(A)

MATH 1903 Lecture 10 Fri 1/9/2017

Infinite series

reference: pp 2·63 – 2·79

- sums of numbers
- delicate interplay between finite & infinite, and between discrete & continuous.

Terminology:

$$a_0, a_1, a_2, a_3, \dots, a_n, \dots \qquad \text{sequence}$$

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + \dots + a_n + \dots \qquad \text{series}$$

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots \qquad \text{power series}$$

A positive real number α is really a series:

$$\alpha = \underbrace{e_k e_{k-1} \dots e_2 e_1}_{\text{integer part}} \underset{\text{decimal point}}{\bullet} \underbrace{d_1 d_2 d_3 \dots d_n \dots}_{\text{digits go on forever}}$$

$$= e_k \dots e_1 + \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \dots + \frac{d_n}{10^n} + \dots$$

$$= \lim_{n \to \infty} \underbrace{e_k \dots e_1 \bullet d_1 \dots d_n}_{\text{rational approximation of } \alpha}$$

e.g. $1{\cdot}111\ldots = 1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots + \left(\frac{1}{10}\right)^n + \cdots$

$= 10/9$ (Why?)

Put

$$s(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

called a <u>geometric series</u>

(with common ratio x)

Then

$$x\,s(x) = x + x^2 + x^3 + x^4 + \cdots + x^{n+1} + \cdots$$

$$= s(x) - 1,$$

So $s(x)\,(1-x) = 1,$

so

$$s(x) = \frac{1}{1-x}.$$

Thus

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots}$$

and this makes sense (converges) iff $|x| < 1$

(see Exercises 1 & 2).

e.g. $1{\cdot}111\ldots = s\left(\frac{1}{10}\right) = \frac{1}{1-1/10} = \frac{1}{9/10} = \frac{10}{9}$

In general,

$$\boxed{a_0 + a_1 + \dots + a_n + \dots \stackrel{\text{def}}{=} \lim_{n\to\infty} (a_0 + a_1 + \dots + a_n)}$$

and this is said to **converge** if the limit exists & is finite.

e.g. $s(1) = 1+1+1+\dots = \infty$

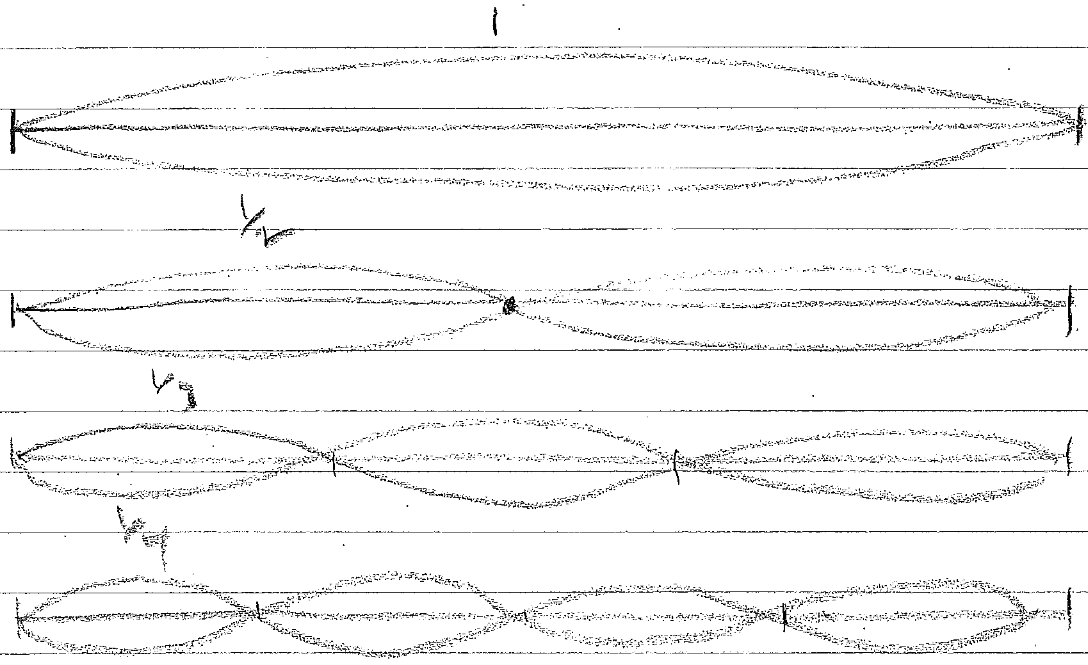
$s(-1) = 1-1+1-1+1-1+\dots$ oscillates

diverge

Harmonic series:

$$\boxed{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} + \dots}$$

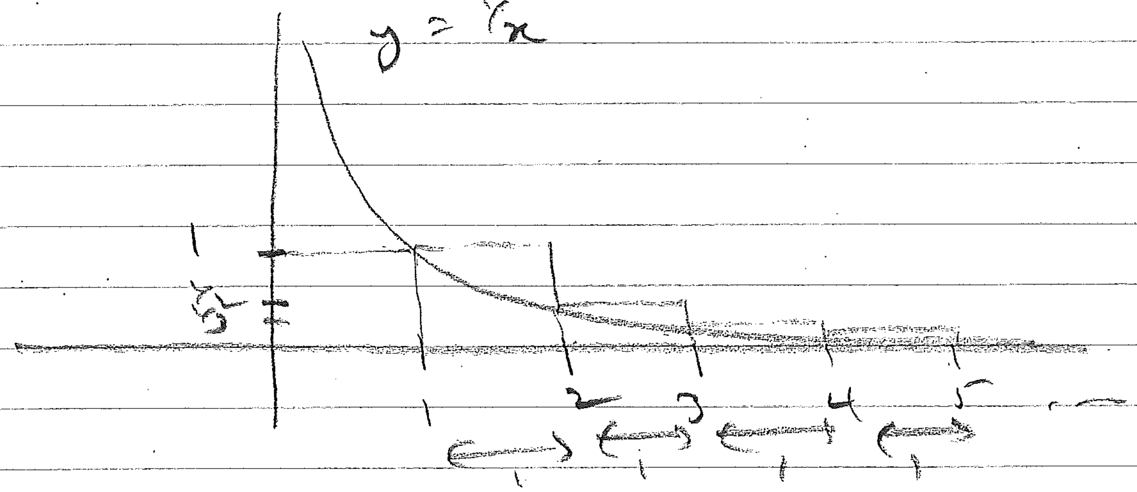
- related to harmonics in music



D

$\boxed{\text{Fact: The harmonic series diverges.}}$

Proof:



$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ is an upper Riemann sum for $\int_1^{n+1} \frac{1}{x}\,dx$.

But $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{x}\,dx = \infty$

(yesterday)

so

$$\underbrace{1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n}}_{\downarrow \atop \infty} \geq \underbrace{\int_1^{n+1} \tfrac{1}{x}\,dx}_{\downarrow \atop \infty} \quad \text{as } n \to \infty$$

by comparison.

$\square$

E

Example of an important function represented by a series:

For $x \in \mathbb{R}$ put

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

goes on forever

RHS in fact converges for all x

(by the Ratio Test, see later),

so this defines a function $f : \mathbb{R} \to \mathbb{R}$.

In fact, f becomes differentiable

(part of a general theory)

and we can differentiate term by term:

$$f'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \dots + \frac{nx^{n-1}}{n!} + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!} + \dots$$

$$= f(x).$$

$$\boxed{\text{Claim: } f(x) = e^x}$$

F

Proof of Claim: Let $y = y(x)$ be any function $: \mathbb{R} \to \mathbb{R}$ such that

$$y' = \frac{dy}{dx} = y,$$

so $\boxed{dy = y\,dx}$,

so $\frac{dy}{y} = dx$ (technique of separating variables)

$\boxed{\text{provided } y \neq 0}$

Hence $\int \frac{dy}{y} = \int dx$

i.e. $\ln|y| = x + C$ $\exists$ constant C

so $|y| = e^{x+C} = e^C e^x$

so $y = \pm e^C e^x = Ke^x$ $\exists$ constant K

Hence (since in our case $f(x) \neq 0$ for all x (Why?))

$\boxed{f(x) = Ke^x}$ $\exists K$

But $f(0) = 1 + 0 + \frac{0}{2!} + \frac{0}{3!} + \dots = 1,$

and $f(0) = Ke^0 = K$, so $K = 1$.

Hence

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots}$$

- series expansion of e^x

and the Claim is proved. $\square$

Ratio Test for convergence: Put

$$L = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

(assuming L exists).

Then

$$\sum_{k=0}^{\infty} a_k$$

(a) converges if $L < 1$;

(b) diverges if $L > 1$.

p 2.77 — convergence of series for e^x for all x

p 2.72 – 2.74 — reason for part (a) of Ratio Test.