Theorem (Möbius Inversion Formula)
Suppose we have $n \in \mathbb{Z}^+$ and numbers a_d for all divisors of n. Then the following system of linear equations in unknowns x_e, e goes through all divisors of n:

$$\sum_{e|d} x_e = a_d \quad \text{for all } d|n \qquad (1)$$

has the unique solution

$$x_e = \sum_{h|e} \mu\left(\frac{e}{h}\right) \cdot a_h \quad \text{for all } e|n \qquad (2)$$

Example: $n=12$. Divisors of 12 are 1, 2, 3, 4, 6, 12.
The system is.

$$\begin{cases} x_1 = a_1 \\ x_1 + x_2 = a_2 \\ x_1 + x_3 = a_3 \\ x_1 + x_2 + x_4 = a_4 \\ x_1 + x_2 + x_3 + x_6 = a_6 \\ x_1 + x_2 + x_3 + x_4 + x_6 + x_{12} = a_{12} \end{cases} \qquad (1)$$

Its solution is

$$x_1 = \mu(1)\, a_1 = a_1$$

$$x_2 = \mu(1)\, a_2 + \mu(2)\, a_1 = a_2 - a_1$$

$$x_3 = \mu(1)\, a_3 + \mu(3)\, a_1 = a_3 - a_1$$

$$x_4 = \mu(1)\, a_4 + \mu(2)\, a_2 + \mu(4)\, a_1 = a_4 - a_2$$

$$x_6 = \mu(1)\, a_6 + \mu(2)\, a_3 + \mu(3)\, a_2 + \mu(6)\, a_1$$

$$= a_6 - a_3 - a_2 + a_1$$

$$x_{12} = \mu(1)a_{12} + \mu(2)a_6 + \mu(3)a_4 + \mu(4)a_3 + \mu(6)a_2 + \mu(12)a_1$$

$$= a_{12} - a_4 - a_6 + a_2.$$

Proof. We solve the system (1) with help of matrices. We can rewrite (1) in the following form:

$$\cancel{x(M)\cdot a}$$

$$M \cdot \underline{x} = \underline{a}$$

where $\underline{x} = (x_e)_{e|n}$ is the vector of unknown numbers

$\underline{a} = (a_d)_{d|n}$ is the vector of known numbers

$M = (m_{de})_{d,e|n}$ is the matrix with

$$m_{d,e} = \begin{cases} 1 & \text{if } e|d \\ 0 & \text{otherwise} \end{cases}$$

Example: for $n = 12$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Notice that M is triangular with 1s on the diagonal.

$\Rightarrow \det M = 1$.

$\Rightarrow$ The system has a unique solution.

$\underline{x} = M^{-1} \cdot \underline{a}$ where M^{-1} is an inverse of M.

We need to check that $M^{-1} = P = (p_{e,h})_{e,h|n}$

where $p_{e,h} = \begin{cases} \mu(\frac{e}{h}) & \text{if } h|e \\ 0 & \text{otherwise.} \end{cases}$

We consider the product $M \cdot P$ and check that it coincides with the identity matrix I.

$$\underbrace{\begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{d1} & \cdots & m_{dn} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}}_{M} \cdot \underbrace{\begin{pmatrix} p_{11} & \cdots & p_{1h} & \cdots & p_{1n} \\ \vdots & & \vdots & & \vdots \\ p_{n1} & \cdots & p_{nh} & \cdots & p_{nn} \end{pmatrix}}_{P}$$

The entry (d,h) of the product is

$$m_{d1} \cdot p_{1h} + m_{d2} \cdot p_{2h} + \ldots + m_{dn} \cdot p_{nh} = \sum_{e|n} m_{de} \cdot p_{eh}$$

$$= \sum_{e|n} \begin{Bmatrix} 1 & \text{if } e|d \\ 0 & \text{otherwise} \end{Bmatrix} \cdot \begin{Bmatrix} \mu(\frac{e}{h}) & \text{if } h|e \\ 0 & \text{otherwise} \end{Bmatrix}$$

$$= \sum_{\substack{\text{all } e \text{ with} \\ e|d,\ h|e}} \mu\left(\frac{e}{h}\right) = \left[\begin{matrix}\text{change variables} \\ e = hk\end{matrix}\right]$$

$$= \sum_{\substack{\text{all } k \text{ with} \\ hk|d}} \mu(k) = \begin{cases} \sum_{k | d/h} \mu(k) \\ 0 \text{ if } h \nmid d \end{cases}$$

$$= [\text{by Proposition}] = \begin{cases} 1 & \text{if } d = h \\ 0 & \text{if } d \neq h. \end{cases}$$

This is exactly the entry (d,h) of I.

$\Rightarrow P = M^{-1}$ $\boxtimes$

Corollary (Möbius Inversion Formula, version 2)
Let two functions $f(n)$, $F(n)$ (not necessarily multiplicative) satisfy

$$F(n) = \sum_{d|n} f(n).$$

Then $$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(n).$$

(We take $a_d = F(d)$, $x_e = f(e)$ and apply MIF for all $n \in \mathbb{Z}^+$).

Examples: (1) $f(n) = 1 \leftarrow$ constant, $F(n) = \tau(n)$

Then $\sum_{d|n} \mu\left(\frac{n}{d}\right) \tau(n) = 1$

(2) $f(n) = \varphi(n)$, $F(n) = n$.

Then MIF implies $\varphi(n) = \sum \mu\left(\frac{n}{d}\right) n$

Then MIF implies $\varphi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \cdot n$.