University of Sydney

MATH 1907

SSP

Assignment 3 - Statistics

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Seminar: New Law Annexe SR 346
Tuesday 4pm

November 2, 2017

1. For the following proofs we assume that all random variables are discrete.

(a) Using the result $X_1 \perp\!\!\!\perp (X_2, X_3)|X_4$, we are required to prove that $X_1 \perp\!\!\!\perp X_2|X_4$. In order to do so, we use the following proof.

$$\begin{aligned}
p(x_1, x_2, x_3|x_4) &= p(x_1|x_4) \cdot p(x_2, x_3|x_4) \\
&\text{Marginalising and normalising over } x_3 \\
\therefore \sum_{x_3} p(x_1, x_2, x_3|x_4) &= \sum_{x_3} p(x_1|x_4) \cdot p(x_2, x_3|x_4) \\
&= \sum_{x_3} p(x_1|x_4) \cdot \sum_{x_3} p(x_2, x_3|x_4) \\
&= p(x_1|x_4) \cdot \sum_{x_3} p(x_2, x_3|x_4) \\
&= p(x_1|x_4) \cdot p(x_2|x_4) \\
&= p(x_1, x_2|x_4)
\end{aligned}$$

This final result gives the desired result, $X_1 \perp\!\!\!\perp X_2|X_4$.

(b) Using the results $X_1 \perp\!\!\!\perp X_2|X_3 \ldots (1)$ and $X_1 \perp\!\!\!\perp X_3|X_2 \ldots (2)$, we are required to show that $X_1 \perp\!\!\!\perp (X_2, X_3)$. In order to do so, we use the following proof.

$$\begin{aligned}
p(x_1|x_2, x_3) &= p(x_1|x_3) \quad \text{from} \quad (1) \\
p(x_1|x_3, x_2) &= p(x_1|x_2) \quad \text{from} \quad (2) \\
p(x_1|x_3) &= p(x_1|x_2) \quad \text{by equating the two results} \\
\therefore \frac{\sum_{x_2} p(x_1, x_2, x_3)}{\sum_{x_1, x_2} p(x_1, x_2, x_3)} &= \frac{\sum_{x_3} p(x_1, x_2, x_3)}{\sum_{x_1, x_3} p(x_1, x_2, x_3)} \\
\therefore \sum_{x_2} p(x_1, x_2, x_3) \cdot \sum_{x_1, x_3} p(x_1, x_2, x_3) &= \sum_{x_3} p(x_1, x_2, x_3) \cdot \sum_{x_1, x_2} p(x_1, x_2, x_3) \\
\therefore p(x_1, x_3) \cdot p(x_2) &= p(x_1, x_2) \cdot p(x_3)
\end{aligned}$$

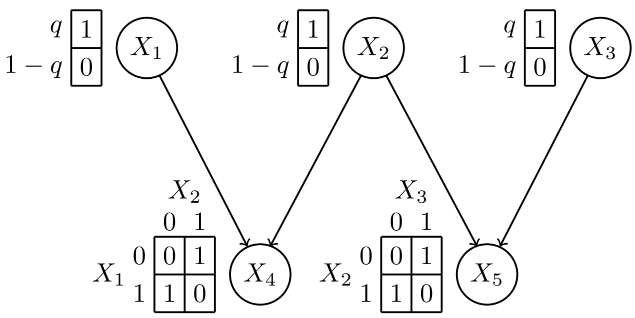
This final result shows that X_2, and X_3 are independent of X_1, through the definition of independence. Thus we have the final result that $X_1 \perp\!\!\!\perp (X_2, X_3)$.

2. For $i = 1, 2, 3$, let X_i be an indicator variable for the event that a coin toss comes up heads (which occurs with probability q). That is, $X_i = 1$ with probability q, and $X_i = 0$ with probability $1 - q$. Assuming that the X_i's are independent, define $X_4 = X_1 \oplus X_2$, $X_5 = X_2 \oplus X_3$, where $\oplus$ denotes addition in modulo 2 arithmetic.

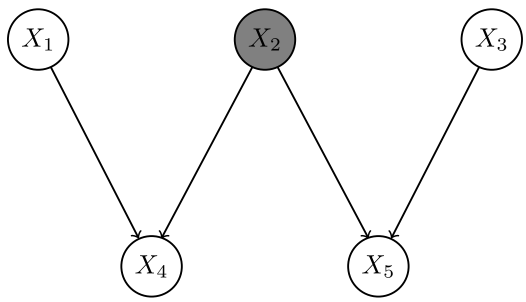
(a) The conditional PMF of (X_2, X_3) given $X_5 = 0$ and $X_5 = 1$ respectively is given by the following results. First, we recall that $X_5 = X_2 \oplus X_3$, and that $X_2 \perp\!\!\!\perp X_3$, as they are coin tosses. Furthermore, if $X_5 = 0$, $\{X_2, X_3\} = \{0, 0\}$ or $\{1, 1\}$. Finally, if $X_5 = 1$, $\{X_2, X_3\} = \{0, 1\}$ or $\{1, 0\}$. Using these results, we have the following proof.

$$\begin{aligned}
p(x_2, x_3 | x_5 = 0) &= p(x_2 = 0) \cdot p(x_3 = 0) + p(x_2 = 1) \cdot p(x_3 = 1) \\
&= q^2 + (1 - q)^2 \\
&= 2q^2 - 2q + 1 \\
p(x_2, x_3 | x_5 = 1) &= p(x_2 = 1) \cdot p(x_3 = 0) + p(x_2 = 0) \cdot p(x_3 = 1) \\
&= q(1 - q) + (1 - q)q \\
&= 2q - q^2
\end{aligned}$$

(b) We are required to draw a directed graphical model for these five random variables, including the conditional probability tables. The graph is as follows below.



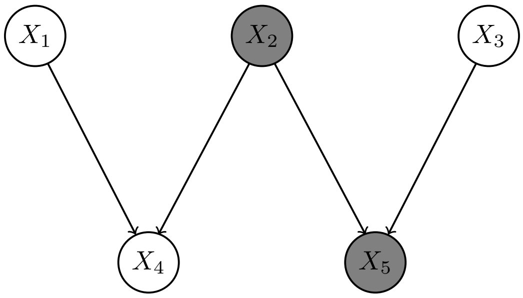
In order to determine at least five conditional independence relations from the directed graphical model, we apply Baye's Ball Theorem, in order to determine conditional independence of discrete random variables. To help with the construction of conditional independence relations, we will use the directed graphical models below.



Thus, it is clear that we have the following four conditional independence relations, derived from the rules that govern the behaviour the theoretical balls within Baye's Ball Theorem.

i. $X_1 \perp\!\!\!\perp X_5 | X_2$
ii. $X_1 \perp\!\!\!\perp X_3 | X_2$
iii. $X_4 \perp\!\!\!\perp X_5 | X_2$
iv. $X_4 \perp\!\!\!\perp X_3 | X_2$

In order to get the fifth result, we use the following directed graphical model.



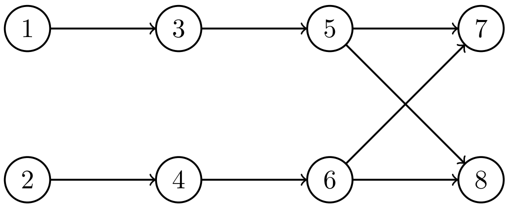
Thus the fifth conditional independence relation is $X_1 \perp\!\!\!\perp X_3 | (X_2, X_5)$. Thus, we have derived five conditional independence relations from the directed graphical model, utilising Baye's Ball Theorem and its governing laws.

(c) We are now required to determine the necessary conditions that exist upon q, in order to have $X_3 \perp\!\!\!\perp X_5$, and $X_1 \perp\!\!\!\perp X_4$. As we have $X_3 \perp\!\!\!\perp X_5$ and $X_1 \perp\!\!\!\perp X_4$, we are unable to apply Baye's Ball Theorem, as it is not conditional independence. Furthermore, independence in this example means that whatever the value that X_1 or X_3 takes has no influence on the value that X_4 or X_5 can take respectively. As a result, X_1 or X_3 must be able to take the values 0 and 1 with equal probability. Thus we have the following consequences.

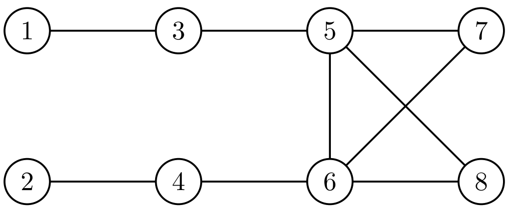
$$\begin{aligned} q &= 1 - q \\ \therefore 2q &= 1 \\ \therefore q &= \frac{1}{2} \end{aligned}$$

These marginal independence assertions are not clear, nor implied from the graph in part (b).

3. Consider the following directed graph.

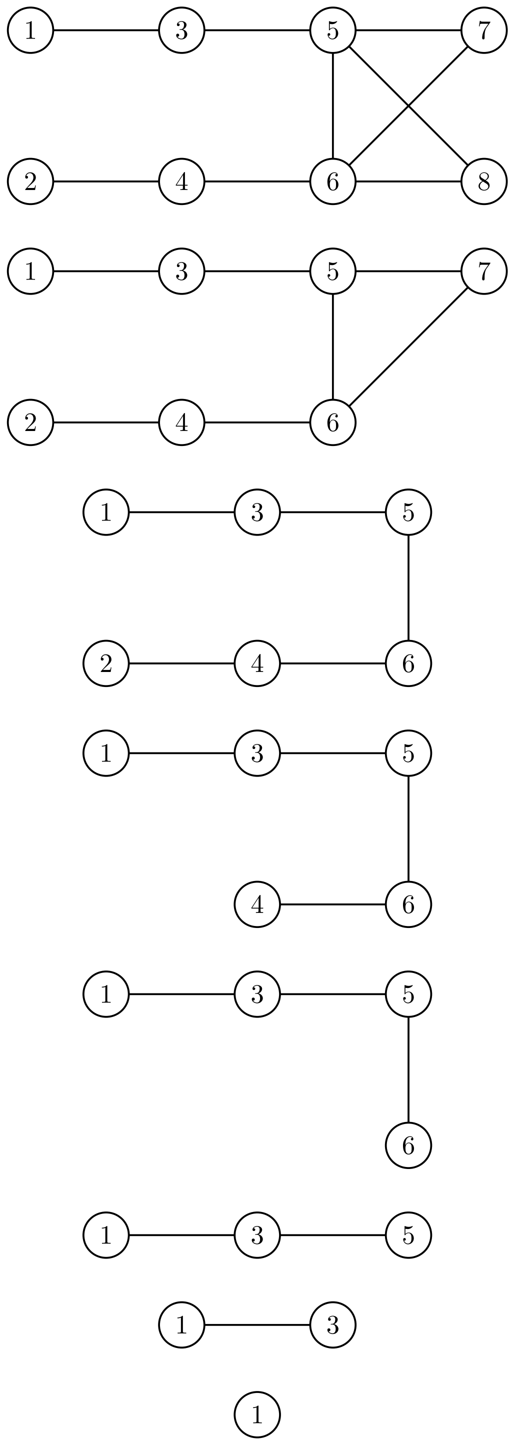


(a) From the directed graph above, we get the following moral graph.

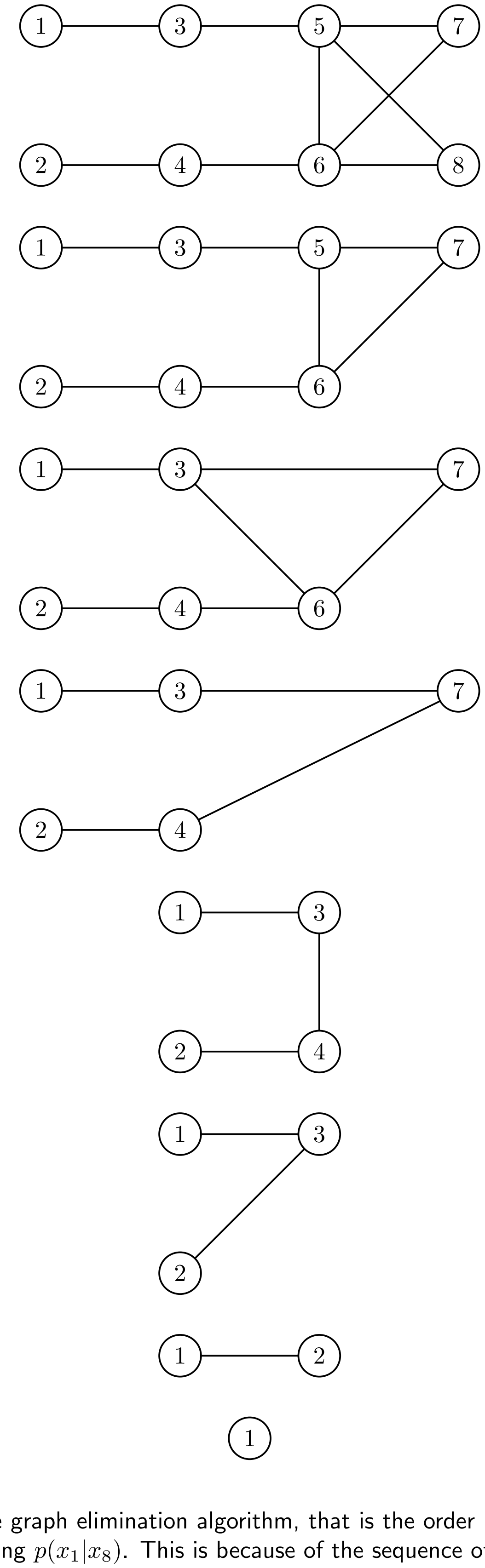


(b) The following sequences provide the results from running the graph elimination algorithm in the given orders.

i. $\{8, 7, 2, 4, 6, 5, 3, 1\}$



ii. $\{8, 5, 6, 7, 4, 3, 2, 1\}$



(c) The first order used in the graph elimination algorithm, that is the order $\{8, 7, 2, 4, 6, 5, 3, 1\}$, is the best order for computing $p(x_1|x_8)$. This is because of the sequence of elimination, that does not add any connections between nodes that did not already exist, rather only removes them from the moral graph. However, using the second order does add connections that did not already exist on the moral graph, and thus is less preferable than the first.