The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Longer Solutions to Selected Exercises for Week 4** 2012

7. (i) First observe that $\overrightarrow{QP} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{QR} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, yielding

$$\overrightarrow{QP} \cdot \overrightarrow{QR} \;=\; 2 + 4 + 2 \;=\; 8 \;>\; 0\,,$$

so that $\angle PQR$ is acute (nonzero, since the vectors are clearly not parallel). Now observe that $\overrightarrow{RQ} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{RS} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, yielding

$$\overrightarrow{RQ} \cdot \overrightarrow{RS} \;=\; -2 - 4 - 2 \;=\; -8 \;<\; 0\,,$$

so that $\angle QRS$ is obtuse (not 180°, since the vectors are clearly not parallel).

(ii) Observe that

$$\overrightarrow{PR} \cdot \overrightarrow{QS} \;=\; (-\mathbf{i} + \mathbf{k}) \cdot (3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \;=\; -3 + 0 + 3 \;=\; 0\,,$$

so PR and QS are mutually perpendicular.

8. The vectors are orthogonal because

$$\begin{aligned}
\mathbf{w} \cdot \left(\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\,\mathbf{w}\right) \;&=\; \mathbf{w}\cdot\mathbf{v} - \mathbf{w} \cdot \left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right) \;=\; \mathbf{w}\cdot\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}(\mathbf{w}\cdot\mathbf{w}) \\
&=\; \mathbf{w}\cdot\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}|\mathbf{w}|^2 \;=\; \mathbf{w}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w} \\
&=\; \mathbf{w}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{v} \;=\; 0\,.
\end{aligned}$$

10. Observe that

$$\begin{aligned}
(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \;&=\; \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} \\
&=\; |\mathbf{a}|^2 - \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b} - |\mathbf{b}|^2 \\
&=\; |\mathbf{a}|^2 - |\mathbf{a}|^2 \;=\; 0\,,
\end{aligned}$$

so that $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are orthogonal.

11. (i) The cosine of the angle between a major diagonal and an edge is

$$\frac{\big|\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\big|}{|\mathbf{i}|\,|\mathbf{i}+\mathbf{j}+\mathbf{k}|} \;=\; \frac{1}{\sqrt{3}}\,,$$

yielding an angle of approximately 55 degrees.

(ii) The cosine of the angle between a major diagonal and a face diagonal is

$$\frac{\left|(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\right|}{|\mathbf{i}+\mathbf{j}|\,|\mathbf{i}+\mathbf{j}+\mathbf{k}|} \;=\; \frac{2}{\sqrt{6}}\,,$$

yielding an angle of approximately 35 degrees.

(iii) The cosine of the angle between diagonals on adjacent faces is

$$\frac{\left|(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{k})\right|}{|\mathbf{i}+\mathbf{j}|\,|\mathbf{i}+\mathbf{k}|} \;=\; \frac{1}{2}\,,$$

yielding an angle of exactly 60 degrees.

(iv) The cosine of the angle between major diagonals is

$$\frac{\left|(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+\mathbf{k})\right|}{|\mathbf{i}+\mathbf{j}+\mathbf{k}|\,|\mathbf{i}-\mathbf{j}+\mathbf{k}|} \;=\; \frac{1}{3}\,,$$

yielding an angle of approximately 71 degrees.

12. (i) If $\mathbf{v}\cdot\mathbf{x} = \mathbf{v}\cdot\mathbf{y} = 0$ then

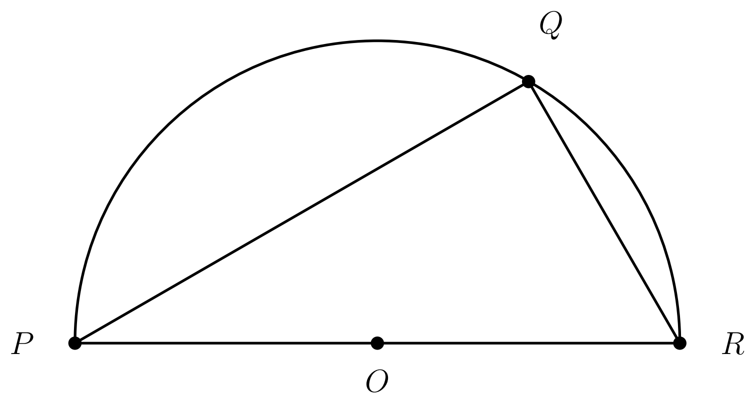
$$\mathbf{v}\cdot(a\mathbf{x}+b\mathbf{y}) \;=\; a\,(\mathbf{v}\cdot\mathbf{x}) + b\,(\mathbf{v}\cdot\mathbf{y}) \;=\; a0 + b0 \;=\; 0\,,$$

for any scalars a and b.

(ii) If $\mathbf{v}\cdot\mathbf{x} = \mathbf{v}\cdot(a\mathbf{x}+b\mathbf{y}) = 0$, where a and b are scalars such that $b \neq 0$, then

$$\begin{aligned}\mathbf{v}\cdot\mathbf{y} \;&=\; \mathbf{v}\cdot\Big(-\frac{a}{b}\mathbf{x}+\frac{a}{b}\mathbf{x}+\mathbf{y}\Big) \;=\; \mathbf{v}\cdot\Big(-\frac{a}{b}\mathbf{x}+\frac{1}{b}(a\mathbf{x}+b\mathbf{y})\Big)\\ &=\; -\frac{a}{b}(\mathbf{v}\cdot\mathbf{x}) + \frac{1}{b}\big(\mathbf{v}\cdot(a\mathbf{x}+b\mathbf{y})\big) \;=\; -\frac{a}{b}0+\frac{1}{b}0 \;=\; 0\,.\end{aligned}$$

13. Consider a semicircle and points O, P, Q, R as shown:



To show the angle at Q is a right-angle, it suffices to show $\overrightarrow{QP}\cdot\overrightarrow{QR} \;=\; 0$. But this follows from Exercise 10, since

$$\overrightarrow{QP} \;=\; \overrightarrow{QO}+\overrightarrow{OP} \qquad\text{and}\qquad \overrightarrow{QR} \;=\; \overrightarrow{QO}+\overrightarrow{OR} \;=\; \overrightarrow{QO}-\overrightarrow{OP}$$

and the fact that $\big|\overrightarrow{QO}\big| \;=\; \big|\overrightarrow{OP}\big|$, the radius of the circle.

14. Note first, by commutativity, that also $\mathbf{j}\cdot\mathbf{i} = \mathbf{k}\cdot\mathbf{i} = \mathbf{k}\cdot\mathbf{j} = 0$. Thus, if $\mathbf{v} = a\,\mathbf{i}+b\,\mathbf{j}+c\,\mathbf{k}$ and $\mathbf{w} = d\,\mathbf{i}+e\,\mathbf{j}+f\,\mathbf{k}$ then, by distributivity and compatibility of scalar multiplication,

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} \;&=\; (a\,\mathbf{i})\cdot(d\,\mathbf{i}) + (a\,\mathbf{i})\cdot(e\,\mathbf{j}) + (a\,\mathbf{i})\cdot(f\,\mathbf{k}) \\
&\qquad + (b\,\mathbf{j})\cdot(d\,\mathbf{i}) + (b\,\mathbf{j})\cdot(e\,\mathbf{j}) + (b\,\mathbf{j})\cdot(f\,\mathbf{k}) \\
&\qquad\qquad + (c\,\mathbf{k})\cdot(d\,\mathbf{i}) + (c\,\mathbf{k})\cdot(e\,\mathbf{j}) + (c\,\mathbf{k})\cdot(f\,\mathbf{k}) \\
&=\; (ad)\,(\mathbf{i}\cdot\mathbf{i}) + (ae)\,(\mathbf{i}\cdot\mathbf{j}) + (af)\,(\mathbf{i}\cdot\mathbf{k}) \\
&\qquad + (bd)\,(\mathbf{j}\cdot\mathbf{i}) + (be)\,(\mathbf{j}\cdot\mathbf{j}) + (bf)\,(\mathbf{j}\cdot\mathbf{k}) \\
&\qquad\qquad + (cd)\,(\mathbf{k}\cdot\mathbf{i}) + (ce)\,(\mathbf{k}\cdot\mathbf{j}) + (cf)\,(\mathbf{k}\cdot\mathbf{k}) \\
&=\; (ad)\,(1) + (ae)\,(0) + (af)\,(0) \\
&\qquad + (bd)\,(0) + (be)\,(1) + (bf)\,(0) \\
&\qquad\qquad + (cd)\,(0) + (ce)\,(0) + (cf)\,(1) \\
&=\; ad + be + cf\ .
\end{aligned}$$

15. Given $\mathbf{v} = a\,\mathbf{i}+b\,\mathbf{j}+c\,\mathbf{k}$ and $\mathbf{w} = d\,\mathbf{i}+e\,\mathbf{j}+f\,\mathbf{k}$ and the geometric definition, we have, applying the Cosine Rule and the length formula,

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} \;&=\; |\mathbf{v}||\mathbf{w}|\cos\theta \;=\; \frac{1}{2}\big(|\mathbf{v}|^2 + |\mathbf{w}|^2 - |\mathbf{v}-\mathbf{w}|^2\big) \\
=\; \frac{1}{2}&\Big(a^2+b^2+c^2+d^2+e^2+f^2 - \big((a-d)^2+(b-e)^2+(c-f)^2\big)\Big) \\
&=\; \frac{1}{2}\big(2ad+2be+2cf\big) \;=\; ad+be+cf\ .
\end{aligned}$$

16. The projection of $\mathbf{u}$ in the direction of $\mathbf{v}$ is

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v} \;=\; \frac{15-6+12}{9+36+4}\,\mathbf{v} \;=\; \frac{3}{7}\big(3\mathbf{i}-6\mathbf{j}+2\mathbf{k}\big)\ .$$

The component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ is

$$\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v} \;=\; \mathbf{u} - \frac{3}{7}\big(3\mathbf{i}-6\mathbf{j}+2\mathbf{k}\big) \;=\; \frac{1}{7}\big(26\mathbf{i}+25\mathbf{j}+36\mathbf{k}\big)\ .$$

Thus, the decomposition of $\mathbf{u}$ as a sum of vectors, the first parallel to $\mathbf{v}$ and the second perpendicular to $\mathbf{v}$, is

$$\mathbf{u} \;=\; \frac{3}{7}\big(3\mathbf{i}-6\mathbf{j}+2\mathbf{k}\big) + \frac{1}{7}\big(26\mathbf{i}+25\mathbf{j}+36\mathbf{k}\big)\ .$$

17. (i) The component of the force in the direction of $-\mathbf{i}+\mathbf{j}$ is

$$\frac{\big(15\mathbf{i}+20\mathbf{j}+6\mathbf{k}\big)\cdot(-\mathbf{i}+\mathbf{j})}{\big|-\mathbf{i}+\mathbf{j}\big|^2}\,(-\mathbf{i}+\mathbf{j}) \;=\; \frac{5}{2}\,(-\mathbf{i}+\mathbf{j}) \ \text{ newtons}\ ,$$

and the component orthogonal to $-\mathbf{i}+\mathbf{j}$ is

$$15\mathbf{i}+20\mathbf{j}+6\mathbf{k} - \frac{5}{2}\,(-\mathbf{i}+\mathbf{j}) \;=\; \frac{1}{2}\big(35\mathbf{i}+35\mathbf{j}+12\mathbf{k}\big) \ \text{ newtons}\ .$$

(ii) The component of the force in the direction of $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is

$$\frac{\big(15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}\big) \cdot \big(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big)}{\big|2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big|^2}\big(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big) \;=\; -\frac{12}{7}\big(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big) \quad \text{newtons} ,$$

and the component orthogonal to $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is

$$15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} + \frac{12}{7}\big(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\big) \;=\; \frac{1}{7}\big(129\mathbf{i} + 104\mathbf{j} + 54\mathbf{k}\big) \quad \text{newtons} .$$

18. If $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular then $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = 0$, so that

$$|\mathbf{a} + \mathbf{b}|^2 \;=\; \big(\mathbf{a} + \mathbf{b}\big) \cdot \big(\mathbf{a} + \mathbf{b}\big) \;=\; \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}$$

$$= \;|\mathbf{a}|^2 + 0 + 0 + |\mathbf{b}|^2 \;=\; |\mathbf{a}|^2 + |\mathbf{b}|^2 .$$

This is just the usual Theorem of Pythagoras where $\mathbf{a}$ and $\mathbf{b}$ label directed edges of a right-angled triangle.

19. Certainly $\mathbf{a}$ and $\mathbf{b}$ are perpendicular since $\mathbf{a} \cdot \mathbf{b} = 10 - 2 - 8 = 0$. Suppose $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so $\mathbf{v} \cdot \mathbf{a} = \mathbf{v} \cdot \mathbf{b} = 0$, yielding

$$2v_1 - v_2 + 4v_3 \;=\; 5v_1 + 2v_2 - 2v_3 \;=\; 0 .$$

Thus $2v_1 - v_2 = -4v_3$ and $5v_1 + 2v_2 = 2v_3$, which solving simultaneously yield

$$v_1 \;=\; -\frac{2}{3}v_3 \quad \text{and} \quad v_2 \;=\; \frac{8}{3}v_3 ,$$

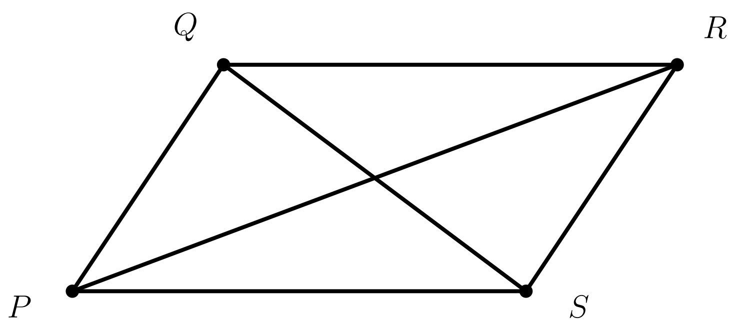
so that

$$\mathbf{v} \;=\; \frac{v_3}{3}(-2\,\mathbf{i} + 8\,\mathbf{j} + 3\,\mathbf{k}) .$$

of length $|v_3|\sqrt{77}/3$. It follows quickly that the only unit vectors perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ are

$$\pm\frac{1}{\sqrt{77}}(2\,\mathbf{i} - 8\,\mathbf{j} - 3\,\mathbf{k}) .$$

20. Consider the following parallelogram, and put $\mathbf{v} = \overrightarrow{PQ}$ and $\mathbf{w} = \overrightarrow{QR}$:



Then the sum of the squares of the lengths of the diagonals is

$$\begin{aligned}
|\overrightarrow{PR}|^2 + |\overrightarrow{QS}|^2 &= \overrightarrow{PR}\cdot\overrightarrow{PR} + \overrightarrow{QS}\cdot\overrightarrow{QS} \\
&= (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) + (\mathbf{w}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{v}) \\
&= \mathbf{v}\cdot\mathbf{v} + 2\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w} - 2\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w} \\
&= 2\mathbf{v}\cdot\mathbf{v} + 2\mathbf{w}\cdot\mathbf{w} \;=\; 2\big(|\mathbf{v}|^2 + |\mathbf{w}|^2\big) ,
\end{aligned}$$

which is the sum of the squares of the lengths of the sides.

21. The diagonals of the parallelogram of the previous exercise are perpendicular if and only if

$$\overrightarrow{QS}\cdot\overrightarrow{PR} = 0 ,$$

that is,

$$(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{w} = |\mathbf{v}|^2 - |\mathbf{w}|^2 = 0 ,$$

that is,

$$|\mathbf{v}|^2 = |\mathbf{w}|^2 ,$$

that is,

$$|\mathbf{v}| = |\mathbf{w}| ,$$

that is, the parallelogram is a rhombus.

22. By algebraic properties of the dot product,

$$\begin{aligned}
&(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{c}) + (\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{a}) + (\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b}) \\
&\quad= \mathbf{a}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d} - \mathbf{b}\cdot\mathbf{a} - \mathbf{c}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{a} \\
&\qquad\qquad + \mathbf{c}\cdot\mathbf{d} - \mathbf{c}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{d} + \mathbf{a}\cdot\mathbf{b} \\
&\quad= \mathbf{a}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} - \mathbf{c}\cdot\mathbf{b} \\
&\qquad\qquad - \mathbf{b}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} - \mathbf{c}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d} \\
&\quad= \mathbf{a}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{c} \\
&\qquad\qquad - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{a} - \mathbf{c}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d} \\
&\quad= 0+0+0+0+0+0 = 0 .
\end{aligned}$$

We apply this identity where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are the position vectors of points A, B, C, D respectively, where ABC is a triangle and D is the point of intersection of the altitudes through A and B. Then

$$\overrightarrow{BA} = \mathbf{a}-\mathbf{b},\quad \overrightarrow{AC} = \mathbf{c}-\mathbf{a},\quad \overrightarrow{CB} = \mathbf{b}-\mathbf{c}.$$

The altitude through A is parallel to $\overrightarrow{AD} = \mathbf{d}-\mathbf{a}$ and the altitude through B is parallel to $\overrightarrow{BD} = \mathbf{d}-\mathbf{b}$, so that

$$(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{a}) = 0 , \quad \text{and} \quad (\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b}) = 0 .$$

The earlier identity immediately implies

$$(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{c}) \;=\; 0\,,$$

so that $\overrightarrow{DC}=\mathbf{d}-\mathbf{c}$ is perpendicular to $\overrightarrow{AB}$. Hence the altitude through C is parallel to $\overrightarrow{DC}$, which implies that D lies on it. This proves that the three altitudes intersect.

23. Suppose that PQR is a triangle and A, B, C are midpoints of QR, PR, PQ respectively. Let D by the intersection of the perpendicular bisectors of PR and QR, so

$$\overrightarrow{BD}\cdot\overrightarrow{PR} \;=\; \overrightarrow{AD}\cdot\overrightarrow{QR} \;=\; 0\,.$$

Then

$$\begin{aligned}
\overrightarrow{CD}\cdot\overrightarrow{PQ} \;&=\; (\overrightarrow{CB}+\overrightarrow{BD})\cdot\overrightarrow{PQ} \;=\; \overrightarrow{CB}\cdot\overrightarrow{PQ}+\overrightarrow{BD}\cdot\overrightarrow{PQ}\\
&=\; \frac{1}{2}\overrightarrow{QR}\cdot\overrightarrow{PQ}+\overrightarrow{BD}\cdot(\overrightarrow{PR}+\overrightarrow{RQ}) \;=\; \frac{1}{2}\overrightarrow{QR}\cdot\overrightarrow{PQ}+\overrightarrow{BD}\cdot\overrightarrow{PR}+\overrightarrow{BD}\cdot\overrightarrow{RQ}\\
&=\; \frac{1}{2}\overrightarrow{QR}\cdot\overrightarrow{PQ}+0+(\overrightarrow{BA}+\overrightarrow{AD})\cdot\overrightarrow{RQ} \;=\; \frac{1}{2}\overrightarrow{QR}\cdot\overrightarrow{PQ}+\overrightarrow{BA}\cdot\overrightarrow{RQ}+\overrightarrow{AD}\cdot\overrightarrow{RQ}\\
&=\; \frac{1}{2}\overrightarrow{QR}\cdot\overrightarrow{PQ}+\frac{1}{2}\overrightarrow{PQ}\cdot\overrightarrow{RQ}+0 \;=\; \frac{1}{2}\overrightarrow{QR}\cdot\overrightarrow{PQ}-\frac{1}{2}\overrightarrow{QR}\cdot\overrightarrow{PQ} \;=\; 0\,,
\end{aligned}$$

so that the perpendicular bisector of PQ is parallel to $\overrightarrow{CD}$. This proves D lies on this perpendicular bisector, so that all three perpendicular bisectors intersect in the common point D.

24. By the Theorem of Pythagoras applied to the right-angled triangles PDB and RDB, and the fact that $|PB|=|BR|$, we have

$$|PD|^2 \;=\; |PB|^2+|BD|^2 \;=\; |BR|^2+|BD|^2 \;=\; |RD|^2\,,$$

so that $|PD|=|RD|$. Similarly $|PD|=|DQ|$, which proves D is equidistant from all three vertices of the triangle.

25. Suppose first that A, B, C, D lie on a plane. The triangle ABC is nondegenerate so we may form axes for the plane through AB and AC. Since D does not lie on either of these axes, it must lie on the interior of one of the quadrants formed by these axes, so that

$$\overrightarrow{AD} \;=\; \lambda\,\overrightarrow{AB}+\mu\,\overrightarrow{AC}$$

for some nonzero scalars λ, μ. But then

$$\overrightarrow{AO}+\overrightarrow{OD} \;=\; \lambda(\overrightarrow{AO}+\overrightarrow{OB})+\mu(\overrightarrow{AO}+\overrightarrow{OC})$$

so that

$$(\lambda+\mu-1)\,\overrightarrow{OA}-\lambda\,\overrightarrow{OB}-\mu\,\overrightarrow{OC}+\overrightarrow{OD} \;=\; \mathbf{0}\,,$$

giving

$$\alpha\,\overrightarrow{OA}+\beta\,\overrightarrow{OB}+\gamma\,\overrightarrow{OC}+\delta\,\overrightarrow{OD} \;=\; \mathbf{0}$$

where

$$\alpha \;=\; \lambda+\mu-1\,,\quad \beta \;=\; -\lambda\,,\quad \gamma \;=\; -\mu\,,\quad \delta \;=\; 1\,.$$

Certainly $\alpha+\beta+\gamma+\delta=0$ and β, γ, δ are nonzero. If $\alpha=0$ then D lies on the line through BC, contradicting that B, C, D are not collinear. Hence α is nonzero also.

Suppose conversely that

$$\alpha\,\overrightarrow{OA} + \beta\,\overrightarrow{OB} + \gamma\,\overrightarrow{OC} + \delta\,\overrightarrow{OD} \;=\; \mathbf{0}$$

where α, β, γ, δ are nonzero scalars such that $\alpha+\beta+\gamma+\delta = 0$. Then

$$\frac{-\beta-\gamma-\delta}{\delta}\,\overrightarrow{OA} + \frac{\beta}{\delta}\,\overrightarrow{OB} + \frac{\gamma}{\delta}\,\overrightarrow{OC} + \overrightarrow{OD} \;=\; \mathbf{0}$$

so that, after rearranging,

$$\overrightarrow{AD} \;=\; \overrightarrow{AO} + \overrightarrow{OD} \;=\; \frac{\beta}{\delta}(\overrightarrow{OA} + \overrightarrow{BO}) + \frac{\gamma}{\delta}(\overrightarrow{OA} + \overrightarrow{CO}) \;=\; \frac{\beta}{\delta}\overrightarrow{BA} + \frac{\gamma}{\delta}\overrightarrow{CA}\,.$$

This proves that D lies in the plane determined by the triangle ABC, so all four points lie on a plane.

Suppose now that the lines through AD, BD, CD cut lines through BC, CA, AB in R, S, T respectively. We have

$$\alpha\,\overrightarrow{OA} + \beta\,\overrightarrow{OB} + \gamma\,\overrightarrow{OC} + \delta\,\overrightarrow{OD} \;=\; \mathbf{0}$$

for some nonzero scalars α, β, γ, δ such that $\alpha+\beta+\gamma+\delta = 0$. Note that because AD intersects BC and A does not lie on the line through BC, the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are not parallel. But the earlier calculation yields

$$\overrightarrow{AD} \;=\; \frac{\beta}{\delta}\overrightarrow{BA} + \frac{\gamma}{\delta}\overrightarrow{CA} \;=\; \frac{\beta}{\delta}\overrightarrow{BA} + \frac{\gamma}{\delta}(\overrightarrow{BA} + \overrightarrow{CB}) \;=\; \frac{\beta+\gamma}{\delta}\overrightarrow{BA} - \frac{\gamma}{\delta}\overrightarrow{BC}\,.$$

If $\beta+\gamma = 0$ then $\overrightarrow{AD} = -\frac{\gamma}{\delta}\overrightarrow{BC}$, contradicting that $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are not parallel. Hence

$$\beta+\gamma \;=\; -\alpha-\delta \;\neq 0\,,$$

and so we may divide through and rearrange the earlier equation to get

$$\frac{\beta\,\overrightarrow{OB} + \gamma\,\overrightarrow{OC}}{\beta+\gamma} \;=\; \frac{\alpha\,\overrightarrow{OA} + \delta\,\overrightarrow{OD}}{\alpha+\delta}\,.$$

But these must represent the position vector of the point R of intersection of AD with BC, so that, from the left-hand side, R divides BC in the ratio $\gamma : \beta$. Similarly S divides CA in the ratio $\alpha : \gamma$ and T divides AB in the ratio $\beta : \alpha$, and the product of these ratios is

$$\frac{\gamma}{\beta}\frac{\alpha}{\gamma}\frac{\beta}{\alpha} \;=\; 1\,,$$

completing the proof of Ceva's Theorem.