THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Practice Questions for Quiz 1

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

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• Quiz 1 is held in **week 6** (starting 3rd September) in tutorials. You have **40 minutes** to complete the quiz. It covers material up to and including lecture 8 (chapters 1, 2, 3 and 4 in the course notes; plus the appendix on integration techniques). The quiz will have considerably fewer questions than this set of practice questions.

• Calculators are **not** permitted, and you will **not** be provided with a table of standard integrals.

There are almost certainly typos in these solutions!

1. Let $f(x) = e^x$, and let $P = \{x_0, \ldots, x_n\}$ be the partition of $[0, 1]$ into n equal parts.

(a) Find a closed formula for the corresponding lower Riemann sum L_n.

Solution: Since P is the partition of $[0, 1]$ into n equal parts we have $x_j = j/n$ for $j = 0, 1, \ldots, n$. Since $f(x) = e^x$ is monotone increasing, and since we are after the lower Riemann sum, we have $x_j^* = x_{j-1} = (j-1)/n$ for $j = 1, \ldots, n$. Thus the Riemann sum is

$$L_P = \sum_{j=1}^{n} f(x_j^*)\Delta x_j = \frac{1}{n}\sum_{j=1}^{n} e^{(j-1)/n} = \frac{1-e}{n(1-e^{1/n})},$$

where we have used the geometric sum formula with ratio $r = e^{1/n}$.

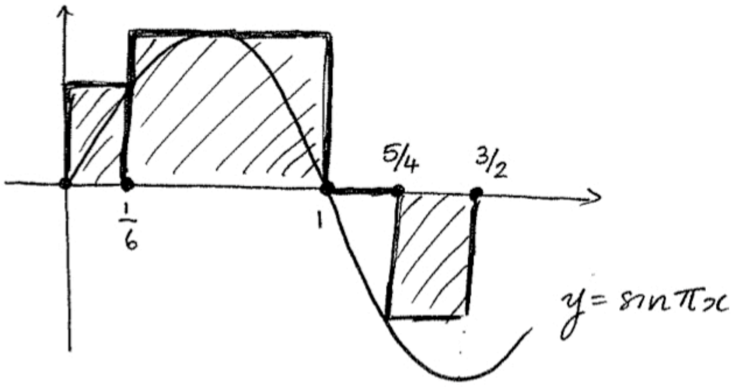
(b) Compute $\lim\limits_{n\to\infty} L_n$.

Solution: We have

$$\lim_{n\to\infty} L_n = (1-e)\lim_{n\to\infty}\frac{(1/n)}{1-e^{1/n}} = (1-e)\lim_{x\to 0^+}\frac{x}{1-e^x} = e-1.$$

2. Compute the upper Riemann sum of $f(x) = \sin \pi x$ over the interval $[0, 3/2]$ using the partition $P = \{0, 1/6, 1, 5/4, 3/2\}$.

Solution: The diagram shows the points in the partition P, and the corresponding rectangles that make up the upper Riemann sum.



Thus

$$U_P = \frac{1}{6}\sin\frac{\pi}{6} + \frac{5}{6}\sin\frac{\pi}{2} + \frac{1}{4}\sin\pi + \frac{1}{4}\sin\frac{3\pi}{4} = \frac{11}{12} - \frac{1}{4\sqrt{2}}.$$

3. Compute $\displaystyle\int_0^{2\pi} \mathrm{Si}(x)\,dx$ where $\displaystyle\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt.$

Solution: Integrating by parts gives

$$\int_0^{2\pi} \mathrm{Si}(x)\,dx = 2\pi\,\mathrm{Si}(2\pi) - \int_0^{2\pi} \sin x\,dx = 2\pi\,\mathrm{Si}(2\pi).$$

4. Given that $\displaystyle f(x) = x\int_0^{2x} te^{-t}\,dt$, find $f''(1)$.

Solution: Using the the product rule, the chain rule, and the Fundamental Theorem of Calculus, we compute

$$\begin{aligned} f'(x) &= \int_0^{2x} te^{-t}\,dt + x\frac{d}{dx}\int_0^{2x} te^{-t}\,dt \\ &= \int_0^{2x} te^{-t}\,dt + x \times 2 \times (2x)e^{-(2x)} = \int_0^{2x} te^{-t}\,dt + 4x^2e^{-2x}. \end{aligned}$$

Then $f''(x) = 2 \times (2x)e^{-(2x)} + 8xe^{-2x} - 8x^2e^{-2x}$, and therefore $f''(1) = 4e^{-2}$.

5. Find the derivative of the function $\displaystyle f(x) = \int_{\sin x}^{3+e^x} \sin t\,dt.$

Solution: Since

$$f(x) = \int_0^{3+e^x} \sin t\,dt - \int_0^{\sin x} \sin t\,dt,$$

the Fundamental Theorem of Calculus and the chain rule imply that

$$f'(x) = e^x\sin(3+e^x) - \cos x\sin(\sin x).$$

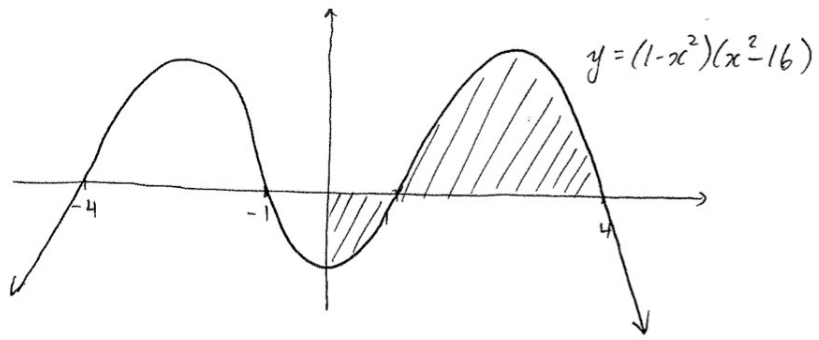
6. Given that $\displaystyle\sin(e^x) - \sin(1) = \int_0^x e^t f(t)\,dt$, find $f(x)$.

Solution: By the Fundamental Theorem of Calculus,

$$e^x \cos(e^x) = e^x f(x), \qquad \text{and so} \qquad f(x) = \cos(e^x).$$

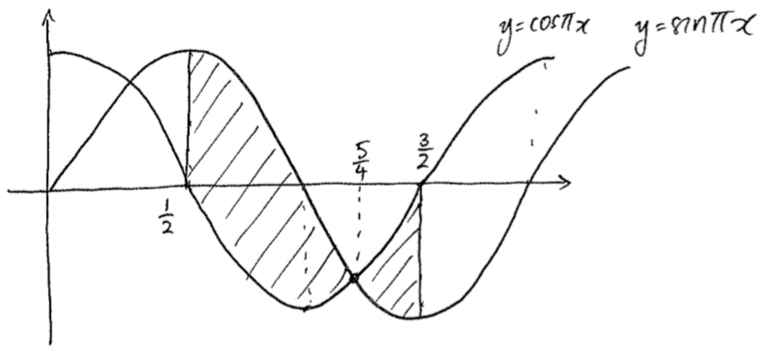
7. Find the value of $x > 0$ which maximises the function $I(x) = \int_0^x (1 - t^2)(t^2 - 16)\, dt$.

Solution: From the picture to see that $x = 4$ gives maximum area.



8. Find the area between the curves $y = \sin \pi x$ and $y = \cos \pi x$ with $1/2 \le x \le 3/2$.

Solution: The sketch is as follows:

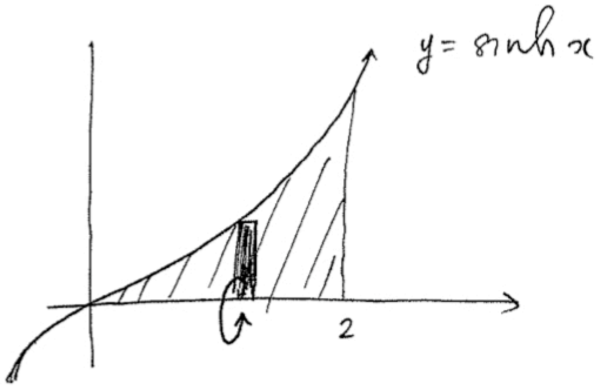


The intersection point in the relevant region is at $x = 5/4$. For $1/2 \le x \le 5/4$ we have $\sin \pi x \ge \cos \pi x$, and for $5/4 \le x \le 3/2$ we have $\cos \pi x \ge \sin \pi x$. Therefore

$$\begin{aligned} A &= \int_{1/2}^{5/4} (\sin \pi x - \cos \pi x)\, dx + \int_{5/4}^{3/2} (\cos \pi x - \sin \pi x)\, dx \\ &= \frac{1}{\pi} (-\cos \pi x - \sin \pi x) \Big|_{1/2}^{5/4} + \frac{1}{\pi} (\sin \pi x + \cos \pi x) \Big|_{5/4}^{3/2} = \frac{2\sqrt{2}}{\pi}. \end{aligned}$$

9. Compute the volume of the solid obtained by rotating about the x-axis the region bounded by the curve $y = \sinh x$, the x-axis, and the line $x = 2$.

Solution: The picture is:



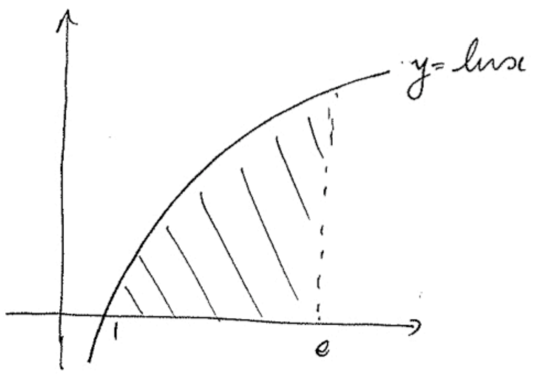
By the disc method,

$$V = \pi \int_0^2 \sinh^2 x \, dx = \frac{\pi}{2} \int_0^2 (\cosh(2x) - 1) \, dx = \frac{\pi}{4} \sinh(4) - \pi.$$

10. Let D be the region bounded by the curve $y = \ln x$, the x-axis, and the line $x = e$.

(a) Compute the area of D.

Solution: The picture is:



and so the area is

$$A = \int_1^e \ln x \, dx = (x \ln x - x) \big|_1^e = 1.$$

(b) Compute the volume of the solid formed by rotating D around the x-axis.

Solution: Using the disc method we have

$$V = \pi \int_1^e (\ln x)^2 \, dx = \pi \left(x(\ln x)^2 \big|_1^e - 2 \int_1^e \ln x \, dx \right) = \pi(e - 2).$$

(c) Find the volume of the solid formed by rotating D around the y-axis.

Solution: By the shell method we have

$$V = 2\pi \int_1^e x \ln x \, dx = \pi x^2 \ln x \big|_1^e - \pi \int_1^e x \, dx = \frac{1}{2}\pi(e^2 + 1).$$

11. Let D be the region bounded by the curve $y = \cosh x$, the x-axis, the y-axis, and the line $x = 1$.

(a) Calculate the perimeter of D.

Solution: The region D has 4 sides, 3 of which are straight line segments, and one which is the part of the graph $y = \cosh x$ between $x = 0$ and $x = 1$. Thus the perimeter is

$$P = \underbrace{1 + 1 + \cosh(1)}_{\text{the straight line segments}} + \int_0^1 \sqrt{1 + \sinh^2 x}\, dx = 2 + \cosh(1) + \sinh(1).$$

(b) Calculate the volume of the solid obtained by revolving D around the x-axis.

Solution: By the disc method,

$$V = \pi \int_0^1 \cosh^2 x\, dx = \frac{\pi}{2} \int_0^1 (1 + \cosh(2x))\, dx = \frac{\pi}{2} + \frac{\pi}{4} \sinh(2).$$

(c) Calculate the volume of the solid obtained by revolving D around the y-axis.

Solution: By the shell method, the volume is

$$2\pi \int_0^1 x \cosh x\, dx = 2\pi \sinh(1) - 2\pi \int_0^1 \sinh x\, dx = 2\pi(\sinh(1) - \cosh(1) + 1).$$

(d) Calculate the surface area of the solid obtained by rotating D around the x-axis. Remember to include the area of the end caps.

Solution: The curved surface area is

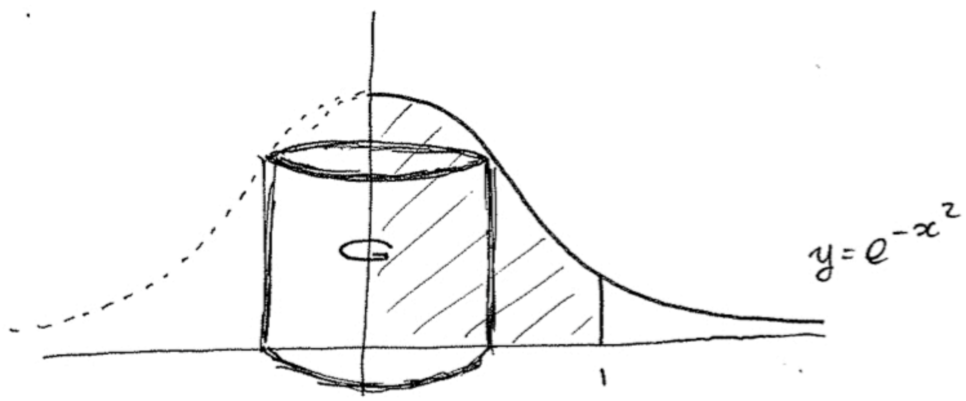
$$2\pi \int_0^1 f(x)\sqrt{1 + f'(x)^2}\, dx = 2\pi \int_0^1 \cosh^2 x\, dx = \pi + \frac{\pi}{2} \sinh(2).$$

Therefore the total surface area (including the two end caps) is

$$A = \pi + \frac{\pi}{2} \sinh(2) + \pi + \pi \cosh^2(1).$$

12. Compute the volume of the solid obtained by rotating about the y-axis the region bounded by the curve $y = e^{-x^2}$, the x-axis, the y-axis, and the line $x = 1$.

Solution: The picture is:



By the shell method,

$$V = 2\pi \int_0^1 x e^{-x^2}\, dx = -\pi e^{-x^2}\Big|_0^1 = \pi(1 - e^{-1}).$$

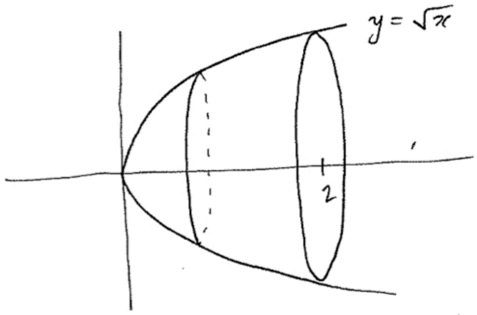
13. Find the length of the curve with parametrisation $x(t) = t - \sin t$ and $y(t) = 1 - \cos t$ with $t \in [0, 2\pi]$.

Solution: Using the formula for the length of a parametrised curve, the length is

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t\, dt} \\
&= \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos t}\, dt \\
&= \sqrt{2} \int_0^{2\pi} \sqrt{2 \sin^2(t/2)}\, dt \\
&= 2 \int_0^{2\pi} \sin(t/2)\, dt \\
&= 8.
\end{aligned}$$

14. Compute the surface area of the solid obtained by revolving the part of the graph of $y = \sqrt{x}$ between $x = 0$ and $x = 2$ around the x-axis. Remember to include any end caps.

Solution: The solid looks like



The curved surface area is

$$A_1 = 2\pi \int_0^2 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx = \pi \int_0^2 \sqrt{1 + 4x}\, dx = \frac{13\pi}{3}.$$

The end cap is a disc of radius $\sqrt{2}$, and therefore has area $A_2 = 2\pi$. Thus

$$A = A_1 + A_2 = \frac{19}{3}\pi.$$

15. Decide if the following improper integrals exist or not:

(a) $\displaystyle\int_1^{\infty} \frac{3 + 2\sin(x^2)}{x}\, dx$

Solution: We have

$$\frac{3 + 2\sin(x^2)}{x} \geq \frac{1}{x} \qquad \text{for all } x > 0,$$

and since $\int_1^\infty \frac{1}{x}\,dx$ diverges to infinity we conclude that the given integral does not exist.

(b) $\displaystyle\int_0^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx$

Solution: This improper integral has definition

$$\int_0^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx = \lim_{\epsilon\to 0^+}\int_\epsilon^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx$$

if the limit exists. But

$$\int_\epsilon^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx = \cos\left(\frac{1}{x}\right)\Big|_\epsilon^1 = \cos(1) - \cos(1/\epsilon).$$

Since $\displaystyle\lim_{\epsilon\to 0^+}\cos(1/\epsilon)$ does not exist, the integral $\displaystyle\int_0^1 \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx$ does not exist.

(c) $\displaystyle\int_0^1 \frac{\cosh x}{\sqrt{x}}\,dx$

Solution: For $x \in (0,1]$ we have

$$\left|\frac{\cosh x}{\sqrt{x}}\right| \le \frac{\cosh(1)}{\sqrt{x}}.$$

Since $\int_0^1 \frac{\cosh(1)}{\sqrt{x}}\,dx$ exists, so does $\int_0^1 \frac{\cosh x}{\sqrt{x}}\,dx$ (by the Comparison Test).

16. Compute the value of the following improper integrals:

(a) $\displaystyle\int_0^\infty e^{-x}\cos x\,dx.$

Solution: The improper integral is defined by

$$\int_0^\infty e^{-x}\cos x\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\cos x\,dx.$$

By integrating by parts twice we have

$$\begin{aligned}\int_0^b e^{-x}\cos x\,dx &= e^{-x}\sin x\Big|_0^b + \int_0^b e^{-x}\sin x\,dx\\ &= e^{-b}\sin b + \left(-e^{-x}\cos x\Big|_0^b - \int_0^b e^{-x}\cos x\,dx\right),\end{aligned}$$

and so

$$2\int_0^b e^{-x}\cos x\,dx = 1 + e^{-b}\sin b - e^{-b}\cos b.$$

Hence

$$\int_0^\infty e^{-x}\cos x\,dx = \frac{1}{2}\lim_{b\to\infty}\left(1 + e^{-b}\sin b - e^{-b}\cos b\right) = \frac{1}{2}.$$

(b) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\, dx.$

Solution: Integrating by parts gives

$$\int_1^b \frac{\ln x}{x^2}\, dx = -\frac{\ln b}{b} + \int_1^b \frac{1}{x^2}\, dx = 1 - \frac{1}{b} - \frac{\ln b}{b}.$$

As $b \to \infty$ we have $\frac{\ln b}{b} \to 0$, and hence

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = 1.$$

(c) $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x}}\, dx.$

Solution: The improper integral is computed by taking a limit of proper integrals:

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{1-x}}\, dx &= \lim_{\epsilon\to 0^+} \int_0^{1-\epsilon} \frac{x}{\sqrt{1-x}}\, dx \\
&= \lim_{\epsilon\to 0^+} \int_1^{\epsilon} \frac{1-u}{\sqrt{u}}(-1)\, du \\
&= \lim_{\epsilon\to 0^+} \int_{\epsilon}^{1} \left(u^{-\frac{1}{2}} - u^{-\frac{1}{2}}\right)\, du \\
&= \lim_{\epsilon\to 0^+} \left(2 - \frac{2}{3} - 2\epsilon^{\frac{1}{2}} + \frac{2}{3}\epsilon^{\frac{3}{2}}\right) \\
&= \frac{4}{3}.
\end{aligned}$$

17. Compute the indefinite integral $\displaystyle\int x^n \ln x\, dx$, where $n \neq -1$.

Solution: Integrating by parts, with $u = \ln x$ and $\frac{dv}{dx} = x^n$ gives

$$\int x^n \ln x\, dx = \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n\, dx = \frac{x^{n+1}(\ln x - 1)}{n+1} + C,$$

where C is the constant of integration.

18. Compute the indefinite integral $\displaystyle\int \frac{x^2}{\sqrt{1+x^2}}\, dx.$

Solution: Set $x = \sinh\theta$. Then $dx = \cosh\theta\, d\theta$, and so

$$\int \frac{x^2}{\sqrt{1+x^2}}\, dx = \int \frac{\sinh^2\theta}{\cosh\theta}\cosh\theta\, d\theta = \int \sinh^2\theta\, d\theta.$$

Since $\sinh^2\theta = \frac{1}{2}(\cosh(2\theta) - 1)$, the integral is

$$\int \frac{x^2}{\sqrt{1+x^2}}\, dx = \frac{1}{2}\int (\cosh(2\theta) - 1)\ d\theta = \frac{1}{4}\sinh(2\theta) - \frac{1}{2}\theta + C.$$

Since $\sinh(2\theta) = 2\sinh\theta\cosh\theta = x\sqrt{1+x^2}$ we have

$$\int \frac{x^2}{\sqrt{1+x^2}}\,dx = \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\sinh^{-1}x + C.$$

19. Find a reduction formula for the integral $I_n = \displaystyle\int x^n \cos x\,dx$.

Solution: Integrating by parts twice gives

$$\begin{aligned}\int x^n \cos x\,dx &= x^n \sin x - n\int x^{n-1}\sin x\,dx \\ &= x^n \sin x + nx^{n-1}\cos x - n(n-1)\int x^{n-2}\cos x\,dx.\end{aligned}$$

Therefore

$$I_n = x^n \sin x + nx^{n-1}\cos x - n(n-1)I_{n-2}$$

is the reduction formula.

20. Calculate the limit $\displaystyle\lim_{x\to 0}\frac{S(x)}{x^3}$, where $S(x) = \displaystyle\int_0^x \sin(t^2)\,dt$.

Solution: This limit is of type $\frac{0}{0}$. By l'Hôpital's rule and the Fundamental Theorem of Calculus we have

$$\lim_{x\to 0}\frac{S(x)}{x^3} = \lim_{x\to 0}\frac{\sin(x^2)}{3x^2} = \frac{1}{3}.$$

21. Decide if the following improper integrals exist or not (either use the Comparison Test, or make a direct limit calculation). If they exist, try to compute their value (this is not always possible!).

(a) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

Solution: Integrating by parts gives

$$\int_1^b \frac{\ln x}{x^2}\,dx = -\frac{\ln}{x}\bigg|_1^b + \int_1^b \frac{1}{x^2}\,dx = -\frac{\ln b}{b} + 1 - b^{-1}.$$

Since $\lim_{b\to\infty}\frac{\ln b}{b} = 0$ we see that $\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = 1$. Therefore the improper integral exists, and equals 1.

(b) $\displaystyle\int_1^\infty \sin(\pi x)\,dx$

Solution: We compute

$$\int_1^b \sin(\pi x)\,dx = -\frac{\cos b\pi}{\pi} - \frac{1}{\pi}.$$

Since $\lim_{b\to\infty}\cos b\pi$ does not exist we see that the improper integral does not exist. Note that it does not diverge to ∞; rather $\int_1^b \cos(\pi x)\,dx$ oscillates, taking values between 0 and $-2/\pi$.

(c) $\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$

Solution: $0 \le e^{-x}/\sqrt{x} \le e^{-x}$ for all $x \ge 1$. Also,

$$\int_1^b e^{-x}\,dx = \Big[-e^{-x}\Big]_1^b = e^{-1} - e^{-b} \to e^{-1} \quad \text{as } b \to \infty.$$

So $\int_1^\infty e^{-x} dx$ converges, so $\int_1^\infty \frac{e^{-x}}{\sqrt{x}} dx$ converges too by the Comparison Test.

(d) $\displaystyle\int_0^\infty \frac{\cosh x}{x^2+1}\,dx$

Solution: Since $\cosh x = \frac{1}{2}(e^x + e^{-x}) \approx \frac{1}{2}e^x$ for large x, we expect that the integral does not exist – the integrand blows up as $x \to \infty$. Indeed, since

$$\lim_{x\to\infty} \frac{\cosh x}{x^2+1} = \infty,$$

there is a number X such that $\frac{\cosh x}{x^2+1} \ge 1$ for all $x > X$. Since

$$\int_X^\infty 1\,dx$$

does not exist, we conclude that the given integral does not exist by the Comparison Test.

(e) $\displaystyle\int_{\pi/4}^{\pi/2} \sec^2 x\,dx$

Solution: Since $\sec^2 x = 1/\cos^2 x \to \infty$ as $x \to \pi/2$, the integrand is unbounded, and the integral is improper. If $0 < \epsilon < \pi/4$,

$$\begin{aligned}
\int_{\pi/4}^{\pi/2-\epsilon} \sec^2 x\,dx &= \int_{\pi/4}^{\pi/2-\epsilon} \frac{d}{dx}\big(\tan x\big)\,dx \\
&= \tan(\pi/2-\epsilon) - \tan(\pi/4) \\
&= \tan(\pi/2-\epsilon) - 1 \\
&\to \infty \quad \text{as } \epsilon \to 0.
\end{aligned}$$

So the improper integral does not exist. More precisely we say that the integral diverges to $+\infty$.

(f) $\displaystyle\int_{-\infty}^{0} e^x \cos x\,dx$

Solution: The dominant behaviour here comes from the e^x, and as $x \to -\infty$ this decays very quickly. So we guess that the integral exists. Indeed,

$$0 \le |e^x \cos x| \le e^x,$$

and since $\int_{-\infty}^0 e^x\,dx = 1$ exists, we conclude that the given integral exists by the Comparison Test.

It is possible to compute the value of the integral. Integrating by parts we have

$$\begin{aligned}\int_a^0 e^x \cos x\,dx &= -e^a \sin a - \int_a^0 e^x \sin x\,dx\\ &= -e^a \sin a + 1 - e^a \cos a - \int_a^0 e^x \cos x\,dx.\end{aligned}$$

Therefore

$$\int_a^0 e^x \cos x\,dx = \frac{1}{2} - \frac{1}{2}e^a \sin a - \frac{1}{2}e^a \cos a \to \frac{1}{2} \qquad \text{as } a \to -\infty.$$

So the improper integral equals $1/2$.

(g) $\displaystyle\int_0^1 \sin\left(\frac{1}{x}\right)\,dx$

Solution: The integrand has a (rather nasty) discontinuity at $x = 0$, and so we need to consider

$$\lim_{a\to 0^+} \int_a^1 \sin\left(\frac{1}{x}\right)\,dx.$$

It is easier to see what is happening after making the change of variable $y = \frac{1}{x}$. Then

$$\int_a^1 \sin\left(\frac{1}{x}\right)\,dx = \int_1^{a^{-1}} \frac{\sin y}{y^2}\,dy,$$

and our improper integral is

$$\int_0^1 \sin\left(\frac{1}{x}\right)\,dx = \lim_{b\to\infty} \int_1^b \frac{\sin y}{y^2}\,dy.$$

This integral exists by comparison with $\int_1^\infty \frac{1}{x^2}\,dx$. It is not easy to compute the exact value of the integral.

(h) $\displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx$

Solution: This integral exists by comparison to $\int_1^\infty \frac{1}{x^2}\,dx$. It is not easy to compute the exact value of this integral using elementary methods. But some "complex analysis" shows that the integral equals $\pi e/2$.

(i) $\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$

Solution: $0 \le e^{-x}/\sqrt{x} \le e^{-x}$ for all $x \ge 1$. Also,

$$\int_1^b e^{-x}\,dx = \Big[-e^{-x}\Big]_1^b = e^{-1} - e^{-b} \to e^{-1} \quad \text{as } b \to \infty.$$

So $\int_1^\infty e^{-x}dx$ converges, so $\int_1^\infty \frac{e^{-x}}{\sqrt{x}}dx$ converges too by the Comparison Test.

(j) $\displaystyle\int_0^\infty x^3 e^{-x}\,dx$

Solution: Integrating by parts we have

$$\int_0^b x^3 e^{-x}\,dx = 6 - (b^3 + 3b^2 + 6b + 6)e^{-b}.$$

Therefore

$$\int_0^\infty x^3 e^{-x}\,dx = 6 - \lim_{b\to\infty}(b^3 + 3b^2 + 6b + 6)e^{-b} = 6.$$

In particular, the integral exists.

Alternatively, we could prove more 'abstractly' that the integral exists (without calculating its value). Note first that $x^3/e^x \to 0$ as $x \to \infty$, as you can see using L'Hôpital's Rule, for example. So there is a number M such that $x^3/e^x \le 1$ once $x \ge M$. Replacing x by $x/2$, we see that $(x/2)^3/e^{x/2} \le 1$ once $x/2 \ge M$. That is, $x^3 \le 8e^{x/2}$ once $x \ge 2M$. So the integrand x^3e^{-x} may be estimated as follows:

$$x^3 e^{-x} \;\le\; 8e^{x/2}e^{-x} \;=\; 8e^{-x/2} \quad \text{once } x \ge 2M.$$

Now $\int_{2M}^\infty 8e^{-x/2}\,dx$ converges, by an easy calculation. So $\int_{2M}^\infty x^3e^{-x}\,dx$ converges by the Comparison Test. For $0 \le x \le 2M$, x^3e^{-x} is continuous, and so $\int_0^{2M} x^3e^{-x}\,dx$ exists. Hence

$$\int_0^\infty x^3e^{-x}\,dx \;=\; \int_0^{2M} x^3e^{-x}\,dx + \int_{2M}^\infty x^3e^{-x}\,dx$$

exists. Using integration by parts, it is easy to calculate its value exactly: it equals 6.

If you prefer to avoid breaking the integral up into two parts as above, you could instead argue as follows: By Calculus, we find that $x^3e^{-x/2}$ takes its maximum value of $C = 216e^{-3}$ at $x = 6$. Thus $x^3 \le Ce^{x/2}$ for all $x \ge 0$. Hence $x^3e^{-x} \le Ce^{-x/2}$ for all $x \ge 0$. Since $\int_0^\infty Ce^{-x/2}\,dx$ converges by an easy direct calculation, so does $\int_0^\infty x^3e^{-x}\,dx$, by the Comparison Test.

(k) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

Solution: We have

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\tan^{-1}(b) = \frac{\pi}{2}.$$

So the integral exists, and equals $\frac{\pi}{2}$.

(l) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Solution: The integrand is unbounded at $x = 1$ and $x = -1$. Therefore we should compute

$$\begin{aligned}\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{a\to1}\int_{-a}^a \frac{1}{\sqrt{1-x^2}}\,dx = 2\lim_{a\to1}\int_0^a \frac{1}{\sqrt{1-x^2}}\,dx\\ &= 2\lim_{a\to1}\sin^{-1}(a) = \pi.\end{aligned}$$

(m) $\displaystyle\int_1^\infty \frac{e^{-x^2}}{\sqrt{x-1}}\,dx$

Solution: We need to worry about both integration limits here. Write

$$\int_1^\infty \frac{e^{-x^2}}{\sqrt{x-1}}\,dx = \int_1^2 \frac{e^{-x^2}}{\sqrt{x-1}}\,dx + \int_2^\infty \frac{e^{-x^2}}{\sqrt{x-1}}\,dx,$$

and treat each integral separately. The second integral exists be a similar argument to that in (i). The first integral exists by comparison with

$$e^{-1}\int_1^2 \frac{1}{\sqrt{x-1}}\,dx.$$

Therefore the given improper integral exists.

(n) $\displaystyle\int_0^1 \sin\left(\frac{1}{x^2}\right)\,dx$

Solution: A change of variable helps here: Let $y = x^{-2}$. Then

$$\int_a^1 \sin\left(\frac{1}{x^2}\right)\,dx = \int_1^{a^{-2}} \frac{\sin y}{y^{3/2}}\,dy.$$

Since $\int_0^\infty \frac{\sin y}{y^{3/2}}\,dy$ exists (by comparison with $\int_1^\infty \frac{1}{y^{3/2}}\,dy$) we see that

$$\int_0^1 \sin\left(\frac{1}{x^2}\right)\,dx = \lim_{a\to 0^+}\int_1^{a^{-2}} \frac{\sin y}{y^{3/2}}\,dy = \int_1^\infty \frac{\sin y}{y^{3/2}}\,dy,$$

and so the improper integral exists. It is not easy to give the value of the integral.

(o) $\displaystyle\int_0^\infty \operatorname{erf}(x)\,dx$

Solution: Since $\operatorname{erf}(x) \to 1$ as $x \to \infty$ it is immediate that the improper integral does not exist (by comparison with $\int_0^\infty 1\,dx$).

(p) $\displaystyle\int_0^\infty \cosh(3x)e^{-4x}\,dx$

Solution: Using the definition of $\cosh(3x)$ we have

$$\int_0^b \cosh(3x)e^{-4x}\,dx = \frac{1}{2}\int_0^b \left(e^{-x} + e^{-7x}\right)\,dx = \frac{1}{2}\left(1 - e^{-b}\right) + \frac{1}{14}\left(1 - e^{-7b}\right).$$

Taking the limit as $b \to \infty$ we see that the improper integral exists, and equals $\frac{1}{2} + \frac{1}{14} = \frac{4}{7}$.

(q) $\displaystyle\int_1^2 \frac{1}{\ln x}\,dx$

Solution: The integrand has a discontinuity at $x = 1$. Notice that

$$\ln x \le x - 1 \qquad \text{for all } x > 0.$$

In particular this is true for all $x \geq 1$. (To see this you can use some calculus: Consider the function $f(x) = \ln x - x + 1$. It has $f'(x) < 0$ for all $x \geq 1$, and $f(1) = 0$, so $f(x) \leq 0$ for all $x \geq 1$). Therefore

$$\frac{1}{\ln x} \geq \frac{1}{x-1} \qquad \text{for all } x > 1.$$

The integral $\int_1^2 \frac{1}{x-1}\,dx$ does not exist, because

$$\int_1^2 \frac{1}{x-1}\,dx = \lim_{a \to 1^+} \int_a^2 \frac{1}{x-1}\,dx = -\lim_{a \to 1^+} \ln(a-1) = \infty.$$

Therefore the given improper integral also does not exist, by the Comparison Test.

(r) $\displaystyle\int_2^\infty \frac{\mathrm{Li}(x)}{x^2}\,dx$

Solution: Integrating by parts gives

$$\int_2^b \frac{\mathrm{Li}(x)}{x^2}\,dx = -\frac{\mathrm{Li}(b)}{b} + \int_2^b \frac{1}{x \ln x}\,dx = -\frac{\mathrm{Li}(b)}{b} + \ln(\ln b) - \ln(\ln 2).$$

By L'Hôpital's Rule we have

$$\lim_{b \to \infty} \frac{\mathrm{Li}(b)}{b} = \lim_{b \to \infty} \frac{\frac{1}{\ln b}}{1} = 0.$$

But since $\ln(\ln b) \to \infty$ as $b \to \infty$ we see that the given improper integral does not exist.