Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

MARKS

1. (a) In the complex z-plane, $z = x + iy$, sketch the set satisfying the inequality, **3**

$$|z - 3 + 2i| \le 2.$$

(b) Factorise the polynomial,

$$P(z) = z^4 - 5z^3 + 5z^2 + 4z + 10,$$

into linear and/or quadratic factors with real coefficients, given that $3 - i$ is one of the roots of the polynomial. **4**

(c) In this part, you may assume the inequality,

$$\cos x < \frac{\sin x}{x} < 1 \qquad (0 < x \le \pi/2),$$

that was proved in lectures.

Hence, or otherwise, prove that the function $f(x) = (\sin x)/x$ is decreasing on the interval $(0, \pi]$. [Hint: take a derivative. Note that the subintervals $(0, \pi/2]$ and $(\pi/2, \pi]$ may require separate handling.] **5**

2. (a) Let $f : \mathbb{R}^2 \backslash \{0, 0\} \to \mathbb{R}$, $(x, y) \mapsto \ln(4x^2 + y^2)$, and let P denote the point $(1, 2)$ in the xy-plane.

(*i*) Calculate the directional derivative $D_{\mathbf{u}}f$ of f at P in the direction of the vector $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$. **3**

(*ii*) Find the unit vector $\hat{\mathbf{v}}$ in the direction in which the directional derivative of f at P is maximised, and give the corresponding value of the maximum directional derivative, that is, $D_{\hat{\mathbf{v}}}f$ at P. **2**

(*iii*) Find the equation of the tangent plane to the graph of $z = f(x, y)$ at the point on the graph vertically above P. Express your answer in the form $z = ax + by + c$. **3**

(b) Use any method to calculate the Taylor polynomial $T_4(x)$ of order 4 about $x = 0$ of the function,

$$f(x) = e^{2x} \cos 3x.$$

[Suggestion: you can use the standard Taylor series for e^x and $\cos x$ on the formula sheet on page 10.] **4**

MARKS

3. (a) Show that the function,

$$g(x) = \ln(2x) - \ln\left(1+\sqrt{1+x^2}\right),$$

has one and only one zero on the interval $[1, 10]$. **4**

(b) Find the following limits, showing the steps of your working clearly, or show that the limit does not exist. (You may use any valid method. Allow $+\infty$ and $-\infty$ as values that a limit can take.)

(*i*) $\displaystyle \lim_{x\to 3} \frac{x^3+x^2-33x+63}{x^3-27x+54}$. **2**

(*ii*) $\displaystyle \lim_{x\to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$.

[Hint: replace $\sinh x$ with its Taylor polynomial $T_3(x)$ about $x = 0$.] **3**

(*iii*) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3+xy+y^3}{x^2+y^2}$. **3**

MARKS

4. (a) Use the Taylor polynomial $T_3(x)$ of order 3 for $f(x) = \sin x$ about $x = 0$ and its remainder $R_3(x)$ to prove that

$$x - \frac{x^3}{6} < \sin x,$$

at least for $0 < x < \pi$. (You may quote the required Taylor polynomial from the formula sheet on Page 10, but you need to supply your own formula for the remainder term.) **4**

(b) Show that the inequality in part (a) holds for all real positive x and deduce that

$$x - \frac{x^3}{6} < \sin x < x$$

for all real positive x. (You can quote a result from the lectures regarding the right-hand inequality.) **3**

(c) Explain briefly why the coefficient $-1/6$ of x^3 on the left-hand side of the inequality in part (b) cannot be replaced by a number between $-1/6$ and zero. **2**

(d) Suppose $0 < \epsilon < 1$. Find the largest value of δ of the form $\delta = A\epsilon^b$ (more precisely, the largest $A > 0$ and smallest $b > 0$) such that the following statement is correct:

The function $(\sin x)/x$ tends to the limit 1 as $x \to 0^+$ because, given $\epsilon > 0$, there exists $\delta > 0$ depending on ϵ such that

$$\left|\frac{\sin x}{x} - 1\right| < \epsilon$$

whenever $0 < x < \delta$. **3**

Standard Derivatives

The following derivatives can be quoted without proof unless a question specifically asks you to show details. These results can be combined with the standard rules of differentiation (not listed here) to differentiate more complicated functions. For example, $(d/dx)\sin(ax+b) = a\cos(ax+b)$. Natural domains common to both sides are assumed.

1. $\dfrac{d}{dx}\,x^k = kx^{k-1} \quad (k \in \mathbb{R})$

2. $\dfrac{d}{dx}\,e^x = e^x$

3. $\dfrac{d}{dx}\,\ln x = \dfrac{1}{x} \quad (x > 0)$

4. $\dfrac{d}{dx}\,\sin x = \cos x$

5. $\dfrac{d}{dx}\,\cos x = -\sin x$

6. $\dfrac{d}{dx}\,\tan x = \sec^2 x$

7. $\dfrac{d}{dx}\,\cot x = -\operatorname{cosec}^2 x$

8. $\dfrac{d}{dx}\,\sec x = \sec x \tan x$

9. $\dfrac{d}{dx}\,\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

10. $\dfrac{d}{dx}\,\sinh x = \cosh x$

11. $\dfrac{d}{dx}\,\cosh x = \sinh x$

12. $\dfrac{d}{dx}\,\tanh x = \operatorname{sech}^2 x$

13. $\dfrac{d}{dx}\,\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$

14. $\dfrac{d}{dx}\,\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$

15. $\dfrac{d}{dx}\,\tan^{-1} x = \dfrac{1}{1+x^2}$

16. $\dfrac{d}{dx}\,\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$

17. $\dfrac{d}{dx}\,\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}} \quad (x > 1)$

18. $\dfrac{d}{dx}\,\tanh^{-1} x = \dfrac{1}{1-x^2} \quad (|x| < 1)$

Standard Taylor Series

The following Taylor series can be quoted without proof unless a question specifically asks you to show details. To get the corresponding Taylor polynomial $T_n(x)$ of order n, terminate the series at the last nonzero term at or before x^n. (Intervals of convergence are not needed.)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots, \qquad \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots,$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots,$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots, \qquad \alpha \in \mathbb{R}.$$

End of Extended Answer Section

This is the last page of the question paper.