The University of Sydney
School of Mathematics and Statistics

MATH1901/1906
Differential Calculus (Advanced)

June 2015 Lecturer: J Parkinson

Time Allowed: **One and a half hours**

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Approved non-programmable calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Marker's use only	

Extended Answer Section

*There are **four** questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

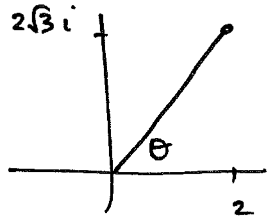
1. (a) (*i*) Write the complex number $2 + 2\sqrt{3}\, i$ in polar form.

(*ii*) Find all solutions $z \in \mathbb{C}$ to the equation

$$z^4 = 2 + 2\sqrt{3}\, i,$$

expressing your final answers in polar form.

Q1 (a) (i)



$|z| = 4$

$\theta = \frac{\pi}{3}$

$$\boxed{z = 4 \operatorname{cis} \frac{\pi}{3}}$$

(ii) $z = r \operatorname{cis} \theta$. So $r^4 \operatorname{cis} 4\theta = 4 \operatorname{cis} \frac{\pi}{3}$.

So $r = \sqrt{2}$, $4\theta = \frac{\pi}{3} + 2k\pi$ $k \in \mathbb{Z}$

$$\theta = \frac{\frac{\pi}{3} + 2k\pi}{4}.$$

So solutions are

$$\boxed{z_k = \sqrt{2} \operatorname{cis} \frac{\frac{\pi}{3} + 2k\pi}{4} \qquad k = 0, 1, 2, 3.}$$

(b) Find all solutions $z \in \mathbb{C}$ of the equation

$$e^{2z} - 1 = i,$$

expressing your final answers in Cartesian form.

Let $z = x + iy$. Then

$$e^{2x} e^{2iy} = 1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$$

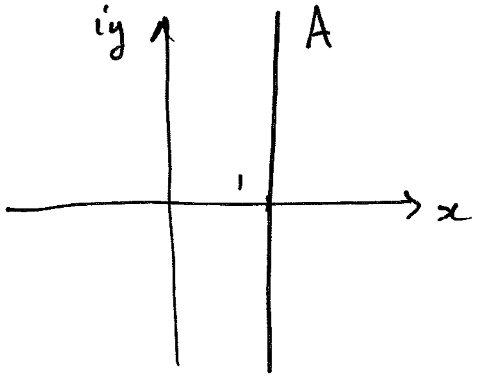
So $$e^{2x} \operatorname{cis}(2y) = \sqrt{2} \operatorname{cis} \frac{\pi}{4}.$$

So $2x = \ln\sqrt{2} \Rightarrow x = \frac{1}{4} \ln 2$

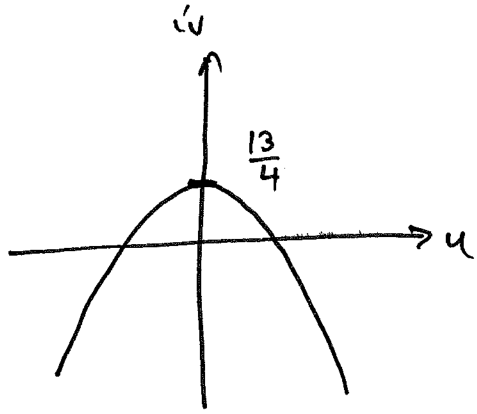
$2y = \frac{\pi}{4} + 2k\pi \Rightarrow y = \frac{\pi}{8} + k\pi \qquad k \in \mathbb{Z}.$

So $\boxed{z = \frac{1}{4}\ln 2 + \left(\frac{\pi}{8} + k\pi\right) i \qquad k \in \mathbb{Z}}$

(c) Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = iz^2 + 3z$ and let $A = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = 1\}$. Sketch the image of A under f in the complex plane.



z-plane



w-plane

$z = 1+it$ with $t \in \mathbb{R}$. So

$$f(z) = f(1+it) = i(1+it)^2 + 3(1+it)$$
$$= i(1+2it-t^2) + 3 + 3it$$
$$= (-2t+3) + i(1-t^2+3t)$$

So $u = 3-2t$, $v = 1-t^2+3t$

So $t = \frac{3-u}{2} \Rightarrow v = 1 - \left(\frac{3-u}{2}\right)^2 + 3\left(\frac{3-u}{2}\right)$

$$\Rightarrow v = 1 - \frac{9-6u+u^2}{4} + \frac{9-3u}{2}$$
$$= -\frac{1}{4}u^2 + \frac{13}{4} \quad \text{(parabola)}.$$

(d) Use the ϵ, δ definition of limits to show that

$$\lim_{x \to 2}(2x - 3) = 1.$$

Let $\epsilon > 0$. Choose $\delta = \epsilon/2$

Then

$$0 < |x-2| < \delta \Rightarrow |f(x) - 1| = |2x-4|$$
$$= 2|x-2|$$
$$< 2\delta = \epsilon.$$

Hence $\lim_{x \to 2}(2x-3) = 1$.

2. (a) Calculate the following limits, or show that they do not exist, showing all of your working. You may use any valid method.

$(i)\quad \lim_{x\to 0} \frac{\ln(1+3x)}{x(2+x^2)}$ $\qquad (ii)\quad \lim_{(x,y)\to(0,0)} \frac{xy+|y|}{x^2+|y|}$

$(iii)\quad \lim_{x\to 2} \frac{\sqrt{x+2}-2\sqrt{x}}{x-2}$ $\qquad (iv)\quad \lim_{(x,y)\to(0,0)} \frac{x^4 \sin y}{x^4+y^4}$

(i) $= \lim_{x\to 0} \frac{\ln(1+3x)}{x} \cdot \frac{1}{2+x^2}$

$= \left(\lim_{x\to 0} \frac{\ln(1+3x)}{x}\right)\left(\lim_{x\to 0} \frac{1}{2+x^2}\right)$ (Limit Law)

$= \left(\lim_{x\to 0} \frac{3}{1+3x}\right) \times \frac{1}{2}$ (L'Hôpital)

$= \frac{3}{2}$

(ii) $f(x,0) = \frac{x\times 0+|0|}{x^2+|0|} = 0 \to 0$

$f(x,x) = \frac{x^2+|x|}{x^2+|x|} = 1 \to 0$

so limit does not exist.

(iii) Does not exist: $\sqrt{x+2}-2\sqrt{x} \to 2-2\sqrt{2}$

$x-2 \to 0$.

(iv) $0 \leq \left|\frac{x^4 \sin y}{x^4+y^4}\right| = \left|\frac{x^4}{x^4+y^4}\right| |\sin y| \leq |\sin y| \to 0$

So $\lim_{(x,y)\to(0,0)} \frac{x^4 \sin y}{x^4+y^4} = 0$ by squeeze Law.

(b) Let $f(x, y) = 1 - 2x + 6y + \sinh(3 - 2x + y)$. Find the equation of the tangent to the level curve $f(x, y) = 3$ at the point $(x, y) = (2, 1)$.

$f_y(x,y) = 6 + \cosh(3-2x+y)$, so

$f_y(2,1) = 6 + \cosh(0) = 7 \neq 0$.

So by IFT, y is defined as a fnctn of x near (2,1), and

$$y'(x) = -\frac{f_x(x,y)}{f_y(x,y)} = -\frac{-2-2\cosh(3-2x+y)}{6+\cosh(3-2x+y)}$$

$$\Rightarrow y'(2) = \frac{2+2}{6+1} = \frac{4}{7}.$$

The tangent is

$$y - 1 = y'(2)\,(x-2)$$

So $\boxed{y - 1 = \frac{4}{7}(x-2)}$

(c) Let $g(x, y) = \sin(x^2 - y) + 4xy + 3$.

(i) Find the tangent plane to the graph $z = g(x, y)$ at the point $(x, y) = (2, 4)$.

(ii) What is the direction of the steepest slope of the graph $z = g(x, y)$ at the point $(x, y) = (2, 4)$.

(iii) What is the slope of the graph $z = g(x, y)$ in the direction $\mathbf{i} + 3\mathbf{j}$ at the point $(x, y) = (2, 4)$?

See tutorial (Wk 13)

QUESTION 3 BEGINS ON THE NEXT PAGE

3. (a) (*i*) State Rolle's Theorem.

(*ii*) Use Rolle's Theorem to show that if $h : [a, b] \to \mathbb{R}$ is continuous with $h'(x) \neq 0$ for all $x \in (a, b)$ then the function h is injective.

(i) See lectures

(ii) ~~Suppose that h is not injective. So there are a < b with h(a) = h(b). By Rolle's Thm there is c ∈ (a,b) with h'(c) = 0,~~

Suppose that h is not injective. So there are numbers $\alpha < \beta$ in $[a, b]$ with

$$h(\alpha) = h(\beta).$$

By Rolle's Thm applied on $[\alpha, \beta]$, there is $\gamma \in (a, b)$ such that

$$h'(\gamma) = 0,$$

a contradiction. Hence h is injective.

(b) Suppose that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable everywhere, and that $f'(x) = g'(x)$ for all $x \in \mathbb{R}$. Use the Mean Value Theorem to prove that $f(x) = g(x) + k$ for all $x \in \mathbb{R}$, where k is a constant.

Let $h(x) = f(x) - g(x)$, and note that

$$h'(x) = f'(x) - g'(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

Let $a < b$. By MVT applied to $h(x)$ on $[a, b]$ there is $c \in (a, b)$ with

$$h'(c) = \frac{h(b) - h(a)}{b - a}$$

Since $h'(c) = 0$ this gives

$$h(b) = h(a).$$

So $h(x) = k$ is a constant function.

Hence $f(x) = g(x) + k$.

(c) Let $f(x) = \sqrt{1+x}$.

(i) Calculate the second order Taylor polynomial $T_2(x)$ for $f(x)$ about $x = 0$.

(ii) Write down a formula for the remainder term $R_2(x) = f(x) - T_2(x)$.

(iii) Hence show that
$$1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 \leq \sqrt{1+x^4} \leq 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12}$$
for all $x \in \mathbb{R}$.

(iv) Hence, or otherwise, compute the limit
$$\lim_{x\to 0} \frac{2\sqrt{1+\sin^4 x} - 2 - \sin^4 x}{\sin^8 x}.$$

See tutorial, Wk 10.

Question 4 begins on the next page

4. (a) Let $f(x) = x^3 - 3x + 1$.

(i) Show that the function $f : [-1, 1] \to [-1, 3]$ is bijective.

(ii) Let $f^{-1} : [-1, 3] \to [-1, 1]$ be the inverse of the function $f : [-1, 1] \to [-1, 3]$. Calculate the third order Taylor polynomial of $f^{-1}(x)$ centred at $x = 1$.

See tutorial, Wk 13

(b) You are given that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(a+b) = \frac{f(a)f(b)}{2} \quad \text{for all } a, b \in \mathbb{R},$$

and that f is differentiable at $x = 0$ with $f'(0) = 5$.

(*i*) Calculate $f(0)$.

(*ii*) Show that f is differentiable everywhere.

(*iii*) Find an explicit formula for the function $f(x)$.

See assignment.

Standard Derivatives

The following derivatives can be quoted without proof unless a question specifically asks you to show details. These results can be combined with the standard rules of differentiation (not listed here) to differentiate more complicated functions. For example, $(d/dx)\sin(ax+b) = a\cos(ax+b)$. Natural domains common to both sides are assumed.

1. $\dfrac{d}{dx}x^k = kx^{k-1} \quad (k \in \mathbb{R})$
2. $\dfrac{d}{dx}e^x = e^x$
3. $\dfrac{d}{dx}\ln x = \dfrac{1}{x} \quad (x > 0)$
4. $\dfrac{d}{dx}\sin x = \cos x$
5. $\dfrac{d}{dx}\cos x = -\sin x$
6. $\dfrac{d}{dx}\tan x = \sec^2 x$
7. $\dfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x$
8. $\dfrac{d}{dx}\sec x = \sec x \tan x$
9. $\dfrac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$
10. $\dfrac{d}{dx}\sinh x = \cosh x$
11. $\dfrac{d}{dx}\cosh x = \sinh x$
12. $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$
13. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$
14. $\dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$
15. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$
16. $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$
17. $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}} \quad (x > 1)$
18. $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2} \quad (|x| < 1)$

End of Extended Answer Section

This is the last page of the question paper.