Week 3

Definition

A function $f(z) = \xi + i\eta$ is said to be analytic at $z = z_0$ if $\zeta'(z) = \frac{df}{dz}$ exists at $z = z_0$ and at all points of some

open neighbourhood at z_0.

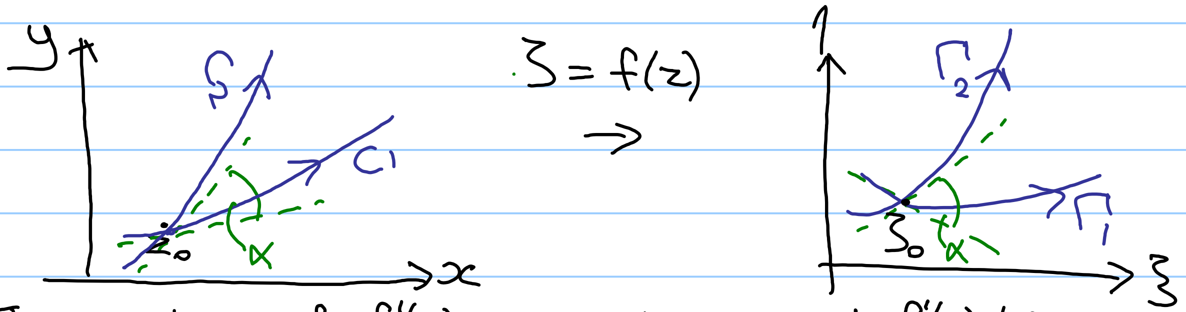
Theorem

$\zeta(x,y)$,

If $f(z)$ is analytic at a point then its real and imaginary components have continuous partial derivatives of all orders at that point.

Conformal transformations

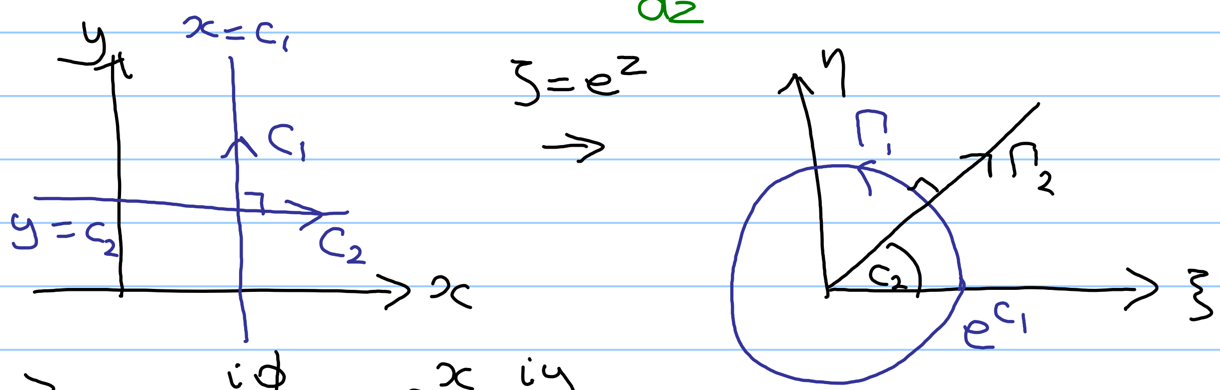
They preserve the angle between line segments



This is true if $f'(z)$ is analytic and $f'(z) \neq 0$.

Example

The transformation $\zeta = e^z$ is conformal in the whole z plane since $\frac{d\zeta}{dz} = e^z \neq 0$



$\zeta = \rho e^{i\phi} = e^x e^{iy}$

If $z = c_1 + iy$, $\zeta = e^{c_1} e^{iy}$

If $z = x + ic_2$, $\zeta = e^x e^{ic_2}$

Thus, if $f(z)$ is analytic the transformation $\zeta = f(z)$ is conformal and a unique inverse transformation can be defined.
The exception is if $f'(z)$ is zero or infinity.
If we want to map a domain whose boundary has corners to a domain with a smooth boundary we must have $f'(z)$ is zero or infinity at the corner.

Definition

A function $\phi(x,y)$ is said to be harmonic in a given domain of the x-y plane if it has continuous partial derivatives (of first and second order) and satisfies the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

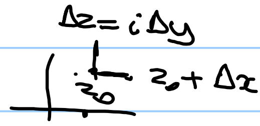
This is known as Laplace's equation.
If $f(z) = \xi + i\eta$ is analytic then ξ and η are harmonic functions.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

Theorem

If $f(z)$ is analytic in a domain D and $f(z) = \xi + i\eta$ then

$$\boxed{f'(z) = \frac{\partial \xi}{\partial x} + i\frac{\partial \eta}{\partial x} = -i\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial y}}$$



Equating real and imaginary parts gives

$$\boxed{(1)\quad \frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y}} \quad \text{and} \quad \boxed{(2)\quad \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x}}$$

These are the Cauchy-Riemann equations

Show ξ and η are harmonic functions:
Differentiate (1) and (2) w.r.t. x

$\xi_{xx} = \eta_{yx}$, $\xi_{yx} = -\eta_{xx}$

Differentiate (1) and (2) w.r.t. y

$\xi_{xy} = \eta_{yy}$, $\xi_{yy} = -\eta_{xy}$

So $\xi_{xx} + \xi_{yy} = 0$ and $\eta_{xx} + \eta_{yy} = 0$.

Laplace's equation

Consider the conformal transformation $\zeta = f(z) = \xi(x,y) + i\eta(x,y)$.
Suppose ϕ satisfies Laplace's equation, i.e.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \qquad \phi(x,y) \rightarrow \phi(\xi,\eta)$$

(Note that since $f(z)$ is analytic ξ and η satisfy the Cauchy–Riemann equations : $\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y}$ and $\frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial x}$.)

$$\phi(x,y) = \phi(x(\xi,\eta), y(\xi,\eta))$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x}, \text{ etc.}$$

The Jacobian of the transformation
$\xi = \xi(x,y)$, $\eta = \eta(x,y)$ is $\qquad \frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y}, \frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial x}$

$$J = \begin{vmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{vmatrix} = \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}$$

$$= \left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2$$

$$= \left(\frac{\partial\eta}{\partial y}\right)^2 + \left(\frac{\partial\eta}{\partial x}\right)^2$$

$$= \left|\frac{d\zeta}{dz}\right|^2 = \left|\frac{df}{dz}\right|^2 \quad \text{since } \left|\frac{df}{dz}\right|^2 = \left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial x}\right)^2 = \left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2$$

etc.

Then $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ becomes $\left(\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2}\right)\left|\frac{d\zeta}{dz}\right|^2 = 0$

(after simplification).

Thus, if $\frac{d\zeta}{dz} \neq 0$ in a domain D then

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} = 0 \text{ in the transformed domain } D^*.$$

So ϕ satisfies Laplace's equation in the transformed domain.