MATH 1903 Lectures

Semester 2, 2012

Week 11

Daniel Daners

Summary:

DE: $ay'' + by' + cy = 0$ $\quad$ (a, b, c const)

auxiliary equation: $a\lambda^2 + b\lambda + c = 0$

Case 1: λ_1, λ_2 two distinct real roots

general solution: $y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

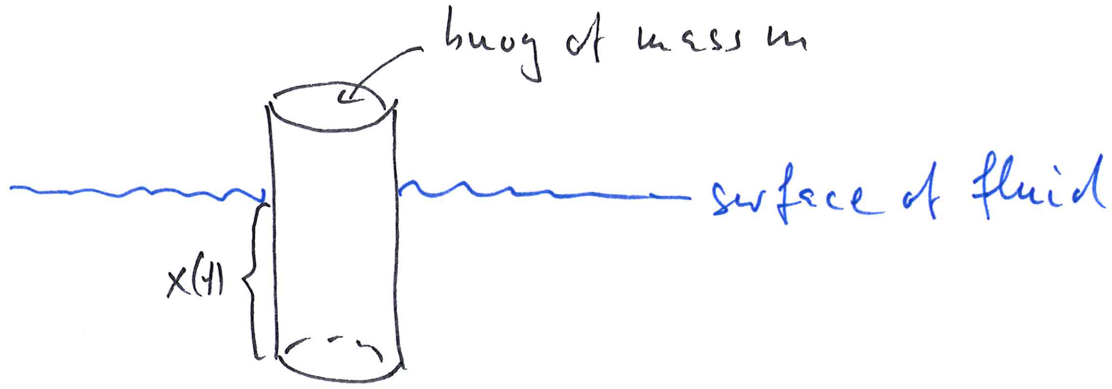
Case 2: $\lambda = \mu \pm i\omega$ pair of complex conjugate roots

general solution: $y(t) = e^{\mu t}(A\cos\omega t + B\sin\omega t)$

Case 3: $\lambda_1 = \lambda_2 = \lambda$ double root (real)

general solution: $y(t) = Ae^{\lambda t} + Bte^{\lambda t}$

Application: Buoy in a fluid



Forces acting on buoy:

- gravitational force
- force of buoyancy: proportional to volume $V(x)$ submerged
- resistive force

$x(t)$ distance the bottom of the buoy is submerged at time t

Newton's second law:

$$m\ddot{x} = mg - \alpha V(x) - R(x, \dot{x})$$

grav. force   force of buoyancy   resistive force

$V(x)$ is volume submerged. Archimedes law: force of buoyancy proportional.

• cylindrical buoy of cross-section A

$$V(x) = Ax \quad \text{volume submerged}$$

• assume the resistive force is proportional to velocity

$$R(x,\dot{x}) = \beta \dot{x} \qquad (\beta > 0)$$

Substitute into de:

$$m\ddot{x} = mg - \alpha A x - \beta \dot{x}$$

$$= -\alpha A\left(x - \frac{mg}{\alpha A}\right) - \beta \dot{x}$$

Transformation: $y = x - \frac{mg}{\alpha A}$. Then, as $\dot{y} = \dot{x}$ and $\ddot{y} = \ddot{x}$ we get

$$m\ddot{y} = -\alpha A y - \beta \dot{y}$$

Homogeneous 2nd order equation

$$m\ddot{y} + \alpha A y + \beta \dot{y} = 0$$

Solve the equation: The solutions of the auxiliary equation

$$m\lambda^2 + \beta\lambda + \alpha A = 0$$

are

$$\lambda = \frac{1}{2m}\left(-\beta \pm \sqrt{\beta^2 - 4m\alpha A}\right)$$

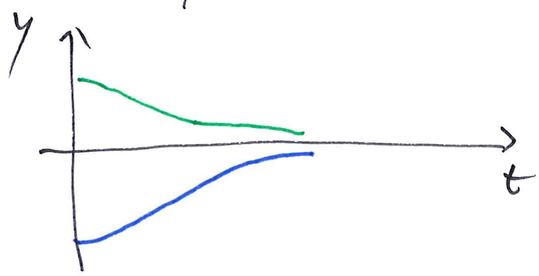
<u>Case 1:</u> $\beta^2 - 4m\alpha A > 0$

Note $\sqrt{\beta^2 - 4m\alpha A} < \beta$

Hence we have two <u>negative</u> roots λ_1, λ_2

This means, the solution

$$y(t) = Ce^{\lambda_1 t} + De^{\lambda_2 t} \longrightarrow 0 \quad \text{as } t \to \infty$$



"bury in a honey pot"

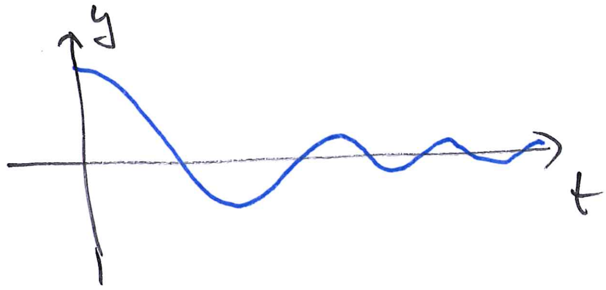
Case 2: If $\beta^2 - 4m\alpha A < 0$ we get a pair of complex conjugate solutions: $\lambda = \mu \pm i\omega$

$$\mu = -\frac{\beta}{2m}, \quad \omega = \frac{1}{2m}\sqrt{4m\alpha A - \beta^2}$$

general solution:

$$y(t) = e^{-\frac{\beta}{2m}t}\left(C\cos\omega t + D\sin\omega t\right)$$

is a damped oscillation (if $\beta > 0$):



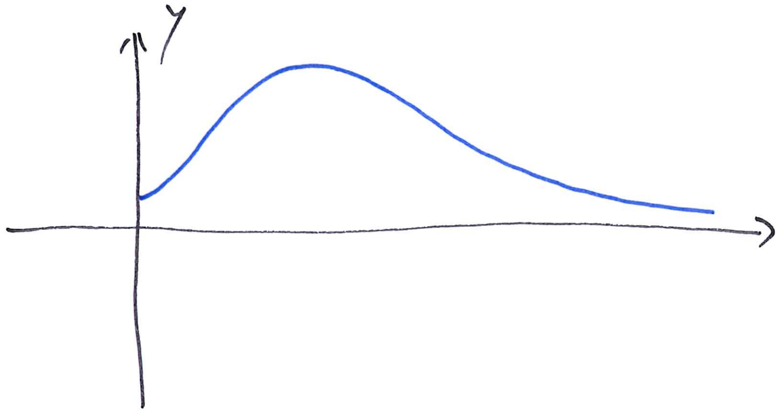
Case 3: one single root: $\beta^2 - 4m\alpha A = 0$

$$\lambda = -\frac{\beta}{2m}$$

general solution:

$$y(t) = C e^{-\frac{\beta}{2m}t} + D t e^{-\frac{\beta}{2m}t}$$

x may increase first, then decay



Inhomogeneous second order equations

$$ay'' + by' + cy = f(x)$$

inhomogeneity

Assume: • $y_p(x)$ is a particular solution of the DE.

• $y_h(x)$ is a solution to the corresponding homogeneous equation $ay'' + by' + cy = 0$

Then

$$y(x) = y_h(x) + y_p(x)$$

is a solution of the inhomogeneous equation

$$a(y_h+y_p)''+b(y_h+y_p)'+c(y_h+y_p)$$
$$=a(y_h''+y_p'')+b(y_h'+y_p')+c(y_h+y_p)$$
$$=\underbrace{(ay_h''+by_h'+cy_h)}_{=0}+\underbrace{(ay_p''+by_p'+cy_p)}_{=f(x)}=f(x)$$

General solution of $ay''+by'+cy=f(x)$:

$$y(x)=y_h(x)+y_p(x)$$

where:

$y_p(x)$ is a particular solution of the inhomogeneous DE.

$y_h(x)$ is the general solution of the homogeneous DE

Question: How to find a sol. to the inhomogeneous DE.

Example:

$$y'' - 5y' + 4y = x^2$$

Since the derivatives of a polynomial are polynomials we try to find a particular solution y_p of the form

$$y_p(x) = A + Bx + Cx^2$$

with A, B C constants to be determined.

Substitute into DE:

$$\underbrace{2C}_{y_p''} - 5\underbrace{(B + 2Cx)}_{y_p'} + 4\underbrace{(A + Bx + Cx^2)}_{y} = x^2$$

Rearrange:

$$(2C - 5B + 4A) + (-10C + 4B)x + 4Cx^2 = x^2$$

Equate coefficients of $1, x, x^2$:

x^2: $4C = 1$, so $C = \frac{1}{4}$

x : $4B - 10C = 0 \rightsquigarrow 2B - 5C = 2B - \frac{5}{4} = 0$, so $B = \frac{5}{8}$

1 : $2C - 5B + 4A = 0$

$$2 \cdot \frac{1}{4} - 5 \cdot \frac{5}{8} + 4A = \frac{4-25}{8} + 4A = -\frac{21}{8} + 4A = 0$$

Hence $A = \frac{21}{32}$

The function

$$y_p(x) = \frac{21}{32} + \frac{5}{8}x + \frac{1}{4}x^2$$

is a particular solution.

Find general solution of the homogeneous equation

$$y'' - 5y' + 4y = 0$$

auxiliary equation: $\lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$,

so the general solution is

$$y_h(x) = ae^{x} + be^{4x}$$

Hence the solution of the non-homogeneous equation is

$$\boxed{y(x) = ae^{x} + be^{4x} + \frac{21}{32} + \frac{5}{8}x + \frac{1}{4}x^2}$$

Example:

$$y'' - 3y' - 10y = 2e^{3t}$$

We try a solution of the form $y_p(t) = Ae^{3t}$

Substitute:

$$\underbrace{9Ae^{3t}}_{y''} - 3(\underbrace{3Ae^{3t}}_{y'}) - 10\underbrace{Ae^{3t}}_{y} = 2e^{3t}$$

Divide by $e^{3t} \neq 0$

$$\cancel{9A} - \cancel{9A} - 10A = 2, \quad \text{so } A = -\frac{1}{5}$$

Hence $y_p(t) = -\frac{1}{5}e^{3t}$ is a particular solution.

auxiliary eq. for hom. problem: $\lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5)$,

so $\lambda = -2, 5$

$$\boxed{\text{general solution } y(t) = ae^{-2t} + be^{5t} - \frac{1}{5}e^{3t}}$$

Example:

$$y'' - 2y' + 5y = \cos 3t$$

Note: up to a constant, the derivatives of $\cos 3t$ are $\sin 3t$ or $\cos 3t$

Hence try a solution of the form

$$y_p(t) = A\cos 3t + B\sin 3t$$

Substitute into equation:

$$\underbrace{(-9A\cos 3t - 9B\sin 3t)}_{y_p''} - 2\underbrace{(-3A\sin 3t + 3B\cos 3t)}_{y_p'}$$

$$+ 5\underbrace{(A\cos 3t + B\sin 3t)}_{y_p} = 8\cos 3t$$

Collect terms:

$-4A - 6B = 1$ (coeff. of cos 3t)

$6A - 4B = 0$ (coeff. of sin 3t)

Solve system of equations

$$\left[\begin{array}{cc|c} -4 & -6 & 1 \\ 6 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & +\frac{3}{2} & -\frac{1}{4} \\ 3 & -2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & -\frac{1}{4} \\ 0 & -\frac{4}{2}-\frac{9}{2} & \frac{3}{4} \end{array}\right] = \left[\begin{array}{cc|c} 1 & \frac{3}{2} & -\frac{1}{4} \\ 0 & -\frac{13}{2} & \frac{3}{4} \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{26} \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{13} \\ 0 & 1 & -\frac{3}{26} \end{array}\right]$$

particular solution $y_p(t) = -\frac{1}{13}\cos 3t - \frac{3}{26}\sin 3t$

Example:

$$y'' + y' - 2y = e^{-2t}$$

Try $y_p = A e^{-2t}$

Substitute:

$$\underbrace{4Ae^{-2t} - 2Ae^{-2t} - 2Ae^{-2t}}_{=0} = e^{-2t}$$

Here the inhomogeneity is a solution of the homogeneous problem. Then we try

$$y_p(t) = A\underset{\substack{\uparrow \\ \text{multiply by } t}}{t}\, e^{-2t}$$

Then $y_p'(t) = A(1-2t)e^{-2t}$

$$y_p''(t) = A\big(-2 - 2(1-2t)\big)e^{-2t} = 4A(t-1)e^{-2t}$$

$$\underbrace{4A(t-1)e^{-2t}}_{y_p''} + \underbrace{A(1-2t)e^{-2t}}_{y_p'} - 2\underbrace{Ate^{-2t}}_{y_p} = e^{-2t}$$

collect terms:

$$\underbrace{(4A-2A-2A)}_{=0}te^{-2t} + (-4A+A)e^{-2t} = e^{-2t}$$

Hence $-4A+A=-3A=1$, so $A=-\frac{1}{3}$

particular solution: $\boxed{y_p(t) = -\frac{t}{3}e^{-2t}}$

to get the general solution compute the general sol of the homogeneous problem.