CHAPTER **EIGHT**

# Graphs

Chapter Overview: This chapter reviews and extends the work done in previous years on graphing functions and relations, including the application of the calculus. In particular the relationships between algebra and geometry are further revealed by examining how graphs are transformed when some common algebraic operations are applied, such as squaring or the taking of reciprocals. The graphs of the transformations of both known and unknown functions are obtained by identifying significant features of both the original function and the transformed function.

The previous work on standard functions and relations done in Years 10 and 11 is assumed knowledge, however Section 8A contains a brief review of this work, omitting the calculus. The following sections deal with superposition (that is, addition), modulation (multiplication), reciprocals, reflections, natural powers, square roots, and composite functions, and the chapter concludes with graphs of relations, where the calculus requires use of implicit differentiation. Each topic is investigated through the use of the relevant parts of the curve sketching menu.

Wherever possible in the text, an accurate graph of a function is shown so that the reader may know what it should look like. However, it must be remembered that the aim of a sketch is not an accurate plot, but rather to show the significant features and the correct general shape. Therefore, students should overlook minor discrepencies when marking their work.

No book could possibly cover the infinite variety of transformations that exist. Thus this text does not attempt to deal with every eventuality. Instead, familiar problems from previous study are combined with pertinant practical examples to intruduce the new work. Comments, observations and deductions about the transformations of functions are confined to general cases, and the functions themselves are assumed to be 'nice'. Thus, for example, if the analysis requires differentiation then it is assumed that the function in question is differentiable. Some exceptions and special cases are presented in the exercises.

## 8A Review

**The Standard Graphs:** The graphs of the common functions and relations studied in both Mathematics Extension 1 and Mathematics Extension 2 courses are assumed knowledge. Where the corresponding equations have more than one standard form, such as $ax+by+c=0$ and $y=mx+b$ for the equation of a straight line, then they too are assumed knowledge. A sample list is given below for convenience.

If any one of these should prove unfamiliar then it should be reviewed in detail before proceeding with the rest of this chapter.

| | |
|---|---|
| Linear | $ax + by + c = 0$ |
| Quadratic | $y = ax^2 + bx + c$ |
| Polynomial | $y = P_n(x)$ |
| Rectangular Hyperbola | $xy = c^2$ |
| Exponential | $y = e^{kx}$ |
| Logarithmic | $y = a \log x$ |
| Trigonometric | $y = \cos x$ |
| Inverse Trigonometric | $y = \sin^{-1} x$ |
| Absolute Value | $y = \lvert x \rvert$ |
| Circle | $x^2 + y^2 = r^2$ |
| Conic | $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ |

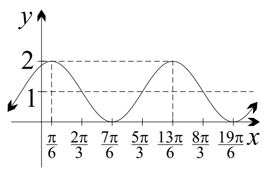
The simplest types of transformations have already been encountered. These are translations, stretches and reflections. They are briefly reviewed here.

**Translations:** A graph may be translated by shifting it horizontally, vertically, or in both directions. When $x$ is replaced by $(x - h)$ in the equation, the graph is shifted $h$ units to the right. When $y$ is replaced by $(y - k)$, the graph is shifted $k$ units up. Thus the parabolas $x^2 = 4ay$ and $(x - h)^2 = 4a(y - k)$ have the same focal length but the latter has been shifted so that its vertex is at $(h, k)$. Similarly $(x - a)^2 + (y - b)^2 = r^2$ is the result of shifting the circle $x^2 + y^2 = r^2$ so that its centre is at $(a, b)$.

**1** HORIZONTAL SHIFT: To shift $h$ units to the right, replace $x$ by $(x - h)$.
VERTICAL SHIFT: To shift $k$ units up, replace $y$ by $(y - k)$.

**WORKED EXERCISE:** Sketch $y = \sin\left(x + \frac{\pi}{3}\right) + 1$.

**SOLUTION:** Rearranging, $(y - 1) = \sin\left(x + \frac{\pi}{3}\right)$.
This is the result of translating $y = \sin x$ left by $\frac{\pi}{3}$ and up by 1, and is sketched on the right. Note the maximum at $x = \frac{\pi}{6}$ and $x$-intercept at $\frac{7\pi}{6}$. Also note that the height is 1 at $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$. The wavelength is $2\pi$ so these features are repeated every $2\pi$.




**Stretches:** A graph may be stretched horizontally, vertically, or in both directions. When $x$ is replaced by $\frac{x}{a}$ in the equation, the graph is stretched horizontally by factor $a$. When $y$ is replaced by $\frac{y}{b}$, the graph is stretched vertically by factor $b$. Thus whilst the sine wave $y = \sin x$ has wavelength $2\pi$, the graph of $y = \sin \pi x$ has wavelength 2, since the wave has been stretched by factor $\frac{1}{\pi}$. Similarly, the circle $x^2 + y^2 = r^2$ stretched vertically by factor $\lambda$ becomes the ellipse $x^2 + \left(\frac{y}{\lambda}\right)^2 = r^2$.

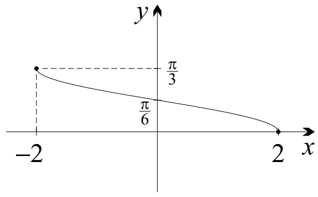
**2** HORIZONTAL STRETCH: To stretch horizontally by factor $a$, replace $x$ by $\frac{x}{a}$.
VERTICAL STRETCH: To stretch vertically by factor $b$, replace $y$ by $\frac{y}{b}$.

**WORKED EXERCISE:** Sketch $y = \frac{1}{3}\cos^{-1}\frac{x}{2}$.

**SOLUTION:** Rearranging, $3y = \cos^{-1}\frac{x}{2}$.

This is the result of stretching $y = \cos^{-1} x$ horizontally by factor 2 and vertically by factor $\frac{1}{3}$. It is graphed on the right. Note the $x$-intercept at $x = 2$ and $y$-intercept at $y = \frac{\pi}{6}$.



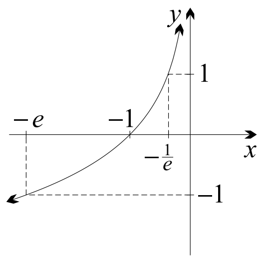

**Reflections in the Axes:** A graph may be reflected in the $y$-axis, in the $x$-axis or in both axes. When $x$ is replaced by $-x$ in the equation, the graph is reflected in the $y$-axis. When $y$ is replaced by $-y$, the graph is reflected in the $x$-axis. Thus the graphs of the exponential functions $y = e^x$ and $y = e^{-x}$ are reflections of each other in the $y$-axis. Likewise, the graphs of $y = \tan x$ and $-y = \tan x$ are reflections of each other in the $x$-axis.

**3**

HORIZONTAL REFLECTION: To reflect in the $y$-axis, replace $x$ by $-x$.
VERTICAL REFLECTION: To reflect in the $x$-axis, replace $y$ by $-y$.

**WORKED EXERCISE:** Sketch $y = \log\left(\dfrac{-1}{x}\right)$.

**SOLUTION:** Rearranging, $-y = \log(-x)$.
This is the result of reflecting the log graph in both axes, and is sketched on the right. Note the $x$-intercept at $-1$, and that the vertical asymptote appears unchanged. Also note that this graph could have been obtained by rotating the log graph by 180° about the origin.




**Odd and Even Functions:** Two special cases of reflections in the axes are odd and even functions. In the case of an even function, the graph is unaltered by a reflection in the $y$-axis. That is $y = f(x)$ and $y = f(-x)$ appear the same, whence $f(-x) = f(x)$. A simple example is $y = x^2$. In the case of an odd function, the graph is unaltered after reflecting in each of the coordinate axes. That is $y = f(x)$ and $-y = f(-x)$ appear the same, whence $f(-x) = -f(x)$. A simple example is $y = x^3$.

**4**

EVEN FUNCTIONS: $f(x)$ is called *even* if $f(-x) = f(x)$, for all $x$ in its domain.
ODD FUNCTIONS: $f(x)$ is called *odd* if $f(-x) = -f(x)$, for all $x$ in its domain.

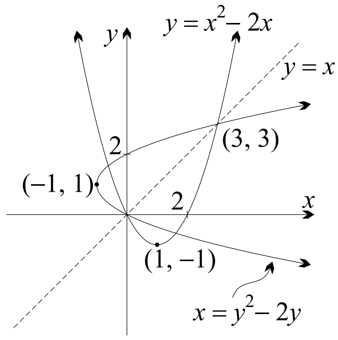
Finally notice that the geometry of odd functions can be interpreted in two ways. By the definition, they are unaltered by a reflection in each of the coordinate axes. Alternatively, they are unaltered by a rotation of 180° about the origin.

**Inverses:** The third type of reflection encountered in the Mathematics Extension 1 course is in the line $y = x$, as used to find the inverse of a function or relation. Algebraically this is achieved by swapping $x$ and $y$ throughout the equation. Thus the parabola $x^2 = 4ay$ and its inverse $y^2 = 4ax$ are symmetric in the line $y = x$. One method to sketch the inverse is to begin by plotting those points which correspond to the significant features of the original relation. Thus reversing the coordinates of a specific point yields the corresponding point on the inverse.

**5** INVERSES: The inverse relation is obtained by reflecting in the line $y = x$. Algebraically, swap $x$ and $y$. On the number plane, reverse each ordered pair.

**WORKED EXERCISE:** (a) Sketch $y = x^2 - 2x$.
(b) Hence sketch $x = y^2 - 2y$.




**SOLUTION:** Part (a) is a parabola, and part (b) is the result when reflected in the line $y = x$. The first one passes through three significant points, the intercepts at $(0, 0)$ and $(2, 0)$, and the vertex at $(1, -1)$. Thus the corresponding points on the inverse are $(0, 0)$, $(0, 2)$ and $(-1, 1)$. Both parabolas are drawn on the right.

**Reflection in the line $x = a$:** A myriad of new transformations can be created by combining translations, stretches and the reflections encountered so far. One combination has a significant application in integration and so is included in this course. When $y = f(x)$ is reflected in the $y$-axis, the result is $y = f(-x)$. If this is then shifted $2a$ units to the right, the result is

$$y = f\big(-(x - 2a)\big)$$

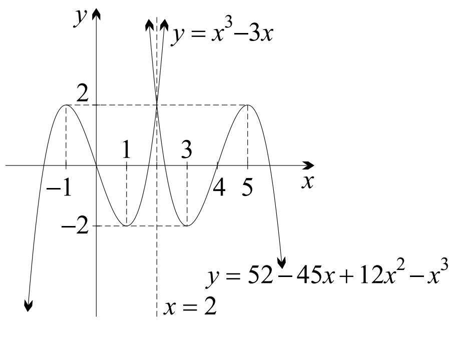
or $\quad y = f(2a - x)$.

This combination of reflection and shift is equivalent to a simple reflection in the line $x = a$, as the following example demonstrates.

The graphs of $y = x^3 - 3x$ and $y = (4-x)^3 - 3(4-x)$ are sketched below the table of values. Note that the latter equation expands to $y = 52 - 45x + 12x^2 - x^3$.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|---|---|---|---|---|---|---|---|---|---|
| $x^3 - 3x$ | $-2$ | $2$ | $0$ | $-2$ | $2$ | $21$ | $52$ | $110$ | $198$ |
| $(4-x)^3 - 3(4-x)$ | $198$ | $110$ | $52$ | $21$ | $2$ | $-2$ | $0$ | $2$ | $-2$ |

It should be clear that the third line of the table of values is just the reverse of the second line. That is, there is symmetry about the middle value $x = 2$. The graph also makes it clear that $y = (4-x)^3 - 3(4-x)$ is obtained by reflecting $y = x^3 - 3x$ in the line $x = 2$. In particular, it shows that the local minima and local maxima are equally spaced either side of the reflection line.




**6** REFLECT IN A VERTICAL LINE: To reflect in the vertical line $x = a$ replace $x$ by $(2a - x)$.

Just as even functions are symmetric in the $y$-axis, there are functions which are symmetric in the vertical line $x = a$ and these functions satisfy the equation

$$f(x) = f(2a - x).$$

There are numerous such functions, but three significant examples have been met in this course, namely the parabola, and the sine and cosine waves. In each case it is easy to prove the symmetry algebraically.

**WORKED EXERCISE:** Prove that $f(x) = \sin x$ is symmetric in the line $x = \frac{\pi}{2}$.

**SOLUTION:** Replacing $x$ with $(\pi - x)$ yields

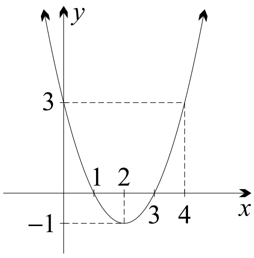
$$\begin{aligned} f(\pi - x) &= \sin(\pi - x) \\ &= \sin\pi\cos x - \cos\pi\sin x \\ &= 0 + \sin x \\ &= f(x)\,. \end{aligned}$$

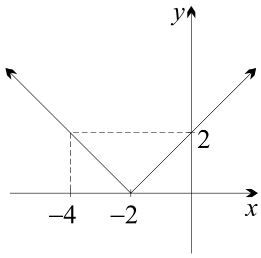
Thus $f(x) = \sin x$ is symmetric in the line $x = \frac{\pi}{2}$.

**7** SYMMETRY IN A VERTICAL LINE: A function which is symmetric in the vertical line $x = a$ satisfies the equation $f(x) = f(2a - x)$.

## Exercise **8A**

**1.** In each case the graph of $y = f(x)$ is given. Sketch the graphs of: (i) $y = f(x+1)$, (ii) $y = f(x) + 1$, (iii) $y = f(\frac{1}{2}x)$, (iv) $y = \frac{1}{2}f(x)$, (v) $y = f(-x)$, (vi) $y = -f(x)$, (vii) $y = f(2 - x)$, (viii) $y = 2 - f(x)$.

(a)



(b)
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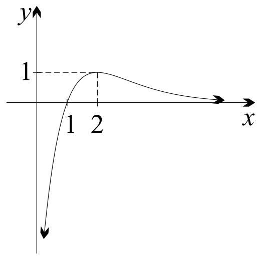


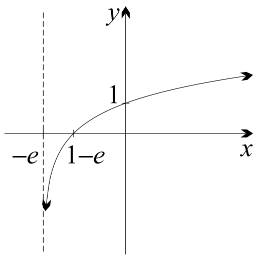

**2.** (a) (i) State the equation of the axis of the parabola with equation $y = 2x - x^2$.

(ii) Prove the result algebraicly by replacing $x$ by $(2-x)$ and showing that the equation is unchanged.

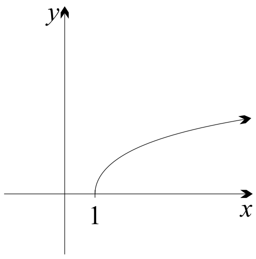
(b) Similarly prove algebraically that each of the following parabolas is symmetric by using an appropriate substitution.

(i) $y = x^2 - 4x + 3$ (ii) $y = 1 - 3x - x^2$

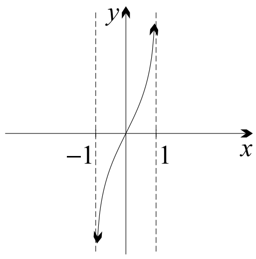
**3.** In each case, the graph of $y = f(x)$ is given. Sketch the graph of $y = f^{-1}(x)$ then determine $f^{-1}(x)$.

(a)



$f(x) = \log(e + x)$

(b)



$f(x) = \log(x + \sqrt{x^2 - 1})$

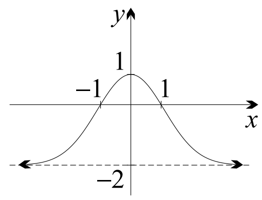
(c)



$f(x) = \log(1+x) - \log(1-x)$

DEVELOPMENT

**4.** For the given sketch of $f(x)$ , sketch the graph of $y = g(x)$ where




(a) $g(x) = \begin{cases} f(x) & \text{for } x \geq 1 \\ f(2-x) & \text{for } x < 1 \end{cases}$

(b) $g(x) = \begin{cases} f(x) & \text{for } x \geq -1 \\ f(-2-x) & \text{for } x < -1 \end{cases}$

**5.** (a) Describe geometrically two ways of transforming the graph of the circle $x^2+(y-1)^2 = 4$ to get the circle $x^2 + (y+1)^2 = 4$ .

(b) Describe geometrically three ways of transforming the graph of the wave $y = \sin(2x)$ to get the wave $y = \sin(2x + \pi)$ .

**6.** Use a suitable substitution to prove that $Q(x) = ax^2 + bx + c$ is symmetric in the line $x = -\frac{b}{2a}$ .

**7.** (a) (i) The graph of $y = \cos x$ is symmetric in the $y$-axis. What other vertical lines are lines of symmetry.

(ii) Prove your result with a suitable substitution.

(b) Do likewise for $y = \sin x$ .

**8.** The function $f(x)$ has the property $f(x) = f(2a-x)$ . Prove algebraicly that this function is symmetric in the line $x = a$ by showing that $f(a+t) = f(a-t)$ .

EXTENSION

**9.** (a) If $f(x)$ is odd then prove that $f'(x)$ is even.

(b) Is the converse true?

(c) Investigate the situation when $f(x)$ is even.

**10.** Show that every function can be written as the sum of an odd and even function.
HINT: Begin by investigating the function $h(x) = f(x) + g(x)$ , where $f(x)$ is even and $g(x)$ is odd.

# 8B Superposition

Superposition is simply the addition of two functions to create a new function. Thus if $f(x)$ and $g(x)$ are two functions then the result will be $h(x) = f(x)+g(x)$ . A simple example might be the sum of the quadratic $f(x) = x^2$ and the linear function $g(x) = 2x$ to obtain $h(x) = x^2 + 2x$ . Thus every quadratic with more than one term is an example of superposition.

When one of the functions is constant, for example $h(x) = f(x)+b$ , the situation reduces to a vertical shift of $b$ units as reviewed in Section 8A, and so will not be considered here. Although this section does not deal explicitly with the difference of two functions, the theory applies equally to differences since the function

$$h(x) = f(x) - g(x)$$

can be written as a sum, as follows:

$$h(x) = f(x) + \Big(-g(x)\Big).$$

Note that in the remainder of this chapter, the function notation will often be dropped for brevity. Thus $h(x) = f(x) + g(x)$ may be written as $h = f + g$ .

**Domain:** The domain of $h$ is the intersection of the individual domains of $f$ and $g$.
Thus if $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x-2}$
then the domain of $h = f + g$ is $x \neq 0, 2$.

**Intercepts:** In most cases, the $y$-intercept is trivially found, provided $x = 0$ is in the domain. Thus there is no further discussion of the $y$-intercept here, nor in the remainder of this chapter.

Since the $x$-intercepts of $y = h(x)$ are solutions of $h = 0$, it follows that

$$f + g = 0$$

or

$$f = -g$$

at these points. That is, the $x$-intercepts occur wherever the ordinates of the constituent functions are opposites or both zero.

**8** INTERCEPTS: The $x$-intercepts of $y = f(x) + g(x)$ occur wherever $f$ and $g$ are opposite or both zero.

**Symmetry:** In general, the sum of two even functions is even, the sum of two odd functions is odd, and a mixture is neither. These results are summarised in the addition table below, and the proofs are left to the exercise.

| + | odd | even |
|---|---|---|
| odd | odd | neither |
| even | neither | even |

Combinations of other functions may yield odd or even symmetry, and there may be other symmetries to investigate. Every function should be routinely checked.

**The Calculus:** At stationary points, $h' = 0$ so

$$f' + g' = 0$$

or

$$f' = -g'$$

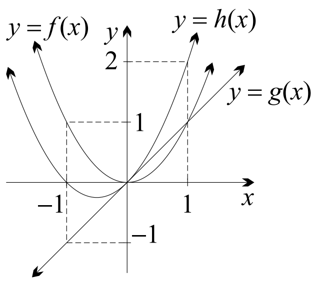
That is, the gradients of the constituent functions are opposite.

**Other:** If $y = f(x)$ and $y = g(x)$ intersect at $x = a$ then $f(a) = g(a)$ and

$$\begin{aligned} h(a) &= f(a) + g(a) \\ &= 2f(a) = 2g(a)\,. \end{aligned}$$

That is, the ordinate of $h$ is double that of $f$ or $g$. Such points should be plotted.

**WORKED EXERCISE:** Let $f(x) = x^2$ and $g(x) = x$, with $h(x) = f(x) + g(x)$. Graph $y = f(x)$ and $y = g(x)$ on the same set of axes and hence draw $y = h(x)$.




**SOLUTION:** At $x = 0$ both $f$ and $g$ are zero so $y = h(x)$ has an intercept there.

At $x = -1$ we find $f = 1$ and $g = -1$ are opposite, so $y = h(x)$ has another intercept at $x = -1$.

At $x = -\frac{1}{2}$ the gradients of $f$ and $g$ are opposite, so $y = h(x)$ has a stationary point there.

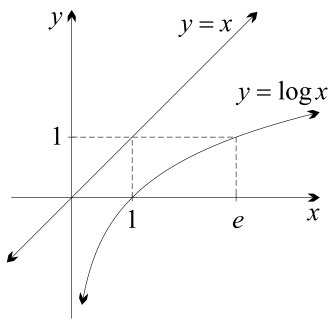
Finally $f$ and $g$ intersect at $(1,1)$, hence $y = h(x)$ passes through $(1,2)$, at double the height.

| | |
|---|---|
| **9** | INTERSECTIONS: At the intersection points of $y = f(x)$ and $y = g(x)$ the height of $y = f(x) + g(x)$ is double. |

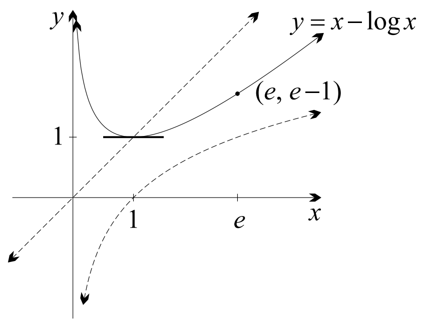
Finally, if $f(x)$ has a zero at $x = a$ then $h(a) = g(a)$, with similar results at the intercepts of $y = g(x)$. Thus it is usual to plot $y = h(x)$ at the $x$-intercepts of both $f$ and $g$.

**WORKED EXERCISE:** (a) Sketch $y = x$ and $y = \log x$ on the same graph.
(b) Hence Sketch $y = x - \log x$.

**SOLUTION:**
(a)




(b)




The graph on the left shows $y = x$ and $y = \log x$. The latter has an intercept at $x = 1$, so the graph of $y = x - \log x$ on the right has height 1.

Also at $x = 1$ both curves on the left have gradient 1, so the difference has a stationary point there. The second derivative is $y'' = x^{-2}$ so the curve is everywhere concave up, and the stationary point is a global minimum.

Since $x > \log x$, it follows that $x - \log x > 0$ for all $x$ in the domain. The domain is $x > 0$, so $y = x - \log x$ must lie entirely in the first quadrant. Lastly, $\log e = 1$ so the point $(e, e-1)$ is in the graph.

| | |
|---|---|
| **10** | INTERCEPTS AGAIN: It is usual to plot $y = f(x) + g(x)$ at the $x$-intercepts of $f$ and $g$ if they are known or easy to find. |

**WORKED EXERCISE:** [A HARD EXAMPLE]
From the sketches of

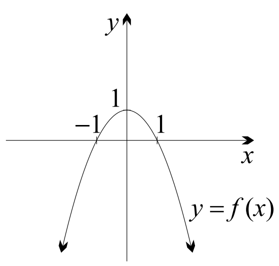
$$y = f(x)$$

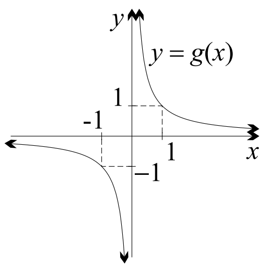
and $\qquad y = g(x)$

given on the right, sketch

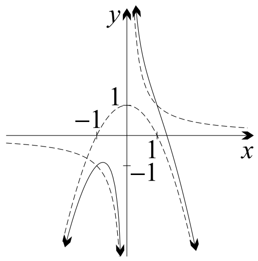
$$y = h(x)$$

where $\quad h(x) = f(x) + g(x)$.




**SOLUTION:** Clearly the domain of $h$ is $x \neq 0$ and hence there is no $y$-intercept.




Since $f(x) = 0$ at $x = -1$ and 1, the graph of $y = h(x)$ crosses $y = g(x)$ at these points.

For $-1 < x < 1$, $f(x) > 0$ so the graph of $y = h(x)$ is above $y = g(x)$ in this region.

For large value of $x$, $g(x) \to 0^+$, so the graph of $y = h(x)$ approaches $y = f(x)$ from above.

For large negative values of $x$, $g(x) \to 0^-$, so the graph of $y = h(x)$ approaches $y = f(x)$ from below.

All these details are shown in the graph of $y = h(x)$ above. The graphs of $y = f(x)$ and $y = g(x)$ are shown dashed on the same set of axes for comparison.

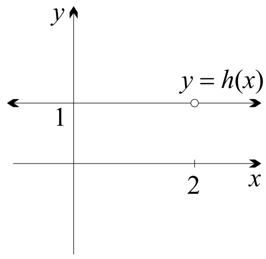
**Discontinuities:** Care must be taken when analysing functions at discontinuities. Consider the following example.

Let $\quad f(x) = 1 + \dfrac{1}{2-x}$

and $\quad g(x) = \dfrac{1}{x-2}$.

Adding
$$\begin{aligned} h(x) &= f(x) + g(x) \\ &= 1 + \frac{1}{2-x} + \frac{1}{x-2} \\ &= 1 - \frac{1}{x-2} + \frac{1}{x-2} \end{aligned}$$

hence $\quad h(x) = 1$.




Thus it appears that $h(x)$ is a continuous function. This is an incorrect conclusion however, and a common mistake to make. Since $h = f + g$ and since neither $f$ nor $g$ are defined at $x = 2$, it follows that the domain of $h$ is $x \neq 2$. Hence the graph of $y = h(x)$ has a hole at $x = 2$, as is shown above. This example demonstrates that the domain of the function should always be checked.

## Exercise **8B**

**1.** (a) (i) Draw the graphs of $y = |x+1|$ and $y = |x-2|$ on the same number plane.

(ii) Hence sketch $y = |x+1| - |x-2|$.

(b) Do likewise for the following.

(i) $y = |x-1| - |x+1|$

(ii) $y = |x-1| + |x-2|$

**2.** In each case, graph the functions $y = f(x)$, $y = -g(x)$ and $y = g(x)$. Hence graph $y = f(x) - g(x)$ and $y = f(x) + g(x)$.

(a) $f(x) = x^2$, $g(x) = 2x$

(b) $f(x) = x$, $g(x) = e^{-x}$

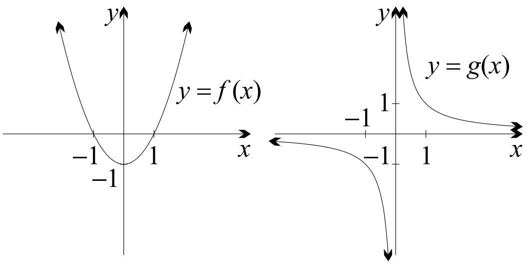
(c) $f(x) = x$, $g(x) = \sin x$

(d) $f(x) = e^x$, $g(x) = e^{-x}$

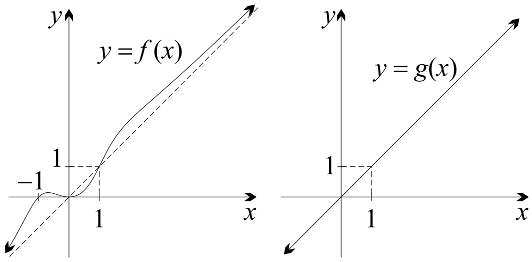
 

**3.** In each case, use the graphs of $y = f(x)$ and $y = g(x)$ to help sketch the required function.

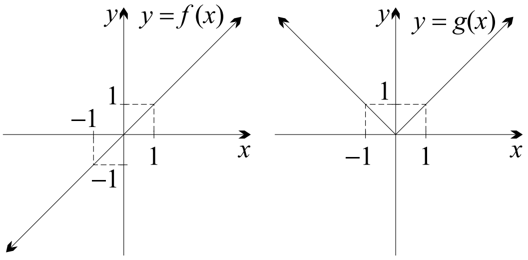
(a) $y = f(x) + g(x)$



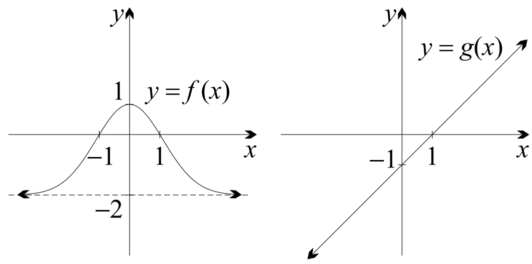

(b) $y = f(x) - g(x)$




(c) $y = f(x) + g(x)$




(d) $y = f(x) - g(x)$




**DEVELOPMENT**

**4.** The piecewise continuous function in Question 1(a) can be written as

$$y = \begin{cases} -3 & \text{for } x < -1 \\ 2x - 1 & \text{for } -1 \le x < 2 \\ 3 & \text{for } x \ge 2 \end{cases}$$

Find similar expressions for the functions in part (b).

**5.** (a) Sketch $y = \log(1 + x)$ and $y = \log(1 - x)$ on the same number plane.

(b) Hence sketch $y = \log(1 + x) - \log(1 - x)$.

**6.** (a) Sketch $y = \cos^{-1} x$ and $y = \sin^{-1} x$ on the same axes, and observe the symmetry in the line $y = \frac{\pi}{4}$.

(b) Hence sketch $y = \cos^{-1} x + \sin^{-1} x$.

**7.** (a) Sketch $f(x) = \dfrac{1}{x}$ for $x > 0$, and $g(x) = \log x$ on the same number plane.

(b) Notice that the behaviour of $y = f(x) + g(x)$ near $x = 0$ cannot be determined from the graph in part (a). Use $y'$ to determine the behaviour of $y$ as $x \to 0^+$.

(c) Hence sketch $y = f(x) + g(x)$.

**8.** This question demonstrates that the domain must be noted when a function is simplified.

(a) Show that $\log(x + \frac{3}{2}) + \log x = \log\left(x^2 + \frac{3}{2}x\right)$ for $x > 0$, but not for $x < 0$.

(b) Graph $y = \log(x + \frac{3}{2}) + \log x$ and $y = \log\left(x^2 + \frac{3}{2}x\right)$ on separate number planes, in each case using the natural domain. Observe that the two graphs differ.

**9.** The addition table for odd and even functions is given in the text. There are essentially three cases: both odd, both even, or one even and one odd.

(a) Find an example to demonstrate each result.

(b) Prove each result in general.

(c) In general, the sum of two odd functions is an odd function. What is the one exception to this rule?

**EXTENSION**

**10.** The Heaviside step function is also called the unit step function, and is defined as follows.

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

(a) Sketch $y = u(x)$ and $y = u(x-1)$ on separate number planes

(b) Hence sketch (i) $y = u(x) - u(x-1)$, and (ii) $y = u(x) + u(x-1)$.

**11.** Use the graphs of $y = x^4 - 1$ and $y = mx$ to determine the number of zeros of the function $f(x) = x^4 + mx - 1$.

# 8C Modulation

Modulation is the product of two functions to create a new function. Thus if $f(x)$ and $g(x)$ are two functions then the result will be $h(x) = f(x) \times g(x)$. A simple example might be the product of the exponential $f(x) = e^x$ and the linear function $g(x) = x$ to obtain $h(x) = xe^x$.

When one of the functions is constant, for example $h(x) = af(x)$, the situation reduces to a vertical stretch by factor $a$ as reviewed in Section 8A, and so will not be considered here. Although this section does not deal explicitly with rational functions, many of the points made here apply to quotients since the function

$$h(x) = \frac{f(x)}{g(x)}$$

can be written as a product, as follows:

$$h(x) = f(x) \times \frac{1}{g(x)}.$$

The Mathematics Extension 1 course includes the study of rational functions, consequently some examples have been included in the exercises. A suitable text book should be consulted for a full exposition of that topic.

**Domain:** As with superposition, the domain of $h$ is the intersection of the individual domains of $f$ and $g$.

**Intercepts:** The $x$-intercepts of $h$ will occur at the $x$-intercepts of $f$ and $g$, provided that those values of $x$ lie in the domain of $h$.

**11** INTERCEPTS: The $x$-intercepts of $h(x) = f(x) \times g(x)$ occur at the $x$-intercepts of $f$ and $g$, provided that those values of $x$ lie in the domain of $h$.

**Symmetry:** The symmetries that result from the products of odd and even functions are summarised in the following multiplication table, and the proofs are left to the exercise.

| $\times$ | odd | even |
|---|---|---|
| odd | even | odd |
| even | odd | even |

These symmetries are very important in the study of integration and should have already been encountered in the Mathematics Extension 1 course. For example:

$$\int_{-\frac{\pi}{5}}^{\frac{\pi}{5}} \cos 2x \sin 3x \, dx = 0$$

since the limits are symmetric and the integrand is odd.

Products of other functions may yield odd or even symmetry, and there may be other symmetries to investigate. For example, the functions $f(x) = e^x$ and $g(x) = e^{-x}$ have no symmetry, yet $h(x) = 1$ which is even. As always, every function should be routinely checked for symmetry.

**The Calculus:** Apart from a few special cases there is little to be said in this course about the calculus of the products of functions.

**Other:** Wherever $f = 1$ the product reduces to $h(x) = g(x)$ and wherever $f = -1$ it reduces to $h(x) = -g(x)$, with similar results wherever $|g| = 1$. Thus whenever possible, points where $|f| = 1$ or $|g| = 1$ should be plotted.

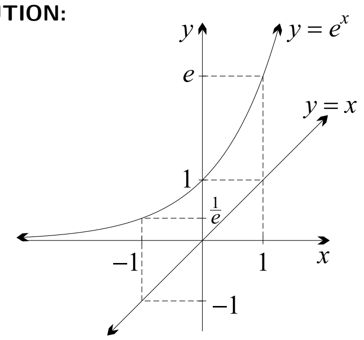
**12** Special points: If the solutions of $|f(x)| = 1$ and $|g(x)| = 1$ are easy to find and in the domain, then plot the corresponding points for $h(x) = f(x) \times g(x)$.

**Worked Exercise:** (a) Sketch $y = x$ and $y = e^x$. (b) Hence sketch $y = xe^x$.

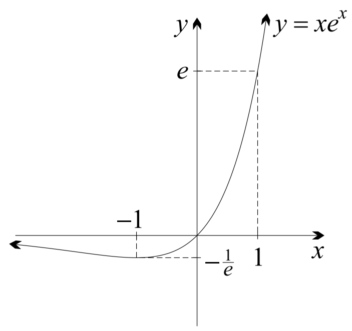
(c) Without the use of calculus, explain why $f(x) = xe^x$ must have a global minimum for a negative value of $x$.

**Solution:**

(a)




(b)




Note that $|x| = 1$ at $x = 1$ or $-1$, and $|e^x| = 1$ at $x = 0$.

Note the corresponding points plotted at $x = -1, 0$ and $1$.

(c) The functions $f(x)$ is continuous and has the three properties:

$$f(0) = 0,$$
$$f(x) < 0 \text{ for } x < 0,$$
and $$\lim_{x \to -\infty} f(x) \to 0^-.$$

Hence there exists a value $x = a$, $-\infty < a < 0$, for which $f(a)$ is a local minimum. Further, since $f(x) \geq 0$ wherever $x \geq 0$, it follows that $f(a)$ is a global minimum. [Calculus reveals that $a = -1$.]

## Exercise **8C**

**1.** (a) Let $f(x) = x$ and $g(x) = \sin x$ , and let $h(x) = f(x) \times g(x)$ .

(i) Show that $h(x)$ is even.

(ii) Find the $x$-intercepts of $y = g(x)$ and hence plot $y = h(x)$ at those values.

(iii) Find the values of $x$ where $g(x) = 1$ and where $g(x) = -1$, and hence plot $y = h(x)$ at those values.

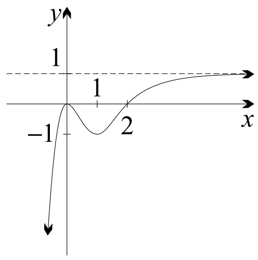
(iv) Complete the graph of $y = h(x)$.

(b) Similiarly sketch $y = h(x)$ when $f(x) = e^{-x}$ and $g(x) = \cos \pi x$ .

**2.** In each case, graph the functions $y = f(x)$ and $y = g(x)$ . Hence graph $y = f(x) \times g(x)$ without the use of calculus.

(a) $f(x) = x^2$ , $g(x) = e^x$ (b) $f(x) = x^2 - 1$ , $g(x) = e^x$

**3.** (a) The diagram on the right shows $y = f(x)$ .
Let $h(x) = xf(x)$ .




(i) Plot $y = h(x)$ at the values of $x$ where $f(x) = 0$ .

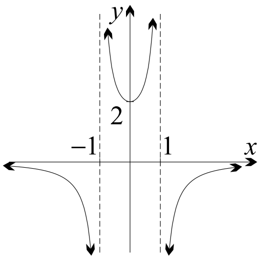
(ii) Locate any point where $|f(x)| = 1$. Hence plot $y = h(x)$ at those values.

(iii) Explain why $h(x) \to x$ as $x \to \infty$.

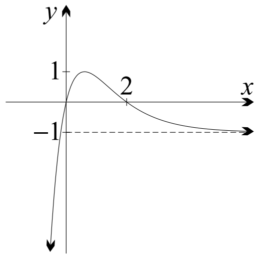
(iv) Hence complete the sketch of $y = h(x)$ .

(b) In each case use a similar approach to part (a) to sketch $y = xf(x)$ for the given graph of $y = f(x)$ .

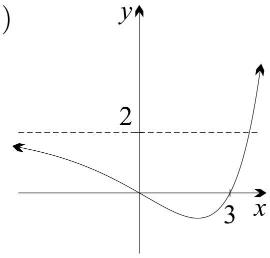
(i)




(ii)

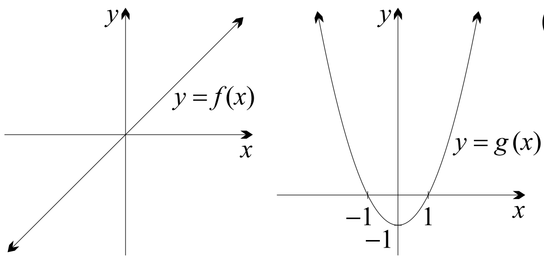



(iii)

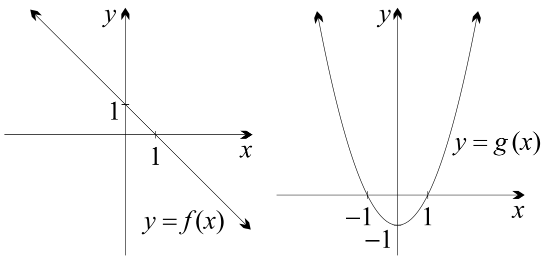



**4.** Use the graphs of $y = f(x)$ and $y = g(x)$ to sketch $y = f(x) \times g(x)$ . In part (f) you may assume that $\lim\limits_{x \to \infty} f(x) \times g(x) = 0$ .

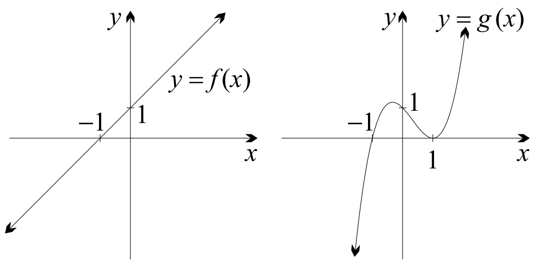
(a)



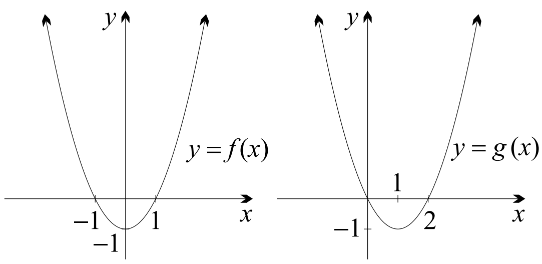

(b)



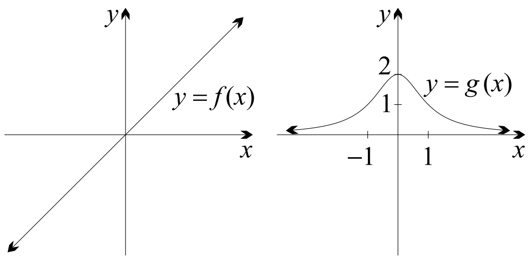

(c)



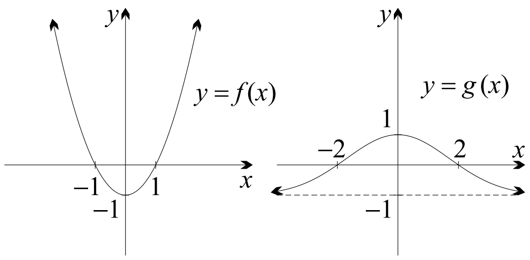

(d)



(e)




(f)




**DEVELOPMENT**

**5.** (a) Graph $y = x^{\frac{1}{3}}$, paying particular attention to the behaviour near the origin.

(b) Hence sketch $y = x^{\frac{1}{3}} e^x$.

**6.** Let $f(x) = \dfrac{x^2}{x^2 - 9}$.

(a) Show that $f(x) = 1 + \dfrac{3}{2}\left(\dfrac{1}{x-3} - \dfrac{1}{x+3}\right)$.

(b) Hence determine the vertical and horiztonal asymptotes of the graph of $y = f(x)$.

(c) Prove that $x^2 - 9 < 0$ for $-3 < x < 3$. Hence show that $f(x)$ has a maximum at $x = 0$. There is no need to resort to the calculus.

(d) Sketch $y = \dfrac{x^2}{x^2 - 9}$.

**7.** (a) Locate and classify the stationary points of $y = \dfrac{2x}{1 + x^2}$

(b) Use the second derivative to show that there is an inflexion point at the origin.

(c) Hence sketch $y = \dfrac{2x}{1 + x^2}$.

**8.** (a) Graph $y = \dfrac{x^2 - 1}{x^2 - 4}$.

(b) Hence solve $\dfrac{x^2 - 1}{x^2 - 4} > 1$.

**9.** Consider $y = \dfrac{(x^2 - 4)(x^2 - 1)}{x^4}$

(a) Write down the $x$-intercepts.

(b) State the equations of any vertical asymptotes.

(c) Show that $y = 1 - \dfrac{5}{x^2} + \dfrac{4}{x^4}$. Hence answer the following.

(i) State the equation of any horiztonal asymptotes.

(ii) Locate any stationary points.

(d) Hence sketch $y = \dfrac{(x^2 - 4)(x^2 - 1)}{x^4}$

(e) For what values of $b$ does the equation $(x^2 - 4)(x^2 - 1) = bx^4$ have four solutions?

(f) What would be a better way of solving this problem?

**10.** The multiplication table for odd and even functions is given in the text. Prove each entry in the multiplication table.

**11.** Let $f(x) = x^3$ and $g(x) = x^2$, with $h(x) = \dfrac{f(x)}{g(x)}$. Explain how the graphs of $y = x$ and $y = h(x)$ differ.

**EXTENSION**

**12.** The Heaviside step function was defined in Section 8B and is

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Graph the following functions

(a) $u(x) \times (x^3 - x)$

(b) $u(x) \times e^{-x}$

(c) $u(x) \times \sin \pi x$

(d) $u(x+1) \times e^{-x}$

**13.** One particular application of modulation is used in sending radio signals. The 'AM' in AM Radio stands for amplitude modulation. A high frequency carrier wave has its amplitude modulated by a signal wave, which forms an envelope around the carrier wave.

(a) Use a computer to graph $y = \sin\left(\frac{x}{2}\right)\cos(5x)$.

(b) Which is the carrier wave?

# 8D Reciprocals

A significant number of problems involve graphing the reciprocal of a function. That is, the graph of $y = g(x)$, where $g(x) = \dfrac{1}{f(x)}$ and either $f(x)$ itself is known or the graph of $y = f(x)$ has been given. The classic example is of course when $f(x) = x$ and $g(x) = \dfrac{1}{x}$, that is, the rectangular hyperbola.

**Domain and Sign:** The domain of $g$ is the domain of $f$ intersecting with $f \neq 0$, since division by zero is undefined. The sign of $g$ is everywhere the same as the sign of $f$.

**Intercepts and Asymptotes:** The zeros of $f$ are the vertical asymptotes of $g$, but the converse is not true. If $y = f(x)$ has a vertical asymptote at $x = a$ then $\lim\limits_{x \to a} g(x) = 0$. That is, the graph approaches the $x$-axis but does not have an intercept at $x = a$.

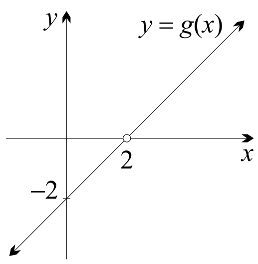
To demonstrate the point, consider the example

$$f(x) = \frac{1}{x-2} \quad \text{with } x \neq 2,$$

for which

$$g(x) = x - 2 \quad \text{with } x \neq 2.$$

The graph of $y = g(x)$ is shown on the right and has a hole at $x = 2$ instead of an $x$-intercept.



**13** INTERCEPTS AND ASYMPTOTES: The zeros of $y = f$ are the asymptotes of $y = \frac{1}{f}$, but if $f(x)$ has a vertical asymptote at $x = a$ then $\lim_{x\to a} \frac{1}{f} = 0$.

If $y = 0$ is a horizontal asymptote of $y = f(x)$ then $|g| \to \infty$, and vice versa. Other horizontal asymptotes behave as expected. As an example let $f(x) = 2 + \frac{1}{x}$. Clearly $f \to 2$ as $x \to \infty$, so $g \to \frac{1}{2}$ as $x \to \infty$.

**Symmetry:** Odd and even symmetry is preserved. The proof is left as an exercise.

**The Calculus:** As before, let $g(x) = \dfrac{1}{f(x)}$. Differentiating the function $g(x)$ yields:

$$\begin{aligned} g' &= \frac{dg}{dx} \\ &= \frac{dg}{df} \times \frac{df}{dx} \quad \text{(by the chain rule)} \\ &= -\frac{1}{f^2} \times \frac{df}{dx} \\ \text{so} \quad g' &= -\frac{f'}{f^2}. \end{aligned}$$

Thus $y = g(x)$ will have stationary points with the same $x$-coordinates as the stationary points of $y = f(x)$, provided $f$ is not simultaneously zero. Further, the above result can be used to show that the nature of the stationary points is reversed. A question in the exercise deals with this.

**14** STATIONARY POINTS: The stationary points of $y = f(x)$ have the same $x$-coordinates as those of $y = g(x) = \dfrac{1}{f(x)}$ provided $f(x) \neq 0$ there.

The nature of the stationary points can also be determined from the second derivative, as follows.

$$g'' = \frac{2(f')^2 - ff''}{f^3} \quad \text{(by the quotient rule)}$$

so at stationary points where $f' = 0$ and $f \neq 0$ this becomes

$$g'' = -\frac{f''}{f^2}.$$

Thus at the stationary points of $f$ the sign of $g''$ is opposite that of $f''$. Hence the types of extrema of $f$ and $g$ are reversed, as asserted earlier. That is, $g$ has a maximum where $f$ has a minimum and vice versa.

**15** TYPES OF STATIONARY POINTS: If $g(x) = \dfrac{1}{f(x)}$ then the nature of the stationary points of $f$ and $g$ are reversed, provided the ordinates are non-zero. That is, $g$ has a maximum where $f$ has a minimum and vice versa.

**Other:** If both $y = f(x)$ and $y = g(x)$ are sketched on the same number plane then they intersect when

$$f = g\,,$$

so $$f = \frac{1}{f}$$

or $$f^2 = 1\,,$$

whence $f(x) = 1$ or $-1$.

Thus it is usual to plot these points if the $x$-coordinates are easy to find.

**WORKED EXERCISE:**

(a) Sketch the graph of $y = \dfrac{2(x^2-1)}{x^2+2}$ without the aid of the calculus.
Use the identity $\dfrac{2(x^2-1)}{x^2+2} = 2 - \dfrac{6}{x^2+2}$ to locate the minimum. Indicate on the sketch the intercepts with the axes, the horizontal asymptote and any points where $y = 1$.

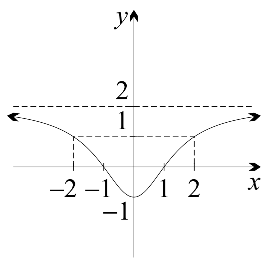
(b) Hence graph $y = \dfrac{x^2+2}{2(x^2-1)}$

**SOLUTION:** (a) Clearly there is no restriction on the domain and the function is even. There are intercepts with the axes at $(-1, 0)$, $(1, 0)$ and $(0, -1)$.

From the given identity, the minimum of $y$ occurs when the denominator is a minimum, that is at the $y$-intercept.

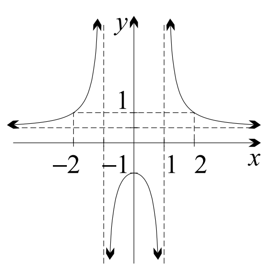
Now $$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{2(1-\frac{1}{x^2})}{1+\frac{2}{x^2}} = 2$$

so $y = 2$ is a horizontal asymptote.




Solving $$\frac{2(x^2-1)}{x^2+2} = 1$$
$$2x^2 - 2 = x^2 + 2$$
thus $$x^2 = 4$$
so $$x = 2 \text{ or } -2\,.$$

Hence $(-2, 1)$ and $(2, 1)$ are on the curve as shown.




(b) From part (a) it is clear that the domain is $x \neq -1, 1$ and the graph is symmetric in the $y$-axis. The $y$-intercept remains unchanged and there is no $x$-intercept. There are three asymptotes, namely $y = \frac{1}{2}$, $x = -1$ and $x = 1$. The behaviour either side of the vertical asymptotes is determined by the sign of $y$, and from part (a) this gives

$$\lim_{x\to 1^+} y \to \infty$$

and $$\lim_{x\to 1^-} y \to -\infty$$

Finally the curve passes through $(-2, 1)$ and $(2, 1)$ as shown.

**16** | INTERSECTION POINTS: The graphs of $y = f$ and $y = \dfrac{1}{f}$ intersect wherever $|f| = 1$.

## Exercise **8D**

**1.** Let $f(x) = x - 1$.

(a) Graph $y = f(x)$ showing the intercepts with the axes and the points where $|f(x)| = 1$.

(b) Hence on the same number plane sketch $y = \dfrac{1}{f(x)}$.

**2.** Let $y = f(x)$ where $f(x) = \frac{1}{3}(x+1)(x-3)$.

(a) Show that $y = 1$ at $x = 1 - \sqrt{7}$ and $x = 1 + \sqrt{7}$. Plot these points.

(b) Complete the graph of $y = f(x)$ showing the vertex, the intercepts with the axes and the points where $f(x) = -1$.

(c) Hence on the same number plane sketch $y = \dfrac{1}{f(x)}$.

**3.** Sketch each polynomial without the aid of calculus and hence sketch its reciprocal.

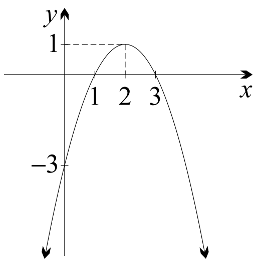
(a) $y = 1 - x^2$ (b) $y = x^3 - x$

**4.** Graph $y = f(x)$ and $y = \dfrac{1}{f(x)}$ for each of the following functions. Take care to show all points where $|f(x)| = 1$.

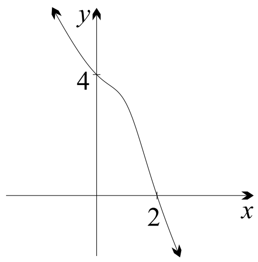
(a) $f(x) = e^x$ (b) $f(x) = \log x$ (c) $f(x) = \cos x$ (d) $f(x) = \tan x$

**5.** Sketch the reciprocal of each function shown.

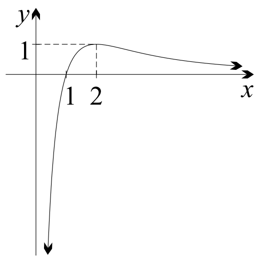
(a)



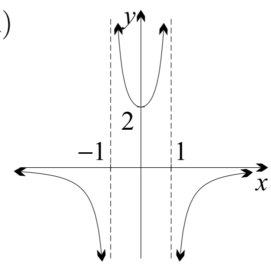

(b)



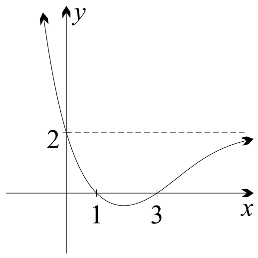

(c)



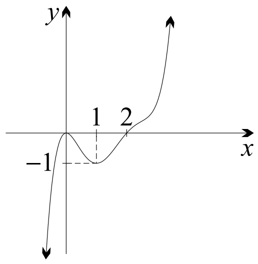

(d)




(e)




(f)




DEVELOPMENT

**6.** (a) Graph $y = \dfrac{2+x}{x}$ by first noting that $y = 1 + \dfrac{2}{x}$.

(b) Hence graph $y = \dfrac{x}{2+x}$.

**7.** (a) Graph $y = \dfrac{(x^2-1)}{x^2+1}$ by first noting that $y = 1 - \dfrac{2}{x^2+2}$.

(b) Hence graph $y = \dfrac{x^2+1}{x^2-1}$.

**8.** Consider the function $f(x) = \frac{1}{2}\left(3x - x^3\right)$.
(a) (i) Show that $f(x)$ is odd.
(ii) Find the $x$-intercepts.
(iii) Show that $f(1) = 1$. Hence solve $f(x) = 1$.
(iv) Use the theory of polynomials to explain why there is a stationary point at $(1, 1)$.
(v) Graph $y = f(x)$, showing all relevant features.
(b) Hence graph $y = \dfrac{1}{f(x)}$.

**9.** Follow similar steps to the previous Question to sketch $y = \dfrac{6}{x^3 - 7x}$.

**10.** If the graph of $y = f(x)$ has a vertical asymptote at $x = a$, then the graph of the reciprocal approaches an $x$-intercept at $x = a$. Further, the nature of the asymptote determines the behaviour of the reciprocal, as the following two examples demonstrates. In each case graph $y = g(x)$ where $g(x) = \dfrac{1}{f(x)}$, paying particular attention to the shape near $x = 2$.
(a) $f(x) = \dfrac{1}{(x-2)^2}$. (b) $f(x) = \dfrac{1}{(x-2)^3}$.

**11.** Prove that odd and even symmetry is preserved for $y = \dfrac{1}{f(x)}$.

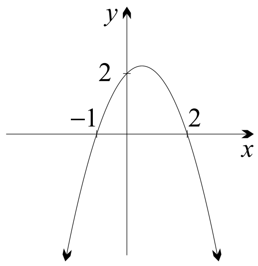
**12.** (a) Given that $g(x) = \dfrac{1}{f(x)}$, use the chain rule to differentiate $g(x)$ and thus explain why the sign of $g'$ is the opposite of the sign of $f'$.
(b) Hence explain why the nature of stationary points is reversed for $y = g(x)$.

**EXTENSION**

**13.** Use differentiation from first principles to show that if $g = \dfrac{1}{f}$ then $g' = -\dfrac{f'}{f^2}$.

# 8E More Reflections

This section examines the seven cases of reflections in the axes that can occur when the absolute value function is applied. In some cases the result is a symmetric graph. As a means of comparison, the same example function, $f(x) = 2 + x - x^2$, is used throughout the theory and is graphed on the right. Other reflections and symmetries may be obtained by combining this section with previous work. Some examples are given in the exercise.

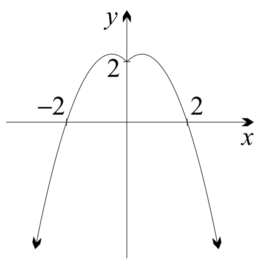



**Symmetry in the $y$-axis:** Let $g(x) = f(|x|)$ then by definition

$$\begin{aligned} g(x) &= f(|x|) \\ &= f(x) \quad \text{for } x \geq 0\,. \end{aligned}$$

Further $g$ is an even function since

$$\begin{aligned} g(-x) &= f(|-x|) \\ &= f(|x|) \\ &= g(x)\,. \end{aligned}$$



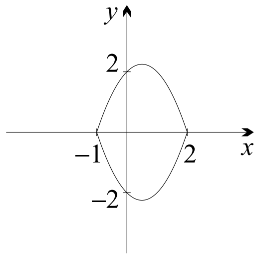
Thus the graph of $y = f(|x|)$ is that part of $y = f(x)$ for which $x \ge 0$, plus its reflection in the $y$-axis. The graph above shows the situation.

**17** SYMMETRY IN THE $Y$-AXIS: The graph of $y = f(|x|)$ is that part of $y = f(x)$ for which $x \ge 0$, plus its reflection in the $y$-axis.

**Symmetry in the *x*-axis:** Consider the relation $|y| = f(x)$. Clearly $|y|$ cannot be negative, thus the domain of $f(x)$ is restricted so that $f(x) \ge 0$. Then, by the definition of the absolute value function

$$y = f(x) \text{ or } -y = f(x)$$

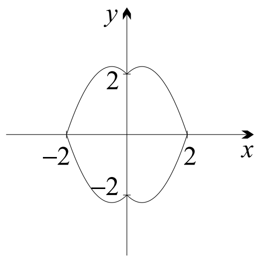
Thus the result is that part of $y = f(x)$ which lies above the $x$-axis, plus its reflection in the $x$-axis, as shown on the right for $f(x) = 2 + x - x^2$.




**18** SYMMETRY IN THE $X$-AXIS: The graph of $|y| = f(x)$ is that part of $y = f(x)$ for which $f(x) \ge 0$, plus its reflection in the $x$-axis.

**Symmetry in Both Axes:** The relation $|y| = f(|x|)$ is identical to $y = f(x)$ whenever both $x$ and $y$ are positive, so begin by graphing that part of $y = f(x)$ which lies in the first quadrant.

As was seen above, the presence of $|x|$ means that the graph is symmetric in the $y$-axis, so add in the second quadrant the reflection of the first quadrant. Lastly, the presence of $|y|$ means that the graph is also symmetric in the $x$-axis, so finally in the third and fourth quadrants add the reflection of what has already been drawn. Thus the result is symmetric in both axes.

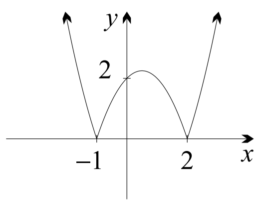



**19** SYMMETRY IN BOTH AXES: The graph of $|y| = f(|x|)$ is that part of $y = f(x)$ which lies in the first quadrant, plus its reflection in the axes.

**Reflection in the *x*-axis:** The function $y = \big|f(x)\big|$ can be written as:

$$y = \begin{cases} f(x) & \text{for } f(x) \ge 0, \\ -f(x) & \text{for } f(x) < 0. \end{cases}$$

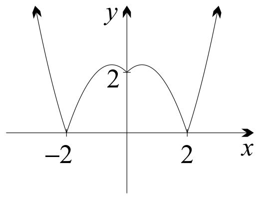
Thus the graph is unchanged wherever $f(x) \ge 0$ and is reflected in the $x$-axis wherever $f(x) < 0$, which is shown in the graph on the right.




**20** REFLECTION IN THE $X$-AXIS: The graph of $y = \big|f(x)\big|$ is the same as $y = f(x)$ at points where $f \ge 0$, and is the result of reflecting $y = f(x)$ in the $x$-axis wherever $f < 0$.

**Reflection in the *x*-axis and Symmetry in the *y*-axis:** The function $y = \left|f(|x|)\right|$ is the same as $y = \left|f(x)\right|$ for $x \geq 0$. The graph of $y = \left|f(|x|)\right|$ is thus the portion of $y = \left|f(x)\right|$ to the right of the $y$-axis, plus its reflection in the $y$-axis, as shown on the right.



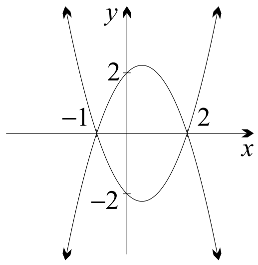

| **21** | REFLECTION IN THE $X$-AXIS AND SYMMETRY IN THE $Y$-AXIS: In order to graph $y = \left|f(|x|)\right|$ first graph $y = |f(x)|$ for $x \geq 0$, then add the reflection in the $y$-axis. |
|---|---|

**Reflection Symmetry in the *x*-axis:** One definition of the absolute value function is $|x| = \sqrt{x^2}$. It is this definition which is most useful to analyse the relation $|y| = \left|f(x)\right|$. Thus begin by squaring both sides to get:

$$y^2 = f^2$$

so $\quad y = f(x)$ or $-f(x)$.

Hence the graph of $|y| = \left|f(x)\right|$ is the graph of $y = f(x)$ plus its reflection in the $x$-axis.

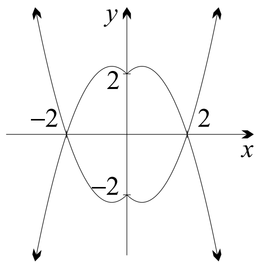



| **22** | REFLECTION SYMMMETRY IN THE $X$-AXIS: The graph of $|y| = \left|f(x)\right|$ is the graph of $y = f(x)$ plus its reflection in the $x$-axis. |
|---|---|

**Symmetry in the *y*-axis and Reflection Symmetry in the *x*-axis:** There are several ways to interpret the relation $|y| = \left|f(|x|)\right|$. One approach is to use the previous case, since

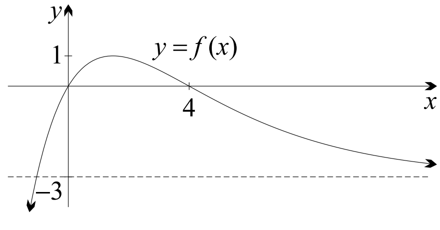
$$\left|f(|x|)\right| = \left|f(x)\right| \text{ for } x \geq 0\,.$$

Thus, begin by graphing $y = f(x)$ for $x \geq 0$. Then add its reflection in the $x$-axis. Next note that $|y| = \left|f(|x|)\right|$ is symmetric in the $y$-axis, so finally add in the second and third quadrants the reflection of what has already been drawn.



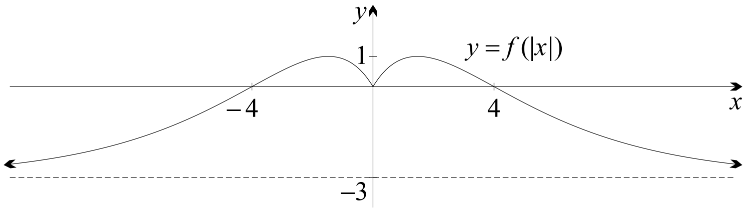

| **23** | SYMMETRY IN THE $Y$-AXIS AND REFLECTION SYMMETRY IN THE $X$-AXIS: In order to graph $|y| = \left|f(|x|)\right|$ first graph $y = f(x)$ for $x \geq 0$, then add the reflection in the $x$-axis. Finally add the reflection in the $y$-axis. |
|---|---|

**WORKED EXERCISE:** The graph of $y = f(x)$ is shown below and has $x$-intercepts at the origin and $(4, 0)$. The maximum is $y = 1$ and $y = -3$ is an asymptote.

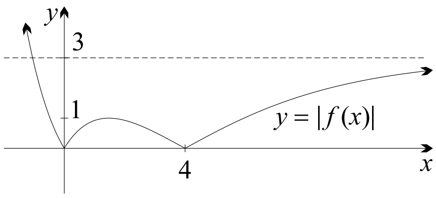



Sketch graphs of: (a) $y = f(|x|)$, (b) $y = |f(x)|$, (c) $|y| = f(|x|)$

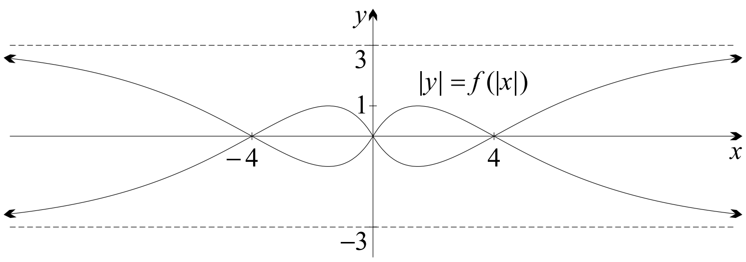
**SOLUTION:** (a) The graph of $y = f(|x|)$ is the same as $y = f(x)$ for $x > 0$, plus its reflection in the $y$-axis, as shown below.




(b) The graph of $y = |f(x)|$ is the result of reflecting $y = f(x)$ in the $x$-axis wherever $f < 0$.



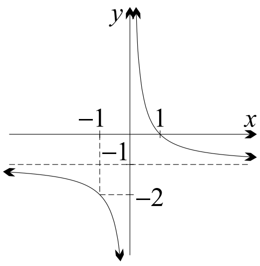

(c) The graph of $|y| = f(|x|)$ is the same as that shown in part (a) plus its reflection in the $x$-axis.



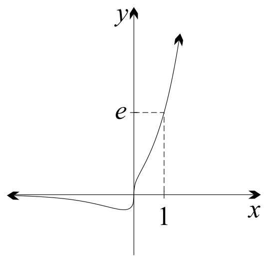

## Exercise **8E**

**1.** Use the given graph of $y = f(x)$ to sketch (i) $|y| = f(x)$, (ii) $y = |f(x)|$, (iii) $|y| = |f(x)|$.

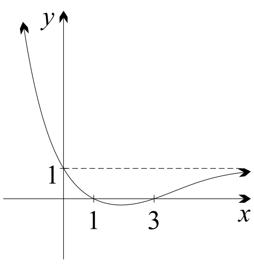
(a)



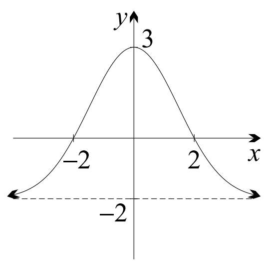

(b)



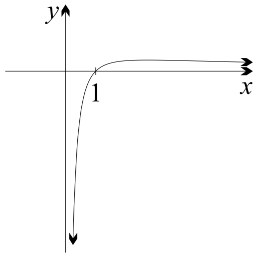

(c)

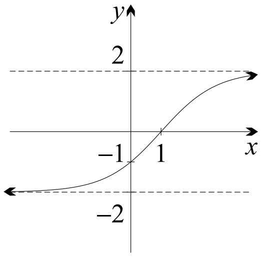


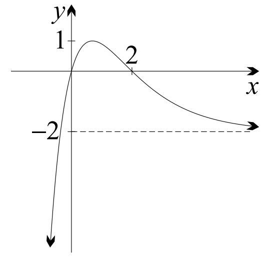

(d)



**2.** In each case use the given graph of $y = f(x)$ to sketch (i) $y = f(|x|)$, (ii) $|y| = f(|x|)$, (iii) $y = \big|f(|x|)\big|$ and (iv) $|y| = \big|f(|x|)\big|$.

(a)



(b)



(c)



**DEVELOPMENT**

**3.** In each case sketch the graphs of (i) $y = f(x)$, (ii) $y = f(|x|)$, (iii) $|y| = f(x)$, (iv) $|y| = f(|x|)$, (v) $y = \big|f(x)\big|$, (vi) $y = \big|f(|x|)\big|$, (vii) $|y| = \big|f(x)\big|$, (viii) $|y| = \big|f(|x|)\big|$.

(a) $f(x) = 2x - x^2$ (b) $f(x) = x^3 - 3x$

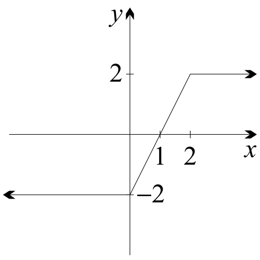
**4.** Repeat Question 3 for the following functions:

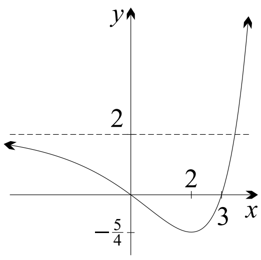
(a) $f(x) = \log x$ (b) $f(x) = 1 - \dfrac{1}{x}$ (c) $f(x) = \sin x$

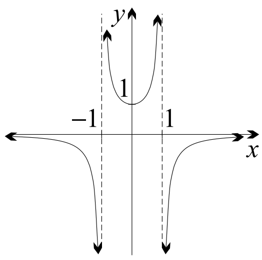
**5.** In each case sketch the graphs of (i) $y = f(x) + \big|f(x)\big|$, and (ii) $y = f(x) - \big|f(x)\big|$.

(a) $f(x) = x^2 - 1$ (b) $f(x) = 1 + \dfrac{1}{x}$ (c) $f(x) = e^x - 1$ (d) $f(x) = \cos x$

**6.** In each case use the given graph of $y = f(x)$ to sketch graphs of (i) $y = f(x) + \big|f(x)\big|$, and (ii) $y = f(x) - \big|f(x)\big|$.

(a)



(b)



(c)



**7.** Sketch the following.

(a) $|x| + |y| = 1$ (b) $|x| - |y| = 1$ (c) $\big||x| - |y|\big| = 1$

**EXTENSION**

**8.** (a) If $f(x)$ is odd, prove that the graphs of $y = |f(x)|$ and $y = \big|f(|x|)\big|$ are identical.
(b) Which other pairs of graphs are identical whenever $f(x)$ is odd?
(c) Prove the result for each pair you find.
(d) Investigate the situation when $f(x)$ is even.
(e) Is it possible for a pair to be identical when the function is neither even nor odd?

# 8F Integer Powers

If the graph of $y = f(x)$ is known then it is possible to determine the shape of the graph of $y = \Big(f(x)\Big)^n$, where $n$ is an integer.

Despite the title of this section, only positive integer powers will be considered. Negative indices can be accounted for by splitting the operation into two steps, a positive integer power and a reciprocal. Thus in order to graph $y = f^{-3}$ first graph $y = f^3$ and then graph its reciprocal.

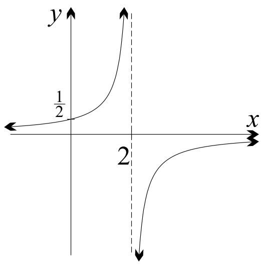
**Domain, Intercepts, Sign and Asymptotes:** If $n$ is a positive integer then the domain, $x$-intercepts and location of vertical asymptotes do not change. Clearly the $y$-intercept and ordinate of any horizontal asymptote will be affected. Note that $f^n \geq 0$ for all $x$ whenever $n$ is even, and when $n$ is odd $f^n$ has the same sign as $f$. Thus the nature of a vertical asymptote may change if $n$ is even. Consider the following example:

if $\quad f(x) = \dfrac{1}{2-x}$

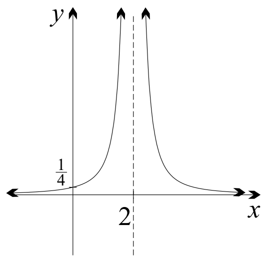
then $\quad f \to -\infty \quad$ as $\quad x \to 2^+$

but $\quad f^2 \to +\infty \quad$ as $\quad x \to 2^+$.

This change in nature is clearly evident in the graphs below.




$y = \dfrac{1}{2-x}$




$y = \dfrac{1}{(2-x)^2}$

**Symmetry:** The symmetries that result from the powers of odd and even functions are summarised in the following table, and the proofs are left to the exercise.

| $f^n$ | $n$ odd | $n$ even |
|---|---|---|
| $f$ odd | odd | even |
| $f$ even | even | even |

Other symmetries should be investigated as the need arises.

**The Calculus:** Let $g(x) = \big(f(x)\big)^n$, then

$$\frac{dg}{dx} = \frac{dg}{df} \times \frac{df}{dx} \quad \text{(by the chain rule)}$$
$$= nf^{n-1} \times f'.$$

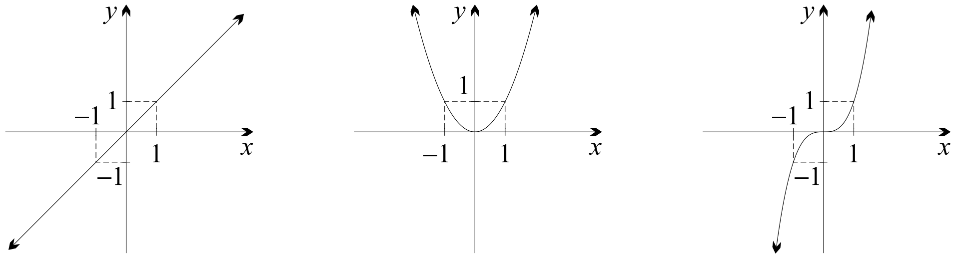
Thus the stationary points of $f$ are also stationary points of $g$. Additionally there are stationary points located at the solutions of $f = 0$, that is at the $x$-intercepts of $y = f(x)$. The nature of a stationary point depends on $n$ and the nature of $f$ and $f'$. Various cases are presented in the exercise questions.

**24** STATIONARY POINTS:

The stationary points of $y = f(x)$ are stationary points of $y = \big(f(x)\big)^n$.

In addition, the zeros of $f(x)$ are also stationary points of $y = \big(f(x)\big)^n$.

This feature of additional stationary points at the $x$-intercepts of powers has been met before, as is demonstrated in the graphs below of $y = x$ , $y = x^2$ and $y = x^3$ .




**Other:** Three other observations can be made about the above graphs of $y = x$ , $y = x^2$ and $y = x^3$ , which apply to all graphs of positive integer powers.

Firstly, if $|f| = 1$ then $|f^n| = 1$ . Thus it is usual to plot points where $|f| = 1$ if the $x$-coordinates are easy to find. The other two features are: if $0 < |f| < 1$ then $|f^n| < |f|$ , and if $|f| > 1$ then $|f^n| > |f|$ . The proofs are in the exercise.

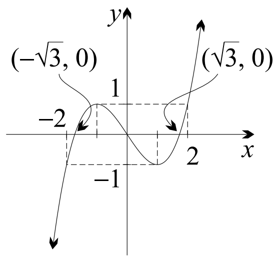
**25**

> RELATIVE HEIGHT: The ordinates of $y = f(x)$ and $y = \big(f(x)\big)^n$ satisfy the following.
> 1. If $|f| = 1$ then $|f^n| = 1$ .
> 2. If $0 < |f| < 1$ then $|f^n| < |f|$ .
> 3. If $|f| > 1$ then $|f^n| > |f|$ .

**WORKED EXERCISE:** (a) Sketch the graph of $y = f(x)$ , where $f(x) = \frac{1}{2}x(x^2 - 3)$ , showing the intercepts with the axes and the stationary points.
(b) Hence sketch the graphs of (i) $y = f^2$ and (ii) $y = f^3$ .

**SOLUTION:** (a) Clearly the intercepts are at the origin, $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$ . It should also be clear that $f(x)$ is odd since it is a polynomial with odd powers.

Now $\quad f'(x) = \frac{3}{2}(x^2 - 1)$
and $\quad f''(x) = 3x$ ,
so $\quad f'(x) = 0 \quad$ at $\quad x = 1$
and $\quad f''(1) = 3$ .

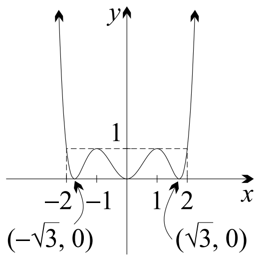



Hence there is a minimum stationary point at $(1, -1)$ and, by the odd symmetry, there is a maximum at $(-1, 1)$ . Finding other points where $f(x) = 1$ leads to

$$x^3 - 3x - 2 = 0 \, .$$

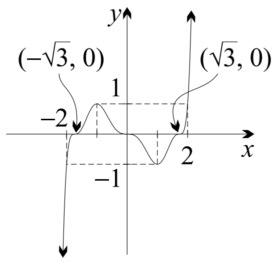
Since there is a stationary point at $(-1, 1)$ , it is evident that $(x+1)^2$ is a factor of the left hand side. Thus by the product of the zeros we know that the remaining zero is $x = 2$ , and $(2, 1)$ is on the curve. Again, symmetry yields the point $(-2, -1)$ on the curve, which is sketched above.

(b) (i) Clearly the resulting function is even and either positive or zero. Thus it is efficient to begin by sketching $y = f^2$ in the first quadrant and then add the reflection in the $y$-axis. There is a stationary point at $(1, 1)$ , and additional stationary points at the $x$-intercepts, $(0, 0)$ and $(\sqrt{3}, 0)$ . The graph also has height 1 at $(2, 1)$ . Here it is.



(ii) Since $y = f$ is odd, its cube is also odd and has the same sign everywhere. Thus it is efficient to begin by sketching $y = f^3$ in the first and fourth quadrants, and then use symmetry to draw the remainder. There is a stationary point at $(1, -1)$, and additional stationary points at the $x$-intercepts, $(0, 0)$ and $(\sqrt{3}, 0)$. The graph has height 1 at $(2, 1)$ and is drawn on the right.




## Exercise 8F

**1.** Let $f(x) = x - 1$. Graph the following.

(a) $y = f(x)$ (b) $y = \big(f(x)\big)^2$ (c) $y = \big(f(x)\big)^3$ (d) $y = \big(f(x)\big)^4$

**2.** (a) Sketch $y = f(x)$, where $f(x) = \frac{1}{4}(4 - x^2)$, showing points where $|y| = 1$.

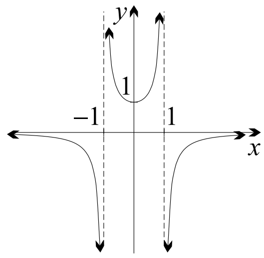
(b) Hence sketch $y = \big(f(x)\big)^2$.

**3.** Use the graph of $y = x^2 - 1$ to help sketch:

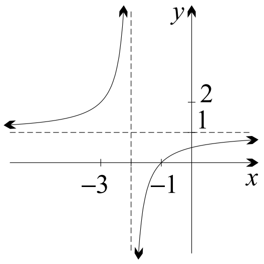
(a) $y = x^4 - 2x^2 + 1$ (b) $y = x^6 - 3x^4 + 3x^2 - 1$

**4.** In each case use the given graph of $y = f(x)$ to help sketch $y = f^2$.

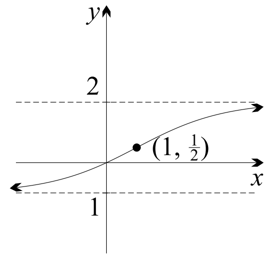
(a)



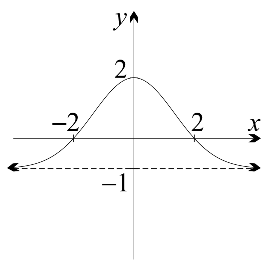

(b)



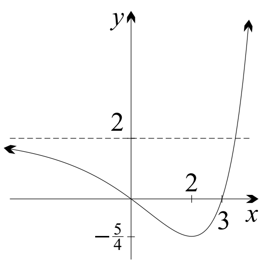

(c)



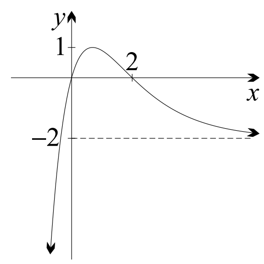

(d)




(e)




(f)




**DEVELOPMENT**

**5.** In each case draw $y = f^2$ for the given funtion. It may help to draw $y = f(x)$ first.

(a) $f(x) = \log x$ (b) $f(x) = \cos x$ (c) $f(x) = e^x - 1$ (d) $f(x) = \sqrt{x + 2}$

**6.** The cubic $f(x) = \frac{1}{4}(x + 1)^2(2 - x)$ has zeros at $x = -1$ and $2$.

(a) Show that the graph of $y = f(x)$ has a maximum turning point at $(1, 1)$. Hence find the other point on the graph where $y = 1$. (Do NOT attempt to find the point where $y = -1$.)

(b) Where is the local minimum?

(c) Graph $y = f(x)$, showing these features. (d) Hence sketch $y = \frac{1}{16}(x + 1)^4(x - 2)^2$.

**7.** The cubic $f(x) = \frac{1}{2}(x-2)^2(x+1)$ has zeros at $x = -1$ and $2$.

(a) Show that the graph of $y = f(x)$ passes through $(1, 1)$ and hence find the other points on the graph where $y = 1$. (Do NOT attempt to find the point where $y = -1$.)

(b) Show that the $y$-intercept is a local maximum. Where is the local minimum?

(c) Graph $y = f(x)$, showing these features.

(d) Hence sketch $y = f^2$.

(e) Hence determine the number of real roots of $\Big(f(x)\Big)^2 = 2$.

**8.** Prove the following two results, as asserted in Box 19.

(a) If $0 < |f| < 1$ then $|f^n| < |f|$.

(b) If $|f| > 1$ then $|f^n| > |f|$.

**9.** Prove the symmetry results in the table for powers of odd and even functions.

**EXTENSION**

**10.** Suppose that the function $f(x)$ is continuous and differentiable everywhere, and that $f(2) = 0$. If the graph of $y = \big(f(x)\big)^n$ is drawn, under what circumstances will the point $(2, 0)$ be:

(a) a local minimum,

(b) a local maximum,

(c) a horiztonal inflexion point?

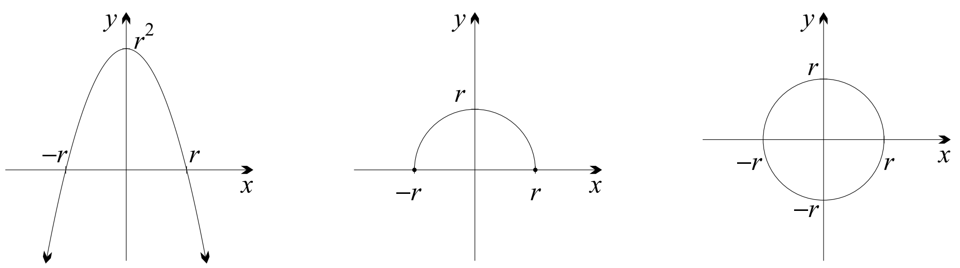
# 8G Square Roots

This section looks at how the graph of $y = \sqrt{f(x)}$ might be determined from the graph of $y = f(x)$. The relation $y^2 = f(x)$ is included here for completeness. Algebraicly, observe that if $y^2 = f(x)$ then

$$y = \sqrt{f(x)} \quad \text{or} \quad -\sqrt{f(x)}\,.$$

Thus the graph of $y^2 = f(x)$ is the union of the graph of $y = \sqrt{f(x)}$ with its reflection in the $x$-axis.

The case when $f(x)$ is a quadratic is significant. Let $f(x) = r^2 - x^2$, then the graph of $y = f(x)$ is a concave down parabola. The graph of $y = \sqrt{f(x)}$ is a semi-circle whilst $y^2 = f(x)$ is the entire circle, centre the origin and radius $r$.




Other fractional indices may be encountered in this course, and candidates should be able to analyse straight forward examples. Consequently, some questions on other fractional indices have been included in the exercise.

**Domain and Intercepts:** Since the square root of a negative is not real, it follows that the domain of $\sqrt{f}$ is the domain of $f$ with the additional restriction that $f \geq 0$. The same is true for $y^2 = f$, since $y^2$ also cannot be negative. This feature of the graphs is evident in the example $f(x) = r^2 - x^2$, above. The domain for both $y = \sqrt{r^2 - x^2}$ and $y^2 = r^2 - x^2$ is clearly $-r \leq x \leq r$.

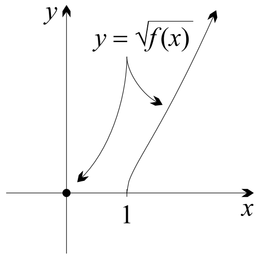
**26** DOMAIN: The domain of $y = \sqrt{f(x)}$ is the domain of $f(x)$ with the additional restriction that $f(x) \geq 0$.

The $x$-intercepts of $y = \sqrt{f}$ are the same as for $y = f(x)$. For some functions this leads to isolated points. Consider the function $f(x) = x^3 - x^2$. Solving $f(x) \geq 0$

$$x^2(x-1) \geq 0$$

so $$x = 0 \text{ or } x \geq 1.$$

Thus the domain of $y = \sqrt{f(x)}$ is the continous region $x \geq 1$ plus the isolated point $x = 0$. Care must be taken to clearly mark these isolated points on the graph, as shown on the right.




**Symmetry:** If $f(x)$ is an even function then so too is $y = \sqrt{f(x)}$. The proof is straight forward and is left as an exercise. If $f(x)$ is an odd function then the graph of $y = \sqrt{f(x)}$ is neither even nor odd. The proof is in the exercise.

The same observations can be made about $y^2 = f(x)$, but this graph has an additional symmetry, namely symmetry in the $x$-axis, as was stated earlier.

**The Calculus:** Let $g(x) = \sqrt{f(x)}$, then

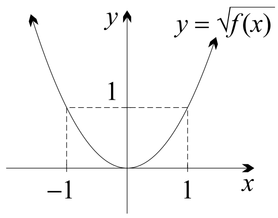
$$\frac{dg}{dx} = \frac{dg}{df} \times \frac{df}{dx} \quad \text{(by the chain rule)}$$
$$= \frac{f'}{2\sqrt{f}}.$$

Thus the stationary points of $f$ have the same $x$-coordinate as the stationary points of $g$, provided $f \neq 0$ simultaneously. If $f(x)$ has a stationary point at $x = a$ and $f(a) = 0$, the precise behaviour of $y = g(x)$ is unclear. In that case, further investigation is required, such as finding $\lim_{x \to a} g'(x)$ or differentiation from first principles.

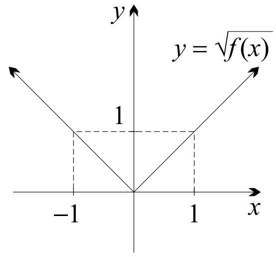
**27** STATIONARY POINTS:
(a) If $y = f(x)$ has a stationary point at $x = a$ and $f(a) \neq 0$ then $y = \sqrt{f(x)}$ also has a stationary point at $x = a$.
(b) If $y = f(x)$ has a stationary point at $x = a$ and $f(a) = 0$ then the behaviour of $y = \sqrt{f(x)}$ requires further investigation.

To demonstrate the problem, consider the following three short examples.

The function $f(x) = x^4$ clearly has a minimum turning point at $x = 0$ and $f(0) = 0$. The function $g(x) = \sqrt{f(x)}$ is identically $g(x) = x^2$. Thus in this case $g(x)$ also has a minimum turning point at $x = 0$.



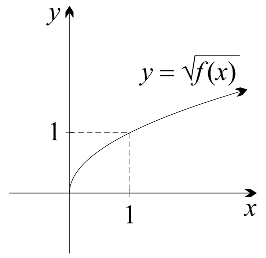
The function $f(x) = x^2$ has a minimum turning point at $x = 0$, but the behaviour of $g(x)$ differs.




$$\begin{aligned} g(x) &= \sqrt{f(x)} \\ &= \sqrt{x^2} \\ &= |x| \quad \text{(by definition)} \end{aligned}$$

Thus $g(x)$ does indeed have a minimum, but its nature is different. It is no longer a stationary point, but instead is a critical point.

To complete the sequence, consider $f(x) = x$, despite the fact that it does not have a stationary point. In this case




$$\begin{aligned} y &= g(x) \\ &= \sqrt{x} \end{aligned}$$

or, rearranging,

$$y^2 = x \text{ for } y \geq 0\,.$$

Thus the graph is the upper half of the parabola $y^2 = x$. In this case the derivative function is undefined at $x = 0$. Nevertheless, from the geometry of the curve, the tangent there is known to be the vertical line $x = 0$.

Strictly speaking, the calculus of $y^2 = f(x)$ requires implicit differentiation, which is considered in detail in Section 8I. Fortunately all that is required in this case is a simple application of the chain rule. Thus

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}f(x)$$

yields $\quad \dfrac{d}{dy}\left(y^2\right)\dfrac{dy}{dx} = f'(x) \quad$ (by the chain rule)

so $\quad 2y\,\dfrac{dy}{dx} = f'(x)$

hence $\quad \dfrac{dy}{dx} = \dfrac{f'(x)}{2y}\,.$

Observe that the derivative is now a function of both $x$ and $y$, and is undefined whenever $y = 0$. Thus, provided $y \neq 0$, it is clear from this that the $x$-coordinates of the stationary points of $y = f(x)$ and $y^2 = f(x)$ are the same.

**Other:** It should be clear that if $y = \sqrt{f(x)}$ then $y = 1$ when $f(x) = 1$. Thus it is usual to plot points where $f(x) = 1$ if the $x$-coordinates are easy to find. Lastly, it should be clear that if $0 < f < 1$ then $\sqrt{f} > f$, and if $f > 1$ then $\sqrt{f} < f$. These features are evident when the graph of $y = x$ is compared with the graph of $y = \sqrt{x}$ above.

**WORKED EXERCISE:** Consider the function $f(x) = 2\big(1 + \cos(\pi x)\big)$ for $0 \leq x \leq 4$.

(a) Sketch $y = f(x)$.

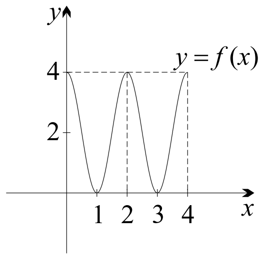
(b) (i) Use the double angle identities to rewrite $f(x)$ in terms of $\cos\left(\frac{\pi}{2}x\right)$.

(ii) Hence sketch $y = \sqrt{f(x)}$.

(iii) At which values of $x$ do the two graphs intersect?

**SOLUTION:**

(a) This is a wave with amplitude 2, centre $y = 2$ and wavelength 2, as shown on the right.

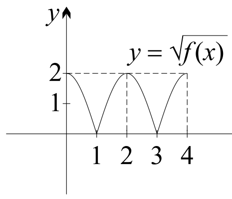



(b) (i) Since $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ it follows that

$$\begin{aligned} f(x) &= 2\big(1 + \cos(\pi x)\big) \\ &= 4\cos^2\left(\tfrac{\pi}{2}x\right) . \end{aligned}$$

(ii) Note that $\sqrt{x^2} = |x|$ , hence

$$\begin{aligned} \sqrt{f(x)} &= \sqrt{4\cos^2\left(\tfrac{\pi}{2}x\right)} \\ &= 2\left|\cos\left(\tfrac{\pi}{2}x\right)\right| . \end{aligned}$$




The is a wave of amplitude 2, centre $y = 0$ and wavelength 4, with those portions below the $x$-axis reflected in the $x$-axis. Here it is sketched on the right. Note that in this case the stationary points at the $x$-intercepts of $y = f(x)$ become corners where the derivative is undefined.

(iii) The two graphs will cross when $y = 1$ . The solutions of this are given by

$$\begin{aligned} &\left|\cos\left(\tfrac{\pi}{2}x\right)\right| = \tfrac{1}{2} \\ \text{so}\quad &\cos\left(\tfrac{\pi}{2}x\right) = \tfrac{1}{2} \text{ or } -\tfrac{1}{2} \\ \text{thus}\quad &\tfrac{\pi}{2}x = \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ or } \tfrac{5\pi}{3} . \\ \text{Hence}\quad &x = \tfrac{2}{3}, \tfrac{4}{3}, \tfrac{8}{3} \text{ or } \tfrac{10}{3} . \end{aligned}$$

The two graphs will also intersect when $y = 0$ , which from the previous graphs occurs at $x = 1$ and $x = 3$ .

## Exercise 8G

**1.** For each of the following functions, graph (i) $y = \sqrt{f(x)}$ , and (ii) $y^2 = f(x)$ .

(a) $f(x) = 9 - x^2$

(b) $f(x) = x + 1$

(c) $f(x) = (x - 2)^2$

(d) $f(x) = \dfrac{2}{x^2 + 1}$

**2.** (a) Let $f(x) = \cos x$ , which is even. Graph $y = \sqrt{f(x)}$ and observe that it is also even.

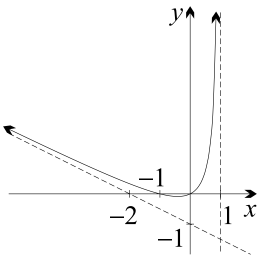
(b) Let $f(x) = \sin x$ , which is odd.

(i) Graph $y = \sqrt{f(x)}$ and observe that it is neither even nor odd.

(ii) Is there any symmetry in this graph?

**3.** In each case use the graph of $y = f(x)$ to sketch (i) $y = \sqrt{f(x)}$ , and (ii) $y^2 = f(x)$ .

(a)

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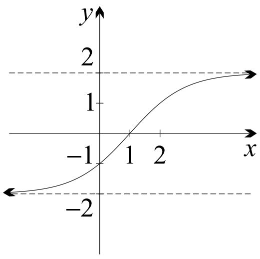

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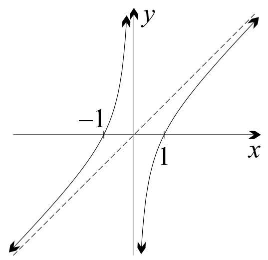


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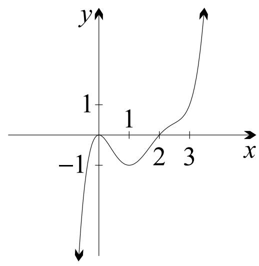



**DEVELOPMENT**

**4.** Consider the function $f(x) = 2\big(1 - \cos(\pi x)\big)$ for $0 \le x \le 4$.
(a) Sketch $y = f(x)$.
(b) (i) Use the double angle identities to rewrite $f(x)$ in terms of $\sin\left(\frac{\pi}{2}x\right)$. Hence sketch the graph of $y = \sqrt{f(x)}$.
(ii) At which values of $x$ do the two graphs intersect?

**5.** Sketch the following graphs for the function $f(x) = -x(1-x)^2$, taking care to clearly identify the isolated points.
(a) $y = \sqrt{f}$
(b) $y^2 = f$

**6.** (a) By first considering the graph of $y = x$, sketch $y = x^{\frac{1}{3}}$, taking care with the shape at the origin. [Do NOT use inverse functions as an aid.]
(b) The equation $y = x^{\frac{2}{3}}$ can be rewritten as $y = \left(x^{\frac{1}{3}}\right)^2$. Use this result and your answer to part (a) to sketch $y = x^{\frac{2}{3}}$.

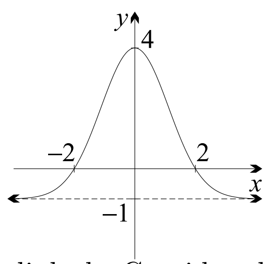
**7.** Sketch $y = (4 - x^2)^{\frac{1}{4}}$, clearly marking the intercepts with the axes.

**8.** Prove that if $f(x)$ is even then so too is $\sqrt{f(x)}$.

**9.** Suppose that $f(a) = 0$ and $f'(a) \ne 0$. Prove that the graph of $y = \sqrt{f(x)}$ has a vertical tangent at $x = a$.

**10.** Use the result in Question 9 to show that $y = \sqrt{\log x}$ has a vertical tangent at $x = 1$. Hence Sketch $y = \sqrt{\log x}$.

**11.** The graph of $y = f(x)$ is on the right. There are $x$-intercepts at $x = 2$ and $-2$, a $y$-intercept at $y = 4$, and $y = -1$ is a horizontal asymptote.
(a) Sketch $y = \sqrt{f}$.
(b) Hence sketch $y = \dfrac{1}{\sqrt{f}}$.




**12.** This question further demonstrates that the conics are intrinsicly linked. Consider the function $f(x) = r^2 - x^2$.
(a) It was noted in the text that the graph of $y^2 = f(x)$ is a circle with centre the origin and radius $r$. Show that the graph of $y^2 = -f(x)$ is a rectangular hyperbola with auxilliary circle radius $r$.
(b) The graph of $y = \sqrt{f(x)}$ is the upper semicircle. What is the graph of $y = \sqrt{-f(x)}$?
(c) Now consider the graphs of $y^2 = f(x)$ and $y^2 = -f(x)$ when $r = 0$.
(i) Draw the two graphs.
(ii) Explain goemetricly how these conics are obtained from the intersection of a plane with a cone which has a vertical axis and a semivertical angle of 45°.

EXTENSION

**13.** Under what circumstances will the graphs of $y = \sqrt{f(x)}$ and $y^2 = f(x)$ be identical?

**14.** Prove that if $f(x)$ is odd then the graph of $y = \sqrt{f(x)}$ is neither even nor odd.

**15.** The curve with equation $(x-1)(x^2+y^2) = x^2$ is one example of a type of curve called the Conchoid of De Sluze.

(a) Show that the equation may be rewritten as $y^2 = \dfrac{x^2(2-x)}{x-1}$.

(b) By noting that squares cannot be negative and denominators cannot be zero, find the domain of this curve.

(c) Graph $y = \dfrac{x^2(2-x)}{x-1}$ for this domain.

(d) Hence sketch $(x-1)(x^2+y^2) = x^2$.

# 8H Composition of Functions

A function which is built up from simpler functions by applying one followed by another is called a *composition of functions*. Thus the function $h(x) = \log(\sin e^x)$ is a composition of functions. It is the result of applying the exponential function followed by the sine function and finally the logarithmic function. The focus of this section is on compositions of just two functions, that is

$$h(x) = g\big(f(x)\big),$$

for some functions $f(x)$ and $g(x)$. For example, the composition of the square root function $g(x) = \sqrt{x}$ and the quadratic function $f(x) = 4 - x^2$ yields

$$\begin{aligned} h(x) &= g\big(f(x)\big) \\ &= g(4-x^2) \\ &= \sqrt{4-x^2}, \end{aligned}$$

which is a semi-circle with radius 2 and centre the origin when graphed.

**Domain:** Care must be taken when determining the domain of the composition of two functions. One approach is to first write down the domain of $f(x)$ and then remove any points were $g(f)$ is undefined.

**WORKED EXERCISE:** Find the domain of $\tan\sqrt{x}$.

**SOLUTION:** The domain of $f(x) = \sqrt{x}$ is $x \ge 0$ and the domain of $g(x) = \tan x$ is $x \ne \frac{(2n+1)\pi}{2}$. Thus the domain of $g\big(f(x)\big)$ is:

$$x \ge 0 \quad \text{and} \quad \sqrt{x} \ne \tfrac{(2n+1)\pi}{2}$$

or $\quad x \ge 0 \quad \text{and} \quad x \ne \left(\frac{(2n+1)\pi}{2}\right)^2$.

**Intercepts:** The $x$-intercepts of $h(x) = g\big(f(x)\big)$ can be found if the zeros of $g(x)$ are known and if $f(x)$ has an inverse function. Suppose that $g(a) = 0$, then $x$ is a zero of $h(x)$ provided

$$f(x) = a$$

or $\quad x = f^{-1}(a)$.

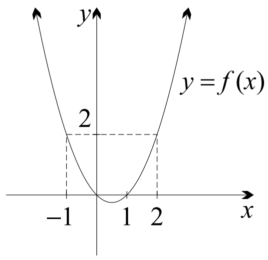
Alternatively, if the graph of $y = f(x)$ is known then add the line $y = a$ and read off the $x$-coordinate at any point of intersection.

**WORKED EXERCISE:** (a) State the zeros of $g(x) = (x-2)^2(x+2)$.

(b) Graph $y = f(x)$ where $f(x) = x(x-1)$.

(c) Hence determine the zeros of $y = g\big(f(x)\big)$.




**SOLUTION:** (a) Clearly the zeros are at $x = 2$ or $-2$.

(b) The parabola is shown on the right.

(c) From the graph it is clear that $f(x) = -2$ has no solution and that $f(x) = 2$ has two solutions, namely $x = -1$ or $2$. These are the zeros of $y = g\big(f(x)\big)$.

**Symmetry:** The symmetries that result from compositions of odd and even functions are summarised in the following table, and the proofs are left to the exercise.

| $g\big(f(x)\big)$ | $g$ odd | $g$ even |
|---|---|---|
| $f$ odd | odd | even |
| $f$ even | even | even |

The careful reader will note that the entries in this table are the same as for integer powers of functions encountered earlier in this chapter, which is hardly surprising since that topic is just a particular example of composite functions.

**The Calculus:** Finally, differentiation of $h(x)$ yields

$$h'(x) = \frac{d}{dx}\Big(g(f)\Big)$$
$$= \frac{dg}{df} \times \frac{df}{dx} \quad \text{(by the chain rule)}$$

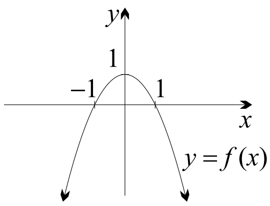
So the stationary points of $f(x)$ are stationary points of $h(x)$ provided $\dfrac{dg}{df}$ is defined at those points. Extra stationary points may be introduced at the zeros of $\dfrac{dg}{df}$, again as was noted in the section on integer powers of functions. Nevertheless, in most examples encountered in this course the calculus is not required.

**WORKED EXERCISE:** (a) Sketch the graph of $y = f(x)$ where $f(x) = 1 - x^2$

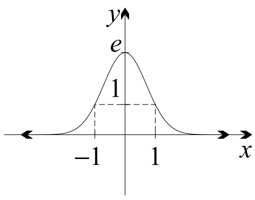
(b) Hence sketch $y = e^{f(x)}$ without resorting to the calculus.

**SOLUTION:**

(a)




(b)




Clearly the $y$-intercept of $y = e^{f(x)}$ is $(0, e)$, and $y > 0$ for all real $x$. Since $f(x)$ is even, it follows that $e^f$ is also even. Hence it is only necessary to determine the features of the graph for $x \ge 0$ and then reflect these in the $y$-axis. If $f(x) = 0$ then $e^f = 1$, hence the point $(1, 1)$ lies on the graph. As $x \to \infty$, $f(x) \to -\infty$ and hence $e^f \to 0^+$. Thus the $x$-axis is an asymptote. The graph is shown above.

## Exercise 8H

1. In each case, use a graph of $y = f(x)$ to help sketch the given composite function. The use of calculus is not required.

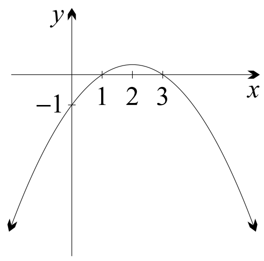
   (a) $y = e^f$, where $f(x) = 2x - x^2$

   (b) $y = e^f$, where $f(x) = \cos \pi x$

   (c) $y = e^f$, where $f(x) = \dfrac{1}{x}$

   (d) $y = \log f$, where $f(x) = e - x^2$

2. In each case, use the given graph of $y = f(x)$ to help sketch $y = e^f$.

   (a)
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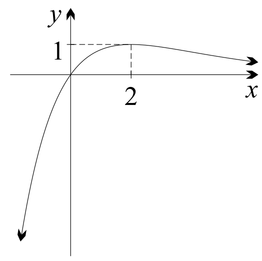
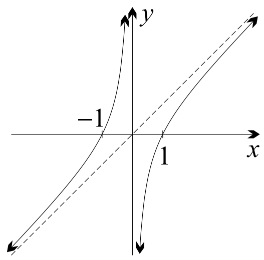
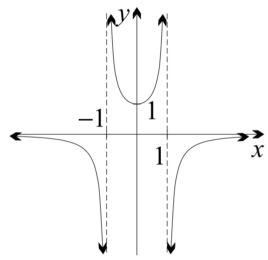
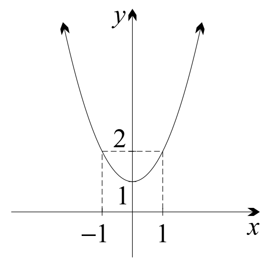



   (c)



   (d)



3. In each case, use the given graph of $y = f(x)$ to help sketch $y = \log f$.

   (a)
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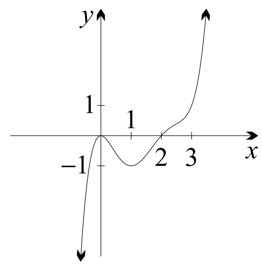


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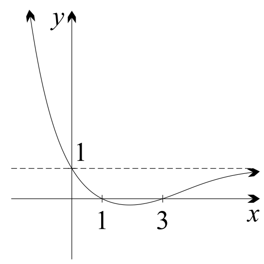


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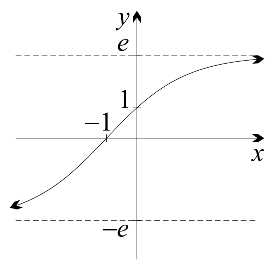



**DEVELOPMENT**

4. Carefully sketch the following composite functions. The use of calculus is not required.

   (a) $y = \cos(2^x \times \frac{\pi}{2})$

   (b) $y = \tan^{-1}\left(\dfrac{1}{x}\right)$

   (c) $y = \log(\sin x)$

   (d) $y = \sin\left(\dfrac{\pi}{x}\right)$

**5.** Sketch the following composite trigonometric functions.

(a) $y = \sin\left(\sin^{-1} x\right)$ (c) $y = \cos\left(\sin^{-1} x\right)$ (e) $y = \sin\left(\cos^{-1} x\right)$

(b) $y = \sin^{-1}(\sin x)$ (d) $y = \sin^{-1}(\cos x)$ (f) $y = \cos^{-1}(\sin x)$

**6.** (a) Carefully graph $y = \log\left(x^2 + \frac{3}{2}x\right)$.

(b) Also graph $y = \log(x + \frac{3}{2}) + \log x$, and hence show that the two functions are not equivalent, unless the domain in part (a) is restricted.

**7.** (a) Graph $y = e^{\tan x}$, clearly showing what happens at odd multiples of $\frac{\pi}{2}$.

(b) Sketch $y = e^{(1-x^2)^{-1}}$, clearly indicating the behaviour at $x = 1$ and $x = -1$.

**8.** Prove the results in the table for compositions of odd and even functions.

**9.** Sketch the following where $f(x) = \dfrac{2x}{1+x^2}$. It may be useful to sketch $y = f(x)$ first.

(a) $y = \log f$ (b) $y = e^f$ (c) $y = \tan^{-1} f$ (d) $y = \sin^{-1} f$

**EXTENSION**

**10.** (a) In Question 7(a), the graph appears to be horizontal in the limit as $x \to \left(\frac{\pi}{2}\right)^+$. Prove that this is the case.

(b) Prove a similar result for the graph in Question 7(b).

**11.** (a) Show that $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = 1$ and that $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x} = -1$.

(b) Hence graph $y = x + \sqrt{x^2+1}$.

(c) Show that $\log(x + \sqrt{x^2+1})$ is odd.

(d) Hence graph $\log(x + \sqrt{x^2+1})$.

(e) What is the inverse of this function?

# 8I Simple Implicit Equations

In many cases it is necessary or convenient to specify the equation of a curve implicitly, that is to say, $y$ is not given as a function of $x$. A familiar example is the equation of a circle, $x^2 + y^2 = r^2$. Like the equation of a circle, an implicit equation often represents the graph of a relation, and thus $y$ cannot be written explicitly as a function of $x$. There are exceptions however, such as the hyperbola $xy = c^2$, which can be written in function form as $y = \dfrac{c^2}{x}$.

**Implicit Differentiation:** When differentiating implicit expressions and equations, the usual rules of differentiation apply, with the added condition that $y$ is treated as an unknown function of $x$.

**WORKED EXERCISE:** Given that $y$ is a function of $x$, differentiate $x^3y$.

**SOLUTION:**
$$\begin{aligned}\frac{d}{dx}\left(x^3y\right) &= \frac{d}{dx}\left(x^3\right) \times y + x^3 \times \frac{d}{dx}(y) \quad \text{(by the product rule)}\\ &= 3x^2y + x^3\frac{dy}{dx}.\end{aligned}$$

Often the chain rule is required to simplify the derivatives of functions of $y$, as in the following worked exercise.

**WORKED EXERCISE:** Given that $y$ is a function of $x$, differentiate $x^2 + xy + y^2$.

**SOLUTION:**
$$\frac{d}{dx}\left(x^2 + xy + y^2\right)$$
$$= 2x + y + x \times \frac{dy}{dx} + \frac{d}{dy}\left(y^2\right)\frac{dy}{dx} \quad \text{(by the chain rule)}$$
$$= 2x + y + xy' + 2yy'.$$

| | |
|---|---|
| **28** | IMPLICIT DIFFERENTIATION: The usual rules of differentiation apply, with the added condition that $y$ is treated as an unknown function of $x$. The chain rule may be required to simplify the derivatives of functions of $y$. |

**Tangents and Normals:** When the derivative is found implicitly the result will generally involve both $x$ and $y$. Thus if the coordinates of a point on the curve are known then the equations of the tangent and normal may be found.

**WORKED EXERCISE:** Consider the hyperbola with equation $x^2 - 7y^2 = 9$.
(a) Find the gradient of the tangent at a point $P(x, y)$ on the hyperbola.
(b) What is the equation of the tangent to this hyperbola at $A(4, 1)$?
(c) Where on the hyperbola is the gradient of the tangent undefined?

**SOLUTION:** (a) Differentiating the equation implicitly yields
$$2x - 14y\frac{dy}{dx} = 0$$
so
$$\frac{dy}{dx} = \frac{x}{7y}.$$
(b) At $A(4, 1)$, $y' = \frac{4}{7}$, hence the equation of the tangent is
$$y - 1 = \tfrac{4}{7}(x - 4)$$
or $\quad 4x - 7y - 9 = 0$.
(c) Clearly the derivative is undefined wherever $y = 0$, that is at the $x$-intercepts. This is expected since the tangents are vertical there.

**Curve Sketching:** Implicit differentiation can be used to help graph the curves of simple implicit equations. In most cases the derivative will be a fraction and, as with functions, it is usual to look for stationary points where the numerator is zero and the denominator is non-zero. The curve will have a vertical tangent at points where the numerator is non-zero and the denominator is zero. There may be other points where the tangent is horizontal or vertical, but these are not dealt with in this text.

| | |
|---|---|
| **29** | HORIZONTAL AND VERTICAL TANGENTS: If the derivative is a fraction then:<br>• the curve has horizontal tangents at points where the numerator is zero and the denominator is non-zero.<br>• the curve has vertical tangents at points where the numerator is non-zero and the denominator is zero.<br>• there may be other points on the curve with horizontal or vertical tangents. |

**WORKED EXERCISE:** Consider the ellipse with equation $25x^2-32xy+16y^2=144$.

(a) (i) The given equation is a quadratic in $y$. Use the discriminant to determine the domain, and find the coordinates of the endpoints.
 (ii) Similarly find the range and the coordinate of the endpoints.
 (iii) Find the intercepts with the axes.

(b) Show that the relation is unchanged when both $x$ and $y$ are replaced by $-x$ and $-y$. What symmetry does this indicate?

(c) (i) Show that $\dfrac{dy}{dx}=\dfrac{25x-16y}{16(x-y)}$.
 (ii) Hence locate the points on the ellipse with horizontal tangents.
 (iii) Investigate the derivative at the end points of the domain.

(d) Sketch the curve.

**SOLUTION:**

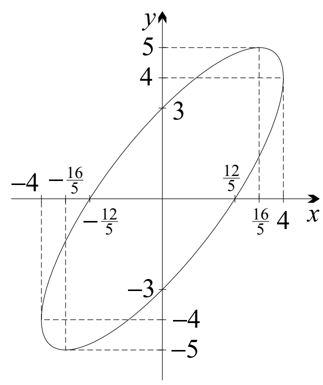
(a) (i) Rearranging,

$$16y^2-32xy+(25x^2-144)=0$$

so $\quad \Delta_y = 32^2x^2-4\times 16\times(25x^2-144)$
$\quad = 576(16-x^2)\ge 0\,,$

hence $\quad -4\le x\le 4\,.$

Substitution yields the endpoints
$(-4,-4)$ and $(4,4)$.




 (ii) Likewise,

$$\Delta_x = 32^2y^2-4\times 25\times(16y^2-144)$$
$$= 576(25-y^2)\ge 0\,,$$

hence $\quad -5\le y\le 5\,.$

Substitution yields the endpoints
$(-\frac{16}{5},-5)$ and $(\frac{16}{5},5)$.

 (iii) At $x=0$

$16y^2=144$

so $\quad y=3$ or $-3$.

At $y=0$

$25x^2=144$

so $\quad x=\frac{12}{5}$ or $-\frac{12}{5}$.

(b) Applying the substitutions,

$$LHS = 25(-x)^2-32(-x)(-y)+16(-y)^2$$
$$= 25x^2-32xy+16y^2$$

and hence the equation is unchanged.
Thus the graph is unchanged by a rotation of $180^\circ$ about the origin.

(c) (i) Differentiating implicitly,

$$50x-32y-32xy'+32yy'=0$$

so $\quad 32y'(x-y)=2(25x-16y)$

or $\quad \dfrac{dy}{dx}=\dfrac{25x-16y}{16(x-y)}\,.$

 (ii) The derivative is zero when the numerator is zero (and the denominator is non-zero). This yeilds $y=\frac{25}{16}x$. Notice that the endpoints of the range lie on this line and so the tangents are horizontal at those two points. From the geometry of the ellipse, we know there are no other points on the curve with horizontal tangents.

(iii) The endpoints of the domain lie on $y = x$. Thus the denominator is zero and the derivative is undefined at these points. This is expected from the geometry of an ellipse. The tangents are vertical at the endpoints of the domain.

(d) The curve is sketched above.

**Deriving Explicit Functions:** Like the circle and other conic sections, some simple implicit equations can be solved for $y$ to obtain a set of explicit functions. These functions can then be investigated further to determine additional information about the graph. For example, in the last worked exercise we may write:

$$9x^2 + 16(x^2 - 2xy + y^2) = 144$$

so
$$16(y-x)^2 = 9(16 - x^2)$$

thus $y = \frac{1}{4}\left(4x + 3\sqrt{16 - x^2}\right)$ or $y = \frac{1}{4}\left(4x - 3\sqrt{16 - x^2}\right)$.

The two functions correspond to the parts of the curve above and below $y = x$.

## Exercise **8I**

**1.** Differentiate the following, where $y$ is an unknown function of $x$.

(a) $y + x$ (d) $y^3 + 3xy$ (g) $y(2x + 3y)$
(b) $xy$ (e) $\log y$ (h) $(x + y)^3$
(c) $x^2 - y^2$ (f) $e^y$ (i) $(x^2 + y^2)^2$

**2.** Solve the following for $y$ to obtain a set of functions to describe the curve.

(a) $x^2 - y^2 = 9$ (c) $x^2 + y^2 - 2y = 0$
(b) $x^2 + y^2 = 4$ (d) $2x^2 + 2xy + y^2 = 1$

**3.** In each case, find an expression for the derivative in terms of $x$ and $y$. List any points on the curve where this derivative is undefined.

(a) $x^2 + y^2 = 36$ (c) $x^2 - 2xy + 2y^2 = 2$
(b) $x^2 - y^2 = 16$ (d) $x(x^2 + y^2) = 2y^2$

**4.** List any points on the curves in Question 4 where it is guaranteed that the tangent is horizontal, where the numerator of the derivative is zero and the denominator is non-zero.

**5.** In each case determine the gradient of the curve at the given point.

(a) $x^2 - y^2 = 9$, $(5, 4)$ (c) $x^4 = 2x^2 - 4y^2$, $(1, \frac{1}{2})$ (e) $x^2 - 2xy - y^2 = 2$, $(-3, -1)$
(b) $x^2 + 2y^2 = 9$, $(1, -2)$ (d) $x^3 + y^3 = 7$, $(-1, 2)$ (f) $x^4 - 5xy^2 + y^4 = 7$, $(2, 3)$

**6.** Find the equation of the tangent to each curve at the given point.

(a) $x^2 + 3y^2 = 12$, $(-3, 1)$ (d) $y^4 - x^4 = 2(xy - 1)$, $(-1, -1)$
(b) $2x^2 - y^2 = 1$, $(5, 7)$ (e) $2y(x^2 - y^2) = x^4 - 10$, $(2, 1)$
(c) $y^4 - 10y^2 - 6 = x^4 - x^2$, $(1, -2)$ (f) $(x^2 - 1)^2 - 4 = y^2(3 + 2y)$, $(-2, 1)$

**DEVELOPMENT**

**7.** Consider the curve with equation $x^2 + 2xy + y^5 = 4$.

(a) Find the equation of the tangent at $(-3, 1)$.
(b) Explain why the curve has no vertical tangents.
(c) Show that if the curve has a horizontal tangent at $(x, y)$ then $x^5 + x^2 + 4 = 0$.
(d) Hence determine how many points on the curve have a horizontal tangent.

**8.** The curve with equation $(x^2 + y^2 + x)^2 = x^2 + y^2$ is an example of a cardioid.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x - (1 + 2x)(x + x^2 + y^2)}{2y(x + x^2 + y^2) - y}$.

(b) Hence find the equation of the tangent to the cardioid at $(0, 1)$.

**9.** Consider the ellipse with equation $x^2 + y^2 + xy = 3$.

(a) The given equation is a quadratic in $y$. Use the discriminant to determine the domain of the relation.

(b) Find the $x$-intercepts.

(c) (i) Show that the curve is symmetric in the line $y = x$.

(ii) Where does the curve cross this line?

(d) Use implicit differentiation to show that $\dfrac{dy}{dx} = -\dfrac{2x + y}{2y + x}$.

(e) Hence find the points on the curve where the tangent is horizontal.

(f) Use symmetry to locate the points where the tangent is vertical.

(g) Sketch the curve, showing all these features.

**10.** Consider the hyperbola with equation $4y(x\sqrt{3} - y) = 3$.

(a) The given equation is a quadratic in $y$. Use the discriminant to determine the domain of the relation.

(b) Are there any intercepts with the axes?

(c) Show that the relation has odd symmetry.

(d) Make $x$ the subject of the given equation, and hence determine the equations of the two asymptotes.

(e) Show that $\dfrac{dy}{dx} = \dfrac{y\sqrt{3}}{2y - x\sqrt{3}}$, and hence show that the derivative is undefined at the end-points of the domain. [In fact, the two tangents are vertical.]

(f) Are there any points on the curve where the tangent is horizontal?

(g) Sketch the curve, showing all these features.

(h) It may have been much simpler to sketch this graph by first sketching the inverse relation. Investigate this possibility.

**EXTENSION**

**11.** A simple example of a lemniscate is the curve with equation $x^4 = x^2 - y^2$.

(a) Determine the domain of the relation.

(b) Find the intercepts with the axes.

(c) Explain why the curve is symmetric in both axes.

(d) Use implicit differentiation to show that $\dfrac{dy}{dx} = \dfrac{x - 2x^3}{y}$.

(e) Hence show that in the first quadrant $\dfrac{dy}{dx} = \dfrac{1 - 2x^2}{\sqrt{1 - x^2}}$, $0 < x < 1$.

(f) Use this last result to show that in the first quadrant $\lim\limits_{x \to 0^+} y' = 1$, and $\lim\limits_{x \to 1^-} y' \to -\infty$.

(g) Find the point in the first quadrant where the tangent is horizontal.

(h) Sketch the curve, showing all these features.

**12.** The folium of Descartes has the equation $x^3 + y^3 = 3xy$.

(a) Determine any intercepts with the axes.

(b) (i) Show that the curve is symmetric in the line $y = x$.

(ii) Where does the curve cross this line?

(c) (i) Show that $\dfrac{dy}{dx} = \dfrac{x^2 - y}{x - y^2}$.

(ii) Hence find where the tangent is guaranteed to be horizontal on this curve.

(d) (i) Show that the curve can be parameterised by $x = \dfrac{3t}{1+t^3}$ and $y = \dfrac{3t^2}{1+t^3}$.

(ii) Hence show that $\dfrac{dy}{dx} = \dfrac{t(2-t^3)}{1-2t^3}$.

(iii) Using this version of the derivative, what is the gradient at the origin?

(e) (i) Applying the symmetry of part (b) to the results in parts (c) and (d), where on the curve is the tangent vertical?

(ii) What do you conclude happens at the origin?

(f) (i) Using the parametric equations, show that $|x| \to \infty$ and $\frac{y}{x} \to -1$ as $t \to -1$.

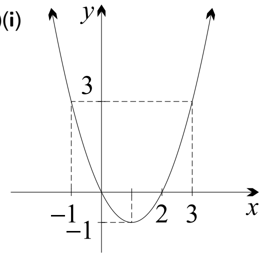
(ii) Show that $(x+y)(\frac{x}{y} - 1 + \frac{y}{x}) = 3$, and hence find the oblique asymptote.

(g) Sketch the curve, showing all these features.

# Chapter Eight

## Exercise **8A** (Page 61)

**1(a)(i)**
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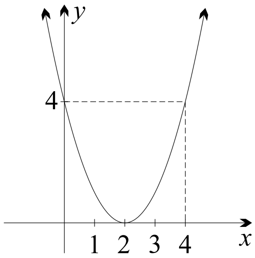


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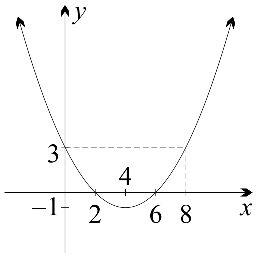


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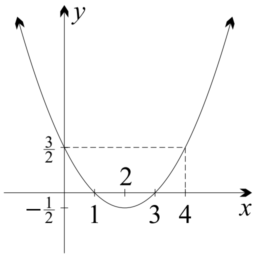


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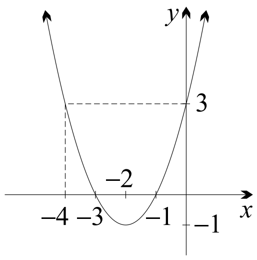


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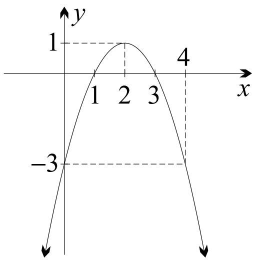


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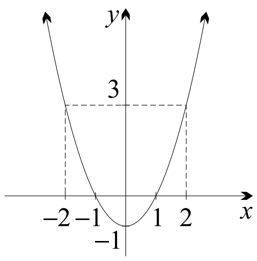


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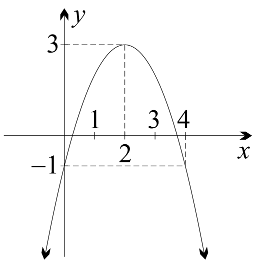
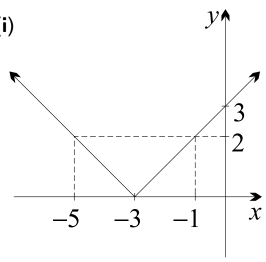



**(b)(i)**
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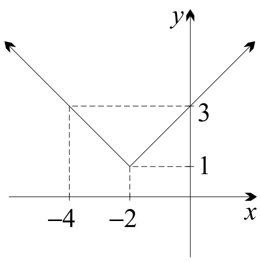


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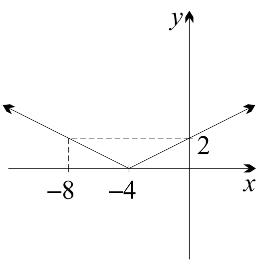


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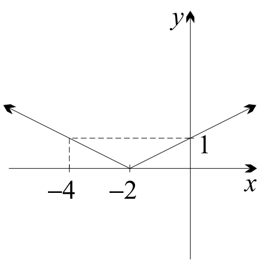


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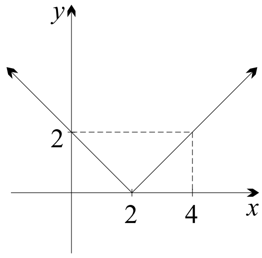


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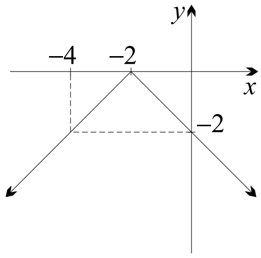
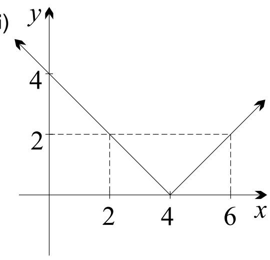
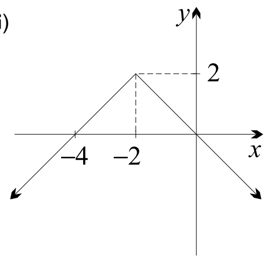
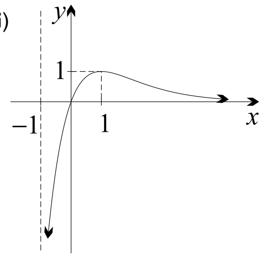
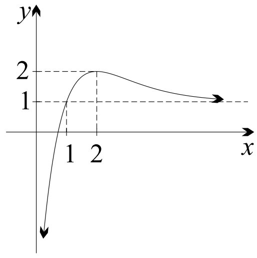
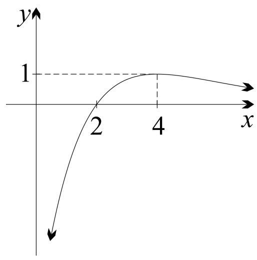


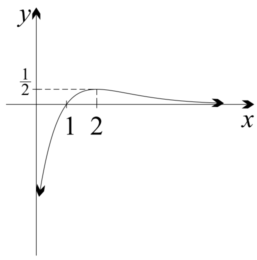

**(vii)**


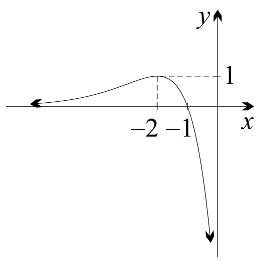

**(viii)**


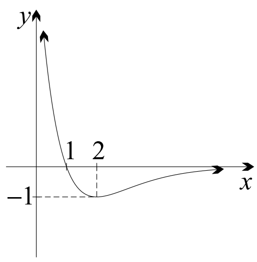

**(c)(i)**


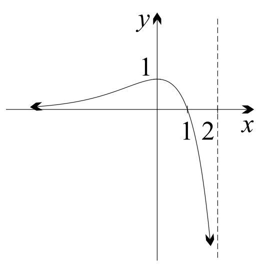

**(ii)**


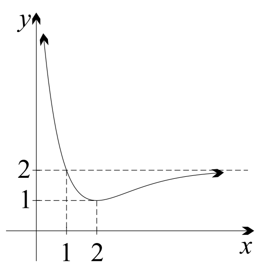

**(iii)**



**(iv)**


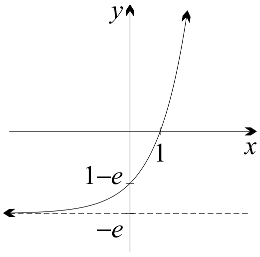

**(v)**



**(vi)**


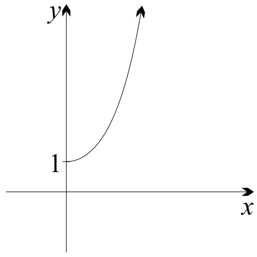

**(vii)**



**(viii)**


**3(a)**

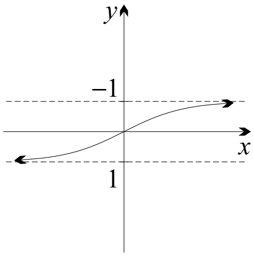



**(b)**



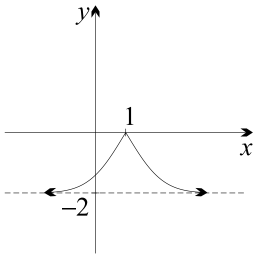

$f^{-1}(x) = e^x - e$ $f^{-1}(x) = \frac{1}{2}(e^x + e^{-x})$

**(c)**

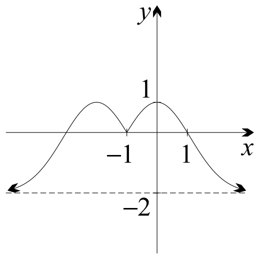



$$f^{-1}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

**4(a)**




**(b)**



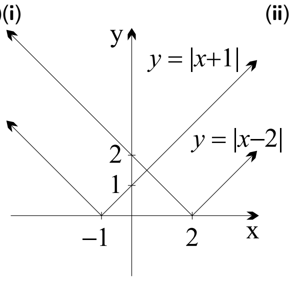

**5(a)** It could be a vertical shift of two down or a reflection in the $x$-axis. **(b)** It could be a shift left by $\frac{\pi}{2}$ or a reflection in the $x$-axis or a reflection in the $y$-axis.

**7(a)(i)** $x = n\pi$ for integer $n$.

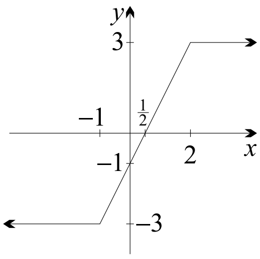
**9(b)** The converse is not true. For example, a primitive of $3x^2$ is $x^3 + 1$ which is neither even nor odd. **(c)** The converse is true in this case.

## Exercise **8B** (Page 65)

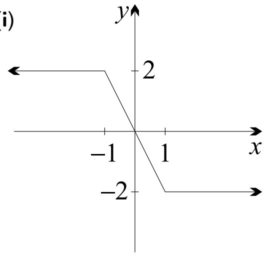
**1(a)(i)**



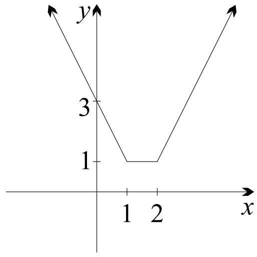

**(ii)**



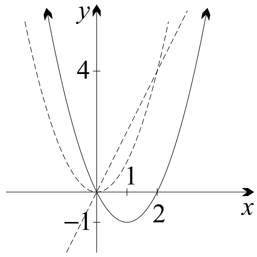

**(b)(i)**

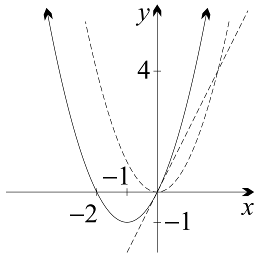



**(ii)**

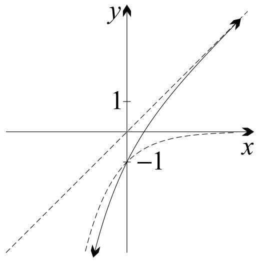


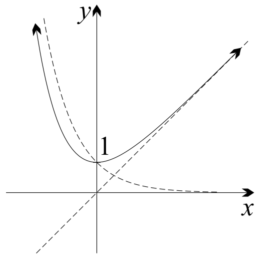

**2(a)**



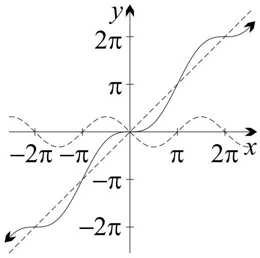



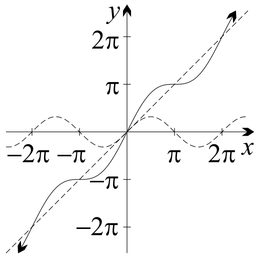

**(b)**



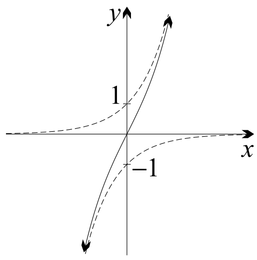



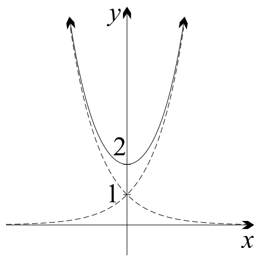

**(c)**



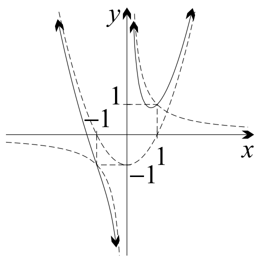




**(d)**

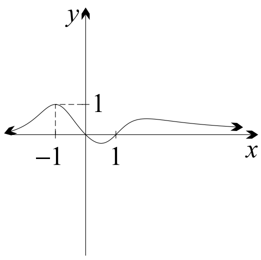


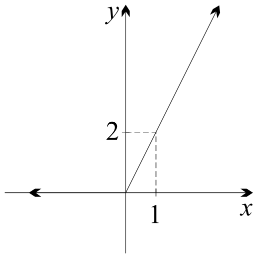

**3(a)**



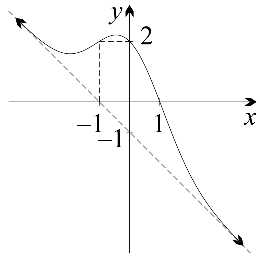

**(b)**



**(c)**



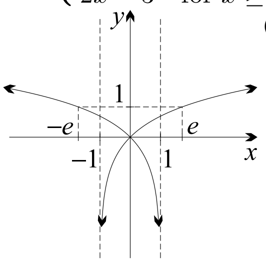

**(d)**



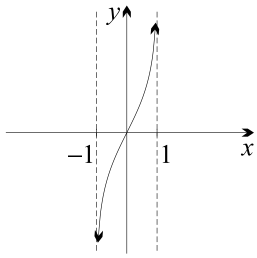

**4(b)(i)** $y = \begin{cases} 2 & \text{for } x < -1 \\ -2x & \text{for } -1 \le x < 1 \\ -2 & \text{for } x \ge 1 \end{cases}$

**(ii)** $y = \begin{cases} 3 - 2x & \text{for } x < 1 \\ 1 & \text{for } 1 \le x < 2 \\ 2x - 3 & \text{for } x \ge 2 \end{cases}$

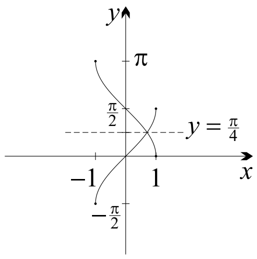
**5(a)**



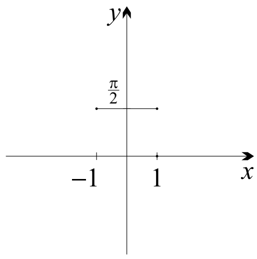

**(b)**



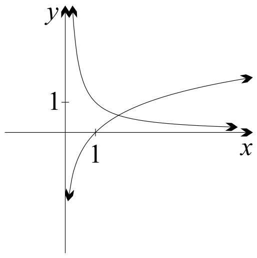

**6(a)**



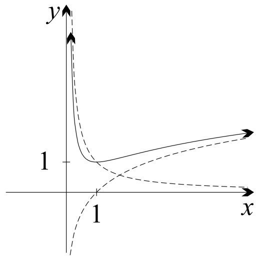

**(b)**



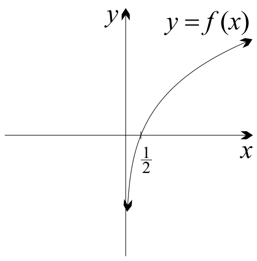

**7(a)**

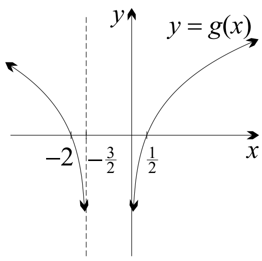



**(c)**



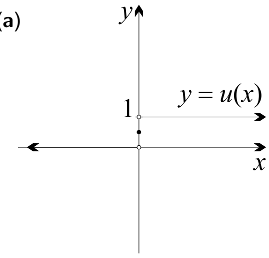

**8(b)**

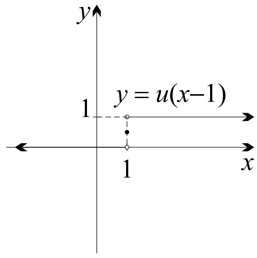


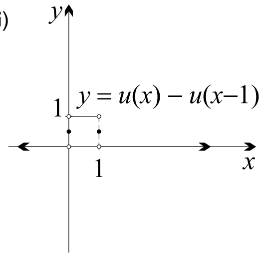

**9(a)** One possible selection is as follows.
Both odd: $f(x) = x$, $g(x) = \sin x$.
Both even: $f(x) = x^2$, $g(x) = \cos x$.
Odd and even: $f(x) = x$, $g(x) = \cos x$.
**(c)** When $g(x) = -f(x)$, $h(x) \equiv 0$ which is even.

**10(a)**

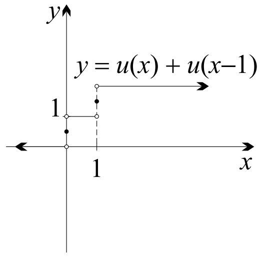



**(b)(i)**



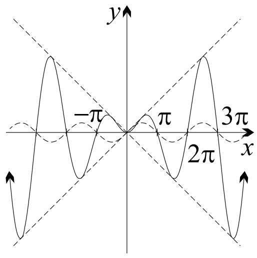

**(ii)**



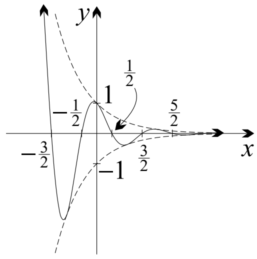

**11** 2

## Exercise **8C** (Page 69)

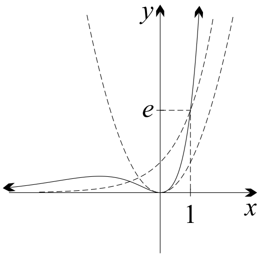
**1(a)**



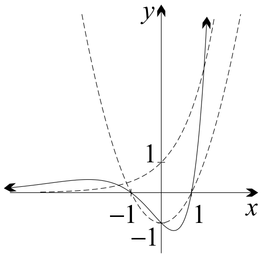

**(b)**



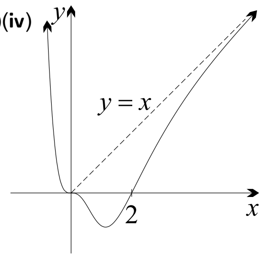

**2(a)**



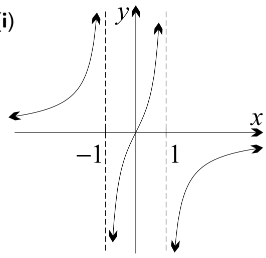

**(b)**



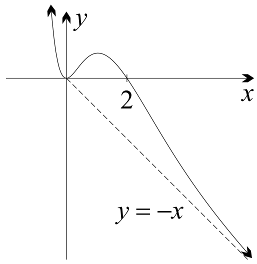

**3(a)(iv)**



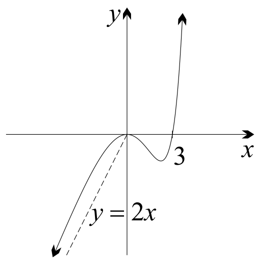

**(b)(i)**



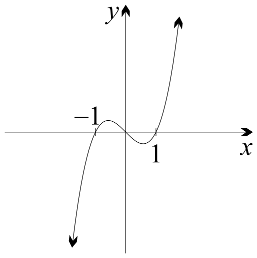

**(ii)**



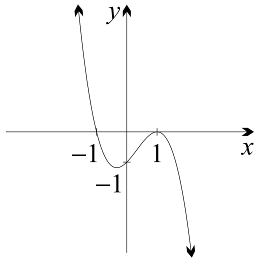

**(iii)**



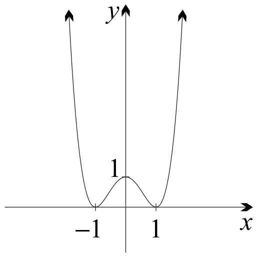
**4(a)**



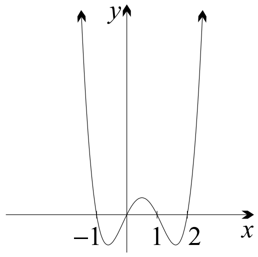

**(b)**



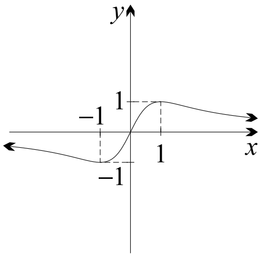

**(c)**



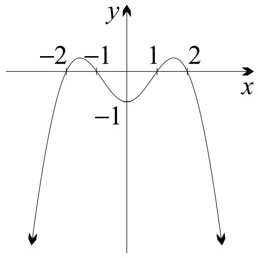

**(d)**



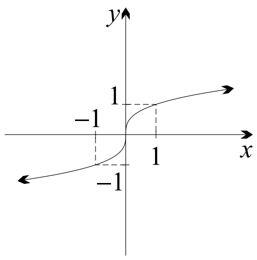

**(e)**



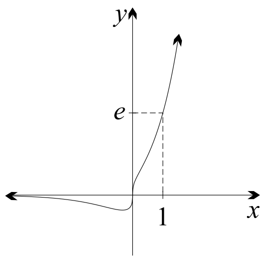

**(f)**



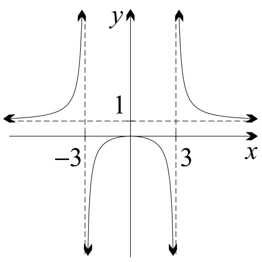

**5(a)**



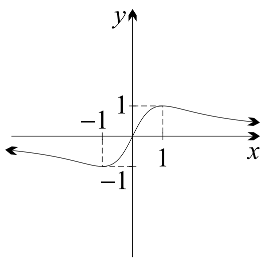

**(b)**



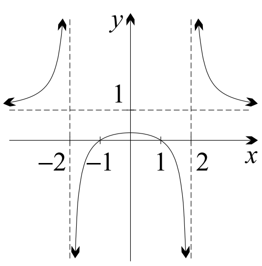

**6**




**7**




**8(a)**




**(b)** $x < -2$ or $x > 2$

**9(a)** $x = -2, -1, 1, 2$

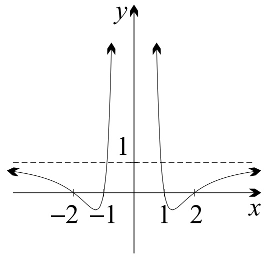
**(b)** $x = 0$

**(c)(i)** $y = 1$

**(ii)** $\left(\frac{\pm 4}{\sqrt{10}}, -\frac{9}{16}\right)$

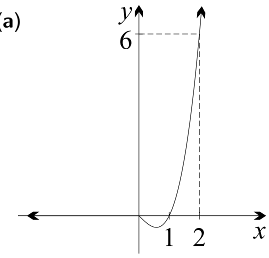
**(d)** $-\frac{9}{16} < b < 1$

**(d)**

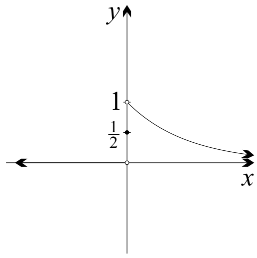



**11** $y = h(x)$ has a hole at the origin.

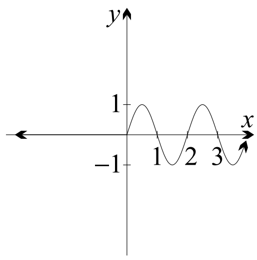
**12(a)**



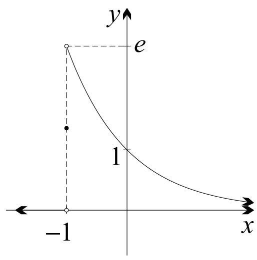

**(b)**



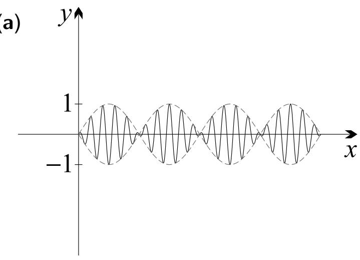

**(c)**




**(d)**



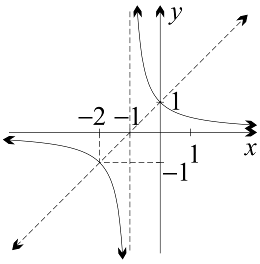

**13(a)**



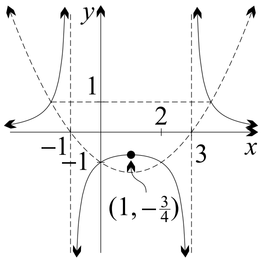

**(b)** $y = \cos 5x$

## Exercise **8D** (Page 73)

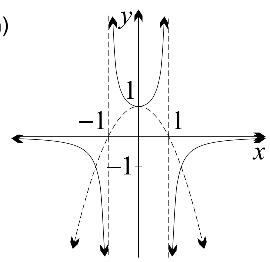
**1**



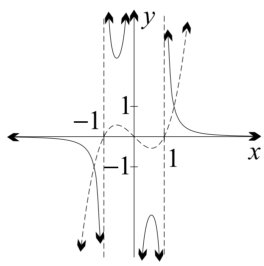

**2**



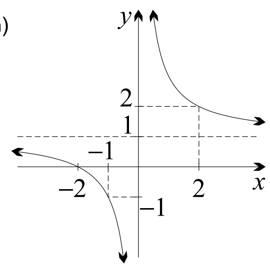
**3(a)**



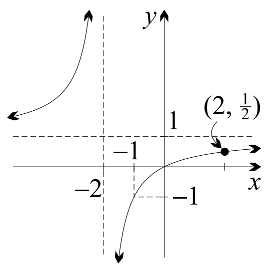

**(b)**



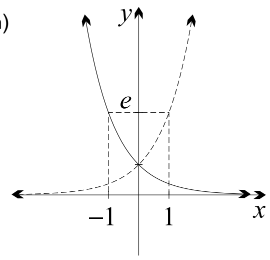

**6(a)**



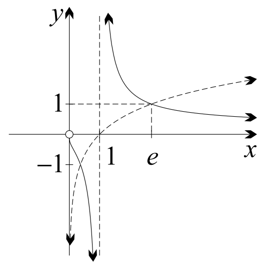

**(b)**



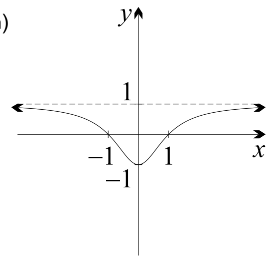

**4(a)**



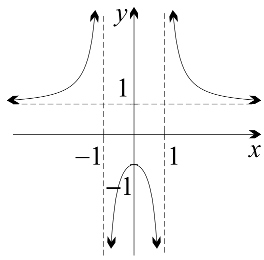

**(b)**



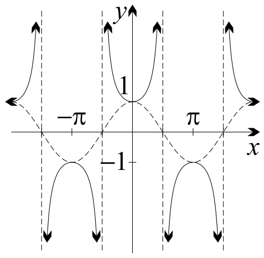

**7(a)**



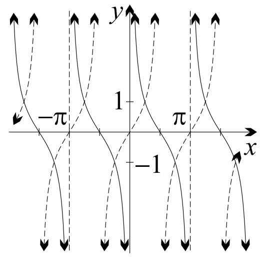

**(b)**




**(c)**



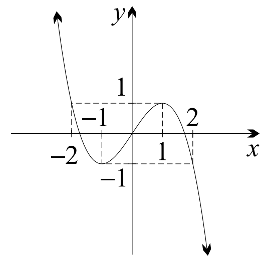

**(d)**



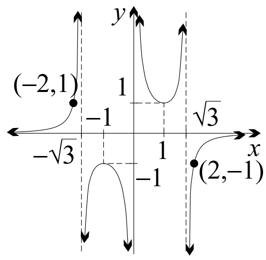

**8(a)(ii)** $x = -\sqrt{3}, 0, \sqrt{3}$ **(iii)** $x = 1, 1, -2$

**(iv)** $f(x) = 1$ has a double root at $x = 1$.

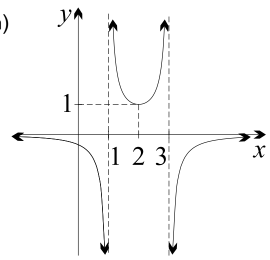
**(v)**



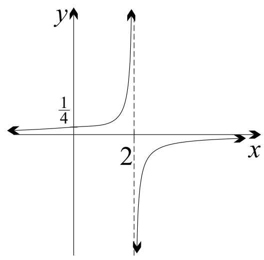

**(b)**



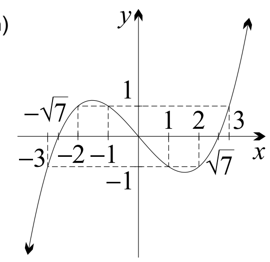

**5(a)**



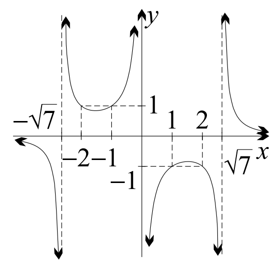

**(b)**



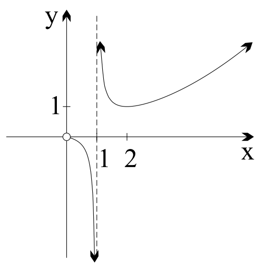

**9(a)**



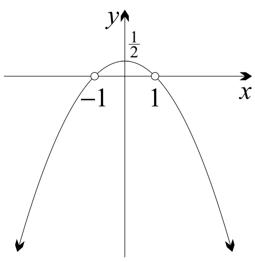

**(b)**



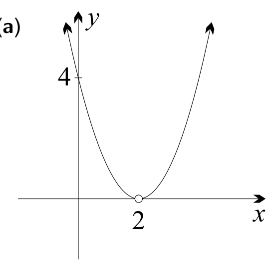

**(c)**



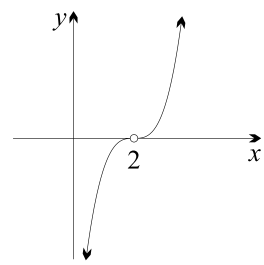

**(d)**



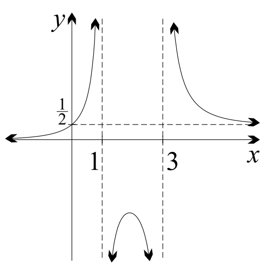

**10(a)**



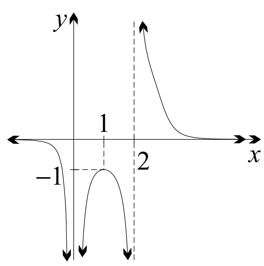

**(b)**




**(e)**



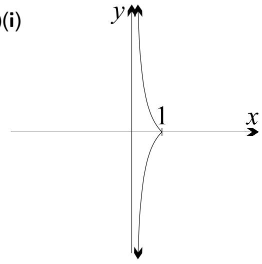

**(f)**



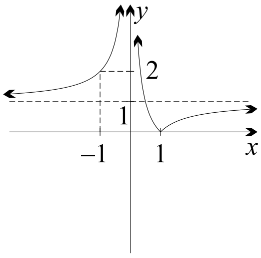

 

## Exercise **8E** (Page 78)

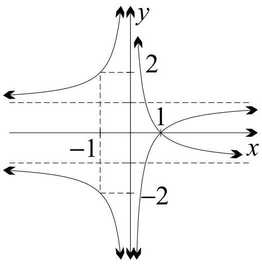
**1(a)(i)**



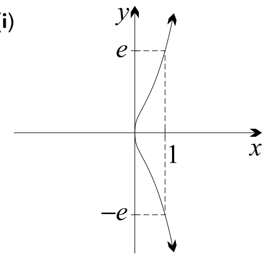

**(ii)**



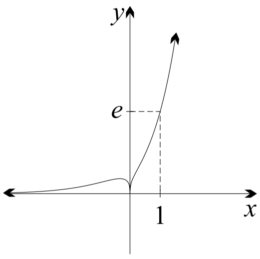

**(iii)**



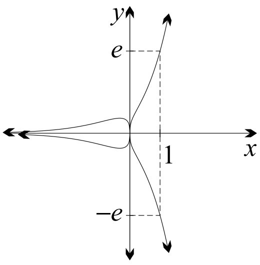

**(b)(i)**



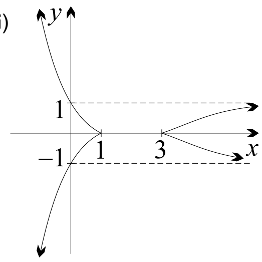

**(ii)**



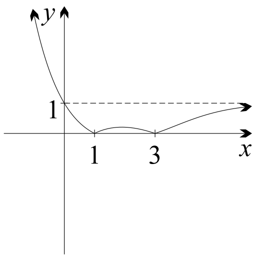

**(iii)**



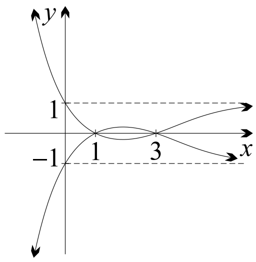

**(c)(i)**



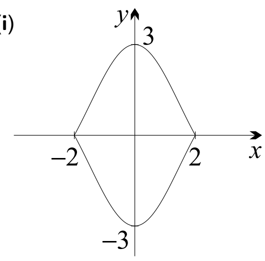

**(ii)**



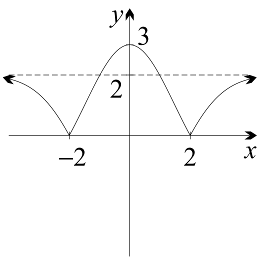

**(iii)**



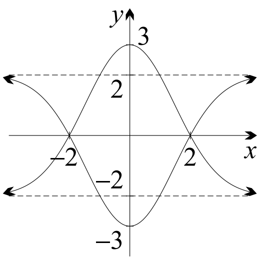

**(d)(i)**



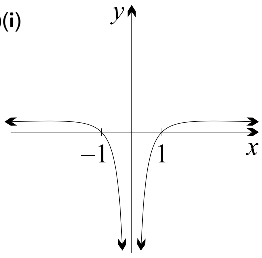

**(ii)**



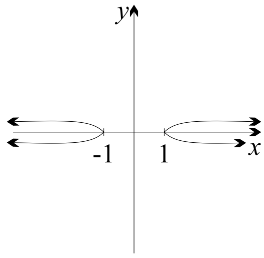

**(iii)**



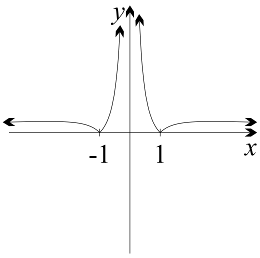

**2(a)(i)**



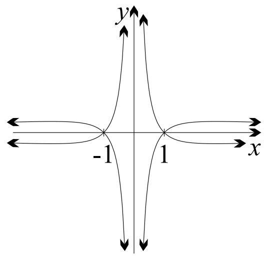

**(ii)**



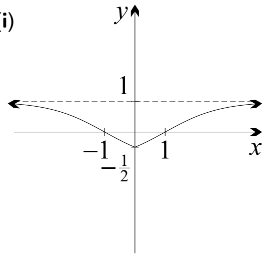

**(iii)**



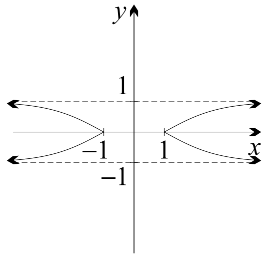

**(iv)**



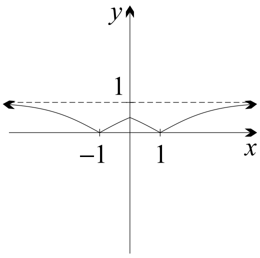

**(b)(i)**



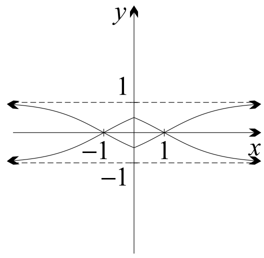

**(ii)**



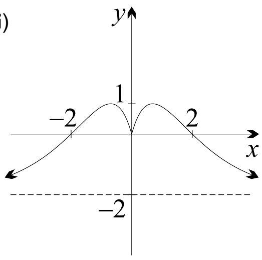

**(iii)**



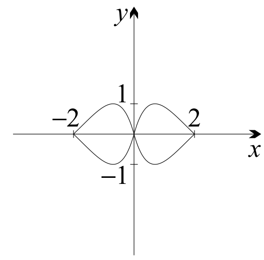

**(iv)**



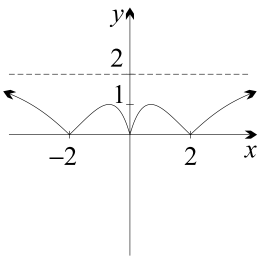

**(c)(i)**



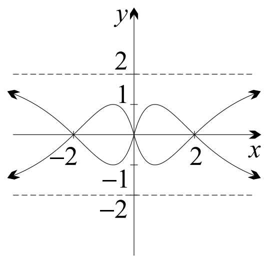

**(ii)**



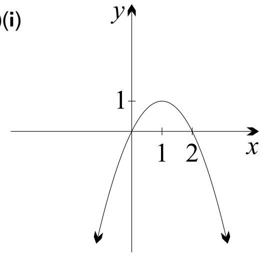

**(iii)**

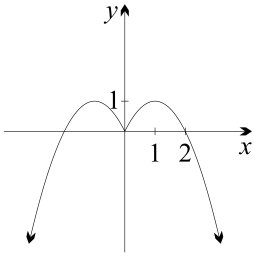


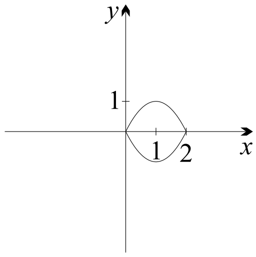

**(iv)**

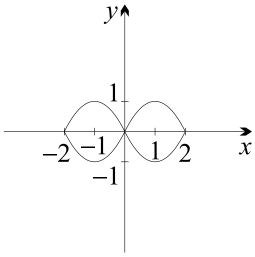


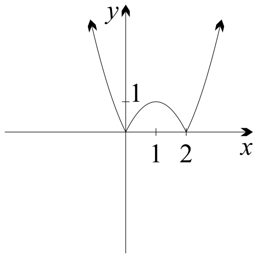

 

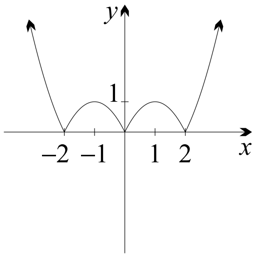
**3(a)(i)**


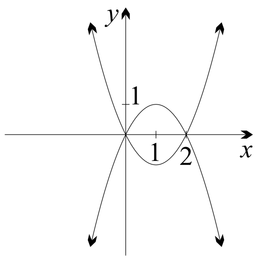

**(ii)**


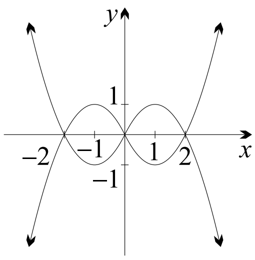

**(iii)**


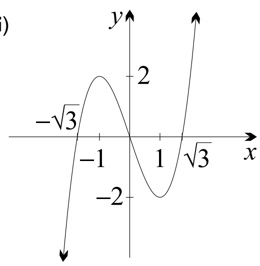

**(iv)**


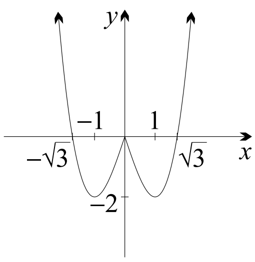

**(v)**


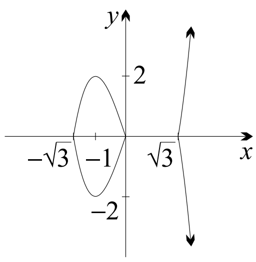

**(vi)**


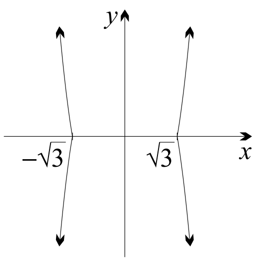

**(vii)**


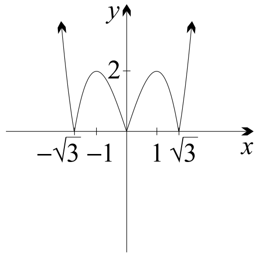

**(viii)**


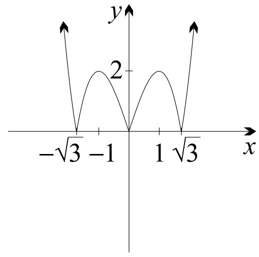

**(b)(i)**


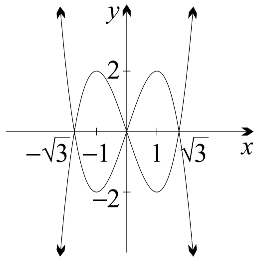

**(ii)**


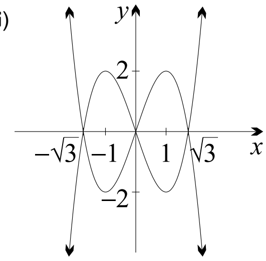

**(iii)**


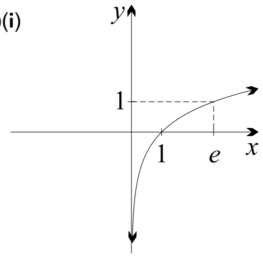

**(iv)**


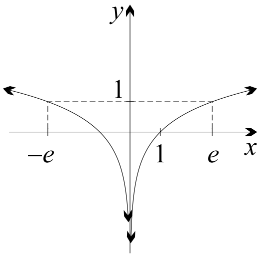

**(v)**


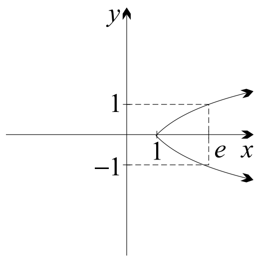

**(vi)**


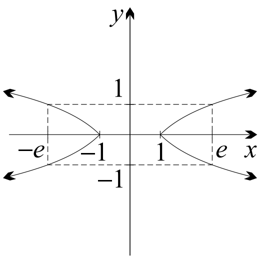

**(vii)**


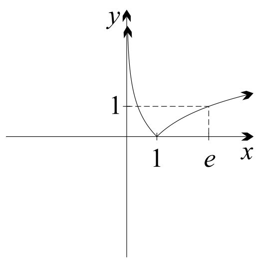

**(viii)**


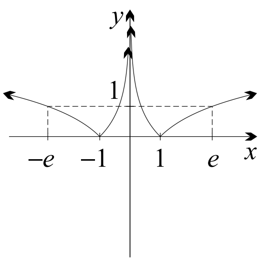

**4(a)(i)**


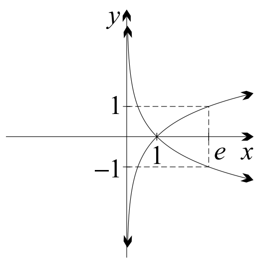

**(ii)**


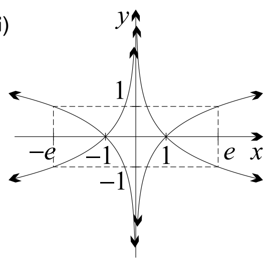

**(iii)**



**(iv)**


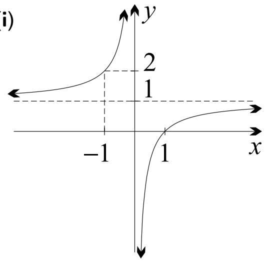

**(v)**


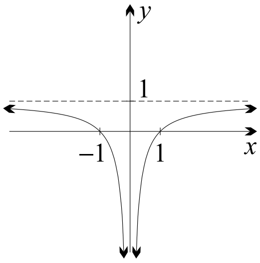

**(vi)**


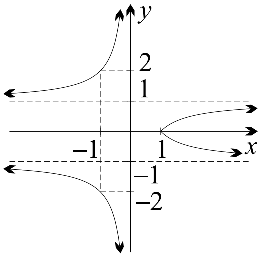

**(vii)**


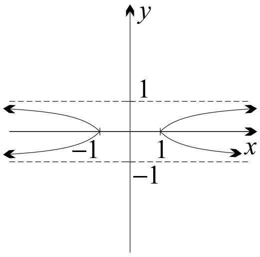

**(viii)**


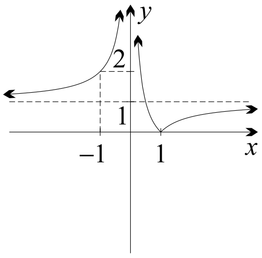

 

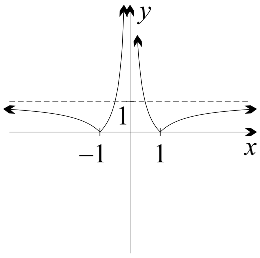
**(b)(i)**


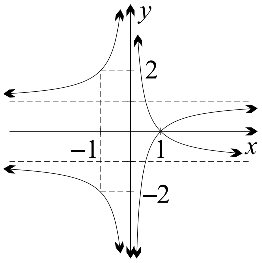

**(ii)**


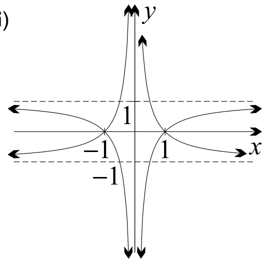

**(iii)**


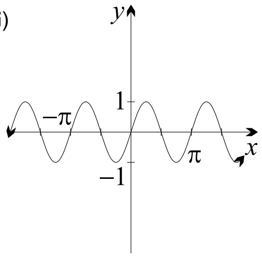

**(iv)**


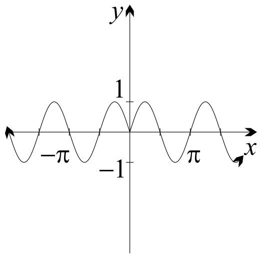

**(v)**


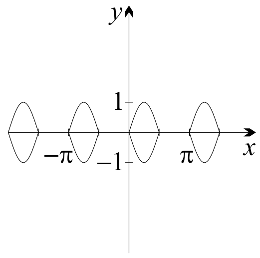

**(vi)**


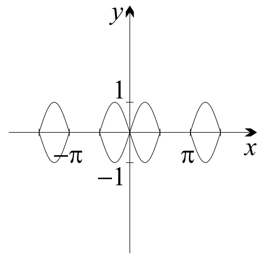

**(vii)**


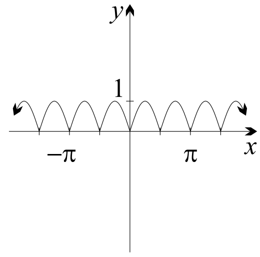

**(viii)**


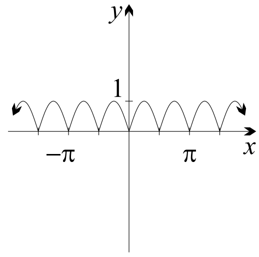

**(c)(i)**


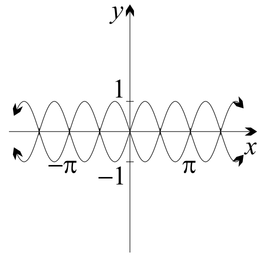

**(ii)**


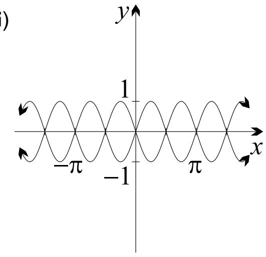

**(iii)**


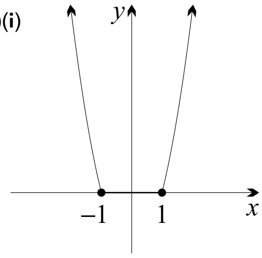

**(iv)**


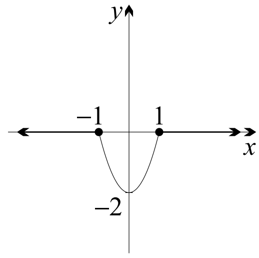

**(v)**


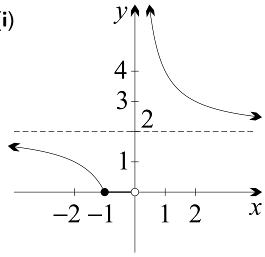

**(vi)**


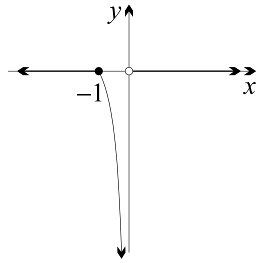

**(vii)**


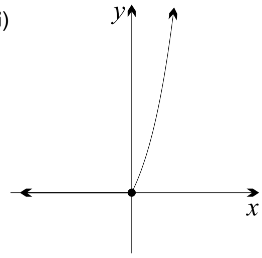

**(viii)**


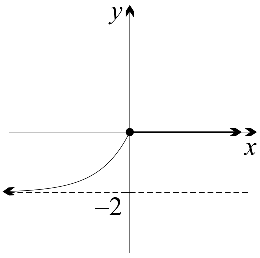

**5(a)(i)**


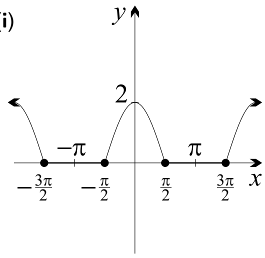

**(ii)**


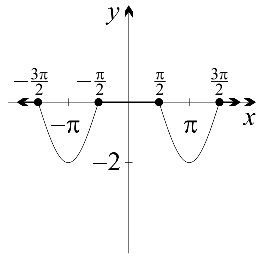

**(b)(i)**



**(ii)**



**(c)(i)**


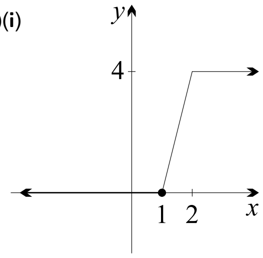

**(ii)**



**(d)(i)**


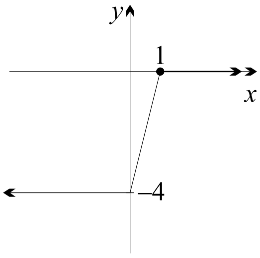

**(ii)**


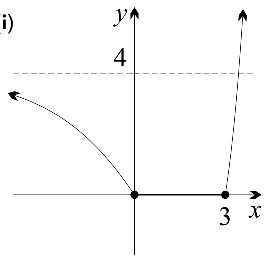
**6(a)(i)**



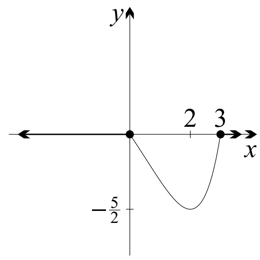

**(ii)**



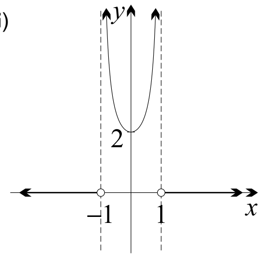

**(b)(i)**



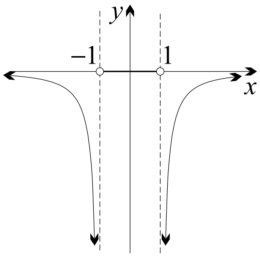

**(ii)**



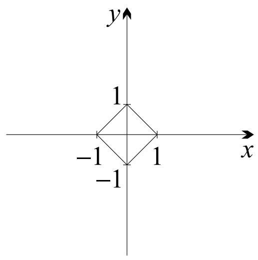

**(c)(i)**



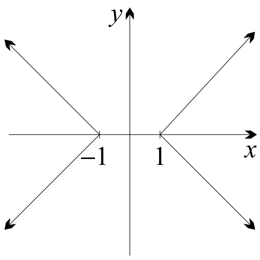

**(ii)**



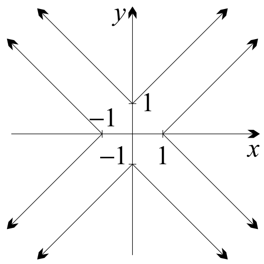

**7(a)**




**(b)**




**(c)**

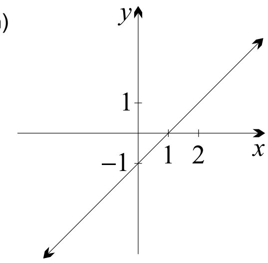



**8(b)** $|y| = |f(x)|$ and $|y| = \Big|f(|x|)\Big|$

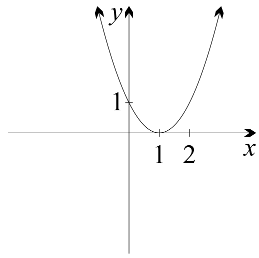
**(e)** Yes: $|y| = f(|x|)$ and $|y| = \Big|f(|x|)\Big|$ are the same if $f(|x|) \ge 0$, for example $f(x) = e^x - 1$.

## Exercise **8F** (Page 82)

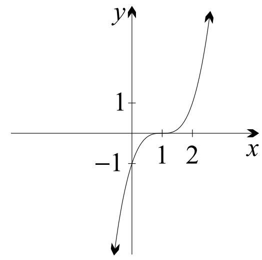
**1(a)**



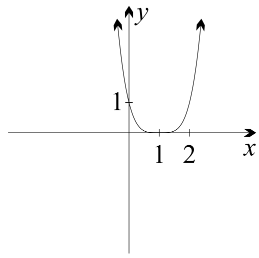

**(b)**



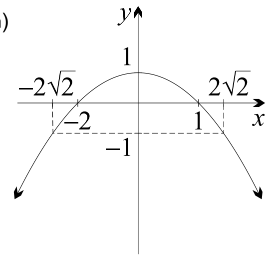

**(c)**



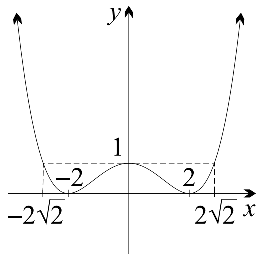

**(d)**



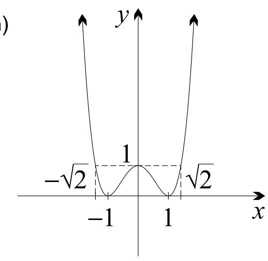

**2(a)**



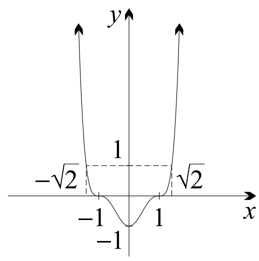

**(b)**



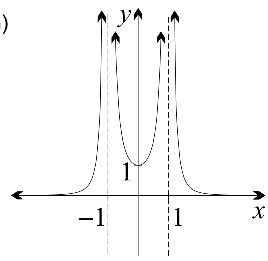

**3(a)**



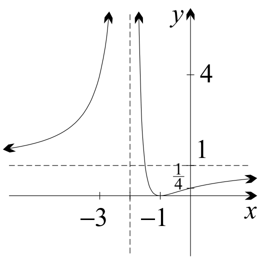

**(b)**



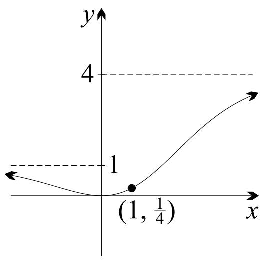

**4(a)**



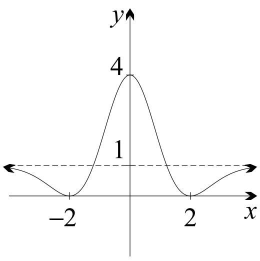

**(b)**



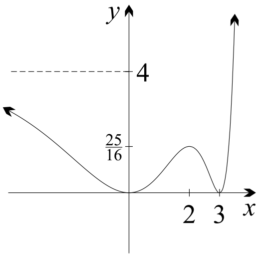

**(c)**

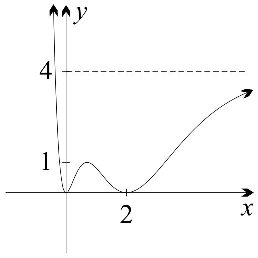


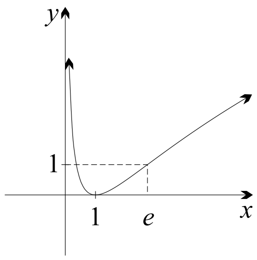

**(d)**

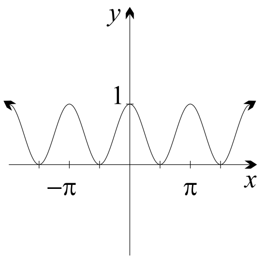


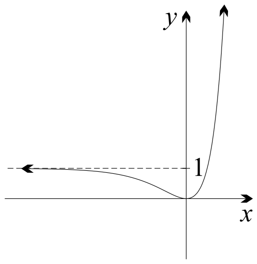

 

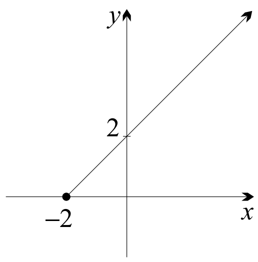
**(e)**



**(f)**


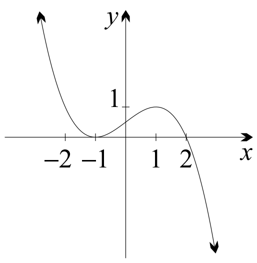

**5(a)**


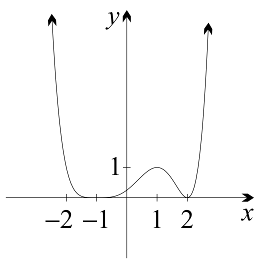

**(b)**



**(c)**


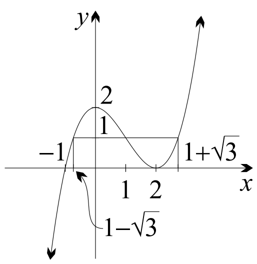

**(d)**


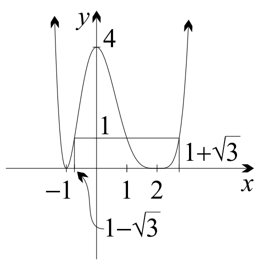

**6(a)** $(-2, 1)$ **(b)** $(-1, 0)$

**(c)**



**(d)**



**7(a)** $(1-\sqrt{3}, 1)$ and $(1+\sqrt{3}, 1)$ **(b)** $(2, 0)$

**(c)**



**(d)**


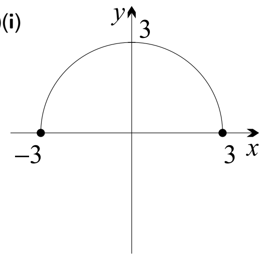

**(e)** $4$

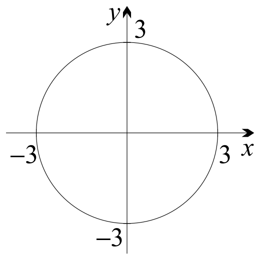
**10(a)** Either $(2, 0)$ is a minimum of $f(x)$, or $n$ is even and $f(x)$ changes sign across $x = 2$.

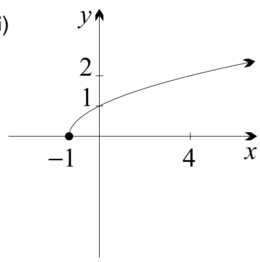
**(b)** $n$ is odd and $f(x)$ has a maximum at $(2, 0)$.

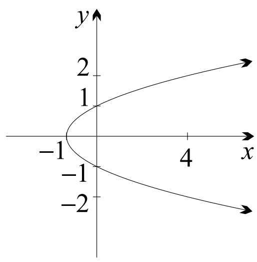
**(c)** $n$ is odd and $f(x)$ changes sign across $x = 2$.

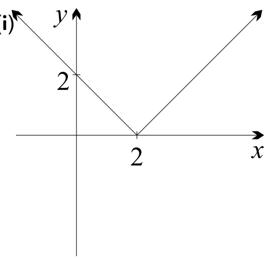
## Exercise **8G** (Page 86)

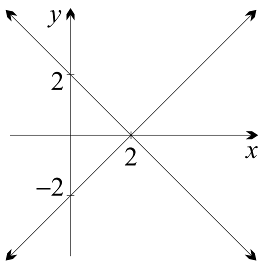
**1(a)(i)**


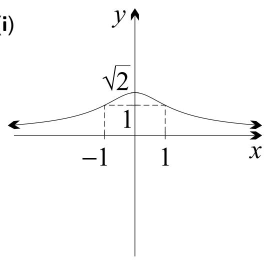

**(ii)**


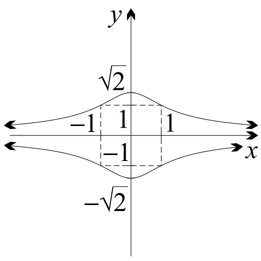

**(b)(i)**


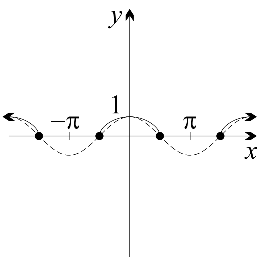

**(ii)**


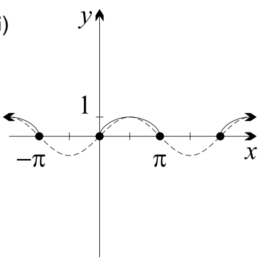

**(c)(i)**



**(ii)**


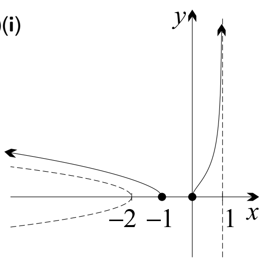

**(d)(i)**


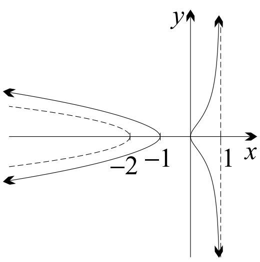

**(ii)**



**2(a)**



**(b)(i)**


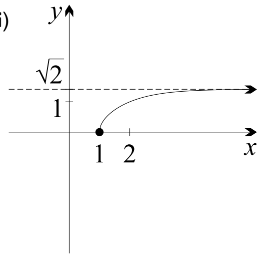

**(ii)** There is symmetry in $x = \frac{\pi}{2}$

**3(a)(i)**


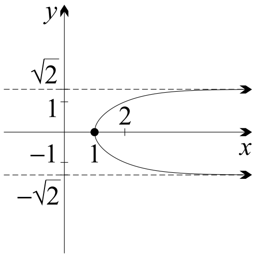

**(ii)**


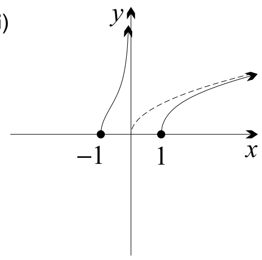
**(b)(i)**



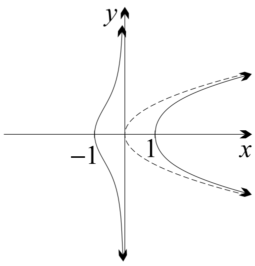

**(ii)**



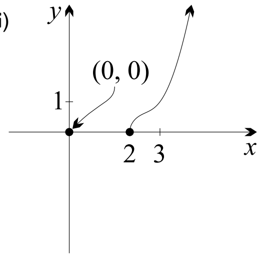

**(c)(i)**



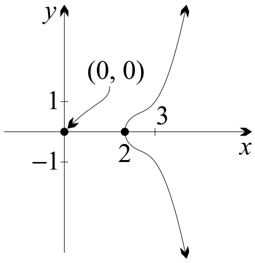

**(ii)**



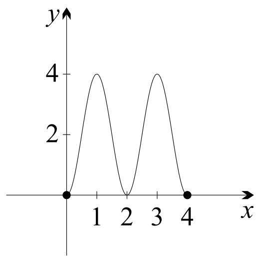

**(d)(i)**



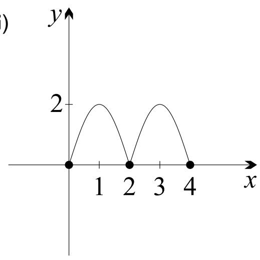

**(ii)**




**4(a)**



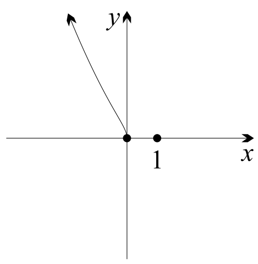

**(b)(i)**



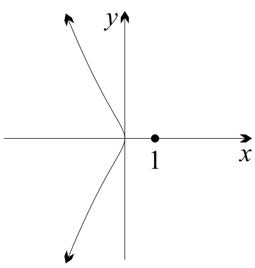

$f(x) = 4\sin^2\left(\frac{\pi}{2}x\right)$

**(ii)** $x = 0, \frac{1}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{11}{3}, 4$

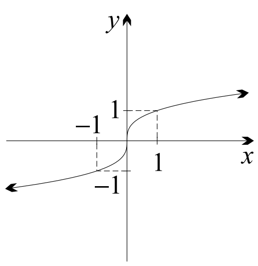
**5(a)**



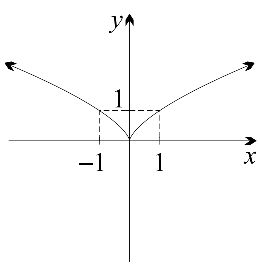

**(b)**



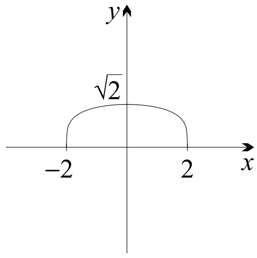

**6(a)**



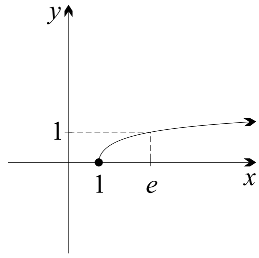

**(b)**



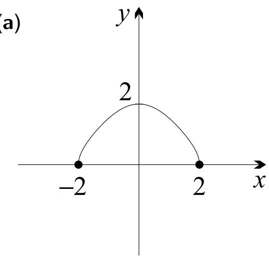

**7**



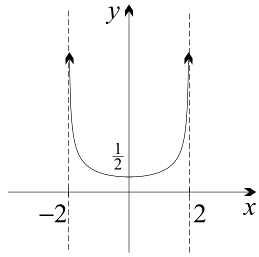

**10**




**11(a)**

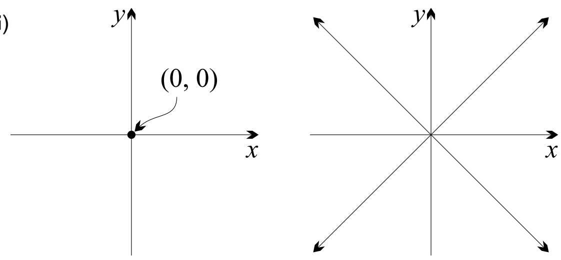



**(b)**




**12(b)** The upper half of the hyperbola in part (a).

**(c)(i)**

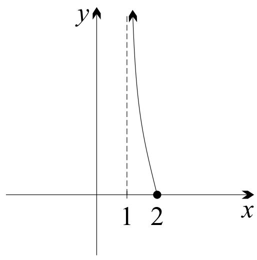



**(ii)** A horizontal plane through the apex yields a solitary point at the origin. A vertical plane through the apex yields a pair of perpendicular lines through the origin.

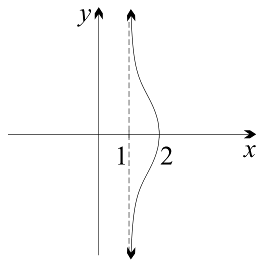
**13** When $f(x) \le 0$ for all $x$ in the natural domain.

**15(b)** $1 < x \le 2$

**(c)**

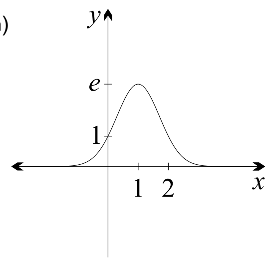



**(d)**

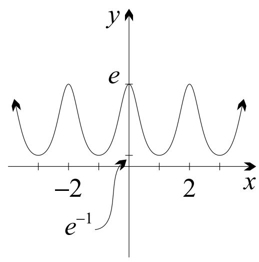


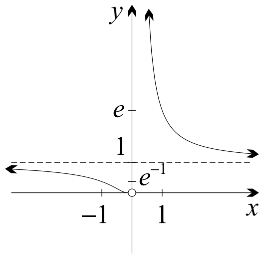

## Exercise **8H** (Page 89)

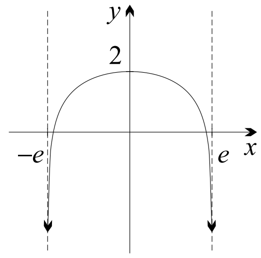
**1(a)**

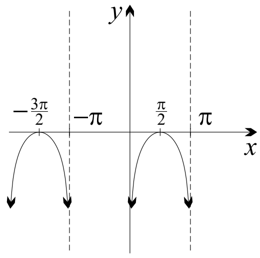


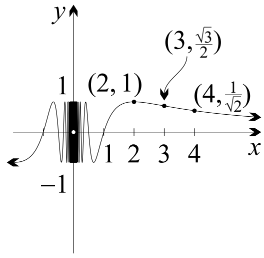

**(b)**

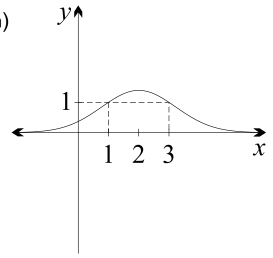


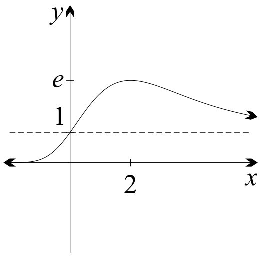
**(c)**


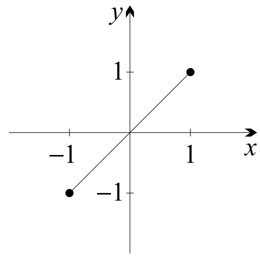

**(d)**


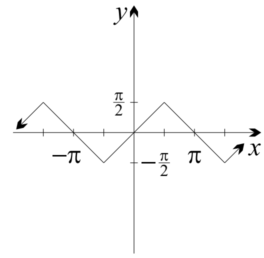

**(c)**


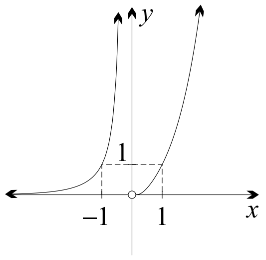

**(d)**


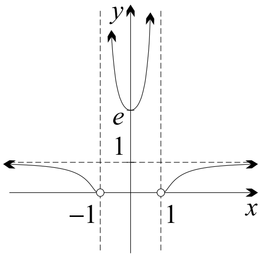

**2(a)**


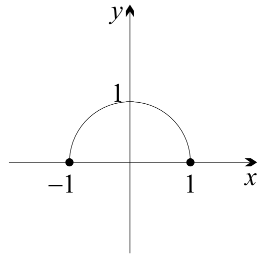

**(b)**


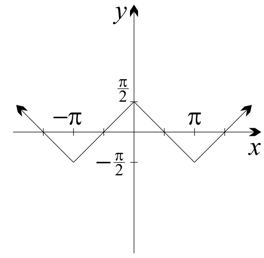

**5(a)**


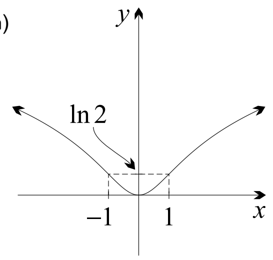

**(b)**


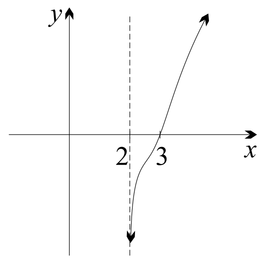

**(c)**


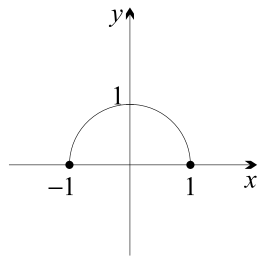

**(d)**


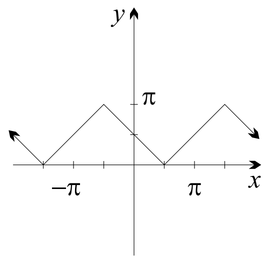

**(c)**


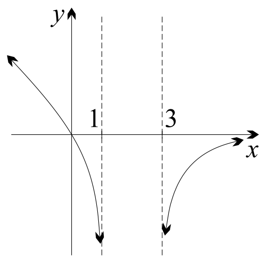

**(d)**


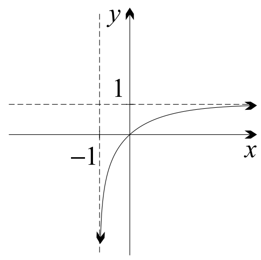

**3(a)**


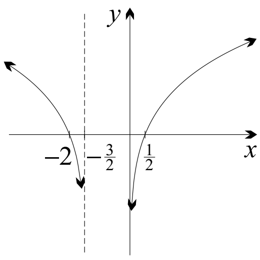

**(b)**


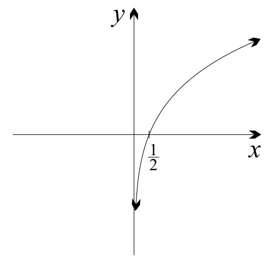

**(e)**


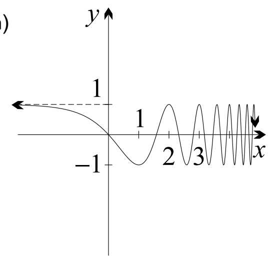

**(f)**


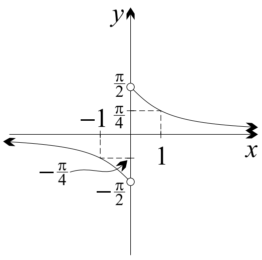

**(c)**


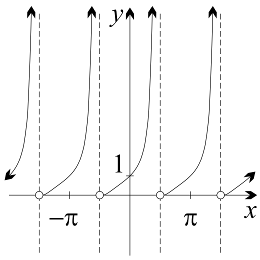

**(d)**


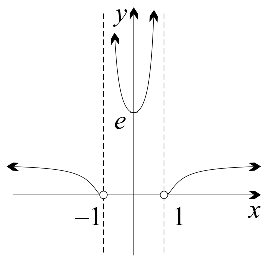

**6(a)**



**(b)**


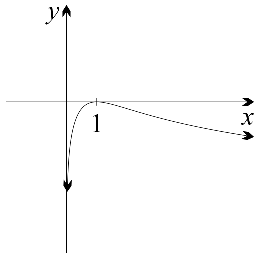

**4(a)**



**(b)**


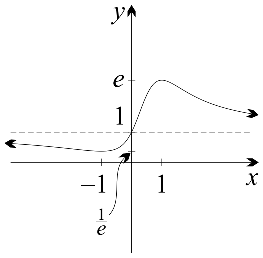

**7(a)**



**(b)**


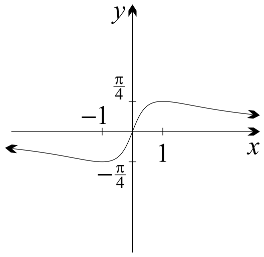
**9(a)**



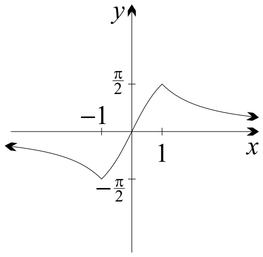

**(b)**



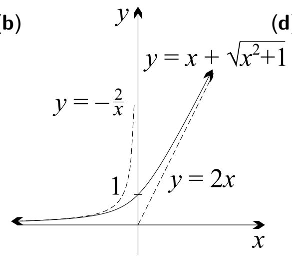

**(c)**



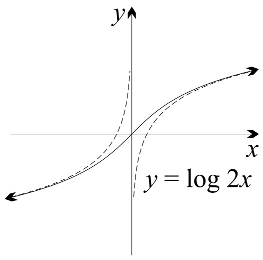

**(d)**




**11(b)**




**(d)**




**(e)** $\sinh x = \dfrac{e^x - e^{-x}}{2}$

## Exercise **8I** (Page 94)

**1(a)** $y' + 1$ **(b)** $y + xy'$ **(c)** $2x - 2yy'$ **(d)** $3y^2y' + 3y + 3xy'$ **(e)** $y^{-1}y'$ **(f)** $e^y y'$ **(g)** $y'(2x + 3y) + y(2 + 3y')$ **(h)** $3(x + y)^2(1 + y')$ **(i)** $4(x^2 + y^2)(x + yy')$

**2(a)** $y = \sqrt{x^2 - 9}$ or $y = -\sqrt{x^2 - 9}$
**(b)** $y = \sqrt{4 - x^2}$ or $y = -\sqrt{4 - x^2}$
**(c)** $y = 1 + \sqrt{1 - x^2}$ or $y = 1 - \sqrt{1 - x^2}$
**(d)** $y = -x + \sqrt{1 - x^2}$ or $y = -x - \sqrt{1 - x^2}$

**3(a)** $y' = -\frac{x}{y}$, $(-6, 0)$, $(6, 0)$ **(b)** $y' = \frac{x}{y}$, $(-4, 0)$, $(4, 0)$ **(c)** $y' = \frac{x-y}{x-2y}$, $(-2, -1)$, $(2, 1)$ **(d)** $y' = \frac{3x^2+y^2}{2y(2-x)}$, $(0, 0)$

**4(a)** $(0, -6)$, $(0, 6)$ **(b)** none **(c)** $(-\sqrt{2}, -\sqrt{2})$, $(\sqrt{2}, \sqrt{2})$ **(d)** none

**5(a)** $\frac{5}{4}$ **(b)** $\frac{1}{4}$ **(c)** $0$ **(d)** $-\frac{1}{4}$ **(e)** $\frac{1}{2}$ **(f)** $\frac{13}{48}$

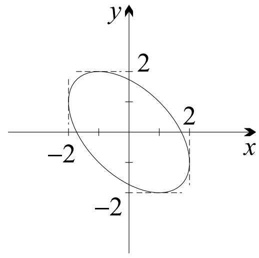
**6(a)** $y = x + 4$ **(b)** $10x - 7y = 1$ **(c)** $x - 2y - 5 = 0$ **(d)** $y = 3x + 2$ **(e)** $y = 12x - 23$ **(f)** $y = 2x - 3$

**7(a)** $4x - 7y + 19 = 0$ **(b)** The denominator of $y'$ is never zero. **(d)** $1$

**8(b)** $y = 1 - x$

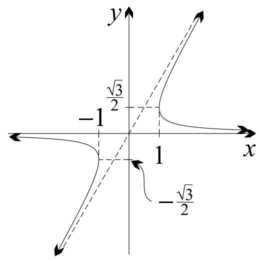
**9(a)** $-2 \le x \le 2$
**(b)** $(-\sqrt{3}, 0)$, $(\sqrt{3}, 0)$
**(c)(ii)** $(-1, -1)$, $(1, 1)$
**(e)** $(-1, 2)$, $(1, -2)$
**(f)** $(-2, 1)$, $(2, -1)$

**(g)**



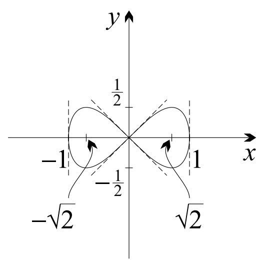

**10(a)** $x \le -1$ or $x \ge 1$
**(b)** no
**(d)** $y = 0$, $y = x\sqrt{3}$
**(f)** no

**(g)**




**11(a)** $-1 \le x \le 1$
**(b)** $(-1, 0)$, $(0, 0)$, $(1, 0)$
**(c)** The relations is even in both $x$ and $y$.
**(g)** $(\frac{1}{\sqrt{2}}, \frac{1}{2})$

**(h)**




**12(a)** $(0, 0)$ **(b)(ii)** $(\frac{3}{2}, \frac{3}{2})$ **(c)(ii)** $(\sqrt[3]{2}, \sqrt[3]{4})$ **(d)(iii)** $0$ **(e)(i)** $(0, 0)$, $(\sqrt[3]{4}, \sqrt[3]{2})$ **(ii)** The curve crosses itself. **(f)(ii)** $x + y + 1 = 0$

**(g)**