Semester 1 **Longer Solutions to Selected Exercises for Week 3** 2012

9. (ii) Observe that $\overrightarrow{PQ} = \overrightarrow{SR} = -2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ so $PQRS$ is a parallelogram. But

$$|\overrightarrow{PQ}| \;=\; |\overrightarrow{PS}| \;=\; 3\,,$$

so $PQRS$ is a rhombus. This rhombus is not a square however because the diagonals have different lengths:

$$|\overrightarrow{PR}| \;=\; |-\mathbf{i}+\mathbf{k}| \;=\; \sqrt{2} \;\neq\; \sqrt{34} \;=\; |-3\mathbf{i}-4\mathbf{j}-3\mathbf{k}| \;=\; |\overrightarrow{SQ}|$$

10. (i) The displacement 300 km southeast is represented by the vector $150\sqrt{2}\,(\mathbf{i}-\mathbf{j})$ and 150 km 30° west of north by the vector $75\,(-\mathbf{i}+\sqrt{3}\,\mathbf{j})$. The net displacement is represented by

$$(150\sqrt{2}-75)\,\mathbf{i} + (75\sqrt{3}-150\sqrt{2})\,\mathbf{j}\,.$$

(ii) The final distance from the starting position is

$$\sqrt{(150\sqrt{2}-75)^2+(75\sqrt{3}-150\sqrt{2})^2} \;\approx\; 160 \quad \text{km}\,.$$

The tangent of the angle south of east is $\dfrac{150\sqrt{2}-75\sqrt{3}}{150\sqrt{2}-75}$ yielding an angle of approximately 31°.

11. Rearranging the equation gives

$$(1-\alpha-\beta)\,\mathbf{v} + \left(\alpha - \frac{\beta}{2}\right)\mathbf{w} \;=\; \mathbf{0}\,,$$

so that, by linear independence, $1-\alpha-\beta \;=\; 0 \;=\; \alpha - \dfrac{\beta}{2}$. Solving simultaneously yields $\alpha = 1/3$, $\beta = 2/3$.

12. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be any three vectors in the plane. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then without loss of generality $\mathbf{u} = \lambda\mathbf{v}$ for some nonzero scalar λ, so that

$$1\mathbf{u} + (-\lambda)\mathbf{v} + 0\mathbf{w} \;=\; \mathbf{0}\,,$$

which proves the vectors are linearly dependent (because the implication in the definition of linear independence fails). Suppose then that $\mathbf{u}$ and $\mathbf{v}$ are not parallel, so when joined tail-to-tail they span a nondegenerate parallelogram $\mathcal{P}$ (with nonzero area). When extending the sides of $\mathcal{P}$ containing the origin indefinitely in all directions, this divides the plane into four quadrants. Then the tip of $\mathbf{w}$ lies in one of the quadrants or lines through $\mathbf{u}$ and $\mathbf{v}$ when all three vectors are joined tail-to-tail at the origin. But then tracing the smallest (possibly degenerate) parallelogram that contains the origin and the tip of $\mathbf{w}$, using sides parallel to the sides of $\mathcal{P}$, we get that $\mathbf{w} = \lambda\mathbf{u} + \mu\mathbf{v}$ for some scalars λ and μ. In this case,

$$\lambda\mathbf{u} + \mu\mathbf{v} + (-1)\mathbf{w} \;=\; \mathbf{0}\,,$$

which again proves linear dependence.

13. Observe that

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BD} \;=\; \overrightarrow{AB} + \frac{\alpha}{\alpha+\beta}\overrightarrow{BC} \;=\; \overrightarrow{AB} + \frac{\alpha}{\alpha+\beta}(\overrightarrow{BA} + \overrightarrow{AC}) \\ &= \overrightarrow{AB} + \frac{\alpha}{\alpha+\beta}(-\overrightarrow{AB} + \overrightarrow{AC}) \;=\; \left(1 - \frac{\alpha}{\alpha+\beta}\right)\overrightarrow{AB} + \frac{\alpha}{\alpha+\beta}\overrightarrow{AC} \\ &= \frac{\beta\,\overrightarrow{AB} + \alpha\,\overrightarrow{AC}}{\alpha+\beta}\,.\end{aligned}$$

If $\alpha < 0$ and $\beta > 0$ then the point D lies outside the triangle on the line through B and D, but on the side beyond B. If $\alpha > 0$ and $\beta < 0$ then the point D again lies outside the triangle on the line through B and D, but on the side beyond D. If both α and β are negative, then this makes sense only in terms of oriented triangles, in which case D would be again on the interior of the line segment BC but the triangle ABC would be oriented anti-clockwise on the page from the point of view of the reader, instead of clockwise as pictured.

14. (i) Observe that $\overrightarrow{PQ} = \mathbf{i}+4\mathbf{j}-2\mathbf{k}$ and $\overrightarrow{PS} = -\mathbf{i}+2\mathbf{j}+(\lambda-2)\mathbf{k}$ so that if $|\overrightarrow{PQ}| = |\overrightarrow{PS}|$ then $\sqrt{21} = \sqrt{5+(\lambda-2)^2}$, giving $(\lambda-2)^2 = 16$, from which it follows quickly that $\lambda = -2$ or 6.

(ii) If $\overrightarrow{PR} = -3\mathbf{i}+6\mathbf{j}-2\mathbf{k}$ is parallel to $\overrightarrow{RS} = 2\mathbf{i}-4\mathbf{j}+\lambda\mathbf{k}$ then $-3/2 = -6/4 = -2/\lambda$, so that $\lambda = 4/3$.

15. (i) We want $D(x,y,z)$ such that $\overrightarrow{AB} = \overrightarrow{DC}$, so that

$$-3\mathbf{i} - \mathbf{j} + 4\mathbf{k} \;=\; -x\,\mathbf{i} + (2-y)\mathbf{j} + (1-z)\mathbf{k}\,,$$

yielding $x = 3$, $y = 3$, $z = -3$. Hence $D = (3, 3, -3)$.

(ii) The coordinates of P are the averages of the respective coordinates of A and C, so $P = (\frac{1}{2}, 2, -1)$ and $\overrightarrow{OP} = \frac{1}{2}\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

(iii) We have $\overrightarrow{BP} = \overrightarrow{PD} = \frac{5}{2}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, so that P must be the midpoint of the line segment joining B and D. Thus the diagonals AC and BD bisect each other.

(iv) We have

$$\left|\overrightarrow{AC}\right| \;=\; |-\mathbf{i} + 4\mathbf{k}| \;=\; \sqrt{17}\,, \qquad \left|\overrightarrow{BD}\right| \;=\; |5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}| \;=\; 3\sqrt{5}\,.$$

Since these lengths are different, the parallelogram $ABCD$ a not a rectangle.

16. We have

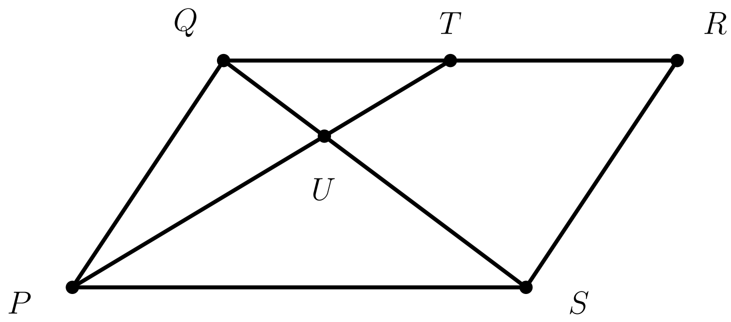
$$\mathbf{v} \;=\; 7\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}\,, \qquad |\mathbf{v}| \;=\; \sqrt{74}\,,$$

so the cosines of the angles made with the x, y and z-axes are

$$\frac{7}{\sqrt{74}}\,, \quad -\frac{4}{\sqrt{74}}\,, \quad \frac{3}{\sqrt{74}}\,,$$

yielding angles of approximately 36°, 118° and 70° respectively.

18. Consider the following parallelogram $PQRS$, and let U be the point of intersection of PT with QS, where T is the midpoint of QR.



Then, for some scalars α and β,

$$\overrightarrow{QU} \;=\; \alpha\overrightarrow{QS}\,, \qquad \overrightarrow{PU} \;=\; \beta\overrightarrow{PT}\,.$$

Put

$$\mathbf{v} \;=\; \overrightarrow{PQ}\,, \qquad \mathbf{w} \;=\; \overrightarrow{PS}\,.$$

On the one hand,

$$\overrightarrow{PU} \;=\; \overrightarrow{PQ} + \overrightarrow{QU} \;=\; \mathbf{v} + \alpha\overrightarrow{QS} \;=\; \mathbf{v} + \alpha\big(\overrightarrow{QP} + \overrightarrow{PS}\big) \;=\; \mathbf{v} + \alpha\big(\mathbf{w} - \mathbf{v}\big)\,,$$

whilst, on the other hand,

$$\overrightarrow{PU} \;=\; \beta\overrightarrow{PT} \;=\; \beta\big(\overrightarrow{PQ} + \overrightarrow{QT}\big) \;=\; \beta\big(\mathbf{v} + \frac{1}{2}\overrightarrow{QR}\big) \;=\; \beta\big(\mathbf{v} + \frac{1}{2}\mathbf{w}\big)\,,$$

whence

$$\mathbf{v} + \alpha\big(\mathbf{w} - \mathbf{v}\big) \;=\; \beta\big(\mathbf{v} + \frac{1}{2}\mathbf{w}\big)\,.$$

By the calculation in Exercise 11,

$$\alpha \;=\; \frac{1}{3}\,, \qquad \beta \;=\; \frac{2}{3}\,.$$

Hence the ratio of the length of QU to the length of US is $1:2$.

An alternative (and faster) solution is to conjecture that the ratio is $1:2$ and simply check that

$$\overrightarrow{PQ} + \frac{1}{3}\overrightarrow{QS} \;=\; \overrightarrow{PQ} + \frac{1}{3}\big(\overrightarrow{QR} + \overrightarrow{RS}\big) \;=\; \overrightarrow{PQ} + \frac{2}{3}\overrightarrow{QT} - \frac{1}{3}\overrightarrow{PQ} \;=\; \frac{2}{3}\big(\overrightarrow{PQ} + \overrightarrow{QT}\big) \;=\; \frac{2}{3}\overrightarrow{PT}\,,$$

which confirms that PT intesects QS one third of the way from Q to S.

19. If $\overrightarrow{PQ} = \gamma\overrightarrow{BC}$ then

$$\gamma(\overrightarrow{AC} - \overrightarrow{AB}) \; = \; \gamma\overrightarrow{BC} \; = \; \overrightarrow{PQ} \; = \; \overrightarrow{AQ} - \overrightarrow{AP} \; = \; \beta\overrightarrow{AC} - \alpha\overrightarrow{AB} \; ,$$

so that, rearranging,

$$(\beta - \gamma)\overrightarrow{AC} \; = \; (\alpha - \gamma)\overrightarrow{AB} \; ,$$

forcing $\beta - \gamma = \alpha - \gamma$, since $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are not parallel, yielding $\alpha = \beta = \gamma$.

20. Applying the ratio formula twice yields

$$\overrightarrow{OQ} \; = \; \frac{-\overrightarrow{OA} + 3\overrightarrow{OB}}{2} \; = \; \frac{7\overrightarrow{OC} - 5\overrightarrow{OD}}{2}$$

where O denotes the origin, so that

$$3\overrightarrow{OB} + 5\overrightarrow{OD} \; = \; \overrightarrow{OA} + 7\overrightarrow{OC} \; .$$

Let P' be the point in space whose position vector is

$$\overrightarrow{OP'} \; = \; \frac{3\overrightarrow{OB} + 5\overrightarrow{OD}}{8} \; = \; \frac{\overrightarrow{OA} + 7\overrightarrow{OC}}{8} \; .$$

By the ratio formula, now in reverse, this implies that P' lies on the line AC, dividing it in the ratio $7 : 1$, and on the line BD, dividing it in the ratio $5 : 3$. But then P' must be P, the point of intersection, and the proof is complete.

21. Observe that

$$\overrightarrow{QT} = \overrightarrow{QP} + \overrightarrow{PT} = -\mathbf{u} + \frac{2}{3}\overrightarrow{PA} = -\mathbf{u} + \frac{2}{3}\frac{1}{2}(\overrightarrow{PQ} + \overrightarrow{PR}) = -\mathbf{u} + \frac{1}{3}(\mathbf{u} + \mathbf{v}) = \frac{1}{3}(\mathbf{v} - 2\mathbf{u}) \; ,$$

and

$$\overrightarrow{QB} \; = \; \overrightarrow{QP} + \overrightarrow{PB} \; = \; -\mathbf{u} + \frac{1}{2}\overrightarrow{PR} \; = \; -\mathbf{u} + \frac{1}{2}\mathbf{v} \; = \; \frac{1}{2}(\mathbf{v} - 2\mathbf{u}) \; ,$$

so that $\overrightarrow{QT}$ and $\overrightarrow{QB}$ are parallel, which means that T lies on the line QB. Similarly T lies on the line RC, and this proves that all three medians intersect at T.

22. Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent vectors, so that

$$\lambda_1\mathbf{v}_1 + \ldots + \lambda_n\mathbf{v}_n \; = \; \mathbf{0}$$

where not all of $\lambda_1, \ldots, \lambda_n$ are zero. Without loss of generality, we may suppose $\lambda_1 \neq 0$ (for otherwise we could reorder the list of vectors so that this is the case). Then, rearranging,

$$\mathbf{v}_1 \; = \; (-\lambda_2/\lambda_1)\mathbf{v}_2 + \ldots + (-\lambda_n/\lambda_1)\mathbf{v}_n \; ,$$

which verifies that $\mathbf{v}_1$ is a linear combination of the other vectors. Suppose conversely that one of the vectors is a linear combination of the other vectors, so without loss of generality, we may suppose

$$\mathbf{v}_1 \; = \; \lambda_2\mathbf{v}_2 + \ldots + \lambda_n\mathbf{v}_n$$

for some scalars $\lambda_2, \ldots, \lambda_n$. Now rearranging gives

$$1\mathbf{v}_1 + (-\lambda_2)\mathbf{v}_2 + \ldots + (-\lambda_n)\mathbf{v}_n \; = \; \mathbf{0} \; ,$$

which verifies that $\mathbf{v}_1, \ldots \mathbf{v}_n$ are not linearly independent (because at least one scalar is nonzero, namely $1 \neq 0$), that is, they are linearly dependent.

23. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{t}$ be any four vectors in space. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in the same plane, when joined together tail-to-tail, then they are linearly dependent by an earlier exercise, so, there exist scalars α, β and γ, not all zero, such that

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} \;=\; \mathbf{0}\;,$$

yielding the equation

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} + 0\mathbf{t} \;=\; \mathbf{0}\;,$$

verifying that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{t}$ are linearly dependent (since the implication in the definition of linear independence fails). Suppose then that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ do not lie in a plane when joined tail-to-tail, so that the tips and the origin span a nondegenerate parallelopiped $\mathcal{P}$ (with nonzero volume). When extending the sides of $\mathcal{P}$ containing the origin indefinitely in all directions, this divides space into eight octants. Then the tip of $\mathbf{t}$ lies inside one of the octants, or in one of the planes through a pair of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, when all four vectors are joined tail-to-tail at the origin. But then tracing the smallest (possibly degenerate) parallelopiped that contains the origin and the tip of $\mathbf{t}$, and whose sides are parallel to the sides of $\mathcal{P}$, we get that $\mathbf{t} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ for some scalars α, β and γ. In this case,

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} + (-1)\mathbf{t} \;=\; \mathbf{0}\;,$$

which again proves linear dependence.

24. If A, B, C lie on a line, then, by the ratio formula

$$\overrightarrow{OA} \;=\; \frac{\mu\,\overrightarrow{OB} + \lambda\,\overrightarrow{OC}}{\lambda + \mu}$$

for some nonzero scalars λ, μ such that $\lambda + \mu \neq 0$, so that

$$\alpha\,\overrightarrow{OA} + \beta\,\overrightarrow{OB} + \gamma\,\overrightarrow{OC} \;=\; \mathbf{0}$$

where $\alpha = -1$, $\beta = \frac{\mu}{\lambda+\mu}$, $\gamma = \frac{\lambda}{\lambda+\mu}$, all of which are nonzero, and $\alpha + \beta + \gamma = 0$. Conversely, if

$$\alpha\,\overrightarrow{OA} + \beta\,\overrightarrow{OB} + \gamma\,\overrightarrow{OC} \;=\; \mathbf{0}$$

for some nonzero scalars α, β, γ such that $\alpha + \beta + \gamma = 0$ then

$$\overrightarrow{OA} \;=\; r\,\overrightarrow{OB} + s\,\overrightarrow{OC}$$

where $r = -\beta/\alpha$ and $s = -\gamma/\alpha$, so that $r + s = 1$ and, by the ratio formula, A divides the line through B and C in the ratio $r : s$, so that, in particular, A, B, C lie on a line.

25. For part (ii), suppose that $f_0, \ldots, f_n$ are linearly dependent, so

$$\lambda_0 f_0 + \ldots + \lambda_n f_n \;=\; \mathbf{0}$$

for some scalars $\lambda_0, \ldots, \lambda_n$ not all zero, where $\mathbf{0}$ denotes the zero function (that takes all reals to zero). Without loss of generality we may suppose $\lambda_n \neq 0$. Then for all real numbers x,

$$\lambda_0 + \lambda_1 x + \ldots + \lambda_n x^n \;=\; 0\;.$$

Consider the polynomial function

$$p(x) \;=\; \lambda_0 + \lambda_1 x + \ldots + \lambda_n x^n \; .$$

Since there are infinitely many real numbers, $p(x)$ has infinitely many roots. We get a contradiction by proving that $p(x)$ has at most n roots, and we do this by induction on the nonnegative integer n. If $n = 0$ then $p(x) = \lambda_0$ is a nonzero constant function, which has no roots, which starts an induction. Suppose $n > 0$. Then the derivative $p'(x)$ is a polynomial with highest term involving x^{n-1}, so, by an induction hypothesis has $\leq n-1$ roots. If $p(x)$ has $> n$ roots then, by Rolle's Theorem from calculus, the derivative must be zero at $\geq n$ places, which is a contradiction. Hence $p(x)$ has at most n roots, and the result now follows by induction.