Extended Answer Section

There are **three** *questions in this section, each with a number of parts. Write your answers in the space provided below each part. If you need more space there are extra pages at the end of the examination paper.*

1. (a) **(6 marks)**

(*i*) Starting from the geometric definition of the dot product of two vectors, or otherwise, show that for any geometric vector $\mathbf{c}$, $\mathbf{c} \cdot \mathbf{c} = |\mathbf{c}|^2$.

(*ii*) Using the result in (i), show that for any two geometric vectors $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}.$$

(*iii*) Using the result in (ii), show that if $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$ then $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

(*iv*) Let $ABCD$ be a parallelogram with adjacent sides given by the vectors $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$. Express the diagonals $\overrightarrow{AC}$ and $\overrightarrow{DB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and hence prove that if these diagonals are of equal length, then $ABCD$ is a rectangle.

(b) **(4 marks)**

(i) Given

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -4 & 1 & -1 \\ 6 & -2 & 1 \end{bmatrix}.$$

Using elementary row operations, or otherwise, find the inverse matrix, A^{-1}.

(*ii*) Using the result in (i), or otherwise, solve the following set of equations for x, y, z:

$$\begin{aligned} x + z &= 3 \\ -4x + y - z &= -2 \\ 6x - 2y + z &= 1. \end{aligned}$$

2. (a) **(2 marks)** Let C be an $n \times n$ matrix, $\mathbf{v}$ be a non-zero vector of length n and λ a scalar. Explain what is meant by the statement "C has an eigenvalue λ with corresponding eigenvector $\mathbf{v}$".
Show that λ satisfies $\det(C - \lambda I) = 0$ where I is the $n \times n$ identity matrix. (You may quote the fact that a square matrix has zero determinant if and only if it is not invertible.)

(b) **(8 marks)** Let B be the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 6 & 0 & 2 \end{bmatrix}.$$

(i) Show that $\det(B - \lambda I) = (1 - \lambda)(2 - \lambda)(4 - \lambda)$, where I is the 3×3 identity matrix. Hence deduce the eigenvalues of B.

(*ii*) Find an eigenvector $\mathbf{v}_3$ corresponding to the eigenvalue $\lambda = 4$.

(*iii*) Using matrix multiplication, or otherwise, verify that

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -4 \\ 6 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

are eigenvectors of B corresponding to $\lambda = 1$ and $\lambda = 2$, respectively.

(*iv*) Write down matrices P and D such that

$$B = PDP^{-1},$$

where D is a diagonal matrix. (**N.B.** You are not required to calculate P^{-1} explicitly.)

(*v*) Explain why the matrices $B - I$, $2I - B$ and $B - 4I$ are not invertible.

3. (a) **(4 marks)** Let p be a plane and Q a point not on the plane. Let R be any point on the plane. Show that the shortest distance from Q to the plane is given by the magnitude of $\overrightarrow{QR} \cdot \widehat{\mathbf{n}}$ where $\widehat{\mathbf{n}}$ is a unit vector perpendicular to p.

Given that the plane p has equation $2x - y + 3z = 2$ and that the point Q has coordinates $(1, -1, 2)$:

(i) Find the shortest distance from Q to p;

(ii) Find the coordinates of the point S on p closest to Q.

(b) **(3 marks)** Let P be a non-zero square matrix such that $P^2 = P$. Show that either $P = I$, where I is the identity matrix, or $\det(P) = 0$.

(c) **(3 marks)** Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three non-zero vectors no two of which are parallel. Show that if

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$$

then

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}.$$

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End of Extended Answer Section