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# MATH1901/1906 SEMESTER 1, 2013

## FINAL EXAM

## SOLUTIONS TO EXTENDED ANSWER SECTION

### Question 1

1(a) Let $z = x+iy$.

Then $z+\bar{z} = (x+iy)+(x-iy) = 2x$

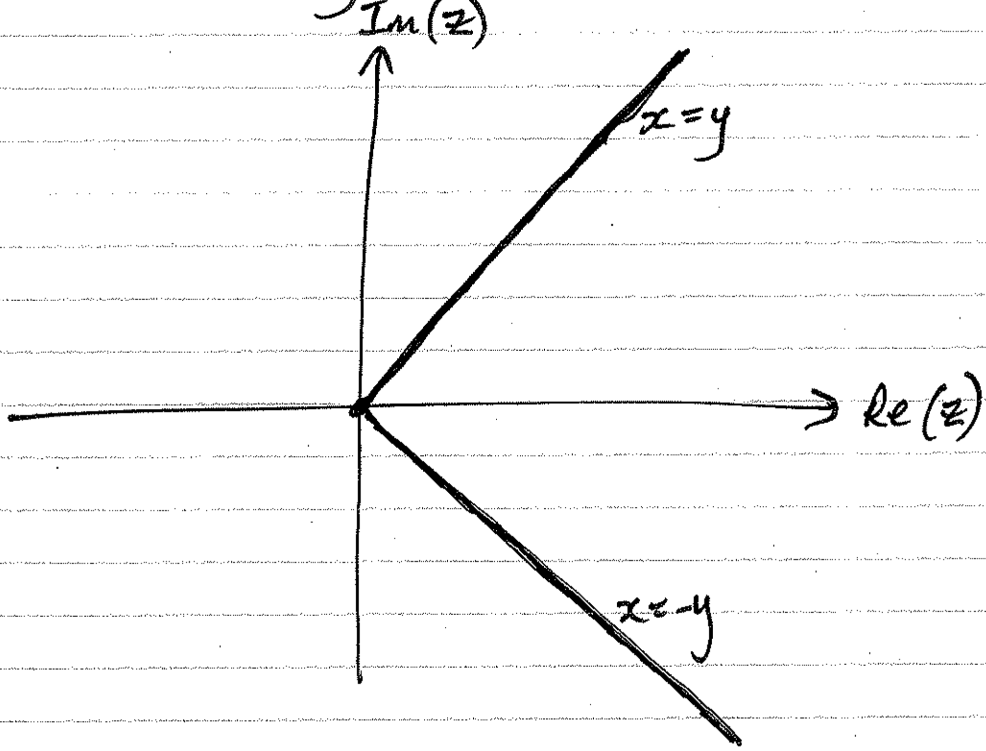
$|z| = \sqrt{x^2+y^2}$

So $2x = \sqrt{2}\sqrt{x^2+y^2}$

$\iff 4x^2 = 2(x^2+y^2)$ and $x \geq 0$

$\iff 2x^2 = 2y^2$ and $x \geq 0$

$\iff x = \pm y$ and $x \geq 0$



1(b) Suppose $f(z_1) = f(z_2)$, $z_1, z_2 \in \mathbb{C}$

Then $$\frac{z_1}{z_1+1} = \frac{z_2}{z_2+1}$$

$$\Rightarrow \quad z_1(z_2+1) = z_2(z_1+1)$$

$$\Rightarrow \quad z_1 z_2 + z_1 = z_2 z_1 + z_2$$

$$\Rightarrow \quad z_1 = z_2.$$

Thus $f$ is injective.

Let $w \in \mathbb{C}$. Then for $z \in \mathbb{C}$

$$f(z) = w$$

$$\Leftrightarrow \quad \frac{z}{z+1} = w$$

$$\Leftrightarrow \quad z = wz + w$$

$$\Leftrightarrow \quad z(1-w) = w$$

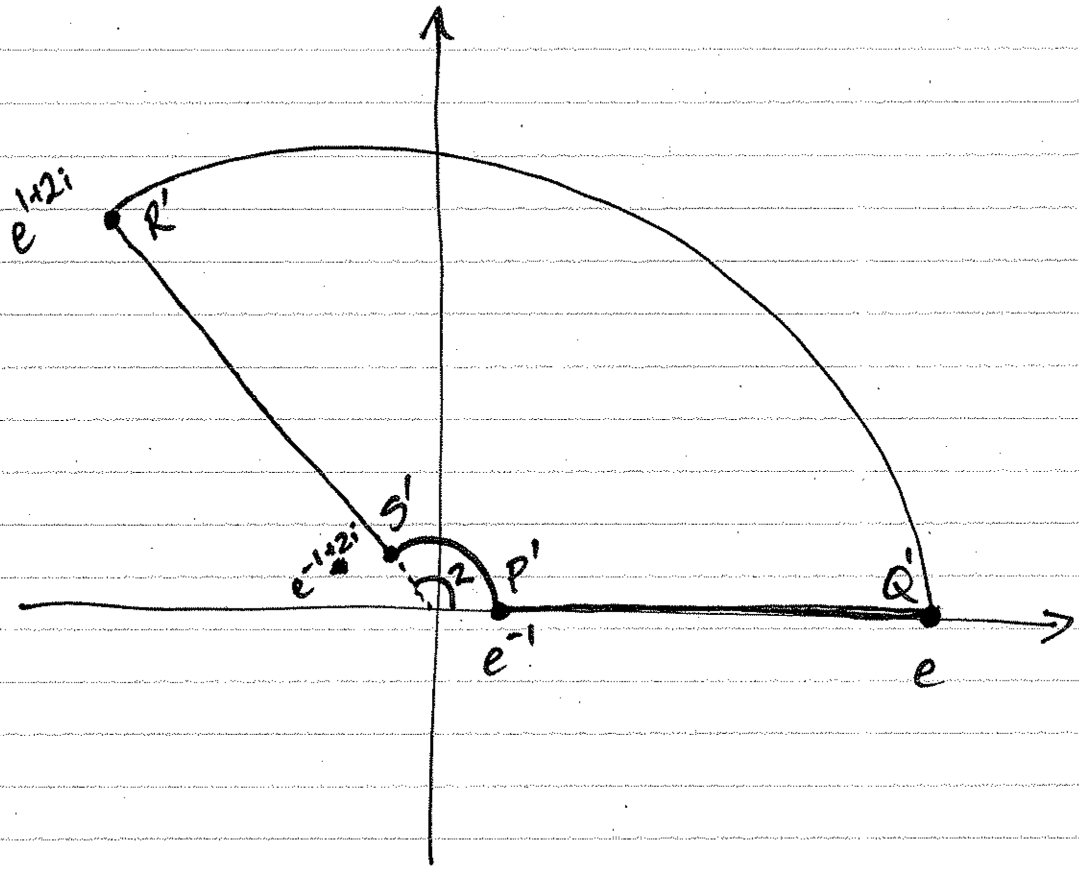
$$\Leftrightarrow \quad z = \frac{w}{1-w}$$

Thus if $w \neq 1$, there is a $z \in \mathbb{C}$ so that $f(z) = w$. However $f(z) = 1$ never occurs since then

$$\frac{z}{z+1} = 1 \Rightarrow z = z+1 \Rightarrow 0 = 1$$

which is impossible. So the range of $f$ is $\mathbb{C} \setminus \{1\} = \{w \in \mathbb{C} \mid w \neq 1\}$.

1(c)




$$e^{-1+2i} = e^{-1}(\cos 2 + i \sin 2)$$

2nd quadrant since $\frac{\pi}{2} < 2 < \pi$

(d) Since $f$ is continuous on $[0,1]$ we may apply the Intermediate Value Theorem.

We have $0 < d < 1$ hence $f(0) < d < f(1)$.

Thus by the IVT there is a $c \in (0,1)$ so that $f(c) = d$.

Alternatively, given $d \in (0,1)$ define $g(x) = f(x) - d$.

Then $g(0) = f(0) - d = 0 - d < 0$

while $g(1) = f(1) - d = 1 - d > 0$ hence as $g$ is continuous on $[0,1]$, $\exists\, c \in (0,1)$ s.t. $g(c) = 0$ thus $f(c) - d = 0$ thus $f(c) = d$.

## Question 2

2(a)(i) $\lim_{t\to 0} \frac{t}{\sqrt{6t}-2} = \frac{0}{\sqrt{6}-2} = 0$

(ii) Let $y = \left(\cos\left(\frac{3}{x}\right)\right)^x$.

Then $\ln y = x \ln\left(\cos\left(\frac{3}{x}\right)\right)$

$$= \frac{\ln\left(\cos\left(\frac{3}{x}\right)\right)}{\frac{1}{x}}$$

By L'Hôpital's Rule for "$\frac{0}{0}$" form

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(\cos\left(\frac{3}{x}\right)\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}\left(\cos\left(\frac{3}{x}\right)\right) / \cos\left(\frac{3}{x}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\left(-\frac{3}{x^2}\right)\left(-\sin\left(\frac{3}{x}\right)\right)}{\left(-\frac{1}{x^2}\right)\cos\left(\frac{3}{x}\right)}$$

$$= \lim_{x\to\infty} \left(-3\tan\left(\frac{3}{x}\right)\right)$$

$$= 0, \quad \text{since } \tan 0 = 0.$$

Take exponential of both sides then

$$\lim_{x\to\infty} \left(\cos\left(\frac{3}{x}\right)\right)^x = e^0 = 1.$$

(iii) Put $x = r\cos\theta$, $y = r\sin\theta$.

Then $\frac{5(x+y)^2}{\sqrt{x^2+y^2}} = \frac{5(r\cos\theta + r\sin\theta)^2}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$

$$= \frac{5r^2(\cos\theta + \sin\theta)^2}{r}$$

$$= 5r(\cos\theta + \sin\theta)^2.$$

Now $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ so

$$0 \le (\cos\theta + \sin\theta)^2 \le 4$$

So

$$0 \le 5r(\cos\theta + \sin\theta)^2 \le 20r$$

Since $\lim_{r\to 0} 20r = 0$, the Squeeze Law then implies

$$\lim_{r\to 0} 5r(\cos\theta + \sin\theta)^2 = 0$$

hence $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5(x+y)^2}{\sqrt{x^2+y^2}} = 0.$

2(b) Since $\lim_{x\to a} g(x) = \ell$, there is a $\delta_1 > 0$ so that if

$0 < |x-a| < \delta_1$, $\quad |g(x) - \ell| < \frac{1}{2}\ell$

$$\Rightarrow \quad \ell - \frac{\ell}{2} < g(x) < \ell + \frac{\ell}{2}$$

$$\Rightarrow \quad \frac{\ell}{2} < g(x)$$

$$\Rightarrow \quad \frac{2}{\ell} > \frac{1}{g(x)}$$

$$\Rightarrow \quad \frac{2}{\ell} > \frac{1}{|g(x)|}.$$

Let $\varepsilon > 0$. Then there is a $\delta_2 > 0$ so that if

$0 < |x-a| < \delta_2$, $\quad |g(x) - \ell| < \frac{\ell^2}{2}\varepsilon.$

Now let $\delta = \min(\delta_1, \delta_2)$. Then if $0 < |x-a| < \delta$

$$\left|\frac{1}{g(x)} - \frac{1}{\ell}\right| = \frac{|g(x) - \ell|}{\ell\,|g(x)|} < \frac{\ell^2}{2}\varepsilon\,\frac{1}{\ell}\,\frac{2}{\ell} = \varepsilon$$

Hence $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{\ell}$ as required.

## Question 3

3(a) $p'(x) = 3(x-a)^2 q(x) + (x-a)^3 q'(x)$ by ~~Chain~~ Product Rule

$p''(x) = 6(x-a)q(x) + 3(x-a)^2 q'(x) + 3(x-a)^2 q'(x) + (x-a)^3 q''(x)$

Since $(x-a)$ is a factor of both $p'(x)$ and $p''(x)$, we have $p'(a)=0$ and $p''(a)=0$, and so $a$ is a root of both $p'$ and $p''$.

(b)(i) $f(0) = \cosh 0 = 1$

$f'(x) = \sinh x \qquad f'(0) = \sinh 0 = 0$

$f''(x) = \cosh x \qquad f''(0) = 1$

$f'''(x) = \sinh x \qquad f'''(0) = 0$

$f^{(4)}(x) = \cosh x \qquad f^{(4)}(0) = 1$

So $T_4(x) = 1 + \frac{f''(0)}{2!}x^2 + \frac{f^{(4)}(0)}{4!}x^4$

$= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4.$

(ii) $T_{17}(x) = 1 + \frac{1}{2}(x^3)^2 + \frac{1}{24}(x^3)^4$

$= 1 + \frac{1}{2}x^6 + \frac{1}{24}x^{12}$

$T_{17}(1) = 1 + \frac{1}{2} + \frac{1}{24}$

$= \frac{37}{24}$

3(b)(iii) By Lagrange's form of the remainder, there is a $c$ between 0 and 1 so that

$$R_{17}(1) = \frac{g^{(18)}(c)}{18!} 1^{18} = \frac{g^{(18)}(c)}{18!}$$

Now $R_{17}(1) = g(1) - T_{17}(1)$

so $$\frac{g^{(18)}(c)}{18!} = \cosh 1^3 - \frac{37}{24} \quad \text{by part (ii)}$$

$$g^{(18)}(c) = 18! \left( \frac{e+e^{-1}}{2} - \frac{37}{24} \right).$$

3(c) $$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{hg(h) - 0}{h} \quad \text{by def}^{n} \text{ of } f$$

$$= \lim_{h \to 0} g(h)$$

$$= g(0) \quad \text{since } g \text{ is continuous at } 0.$$

Thus $f$ is differentiable at 0 and

$$f'(0) = g(0).$$

Question 4

4(a) Domain: $x^2+y^2 \neq 0 \iff (x,y) \neq (0,0)$

So domain is $R^2 \setminus \{(0,0)\} = \{ (x,y) \mid (x,y) \neq (0,0) \}$

Level curves: $$\frac{y^2-x^2}{x^2+y^2} = \frac{1}{2}$$

$$y^2 - x^2 = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

$$\frac{1}{2}y^2 = \frac{3}{2}x^2$$

$$y^2 = 3x^2 \qquad y = \pm\sqrt{3}x$$

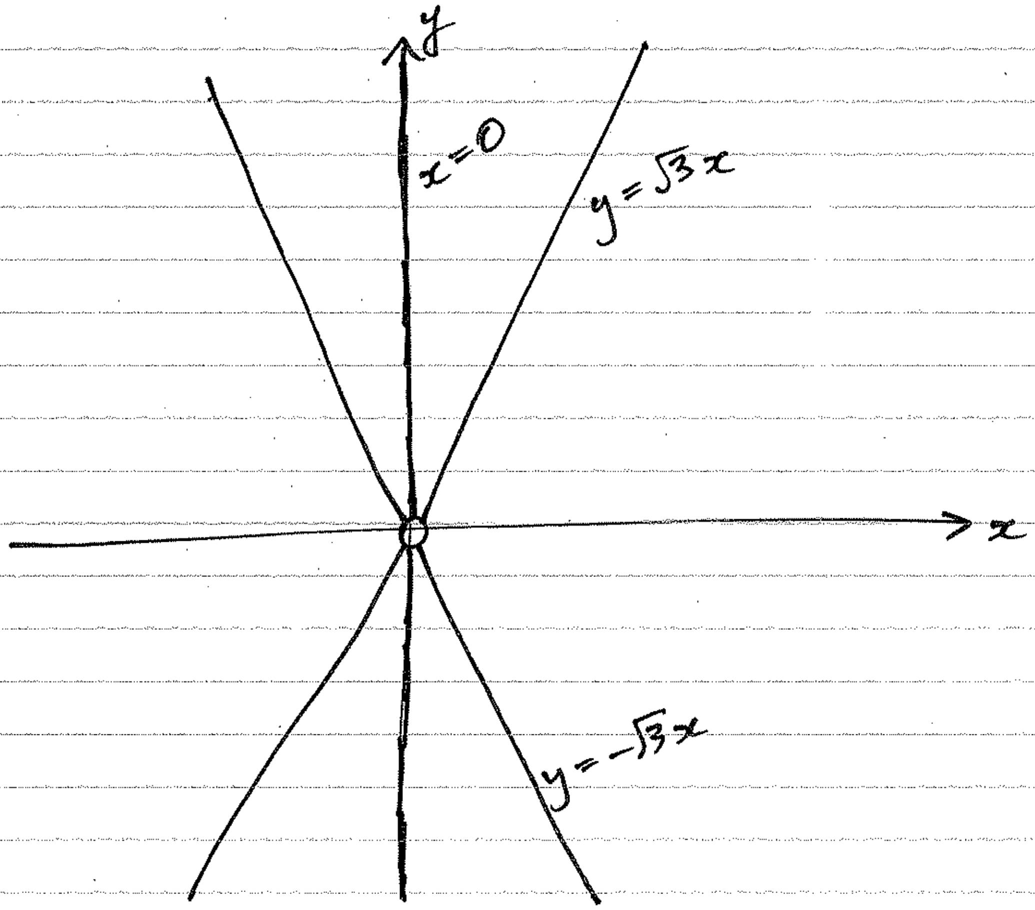
SOLUTIONS 

Height 1: $\frac{y^2-x^2}{x^2+y^2}=1$

$$y^2-x^2=x^2+y^2$$

$$2x^2=0$$

$$x=0$$




4(b) $x\cos y - y\sin x = \frac{\pi}{2}$

Let $f(x,y) = x\cos y - y\sin x - \frac{\pi}{2}$

Then $\frac{\partial f}{\partial x} = \cos y - y\cos x \qquad \frac{\partial f}{\partial y} = -x\sin y - \sin x$

So $\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-\cos y + y\cos x}{-x\sin y - \sin x}$

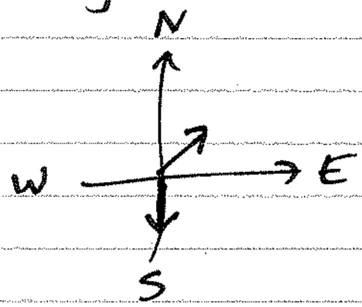
At the point $(\frac{\pi}{2},0)$ $\frac{dy}{dx} = \frac{-\cos 0 + 0}{-\frac{\pi}{2}\sin 0 - \sin\frac{\pi}{2}} = \frac{-1}{-1} = 1$

So tangent line is $y-0 = 1(x-\frac{\pi}{2})$ i.e. $y = x-\frac{\pi}{2}$.

4(c)(i) $\underset{\sim}{\nabla} h(x,y) = -\frac{2}{100}(x-14)\,\underset{\sim}{i} - \frac{2}{25}(y+43)\,\underset{\sim}{j}$

$\underset{\sim}{\nabla} h(64,57) = -\frac{1}{50}(50)\,\underset{\sim}{i} - \frac{2}{25}(100)\,\underset{\sim}{j}$

$= -\underset{\sim}{i} - 8\underset{\sim}{j}$




Due south: $\underset{\sim}{u} = (0,-1)$

$D_{\underset{\sim}{u}} h(64,57) = \underset{\sim}{\nabla} h(64,57) \cdot \underset{\sim}{u}$

$= (-1,-8)\cdot(0,-1)$

$= 8 > 0$ so you start to go up

North-east: $\underset{\sim}{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$D_{\underset{\sim}{u}} h(64,57) = (-1,-8)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= -\frac{1}{\sqrt{2}} - \frac{8}{\sqrt{2}} < 0$ so you start to go down

(ii) Direction in which slope is greatest is

$\underset{\sim}{\nabla} h(64,57) = (-1,-8)$.

2 directions to walk and stay level:

$(8,-1)$ and $(-8,1)$.

(iii) $Q = (14, -43, 2228)$

Tangent plane at $Q$: $z = 2228$

Tangent plane is horizontal $\Leftrightarrow h_x = 0$ and $h_y = 0$.

$h_x = -\frac{1}{50}(x-14) = 0 \Leftrightarrow x = 14$

$h_y = -\frac{2}{25}(y+43) = 0 \Leftrightarrow y = -43$. So tangent plane at $Q$ is horizontal, and there are no other horizontal tangent planes.