The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Exercises for Week 13 (beginning 04 June)** 2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Let M be a square matrix, $\mathbf{x}$ a nonzero column vector and λ a scalar such that

$$M\mathbf{x} = \lambda\mathbf{x} .$$

Then λ is called an *eigenvalue* of M and $\mathbf{x}$ is called an *eigenvector* of M associated with the eigenvalue λ.

(ii) The *eigenspace* of M associated with an eigenvalue λ is the collection

$$\left\{ \mathbf{v} \;\middle|\; M\mathbf{v} = \lambda\mathbf{v} \right\} = \left\{ \mathbf{v} \;\middle|\; (M - \lambda I)\mathbf{v} = \mathbf{0} \right\}$$

comprising all the eigenvectors of M associated with λ and the zero vector (which is never an eigenvector).

(iii) A scalar λ is an eigenvalue of a square matrix M if and only if

$$\det(M - \lambda I) = 0 .$$

(iv) The expression $\det(M - \lambda I)$ is always a polynomial in λ and is called the *characteristic polynomial* of M. Thus the eigenvalues of a matrix are precisely the roots of its characteristic polynomial.

(v) Finding the eigenspace corresponding to the eigenvalue λ of a matrix M is equivalent to solving the homogeneous system with coefficient matrix $M - \lambda I$. After the eigenspace has been found, substituting particular values of the parameters yields particular eigenvectors.

(vi) The eigenvalues of a triangular matrix are simply the diagonal entries.

(vii) A square matrix D is *diagonal* if all entries off the diagonal are zero. If D and E are diagonal then DE is also diagonal, and its diagonal entries are simply the products of corresponding diagonal entries of D and E. Thus the diagonal elements of D^n are just the nth powers of the diagonal elements of D.

(viii) Let M be a square $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then

$$MP = PD$$

where D is the diagonal matrix with eigenvalues down the diagonal and P the matrix with corresponding eigenvectors as columns. If P is invertible then

$$M = PDP^{-1} \qquad \text{and} \qquad D = P^{-1}MP .$$

In this case we say that M is *diagonalisable*.

(ix) In the preceding discussion, if the eigenvalues are all different then P is invertible and M is diagonalisable.

(x) If M is diagonalisable then powers of M can be found easily by the formula

$$M^n \; = \; PD^nP^{-1} \; .$$

(xi) The Fundamental Theorem of Algebra: Every nonzero polynomial with complex number coefficients has a root in the complex numbers.

(xii) The Cayley-Hamilton Theorem: Every square matrix is a root of its own characteristic polynomial.

Preparatory Exercises:

1. Find $A\mathbf{v}$ and $A\mathbf{w}$ where

$$A \; = \; \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \; , \quad \mathbf{v} \; = \; \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; , \quad \mathbf{w} \; = \; \begin{bmatrix} -1 \\ 1 \end{bmatrix} \; .$$

By inspection, write down the two eigenvalues of A. Now factorise the determinant

$$\begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} \; ,$$

which is a quadratic in λ, and compare your answers.

2. Find $B\mathbf{v}_1$, $B\mathbf{v}_2$ and $B\mathbf{v}_3$ where

$$B \; = \; \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \; , \quad \mathbf{v}_1 \; = \; \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \; , \quad \mathbf{v}_2 \; = \; \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \; , \quad \mathbf{v}_3 \; = \; \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \; .$$

By inspection, write down the three eigenvalues of B. Now factorise the determinant

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \; ,$$

which is a cubic in λ, and compare your answers.

3. Find the characteristic polynomial $\det(M-\lambda I)$, the eigenvalues of M and corresponding eigenspaces in each case:

(i) $M \; = \; \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (ii) $M \; = \; \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (iii) $M \; = \; \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

4. Write down the eigenvalues immediately for the following triangular matrices, and then find all of the corresponding eigenspaces.

(i) $M \; = \; \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (ii) $M \; = \; \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$ (iii) $M \; = \; \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

Tutorial Exercises:

5. Find the eigenvalues and corresponding eigenvectors for $M = \begin{bmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{bmatrix}$.

6. The matrix $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalues 2 and 4 with corresponding eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

(i) Write down an invertible matrix P and a diagonal matrix D such that

$$B = PDP^{-1}.$$

(ii) Find a formula for B^n, and use it to find B^3 and B^4.

7. The matrix $C = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has eigenvalues 0, 1 and 3 with corresponding eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ respectively.

(i) Write down an invertible matrix P and diagonal matrix D such that

$$C = PDP^{-1}.$$

(ii) Find a formula for C^n, and use it to find C^4.

8. (suitable for group discussion) Verify that if A is invertible and λ is an eigenvalue of A, then $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A^{-1}. What can be said about eigenvalues of A^k where k is any integer?

9. (suitable for group discussion) Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for a matrix M corresponding to different eigenvalues λ_1 and λ_2. Explain why $\mathbf{v}_1$ cannot be a scalar multiple of $\mathbf{v}_2$.

10. (suitable for group discussion) Use the multiplicative property of the determinant to verify that if A and B are square matrices of the same size, and B is invertible, then A and $B^{-1}AB$ have the same eigenvalues.

11.* Suppose that $0 \leq \theta \leq \pi$. Verify that $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has real eigenvalues if and only if $\theta = 0$ or π. Interpret this result geometrically.

12.* Let A be a square matrix with eigenvalue λ. Prove the following implications:

(i) $A^2 = 0 \implies \lambda = 0$

(ii) $A^2 = A \implies \lambda = 0$ or $\lambda = 1$

(iii) $A^2 = I \implies \lambda = 1$ or $\lambda = -1$

13.* Three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are said to be *linearly independent* if

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = \mathbf{0} \qquad \Longrightarrow \qquad \alpha = \beta = \gamma = 0\,,$$

where α, β, γ are scalars. Explain why three eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ corresponding to three different eigenvalues λ_1, λ_2, λ_3 of a matrix M must be linearly independent.

Further Exercises:

14. Find eigenvalues and corresponding eigenvectors for $M = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

15. Write down an invertible matrix P and a diagonal matrix D such that

$$M = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = PDP^{-1}\,.$$

16. Evaluate

$$M^n = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n = PD^nP^{-1}$$

for any positive integer n. Use your answer to find M^4.

17. Diagonalise $M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and find M^n for any positive integer n.

18. Diagonalise $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ and find M^n for any positive integer n.

19.* Prove that $M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not diagonalisable.

20.* Verify that a square matrix A has the same eigenvalues as its transpose A^T.

21. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Verify that the characteristic polynomial of A is

$$\lambda^2 - (a+d)\lambda + ad - bc\,.$$

Now also verify that

$$A^2 - (a+d)A + (ad-bc)I = 0\,.$$

This verifies the 2×2 case of the Cayley-Hamilton Theorem.

22.* Find the characteristic polynomial of the matrix

$$M = \begin{bmatrix} -7 & -2 & 6 \\ -2 & 1 & 2 \\ -10 & -2 & 9 \end{bmatrix},$$

and use the Cayley-Hamilton Theorem, and manipulate a matrix equation, to find M^{-1}.

23.* Consider the matrix $M = \begin{bmatrix} 1/2 & 2/5 \\ 1/2 & 3/5 \end{bmatrix}$, whose entries are positive and the columns add to 1. It is an example of a *regular stochastic* matrix. It is a theorem about regular stochastic matrices M that

$$\lim_{n\to\infty} M^n = \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix}$$

where $\mathbf{v}$ is the unique *steady state vector* of M, that is, $\mathbf{v}$ is the unique eigenvector corresponding to eigenvalue 1 whose entries add up to 1. Diagonalise M and verify this limiting behaviour in this particular example.

24.* The sequence of *Fibonacci numbers* is obtained by writing down 1 twice, and obtaining each successive number by adding the previous two numbers together:

$$1\,,\ 1\,,\ 2\,,\ 3\,,\ 5\,,\ 8\,,\ 13\,,\ 21\,,\ 34\,,\ 55\,,\ 89\,,\ 144\,,\ \ldots$$

If we let x_n denote the nth Fibonacci number then

$$x_1 = x_2 = 1\,,\quad x_n = x_{n-1} + x_{n-2} \quad \text{for } n \geq 3\,,$$

so that

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Diagonalise $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ to find a general formula for the nth Fibonacci number.

25.** Two matrices A and B are similar if there is an invertible matrix P such that $A = PBP^{-1}$. Prove that every 2×2 complex matrix is similar to a diagonal matrix or to a matrix of the form

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

for some $\lambda \in \mathbb{C}$. Deduce that every 2×2 real matrix is similar to a diagonal matrix or a matrix of the above form for some $\lambda \in \mathbb{R}$, or a scalar multiple of a rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some $\theta \in \mathbb{R}$. These results are special cases of a more general *Jordan Canonical Form Theorem* discussed next year.

Short Answers to Selected Exercises:

1. $\begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \ 5\,,\ -3\,,\ (\lambda - 5)(\lambda + 3)$

2. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}, \ 0\,,\ 1\,,\ 3\,,\ \lambda(\lambda - 1)(3 - \lambda)$

3. (i) $(\lambda - 1)(\lambda - 2)\,,\ 1\,,\ 2 \quad \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$

(ii) $(\lambda-1)(\lambda+1)$, 1 , -1 $\quad \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

(iii) $(\lambda+3)(\lambda-2)$, -3 , 2 $\quad \left\{ \begin{bmatrix} -3t \\ 2t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

4. (i) eigenvalue 1 with eigenspace $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

(ii) eigenvalues 2, -1 with eigenspaces $\left\{ \begin{bmatrix} -3t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

(iii) eigenvalues 3, 5 with eigenspaces $\left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \left\{ \begin{bmatrix} 3t \\ 2t \\ 4t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

5. $3\,, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; 1\,, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \; -1\,, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

6. $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 2^{n-1}+2(4^{n-1}) & -2^{n-1}+2(4^{n-1}) \\ -2^{n-1}+2(4^{n-1}) & 2^{n-1}+2(4^{n-1}) \end{bmatrix}, \begin{bmatrix} 36 & 28 \\ 28 & 36 \end{bmatrix}, \begin{bmatrix} 136 & 120 \\ 120 & 136 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1+3^{n-1} & 3^{n-1} & -1 \\ -1+2(3^{n-1}) & 2(3^{n-1}) & 1 \\ 3^{n-1} & 3^{n-1} & 0 \end{bmatrix} \begin{bmatrix} 28 & 27 & -1 \\ 53 & 54 & 1 \\ 27 & 27 & 0 \end{bmatrix}$

8. Suppose $\mathbf{v}$ is an eigenvector for invertible A corresponding to λ. If $\lambda = 0$ then $\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v} = A^{-1}0\mathbf{v} = \mathbf{0}$, a contradiction. If k is any integer, $A^k\mathbf{v} = \lambda^k\mathbf{v}$.

9. Argue by contradiction. Suppose $\mathbf{v}_1 = \alpha\mathbf{v}_2$ and apply M to both sides.

10. $\det(B^{-1}AB - \lambda I) = \det(B^{-1}(A-\lambda I)B) = \det B^{-1}\det(A-\lambda I)\det B = \det(A-\lambda I)$

11. eigenvalues are $\operatorname{cis}(\pm\theta)$, which are real if and only if $\theta = 0$ or π .

12. Let $\mathbf{v}$ be an eigenvector of A corresponding to λ.

(i) If $A^2 = 0$ and $\lambda \neq 0$ then $\mathbf{v} = \lambda^{-2}\lambda^2\mathbf{v} = \lambda^{-2}A^2\mathbf{v} = \lambda^{-2}0\mathbf{v} = \mathbf{0}$, a contradiction.

(ii) If $A^2 = A$ and $\lambda \neq 0$ then $\mathbf{v} = \lambda^{-1}\lambda\mathbf{v} = \lambda^{-1}A\mathbf{v} = \lambda^{-2}A^2\mathbf{v} = \lambda^{-1}\lambda^2\mathbf{v} = \lambda\mathbf{v}$, so that $(1-\lambda)\mathbf{v} = \mathbf{0}$, yielding $1-\lambda = 0$, so that $\lambda = 1$.

(iii) If $A^2 = I$ then $\mathbf{v} = A^2\mathbf{v} = \lambda^2\mathbf{v}$, so that $(1-\lambda^2)\mathbf{v} = \mathbf{0}$, yielding $1-\lambda^2 = 0$, so that $\lambda = 1$ or -1 .

13. Suppose $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = \mathbf{0}$, apply M twice and rearrange to deduce that one of the scalars is zero. Reduce to an earlier exercise to deduce that the other scalars are zero.

14. $1\,, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \; 2\,, \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}, \; -1\,, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

15. $\begin{bmatrix} -1 & 5 & 1 \\ 1 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

16. $\frac{1}{6}\begin{bmatrix} -3+5(2^{n+1})-(-1)^n & -9+5(2^{n+1})-(-1)^n & -12+5(2^{n+1})+2(-1)^n \\ 3-6(2^n)+3(-1)^n & 9-6(2^n)+3(-1)^n & 12-6(2^n)-6(-1)^n \\ 2^{n+1}-2(-1)^n & 2^{n+1}-2(-1)^n & 2^{n+1}+4(-1)^n \end{bmatrix}$,

$\begin{bmatrix} 26 & 25 & 25 \\ -15 & -14 & -15 \\ 5 & 5 & 6 \end{bmatrix}$

17. $\begin{bmatrix} 2^n & 2^n-1 \\ 0 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2^n-1 & 2^n-1 \\ 0 & 2^n & 2^n-3^n \\ 0 & 0 & 3^n \end{bmatrix}$

19. Suppose $P^{-1}MP$ is diagonal where $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Deduce that $ad-bc=0$, contradicting that P is invertible.

20. $\det(A-\lambda I) = \det(A-\lambda I)^T = \det(A^T-\lambda I^T) = \det(A^T-\lambda I)$

22. $\lambda^3 - 3\lambda^2 - \lambda + 3$,

$M^3 - 3M^2 - M + 3I = 0$, so $M^{-1} = -\frac{1}{3}(M^2-3M-I) = \frac{1}{3}\begin{bmatrix} -13 & -6 & 10 \\ 2 & 3 & -2 \\ -14 & -6 & 11 \end{bmatrix}$

23. $\mathbf{v} = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix}$, $M^n = \frac{1}{9}\begin{bmatrix} 4+5(1/10)^n & 4-4(1/10)^n \\ 5-5(1/10)^n & 5+4(1/10)^n \end{bmatrix} \to [\ \mathbf{v}\ \ \mathbf{v}\ ]$

24. eigenvalues of M are $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$,

$$M^n = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{n+1}-\lambda_2^{n+1} & \lambda_1\lambda_2^{n+1}-\lambda_2\lambda_1^{n+1} \\ \lambda_1^n-\lambda_2^n & \lambda_1\lambda_2^n-\lambda_2\lambda_1^n \end{bmatrix},$$

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$