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THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education, Engineering and Science

MATH1902

Linear Algebra (Advanced)

June/July 2002

Time Allowed: One and a half hours

Lecturers: TM Gagen, DJ Ivers

This Examination has 3 Printed Components.

(1) An Extended Answer Question Paper (this booklet, Green 80/16A): 3 pages numbered 1 to 3; 5 questions numbered 1 to 5.

(2) A Multiple Choice Question Paper (Yellow 80/16B): 4 pages numbered 1 to 4; 15 questions numbered 1 to 15.

(3) A Multiple Choice Answer Sheet (White 80/16C): 1 page.

Components 2 and 3 MUST NOT be removed from the examination room.

This Examination has 2 Sections: ***Extended Answer*** *and* ***Multiple Choice****.*

The ***Extended Answer Section*** *is worth 75% of the total marks for the paper: all questions may be attempted; questions are of equal value; working must be shown.*

The ***Multiple Choice Section*** *is worth 25% of the total marks for the paper: all questions may be attempted; questions are of equal value; answers must be coded onto the* ***Multiple Choice Answer Sheet****.*

Calculators will be supplied; no other electronic calculators are permitted.

1. (*i*) Find the parametric vector equation of the line ℓ through the points $P(2,3,-1)$ and $Q(3,2,-3)$.

(*ii*) Write down the vector equation of the line in (*i*) in cartesian form.

(*iii*) Consider the three points $A(1,2,1)$, $B(4,3,2)$ and $C(3,-2,4)$. Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and hence find $\mathbf{u} = \overrightarrow{AB} \times \overrightarrow{AC}$.

(*iv*) Using the results of the previous part or otherwise find the equation of the plane through the points A, B and C.

(*v*) Show that the line in part (*i*) is perpendicular to the plane in part (*iv*).

2. (*i*) Suppose that $\mathbf{p} = \overrightarrow{OP} = 2\mathbf{i} - 3\mathbf{j}$ and that $Q = (5,1)$ are points in the plane. Find the vector equation of the tangent at the point Q to the circle with centre P and radius 5.

(*ii*) A tetrahedron has four vertices O, A, B and C and four triangular faces. Let K, L, M, N respectively divide the sides OA, OB, CB and CA in the ratio $\alpha : \beta$. Let $\mathbf{a} = \overrightarrow{OA}$, etc.

(a) Express $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(b) Show that KM and LN intersect at their midpoints.

3. (*i*) Give an example of a system of 3 linear equations in 2 unknowns, x and y, which is:

(a) consistent with a unique solution;

(b) consistent with infinitely many solutions.

(*ii*) Determine a row-echelon form for the augmented matrix of the system of equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & a^2z & = & 2 \\ x & + & 3y & + & 4z & = & 1 \\ 2x & + & ay & + & 8z & = & 6 \end{array}$$

for the values of a for which there is a unique solution. (Do not find the unique solution.)

(*iii*) Show that the equations are inconsistent if $a = -2$. Is this the only value of a for which the equations are inconsistent?

4. (i) Find infinitely many matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $A^2 = I$.

(ii) Use elementary row operations to show that

$$\det\begin{pmatrix} a+b & a & a & a \\ a & a+b & a & a \\ a & a & a+b & a \\ a & a & a & a+b \end{pmatrix} = \det\begin{pmatrix} 4a+b & 4a+b & 4a+b & 4a+b \\ a & a+b & a & a \\ a & a & a+b & a \\ a & a & a & a+b \end{pmatrix}.$$

(iii) Hence use elementary row operations to evaluate

$$\det\begin{pmatrix} a+b & a & a & a \\ a & a+b & a & a \\ a & a & a+b & a \\ a & a & a & a+b \end{pmatrix}.$$

(iv) Hence find the eigenvalues and eigenspaces of the matrix

$$J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

5. (i) If $\mathbf{v}$ is an eigenvector of the matrix A corresponding to the eigenvalue λ, show that $\mathbf{v}$ is an eigenvector of $A + aI$ for any scalar a and find the eigenvalue.

(ii) If $\mathbf{v}$ is an eigenvector of the matrix A corresponding to the eigenvalue λ, show that $\mathbf{v}$ is an eigenvector of bA for any scalar b and find the eigenvalue.

(iii) Let A and $\mathbf{v}$ be the $n \times n$ and $n \times 1$ matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \dots & 1 & 0 & 1 \\ 0 & & \dots & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} \sin\theta \\ \sin 2\theta \\ \sin 3\theta \\ \vdots \\ \sin n\theta \end{pmatrix}.$$

Given that $\sin(j-1)\theta + \sin(j+1)\theta = 2\cos\theta\ \sin j\theta$, show that $\mathbf{v}$ is an eigenvector of A if $\theta = k\pi/(n+1)$, where k is an integer, $1 \le k \le n$, and find the corresponding eigenvalue.

(iv) Find all the eigenvalues and eigenvectors of the matrix

$$M = bA + aI = \begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix}.$$

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