THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1	**Tutorial Week 4**	2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

For each of the following functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, sketch its graph and decide whether the function is injective, whether it is surjective, and whether it is bijective.

(a) $f(x) = |x|$, (b) $f(x) = x^3 + 1$, (c) $f(x) = \begin{cases} -x^2, & x < 0, \\ 0, & 0 \le x \le 1, \\ x - 1, & x > 1. \end{cases}$

Questions for the tutorial

2. Each formula below belongs to a function defined on some subset of $\mathbb{R}$. For each one, find the natural domain of the function (that is, the largest subset of $\mathbb{R}$ for which the rule makes sense) and the corresponding range. For the domain and range you found, decide if the function is a bijection and if so, find a formula for the inverse function.

(a) $f(x) = \dfrac{x-2}{x+2}$, (b) $f(x) = \sqrt{2+5x}$, (c) $f(x) = x|x| + 1$.

3. Explain why the functions given by the formulas and domains below are injective. Find their ranges and formulas for their inverses.

(a) $f(x) = x^2 + x, \ x \ge -\frac{1}{2}$. (b) $g(x) = \sqrt[4]{x}, \ x \ge 0$.

(c) $h(x) = \dfrac{1+e^x}{1-e^x}, \ x \ne 0$. (d) $f(x) = \ln(3 + \sqrt{x-4}), \ x \ge 5$.

4. For what values of the constants a, b, c (with $b \ne 0$) is the function with formula

$$f(x) = \frac{x-a}{bx-c}, \qquad \text{and domain } x \ne \frac{c}{b},$$

equal to its own inverse? (*Hint*: It may help to draw the graph.)

5. The function $\cosh : (0, \infty) \longrightarrow (1, \infty)$ is a bijection and the function $\cosh : (-\infty, 0) \longrightarrow (1, \infty)$ is also a bijection.

(a) Let $\cosh^{-1}$ denote the inverse function of the first-mentioned bijection. Show that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$.

(b) How would the answer change if you used $(-\infty, 0)$ as the domain of cosh?

6. Show that

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y, \text{ and}$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

for all x, $y \in \mathbb{R}$.

7. (a) Suppose that $g(x)$ is (strictly) increasing on the domain $D \subseteq \mathbb{R}$, i.e. if $x_1, x_2 \in D$ and $x_1 < x_2$, then $g(x_1) < g(x_2)$. Suppose that $f(x)$ is increasing on $E \subseteq \mathbb{R}$, and that $f(E) \subseteq D$. Prove that the composite function $g(f(x))$ is increasing on E.

(b) Using the result of the previous part, and the fact that e^x is increasing on the whole real line, prove that $\cosh x$ is increasing on $[0, \infty)$.

8. If $f : A \to B$ and $g : B \to C$ are injective functions, then the composite function $g \circ f : A \to C$ is also injective. In this question you are asked to construct a proof of this result, starting with the sentence:

"Let $a_1, a_2 \in A$, and assume that $a_1 \neq a_2$."

Complete the proof using five of the following eight phrases (in some order), together with appropriate logical connecting words (e.g. "because", "so", "therefore").

(1) f is injective
(2) g is injective
(3) $g \circ f$ is injective
(4) $b_1, b_2 \in B$
(5) $b_1 \neq b_2$
(6) $f(a_1) \neq f(a_2)$
(7) $g(b_1) \neq g(b_2)$
(8) $g(f(a_1)) \neq g(f(a_2))$

Extra Questions

9. Is the following statement true or false: "a function $f : \mathbb{R} \to \mathbb{R}$ is injective if and only if f is either strictly increasing or strictly decreasing"?
If you think it is true, give a proof. If you think it is false, give a counter-example.

10. An (infinite) set $\mathcal{C}$ is said to be *countable* if there exists a bijection between $\mathcal{C}$ and the set $\mathbb{N}$ of natural numbers. This means that the elements of $\mathcal{C}$ can be listed as $c_1, c_2, c_3, \cdots$ so that every element of $\mathcal{C}$ will appear, sooner or later, in the list. For example, $\mathbb{Z}$ is countable, because you can list all the integers in the order $0, 1, -1, 2, -2, 3, -3, \cdots$.
Is the set $\mathbb{Q}$ of all rational numbers countable? How about $\mathbb{R}$ and $\mathbb{C}$?

Solution to Question 1

(a) The function is neither injective nor surjective (hence not bijective).
(b) Each horizontal line meets the graph of f at exactly one point. Hence f is bijective.
(c) Each horizontal line meets the graph in at least one point; the function is surjective. But the horizontal line $y = 0$ meets the graph at infinitely many points, hence f is not injective.