(A)

MATH1902 1998 exam solutions 11/6/2010

Q1/ $\mathbf{u} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

(i) $\mathbf{u} \cdot \mathbf{v} = 2 + 1 + 2 = 5$

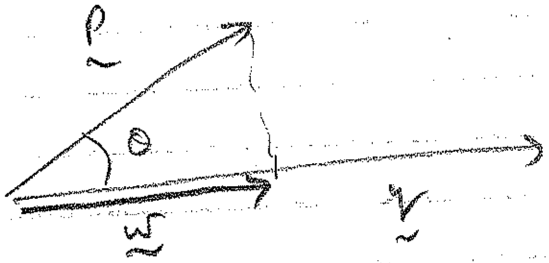
(ii) $$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{5}{\sqrt{1+1+4}\,\sqrt{4+1+1}} = \frac{5}{6}$$

(iii) $$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 2 & 1 & -1 \end{vmatrix} = \mathbf{i} - 3\mathbf{j} - \mathbf{k}$$

(iv) $$\widehat{\mathbf{u}\times\mathbf{v}} = \frac{1}{\sqrt{1+9+1}}\,\mathbf{u}\times\mathbf{v} = \frac{1}{\sqrt{11}}(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$$

(v) $$(3\mathbf{u} - 2\mathbf{v}) \times (\mathbf{u} + 5\mathbf{v}) = 15(\mathbf{u}\times\mathbf{v}) - 2(\mathbf{v}\times\mathbf{u})$$
$$= 17(\mathbf{u}\times\mathbf{v}) = \frac{17}{\sqrt{11}}(\mathbf{i} - 3\mathbf{j} - \mathbf{k}).$$

(vi)



If $\mathbf{w}$ is the vector projection of $\mathbf{p}$ in the direction of $\mathbf{q}$ then
$$|\mathbf{w}| = \big||\mathbf{p}|\cos\theta\big| = \frac{|\mathbf{p}||\mathbf{q}||\cos\theta|}{|\mathbf{q}|} = \frac{|\mathbf{p}\cdot\mathbf{q}|}{|\mathbf{q}|}$$

and $$\mathbf{w} = |\mathbf{w}|\,\hat{\mathbf{w}} = \begin{cases} |\mathbf{w}|\,\hat{\mathbf{q}} & \text{if } 0 \le \theta \le \pi/2 \\ -|\mathbf{w}|\,\hat{\mathbf{q}} & \text{if } \pi/2 < \theta \le \pi \end{cases} = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}|^2}\,\mathbf{q}$$
as required.

(noting $\hat{\mathbf{q}} = \frac{1}{|\mathbf{q}|}\mathbf{q}$)

Q1/ (vi) (cont.)

Hence the projection of $\underline{u}$ in the direction of $\underline{v}$ is

$$\frac{\underline{u}\cdot\underline{v}}{|\underline{v}|^2}\,\underline{v} = \frac{5}{6}\,(2\underline{i}+\underline{j}-\underline{k}).$$

(vii) $\underline{u} = \underline{a}+\underline{b}$ where $\underline{a} = \frac{5}{6}(2\underline{i}+\underline{j}-\underline{k})$ is parallel to $\underline{v}$ and

$$\underline{b} = \underline{u}-\underline{a} = (\underline{i}+\underline{j}-2\underline{k}) - \tfrac{5}{6}(2\underline{i}+\underline{j}-\underline{k})$$

$$= \frac{1}{6}(-4\underline{i}+\underline{j}-7\underline{k})$$

(viii) Suppose $\underline{u} = \underline{a}+\underline{b} = \underline{c}+\underline{d}$

where $\underline{a}, \underline{c} \parallel \underline{v}$ and $\underline{b}, \underline{d} \perp \underline{v}$.

Then $(\underline{a}+\underline{b})\cdot\underline{v} = (\underline{c}+\underline{d})\cdot\underline{v}$,

so $\underline{a}\cdot\underline{v}+\underline{b}\cdot\underline{v} = \underline{c}\cdot\underline{v}+\underline{d}\cdot\underline{v}$,

giving $\underline{a}\cdot\underline{v} = \underline{c}\cdot\underline{v}$, since $\underline{b}\cdot\underline{v} = \underline{d}\cdot\underline{v} = 0$

so $\lambda\underline{v}\cdot\underline{v} = \mu\underline{v}\cdot\underline{v}$, where $\underline{a} = \lambda\underline{v}$, $\underline{c} = \mu\underline{v}$,

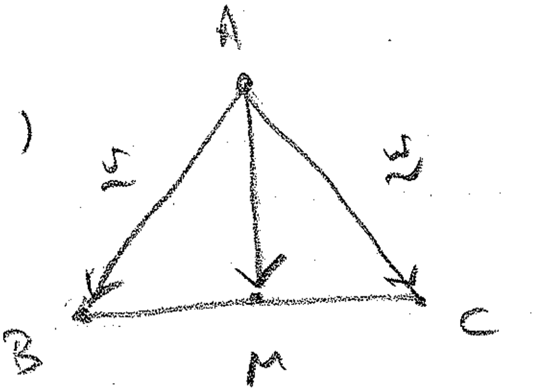
giving $\lambda|\underline{v}|^2 = \mu|\underline{v}|^2$, so $(\lambda-\mu)|\underline{v}|^2 = 0$,

so $\lambda-\mu = 0$ (since $\underline{v}\neq\underline{0}$), so $\underline{a} = \lambda\underline{v} = \mu\underline{v} = \underline{c}$.

Then $\underline{b} = \underline{u}-\underline{a} = \underline{u}-\underline{c} = \underline{c}+\underline{d}-\underline{c} = \underline{d}$, as required.

c

Q2/(i)



$\underline{v} = \overrightarrow{AB}$

$\underline{w} = \overrightarrow{AC}$

$|\underline{v}| = |\underline{w}|$

$$\overrightarrow{AM} \cdot \overrightarrow{BC} = (\overrightarrow{AB} + \overrightarrow{BM}) \cdot (\overrightarrow{BA} + \overrightarrow{AC})$$

$$= (\underline{v} + \tfrac{1}{2}\overrightarrow{BC}) \cdot (-\underline{v} + \underline{w})$$

$$= (\underline{v} + \tfrac{1}{2}(-\underline{v} + \underline{w})) \cdot (\underline{w} - \underline{v})$$

$$= \tfrac{1}{2}(\underline{w} + \underline{v}) \cdot (\underline{w} - \underline{v})$$

$$= \tfrac{1}{2}(\underline{w}\cdot\underline{w} - \underline{v}\cdot\underline{v}) = \tfrac{1}{2}(|\underline{w}|^2 - |\underline{v}|^2)$$

$= 0$, since $|\underline{v}| = |\underline{w}|$,

showing $AM \perp BC$, as required.

(ii) $l_1 : \underline{r} - (2\underline{i} + 3\underline{j} - \underline{k}) = t(\underline{i} - \underline{j} + 3\underline{k})$

$l_2 : x - 1 = 4 - y = 2z + 8$

i.e. $x - 1 = \dfrac{y-4}{-1} = \dfrac{z+4}{\frac{1}{2}}$

(a) l_1 has parametric equations $\begin{cases} x = 2 + t \\ y = 3 - t \\ z = -1 + 3t \end{cases}$

Q2/ (ii) (a) (cont.)

and l_2 has parametric equations $\begin{cases} x = 1+s \\ y = 4-s \\ z = -4+s/2 \end{cases}$

so the intersection point Q occurs when

$$\begin{cases} 2+t = 1+s \\ 3-t = 4-s \\ -1+3t = -4+s/2 \end{cases}$$

so $\begin{cases} s-t = 1 \\ s-t = 1 \\ s/2 - 3t = 3 \end{cases}$, i.e. $s-6t=6$,

so $5t=-5$, $t=-1$, $s=0$, yielding

$$\boxed{Q = (1,4,-4)}.$$

(b) Direction vectors are $\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ for l_1

and $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$ for l_2, so a normal to the

plane is $\underset{\sim}{u} \times 2\underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -1 & 3 \\ 2 & -2 & 1 \end{vmatrix} = 5\underset{\sim}{i} + 5\underset{\sim}{j} = 5(\underset{\sim}{i}+\underset{\sim}{j})$

so a Cartesian equation is $x+y = 2+3 = 5$

i.e. $\boxed{1x + 1y + 0z + (-5) = 0}$, as required.

$(A=1,\ B=1,\ C=0,\ D=-5)$

Q4/ (i) $\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 3 & 3 & 4 & 4 & 5 & 5 \\ -3 & -3 & -3 & -1 & -1 & 1 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 4 & 4 \end{array}\right]$

$\sim \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

which is reduced row echelon.

(ii) This becomes $u + v + w \quad - z = -1$

$x + y + 2z = 2$

so put $v = s_1$, $w = s_2$, $y = t_1$, $z = t_2$.

giving $x = 2 - y - 2z = 2 - t_1 - 2t_2$

$u = -1 + z - v - w = -1 + t_2 - s_1 - s_2$

and general solution

$$\boxed{(u, v, w, x, y, z) = (-1 + t_2 - s_1 - s_2, s_1, s_2, 2 - t_1 - 2t_2, t_1, t_2)}$$

where $s_1, s_2, t_1, t_2 \in \mathbb{R}$.

(P)

Q4/ (iii) $$M = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

If we have to assign 6 parameters then there are 3 leading variables, so 3 equations corresponding to the row echelon form. Since M is already in row echelon form, there must be $\boxed{\text{2 rows of zeros}}$.

Q5/ (i) C is the <u>inverse</u> of A if $AC = CA = I_n$ for some square identity matrix I_n.

(ii) Given $AC = CA = BD = DB = I_n$,

$$(AB)(DC) = A(BD)C = AI_nC = AC = I_n$$

and $$(DC)(AB) = D(CA)B = DI_nB = DB = I_n,$$

which shows $DC = (AB)^{-1}$.

(iii) The number of leading variables in the system corresponding to row echelon form is $\leq r < n$, so there are $\geq n-r > 0$ nonleading variables, to which parameters are assigned, giving infinitely many solutions.

Q5/ (iv) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^{-1}$ does not exist because

the matrix is not square. (I think?)

(The intention of the question is that

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

does not have a unique solution, by (iii), and the existence of $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^{-1}$ would imply a unique solution, a contradiction.)

Q6/ (i) $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ 3 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ R_2/4 \\ R_3/3 \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 - R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 - R_3$

so $E_1 E_2 E_3 E_4 E_5 E_6 A = I$ for elementary matrices

$E_1, \dots, E_6$, and working backwards

$A = E_6^{-1} \dots E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q6/ (ii) $\det A = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ 3 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 4 & 0 \\ 3 & 0 & 3 \end{vmatrix}$

$= (1)(4)(3) = 12$.

(The intention is to multiply the determinants of $E_6^{-1}, \ldots, E_1^{-1}$ together, which quickly yields 12. ✓)

Q7/ (i)(a) $\underline{v}$ is an eigenvector for M and λ the corresponding eigenvalue if $\underline{v} \neq \underline{0}$ and

$$M\underline{v} = \lambda \underline{v}.$$

(b) $M\underline{v} = \lambda\underline{v}$ starts the induction when $n = 1$.

Suppose $M^k \underline{v} = \lambda^k \underline{v}$ as inductive hypothesis.

Then $M^{k+1}\underline{v} = M M^k \underline{v} = M \lambda^k \underline{v} = \lambda^k M \underline{v}$

$= \lambda^k \lambda \underline{v} = \lambda^{k+1} \underline{v}$,

and the result follows by mathematical induction.

Q7/ (i) (c) Given M^{-1} exists and λ is an eigenvalue of M with eigenvector $\underline{v}$ say.

Then $M\underline{v} = \lambda \underline{v}$, so $M^{-1}M\underline{v} = M^{-1}\lambda\underline{v} = \lambda M^{-1}\underline{v}$,

so $I\underline{v} = \lambda M^{-1}\underline{v}$, so

$$M^{-1}\underline{v} = \frac{1}{\lambda} I \underline{v} = \frac{1}{\lambda}\underline{v},$$

which shows $\underline{v}$ is also an eigenvector for M^{-1} with eigenvalue $\frac{1}{\lambda}$, provided $\lambda \neq 0$.

If $\lambda = 0$ then $M\underline{v} = 0\underline{v} = \underline{0}$, so

$$\underline{v} = M^{-1}M\underline{v} = M^{-1}\underline{0} = \underline{0},$$

contradicting that $\underline{v} \neq \underline{0}$ (being an eigenvector)

(d) Given $PM = MP$ and $\underline{v}$ is an eigenvector of M with eigenvalue λ say.

Then
$$M(P\underline{v}) = (MP)\underline{v} = (PM)\underline{v} = P(M\underline{v})$$
$$= P\lambda\underline{v} = \lambda(P\underline{v}),$$

which verifies that $P\underline{v}$ is also an eigenvector of M (also with eigenvalue λ).

Q7/ (ii) Put $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Then $\det(M - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$

$= (\cos\theta - \lambda)^2 + \sin^2\theta = \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$

$= \lambda^2 - (2\cos\theta)\lambda + 1 = 0$

$\iff \quad \lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \cos\theta \pm \sqrt{\cos^2\theta - 1}$

$= \cos\theta \pm i\sin\theta \qquad i = \sqrt{-1}$

$= \operatorname{cis}(\pm\theta) = e^{\pm i\theta}$

Thus the eigenvalues of M are $e^{\pm i\theta}$.

(iii) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$

$= (1-\lambda)\left[(2-\lambda)^2 - 1\right] = (1-\lambda)(\lambda^2 - 4\lambda + 4 - 1) = (1-\lambda)(\lambda^2 - 4\lambda + 3)$

$= (1-\lambda)(\lambda - 3)(\lambda - 1) = (\lambda - 1)^2(3 - \lambda)$

So the eigenvalues are $\lambda = 1, 3$.

Q7/ (iii) (cont.)

$$A - I = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad y + z = 0$$

so eigenspace for $\lambda = 1$ is

$$\left\{ \begin{bmatrix} s \\ -t \\ t \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$$A - 3I = \begin{bmatrix} -2 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x - 2z = 0 \qquad y - z = 0$$

So eigenspace for $\lambda = 3$ is

$$\left\{ \begin{bmatrix} 2t \\ t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

giving eigenvector $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Put $T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ so $AT = T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

and T^{-1} exists since $\det T = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0$,

so $T^{-1}AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonal, as required.