THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial for Week 7

MATH1903: Integral Calculus and Modelling (Advanced) Semester 1, 2012

Web Page: http://www.maths.usyd.edu.au/u/UG/JM/MATH1903/
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Topics covered

In lectures last week:

- ☐ Taylor polynomials and the remainder term.
- ☐ Taylor series. Examples: $e^x, \cos x, \sin x, \cosh x, \sinh x, \ln(1+x), \tan^{-1} x, (1+x)^\alpha$.

Objectives

After completing this tutorial sheet you will be able to:

- ☐ Compute Taylor polynomials.
- ☐ Understand that error bounds are an essential part to any good approximation.
- ☐ Be able to use the remainder term to find polynomial bounds for a function.
- ☐ Show that certain Taylor series converge by showing that $R_n(x)$ tends to 0.
- ☐ Find Taylor series of complicated functions by using the Taylor series of the basic building blocks of the function.
- ☐ Approximate integrals using Taylor polynomials and series.

Preparation questions to do *before* class

1. (a) Compute the Taylor series for $\cos x$ about $x = 0$. Show that the Taylor series converges to $\cos x$ for all $x \in \mathbb{R}$.

Solution: We compute all the derivatives, and evaluate them at 0:

$$\begin{aligned} f(x) &= \cos x \\ f^{(1)}(x) &= -\sin x \\ f^{(2)}(x) &= -\cos x \\ f^{(3)}(x) &= \sin x \\ f^{(4)}(x) &= \cos x \\ &\vdots \end{aligned} \quad \Longrightarrow \quad \begin{aligned} f(0) &= 1 \\ f^{(1)}(0) &= 0 \\ f^{(2)}(0) &= -1 \\ f^{(3)}(0) &= 0 \\ f^{(4)}(0) &= 1 \\ &\vdots \end{aligned}$$

So we see that $f^{(k)}(0)$ repeats: $1, 0, -1, 0, 1, 0, -1, 0, \ldots$. Therefore the Taylor series is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

At this stage we need to resist the temptation of writing "$\cos x =$ Taylor series". This is something we need to *prove*. To do this we need to show that

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \text{for all } x \in \mathbb{R}.$$

By Taylor's Theorem we have

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}\left(\frac{d^{n+1}}{dx^{n+1}}\cos x\right)\Bigg|_{x=c} \qquad \text{for some } c \text{ between } 0 \text{ and } x.$$

But the $(n+1)$-th derivative of $\cos x$ is one of $\cos x, -\cos x, \sin x, -\sin x$, and hence

$$\left|\frac{d^{n+1}}{dx^{n+1}}\cos x\right| \le 1 \qquad \text{for all } x \in \mathbb{R}.$$

Therefore

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \quad \text{for all } x \in \mathbb{R},$$

and by the ratio test for sequences this converges to 0 for each fixed $x \in \mathbb{R}$. Hence $R_n(x) \to 0$ for each x, and so the Taylor series converges to $\cos x$ for all $x \in \mathbb{R}$. So now we can write the equality

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{for all } x \in \mathbb{R}.$$

(b) Write down the Taylor series for $\cos(x^3)$.

Solution: Just replace x by x^3 in the Taylor series for $\cos x$. The Taylor series is

$$1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \cdots,$$

and this series converges to $\cos(x^3)$ for all $x \in \mathbb{R}$.

2. Approximate, with error bounds, the integral $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$.

Solution: By Taylor's Theorem we have

$$\sin x = T_6(x) + R_6(x),$$

where

$$T_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad \text{and} \qquad R_6(x) = \frac{\sin'''''''(c)}{7!}x^7$$

for some c between 0 and x. Since $\sin'''''''(c) = -\cos c$ we have

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{\cos c}{7!}x^6$$

for some c between 0 and x (this formula is also true when $x = 0$, if the left hand side is interpreted as a limit). Since $-1 \le \cos c \le 1$ we have

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \le \frac{\sin x}{x} \le 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!}.$$

Integrating between $x = 0$ and $x = 1$ gives

$$1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} \le \int_0^1 \frac{\sin x}{x}\,dx \le 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} + \frac{1}{7\cdot 7!},$$

and so

$$0.94608\ldots \le \int_0^1 \frac{\sin x}{x}\,dx \le 0.94613\ldots,$$

and hence

$$\int_0^1 \frac{\sin x}{x}\,dx = 0.946\ldots$$

with the first 3 decimals correct.

Questions to attempt in class

3. (a) Compute the nth order Taylor polynomial of $f(x) = \ln(1 + x)$ about $x = 0$.

Solution: The derivatives of $f(x)$ are:

$$\begin{aligned}
f(x) &= \ln(1+x) & & & f(0) &= 0 \\
f^{(1)}(x) &= (1+x)^{-1} & & & f^{(1)}(0) &= 1 \\
f^{(2)}(x) &= -1!(1+x)^{-2} & & & f^{(2)}(0) &= -1! \\
f^{(3)}(x) &= 2!(1+x)^{-3} & &\Longrightarrow & f^{(3)}(0) &= 2! \\
f^{(4)}(x) &= -3!(1+x)^{-4} & & & f^{(4)}(0) &= -3! \\
&\vdots & & & &\vdots \\
f^{(n)}(x) &= (-1)^{n-1}(n-1)!(1+x)^{-n} & & & f^{(n)}(0) &= (-1)^{n-1}(n-1)!
\end{aligned}$$

Therefore the nth order Taylor polynomial is

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}.$$

(b) Use Taylor's Theorem to write down an expression for the remainder term.

Solution: By Taylor's Theorem

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \qquad \text{for some } c \text{ between } 0 \text{ and } x.$$

Since $f^{(n+1)}(x) = (-1)^n n!(1+x)^{-n-1}$, we have

$$R_n(x) = (-1)^n \frac{x^{n+1}}{n+1}(1+c)^{-n-1} \quad \text{for some } c \text{ between } 0 \text{ and } x.$$

(c) Deduce that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for all } x \in [0,1].$$

(This equation actually holds for $x \in (-1, 1]$).

Solution: We need to show that $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in [0,1]$. Since $0 \le c \le 1$ we have $(1+c)^{-n-1} \le 1$. Therefore for $x \in [0,1]$ we have

$$|R_n(x)| \le \frac{x^{n+1}}{n+1}.$$

This tends to zero for each fixed $x \in [0,1]$, and so the Taylor series converges to $f(x)$ for all $x \in [0,1]$.

Remark: The series actually converges for all $x \in (-1, 1]$.

4. Recall that the error function is $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt$. Assuming that any reasonable series manipulations are valid, derive a series formula for $\operatorname{erf}(x)$.

Solution: Using the Taylor series formula for e^{-t}, and replacing t by t^2, we get

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k} \right) dt \\
&= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{k!} \int_0^x t^{2k}\, dt \right) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)\cdot k!} x^{2k+1}
\end{aligned}$$

In the move from line 1 to line 2 we have assumed that we can integrate the series term-by-term. Indeed we can do this, but it requires justification. Question 10 shows you how to do this.

5. (a) Use Taylor's Theorem to show that for all $x \geq 0$

$$1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \leq \frac{1}{\sqrt{1+x}} \leq 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4.$$

Solution: The strategy here is to write

$$f(x) = T_3(x) + R_3(x)$$

and then use Taylor's Theorem to bound $R_3(x)$.

First, let's compute the Taylor polynomial $T_3(x)$. You could use what you know about the binomial series from lectures, but let's just do it by hand in this question. We'll compute the first 4 derivatives (because we will need the 4th one for the remainder $R_3(x)$).

$$\begin{aligned}
f(x) &= (1+x)^{-1/2} & & & f(0) &= 1 \\
f^{(1)}(x) &= -\frac{1}{2}(1+x)^{-3/2} & & & f^{(1)}(0) &= -\frac{1}{2} \\
f^{(2)}(x) &= \frac{3}{4}(1+x)^{-5/2} & &\Longrightarrow & f^{(2)}(0) &= \frac{3}{4} \\
f^{(3)}(x) &= -\frac{15}{8}(1+x)^{-7/2} & & & f^{(3)}(0) &= -\frac{15}{8} \\
f^{(4)}(x) &= \frac{105}{16}(1+x)^{-9/2} & & & &
\end{aligned}$$

Therefore

$$T_3(x) = 1 - \frac{1}{2}x + \frac{3}{4}\frac{x^2}{2!} - \frac{15}{8}\frac{x^3}{3!} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3.$$

Taylor's Theorem gives

$$R_3(x) = \frac{105}{16}\frac{x^4}{4!}(1+c)^{-9/2} \qquad \text{for some } c \text{ between } 0 \text{ and } x.$$

Therefore if $x \geq 0$ we have $c \geq 0$, and so

$$0 \leq R_3(x) = \frac{35}{128}x^4(1+c)^{-9/2} \leq \frac{35}{128}x^4.$$

This establishes the required inequalities.

(b) Hence give upper and lower bounds for the integral $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1+x^3}}\,dx$.

Solution: Replacing x by x^3 in the inequalities from (a) gives

$$A \le \int_0^{1/2} \frac{1}{\sqrt{1+x^3}}\,dx \le B,$$

where

$$\begin{aligned} A &= \int_0^{1/2} \left(1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9\right) dx = 0.4925755\ldots \\ B &= \int_0^{1/2} \left(1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \frac{35}{128}x^{12}\right) dx = 0.4925780\ldots \end{aligned}$$

Therefore the integral equals $0.49257\ldots$ with the first 5 places correct.

Questions for extra practice

6. Derive a series formula for $\displaystyle\int_0^x \frac{e^t - 1}{t}\,dt$.

Solution: Using the series expansion for e^t we get

$$\frac{e^t - 1}{t} = \sum_{k=1}^{\infty} \frac{1}{k!} t^{k-1},$$

and so

$$\begin{aligned} \int_0^x \frac{e^t - t}{t}\,dt &= \int_0^x \left(\sum_{k=1}^{\infty} \frac{1}{k!} t^{k-1}\right) dt \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{k!}\int_0^x t^{k-1}\,dt\right) = \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}. \end{aligned}$$

Again, the move from line 1 to line 2 actually needs some justification. You could model such a justification on Question 10.

7. (a) Compute the Taylor series of $f(x) = \sinh x$ about $x = 0$, and show that the series converges to $\sinh x$ for all $x \in \mathbb{R}$.

Solution: Calculating the derivatives gives

$$\begin{array}{rcr} f(x) = \sinh x & & f(0) = 0 \\ f^{(1)}(x) = \cosh x & & f^{(1)}(0) = 1 \\ f^{(2)}(x) = \sinh x & \Longrightarrow & f^{(2)}(0) = 0 \\ f^{(3)}(x) = \cosh x & & f^{(3)}(0) = 1 \\ \vdots & & \vdots \end{array}$$

and so $f^{(n)}(0)$ has pattern $0, 1, 0, 1, 0, 1, \ldots$ Therefore the Taylor series is (as you no doubt knew):

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

Let's show that this series converges to $\sinh x$ for all $x \in \mathbb{R}$. To do this we show that the remainder term $R_n(x)$ tends to 0 for all $x \in \mathbb{R}$, and to do this we use Taylor's Theorem to write down a formula for the remainder term:

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}\left(\frac{d^{n+1}}{dx^{n+1}}\sinh x\right)\Bigg|_{x=c} \quad \text{for some } c \text{ between 0 and } x.$$

But the $(n+1)$-th derivative of $\sinh x$ is either $\sinh x$ or $\cosh x$. Using the definitions of sinh and cosh gives the inequalities

$$|\sinh x| \le e^{|x|} \qquad \text{and} \qquad |\cosh x| \le e^{|x|} \qquad \text{for all } x \in \mathbb{R},$$

and therefore

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}e^{|c|} \quad \text{for some } c \text{ between 0 and } x.$$

Since c is between 0 and x we have $0 \le |c| \le |x|$, and hence $e^{|c|} \le e^{|x|}$. Therefore

$$|R_n(x)| \le e^{|x|}\frac{|x|^{n+1}}{(n+1)!}.$$

Therefore the remainder tends to 0 as $n \to \infty$ (for each fixed x) by the ratio test for sequences. This shows that the Taylor series converges to $\sinh x$ for all $x \in \mathbb{R}$.

(b) Derive series formulas for $\displaystyle\int_0^1 \sinh(x^2)\,dx$ and $\displaystyle\int_0^1 \frac{\sinh x}{x}\,dx$.

Solution: We have

$$\begin{aligned}\int_0^1 \sinh(x^2)\,dx &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{x^{4n+2}}{(2n+1)!}\right) dx \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\int_0^1 x^{4n+2}\,dx \\ &= \sum_{n=0}^{\infty} \frac{1}{(4n+3)(2n+1)!}.\end{aligned}$$

Similarly, using $\frac{\sinh x}{x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots$ gives

$$\int_0^1 \frac{\sinh x}{x}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}\right) dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+1)!}.$$

Both of these series converge very quickly, and so can be used to give accurate approximations to the integrals.

Remark: You'll learn more about interchanging the order of integration and summation in later mathematics courses. But you could make these manipulations rigorous by following the strategy of Question 10.

8. The Taylor series for $\tan^{-1} x$ is hard to find directly; here's an indirect method.

(a) Show that $\dfrac{1}{1+t^2} = \displaystyle\sum_{k=0}^{n-1}(-1)^k t^{2k} + \dfrac{(-1)^n t^{2n}}{1+t^2}$ for all $t \in \mathbb{R}$, and deduce that

$$\tan^{-1} x = \sum_{k=0}^{n-1}(-1)^k \frac{x^{2k+1}}{2k+1} + E_n(x), \quad \text{where} \quad E_n(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt.$$

Solution: The first statement is just a rearrangement of the geometric sum formula

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

with $r = -t^2$. The formula is valid whenever $r \neq 1$. That is, $t^2 \neq -1$, and so the formula is valid for all $t \in \mathbb{R}$.

Integrating this formula between 0 and x proves the second claim.

(b) Show that $|E_n(x)| \leq \int_0^{|x|} t^{2n}\,dt = \frac{|x|^{2n+1}}{2n+1}$, and conclude that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for all } -1 \leq x \leq 1.$$

Solution: Notice that $\frac{1}{1+t^2} \leq 1$ for all $t \in \mathbb{R}$. If $x \geq 0$ then we get

$$|E_n(x)| = \int_0^x \frac{t^{2n}}{1+t^2}\,dt \leq \int_0^x t^{2n}\,dt = \frac{x^{2n+1}}{2n+1}.$$

If $x < 0$

$$E_n(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt = -(-1)^n \int_x^0 \frac{t^{2n}}{1+t^2}\,dt = -(-1)^n \int_0^{-x} \frac{t^{2n}}{1+t^2}\,dt,$$

where we have used the fact that the integrand is even. Hence if $x < 0$ we have

$$|E_n(x)| = \int_0^{|x|} \frac{t^{2n}}{1+t^2}\,dt \leq \int_0^{|x|} t^{2n}\,dt = \frac{|x|^{2n+1}}{2n+1}.$$

Therefore if $|x| \leq 1$ we have $E_n(x) \to 0$ as $n \to \infty$. This establishes the formula

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for all } -1 \leq x \leq 1.$$

Challenging problems

9. From Question 3 we have the formula

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Unfortunately this converges pathetically slowly - it turns out that you need 1565238 terms to get $\ln 2$ correct to 6 decimal places! We can do much better using the function

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)$$

and noticing that $f(1/3) = \ln 2$.

(a) Find the Taylor series of $f(x)$ about $x = 0$. *Hint:* $f(x) = \ln(1 + x) - \ln(1 - x)$.

Solution: Writing $f(x) = \ln(1 + x) - \ln(1 - x)$ makes it clear that

$$f^{(n)}(x) = \frac{d^n}{dx^n}\ln(1+x) - \frac{d^n}{dx^n}\ln(1-x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} + \frac{(n-1)!}{(1-x)^n}.$$

Evaluating at $x = 0$, we get

$$f^{(n)}(0) = (n-1)!\big((-1)^{n-1} + 1\big),$$

and so

$$\frac{f^{(n)}(0)}{n!} = \frac{1 + (-1)^{n-1}}{n} = \begin{cases} 2/n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Hence the Taylor series for $f(x)$ is

$$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots\right).$$

(b) Use the Taylor polynomial $T_6(1/3)$ to approximate $\ln 2$. Estimate the size of the remainder term $R_6(1/3)$. Deduce that you have $\ln 2$ correct to 2 decimals.

Solution: The Taylor polynomial $T_6(x)$ is

$$T_6(x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right)$$

(the coefficient of x^6 is zero). Therefore $T_6(1/3) = 0.693004115\ldots$
We will now approximate the error. By Taylor's Theorem we have

$$R_6(1/3) = \frac{f^{(7)}(c)}{7!}(1/3)^7 \qquad \text{for some } c \text{ between } 0 \text{ and } 1/3,$$

and so by the formula for $f^{(n)}(x)$ given above we have

$$R_6(1/3) = \frac{1}{3^7 \cdot 7}\left(\frac{1}{(1+c)^7} + \frac{1}{(1-c)^7}\right) \quad \text{for some } 0 \le c \le 1/3.$$

Therefore $R_6(1/3) \ge 0$, and since $0 \le c \le 1/3$ we have

$$\frac{1}{(1+c)^7} \le 1 \qquad \text{and} \qquad \frac{1}{(1-c)^7} \le \left(\frac{3}{2}\right)^7.$$

Putting this together gives

$$0 \le R_6(1/3) \le \frac{1}{7}\left(\frac{1}{3^7} + \frac{1}{2^7}\right) = 0.00118139\ldots$$

Therefore we know that $\ln 2 = 0.69\ldots$ with the first 2 places correct. To get 6 decimals correct you only need to use 6 terms in the series. This is a lot better than 1565238 terms!

10. Here we use Taylor's Theorem to justify the manipulations made in Question 4.

(a) Use Taylor's Theorem to show that

$$e^{-t^2} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} t^{2k} + E_n(t), \qquad \text{where} \qquad |E_n(t)| \le \frac{t^{2n+2}}{(n+1)!}$$

Solution: Applying Taylor's Theorem to e^{-x}, we have

$$e^{-x} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} x^k + R_n(x),$$

where the Lagrange formula for the remainder $R_n(x)$ is

$$R_n(x) = (-1)^{n+1} \frac{e^{-c}}{(n+1)!} x^{n+1}$$

for some c between 0 and x. So writing $x = t^2$ gives

$$e^{-t^2} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} t^{2k} + E_n(t),$$

where

$$E_n(t) = (-1)^{n+1} \frac{e^{-c}}{(n+1)!} t^{2n+2} \quad \text{for some } c \text{ between 0 and } t^2.$$

Since $t^2 \ge 0$ we have $0 \le c \le t^2$, and so $e^{-c} \le 1$, and hence

$$|E_n(t)| \le \frac{|t|^{2n+2}}{(n+1)!} = \frac{t^{2n+2}}{(n+1)!}.$$

(b) Hence show that $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cdot k!} x^{2k+1}$ for all $x \in \mathbb{R}$.

Solution: By the previous part we have

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)\cdot k!} x^{2k+1} + \frac{2}{\sqrt{\pi}} \int_0^x E_n(t)\, dt.$$

But

$$\left| \int_0^x E_n(t)\, dt \right| \le \int_0^{|x|} |E_n(t)|\, dt \le \int_0^{|x|} \frac{t^{2n+2}}{(n+1)!}\, dt = \frac{|x|^{2n+3}}{(2n+3)(n+1)!},$$

and so $\displaystyle\lim_{n\to\infty} \int_0^x E_n(x)\, dt = 0$ for each x, and so

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cdot k!} x^{2k+1} \qquad \text{for each } x \in \mathbb{R}.$$

11. Give another proof of the Lagrange formula for the remainder term $R_n(x;a)$: Suppose that $f(x)$ is $(n+1)$-times differentiable, and (rather cleverly) let

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} (x-t)^k - R_n(x;a) \frac{(x-t)^{n+1}}{(x-a)^{n+1}}.$$

(a) Show that $g(a) = 0$ and $g(x) = 0$.

Solution: We have

$$\begin{aligned} g(a) &= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k - R_n(x;a) \\ &= f(x) - (T_n(x;a) + R_n(x;a)) = 0, \end{aligned}$$

because (by the definition of the remainder term) $f(x) = T_n(x;a) + R_n(x;a)$. Since

$$\sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k = f(t) + (\text{terms involving } (x-t))$$

we see that when we set $t = x$ the sum equals $f(x)$. Thus

$$g(x) = f(x) - f(x) = 0.$$

(b) Apply the Mean Value Theorem to $g(t)$ to show that there is a c strictly between a and x such that

$$R_n(x;a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Solution: By the Mean Value Theorem applied to $g(t)$ there is a number c strictly between a and x such that

$$g'(c) = \frac{g(x) - g(a)}{x-a} = 0.$$

We now compute $g'(t)$. First note that

$$\begin{aligned} \frac{d}{dt}\sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k &= f'(t) + \frac{d}{dt}\sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \\ &= f'(t) + \sum_{k=1}^{n}\left(\frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}\right) \\ &= f'(t) + \frac{f^{(n+1)}(t)}{n!}(x-t)^n - f'(t) \\ &= \frac{f^{(n+1)}(t)}{n!}(x-t)^n, \end{aligned}$$

where we have noticed that the sum collapses. Therefore

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x;a)\frac{(x-t)^n}{(x-a)^{n+1}}.$$

Since $g'(c) = 0$ we have

$$0 = g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x;a)\frac{(x-c)^n}{(x-a)^{n+1}}.$$

Rearranging this formula gives

$$R_n(x;a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ strictly between } a \text{ and } x,$$

which is the Lagrange formula for the remainder term.

12. From Question 8 we have *Leibnitz's Formula*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

This series is essentially useless for the purpose of approximating π (try it!). But there is something clever we can do. Recall the identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \qquad \text{valid for } xy < 1.$$

(a) Show that $4\tan^{-1}(\frac{1}{5}) = \tan^{-1}(\frac{120}{119})$ and $\tan^{-1} 1 + \tan^{-1}(\frac{1}{239}) = \tan^{-1}(\frac{120}{119})$.

Solution: We have

$$2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{25}}\right) = \tan^{-1}\left(\frac{5}{12}\right),$$

and therefore

$$4\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{5}{12}\right) = \tan^{-1}\left(\frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{25}{144}}\right) = \tan^{-1}\left(\frac{120}{119}\right).$$

We also have

$$\tan^{-1}(1) + \tan^{-1}\left(\frac{1}{239}\right) = \tan^{-1}\left(\frac{1+\frac{1}{239}}{1-\frac{1}{239}}\right) = \tan^{-1}\left(\frac{120}{119}\right).$$

(b) Hence prove *Machin's formula*: $\pi = 16\tan^{-1}(1/5) - 4\tan^{-1}(1/239)$. Use the first five terms from the $\tan^{-1} x$ series from Question 8 to approximate π.

Solution: From the previous part we have

$$\frac{\pi}{4} = \tan^{-1}(1) = \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right) = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

Now multiply by 4.

Using the Taylor series for $\tan^{-1} x$ (with $x = \frac{1}{5}$ and $x = \frac{1}{239}$) we approximate

$$\begin{aligned}\pi \approx 16&\left(\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \frac{1}{9\cdot 5^9}\right)\\ &- 4\left(\frac{1}{239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 239^5} - \frac{1}{7\cdot 239^7} + \frac{1}{9\cdot 239^9}\right)\\ &= 3.141592652615\ldots\end{aligned}$$

The first 8 decimals are correct. You would need millions of terms in Leibnitz's formula to obtain this accuracy.

Remark: There actually was a time before pocket calculators. A time when only a few decimal places of π were known. Now-days some of this "computing stuff by hand" seems a tad outdated, but think of how happy people (mathematicians, engineers, physicists) must have been when John Machin discovered his quickly converging series for π in 1706. He used this series to compute π to 100 decimal places by hand (fun guy to have at a dinner party). At the time this was the 'world record'. In 1945 only 527 decimals were known - so Machin really did a pretty good job. Now more than 2,000,000,000,000 decimals are known. This kind of accuracy is a bit overwhelming: Knowing π to 50 decimal places is sufficient to compute the circumference (given the diameter) of the known universe to within the thickness of an electron.

13. This question shows that we need to be careful when rearranging the terms of a series. Recall that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

Consider the rearrangement

$$S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \cdots$$

(a) Express the partial sums of S in terms of the harmonic numbers H_m.

Solution: Let S_n be the nth partial sum of S. Then

$$\begin{aligned}
S_{3n} &= 1 + \frac{1}{3} - \frac{1}{2} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \\
&= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{4n-1} - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= 1 + \frac{1}{3} + \cdots + \frac{1}{4n-1} - \frac{1}{2}H_n \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{4n} - \left(\frac{1}{2} + \cdots + \frac{1}{4n}\right) - \frac{1}{2}H_n \\
&= H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n.
\end{aligned}$$

(b) Calculate the value of the series (*Hint:* $H_n - \ln n \to \gamma$ *as* $n \to \infty$).

Solution: We use the fact that $\lim_{n\to\infty} (H_n - \ln n) = \gamma$ (as proved in lectures). Therefore,

$$\begin{aligned}
S_{3n} &= H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n \\
&= (H_{4n} - \ln(4n)) - \frac{1}{2}(H_{2n} - \ln(2n)) - \frac{1}{2}(H_n - \ln n) \\
&\qquad + \ln(4n) - \frac{1}{2}\ln(2n) - \frac{1}{2}\ln n \\
&= (H_{4n} - \ln(4n)) - \frac{1}{2}(H_{2n} - \ln(2n)) - \frac{1}{2}(H_n - \ln n) + \ln 4 - \frac{1}{2}\ln 2.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} S_{3n} = \gamma - \frac{1}{2}\gamma - \frac{1}{2}\gamma + \frac{3}{2}\ln 2 = \frac{3}{2}\ln 2.$$

Since $|S_{3n} - S_{3n+1}| \to 0$ and $|S_{3n} - S_{3n+2}| \to 0$ it follows that $S_n \to \frac{3}{2}\ln 2$, and so

$$S = \frac{3}{2}\ln 2.$$

In particular, notice that this is not equal to $\ln 2$.

14. Use 'reasonable' series manipulations and the Euler series formula to show that

$$\int_0^\infty \frac{x}{e^x - 1}\,dx = \frac{\pi^2}{6}.$$

Solution: Using the geometric series formula we have

$$\begin{aligned}\int_0^\infty \frac{x}{e^x-1}\,dx &= \int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,dx\\ &= \int_0^\infty xe^{-x}\left(1+e^{-x}+e^{-2x}+\cdots\right)\,dx.\end{aligned}$$

Assuming that we can change the order of integration and summation (we can, but to justify this you will have to wait until next year) we have

$$\int_0^\infty \frac{x}{e^x-1}\,dx = \sum_{n=1}^\infty \left(\int_0^\infty xe^{-nx}\,dx\right).$$

Integrating by parts shows that

$$\begin{aligned}\int_0^\infty xe^{-nx}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-nx}\,dx\\ &= \lim_{b\to\infty}\left(-\frac{be^{-nb}}{n}+\frac{1}{n}\int_0^b e^{-nx}\,dx\right)\\ &= \lim_{b\to\infty}\left(-\frac{be^{-nb}}{n}-\frac{e^{-nb}}{n^2}+\frac{1}{n^2}\right)=\frac{1}{n^2}.\end{aligned}$$

Therefore

$$\int_0^\infty \frac{x}{e^x-1}\,dx = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6},$$

where we have used Euler's series formula (from class).