MATH1903/1907 Lectures

Week 10, Semester 2, 2017

Daniel Daners

Linear 2nd order DE's with constant coefficients

$$ay'' + by' + cy = f(t) \qquad (t \in \mathbb{R}\ ,\ a, b, c \text{ constants})$$

↑ inhomogeneity

Corresponding homogeneous equation

$$ay'' + by' + cy = 0$$

Equation is linear in y, y', y''

$$\boxed{\text{Superposition principle: If } u, v \text{ are solutions, then every linear combination } Au + Bv \text{ is a solution } (A, B \text{ constants})}$$

$$a(Au+Bv)'' + b(Au+Bv)' + c(Au+Bv)$$
$$= a(Au'' + Bv'') + b(Au' + Bv') + c(Au + Bv)$$
$$= A\underbrace{(au'' + bu' + cu)}_{=0\ (u,v \text{ are solutions})} + B\underbrace{(av'' + bv' + cv)}_{=0} = 0$$

Try to find a solution of the form

$$e^{\lambda t}, \quad \lambda \text{ to be determined.}$$

Substitute into the equation:

$$a(e^{\lambda t})'' + b(e^{\lambda t})' + ce^{\lambda t} \overset{\text{want}}{=} 0$$

$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0$$

As $e^{\lambda t} \neq 0$ we can divide the equation by $e^{\lambda t}$:

$$\boxed{a\lambda^2 + b\lambda + c = 0}$$ "auxiliary equation"

This equation determines all values of λ so that $e^{\lambda t}$ is a solution.

To obtain the auxiliary equation replace $y^{(k)}$ by λ^k

Example

$y'' + 3y' - 10y = 0$

auxiliary equation: $\lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2) = 0$

Solutions: $\lambda_1 = -5$, $\lambda_2 = 2$

Solutions of de: $e^{\lambda_1 t} = e^{-5t}$ and $e^{\lambda_2 t} = e^{2t}$

General solution is a superposition (linear combination)

$$y(t) = Ae^{-5t} + Be^{2t}$$

This is a second order equation, so we need two initial conditions:

$$y(0) = 1, \quad y'(0) = -1$$

$$y(0) = Ae^{-5\cdot 0} + Be^{2\cdot 0} = 1 = A + B$$

$$y'(t) = -5Ae^{-5t} + 2Be^{2t}$$

$$y'(0) = -5A + 2B = -1$$

We obtain a system of two equations for A, B:

$$A + B = 1$$
$$-5A + 2B = -1$$

Linear system. We can write it as

$$\begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Use method from linear algebra to solve the system.

Row reduce augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -5 & 2 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 5R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{R_2}{7}} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{4}{7} \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{7} \\ 0 & 1 & \frac{4}{7} \end{array}\right]$$

Hence $A = \frac{3}{7}$, $B = \frac{4}{7}$

Solution $y(t) = \frac{3}{7} e^{-5t} + \frac{4}{7} e^{2t}$

<u>Alternative solution:</u> solve for A, B using substitution method.

Example:

$$y'' - 4y' + 13y = 0$$

auxiliary equation: $\lambda^2 - 4\lambda + 13 = 0$

$$\lambda = \frac{1}{2}\left(4 \pm \sqrt{4^2 - 4 \times 13}\right)$$

$$= 2 \pm \sqrt{4 - 13} = 2 \pm \sqrt{-9} = 2 \pm 3i$$

Pair of complex conjugate roots $\lambda = 2 \pm 3i$

$e^{(2+3i)t}$, $e^{(2-3i)t}$ are complex valued!

How to differentiate complex valued functions?

$$f : \mathbb{R} \to \mathbb{C}$$

$f(t) = u(t) + i\,v(t)$, u, v real valued.

$f'(t) = u'(t) + i\,v'(t)$

Exponential from before:

$$f(t) = e^{(2+3i)t} = e^{2t} e^{3it} = e^{2t}(\cos 3t + i \sin 3t)$$

$$= \underbrace{e^{2t}\cos 3t}_{u(t)} + i\, \underbrace{e^{2t}\sin 3t}_{v(t)}$$

Differentiate:

$$f'(t) = u'(t) + i v'(t)$$
$$= 2e^{2t}\cos 3t - 3e^{2t}\sin 3t$$
$$\quad + i\left(2e^{2t}\sin 3t + 3e^{2t}\cos 3t\right)$$
$$= (2+3i)\left(e^{2t}\cos 3t + i e^{2t}\sin 3t\right)$$
$$= (2+3i)\, e^{(2+3i)t}$$

Generalising to arbitrary $\lambda \in \mathbb{C}$ we get

$$\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t} \quad (\text{similar to } \lambda \in \mathbb{R})$$

Consequence: Even if the auxiliary equation has complex roots λ, then $e^{\lambda t}$ solves the differential equation

In our example the roots were $\lambda_1 = 2+3i$, $\lambda_2 = 2-3i$

Hence

$$e^{(2+3i)t} \quad \text{and} \quad e^{(2-3i)t}$$

are solutions of the d.e.

By the superposition principle we can obtain other solutions:

$$\frac{1}{2}e^{(2+3i)t} + \frac{1}{2}e^{(2-3i)t} = e^{2t}\cos 3t$$

(real part of complex solution)

$$\frac{1}{2i}e^{(2+3i)t} - \frac{1}{2i}e^{(2-3i)t} = e^{2t}\sin 3t$$

(imaginary part of complex solution)

Hence we have the two real solutions

$$e^{2t}\cos 3t \quad \text{and} \quad e^{2t}\sin 3t$$

General solution $y(t) = Ae^{2t}\cos 3t + Be^{2t}\sin 3t$.

General situation:

If $\lambda = \mu \pm i\omega$ is a pair of complex conjugate solutions of the auxiliary equation of

$$ay'' + by' + cy = 0,$$

then

$$\begin{aligned} y(t) &= Ae^{\mu t}\cos \omega t + Be^{\mu t}\sin \omega t \\ &= e^{\mu t}(A\cos \omega t + B\sin \omega t) \end{aligned}$$

is the (real form) of the general solution.

Example:

$$y'' - 4y' + 4y = 0$$

auxiliary equation: $\lambda^2 - 4\lambda + 4 = 0$

$$(\lambda - 2)^2 = 0$$

Double root $\lambda_1 = \lambda_2 = 2$, so our method only provides one solution

$$e^{2t}.$$

For a second solution, try $y(t) = \underline{t}\, e^{2t}$

$$y'(t) = e^{2t} + 2t e^{2t}$$

$$y''(t) = 2e^{2t} + 2e^{2t} + 4te^{2t} = 4e^{2t} + 4te^{2t}$$

Substitute into de:

$$\begin{aligned} y'' - 4y' + 4y &= (4e^{2t} + 4te^{2t}) - 4(e^{2t} + 2te^{2t}) + 4te^{2t} \\ &= e^{2t}(\cancel{4} - \cancel{4}) + te^{2t}(\cancel{4} - \cancel{8} + \cancel{4}) = 0 \end{aligned}$$

The general solution is

$$y(t) = Ae^{2t} + Bte^{2t} = e^{2t}(A + Bt)$$

Summary

D.E. $ay'' + by' + cy = 0$ (a, b, c constants)

auxiliary equation: $a\lambda^2 + b\lambda + c = 0$

Case 1: λ_1, λ_2 two distinct real roots

general solution: $y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

Case 2: $\mu \pm i\omega$ pair of complex conjugate roots

general solution: $y(t) = e^{\mu t}(A\cos\omega t + B\sin\omega t)$

Case 3: $\lambda_1 = \lambda_2 = \lambda$ double root (real)

general solution: $y(t) = e^{\lambda t}(A + Bt)$

Inhomogeneous second order equations

$$ay'' + by' + cy = f(t)$$

$\uparrow$ inhomogeneity or "forcing term"

Assume:

- $y_p(t)$ is one (arbitrary) particular solution
- $y_h(t)$ general solution of the corresponding homogeneous equation $ay'' + by' + cy = 0$

Then

$$y(t) = y_p(t) + y_h(t)$$

is the general solution of the inhomogeneous equation.

Verify this:

$$\begin{aligned}
& a(y_p + y_h)'' + b(y_p + y_h)' + c(y_p + y_h) \\
&= a(y_p'' + y_h'') + b(y_p' + y_h') + c(y_p + y_h) \\
&= \underbrace{(ay_p'' + by_p' + cy_p)}_{= f(t)} + \underbrace{(ay_h'' + by_h' + cy_h)}_{= 0} \\
&= f(t)
\end{aligned}$$

General solution of $ay'' + by' + cy = f(t)$:

$$y(t) = y_p(t) + y_h(t)$$

where

- y_p is a particular solution of the inhomogeneous eq.
- y_h is the general solution of the homogeneous eq.

<u>Question</u>: How to find a particular solution of the inhomogeneous equation?

<u>Example</u>:

$y'' - 5y' + 4y = t^2$

Since derivatives of polynomials are polynomials, we try a polynomial:

$$y_p(t) = A + Bx + Cx^2$$