MATH1903/1907 Lectures

Week 8, Semester 2, 2017

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The differential equations we looked at were <u>separable</u>: They are of the form

$$y'(x) = f(x)g(y) \quad \left(= F(x,y)\right)$$

<u>Question</u>: Existence & Uniqueness of solutions.

Assume we have a solution $y(x)$, find a formula for it.

Divide by $g(y)$ (assume $\neq 0$)

$$\frac{y'(x)}{g(y(x))} = f(x), \quad \text{then integrate}$$

$$\int_{x_0}^{x} \frac{y'(\xi)}{g(y(\xi))} d\xi = \int_{x_0}^{x} f(\xi)\, d\xi \quad \text{substitute } z = y(\xi)$$

$$\int_{y(x_0)}^{y(x)} \frac{1}{g(z)} dz = \int_{0}^{x} f(\xi)\, d\xi$$

If we set

$$G(y) := \int_{y(x_0)}^{y} \frac{1}{g(z)}\, dz \quad , \quad F(x) = \int_{x_0}^{x} f(s)\, ds \, ,$$

then the solution of $y' = f(x) g(y)$ satisfies

$$G(y(x)) = F(x) \quad \text{implicit eq. for } y$$

We integrate $\frac{1}{g(z)}$, so we need to assume (or know) that $g(z) \neq 0$. Assume for initial condition $y(x_0) = y_0$

$$\boxed{g(y_0) \neq 0}$$

If a solution to $G(y) = F(x)$ exists, then

$$\frac{dG}{dy} = \frac{d}{dy} \int_{y_0}^{y} \frac{1}{g(z)}\, dz = \frac{1}{g(y)}$$

by the fundamental theorem of calculus. In particular:

$$\frac{dG}{dy}(y_0) = \frac{1}{g(y_0)} \neq 0 \quad \text{by assumption.}$$

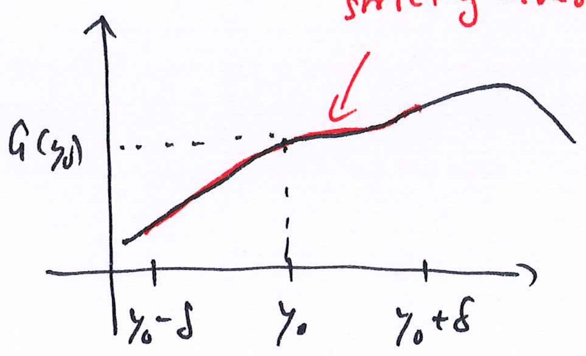
Assuming that g is continuous, either

$$G'(y) > 0 \quad \text{or} \quad G'(y) < 0$$

in some interval $(y_0 - \delta, y_0 + \delta)$

Hence G is strictly monotone on $(y_0 - \delta, y_0 + \delta)$

A monotone function has an inverse



By the intermediate value thm there is a (local) inverse G^{-1} of G. Then

$$y(x) = G^{-1}(F(x))$$

should be a solution of the d.e. $y'(x) = f(x)g(y)$ for x near x_0.

<u>Consequence</u>: Local existence and uniqueness of solutions to separable equations

$$y'(x) = f(x)\,g(y(x)), \quad y(x_0) = y_0$$

If $g(y_0) \neq 0$, then the d.e. has a unique solution in a (possibly small) neighbourhood of x_0.

That solution is implicitly given by

$$\int_{y_0}^{y} \frac{1}{g(z)}\,dz = \int_{x_0}^{x} f(\xi)\,d\xi$$

This justifies the method of <u>separation of variables</u>:

Applies to equations of the form

$$\frac{dy}{dx} = f(x)\, g(y)$$

<u>Step 1</u>: Separate variables:

$$\frac{dy}{g(y)} = f(x)\, dx$$

<u>Step 2</u>: Integrate: (involves an integration constant)

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

sometimes more efficient to interchange

<u>Step 3</u>: solve for y (if possible)

<u>Step 4</u>: Determine the integration constant from the initial conditions

Note: Solutions do not need to exist for all $x \in \mathbb{R}$:

Example $y' = y^2$ or $\dfrac{dy}{dx} = y^2$

Step 1: $\dfrac{dy}{y^2} = dx$

Step 2: $\displaystyle\int \frac{dy}{y^2} = \int dx = x + C$

$\parallel$

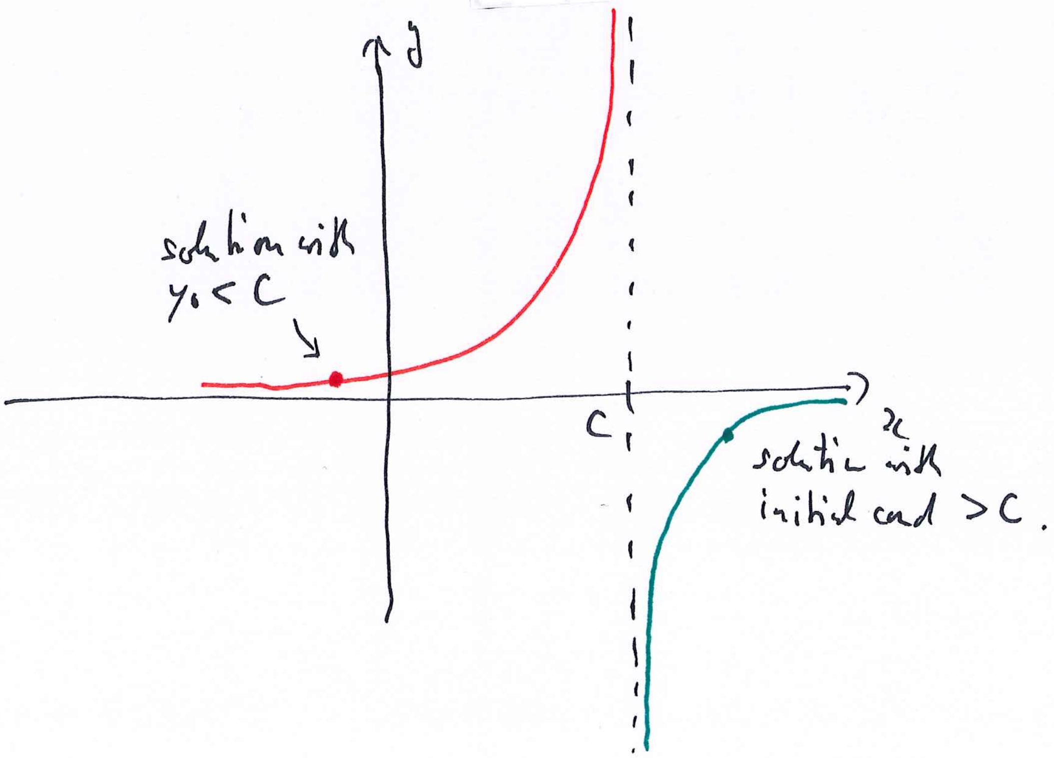
$-\dfrac{1}{y}$, so $-\dfrac{1}{y} = x + C$ (implicit equation)

Step 3: Solve for y:

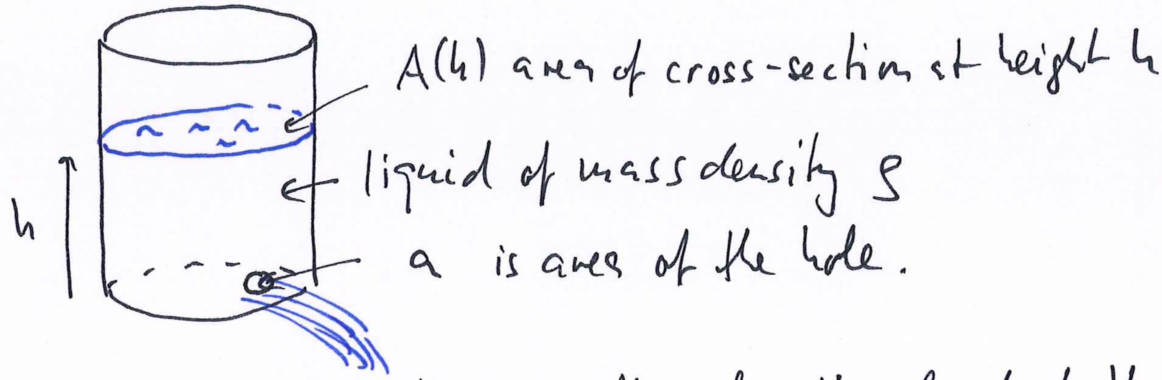
$$y = -\frac{1}{x+C}$$

For convenience change constant $C \to -C$

$$y = \frac{1}{C-x}$$ "general solution"



Non-uniqueness of solutions: The leaky bucket



Derive a differential equation for the level of the water, h, from first principles.

Use conservation laws:

- conservation of energy
- conservation of mass

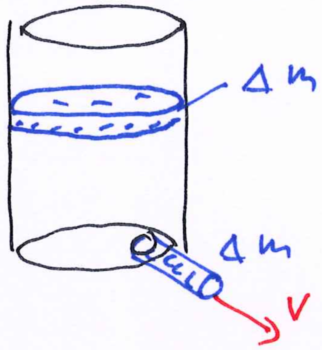
Balance of energy: loss of mass Δm

potential energy $=$ kinetic energy

$$\Delta m h g = \frac{1}{2} v^2 \Delta m$$

g gravitational constant

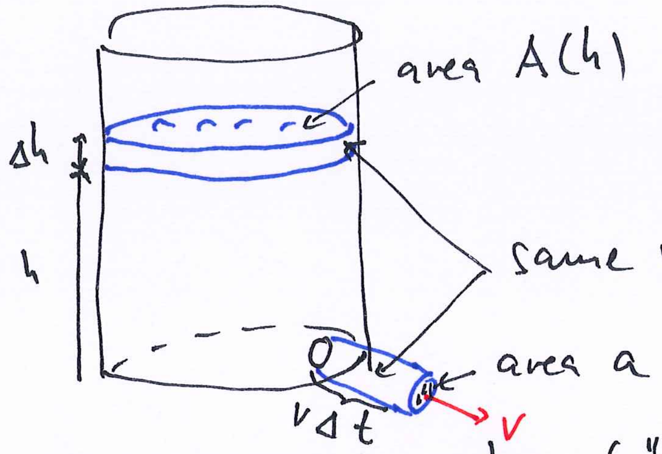
v velocity of water leaving the bucket through the hole.



Hence:

$$\boxed{v^2 = 2gh}$$

Balance of mass: consider small time interval Δt



layer of thickness Δh: ("top layer")

$$\Delta m = \rho A(h)\Delta h = -\rho A(h)\big(\underbrace{h(t+\Delta t) - h(t)}_{\Delta h}\big)$$

cylinder of cross-section a: (water flowing out)

$$\Delta m = \rho a v \Delta t$$

Equate the two (conservation of mass)

$$\Delta m = -\rho A(h)\big(h(t+\Delta t) - h(t)\big) = \rho a v \Delta t$$

Divide by Δt and let $\Delta t \to 0$:

$$\frac{h(t+\Delta t) - h(t)}{\Delta t} = -\frac{a}{A(h)} v \underset{\uparrow}{=} -\frac{a}{A(h)}\sqrt{2gh}$$

from energy balance

$$\Big\downarrow \Delta t \to 0$$

$$\frac{dh}{dt} = -\frac{a}{A(h)}\sqrt{2gh}$$

Hence h satisfies the differential equation

$$\boxed{\frac{dh}{dt} = -\frac{a}{A(h)}\sqrt{2gh}}$$

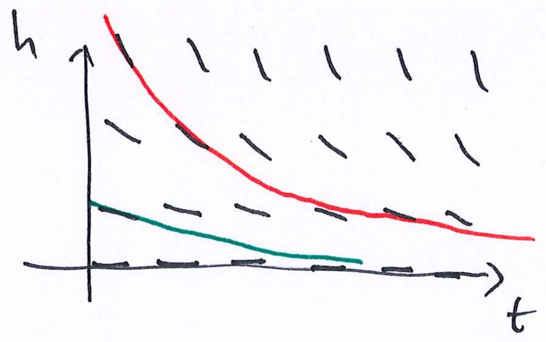
Toricelli's Law

We now assume $A(h) = \text{const}$ (bucket is cylindrical)

Then we need to solve a d.e. of the form

$$\frac{dh}{dt} = -k\sqrt{h} \qquad (k>0 \text{ constant})$$

Direction field:



Note: We cannot see from the direction field whether or not $h(t)=0$ for some finite time t.

Need to solve the equation.

Solve by separation of variables:

$$\frac{dh}{\sqrt{h}} = -k\,dt$$

Integrate:

$$\int \frac{dh}{\sqrt{h}} = -k \int dt = -kt + C$$

$$2\sqrt{h} = -kt + C \quad \underline{\geq 0}$$

only left branch of parabola is a solution!

Solve for h:

$$h = \frac{1}{4}(-kt + C)^2$$

$$\text{Solution: } h(t) = \begin{cases} \frac{1}{4}(-kt+C)^2 & \text{if } t \leq \frac{C}{k} \\ 0 & \text{if } t > \frac{C}{k} \end{cases}$$

<u>Note</u>:
- h is differentiable for all $t \in \mathbb{R}$
- The solution is unique in the forward time direction, but <u>not</u> in the backwards direction.