THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 8

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Intermediate Value Theorem.
- ☐ Global maximum and minimum values of a function.
- ☐ Extreme Value Theorem.
- ☐ Differentiation and the differentiation laws.
- ☐ Rolle's Theorem.

Outcomes

After completing this tutorial you should

- ☐ understand and be able to apply the Intermediate Value Theorem;
- ☐ understand the conditions under which the Extreme Value Theorem guarantees existence of global max/min;
- ☐ understand and apply the definition of differentiability;
- ☐ be able to use the Intermediate Value Theorem to prove existence of solutions;
- ☐ be able to use Rolle's Theorem to prove existence of solutions;

Summary of essential material

Intermediate Value Theorem: If $f\colon [a,b] \to \mathbb{R}$ is continuous on the and if either $f(a) < k < f(b)$ of $f(b) < k < f(a)$, then there exists $c \in (a,b)$ with $f(c) = k$. It is important that $[a,b]$ be a closed interval.

Extreme Value Theorem: If $f\colon [a,b] \to \mathbb{R}$ is continuous, then there exist $c, d \in [a,b]$ such that $M := f(c)$ is the global maximum value of $f(x)$ on $[a,b]$, and $m := f(d)$ is the global minimum value of $f(x)$ on $[a,b]$. It is essential that $[a,b]$ is a closed and bounded interval!

Derivatives: The derivative of a function f at x_0 is the slope of the *tangent* to the graph of f at the point $(x_0, f(x_0))$. If it exists, that slope is defined to be the limit of the slopes of the secants through $(x_0, f(x_0))$ and $(x, f(x))$ as $x \to x_0$. The slope of the secants,

$$m_{x_0}(x) = \frac{f(x) - f(x_0)}{x - x_0},$$

is called the *difference quotient*. If it exists, the limit of $m_{x_0}(x)$ as $x \to x_0$ is called the *derivative* of f at x_0 and is denoted by $\dfrac{df}{dx}(x_0)$ or $f'(x_0)$, and f is called *differentiable* at x_0. Alternatively we can say $f\colon (a,b) \to \mathbb{R}$ is differentiable at x_0 if there exists a function $m_{x_0}\colon (a,b) \to \mathbb{R}$ that is continuous at x_0 and

$$f(x) = f(x_0) + m_{x_0}(x)(x - x_0)$$

for all $x \in (a,b)$. In that case $f'(x_0) = m_{x_0}(x_0)$. (This is *Carathéodory's characterisation* from about 1950).

Differentiation Laws: If $f(x)$ and $g(x)$ are differentiable, then the following functions are also differentiable, with derivatives as stated:

(1) $(\alpha f)' = \alpha f'$ for $\alpha \in \mathbb{R}$

(2) $(f + g)'(x) = f' + g'$

(3) $(fg)' = f'g + fg'$ (product rule)

(4) $\left(\frac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$ (if $g'(x) \neq 0$, quotient rule)

(5) $(f \circ g)'(x) = g'(x) f'\big(g(x)\big)$ (chain rule)

Rolle's Theorem/Mean Value Theorem: Let $f : [a,b] \to \mathbb{R}$ be continuous and $f : (a,b) \to \mathbb{R}$ differentiable.

Rolle's Theorem: If $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Mean Value Theorem: There exists $c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

Questions to complete during the tutorial

1. Use the definition of the hyperbolic functions to show that $\dfrac{d}{dx}\cosh x = \sinh x$ and $\dfrac{d}{dx}\sinh x = \cosh x$.

2. Let $f(x)$ be differentiable for all $x \in \mathbb{R}$. Suppose that $f(0) = -3$ and that $f'(x) \le 5$ for all $x \in \mathbb{R}$. Use the Mean Value Theorem to show that $f(2) \le 7$.

3. Use the Intermediate Value Theorem to prove that if $f : [0,1] \to [0,1]$ is continuous, then there is exists $c \in [0,1]$ such that $f(c) = c$.

4. Briefly explain why Rolle's theorem implies that there is a stationary point between any two zeros of a differentiable function. Hence use the Intermediate Value Theorem to show that the equation

$$x^2 - x\sin x - \cos x = 0$$

has exactly two distinct real solutions.

5. Use the Mean Value Theorem to prove the following inequalities:

(a) $\sinh x > x$ for all $x > 0$.

(b) $e^x \ge 1 + x$ for all $x \in \mathbb{R}$

6. Let $n \in \mathbb{N}$, $n \ge 1$.

(a) Show that $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$ for all $x, y \in \mathbb{R}$.

(b) Hence, given $x_0 \in \mathbb{R}$ write $f(x) := x^n$ in the form $f(x) = f(x_0) + m_{x_0}(x)(x - x_0)$ and deduce that $f'(x_0) = nx_0^{n-1}$.

(c) Prove that the above formula for the derivative of x^n is valid for $n \in \mathbb{Z}$. First consider $n = -1$, then use the chain rule.

7. Let $f, g : (a,b) \to \mathbb{R}$ and assume that f and g are differentiable at $x_0 \in (a,b)$.

(a) Prove the product rule, that is, show that $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
Hint: Use Carathéodory's characterisation of the derivative.

(b) Use mathematical induction by n to prove the following formula for the n-th derivative of a product:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

where $f^{(k)}$ denotes the k-th derivative. For convenience we set $f^{(0)} = f$ (the "zeroth" derivative, that is, no derivative). The formula is called the *Leibniz formula* for the n-th derivative. It looks like the binomial formula, and its proof works very similarly as well.

(c) Use the Leibniz formula to compute the third derivative $(x^3e^x)'''$.

Extra questions for further practice

8. Consider the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ e^{ax+b} & \text{if } x > 1. \end{cases}$$

(a) Determine for which values of $a, b \in \mathbb{R}$ the function f is continuous at $x = 1$.

(b) Determine for which values of $a, b \in \mathbb{R}$ the function f is differentiable at $x = 1$.

9. Use Rolle's Theorem and the Intermediate Value Theorem to show that the equation

$$e^x + x^3 = 2$$

has a unique real root α.

10. Use the Intermediate Theorem and Rolle's Theorem to show that every positive real number has a unique positive real square root.

11. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ differentiable such that $f'(x) \neq 0$ for all $x \in (a, b)$. Use Rolle's Theorem to show that f is injective.

12. A mountaineer leaves home at 7am and hikes to the top of the mountain, arriving at 7pm. The following morning, he starts out at 7am from the top of the mountain and takes the same path back, arriving home at 7pm. Use the Intermediate Value Theorem to show that there is a point on the path that he will cross at exactly the same time of day on both days.

13. The road between two towns, A and B, is 110 km long, with a speed limit of 100 km/h. You left town A to drive to town B at the same time as I left town B to drive to town A. We met exactly 30 minutes later.

(a) Use the Mean Value Theorem to prove that at least one of us exceeded the speed limit by 10 km/h or more.

(b) You know that you never travelled faster than 90 km/hr over your journey. How fast can you be sure that I travelled at some point in my journey?

14. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable for all $x \in \mathbb{R}$. Use the Mean Value Theorem to show that if $f'(x) < 0$ for all $x \in \mathbb{R}$ then f is strictly decreasing.

15. Find the global maximum and global minimum values of $f(x) = ||x| - 1|$ on the interval $[-1, 1]$.

16. For the functions given by the following formulas, find the global maximum and global minimum values on the indicated intervals.

(a) $f(x) = \dfrac{e^x}{x+1}$ on $[2, 3]$ (b) $f(x) = \dfrac{x}{x^2+1}$ on $[-2, 0]$ (c) $f(x) = e^{x^2-1}$ on $[-1, 1]$

17. Use the Intermediate Value Theorem and Rolle's Theorem to show that the equation

$$x + \sin x = 1$$

has exactly one real solution. Use your calculator to find this solution correct to 1 decimal place.

18. Define a function f by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is differentiable at 0.

Challenge questions (optional)

19. Using Rolle's Theorem, prove that a polynomial of degree $n > 0$ has at most n real roots.