THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 12

MATH1901: Differential Calculus (Advanced) Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
Lecturer: Daniel Daners

Material covered

- ☐ Partial derivatives of functions $f(x, y)$.
- ☐ The formula of the tangent plane to the graph $z = f(x, y)$.
- ☐ The Mixed Derivatives Theorem (also known as Clairaut's Theorem).
- ☐ The Chain Rule for functions $f(x, y)$.

Outcomes

After completing this tutorial you should

- ☐ quickly and efficiently compute partial derivatives and equations of tangent planes;
- ☐ appreciate the statement of the Mixed Derivatives Theorem, and understand its limitations.
- ☐ calculate partial derivatives directly from the limit definition in relevant cases.
- ☐ use the chain rule to compute partial and total derivatives,
- ☐ appreciate the subtleties involved in defining the notion of differentiability for functions $f(x, y)$.

Summary of essential material

Definition of partial derivatives. The *partial derivative* of $f(x, y)$ with respect to x is the derivative of f obtained by fixing y and differentiating with respect to x. By first principles it is the limit

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

provided the limit exists. Similarly, the *partial derivative* of $f(x, y)$ with respect to y is the derivative of f obtained by fixing x and differentiating with respect to x. By first principles it is the limit

$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

provided the limit exists. Geometrically, the partial derivative $f_x(x_0, y_0)$ is the slope of the curve obtained by intersecting the graph of f with the plane parallel to the xz-axis through the point (x_0, y_0) at x_0. A similar interpretation holds for $f_y(x_0, y_0)$.

Calculating partial derivatives. To calculate the x-partial derivative $f_x(x, y)$ we fix y (that is, consider y to be a constant) and differentiate with respect to x as usual. To calculate the y-partial derivative, we fix x and differentiate with respect to y.

Tangent planes to graphs. The graph of a real valued function of two variables is a surface. The equation of the *tangent plane* to the graph $z = f(x, y)$ at the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Questions to complete during the tutorial

No tutorial questions due to quiz.

Extra questions for further practice

1. (a) Find the equation of the tangent plane to $z = \sin(x^2 - y) + 4xy + 3$ at the point $(x, y) = (2, 4)$.

(b) Find the equation of the tangent plane to the surface $z = e^x \ln y$ at $(3, 1, 0)$.

2. Find all points at which the tangent plane to the surface $z = x^2 + 2xy + 2y^2 - 6x + 8y$ is horizontal.

3. Define a function f of two variables by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Find $f_x(x, y)$ and $f_y(x, y)$. To find $f_x(0, 0)$ and $f_y(0, 0)$ you will need to use the definition of partial derivatives in terms of limits.

(b) Find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$. Again, you will need to use the limit definitions.

(c) Observe that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. Why does this not contradict the Mixed Derivatives Theorem?

4. Partial derivatives are functions of x and y again. Hence we can take further partioal derivatives. Let $f(x, y) = 1 + x^2 + 2y^2 + 2y + x^2 y$. Calculate

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial y}.$$

Note that the mixed derivatives are equal.

Challenge questions (optional)

5. In this question we investigate what the definition of *differentiability* should be for a function $f(x, y)$. In class we considered the function $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$g(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Both of the first order partial derivatives of this function exist, indeed $g_x(0, 0) = g_y(0, 0) = 0$. However we certainly do not want to call this function differentiable at $(x, y) = (0, 0)$, it is not even continuous there!!! Thus defining "differentiability" of a function $f(x, y)$ at $(x, y) = (a, b)$ to simply mean that $f_x(a, b)$ and $f_y(a, b)$ exist is not appropriate.

Instead a better approach is to define a function $f(x, y)$ to be differentiable at the point $(x, y) = (a, b)$ if $f(x, y)$ is "well approximated" by a tangent plane at $(x, y) = (a, b)$. That is, there is a plane $z = f(a, b) + m_1(x - a) + m_2(y - b)$ such that

$$f(x, y) - [f(a, b) + m_1(x - a) + m_2(y - b)]$$

is very small for all (x, y) close to (a, b). How small? We will insist that the difference between $f(x, y)$ and the tangent plane is considerably smaller than the distance from (x, y) to (a, b). A way of quantifying this is to require:

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - [f(a, b) + m_1(x - a) + m_2(y - b)]}{\sqrt{(x - a)^2 + (y - b)^2}} = 0. \tag{2}$$

Thus we have arrived at a definition: A function $f(x, y)$ is *differentiable* at $(x, y) = (a, b)$ if there are numbers $m_1, m_2 \in \mathbb{R}$ such that (2) holds.

6. Show that the function $g(x, y)$ in (1) is not differentiable at $(x, y) = (0, 0)$.

7. Show that if $f(x, y)$ is differentiable at $(x, y) = (a, b)$, then it is continuous at $(x, y) = (a, b)$.

8. Show that if $f(x, y)$ is differentiable at $(x, y) = (a, b)$ then $m_1 = f_x(a, b)$ and $m_2 = f_y(a, b)$. In particular, differentiability implies that the partial derivatives exist. (The example $g(x, y)$ shows that the converse is false).

9. *Very challenging!* Show that if $f_x(x, y)$ and $f_y(x, y)$ exist around $(x, y) = (a, b)$ and are continuous at $(x, y) = (a, b)$, then $f(x, y)$ is differentiable at $(x, y) = (a, b)$. So most reasonable functions are differentiable, which is reassuring.

Hint: It helps to write $f(x, y) - f(a, b) = [f(x, y) - f(a, y)] + [f(a, y) - f(a, b)]$.