The University of Sydney

Faculties of Arts, Economics, Education, Engineering and Science

MATH1901/1906

Differential Calculus (Advanced)

June 2013 Lecturer: Anne Thomas

Time Allowed: One and a half hours

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

Marker's use only	

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination; there are 20 questions; the questions are of equal value; all questions may be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination; there are 4 questions; the questions are of equal value; all questions may be attempted; working must be shown.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

MARKS

1. (a) Sketch the set $\{z \in \mathbb{C} \mid z + \overline{z} = \sqrt{2}|z|\}$. **3**

(b) Let $A = \{z \in \mathbb{C} \mid z \neq -1\}$. Show that the function $f : A \to \mathbb{C}$ given by $f(z) = \dfrac{z}{z+1}$ is injective, and find the range of f. **3**

(c) Consider the set B of complex numbers which forms the boundary of the square with corners $P = -1$, $Q = 1$, $R = 1 + 2i$ and $S = -1 + 2i$. Sketch the image of B under the exponential function $z \mapsto e^z$. Your sketch must indicate the images P', Q', R' and S' of P, Q, R and S respectively. **4**

(d) Suppose that a function $f : [0,1] \to \mathbb{R}$ is continuous, with $f(0) = 0$ and $f(1) = 1$. Prove that for every $d \in (0,1)$, there is a $c \in (0,1)$ such that $f(c) = d$. **2**

2. (a) Find the following limits, showing the steps of your working clearly, or show that the limit does not exist. You may use any valid method. Allow $+\infty$ and $-\infty$ as values that a limit can take.

(*i*) $\displaystyle \lim_{t \to 0} \frac{t}{\sqrt{6-t} - 2}$ **2**

(*ii*) $\displaystyle \lim_{x \to \infty} \left(\cos\left(\frac{3}{x}\right)\right)^x$ **3**

(*iii*) $\displaystyle \lim_{(x,y) \to (0,0)} \frac{5(x+y)^2}{\sqrt{x^2+y^2}}$ **3**

(b) The following statement is a special case of the Quotient Law for limits:

$$\text{if } \lim_{x \to a} g(x) = \ell \text{ and } \ell > 0 \text{ then } \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{\ell}.$$

Prove this statement using the ϵ, δ definition of limit. You may assume that $g(x)$ is never equal to 0. **4**

MARKS

3. (a) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial and let $a \in \mathbb{R}$. Suppose there is some polynomial $q : \mathbb{R} \to \mathbb{R}$ such that $p(x) = (x-a)^3 q(x)$. Show that a is a root of both p' and p''. **2**

(b) Let $f(x) = \cosh x$ and $g(x) = \cosh(x^3)$.

(*i*) Compute the Taylor polynomial $T_4(x)$ of order 4 about 0 for the function f. **2**

(*ii*) Let $T_{17}(x)$ be the Taylor polynomial of order 17 about 0 for the function g. Use your answer to (*i*) to write down $T_{17}(x)$, and deduce that $T_{17}(1) = 37/24$. You do not need to give any reasons for $T_{17}(x)$ being the required Taylor polynomial. **2**

(*iii*) Use part (*ii*) and Lagrange's form of the remainder to show that there is a c between 0 and 1 such that **3**

$$g^{(18)}(c) = 18!\left(\frac{e+e^{-1}}{2} - \frac{37}{24}\right).$$

(c) Suppose that a function $g : \mathbb{R} \to \mathbb{R}$ is continuous at 0. Let $f(x) = xg(x)$. Use the definition of the derivative as a limit to prove that f is differentiable at 0 and find $f'(0)$ in terms of g. **3**

4. (a) Consider the function $f(x,y) = \dfrac{y^2 - x^2}{x^2 + y^2}$.
Find the domain of f, and draw the level curves of f at heights $1/2$ and 1. **3**

(b) Consider the function $g(x,y) = (x\cos y - y\sin x)$.
Let $g(x,y) = \pi/2$ be an implicit relationship between x and y. Find $\dfrac{dy}{dx}$ and hence find the equation of the tangent line to the curve $g(x,y) = \pi/2$ at the point $(\pi/2, 0)$. **2**

(c) Consider the function $h(x,y) = 2228 - \dfrac{1}{100}(x-14)^2 - \dfrac{1}{25}(y+43)^2$.
Suppose a hill has shape given by the equation $z = h(x,y)$. You are standing at the point P with coordinates $(64, 57, 1803)$. The positive x-axis points east and the positive y-axis points north.

(*i*) If you walk due south, will you start to go up or go down? If you walk north-east, will you start to go up or go down? **3**

(*ii*) Find the direction in which the slope is greatest, and the two directions in which you can walk and, at least initially, stay at the same height. Leave your answers as vectors. **2**

(*iii*) Write down the coordinates of Q, the highest point on the hill, and the equation of the tangent plane to the hill at Q. Show that there are no other points R on the hill so that the tangent plane to the hill at R is parallel to the tangent plane to the hill at Q. **3**

Standard Derivatives

The following derivatives can be quoted without proof unless a question specifically asks you to show details. These results can be combined with the standard rules of differentiation (not listed here) to differentiate more complicated functions. For example, $(d/dx)\sin(ax+b) = a\cos(ax+b)$. Natural domains common to both sides are assumed.

1. $\dfrac{d}{dx}\,x^k = kx^{k-1} \quad (k \in \mathbb{R})$

2. $\dfrac{d}{dx}\,e^x = e^x$

3. $\dfrac{d}{dx}\ln x = \dfrac{1}{x} \quad (x > 0)$

4. $\dfrac{d}{dx}\sin x = \cos x$

5. $\dfrac{d}{dx}\cos x = -\sin x$

6. $\dfrac{d}{dx}\tan x = \sec^2 x$

7. $\dfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

8. $\dfrac{d}{dx}\sec x = \sec x \tan x$

9. $\dfrac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

10. $\dfrac{d}{dx}\sinh x = \cosh x$

11. $\dfrac{d}{dx}\cosh x = \sinh x$

12. $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$

13. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$

14. $\dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$

15. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$

16. $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$

17. $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}} \quad (x > 1)$

18. $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2} \quad (|x| < 1)$

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.