School of Mathematics University of New South Wales

MATH2701: Abstract Algebra and Fundamental Analysis

Test

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1. For functions $f(x), g(x)$, $f(x) = \Theta(g(x))$ as $x \to \infty$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$ as $x \to \infty$. Thus, we break this proof into two parts, firstly proving

$$\sum_{n=N}^{\infty} \frac{1}{n^3} = O\left(\frac{1}{N^2}\right) \ldots (A),$$

and secondly proving

$$\frac{1}{N^2} = O\left(\sum_{n=N}^{\infty} \frac{1}{n^3}\right) \ldots (B).$$

In order to prove result (A), we must show there exists some $M_2 \neq 0$, and N_2, such that for all $N > N_2$,

$$\left|\sum_{n=N}^{\infty} \frac{1}{n^3}\right| \leq M_2 \left|\frac{1}{N^2}\right|.$$

Working with Riemann Sums, its clear that for $N > 1$, we have

$$\begin{aligned}
\sum_{n=N}^{\infty} \frac{1}{n^3} &\leq \int_{N-1}^{\infty} \frac{1}{x^3} dx \\
&= \lim_{k\to\infty} \int_{N-1}^{k} \frac{1}{x^3} dx \\
&= \lim_{k\to\infty} \left[\frac{-1}{2x^2}\right]_{N-1}^{k} \\
&= \lim_{k\to\infty} \left[\frac{-1}{2k^2} + \frac{-1}{2(N-1)^2}\right] \\
&= \frac{1}{2(N-1)^2} \\
&\leq \frac{10}{N^2} \text{ for } N > 1 \\
\therefore \left|\sum_{n=N}^{\infty} \frac{1}{n^3}\right| &\leq 10 \left|\frac{1}{N^2}\right| \text{ as } N > 1.
\end{aligned}$$

Clearly, $M_2 = 10$, $N_2 = 1$, and so

$$\sum_{n=N}^{\infty} \frac{1}{n^3} = O\left(\frac{1}{N^2}\right).$$

To prove result (B), it is equivalent to showing there exists some $M_1 \neq 0$, and N_1, such that for all $N > N_1$,

$$\left|\sum_{n=N}^{\infty} \frac{1}{n^3}\right| \geq M_1 \left|\frac{1}{N^2}\right|.$$

Working with Riemann Sums again, its clear that for $N > 1$, we have

$$\begin{aligned}\sum_{n=N}^{\infty} \frac{1}{n^3} &\geq \int_N^{\infty} \frac{1}{x^3} dx \\ &= \lim_{k\to\infty} \int_N^k \frac{1}{x^3} dx \\ &= \lim_{k\to\infty} \left[\frac{-1}{2x^2}\right]_{N-1}^k \\ &= \lim_{k\to\infty} \left[\frac{-1}{2k^2} + \frac{-1}{2N^2}\right] \\ &= \frac{1}{2N^2} \\ \therefore \left|\sum_{n=N}^{\infty} \frac{1}{n^3}\right| &\geq \frac{1}{2}\left|\frac{1}{N^2}\right| \quad \text{as } N > 1.\end{aligned}$$

Clearly, $M_1 = \frac{1}{2}$, $N_1 = 1$, and so

$$\frac{1}{N^2} = O\left(\sum_{n=N}^{\infty} \frac{1}{n^3}\right).$$

Thus,

$$\sum_{n=N}^{\infty} \frac{1}{n^3} = \Theta\left(\frac{1}{N^2}\right).$$

2. We shall prove that $\|\cdot\|_X$ is the dual norm of $\|\cdot\|_Y$. By definition, for fixed $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$, the dual norm of $\|\cdot\|_Y$ is

$$\|\mathbf{x}\|_X = \sup_{\mathbf{y}\in\mathbb{R}^{\mathbf{n}}} \frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|_Y}.$$

Thus, it suffices to show that $\|\mathbf{x}\|_X$ is an upper bound for $\frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|_Y}$ and that equality is attained for some $\mathbf{y} \in \mathbb{R}^{\mathbf{n}}$. By the first property of the norms provided,

$$\begin{aligned}|\mathbf{x}\cdot\mathbf{y}| &\leq \|\mathbf{x}\|_X \|\mathbf{y}\|_Y \\ \therefore \|\mathbf{x}\|_X &\geq \frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|_Y}.\end{aligned}$$

Thus, $\|\mathbf{x}\|_X$ is an upper bound for $\frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|_Y}$. By the second property, there exists a $\mathbf{y} \in \mathbb{R}^{\mathbf{n}}$ such that for all $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$,

$$\begin{aligned}|\mathbf{x}\cdot\mathbf{y}| &= \|\mathbf{x}\|_X \|\mathbf{y}\|_Y \\ \therefore \|\mathbf{x}\|_X &= \frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|_Y}.\end{aligned}$$

Thus, $\|\mathbf{x}\|_X$ is a least upper bound for $\frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{y}\|_Y}$, and so $\|\mathbf{x}\|_X$ is the dual norm of $\|\mathbf{y}\|_Y$. Recall that the dual norm of a dual norm is the original norm, so $\|\cdot\|_Y^* = \|\cdot\|_X$ implies that $\|\cdot\|_X^* = \|\cdot\|_Y$. If $\frac{1}{p} + \frac{1}{q} = 1$, then we can apply Hoelders Inequality, and its clear that $\|\cdot\|_p$ and $\|\cdot\|_q$ satisfy the properties of the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Thus, $\|\cdot\|_p$ and $\|\cdot\|_q$ are the dual norms of each other.

3. Consider $\mathbf{y} = (u, v, w) \in K^\circ$. By definition, $ux + vy + wz \le 1$, for all $\mathbf{x} = (x, y, z) \in K$. From Cauchy-Schwarz,

$$\begin{aligned} ux + vy + wz &\le (u^2 + v^2)^{1/2}(x^2 + y^2)^{1/2} + wz \\ &\le \sqrt{u^2 + v^2} + wz. \end{aligned}$$

Examining the condition for equality in the Cauchy-Schwarz inequality, we make the claim that $\sqrt{u^2 + v^2} \le 1 - cz$. As $|z| \le 1$, the previous result yields

$$\sqrt{u^2 + v^2} \le 1 - w \text{ and } \sqrt{u^2 + v^2} \le 1 + w.$$

Clearly, our polar body is given by

$$K^\circ = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 \le \min\{(1 - w)^2, (1 + w^2)\}\}.$$

As K is a cylinder with radius 1 and height 2, vol$(K) = 2\pi$. Considering K° as concentric circles of radius either $(1 - w)^2$ or radius $(1 + w^2)$, we have

$$\text{vol}(K^\circ) = \int_{-1}^{0} \pi(1 + w)^2 dw + \int_{0}^{1} \pi(1 - w)^2 dw = \frac{2\pi}{3}.$$

Thus, $M(K) = \text{vol}(K)\text{vol}(K^\circ) = \dfrac{4\pi^2}{3}$.

4. Suppose $\beta \in \mathbb{R}$ is an upper bound for S. Since S is non-empty, there exists an $s \in S$ which is itself not empty. Since $s \subset \alpha$, α is non-empty. Further since $\alpha \subset \beta$, (as $s \subset \beta$ for every $s \in S$), $\alpha \neq \mathbb{Q}$. To show that α satisfies all properties of a cut, we fix $p \in \alpha$. Then we must have $p \in s_0$ for some $s_0 \in S$ and so for some $q < p$ we have $q \in s_0$ and consequently $q \in \alpha$. Subsequently, if $r \in s$ is chosen so that $p < r$, which is possible since s_0 has no largest element, then $r \in \alpha$. Hence $\alpha \in \mathbb{R}$. It is also clear that α is an upper bound of S since $s \le \alpha$ for every $s \in S$. Suppose $\delta < \alpha$, that is there exists $p \in \alpha$ with $p \notin \delta$. Since $p \in \alpha$, $p \in s_0$ for some $s_0 \in S$. Hence $\delta < s_0$ and so δ can't be an upper bound of S. This shows that the cut α is the least upper bound of the set S.

5. To find the first four digits to the left of the decimal point of $(\ldots 333.3)^2$, we perform the long multiplication $(\ldots 333333) \times (\ldots 333333)$, adding the decimal point back in at the end.

$$\begin{array}{r} \ldots 333333 \\ \times \ldots 333333 \\ \hline \ldots 111104 \\ + \ldots 111040 \\ + \ldots 110400 \\ + \ldots 104000 \\ + \ldots 040000 \\ + \ldots 400000 \\ \hline \ldots 432044 \\ = \ldots 4320.44 \end{array}$$

Thus, the first four digits to the left of the decimal point are 4, 3, 2, and 0.

6. (a) We can write $k!$ as a product, which can be rewritten to involve p,

$$k! = 1 \times 2 \times \cdots \times (k - 1) \times k = 1 \times 2 \times \cdots \times p \times \cdots \times 2p \times \cdots \times ap \times k,$$

for some $a \in \mathbb{Z}$. Clearly, this gives the p-adic absolute value of $k!$ as $|k!|_p = p^{-a}$. As $a \in \mathbb{Z}$, we can write $a = \left\lfloor \frac{k}{p} \right\rfloor$. Note that,

$$\begin{aligned}
\left\lfloor \frac{k}{p} \right\rfloor &= \left\lfloor \frac{k}{p-1} - \frac{k}{p(p-1)} \right\rfloor \\
&\geq \left\lfloor \frac{k}{p-1} \right\rfloor - \left\lfloor \frac{k}{p(p-1)} \right\rfloor \\
&= \frac{k}{p-1} + c - \left\lfloor \frac{k}{p(p-1)} \right\rfloor \quad \text{for some } c \in [0,1) \\
\therefore -a &\leq -\frac{k}{p-1} - c + \left\lfloor \frac{k}{p(p-1)} \right\rfloor \\
\therefore -a &= -\frac{k}{p-1} + O(logk) \\
\therefore |k!|_p &= p^{\left(-\frac{k}{p-1} + O(logk)\right)}.
\end{aligned}$$

(b) For $x \in \mathbb{Q}$ we have $x = \frac{p^l b}{c}$, with $l \in \mathbb{Z}$, and $p \nmid bc$. Using the results provided in the question, and the previous part, we have,

$$\begin{aligned}
\left| \frac{x^k}{k!} \right|_p &= \frac{\left| \frac{p^{kl} b^k}{c^k} \right|_p}{|k!|_p} \\
&= \frac{p^{-kl}}{p^{-\frac{k}{p-1} + O(logk)}} \\
&= p^{-kl + \frac{k}{p-1} - O(logk)}.
\end{aligned}$$

Consider first the case where $p = 2$. The above result gives,

$$\begin{aligned}
\left| \frac{x^k}{k!} \right|_2 &= 2^{-kl + \frac{k}{2-1} - O(logk)} \\
&= 2^{-kl + k - O(logk)} \\
&= 2^{-k(l-1) - O(logk)}.
\end{aligned}$$

Taking the limit of the LHS as $k \to \infty$,

$$\lim_{k \to \infty} \left| \frac{x^k}{k!} \right|_2 = \lim_{k \to \infty} 2^{-k(l-1) - O(logk)},$$

which is equal to 0 when $(l-1) \geq 0$, or equivalently $l \geq 1$. Examining the second case, where $p \geq 3$, we have,

$$\begin{aligned}
\left| \frac{x^k}{k!} \right|_p &= p^{-kl + \frac{k}{p-1} - O(logk)} \\
&= p^{-k\left(l - \frac{1}{p-1}\right) - O(logk)}.
\end{aligned}$$

Taking the limit of the LHS as $k \to \infty$,

$$\lim_{k\to\infty} \left| \frac{x^k}{k!} \right|_p = \lim_{k\to\infty} p^{-k\left(l-\frac{1}{p-1}\right)-O(logk)},$$

which is equal to 0 when $\left(l - \frac{1}{p-1}\right) \geq 0$. As $p \geq 3$,

$$\begin{aligned} \left(l - \frac{1}{p-1}\right) &\geq \left(l - \frac{1}{3-1}\right) \\ &= \left(l - \frac{1}{2}\right). \end{aligned}$$

The condition for the limit to be equal to 0 now becomes $\left(l - \frac{1}{2}\right) \geq 0$, or equivalently $l \geq 1$, as $l \in \mathbb{Z}$. Thus,

$$\lim_{k\to\infty} \left| \frac{x^k}{k!} \right|_p = 0$$

iff $l \geq 1$ for $p \geq 3$ and $l \geq 2$ for $p = 2$.