THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 9

MATH1901: Differential Calculus (Advanced) Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ L'Hôpital's Rule;
- ☐ Taylor Polynomials;
- ☐ Differentiability.

Outcomes

After completing this tutorial you should

- ☐ use L'Hôpital's Rule to compute limits;
- ☐ construct Taylor polynomials of various functions;
- ☐ understand practical and theoretical properties of derivatives.

Summary of essential material

L'Hôpital's Rule: Suppose that f and g are differentiable in a neighbourhood of a but not necessarily at $x = a$. Further assume that either $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or $f(x) \to \pm\infty$ and $g(x) \to \infty$ as $x \to a$. (We say $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is of is of type $0/0$ or $\pm\infty/\infty$.) If $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists (or is $\pm\infty$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

If $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ is still of type $0/0$, then we can apply L'Hôpital's rule again: If $\lim\limits_{x\to a} \dfrac{f''(x)}{g''(x)}$ exists (or is $\pm\infty$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}.$$

More applications are possible if necessary. The given limits are not always in the form of a ratio, but need to be brought into that form. Commonly used methods:

- $fg = \dfrac{f}{1/g}$
- $f(x)^x = e^{x\ln f(x)}$, then compute the limit of the exponent $x \ln f(x) = \frac{\ln f(x)}{1/x}$ and use the continuity of the exponential function. This method can also be used for limits of the form $f(x)^{g(x)}$.

Taylor Polynomials: Let $f(x) : (a, b) \to \mathbb{R}$ be a function differentiable at least n times at $x = x_0$. The *n-th order Taylor polynomial* of $f(x)$ centred at $x = x_0$ is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

where, by convention, $f^{(0)}(x) = f(x)$ and $0! = 1$.

Note: The n-th order Taylor polynomial provides the best approximation of the function f near x_0 by a polynomial of order n. In particular, it is uniquely determined by the condition

$$f^{(k)}(x_0) = T_n^{(k)}(x_0) \qquad \text{for } k = 0, 1, \dots, n.$$

(All derivatives up to order n coincide with those of f.)

Questions to complete during the tutorial

1. Find the following limits. Some need L'Hôpital's rule, others can be done without.

(a) $\displaystyle\lim_{x\to -1} \frac{x^6+x^4-2}{x^4-1}$

(b) $\displaystyle\lim_{x\to \pi} \frac{\tan x}{x-\pi}$

(c) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\ln(\ln x)}$

(d) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$

(e) $\displaystyle\lim_{x\to \infty} \frac{x}{\sqrt{x^2+1}}$

(f) $\displaystyle\lim_{x\to \infty} x^{1/x}$

2. Find the Taylor polynomial $T_5(x)$ of order five about $x=0$ for each of the following functions.

(a) $f(x) = \cosh x$

(b) $f(x) = \ln(1+x)$

(c) $f(x) = \sqrt{1+x}$

3. Let $\alpha > 0$. Show that $\displaystyle\lim_{x\to 0+} x^\alpha \ln x = 0$.

4. Find the n-th order Taylor polynomial of $f(x) = \dfrac{1}{1-x}$.

5. We know that the n-th order Taylor polynomial of a function f centred at x_0 is the unique polynomial T_n such that $f^{(k)}(x_0) = T_n^{(k)}(x_0)$ for $k = 0, 1, \dots, n$. Use this characterisation to derive the following facts.

(a) Suppose that T_n is the n-th order Taylor polynomial of f centred at x_0. Let $g := f'$. Show that T_n' is the Taylor polynomial g of order $(n-1)$ centred at x_0.

(b) How can you find the Taylor polynomial of f if you have the one for $g = f'$?

(c) Suppose that T_n is the n-th order Taylor polynomial of f centred at 0. Let $g(x) := f(ax^2)$ with $a \in \mathbb{R}$. Show that $T_n(ax^2)$ is the $2n$-th order Taylor polynomial of g centred at 0.

(d) Use the above facts to find the Taylor polynomials of order n centred at 0 for the following functions. In each case think about why it is easier than a direct computation.

(i) e^{-x^2} using the Taylor polynomial of e^x.

(ii) $\ln(1-x)$ using the Taylor polynomial of the derivative.

(iii) $\dfrac{1}{1+x^2}$ using the Taylor polynomial of $\dfrac{1}{1-x}$

(iv) $\tan^{-1}(x)$ using the Taylor polynomial of the derivative.

(v) $\cos x$ using the Taylor polynomial of $\sin x$.

6. Define a function f by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable everywhere and that f' is not continuous at 0. Thus we cannot compute $f'(0)$ by using the formula $x^2 \sin \frac{1}{x}$ to calculate $f'(x)$ for $x \neq 0$ and then taking a limit.

7. The derivative of a function does not need to be continuous as the example in Question 6 shows. However, the nature of such a discontinuity must be quite complicated as the following facts show.

(a) Assume that $f : (a,b) \to \mathbb{R}$ is differentiable and that $\lim_{x\to x_0} f'(x) = L$ exists. Use L'Hôpital's rule to prove that f' is continuous at x_0. (Such a statement is certainly not true for arbitrary functions!)

(b) Hence show that the function given by $f(x) := 1$ for $x \neq 0$ and $f(0) := -1$ on $\mathbb{R}$ cannot be the derivative of any function.

Extra questions for further practice

8. Compute the following limits.

(a) $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$

(b) $\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x}$

(c) $\lim_{x \to 0^+} (\sinh \frac{4}{x})^x$

(d) $\lim_{x \to 0} \dfrac{2^x - 1}{x}$

(e) $\lim_{x \to \infty} (1 + e^{-x})^x$

(f) $\lim_{x \to \infty} \dfrac{x^{-1/2} + x^{-3/2}}{x^{-1/2} - x^{-3/2}}$

9. Use induction on n and L'Hôpital's rule to prove that $\lim_{x \to 0^+} x(\ln x)^n = 0$ for $n \in \mathbb{N}$.

10. Using the 5th order Taylor polynomial of $f(x) = \ln(1 + x)$ (see Question 2) to approximate $\ln 2$ we get

$$\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.78\dot{3}.$$

This is not so impressive, because $\ln 2 = 0.693147\ldots$ In fact it turns out that you need to use the Taylor polynomial of order 1565237 to get $\ln 2$ correct to only 6 decimal places! We can do much better using the function

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)$$

and noticing that $f(1/3) = \ln 2$.

(a) Find the general formula of the Taylor polynomial of $f(x)$ about $x = 0$.
Hint: $f(x) = \ln(1 + x) - \ln(1 - x)$.

(b) Use the Taylor polynomial $T_5(1/3)$ to approximate $\ln 2$.

11. Let f and g be differentiable at $x = a$, with $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. A proposed "converse" to L'Hôpital's Rule reads as follows:

"If $\lim_{x \to a} \dfrac{f'(x)}{g'(x)}$ does not exist, then $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ does not exist."

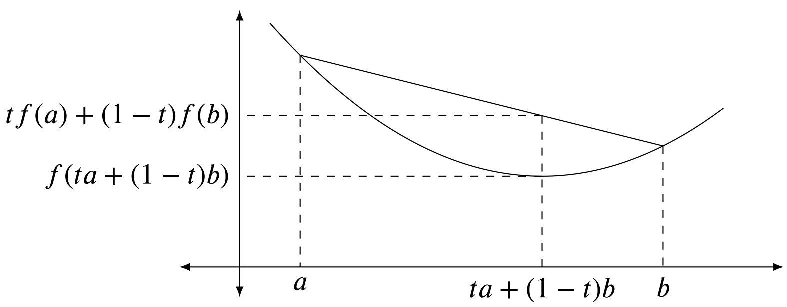
By considering $f(x) = x^2 \sin(1/x)$ and $g(x) = x$, sh that the above statement is false.

Challenge questions (optional)

12. (Very challenging!) Use the Mean Value Theorem to show that if $f''(x) \geq 0$ for all $x \in [a, b]$ then

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \qquad \text{for all } t \in [0, 1].$$

Geometrically this says that f is concave up on $[a, b]$:



Hint: Let $p_t = ta + (1-t)b$. Apply MVT twice – once on $[a, p_t]$, and also on $[p_t, b]$.