The directional derivative of $f(x,y)$ in the direction of a vector $\hat{\underline{u}}$ (unit vector)

$$D_{\hat{\underline{u}}} f = \nabla f \cdot \hat{\underline{u}}$$

where $\nabla f = f_x i + f_y j$

The directional derivative of $f(x,y,z)$ is

$$D_{\hat{\underline{u}}} f = \nabla f \cdot \hat{\underline{u}}$$

where $\nabla f = f_x i + f_y j + f_z k$.

On the RHS, $\hat{\underline{u}}$ must be a unit vector. We may define

$$D_{\hat{\underline{u}}} f$$

to be the directional derivative of f in the direction of a non-unit vector $\underline{u}$, but

$$D_{\underline{u}} f \equiv D_{\hat{\underline{u}}} f, \quad \hat{\underline{u}} = \frac{\underline{u}}{|\underline{u}|}$$

E.g. 2010 exam Q2.(a)

$$f: \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \mapsto \tan^{-1}(x^2+3y^2)$$

P is the point $(2,1)$

(i) The directional derivative of f at P in the direction of $\underline{u} = 4\underline{i} - \underline{j}$

$D_{\underline{u}} f$ at $(2,1)$

$$D_{\underline{u}} f = D_{\hat{\underline{u}}} f = \nabla f \cdot \hat{\underline{u}} = \nabla f \, \frac{\langle 4, -1 \rangle}{\sqrt{17}}$$

$$\nabla f = \frac{\partial f}{\partial x} \underline{i} + \frac{\partial f}{\partial y} \underline{j}$$
$$= \left\langle \frac{2x}{1+(x^2+3y^2)}, \frac{6y}{1+(x^2+3y^2)} \right\rangle$$
$$= \left\langle \frac{2}{25}, \frac{3}{25} \right\rangle$$

$$D_{\hat{u}}f = \nabla f \cdot \hat{u} = \frac{\langle \frac{2}{25}, \frac{3}{25} \rangle \langle 4, -1 \rangle}{\sqrt{17}} = \frac{5}{5\cancel{25}\sqrt{17}} = \frac{1}{5\sqrt{17}}$$

<u>Maximum directional derivative</u>

Suppose f and (x_0, y_0) are given and $\hat{u}$ varies: directional derives in different direction

$$\begin{aligned} D_{\hat{u}}f &= \nabla f \cdot \hat{u} \\ &= |\nabla f| |\hat{u}| \cos\theta \\ &= |\nabla f| \cos\theta \end{aligned}$$

where θ is the angle between the vectors ∇f and $\hat{u}$
$|\nabla f|$ is fixed here. θ is varying.
$D_{\hat{u}}f$ is maximised when $\theta = 0$: $\hat{u}$ is in the same direction as ∇f.

$$\hat{u} = \frac{\nabla f}{|\nabla f|}$$

Then the value of the maximum directional derivative is

$$D_{\hat{u}}f \quad \left(\text{where } \hat{u} = \frac{\nabla f}{|\nabla f|}\right)$$

$$= \nabla f \cdot \frac{\nabla f}{|\nabla f|}$$

$$= |\nabla f|^2 / |\nabla f|$$

$$= |\nabla f|$$

The value of the maximum directional derivative is

$$\frac{\partial f}{\partial n} = |\nabla f|$$

The notation on RHS is called <u>normal</u> derivative.

Consider the family of level curve of $f(x,y)$:

$$f(x,y) = \text{constant}$$

The curve through (x_0, y_0) has equation

$$f(x,y) = f(x_0, y_0)$$

This can solved implicitly for y as a function of x near x_0 provided $\partial f/\partial y \neq 0$ at x_0.

The slope of tangent is the implicit derivative

$$\frac{dy}{dx} = -\frac{f_x}{f_y}\bigg|_{(x_0,y_0)}$$

The vector $\underline{i} - \frac{f_x}{f_y}\underline{j}$ is tangent to the level curve. So also $f_y\underline{i} - f_x\underline{j}$.
(This allow vertical tangents also)

Consider $\underline{I} = f_y\underline{i} - f_x\underline{j}$

$$D_{\underline{I}} f = \nabla f \cdot \hat{\underline{I}}$$

$$= (f_x\underline{i} + f_y\underline{j})\frac{(f_y\underline{i} - f_x\underline{j})}{\sqrt{f_x^2 + f_y^2}}$$

$$= 0$$

The directional derivative of f is zero in the direction of a tangent to a level curve.

The normal to the level curve is

$$\nabla f = f_x\underset{\sim}{i} + f_y\underset{\sim}{j} = \underset{\sim}{n}$$

$(\underset{\sim}{n} \perp \underset{\sim}{T},\ f_x i + f_y j \perp f_y i - f_x j)$

The direction of maximum directional derivative is normal to the level curves.
The normal derivative is the derivative in the direction of the normal.

$$\frac{\partial f}{\partial n} = \nabla f \cdot \hat{\underset{\sim}{n}} = |\nabla f|$$

Return to

$$D_{\underset{\sim}{u}} f = |\nabla f| \cos\theta$$

max when $\theta = 0$: $\hat{\underset{\sim}{u}} = \hat{\underset{\sim}{n}} = \frac{\nabla f}{|\nabla f|}$

min when $\theta = \pi$: $\hat{\underset{\sim}{u}} = -\hat{\underset{\sim}{n}}$

Zero when $\theta = \pm\frac{1}{2}\pi$: $\hat{\underset{\sim}{u}} = \pm\hat{\underset{\sim}{T}}$ tangent.

$\boxed{\times}$ Level surfaces of functions of 3 variables

Suppose

f: part of $R^3 \to R$

$(x, y, z) \longrightarrow f(x, y, z)$

- The directional derivative is

$\nabla f \cdot \hat{\underset{\sim}{u}} = |\nabla f| \cos\theta$

$\Rightarrow$

where θ is the angle between

$$\nabla f = \langle f_x, f_y, f_z \rangle \quad \text{and} \quad \hat{u}$$

$D_{\hat{u}} f$ is maximised when $\theta = 0$.

when $\hat{u} = \frac{\nabla f}{|\nabla f|}$ then

$$D_{\hat{u}} f \big|_{max} = |\nabla f|$$

we want to show that this

$$\frac{\partial f}{\partial n} = \nabla f \cdot \hat{n}$$

when $\hat{n}$ is normal to the level surface of f.

Start with

$$f(x, y, z) = \text{constant}$$

Let this define z implicitly in terms of x and y near a point where $f_z \neq 0$

Implicitly differentiate w.r.t x:

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial z}\frac{dz}{dx} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$$

similarly

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

On the surface we have tangent lines in the direction of

$$i - \frac{f_x}{f_z} k$$

$$j - \frac{f_y}{f_z} k$$

or

$f_z i - f_x k$
$f_z j - f_y k$
(This allow vertical tangent)

The normal to these two vectors is

$(f_z i - f_x k) \times (f_z j - f_y k)$

$= f_x f_z i + f_y f_z j + f_{x}^{2} k$

$= \langle f_z | f_x, f_z | f_y, f_z | f_z \rangle$

So

$\nabla f = \langle f_x, f_y, f_z \rangle$

is normal to the surface.

Hence, the maximum directional derivative of

$D_u f$ is the normal derivative.

$$\frac{\partial f}{\partial n} = \nabla f \cdot \hat{\underset{\sim}{n}} = |\nabla f|$$

The normal at (x_0, y_0, z_0) can also be seen from the tangent plane. Consider the case.

$$z = f(x, y)$$

The equation of the tangent plane at (x_0, y_0) is

$$z = f(x_0, y_0) + f_x (x_0, y_0)(x - x_0) + f_y (x_0, y_0)(y - y_0)$$

from the equation of a plane in Cartesian words, we can read off the normal.

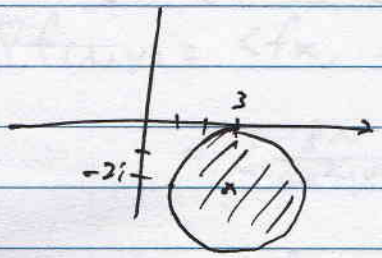
$\langle -f_x, -f_y, 1\rangle$ at (x_0, y_0) This is the gradient of the scalar $z - f(x,y)$

$\nabla(z-f(x,y)) = \langle -f_x, -f_y, 1\rangle$

This method works for $f(x,y,z)=c$ using implicit differentiation.

2011 exam

Q1 (a) Sketch the set $|z-3+2i| \leq 2$ [closed disk]



1(b) Factorise

$P(z) = z^4 - 5z^3 + 5z^2 + 4z + 10$.

given that $3-i$ is one of the root

$3+i$ is a root too,

$3-i \Rightarrow$ also $3+i$ is a root.

$z-(3-i)$ and $z-(3+i)$ are factors.

$[z-3+i][z-3-i] = z^2-6z+10$

method I.

$$\begin{array}{r|l} & \quad\quad\quad z^2 + z + 1 \\ \hline z^2-6z+10 & z^4 - 5z^3 + 5z^2 + 4z + 10 \\ & z^4 - 6z^3 + 10z^2 \\ \hline & \quad\quad z^3 - 5z^2 + 4z \\ & \quad\quad z^3 - 6z^2 + 10z \\ \hline & \quad\quad\quad z^2 - 6z + 10 \\ & \quad\quad\quad z^2 - 6z + 10 \\ \hline & \quad\quad\quad\quad\quad 0 \end{array}$$

$P(z) = (z^2-6z+10)(z^2+z+1)$

method 2

$$z^4-6z^3+5z^2+4z+10 = (z^2-6z+10)(z^2+az+b)$$

$10 = 10b \Rightarrow b=1$

$a-6=5 \Rightarrow a=1$

$\therefore \quad z^2+az+b = z^2+z+1$

$P(z) = (z^2-6z+10)(z^2+z+1)$

1 (c) Show that $f(x)=\frac{\sin x}{x}$ is decreasing on $(0,\pi]$.
Hints: $\cos x < \frac{\sin x}{x} < 1$ on $(0, \frac{1}{2}\pi]$

Proof:

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

< 0 on $(0, \frac{1}{2}\pi)$

by the hint,

on $[\frac{1}{2}\pi, \pi]$ both terms are ≤ 0

$\therefore \ f'(x) < 0$ on $[0, \pi]$

$\therefore \ f(x)$ is decreasing on $(0, \pi]$

2(a) $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$

$(x,y) \longmapsto \ln(4x^2+y^2)$

P point $(1,2)$

Solution: $z = f(x,y)$

$= \ln(4x^2+y^2)$

(i) directional derivative of z at P in the direction of $\underline{u} = \langle 3,-2 \rangle$

Solution: $D_u f|_{(1,2)} = \nabla f \cdot \hat{u}$

$\nabla f_{(x,y)} = \langle f_x, f_y \rangle$ $\qquad\qquad \hat{u} = \frac{1}{\sqrt{13}}\langle 3,-2 \rangle$

$= \left\langle \frac{8x}{4x^2+y^2}, \frac{2y}{4x^2+y^2} \right\rangle$

$\nabla f_{(1,2)} = \left\langle \frac{8}{8}, \frac{4}{8} \right\rangle$

$= \left\langle 1, \frac{1}{2} \right\rangle$

$\therefore$ At $P(1,2)$, $D_u f|_{(1,2)} = \langle 1, \frac{1}{2} \rangle (\frac{1}{\sqrt{13}}) \langle 3,-2 \rangle$

$= \frac{1}{\sqrt{13}}(3-1)$

$= \frac{2}{\sqrt{13}}$

(ii) Unit vector $\hat{v}$ in the direction of the maximum directional derivative, and the value of the maximum directional derivative.

Solution

$\hat{v} = \frac{\nabla f}{|\nabla f|} = \frac{2}{\sqrt{5}}\langle 1, \frac{1}{2} \rangle$

Value $\frac{\partial f}{\partial n} = |\nabla f| = \frac{1}{2}(\sqrt{5})$

(iii) Equation of the tangent plane at P.

$z = ax + by + c$

solution

$\Delta z = f_x \Delta x + f_y \Delta y$

$z - z_0 = f_{x(x_0,y_0)}(x - x_0) + f_{y(x_0,y_0)}(y - y_0)$

$z - \ln 8 = 1(x-1) + \frac{1}{2}(y-2)$

$z = x + \frac{1}{2}y + 3\ln 2 - 2$

2(b) Any method to calculate Taylor polynomial $T_4(x)$ for $f(x) = e^{2x}\cos(3x)$

~~Sub int $T_4(x)$ of e^{2x}~~

~~Sub in $T_4(x)$ of $\cos(3x)$~~.

~~Diff.~~ Sol.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$x \to 2x$:

$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots$ T_4 the power of x

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} = \dots$

$\cos(3x) = 1 - \frac{9x^2}{2} + \frac{27}{8}x^4 - \dots$

Multiply. truncating at x^4:

$T_4(x) =$ for $e^{2x}\cos(3x)$

$= (1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4)(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \dots)$

$= 1 + 2x + (2 - \frac{9}{2})x^2 + (\frac{4}{3} - 9)x^3 + (\frac{2}{3} - 9 + \frac{27}{8})x^4 + \dots$

$= 1 + 2x - \frac{5}{2}x^2 + \left(\frac{-23}{3}\right)x^3 + \left(\frac{-119}{24}\right)x^4 + \dots$

Q3(a) Show that

$$g(x) = \ln(2x) - \ln(1+\sqrt{1+x^2})$$

has one and only one zero on $[1, 10]$

Method I [direct. solution not intended sol]

$$\ln(2x) - \ln(1+\sqrt{1+x^2}) = 0$$

$$\frac{2x}{1+\sqrt{1-x^2}} = 1$$

$$2x = 1+\sqrt{1+x^2}$$

$$(2x-1)^2 = 1+x^2$$

$$4x^2-4x+1 = 1+x^2$$

$$3x^2-4x = 0$$

$$x(3x-4) = 0$$

$x=0$ is not admissible.

The unique root is $x = 4/3$

It is on $[1, 10]$

Method II. Intermediate Value Theorem.

IVT

$$g(1) = \ln(2) - \ln(1+\sqrt{2}) \quad \cancel{\geq 0} < 0$$

$$= \ln 2 - \ln(2.414)$$

$$g(10) = \ln(20) - \ln(1+\sqrt{101}) \quad > 0$$

The sign change (and continuity of $g(x)$ $\Rightarrow$ at least one root on $[1, 10]$.

To prove that the root is unique., take derivative.

$$g'(x) = \frac{1}{x} - \frac{\frac{x}{\sqrt{1+x^2}}}{1+\sqrt{1+x^2}} = \frac{1}{x} - \frac{x}{1+x^2+\sqrt{1+x^2}} > \frac{1}{x} - \frac{x}{1+x^2} > 0$$

$\therefore$ $g(x)$ is increasing for $x>0$

M2: Cancelling common factors.

$$\lim_{x \to 3} \frac{x^3 + x^2 - 33x + 63}{x^3 - 27x + 54} \qquad (x-3)$$

Q3 (b) $\lim_{x\to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$

Hint: replace $\sinh x$ with its taylor polynomial $T_3(x)$ about $x=0$.

Method 1 Taylor.

$T_3(x)$ for $\sinh x$

$= x + \frac{x^3}{6}$

$T_2(x)$ for $\frac{\sinh(x)}{x} = 1 + \frac{x^2}{6}$

Higher power do not contribute to the limit so.

$\lim_{x\to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2} = \lim_{x\to 0} \left(1+\frac{x^2}{6}\right)^{1/x^2}$

$= \lim_{n\to\infty} \left(1+\frac{1}{6n}\right)^n$ $n = 1/x^2$

$e^0 = (1+\frac{1}{n})^n$

$= e^{1/6}$

Method 2 ~~Tay~~ L'Hôpital

Let $L = \lim_{x\to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$.

Then $\ln L = \lim_{x\to 0} \frac{\ln\left(\frac{\sinh x}{x}\right)}{x^2}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{\ln \sinh x - \ln x}{x^2}$

$= \lim_{x\to 0} \frac{\coth x - \frac{1}{x}}{2x}$

$= \lim_{x\to 0} \frac{x\cosh x - \sinh x}{2x^2 \sinh x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{x \sinh x}{2x^2\cosh x + 4x\sinh x}$

$= \lim_{x\to 0} \frac{\sinh x}{2x\cosh x + 4\sinh x}$

$= \lim_{x\to 0} \frac{\cosh x}{2x\sinh x + 6\cosh x}$

$= \frac{1}{6}$

$\therefore L = e^{1/6}$

3(b) (iii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+xy+y^3}{x^2+y^2}$ prove Does not exist
even then to prove exist.

show fast the numerator goes to zero.

Consider 2 paths
The term $xy \to 0$ at about the the same speed as the denominator.

we will prove the limit does not exist.

Look for 2 paths to $(0,0)$ on which different limit occurs.

x-axis $y=0$

$$\lim_{x\to 0} \frac{x^3+0+0}{x^2+0}$$

$$= \lim_{x\to 0} x$$

$$= 0$$

y-axis not considered due to symmetry.

$y=x$

$$\lim_{x\to 0} \frac{x^3+x^2+x^3}{2x^2}$$

$$= \lim_{x\to 0} \frac{1+2x}{2}$$

$$= \frac{1}{2}$$

Different limits $\Rightarrow$ the original limit does not exist.

4 (a) Use $T_3(x)$ for $\sin x$ and $R_3(x)$ to prove.
(about $x=0$) to prove.

$$x-\frac{x^3}{6}<\sin x$$

at least for $0<x<\pi$

$\sin x = T_3(x) + R_3(x)$ where

$$T_3(x) = x-\frac{x^3}{3!} = x-\frac{x^3}{6}$$

$$R_3(x) = \frac{f^{(4)}(c)}{4!}x^4$$

$$= \frac{d^4}{dx^4}\sin x\Big|_{x=c}$$

$$R_3(x) = \frac{\sin(c)}{4!}x^4.$$

$0<c<x.$

Note: If $0<\cancel{x}\leq\pi,$
then $0<c<\pi$ and
$\sin c>0.$
$\therefore R_3(x)>0, \quad 0<x<\pi.$

This implies $x-\frac{x^3}{6}<\sin x$ at least for $0<x\leq\pi.$

4(b). We want to upgrade

$$x - \frac{x^3}{6} < \sin x$$

to all positive x.

so far proved for $0 < x \leq \pi$

Consider $x \geq 3$ (which overlaps $0 < x \leq \pi$)

at $x = 3$. $x - \frac{x^3}{6} = 3 - \frac{27}{6} = -\frac{3}{2} < -1$

$\sin x \geq -1$

$\therefore \sin x > x - \frac{x^3}{6}$ at $x = 3$

for $x \geq 3$, $x - \frac{x^3}{6}$ is decreasing because $1 - \frac{x^2}{2} < 0$

$\therefore x - \frac{x^3}{6} < \sin x$

for all $x \geq 3$

together with part (a)

$x - \frac{x^3}{6} < \sin x$

for all $x > 0$

We also want to show

$\sin x < x$

for all $x > 0$

In lectures

$\sin x < x < \tan x, \quad 0 < x < \frac{\pi}{2}$

for $x \geq \frac{\pi}{2}$, LHS ≤ 1, RHS > 1.5.

$\therefore \sin x < x$ for all $x > 0$.

$\frac{d}{dx}(x - \sin x) = 1 - \cos x \geq 0$

$\therefore x - \sin x$ is increasing for all $x \in \mathbb{R}$

0 at origin $\Rightarrow > 0$ for $x > 0$

$\therefore x - \sin x > 0$ for all $x > 0$

4(c) explain briefly why $-\frac{1}{6}$ cannot be replaced by a number between $-\frac{1}{6}$ and 0.

The part (a) shows $T_3(x) < \sin x < T_5(x)$ at least for $0 < x < \frac{\pi}{2}$

$x - \frac{x^3}{6} < \sin x$

we want to replace $T_3(x)$ by

$x - \alpha x^3$

$x - \alpha x^3 < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$

$-\alpha < -\frac{1}{6} + \frac{x^2}{120}$

$\alpha > \frac{1}{6} - \frac{x^2}{120}$

when x is small, this forces

$\alpha \geq \frac{1}{6}$.

so we cannot replace $x - \frac{1}{6}x^3$ with $x - \alpha x^3$ and with $\frac{1}{6} < \alpha < 0$.

$$R_n(x) = \frac{f^{n+1}(c)}{n+1!}(x)^{n+1}$$

3(b) limits

$$\lim_{x \to 3} \frac{x^3+x^2-33x+17}{x^3-27x+54} \quad \left(\frac{0}{0}\right)$$

method I L'Hôpital's Rule.