The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 2

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

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Topics covered

In lectures last week:

- ☐ Partitions $P = \{x_0, x_1, \ldots, x_n\}$, sample points $x_j^* \in [x_{j-1}, x_j]$, and $\|P\|$.
- ☐ Riemann sums $\sum_{j=1}^n f(x_j^*)\Delta x_j$
- ☐ Upper and lower Riemann sums U_P and L_P.
- ☐ Riemann integrability and Riemann integrals.
- ☐ Continuous functions are Riemann integrable.

Objectives

After completing this tutorial sheet you will be able to:

- ☐ Work with partitions, and compute some Riemann sums.
- ☐ Understand the connection between Riemann sums and Riemann integrals.
- ☐ Be able to work with inequalities involving Riemann sums.
- ☐ Critically evaluate a mathematical statement.
- ☐ Develop summation techniques.
- ☐ Prove theorems about the Riemann integral.

Preparation questions to do *before* class

1. Let P be the partition $P = \{-2, 0, 1, 2\}$ of $[-2, 2]$, and let $f(x) = xe^{-2x^2}$.

(a) What is $\|P\|$?

(b) Find all local maxima and minima of $f(x)$.

(c) Compute the upper and lower Riemann sums for $f(x)$ on the interval $[-2, 2]$.

2. Let $n \geq 1$ and let $S(n)$ be the sum

$$S(n) = \sum_{j=1}^{n} \big(j^2 - (j-1)^2\big).$$

Compute this sum in 2 ways: One way by realising that the sum 'collapses', and another way by writing $j^2 - (j-1)^2 = 2j - 1$. Hence find a formula for $\sum_{j=1}^n j$.

Questions to attempt in class

3. (a) Let $P = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[a, b]$ into n equal parts, and choose $x_j^* = x_j$. Write down the corresponding Riemann sum for $f(x) = x^2$.

(b) Generalise the technique of Question 2 to find a closed formula for $\sum_{j=1}^{n} j^2$, and hence find a closed formula for the Riemann sum in Question 3(a).

(c) Compute the limit as $n \to \infty$ in your formula, and explain why the answer is what it is using a theorem from class.

4. Suppose that $f(x)$ is monotonically increasing on $[a, b]$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$ into n subintervals.

(a) Write down expressions for the upper and lower Riemann sums U_P and L_P.

(b) Show that $U_P - L_P \leq (f(b) - f(a))\|P\|$, where $\|P\| = \max\{\Delta x_1, \ldots, \Delta x_n\}$.

5. Let $f(x) = x^{-2}$ and let $0 < a < b$.

(a) Let $P = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition of $[a, b]$, and make the clever choice $x_j^* = \sqrt{x_{j-1}x_j}$. Compute the corresponding Riemann sum of f.

(b) Explain how you know that the Riemann integral $\int_a^b f(x)\,dx$ exists. What is its value?

Discussion question

6. Let $f(x)$ be the function

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

Is f continuous? Is f differentiable at any point? Is f Riemann integrable on $[0, 1]$?

Questions for extra practice

7. Let $f(x) = e^x$, and let $P = \{x_0, \ldots, x_n\}$ be the partition of $[0, 1]$ into n equal parts. Choose sample points $x_j^* = x_j$.

(a) Compute the Riemann sum $\sum_{j=1}^{n} f(x_j^*)\Delta x_j$.

(b) Find the limit of your Riemann sum as $n \to \infty$, and explain why your answer is what it is using a theorem from class.

8. Let $\alpha \in \mathbb{R}$ and let $0 < a < b$. Use the partition $P = \{a, ar, \ldots, ar^n\}$ with $r = \sqrt[n]{b/a}$ to compute the integral $\displaystyle\int_a^b x^\alpha\,dx$ from first principles. (Treat $\alpha = -1$ separately).

9. (a) Use the identity $\cos(A-B)-\cos(A+B)=2\sin A\sin B$ to prove that

$$2\sin(j\theta)\sin\left(\tfrac{1}{2}\theta\right)=\cos\left((j-\tfrac{1}{2})\theta\right)-\cos\left((j+\tfrac{1}{2})\theta\right).$$

(b) Deduce that

$$\sum_{j=1}^{n}\sin(j\theta)=\frac{\cos\left(\frac{1}{2}\theta\right)-\cos\left((n+\frac{1}{2})\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}\qquad\text{if } \theta \text{ is not a multiple of } 2\pi.$$

(c) Let $a>0$ and let $\{x_0,\ldots,x_n\}$ be a partition of $[0,a]$ into n subintervals of length a/n. Let $x_j^*=x_j$ for each j. Show that

$$\sum_{j=1}^{n}\sin(x_j^*)\Delta x_j=\frac{a/(2n)}{\sin(a/(2n))}\left[\cos\left(\frac{a}{2n}\right)-\cos\left(a+\frac{a}{2n}\right)\right].$$

Show that this tends to $1-\cos a$ as $n\to\infty$. Explain this using a theorem.

Challenging problems

10. (a) Find a closed formula for the sum $\displaystyle\sum_{j=0}^{n-1} jr^j$, where $r\neq 1$.

Hint: Use the geometric sum formula on $\sum_{j=0}^{n-1} r^j$, and differentiate with respect to r.

(b) Hence find a closed formula for the Riemann sum of $f(x)=x2^x$ over the interval $[0,1]$ using the partition $P=\{x_0,x_1,\ldots,x_n\}$ of $[0,1]$ into n equal parts, with $x_j^*=x_{j-1}$.

(c) Calculate the limit of this Riemann sum as $n\to\infty$.

11. Suppose that f is an unbounded positive function on the interval $[a,b]$. Show that f is not Riemann integrable on $[a,b]$.

Hint: Let M be a given (big) number. Show that there is a partition P and sample points x_j^* such that $\|P\|$ small and such that $\sum_{j=1}^{n} f(x_j^*)\Delta x_j>M$.

12. Suppose that f is continuous on $[a,b]$. Show that if $f\geq 0$ and $\int_a^b f(x)dx=0$ then $f(x)=0$ for all $x\in[a,b]$. What happens if we drop the assumption of continuity?

Hint: Continuity implies that if $f(\alpha)>\epsilon>0$ for some α, then $f(x)>\epsilon/2$ for all x in some (small) interval containing α.

13. This question shows how the famous "22/7 approximation" for π is related to integrals. By directly computing the integral, show that

$$\int_0^1\frac{x^4(1-x)^4}{1+x^2}\,dx=\frac{22}{7}-\pi.$$

Deduce that $3.1412<\pi<3.1429$. Find better bounds for π using the integral

$$\int_0^1\frac{x^8(1-x)^8}{1+x^2}\,dx.$$