MATH 185, LECTURE 3
MONDAY, JANUARY 24, 2011

Stereographic Projection.

"Stereographic projection" is a function which maps the complex plane onto the unit sphere, excluding the north pole. Including the north pole on the sphere will correspond to adding "the point at infinity" to the complex plane. Here I'll provide the outline (in class: a live demonstration and more details).

In $\mathbb{R}^3$ consider the unit sphere (called "the Riemann sphere") centered at the origin:

$$S^2 = \{(X, Y, Z) \in \mathbb{R}^3;\ X^2 + Y^2 + Z^2 = 1\}.$$

We say a point $(X, Y, Z) \in S^2$ corresponds with a point $(x, y, 0) \in \mathbb{R}^3$ if $(X, Y, Z,)$, $(x, y, 0)$, and the north pole $(0, 0, 1)$ all lie on the same straight line. Using high school math, one can derive a formula for this correspondence:

$$(X, Y, Z) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

Identifying $\{x, y, 0;\ x, y \in \mathbb{R}\}$ with $\mathbb{C}$ in the usual way gives the formula:

$$(X, Y, Z) = \left(\frac{2\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\mathrm{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

Either the mapping $(x, y, 0) \mapsto (X, Y, Z)$ or its inverse $(X, Y, Z) \mapsto (x, y, 0)$ is called stereographic projection. (It should be clear by context.) I drew some pictures and looked at what sets correspond to what sets. You should do it, too–it's fun!

Advantage of looking at the Riemann sphere: "∞" is no longer distinguished and is no longer mysterious. Now it's just another point, same as any other.

Disadvantage of looking at the Riemann sphere: There is no longer a simple interpretation (as far as I see) of addition and multiplication.

Advantage of having advantages and disadvantages: Sometimes something is easier from another point of view.

Here's a pretty fact about stereographic projection: circles and lines in the complex plane are really just circles on the Riemann sphere. (A line is a circle passing through infinity!)

Theorem 1. *Any circle on the sphere corresponds to a circle or straight line in the plane and vice versa.*

Proof. Any circle on S^2 lies in a plane

$$\alpha_1 X + \alpha_2 Y + \alpha_3 Z = \alpha_0.$$

To visualize this, we may assume that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $0 \leq \alpha_0 < 1$ (what does $\alpha_0 = 1$ correspond to? Draw a picture!). Now stereographically project to $\mathbb{C}$. Then the equation for the plane, in the complex plane's coordinates, becomes

$$2\alpha_1 x + 2\alpha_2 y + \alpha_3(x^2 + y^2 - 1) = \alpha_0(x^2 + y^2 + 1).$$

That is,

$$(\alpha_0 - \alpha_3)(x^2 + y^2) - 2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0.$$

If $\alpha_0 \neq \alpha_3$, this is the equation for a circle.
If $\alpha_0 = \alpha_3$, this is the equation for a line.

The converse: an optional exercise. □

Functions of a Complex Variable. [See the textbook.]

First big example: $f(z) = z^2$. Think geometrically. The inverse images of horizontal lines and vertical lines are hyperbolas (we found algebraic formulas and drew the pictures). The images of horizontal lines and vertical lines are parabolas. It's also useful to think of this function in terms of the polar representation.

Mapping by the Exponential Function. Define

$$f(z) = e^z, \qquad z = x + iy,$$

to be

$$f(x + iy) = e^x(\cos y + i \sin y).$$

Note: $e^z e^w = e^{z+w}$. (We did this calculation last time.) And note that, if $y = 0$, $f(x) = e^x$. We will give further support for the "e" notation when we get to power series.

Geometrically: Write $f(z) = e^z = w$. Then

$$\{\mathrm{Re}(z) \text{ fixed, } \mathrm{Im}(z) \text{ variable}\} \to \{|w| \text{ fixed, } \arg(w) \text{ variable}\}$$

and

$$\{\mathrm{Re}(z) \text{ variable, } \mathrm{Im}(z) \text{ fixed}\} \to \{|w| \text{ variable, } \arg(w) \text{ fixed}\}.$$

[Draw the picture.]

In fact, $f(z) = e^z$ takes the strip

$$\{z = x + iy;\ x \in \mathbb{R},\ y \in [0, 2\pi)\}$$

onto $\mathbb{C}$.

Hints about the future:

(1) Vertical lines are orthogonal to horizontal lines. Are the images still orthogonal? Look at the above examples.

(2) Does the function $f(z) = e^z$ have an *inverse* function?