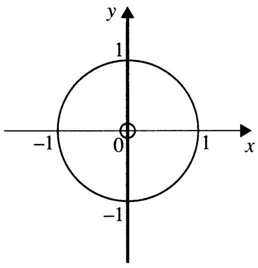
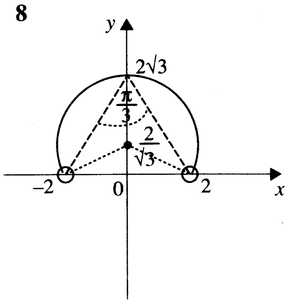


$x = 0, y = 0$ or $x^2 + y^2 = 1$

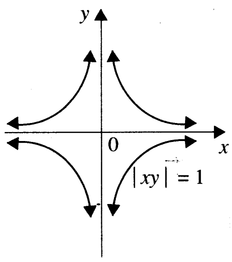
8



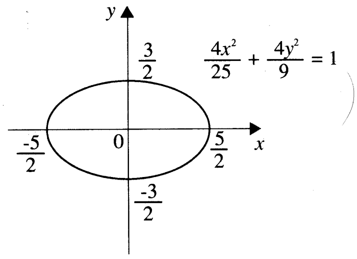
Centre $(0, \frac{2}{\sqrt{3}})$

Radius $\frac{4}{\sqrt{3}}$

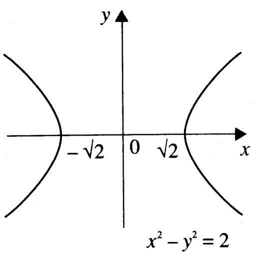
9



10(a)



10(b)



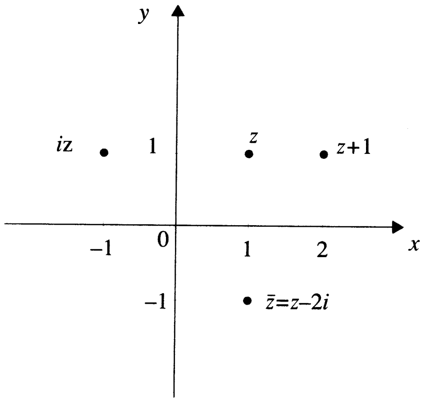
Diagnostic Test 2

1 $2 + 2i, 2, -1 + 2i, 1 - 2i; 6 + 4i, 2 - 2i, 5 + 14i, \frac{11}{13} - \frac{10}{13}i$ **2** $3, 0, 3; 0, 4, -4i; 3, 4, 3 - 4i$

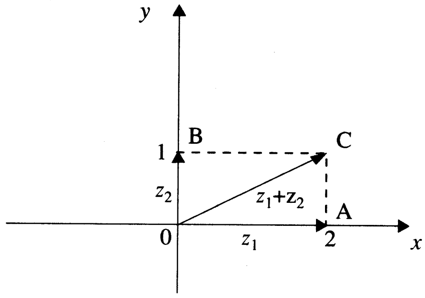
3 $x = 2, y = 1$ or $x = -2, y = -1$ **4** $x = -1 \pm i; x = -2, i$ **5** $2, 0; 2, \frac{\pi}{2}; 2, \frac{\pi}{3}; 2, \frac{-5\pi}{6}$

6 $\sqrt{2} \operatorname{cis} \frac{3\pi}{4}, \sqrt{2} \operatorname{cis} \left(\frac{-\pi}{4}\right)$ **7** $-2 + 2\sqrt{3}i, \sqrt{3} - i$ **8** $2\sqrt{2}, \frac{\pi}{12}; \sqrt{2}, \frac{7\pi}{12}$ **9** $32, \frac{\pi}{2}$

10



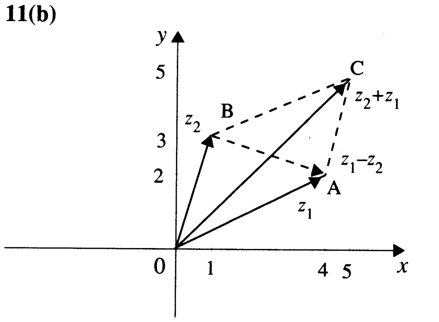
11(a)



$\overrightarrow{BA} = z_1 - z_2$

$\overrightarrow{AB} = z_2 - z_1$

11(b)



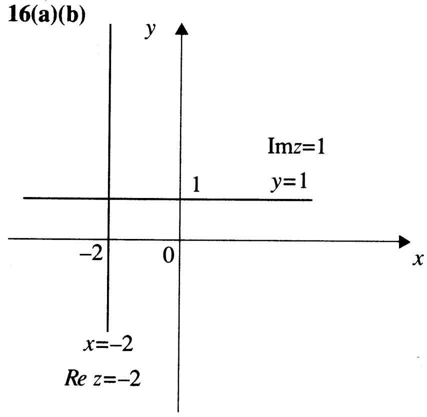
$\overrightarrow{BA} = z_1 - z_2$

$\overrightarrow{AB} = z_2 - z_1$

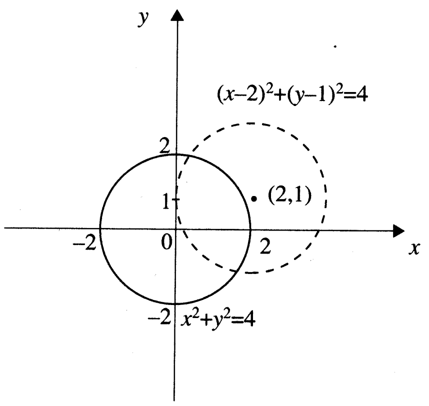
12 cis 4θ **13** $(\text{cis }\theta)^{-2}$

15 $8 \text{ cis } \frac{\pi}{4}$; $\pm\, 2\sqrt{2} \text{ cis } \frac{\pi}{8}$

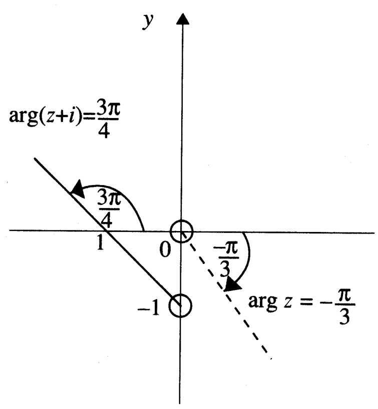
16(a)(b)



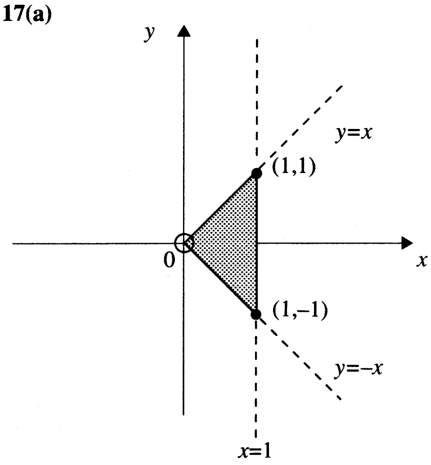
16(c)(d)



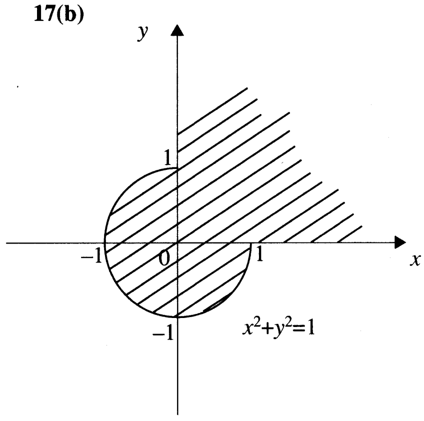
16(e)(f)



17(a)



17(b)



Further questions 2

1 $7 + 22i$, $7 - 22i$; $7^2 + 22^2 = (3^2 + 2^2)(5^2 + 4^2)$ **2** $a = 4$, $b = -5$ **3** $a = -3$, $b = -1$
4 $-2 + i$; $k = 5i$ **5** $a = 3$, $b = 1$; $a = -3$, $b = -1$ **6** $x = 4 + i$, $x = -i$

7 $2, \frac{\pi}{2}; 2, \frac{\pi}{3}; \frac{5\pi}{12}, \frac{11\pi}{12}$ **9** $\frac{1}{2r\cos\theta}$ **12** $\text{cis}\left(\pm\frac{2\pi}{9}\right), \text{cis}\left(\pm\frac{4\pi}{9}\right), \text{cis}\left(\pm\frac{8\pi}{9}\right)$

13 $\frac{-\pi}{2} < \arg z < \frac{\pi}{2}; \frac{18}{5} \pm \frac{24}{5}i$ **14** $x = 1$ and $y = \sqrt{(4 - x^2)}$; $1 + \sqrt{3}i$

3 Conics

Exercise 3.1

1 $\frac{x^2}{9} + \frac{y^2}{8} = 1, \frac{x^2}{3} + \frac{y^2}{4} = 1, \frac{x^2}{9} - \frac{y^2}{72} = 1, \frac{y^2}{4} - \frac{x^2}{12} = 1$
2 $\frac{3}{5}$, $(\pm 3, 0)$, $x = \pm\frac{25}{3}$; $\frac{3}{5}$, $(0, \pm 3)$, $y = \pm\frac{25}{3}$; $\frac{1}{\sqrt{3}}$, $(\pm 1, 0)$, $x = \pm 3$; $\frac{1}{\sqrt{2}}$, $(\pm\sqrt{2}, 0)$, $x = \pm 2\sqrt{2}$
3 $\frac{5}{3}$, $(\pm 5, 0)$, $x = \pm\frac{9}{5}$, $y = \pm\frac{4}{3}x$; $\frac{5}{4}$, (0 ± 5), $y = \pm\frac{16}{5}$, $y = \pm\frac{4}{3}x$;

$\sqrt{3}$, $(\pm\sqrt{6}, 0)$, $x = \pm\sqrt{\frac{2}{3}}$, $y = \pm\sqrt{2}x$; $\sqrt{2}$, $(\pm 2\sqrt{2}, 0)$, $x = \pm\sqrt{2}$, $y = \pm x$
4 $\frac{x^2}{25} + \frac{y^2}{9} = 1; \frac{x^2}{36} + \frac{y^2}{20} = 1$ **5** $\frac{x^2}{16} - \frac{y^2}{9} = 1; \frac{x^2}{36} - \frac{y^2}{45} = 1$ **6** $\frac{x^2}{16} + \frac{y^2}{12} = 1$ **7** 8; 2 or 14

Exercise 3.2

1 $x = 4\cos\theta,\ y = 3\sin\theta;\ x = 2\cos\theta,\ y = \sin\theta;\ x = 4\sec\theta,\ y = 5\tan\theta;\ x = 2\sec\theta,$
$y = 2\tan\theta$

2 $\frac{x^2}{9} + \frac{y^2}{4} = 1;\ \frac{x^2}{25} + \frac{y^2}{16} = 1;\ \frac{x^2}{9} - \frac{y^2}{16} = 1;\ \frac{x^2}{4} - \frac{y^2}{25} = 1$

9 $(2\sqrt{3}, -3\sqrt{3})$ or $\left(\frac{-14\sqrt{3}}{13}, \frac{9\sqrt{3}}{13}\right)$

Exercise 3.3

1 $x + y = 5,\ x - y = 1;\ x - 2y = 8,\ 2x + y = 1;\ 2x - y = 4,\ x + 2y = 7;\ 3x + y = 3,$
$x - 3y = 11$

2 $\sqrt{3}x + 3y = 12,\ 3x - \sqrt{3}y = 8\sqrt{3};\ x - 2y = 4\sqrt{2},\ 2x + y = 3\sqrt{2};\ 3x - \sqrt{3}y = 3,$
$x + \sqrt{3}y = 13;\ 2\sqrt{2}x + y = 4,\ x - 2\sqrt{2}y = 10\sqrt{2}$

3 $5x + 6y = 15;\ 9x + 8y = 24;\ 2x - 3y = 12;\ 9x - 4y = 18$

9 $y = -4x + 7, \left(\frac{16}{7}, \frac{-15}{7}\right);\ y = 2x + 1, (-8, -15)$ **10** $5x - 4y = 20$, (as in **9**)

11 $x^2 + \left(y - \frac{3\sqrt{2}}{4}\right)^2 = \frac{73}{8}$

Exercise 3.4

1 $\sqrt{2}, (\pm 4, \pm 4), x + y = \pm 4, x = 0, y = 0;\ \sqrt{2}, (\pm 4\sqrt{2}, \pm 4\sqrt{2}), x + y = \pm 4\sqrt{2}, x = 0,$
$y = 0.$

2 $x = 2t,\ y = \frac{2}{t};\ x = 5t,\ y = \frac{5}{t}$ **3** $xy = 16;\ xy = 9$

4 $x + 2y = 8,\ 2x - y = 6;\ 4x + 3y = -24,\ 3x - 4y = 7;\ x + 16y = 16,\ 32x - 2y = 255;$
$x + y = -6,\ x - y = 0$

5 $x + 2y = 20;\ 2x - y = -12$ **6** $x + y = -4, (-2, -2);\ 9x + y = -12, \left(\frac{-2}{3}, -6\right)$

7 $3x + y = -8$ (as in **6**) **11** $x + 3y = 9$ **12** $y = \frac{2c^2}{a}$

Diagnostic Test 3

1 $\frac{1}{2}, (\pm 1, 0), x = \pm 4$ **2** $2, (\pm 4, 0), x = \pm 1, y = \pm\sqrt{3}x$ **3** 4 **4** $\frac{x^2}{16} - \frac{y^2}{48} = 1$

5 $x = 3\cos\theta,\ y = 2\sin\theta;\ x = 3\sec\theta,\ y = 4\tan\theta$ **6** $\frac{x^2}{16} + \frac{y^2}{9} = 1;\ \frac{x^2}{16} - \frac{y^2}{25} = 1$

9 $x + 2y = 4,\ 2x - y = 3;\ x + 2\sqrt{3}y = 8,\ 6x - \sqrt{3}y = 9;\ 3x - y = 9,\ x + 3y = 13;$
$4x - y = 6\sqrt{3},\ x + 4y = 10\sqrt{3}$

10 $x + 3y = 2$; $9x - 2y = 54$ **13** $\sqrt{2}$, $(\pm 6, \pm 6)$, $x + y = \pm 6$, $x = 0$, $y = 0$

14 $x = 3t$, $y = \dfrac{3}{t}$; $xy = 25$

15 $2x + 3y = 12$, $3x - 2y = 5$; $x + 4y = 16$, $4x - y = 30$; $x - 2y = -8$

Further questions 3

7 $x + 6y = \pm 19$, $(\pm 1, \pm 3)$ **8** $8x - 9y = \pm 5$, $(\pm 4, \pm 3)$ **9** $4x - 3y = \pm 7$, $(\pm 4, \pm 3)$

10 $16x + 3y = 35$, $(2, 1)$; $8x - 9y = -35$, $(-1, 3)$ **11** $x = 3$, $(3, 0)$; $5x - 12y = -9$, $\left(-5, \dfrac{-4}{3}\right)$

18 $Q\left(\dfrac{-c}{t^3}, -ct^3\right)$, $R\left(-ct, \dfrac{-c}{t}\right)$ **19** $(2x - a)^2 - (2y)^2 = a^2$ **20** $4x^3y^3 + c^2(x^2 - y^2)^2 = 0$

4 Polynomials

Exercise 4.1

1 $\pm 1, \pm 2$ over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$; ± 1 over $\mathbb{Q}$, $\pm 1, \pm\sqrt{2}$ over $\mathbb{R}$ or $\mathbb{C}$; ± 1 over $\mathbb{Q}$ or $\mathbb{R}$, $\pm 1, \pm 2i$ over $\mathbb{C}$
2 $\pm\sqrt{2}, \pm\sqrt{3}$ over $\mathbb{R}$ or $\mathbb{C}$; $\pm\sqrt{2}$ over $\mathbb{R}$, $\pm\sqrt{2}, \pm i$, over $\mathbb{C}$; $\pm i, \pm 2i$ over $\mathbb{C}$
5 $(x + 1)(x^2 - 3)$ over $\mathbb{Q}$, $(x + 1)(x + \sqrt{3})(x - \sqrt{3})$ over $\mathbb{R}$ or $\mathbb{C}$; $(x - 2)(x^2 + 4)$ over $\mathbb{Q}$ or $\mathbb{R}$, $(x - 2)(x + 2i)(x - 2i)$ over $\mathbb{C}$
6 $(x - 1)(x + 4)(x^2 - 2)$ over $\mathbb{Q}$, $(x - 1)(x + 4)(x + \sqrt{2})(x - \sqrt{2})$ over R or $\mathbb{C}$; $(x + 2)(x - 3)(x^2 + 1)$ over $\mathbb{Q}$ or $\mathbb{R}$, $(x + 2)(x - 3)(x + i)(x - i)$ over $\mathbb{C}$

7 $x^3 - x^2 - 16x - 20$; $x^4 - 8x^3 + 18x^2 - 27$ **8** 2, 2 **9** -1, 3

10 $\dfrac{1}{2}, \dfrac{1}{2}, -4$; $(2x - 1)^2 (x + 4)$ **11** 2, 2, 2, -3; $(x - 2)^3 (x + 3)$

12 $c = -5$, $(x + 1)^2 (x - 5)$ or $c = 27$, $(x - 3)^2 (x + 3)$ **13** $c = -32$, $(x + 2)^3 (x - 4)$

17 $\dfrac{3}{2}$, $(2x - 3)(x^2 + 1)$; $-\dfrac{1}{2}$, $(2x + 1)(x + \sqrt{2})(x - \sqrt{2})$

8 $\dfrac{1}{2}, -\dfrac{3}{2}$, $(2x - 1)(2x + 3)(x^2 + x + 1)$

Exercise 4.2

1 $-1 + i$; $-x - 1$ **2** $-49 + 36i$; $18x - 49$ **3** -2 **4** $\dfrac{1}{2}$ **5** $a = 2$, $b = -1$

6 $a = -\dfrac{1}{4}$, $b = -3$ **9** $-2, 1, -i, i$; $(x + 2)(x - 1)(x^2 + 1)$

10 $-1, 2, 2 \pm i$; $(x + 1)(x - 2)(x^2 - 4x + 5)$ **11** $\pm i$, $1 \pm 2i$; $(x^2 + 1)(x^2 - 2x + 5)$
12 $(x + 3)(x + 1)(x^2 + 1)$ **13** $(x + \sqrt{2})(x - \sqrt{2})(x^2 + 4)$
14 $(x + 1)(x - 2)(x + \sqrt{2})(x - \sqrt{2})$

Exercise 4.3

1 $x^3 - 6x^2 + 11x - 6$ **2** $x^4 - 10x^2 + 9$ **3** $a = -16; \frac{1}{3}, 3, 2$ **4** $a = 12; -2, 2, 3$

7 $x = -\frac{4}{3}, -\frac{1}{2}, \frac{1}{3}$ **8** $a = 3; x = -1, 2, 5$ **10** $x = \frac{1}{2}, -2, 8$ **11** $a = 39; x = 1, 3, 9$

12 $x^3 + 6x^2 - 8x - 16 = 0; x^3 + 9x^2 + 22x + 14 = 0; 2x^3 + 2x^2 - 3x - 1 = 0;$
$x^3 - 13x^2 + 16x - 4 = 0$

13 $x^4 + 8x^3 - 12x^2 - 32x + 32 = 0; x^4 + 12x^3 + 45x^2 + 64x + 30 = 0;$
$2x^4 - 4x^3 - 3x^2 + 4x + 1 = 0; x^4 - 22x^3 + 45x^2 - 28x + 4 = 0$

14 $x^3 + 3x^2 - 18x - 81 = 0; x^3 + 4x^2 + 3x - 3 = 0$

15 $x^3 + 2x^2 + 1 = 0; x^3 + 4x^2 + 4x - 1 = 0; x^3 - 4x^2 - 4x - 1 = 0$

16 $rx^3 + px + 1 = 0; x^3 - p^2x^2 - 2prx - r^2 = 0; r^2x^3 + 2prx^2 + p^2x - 1 = 0$

17 $-1, 1, -7, 9$ **18** $-2q, -3r, 5qr$

Exercise 4.4

1 $(x + 2)(x - 2)(x + i)$ **2** $(x + \sqrt{3})(x - \sqrt{3})(x - 2i)$

3 $x = -3, -\frac{1}{3}, \pm i; (x + 3)(3x + 1)(x^2 + 1)$

4 $x = -2, \frac{1}{2}, -1 \pm \sqrt{2}; (x + 2)(2x - 1)(x + 1 + \sqrt{2})(x + 1 - \sqrt{2})$

5 $(x^2 + x + 1)(x^2 - x + 1)$ **6** $(x^2 + \sqrt{3}x + 1)(x^2 - \sqrt{3}x + 1)$ **7** $z = 0, \pm \sqrt{2}, \pm\sqrt{2}i$

8 $z = 0, \pm \frac{1}{2}(1 + i), \pm \frac{1}{2}(1 - i)$

9 $x = \pm \frac{1}{2}\sqrt{(2 + \sqrt{2})}, \pm \frac{1}{2}\sqrt{(2 - \sqrt{2})}; \cos\frac{\pi}{8} = \frac{1}{2}\sqrt{(2 + \sqrt{2})}, \cos\frac{5\pi}{8} = -\frac{1}{2}\sqrt{(2 - \sqrt{2})};$
$x = \pm \frac{1}{2}\sqrt{(2 + \sqrt{3})}, \pm \frac{1}{2}\sqrt{(2 - \sqrt{3})}; \cos\frac{\pi}{12} = \frac{1}{2}\sqrt{(2 + \sqrt{3})}, \cos\frac{5\pi}{12} = \frac{1}{2}\sqrt{(2 - \sqrt{3})}$

10 $1, \cos\frac{2\pi}{5}, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{4\pi}{5}; \cos\frac{2\pi}{5} = \frac{1}{4}(-1 + \sqrt{5}), \cos\frac{4\pi}{5} = -\frac{1}{4}(1 + \sqrt{5});$
$x = \frac{1}{2}, \cos\frac{\pi}{15}, \cos\frac{7\pi}{15}, \cos\frac{13\pi}{15}, \cos\frac{19\pi}{15}$

Exercise 4.5

1 $\frac{3}{x - 1} - \frac{1}{x + 3}$ **2** $\frac{1}{x + 1} + \frac{2}{2x + 3}$ **3** $\frac{2}{x - 2} - \frac{4}{2x - 1}$ **4** $1 + \frac{2}{x} - \frac{2}{x + 1}$

5 $\frac{1}{x - 2} - \frac{x}{x^2 + 4}$ **6** $\frac{1}{2x - 1} + \frac{x - 1}{x^2 + 1}$ **7** $1 + \frac{1}{x + 1} + \frac{2}{x^2 + 4}$ **8** $\frac{2}{2x + 3} - \frac{x - 1}{x^2 + 1}$

9 $\frac{2}{x^2 + 1} - \frac{1}{x^2 + 4}$ **10** $\frac{x}{x^2 + 1} - \frac{x}{x^2 + 4}$

Diagnostic Test 4

1 ± 1 over $\mathbb{Q}$, ± 1, $\pm\sqrt{3}$ over $\mathbb{R}$ or $\mathbb{C}$; $\pm\sqrt{3}$ over $\mathbb{R}$, $\pm\sqrt{3}$, $\pm i$ over $\mathbb{C}$
2 $(x+5)(x-1)(x^2+2)$ over $\mathbb{Q}$ or $\mathbb{R}$, $(x+5)(x-1)(x+\sqrt{2}i)(x-\sqrt{2}i)$ over $\mathbb{C}$
3 $-2, -2, \frac{1}{4}$; $(x+2)^2(4x-1)$ **4** $\frac{1}{2}, \pm\sqrt{3}$; $(2x-1)(x+\sqrt{3})(x-\sqrt{3})$
5 $-3-i$; $-x-3$ **6** $a=-1, b=1$
7 $\pm\sqrt{3}, 1\pm i$; $(x+\sqrt{3})(x-\sqrt{3})(x^2-2x+2)$
8 $(x+\sqrt{2})(x-\sqrt{2})(x+\sqrt{3})(x-\sqrt{3})$
9 $a=-6$; $x=-2, 1, 4$
10 $x^3+2x^2-8x-24=0$; $8x^3+4x^2-4x-3=0$; $x^3+7x^2+14x+5=0$; $x^3-5x^2+6x-3=0$
11 $rx^3+qx^2+1=0$; $x^3+2qx^2+q^2x-r^2=0$ **12** $0, -4, -3, 8$
13 $x=-1, -1, \frac{1}{3}, 3$; $(x+1)^2(3x-1)(x-3)$ **14** $0, \pm 2, \pm 2i$
15 $x=-1, 2\pm\sqrt{3}$; $\tan\frac{\pi}{12}=2-\sqrt{3}$, $\tan\frac{5\pi}{12}=2+\sqrt{3}$
16 $x=0, \pm\sqrt{\left[\frac{1}{8}(5+\sqrt{5})\right]}, \pm\sqrt{\left[\frac{1}{8}(5-\sqrt{5})\right]}$; $\cos\frac{\pi}{10}=\sqrt{\left[\frac{1}{8}(5+\sqrt{5})\right]}$, $\cos\frac{3\pi}{10}=\sqrt{\left[\frac{1}{8}(5-\sqrt{5})\right]}$
17 $\frac{2}{x+2}+\frac{1}{x-3}$ **18** $\frac{2}{x-4}+\frac{x-2}{x^2+1}$

Further questions 4

1 $x=\sqrt{7}, \frac{1}{2}(-\sqrt{7}\pm\sqrt{5})$ **2** $(x^2+1)(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$
3 $x=\frac{1}{5}(3\pm 4i), \frac{1}{2}(1\pm\sqrt{3}i)$; $(5x^2-6x+5)(x^2-x+1)$
4 $x=2\pm i\sqrt{3}, 2\pm i\sqrt{3}$; $(x^2-4x+7)^2$ **5** $z=1\pm i, -1\pm 2i$ **6** $2x^3+3x^2+8=0$
12 $a_1=-\frac{1}{2}n(n+1)$, $a_n=(-1)^n n!$

5 Integration (constant of integration omitted)

Exercise 5.1

1 $\frac{1}{2}\ln(1+x^2)$ **2** $\frac{-1}{2(1+x^2)}$ **3** $e^{\sin x}$ **4** $-\cos(e^x)$ **5** $\frac{1}{3}(1+x^2)^{3/2}$ **6** $\sin^{-1}\frac{x}{2}$
7 $\frac{1}{2}\tan^{-1}2x$ **8** $\frac{1}{4}\tan^4 x$ **9** $\frac{1}{2}\tan x^2$ **10** $2e^{\sqrt{x}}$ **11** $\frac{1}{3}\sec^3 x$ **12** $\ln(2+\sin^2 x)$

13 $\frac{1}{2}\ln 2$ **14** $\sin 1$ **15** $\frac{\pi}{8}$ **16** $\ln(1+\sqrt{2})$ **17** $\ln(1+\sqrt{2})$ **18** $\frac{\pi}{4}$ **19** $\frac{1}{2}$
20 $\ln 2$

Exercise 5.2

1 $\tan^{-1}(x+1)$ **2** $\sin^{-1}(x-1)$ **3** $\frac{1}{2}\ln(x^2+1)-\tan^{-1}x$ **4** $x^2-x+\ln|x+1|$

5 $\ln\left|\frac{x}{\sqrt{(2x+1)}}\right|$ **6** $x+\ln\left|\frac{x+1}{(x+2)^4}\right|$ **7** $\ln(x^2+2x+5)+\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$

8 $\ln\{(x+1)^4(x-3)^2\}$ **9** $\sqrt{2}\ln\left|\frac{x-1-\sqrt{2}}{x-1+\sqrt{2}}\right|$ **10** $2\tan^{-1}\frac{x}{2}-\ln|x+1|$

11 $\ln\left|\frac{x-1}{\sqrt{(x^2+9)}}\right|-\frac{1}{3}\tan^{-1}\frac{x}{3}$ **12** $\tan^{-1}x-\frac{1}{2}\tan^{-1}\frac{x}{2}$ **13** $\frac{\pi}{8}$ **14** $\frac{\pi}{6}$ **15** $\frac{1}{2}\ln 2+\frac{\pi}{8}$

16 $4+\ln 2$ **17** $\ln 2-\frac{\pi}{4}$ **18** $3+\ln 2$ **19** $\frac{1}{2}\ln 20$ **20** $\frac{5\pi}{18}$

Exercise 5.3

1 $2\sin^{-1}\sqrt{x}$ **2** $\frac{1}{4}\ln\left|\frac{x^2-1}{x^2+1}\right|$ **3** $\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}$

4 $\frac{2}{7}(x-1)^{7/2}+\frac{4}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}$ **5** $x-\ln(e^x+1)$ **6** $\frac{1}{2}\ln(e^{2x}+1)+\tan^{-1}e^x$

7 $2\sqrt{x}-2\tan^{-1}\sqrt{x}$ **8** $\sin^{-1}\sqrt{x}-\sqrt{[x(1-x)]}$ **9** $\frac{-\sqrt{(16-x^2)}}{x}-\sin^{-1}\frac{x}{4}$

10 $-\cos^{-1}x-\sqrt{(1-x^2)}$ **11** $\ln\left|\tan\frac{x}{2}\right|$ **12** $\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$ **13** $\frac{\pi}{8}$ **14** $\ln\frac{3}{2}$

15 $\frac{96}{5}$ **16** $2-\sqrt{2}$ **17** $2-\frac{\pi}{2}$ **18** $\pi-2$ **19** $1+\sqrt{3}$ **20** $\frac{1}{4}\ln 3$

Exercise 5.4

1 $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x$ **2** $-\frac{1}{7}\cos^7 x+\frac{2}{5}\cos^5 x-\frac{1}{3}\cos^3 x$ **3** $-\sin x-\operatorname{cosec} x$

4 $-\frac{1}{2}\sec^2 x+\frac{1}{4}\sec^4 x$ **5** $\frac{3}{4}(\sin x)^{4/3}-\frac{3}{10}(\sin x)^{10/3}$ **6** $\frac{1}{16}\sin 8x+\frac{1}{8}\sin 4x$

7 $\frac{1}{8}\sin 4x-\frac{1}{16}\sin 8x$ **8** $-\frac{1}{8}\cos 4x-\frac{1}{4}\cos 2x$ **9** $-\frac{1}{8}\cos 4x+\frac{1}{4}\cos 2x$

10 $\frac{1}{12}\sin 6x+\frac{1}{4}\sin 2x$ **11** $-\frac{1}{14}\cos 7x-\frac{1}{2}\cos x$ **12** $\frac{1}{6}\cos 3x-\frac{1}{14}\cos 7x$ **13** $\frac{1}{2}$

14 $\frac{-7}{24}$ **15** $\frac{8}{21}$ **16** $\frac{1}{3}$

Exercise 5.5

1 $xe^x - e^x$ **2** $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3$ **3** $-x\cos x + \sin x$ **4** $x^2 \sin x + 2x\cos x - 2\sin x$
5 $\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x$ **6** $x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2)$

7 $\frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x$ **8** $\frac{1}{2}e^x(\cos x + \sin x)$ **9** $x\sec x - \ln|\sec x + \tan x|$

10 $\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x|$ **11** $\frac{1}{4}(e^2 + 1)$ **12** $e - 2$ **13** $\frac{\pi}{2} - 1$ **14** $\frac{\pi}{8}$

19 $\left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720$

20 $\dfrac{3^n\, n!}{(3n + 2)(3n - 1)\ldots 8.5.2}$

Exercise 5.6

2 $1 - \frac{\pi}{4}; \frac{\pi}{4}$ **4** 1; 2

Diagnostic test 5

1 $\frac{1}{2}(\ln x)^2$ **2** $-e^{1/x}$ **3** $\ln(2 + \sin x)$ **4** $\sqrt{5} - 1$ **5** $\frac{1}{2}\ln(x^2 + 9) + \frac{1}{3}\tan^{-1}\frac{x}{3}$

6 $\ln\left|\frac{(x - 1)^2}{(x + 3)}\right|$ **7** $\ln|(x - 2)\sqrt{(x^2 + 1)}|$ **8** $48 + \ln 2$ **9** $2\tan^{-1}\sqrt{x}$

10 $\frac{2}{3}(x + 1)^{3/2} - 2(x + 1)^{1/2}$ **11** $\frac{\pi}{12}$ **12** 1 **13** $\frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2}$
14 $-\frac{1}{10}\cos 5x - \frac{1}{6}\cos 3x$ **15** $x^2e^x - 2xe^x + 2e^x$ **16** $\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x$

17 $\frac{\pi}{4} - \frac{1}{2}\ln 2$ **18** $9e - 24$ **19** 0

Further questions 5

1 $\frac{1}{2}(\tan^{-1}x)^2$ **2** $\frac{1}{2}\ln\left|\frac{x + 1}{x + 3}\right|$ **3** $\ln(e^x + 1)$ **4** $\tan^{-1}(x + 2)$

5 $x\ln|x^2 - 1| + \ln\left|\frac{x + 1}{x - 1}\right| - 2x$ **6** $\tan x + \sec x$ **7** $-\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x$

8 $\frac{1}{4}\ln\left|\frac{1 + x}{3 - x}\right|$ **9** $\tan^{-1}e^x$ **10** $x\ln(x^2 + 1) + 2\tan^{-1}x - 2x$ **11** $\ln|\tan x|$

12 $\sin^{-1}\left(\frac{x - 1}{2}\right)$ **13** $\frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x$ **14** $x - \frac{1}{2}\ln\left|\frac{x + 1}{x - 1}\right|$

15 $\frac{1}{2}x\sqrt{(4 - x^2)} + 2\sin^{-1}\frac{x}{2}$ **16** $\tan(\ln x)$ **17** $-\cot\frac{x}{2}$

18 $\ln(x^2 + 2x + 2) - \tan^{-1}(x + 1)$ **19** $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x$

20 $\frac{1}{2}x\sqrt{(16 + x^2)} + 8\ln\{x + \sqrt{(16 + x^2)}\}$ **21** $e^{\sin^{-1}x}$ **22** $\ln|(x - 3)(x^2 + 1)|$

23 $\tan^{-1}\sqrt{(x - 1)} + \frac{1}{x}\sqrt{(x - 1)}$ **24** $2\ln|x - 2| - \frac{1}{3}\tan^{-1}\frac{x}{3}$

25 $\sin^{-1}(e^x)$ **26** $\frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{1}{4}\tan^{-1}\frac{x}{4}$ **27** $\cos x + \sec x$ **28** $\frac{1}{2}\ln 2$

29 $\frac{1}{2}\ln 2 + \frac{\pi}{4}$ **30** 2 **31** $2\ln 2 - \pi$ **32** $\frac{1}{6}$ **33** $\frac{\pi}{4} - \frac{3}{2}\ln 2$ **34** $-\frac{1}{n}, 0, \frac{1}{n}$

35 $0, \pi$ **36** $\frac{\pi}{18}$ **37** $\frac{1}{6}\ln(2 + \sqrt{3})$ **38** $\frac{2\pi}{3}$ **39** $\frac{\pi}{6}$

40 $x\ln x - x, \frac{1}{2}(\ln x)^2, \frac{1}{(n - 1)x^{n-1}}\left\{\frac{1}{1 - n} - \ln x\right\}$

41 $-\frac{1}{2(x^2 + 1)} + \frac{1}{4(x^2 + 1)^2} + \frac{x^4}{4(x^2 + 1)^2} + c$ **43** $\frac{\pi}{2}$ **44** $(-1)^n\frac{\pi}{2}$

47 $\frac{1}{2}(\pi^3 - 24\pi + 48)$ **48** $\frac{63\pi}{512}, \frac{63\pi}{128}$

6 Volumes (u^3 omitted)

Exercise 6.1

1 $\frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{15}$ **2** $\frac{224\pi}{15}, 8\pi, \frac{8\pi}{3}, \frac{32\pi}{5}$ **3** $\frac{8\pi}{3}$ **4** 36π **5** $\frac{\pi^2}{4}$ **6** $2\pi - \frac{\pi^2}{4}$
7 $2\pi^2$ **8** $4\pi^2$

Exercise 6.2

1, 2, 3, 4 As for Exercise 6.1 **5** 2π **6** 2π **7, 8** As for Exercise 6.1.

Exercise 6.3

1 $144, 18\pi$. **2** $\frac{8\sqrt{3}}{3}, \frac{8}{3}$ **5** $8, 2\sqrt{3}$ **6** $16\pi, 32$ **7** $\frac{256}{9}$ **8** $\frac{32\pi}{3}$

Diagnostic Test 6

1 $\frac{\pi}{6}, \frac{\pi}{2}$ **2** $\frac{4\pi}{3}$ **3** $\frac{4\pi}{5}, \frac{8\pi}{15}$ **4** $\frac{\pi}{30}$ **5** 36 **6** $\frac{32}{3}$

Further questions 6

1 $\pi(e-2), \frac{\pi}{2}(e^2+1)$ **2** $\frac{\pi}{2}(-e^2+4e-1), \pi(4-e)$ **3** $\pi \ln 2$ **4** $\pi(1-e^{-1})$

5 $\frac{\pi}{2}(\pi-2)$ **6** $\frac{\pi a^3}{15}$ **7** $\frac{32\pi a^3}{15}$ **8** $2ba^2\pi^2$ **9** $\frac{16a^3}{3}, \frac{4\pi a^3}{3}$ **10** $\frac{4ab^2\sqrt{3}}{3}, \frac{4ab^2}{3}$

7 Mechanics

Exercise 7.1

1 $v = 5 - \frac{2}{t^2}$ **2** 20 m **4** $v = Ve^{-kx}, v = \frac{V}{1+kVt}, x = \frac{1}{k}\ln\{1+kVt\}$

5 $A = \frac{1}{80}, B = \frac{1}{6400}; x = 6400\ln\left(1+\frac{t}{80}\right), v = 80\, e^{\frac{-x}{6400}}$ **7** $\frac{1}{2}\ln\left(1+\frac{1}{k}u^2\right); \frac{1}{\sqrt{k}}\tan^{-1}\frac{u}{\sqrt{k}}$

8 $\frac{1}{2k}\ln\frac{5}{2}, \frac{1}{ck}\tan^{-1}\frac{1}{3}; \frac{1}{2k}\ln 2, \frac{\pi}{4kc}$ **9** $v = \sqrt{\left[u^2 - 2gr\left(1-\frac{r}{x}\right)\right]}$ **10** 44 s

11 9.803 ms^{-2} **12** 7.06 am **13** 7.40 am − 11.50 am; 0.034 m min^{-1}

Exercise 7.2

2 $\frac{V^2}{2g}\ln\frac{4}{3}, \frac{V}{2g}\ln 3$ **3** $\frac{g}{8k^2}\{8\ln 2 - 1\}$ **4** $\frac{4v_1}{\sqrt{7}}$

Exercise 7.3

2 $2h\cot\alpha, \sqrt{\left(\frac{2gh}{3}\right)}\operatorname{cosec}\alpha$ **5** $\sqrt{2}$ s, $(20-g)$ m

6 $X = (V_1\cos\theta_1 + V_2\cos\theta_2)\,T$, where T is the time to collision; 2 s, 52 m

7 $32\sqrt{10}\ ms^{-1}$, at angle of elevation $\tan^{-1}\frac{1}{3}$ **8** 1 s after projection of second particle

9 3 s, 90 m; $15\sqrt{5}\ ms^{-1}$, at angle of elevation $\tan^{-1}\frac{1}{2}$

10 $(U\sin\alpha)\,T = d$, where T is the time to collision; 2.5 s, 28.75 m

11 $3(2-\sqrt{2})$ m; $10(\sqrt{2}-1)\ ms^{-1}$ **12** $\sqrt{\left(\frac{dg}{\sin 2\alpha}\right)}, \frac{d}{U}\sin\alpha\operatorname{cosec}(\beta-\alpha)$

Exercise 7.4

1 $32\ \text{ms}^{-2}$, 4 **2** 32 N, $\frac{1}{4}g$ N; 20 rad s^{-1}. **3** $16\pi^2$ N, $\sqrt{20g}\ \text{ms}^{-1}$. **4** $g - \frac{1}{2}\pi^2$ N, π^2 N

6 $\frac{9mv^2}{8\ell}$, $m\left(g - \frac{3v^2}{8\ell}\right)$ **7** 15, 25; $r = 2$, $k = 480$; $0 \leqslant k \leqslant 80$

Exercise 7.5

1 $12g$ N, $\sqrt{g}$ rad s^{-1} **2** AB $\frac{1}{2}m\ell\omega^2 + mg$, BC $\frac{1}{2}m\ell\omega^2 - mg$, $\sqrt{\left(\frac{2g}{\ell}\right)}$ **3** $m = 2M$; $\sqrt{\left(\frac{g}{\ell}\right)}$

4 mg, $(2 - \sqrt{3})\,mg$, $\sqrt{\left[\frac{2g}{\ell}(2 - \sqrt{3})\right]}$ **5** $\frac{mMv^2}{g(m^2 - M^2)}$, $\cos^{-1}\left(\frac{M}{m}\right)$; $\frac{Mv^2}{g\sqrt{(m^2 - M^2)}}$

6 OA $10\,mg$, AB $6\,mg$, $\sqrt{\left(\frac{8g}{\ell}\right)}$

7 AB $2\omega^2 + \frac{2g}{\sqrt{3}}$, BC $2\omega^2 - \frac{2g}{\sqrt{3}}$, $2g - \omega^2\sqrt{3}$; the ring lifts off the ledge

Exercise 7.6

1 $14.4\ \text{ms}^{-1}$ **2** 24.7° **3** $11.8\ \text{ms}^{-1}$ **4** 14.3° **5** 28.4° **6** 18.9 mm **7** 38.3 mm

Diagnostic test 7

1 u^3, $\frac{3}{4}u^4$ **3** $4 - 2\ln 3$, $\ln 3$ **5** 0.994 m; gains 486 s; 1.005 m

6 6.29 pm and 10.11 pm; 7.59 am. **8** $\frac{V^2}{g}\ln 2 - \frac{V^2}{2g}$; $\frac{V}{g}\ln 2$ **9** $\tan^{-1}\frac{3}{4}$; $3\sqrt{3} - 4$ s

10 $\frac{6}{\sqrt{5}} - 2$ **13** 36 N; $16\ \text{ms}^{-1}$ **14** $r = 2$, $k = 160$; 10, 30; $0 < k \leqslant 40$

15 AP $\frac{5mv^2}{9\ell} + \frac{5mg}{8}$, BP $\frac{5mv^2}{9\ell} - \frac{5mg}{8}$ **16** $18 + \frac{20}{\sqrt{3}}$ N, $18 - \frac{20}{\sqrt{3}}$ N; $20 - 9\sqrt{3}$ N

18 $19.8\ \text{ms}^{-1}$

Further questions 7

1 $e^{-1/2} - e^{-1}$, $1 - e^{-1/2}$; $1 - 2e^{-1}$ **2** $\frac{1}{\sqrt{(2k)}}\left\{2\sqrt{3} - \sqrt{2} + \ln\frac{2 + \sqrt{3}}{1 + \sqrt{2}}\right\}$

5 $\frac{c}{g}\tan^{-1} 2$; $\frac{2c}{\sqrt{5}}$ **6** $\frac{V}{g}\{\tan^{-1} n + \ln(n + \sqrt{(n^2 + 1)})\}$; $V\sqrt{\left(\frac{n^2}{n^2 + 1}\right)}$

9 $\dfrac{2V^2 \cos\theta \sin(\theta - \alpha)}{g \cos^2 \alpha}$; $\dfrac{V^2}{g(1 + \sin\alpha)}$ **10** $\dfrac{V}{g}\sqrt{(V^2 + 2gh)}$

8 Harder 3 Unit Topics

Further questions 8

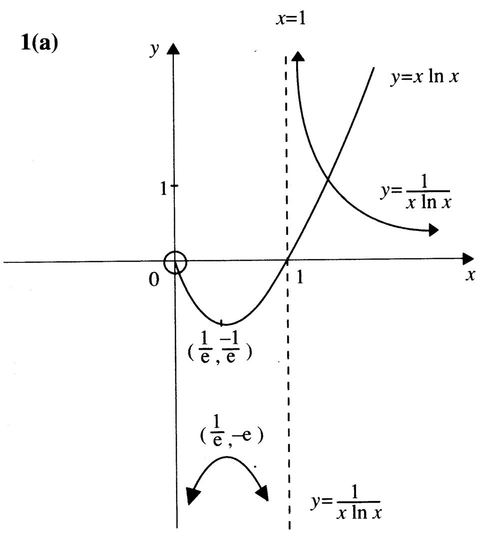
4 $\dfrac{1}{30}$; $36 \ln \dfrac{3}{2} - \dfrac{175}{12}$ **5** $\dfrac{1}{630}$; $\dfrac{22}{7} - \pi$

Appendix 1 Specimen Papers

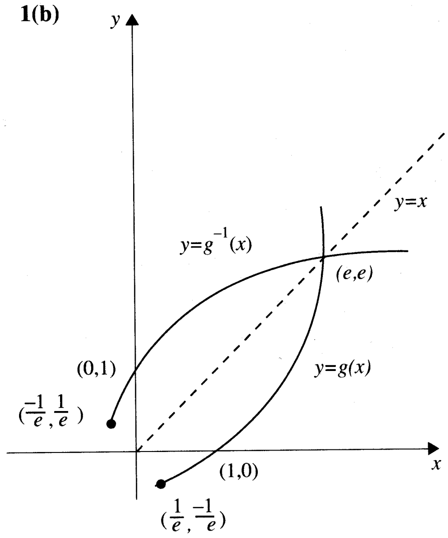
Specimen Paper 1

1 (see figure below); $\dfrac{1}{4}(e^2 + 1)$, $\dfrac{1}{2}(e^2 - 1)$

1(a)



1(b)

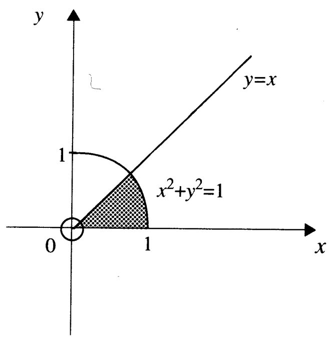


2 $e^x - \ln(e^x + 1)$; $\dfrac{1}{3}\sec^3 x - \sec x$; $\dfrac{1}{3}\ln 2$; $\pi\sqrt{2} - 4$ **3** Midpoint of AB

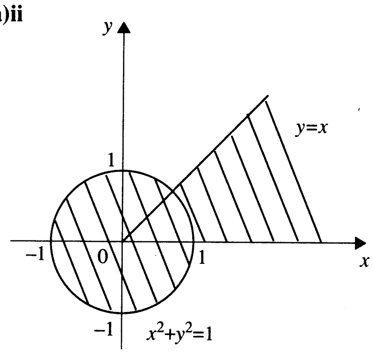
4 $a = \pm 1$, $b = \pm 2$, $z = 2 + i$, $1 - i$; $2 \operatorname{cis}\dfrac{\pi}{3}$, $2 \operatorname{cis}\left(-\dfrac{\pi}{3}\right)$, -1024; $y = 2x$, $(1, 2)$

5 (see figure below); $\frac{4\pi}{3}$ $(4\pi - 3\sqrt{3})$

5(a)i



5(a)ii

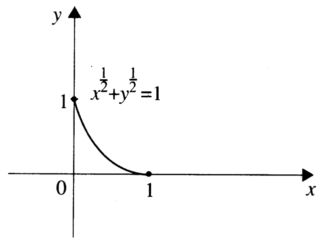


6 $mr\omega^2 \sin\theta + mg\cos\theta$, $mr\omega^2\cos\theta - mg\sin\theta$
7 $k = 1$, $-x + 1$; midnight to 3.10 am, 11.50 am to 4.10 pm

Specimen Paper 2

1 $\{x: 0 \leqslant x \leqslant 1\}$, $\{y: 0 \leqslant y \leqslant 1\}$, (see figure below); e, $k > e$

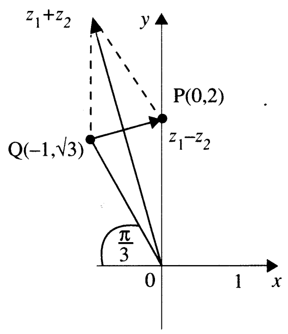
1(a)i



2 $b = 2$, $a = 1$, π; $x\tan x + \ln|\cos x|$; $\ln 2$, $\frac{\pi}{4}\ln 2$

4 $2\text{ cis }\frac{\pi}{2}$, $2\text{ cis }\frac{2\pi}{3}$; (see figure below); $\frac{7\pi}{12}$, $\frac{\pi}{12}$; $(0, 2)$, $2\sqrt{2}$; $a = 2$, $P(x) = (x + 1)(x - 1)(x + 2)$

4(a)

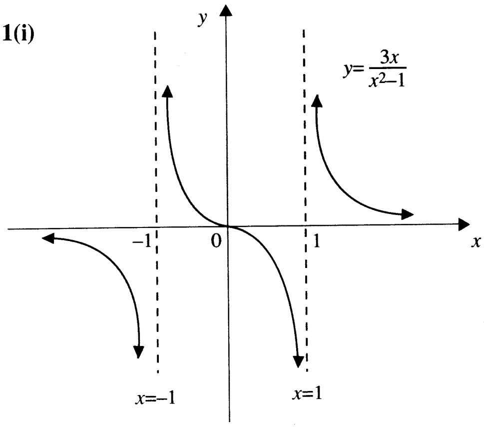


5 3π **6** $\sqrt{\left(\frac{g}{x}\right)}$, particle loses contact with floor; $\frac{1}{4}g$; $\frac{1}{4}\sqrt{5g}$ **7** 0, $a^{n-1}x - a^n$; 80 m

Specimen Paper 3

1 (see figure below); $x = -\frac{1}{2}$, 2, $-1 < x < -\frac{1}{2}$ or $1 < x < 2$; -3, $-3 < k < 0$; $\frac{3}{2} \ln \frac{3}{4}$

1(i)



2 $\ln\left|\frac{x+1}{\sqrt{(x^2+1)}}\right| + 2\tan^{-1} x$; π; $9e - 24$, $\frac{1}{32}(9e - 24)$ **3** $R\left(ct^9, \frac{c}{t^9}\right)$; $2c^2 xy = c^4 - y^4$

4 $z = 3 - i$, $3 + 3i$; -1, $\text{cis}\left(\pm\frac{\pi}{5}\right)$, $\text{cis}\left(\pm\frac{3\pi}{5}\right)$; $\cos\frac{\pi}{5} = \frac{1}{4}(1 + \sqrt{5})$, $\cos\frac{3\pi}{5} = \frac{1}{4}(1 - \sqrt{5})$

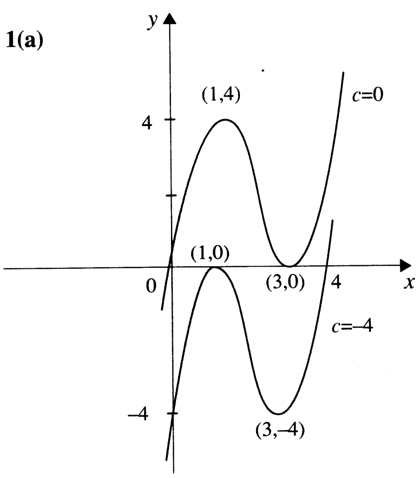
5 $m = -1$, $n = -2$; $3q$, pq; $\frac{64}{\sqrt{3}}$ **6** 18.1 ms^{-1}, $\frac{1}{2k}\ln(1 + u^2)$

8 $4\sqrt{6}$ cm, $\sqrt{2}$; $\frac{5}{324}, \frac{63}{64}, \frac{5}{192}, \frac{5}{216}$

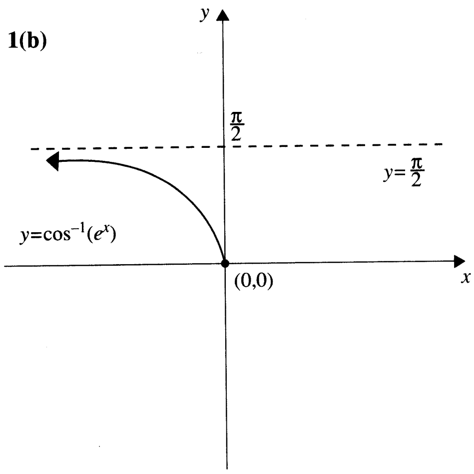
Specimen Paper 4

1 $x = 1$, $c = -4$ or $x = 3$, $c = 0$, (see figure below), $c < -4$ or $c > 0$; $\{x: x \leqslant 0\}$, $\left\{y: 0 \leqslant y < \frac{\pi}{2}\right\}$, (see figure below).

1(a)



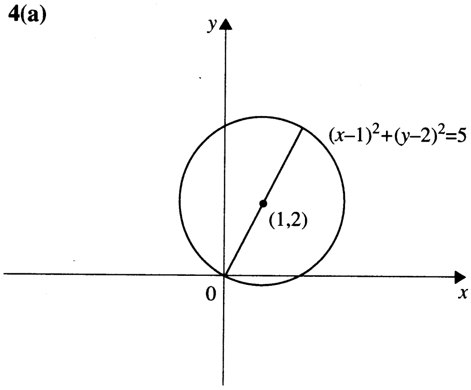
1(b)



2 $a = 1, b = 2, \frac{\pi}{8}; 2\sqrt{x}\ln x - 4\sqrt{x}; \frac{\ln 2}{\pi}$ **3** $x^2 + y^2 = 5$

4 $1 + 2i$, (see figure below), $2\sqrt{5}; -1, 1, \frac{\pi}{4} \leqslant \arg z \leqslant \frac{3\pi}{4}; x^3 + 3x^2 - 10x + 6 = 0, 1, -2 \pm \sqrt{10}$

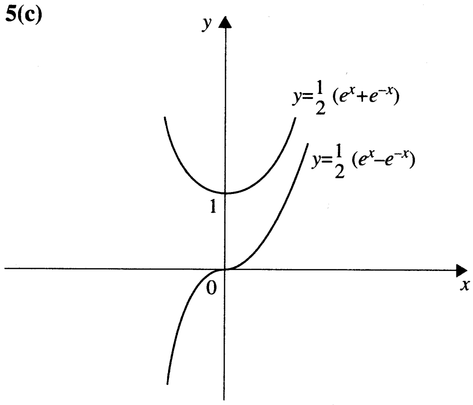
4(a)



5 $1 + 6ic - 15c^2 - 20ic^3 + 15c^4 + 6ic^5 - c^6, c = 0, \pm\frac{1}{\sqrt{3}}, \pm\sqrt{3}; 2 - i, 1 + 3i, b = -6, e = 50;$

(see figure below), $2\pi\left(1 - \frac{2}{e}\right)$

5(c)



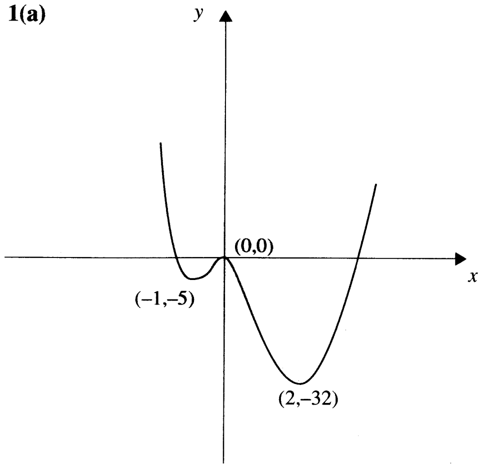
6 $\ddot{x} = -g - kv^2; \ddot{x} = g - kv^2, \sqrt{\left(\frac{g}{k}\right)}$ **7** $1 + \sqrt{2}, 2 + \sqrt{3}$; BD diameter

Specimen Paper 5

1 (see figure below); $k < -32; -32 < k < -5$ or $k > 0; -5 < k < 0; k = -13, 2 < \alpha < 3,$

$\sqrt{\left(\frac{13}{3\alpha}\right)}, \sqrt{\left(\frac{13}{9}\right)} < |z| < \sqrt{\left(\frac{13}{6}\right)}, \frac{1}{6} - \frac{\alpha}{2}, \frac{-4}{3} < \text{Re } z < \frac{-5}{6}$

1(a)



2 $x + \ln\left|\dfrac{(x-2)^4}{x-1}\right|$; $\sin x - \dfrac{1}{3}\sin^3 x$, $\dfrac{1}{2}(\sin^{-1} x)^2$; $2\ln 2 - 4 + \pi$

3 $2x - y = 3$; $(2, 1)$, $x + y = 3$; $\left(\dfrac{2}{3}, -\dfrac{5}{3}\right)$, $x - 5y = 9$

4 $64, \pi$; $\dfrac{1}{2}, \dfrac{\pi}{6}$; $32, \dfrac{-5\pi}{6}$; $p = 2$, $q = -1$, $(x + 1 + \sqrt{2})(x + 1 - \sqrt{2})(x^2 + 1)$,
$(x + 1 + \sqrt{2})(x + 1 - \sqrt{2})(x + i)(x - i)$

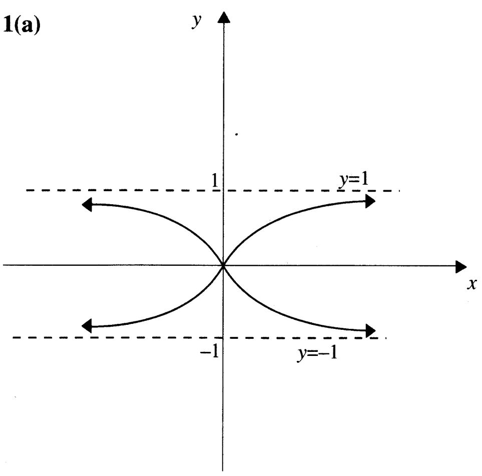
5 $32 + 47i$, $32 - 47i$, 61.53; $x^3 + 2x + 1 = 0$, $x^6 + 4x^4 + 4x^2 - 1 = 0$, $x^3 + 4x^2 + 4x - 1 = 0$; $\dfrac{8\pi}{3}$

8 $2\sqrt{7}$, $\dfrac{\sqrt{3}}{\sqrt{7}}$; $\dfrac{12}{91}$, $\dfrac{89}{91}$

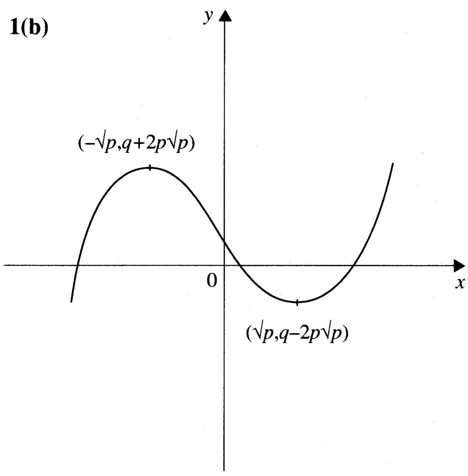
Specimen Paper 6

1 $y = \pm 1$, $y = \pm x$, (see figure below)

1(a)



1(b)



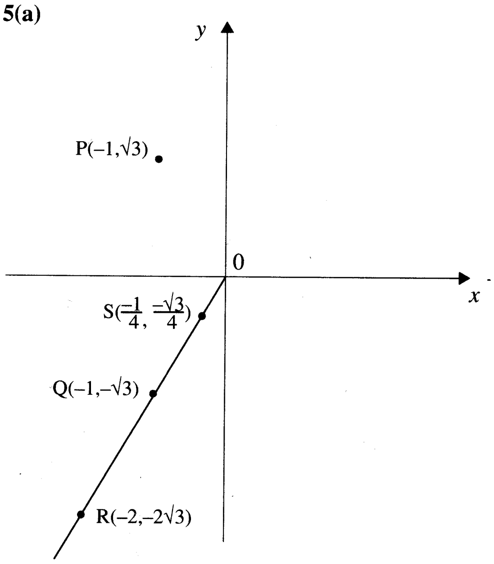
2 $\sqrt{2} \ln \left| \dfrac{x + 1 - \sqrt{2}}{x + 1 + \sqrt{2}} \right|$; 0

3 $Q\left(\dfrac{a \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}, \dfrac{b \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} \right)$, $R\left(\dfrac{a \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}, \dfrac{b \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} \right)$; $\sqrt{(a^2 + b^2)}$

4 $x^3 + 2px^2 + p^2x - q^2 = 0$; $\theta = \dfrac{(4n + 1)\pi}{16}$, n integral, $\tan \dfrac{\pi}{16}$, $\tan \dfrac{5\pi}{16}$, $-\tan \dfrac{3\pi}{16}$, $-\tan \dfrac{7\pi}{16}$

5 $2 \text{ cis } \dfrac{2\pi}{3}$ (see figure below); $-3 + i$, $-1 + 7i$; $\dfrac{32\pi}{3}$

5(a)



6 $\sqrt{\left(\dfrac{R}{g}\right)\left(\dfrac{\pi}{2} + 1\right)}$

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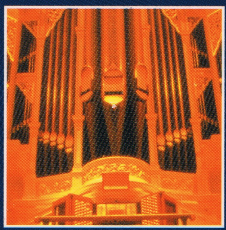
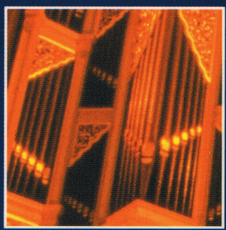


This comprehensive 4-Unit book has been purpose written to meet the needs of New South Wales HSC students studying the revised syllabus for 4-Unit Mathematics. It includes very clearly written theory and fully worked examples that enable students to work independently. There are extension questions at the end of each chapter, providing an ample set of challenges for advanced students. Also included are diagnostic tests and six specimen HSC papers, so students using this book will be fully equipped to prepare for their examinations.



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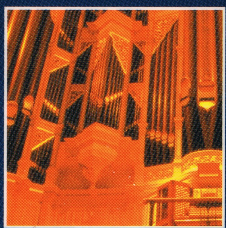
- relevant theory, presented in a clear, easy to follow manner;
- teaching by example;
- numerous worked examples and practice questions.



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