Extended Answer Section

*There are **four** questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

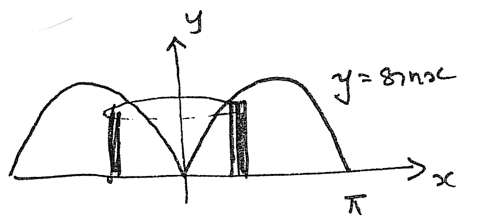
MARKS

1. (a) Calculate the volume of the solid obtained by revolving the region of $\mathbb{R}^2$ bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$ and $y = 0$ about the y-axis. **2**

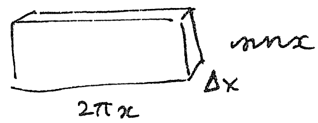
   (b) Calculate the length of the curve in $\mathbb{R}^2$ with parametric equations **3**

$$x(t) = 3t^2 + 2, \qquad y(t) = 4 - t^3, \qquad \text{with } t \in [0, 1].$$

(a)



$\Delta V =$



$= 2\pi x \sin x \, \Delta x$

$V = \int_0^\pi 2\pi x \sin x \, dx \qquad u = x, \ \frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \sin x, \ v = -\cos x$

$= 2\pi\left[-x\cos x \Big|_0^\pi + \int_0^\pi \cos x \, dx\right] = \pi \times 2\pi = 2\pi^2.$

(b) $L = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} \, dt$

$= \int_0^1 \sqrt{(6t)^2 + (3t^2)^2} \, dt$

$= 3 \int_0^1 t\sqrt{4 + t^2} \, dt \qquad u = 4 + t^2, \ \frac{du}{dt} = 2t$

$= \frac{3}{2} \int_4^5 \sqrt{u} \, du$

$= u^{3/2} \Big|_4^5$

$= 5^{3/2} - 4^{3/2}$

$= 5\sqrt{5} - 8$

MARKS

(c) Calculate the value of the improper integral 3

$$\int_0^\infty \frac{1}{(x+1)(x+2)}\,dx.$$

(d) Find $\dfrac{d}{dx}\displaystyle\int_x^{e^x} \ln(1+t^2)\,dt.$ 3

(c) $\dfrac{1}{(x+1)(x+2)} = \dfrac{1}{x+1} - \dfrac{1}{x+2}$

$$\int_0^\infty \frac{1}{(x+1)(x+2)}\,dx = \lim_{b\to\infty} \int_0^b \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$$

$$= \lim_{b\to\infty} \left(\ln(b+1) - \ln(b+2) - \ln 1 + \ln 2\right)$$

$$= \lim_{b\to\infty} \ln\frac{b+1}{b+2} + \ln 2 = \ln 2 + \lim_{b\to\infty} \ln\frac{1+\frac{1}{b}}{1+\frac{2}{b}}$$

$$= \ln 2$$

(d) $$\frac{d}{dx}\int_x^{e^x} \ln(1+t^2)\,dt = \frac{d}{dx}\left(-\int_0^x \ln(1+t^2)\,dt + \int_0^{e^x} \ln(1+t^2)\,dt\right)$$

$$= -\ln(1+x^2) + e^x \ln(1+e^{2x}).$$

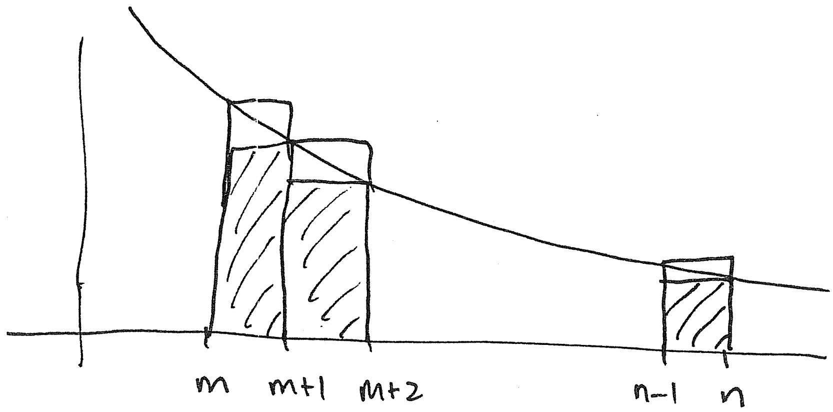
MARKS

2. (a) (*i*) Let m, n be integers with $m < n$, and let $f(x)$ be a monotone decreasing continuous function with $f(x) \geq 0$ for all x. Use upper and lower Riemann sums on the interval $[m, n]$ to show that **3**

$$f(n) \leq \sum_{k=m}^{n} f(k) - \int_m^n f(x)\, dx \leq f(m).$$

(*ii*) Hence, or otherwise, show that the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges. **2**

(a) (i)



$$L \leq \int_m^n f(x)\,dx \leq U$$

$$f(m+1) + \ldots + f(m) \leq \int_m^n f(x)\,dx \leq f(m) + \ldots + f(n-1)$$

Hence $$\sum_{k=m}^{n} f(k) - f(m) \leq \int_m^n f(x)\,dx \qquad (1)$$

and $$\int_m^n f(x)\,dx \leq \sum_{k=m}^{n} f(k) - f(n). \qquad (2)$$

So $$f(n) \overset{(2)}{\leq} \sum_{k=m}^{n} f(k) - \int_m^n f(x)\,dx \overset{(1)}{\leq} f(m)$$

(ii) $f(x) = \frac{1}{x \ln x}$ is monotone decreasing on $[2, m]$

(because $x \ln x$ is monotone increasing).

By part (i) we have

$$\int_2^n \frac{1}{x \ln x}\,dx + \frac{1}{n \ln n} \leqslant \sum_{k=2}^{n} \frac{1}{k \ln k}$$

But

$$\int_2^n \frac{1}{x \ln x}\,dx = \int_{\ln 2}^{\ln n} \frac{1}{u}\,du \qquad \left(\begin{array}{c} u = \ln x \\ \frac{du}{dx} = \frac{1}{x} \end{array}\right)$$

$$= \ln\ln n - \ln\ln 2.$$

Hence

$$\ln\ln 2 - \ln\ln 2 + \frac{1}{n \ln n} \leqslant \sum_{k=2}^{n} \frac{1}{k \ln k}.$$

Hence $\lim_{n \to \infty} \sum_{k=2}^{n} \frac{1}{k \ln k} = \infty$, and so the

series diverges.

MARKS

(b) You are given that the equation

$$ye^y = x$$

implicitly defines a function $y = y(x)$ with domain $x \geq -e^{-1}$ and range $y \geq -1$, and that this function can be differentiated any number of times.

(i) Calculate the integral **3**

$$\int_0^e \frac{1}{1+y(x)}\,dx.$$

(ii) Find the second order Taylor polynomial for $y(x)$ about $x = 0$. **3**

(i) $dx = (1+y)e^y dy$ since $x = ye^y$.

If $x = 0$, then $ye^y = 0$, so $y = 0$

If $x = e$, then $ye^y = e$, so $y = 1$

Hence

$$\int_0^e \frac{1}{1+y(x)}dx = \int_0^1 \frac{(1+y)e^y}{1+y}dy = \int_0^1 e^y dy = e^y\Big|_0^1$$

$$= \underline{e-1}$$

(ii) We know already

$y(0) = 0$

By implicit differentiation:

$1 = y'e^y + yy'e^y = y'$ if $x=0$, so $y'(0) = 1$

$0 = y''e^y + 2y'y'e^y + y(y'e^y)' =$

$= y'' + 2$ if $x=0$, so $y''(0) = -2$

Hence Taylor polynomial is

$T_2(x) = 0 + x - \frac{2}{2}x^2 = \underline{x - x^2}$

MARKS

3. (a) Find the general solution to the differential equation **2**

$$y' \cos^2 x = y^2(1 - \sin x).$$

Separate and integrate

$$-\frac{1}{y} = \int \frac{1}{y^2}\, dy = \int \frac{1 - \sin x}{\cos^2 x}\, dx = \int \sec^2 x - \tan x \sec x \, dx$$

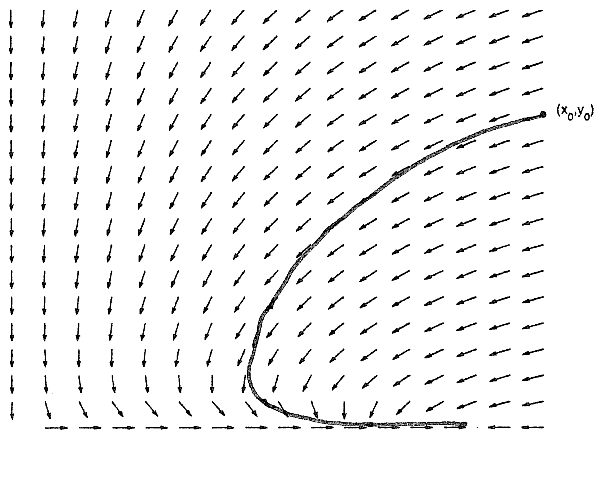
$$= \tan x - \sec x - C$$

Hence

$$y = \frac{1}{-\tan x + \sec x + C}$$

is the general solution.

(b) The diagram below shows a vector field of a system of two differential equations. In that diagram, draw the trajectory of the solution starting at the point (x_0, y_0) marked in the diagram. **1**



MARKS

(c) (i) Find the general solution of homogeneous second order differential equation **2**

$$\ddot{x} + \dot{x} - 6x = 0.$$

(ii) Find a particular solution of the inhomogeneous differential equation **3**

$$\ddot{x} + \dot{x} - 6x = e^{2t}.$$

(i) Auxiliary equation is $\lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3) = 0$

roots are $2, -3$, so general solution is

$$x(t) = Ae^{2t} + Be^{-3t}$$

(ii) As e^{2t} solves homogeneous problem try

$y_p(t) = kte^{2t}$:

$y_p'(t) = ke^{2t} + 2kte^{2t}$

$y_p''(t) = 4tke^{2t} + 4ke^{2t}$

Hence

$$\cancel{4kte^{2t}} + 4ke^{2t} + ke^{2t} + \cancel{2kte^{2t}} - \cancel{6kte^{2t}} = e^{2t}$$

$$5ke^{2t} = e^{2t}, \text{ so } k = \frac{1}{5}$$

Hence

$$y_p(t) = \frac{t}{5}e^{2t}$$

is a particular solution.

QUESTION 3 CONTINUES ON THE NEXT PAGE

MARKS

(d) Solve the initial value problem **3**

$$u' = 2xu + x^3, \qquad u(0) = 2.$$

Integrating factor is $e^{-\int 2x\,dx} = e^{-x^2}$, so

$$u e^{-x^2} = \int x^3 e^{-x^2} dx$$

Integration by parts

$$\int x^3 e^{-x^2} dx = -\frac{1}{2}\int x^2 \left(-2x e^{-x^2}\right) dx$$

$$= -\frac{1}{2} x^2 e^{-x^2} + \frac{1}{2}\int 2x e^{-x^2} dx = -\frac{x^2}{2} e^{-x^2} - \frac{1}{2} e^{-x^2} + C$$

Hence

$$u(x) = e^{x^2}\left(-\frac{x^2}{2} e^{-x^2} - \frac{1}{2} e^{-x^2} + C\right)$$

$$= -\frac{1}{2}(x^2+1) + C e^{x^2}$$

Determine C using initial condition $u(0) = 2$:

$$2 = u(0) = -\frac{1}{2} + C, \text{ so } C = \frac{5}{2}$$

Solution is $u(x) = -\frac{1}{2}(x^2+1) + \frac{5}{2} e^{x^2}$

MARKS

4. (a) By infusion, the glucose concentration of blood is increased at a constant rate measured in mg/minute. At the same time, the glucose is converted and excreted from the blood at a rate proportional to the present concentration of the glucose.

(*i*) Carefully define all dependent and independent variables needed to model the concentration of the glucose in the blood. **1**

(*ii*) Derive a differential equation describing the concentration of the glucose as a function of time. Use the variables you introduced in (*i*). **2**

(i) $g(t)$: glucose concentration at time t in mg/vol. (dependent variable)

t : time in minutes (independent variable)

I : rate of infusion mg/min

k : constant of proportionality

(ii)
$$\underbrace{\frac{d}{dt}g(t)}_{\text{total rate of change.}} = \underbrace{I}_{\text{increase due to infusion.}} - \underbrace{k\,g(t)}_{\text{decrease due to excretion.}}$$

MARKS

(b) Consider the nonlinear differential equation

$$xy' = y + ax\sqrt{x^2 + y^2}, \qquad x > 0,$$

where $a > 0$ is a constant.

(i) Show that $v := yx^{-1}$ satisfies the separable differential equation **2**

$$v' = a\sqrt{1 + v^2}$$

(ii) Use the differential equation in part (i) to get the general solution to the original differential equation. (Note the table of standard integrals.) **2**

(i) As $y = xv$ we have, using the equation for y:

$$y' = v + xv' = \frac{1}{x}\left(y + ax\sqrt{x^2+y^2}\right)$$

$$= \frac{y}{x} + a\,x\sqrt{1 + \frac{y^2}{x^2}} = v + ax\sqrt{1+v^2}.$$

Hence $xv' = ax\sqrt{1+v^2}$, so $v' = a\sqrt{1+v^2}$

(ii) Separate variables, integrate and use standard integral:

$$\int \frac{dv}{\sqrt{1+v^2}} = \int a\,dx$$

$$\sinh^{-1}(v) = ax + C$$

$$v = \sinh(ax + C)$$

$$y = xv = x\sinh(ax + C)$$

MARKS

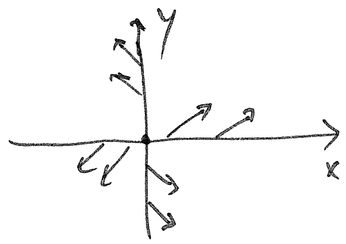
(c) Consider the system of differential equations

$$x' = 2x - y$$
$$y' = x + 2y$$

(i) Determine the stability of the zero solution $x = y = 0$. **1**

(ii) Find the solution of the system for the initial values $x(0) = 0$ and $y(0) = -1$. **3**

(i) Direction field near zero: if $x>0$, $y=0$, then $x', y' > 0$:



points away from $(0,0)$, so $(0,0)$ is unstable.

(ii) Differentiate first equation; then use second:

$$x'' = 2x' - y' = 2x' - x - 2y$$

From first equation

$$y = 2x - x', \qquad (*)$$

so

$$x'' = 2x' - x - 4x + 2x' = 4x' - 5x.$$

Hence

$$x'' - 4x' + 5x = 0$$

auxiliary equation $\lambda^2 - 4\lambda + 5 = 0$ has solutions $2 \pm i$.

Therefore

$$x = e^{2t}(A\cos t + B\sin t)$$

As $x(0) = 0 = A$ we have

$$\boxed{x(t) = e^{2t} B \sin t}$$

From $(*)$

$$y(t) = 2x(t) - x'(t)$$
$$= 2Be^{2t}\sin t - 2Be^{2t}\sin t - e^{2t} B \cos t$$

As $y(0) = -1 = -B$ we have

$$\boxed{y(t) = -e^{2t}\cos t}$$