THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1	**Tutorial Week 3**	2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Express the following complex numbers in Cartesian form:

(a) $2\operatorname{cis}\frac{\pi}{4}$ (b) $-4\operatorname{cis}\frac{\pi}{3}$

(c) $\operatorname{cis}\frac{\pi}{2}\operatorname{cis}\frac{\pi}{3}\operatorname{cis}\frac{\pi}{6}$ (d) $e^{-i\pi}$

(e) $e^{\ln 2+i\pi}$ (f) $e^{1+i}e^{1-i}e^{-2-i\pi}$

Questions for the tutorial

2. Solve the following equations (leaving your answers in polar form) and plot the solutions in the complex plane:

(a) $z^5 = 1$

(b) $z^6 = -1$

(c) $z^3 + i = 0$

(d) $z^4 = 8\sqrt{2} + 8\sqrt{2}\,i$

(e) $z^5 + z^3 - z^2 - 1 = 0$, given that $z = i$ is a solution.

3. The complex sine and cosine functions are defined by the formulas

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}.$$

(a) Show that when z is real ($z = x$), $\sin z$ and $\cos z$ reduce to the familiar real sine and cosine functions.

(b) Show that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.

(c) Is it true that $|\sin z| \le 1$ and $|\cos z| \le 1$, for all $z \in \mathbb{C}$? (*Hint*: You know these are true when z is real. See what happens when z is purely imaginary, $z = iy$.)

4. Find all solutions of the following equations:

(a) $e^z = i$ (b) $e^z = -10$

(c) $e^z = -1 - i\sqrt{3}$ (d) $e^{2z} = -i$

5. Sketch and describe the following sets and their images under the function $z \mapsto z^2$.

(a) The set of all points of the form $z = x + 2i$.

(b) The set of all points of the form $z = x + 2xi$.

(c) The set of all points on the upper half of the unit circle centred at the origin, that is, points z with polar coordinates (r, θ) such that $r = 1$ and $0 \le \theta \le \pi$.

(d) The set of all points on the unit circle centred at the origin, that is, points z with polar coordinates (r, θ) such that $r = 1$ and $-\pi < \theta \le \pi$.

6. Sketch the following sets and their images under the function $z \mapsto e^z$:

(a) $\{z = x + iy \in \mathbb{C} \mid 0 < x < 2,\ y = \dfrac{\pi}{2}\}$;

(b) $\{z \in \mathbb{C} \mid x = 1,\ |y| < \dfrac{\pi}{2}\}$;

(c) $\{z \in \mathbb{C} \mid x < 0,\ \dfrac{\pi}{3} < y < \pi\}$;

(d) $\{z = (1+i)t \mid t \in \mathbb{R}\}$.

7. (a) Sketch the set $\{z \in \mathbb{C} \mid \dfrac{1}{2} < |z| < 4,\ 0 \leq \operatorname{Arg}(z) \leq \dfrac{\pi}{4}\}$.

(b) Sketch the image of the set in the w-plane under the function $z \mapsto w = \dfrac{1}{z}$.

(c) An insect is crawling clockwise around the boundary of the set in the z-plane. Is its image crawling clockwise, or anticlockwise, in the w-plane? (If clockwise, we say the transformation is *orientation-preserving*; if anticlockwise, we say it is *orientation-reversing.*)

(d) Now consider the function $z \mapsto w = \bar{z}$, the complex conjugate of z. Is it orientation-preserving or orientation-reversing?

8. Find all solutions of the equation $e^{2z} - (1+3i)e^z + i - 2 = 0$.

Extra Questions

9. This question demonstrates that complex numbers can be useful in solving cubic equations, even when all the solutions are real.

(a) Show that for any complex number w, there exists a nonzero complex number z such that $z + \dfrac{1}{z} = w$.

(b) Use this substitution to solve the equation $w^3 - 3w - 1 = 0$.

10. Let n be a given positive integer. By a *primitive nth root of unity* we mean a solution η of $z^n = 1$ which has the property that its powers $\eta, \cdots, \eta^{n-1}, \eta^n (= 1)$ are exactly the solutions of this equation in $\mathbb{C}$. For example, $e^{i\frac{2\pi}{n}}$ is a primitive nth root of unity.

(a) Find all primitive 6th roots of unity.

(b) Find all primitive 5th roots of unity.

(c) For which values of k, $0 \leq k \leq n-1$, is $e^{i\frac{2\pi k}{n}}$ a primitive nth root of unity?

Solution to Question 1

1. (a) $2\operatorname{cis}\pi/4 \;=\; \sqrt{2} + \sqrt{2}\,i$ (b) $-4\operatorname{cis}\pi/3 \;=\; -2 - 2\sqrt{3}\,i$

(c) $\operatorname{cis}(\pi/2)\ \operatorname{cis}(\pi/3)\ \operatorname{cis}(\pi/6) \;=\; \operatorname{cis}\pi \;=\; -1$ (d) $e^{-i\pi} \;=\; \operatorname{cis}(-\pi) \;=\; -1$

(e) $e^{\ln 2 + i\pi} \;=\; e^{\ln 2}\operatorname{cis}\pi \;=\; -2$ (f) $e^{1+i}e^{1-i}e^{-2-i\pi} \;=\; e^{1+i+1-i-2-i\pi} \;=\; e^{-i\pi} \;=\; -1$