MATH3701
Higher Topology and Differential Geometry
Assignment

1. Consider $d(l\gamma(u)), u \in I$. Note that l is linear and so $dl = l$.

$$\begin{aligned} d(l \circ \gamma(u)) &= d\gamma(u) \cdot dl(\gamma(u)) \\ &= d\gamma(u) \cdot l \circ \gamma(u) \quad \text{as } l \text{ is linear} \\ &= d\gamma(u) \cdot c. \end{aligned}$$

We know that γ sends u into C, which is inside the plane P. For any $x = \gamma(u)$ inside the plane, by definition, $l \circ \gamma(u) = l(x) = c$. From above, we have $d(l \circ \gamma(u)) = d\gamma(u) \cdot c$. Alternatively, we can differentiate as follows:

$$\begin{aligned} \therefore d(l \circ \gamma(u)) &= d(l(x)) \\ &= d(c) \\ &= 0 \\ \therefore d\gamma(u) \cdot c &= 0 \\ d\gamma(u) &= 0. \end{aligned}$$

Thus all derivatives of γ are zero, so they cannot be linearly independent. Hence γ is not Frenet. The curve γ lies within a plane, and so it can only curve within a strict range. Given we know $d\gamma(u) = 0$, then the curve is not moving out of this plane and so the derivatives must be scalar multiples, and hence γ is not Frenet.

2.

$$\begin{aligned} \gamma(u) &= (-u, \sin 2u, \cos 2u)^{\mathsf{T}} \\ \dot\gamma(u) &= (-1, 2\cos 2u, -2\sin 2u)^{\mathsf{T}} \\ \ddot\gamma(u) &= (0, -4\sin 2u, -4\cos 2u)^{\mathsf{T}}. \end{aligned}$$

Clearly these are independent and so γ is Frenet. For the Frenet Frame we have:

$$\varepsilon_1(u) = \frac{\dot\gamma(u)}{|\dot\gamma(u)|} = \frac{1}{\sqrt{5}}(-1, 2\cos 2u, -2\sin 2u)^{\mathsf{T}}.$$

Notice that $\langle \varepsilon_1(u), \ddot\gamma(u) \rangle = 0$ and so $\ddot\gamma(u)$ is perpendicular to $\varepsilon_1(u)$. Hence for $\varepsilon_2(u)$ we simply normalise $\ddot\gamma(u)$.

$$\varepsilon_2(u) = (0, -\sin 2u, -\cos 2u)^{\mathsf{T}}.$$

For $\varepsilon_3(u)$ we take the cross product of the first two vectors in the distinguished Frenet frame.

$$\begin{aligned} \varepsilon_3(u) &= \varepsilon_1(u) \times \varepsilon_2(u) \\ &= \frac{1}{\sqrt{5}}(-1, 2\cos 2u, -2\sin 2u)^{\mathsf{T}} \times (0, -\sin 2u, -\cos 2u)^{\mathsf{T}} \\ &= \frac{1}{\sqrt{5}}(-2, -\cos 2u, \sin 2u)^{\mathsf{T}}. \end{aligned}$$

Note $\det(\varepsilon_1(u), \varepsilon_2(u), \varepsilon_3(u)) = -1$, we negate $\varepsilon_3(u)$ yielding a determinant of $+1$. Thus the distinguished Frenet frame for γ is:

$$\begin{aligned} \varepsilon_1(u) &= \frac{1}{\sqrt{5}}(-1, 2\cos 2u, -2\sin 2u)^{\mathsf{T}} \\ \varepsilon_2(u) &= (0, -\sin 2u, -\cos 2u)^{\mathsf{T}} \\ \varepsilon_3(u) &= \frac{1}{\sqrt{5}}(2, \cos 2u, -\sin 2u)^{\mathsf{T}}. \end{aligned}$$

3.

(a) For any i, j, we have $\langle \varepsilon_i(u), \varepsilon_j(u) \rangle = 0$ or 1. In either case we know,

$$\frac{d}{du} \langle \varepsilon_i(u), \varepsilon_j(u) \rangle = 0.$$

Using the product rule:

$$\begin{aligned} \langle \dot{\varepsilon}_i(u), \varepsilon_j(u) \rangle + \langle \varepsilon_i(u), \dot{\varepsilon}_j(u) \rangle &= 0 \\ \langle \dot{\varepsilon}_i(u), \varepsilon_j(u) \rangle &= -\langle \varepsilon_i(u), \dot{\varepsilon}_j(u) \rangle \\ \langle \dot{\varepsilon}_i(u), \varepsilon_j(u) \rangle &= -\langle \dot{\varepsilon}_j(u), \varepsilon_i(u) \rangle \\ \therefore w_{ij} &= -w_{ji}.. \end{aligned}$$

(b) Proposition 2 tells us that for $1 \leq i \leq m-1$, then $Span(\varepsilon_1(u), ..., \varepsilon_i(u)) = Span(\dot{\gamma}(u), \ddot{\gamma}(u), ..., \gamma^{(i)}(u))$. Clearly $\varepsilon_i(u) \in Span(\varepsilon_1(u), ..., \varepsilon_i(u)) = Span(\dot{\gamma}(u), \ddot{\gamma}(u), ..., \gamma^{(i)}(u))$. Hence we can write,

$$\varepsilon_i(u) = \alpha_1 \dot{\gamma}(u) + \alpha_2 \ddot{\gamma}(u) + ... + \alpha_i \gamma^{(i)}(u).$$

Differentiating yields:

$$\begin{aligned} \dot{\varepsilon}_i(u) &= \alpha_1 \ddot{\gamma}(u) + ... + \alpha_i \gamma^{(i+1)}(u). \\ \implies \dot{\varepsilon}_i(u) &\in Span(\ddot{\gamma}(u), ..., \gamma^{(i+1)}(u)). \end{aligned}$$

Note that $Span(\ddot{\gamma}(u), ..., \gamma^{(i+1)}(u))$ is a subset of $Span(\dot{\gamma}(u), \ddot{\gamma}(u), ..., \gamma^{(i+1)}(u)) = Span(\varepsilon_1(u), ..., \varepsilon_{i+1}(u))$.

$$\therefore \dot{\varepsilon}_{i+1}(u) \in Span(\dot{\gamma}(u), \ddot{\gamma}(u), ..., \gamma^{(i+1)}(u)).$$

(c) First note that given $\dot{\varepsilon}_i(u) \in Span(\varepsilon_1(u), \varepsilon_2(u), ..., \varepsilon_{i+1}(u))$ then we can write.

$$\begin{aligned} \dot{\varepsilon}_i(u) &= \alpha_1 \varepsilon_1(u) + \alpha_2 \varepsilon_2(u) + ... + \alpha_{i+1} \varepsilon_{i+1}(u). \\ \implies w_{ij} &= \langle \dot{\varepsilon}_i(u), \varepsilon_j(u) \rangle \\ &= \langle \alpha_1 \varepsilon_1(u) + \alpha_2 \varepsilon_2(u) + ... + \alpha_{i+1} \varepsilon_{i+1}(u), \varepsilon_j(u) \rangle \\ &= \begin{cases} 0 & j > i+1 \\ \alpha_j & j \leq i+1. \end{cases} \end{aligned}$$

We can obtain a similar expression for w_{ji},

$$w_{ji} = \begin{cases} 0 & i > j+1 \\ \beta_i & i \leq j+1. \end{cases}$$

We also know that $w_{ij} = -w_{ji}$. Hence we have the following cases:
(1) $i > j+1$ or $j > i+1$:
In this case either $w_{ij} = 0$ or $w_{ji} = 0$. Either way, as $w_{ij} = -w_{ji}$, we know that $w_{ij} = 0$.
(2) $i \leq j+1$ and $j \leq i+1$, this is equivalent to saying $|i-j| = 1$:
$w_{ij} = \alpha_j$ and $w_{ji} = \beta_i$. Hence we also have the relationship $\alpha_j = -\beta_i$.
Thus $w_{ij} = 0$ unless $|i-j| = 1$.

4. Note we have the following distinguished Frenet frame for γ.

$$\begin{aligned} \varepsilon_1(u) &= \frac{1}{\sqrt{5}}(-1, 2\cos 2u, -2\sin 2u)^\mathsf{T} \\ \varepsilon_2(u) &= (0, -\sin 2u, -\cos 2u)^\mathsf{T} \\ \varepsilon_3(u) &= \frac{1}{\sqrt{5}}(2, \cos 2u, -\sin 2u)^\mathsf{T}. \end{aligned}$$

Computing $\kappa_1(u)$ and $\kappa_2(u)$:

$$\begin{aligned}
\kappa_1(u) &= \frac{\langle \dot{\varepsilon}_1(u), \varepsilon_2(u)\rangle}{|\dot{\gamma}(u)|} \\
&= \frac{\langle \frac{1}{\sqrt{5}}(0, -4\sin 2u, -4\cos 2u)^{\mathsf{T}}, (0, -\sin 2u, -\cos 2u)^{\mathsf{T}}\rangle}{\sqrt{5}} \\
\therefore \kappa_1(u) &= \frac{4}{5}. \\
\kappa_2(u) &= \frac{\langle \dot{\varepsilon}_2(u), \varepsilon_3(u)\rangle}{|\dot{\gamma}(u)|} \\
&= \frac{\langle (0, -2\cos 2u, 2\sin 2u)^{\mathsf{T}}, \frac{1}{\sqrt{5}}(2, \cos 2u, -\sin 2u)^{\mathsf{T}}\rangle}{\sqrt{5}} \\
\therefore \kappa_2(u) &= -\frac{2}{5}.
\end{aligned}$$

5. Given a unit speed Frenet curve $\gamma : I \to \mathbb{R}^m$, we know that $|\dot{\gamma}(u)| = 1$ and,

$$C(u) = (\dot{\gamma}(u), \ddot{\gamma}(u), ..., \gamma^{(m)}(u)).$$

We can also express each column $\gamma^{(i)}(u)$ as a linear combination of the first i vectors in the Frenet frame.

$$\gamma^{(i)}(u) = \alpha_1\varepsilon_1(u) + \alpha_2\varepsilon_2(u) + ... + \alpha_i\varepsilon_i(u).$$