CHAPTER **ELEVEN**

Integration

The calculation of areas has so far been restricted to regions bounded by straight lines or parts of circles. *Integration* is the second of the two basic processes of calculus (the first being differentiation), and it extends the study of areas to regions bounded by more general curves — for example we will be able to calculate the area bounded by the parabola $x^2 = 4ay$ and its latus rectum. We will also be able to find the volume of the solid generated by rotating a region in the coordinate plane about one of the axes.

The surprising result at the centre of this section is, as mentioned before, that finding tangents and finding areas are inverse processes, so that integration is the inverse process of differentiation. This result is called the *fundamental theorem of calculus* because it is the basis of the whole theory of differentiation and integration. It will greatly simplify the calculations required.

Study Notes: In Section 11A some simple areas are calculated by a limiting process involving infinite dissections, and the definite integral is defined for functions with positive values. The fundamental theorem is proven and applied in Section 11B. In Section 11C the definite integral is extended to functions with negative values, and some simple theorems on the definite integral are established by dissection. This allows the standard methods of integration to be developed and applied to areas and volumes in Sections 11D–11G. Approximation methods are left until Sections 11I and 11J at the end of the chapter after the exact theory has been developed. The reverse chain rule is developed in Section 11H — this work may prove a little too hard for a first treatment of integration, and could be left until later. Computers or graphics calculators could be used to reinforce the definition of the definite integral in terms of areas, and they are of course particularly suited to the approximation methods. Computer graphics of volumes of rotation, or models constructed on a lathe, would be an effective way of making these solids a little more visible.

11 A Finding Areas by a Limiting Process

All work on areas must rest eventually on the basic definition of area, which is that the area of a rectangle is length times breadth. Any region bounded by straight lines, such as a triangle or a trapezium, can be rearranged into rectangles with a few well chosen cuttings and pastings, but any dissection of a curved region into rectangles must involve an infinite number of rectangles, and so must be a limiting process, like differentiation.

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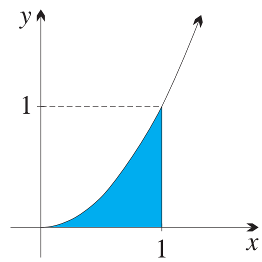
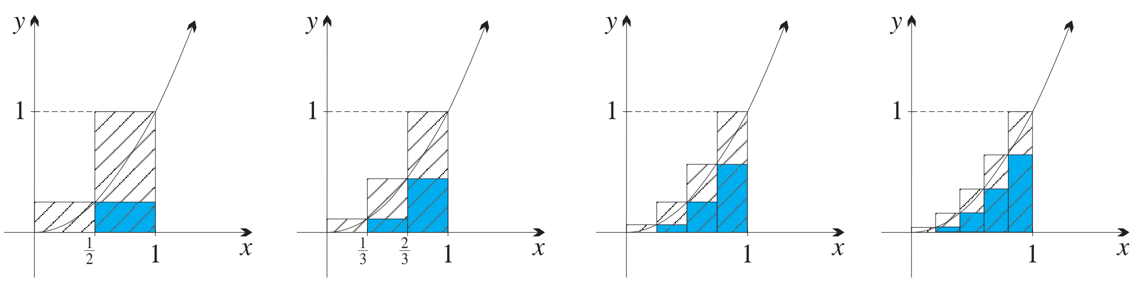
Example — The Area Under the Parabola $y = x^2$: These limiting calculations are much more elaborate than first-principles differentiation, and the fundamental theorem of calculus will soon make them unnecessary. However, it is advisable to carry out a very few such calculations in order to understand what is being done. The example below illustrates the technique, which in different forms was already highly developed by the Greeks independently of our present ideas about functions and graphs. We shall find the shaded area A of the region 'under the curve' $y = x^2$ from $x = 0$ to $x = 1$.



To begin the process, notice that the area is completely contained within the unit square, so we know that

$$0 < A < 1.$$

In the four pictures below, the region has been sliced successively into two, three, four and five strips, then 'upper' and 'lower' rectangles have been constructed so that the region is trapped, or sandwiched, between the upper and lower rectangles. Calculating the areas of these upper and lower rectangles provides tighter and tighter bounds on the area A.



In the first picture,

$$\begin{aligned} \tfrac{1}{2} \times (\tfrac{1}{2})^2 &< A < \tfrac{1}{2} \times (\tfrac{1}{2})^2 + \tfrac{1}{2} \times 1^2 \\ \tfrac{1}{8} &< A < \tfrac{5}{8}. \end{aligned}$$

In the second picture,

$$\begin{aligned} \tfrac{1}{3} \times (\tfrac{1}{3})^2 + \tfrac{1}{3} \times (\tfrac{2}{3})^2 &< A < \tfrac{1}{3} \times (\tfrac{1}{3})^2 + \tfrac{1}{3} \times (\tfrac{2}{3})^2 + \tfrac{1}{3} \times 1^2 \\ \tfrac{5}{27} &< A < \tfrac{14}{27}. \end{aligned}$$

In the third picture,

$$\begin{aligned} \tfrac{1}{4}\left((\tfrac{1}{4})^2 + (\tfrac{2}{4})^2 + (\tfrac{3}{4})^2\right) &< A < \tfrac{1}{4}\left((\tfrac{1}{4})^2 + (\tfrac{2}{4})^2 + (\tfrac{3}{4})^2 + (\tfrac{4}{4})^2\right) \\ \tfrac{14}{64} &< A < \tfrac{30}{64}. \end{aligned}$$

In the fourth picture,

$$\begin{aligned} \tfrac{1}{5}\left((\tfrac{1}{5})^2 + (\tfrac{2}{5})^2 + (\tfrac{3}{5})^2 + (\tfrac{4}{5})^2\right) &< A < \tfrac{1}{5}\left((\tfrac{1}{5})^2 + (\tfrac{2}{5})^2 + (\tfrac{3}{5})^2 + (\tfrac{4}{5})^2 + (\tfrac{5}{5})^2\right) \\ \tfrac{30}{125} &< A < \tfrac{55}{125}. \end{aligned}$$

The Limiting Process: The bounds on the area are getting tighter, but the exact value of the area can only be obtained if this sandwiching process can be turned into a limiting process. The calculations below will need the formula for the sum of the first n squares proven in Exercise 6N by induction:

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

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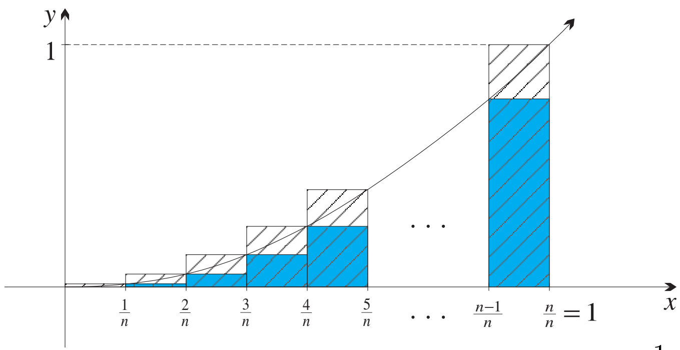

A. Divide the interval $0 \le x \le 1$ into n subintervals, each of width $\dfrac{1}{n}$.

On each subinterval form the upper rectangle and the lower rectangle. Then the required region is entirely contained within the upper rectangles, and, in turn, the lower rectangles are entirely contained within the required region. So however many strips the region has been dissected into,

$$\text{(sum of lower rectangles)} \le A \le \text{(sum of upper rectangles)}.$$

B. The heights of the successive upper rectangles are $\dfrac{1^2}{n^2}, \dfrac{2^2}{n^2}, \dots, \dfrac{n^2}{n^2}$, and so, using the formula quoted above,

$$\begin{aligned}
\text{sum of upper rectangles} &= \frac{1}{n}\left(\frac{1^2}{n^2}+\frac{2^2}{n^2}+\cdots+\frac{n^2}{n^2}\right)\\
&= \frac{1}{n^3}\left(1^2+2^2+\cdots+n^2\right)\\
&= \frac{1}{n^3}\times\frac{n(n+1)(2n+1)}{6}\\
&= \frac{1}{3}\times\frac{n}{n}\times\frac{n+1}{n}\times\frac{2n+1}{2n}\\
&= \frac{1}{3}\times\left(1+\frac{1}{n}\right)\times\left(1+\frac{1}{2n}\right),
\end{aligned}$$

hence the sum of the upper rectangles has limit $\frac{1}{3}$ as $n \to \infty$.

C. The heights of the successive lower rectangles are $0, \dfrac{1^2}{n^2}, \dfrac{2^2}{n^2}, \dots, \dfrac{(n-1)^2}{n^2}$, so substituting $n-1$ for n into the quoted formula,

$$\begin{aligned}
\text{sum of lower rectangles} &= \frac{1}{n}\left(0+\frac{1^2}{n^2}+\frac{2^2}{n^2}+\cdots+\frac{(n-1)^2}{n^2}\right)\\
&= \frac{1}{n^3}\left(1^2+2^2+\cdots+(n-1)^2\right)\\
&= \frac{1}{n^3}\times\frac{(n-1)n(2n-1)}{6}\\
&= \frac{1}{3}\times\frac{n}{n}\times\frac{n-1}{n}\times\frac{2n-1}{2n}\\
&= \frac{1}{3}\times\left(1-\frac{1}{n}\right)\times\left(1-\frac{1}{2n}\right),
\end{aligned}$$

hence the sum of the lower rectangles also has limit $\frac{1}{3}$ as $n \to \infty$.

D. Finally, since (sum of lower rectangles) $\le A \le$ (sum of upper rectangles), and since both these sums have the same limit $\frac{1}{3}$, it follows that $A = \frac{1}{3}$.

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The Definite Integral: More generally, suppose that $f(x)$ is a function that is continuous in a closed interval $a \le x \le b$. For the moment (that is, in Sections 11A and 11B), suppose that $f(x)$ is never negative anywhere within this interval. Then the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is called the *definite integral of $f(x)$ from $x = a$ to $x = b$*, and is given the symbol $\displaystyle\int_a^b f(x)\,dx$. The function $f(x)$ is called the *integrand*, and the numbers a and b are called the *lower bound* and the *upper bound* of the integral.

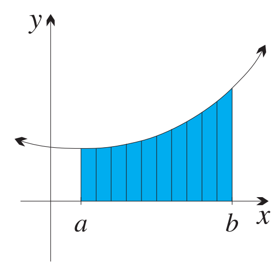
With this notation, the calculations above mean that

$$\int_0^1 x^2\,dx = \tfrac{1}{3},$$

and the process which was carried out can be described as follows:

1 INTEGRATION BY FIRST PRINCIPLES: To find a definite integral by first principles, dissect the interval into n equal subintervals, construct upper and lower rectangles on each subinterval, and find the sums of the upper and lower rectangles. Then their common limit will be the value of the integral.

Leibniz's striking notation for the definite integral arises from the intuitive understanding of the previous limiting process. Dissect the region $a \le x \le b$ into infinitely many slices, each of infinitesimal width dx. Each slice, being infinitesimally thin, is essentially a rectangle of width dx and height $f(x)$, so the area of each slice is $f(x)\,dx$. Now sum these slices from $x = a$ to $x = b$. The symbol $\int$ is an old form of the letter S, and thus $\displaystyle\int_a^b f(x)\,dx$ becomes the symbol for the sum of all the areas.



Contrast the 'smooth sum' of the integral with the 'jagged sum' of the sigma notation $\displaystyle\sum_{n=a}^{b} u_n$ introduced in Chapter Six, where the symbol $\sum$ is the Greek letter capital sigma, also meaning S and also standing for sum.

The name 'integration' suggests putting parts together to make a whole, and the approach and the notation both arise from building up a region in the plane out of an infinitely large number of infinitesimally thin strips. So integration is indeed 'making a whole' of these thin slices. Notice that the definite integral has been defined geometrically in terms of areas associated with the graph of a function, and that the language of functions has now been brought into the study of areas.

Further Examples of Definite Integrals: When the function is linear, the integral can be calculated using area formulae from mensuration, as in the first two examples (but a quicker method will be developed later). The last example involves a circular function.

WORKED EXERCISE: Evaluate using a graph and area formulae:

(a) $\displaystyle\int_1^4 (x-1)\,dx$ (b) $\displaystyle\int_2^4 (x-1)\,dx$ (c) $\displaystyle\int_{-a}^{a} |x|\,dx$ (d) $\displaystyle\int_{-5}^{5} \sqrt{25-x^2}\,dx$

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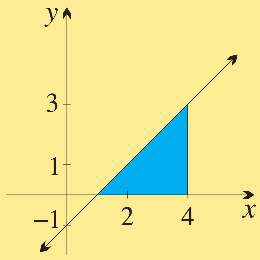
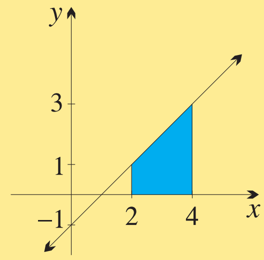
SOLUTION:

(a)



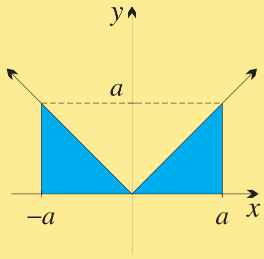
$$\int_1^4 (x-1)\,dx = \tfrac{1}{2} \times 3 \times 3$$
$$= 4\tfrac{1}{2}$$

(b)



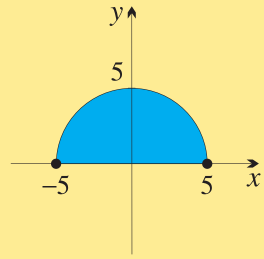
$$\int_2^4 (x-1)\,dx = 2 \times \tfrac{1}{2}(1+3)$$
$$= 4$$

(c)



$$\int_{-a}^{a} |x|\,dx = 2 \times \tfrac{1}{2}a \times a$$
$$= a^2$$

(d)



$$\int_{-5}^{5} \sqrt{25-x^2}\,dx = \tfrac{1}{2} \times 5^2 \times \pi$$
$$= 12\tfrac{1}{2}\pi$$

NOTE: Three area formulae from earlier mensuration have been used here:

2

FOR A TRIANGLE:	FOR A TRAPEZIUM:	FOR A CIRCLE:
$A = \frac{1}{2}bh$	$A = \frac{1}{2}h(a+b)$	$A = \pi r^2$

Exercise 11A

1. Sketch a graph of each integral, then use area formulae to evaluate it:

(a) $\displaystyle\int_4^8 (x-4)\,dx$

(b) $\displaystyle\int_{-1}^{3} (x+5)\,dx$

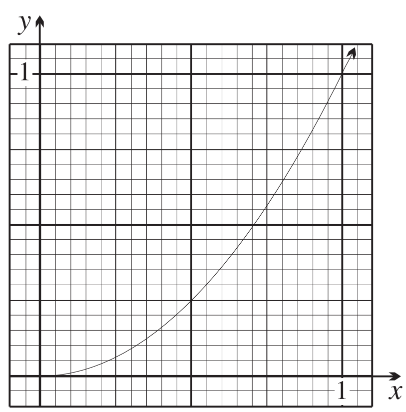
(c) $\displaystyle\int_{-4}^{4} \sqrt{16-x^2}\,dx$

(d) $\displaystyle\int_{-5}^{0} \sqrt{25-x^2}\,dx$

(e) $\displaystyle\int_3^6 (2x+1)\,dx$

(f) $\displaystyle\int_0^3 5\,dx$

2. The notes above this exercise used arguments involving limits to prove that $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$. The diagram on the right shows the graph of $y = x^2$ from $x = 0$ to $x = 1$, drawn with a scale of 20 little divisions to 1 unit. This means that 400 little squares make up 1 square unit.



(a) Count how many little squares there are under the graph from $x = 0$ to $x = 1$ (keeping reasonable track of fragments of squares), then divide by 400 to check how close this result is to $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$.

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(b) Question 4(a) below establishes the more general result $\int_0^a x^2\,dx = \frac{1}{3}a^3$. By counting the appropriate squares, check this result for $a = \frac{1}{4}$, $a = \frac{1}{2}$, $a = \frac{3}{5}$, $a = \frac{3}{4}$ and $a = \frac{4}{5}$.

DEVELOPMENT

3. Using exactly the same setting out as in the example in the notes above this exercise, show by first principles that $\int_0^1 x^3\,dx = \frac{1}{4}$, using upper and lower rectangles and taking the limit as the number of rectangular strips becomes infinite.
NOTE: This calculation will need the formula $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for the sum of the first n cubes, proven in Section 6N using mathematical induction.

4. Prove the following two definite integrals by first principles, using upper and lower rectangles and taking the limit as the number of rectangular strips becomes infinite:

(a) $\displaystyle\int_0^a x^2\,dx = \frac{a^3}{3}$ (b) $\displaystyle\int_0^a x^3\,dx = \frac{a^4}{4}$

EXTENSION

5. Draw a large sketch of $y = x^2$ for $0 \le x \le 1$, and let U be the point $(1, 0)$. For some positive integer n, let $P_0 (= O)$, P_1, P_2, …, P_n be the points on the curve with x-coordinates $x = 0$, $x = \dfrac{1}{n}$, $x = \dfrac{2}{n}$, …, $x = \dfrac{n}{n} = 1$. Join the chords P_0P_1, P_1P_2, … and $P_{n-1}P_n$, and join P_nU.

(a) Use the area formula for a trapezium to find the area of the polygon $P_0P_1P_2 \ldots P_nU$.

(b) Explain geometrically why this area is always greater than $\int_0^1 x^2\,dx$.

(c) Show that its limit as $n \to \infty$ is equal to $\frac{1}{3}$, that is, equal to $\int_0^1 x^2\,dx$.

11 B The Fundamental Theorem of Calculus

The fundamental theorem will allow us to evaluate definite integrals quickly using a quite straightforward algorithm based on primitives. Because the details of the proof are rather demanding, the algorithm is presented first, by means of some worked examples, and the proof is left to the end of the section.

Statement of the Fundamental Theorem (Integral Form): The integral form of the fundamental theorem, stated below, is essentially a formula for evaluating a definite integral.

3 THE FUNDAMENTAL THEOREM (INTEGRAL FORM): Let $f(x)$ be a function continuous in the closed interval $a \le x \le b$, and let $F(x)$ be a primitive of $f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This means that a definite integral can be evaluated by writing down any primitive $F(x)$ of $f(x)$, then substituting the upper and lower bounds into it and subtracting.

Using the Fundamental Theorem to Evaluate an Integral: The conventional way to set out these calculations is to enclose the primitive in square brackets, writing the upper and lower bounds as superscript and subscript respectively.

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Worked Exercise:

(a) $\displaystyle\int_0^1 x^2\,dx = \Big[\tfrac{1}{3}x^3\Big]_0^1$
$= \tfrac{1}{3} - 0$
$= \tfrac{1}{3}$
(as calculated in Section 11A)

(b) $\displaystyle\int_2^4 (x-1)\,dx = \Big[\tfrac{1}{2}x^2 - x\Big]_2^4$
$= (8-4) - (2-2)$
$= 4$
(as calculated in Section 11A)

(c) $\displaystyle\int_{-2}^2 (x^3+8)\,dx = \Big[\tfrac{1}{4}x^4 + 8x\Big]_{-2}^2$
$= (4+16) - (4-16)$
$= 32$

(d) $\displaystyle\int_1^2 x^{-2}\,dx = \Big[-x^{-1}\Big]_1^2$
$= -\tfrac{1}{2} + 1$
$= \tfrac{1}{2}$

Change of Pronumeral: The rest of this section is an exposition of the proof of the fundamental theorem. First, notice that the pronumeral in the definite integral notation is a *dummy variable*, meaning that it can be replaced by any other pronumeral. For example, the four integrals

$$\int_0^1 x^2\,dx = \int_0^1 t^2\,dt = \int_0^1 y^2\,dy = \int_0^1 \lambda^2\,d\lambda$$

all have the same value $\frac{1}{3}$ — the letter used for the variable has changed, but the function remains the same and so the area involved remains the same. Similarly the pronumeral in sigma notation is a dummy variable. For example,

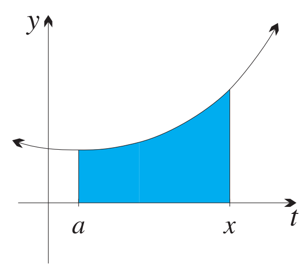
$$\sum_{n=1}^{4} n^2 = \sum_{r=1}^{4} r^2 = \sum_{x=1}^{4} x^2 = \sum_{\lambda=1}^{4} \lambda^2$$

all have the same value $1+4+9+16 = 30$.

The Definite Integral as a Function of its Upper Bound: The value of the definite integral $\displaystyle\int_a^b f(x)\,dx$ changes when the value of a or b is changed. This means that it is a function both of its upper bound b and of its lower bound a. In order to suggest the functional relationship with the upper bound b more closely, we shall replace the letter b by the letter x, which is conventionally the variable of a function. The original letter x needs to be replaced in turn by some other letter — a suitable choice is t, which is also conventionally a variable. Then we can speak clearly about the definite integral

$$A(x) = \int_a^x f(t)\,dt$$

being a function of its upper bound x. This integral is represented in the sketch above.



The Fundamental Theorem of Calculus — Differential Form: There are two forms of the theorem, the *integral form* stated above, and the following *differential form* which will be proven first. The differential form claims that this definite integral $A(x) = \displaystyle\int_a^x f(t)\,dt$ is a primitive of $f(x)$. It has been stated so as to make clear that the two processes of differentiation and integration are inverse processes and cancel each other out.

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4 THE FUNDAMENTAL THEOREM (DIFFERENTIAL FORM): Let $f(x)$ be a function that is continuous in the closed interval $a \le x \le b$. Then the definite integral $\displaystyle\int_a^x f(t)\,dt$, regarded as a function of its upper bound x, is a primitive of $f(x)$:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x), \quad \text{for } a < x < b.$$

Examples of the Fundamental Theorem: Before proving the theorem, let us look at two examples which should make the statement of the theorem clear.

First, we have shown in Exercise 11A that $\displaystyle\int_0^a x^2\,dx = \tfrac{1}{3}a^3$.

Changing the variables as described above, $\displaystyle\int_0^x t^2\,dt = \tfrac{1}{3}x^3$,

and differentiating, $\displaystyle\frac{d}{dx}\int_0^x t^2\,dt = x^2$, as the theorem says.

As a second example, we showed also that $\displaystyle\int_0^a x^3\,dx = \tfrac{1}{4}a^4$.

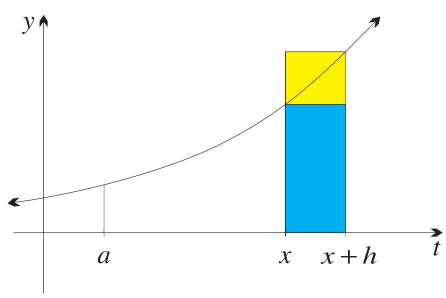
Changing variables and differentiating, $\displaystyle\frac{d}{dx}\int_0^x t^3\,dt = x^3$, as the theorem says.

Proof of the Differential Form: The proof is based on the sandwiching technique. Notice that in constructing this proof, we are still working under the assumption that $f(x)$ is never negative anywhere within the interval $a \le x \le b$.

Let $\displaystyle A(x) = \int_a^x f(t)\,dt$, then we must prove that $A'(x) = f(x)$.

Recall the definition of the derivative as a limit:

$$A'(x) = \lim_{h\to 0}\frac{A(x+h) - A(x)}{h}.$$



Now $\displaystyle A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$

$$= \int_x^{x+h} f(t)\,dt,$$

so $\displaystyle A'(x) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt.$

Suppose that $f(t)$ is increasing in the interval $x \le t \le x+h$, as in the diagram above.
Then the lower rectangle on the interval $x \le t \le x+h$ has height $f(x)$,
and the upper rectangle on the interval $x \le t \le x+h$ has height $f(x+h)$,

so, using areas, $\displaystyle h \times f(x) \le \int_x^{x+h} f(t)\,dt \le h \times f(x+h)$

$\boxed{\div h}$ $\displaystyle f(x) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x+h).$

Since $f(x)$ is continuous, $f(x+h) \to f(x)$ as $h \to 0$,

and so $\displaystyle\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$, meaning that $A'(x) = f(x)$, as required.

If $f(x)$ is decreasing in $x \le t \le x+h$, the argument applies with inequalities reversed.

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Proof of the Integral Form: It is given that $F(x)$ is a primitive of $f(x)$,

and the fundamental theorem says that $\displaystyle\int_a^x f(t)\,dt$ is also a primitive of $f(x)$,

so the primitives $\displaystyle\int_a^x f(t)\,dt$ and $F(x)$ must differ only by a constant,

that is, $$\int_a^x f(t)\,dt = F(x) + C, \text{ for some constant } C.$$

Substituting $x = a$, $$\int_a^a f(t)\,dt = F(a) + C,$$

but $\displaystyle\int_a^a f(t)\,dt = 0$, because the area in this definite integral has zero width,

so $$0 = F(a) + C.$$

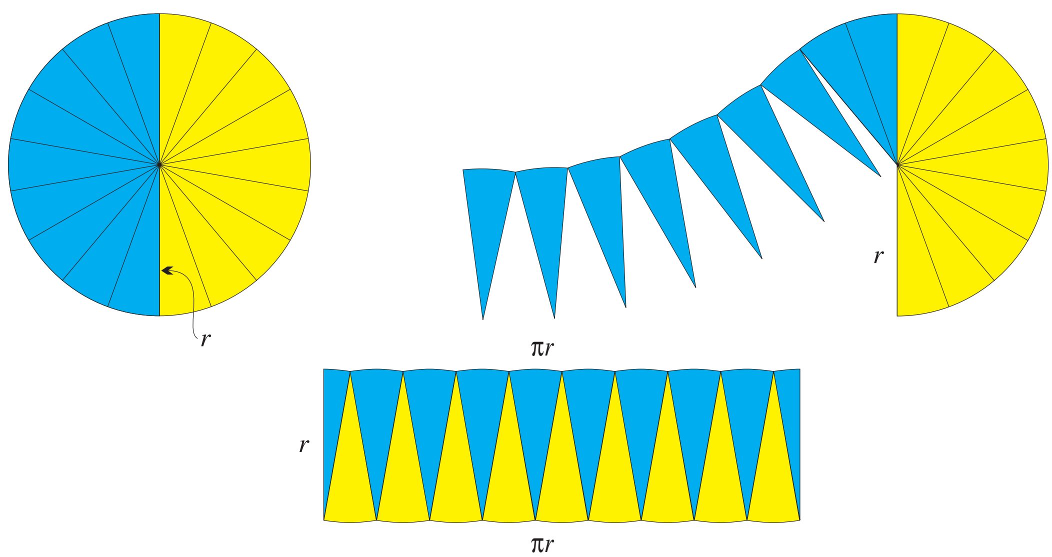
Hence $C = -F(a)$, and $\displaystyle\int_a^x f(t)\,dt = F(x) - F(a)$.

Changing letters, $$\int_a^b f(x)\,dx = F(b) - F(a).$$

NOTE: Some readers who may have noticed a lack of rigour in the preceding arguments could benefit from the treatment in a more advanced text. In particular, we assumed in the proof of the differential form that $f(t)$ was either increasing or decreasing in the closed interval $x < t < x + h$, and we also assumed that h was positive. Secondly, the proof of the integral form actually requires that $A(x)$ be differentiable in the closed interval $a \le x \le b$, which in turn requires constructing a definition of *one-sided derivatives* at the endpoints, in a manner similar to the definition of continuity in a closed interval.

More fundamentally, the definite integral was defined in terms of area, but it is not at all clear that a region bounded by curves actually has an area, since area had previously only been defined for rectangles and then by dissection for regions bounded by straight lines. More rigorous treatments turn a generalisation of the 'first principles' calculation of integrals into the definition of the definite integral, and then define area in terms of the definite integral.

The Area of a Circle:



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In earlier years, the formula $A = \pi r^2$ for the area of a circle was proven. Because the boundary is a curve, some limiting process had to be used in that proof. For comparison with the arguments used above about the definite integral, here is the most common version of that argument — a little rough in its logic, but very quick. It involves dissecting the circle into infinitesimally thin sectors, and then rearranging them into a rectangle.

The height of the rectangle in the lower diagram is r. Since the circumference $2\pi r$ is divided equally between the top and bottom sides, the length of the rectangle is πr. So the rectangle has area πr^2, which is therefore the area of the circle.

Exercise 11B

1. Evaluate the following definite integrals using the fundamental theorem:

(a) $\int_1^4 2x\,dx$ (d) $\int_1^2 10x^4\,dx$ (g) $\int_{-1}^1 (4x^3+3x^2+1)\,dx$

(b) $\int_0^5 x^2\,dx$ (e) $\int_3^6 (2x+1)\,dx$ (h) $\int_0^3 (x+x^2+x^3)\,dx$

(c) $\int_2^3 3x^2\,dx$ (f) $\int_4^7 dx$ (i) $\int_{-3}^1 (2x^2-7x+5)\,dx$

2. (a) Evaluate the following definite integrals:

(i) $\int_5^{10} x^{-2}\,dx$ (ii) $\int_2^3 2x^{-3}\,dx$ (iii) $\int_{\frac{1}{2}}^1 x^{-5}\,dx$

(b) By writing them with negative indices, evaluate the following definite integrals:

(i) $\int_1^2 \frac{dx}{x^2}$ (ii) $\int_1^4 \frac{dx}{x^3}$ (iii) $\int_{\frac{1}{2}}^1 \frac{3}{x^4}\,dx$

3. By expanding the brackets where necessary, evaluate the following definite integrals:

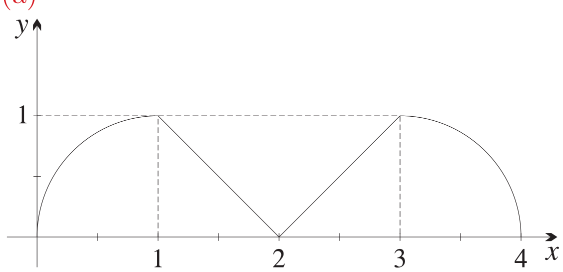
(a) $\int_0^1 3x(2+x)\,dx$ (c) $\int_1^2 \frac{1+x^2}{x^2}\,dx$ (e) $\int_{-1}^0 (1-x^2)^2\,dx$

(b) $\int_1^3 (x+2)^2\,dx$ (d) $\int_1^3 \left(x+\frac{1}{x}\right)^2\,dx$ (f) $\int_4^9 \left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)\,dx$

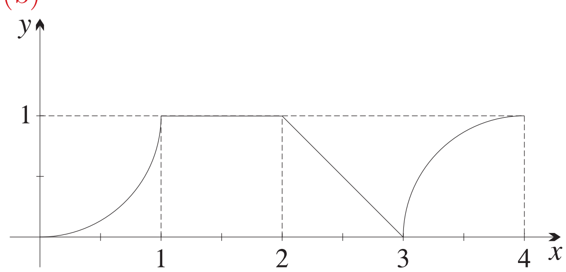
DEVELOPMENT

4. Use area formulae to find $\int_0^4 f(x)\,dx$ in each sketch of $f(x)$:

(a)



(b)



5. Find the value of k if:

(a) $\int_2^k \frac{3}{x^2}\,dx = \frac{9}{10}$ (b) $\int_0^3 kx^2\,dx = 4$ (c) $\int_{-1}^2 (3x^2+4x+k)\,dx = 30$

6. Sketch the integrand and explain why this calculation is invalid:

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2.$$

EXTENSION

7. (a) Find $\dfrac{d}{dx}\displaystyle\int_3^x (4t^3 - 3t^2 + t - 1)\, dt$.

(b) Find $\dfrac{d}{dx}\displaystyle\int_x^2 (7 - 6t)^4\, dt$.

(c) The derivative of the function $U(x)$ is $u(x)$.

(i) Find $V'(x)$, where $V(x) = (a - x)U(x) + \displaystyle\int_0^x U(t)\, dt$ and a is a constant.

(ii) Hence prove that $\displaystyle\int_0^a U(x)\, dx = aU(0) + \int_0^a (a - x)u(x)\, dx$.

8. Is it possible to have a non-negative function $f(x)$ defined on the interval $0 \le x \le 1$ such that $f(c) > 0$ for some c such that $0 < c < 1$, but $\displaystyle\int_0^1 f(x)\, dx = 0$?

11 C The Definite Integral and its Properties

This section will first extend the theory to functions with negative values. Then some properties of the definite integral will be established using fairly obvious arguments about the dissection of the area associated with the integral.

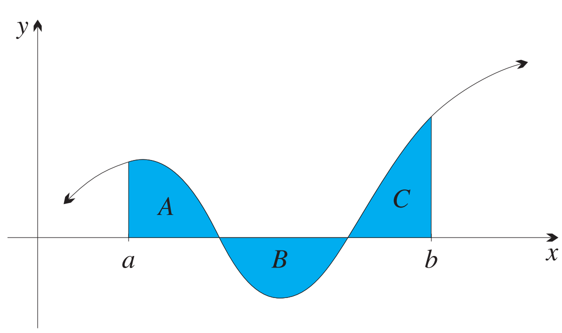
Integrating Functions with Negative Values: When a function has negative values, its graph is below the x-axis, so the 'heights' of the little rectangles in the dissection are negative numbers. This means that areas below the x-axis should contribute negative values to the final integral. The fundamental theorem will then allow these integrals to be evaluated in the usual way.

5

DEFINITION: Let $f(x)$ be a function which is continuous in some closed interval $a \le x \le b$. The *definite integral* $\displaystyle\int_a^b f(x)\, dx$ is the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$, with areas above the x-axis counted as positive and areas below the x-axis counted as negative.

In the diagram to the right, the region B is below the x-axis, and so is counted as negative in the definite integral:

$$\int_a^b f(x)\, dx = \text{area } A - \text{area } B + \text{area } C.$$



Because areas under the x-axis are counted as negative, the definite integral is sometimes referred to as the *signed area* under the curve, to distinguish it from area, which is always positive.

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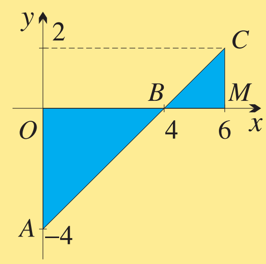
WORKED EXERCISE: Evaluate and sketch: (a) $\displaystyle\int_0^4 (x-4)\,dx$ (b) $\displaystyle\int_0^6 (x-4)\,dx$

SOLUTION:

(a) $$\int_0^4 (x-4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_0^4$$
$$= (8-16)-(0-0)$$
$$= -8$$



(b) $$\int_0^6 (x-4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_0^6$$
$$= (18-24)-(0-0)$$
$$= -6$$

Notice that the area of the shaded triangle OAB below the x-axis has area 8, and accordingly the first integral is -8. The shaded triangle BCM above the x-axis has area 2, and accordingly the second integral is $-8+2=-6$.

Odd and Even Functions: The first example below shows the graph of $y = x^3 - 4x$. Because the function is odd, the area of each shaded hump is the same, so the whole integral from $x=-2$ to $x=2$ is zero, because the equal humps above and below the x-axis cancel out. But if the function is even, as in the second example, there is a doubling instead of a cancelling:

6

ODD FUNCTIONS: If $f(x)$ is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

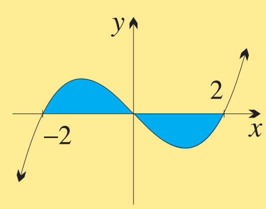
EVEN FUNCTIONS: If $f(x)$ is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

WORKED EXERCISE: Sketch, then evaluate using symmetry:

(a) $\displaystyle\int_{-2}^{2} (x^3-4x)\,dx = 0$ (b) $\displaystyle\int_{-2}^{2} (x^2+1)\,dx$

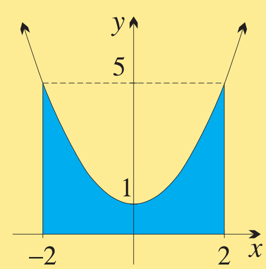
SOLUTION:

(a) $\displaystyle\int_{-2}^{2} (x^3-4x)\,dx = 0,$ since the integrand is odd.



(Without this simplification the calculation is
$$\int_{-2}^{2} (x^3-4x)\,dx = \left[\tfrac{1}{4}x^4 - 2x^2\right]_{-2}^{2}$$
$$= (4-8)-(4-8)$$
$$= 0,\ \text{as before.})$$

(b) Since the integrand is even,
$$\int_{-2}^{2} (x^2+1)\,dx = 2\int_0^2 (x^2+1)\,dx$$
$$= 2\left[\tfrac{1}{3}x^3 + x\right]_0^2$$
$$= 2(2\tfrac{2}{3}+2)-(0+0)$$
$$= 9\tfrac{1}{3}.$$



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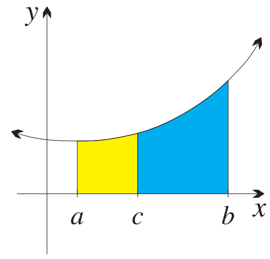
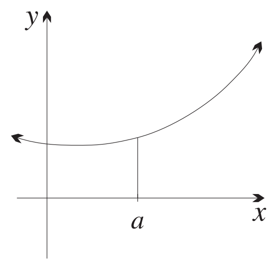


Dissection of the Interval: In the work so far, we have routinely dissected the region by dissecting the interval over which the integration is being performed. If $f(x)$ is continuous in the interval $a \le x \le b$ and the number c lies in this interval, then:

7 DISSECTION: $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Intervals of Zero Width: In the course of the proof in Section 11B of the fundamental theorem, the trivial remark was made that if the interval has width zero, then the integral is zero. Provided that a function $f(x)$ is defined at $x = a$, then:

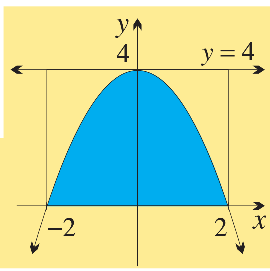
8 INTERVALS OF ZERO WIDTH: $$\int_a^a f(x)\,dx = 0$$

Inequalities with Definite Integrals: If $f(x)$ and $g(x)$ are two functions continuous in the interval $a \le x \le b$, with $f(x) \le g(x)$ throughout the interval, then the integral of $f(x)$ is less than the integral of $g(x)$.

9 INEQUALITY: If $f(x) \le g(x)$ in the interval $a \le x \le b$, then
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

When both functions are positive, then the region under the curve $y = f(x)$ is contained within the region under the curve $y = g(x)$. If one or both functions become negative, then the inequality still holds because of the qualification that areas under the x-axis are counted as negative.

WORKED EXERCISE: Sketch the graph of $f(x) = 4 - x^2$ for $-2 \le x \le 2$, and explain why $0 \le \displaystyle\int_{-2}^{2} (4 - x^2)\,dx \le 16$.



SOLUTION: Since $0 \le 4 - x^2 \le 4$ in the interval $-2 \le x \le 2$, it follows that the region associated with the integral is inside the square of side length 4 in the diagram opposite.

Running an Integral Backwards: A further small qualification must be made to the definition of the definite integral. Suppose that the function $f(x)$ is defined in the closed interval $a \le x \le b$. Then:

10 REVERSING THE INTERVAL: $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

So if the integral 'runs backwards' over the interval, then the integral reverses in sign. This agrees perfectly with the fundamental theorem, because

$$F(a) - F(b) = -\big(F(b) - F(a)\big).$$

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WORKED EXERCISE: By this definition,

$$\begin{aligned}\int_2^{-2}(x^2+1)\,dx &= -\int_{-2}^{2}(x^2+1)\,dx\\ &= -9\tfrac{1}{3} \quad \text{(the last integral was calculated above).}\end{aligned}$$

Alternatively, calculating the integral directly,

$$\begin{aligned}\int_2^{-2}(x^2+1)\,dx &= \Big[\tfrac{1}{3}x^3+x\Big]_2^{-2}\\ &= (-2\tfrac{2}{3}-2)-(2\tfrac{2}{3}+2)\\ &= -9\tfrac{1}{3}.\end{aligned}$$

Linear Combinations of Functions: When two functions are added, the two regions are piled on top of each other, so that:

11 INTEGRAL OF A SUM: $\displaystyle\int_a^b \Big(f(x)+g(x)\Big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Similarly, when a function is multiplied by a constant, the region is expanded vertically by that constant, so that:

12 INTEGRAL OF A MULTIPLE: $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

EXAMPLE: Using these rules about linear combinations,

$$\begin{aligned}\int_0^1(3x^2+2x)\,dx &= \int_0^1 3x^2\,dx + \int_0^1 2x\,dx\\ &= 3\int_0^1 x^2\,dx + 2\int_0^1 x\,dx\\ &= 3\times\tfrac{1}{3}+2\times\tfrac{1}{2}, \quad \text{since } \int_0^1 x^3\,dx=\tfrac{1}{3} \text{ and } \int_0^1 x^2\,dx=\tfrac{1}{2},\\ &= 2, \quad \text{as before.}\end{aligned}$$

This result should be checked by the simpler direct evaluation of the integral.

Exercise 11C

1. Calculate each definite integral using a graph and area formulae. In computing the integral, recall that areas above the x-axis are counted as positive, and areas below the x-axis are counted as negative.

(a) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ (c) $\displaystyle\int_2^4 (3-2x)\,dx$ (e) $\displaystyle\int_{-3}^0 (x+2)\,dx$

(b) $\displaystyle\int_{-3}^3 \Big(-\sqrt{9-x^2}\Big)\,dx$ (d) $\displaystyle\int_0^4 (-2)\,dx$ (f) $\displaystyle\int_2^{10} (5-x)\,dx$

2. Evaluate the following definite integrals using the fundamental theorem:

(a) $\displaystyle\int_{-1}^2 x^3\,dx$ (c) $\displaystyle\int_0^6 (x^2-6x)\,dx$ (e) $\displaystyle\int_0^2 (4x^3-2x)\,dx$

(b) $\displaystyle\int_{-1}^1 6x^5\,dx$ (d) $\displaystyle\int_{-1}^1 (x^3-x)\,dx$ (f) $\displaystyle\int_{-2}^{10} (12-3x)\,dx$

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3. (a) Evaluate the following definite integrals:

(i) $\displaystyle\int_0^1 x^{\frac{1}{2}}\,dx$ (ii) $\displaystyle\int_1^4 x^{-\frac{1}{2}}\,dx$ (iii) $\displaystyle\int_1^8 x^{\frac{1}{3}}\,dx$

(b) By writing them with fractional indices, evaluate the following definite integrals:

(i) $\displaystyle\int_1^4 \sqrt{x}\,dx$ (ii) $\displaystyle\int_1^9 x\sqrt{x}\,dx$ (iii) $\displaystyle\int_1^9 \frac{dx}{\sqrt{x}}$

4. By expanding the brackets where necessary, evaluate the following definite integrals:

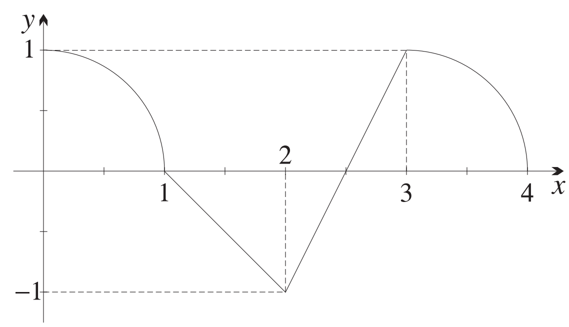
(a) $\displaystyle\int_0^2 x(1-x)\,dx$ (c) $\displaystyle\int_1^4 \frac{x^3-3}{x^3}\,dx$ (e) $\displaystyle\int_1^4 \left(2\sqrt{x}-\frac{3}{\sqrt{x}}\right)dx$

(b) $\displaystyle\int_{-2}^2 (2-x)(1+x)\,dx$ (d) $\displaystyle\int_0^5 x(x+1)(x-1)\,dx$ (f) $\displaystyle\int_1^8 \left(\sqrt[3]{x}+x\right)dx$

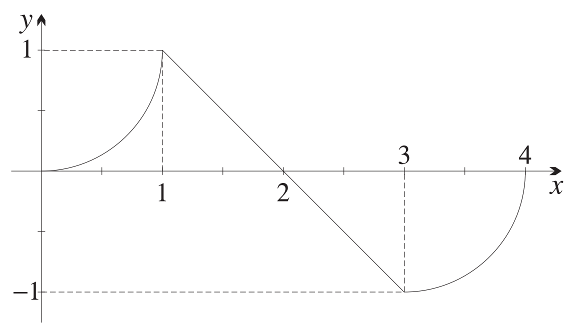
DEVELOPMENT

5. Use area formulae to find $\displaystyle\int_0^4 f(x)\,dx$ in each sketch of $f(x)$:

(a)



(b)



6. Find the value of k if:

(a) $\displaystyle\int_1^k (x+1)\,dx = 6$ (b) $\displaystyle\int_1^k (k+3x)\,dx = \tfrac{13}{2}$ (c) $\displaystyle\int_1^9 \frac{k}{\sqrt{x}}\,dx = 4$

7. Use the properties of the definite integral to evaluate, stating reasons:

(a) $\displaystyle\int_3^3 \sqrt{9-x^2}\,dx$ (c) $\displaystyle\int_{-1}^1 x^3\,dx$ (e) $\displaystyle\int_{-3\cdot47}^{3\cdot47} \frac{1}{5x^3-7x}\,dx$

(b) $\displaystyle\int_4^4 (x^3-3x^2+5x-7)\,dx$ (d) $\displaystyle\int_{-5}^5 (x^3-25x)\,dx$ (f) $\displaystyle\int_{-2}^2 \frac{x}{1+x^2}\,dx$

8. Using the properties of the definite integral, explain why: (a) $\displaystyle\int_{-\alpha}^{\alpha} (ax^5+cx^3+ex)\,dx = 0$

(b) $\displaystyle\int_{-\alpha}^{\alpha} (ax^5+bx^4+cx^3+dx^2+ex+f)\,dx = 2\int_0^{\alpha} (bx^4+dx^2+f)\,dx$

9. (a) On one set of axes, sketch $y = x^2$ and $y = \sqrt{x}$, showing the point of intersection.

(b) Hence explain why $0 < \displaystyle\int_0^1 x^2\,dx < \int_0^1 \sqrt{x}\,dx < 1$.

10. (a) Calculate using a graph and area formulae: (i) $\displaystyle\int_0^5 1\,dx$ (ii) $\displaystyle\int_0^5 x\,dx$

(b) Using these results, and the properties of integrals of sums and multiples, evaluate:

(i) $\displaystyle\int_0^5 2x\,dx$ (ii) $\displaystyle\int_0^5 (x+1)\,dx$ (iii) $\displaystyle\int_0^5 (3x-2)\,dx$

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11. (a) Calculate the following definite integrals using a graph and area formulae:

(i) $\displaystyle\int_{-\frac{1}{4}}^{\frac{1}{4}} (1-4x)\,dx$ (ii) $\displaystyle\int_{-5}^{1} |x+5|\,dx$ (iii) $\displaystyle\int_{0}^{2} (|x|+3)\;dx$

(b) Hence write down the values of the following definite integrals:

(i) $\displaystyle\int_{\frac{1}{4}}^{-\frac{1}{4}} (1-4x)\,dx$ (ii) $\displaystyle\int_{1}^{-5} |x+5|\,dx$ (iii) $\displaystyle\int_{2}^{0} (|x|+3)\;dx$

12. (a) (i) Show that $\displaystyle\int_3^4 dx = \int_2^3 dx = \int_1^2 dx = 1.$

(ii) Show that $\displaystyle\tfrac{2}{7}\int_3^4 x\,dx = \tfrac{2}{5}\int_2^3 x\,dx = \tfrac{2}{3}\int_1^2 x\,dx = 1.$

(iii) Show that $\displaystyle\tfrac{3}{37}\int_3^4 x^2\,dx = \tfrac{3}{19}\int_2^3 x^2\,dx = \tfrac{3}{7}\int_1^2 x^2\,dx = 1.$

(b) Using the above results and the theorems on the definite integral in Section 11C only, calculate the following:

(i) $\displaystyle\int_1^4 dx$ (iii) $\displaystyle\int_2^1 x^2\,dx$ (v) $\displaystyle\int_1^3 7x^2\,dx$

(ii) $\displaystyle\int_1^3 x\,dx$ (iv) $\displaystyle\int_1^2 (x^2+1)\,dx$ (vi) $\displaystyle\int_1^4 (3x^2-6x+5)\,dx$

EXTENSION

13. State with reasons whether the following statements are true or false:

(a) $\displaystyle\int_{-90}^{90} \sin^3 x^\circ\,dx = 0$ (d) $\displaystyle\int_0^1 2^x\,dx < \int_0^1 3^x\,dx$

(b) $\displaystyle\int_{-30}^{30} \sin 4x^\circ \cos 2x^\circ\,dx = 0$ (e) $\displaystyle\int_{-1}^0 2^x\,dx < \int_{-1}^0 3^x\,dx$

(c) $\displaystyle\int_{-1}^1 2^{-x^2}\,dx = 0$ (f) $\displaystyle\int_0^1 \frac{dt}{1+t^n} \le \int_0^1 \frac{dt}{1+t^{n+1}}$, where $n = 1, 2, 3, \ldots$.

14. First evaluate the integrals, then establish the following results. Give a sketch of each situation.

(a) $\displaystyle\int_1^N \frac{dx}{x^2}$ converges to 1 as $N \to \infty$. (c) $\displaystyle\int_1^N \frac{dx}{\sqrt{x}}$ diverges to ∞ as $N \to \infty$.

(b) $\displaystyle\int_\varepsilon^1 \frac{dx}{x^2}$ diverges to ∞ as $\varepsilon \to 0^+$. (d) $\displaystyle\int_\varepsilon^1 \frac{dx}{\sqrt{x}}$ converges to 2 as $\varepsilon \to 0^+$.

11 D The Indefinite Integral

Having established primitives as the key to calculating definite integrals, we now turn again to the calculation of primitives.

The Indefinite Integral: Because of the close connection established by the fundamental theorem between primitives and definite integrals, the term *indefinite integral* is often used for the primitive, and the usual notation for the primitive of a function $f(x)$ is an integral sign without any limits of integration. For example, the primitive or indefinite integral of x^2+1 is

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$$\int (x^2+1)\,dx = \tfrac{1}{3}x^3 + x + C, \text{ for some constant } C.$$

The word 'indefinite' implies that the integral cannot be evaluated further because no limits of integration have yet been specified. Whereas a definite integral is a pure number whose pronumeral is a 'dummy variable', an indefinite integral is a function of x whose pronumeral is carried across to the answer. The constant is an important part of the answer, despite being a nuisance to write every time, and it must always be written down. In most problems, it will not be zero.

Standard Forms for Integration: The previous rules in Section 10 J for finding primitives can now be restated a little more elegantly in this new notation.

13 STANDARD FORMS: (a) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, for some constant C.

(b) $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$, for some constant C.

Integrals are usually found using the fundamental theorem, so the word 'integration' is commonly used to refer both to the finding of a primitive and to the evaluating of a definite integral.

WORKED EXERCISE: Here are some examples of integration techniques.

(a) $\displaystyle\int (5-2x)^2\,dx$

$\displaystyle = \frac{(5-2x)^3}{(-2)\times 3} + C$

$= -\frac{1}{6}(5-2x)^3 + C$

(b) $\displaystyle\int \frac{dx}{\sqrt{9-2x}}$

$\displaystyle = \int (9-2x)^{-\frac{1}{2}}\,dx$

$= -\frac{1}{2}\times\frac{2}{1}\times(9-2x)^{\frac{1}{2}} + C$

$= -\sqrt{9-2x} + C$

(c) $\displaystyle\int \left(x-\frac{1}{x}\right)^2 dx$

$\displaystyle = \int \left(x^2 - 2 + x^{-2}\right) dx$

$\displaystyle = \tfrac{1}{3}x^3 - 2x - \frac{1}{x} + C$

(d) $\displaystyle\int \frac{x^2-1}{\sqrt{x}}\,dx$

$\displaystyle = \int (x^{1\frac{1}{2}} - x^{-\frac{1}{2}})\,dx$

$= \frac{2}{5}x^{2\frac{1}{2}} - 2x^{\frac{1}{2}} + C$

Exercise 11D

1. Find the indefinite integral of each of the following:

(a) 4 (d) 0 (g) $7x^{13}+3x^8$ (j) ax^2+bx

(b) $2x$ (e) $4x^5$ (h) $4-3x$ (k) x^a, $a\neq -1$

(c) $3x^2$ (f) $x^{0\cdot 4}$ (i) $3x^2-8x^3+7x^4$ (l) ax^a+bx^b, $a,b\neq -1$

2. Write these functions using negative indices, then find the indefinite integrals, giving your answers in fractional form:

(a) $\dfrac{1}{x^2}$ (c) $-\dfrac{1}{5x^3}$ (e) $\dfrac{1}{x^a}$, $a\neq 1$ (g) $\dfrac{ax^a}{x}$, $a\neq 0$

(b) $\dfrac{3}{x^4}$ (d) $\dfrac{1}{x^2}-\dfrac{1}{x^5}$ (f) $\dfrac{x^a}{x^b}$, $a-b\neq -1$ (h) $\dfrac{x^a+x^b}{x^a}$, $b-a\neq -1$

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3. Write these functions with fractional indices and find the indefinite integrals:

(a) $\sqrt{x}$ (b) $\sqrt[3]{x}$ (c) $\dfrac{1}{\sqrt{x}}$ (d) $\sqrt[3]{x^2}$

4. By expanding the brackets where necessary, perform the following integrations:

(a) $\displaystyle\int x^2(5-3x)\,dx$ (d) $\displaystyle\int \sqrt{x}\left(3\sqrt{x}-x\right)\,dx$ (g) $\displaystyle\int \frac{x^7+x^4}{x^6}\,dx$

(b) $\displaystyle\int (2x+1)^2\,dx$ (e) $\displaystyle\int \left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)\,dx$ (h) $\displaystyle\int \frac{2x^3-x^4}{4x}\,dx$

(c) $\displaystyle\int (1-x^2)^2\,dx$ (f) $\displaystyle\int \left(2\sqrt{x}-1\right)^2\,dx$ (i) $\displaystyle\int \frac{x^2+2x}{\sqrt{x}}\,dx$

DEVELOPMENT

5. By using the formula $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$, find:

(a) $\displaystyle\int (x+1)^5\,dx$ (d) $\displaystyle\int (3x+1)^4\,dx$ (g) $\displaystyle\int 2(2x-1)^{10}\,dx$

(b) $\displaystyle\int (x+2)^3\,dx$ (e) $\displaystyle\int \left(1-\frac{x}{5}\right)^3\,dx$ (h) $\displaystyle\int \frac{8}{(4x+1)^5}\,dx$

(c) $\displaystyle\int (4-x)^4\,dx$ (f) $\displaystyle\int \frac{1}{(x+1)^3}\,dx$ (i) $\displaystyle\int \frac{4}{5(1-4x)^2}\,dx$

6. Find each of the following indefinite integrals:

(a) $\displaystyle\int \sqrt{x+1}\,dx$ (d) $\displaystyle\int \sqrt[3]{4x-1}\,dx$ (g) $\displaystyle\int \left(\frac{1}{\sqrt{x+1}}+\frac{1}{\sqrt{x+2}}\right)\,dx$

(b) $\displaystyle\int \sqrt{2x-1}\,dx$ (e) $\displaystyle\int \frac{1}{\sqrt{3x+5}}\,dx$ (h) $\displaystyle\int \left(\sqrt{4-x}+\frac{1}{\sqrt{4-x}}\right)\,dx$

(c) $\displaystyle\int \sqrt{7-4x}\,dx$ (f) $\displaystyle\int \sqrt{1-\tfrac{1}{2}x}\,dx$ (i) $\displaystyle\int \left(\sqrt{ax}+\frac{1}{\sqrt{ax}}\right)\,dx$

7. Evaluate the following:

(a) $\displaystyle\int_0^2 (x+1)^4\,dx$ (e) $\displaystyle\int_0^1 (ax+b)^2\,dx$ (i) $\displaystyle\int_1^2 \left(2x+7(3x-4)^6\right)\,dx$

(b) $\displaystyle\int_2^3 (2x-5)^3\,dx$ (f) $\displaystyle\int_0^1 \sqrt{9-8x}\,dx$ (j) $\displaystyle\int_{-1}^2 \left(\frac{1}{\sqrt{x+2}}+\sqrt{x+2}\right)\,dx$

(c) $\displaystyle\int_{-2}^2 (1-x)^5\,dx$ (g) $\displaystyle\int_2^7 \frac{dx}{\sqrt{x+2}}$ (k) $\displaystyle\int_1^5 \sqrt{3x+1}\,dx$

(d) $\displaystyle\int_0^5 \left(1-\frac{x}{5}\right)^4\,dx$ (h) $\displaystyle\int_0^7 \sqrt[3]{x+1}\,dx$ (l) $\displaystyle\int_{-3}^0 \sqrt{1-5x}\,dx$

EXTENSION

8. (a) If u and v are differentiable functions in x, prove that $\displaystyle\int u\,\frac{dv}{dx}\,dx = uv - \int v\,\frac{du}{dx}\,dx$.

(b) Hence find $\displaystyle\int x\sqrt{1+x}\,dx$.

9. A question in Exercise 11B asked for an explanation, with a sketch, why this calculation is invalid:

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -1-1=-2.$$

Given that it is invalid, can any meaning nevertheless be given to the final result?

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11 E Finding Area by Integration

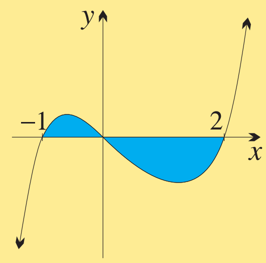
The aim of this section and the next is to use definite integrals to find the areas of regions bounded by curves, lines and axes.

Integral and Area: A definite integral is a pure number, which can be positive or negative because areas of regions below the x-axis are counted as negative. An area has units (called 'square units' or u^2 in the absence of any physical interpretation), and cannot be negative. Any problem on areas requires some care when finding the correct integral or combination of integrals required. Some particular techniques are listed below, but the general rule is to draw a picture first to see which bits need to be added or subtracted.

14 FINDING AN AREA: When using integrals to find the area of a region, first draw a sketch of the curves. The sketch will usually need to show any intercepts and any points of intersection.

Area Above and Below the x-axis: When a curve crosses the x-axis, the area of the region between the curve and the x-axis cannot usually be found by a single integral, because the integral represents areas of regions under the x-axis as negative.

WORKED EXERCISE: Find the area between the x-axis and $y = (x+1)x(x-2)$.



SOLUTION: Expanding, $y = x^3 - x^2 - 2x$.

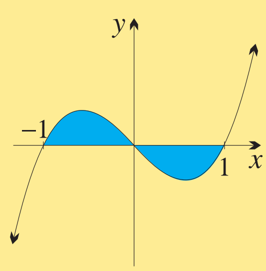
For the region above the x-axis,
$$\begin{aligned}\text{area} &= +\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx \\ &= \Big[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - x^2\Big]_{-1}^{0} \\ &= (0-0-0) - (\tfrac{1}{4} + \tfrac{1}{3} - 1) \\ &= \tfrac{5}{12} \text{ square units.}\end{aligned}$$

For the region below the x-axis,
$$\begin{aligned}\text{area} &= -\int_{0}^{2} (x^3 - x^2 - 2x)\,dx \\ &= -\Big[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - x^2\Big]_{0}^{2} \\ &= -(4 - 2\tfrac{2}{3} - 4) + (0-0-0) \\ &= 2\tfrac{2}{3} \text{ square units.}\end{aligned}$$

Adding these, total area $= 3\frac{1}{12}$ square units.

Using Symmetry: As always, symmetry should be exploited whenever possible, but some explanation of what has been done needs to be given.

WORKED EXERCISE: Find the area between the curve $y = x^3 - x$ and the x-axis.



SOLUTION: Factoring, $y = x(x-1)(x+1)$,
and the two regions have equal area, since the function is odd.

Now area below the x-axis
$$\begin{aligned} &= -\int_{0}^{1} (x^3 - x)\,dx \\ &= -\Big[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\Big]_{0}^{1} \\ &= -(\tfrac{1}{4} - \tfrac{1}{2}) + (0-0) \\ &= \tfrac{1}{4} \text{ square units.}\end{aligned}$$

Doubling, total area $= \frac{1}{2}$ square units.

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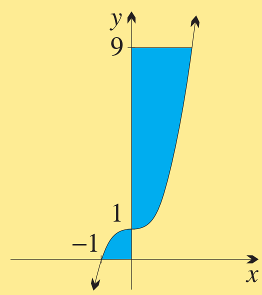
Area Between a Graph and the y-axis: When x is a function of y, the definite integral with respect to y will find the area of the region between the curve and the y-axis, with areas of regions to the left of the y-axis being counted as negative. The limits of integration are then values of y rather than of x. This technique can often avoid a subtraction.

15 DEFINITION: Suppose that x is a function of y in some closed interval $a \le y \le b$. Then $\displaystyle\int_a^b x\,dy$ is the area of the region between the curve and the y-axis from $y = a$ to $y = b$, with areas right of the y-axis counted as positive and areas left of the y-axis counted as negative.

WORKED EXERCISE: Use integration with respect to y to find the area of the region between the cubic $y = x^3 + 1$, the y-axis, the x-axis and the line $y = 9$.

SOLUTION: The cubic crosses the y-axis at $(0, 1)$.

Solving for x, the equation of the cubic is $x = (y-1)^{\frac{1}{3}}$.

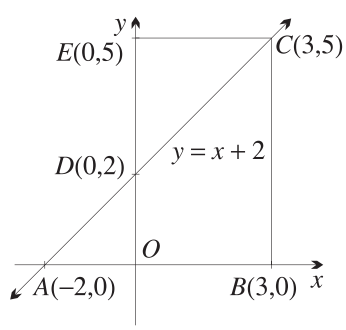


For the region left of the y-axis,
$$\begin{aligned}\text{area} &= -\int_0^1 (y-1)^{\frac{1}{3}}\,dy\\ &= -\tfrac{3}{4}\left[(y-1)^{\frac{4}{3}}\right]_0^1\\ &= -\tfrac{3}{4}(0-1)\\ &= \tfrac{3}{4}\text{ square units.}\end{aligned}$$

For the region right of the y-axis,
$$\begin{aligned}\text{area} &= +\int_1^9 (y-1)^{\frac{1}{3}}\,dy\\ &= \tfrac{3}{4}\left[(y-1)^{\frac{4}{3}}\right]_1^9\\ &= \tfrac{3}{4}(16-0)\\ &= 12\text{ square units.}\end{aligned}$$

Adding these, total area $= 12\frac{3}{4}$ square units.

Exercise 11E



1. (a) In the figure to the right, find the area of triangle AOD:

(i) by using the formula for the area of a triangle,

(ii) by evaluating $\displaystyle\int_{-2}^{0} (x+2)\,dx$.

(b) Find the area of the trapezium $OBCD$:

(i) by using the formula for the area of a trapezium,

(ii) by evaluating $\displaystyle\int_{0}^{3} (x+2)\,dx$.

(c) Find the area of triangle CDE:

(i) by using the formula for the area of a triangle,

(ii) by evaluating $\displaystyle\int_{2}^{5} (y-2)\,dy$.

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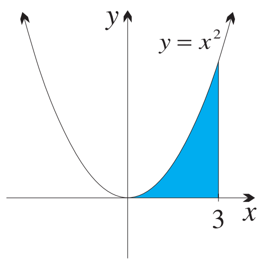
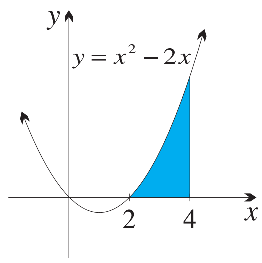
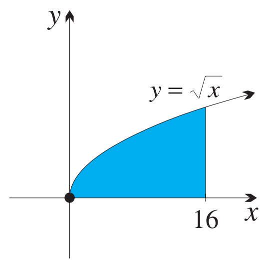
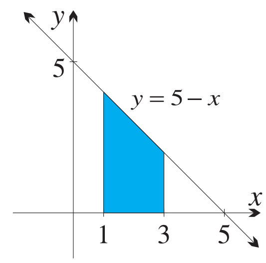
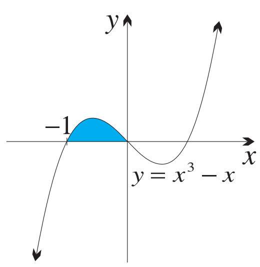
2. Find the area of each shaded region below by evaluating the appropriate integral:

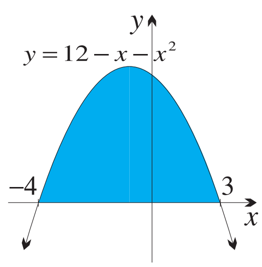
(a)


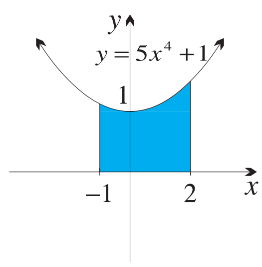
(b)


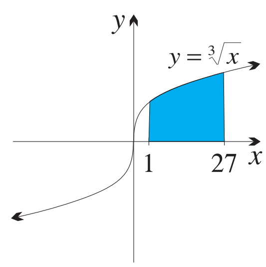
(c)


(d)


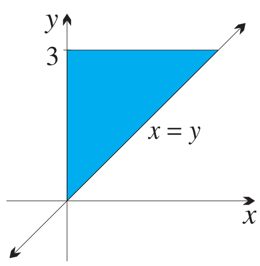
(e)


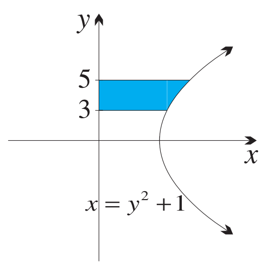
(f)


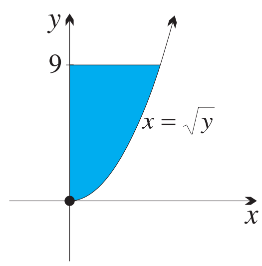
(g)


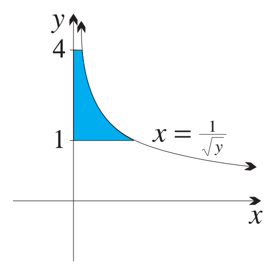
(h)


3. Find the area of each shaded region below by evaluating the appropriate integral:

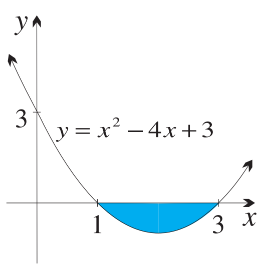
(a)


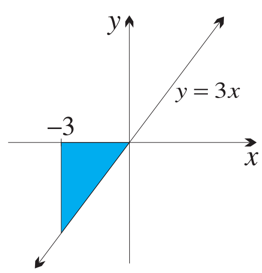
(b)


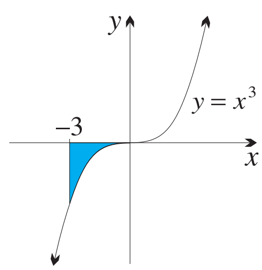
(c)


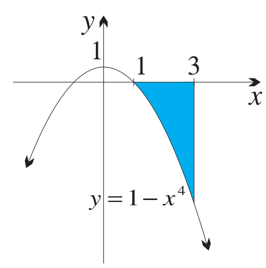
(d)


4. Find the area of each shaded region below by evaluating the appropriate integral:

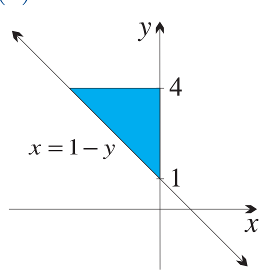
(a)


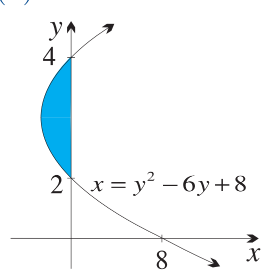
(b)


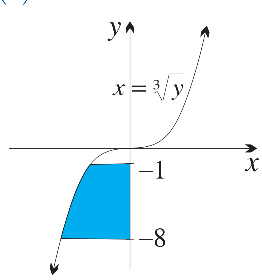
(c)


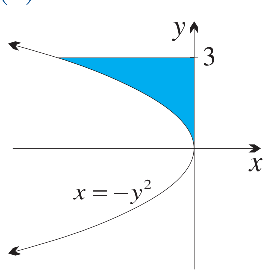
(d)


5. Find the area of each shaded region below by evaluating the appropriate integral:

(a)


(b)


(c)


(d)


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DEVELOPMENT

6. Find the area bounded by the curve $y = |x+2|$ and the x-axis for $-2 \le x \le 2$.

7. Find the area bounded by the graph of the given function and the x-axis between the specified values. You should draw a diagram for each question and check to see whether the area is above or below the x-axis.

(a) $y = x^2$, $x = -3$ and $x = 2$
(b) $y = 2x^3$, $x = -4$ and $x = 1$
(c) $y = 3x(x-2)$, $x = 0$ and $x = 2$
(d) $y = x - 3$, $x = -1$ and $x = 4$
(e) $y = 4 - x$, $x = 0$ and $x = 8$
(f) $y = x(3-x)^2$, $x = 0$ and $x = 3$
(g) $y = x(x-1)(x+3)$, $x = 1$ and $x = -3$
(h) $y = (x-1)(x+3)(x-2)$, $x = -3$ and $x = 2$
(i) $y = x(1-x)$, $x = -2$ and $x = 2$
(j) $y = x^4 - 4x^2$, $x = 0$ and $x = 3$

8. Find the area bounded by the graph of the given function and the y-axis between the specified values. You should draw a diagram for each question and check to see whether the area is to the right or left of the y-axis.

(a) $x = y - 5$, $y = 0$ and $y = 6$
(b) $x = 3 - y$, $y = 2$ and $y = 5$
(c) $x = y^2$, $y = -1$ and $y = 3$
(d) $x = (y-1)(y+1)$, $y = -2$ and $y = 0$

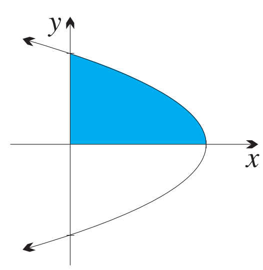
9. In these questions you should draw a graph and look carefully for any symmetries that will simplify the calculation.

(a) Find the area bounded by the curve and the x-axis: (i) $y = x^7$, $-2 \le x \le 2$
(ii) $y = x^3 - 16x$, $-4 \le x \le 4$ (iii) $y = x^4 - 3x^2$, $-\sqrt{3} \le x \le \sqrt{3}$

(b) Find the area bounded by the curve and the y-axis: (i) $x = 2y$, $-5 \le y \le 5$
(ii) $x = y^2$, $-3 \le y \le 3$ (iii) $x = 4 - y^2$, $-2 \le y \le 2$

10. Find the area bounded by the graph of the given function and the coordinate axes:

(a) $y = (x+2)^3$
(b) $y = (3x-4)^4$
(c) $y = \sqrt{x+3}$
(d) $y = \sqrt{5-2x}$



11. The diagram shows a graph of $y^2 = 16(2-x)$.

(a) Find the x and y intercepts.

(b) Find the magnitude of the shaded area by considering:

(i) the area between the curve and the x-axis,

(ii) the area between the curve and the y-axis.

12. From the point $A(2,2)$ on the curve $y = \frac{1}{4}x^3$, a line is drawn parallel to the y-axis to meet the x-axis at B and a line is drawn parallel to the x-axis to meet the y-axis at C. Show that the curve divides the resultant rectangle in the ratio $1:3$.

13. (a) The gradient of a curve is given by $f'(x) = x^2 - 2x - 3$, and the curve passes through the origin. Find its equation and sketch its graph, indicating all stationary points.

(b) Find the area enclosed between the curve and the x-axis between the turning points.

14. Sketch $y = x^2$ and mark the points $A(a, a^2)$, $B(-a, a^2)$, $P(a, 0)$ and $Q(-a, 0)$.

(a) Show that $\displaystyle\int_0^a x^2\,dx = \tfrac{2}{3}$ (area of $\triangle OAP$).

(b) Show that $\displaystyle\int_{-a}^a x^2\,dx = \tfrac{1}{3}$ (area of rectangle $ABQP$).

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15. Given positive real numbers a and n, let A, P and Q be the points (a, a^n), $(a, 0)$ and $(0, a^n)$ respectively. Find the ratios:

(a) $\displaystyle\int_0^a x^n\,dx$: (area of $\triangle AOP$) (b) $\displaystyle\int_0^a x^n\,dx$: (area of rectangle $OPAQ$)

16. (a) Show that $x^4 - 2x^3 + x = x(x-1)(x^2 - x - 1)$. Then sketch a graph of the function $y = x^4 - 2x^3 + x$ and shade the three regions bounded by the graph and the x-axis.

(b) If $a = \frac{1}{2}\left(1 + \sqrt{5}\right)$, evaluate a^2, a^4 and a^5.

(c) Show that the area of one shaded region equals the sum of the areas of the other two.

17. (a) Find the area bounded by the curve $y = ax^2 + bx + c$ and the x-axis between $x = h$ and $x = -h$, where $y > 0$ for $-h \le x \le h$.

(b) [Simpson's rule — see Section 11J] Hence show that if $y = y_0$, y_1 and y_2 when $x = -h$, 0 and h respectively, then the area is given by $\frac{1}{3}h(y_0 + 4y_1 + y_2)$.

EXTENSION

18. Consider the function $G(x) = \displaystyle\int_0^x g(u)\,du$, where $g(u) = \begin{cases} 4 - \frac{4}{3}u, & \text{for } 0 \le u < 6, \\ u - 10, & \text{for } 6 \le u \le 12. \end{cases}$

(a) Sketch a graph of $g(u)$.

(b) Find the stationary points of the function $y = G(x)$ and determine their nature.

(c) Find those values of x for which $G(x) = 0$.

(d) Sketch the curve $y = G(x)$, indicating all important features.

(e) Find the area bounded by the curve $y = G(x)$ and the x-axis for $0 \le x \le 6$.

19. (a) Show that for $n < -1$, $\displaystyle\int_1^N x^n\,dx$ converges as $N \to \infty$, and find the limit.

(b) Show that for $n > -1$, $\displaystyle\int_\varepsilon^1 x^n\,dx$ converges as $\varepsilon \to 0^+$, and find the limit.

(c) Interpret these two results as areas.

11 F Area of a Compound Region

When a region is bounded by two or more different curves, some dissection process may need to be employed before its area can be calculated using definite integrals. A preliminary sketch of the region therefore becomes all the more important.

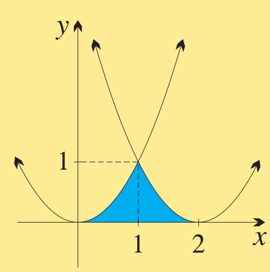
Area Under a Combination of Curves: Sometimes a region is bounded by different curves in different parts of the x-axis.

WORKED EXERCISE: Find the area bounded by $y = x^2$, $y = (x-2)^2$ and the x-axis.

SOLUTION: The two curves intersect at $(1, 1)$.

First, $$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}.$$

Secondly, $$\int_1^2 (x-2)^2\,dx = \left[\tfrac{1}{3}(x-2)^3\right]_1^2 = 0 - (-\tfrac{1}{3}) = \tfrac{1}{3}.$$



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Combining these, $\quad \text{area} = \frac{1}{3} + \frac{1}{3}$
$\qquad\qquad\qquad\qquad = \frac{2}{3}$ square units.

Area Between Curves: Let $y = f(x)$ and $y = g(x)$ be two curves with $f(x) \le g(x)$ in the interval $a \le x \le b$, so that $y = f(x)$ is never above $y = g(x)$. Then the area of the region contained between the curves can be found by subtraction.

16 AREA BETWEEN CURVES: If $f(x) \le g(x)$ in the interval $a \le x \le b$, then

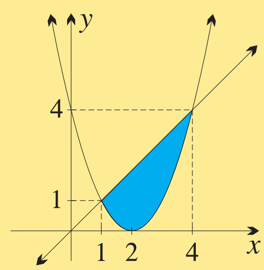
$$\text{area between the curves} = \int_a^b \Big(g(x) - f(x)\Big)\,dx.$$

That is, take the integral of the top curve minus the bottom curve.

The assumption that $f(x) \le g(x)$ is important. If the curves cross each other, then separate integrals will need to be taken or else the areas where different curves are on top will begin to cancel each other out.

WORKED EXERCISE: Find the area between the two curves $y = (x-2)^2$ and $y = x$.

SOLUTION: The two curves intersect at $(1, 1)$ and $(4, 4)$, and in the shaded region, the line is above the parabola.

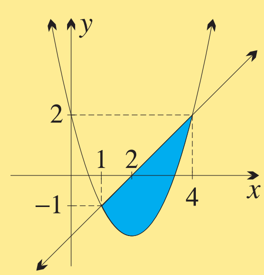


$$\begin{aligned}
\text{Area} &= \int_1^4 \Big(x - (x-2)^2\Big)\,dx \\
&= \int_1^4 (-x^2 + 5x - 4)\,dx \\
&= \Big[-\tfrac{1}{3}x^3 + \tfrac{5}{2}x^2 - 4x\Big]_1^4 \\
&= (-21\tfrac{1}{3} + 40 - 16) - (-\tfrac{1}{3} + 2\tfrac{1}{2} - 4) \\
&= 4\tfrac{1}{2} \text{ square units.}
\end{aligned}$$

NOTE: This formula for the area of the region between two curves holds even if the region crosses the x-axis. To illustrate this, the next example is the previous example pulled down 2 units so that the region between the line and the parabola crosses the x-axis. The area of course remains the same — and notice how the formula still gives the correct answer.

WORKED EXERCISE: Find the area between $y = x^2 - 4x + 2$ and $y = x - 2$.

SOLUTION: The two curves intersect at $(1, -1)$ and $(4, 2)$.



$$\begin{aligned}
\text{Hence area} &= \int_1^4 \Big((x-2) - (x^2 - 4x + 2)\Big)\,dx \\
&= \int_1^4 (-x^2 + 5x - 4)\,dx \\
&= \Big[-\tfrac{1}{3}x^3 + \tfrac{5}{2}x^2 - 4x\Big]_1^4 \\
&= (-21\tfrac{1}{3} + 40 - 16) - (-\tfrac{1}{3} + 2\tfrac{1}{2} - 4) \\
&= 4\tfrac{1}{2} \text{ square units.}
\end{aligned}$$

Area Between Curves that Cross: Now suppose that one curve $y = f(x)$ is sometimes above and sometimes below another curve $y = g(x)$ in the region where areas are being calculated. In this case, separate integrals will need to be taken.

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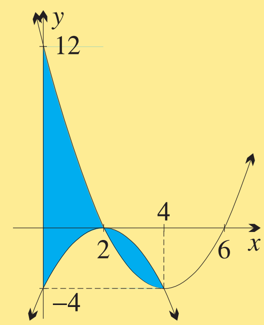
WORKED EXERCISE: The diagram below shows the curves $y = -x^2 + 4x - 4$ and $y = x^2 - 8x + 12$ meeting at the points $(2, 0)$ and $(4, -4)$. Find the shaded area.

SOLUTION: In the left-hand region, the second curve is above the first,

so $$\begin{aligned} \text{area} &= \int_0^2 \big((x^2 - 8x + 12) - (-x^2 + 4x - 4)\big)\,dx \\ &= \int_0^2 (2x^2 - 12x + 16)\,dx \\ &= \Big[\tfrac{2}{3}x^3 - 6x^2 + 16x\Big]_0^2 \\ &= 5\tfrac{1}{3} - 24 + 32 \\ &= 13\tfrac{1}{3} \text{ square units.} \end{aligned}$$



In the right-hand region, the first curve is above the second,

so $$\begin{aligned} \text{area} &= \int_2^4 \big((-x^2 + 4x - 4) - (x^2 - 8x + 12)\big)\,dx \\ &= \int_2^4 (-2x^2 + 12x - 16)\,dx \\ &= \Big[-\tfrac{2}{3}x^3 + 6x^2 - 16x\Big]_2^4 \\ &= (-42\tfrac{2}{3} + 96 - 64) - (-5\tfrac{1}{3} + 24 - 32) \\ &= 2\tfrac{2}{3} \text{ square units.} \end{aligned}$$

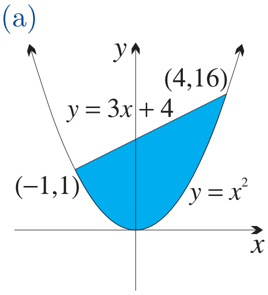
So total area $= 16$ square units.

NOTE: When one of the curves is the x-axis, then the region involved is the region between the other curve and $y = 0$. So the calculations of areas between a curve and the x-axis in the previous section were special cases of the areas between curves in this section.

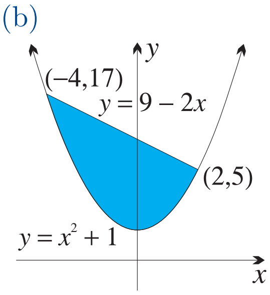
Exercise 11F

1. Find the magnitude of each shaded area in the diagrams below:

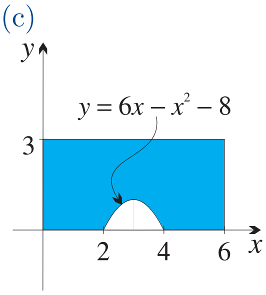
(a)



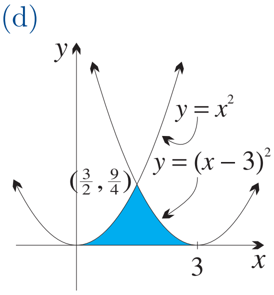
(b)



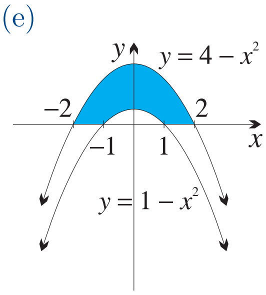
(c)



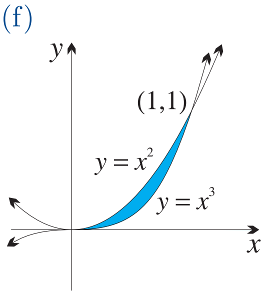
(d)



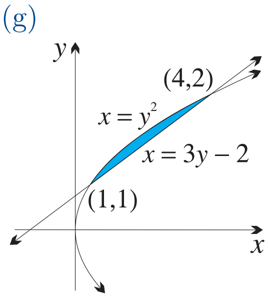
(e)



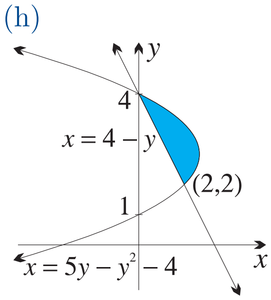
(f)



(g)



(h)



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2. (a) By solving the equations simultaneously, show that the curves $y = x^2 + 4$ and $y = x + 6$ intersect at the points $(-1, 5)$ and $(2, 8)$. Then sketch the curves on the same number plane and shade the area enclosed between them.

(b) Show that this area is equal to $\int_{-1}^{2} (x - x^2 + 2)\, dx$ and evaluate the integral.

3. (a) Find the points of intersection of the curves $y = (x - 3)^2$ and $y = 14 - 2x$, then sketch the curves on the same number plane and shade the area enclosed between them.

(b) Show that this area is equal to $\int_{-1}^{5} (4x + 5 - x^2)\, dx$ and evaluate the integral.

DEVELOPMENT

4. Sketch graphs of each pair of functions, showing their points of intersection. By evaluating the appropriate integral in each case, find the area enclosed between the two curves.

(a) $y = 9 - x^2$ and $y = 3 - x$

(b) $y = x + 10$ and $y = (x - 3)^2 + 1$

(c) $y = x^4$ and $y = x^2$

(d) $y = 3x^2$ and $y = 6x^3$

5. (a) By solving the equations simultaneously, show that the curves $y = x^2 + 2x - 8$ and $y = 2x + 1$ intersect when $x = 3$ and $x = -3$. Then sketch both curves on the same number plane and shade the region enclosed between them.

(b) Despite the fact that it crosses the x-axis, the area of the region between the curves is given by $\int_{-3}^{3} \Big((2x + 1) - (x^2 + 2x - 8)\Big)\, dx$. Evaluate the integral and hence find the magnitude of the area enclosed between the curves.

6. Find the area bounded by the lines $y = \frac{1}{4}x$ and $y = -\frac{1}{2}x$ between $x = 1$ and $x = 4$.

7. Sketch graphs of each pair of functions, clearly indicating their points of intersection, then find the area enclosed between the two curves in each case:

(a) $y = x^2 - 6x + 5$ and $y = x - 5$

(b) $y = x$ and $y = x^3$

(c) $y = -3x$ and $y = 4 - x^2$

(d) $y = x^2 - 1$ and $y = 7 - x^2$

8. (a) On the same number plane sketch graphs of the functions $y = x^2$ and $x = y^2$, clearly indicating the points of intersection.

(b) Find the magnitude of the area bounded by the two curves.

9. Consider the function $x^2 = 8y$. Tangents are drawn at the points $A(4, 2)$ and $B(-4, 2)$ and intersect on the y-axis. Find the area bounded by the curve and the tangents.

10. (a) Show that the equation of the tangent to the curve $y = x^3$ at the point where $x = 2$ is $y - 12x + 16 = 0$.

(b) Show that the tangent and the curve meet again at the point $(-4, -64)$.

(c) Find the magnitude of the area enclosed between the curve and the tangent.

11. Sketch graphs of each pair of functions, showing their points of intersection, then find the area enclosed between the two curves in each case.

(a) $y = \sqrt{x + 2}$ and $5y = x + 6$

(b) $y = \sqrt{3 - x}$ and $2x + 3y - 6 = 0$

12. (a) Given the two functions $f(x) = (x + 1)(x - 1)(x - 3)$ and $g(x) = (x - 1)(x + 1)$, for what values of x is $f(x) > g(x)$?

(b) Sketch a graph of the two functions on the same number plane and find the magnitude of the area between them.

13. Find the area bounded by the curves $y = x^2(1 - x)$ and $y = x(1 - x)^2$.

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14. (a) Sketch a graph of the function $y = 12x - 32 - x^2$, clearly indicating the x-intercepts.
(b) Find the equation of the tangent to the curve at the point A where $x = 5$.
(c) If the tangent meets the x-axis at B, and C is the x-intercept of the parabola closer to the origin, find the area of the region bounded by AB, BC and the arc CA.

EXTENSION

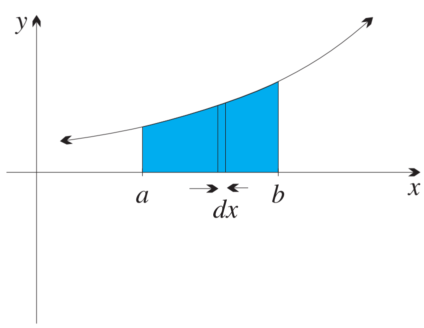
15. Find the value of k for which the line $y = kx$ bisects the area enclosed by the curve $4y = 4x - x^2$ and the x-axis.

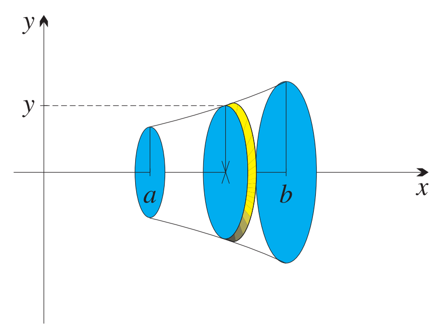
16. The average value of a continuous function $f(x)$ over an interval $a \le x \le b$ is defined to be $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ if $a \ne b$, or $f(a)$ if $a = b$. If k is the average value of $f(x)$ on the interval $a \le x \le b$, show that the area of the region bounded by $f(x)$ above the line $y = k$ is equal to the area of the region bounded by $f(x)$ below the line $y = k$.

11 G Volumes of Solids of Revolution

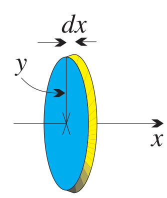
If a region in the coordinate plane is rotated about either the x-axis or the y-axis, a solid region in three dimensions is generated, called the *solid of revolution*. The process is similar to shaping a piece of wood on a lathe, or making pottery on a wheel, because such shapes have rotational symmetry and circular cross sections. The volumes of such solids can be found using a simple integration formula. The well-known formulae for the volumes of cones and spheres can finally be proven by this method.

Rotating a Region about the x-axis: The first diagram below shows the region under the curve $y = f(x)$ in the interval $a \le x \le b$, and the second shows the solid generated when this region is rotated about the x-axis.





Imagine the solid sliced like salami perpendicular to the x-axis into infinitely many circular slices, each of width dx. One of the slices is shown below, and the vertical strip on the first diagram above is what generates this slice when it is rotated about the x-axis.



Now radius of circular slice $= y$, the height of the strip,
so area of circular slice $= \pi y^2$, because it is a circle.
But the slice is essentially a very thin cylinder of thickness dx,
so volume of circular slice $= \pi y^2\,dx$ (area $\times$ thickness).
To get the total volume, we simply sum all the slices from $x = a$ to $x = b$,

so volume of solid $= \displaystyle\int_a^b \pi y^2\,dx$.

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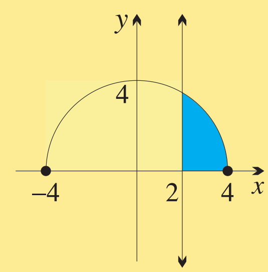
17 **VOLUMES OF REVOLUTION ABOUT THE x- AXIS:** The volume generated when the region between a curve and the x-axis from $x = a$ to $x = b$ is rotated about the x-axis is $\int_a^b \pi y^2 \, dx$ cubic units.

Notice that if the curve is below the x-axis, so that y is negative, the volume calculated is still positive, because it is y^2 rather than y which occurs in the formula. Unless other units are specified, 'cubic units' (u^3), should be used, by analogy with the areas of regions disussed in Section 11E.

WORKED EXERCISE: The shaded region cut off the semicircle $y = \sqrt{16 - x^2}$ by the line $x = 2$ is rotated about the x-axis. Find the volume generated.

SOLUTION: Squaring, $y^2 = 16 - x^2$,

so
$$\begin{aligned} \text{volume} &= \int_2^4 \pi y^2 \, dx \\ &= \pi \int_2^4 (16 - x^2) \, dx \\ &= \pi \left[16x - \tfrac{1}{3}x^3 \right]_2^4 \\ &= \pi(64 - \tfrac{64}{3} - 32 + \tfrac{8}{3}) \\ &= \tfrac{40\pi}{3} \text{ cubic units.} \end{aligned}$$



Volumes of Revolution about the y-axis:

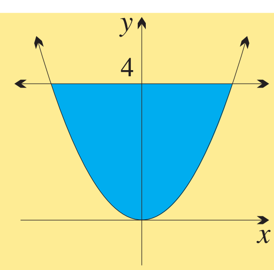
Calculating the volume when some region is rotated about the y-axis is simply a matter of exchanging x and y.

18 **VOLUMES OF REVOLUTION ABOUT THE y-AXIS:** The volume generated when the region between a curve and the y-axis from $y = a$ to $y = b$ is rotated about the y-axis is $\int_a^b \pi x^2 \, dy$.

When y is given as a function of x, the equation will need to be written with x^2 as the subject.

WORKED EXERCISE: Find the volume of the solid formed by rotating the region between $y = x^2$ and the line $y = 4$ about the y-axis.

SOLUTION: Since $x^2 = y$,
$$\begin{aligned} \text{volume} &= \int_0^4 \pi x^2 \, dy \\ &= \pi \int_0^4 y \, dy \\ &= \pi \left[\tfrac{1}{2}y^2 \right]_0^4 \\ &= 8\pi \text{ cubic units.} \end{aligned}$$



Volume by Subtraction:

When rotating the region between two curves lying above the x-axis, the two integrals need to be subtracted, as if the outer volume has first been formed, and the inner volume cut away from it. The two volumes can always be calculated separately and subtracted, but if the two integrals have the same limits of integration, it may be more convenient to combine them.

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19 ROTATING THE REGION BETWEEN CURVES: The volume generated when the region between two curves from $x = a$ to $x = b$ is rotated about the x-axis is

$$\int_a^b \pi({y_2}^2 - {y_1}^2)\,dx \quad (\text{where } y_2 > y_1 > 0).$$

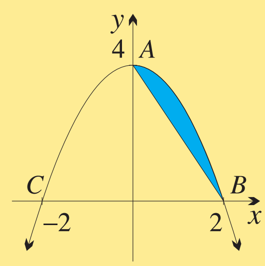
Similarly, $\int_a^b \pi({x_2}^2 - {x_1}^2)\,dy$ is the volume of the solid generated when the region between two curves from $y = a$ to $y = b$ is rotated about the y-axis (where $x_2 > x_1 > 0$).

WORKED EXERCISE: The curve $y = 4 - x^2$ meets the y-axis at $A(0, 4)$ and the x-axis at $B(2, 0)$ and $C(-2, 0)$. Find the volume generated when the region between the curve and the line AB is rotated: (a) about the x-axis, (b) about the y-axis.

SOLUTION:

(a) The parabola is $y_2 = 4 - x^2$, and the line AB is $y_1 = 4 - 2x$,

$$\begin{aligned}
\text{so volume} &= \int_0^2 \pi({y_2}^2 - {y_1}^2)\,dx \\
&= \pi \int_0^2 \Big((16 - 8x^2 + x^4) - (16 - 16x + 4x^2)\Big)\,dx \\
&= \pi \int_0^2 (x^4 - 12x^2 + 16x)\,dx \\
&= \pi \Big[\tfrac{1}{5}x^5 - 4x^3 + 8x^2\Big]_0^2 \\
&= \pi(\tfrac{32}{5} - 32 + 32) \\
&= \tfrac{32\pi}{5} \text{ cubic units.}
\end{aligned}$$



(b) The parabola is ${x_2}^2 = 4 - y$, and the line is $x_1 = 2 - \frac{1}{2}y$,

$$\begin{aligned}
\text{so volume} &= \int_0^4 \pi({x_2}^2 - {x_1}^2)\,dy \\
&= \pi \int_0^4 \Big((4 - y) - (4 - 2y + \tfrac{1}{4}y^2)\Big)\,dy \\
&= \pi \int_0^4 (-\tfrac{1}{4}y^2 + y)\,dy \\
&= \pi \Big[-\tfrac{1}{12}y^3 + \tfrac{1}{2}y^2\Big]_0^4 \\
&= \pi(-\tfrac{16}{3} + 8) \\
&= \tfrac{8\pi}{3} \text{ cubic units.}
\end{aligned}$$

NOTE: One would not normally expect the volumes of revolution about the two different axes to be equal. This is because an element of area will generate a larger element of volume if it is moved further away from the axis of rotation.

Cones and Spheres: The formulae for the volumes of cones and spheres may have been learnt earlier, but they cannot be proven without arguments involving integration. The proofs of both results are developed in the following exercise, and these questions should be carefully worked.

20 VOLUME OF A CONE: $V = \frac{1}{3}\pi r^2 h$ VOLUME OF A SPHERE: $V = \frac{4}{3}\pi r^3$

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Exercise 11G

1. (a) Sketch the region bounded by the line $y = 3x$ and the x-axis between $x = 0$ and $x = 3$.

(b) When this region is rotated about the x-axis, a right circular cone will be formed. Find the radius and height of the cone and hence find its volume.

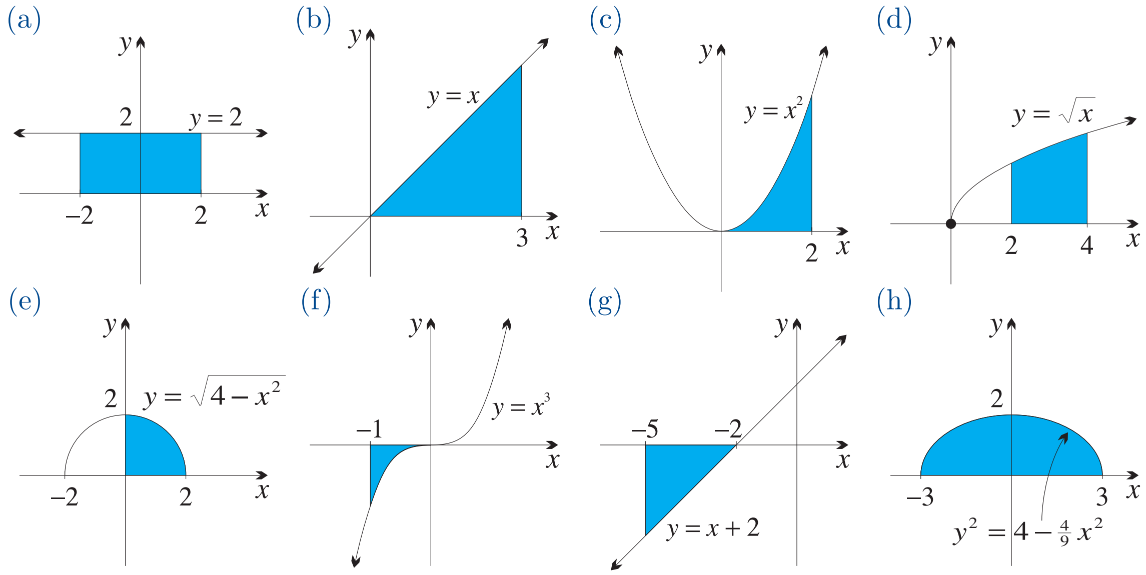
(c) Evaluate $\displaystyle\int_0^3 \pi y^2\,dx = \pi \int_0^3 9x^2\,dx$ in order to check your answer.

2. (a) Sketch a graph of the region bounded by the curve $y = \sqrt{9 - x^2}$ and the x-axis between $x = -3$ and $x = 3$.

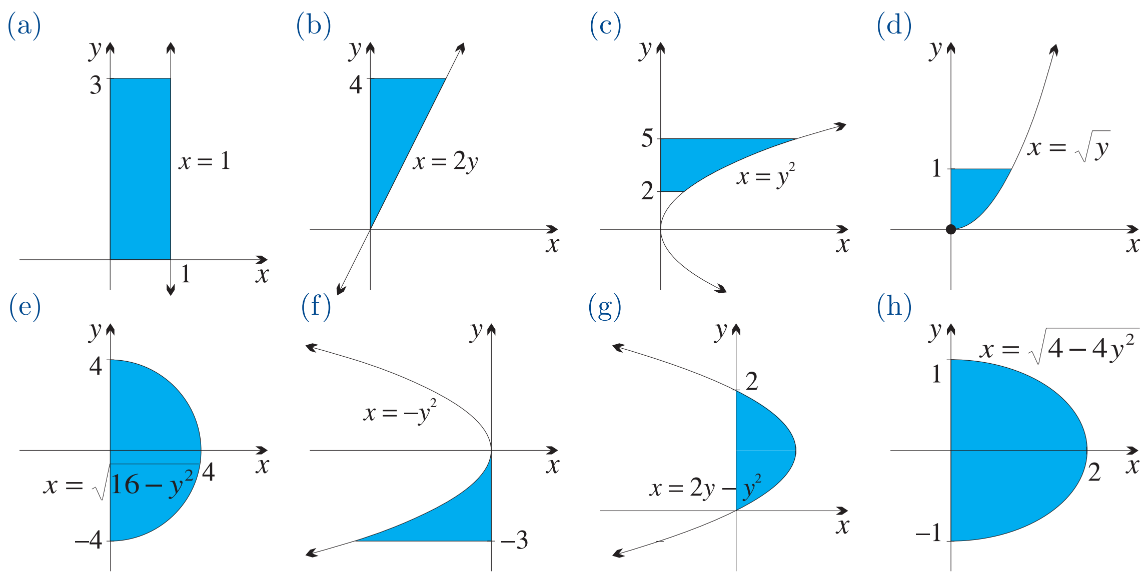
(b) When this region is rotated about the x-axis, a sphere will be formed. Find the radius of the sphere and hence find its volume.

(c) Evaluate $\displaystyle\int_{-3}^3 \pi y^2\,dx = \pi \int_{-3}^3 (9 - x^2)\,dx$ in order to check your answer.

3. Calculate the volume generated when each shaded region is rotated about the x-axis:



4. Calculate the volume generated when each shaded region is rotated about the y-axis:



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DEVELOPMENT

5. Find the volume of the solid generated by rotating the region bounded by the following curves about the x-axis. Sketch a graph of each region and when finding y^2, recall that $(a+b)^2 = a^2 + 2ab + b^2$.

(a) $y = x + 3$, $x = 3$, $x = 5$ and $y = 0$

(b) $y = 1 + \sqrt{x}$, $x = 1$, $x = 4$ and $y = 0$

(c) $y = 5x - x^2$ and $y = 0$

(d) $y = x^3 - x$ and $y = 0$

6. Find the volume of the solid generated by rotating the region bounded by the following curves about the y-axis. Preliminary sketches will be needed.

(a) $x = y - 2$, $y = 1$ and $x = 0$

(b) $x = y^2 + 1$, $y = 0$, $y = 1$ and $x = 0$

(c) $x = y(y - 3)$ and $x = 0$

(d) $y = 1 - x^2$ and $y = 0$

7. (a) Sketch the region bounded by $y = x^2$ and the x-axis between $x = 0$ and $x = 4$.

(b) Find the volume of the solid generated when this region is rotated about the x-axis.

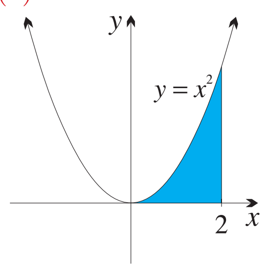
(c) Find the volume V_1 of the cylinder formed when the line $x = 4$ between $y = 0$ and $y = 16$ is rotated about the y-axis.

(d) Evaluate $V_2 = \displaystyle\int_0^{16} \pi x^2\, dy = \pi \int_0^{16} y\, dy$.

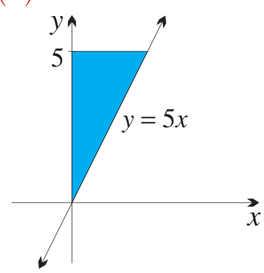
(e) Hence evaluate the volume $V = V_1 - V_2$ when the region defined in part (a) is rotated about the y-axis.

8. Find the volume of the solid generated by rotating each region about: (i) the x-axis, (ii) the y-axis. [HINT: In some cases a subtraction of volumes will be necessary.]

(a)



(b)



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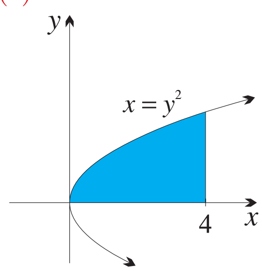

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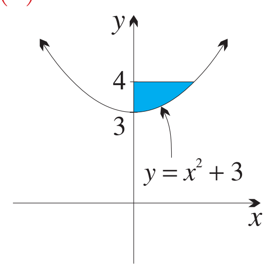

9. By evaluating the appropriate integral, find the volume of the sphere generated when the region inside the circle $x^2 + y^2 = 64$ is rotated about the x-axis.

10. A vase is formed by rotating the portion of the curve $y^2 = x - 6$ between $y = 6$ and $y = -6$ about the y-axis. Find the volume of the vase.

11. The region bounded by the curve $y = 3 - x^2$ and the lines $y = 3$ and $x = \sqrt{3}$ is rotated about the y-axis. Sketch this region on a number plane and calculate the volume of the solid that is generated.

12. (a) By solving the equations by substitution, show that the curves $x^2 = 16y$ and $4y^2 = x$ intersect at the point $(4, 1)$.

(b) Calculate the area of the region bounded by the two curves.

(c) Find the volume of the solid formed when this region is rotated about:

(i) the x-axis, (ii) the y-axis.

13. (a) Sketch a graph of the region bounded by the curves $y = x^2$ and $y = x^3$.

(b) Find the volume of the solid generated when this region is rotated about:

(i) the x-axis, (ii) the y-axis.

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14. (a) On the same number plane sketch graphs of the functions $xy = 5$ and $x + y = 6$, clearly indicating their points of intersection.

(b) Find the volume of the solid generated when the region bounded by the two curves is rotated about the x-axis.

15. In this question a number of standard volume formulae will be proven.

(a) (i) A right circular cone of height h and radius r is generated by rotating the straight line $y = \dfrac{r}{h}x$ between $x = 0$ and $x = h$ about the x-axis. Show that the volume of the cone is given by $\frac{1}{3}\pi r^2 h$.

(ii) Find the volume of the frustrum of the cone in the interval $a \le x \le b$.

(b) A cylinder of height h and radius r is generated by rotating the line $y = r$ between $x = 0$ and $x = h$ about the x-axis. Show that the volume of the cylinder is $\pi r^2 h$.

(c) (i) A sphere of radius r is generated by rotating the semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis. Show that the volume of the sphere is given by $\frac{4}{3}\pi r^3$.

(ii) A spherical cap of radius r and height h is formed by rotating the semicircle $y = \sqrt{r^2 - x^2}$ between $x = r$ and $x = r - h$ about the x-axis. Show that the volume of the cap is given by $\frac{1}{3}\pi h^2(3r - h)$.

16. Find the volume of the solid generated when:

(a) the region between the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the x-axis is rotated about the x-axis,

(b) the region between the parabola $x^2 = 4ay$ and the y-axis between $y = 0$ and $y = h$ is rotated about the y-axis.

17. Sketch $x^2 = 4ay$, and shade the region bounded by the curve and the latus rectum. Find the volume of the solid generated when this region is rotated about:

(a) the axis of symmetry, (b) the tangent at the origin.

18. (a) State the domain and range of the function $y = \sqrt{9 - x}$.

(b) Sketch a graph of the function.

(c) Calculate the area bounded by the curve and the coordinate axes in the first quadrant.

(d) Calculate the volume of the solid generated when this region is rotated about:

(i) the x-axis, (ii) the y-axis.

19. (a) Find the equation of the tangent to the curve $y = x^3 + 2$ at the point where $x = 1$.

(b) Draw a diagram showing the region bounded by the curve, the tangent and the y-axis.

(c) Calculate the volume of the solid generated when this region is rotated about:

(i) the x-axis, (ii) the y-axis.

20. (a) On the same set of axes sketch the curves $y = \sqrt{9 - x^2}$ and $y = 18 - 2x^2$.

(b) Find the area bounded by the two curves.

(c) Find the volume of the solid formed when this region is rotated about the x-axis.

21. (a) Find all stationary points of the function $y = x + \dfrac{1}{x}$ and sketch its graph.

(b) Show that the line $y = \frac{5}{2}$ intersects the curve when $x = \frac{1}{2}$ or $x = 2$.

(c) Find the volume of the solid generated when the region between the curve and the line $y = \frac{5}{2}$ is rotated about the x-axis.

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22. Consider the curves $f(x) = x^n$ and $f(x) = x^{n+1}$, where n is a positive integer.

(a) Find the points of intersection of the curves.

(b) Show that the volume V_n of the solid generated when the region bounded by the two curves is rotated around the x-axis is given by $\pi\left(\dfrac{1}{2n+1} - \dfrac{1}{2n+3}\right)$ cubic units.

(c) By means of a diagram, show that the solid whose volume is given by $V_1 + V_2 + V_3 + \cdots$ will be the cone formed by rotating the line $y = x$ between $x = 0$ and $x = 1$ about the x-axis.

(d) Find the volume of the cone in part (c).

(e) Hence show that $\dfrac{1}{3\times 5} + \dfrac{1}{5\times 7} + \dfrac{1}{7\times 9} + \cdots = \dfrac{1}{6}$.

EXTENSION

23. (a) Sketch a graph of the function $y = \sqrt{x}(1-x)$, clearly indicating any stationary points and intercepts with the axes.

(b) Hence sketch a graph of the function $y^2 = x(1-x)^2$.

(c) Find the area contained in the loop between $x = 0$ and $x = 1$.

(d) Find the volume of the solid formed when this region is rotated about the x-axis.

24. (a) Sketch a graph of the region bounded by $y = -x^2+6x-8$ and the x-axis for $2 \le x \le 4$.

(b) Use the quadratic formula to show that the equation of the curve is $x = 3 + \sqrt{1-y}$ for $3 \le x \le 4$, and $x = 3 - \sqrt{1-y}$ for $2 \le x \le 3$.

(c) The region in part (a) is rotated about the y-axis. Show that the volume of the solid formed is given by $V = \displaystyle\int_0^1 12\pi\sqrt{1-y}\,dy$ and evaluate the integral.

25. (a) Sketch a graph of the circle $(x-3)^2 + y^2 = 1$.

(b) The region inside the circle is rotated about the y-axis. Show that the volume of the solid formed is given by $V = 24\pi\displaystyle\int_0^1 \sqrt{1-y^2}\,dy$ and evaluate the integral.

11 H The Reverse Chain Rule

The result of a chain-rule differentiation is a product, so some products should be able to be integrated using a reverse form of the chain rule.

Running a Chain Rule Derivative Backwards: Once a derivative has been calculated using the chain rule, the result can be reversed to give an integral. There may then need to be some juggling with constants.

WORKED EXERCISE: Differentiate $(x^2+1)^4$, and hence integrate $x(x^2+1)^3$.

SOLUTION: Let $y = (x^2+1)^4$.

Then
$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$
$$= 2x \times 4(x^2+1)^3$$
$$= 8x(x^2+1)^3.$$

Let $u = x^2 + 1$,
then $y = u^4$.
So $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = 4u^3$.

Hence $\dfrac{d}{dx}(x^2+1)^4 = 8x(x^2+1)^3$, and writing this as an indefinite integral,

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$$\int 8x(x^2+1)^3\,dx = (x^2+1)^4 + C, \text{ for some constant } C,$$

$$\boxed{\div 8} \quad \int x(x^2+1)^3\,dx = \tfrac{1}{8}(x^2+1)^4 + D, \text{ where } D \text{ is a constant.}$$

The Reverse Chain Rule: Obtaining this same result without having first been given a derivative to calculate is a little more sophisticated. The chain rule says that if $y = u^{n+1}$ is a power of u, where u is a function of x, then $\dfrac{dy}{dx} = (n+1)u^n \times \dfrac{du}{dx}$. Turning this around gives:

21 THE REVERSE CHAIN RULE: $\displaystyle\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C$, for some constant C.

Or, using function notation, $\displaystyle\int f'(x)\big(f(x)\big)^n\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1} + C.$

Let us illustrate this by finding the previous integral directly.

WORKED EXERCISE: Find $\displaystyle\int x(x^2+1)^3\,dx$, using the substitution $u = x^2+1$.

NOTE: The key to the whole process is to identify clearly the intermediate function u (or $f(x)$ in function notation).

SOLUTION:

$$\int x(x^2+1)^3\,dx = \tfrac{1}{2}\int 2x(x^2+1)^3\,dx$$
$$= \tfrac{1}{8}(x^2+1)^4 + C,$$

as calculated in the last worked exercise.

Let $u = x^2+1$.
Then $\dfrac{du}{dx} = 2x$.
$\displaystyle\int u^3\frac{du}{dx}\,dx = \tfrac{1}{4}u^4$

Or, using function notation,

$$\int x(x^2+1)^3\,dx = \tfrac{1}{2}\int 2x(x^2+1)^3\,dx$$
$$= \tfrac{1}{8}(x^2+1)^4 + C.$$

Let $f(x) = x^2+1$.
Then $f'(x) = 2x$.
$\displaystyle\int f'(x)\big(f(x)\big)^3\,dx = \tfrac{1}{4}\big(f(x)\big)^4$

Exercise 11H

1. (a) Find $\dfrac{d}{dx}(x^2+3)^4$. (b) Hence find: (i) $\displaystyle\int 8x(x^2+3)^3\,dx$ (ii) $\displaystyle\int x(x^2+3)^3\,dx$

2. (a) Find $\dfrac{d}{dx}(x^3+3x^2+5)^4$.

(b) Hence find: (i) $\displaystyle\int 12(x^2+2x)(x^3+3x^2+5)^3\,dx$ (ii) $\displaystyle\int (x^2+2x)(x^3+3x^2+5)^3\,dx$

3. (a) Find $\dfrac{d}{dx}(5-x^2-x)^7$.

(b) Hence find: (i) $\displaystyle\int (-14x-7)(5-x^2-x)^6\,dx$ (ii) $\displaystyle\int (2x+1)(5-x^2-x)^6\,dx$

4. (a) Find $\dfrac{d}{dx}(x^3-1)^5$. (b) Hence find: (i) $\displaystyle\int 15x^2(x^3-1)^4\,dx$ (ii) $\displaystyle\int 3x^2(x^3-1)^4\,dx$

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5. (a) Find $\dfrac{d}{dx}\sqrt{2x^2+3}$. (b) Hence find: (i) $\displaystyle\int \frac{2x}{\sqrt{2x^2+3}}\,dx$ (ii) $\displaystyle\int \frac{x}{\sqrt{2x^2+3}}\,dx$

6. (a) Find $\dfrac{d}{dx}\left(\sqrt{x}+1\right)^3$. (b) Hence find: (i) $\displaystyle\int \frac{3\left(\sqrt{x}+1\right)^2}{2\sqrt{x}}\,dx$ (ii) $\displaystyle\int \frac{\left(\sqrt{x}+1\right)^2}{\sqrt{x}}\,dx$

DEVELOPMENT

7. Find the following indefinite integrals using the reverse chain rule. Some hints are given for the substitution.

(a) $\displaystyle\int 10x(5x^2+3)^2\,dx$ (Let $u=5x^2+3$.)

(b) $\displaystyle\int 2x(x^2+1)^3\,dx$ (Let $u=x^2+1$.)

(c) $\displaystyle\int 12x^2(1+4x^3)^5\,dx$ (Let $u=1+4x^3$.)

(d) $\displaystyle\int x(1+3x^2)^4\,dx$ (Let $u=1+3x^2$.)

(e) $\displaystyle\int (x-2)(x^2-4x-5)^3\,dx$

(f) $\displaystyle\int x^3(1-x^4)^7\,dx$

(g) $\displaystyle\int 3x^2\sqrt{x^3-1}\,dx$ (Let $u=x^3-1$.)

(h) $\displaystyle\int x\sqrt{5x^2+1}\,dx$ (Let $u=5x^2+1$.)

(i) $\displaystyle\int \frac{x}{\sqrt{x^2+3}}\,dx$ (Let $u=x^2+3$.)

(j) $\displaystyle\int \frac{x+1}{\sqrt{4x^2+8x+1}}\,dx$

(k) $\displaystyle\int \frac{x}{(x^2+5)^3}\,dx$

(l) $\displaystyle\int \frac{(\sqrt{x}-3)^4}{\sqrt{x}}\,dx$ (Let $u=\sqrt{x}-3$.)

(m) $\displaystyle\int px(qx^2-3)^3\,dx$

(n) $\displaystyle\int rx^2(px^3+q)^4\,dx$

8. Calculate the following definite integrals using the reverse chain rule:

(a) $\displaystyle\int_{-1}^{1} x^2(x^3+1)^4\,dx$

(b) $\displaystyle\int_0^1 \frac{x}{(5x^2-1)^3}\,dx$

(c) $\displaystyle\int_0^{\frac{1}{2}} x\sqrt{1-4x^2}\,dx$

(d) $\displaystyle\int_{-3}^{-1} (x+5)(x^2+10x+3)^2\,dx$

(e) $\displaystyle\int_0^a \frac{x}{(ax^2+1)^2}\,dx$

(f) $\displaystyle\int_{b^2}^{4b^2} \frac{(\sqrt{x}+b)^2}{\sqrt{x}}\,dx$, where $b>0$

9. (a) What is the domain of the function $f(x)=x\sqrt{x^2-1}$?
(b) Find $f'(x)$ and hence show that the function has no stationary points in its domain.
(c) Show that the function is odd, and hence sketch its graph.
(d) By evaluating the appropriate integral, find the area enclosed by the curve and the x-axis between $x=1$ and $x=3$.

10. (a) Sketch $y=x(7-x^2)^3$, indicating all stationary points and intercepts with the axes.
(b) Find the area enclosed between the curve and the x-axis.

EXTENSION

11. (a) Explain why the graphs of $y=f(x)$ and $y=f(a-x)$ are reflections of each other in the vertical line $x=\frac{1}{2}a$. (b) Deduce $\displaystyle\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$, and hence evaluate:

(i) $\displaystyle\int_0^4 x(4-x)^4\,dx$ (ii) $\displaystyle\int_0^1 x^2\sqrt{1-x}\,dx$

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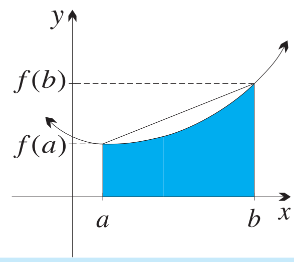
11 I The Trapezoidal Rule

Approximation methods for definite integrals become necessary when exact calculation through the primitive is not possible. This can happen for two major reasons. First, the primitives of many important functions cannot be written down in a form suitable for calculation. Secondly, some values of the function may be known from experiments, but the function itself may still be unknown.



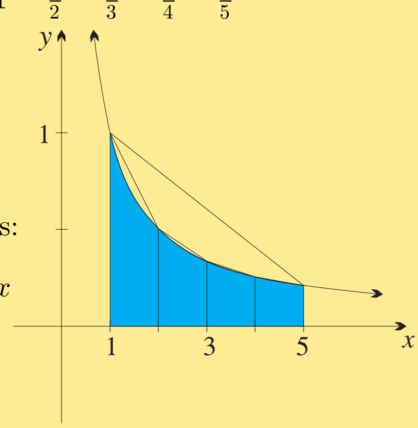
The Trapezoidal Rule: The most obvious way to approximate an integral is to replace the curve by a straight line. The resulting region is then a trapezium, and so the approximation method is called the *trapezoidal rule*. The width of the trapezium is $b-a$, and the average of the sides is $\frac{1}{2}\big(f(a)+f(b)\big)$. Hence, using the area formula for a trapezium:

22 TRAPEZOIDAL RULE: $\displaystyle\int_a^b f(x)\,dx \doteqdot \frac{b-a}{2}\Big(f(a)+f(b)\Big)$,
with equality when the function is linear.

WORKED EXERCISE: Find approximations for $\displaystyle\int_1^5 \frac{1}{x}\,dx$ using the trapezoidal rule with (a) one application, (b) four applications.

SOLUTION: Constructing a table of values:

x	1	2	3	4	5
$1/x$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$



(a) $$\begin{aligned}\int_1^5 \frac{1}{x}\,dx &\doteqdot \frac{5-1}{2}\Big(f(1)+f(5)\Big)\\ &\doteqdot 2\times\left(1+\tfrac{1}{5}\right)\\ &\doteqdot 2\tfrac{2}{5}\end{aligned}$$

(b) Dividing the interval $1 \le x \le 5$ into four subintervals:
$$\begin{aligned}\int_1^5 \frac{1}{x}\,dx &= \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \int_3^4 \frac{1}{x}\,dx + \int_4^5 \frac{1}{x}\,dx\\ &\doteqdot \tfrac{1}{2}\left(\tfrac{1}{1}+\tfrac{1}{2}\right)+\tfrac{1}{2}\left(\tfrac{1}{2}+\tfrac{1}{3}\right)+\tfrac{1}{2}\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\tfrac{1}{2}\left(\tfrac{1}{4}+\tfrac{1}{5}\right)\\ &\doteqdot 1\tfrac{41}{60}\end{aligned}$$

NOTE: Because the curve is concave up, every approximation found using the trapezoidal rule is greater than the value of the integral. Similarly if a curve is concave down, every trapezoidal rule approximation is less that the integral. For linear functions the rule gives the exact value.

Exercise 11I

1. Show, by means of a diagram, that the trapezoidal rule will:

(a) overestimate $\displaystyle\int_a^b f(x)\,dx$ if $f''(x)>0$ for $a\le x\le b$,

(b) underestimate $\displaystyle\int_a^b f(x)\,dx$ if $f''(x)<0$ for $a\le x\le b$.

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2. (a) Complete this table for the function $y = x(4-x)$:

x	0	1	2	3	4
y					

(b) Using the the trapezoidal rule with these five function values, estimate $\int_0^4 x(4-x)\,dx$.

(c) What is the exact value of $\int_0^4 x(4-x)\,dx$, and why does it exceed the aproximation?

(d) Calculate the percentage error in the approximation (that is, divide the error by the correct answer and convert to a percentage).

3. (a) Complete this table for the function $y = \dfrac{6}{x}$:

x	1	2	3	4	5
y					

(b) Use the trapezoidal rule with the five function values above to estimate $\int_1^5 \dfrac{6}{x}\,dx$.

(c) Find the second derivative of $y = \dfrac{6}{x}$ and use it to explain why the estimate will exceed the exact value of the integral.

4. (a) Complete this table for the function $y = \sqrt{x}$:

x	9	10	11	12	13	14	15	16
y								

(b) Use the trapezoidal rule with the eight function values above to estimate $\int_9^{16} \sqrt{x}\,dx$. Give your answer correct to three significant figures.

(c) What is the exact value of $\int_9^{16} \sqrt{x}\,dx$? Find the second derivative of $y = x^{\frac{1}{2}}$ and use it to explain why the estimate will be less than the exact value of the integral.

DEVELOPMENT

5. (a) Show that the function $y = \dfrac{1}{1+x^2}$ has a stationary point at $(0,1)$.

(b) Sketch a graph of the function, showing all important features.

(c) Use the trapezoidal rule with five function values to estimate $\int_0^2 \dfrac{1}{1+x^2}\,dx$.

6. Use the trapezoidal rule with three function values to approximate each of these integrals. Answer correct to three decimal places. (a) $\int_0^1 2^{-x}\,dx$ (b) $\int_1^3 \sqrt[3]{9-2x}\,dx$.

7. (a) Use the trapezoidal rule with five function values to estimate $\int_0^1 \sqrt{1-x^2}\,dx$ to four decimal places.

(b) Use part (a) and the fact that $y = \sqrt{1-x^2}$ is a semicircle to estimate π. Give your answer to three decimal places, and explain why your estimate is less than π.

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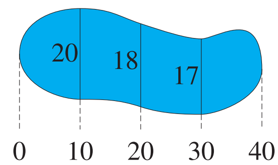
8. An object is moving along the x-axis with values of the velocity v in m/s at time t given in the table on the right. Given that the distance travelled may be found by calculating the area under a velocity/time graph, use the trapezoidal rule to estimate the distance travelled by the particle in the first five seconds.

t	0	1	2	3	4	5
v	1·5	1·3	1·4	2·0	2·4	2·7

9. The diagram on the right shows the width of a lake at 10 metre intervals. Use the trapezoidal rule to estimate the surface area of the water.



10. (a) A cone is generated by rotating the line $y = 2x$ about the x-axis between $x = 0$ and $x = 3$. Using the trapezoidal rule with four function values, estimate the volume of the cone.

(b) Calculate the exact volume of the cone and hence find the percentage error in the approximation.

11. The region under the graph $y = 2^{x+1}$ between $x = 1$ and $x = 3$ is rotated about the x-axis. Using the trapezoidal rule with five function values, estimate the volume of the solid formed.

EXTENSION

12. (a) Show that the function $y = \sqrt{x}$ is increasing for all $x > 0$.

(b) By dividing the area under the curve $y = \sqrt{x}$ into n equal subintervals, show that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \ge \int_0^n \sqrt{x}\,dx = \frac{2n\sqrt{n}}{3}.$$

(c) Use mathematical induction to prove that for all integers $n \ge 1$,

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \le \frac{\sqrt{n}(4n+3)}{6}.$$

(d) Give an alternative proof of part (c) using the trapezoidal rule.

(e) Hence estimate $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{12\,000}$, correct to the nearest hundred.

11 J Simpson's Rule

The trapezoidal rule approximates the function by a linear function, which is a polynomial of degree 1. The next most obvious method is to approximate the function by a polynomial of degree 2, that is, by a quadratic function. This is called *Simpson's rule*, and geometrically, it approximates the curve with a parabola.

Simpson's Rule: To approximate a definite integral using Simpson's rule, the value at the midpoint as well as the values at the endpoints must be known.

23 SIMPSON'S RULE: $$\int_a^b f(x)\,dx \doteqdot \frac{b-a}{6}\Big(f(a) + 4f(\tfrac{a+b}{2}) + f(b)\Big),$$
with equality holding when the function is quadratic.

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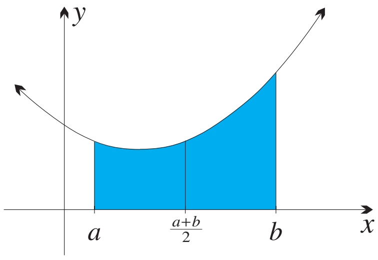
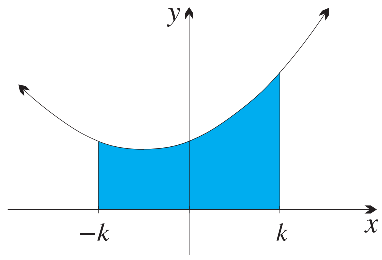
PROOF:





Shifting the origin to the midpoint of the interval $a \le x \le b$ does not change the value of the integral, so we need only deal with the case where the interval is $-k \le x \le k$. We must therefore prove that if $f(x) = Ax^2 + Bx + C$ is any quadratic function, then

$$\int_{-k}^{k} f(x)\,dx = \frac{k}{3}\Big(f(-k) + 4f(0) + f(k)\Big).$$

$$\begin{aligned} \text{LHS} &= \Big[\tfrac{1}{3}Ax^3 + \tfrac{1}{2}Bx^2 + Cx\Big]_{-k}^{k} \\ &= \tfrac{2}{3}Ak^3 + 2Ck \\ \text{RHS} &= \tfrac{1}{3}k\Big((Ak^2 - Bk + C) + 4C + (Ak^2 + Bk + C)\Big) \\ &= \tfrac{2}{3}Ak^3 + 2Ck, \quad \text{as required.} \end{aligned}$$

NOTE: Simpson's rule also gives the exact answer for cubic functions. This can be seen from the proof above, if one imagines a term Dx^3 being added to the quadratic. Being an odd function, Dx^3 would not affect the value of the integral on the LHS, and would also cancel out of the RHS when k and $-k$ were substituted.

WORKED EXERCISE: Use Simpson's rule to find an approximation to $\displaystyle\int_1^5 f(x)\,dx$, given the following table of values:

x	1	2	3	4	5
$f(x)$	2·31	4·56	5·34	3·02	0·22

SOLUTION: The best use of the data is to apply Simpson's rule on each of the intervals $1 \le x \le 3$ and $3 \le x \le 5$, and then add the results.

First, $\displaystyle\int_1^3 f(x)\,dx \doteqdot \frac{3-1}{6} \times (2{\cdot}31 + 4 \times 4{\cdot}56 + 5{\cdot}34) \doteqdot 8{\cdot}63.$

Secondly, $\displaystyle\int_3^5 f(x)\,dx \doteqdot \frac{5-3}{6} \times (5{\cdot}34 + 4 \times 3{\cdot}02 + 0{\cdot}22) \doteqdot 5{\cdot}88.$

Combining these gives $\displaystyle\int_1^5 f(x)\,dx \doteqdot 14{\cdot}51.$

Exercise 11J

1. (a) Complete this table for the function $y = \dfrac{2}{x}$:

x	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
y					

(b) Use Simpson's rule with three function values to estimate $\displaystyle\int_1^2 \frac{2}{x}\,dx$.

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(c) Use Simpson's rule with three function values to estimate $\int_2^3 \frac{2}{x}\,dx$.

(d) Hence use Simpson's rule with five function values to estimate $\int_1^3 \frac{2}{x}\,dx$.

2. (a) Complete this table for the function $y = \sqrt{x+5}$:

x	-4	-3	-2
y			

(b) Use Simpson's rule to estimate $\int_{-4}^{-2} \sqrt{x+5}\,dx$ correct to three significant figures.

3. (a) Sketch a graph of the function $y = \sqrt{9-x^2}$.

(b) Hence evaluate $\int_{-3}^{3} \sqrt{9-x^2}\,dx$ correct to three decimal places.

(c) Complete this table for $y = \sqrt{9-x^2}$:

x	-3	$-1{\cdot}5$	0	$1{\cdot}5$	3
y					

(d) Using five function values, estimate $\int_{-3}^{3} \sqrt{9-x^2}\,dx$ correct to three decimal places:

(i) by the trapezoidal rule, (ii) by Simpson's rule.

4. (a) Complete this table for $y = 3 - 2x - x^2$:

x	-3	-2	-1	0	1
y					

(b) Use Simpson's rule with five function values to approximate $\int_{-3}^{1} (3 - 2x - x^2)\,dx$.

(c) Evaluate $\int_{-3}^{1} (3 - 2x - x^2)\,dx$. How does this compare with the answer obtained in part (b)? Why is this the case?

DEVELOPMENT

5. Use Simpson's rule with three function values to approximate:

(a) $\int_1^3 \frac{dx}{x^2+1}$ (b) $\int_{-1}^{1} 3^{-x}\,dx$

6. Use Simpson's rule with five function values to approximate the following integrals. (Give the approximation correct to four significant figures where necessary.)

(a) $\int_3^5 \sqrt{x^2-1}\,dx$ (b) $\int_{-1}^{1} \log_{10}(x+3)\,dx$ (c) $\int_2^6 3^x\,dx$

7. (a) Use Simpson's rule with five function values to estimate $\int_0^1 \sqrt{1-x^2}\,dx$. Give your answer correct to four decimal places.

(b) Use part (a) and the fact that $y = \sqrt{1-x^2}$ is a semicircle to estimate π. Give your answer correct to three decimal places.

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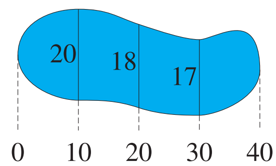
8. An object is moving along the x-axis with values of the velocity v in m/s at time t given in the table on the right. Given that the distance travelled may be found by calculating the area under a velocity/time graph, use Simpson's rule to estimate the distance travelled by the particle in the first four seconds.

t	0	1	2	3	4
v	1·5	1·3	1·4	2·0	2·4

9. The diagram on the right shows the width of a lake at 10 metre intervals. Use Simpson's rule to estimate the surface area of the water.



10. The region bounded by the curve $y = 3^{x-1}$ and the x-axis between $x = 1$ and $x = 3$ is rotated about the x-axis. Use Simpson's rule with five function values to approximate the volume of the solid that is formed. Give your answer correct to two decimal places.

11. Consider the function $f(x) = \sqrt{x}(2 - x)$.

(a) Find $f'(x)$, and hence find the coordinates of any stationary points.

(b) Sketch a graph of the function, indicating all important features.

(c) Use Simpson's rule with five function values to approximate the area enclosed by the curve and the x-axis between its two zeroes. Answer correct to two decimal places.

(d) Use Simpson's rule with five function values to approximate the volume of the solid formed when the region in part (c) is rotated about the x-axis. Answer correct to two decimal places.

(e) Check your answers to parts (c) and (d) by evaluating the appropriate integrals.

12. $L = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx$ is the length of the arc of a curve $y = f(x)$ between $x = a$ and $x = b$. Using Simpson's rule with five function values, estimate the length of the graph $y = x^2$ between $x = 0$ and $x = 2$. Answer correct to two decimal places.

EXTENSION

13. Machine computation of approximations to an integral $\int_a^b f(x)\,dx$ is more efficient with extended forms of the trapezoidal rule and Simpson's rule. Divide the interval $a \le x \le b$ into n subintervals, each of width h, so that $h = \dfrac{b-a}{n}$. Define $y_0, y_1, \ldots, y_n$ by

$$y_0 = f(a), \quad y_1 = f(a+h), \quad y_2 = f(a+2h), \quad \ldots, \quad y_n = f(b).$$

(a) Explain why the two rules, using all these function values, can be written as

TRAPEZOIDAL RULE: $\displaystyle\int_a^b f(x)\,dx = \frac{h}{2}\Big(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\Big)$

SIMPSON'S RULE: $\displaystyle\int_a^b f(x)\,dx = \frac{h}{3}\Big(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n\Big)$

(b) Apply these formulae with various values of n to find estimates of $\int_1^2 \frac{1}{x}\,dx$ and $\int_0^1 2^x\,dx$. Use a computer or programmable calculator if these are available. To test the accuracy of your calculations, you need to know that according to the methods developed in the next two chapters, the values of these integrals are reciprocals of each other, being 0·693 147 and 1·442 695 respectively, correct to six decimal places.

GO Online Multiple Choice Quiz

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