THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Quiz 1: Practice Questions

MATH1901/1906: Differential Calculus (Advanced) — Semester 1, 2017

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Some information about the Quiz:

(1) The quiz covers material up to and including Week 5 lectures. That is, material from the tutorials in weeks 2–6.

(2) The quiz runs for 40 minutes.

(3) You must write your answers in pen, not pencil.

(4) The format of the real quiz is mostly “short-answer” questions. There will be answer boxes provided below each question where you should write your final answers. In some questions your working will be considered, so please write neatly.

(5) There will be 10 questions in the real quiz, each worth the same amount. This practice quiz is considerably longer than the real quiz.

(6) The quiz is a closed book examination. No notes or books are allowed.

(7) Non-programmable non-graphics calculators are allowed, but are not needed.

The questions provided here are really just for additional practice (building from the lectures and tutorials). Therefore you should not expect that doing only these practice questions will be adequate preparation – it is strongly advised that you also revise the tutorials from Weeks 2–6, and make your own summary notes of the lecture material.

1. Let $A = \{-1, 2, 5, 9\}$, $B = \{-2, -1, 2, 9\}$, and $C = \{-1, 0, 5\}$.

(a) Find $A \cap B$ and $B \backslash C$.

(b) Explain why every injective function $f : A \to B$ is necessarily also surjective.

2. Let $z = 12 + 5i$ and $w = 2 - 3i$. Calculate the following complex numbers in Cartesian form:

$$5w, \qquad z - 5w, \qquad zw, \qquad z\bar{w}, \qquad |z - w|, \qquad \frac{z}{w}, \qquad w^4, \qquad e^z$$

3. Evaluate the following complex numbers. Express your final answers in Cartesian form.

(a) $(1 + i)^{23}$ (b) $(-1 + i\sqrt{3})^{23}$

4. Find all solutions to the following equations:

(a) $z^7 = -1 + \sqrt{3}i$ (b) $(2 + i)e^z = 1 + 3i$

5. Sketch the following sets in the complex plane:

(a) $\{z \in \mathbb{C} \mid |z - 3 + i| < 5\}$ (b) $\{z \in \mathbb{C} \mid \mathrm{Re}((1 + 2i)z) > 2\}$

6. Sketch the image of the set $A = \{z \in \mathbb{C} \mid 1 \leq \mathrm{Re}(z) \leq 2\}$ under the function $f(z) = e^z$ in the complex plane.

7. Given that $z = 3 - i$ and $z = -1 + i$ are roots of the degree 5 equation

$$z^5 - 3z^4 - 4z^3 + 8z^2 + 28z + 20 = 0,$$

find all of the roots.

8. Let

$$A = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0 \text{ and } 0 \leq \mathrm{Im}(z) \leq \pi\}$$
$$B = \{w \in \mathbb{C} \mid \mathrm{Im}(w) \geq 0 \text{ and } |w| \geq 1\}.$$

You are given that the function $f : A \to B$ with formula $f(z) = e^z$ is bijective. Find a formula for the inverse function $f^{-1} : B \to A$.

9. Find the natural domains of the following functions:

$$\sqrt{4 - x^2}, \quad (4 - x^2)^{-1/2}, \quad \ln x, \quad \sqrt{\ln x}, \quad \frac{\sin x}{x}, \quad \ln(\ln(\ln x))$$

10. Decide which of the following functions $f : A \to B$ are surjective, injective, or bijective:

(a) $f : [-1, 1] \to \mathbb{R}$ given by $x \mapsto \sinh x$.

(b) $f : \mathbb{R} \to [-1, 1]$ given by $x \mapsto \cos x$.

(c) $f : (0, \infty) \to \mathbb{R}$ given by $x \mapsto \ln\big(x\sqrt{x^2 + 2}\big)$.

(d) $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto e^z$.

11. Evaluate the following limits or show that the limit does not exist:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x}$

(b) $\displaystyle\lim_{x\to 0} \frac{\cos 3x}{x}$

(c) $\displaystyle\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3}$

(d) $\displaystyle\lim_{x\to 0^+} \frac{\sin(1/x)}{\ln x}$

(e) $\displaystyle\lim_{x\to\infty} x\left(\sqrt{x^2+1} - \sqrt{x^2-1}\right)$

(f) $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2}$

12. Find all constants $k \in \mathbb{R}$ such that the function

$$f(x) = \begin{cases} k^2 \cosh x & \text{if } x < 0 \\ x + 3 & \text{if } x \geq 0. \end{cases}$$

is continuous at every $x \in \mathbb{R}$.

13. Suppose that $\lim_{x\to a} f(x) = \ell$ and let $k \neq 0$ be a constant. Prove, using and (ϵ, δ) argument, that

$$\lim_{x\to a}(kf(x)) = k\ell.$$

14. Show, using the squeeze law, that

$$\lim_{x\to a} x^{1/4} = a^{1/4} \quad \text{for all } a > 0.$$