The University of Sydney

Faculties of Arts, Economics, Education, Engineering and Science

MATH1901/1906

Differential Calculus (Advanced)

June 2007

Lecturers: Anthony Henderson and Charlie Macaskill

Time Allowed: **One and a half hours**

Name: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 25% of the total examination; there are 15 questions; the questions are of equal value; all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 75% of the total examination; there are 5 questions; the questions are of equal value; all questions may be attempted; working must be shown.

Calculators will be supplied; no other calculators are permitted.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

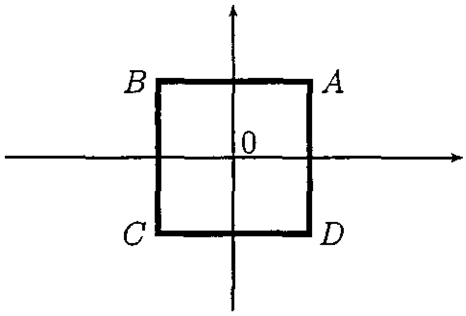
Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

1. (a) Find and sketch the set $\{z \in \mathbb{C} \mid 2|z| = z + \overline{z} + 1\}$. (3 marks)

(b) Prove that the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $z \mapsto z - \dfrac{1}{z}$,
is surjective but not injective. (3 marks)

(c) Consider the set S of complex numbers forming a square with corners $A = 1 + i$, $B = -1 + i$, $C = -1 - i$, and $D = 1 - i$.



Sketch the image of S under the function $z \mapsto e^z$.
Your sketch must indicate the images A', B', C', and D' of A, B, C, and D respectively. (4 marks)

2. Find the following limits, showing the steps of your working clearly. You may use any valid method.

(a) $\displaystyle\lim_{t \to 0} \frac{\sqrt{4+t} - 2}{3t}$ (b) $\displaystyle\lim_{x \to \infty} \left(\frac{x^2+1}{x-2} - \frac{x^2+1}{x+2} \right)$ (4 marks)

(c) $\displaystyle\lim_{y \to 0^+} \left(\cosh \frac{3}{y} \right)^y$ (d) $\displaystyle\lim_{(x,y) \to (0,0)} \frac{(x+y)^2}{\sqrt{x^2+y^2}}$ (6 marks)

3. (a) Let $h : (1, \infty) \to (0, \infty)$ be the inverse of the bijective function $\cosh : (0, \infty) \to (1, \infty)$. Assume that h is differentiable on $(1, \infty)$.

(i) Show that for all $x > 1$, $h'(x) = \dfrac{1}{\sqrt{x^2 - 1}}$. (2 marks)

(ii) Using the Mean Value Theorem, show that for all $x > \cosh 1$,
$$h(x) < 1 + \frac{x - \cosh 1}{\sinh 1}.$$ (4 marks)

(b) A special case of the Product Law for limits is the statement:
$$\text{if } \lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0, \text{ then } \lim_{x \to a} f(x)g(x) = 0.$$
Prove this special case, using the ϵ, δ definition of limit. (4 marks)

4. (a) Consider the function $f(x,y) = (x+2y)e^{x^2}$.

   (*i*) Find $\nabla f(x,y)$. (1 mark)

   (*ii*) Let $f(x,y) = 2$ be an implicit relation between x and y. For the given f, find dy/dx.

   Hence find the slope of the tangent to the curve $f(x,y) = 2$ at the point $(0,1)$.

   (2 marks)

   (*iii*) For the surface $z = f(x,y)$ at the point $(0,1,2)$, find the greatest slope and the two directions in which the surface is initially level. (2 marks)

   (b) (*i*) For any two functions $f(x,y)$ and $g(x,y)$ show that

$$\nabla(fg) = f\nabla g + g\nabla f,$$

   where you may assume that f_x, f_y, g_x and g_y are all well-defined. (3 marks)

   (*ii*) In the special case where f is a function of x only and g is a function of y only, show that the only solutions of $\nabla(fg) = 0$ are $f = 0$, $g = 0$, or both f and g constant. (2 marks)

5. (a) Consider the function $h : D \to \mathbb{R}$, $h(x,y) = \dfrac{x^2}{x^2+y^2}$. Find the domain and range of $h(x,y)$ and draw level curves for $z = 1/2$ and $z = 1/5$. (3 marks)

   (b) Taylor's formula for $f(x)$ about the point a is:

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots \\ &\quad + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(c) \end{aligned}$$

   for some c between a and x.

   (*i*) Write down Taylor's formula for e^x when $a = 0$. (1 mark)

   (*ii*) Show by substituting $x = 1$ in your formula for e^x in the previous part (*i*) that the remainder term, R_n, then satisfies the inequalities

$$\frac{1}{(n+1)!} < R_n < \frac{3}{(n+1)!}.$$

   (2 marks)

   (*iii*) For $n \geq 2$ show that $n!e$ is not an integer. (2 marks)

   (*iv*) Hence, using the result from the previous part (*iii*), prove that e is irrational by assuming that e is indeed rational and showing that this assumption leads to a contradiction. (2 marks)

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.