THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 13

MATH1901: Differential Calculus (Advanced) Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Implicit Function Theorem.
- ☐ Tangents to level curves.
- ☐ Directional derivatives.
- ☐ The gradient vector.
- ☐ Properties of the gradient vector.

Outcomes

After completing this tutorial you should

- ☐ apply the implicit function theorem;
- ☐ calculate tangents to level curves;
- ☐ compute directional derivatives;
- ☐ understand the properties of the gradient vector.

Summary of essential material

Mixed Derivatives Theorem. Suppose $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$. If $(x_0, y_0) \in D$ that $f_{xy}(x, y)$ and $f_{yx}(x, y)$ both exist in a disc around (x_0, y_0) and are both continuous at (x_0, y_0), then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0),$$

that is, mixed partial derivatives can be computed in any order. For efficient computation choose the order that requires minimal effort.

Chain Rule. Let $z = f(x, y)$, and suppose that $x = x(t)$ and $y = y(t)$. If $x(t)$ and $y(t)$ are continuous at $t = t_0$, and if f_x and f_y are continuous at $(a, y_0) = (x(t_0), y(t_0))$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

The chain rule generalises to functions $x = x(u, v)$ and $y = y(u, v)$ of two variables (if we fix one of them we are back to the one variable case):

$$\frac{\partial z}{du} = \frac{\partial f}{\partial x}\frac{\partial x}{du} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \text{and} \qquad \frac{\partial z}{dv} = \frac{\partial f}{\partial x}\frac{\partial x}{dv} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Implicit Function Theorem. Suppose that (x_0, y_0) lies on the level curve C given by $f(x, y) = c$. If f has continuous partial derivatives f_x and f_y in a neighbourhood of (x_0, y_0), and if $f_y(x_0, y_0) \neq 0$, then there is a disc D around (x_0, y_0) such that the part of C inside D is the graph of a differentiable function $y = y(x)$, and the derivative of $y(x)$ at $(x, y) = (x_0, y_0)$ is

$$y'(x_0) = -\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)}.$$

Gradient Vectors. The *gradient vector* of $f(x, y)$ is

$$\nabla f(x, y) := f_x(x, y)\boldsymbol{i} + f_y(x, y)\boldsymbol{j}.$$

Properties of $\nabla f(x, y)$: Suppose that $f(x, y)$ has continuous partial derivatives at $(x, y) = (x_0, y_0)$. Then

(1) $\nabla f(x_0, y_0)$ points in the direction of the steepest increase of $z = f(x, y)$ at $(x, y) = (x_0, y_0)$.

(2) $|\nabla f(x_0, y_0)| = \sqrt{f_x(x_0, y_0)^2 + f_y(x_0, y_0)^2}$ is the magnitude of this steepest increase.

(3) If (x_0, y_0) lies on the level curve $f(x, y) = k$, then $\nabla f(x_0, y_0)$ is perpendicular to this level curve at (x_0, y_0).

Directional Derivatives. Let $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ be a *unit vector*. The *directional derivative* is the slope of the tangent of the curve obtained by intersecting graph $z = f(x, y)$ with the plane through (x_0, y_0) parallel to the z-axis and containing $\boldsymbol{u}$. If $f(x, y)$ has continuous partial derivatives, then it is given by

$$\frac{\partial f}{\partial \boldsymbol{u}} := D_{\boldsymbol{u}} f(x, y) = \nabla f(x, y) \cdot \boldsymbol{u} = f_x(x, y)u_1 + f_y(x, y)u_2 \qquad \text{(the “}\cdot\text{” denotes the dot product).}$$

Special cases are the partial derivatives: they are the directional derivatives in the direction of $\boldsymbol{i}$ and $\boldsymbol{j}$.

From first principles, the directional derivative of $f(x, y)$ in the direction of $\boldsymbol{u}$ is

$$\frac{\partial f}{\partial \boldsymbol{u}}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

if the limit exists.

Questions to complete during the tutorial

1. Find the equation of the tangent to the level curve

$$1 - 2x + 6y + \sinh(3 - 2x + y) = 3$$

at the point $(x, y) = (2, 1)$.

Solution: Let $f(x, y) = 1 - 2x + 6y + \sinh(3 - 2x + y)$. We have $f_y(x, y) = 6 + \cosh(3 - 2x + y)$ and so $f_y(2, 1) = 6 + 1 = 7 \neq 0$. Therefore the equation $f(x, y) = 3$ implicitly defines y as a differentiable function of x around $(x, y) = (2, 1)$, and

$$y'(x) = -\frac{f_x(x, y)}{f_y(x, y)} = -\frac{-2 - 2\cosh(3 - 2x + y)}{6 + \cosh(3 - 2x + y)} = \frac{2(1 + \cosh(3 - 2x + y))}{6 + \cosh(3 - 2x + y)}.$$

Thus $y'(2) = 4/7$, and so the tangent is $y - 1 = (4/7)(x - 2)$, or $4x - 7y = 1$.

2. Let $f(x, y) = \sin(x^2 - y) + 4xy + 3$.

(a) Find the tangent plane to $z = f(x, y)$ at the point $(x, y) = (2, 4)$.

Solution: We have $f_x(x, y) = 2x\cos(x^2 - y) + 4y$ and $f_y(x, y) = -\cos(x^2 - y) + 4x$. So $f_x(2, 4) = 4 + 16 = 20$ and $f_y(2, 4) = -1 + 8 = 7$. Thus the tangent plane has equation

$$z = 32 + 20(x - 2) + 7(y - 4).$$

(b) What is the direction of the steepest slope to the graph $z = f(x, y)$ at the point $(x, y) = (2, 4)$? What is the magnitude of this slope?

Solution: The direction is $\nabla f(2, 4) = 20\boldsymbol{i}+7\boldsymbol{j}$. The magnitude of the steepest slope is $|\nabla f(2, 4)| = \sqrt{20^2 + 7^2} = \sqrt{449}$.

(c) What is the slope of the graph $z = f(x, y)$ in the direction $\boldsymbol{i} + 3\boldsymbol{j}$ at the point $(x, y) = (2, 4)$?

Solution: First replace $\boldsymbol{u}$ by the unit vector $\hat{\boldsymbol{u}} = \frac{1}{\sqrt{10}}\boldsymbol{i} + \frac{3}{\sqrt{10}}\boldsymbol{j}$. The slope that we are after is

$$D_{\hat{\boldsymbol{u}}} f(2, 4) = \nabla f(2, 4) \cdot \hat{\boldsymbol{u}} = \frac{1}{\sqrt{10}}(20\boldsymbol{i} + 7\boldsymbol{j}) \cdot (\boldsymbol{i} + 3\boldsymbol{j}) = \frac{41}{\sqrt{10}}.$$

3. Let $f(x, y) = e^y - x\sin(x + y)$. Show that the equation $f(x, y) = 1$ implicitly defines $y = y(x)$ as a function of x in a disc around $(x, y) = (\pi, 0)$. Compute $y'(x)$, and hence find the equation of the tangent line to the level curve $f(x, y) = 1$ at the point $(x, y) = (\pi, 0)$.

Solution: Since $f_y(x, y) = e^y - x\cos(x + y)$ we have $f_y(\pi, 0) = 1 + \pi \neq 0$, and so $f(x, y) = 1$ implicitly defines y as a function of x near the point $(x, y) = (\pi, 0)$. Then

$$y'(x) = -\frac{f_x(x, y)}{f_y(x, y)} = \frac{\sin(x + y) + x\cos(x + y)}{e^y - x\cos(x + y)}.$$

Thus

$$y'(\pi) = -\frac{\pi}{1+\pi}.$$

So the tangent line is $y - 0 = y'(\pi)(x - \pi)$, and so

$$y = -\frac{\pi}{1+\pi}x + \frac{\pi^2}{1+\pi}.$$

4. An ant is standing on the kitchen floor. The floor has coordinates such that the x-axis points east, and the y-axis points north. The temperature on the kitchen floor at the point (x, y) is given by the formula $T(x, y) = x^2 - 2y^2 + 4xy$. The ant is currently at the point $(x, y) = (2, 1)$ on the floor.

(a) In which direction should the ant walk to initially increase temperature most rapidly? What is the rate of change in temperature that the ant will experience if it walks in this direction?

Solution: The ant should walk in the direction of the gradient vector. Since

$$\nabla f(x, y) = f_x(x, y)\boldsymbol{i} + f_y(x, y)\boldsymbol{j} = (2x + 4y)\boldsymbol{i} + (-4y + 4x)\boldsymbol{j}.$$

Thus $\nabla f(2, 1) = 8\boldsymbol{i} + 4\boldsymbol{j}$ is the direction. The rate of change in temperature that the ant will experience is $|\nabla f(2, 1)| = \sqrt{8^2 + 4^2} = \sqrt{80}$

(b) What is the rate of change in temperature that the ant initially experiences if it walks in the direction $\boldsymbol{i} - \boldsymbol{j}$?

Solution: Let $\boldsymbol{u} = (\boldsymbol{i} - \boldsymbol{j})/\sqrt{2}$ be the unit vector in the given direction. The rate of change experienced is the directional derivative,

$$D_{\boldsymbol{u}} f(2, 1) = \nabla f(2, 1) \cdot \boldsymbol{u} = (8\boldsymbol{i} + 4\boldsymbol{j}) \cdot (\boldsymbol{i} - \boldsymbol{j})/\sqrt{2} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

5. (a) Use two different methods to calculate $\frac{\partial z}{\partial t}$ if $z = \sqrt{x^2 + y^2}$, $x = e^{st}$ and $y = 1 + s^2 \cos t$.

Solution: First substitute for x and y into the formula for z, to obtain z as a function of s and t directly. We obtain

$$z = \sqrt{e^{2st} + (1 + s^2 \cos t)^2},$$

and hence

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\frac{\partial}{\partial t}\left(e^{2st} + (1 + s^2 \cos t)^2\right)}{2\sqrt{e^{2st} + (1 + s^2 \cos t)^2}} \\
&= \frac{2se^{2st} + 2(1 + s^2 \cos t)(-s^2 \sin t)}{2\sqrt{e^{2st} + (1 + s^2 \cos t)^2}} \\
&= \frac{se^{2st} - s^2 \sin t(1 + s^2 \cos t)}{\sqrt{e^{2st} + (1 + s^2 \cos t)^2}}
\end{aligned}$$

Alternatively, use the chain rule:

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= \frac{x}{\sqrt{x^2 + y^2}} \times se^{st} + \frac{y}{\sqrt{x^2 + y^2}} \times -s^2 \sin t \\
&= \frac{e^{st} se^{st} + (1 + s^2 \cos t)(-s^2 \sin t)}{\sqrt{x^2 + y^2}} \\
&= \frac{se^{2st} - s^2 \sin t(1 + s^2 \cos t)}{\sqrt{e^{2st} + (1 + s^2 \cos t)^2}}.
\end{aligned}$$

(b) Use two different methods to calculate $\frac{\partial z}{\partial u}$ if $z = \tan^{-1}(x/y)$, $x = 2u + v$ and $y = 3u - v$.

Solution: As in the previous part, we start with $z = \tan^{-1}\left(\frac{2u+v}{3u-v}\right)$ to obtain

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{2(3u-v) - 3(2u+v)}{(3u-v)^2[1 + \left(\frac{2u+v}{3u-v}\right)^2]} \\ &= \frac{-5v}{(3u-v)^2 + (2u+v)^2}.\end{aligned}$$

Using the chain rule,

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= \frac{1/y}{1 + \frac{x^2}{y^2}} \times 2 + \frac{-x/y^2}{1 + \frac{x^2}{y^2}} \times 3 \\ &= \frac{2y - 3x}{y^2 + x^2} \\ &= \frac{-5v}{(3u-v)^2 + (2u+v)^2}.\end{aligned}$$

6. An object moves on the surface $z = (x-1)^2 + y^2$. The projection of the object's path onto the xy-plane is given by the parametric equations $x = 2\cos t$, $y = 2\sin t$ where $t \geq 0$ represents time. Use the chain rule to find the rate of change of height of the object above the xy plane. Hence find the maximum height achieved by the object.

Solution: By the chain rule

$$\frac{dz}{dt} = 2(x-1)(-2\sin t) + 2y(2\cos t) = 2(2\cos t - 1)(-2\sin t) + 2(2\sin t)(2\cos t) = 4\sin t.$$

Thus $\frac{dz}{dt} = 0$ exactly when t is a multiple of π, corresponding to the points $(2, 0, 1)$ and $(-2, 0, 9)$ on the surface. Thus the maximum height equals 9, which occurs whenever t is an odd multiple of π.

7. Let $z = f(x, y)$, and suppose that $x = r\cos\theta$ and $y = r\sin\theta$. Show that

$$r^2\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right).$$

You may assume that $f(x, y)$ is nice enough so that the Chain Rule applies.

Solution: This is a calculation using the chain rule. Let's use the notation z_r and z_θ for $\partial z/\partial r$ and $\partial z/\partial \theta$, and similarly for x_r, x_θ, y_r, y_θ, f_x, and f_y. Then the chain rule gives

$$\begin{aligned}z_r &= f_x x_r + f_y y_r = f_x \cos\theta + f_y \sin\theta \\ z_\theta &= f_x x_\theta + f_y y_\theta = -f_x r\sin\theta + f_y r\cos\theta.\end{aligned}$$

Therefore

$$\begin{aligned}r^2 z_r^2 + z_\theta^2 &= r^2(f_x^2\cos^2\theta + 2f_x f_y\cos\theta\sin\theta + f_y^2\sin^2\theta) + r^2 f_x^2\sin^2\theta - 2r^2 f_x f_y\cos\theta\sin\theta + r^2 f_y^2\cos^2\theta \\ &= r^2(\cos^2\theta + \sin^2\theta)f_x^2 + r^2(\cos^2\theta + \sin^2\theta)f_y^2 \\ &= r^2(f_x^2 + f_y^2),\end{aligned}$$

which is what we were trying to prove.

Extra questions for further practice

8. Let $f(x) = x^3 - 3x + 1$.

(a) Show that the function $f : [-1,1] \to [-1,3]$ is bijective.

Solution: We have $f'(x) = 3x^2 - 3 < 0$ on $(-1,1)$. Therefore $f(x)$ is strictly decreasing on the open interval $(-1,1)$. Therefore $f(x)$ is injective on $[-1,1]$, and since $f(-1) = -1 + 3 + 1 = 3$ and $f(1) = 1 - 3 + 1 = -1$ the image of $[-1,1]$ under f is $[-1,3]$. Thus $f : [-1,1] \to [-1,3]$ is bijective.

(b) Let $f^{-1} : [-1,3] \to [-1,1]$ be the inverse function. Calculate the third order Taylor polynomial of $f^{-1}(x)$ centred at $x = 1$.

Solution: The inverse function $y = f^{-1}(x)$ satisfies $y^3 - 3y + 1 = x$, and we write this as $g(x,y) = 0$ where $g(x,y) = y^3 - 3y + 1 - x$. At $x = 1$ we have $y^3 - 3y = 0$, and so $y(y^2 - 3) = 0$, and so $y = 0$ or $y = \pm\sqrt{3}$. Thus $y(1) = 0$ (because $\pm\sqrt{3} \notin [-1,1]$). Then $g_y(x,y) = 3y^2 - 3$, giving $g_y(1,0) = -3 \neq 0$, and so the equation $g(x,y) = 0$ implicitly defines y as a differentiable function of x for (x,y) near $(1,0)$. Then

$$y'(x) = -\frac{g_x(x,y)}{g_y(x,y)} = \frac{1}{3}(y^2 - 1)^{-1}.$$

In particular, $y'(1) = -\frac{1}{3}$. Then $y''(x) = -\frac{1}{3}2y'y(y^2 - 1)^{-2}$, and so $y''(1) = 0$. Similarly, we compute

$$y'''(1) = -\frac{2}{27}.$$

Therefore the Taylor polynomial is

$$T_3(x) = -\frac{1}{3}(x-1) - \frac{1}{81}(x-1)^3.$$

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be function, and suppose that f satisfies

$$f(ta, tb) = tf(a,b) \qquad \text{for all} \quad (a,b) \in \mathbb{R}^2 \quad \text{and all} \quad t \in \mathbb{R}.$$

In this question you may assume that f is sufficiently smooth so that the chain rule, and any other theorems from lectures, apply.

(a) Use the chain rule to show that

$$f_x(a,b)a + f_y(a,b)b = f(a,b) \qquad \text{for all} \quad (a,b) \in \mathbb{R}^2.$$

Solution: Let $x(t) = at$ and $y(t) = bt$. Differentiating the equation $f(x(t), y(t)) = tf(a,b)$ with respect to t using the chain rule gives

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f(a,b),$$

which reads

$$f_x(at,bt)a + f_y(at,bt)b = f(a,b).$$

Now set $t = 1$.

(b) Show that the origin $(0,0,0) \in \mathbb{R}^3$ lies on every tangent plane to the surface $z = f(x,y)$.

Solution: Let $(a,b) \in \mathbb{R}^2$. The tangent plane to the surface $z = f(x,y)$ at the point $(x,y) = (a,b)$ is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b),$$

and using the formula from (a) this reduces to

$$z = f_x(a,b)x + f_y(a,b)y,$$

and (0, 0, 0) lies on this plane.

(c) You are given that $f(2,1) = 4$. Find the value of the directional derivative $D_{\boldsymbol{u}} f(6,3)$ in the direction $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j}$.

Solution: We have $\hat{\boldsymbol{u}} = \frac{1}{\sqrt{5}}(2\boldsymbol{i} + \boldsymbol{j})$, and the directional derivative is

$$D_{\boldsymbol{u}} f(6,3) = \frac{1}{\sqrt{5}}(f_x(6,3) \times 2 + f_y(6,3) \times 1).$$

Using the formula from the working in (a) (with $(a,b) = (2,1)$ and $t = 3$) we see that

$$D_{\boldsymbol{u}} f(6,3) = \frac{1}{\sqrt{5}} f(2,1) = \frac{4}{\sqrt{5}}.$$

10. Suppose that $f(x,y) = ye^{xy}$.

(a) Use the implicit function theorem to show that $f(x,y) = 1$ defines y implicitly as a function of x in a neighbourhood of the point $(0,1)$.

Solution: First note that $f(0,1) = 1e^0 = 1$. Moreover,

$$f_y(x,y) = e^{xy} + xye^{xy},$$

so $f_y(0,1) = 1 + 0 = 1 \neq 0$. Hence the implicit function theorem applies and y can be represented implicitly as a function of x in a neighbourhood of $(0,1)$.

(b) Use implicit differentiation to find the second order Taylor polynomial of the implicity defined function $y = y(x)$ centred at $x = 0$.

Solution: Implicit differentiation with respect to x gives

$$0 = y'e^{xy} + xyy'e^{xy} + y^2e^{xy} = y'(1 + xy)e^{xy} + y^2e^{xy}.$$

Substitution by $x = 0$ and $y = 1$ gives $0 = y' + 1$, so $y'(0) = -1$. To compute the second derivative we differentiate implicitly again.

$$\begin{aligned} 0 &= \left(y'(1 + xy)e^{xy} + y^2e^{xy}\right)' \\ &= y''(1 + xy)e^{xy} + y'(y + xy') + y'(1 + xy)(y + xy')e^{xy} + 2yy'e^{xy} + y^2(y + xy')e^{xy}. \end{aligned}$$

If we set $x = 0$, $y = 1$ and $y' = -1$ we obtain

$$0 = y'' - 1 - 1 - 2 + 1 = y'' - 3$$

Hence $y''(0) = 3$. The second order Taylor polynomial therefore is

$$T_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 = 1 - x + \frac{3}{2}x^2.$$