(A)

MATH 1903 Lecture 12 Fri 8/9/2017

More examples with series

(1) Let $f(x) = \tan^{-1} x = ?$ (as a series?)

Then $f'(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

which converges for $|-x^2| = |x|^2 < 1$, i.e. $|x|<1$,

so this series has radius of convergence $R=1$.

Antidifferentiating gives

$$\tan^{-1}(x) = f(x) = x - x^3/3 + x^5/5 - x^7/7 + \frac{x^9}{9} - \cdots + C,$$

for some constant C. But

$$0 = \tan^{-1}(0) = 0 - 0 + 0 - \cdots + C,$$

so $C=0$. Thus

$$\boxed{\begin{aligned}\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k+1}}{2k+1}\end{aligned}}$$

for $-1 < x < 1$ (also with radius of convergence $R=1$).

(B)

(2) Let $f(x) = \tan x = \frac{\sin x}{\cos x} = ?$ (as a series?)

Using the series for $\sin x$ and $\cos x$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \frac{1}{1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots\right)}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots\right) + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots\right)^2 + \cdots\right)$$

higher terms

$$= x + x^3\left(-\frac{1}{3!} + \frac{1}{2!}\right) + x^5\left(\frac{1}{5!} - \frac{1}{4!} - \frac{1}{3!\,2!} + \left(\frac{1}{2!}\right)^2\right) + \cdots$$

$$= x + x^3\left(\frac{1}{2} - \frac{1}{6}\right) + x^5\left(\frac{1}{120} - \frac{1}{24} - \frac{1}{12} + \frac{1}{4}\right) + \cdots$$

$$= x + \frac{x^3}{3} + x^5\left(\frac{1 - 5 - 10 + 30}{120}\right) + \cdots$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

(there is a pattern but very difficult to write down).

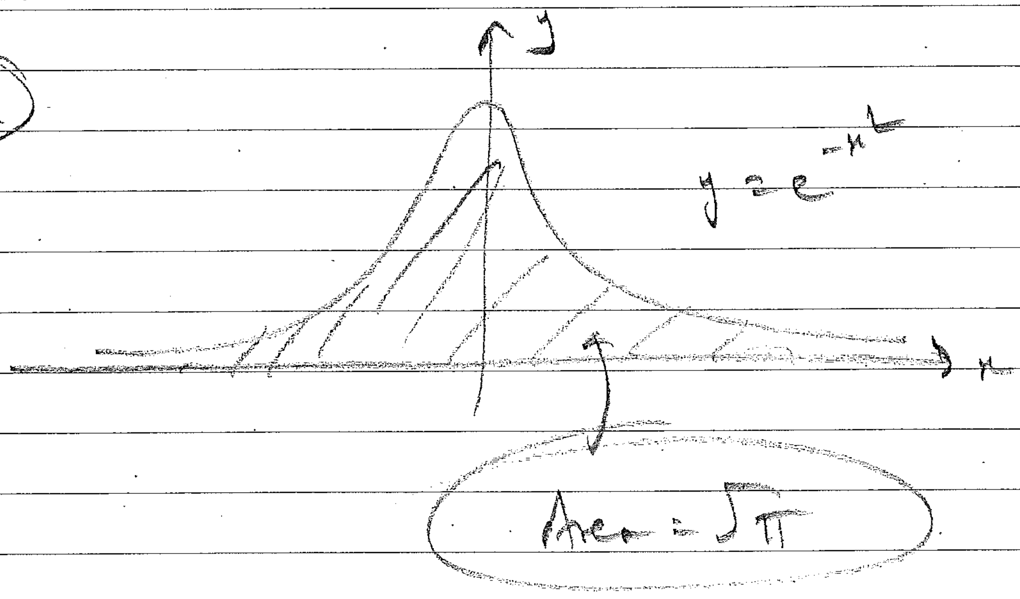
(c)

Revisiting an important improper integral:

- extra notes pp 3.1 – 3.12
- not examinable, but worthwhile to follow, & links with second year calculus.

$$\boxed{\textbf{Theorem:} \quad \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

p 3.1

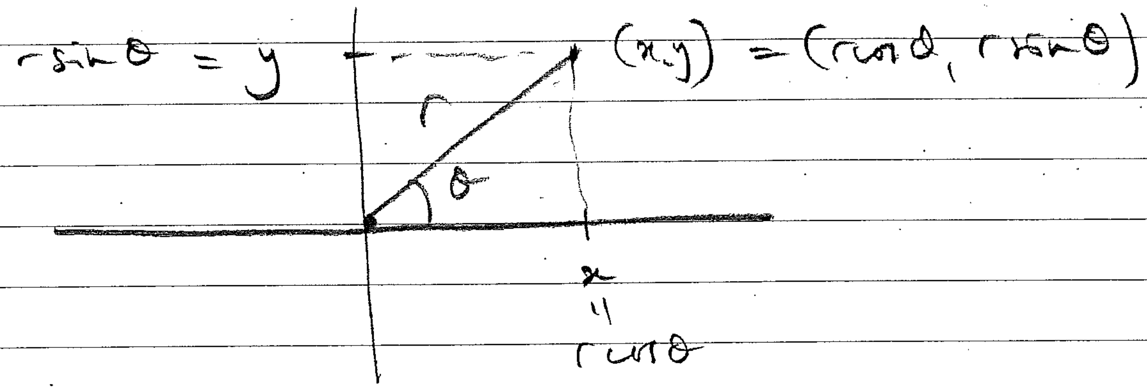


- existence pp 3.2, 3.3 ✓
- use of functions of two variables & double integrals

pp 3.3 – 3.12 ✓

D

Connections between areas & determinants:



We have the following connection between cartesian & polar coordinates: $\boxed{x = r\cos\theta \quad , \quad y = r\sin\theta}$,

so $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$,

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$.

Form the Jacobian matrix

$$M = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

and the Jacobian determinant

$$\det M = \cos\theta(r\cos\theta) - \sin\theta(-r\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r,$$

The area differentials are connected by multiplication by $\det M$:

$$\boxed{dx\,dy = (\det M)\,dr\,d\theta = r\,dr\,d\theta}$$

as calculated earlier