THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 10

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Taylor's Theorem.
- ☐ Curves in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Outcomes

After completing this tutorial you should

- ☐ use Taylor's Theorem to approximate functions and integrals;
- ☐ use Taylor's Theorem to compute limits;
- ☐ sketch curves and find parametrisations in simple cases.

Summary of essential material

Taylor's Theorem: Let $f(x)$ be n times differentiable at $x = x_0$. Then $f(x) = T_n(x) + R_n(x)$ where

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is the n-th order Taylor polynomial of f centred at $x = x_0$. It is the *best polynomial approximation* of order n near x_0. If f has at least $n + 1$ derivatives in a neighbourhood of x_0, then the *remainder* can be written in the form

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \qquad \text{for some } c \text{ strictly between } x_0 \text{ and } x.$$

This is called the *Lagrange form* of the remainder. The remainder looks like the general term in $T_n(x)$, but $f^{(n+1)}$ is evaluated at an intermediate c value between x and x_0 and not at x_0. A special case is the Mean Value Theorem ($n = 0$).

To estimate $R_n(x)$ we maximise $|f^{(n+1)}(c)|$ over all c between x and x_0. A common method is to use the "worst case scenario" for an estimate:

- If $|f^{(n)}(c)|$ is monotone in c set $c = 0$ or $c = x_0$ depending on whether it is increasing or decreasing;
- If $|f^{(n)}(c)|$ is a fraction maximise the numinator and minimize the denominator, often by setting $c = 0$ or $c = x_0$.

Often this is achieved by setting $c = 0$ or $c = x_0$, or if it is a fraction to minimize the denominator and maximise the numerator. One can also use calculus to find the maximum, but that is not necessary most of the time.

Curves: A *curve* in $\mathbb{R}^2$ is a function $C\colon [a, b] \to \mathbb{R}^2$. We are often a little relaxed here, and either regard the curve as the the actual function, or as the image of $[a, b]$ under C in $\mathbb{R}^2$. A curve is given by its "component functions"

$$\big(x(t), y(t)\big) \quad \text{or} \quad x(t)\boldsymbol{i} + y(t)\boldsymbol{j} \quad \text{or} \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad t \in [a, b].$$

The points $\big(x(t), y(t)\big)$ trace out a curve as t increases from a to b. Similarly, a curve in $\mathbb{R}^3$ is a function $C\colon [a, b] \to \mathbb{R}^3$ with the same interpretation as above, but three components.

Questions to complete during the tutorial

Questions marked with * are more difficult.

1. Prove that for all $n \geq 0$, we have the estimate $\left| \sin(1) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \right| \leq \frac{1}{(2n+2)!}$.

2. Let $p(x) := x^4 - 7x^3 + 13x^2 + 2x - 10$ be a polynomial.

(a) Determine the Taylor polynomials of order n about $x = 0$ for $n \geq 0$.

(b) Express the polynomial p as a polynomial in terms of $(x - 3)$ by computing a relevant Taylor polynomial.

3. (a) Calculate the second order Taylor polynomial $T_2(x)$ for $f(x) = \sqrt{1+x}$ about $x = 0$, and write down a formula for the remainder term $R_2(x) = f(x) - T_2(x)$. Hence show that

$$1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 \leq \sqrt{1+x^4} \leq 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} \quad \text{for all } x \in \mathbb{R}$$

(b) Hence, or otherwise, calculate $\lim_{x\to 0} \frac{2\sqrt{1+x^4} - 2 - x^4}{x^8}$.

4. Let $f(x) := (1+x)^\alpha$, where $\alpha \in \mathbb{R}$ is fixed. We define

$$\binom{\alpha}{0} := 1 \qquad \text{and} \qquad \binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-3)\cdots(\alpha-k+1)}{k!} \quad \text{for all integers } k \geq 1.$$

If $\alpha \geq 1$ is an integer these coincide with the binomial coefficients.

(a) Show that the n-th Taylor polynomial of f about $x = 0$ is given by $T_n(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k$.

(b) Hence, write down the 4th order Taylor polynomial of $\frac{1}{\sqrt{1+x}}$ about $x = 0$.

(c) This part shows that the remainder $R_n(x)$ approaches zero as n gets large provided that $0 \leq x < 1$.

*(i) Fix $x \in \mathbb{R}$ with $|x| < 1$. Using the Lagrange form of the remainder, show that

$$R_n(x) = (-1)^n \alpha \frac{(1+c)^\alpha}{n+1} \Big(1 - \alpha\Big)\Big(1 - \frac{\alpha}{2}\Big)\Big(1 - \frac{\alpha}{3}\Big) \cdots \Big(1 - \frac{\alpha}{n}\Big)\Big(\frac{x}{1+c}\Big)^{n+1}$$

for some c strictly between 0 and x. Derive that for $1 \leq m < n$ and $x \in (0,1)$

$$|R_n(x)| \leq \frac{2^\alpha}{n+1}\Big(1 + |\alpha|\Big)^m \Big(1 + \frac{|\alpha|}{m}\Big)^{n+1} |x|^{n+1}.$$

(ii) Hence show that $R_n(x) \to 0$ for all $x \in (0,1)$, that is, T_n approximates $(1+x)^\alpha$ well if $x \in (0,1)$. One can show that the same is true if $x \in (-1,0)$, but this is much harder.

5. Find parametrisations of the following curves.

(a) The line segment in $\mathbb{R}^3$ between $A(1,3,2)$ and $B(5,2,4)$ going from B to A.

(b) The circle in $\mathbb{R}^2$ centred at the origin with radius r, clockwise starting from the positive y-axis.

6. (a) Find the points of intersection of the helix whose general point is given parametrically as $(\cos t, \sin t, t)$, $t \in \mathbb{R}$, with the sphere whose cartesian equation is $x^2 + y^2 + z^2 = 4$.

(b) Find all points common to the helices C_1 and C_2, where

$$C_1(t) = (\cos t, \sin t, t), \quad t \in \mathbb{R}, \qquad C_2(s) = (\cos s, s, \sin s), \quad s \in \mathbb{R}.$$

Extra questions for further practice

7. Let $f(x) = \ln(1+x)$.

(a) Calculate the fourth order Taylor polynomial $T_4(x)$ for $f(x)$ centred at 0.

(b) Use Taylor's Theorem to write down a formula for the fourth remainder term $R_4(x)$, and deduce that

$$\frac{x^5}{5(1+x)^5} \le f(x) - T_4(x) \le \frac{x^5}{5} \qquad \text{for all } x > 0.$$

(c) Use the previous part to compute the limit

$$\lim_{x\to 0^+} \frac{12\ln(1+x) - 12x + 6x^2 - 4x^3 + 3x^4}{\sin^5 x}$$

8. Show that the curve C with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$, $t \in \mathbb{R}$, passes through $(1, 4, 0)$ and $(9, -8, 28)$ but not $(4, 7, -6)$.

9. Sketch the curves in $\mathbb{R}^3$ given by the following parametric equations.

(a) $x = t\cos t,\ y = t\sin t,\ z = 5t, \quad t \in [0, 100]$.

(b) $x = 2\cos t,\ y = \sin t,\ z = e^{-t}, \quad t \ge 0$.

10. Suppose that f has at least $n+1$ derivatives in a neighbourhood of x_0, and that $f^{(n+1)}$ is bounded near x_0. Let T_n be the n-th order Taylor theorem about x_0 and let R_n be the remainder term. We know that

$$\lim_{x\to x_0} \frac{f(x) - T_n(x)}{(x-x_0)^n} = \lim_{x\to x_0} \frac{R_n(x)}{(x-x_0)^n} = 0.$$

Show that T_n is the only polynomial with the above property, that is, if $P(x)$ is a polynomial of at most degree n and

$$\lim_{x\to x_0} \frac{f(x) - P(x)}{(x-x_0)^n} = 0,$$

then $P = T_n$. Proceed as follows:

(a) Show that $\displaystyle\lim_{x\to x_0} \frac{T_n(x) - P(x)}{(x-x_0)^k} = 0$ for every $k = 0, 1, \dots, n$.

(b) Setting $Q(x) := T_n(x) - P(x)$ and writing it as a polynomial in $(x - x_0)$, use induction by k to show that the coefficient a_k of x^k is zero for every $k = 0, \dots, n$. Conclude that $T_n = P$.

Challenge questions (optional)

11. The *error function* from probability and statistics is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Show that for each $n \ge 1$ and $x \ge 0$ we have

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)k!} x^{2k+1} + E_n(x) \quad \text{with} \quad |E_n(x)| \le \frac{2x^{2n+3}}{\sqrt{\pi}(2n+3)(n+1)!},$$

and hence find a numerical approximation α to $\operatorname{erf}(1)$ such that

$$|\operatorname{erf}(1) - \alpha| < \frac{1}{1000}.$$

12. Use Question 1 to prove that $\sin(1)$ is irrational. Show also that $\cos(1)$ is irrational.

13. There are limitations to Taylor polynomials. This question describes a function whose Taylor polynomials are all identically zero. That is, $T_n(x) = 0$ for all n, and so $R_n(x) = f(x) - T_n(x) = f(x)$. Thus this function is "all remainder"!

Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show, by induction, that for all $n \geq 0$ and for $x \neq 0$,

$$f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$$

where $P_n(t)$ is a polynomial with integer coefficients.

(b) Show that $\lim_{x \to 0} |x|^{-k} e^{-1/x^2} = 0$ for all integers $k \geq 0$.

(c) Show, by induction, that $f(x)$ is differentiable as many times as we please at the point $x = 0$, and that $f^{(n)}(0) = 0$. Thus the nth order Taylor polynomial of $f(x)$ centred at $x = 0$ is identically zero.