(A)

MATH 1902 1999 exam solutions 11/6/2010

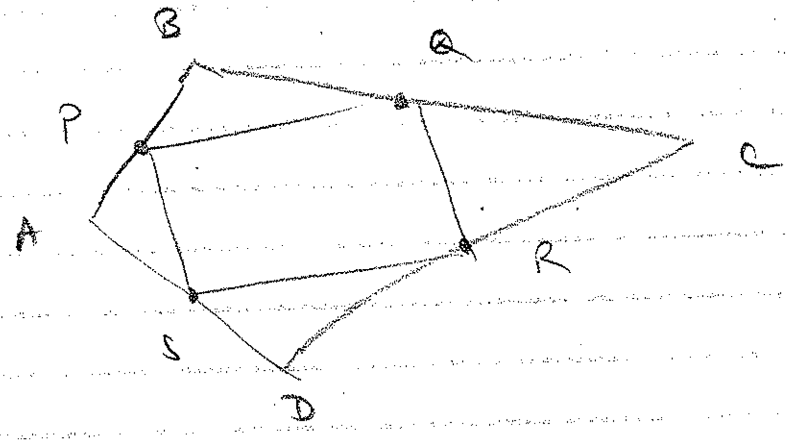
Q1/ (i) $\underset{\sim}{p} = \overrightarrow{OP}$, $\underset{\sim}{q} = \overrightarrow{OQ}$. If M is the midpoint of PQ then its position vector is

$$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM} = \underset{\sim}{p} + \tfrac{1}{2}(\overrightarrow{PQ}) = \underset{\sim}{p} + \tfrac{1}{2}(\overrightarrow{PO} + \overrightarrow{OQ})$$

$$= \underset{\sim}{p} + \tfrac{1}{2}(-\overrightarrow{OP} + \overrightarrow{OQ}) = \underset{\sim}{p} + \tfrac{1}{2}(-\underset{\sim}{p} + \underset{\sim}{q})$$

$$= \tfrac{1}{2}(\underset{\sim}{p} + \underset{\sim}{q}),$$ as required.

(ii)



Let P, Q, R, S be midpoints of AB, BC, CD, DA respectively.

Then $\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC})$

$= \frac{1}{2}\overrightarrow{AC}$

and similarly $\overrightarrow{SR} = \frac{1}{2}\overrightarrow{AC}$, so $\overrightarrow{PQ} = \overrightarrow{SR}$.

This is enough to prove PQRS is a parallelogram.

(B)

Q3/ $P_1: 3x+2y-z=1$, $P_2: \underline{r}\cdot(\underline{i}-\underline{j}-\underline{k})=3$

(i) $\underset{\sim}{s}_1 = 3\underline{i}+2\underline{j}-\underline{k}$ is perpendicular to P_1

(ii) $\underset{\sim}{s}_2 = \underline{i}-\underline{j}-\underline{k}$ " " " P_2

(iii) P_2 becomes $x-y-z=3$

Putting $x=0$, equations become $2y-z=1$
$-y-z=3$

so $3y=-2$, so $y=-2/3$, $z=-7/3$, so

$P=(0,-2/3,-7/3)$ lies on both planes.

(iv) The line of intersection contains P and has

direction vector $\underline{s}_1\times\underline{s}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -1 \\ 1 & -1 & -1 \end{vmatrix} = -3\underline{i}+2\underline{j}-5\underset{\sim}{k}$

so has vector equation $\boxed{\underline{r}=-2/3\,\underline{j}-7/3\,\underline{k}+t(-3\underline{i}+2\underline{j}-5\underline{k})}$

Q4/(i) B is reduced row-echelon if

(a) zero rows are at the bottom

(b) leading entries of nonzero rows are 1
and appear successively to the right as
one moves down the matrix

(c) entries above & below leading entries are 0.

Q4/ (ii) Let A be an $r \times n$ matrix.

When we row reduce A to reduced row echelon form B, say, we apply successive row operations that correspond to elementary matrices $E_1, \dots, E_k$, say, and successive multiplications on the left, so

$$B = E_k \cdots E_1 A.$$

But $E_1, \dots, E_k$ are $r \times r$ matrices (being the effect of row operations on the $r \times r$ identity matrix)

so $B = MA$ where $M = E_k \cdots E_1$.

But M is invertible, since it is a product of invertible matrices (each elementary matrix being invertible).

Q5/ (ii)

$$\det A = \begin{vmatrix} 2 & 2 & 4 & -2 \\ 3 & 3 & 6 & 5 \\ 4 & 4 & 11 & 1 \\ 1 & 2 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2 & 0 & 0 \\ 0 & -3 & 0 & 8 \\ 0 & -4 & 3 & 5 \\ 1 & 2 & 2 & -1 \end{vmatrix} = -\begin{vmatrix} -2 & 0 & 0 \\ -3 & 0 & 8 \\ -4 & 3 & 5 \end{vmatrix} = 2\begin{vmatrix} 0 & 8 \\ 3 & 5 \end{vmatrix}$$

$$= 2(0-24) = -48.$$

(iii) $B(C+AD) = B+C \Rightarrow C+AD = B^{-1}(B+C) = I + B^{-1}C$

$\Rightarrow AD = I + B^{-1}C - C \Rightarrow A = D^{-1} + B^{-1}CD^{-1} - CD^{-1}.$

Q6/ (i) (a) A is $n \times n$. To say B is the <u>inverse</u> of A

means $AB = BA = I_n$.

(b) Given $AB = BA = AC = CA = I_n$ then

$$B = BI = B(AC) = (BA)C = IC = C.$$

(ii) (a)

$$[M|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 6 & 4 & 0 & 1 & 0 \\ -2 & -5 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & -1 & 2 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 0 & -1 \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 5 & 0 & 2 \\ 0 & 1 & 0 & -8 & 2 & -1 \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 20 & -5 & 2 \\ 0 & 1 & 0 & -8 & 2 & -1 \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right] \quad \text{so} \quad M^{-1} = \begin{bmatrix} 20 & -5 & 2 \\ -8 & 2 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

(E)

Q6/ (ii) (b) $XM = \begin{bmatrix} -1 & 3 & 4 \\ 1 & -1 & 5 \end{bmatrix}$

$$\Rightarrow X = \begin{bmatrix} -1 & 3 & 4 \\ 1 & -1 & 5 \end{bmatrix} M^{-1}$$

$$= \begin{bmatrix} -1 & 3 & 4 \\ 1 & -1 & 5 \end{bmatrix} \begin{pmatrix} 20 & -5 & 2 \\ -8 & 2 & -1 \\ -3 & 1 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} -56 & 15 & -5 \\ 13 & -2 & 3 \end{bmatrix}.$$

Q7/ (i) $p(\lambda) = \det(M - \lambda I)$ is a polynomial of degree 7, with leading term 1, so $\lim_{\lambda \to \infty} p(\lambda) = \infty$, $\lim_{\lambda \to -\infty} p(\lambda) = -\infty$,

So by the Intermediate Value Theorem, $p(\lambda) = 0$ for some $\lambda \in \mathbb{R}$, which proves A has at least one real eigenvalue.

(ii) Given $A^2 + A = cI$. Suppose λ is an eigenvalue of A, so $A\underline{v} = \lambda \underline{v}$ for some $\underline{v} \neq \underline{0}$. Then

$(A^2 + A)\underline{v} = A^2\underline{v} + A\underline{v} = A\lambda\underline{v} + \lambda\underline{v} = \lambda A\underline{v} + \lambda\underline{v} = \lambda^2\underline{v} + \lambda\underline{v}$,

and $cI\underline{v} = c\underline{v}$, so $\lambda^2\underline{v} + \lambda\underline{v} = c\underline{v}$, so

F

Q7/ (ii) (cont.)

$$(\lambda^2+\lambda-c)\,\underline{v} = \underline{0},$$

so $\lambda^2+\lambda-c=0$, since $\underline{v}\neq\underline{0}$,

so $\lambda^2+\lambda=c$, as required.

(iii) Let λ be the real eigenvalue from (i),

so $\lambda^2+\lambda-c=0$, so $\lambda = \dfrac{-1\pm\sqrt{1+4c}}{2}$,

which forces $1+4c\geq 0$, so $4c\geq -1$,

so $c\geq -\frac{1}{4}$, as required.