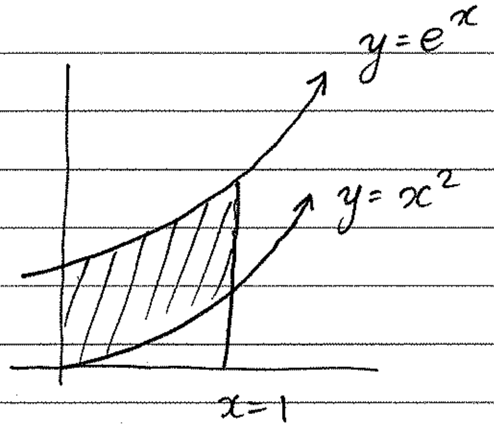
Extended Answer Section

There are **four** *questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

MARKS

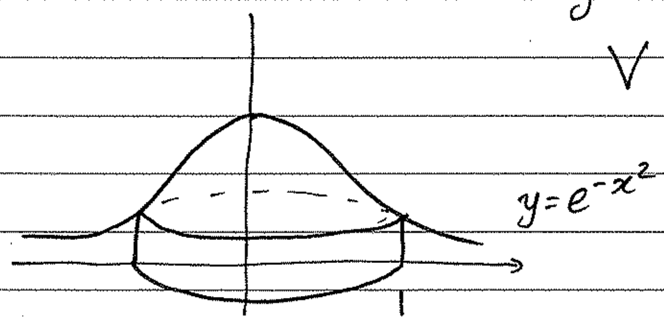
1. (a) Compute the area of the region in the first quadrant bounded by the curve $y = e^x$, the curve $y = x^2$, the y-axis, and the line $x = 1$. **2**



$$A = \int_0^1 (e^x - x^2)\,dx$$
$$= \left(e^x - \tfrac{1}{3}x^3\right)\Big|_0^1$$
$$= e - \frac{4}{3}$$

(b) Compute the volume of the solid obtained by rotating about the y-axis the region bounded by the curve $y = e^{-x^2}$, the y-axis, the x-axis, and the line $x = 1$. **2**

By shells



$$V = 2\pi \int_0^1 x e^{-x^2}\,dx$$
$$= -\pi e^{-x^2}\Big|_0^1$$
$$= \pi(1 - e^{-1}).$$

MARKS

(c) Find the length of the curve given by parametric equations 2

$$x(t) = e^t \cos t \qquad \text{and} \qquad y(t) = e^t \sin t$$

with $t \in [0, 2\pi]$.

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, e^t\, dt$$

$$= \sqrt{2}\,(e^{2\pi} - 1)$$

(d) Compute the limit as $n \to \infty$ of the sequence 2

$$a_n = \frac{(2n)!2^{2n}}{(n!)^2(2n+1)5^{2n}}.$$

Use the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{4}{25} \cdot \frac{(2n+1)(2n+2)(2n+1)}{(n+1)(n+1)(2n+3)}$$

$$= \frac{16}{25} < 1$$

Therefore $a_n \to 0$ as $n \to \infty$, by the ratio test for sequences

MARKS

(e) Compute the value of the improper integral $\int_0^\infty e^{-x} \sin x \, dx$ by taking the limit of an appropriate proper integral. **2**

Let $I_b = \int_0^b e^{-x} \sin x \, dx$ $u = e^{-x}$ $\frac{dv}{dx} = \sin x$ $\frac{du}{dx} = -e^{-x}$ $v = -\cos x$

$= -e^{-x} \cos x \Big|_0^b - \int_0^b e^{-x} \cos x \, dx$

$= 1 - e^{-b} \cos b - \int_0^b e^{-x} \cos x \, dx$ $u = e^{-x}$ $\frac{dv}{dx} = \cos x$ $\frac{du}{dx} = -e^{-x}$ $v = \sin x$

$= 1 - e^{-b} \cos b - e^{-b} \sin b - \int_0^b e^{-x} \sin x \, dx$

Hence $I_b = \frac{1}{2} - \frac{1}{2} e^{-b} (\cos b + \sin b)$

Therefore (by squeeze law)

$\int_0^\infty e^{-x} \sin x \, dx = \lim_{b \to \infty} I_b = \frac{1}{2}$.

MARKS

2. (a) Write down the first and second order Taylor polynomials $T_1(x)$ and $T_2(x)$ for the function $f(x) = \ln(1+x)$ about $x = 0$, and use Taylor's Theorem to write down formulas for the first and second order remainder terms $R_1(x) = \ln(1+x) - T_1(x)$ and $R_2(x) = \ln(1+x) - T_2(x)$. 3

$f(x) = \ln(1+x) \implies f(0) = 0$

$f^{(1)}(x) = (1+x)^{-1} \implies f^{(1)}(0) = 1$

$f^{(2)}(x) = -(1+x)^{-2} \implies f^{(2)}(0) = -1$

$f^{(3)}(x) = 2(1+x)^{-3}$

So $T_1(x) = f(0) + \frac{f^{(1)}(0)}{1!}x = x$

$T_2(x) = T_1(x) + \frac{f^{(2)}(0)}{2!}x^2 = x - \frac{1}{2}x^2$

$R_1(x) = -\frac{1}{2}(1+c)^{-2}x^2$ for some c between 0 and x

$R_2(x) = \frac{1}{3}(1+d)^{-3}x^3$ for some d between 0 and x

MARKS

(b) Hence, or otherwise, prove that 2

$$x - \frac{1}{2}x^2 \leq \ln(1+x) \leq x \qquad \text{for all } x \geq 0.$$

If $x \geq 0$ then by the formulas in (a) we have

$R_1(x) \leq 0$ and $R_2(x) \geq 0$.

Therefore

$\ln(1+x) = T_1(x) + R_1(x) \leq T_1(x)$

$\ln(1+x) = T_2(x) + R_2(x) \geq T_2(x)$

So:

$x - \frac{1}{2}x^2 \leq \ln(1+x) \leq x \quad \forall x \geq 0.$

(c) Show that 2

$$\ln(n+1) = \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) \qquad \text{for } n \geq 1.$$

RHS $= \sum_{k=1}^{n} \ln(1+\frac{1}{k})$

$= \ln \prod_{k=1}^{n} (1+\frac{1}{k})$

$= \ln\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{n}{n-1} \cdot \frac{n+1}{n}\right)$

$= \ln(n+1) =$ LHS.

MARKS

(d) Use parts (b) and (c) and an appropriate comparison test to deduce that the sequence **3**

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1)$$

converges as $n \to \infty$.

$\gamma_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) \right)$

$= \sum_{k=1}^{n} a_k$, where $a_k = \frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)$

Since $x - x^2/2 \leq \ln(1+x) \leq x \quad \forall x \geq 0$,

we have $0 \leq a_k \leq \frac{1}{2k^2}$.

Therefore $|a_k| \leq \frac{1}{2k^2}$ for all $k = 1, 2, \ldots$

and since $\sum \frac{1}{2k^2}$ converges, so

does $\sum a_k$ (by comparison test)

Hence $\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exists.

MARKS

3. (a) Use an integrating factor to find the general solution of the differential equation **2**

$$\frac{dy}{dx} = \alpha + \frac{y}{x}.$$

$$I = e^{\int -\frac{1}{x}\,dx} = \frac{1}{|x|}$$

$$\frac{d}{dx}\left(\frac{1}{|x|}y\right) = \frac{\alpha}{|x|}, \text{ so}$$

$$\frac{y}{|x|} = \alpha \ln|x| + C$$

$$\Rightarrow \quad y = \cancel{\alpha}\,|x|\,(\alpha \ln|x| + C)$$

(absolute value signs NOT needed in marking)

(b) Solve the initial value problem **2**

$$\frac{d^2y}{dx^2} + y = -2\sin x$$

where $y(0) = 0$ and $y'(0) = 1$.

Homog: Aux: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$y_h = A\sin x + B\cos x$

InHomog: Try $y_p = x(C_1 \sin x + C_2 \cos x)$

$\Rightarrow C_1 = 0$ and $C_2 = 1$

So $y_p = x\cos x$

Then $y = y_h + y_p = A\sin x + B\cos x + x\cos x$.

Initial condition $\Rightarrow$ $A = 0$ and $B = 0$; so

$y = x\cos x$ is solution.

MARKS

(c) Consider a differential equation of the form **3**

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Introduce a new dependent variable z by $y = xz$ and show that the resulting differential equation for $z(x)$ can be written in separable form $dz/dx = G(x)/F(z)$. Use this method to again solve the differential equation from Question 3(a).

$y = xz$

$\Rightarrow \quad \frac{dy}{dx} = z + x\frac{dz}{dx} = f(z).$

So $\quad \frac{dz}{dx} = \frac{f(z) - z}{x}$ is separable.

In Q 3(a) $\quad f(z) = \alpha + z$

$\Downarrow$

$\frac{dz}{\alpha} = \frac{dx}{x}$

$\Rightarrow \quad z = \alpha \log x + C$

So $\quad y = x(\alpha \ln x + C).$

MARKS

(d) Let n be a positive integer and a, b, c, d real constants with $ad - bc \neq 0$. Find a transformation to a new dependent variable such that the differential equation **3**

$$\frac{1}{x^{n-1}}\frac{dy}{dx} = \frac{ay + bx^n}{cy + dx^n}$$

becomes separable and derive the equation in separated form (do not solve the DE).

Let $y = x^n z$

Then $\frac{dy}{dx} = n x^{n-1} z + x^n \frac{dz}{dx}$

$\Rightarrow \quad nz + x\frac{dz}{dx} = \frac{az+b}{cz+d}$

Hence $\frac{dz}{dx} = \frac{\left(\frac{az+b}{cz+d} - nz\right)}{x}$

is separable.

MARKS

4. A skydiver with mass M jumps out of a plane at $t = 0$ with initial vertical velocity $v(0) = 0$. Newton's second law states that mass times acceleration equals force. The force of gravity is $-Mg$, where g is the gravitational acceleration, and the resistive force due to turbulent drag is proportional to the velocity squared. (In this question the positive direction is "up").

(a) Derive the differential equation for $v(t)$ and solve it for the given initial condition. Find the terminal speed $v_\infty = \left|\lim_{t\to\infty} v(t)\right|$. If the ratio of masses of two skydivers is $\rho = M_1/M_2$, what is the ratio of their terminal velocities? 4

$$M\frac{dv}{dt} = -Mg + cv^2$$

$$\frac{dv}{dt} = -g + \frac{c}{M}v^2$$

$$\frac{1}{\frac{c}{M}(v^2-\alpha^2)}\frac{dv}{dt} = 1 \quad \text{where } \alpha = \sqrt{\frac{gM}{c}}$$

$$-\frac{M}{c}\cdot\frac{1}{2\alpha}\ln\left|\frac{\alpha+v}{\alpha-v}\right| + \tilde{C} = t$$

$$\frac{\alpha+v}{\alpha-v} = e^{-\frac{2c\alpha}{M}(t-\tilde{C})} = \beta, \text{ say}$$

So
$$v = \alpha\tanh\left(-\frac{c\alpha}{M}(t-\tilde{C})\right)$$

at $t=0$, $v=0$; $\tanh(0)=0 \Rightarrow \tilde{C}=0$.

$$v = \alpha\tanh\left(-\frac{c\alpha}{M}t\right)$$

$$\left|\lim_{t\to\infty} v(t)\right| = \alpha = \sqrt{\frac{Mg}{c}}. \qquad \sqrt{\rho} = \sqrt{\frac{M_1}{M_2}}$$

is the ratio

MARKS

(b) Using the result of part (a), find the time T it takes to fall a height of h when initially $v(0) = 0$. **3**

$$h(t) = -\int_0^t v(s)\,ds$$

$$= v_\infty \int_0^t \tanh\left(\frac{gs}{v_\infty}\right) ds \qquad (v_\infty = \alpha)$$

$$= \frac{v_\infty}{g} \ln \cosh \frac{gt}{v_\infty}$$

$$\therefore T = \frac{v_\infty}{g} \cosh^{-1}\left(\exp\left(\frac{hg}{v_\infty}\right)\right).$$

(c) For small times $h(t) \approx gt^2/2$. Using the Taylor series of $h(t)$ about $t = 0$, find the first non-zero correction term to this expression due to air resistance. **3**

$h(0) = 0$

$h' = -v$ $\qquad h'(0) = -v(0) = 0$

$h'' = -v' = g - \frac{c}{M} v^2$ $\qquad h''(0) = g$

$h''' = -2\frac{c}{M} vv'$ $\qquad h'''(0) = 0$

$h'''' = -2\frac{c}{M}(v'^2 + vv'')$ $\qquad h''''(0) = -2\, g^3/v_\infty^2$

Hence $h(t) = g\frac{t^2}{2!} - \frac{2}{4!}\frac{g^3}{v_\infty^2} t^4$

$$= g\frac{t^2}{2}\left(1 - \frac{g^2}{6v_\infty^2} t^2\right).$$