The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Board Tutorial for Week 7** 2017

(ii) A *system* of linear equations has the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

The system is *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$.

(iii) Every system of linear equations has either no solutions, one solution, or infinitely many solutions. If it has no solutions then the system is called *inconsistent*. If it has at least one solution then the system is called *consistent*.

(iv) There are three types of *elementary row operations* performed on augmented matrices:
   (a) interchanging the ith and jth rows (denoted by $R_i \leftrightarrow R_j$)
   (b) multiplying the ith row through by a nonzero constant λ (denoted by $R_i \to \lambda R_i$)
   (c) adding a multiple of the jth row to the ith row (denoted by $R_i \to R_i + \lambda R_j$)

(v) A matrix is in *row echelon form* if
   (a) rows of zeros appear at the bottom,
   (b) first nonzero (*leading*) entries of consecutive rows appear further to the right,
   (c) leading entries of rows are equal to 1;

   and in *reduced row echelon form* if, in addition,
   (d) entries above (and below) leading entries are zero.

(vi) The process of *Gaussian elimination* applies elementary row operations (*row reduction*) to transform the augmented matrix of a system into row echelon form, after which the associated system is solved using *back substitution*:
   (a) the *leading variables* corresponding to leading entries are evaluated one equation at a time from the bottom towards the top,
   (b) parameters are assigned to each nonleading variable (if any).

(vii) A system is inconsistent if and only if at some stage in the process of row reduction a row of the following form is produced for some nonzero real number k:

$$\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & k \end{array}$$

(viii) The process of *Gauss-Jordan elimination* row reduces the augmented matrix to reduced row echelon form, after which the process of back substitution simplifies. However, Gauss-Jordan elimination is usually less efficient in terms of the overall number of arithmetic operations used than Gaussian elimination.

(ix) The reduced row echelon form of a given matrix is unique.

Tutorial Exercises:

6. Solve the following homogeneous systems of equations:

(i)
$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 0 \\ 3x & + & 2y & + & z & = & 0 \end{array}$$

(ii)
$$\begin{array}{rcrcrcrcl} -x & + & y & + & z & - & w & = & 0 \\ 2x & & & + & z & + & w & = & 0 \\ x & - & 2y & + & z & + & 3w & = & 0 \end{array}$$

7. Solve the following homogeneous system of equations:

$$\begin{array}{rcrcrcrcrcl} 2x_1 & + & 3x_2 & + & x_3 & - & x_4 & + & 4x_5 & = & 0 \\ -2x_1 & - & 3x_2 & + & x_3 & + & 2x_4 & - & 3x_5 & = & 0 \\ 2x_1 & + & 3x_2 & + & 2x_3 & + & 2x_4 & + & 2x_5 & = & 0 \end{array}$$

8. Give a very brief reason why a homogeneous system can never be inconsistent.

9. Solve the following systems of equations:

(i)
$$\begin{array}{rcrcrcl} x & + & 2y & + & 7z & = & 5 \\ x & + & y & + & 4z & = & 3 \\ 2x & + & 3y & + & 11z & = & 7 \end{array}$$

(ii)
$$\begin{array}{rcrcrcrcr} x & + & 2y & + & z & - & w & = & 4 \\ 2x & + & 4y & - & z & + & 4w & = & -1 \\ -x & - & 2y & + & 2z & - & 5w & = & 5 \end{array}$$

10. A cubic polynomial in x takes the value -2 when $x = 1$ and the value -10 when $x = -1$. Its derivative takes the value 0 when $x = 1$ and the value 12 when $x = -1$. Find the polynomial.

11. The Jones family consists of Ann and Bill and their two children Cathy and Daniel. Their current ages add up to 70. Bill is three times as old as the present total age of Cathy and Daniel. In 10 years time, Ann's age will be 20 less than twice the total of the then ages of Cathy and Daniel. Four years ago, Cathy's age was equal to Ann's age minus Bill's age. Find the present ages of Ann, Bill, Cathy and Daniel.

12. Find A, B, C and D such that

$$\frac{x^3}{(x-1)^4} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4}\,.$$

13.* Catalogue all reduced row echelon 2×3 matrices.

14. Describe the possible reduced row echelon forms for augmented matrices for the following homogeneous system in α, β, γ:

$$\begin{array}{rcrcrcl} u_1\alpha & + & v_1\beta & + & w_1\gamma & = & 0 \\ u_2\alpha & + & v_2\beta & + & w_2\gamma & = & 0 \end{array}$$

Find a simple connection that explains why any three geometric vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in the plane must be linearly dependent.

15.* Find the values of λ such that the following system (i) is inconsistent; (ii) has infinitely many solutions; (iii) has a unique solution:

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & -3 \\ -2x & - & \lambda y & + & z & = & 2 \\ x & + & 2y & + & \lambda z & = & 1 \end{array}$$