Math2221
Higher Theory and Applications of Differential Equations

Tutorial Problem Set 1

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Standard Questions

Before the tutorial, you should at least attempt the recommended questions (or parts of questions), which are numbered in **bold**. Answers and hints (including some with working) are given at the end.

1. (Revision) Classify each of the following first-order ODEs as separable, linear, exact or homogeneous and then find the general solution.

   (i) $4xy' = 5y$

   (ii) $(x+1)^2 y' + 3(x+1)y = 4$

   (iii) $(x^2+y^2)y' + 2xy = 0$

   (iv) $2xy\,dx + x^2\,dy = 0$

2. (Revision) Find the general solution of each of the following second-order ODEs.

   (i) $u'' - 4u = 0$

   (ii) $u'' + 4u = 0$

   (iii) $u'' + 2u' + 2u = 0$

   (iv) $u'' + 6u' + 9u = 0$

3. (Revision) Solve each of the following initial value problems.

   (i) $\dfrac{dx}{dt} = 3t^2x^2$ with $x(0) = 1$.

   (ii) $t\,\dfrac{dx}{dt} = x + \sqrt{t^2+x^2}$ with $x(1) = 0$.

   (iii) $2xe^y + e^x + (x^2+1)e^y\,\dfrac{dy}{dx} = 0$ with $y = 0$ when $x = 0$.

   (iv) $\dfrac{dy}{d\theta} + y\,\cos\theta = \frac{1}{2}\sin 2\theta$ with $y(\pi/2) = 4$.

   (v) $\ddot{x} + 4x = \sin 2t$ with $x = 0$ and $\dot{x} = 1$ when $t = 0$. (The dot means d/dt.)

   (vi) $u'' - 5u' + 4u = 2e^{2x}$ with $u = 1$ and $u' = 3$ at $x = 0$.

4. (Revision) Find the general solution to each of the following inhomogeneous ODEs.

   (i) $u'' + 4u = 8x^2 + 13e^{3x} + 16\cos 2x$

   (ii) $u'' + 2u' + 5u = 12e^{-x}\cos x$

In both cases, solve the initial-value problem for the ODE with initial conditions $u(0) = 0$ and $u'(0) = 0$.

5. Verify that $u_1 = \sin x^2$ is a solution of

$$xu'' - u' + 4x^3 u = 0$$

and use *reduction of order* to find a second, linearly independent solution.

6. Verify that $u_1 = x + 1$ is a solution of

$$xu'' - (x + 1)u' + u = 0$$

and use reduction order to find the general solution.

7. For each of the following polynomials $p(z)$, write down the general solution of the homogeneous equation $p(D)u = 0$.

   (i) $p(z) = z^3 - z^2 + z - 1$

   (ii) $p(z) = z^4 - 2z^3 - 2z^2 + 8 = (z - 2)^2(z^2 + 2z + 2)$

8. State whether the following are true or false.

   (i) If the Wronskian of a set of functions is identically zero, the functions must be linearly dependent.

   (ii) If a set of functions is linearly dependent, the Wronskian must be identically zero.

   (iii) If the Wronskian of a set of functions is zero at some points and not zero at some points, the functions are linearly independent.

9. Compute the Wronskian of the set of functions

$y_1(x) := x^2 - x, \qquad y_2(x) := x^2 + x, \qquad y_3(x) := x^2.$

Is the set of functions linearly dependent or linearly independent?

10. For each of the following ODEs, write down the linear, first-order ODE satisfied by the Wronskian W of any two, linearly independent solutions, and hence find the form of W.

   (i) $(1-x^2)u'' - 2xu' + \nu(\nu+1)u = 0$ (Legendre's equation of order ν).

   (ii) $x^2u'' + xu' + x^2u = 0$ (Bessel's equation of order 0).

Advanced Questions

The following questions are more challenging. In some cases they may require you to do additional research and/or draw on material related to non-prerequisite courses.

11. Prove that if the polynomial $p(z)$ can be factorised as $p(z) = q(z)r(z)$ then $p(D) = q(D)r(D) = r(D)q(D)$.

12. Suppose that L and M are linear differential operators with orders r and s, respectively,

$$Lu(x) = \sum_{j=0}^{r} a_j(x)\, D^j u \quad \text{and} \quad Mu(x) = \sum_{k=0}^{s} b_k(x)\, D^k u.$$

   (i) Use induction on r to show that LM is a linear differential operator of order $r + s$.

   (ii) What is the leading coefficient of LM?

   (iii) Deduce that the *commutator* $LM - ML$ is a linear differential operator of order $r + s - 1$.

13. Suppose that the polynomial $p(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ has distinct zeros λ_1, λ_2, ..., λ_m. Thus, $u_j(x) = e^{\lambda_j x}$ is a solution of the homogeneous equation $p(D)u = 0$, for $j = 1$, 2, ..., m.

   (i) Show that the Wronskian of u_1, u_2, ..., u_m is

$$W(x) = e^{(\lambda_1+\lambda_2+\cdots+\lambda_m)x} V(\lambda_1, \lambda_2, \ldots, \lambda_m)$$

   where V is the *Vandermonde determinant*:

$$V(\lambda_1, \lambda_2, \ldots, \lambda_m) = \det[\lambda_i^{j-1}] = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} \end{vmatrix}.$$

   (ii) Use column operations to show that

$$V(\lambda_1, \lambda_2, \cdots, \lambda_m) = V(\lambda_2, \ldots, \lambda_m) \prod_{i=2}^{m} (\lambda_i - \lambda_1).$$

   (iii) Deduce that

$$V(\lambda_1, \lambda_2, \ldots, \lambda_m) = \prod_{1 \le j < i \le m} (\lambda_i - \lambda_j)$$

   and that u_1, u_2, ..., u_m are linearly independent.

Answers and Hints

1. (i) linear, separable, $y^4 = Cx^5$ (ii) linear, $y = \dfrac{2}{x+1} + \dfrac{C}{(x+1)^3}$ (iii) homogeneous, $y(y^2 + 3x^2) = C$ (iv) separable, linear and exact, $x^2 y = C$

2. (i) $u = Ae^{2x} + Be^{-2x}$ (ii) $u = A\cos 2x + B\sin 2x$
(iii) $u = e^{-x}(A\cos x + B\sin x) = Ce^{-x}\sin(x - \phi)$ (iv) $u = e^{-3x}(A + Bx)$

3. (i) $x = 1/(1 - t^3)$ (ii) $x = \frac{1}{2}(t^2 - 1)$ (iii) $(x^2 + 1)e^y + e^x = 2$
(iv) $y = \sin\theta - 1 + 4e^{1-\sin\theta}$ (v) $x = \frac{5}{8}\sin 2t - \frac{1}{4}t\cos 2t$
(vi) $u = e^x + e^{4x} - e^{2x}$

4. (i) $u = A\cos 2x + B\sin 2x + 2x^2 - 1 + e^{3x} + 4x\sin 2x,\ A = 0,\ B = -3/2$
(ii) $u = e^{-x}(A\cos 2x + B\sin 2x + 4\cos x),\ A = -4,\ B = 0$

5. $u = \cos x^2$

6. $u = Ae^x + B(x + 1)$

7. (i) $Ae^x + B\cos x + C\sin x$ (ii) $Ae^{2x} + Bxe^{2x} + Ce^{-x}\cos x + Ee^{-x}\sin x$

8.

(i) False. $W(x) = 0$ for all x is a necessary condition for linear dependence but it is not sufficient. This is not quite obvious, and we give a counter example of two linearly independent functions for which the Wronskian vanishes for all x: Let

$$f_1(x) := x^2, \qquad f_2(x) := x\,|x|.$$

To compute the Wronskian we have to distinguish the two cases (i) $x \geq 0$ and (ii) $x < 0$.

(i) For $x \geq 0$, $f_2(x) = x^2$ and the Wronskian is given by

$$W(x) = \det\begin{pmatrix} x^2 & x^2 \\ 2x & 2x \end{pmatrix} = 2x^3 - 2x^3 = 0.$$

(ii) For $x < 0$, $f_2(x) = -x^2$ and the Wronskian is given by

$$W(x) = \det\begin{pmatrix} x^2 & -x^2 \\ 2x & -2x \end{pmatrix} = -2x^3 + 2x^3 = 0.$$

Clearly the Wronskian always vanishes, but the functions f_1 and f_2 are linearly independent. To prove the linear independence of f_1 and f_2, we write down

$$c_1\, f_1(x) + c_2\, f_2(x) = c_1\, x^2 + c_2\, x\, |x| = 0.$$

For $x = 1$ we obtain $c_1 + c_2 = 0$ and for $x = -1$ we get $c_1 - c_2 = 0$. Thus $c_1 = c_2 = 0$ and the functions are linearly independent.

(ii) True. Because the functions are linearly dependent the columns in the matrix in the definition of the Wronskian are linearly dependent and thus $W(x) = 0$.

(iii) True. If there exists $x \in I$ such that $W(x) \neq 0$ then the functions are linearly independent. (Note that this statement is the negation of the statement in (b).)

9. We compute the Wronskian:

$$W(x) = \det \begin{pmatrix} x^2 - x & x^2 + x & x^2 \\ 2x - 1 & 2x + 1 & 2x \\ 2 & 2 & 2 \end{pmatrix}$$

$$\begin{aligned} W(x) &= (x^2 - x)[2(2x + 1) - 2(2x)] + (x^2 + x)[2(2x) - 2(2x - 1)] + ... \\ &\qquad x^2[2(2x - 1) - 2(2x + 1)] \\ &= 0 \end{aligned} \tag{1}$$

This is a necessary condition for linear dependence. However it is not sufficient! Note that

$$y_3(x) = x^2 = \frac{1}{2}(x^2 - x) + \frac{1}{2}(x^2 + x) = \frac{1}{2}\, y_1(x) + \frac{1}{2}\, y_2(x);$$

so in this case the functions are linearly dependent.

10. (i) $W = C/(1 - x^2)$ for $x \neq \pm 1$. (ii) $W = C/x$
12. (ii) $a_r(x)b_s(x)$

13. (ii) For $j = 2, 3, ..., m$, replace C_j with $C_j - \lambda_1 C_{j-1}$.