THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Assignment 2

MATH1903/1907: Integral Calculus and Modelling (Advanced) — Semester 2, 2017

Web Page: http://sydney.edu.au/science/maths/u/UG/JM/MATH1903/
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1. Consider a differentiable function $f : I \to \mathbb{R}$ on the open interval I. Fix a point $a \in I$. The fundamental theorem of calculus asserts that

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

In the questions below assume all derivatives required exist and are continuous.

(a) Use integration by parts to show that

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt.$$

Solution: If we set $u(t) = f'(t)$ and $v(t) = -(x-t)$, then $u'(t) = f''(t)$ and $v'(t) = 1$. Hence, using the integration by parts formula

$$\begin{aligned}\int_a^x f'(t)\,dt &= \int_a^x 1\cdot f'(t)\,dt\\ &= \Big[-(x-t)f'(t)\Big]_a^x + \int_a^x (x-t)f''(t)\,dt\\ &= -(x-x)f'(x) + (x-a)f'(a) + \int_a^x (x-t)f''(t)\,dt\\ &= f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt.\end{aligned}$$

Hence,

$$f(x) = f(a) + \int_0^x f'(t)\,dt = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt.$$

(b) Use induction by n to show that for $n \in \mathbb{N}$ we have

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\dots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt.$$

Solution: The case $n = 1$ is given by part (a). Hence we assume that the formula holds for n. We then apply the integration by parts formula with $u(t) = f^{(n+1)}(t)$ and $v(t) = -\dfrac{(x-t)^{n+1}}{(n+1)!}$, then $u'(t) = f^{(n+2)}(t)$ and $v'(t) = \dfrac{(x-t)^n}{n!}$ so that

$$\begin{aligned}\int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt &= \Big[-\frac{(x-t)^{n+1}}{(n+1)!}f^{(n+1)}(t)\Big]_a^x + \int_a^x -\frac{(x-t)^{n+1}}{(n+1)!}f^{(n+2)}(t)\,dt\\ &= -\frac{(x-x)^{n+1}}{(n+1)!}f^{(n+1)}(x) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(a) + \int_a^x \frac{(x-t)^{n+1}}{(n+1)!}f^{(n+2)}(t)\,dt\\ &= \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \int_a^x \frac{(x-t)^{n+1}}{(n+1)!}f^{(n+2)}(t)\,dt\end{aligned}$$

Hence, using the induction assumption we obtain

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &\quad + \int_a^x \frac{f^{(n+1)}(t)}{n!}(t-a)^n\,dt \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &\quad + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \int_a^x \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)\,dt \end{aligned}$$

which is the statement for $n+1$. Hence by mathematical induction the formula is true for all $n \in \mathbb{N}$. In particular, the formula shows that as an alternative to the Lagrange form, the remainder term in the Taylor polynomial can be represented as an integral in the form

$$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(t-a)^n\,dt.$$

2. Consider the differential equation

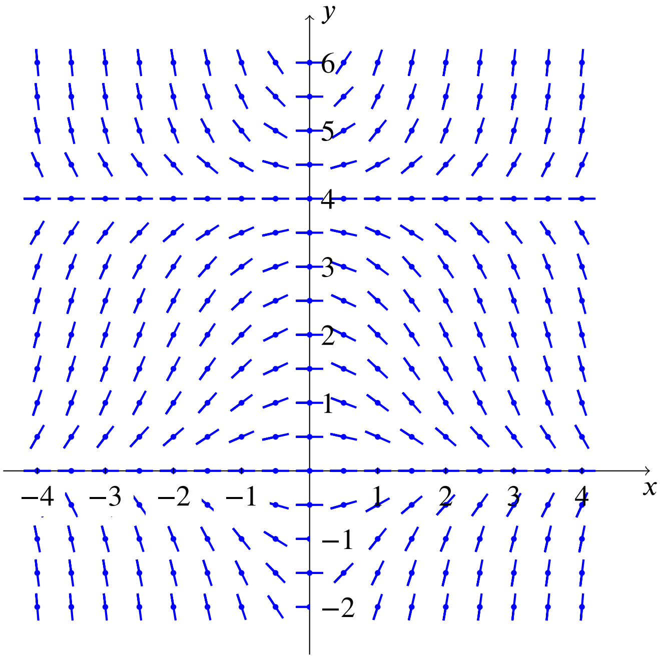
$$y' = xy(y-4)/4.$$

(a) Sketch the direction field of the differential equation between $-4 \leq x \leq 4$ and $-2 \leq y \leq 6$. Briefly explain how you found it.

Solution: To find the direction field we look at the sign of $xy(y-4)$ to find the direction field:

- Note that $y(y-4)$ is a concave up parabola with intercepts $y=0$ and $y=4$. Hence $y(y-4) > 0$ if $y < 0$ or $y > 4$ and $y(y-4) < 0$ if $0 < y < 4$.
- If $x > 0$, then from the above, $y' > 0$ if $y < 0$ or $y > 4$ and $y' < 0$ if $0 < y < 4$. Moreover the slope becomes steeper with x increasing and flatter with y closer to 0 and 4.
- If $x = 0$, then $y' = 0$.
- If $x < 0$, then from the above, $y' < 0$ if $y < 0$ or $y > 4$ and $y' > 0$ if $0 < y < 4$. Moreover the slope becomes steeper with $-x$ increasing and flatter with y closer to 0 and 4.
- The direction field is symmetric with respect to the y-axis.

Hence a sketch looks as follows:



(b) Find the general solution of the differential equation.

Solution: The equation is separable and we can write

$$\frac{dy}{y(4-y)} = \frac{x}{4}\,dx$$

To integrate the left hand side use the partial fraction decomposition

$$\frac{1}{y(4-y)} = \frac{1}{4}\Big(\frac{1}{y-4} - \frac{1}{y}\Big).$$

Integration yields

$$\int \frac{1}{4}\Big(\frac{1}{y-4} - \frac{1}{y}\Big)\,dy = \frac{1}{4}\big(\ln|y-4| - \ln|y|\big)$$
$$= \frac{1}{4}\ln\Big|\frac{y-4}{y}\Big| = \frac{1}{4}\int x\,dx = \frac{1}{4}\frac{x^2}{2} + C.$$

Next we cancel the factor $1/4$ and take exponentials on both sides:

$$\frac{y-4}{y} = \pm e^C e^{x^2/2} = Ae^{x^2/2} \tag{1}$$

for some constant A (of arbitrary sign determined by the initial conditions). Solving the equation for y we obtain

$$y(x) = \frac{4}{1-Ae^{x^2/2}} = \frac{4e^{-x^2/2}}{e^{-x^2/2} - A}.$$

(c) Find the particular solution with initial value $y(0) = 2$.

Solution: The most efficient way to find the constant A in the general solution given $y(0) = 2$ is to use (1):

$$\frac{y(0)-4}{y(0)} = \frac{2-4}{2} = -1 = Ae^{0^2/2} = A.$$

Hence $A = 1$ and the particular solution is

$$y(x) = \frac{4}{1 - e^{x^2/2}} = \frac{4e^{-x^2/2}}{e^{-x^2/2} - 1}.$$

Alternatively, use the solution and solve

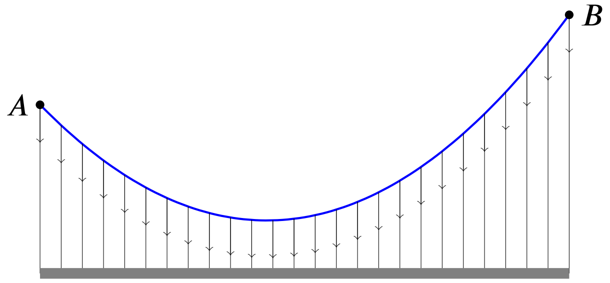
$$y(0) = 2 = \frac{4}{1 - Ae^{0^2/2}} = \frac{4}{1 - A}$$

and solve for A. Note however, that computing A from the above equation means reversing the computation done to obtain the solution y from (1), meaning it is much more efficient in terms of effort to use (1) to compute the integration constant A.

(d) Determine the equilibrium solutions and comment on their stability properties.

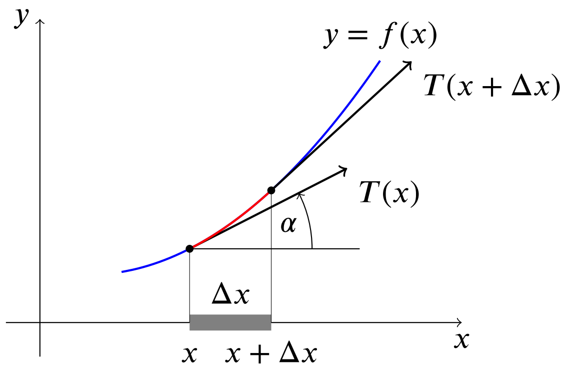
Solution: Equilibrium solutions are by definition those for which $y' = \dfrac{xy(y-4)}{4} = 0$ for all x. This is the case for $y = 0$ and $y = 4$. From the direction field it is evident that $y = 0$ is stable and $y = 4$ is unstable.

3. A perfectly flexible and weightless cable is suspended at points A and B. A uniformly distributed load of density ρ per length is supended from the cable exerting a force in the vertical direction as shown below. A real world example would be a cable suspension bridge.



Uniformly distributed load

The aim of this question is to find the shape of the cable. Assume that the shape can be described by the graph of a smooth function $f : (a, b) \to \mathbb{R}$. To find an equation for the shape we need to look at the balance of forces on the section of the cable between x and $x + \Delta x$. We let $T(x)$ be the magnitude of the tension force of the cable in the direction of the tangent of the cable at $(x, f(x))$.



Let $\alpha = \alpha(x)$ be angle the tangent makes with the horizontal and note that $\tan(\alpha) = f'(x)$.

(a) Explain why the force acting in the horizontal direction on the cable is constant in x and deduce that $T(x) = H\sqrt{1 + \big(f'(x)\big)^2}$ for some constant H.

Solution: The only force acting in the horizontal direction is the tension force. In order for the cable not to shift horizontally the component in the horizontal direction needs to be constant. We have $\tan\alpha = f'(x)$ and hence

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+\big(f'(x)\big)^2}}.$$

Hence

$$T(x)\cos(\alpha(x)) = \frac{T(x)}{\sqrt{1+\big(f'(x)\big)^2}} = H = \text{const},$$

that is,

$$T(x) = H\sqrt{1+\big(f'(x)\big)^2}.$$

(b) Explain why the forces acting in the vertical direction on the segment of the cable between x and $x + \Delta x$ satisfy the equation

$$H\big(f'(x+\Delta x) - f'(x)\big) = \rho g \Delta x,$$

where g is the gravitational constant.

Solution: Here we need the balance of forces in the vertical direction. As argued in (a) we have that

$$\sin(\alpha) = \tan(\alpha)\cos(\alpha) = \frac{f'(x)}{\sqrt{1+\big(f'(x)\big)^2}}.$$

Again the tension forces act in opposite directons on the different ends of the segement of cable between x and $x + \Delta x$. The gravitational force in the downwards direction is $\rho g \Delta x$ (the mass of the segment attached is $\rho\Delta x$). Using the result from (a) we therefore have

$$\begin{aligned}
\rho g \Delta x &= T(x+\Delta x)\sin(\alpha(x+\Delta x)) - T(x)\sin(\alpha(x)) \\
&= \frac{T(x+\Delta x) f'(x+\Delta x}{\sqrt{1+\big(f'(x+\Delta x)\big)^2}} - \frac{T(x)f'(x)}{\sqrt{1+\big(f'(x)\big)^2}} \\
&= H\big(f'(x+\Delta x) - f'(x)\big)
\end{aligned}$$

(c) Hence determine $f(x)$. What shape does the cable have?

Solution: Dividing the identity in (b) by Δx and letting Δx go to zero yields

$$\lim_{\Delta x\to 0} \frac{H\big(f'(x+\Delta x) - f'(x)\big)}{\Delta x} = Hf''(x) = \rho g.$$

This is a differential equation we can easily solve:

$$\begin{aligned}
f'(x) &= \frac{\rho g}{H}x + C \\
f(x) &= \frac{\rho g}{2H}x^2 + Cx + D
\end{aligned}$$

with C and D constants. (They depend on the length L of the cable, the distance $b - a$, difference of heights either end is suspended as well as ρ, g and H.)

(d) Show that $\int_a^b T(x)\,dx = HL$, where L is the length of the cable.

Solution: By part (a) we have

$$\int_a^b T(x)\,dx = H \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx.$$

The integral on the right hand side is the formula for the lenght of a curve, so equals L.