The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Board Tutorial for Week 6** 2017

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) A line in space is determined by two points, or by one point and a direction.

(ii) A plane in space is determined either by three non-collinear points, or by one point and a perpendicular (normal) direction.

(iii) If the vector $\mathbf{v}$ points in the direction of a line $\mathcal{L}$ containing the point P_0, then the *parametric vector equation* of $\mathcal{L}$ is

$$\mathbf{r} - \mathbf{r}_0 \;=\; t\mathbf{v} \qquad \text{or equivalently} \qquad \mathbf{r} \;=\; \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}$ is the position vector of a typical point on $\mathcal{L}$, $\mathbf{r}_0$ is the position vector of P_0 and t is a parameter which varies over all real numbers.

(iv) If the vector $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ points in the direction of a line $\mathcal{L}$ containing the point $P_0(x_0, y_0, z_0)$, then the *parametric scalar equations* of $\mathcal{L}$ are

$$\left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}\right\} \; t \in \mathbb{R}$$

and the *Cartesian equations* are (in the case that a, b, c are all nonzero):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}\,.$$

(v) The shortest distance d from a point P to a line containing the point Q and pointing in the direction of $\mathbf{v}$ is

$$d \;=\; \frac{|\mathbf{v} \times \overrightarrow{PQ}|}{|\mathbf{v}|}\,.$$

(vi) If the vector $\mathbf{n}$ is normal to a plane $\mathcal{P}$ containing the point P_0, then the *vector equation* of $\mathcal{P}$ is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} \;=\; 0 \qquad \text{or equivalently} \qquad \mathbf{r} \cdot \mathbf{n} \;=\; \mathbf{r}_0 \cdot \mathbf{n}$$

where $\mathbf{r}$ is the position vector of a typical point and $\mathbf{r}_0$ is the position vector of P_0.

(vii) If the vector $\mathbf{n} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is normal to the plane $\mathcal{P}$ containing the point $P_0(x_0, y_0, z_0)$, then the *Cartesian equation* of $\mathcal{P}$ is

$$ax + by + cz \;=\; d$$

where $d = ax_0 + by_0 + cz_0\,$.

(viii) If P_1, P_2, P_3 are non-collinear points on a plane, then a normal vector to the plane is

$$\mathbf{n} \;=\; \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\;.$$

(ix) The shortest distance d from a point P to a plane containing the point Q and with normal vector $\mathbf{n}$ is

$$d \;=\; \frac{|\mathbf{n}\cdot\overrightarrow{PQ}|}{|\mathbf{n}|}\;.$$

Tutorial Exercises:

8. For each of (i)–(vii), find two matching descriptions from (a)–(n).

(i) line containing $(0,0,0)$ in the direction of $\mathbf{i}+\mathbf{j}+\mathbf{k}$
(ii) line containing $(-1,2,-1)$ in the direction of $-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$
(iii) line containing $(-1,2,-1)$ and $(0,0,-2)$
(iv) plane containing $(0,0,0)$ with normal vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$
(v) plane containing $(-1,2,-1)$ with normal vector $-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$
(vi) plane containing $(-1,2,-1)$, $(0,0,-2)$ and $(1,3,3)$
(vii) plane containing $(-1,2,-1)$, $(0,0,-2)$ and $(1,3,2)$

(a) $x+y+z=0$ (b) $x=y=z$ (c) $x+y-z=2$

(d) $x+1=\dfrac{y-2}{-2}=\dfrac{z+1}{-2}$ (e) $7x+6y-5z=10$

(f) $x+1=\dfrac{y-2}{-2}=\dfrac{z+1}{-1}$ (g) $x-2y-2z=-3$

(h) $\big(\mathbf{r}+2\mathbf{k}\big)\cdot\big(7\mathbf{i}+6\mathbf{j}-5\mathbf{k}\big)=0$ (i) $\mathbf{r}=\mathbf{i}-2\mathbf{j}-5\mathbf{k}+t(\mathbf{i}-2\mathbf{j}-2\mathbf{k})$

(j) $\big(\mathbf{r}+2\mathbf{k}\big)\cdot\big(\mathbf{i}+\mathbf{j}-\mathbf{k}\big)=0$ (k) $\mathbf{r}\cdot\big(\mathbf{i}+\mathbf{j}+\mathbf{k}\big)=0$

(l) $\big(\mathbf{r}+3\mathbf{i}\big)\cdot\big(\mathbf{i}-2\mathbf{j}-2\mathbf{k}\big)=0$ (m) $\mathbf{r}=\mathbf{i}-2\mathbf{j}-3\mathbf{k}+t(\mathbf{i}-2\mathbf{j}-\mathbf{k})$

(n) $\mathbf{r}=\mathbf{i}+\mathbf{j}+\mathbf{k}-t(\mathbf{i}+\mathbf{j}+\mathbf{k})$

9. Consider the following points in space:

$$P(1,1,1)\;,\quad Q(5,-5,-3)\;,\quad R(6,-3,-1)\;,\quad S(2,3,3)\;.$$

(i) Find the parametric vector, parametric scalar and Cartesian equations of the line $\mathcal{L}_1$ passing through P and R, and also the line $\mathcal{L}_2$ passing through Q and S.
(ii) Find the intersection point T of $\mathcal{L}_1$ and $\mathcal{L}_2$.
(iii) Verify that T is the midpoint of both PR and QS. Are you surprised?

10. The following planes intersect in a line:

$$x+y+z=2 \qquad \text{and} \qquad x-y+3z=0\;.$$

Find a point on this line of intersection and its direction. Now write down parametric and Cartesian equations for this line.

11. What do we mean by the angle between two planes? Find the cosine of the angle between the two planes given by equations

$$x + y + z = 6 \qquad \text{and} \qquad x - 2y - z = 3\,.$$

12. Let P and Q be fixed points in space. Suppose R is a point in space (which varies) such that

$$\overrightarrow{OR} \;=\; \lambda\,\overrightarrow{OP} \;+\; \mu\,\overrightarrow{OQ}$$

for real numbers λ and μ, subject to the constraint

$$\lambda + \mu \;=\; 1\,.$$

Prove that R varies over the line that passes through P and Q. Describe the values of λ such that R is located:

(i) somewhere on the line segment joining P to Q

(ii) somewhere on the line beyond P on the side away from Q

(iii) somewhere on the line beyond Q on the side away from P

(iv) twice as far from P as it is from Q.

13. Let r be a fixed positive real number. Describe geometrically the configuration $\mathcal{S}$ in space of points whose position vectors $\mathbf{r}$ satisfy the equation

$$|\mathbf{r}| \;=\; r\,.$$

Let $P(x_0, y_0, z_0)$ be a point on $\mathcal{S}$. Find the Cartesian equation of the tangent plane to $\mathcal{S}$ at P.

14. Suppose that P is a point in space and $\mathcal{P}$ is a plane not containing P. Let Q be any point on $\mathcal{P}$ and let R be the closest point on $\mathcal{P}$ to P. Explain why the dot product $\overrightarrow{PQ} \cdot \overrightarrow{PR}$ must be positive.

15.* Find the distance from $P(3, 0, -1)$ to the plane $\mathcal{P}$ described by the equation

$$4x + 2y - z = 6\,.$$

Find the closest point to P which lies on $\mathcal{P}$.