THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 8

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Intermediate Value Theorem.
- ☐ Global maximum and minimum values of a function.
- ☐ Extreme Value Theorem.
- ☐ Differentiation and the differentiation laws.
- ☐ Rolle's Theorem.

Outcomes

After completing this tutorial you should

- ☐ understand and be able to apply the Intermediate Value Theorem;
- ☐ understand the conditions under which the Extreme Value Theorem guarantees existence of global max/min;
- ☐ understand and apply the definition of differentiability;
- ☐ be able to use the Intermediate Value Theorem to prove existence of solutions;
- ☐ be able to use Rolle's Theorem to prove existence of solutions;

Summary of essential material

Intermediate Value Theorem: If $f\colon [a,b]\to\mathbb{R}$ is continuous on the and if either $f(a)<k<f(b)$ of $f(b)<k<f(a)$, then there exists $c\in(a,b)$ with $f(c)=k$. It is important that $[a,b]$ be a closed interval.

Extreme Value Theorem: If $f\colon [a,b]\to\mathbb{R}$ is continuous, then there exist $c,d\in[a,b]$ such that $M:=f(c)$ is the global maximum value of $f(x)$ on $[a,b]$, and $m:=f(d)$ is the global minimum value of $f(x)$ on $[a,b]$. It is essential that $[a,b]$ is a closed and bounded interval!

Derivatives: The derivative of a function f at x_0 is the slope of the *tangent* to the graph of f at the point $(x_0,f(x_0))$. If it exists, that slope is defined to be the limit of the slopes of the secants through $(x_0,f(x_0))$ and $(x,f(x))$ as $x\to x_0$. The slope of the secants,

$$m_{x_0}(x)=\frac{f(x)-f(x_0)}{x-x_0},$$

is called the *difference quotient*. If it exists, the limit of $m_{x_0}(x)$ as $x\to x_0$ is called the *derivative* of f at x_0 and is denoted by $\dfrac{df}{dx}(x_0)$ or $f'(x_0)$, and f is called *differentiable* at x_0. Alternatively we can say $f\colon(a,b)\to\mathbb{R}$ is differentiable at x_0 if there exists a function $m_{x_0}\colon(a,b)\to\mathbb{R}$ that is continuous at x_0 and

$$f(x)=f(x_0)+m_{x_0}(x)(x-x_0)$$

for all $x\in(a,b)$. In that case $f'(x_0)=m_{x_0}(x_0)$. (This is *Carathéodory's characterisation* from about 1950).

Differentiation Laws: If $f(x)$ and $g(x)$ are differentiable, then the following functions are also differentiable, with derivatives as stated:

(1) $(\alpha f)'=\alpha f'$ for $\alpha\in\mathbb{R}$

(2) $(f+g)'(x)=f'+g'$

(3) $(fg)'=f'g+fg'$ (product rule)

(4) $\left(\frac{f}{g}\right)'=\dfrac{f'g-fg'}{g^2}$ (if $g'(x)\neq 0$, quotient rule)

(5) $(f\circ g)'(x)=g'(x)f'\big(g(x)\big)$ (chain rule)

Rolle's Theorem/Mean Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous and $f : (a, b) \to \mathbb{R}$ differentiable.

Rolle's Theorem: If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem: There exists $c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

Questions to complete during the tutorial

1. Use the definition of the hyperbolic functions to show that $\dfrac{d}{dx} \cosh x = \sinh x$ and $\dfrac{d}{dx} \sinh x = \cosh x$.

Solution: Using that $\frac{d}{dx} e^{ax} = ae^x$ we have

$$\frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} \sinh x$$

and

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} \cosh x.$$

2. Let $f(x)$ be differentiable for all $x \in \mathbb{R}$. Suppose that $f(0) = -3$ and that $f'(x) \leq 5$ for all $x \in \mathbb{R}$. Use the Mean Value Theorem to show that $f(2) \leq 7$.

Solution: We are given that f is differentiable on $\mathbb{R}$, and therefore f is continuous on $\mathbb{R}$. In particular, f is continuous on the closed interval $[0, 2]$ and differentiable on the open interval $(0, 2)$, as is required for application of the Mean Value Theorem for $f(x)$ on the interval $[0, 2]$. Applying the Mean Value Theorem to $[0, 2]$, we see that there is a number c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{f(2) + 3}{2}.$$

Therefore $f(2) = 2f'(c) - 3 \leq 2 \times 5 - 3 = 7$.

3. Use the Intermediate Value Theorem to prove that if $f : [0, 1] \to [0, 1]$ is continuous, then there is exists $c \in [0, 1]$ such that $f(c) = c$.

Solution: If $f(0) = 0$ or $f(1) = 1$ we are done. So suppose that $f(0) \neq 0$, so that $0 < f(0) \leq 1$, and that $f(1) \neq 1$, so that $0 \leq f(1) < 1$. Let $g(x) = x - f(x)$. Then g is continuous on $[0, 1]$,

$$g(0) = 0 - f(0) < 0 \quad (\text{as } f(0) > 0))$$

and

$$g(1) = 1 - f(1) > 0 \quad (\text{as } f(1) < 1).$$

By the intermediate value theorem, there exists a number $c \in (0, 1)$ such that $g(c) = 0$, that is, $f(c) = c$.

Remark: Note that we invented a function $g(x)$ to make the proof work nicely. This is a common theme – it is easier to think about things if we are trying to show that $g(x) = 0$ has a solution, rather than if $f(x) = x$ has a solution (however, of course, they are equivalent).

4. Briefly explain why Rolle's theorem implies that there is a stationary point between any two zeros of a differentiable function. Hence use the Intermediate Value Theorem to show that the equation

$$x^2 - x \sin x - \cos x = 0$$

has exactly two distinct real solutions.

Solution: If $x_1 < x_2$ satisfy $f(x_1) = f(x_2) = 0$, then Rolle's theorem implies that there exists $c \in (x_1, x_2)$ such that $f'(c) = 0$, that is, c is a stationary point between x_1 and x_2.

Let $f(x) = x^2 - x \sin x - \cos x$. This function is both continuous and differentiable everywhere. We determine the stationary points by looking at the first derivative

$$f'(x) = 2x - \sin x - x \cos x + \sin x = x(2 - \cos x).$$

Thus the only critical point is at $x = 0$. Now observe that $f(-\pi) = \pi^2 + 1 > 0$, $f(0) = -1 < 0$ and $f(\pi) = \pi^2 + 1 > 0$. By the Intermediate Value Theorem, there must be a solution in $(-\pi, 0)$ and another solution in $(0, \pi)$. That is, the equation has exactly two solutions, since if there were more, then between each pair their would be another stationary point. However, there is noly one stationary point.

5. Use the Mean Value Theorem to prove the following inequalities:

(a) $\sinh x > x$ for all $x > 0$.

Solution: Apply the Mean Value Theorem to $f(t) = \sinh t$ on the interval $[0, x]$. (Note that f is continuous and differentiable everywhere so the hypotheses of the Mean Value Theorem hold.) There exists $c \in (0, x)$ such that

$$\frac{\sinh x - \sinh 0}{x - 0} = f'(c) = \cosh c > 1.$$

Thus $\sinh x > x$ as required.

(b) $e^x \geq 1 + x$ for all $x \in \mathbb{R}$

Solution: We will need to distinguish between the cases $x < 0$, $x = 0$, and $x > 0$. When $x = 0$, both sides of the inequality are 1. Suppose that $x < 0$. Apply the Mean Value Theorem to $f(t) = e^t$ on the interval $[x, 0]$. (Note that f is continuous and differentiable everywhere so the hypotheses of the Mean Value Theorem hold.) There exists $c \in (x, 0)$ such that

$$\frac{e^0 - e^x}{0 - x} = f'(c) = e^c < 1.$$

That is, $1 - e^x < -x$, so $e^x > 1 + x$.
If $x > 0$, apply the Mean Value Theorem to $f(t) = e^t$ on the interval $[0, x]$. There exists $c \in (0, x)$ such that

$$\frac{e^x - e^0}{x - 0} = f'(c) = e^c > 1.$$

That is, $e^x - 1 > x$, so $e^x > 1 + x$ as before. We conclude that for all x, $e^x \geq 1 + x$.

6. Let $n \in \mathbb{N}$, $n \geq 1$.

(a) Show that $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$ for all $x, y \in \mathbb{R}$.

Solution: Expanding the right hand side we have

$$\begin{aligned}(x - y)&(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1} \\ &= x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + x^2y^{n-2} + xy^{n-1} \\ &\quad - x^{n-1}y - x^{n-2}y^2 - \cdots + x^2y^{n-2} - xy^{n-1} - y^n = x^n - y^n\end{aligned}$$

since all middle terms cancel out.

(b) Hence, given $x_0 \in \mathbb{R}$ write $f(x) := x^n$ in the form $f(x) = f(x_0) + m_{x_0}(x)(x - x_0)$ and deduce that $f'(x_0) = nx_0^{n-1}$.

Solution: Using the formula from part (a)

$$x^n = x_0^n + x^n - x_0^n = x_0^n + (x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \cdots + xx_0^{n-2} + x_0^{n-1})(x - x_0)$$

The coefficient of $(x - x_0)$ has n term, and its limit as $x \to x_0$ is nx_0^{n-1}. According to Carathéodory's characterisation we have $f'(x_0) = nx^{n-1}$.

(c) Prove that the above formula for the derivative of x^n is valid for $n \in \mathbb{Z}$. First consider $n = -1$, then use the chain rule.

Solution: We can write

$$\frac{1}{x} = \frac{1}{x_0} + \frac{1}{x} - \frac{1}{x_0} = \frac{1}{x_0} - \frac{1}{xx_0}(x - x_0)$$

for all $x \in \mathbb{R}$. Letting $x \to x_0$ we have $-\dfrac{1}{xx_0} \to -\dfrac{1}{x_0^2}$ as $x \to x_0$ and hence $\dfrac{d}{dx}x^{-1} = -\dfrac{1}{x^2}$. If $n = 0$ the formula is obvious, so we only need to look at negative powers. By the chain rule

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\frac{1}{x^n} = -\frac{nx^{n-1}}{x^{2n}} = -n\frac{1}{x^{n+1}} = -nx^{-n-1}.$$

7. Let $f, g\colon (a,b) \to \mathbb{R}$ and assume that f and g are differentiable at $x_0 \in (a,b)$.

(a) Prove the product rule, that is, show that $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
Hint: Use Carathéodory's characterisation of the derivative.

Solution: According to Carathéodory's characterisation of the derivative there exist functions $m_{x_0}, \tilde{m}_{x_0}\colon (a,b) \to \mathbb{R}$ continuous at x_0 such that

$$f(x) = f(x_0) + m_{x_0}(x)(x - x_0),$$
$$g(x) = g(x_0) + \tilde{m}_{x_0}(x)(x - x_0),$$

and such that $m_{x_0}(x_0) = f'(x_0)$ and $\tilde{m}_{x_0}(x_0) = g'(x_0)$. Hence we have that

$$\begin{aligned} f(x)g(x) &= \big(f(x_0) + m_{x_0}(x)(x - x_0)\big)\big(g(x_0) + \tilde{m}_{x_0}(x)(x - x_0)\big) \\ &= f(x_0)g(x_0) + \big[f(x_0)\tilde{m}_{x_0}(x) + m_{x_0}(x)g(x_0) + m_{x_0}(x)\tilde{m}_{x_0}(x)(x - x_0)\big](x - x_0). \end{aligned}$$

It follows that

$$\begin{aligned} (fg)'(x_0) &= \lim_{x\to x_0}\big[f(x_0)\tilde{m}_{x_0}(x) + m_{x_0}(x)g(x_0) + m_{x_0}(x)\tilde{m}_{x_0}(x)(x - x_0)\big] \\ &= f(x_0)g'(x_0) + f'(x_0)g(x_0) + f'(x_0)g'(x_0)0 = f(x_0)g'(x_0) + f'(x_0)g(x_0) \end{aligned}$$

as claimed.

(b) Use mathematical induction by n to prove the following formula for the n-th derivative of a product:

$$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n-k)}$$

where $f^{(k)}$ denotes the k-th derivative. For convenience we set $f^{(0)} = f$ (the "zeroth" derivative, that is, no derivative). The formula is called the *Leibniz formula* for the n-th derivative. It looks like the binomial formula, and its proof works very similarly as well.

Solution: The base case $n = 1$ is given by the product rule. Hence assume that the formula is valid for some $n \geq 1$ (induction assumption). We show it is valid for $n + 1$. Using the induction assumption and the product rule we have

$$(fg)^{(n+1)} = \frac{d}{dx}(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}\frac{d}{dx}\big(f^{(k)}g^{(n-k)}\big) = \sum_{k=0}^{n}\binom{n}{k}\big(f^{(k+1)}g^{(n-k)} + f^{(k)}g^{(n-k+1)}\big).$$

Splitting the sum and replacing the index k in the first one by $k - 1$ we see that

$$\begin{aligned} (fg)^{(n+1)} &= \sum_{k=1}^{n+1}\binom{n}{k-1}f^{(k)}g^{(n-k+1)} + \sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n+1-k)} \\ &= f^{(n+1)}g + fg^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k-1} + \binom{n}{k}\right)f^{(k)}g^{(n-k+1)}. \end{aligned}$$

Now by definition of the binomial coefficients

$$\begin{aligned} \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!k + n!(n+1-k)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}. \end{aligned}$$

Hence,

$$
\begin{aligned}
(fg)^{(n+1)} &= f^{(n+1)}g + fg^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) f^{(k)} g^{(n-k+1)} \\
&= f^{(n+1)}g + fg^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)} g^{(n-k+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)},
\end{aligned}
$$

which is the required formula for $n+1$.

(c) Use the Leibniz formula to compute the third derivative $(x^3 e^x)'''$.

Solution: Using the formula yields

$$
(x^3 e^x)''' = 6e^x + 3(6xe^x) + 3(3x^2 e^x) + x^3 e^x = (6 + 18x + 9x^2 + x^3)e^x.
$$

Extra questions for further practice

8. Consider the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ e^{ax+b} & \text{if } x > 1. \end{cases}$$

(a) Determine for which values of $a, b \in \mathbb{R}$ the function f is continuous at $x = 1$.

Solution: The function is continuous at $x = 1$ if and only if $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$, that is, $1 = e^{a+b}$. So f is continuous at $x = 1$ if and only if $b = -a$.

(b) Determine for which values of $a, b \in \mathbb{R}$ the function f is differentiable at $x = 1$.

Solution: Since a function which is differentiable at $x = 1$ is also continuous at $x = 1$, we certainly must have $a = -b$. The left hand derivative is

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)^2 - 1}{h} = \lim_{h\to 0^-} \frac{2 + h + h^2}{h} = 1,$$

and the right hand derivative is

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{e^{a(1+h)-a} - 1}{h} = \lim_{h\to 0^+} \frac{e^{ah} - 1}{h} = a \lim_{h\to 0^+} \frac{e^{ah} - 1}{ah} = a.$$

For the last limit we used a derivative by first principles, namely

$$\lim_{x\to 0} \frac{e^x - 1}{1} = \lim_{x\to 0} \frac{e^x - e^0}{x - 0} = \frac{d}{dx} e^x |_{x=0} = 1$$

Since f is differentiable at $x = 1$ if and only the left and right derivatives are equal, we see that f is differentiable at $x = 1$ if and only if $a = -b = 2$.

9. Use Rolle's Theorem and the Intermediate Value Theorem to show that the equation

$$e^x + x^3 = 2$$

has a unique real root α.

Solution: Let $f(x) = e^x + x^3 - 2$. This function is continuous and differentiable everywhere. Since $f(0) = -1 < 0$ and $f(1) = e - 1 > 0$ the Intermediate Value Theorem, applied to f on the interval $[0, 1]$, implies that there exists $\alpha \in (0, 1)$ with $f(\alpha) = 0$.

Suppose that there are at least two real solutions. Then there exist $a < b$ with $f(a) = f(b) = 0$. Hence by Rolle's Theorem applied to the interval $[a, b]$ there exists $c \in (a, b)$ with $f'(c) = 0$. However

$$f'(x) = e^x + 3x^2 > 0,$$

a contradiction. Hence there is precisely 1 real root.

10. Use the Intermediate Theorem and Rolle's Theorem to show that every positive real number has a unique positive real square root.

Solution: Let $\alpha > 0$ be a fixed number. We are attempting to show that the equation $x^2 = \alpha$ has a unique solution with $x \ge 0$.

We'll first show that there is at least one solution. Let $f(x) = x^2 - \alpha$. This function is continuous and differentiable everywhere. Note that $f(0) = -\alpha < 0$. If $\alpha < 1$ then $f(1) = 1 - \alpha > 0$, and if $\alpha = 1$ then $f(2) = 4 - 1 = 3 > 0$, and if $\alpha > 1$ then $\alpha^2 > \alpha$, and so $f(\alpha) = \alpha^2 - \alpha > 0$. Thus in all case we see that there is a number $a > 0$ with $f(a) > 0$. Hence the Intermediate Value Theorem applied to the interval $[0, a]$ tells us that there is a number $0 < \beta < a$ such that $f(\beta) = 0$. That is, $\beta^2 = \alpha$, and so there is at least one square root.

Suppose that there are two positive square roots: $0 < \beta < \gamma$ with $f(\beta) = f(\gamma) = 0$. Then by Rolle's Theorem there is $c \in (\beta, \gamma)$ with $f'(c) = 0$. However $f'(x) = 2x$, and so $c = 0$, a contradiction (as $c > 0$). Thus there is a unique positive square root.

11. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ differentiable such that $f'(x) \neq 0$ for all $x \in (a, b)$. Use Rolle's Theorem to show that f is injective.

Solution: Assume that $f(x) = f(y)$ for some $x, y \in [a, b]$ with $x \neq y$. Then Rolle's Theorem guarantees that there exists c between x and y such that $f'(c) = 0$. This contradicts the assumption that $f(x) \neq 0$ for all $x \in (a, b)$.

12. A mountaineer leaves home at 7am and hikes to the top of the mountain, arriving at 7pm. The following morning, he starts out at 7am from the top of the mountain and takes the same path back, arriving home at 7pm. Use the Intermediate Value Theorem to show that there is a point on the path that he will cross at exactly the same time of day on both days.

Solution: Define $u(t)$ to be the mountaineer's distance from home, measured along the path that he takes, as a function of time (measured in hours), on the first day. In a similar fashion, define $d(t)$ to be the mountaineer's distance from home, as a function of time, on the second day. Let D be the total length of the path from home to the top of the mountain. We know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$.

We may assume that u and d are continuous functions on the interval $[0, 12]$. Now consider the function $u - d$, which is also continuous on $[0, 12]$. We have $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there is some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0$, or equivalently, $u(t_0) = d(t_0)$. Thus, at the same time t_0 after 7am, the mountaineer will be at the same place on both days.

It is not actually necessary to have the start times the same or the finish times the same. We just need the time intervals to overlap. Then the graph of $y = u(t)$ must cross the graph of $y = d(t)$ somewhere on the overlap.

13. The road between two towns, A and B, is 110 km long, with a speed limit of 100 km/h. You left town A to drive to town B at the same time as I left town B to drive to town A. We met exactly 30 minutes later.

(a) Use the Mean Value Theorem to prove that at least one of us exceeded the speed limit by 10 km/h or more.

Solution: Let $f(t)$ km be the distance that you have travelled and $g(t)$ km the distance I have travelled t hours after departure from our starting points. Since we meet after 30 minutes we have $f(0.5) + g(0.5) = 110$, so either $f(0.5)$ or $g(0.5)$ is at least 55. We take f and g to be continuous on $[0, 0.5]$ and differentiable on $(0, 0.5)$ so the hypotheses of the Mean Value Theorem hold. Then

$$f'(c_1) = \frac{f(0.5) - f(0)}{0.5 - 0} = 2f(0.5)$$

and

$$g'(c_2) = \frac{g(0.5) - g(0)}{0.5 - 0} = 2g(0.5)$$

for some numbers $c_1, c_2 \in (0, 0.5)$. Hence either $f'(c_1) \geq 2 \times 55 = 110$ or $g'(c_2) \geq 2 \times 55 = 110$, which means that one (or both) of us travelled at a speed of at least 110 km/hour at some moment before we met.

(b) You know that you never travelled faster than 90 km/hr over your journey. How fast can you be sure that I travelled at some point in my journey?

Solution: In the notation of the previous solution, we now have $g(0.5) \leq 90/2 = 45$. Since $f(0.5) + g(0.5) = 110$ we have $f(0.5) = 110 - g(0.5) \geq 110 - 45 = 65$. Therefore

$$f'(c_1) = 2f(0.5) \geq 2 \times 65 = 130,$$

so you can be sure that at some point I drove at 130 km/hr or faster.

Remark: Speeding is illegal and dangerous, so please refrain from doing it.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable for all $x \in \mathbb{R}$. Use the Mean Value Theorem to show that if $f'(x) < 0$ for all $x \in \mathbb{R}$ then f is strictly decreasing.

Solution: Let $a, b \in \mathbb{R}$ with $b > a$. Then f is continuous on $[a, b]$ and differentiable on (a, b), and so the Mean Value Theorem tells us that there is a number $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f'(c) < 0$ we have $(f(b) - f(a))/(b - a) < 0$, and so $f(b) - f(a) < 0$, and so $f(b) < f(a)$. Thus $f(x)$ is strictly decreasing.

15. Find the global maximum and global minimum values of $f(x) = ||x| - 1|$ on the interval $[-1, 1]$.

Solution: This function is not differentiable at the points $x = -1, 0, 1$, and so we can't use the usual technique of finding critical points. Instead it is best to draw this graph directly. We have

$$f(x) = \begin{cases} |x - 1| & \text{if } x \geq 0 \\ |x + 1| & \text{if } x < 0 \end{cases}$$

and so from the graph we see that the global maximum is at $x = 0$, and the global minimum is at $x = -1$ and $x = 1$.

16. For the functions given by the following formulas, find the global maximum and global minimum values on the indicated intervals.

(a) $f(x) = \dfrac{e^x}{x + 1}$ on $[2, 3]$

Solution: The given function is continuous on the closed interval $[2, 3]$ and differentiable on the open interval $(2, 3)$. Thus we can compute the global maximum and minimum values using the method from class.
We have $f'(x) = \frac{xe^x}{(x+1)^2}$. As this is zero only when $x = 0$, there are no critical points in $[2, 3]$. The maximum and minimum values therefore occur at the endpoints. We find that $f(2) = \frac{e^2}{3} \approx 2.463$ is the minimum value and $f(3) = \frac{e^3}{4} \approx 5.021$ is the maximum value.

(b) $f(x) = \dfrac{x}{x^2 + 1}$ on $[-2, 0]$

Solution: The given function is continuous on the closed interval $[-2, 0]$ and differentiable on the open interval $(-2, 0)$. Thus we can compute the global maximum and minimum values using the method from class.
We have $f'(x) = \frac{1-x^2}{(1+x^2)^2}$. There is a critical point at $x = -1$. We find that $f(-2) = -\frac{2}{5}$, $f(-1) = -\frac{1}{2}$, $f(0) = 0$. Thus the maximum value is 0 and the minimum value is $-\frac{1}{2}$.

(c) $f(x) = e^{x^2-1}$ on $[-1, 1]$

Solution: The given function is continuous on the closed interval $[-1, 1]$ and differentiable on the open interval $(-1, 1)$. Thus we can compute the global maximum and minimum values using the method from class.
We have $f'(x) = 2xe^{x^2-1}$ and so the only critical point is at $x = 0$. We have $f(0) = e^{-1}$, $f(-1) = 1$, and $f(1) = 1$. Thus the global maximum value is 1 occurring at $x = -1$ and $x = 1$, and the global minimum value is e^{-1} occurring at $x = 0$.

17. Use the Intermediate Value Theorem and Rolle's Theorem to show that the equation

$$x + \sin x = 1$$

has exactly one real solution. Use your calculator to find this solution correct to 1 decimal place.

Solution: Let $f(x) = x + \sin x - 1$. This function is both continuous and differentiable for all $x \in \mathbb{R}$. Since $f(0) = -1 < 0$ and $f(1) = \sin 1 \approx 0.841471 > 0$, by the Intermediate Value Theorem there is $\alpha \in (0, 1)$ with $f(\alpha) = 0$.

If $x > 2$ then $f(x) = x - 1 + \sin x > 1 + \sin x > 0$, and if $x < 0$ then $f(x) < -1 + \sin x < 0$. Thus if $f(x) = 0$ then $x \in [0, 2]$. Suppose that there are at least 2 solutions. Hence there are numbers $0 \le a < b \le 2$ with $f(a) = f(b) = 0$. By Rolle's Theorem applied to the interval $[0, 2]$ there is $c \in (a, b)$ such that $f'(c) = 0$. However

$$f'(x) = 1 + \cos x,$$

and so $f'(c) \neq 0$ for $c \in (0, 2)$. Thus there is a unique solution.

Since $f(1/2) \approx -0.0205 < 0$ there is a root $\alpha \in (1/2, 1)$. Then, since $f(3/4) = 0.431 > 0$ there is a root $\alpha \in (1/2, 3/4)$. Since $f(5/8) = 0.210 > 0$ there is a root $\alpha \in (1/2, 5/8)$. Since $f(9/16) = 0.095 > 0$ there is a root $\alpha \in (1/2, 9/16)$. Thus there is a root with $0.5 < \alpha < 9/16 = 0.5625$. Thus $\alpha = 0.5$ correct to 1 decimal place.

18. Define a function f by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is differentiable at 0.

Solution: We have

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

(since $f(0) = 0$). Now,

$$0 \le \left| \frac{f(h)}{h} \right| \le \frac{|h^2|}{|h|} = |h|,$$

and so by the Squeeze Law $\lim_{h \to 0} f(h)/h = 0$. Thus

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = 0,$$

and so f is differentiable at 0, with $f'(0) = 0$.

Challenge questions (optional)

19. Using Rolle's Theorem, prove that a polynomial of degree $n > 0$ has at most n real roots.

Solution: Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial of degree $n > 0$, so $a_n \neq 0$. If $n = 1$ then $p(x) = a_0 + a_1 x$ which has the unique root $-\frac{a_0}{a_1}$. This starts an induction process. Suppose (as the inductive hypothesis) that $n > 1$ and that the claim holds for any polynomial of degree less than n. In particular, the derivative $p'(x)$ is a polynomial of degree $n - 1$, so has at most $n - 1$ real roots.

We argue by contradiction. Suppose that $p(x)$ has at least $n + 1$ real roots, so there exist

$$x_1 < x_2 < \cdots < x_n < x_{n+1}$$

such that $p(x_i) = 0$ for each $i = 1, \ldots, n + 1$. By Rolle's Theorem, for each $i = 1, \ldots, n$ there exists $y_i \in (x_i, x_{i+1})$ such that $p'(y_i) = 0$. But

$$y_1 < y_2 < \cdots < y_n.$$

That is, all the y_i are distinct real roots of $p'(x) = 0$. This contradicts the assumption that $p'(x)$ has at most $n - 1$ real roots. We conclude that $p(x)$ has at most n real roots, which completes the inductive step.