The University of Sydney
School of Mathematics and Statistics

Solutions to Quiz 2: Practice Questions

MATH1901/1906: Differential Calculus (Advanced) Semester 1, 2017

Some information for the Quiz:

(1) The quiz covers material up to (and including) week 10 lectures. That is, material from the tutorials in weeks 2–11. The focus is primarily on the material from the lectures in weeks 6–10 (tutorial weeks 7–11), however you need to know the concepts from the earlier weeks too.

(2) The quiz runs for 40 minutes.

(3) You must write your answers in pen, not pencil.

(4) The format of the real quiz is mostly "short-answer" questions. There will be answer boxes provided below each question where you should write your final answers. Your working may be considered, so please write neatly. The format is similar to Quiz 1.

(5) There will be 10 questions in the real quiz, each worth the same amount.

(6) The quiz is a closed book examination. No notes or books are allowed.

(7) Non-programmable non-graphics calculators are allowed, but are not needed.

The questions provided here are for additional practice (building from the lectures and tutorials). It is strongly advised that you also revise the material from lectures and tutorials.

1. Find the third order Taylor polynomial $T_3(x)$ centred at $x = 0$ for the function $f(x) = \sqrt{x+1}$, and use Taylor's Theorem to find a formula for the remainder $R_3(x) = f(x) - T_3(x)$.

Solution: Very similar questions are done in tutorials – see those solutions.

2. Let $T_{2n}(x)$ be the Taylor polynomial of the function $f(x) = \cosh x$ of order $2n$ centred at $x = 0$, and let $R_{2n}(x) = f(x) - T_{2n}(x)$ be the corresponding remainder term. According to Taylor's Theorem, there is a number c between 0 and x such that (circle the correct answer):

(a) $R_{2n}(x) = \dfrac{x^{2n}}{(2n)!} \sinh c$

(b) $R_{2n}(x) = \dfrac{x^{2n+1}}{(2n+1)!} \sinh c$

(c) $R_{2n}(x) = \dfrac{c^{2n+1}}{(2n+1)!} \sinh c$

(d) $R_{2n}(x) = \dfrac{c^{2n}}{(2n)!} \sinh c$

Solution: The correct answer is (b).

3. You are given that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, and that $f(1) = 3$ and $-2 \le f'(x) \le 4$ for all $x \in \mathbb{R}$. Which one of the following statements is necessarily true?

(a) $-1 \le f(3) \le 11$

(b) $-3 \le f(3) \le 6$

(c) $3 \le f(3) \le 13$

(d) $5 \le f(3) \le 14$

Solution: By the MVT there is $c \in (1,3)$ such that $f'(c) = (f(3) - f(1))/(3-1)$. Since $f(1) = 3$ we have $f'(c) = (f(3) - 3)/2$, and so $f(3) = 2f'(c) + 3$. Since $-2 \le f'(c) \le 4$ we have $-1 \le f(3) \le 11$, and so (a) is the correct answer.

4. Calculate the following limits

(a) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3}$

Solution: By L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3} &= \lim_{x\to 0} \frac{(1+x^2)^{-1} - 1}{3x^2} \\ &= \lim_{x\to 0} \frac{-x^2}{3x^2} \\ &= -\frac{1}{3}.\end{aligned}$$

(b) $\displaystyle\lim_{x\to 0} (\cos x)^{1/x^2}$

Solution: Write $(\cos x)^{1/x^2} = e^{\ln(\cos x)/x^2}$. Now,

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x \cos x} = -\frac{1}{2}\left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \cos x\right) = -\frac{1}{2},$$

where we used L'Hôpital's Rule. Thus

$$\lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2} = 1/\sqrt{e}.$$

5. Let $f : (-1, \infty) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions

$$f(x) = |x| \ln(1+x) \qquad \text{and} \qquad g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$$

Which one of the following statements is true?

(a) Neither $f(x)$ nor $g(x)$ are differentiable at $x = 0$.

(b) Both $f(x)$ and $g(x)$ are differentiable once and only once at $x = 0$.

(c) $f''(x)$ is continuous at $x = 0$, and $g'(0) = 0$.

(d) $f'(x)$ is continuous at $x = 0$, and $g''(0) = 2$.

Solution: The correct answer is (b).

To see that $f(x)$ is differentiable at $x = 0$, note that

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| \ln(1+h)}{h} = 0,$$

where we have used the squeeze law. In a similar way you can check that $g(x)$ is differentiable at $x = 0$. We claim that $f(x)$ is not differentiable twice at $x = 0$. We have

$$f'(x) = \begin{cases} \ln(1+x) + \frac{x}{1+x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\ln(1+x) - \frac{x}{1+x} & \text{if } x < 0. \end{cases}$$

Therefore

$$\begin{aligned} \lim_{h \to 0^+} \frac{f'(h) - f'(0)}{h} &= \lim_{h \to 0^+} \frac{\ln(1+h) + \frac{h}{1+h}}{h} \\ &= \lim_{h \to 0^+} \left(\frac{\ln(1+h)}{h} + \frac{1}{1+h} \right) \\ &= \lim_{h \to 0^+} \frac{\ln(1+h)}{h} + \lim_{h \to 0^+} \frac{1}{1+h} \\ &= \lim_{h \to 0^+} \frac{1}{1+h} + 1 \\ &= 2, \end{aligned}$$

where we have used L'Hôpital's Rule. A similar calculation shows that

$$\lim_{h \to 0^-} \frac{f'(h) - f'(0)}{h} = -2,$$

and so $f''(0)$ does not exist. Similar (and easier) considerations show that $g''(0)$ does not exist. This shows that (b) is true, and eliminates the possibilities (c) and (d).

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the real valued function of 2 real variables given by the formula

$$f(x, y) = \frac{\sin y}{\cosh(x - y^2)}.$$

Which one of the following statements is true?

(a) f is injective, and the range of f is $[-1, 1]$.

(b) f is not injective, and the range of f is $[-1, 1]$.

(c) f is injective, and the range of f is $\mathbb{R}$.

(d) f is not injective, and the range of f is $\mathbb{R}$.

Solution: This function is certainly not injective, because $f(x,0) = 0$ for all $x \in \mathbb{R}$. Note that

$$|f(x,y)| = \left|\frac{\sin y}{\cosh(x-y^2)}\right| \leq \frac{1}{\cosh(x-y^2)} \leq 1,$$

because $|\sin y| \leq 1$ and $\cosh(x-y^2)| \geq 1$. This shows that range$(f) \subseteq [-1,1]$. We have $f(y^2,y) = \sin y$, which as y varies takes all values between -1 and 1, and hence $[-1,1] \subseteq$ range(f). Thus range$(f) = [-1,1]$, and so the correct answer is (b).

7. Let C_1 and C_2 be the curves in $\mathbb{R}^3$ with parametric equations

$$C_1(t) = (1+t, 1+2t^2, 2+2t^2) \qquad t \in \mathbb{R}$$
$$C_2(s) = (1+s, 3-2s^2, 4-2s^2) \qquad s \in \mathbb{R}.$$

Which one of the following statements is true?

(a) The curves C_1 and C_2 are both contained in a common plane.

(b) There is no plane containing both of the curves C_1 and C_2.

(c) The curves C_1 and C_2 do not intersect.

(d) The curves C_1 and C_2 intersect at exactly one point.

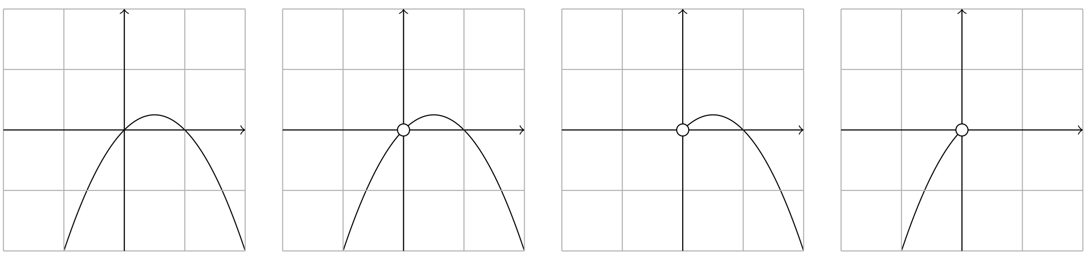
Solution: These curves both lie on the plane $y - z = -1$, because $(1+2t^2) - (2+2t^2) = -1$ and $(3-2s^2) - (4-2s^2) = -1$. So (a) is correct.

To check that (c) and (d) are false, note that $C_1(t) = C_2(s)$ if and only if $1+t = 1+s$ and $1+2t^2 = 3-2s^2$ and $2+2t^2 = 4-2s^2$, and these equations hold if and only if $s = t$ and $1+2t^2 = 3-2t^2$ and $2+2t^2 = 4-2t^2$, and these equations hold if and only if $s = t = \pm 1/\sqrt{2}$. Therefore there are two intersection points.

8. Let

$$f(x,y) = \frac{x}{\sqrt{x-y}}.$$

Which of the following best represents the level curve of $f(x,y)$ of height $z = 1$?



(a) (b) (c) (d)

Solution: The domain of the function is $\{(x,y) \mid y < x\}$. The level curve at height $z = 1$ consists of those points in the domain satisfying $x = \sqrt{x-y}$, that is, $x^2 = x - y$ with $y > x$ and $x \geq 0$ (we need to include $x \geq 0$ here because $x = \sqrt{x-y} \geq 0$, and we loose this information when we square the equation). So $y = x(1-x)$ with $y < x$ and $x \geq 0$, and so (c) is the correct answer.

9. For each of the following cases, write down the Taylor polynomial $T_n(x)$ of order n centred at $x = a$ for the functions $f(x)$, and use Taylor's Theorem to write down a formula for the associated remainder term $R_n(x)$.

(a) $f(x) = \cos x$, $n = 2$, and $a = \pi/4$.

Solution: We have $f(x) = \cos x$, $f^{(1)}(x) = -\sin x$, and $f^{(2)}(x) = -\cos x$. Therefore $f(\pi/4) = 1/\sqrt{2}$, $f^{(1)}(\pi/4) = -1/\sqrt{2}$, and $f^{(2)}(\pi/4) = -1/\sqrt{2}$. Thus the Taylor polynomial is

$$T_2(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \pi/4) - \frac{1}{2\sqrt{2}}(x - \pi/4)^2.$$

Since $f^{(3)}(x) = \sin x$, Taylor's Theorem says that

$$R_2(x) = \frac{f^{(3)}(c)}{3!}(x - \pi/4)^3 = \frac{\sin c}{6}(x - \pi/4)^3$$

for some number c between $\pi/4$ and x.

(b) $f(x) = \ln(1 + x)$, $n = 3$, and $a = 1$.

Solution: We have $f(x) = \ln(1 + x)$, $f^{(1)}(x) = (1 + x)^{-1}$, $f^{(2)}(x) = -(1 + x)^{-2}$, and $f^{(3)}(x) = 2(1 + x)^{-3}$. Thus $f(1) = \ln 2$, $f^{(1)}(1) = 1/2$, $f^{(2)}(1) = -1/4$, and $f^{(3)}(1) = 1/4$. Thus

$$T_3(x) = \ln 2 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{24}(x - 1)^3.$$

Since $f^{(4)}(x) = -6(1 + x)^{-4}$, we have

$$R_3(x) = -\frac{(x - 1)^4}{4(1 + c)^4}$$

for some c between 1 and x.

10. Let $f(x, y) = \ln(1 - \sqrt{x + y})$. Find the natural domain and the corresponding range of f.

Solution: The rule of f makes sense provided that $x + y \geq 0$ and $1 - \sqrt{x + y} > 0$. Thus $y \geq -x$ and $\sqrt{x + y} < 1$, and so $y \geq -x$ and $x + y < 1$. Thus the natural domain is

$$\{(x, y) \in \mathbb{R}^2 \mid -x \leq y < 1 - x\}.$$

The corresponding range is $(-\infty, 0]$. To see this, note that for all (x, y) in the domain of f,

$$0 < 1 - \sqrt{x + y} \leq 1,$$

and thus $-\infty < \ln(1 - \sqrt{x + y}) \leq 0$. Moreover, each point $(x, y) = (x, a - x)$ with $0 \leq a < 1$ is in the domain of f, and $f(x, a - x) = \ln(1 - \sqrt{a})$. Thus we see that by moving $a \in [0, 1)$ we can obtain all values between $-\infty$ and 0 (including 0), and hence the range is $(-\infty, 0]$.

11. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = xg(x)$.

(a) Show that f is differentiable at $x = 0$, with $f'(0) = g(0)$.

Solution: We have

$$\lim_{h\to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h\to 0} \frac{hg(h) - 0}{h} = \lim_{h\to 0} g(h).$$

Thus, since $g(x)$ is continuous, we have

$$\lim_{h\to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h\to 0} g(h) = g(0),$$

and so f is differentiable at $x = 0$, with $f'(0) = g(0)$.

(b) If g is no longer assumed to be continuous, is it still true that f is differentiable at 0?

Solution: No. For example, take $g(x) = 1$ if $x \geq 1$, and $g(x) = -1$ if $x < 0$. Then $f(x) = xg(x) = |x|$, which is not differentiable at 0.

12. Calculate the following limits, or show that they do not exist.

(a) $\lim_{x\to 1} \dfrac{x^3 - 5x + 4}{x^3 - 4x + 3}$

Solution: As $x \to 1$ both the numerator and the denominator tend to 0, and so we can apply L'Hôpital's Rule. Thus

$$\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3} = \lim_{x\to 1} \frac{3x^2 - 5}{3x^2 - 4} = \frac{-2}{-1} = 2.$$

(b) $\lim_{x\to 0^+} (\sinh x)^{1/x}$

Solution: We write

$$\lim_{x\to 0^+} (\sinh x)^{1/x} = \lim_{x\to 0^+} e^{\frac{\ln(\sinh x)}{x}}.$$

As $x \to 0^+$ we have $\sinh x \to 0^+$, and so $\ln(\sinh x) \to -\infty$ (note that this is not a L'Hôpital's Rule question!). Thus

$$\lim_{x\to 0^+} (\sinh x)^{1/x} = 0.$$

(c) $\lim_{x\to\infty} (\sinh x)^{1/x}$

Solution: In a similar way to the previous question, we have

$$\lim_{x\to\infty} (\sinh x)^{1/x} = e^{\lim_{x\to\infty} \frac{\ln(\sinh x)}{x}}.$$

This time the limit in the exponent is of type ∞/∞, and so we may apply L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln(\sinh x)}{x} = \lim_{x\to\infty} \frac{\cosh x}{\sinh x}.$$

If we try to compute this last limit using L'Hôpital's Rule we will go around and around in circles. Instead, using the definition of $\cosh x$ and $\sinh x$ we have

$$\lim_{x\to\infty} \frac{\cosh x}{\sinh x} = \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x\to\infty} \frac{1 + e^{-2x}}{1 - e^{-2x}} = 1.$$

Therefore

$$\lim_{x\to\infty} (\sinh x)^{1/x} = e^1 = e.$$

(d) $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$

Solution: The numerator and denominator both tend to 0 as $x \to 0$, and so it is appropriate to try L'Hôpitals Rule:

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}.$$

13. Calculate the following limits, or show that they do not exist.

(a) $\lim_{(x,y)\to(0,0)} \dfrac{x^3 + 2y^3}{x^2 + y^2}$

Solution: Put $(x, y) = (r\cos\theta, r\sin\theta)$, and note that $(x, y) \to (0, 0)$ if and only if $r \to 0^+$. Then

$$0 \le \left|\frac{x^3 + 2y^3}{x^2 + y^2}\right| = \left|\frac{r^3(\cos^3\theta + 2\sin^3\theta)}{r^2}\right| = r|\cos^3\theta + 2\sin^3\theta| \le 3r.$$

Since $3r \to 0$ as $r \to 0^+$, the Squeeze Law implies that

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + 2y^3}{x^2 + y^2} = 0.$$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$

Solution: Along the path of approach $y = 0$, we have

$$f(x,0) = \frac{\sin(0)}{x^2+0^2} = 0,$$

and so $f(x,y) \to 0$ along the path $y = 0$. On the other hand, approaching along the path $y = x$ we have

$$f(x,x) = \frac{\sin(x^2)}{2x^2},$$

and by L'Hôpital's Rule

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{x\to 0} \frac{2x\cos(x^2)}{2x} = 1.$$

Therefore $f(x,y) \to 1$ along the path $y = x$, and thus the limit does not exist.

(c) $\lim_{(x,y)\to(\pi,0)} \dfrac{y^2+\cos x}{x^2+y^2}$

Solution: By the limit laws we compute

$$\lim_{(x,y)\to(\pi,0)} \frac{y^2+\cos x}{x^2+y^2} = \frac{0^2+\cos\pi}{\pi^2+0^2} = -\frac{1}{\pi^2}.$$

(d) $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$.

Solution: Along the path $y = x$ we have

$$f(x,x) = \frac{x^3}{x^4+x^2} = \frac{x}{x^2+1} \to 0.$$

Along the path $y = x^2$ we have

$$f(x,x^2) = \frac{x^4}{x^4+x^4} = \frac{1}{2} \to \frac{1}{2}.$$

Therefore the limit does not exist.

14. Draw the level curves of heights $c = 0, 1, 2$ of the functions:

(a) $f(x,y) = e^{x^2+y^2}$

Solution: The level curve of height c is $f(x,y) = c$. Thus $e^{x^2+y^2} = c$. If $c < 1$ then the level curve is empty (since $e^t \geq 1$ for all $t \geq 0$). If $c \geq 1$ then the level curve is $x^2+y^2 = \ln(c)$. Thus the level curves of heights $c = 1, 2$ are circles centred at $(0,0)$ of radius $0, \ln 2$ (respectively). [The picture is omitted from these brief solutions]

(b) $g(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

Solution: The domain of g is $\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$.
If $g(x,y) = c$ then $x^2-y^2 = c(x^2+y^2)$, and so $(c-1)x^2+(c+1)y^2 = 0$.
If $c = 0$ then $-x^2+y^2 = 0$, and so $y = \pm x$. Thus the level curve consists of the lines $y = x$ and $y = -x$, with the point $(x,y) = (0,0)$ removed (because this point is not in the domain of $g(x,y)$).
If $c = 1$ then $2y^2 = 0$, and so $y = 0$, and so the level curve is the x-axis, again with the point $(x,y) = (0,0)$ removed since this point is not in the domain of $g(x,y)$.
If $c = 2$ then $x^2+3y^2 = 0$, and so $x = y = 0$ (because $x^2+3y^2 \geq 0$). However $(x,y) = (0,0)$ is not in the domain of $g(x,y)$, and so the level curve is empty.
[The pictures are omitted from these brief solutions]

15. Suppose that f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Which of the following statements are true, and which are false?

(a) There exists a point $c \in (0, 1)$ such that $f'(c) = 0$.

Solution: False: For example, $f(x) = x$. (If the extra assumption $f(0) = f(1)$ is included, then the result is true, by Rolle's Theorem).

(b) There exists a point $c \in (0, 1)$ such that $f'(c) = f(1) - f(0)$.

Solution: True, by The Mean Value Theorem on the interval $[0, 1]$.

(c) There exists a unique point $c \in (0, 1)$ such that $f'(c) = f(1) - f(0)$.

Solution: False: There could be many such points. For example, $f(x) = \sin(2\pi x)$ has $f'(1/4) = 0 = f(1) - f(0)$ and $f'(3/4) = 0 = f(1) - f(0)$.

(d) If $f'(x) = 0$ for all $x \in (0, 1)$ then f is a constant function on $[0, 1]$.

Solution: True: By a corollary to the Mean Value Theorem if $f'(x) = 0$ on $(0, 1)$ then $f(x)$ is constant on $(0, 1)$. Then since f is continuous on $[0, 1]$ it follows that f is constant on $[0, 1]$.

16. Let $f(x) = x^3 - x$. Find all points $c \in (-2, 2)$ that satisfy the conclusion of the Mean Value Theorem when it is applied to f on the interval $[-2, 2]$.

Solution: We need to find the points $c \in (-2, 2)$ which satisfy

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)}.$$

That is,

$$3c^2 - 1 = \frac{6 + 6}{4} = 3.$$

Thus $c^2 = 4/3$, and so $c = \pm 2/\sqrt{3}$.

17. Let $f(x, y) = x^2 + y^2 + 1$ and consider the surface $z = f(x, y)$. Which of the following is true?

(a) This surface intersects the xz-plane in a straight line.

Solution: False: The xz-plane is the plane $y = 0$, and thus $z = x^2 + 0^2 + 1 = x^2 + 1$. This is a parabola, not a straight line.

(b) This surface intersects the plane $x = 3$ in a circle.

Solution: False: If $x = 3$ then $z = 3^2 + y^2 + 1 = 10 + y^2$. This is a parabola, not a circle.

(c) Every point on the curve $C(t) = (t \cos t, t \sin t, t^2 + 1)$ lies on this surface.

Solution: True: Each point $(x, y, z) = (t \cos t, t \sin t, t^2 + 1)$ satisfies the equation $z = x^2 + y^2 + 1$, and thus every point of the curve lies on the surface.

(d) For all $c > 1$, the level curve of this surface at height $z = c$ is a parabola.

Solution: False: If $z = c > 1$ then the level curve is $x^2 + y^2 + 1 = c$, and so $x^2 + y^2 = c - 1$. This is a circle, not a parabola.