CHAPTER **FIVE**

Coordinate Geometry

Coordinate geometry is geometry done in a number plane, where points are represented by ordered pairs of numbers, lines are represented by linear equations, and circles, parabolas and other curves are represented by more complicated equations. This chapter establishes the methods used in coordinate geometry to deal with intervals and lines.

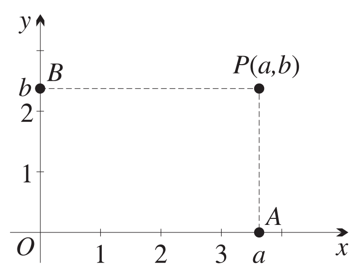
STUDY NOTES: Much of this work will be a consolidation of material from earlier years. Three topics, however, are quite new: the ratio division formula, including external division, in Section 5A, the perpendicular distance from a point to a line in Section 5E, and lines through the intersection of two given lines in Section 5F. The final Section 5G uses the methods of coordinate geometry to develop alternative proofs of theorems from geometry. The last two Sections 5F and 5G could be delayed if they seem too demanding at this stage.

5 A Points and Intervals

The first task is to set up the coordinate plane, and to develop the distance formula, the midpoint formula and the ratio division formula for intervals.

Representing Points by Ordered Pairs: A blank plane in Euclidean geometry can be made into a *coordinate plane* by constructing a pair of axes in it:

1. Any pair of perpendicular lines can be chosen as the axes. Their intersection is called the *origin*, and given the symbol O.
2. Each line must be made into a number line, with zero at the origin, and with the same scale on both axes.
3. The x-axis and the y-axis can be distinguished from each other, because a rotation of 90° anticlockwise about O rotates the x-axis onto the y-axis.



Any point P in the plane can now be given a unique pair of *coordinates*. Construct the rectangle $OAPB$ in which A lies on the x-axis and B lies on the y-axis, and let a and b be the real numbers on the axes associated with A and B respectively. Then the point P is identified with the ordered pair (a, b).

Every point P now corresponds to a single ordered pair (a, b) of real numbers, and every ordered pair (a, b) of real numbers corresponds to a single point P. There is therefore no need to distinguish between the points and the ordered pairs, and we will write statements like 'Let $P = (3, 5)$'.

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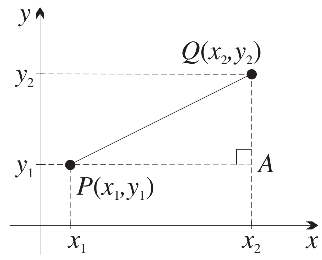
The Distance Formula: The formula for distance on the number plane is Pythagoras' theorem. Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane. Form the right triangle PQA, where A is the point (x_2, y_1). Then $PA = |x_2 - x_1|$ and $QA = |y_2 - y_1|$, and so by Pythagoras' theorem the square of the hypotenuse PQ is given by:

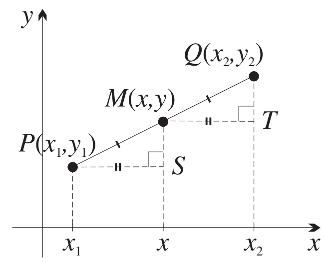


1 DISTANCE FORMULA: $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

NOTE: The distance formula is better understood as a formula for the *square of the distance*, rather than for the distance itself. In applying the formula, first find the square of the distance, then write down the distance as the final step if it is required.

The Midpoint Formula: The midpoint of an interval can be found by averaging the coordinates of the two points. Congruence is the basis of the proof.

Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, and let $M(x, y)$ be the midpoint of PQ. Then $\triangle PMS$ is congruent to $\triangle MQT$, and so $PS = MT$. Algebraically,



$$\begin{aligned} x - x_1 &= x_2 - x \\ 2x &= x_1 + x_2 \\ x &= \frac{x_1 + x_2}{2}\,. \end{aligned}$$

The calculation for the y-coordinate is similar, so:

2 MIDPOINT FORMULA: $x = \dfrac{x_1 + x_2}{2}$ and $y = \dfrac{y_1 + y_2}{2}$

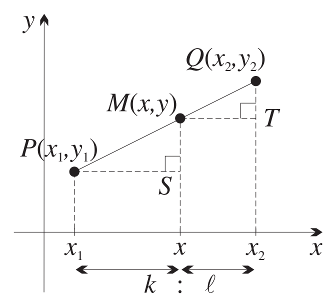
WORKED EXERCISE: The interval joining $A(3, -7)$ and $B(-6, 2)$ is a diameter of a circle. Find the centre and radius of the circle.

SOLUTION:
$$\begin{aligned} AB^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (-9)^2 + 9^2 \\ &= 2 \times 9^2 \\ AB &= 9\sqrt{2}\,, \end{aligned}$$
so the radius is $\frac{9}{2}\sqrt{2}$.

$$\begin{aligned} \text{Centre} &= \text{midpoint of } AB \\ &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3 - 6}{2}, \frac{-7 + 2}{2}\right) \\ &= \left(-1\tfrac{1}{2}, -2\tfrac{1}{2}\right). \end{aligned}$$

The Ratio Division Formula: Often an interval needs to be divided in some ratio other than $1 : 1$. Suppose then that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, and let $M(x, y)$ be the point dividing PQ in some ratio $k : \ell$. Then $\triangle PMS$ is similar to $\triangle MQT$, hence $PS : MT = k : \ell$, so that

$$\begin{aligned} \frac{x - x_1}{x_2 - x} &= \frac{k}{\ell} \\ \ell x - \ell x_1 &= k x_2 - k x \\ (k + \ell)x &= \ell x_1 + k x_2 \\ x &= \frac{\ell x_1 + k x_2}{k + \ell}\,. \end{aligned}$$



The calculation for the y-coordinate is similar, so:

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3 RATIO DIVISION FORMULA: $x = \dfrac{\ell x_1 + kx_2}{k + \ell}$ and $y = \dfrac{\ell y_1 + ky_2}{k + \ell}$

WORKED EXERCISE: Given the points $A(1, 3)$ and $B(6, 28)$, find the point P dividing the interval AB in the ratio $2 : 3$.

SOLUTION:

$$x = \frac{\ell x_1 + kx_2}{k + \ell} \qquad\qquad y = \frac{\ell y_1 + ky_2}{k + \ell}$$
$$= \frac{3 \times 1 + 2 \times 6}{5} \qquad\qquad = \frac{3 \times 3 + 2 \times 28}{5}$$
$$= 3, \qquad\qquad = 13,$$

and so P is the point $(3, 13)$.

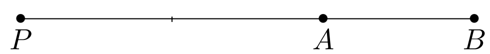
WORKED EXERCISE: The point $P(6, 13)$ divides the interval AB in the ratio $2 : 3$. Find the coordinates of A if $B = (-9, 25)$.

SOLUTION:

$$x = \frac{\ell x_1 + kx_2}{k + \ell} \qquad\qquad y = \frac{\ell y_1 + ky_2}{k + \ell}$$
$$6 = \frac{3x_1 + 2 \times (-9)}{5} \qquad\qquad 13 = \frac{3y_1 + 2 \times 25}{5}$$
$$30 = 3x_1 - 18 \qquad\qquad 65 = 3y_1 + 50$$
$$x_1 = 16, \qquad\qquad y_1 = 5,$$

and so A is the point $(16, 5)$.

External Division of an Interval: The diagram below can be described by saying that 'A divides PB in the ratio $2 : 1$'.



But since $AP : PB = 2 : 3$, we shall also describe it by the statement 'P divides AB *externally* in the ratio $2 : 3$'. It turns out that if we use negative numbers in the ratio, and say

'P divides AB in the ratio $-2 : 3$ (or in the ratio $2 : -3$)',

then the formula for ratio division will give the coordinates of P, provided that a negative sign is first applied to one of the numbers in the ratio.

4 EXTERNAL DIVISION: If P divides AB *externally* in some ratio, for example $2 : 3$, then P divides AB in the ratio $-2 : 3$, or equivalently $2 : -3$.

WORKED EXERCISE: Find the point P which divides the interval AB externally in the ratio $2 : 5$, where $A = (-3, -5)$ and $B = (3, 7)$.

SOLUTION: The point P divides the interval AB in the ratio $-2 : 5$. Using the ratio division formula with $k = -2$ and $\ell = 5$, the point $P(x, y)$ is given by

$$x = \frac{5 \times (-3) + (-2) \times 3}{-2 + 5} \qquad\qquad y = \frac{5 \times (-5) + (-2) \times 7}{-2 + 5}$$
$$= -7, \qquad\qquad = -13,$$

and so P is the point $(-7, -13)$.

NOTE: We could equally well have taken the ratio as $2 : -5$, in which case the top and the bottom of each fraction would have been opposite, but the final result would be the same.

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Testing for Special Quadrilaterals: Euclidean geometry will be reviewed in the Year 12 Volume, but many questions in this chapter ask for proofs that a quadrilateral is of a particular type. The most obvious way is to test the definition itself.

5

DEFINITIONS OF THE SPECIAL QUADRILATERALS:
A *trapezium* is a quadrilateral in which a pair of opposite sides are parallel.
A *parallelogram* is a quadrilateral in which the opposite sides are parallel.
A *rhombus* is a parallelogram with a pair of adjacent sides equal.
A *rectangle* is a parallelogram with one angle a right angle.
A *square* is both a rectangle and a rhombus.

There are, however, several further standard tests which the exercises assume (tests involving angles are omitted, being irrelevant here).

6

A QUADRILATERAL IS A PARALLELOGRAM:
- if the opposite sides are equal, or
- if one pair of opposite sides are equal and parallel, or
- if the diagonals bisect each other.

A QUADRILATERAL IS A RHOMBUS:
- if all sides are equal, or
- if the diagonals bisect each other at right angles.

A QUADRILATERAL IS A RECTANGLE:
- if the diagonals are equal and bisect each other.

Exercise 5A

NOTE: Diagrams should be drawn wherever possible.

1. Find the distance between each pair of points (find AB^2 first):

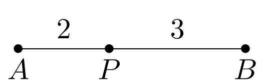
(a) $A(1,4)$, $B(5,1)$ (c) $A(-5,-2)$, $B(3,4)$ (e) $A(-4,-1)$, $B(4,3)$
(b) $A(-2,7)$, $B(3,-5)$ (d) $A(3,6)$, $B(5,4)$ (f) $A(5,-12)$, $B(0,0)$

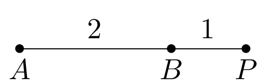
2. Find the midpoint of each pair of points in the previous question.

3. Find the points dividing each interval AB in the given ratios:

(a) $A(1,2)$ and $B(7,5)$	(i) $1:2$	(ii) $2:1$	(iii) $4:-1$	(iv) $-4:1$
(b) $A(-1,1)$ and $B(3,-1)$	(i) $1:3$	(ii) $3:1$	(iii) $-1:3$	(iv) $-3:1$
(c) $A(-3,2)$ and $B(7,-3)$	(i) $1:4$	(ii) $3:2$	(iii) $7:-2$	(iv) $-4:3$
(d) $A(-7,5)$ and $B(-1,-7)$	(i) $1:5$	(ii) $1:1$	(iii) $1:-3$	(iv) $-1:5$

4. Write down the ratio in which P divides each interval AB:

(a)


(c)


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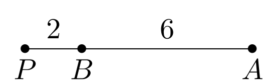

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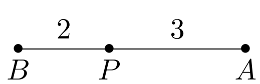

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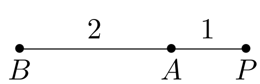

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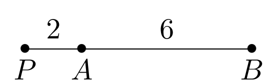

5. For each diagram in the previous question, write down the ratio in which:
(i) A divides PB, (ii) B divides AP.

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6. Given $A(1,1)$ and $B(5,3)$, find the coordinates of E if it divides the interval AB externally in the following ratios:
(a) $3:2$ (b) $2:3$ (c) $1:3$ (d) $7:5$

7. The points $A(3,1)$, $B(2,4)$, $C(-1,3)$ and $D(-1,-2)$ are the vertices of a quadrilateral. Find the lengths of all four sides, and show that two pairs of adjacent sides are equal (you may know a common name for this sort of quadrilateral).

8. (a) Find the side lengths of the triangle formed by $X(0,-4)$, $Y(4,2)$ and $Z(-2,6)$, and show that it is a right isosceles triangle by showing that the side lengths satisfy Pythagoras' theorem.
(b) Hence find the area of this triangle.

DEVELOPMENT

9. Each set of points given below comprises the vertices of: an isosceles triangle, an equilateral triangle, a right triangle, or none of these. Find the side lengths of each triangle and hence determine its type.
(a) $A(-1,0)$, $B(1,0)$, $C(0,\sqrt{3})$
(b) $P(-1,1)$, $Q(0,-1)$, $R(3,3)$
(c) $D(1,1)$, $E(2,-2)$, $F(-3,0)$
(d) $X(-3,-1)$, $Y(0,0)$, $Z(-2,2)$

10. (a) The interval joining $G(2,-5)$ and $H(-6,-1)$ is divided into four equal subintervals by the three points A, B and C. Find their coordinates by repeatedly taking midpoints.
(b) Find the coordinates of the four points A, B, C and D which divide the interval joining $S(-2,3)$ and $T(8,18)$ into five equal subintervals. (You will need the ratio division formula.)

11. The quadrilateral $ABCD$ has vertices at $A(1,0)$, $B(3,1)$, $C(4,3)$ and $D(2,2)$.
(a) Show that the intervals AC and BD bisect each other, by finding the midpoint of each and showing that these midpoints coincide. What can you now conclude about the type of quadrilateral $ABCD$ is?
(b) Show that $AB = AD$. What can you now conclude about the quadrilateral $ABCD$?

12. Show that the points $A(1,4)$, $B(2,\sqrt{13})$, $C(3,2\sqrt{2})$ and $D(4,1)$ lie on a circle with centre the origin. What are the radius, diameter, circumference and area of this circle?

13. As discussed in Chapter Two, the circle with centre (h,k) and radius r has equation $(x-h)^2 + (y-k)^2 = r^2$. By identifying the centre and radius, find the equations of:
(a) the circle with centre $(5,-2)$ passing through $(-1,1)$,
(b) the circle with $K(5,7)$ and $L(-9,-3)$ as endpoints of a diameter.

14. (a) If $A(-1,2)$ is the midpoint of $S(x,y)$ and $T(3,6)$, find the coordinates of S.
(b) The midpoint of PQ is $M(2,-7)$. Find P if: (i) $Q = (5,3)$, (ii) $Q = (-3,-7)$.
(c) If AB is a diameter of a circle with centre $Q(4,5)$ and $A = (8,3)$, find B.
(d) Given that $P(4,7)$ is one vertex of a square $PQRS$ and the centre of the square is $M(8,-1)$, find the coordinates of R.

15. (a) Given the point $A(7,8)$, find the coordinates of three points P with integer coordinates such that $AP = \sqrt{5}$.
(b) If the distance from $U(3,7)$ to $V(1,y)$ is $\sqrt{13}$, find the two possible values of y.
(c) Find a, if the distance from $A(a,0)$ to $B(1,4)$ is $\sqrt{18}$ units.

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16. A triangle has vertices $A(-2,2)$, $B(-4,-3)$ and $C(6,-2)$.

(a) Find the midpoint P of BC, then find the coordinates of the point M dividing the interval AP (called a *median*) in the ratio $2:1$.

(b) Do likewise for the medians BQ and CR and confirm that the same point is obtained each time (this point is called the *centroid* of the triangle).

17. (a) Find the ratio in which the point $M(3,5)$ divides the interval joining $A(-4,-9)$ and $B(5,9)$. [HINT: Let M divide AB in the ratio $k:1$, and find k.]

(b) Given the collinear points $P(-2,-11)$, $Q(1,-2)$ and $R(3,4)$, find: (i) the ratio in which P divides QR, (ii) the ratio in which Q divides RP, (iii) the ratio in which R divides PQ.

18. If the point M divides the interval AB in each ratio given below, draw a diagram and find in what ratio B divides AM:

(a) $AM:MB = 1:1$ (b) $AM:MB = 1:3$ (c) $AM:MB = 7:4$ (d) $AM:MB = 2:-1$ (e) $AM:MB = -1:3$ (f) $AM:MB = 4:-3$

19. (a) Given the four collinear points $P(1,-8)$, $Q(5,-2)$, $R(7,1)$ and $S(13,10)$, show that:

(i) Q divides PR internally in the same ratio as S divides PR externally,

(ii) R divides QS internally in the same ratio as P divides QS externally.

(b) Prove in general that if P, Q, R and S are four collinear points such that Q divides PR internally in the same ratio as S divides PR externally, then R divides QS internally in the same ratio as P divides QS externally. [HINT: Let $PQ = a$, $QR = b$ and $RS = c$.]

20. (a) Given $K(3,-1)$ and $L(-4,2)$, find two positions of A on KL such that $KA = 2\times KL$.

(b) The point $Q(1,-2)$ divides the interval $R(x,y)$ to $S(4,2)$ in the ratio $1:4$. Find R.

21. The point P divides the interval joining $A(-1,4)$ and $B(2,-2)$ in the ratio $k:1$.

(a) Write down the coordinates of P.

(b) Given that P lies on the line $2y-x+1=0$, find k, and hence find the coordinates of P.

22. (a) Given that $C(x,y)$ is equidistant from each of the points $P(1,5)$, $Q(-5,-3)$ and $R(2,-2)$, use the distance formula to form two equations in x and y and solve them simultaneously to find the coordinates of C.

(b) Find the coordinates of the point $M(x,y)$ which is equidistant from each of the points $P(4,3)$ and $Q(3,2)$, and is also equidistant from $R(6,1)$ and $S(4,0)$.

23. (a) The point $F(0,1)$ divides PQ in the ratio $t:\frac{1}{t}$, where P is $(2t,t^2)$. Find Q.

(b) The origin O divides RS externally in the ratio $r:\frac{1}{r}$, where R is (a,b).

(i) Find the coordinates of S. (ii) What are these coordinates if $r = OR$?

24. Suppose that A, B and P are the points $(0,0)$, $(3a,0)$ and (x,y) respectively. Use the distance formula to form an equation in x and y for the point P, and describe the curve so found if: (a) $PA = PB$, (b) $PA = 2PB$.

EXTENSION

25. The point M on the line through $P(x_1,y_1)$ and $Q(x_2,y_2)$ which divides PQ into the ratio $k:1$ has coordinates $\left(\dfrac{x_1+kx_2}{1+k}, \dfrac{y_1+ky_2}{1+k}\right)$.

(a) Which point on the line PQ cannot be expressed in this manner?

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(b) What range of values of k will result in: (i) M between P and Q, (ii) M on the opposite side of Q from P, and (iii) M on the opposite side of P from Q. (iv) What happens as $k \to (-1)^+$ and as $k \to (-1)^-$?

26. The point $M(x, y)$ divides the interval joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $k : \ell$.

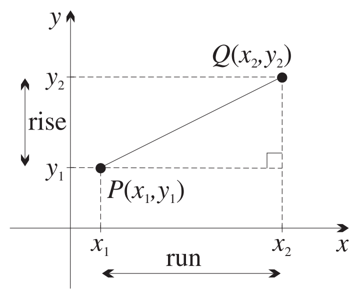
(a) Show by geometry that $\dfrac{k}{\ell} = \dfrac{y - y_1}{y_2 - y}$.

(b) Hence show that an equation of the line PQ is $(x - x_1)(y - y_2) = (y - y_1)(x - x_2)$.

(c) Alternatively, justify the equation above by showing that P and Q satisfy it.

5 B Gradients of Intervals and Lines

Gradient is the key to bringing lines and their equations into coordinate geometry. The gradient of an interval is easy to define, but we need to use similarity to define the gradient of a line.



The Gradient of an Interval: Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points in the number plane.

Define the *rise* from P to Q as the change $y_2 - y_1$ in y from P to Q, so that the rise is positive when Q is above P, and negative when Q is below P.

Define the *run* from P to Q as the change $x_2 - x_1$ in x, so that the run is positive when Q is on the right of P, and negative when Q is on the left of P.

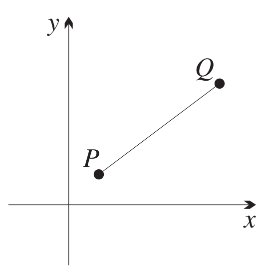
The *gradient* of PQ is the ratio of these two changes.

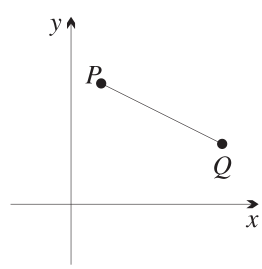
7 GRADIENT FORMULA: gradient of $PQ = \dfrac{\text{rise}}{\text{run}} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Intervals have gradient zero if and only if they are horizontal, because only horizontal intervals have zero rise. Vertical intervals, on the other hand, don't have a gradient, because their run is always zero and so the fraction is undefined.

Positive and Negative Gradients: If the rise and the run have the same sign, then the gradient will be positive, as in the first diagram below — in this case the interval slopes *upwards* as one moves from left to right.

If the rise and run have opposite signs, then the gradient will be negative, as in the second diagram — now the interval slopes *downwards* as one moves from left to right.





Notice that if the points P and Q are interchanged, then both rise and run change signs, but the gradient remains the same.

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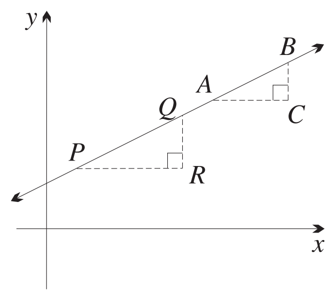
The Gradient of a Line: The *gradient of a line* is defined to be the gradient of any interval within the line. This definition makes sense because any two intervals on the same line always have the same gradient.

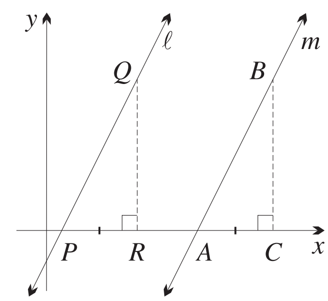


To prove this, suppose that PQ and AB are two intervals on the same line ℓ. Construct right triangles PQR and ABC underneath the intervals, with sides parallel to the axes. Because these two triangles are similar (by the AA similarity test), the ratios of their heights and bases are the same, which means that the two intervals AB and PQ have the same gradient.

A Condition for Two Lines to be Parallel: The condition for two lines to be parallel is:

8 PARALLEL LINES: Two lines are parallel if and only if they have the same gradient (or are both vertical).

To prove this, let ℓ and m be two lines meeting the x-axis at P and A respectively, and construct the two triangles PQR and ABC as shown, with the two runs PR and AC equal.



If the lines are parallel, then the corresponding angles $\angle P$ and $\angle A$ are equal. Hence the two triangles are congruent by the AAS test, and so the rises RQ and CB must be equal.

Conversely, if the gradients are equal, then the rises RQ and CB are equal. Hence the triangles are congruent by the SAS test, so the corresponding angles $\angle P$ and $\angle A$ are equal, and so the lines must be parallel.

WORKED EXERCISE: Show that the points $A(3,6)$, $B(7,-2)$, $C(4,-5)$ and $D(-1,5)$ form a trapezium with $AB \parallel CD$.

SOLUTION: gradient of $AB = \dfrac{-2-6}{7-3}$ $= -2$, gradient of $CD = \dfrac{5+5}{-1-4}$ $= -2$.

So $AB \parallel CD$, and $ABCD$ is a trapezium.

Testing for Collinear Points: Three distinct points are called *collinear* if they all lie on the same line. To test whether three given points A, B and C are collinear, the most straightforward method is to find the gradients of AB and AC. If these gradients are equal, then the three points must be collinear, because then AB and AC are parallel lines passing through a common point A.

WORKED EXERCISE: Test whether $A(-2,5)$, $B(1,3)$ and $C(7,-1)$ are collinear.

SOLUTION: gradient of $AB = \dfrac{3-5}{1+2}$ $= -\frac{2}{3}$, gradient of $AC = \dfrac{-1-5}{7+2}$ $= -\frac{2}{3}$.

Since the gradients are equal, the points are collinear.

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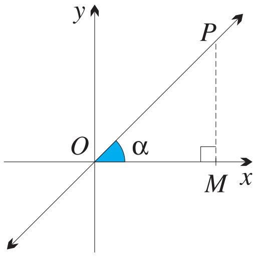
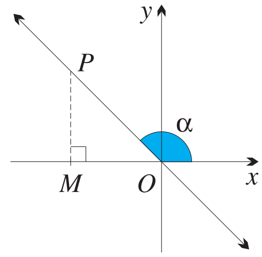
Gradient and the Angle of Inclination: Another very natural way of measuring the steepness of a line is to look at its *angle of inclination*, which is the angle between the upward direction of the line and the positive direction of the x-axis. The two diagrams below show that lines with positive gradient have acute angles of inclination, and lines with negative gradient have obtuse angles of inclination.





These two diagrams also illustrate the trigonometric relationship between the gradient and angle of inclination α.

9 ANGLE OF INCLINATION: gradient $= \tan\alpha$

PROOF: When α is acute, as in the first diagram, then the rise MP and the run OM are just the opposite and adjacent sides of the triangle POM, so

$$\tan\alpha = \frac{MP}{OM} = \text{gradient } OP.$$

When α is obtuse, as in the second diagram, then $\angle POM = 180^\circ - \alpha$, so

$$\tan\alpha = -\tan\angle POM = -\frac{MP}{OM} = \text{gradient } OP.$$

WORKED EXERCISE: Given the points $A(-3, 5)$, $B(-6, 0)$ and $O(0, 0)$, find the angles of inclination of AB and AO, and show that they are supplementary. What sort of triangle is $\triangle ABO$?

SOLUTION: gradient of $AB = \dfrac{0-5}{-6+3}$

$= \frac{5}{3}$,

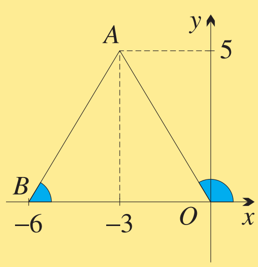
so angle of inclination $\doteqdot 59^\circ$.

gradient of $AO = \dfrac{5-0}{-3-0}$

$= -\frac{5}{3}$.

so angle of inclination $\doteqdot 121^\circ$.

Hence the angles of inclination are supplementary, and $\triangle ABO$ is isosceles.



A Condition for Lines to be Perpendicular: The condition for two lines to be perpendicular is:

10 PERPENDICULARITY: Two lines are perpendicular if and only if the product of their gradients is -1 (or one is vertical and the other horizontal).

PROOF: We can shift each line sideways without rotating it so that it passes through the origin (remember that parallel lines have the same gradient). Also, one line must have positive gradient and the other negative gradient, otherwise one of the angles between them would be acute.

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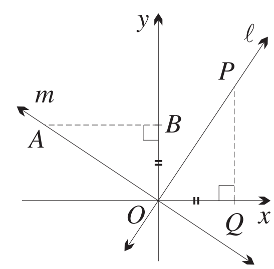
So let ℓ and m be two lines through the origin with positive and negative gradients respectively, and construct the two triangles POQ and AOB as shown, with the run OQ of ℓ equal to the rise OB of m. Then



$$\text{gradient of } \ell \times \text{gradient of } m = -\frac{QP \times OB}{OQ \times AB} = -\frac{QP}{AB}.$$

If the lines are perpendicular, then $\angle AOB = \angle POQ$. Hence the triangles are congruent by the AAS test, so $QP = AB$, with the result that the product of the gradients is -1.

Conversely, if the product of the gradients is -1, then $QP = AB$. Hence the triangles are congruent by the SAS test, so $\angle AOB = \angle POQ$ and the lines are perpendicular.

WORKED EXERCISE: Given the four points $A(-1,1)$, $B(7,11)$, $C(0,8)$ and $D(a,-1)$, find a and the coordinates of D if $AB \perp CD$.

SOLUTION:

$$\begin{aligned}\text{gradient of } AB &= \frac{11-1}{7+1}\\ &= \tfrac{5}{4}\\ \text{gradient of } CD &= \frac{-1-8}{a-0}\\ &= -\frac{9}{a}\end{aligned}$$

Since AB and CD are perpendicular,
gradient of $CD\times$gradient of $AB = -1$

$$\begin{aligned}-\frac{9}{a} \times \frac{5}{4} &= -1\\ 4a &= 45\\ a &= \tfrac{45}{4}.\end{aligned}$$

So D is the point $(11\frac{1}{4}, -1)$.

Exercise 5B

NOTE: Diagrams should be drawn wherever possible.

1. Find the gradient of a line (i) parallel to, (ii) perpendicular to a line with gradient:

(a) 2 (b) -1 (c) $\dfrac{3}{4}$ (d) $-\dfrac{p}{q}$

2. Find the gradient of each interval AB, then find the gradient of a line perpendicular to it:

(a) $(1,4)$, $(5,0)$
(b) $(-2,-7)$, $(3,3)$
(c) $(-5,-2)$, $(3,2)$
(d) $(3,6)$, $(5,5)$
(e) $(-1,-2)$, $(1,4)$
(f) $(-a,b)$, $(3a,-b)$

3. Find the gradient, to two decimal places, of a line with angle of inclination:

(a) 15° (b) 135° (c) $22\frac{1}{2}^\circ$ (d) 72°

4. (a) What angle of inclination (to the nearest degree where necessary) does a line with each gradient make with the x-axis? Does the line slope upwards or downwards?

(i) 1 (ii) $-\sqrt{3}$ (iii) 4 (iv) $\dfrac{1}{\sqrt{3}}$

(b) Find the acute angle made by each line in part (a) with the y-axis.

5. Find the points A and B where each line meets the x-axis and y-axis respectively. Hence find the gradient of AB and its angle of inclination α (to the nearest degree):

(a) $y = 3x + 6$
(b) $y = -\frac{1}{2}x + 1$
(c) $3x + 4y + 12 = 0$
(d) $\dfrac{x}{3} - \dfrac{y}{2} = 1$
(e) $4x - 5y - 20 = 0$
(f) $\dfrac{x}{2} + \dfrac{y}{5} = 1$

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6. Given $A = (2, 3)$, write down the coordinates of any three points P such that AP has gradient 2.

DEVELOPMENT

7. Find the gradient, to two decimal places, of a line sloping upwards if its acute angle with the y-axis is:

(a) $15°$ (b) $45°$ (c) $22\frac{1}{2}°$ (d) $72°$

8. Find the gradients of PQ and QR, and hence determine whether P, Q and R are collinear:

(a) $P(-2, 7)$, $Q(1, 1)$, $R(4, -6)$ (b) $P(-5, -4)$, $Q(-2, -2)$, $R(1, 0)$

9. (a) Triangle ABC has vertices $A(-1, 0)$, $B(3, 2)$ and $C(4, 0)$. Calculate the gradient of each side and hence show that $\triangle ABC$ is a right-angled triangle.

(b) Do likewise for the triangles with the vertices given below. Then find the lengths of the sides enclosing the right angle, and calculate the area of each triangle:

(i) $P(2, -1)$, $Q(3, 3)$, $R(-1, 4)$ (ii) $X(-1, -3)$, $Y(2, 4)$, $Z(-3, 2)$

10. Quadrilateral $ABCD$ has vertices $A(-1, 1)$, $B(3, -1)$, $C(5, 3)$ and $D(1, 5)$.

(a) Show that it has two pairs of parallel sides. (b) Confirm that $AB \perp BC$.

(c) Show also that $AB = BC$. (d) What type of quadrilateral is $ABCD$?

11. Use gradients to show that each quadrilateral $ABCD$ below is a parallelogram. Then show that it is:

(a) a rhombus, for the vertices $A(2, 1)$, $B(-1, 3)$, $C(1, 0)$ and $D(4, -2)$,

(b) a rectangle, for the vertices $A(4, 0)$, $B(-2, 3)$, $C(-3, 1)$ and $D(3, -2)$,

(c) a square, for the vertices $A(3, 3)$, $B(-1, 2)$, $C(0, -2)$ and $D(4, -1)$.

12. The interval PQ has gradient -3. A second line passes through $A(-2, 4)$ and $B(1, y)$. Find the value of y if: (a) AB is parallel to PQ, (b) AB is perpendicular to PQ.

13. Find λ for the points $X(-1, 0)$, $Y(1, \lambda)$ and $Z(\lambda, 2)$, if $\angle YXZ = 90°$.

14. For the four points $P(k, 1)$, $Q(-2, -3)$, $R(2, 3)$ and $S(1, k)$, it is known that PQ is parallel to RS. Find the possible values of k.

15. On a number plane, mark the origin O and the points $A(2, 1)$ and $B(3, -1)$.

(a) Find the gradients of OA and AB and hence show that they are perpendicular.

(b) Show that $OA = AB$. (c) Find the midpoint D of OB.

(d) Given that D is the midpoint of AC, find the coordinates of C.

(e) What shape best describes quadrilateral $OABC$?

16. Answer the following questions for the points $W(2, 3)$, $X(-7, 5)$, $Y(-1, -3)$ and $Z(5, -1)$.

(a) Show that WZ is parallel to XY.

(b) Find the lengths WZ and XY. Hence deduce the type of the quadrilateral $WXYZ$.

(c) Show that the diagonals WY and XZ are perpendicular.

17. Quadrilateral $ABCD$ has vertices $A(1, -4)$, $B(3, 2)$, $C(-5, 6)$ and $D(-1, -2)$.

(a) Find the midpoints P of AB, Q of BC, R of CD, and S of DA.

(b) Prove that $PQRS$ is a parallelogram by showing that $PQ \parallel RS$ and $PS \parallel QR$.

18. (a) $A(1, 4)$, $B(5, 0)$ and $C(9, 8)$ form the vertices of a triangle. Find the coordinates of P and Q if they divide the sides AB and AC respectively in the ratio $1 : 3$.

(b) Show that PQ is parallel to BC and is one quarter of its length.

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19. Given the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$, find and simplify the gradient of PQ.

20. The points $A(4, -2)$, $B(-4, 4)$ and $P(x, y)$ form a right angle at P. Form an equation in x and y, and hence find the equation of the curve on which P lies. Describe this curve.

21. The points $O(0, 0)$, $P(4, 0)$ and $Q(x, y)$ form a right angle at Q and $PQ = 1$ unit.
(a) Form a pair of equations for x and y.
(b) Solve them simultaneously to find the coordinates of the two possible locations of Q.

EXTENSION

22. (a) The points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x, y)$ are collinear. Use the gradient formula to show that

$$(x - x_1)(y - y_2) = (y - y_1)(x - x_2).$$

(b) If AB is the diameter of a circle and P another point on the circumference then Euclidean geometry tells us that $\angle APB = 90^\circ$. Use this fact to show that the equation of the circle whose diameter has endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

23. (a) Three points $A_1(a_1, b_1)$, $A_2(a_2, b_2)$, $A_3(a_3, b_3)$ form a triangle. By dropping perpendiculars to the x-axis and taking the areas of the resulting trapeziums, show that the area Δ of the triangle $A_1A_2A_3$ is

$$\Delta = \tfrac{1}{2}|a_1b_2 - a_2b_1 + a_2b_3 - a_3b_2 + a_3b_1 - a_1b_3|,$$

with the expression inside the absolute value sign positive if and only if the vertices A_1, A_2 and A_3 are in anticlockwise order.
(b) Use part (a) to generate a test for A_1, A_2 and A_3 to be collinear.
(c) Generate the same test by putting gradient A_1A_2 = gradient A_2A_3.

24. Consider the points $P(2p, p^2)$, $Q(-\frac{2}{p}, \frac{1}{p^2})$ and $T(x, -1)$. Find the x-coordinate of T if:
(a) the three points are collinear, (b) PT and QT are perpendicular.

25. The points $P(p, 1/p)$, $Q(q, 1/q)$, $R(r, 1/r)$ and $S(s, 1/s)$ lie on the curve $xy = 1$.
(a) If $PQ \parallel RS$, show that $pq = rs$.
(b) Show that $PQ \perp RS$ if and only if $pqrs = -1$.
(c) Use part (b) to conclude that if a triangle is drawn with its vertices on the rectangular hyperbola $xy = 1$, then the altitudes of the triangle intersect at a common point which also lies on the hyperbola (an altitude of a triangle is the perpendicular from a vertex to the opposite side).

5 C Equations of Lines

In coordinate geometry, a line in the number plane is represented by an equation in x and y. This section and the next summarise that theory from earlier years, and develop various useful forms for the equation of the line.

Horizontal and Vertical Lines: In a vertical line, all points on the line have the same x-coordinate, but the y-coordinate can take any value.

11 VERTICAL LINES: The vertical line through $P(a, b)$ has equation $x = a$.

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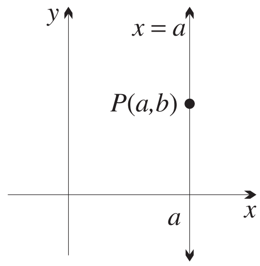
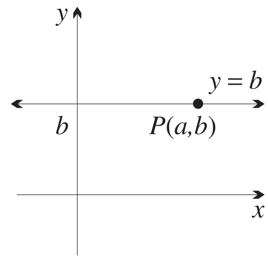
In a horizontal line, all points on the line have the same y-coordinate, but the x-coordinate can take any value.

12 HORIZONTAL LINES: The horizontal line through $P(a, b)$ has equation $y = b$.





Gradient–Intercept Form: The problem here is to find a formula for the equation of a line when its gradient and y-intercept are known. Suppose that ℓ has gradient m, and that it has y-intercept b, passing through the point $B(0, b)$. If $Q(x, y)$ is any other point in the plane, then the condition that Q lie on the line ℓ is

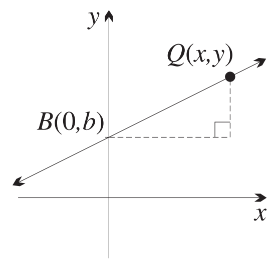
gradient of $BQ = m$,

that is, $\dfrac{y-b}{x} = m$,

or $y - b = mx$.

Hence $y = mx + b$,

which is the equation of BQ in *gradient–intercept form*.



13 GRADIENT–INTERCEPT FORM: $y = mx + b$

WORKED EXERCISE:

(a) Write down the gradient and the y-intercept of the line ℓ: $y = 3x - 2$.

(b) Hence find the equations of the lines through $B(0, 5)$ which are parallel and perpendicular to ℓ.

SOLUTION:

(a) The line ℓ has gradient 3 and y-intercept -2.

(b) The line through B parallel to ℓ has gradient 3 and y-intercept 5, so its equation is $y = 3x + 5$.
The line through B perpendicular to ℓ has gradient $-\frac{1}{3}$ and y-intercept 5, so its equation is $y = -\frac{1}{3}x + 5$.

General Form: It is often useful to have the equation of a line in a standard simplified form, with everything on the LHS. The *general form* of the equation of a line is:

14 GENERAL FORM: $ax + by + c = 0$

When an equation is given in general form, it should be simplified by multiplying out all fractions and dividing out all common factors. It may also be convenient to make the coefficient of x positive.

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WORKED EXERCISE:

(a) Find the gradient and y-intercept of the line ℓ: $2x - 3y + 4 = 0$.

(b) Find in general form the equations of the lines passing through $B(0, -2)$ and:
(i) parallel to ℓ, (ii) perpendicular to ℓ, (iii) having angle of inclination 60°.

SOLUTION:

(a) Solving the line ℓ for y,

$$3y = 2x + 4$$
$$y = \tfrac{2}{3}x + \tfrac{4}{3},$$

so ℓ has gradient $\frac{2}{3}$ and y-intercept $\frac{4}{3}$.

(b) (i) The line through B parallel to ℓ has gradient $\frac{2}{3}$ and y-intercept -2,

so its equation is $y = \frac{2}{3}x - 2$

$\boxed{\times 3}$ $3y = 2x - 6$

$$2x - 3y - 6 = 0.$$

(ii) The line through B perpendicular to ℓ has gradient $-\frac{3}{2}$ and y-intercept -2,

so its equation is $y = -\frac{3}{2}x - 2$

$\boxed{\times 2}$ $2y = -3x - 4$

$$3x + 2y + 4 = 0.$$

(iii) The line through B with angle of inclination 60° has gradient $\tan 60^\circ = \sqrt{3}$,

so its equation is $y = x\sqrt{3} - 2$

$$x\sqrt{3} - y - 2 = 0.$$

Exercise 5C

1. Determine by substitution whether the point $A(3, -2)$ lies on the line:
(a) $y = 4x - 10$ (b) $8x + 10y - 4 = 0$ (c) $x = 3$

2. Write down the coordinates of any three points on the line $2x + 3y = 4$.

3. Write down the equations of the vertical and horizontal lines through:
(a) $(1, 2)$ (b) $(-1, 1)$ (c) $(3, -4)$ (d) $(5, 1)$ (e) $(-2, -3)$ (f) $(-4, 1)$

4. Write down the gradient and y-intercept of each line:
(a) $y = 4x - 2$ (b) $y = \frac{1}{5}x - 3$ (c) $y = 2 - x$

5. Use the formula $y = mx + b$ to write down the equation of each of the lines specified below, then put that equation into general form:
(a) with gradient 1 and y-intercept 3
(b) with gradient -2 and y-intercept 5
(c) with gradient $\frac{1}{5}$ and y-intercept -1
(d) with gradient $-\frac{1}{2}$ and y-intercept 3

6. Solve each equation for y and hence write down the gradient and y-intercept:
(a) $x - y + 3 = 0$
(b) $y + x - 2 = 0$
(c) $2x - y = 5$
(d) $x - 3y + 6 = 0$
(e) $3x + 4y = 5$
(f) $2y - 3x = -4$

7. For each line in question 6, substitute $y = 0$ and $x = 0$ to find the points A and B where the line intersects the x-axis and y-axis respectively, and hence sketch the curve.

8. For each line in question 6, use the formula gradient $= \tan\alpha$ to find its angle of inclination, to the nearest minute where appropriate.

9. Show by substitution that the line $y = mx + b$ passes through $A(0, b)$ and $B(1, m + b)$. Then show that the gradient of AB, and hence of the line, is m.

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DEVELOPMENT

10. Find the gradient of each line below, and hence find in gradient–intercept form the equation of a line passing through $A(0,3)$ and (i) parallel to it, (ii) perpendicular to it:

(a) $2x + y + 3 = 0$ (b) $5x - 2y - 1 = 0$ (c) $3x + 4y - 5 = 0$

11. In each part below, the angle of inclination α and the y-intercept A of a line are given. Use the formula gradient $= \tan\alpha$ to find the gradient of each line, then find its equation in general form:

(a) $\alpha = 45°$, $A = (0,3)$
(b) $\alpha = 60°$, $A = (0,-1)$
(c) $\alpha = 30°$, $A = (0,-2)$
(d) $\alpha = 135°$, $A = (0,1)$

12. A triangle is formed by the x-axis and the lines $5y = 9x$ and $5y + 9x = 45$. Find (to the nearest degree) the angles of inclination of the two lines, and hence show that the triangle is isosceles.

13. Consider the two lines ℓ_1: $3x - y + 4 = 0$ and ℓ_2: $x + ky + \ell = 0$. Find the value of k if:

(a) ℓ_1 is parallel to ℓ_2, (b) ℓ_1 is perpendicular to ℓ_2.

14. [HINT: In each part of this question, draw a diagram of the situation, then use congruent or similar triangles to find the gradient and y-intercept of the line.] Find the equation of the line through $M(-1,2)$ if:

(a) M is the midpoint of the intercepts on the x-axis and y-axis,
(b) M divides the intercepts in the ratio $2:1$ (x-intercept to y-intercept),
(c) M divides the intercepts in the ratio $-2:5$ (x-intercept to y-intercept).

EXTENSION

15. Find the equations of the four circles which are tangent to the x-axis, the y-axis, and the line $x + y = 2$.

16. (a) Show that the four lines $y = 2x - 1$, $y = 2x + 1$, $y = 3 - \frac{1}{2}x$ and $y = k - \frac{1}{2}x$ enclose a rectangle. (b) Find the possible values of k if they enclose a square.

5 D Further Equations of Lines

This section introduces two further standard forms of the equations of lines, namely point–gradient form and the two-intercept form. It also deals with lines through two given points, and the point of intersection of two lines.

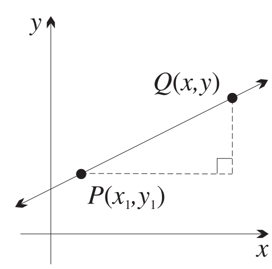
Point–Gradient Form: The problem here is to find a formula for the equation of a line ℓ when we know that ℓ has a particular gradient m and passes through a particular point $P(x_1, y_1)$. If $Q(x,y)$ is any other point in the plane, then the condition that Q lie on the line ℓ is

gradient of $PQ = m$,

that is, $\dfrac{y - y_1}{x - x_1} = m$,

or $y - y_1 = m(x - x_1)$,

which is the equation of PQ in *point–gradient form.*



15 POINT–GRADIENT FORM: $y - y_1 = m(x - x_1)$

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NOTE: Careful readers will realise that the equation $\dfrac{y-y_1}{x-x_1} = m$ actually describes a line with the point P itself removed, because substituting $P(x_1, y_1)$ into the LHS yields $\frac{0}{0}$. However, when both sides are multiplied by $x - x_1$ to give $y - y_1 = m(x - x_1)$, then the point P is now included, because substitution into either side gives 0.

WORKED EXERCISE:

(a) Find the equation of a line through $(-2, -5)$ perpendicular to $y = 3x + 2$.

(b) Express the answer in gradient–intercept form, and hence write down its y-intercept.

SOLUTION:

(a) The given line has gradient 3, so the perpendicular gradient is $-\frac{1}{3}$.
Hence, using point–gradient form, the required line is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 5 &= -\tfrac{1}{3}(x + 2) \\ y &= -\tfrac{1}{3}x - 5\tfrac{2}{3}. \end{aligned}$$

(b) This is gradient–intercept form, and so its y-intercept is $-5\frac{2}{3}$.

The Line through Two Given Points: Given two distinct points, there is just one line passing through them both. Its equation is best found by a two-step approach.

16 THE LINE THROUGH TWO GIVEN POINTS:
1. Find the gradient of the line.
2. Use point–gradient form to find the equation of the line.

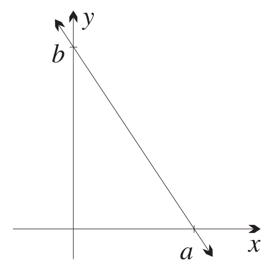
WORKED EXERCISE: Find the equation of the line through $A(1, 5)$ and $B(4, -1)$.

SOLUTION: First, gradient $AB = \dfrac{-1-5}{4-1}$
$= -2.$

Then, using point–gradient form for a line with gradient -2 through $A(1, 5)$, the line AB is

$$\begin{aligned} y - y_1 &= -2(x - x_1) \\ y - 5 &= -2(x - 1) \\ y &= -2x + 7. \end{aligned}$$

Two-Intercept Form: The problem here is to find the equation of a line ℓ whose x-intercept is a and whose y-intercept is b. This time, the result is very obvious once it is written down:



17 TWO-INTERCEPT FORM: $\dfrac{x}{a} + \dfrac{y}{b} = 1$

PROOF: The line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ passes through the two points $(a, 0)$ and $(0, b)$, because both points satisfy the equation. Notice that if the line passes through the origin, then $a = b = 0$ and the equation fails. It also fails if the line is vertical (when it has no y-intercept), or horizontal (when it has no x-intercept).

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WORKED EXERCISE: Given $A(6,0)$ and $B(0,9)$, find in general form the equations of:

(a) the line AB,

(b) the line ℓ_1 perpendicular to AB through A,

(c) the line ℓ_2 perpendicular to AB through B.

SOLUTION:

(a) Using the two-intercept form,

$$AB \text{ is } \quad \frac{x}{6} + \frac{y}{9} = 1$$

$$\boxed{\times 18} \quad 3x + 2y - 18 = 0.$$

(b) Since AB has gradient $-\frac{3}{2}$,
the gradient perpendicular to AB is $\frac{2}{3}$,
so using point–gradient form,

$$\ell_1 \text{ is } \quad y - 0 = \tfrac{2}{3}(x - 6)$$

$$2x - 3y - 12 = 0.$$

(c) Using gradient–intercept form,

$$\ell_2 \text{ is } \quad y = \tfrac{2}{3}x + 9$$

$$2x - 3y + 27 = 0.$$

Intersection of Lines — Concurrent Lines: The point where two distinct lines intersect can be found using simultaneous equations, as discussed in Chapter One.

Three distinct lines are called *concurrent* if they all pass through the same point. To test whether three given lines are concurrent, the most straightforward method is to find the point where two of them intersect, then test by substitution whether this point lies on the third line.

WORKED EXERCISE: Test whether the three lines ℓ_1: $5x-y-10=0$, ℓ_2: $x+y-8=0$ and ℓ_3: $2x-3y+9=0$ are concurrent.

SOLUTION: First we solve ℓ_1 and ℓ_2 simultaneously.

Adding ℓ_1 and ℓ_2, $\quad 6x - 18 = 0$

$$x = 3$$

and substituting into ℓ_2, $\quad 3 + y - 8 = 0$

$$y = 5$$

so ℓ_1 and ℓ_2 intersect at $(3,5)$.

Then substituting $(3,5)$ into the third line ℓ_3,

$$\begin{aligned} \text{LHS} &= 6 - 15 + 9 \\ &= 0 \\ &= \text{RHS}, \end{aligned}$$

so the three lines are concurrent, meeting at $(3,5)$.

Some Consequences: It may be useful to know some of the obvious consequences of all these formulae. Here are a few remarks, chosen from many more that could usefully be made. Most problems, however, are best done without quoting these consequences, which can easily be derived on the spot if needed.

1. The gradient of the line $ax + by + c = 0$ is $-\dfrac{a}{b}$, its y-intercept is $-\dfrac{c}{b}$, and its x-intercept is $-\dfrac{c}{a}$.
2. Any line parallel to $ax + by + c = 0$ has the form $ax + by + k = 0$, for some constant k determined by the particular circumstances.
3. Any line perpendicular to $ax + by + c = 0$ has the form $bx - ay + k = 0$, for some constant k determined by the particular circumstances.

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4. Two lines $ax + by + c = 0$ and $Ax + By + C = 0$:
   (a) intersect in a single point if their gradients $-a/b$ and $-A/B$ are unequal,
   (b) are parallel if their gradients $-a/b$ and $-A/B$ are equal, but their y-intercepts $-c/b$ and $-C/B$ are unequal,
   (c) are the same line if their gradients $-a/b$ and $-A/B$ are equal, and their y-intercepts $-c/b$ and $-C/B$ are equal.
5. Two lines $ax + by + c = 0$ and $Ax + By + C = 0$ are perpendicular if and only if the products of their gradients $-a/b$ and $-A/B$ is -1.

Exercise **5D**

NOTE: Selection of questions from this long exercise will depend on students' previous knowledge.

1. Use point–gradient form $y - y_1 = m(x - x_1)$ to find in general form the equation of the line:
(a) through $(1, 1)$ with gradient 2,
(b) with gradient -1 through $(3, 1)$,
(c) with gradient 3 through $(-5, -7)$,
(d) through $(0, 0)$ with gradient -5,
(e) through $(-1, 3)$ with gradient $-\frac{1}{3}$,
(f) with gradient $-\frac{4}{5}$ through $(3, -4)$.

2. Find the gradient of the line through each pair of given points, and hence find its equation:
(a) $(3, 4)$, $(5, 8)$ (b) $(-1, 3)$, $(1, -1)$ (c) $(-4, -1)$, $(6, -6)$ (d) $(5, 6)$, $(-1, 4)$

3. Write down the equation of the line with the given intercepts, then rewrite it in general form:
(a) $(-1, 0)$, $(0, 2)$ (b) $(2, 0)$, $(0, 3)$ (c) $(0, -1)$, $(-4, 0)$ (d) $(0, -3)$, $(3, 0)$

4. (a) Find the point M of intersection of the lines $\ell_1\colon x + y = 2$ and $\ell_2\colon 4x - y = 13$.
(b) Show that M lies on $\ell_3\colon 2x - 5y = 11$, and hence that ℓ_1, ℓ_2 and ℓ_3 are concurrent.
(c) Use the same method to test whether each set of lines is concurrent:
   (i) $2x + y = -1$, $x - 2y = -18$ and $x + 3y = 15$
   (ii) $6x - y = 26$, $5x - 4y = 9$ and $x + y = 9$

5. Find the gradient of each line below and hence find, in gradient–intercept form, the equation of a line: (i) parallel to it passing through $A(3, -1)$, (ii) perpendicular to it passing through $B(-2, 5)$.
(a) $2x + y + 3 = 0$ (b) $5x - 2y - 1 = 0$ (c) $4x + 3y - 5 = 0$

6. Given the points $A(1, -2)$ and $B(-3, 4)$, find in general form the equation of:
(a) the line AB, (b) the line through A perpendicular to AB.

DEVELOPMENT

7. The angle of inclination α and a point A on a line are given below. Use the formula gradient $= \tan\alpha$ to find the gradient of each line, then find its equation in general form:
(a) $\alpha = 45°$, $A = (1, 0)$
(b) $\alpha = 120°$, $A = (-1, 0)$
(c) $\alpha = 30°$, $A = (4, -3)$
(d) $\alpha = 150°$, $A = (-2, -5)$

8. Explain why the four lines $\ell_1\colon y = x + 1$, $\ell_2\colon y = x - 3$, $\ell_3\colon y = 3x + 5$ and $\ell_4\colon y = 3x - 5$ enclose a parallelogram. Then find the vertices of this parallelogram.

9. Triangle ABC has vertices $A(1, 0)$, $B(6, 5)$ and $C(0, 2)$. Show that it is right-angled, then find the equation of each side.

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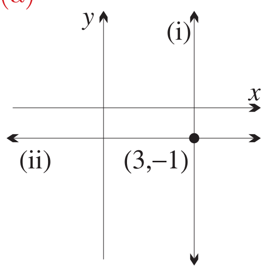
10. Determine in general form the equation of each straight line sketched:

(a)


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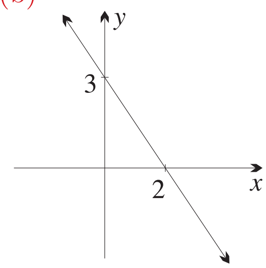

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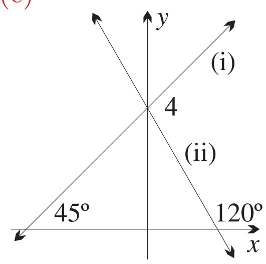

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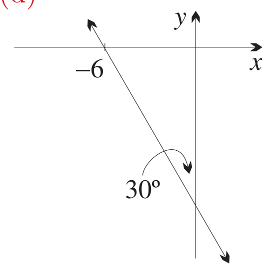
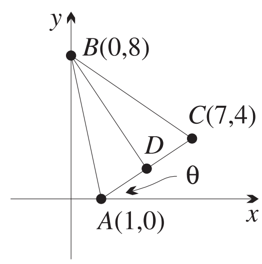


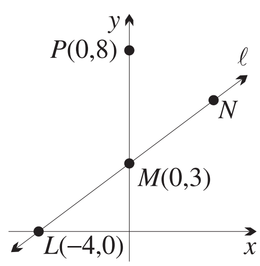
11. The points A, B and C have coordinates $(1,0)$, $(0,8)$ and $(7,4)$, and the angle between AC and the x-axis is θ.



(a) Find the gradient of the line AC and hence determine θ to the nearest degree. (b) Derive the equation of AC.
(c) Find the coordinates of D, the midpoint of AC.
(d) Show that AC is perpendicular to BD.
(e) What type of triangle is ABC?
(f) Find the area of this triangle. (g) Write down the coordinates of a point E such that $ABCE$ is a rhombus.

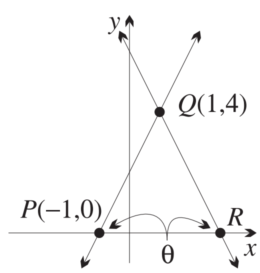
12. (a) On a number plane, plot the points $A(4,3)$, $B(0,-3)$ and $C(4,0)$.
(b) Find the equation of BC. (c) Explain why $OABC$ is a parallelogram.
(d) Find the area of $OABC$ and the length of the diagonal AB.

13. The line ℓ has intercepts at $L(-4,0)$ and $M(0,3)$. N is a point on ℓ and P has coordinates $(0,8)$.



(a) Copy the sketch. (b) Find the equation of ℓ.
(c) Find the lengths of ML and MP and hence show that LMP is an isosceles triangle.
(d) If M is the midpoint of LN, find the coordinates of N.
(e) Show that $\angle NPL = 90°$.
(f) Write down the equation of the circle through N, P and L.

14. The vertices of the triangle are $P(-1,0)$ and $Q(1,4)$ and R, where R lies on the x-axis and $\angle QPR = \angle QRP = \theta$.



(a) Find the coordinates of the midpoint of PQ.
(b) Find the gradient of PQ and show that $\tan\theta = 2$.
(c) Show that PQ has equation $y = 2x + 2$.
(d) Explain why QR has gradient -2, and hence find its equation.
(e) Find the coordinates of R and hence the area of triangle PQR.
(f) Find the length QR, and hence find the perpendicular distance from P to QR.

15. Find k if the lines ℓ_1: $x + 3y + 13 = 0$, ℓ_2: $4x + y - 3 = 0$ and ℓ_3: $kx - y - 10 = 0$ are concurrent. [HINT: Find the point of intersection of ℓ_1 and ℓ_2 and substitute into ℓ_3.]

16. Consider the two lines ℓ_1: $3x + 2y + 4 = 0$ and ℓ_2: $6x + \mu y + \lambda = 0$.
(a) Write down the value of μ if: (i) ℓ_1 is parallel to ℓ_2, (ii) ℓ_1 is perpendicular to ℓ_2.
(b) Given that ℓ_1 and ℓ_2 intersect at a point, what condition must be placed on μ?
(c) Given that ℓ_1 is parallel to ℓ_2, write down the value of λ if: (i) ℓ_1 is the same line as ℓ_2, (ii) the distance between the y-intercepts of the two lines is 2.

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17. (a) Write down, in general form, the equation of a line parallel to $2x - 3y + 1 = 0$.
(b) Hence find the equation of the line if it passes through: (i) $(2, 2)$ (ii) $(3, -1)$

18. (a) Write down, in general form, the equation of a line perpendicular to $3x + 4y - 3 = 0$.
(b) Hence find the equation of the line if it passes through: (i) $(-1, -4)$ (ii) $(-2, 1)$

19. (a) What is the natural domain of the relation $\dfrac{y}{x-1} = 3$? (b) Graph this relation and indicate how it differs from the graph of the straight line $y = 3x - 3$.

20. Explain how $x + y = 1$ may be transformed into $\frac{x}{a} + \frac{y}{b} = 1$ by stretching.

21. Find the point of intersection of $px + qy = 1$ and $qx + py = 1$, and explain why these lines intersect on the line $y = x$.

22. Determine the equation of the line through $M(4, 3)$ if M is the midpoint of the intercepts on the x-axis and y-axis. [HINT: This time, let the gradient of the line be m, and begin by writing down the equation of the line in point–gradient form.]

23. Write down the equation of the line through $M(-1, 2)$ with gradient m. Hence determine the equation of the line through M if:
(a) M is the midpoint of the intercepts on the x-axis and y-axis,
(b) M divides the intercepts in the ratio $2 : 1$ (x-intercept to y-intercept),
(c) M divides the intercepts in the ratio $-2 : 5$ (x-intercept to y-intercept).

24. Two distinct linear functions $f(x)$ and $g(x)$ have zeroes at $x = a$ and have gradients ℓ and m respectively. Show that $f(x) : g(x) = \ell : m$, for $x \neq a$.

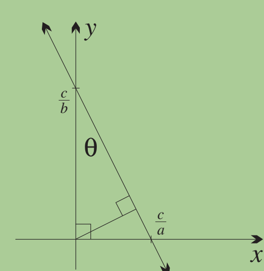
EXTENSION

25. The line passing through $M(a, b)$ intersects the x-axis at A and the y-axis at B. Find the equation of the line if: (a) M bisects AB, (b) M divides AB in the ratio $2 : 1$, (c) M divides AB in the ratio $k : \ell$.

26. The tangent to a circle is perpendicular to the radius at the point of contact. Use this fact to show that the tangent to $x^2 + y^2 = r^2$ at the point (a, b) has equation $ax + by = r^2$.

27. Show that the parametric equations $x = t\cos\alpha + a$ and $y = t\sin\alpha + b$ represent a straight line through (a, b) with gradient $m = \tan\alpha$.

28. [Perpendicular form of a line] Consider the line ℓ with equation $ax + by = c$ where, for the sake of convenience, the values of a, b and c are positive. Suppose that this line makes an acute angle θ with the y-axis as shown.



(a) Show that $\cos\theta = \dfrac{a}{\sqrt{a^2 + b^2}}$ and $\sin\theta = \dfrac{b}{\sqrt{a^2 + b^2}}$.
(b) The *perpendicular form* of the line ℓ is

$$\frac{a}{\sqrt{a^2 + b^2}}\,x + \frac{b}{\sqrt{a^2 + b^2}}\,y = \frac{c}{\sqrt{a^2 + b^2}}\,.$$

Use part (a) to help show that the RHS of this equation is the perpendicular distance from the line to the origin.
(c) Write these lines in perpendicular form and hence find their perpendicular distances from the origin: (i) $3x + 4y = 5$ (ii) $3x - 2y = 1$

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5 E Perpendicular Distance

If P is a point and ℓ is a line not passing through P, then the shortest distance from P to ℓ is the perpendicular distance. It is useful to have a formula for this perpendicular distance, rather than having to find the equation of the perpendicular line and the coordinates of the closest point. This formula, and the method developed in the next section, both make use of the general form of the equation of a line.

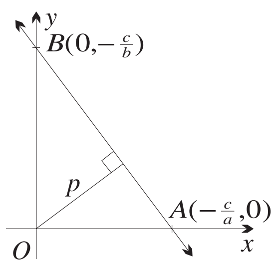
A Formula for the Distance from the Origin: The first step is to develop a formula for the perpendicular distance p of a given line ℓ: $ax + by + c = 0$ from the origin. This can be done very quickly by looking at the triangle OAB formed by the line ℓ and the two axes. We find two different expressions for the area of this triangle OAB and equate them.

The line $ax + by + c = 0$ has x-intercept $-c/a$ and y-intercept $-c/b$,

so OA has length $\left|\dfrac{c}{a}\right|$, and OB has length $\left|\dfrac{c}{b}\right|$.

Using OA as the base, area of $\triangle OAB = \frac{1}{2} \times \left|\dfrac{c}{a}\right| \times \left|\dfrac{c}{b}\right|$

$$= \tfrac{1}{2}\left|\frac{c^2}{ab}\right|.$$



The length of the hypotenuse AB is $AB^2 = \dfrac{c^2}{a^2} + \dfrac{c^2}{b^2}$

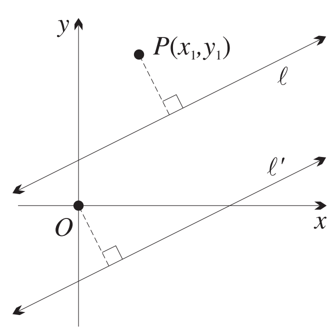
$$= \frac{c^2(a^2+b^2)}{a^2b^2},$$

$$AB = \left|\frac{c}{ab}\right|\sqrt{a^2+b^2},$$

so, using AB as the base, area of $\triangle OAB = \frac{1}{2}p\left|\dfrac{c}{ab}\right|\sqrt{a^2+b^2}$.

Equating the two expressions for area, $p = \dfrac{|c|}{\sqrt{a^2+b^2}}$.

The Perpendicular Distance Formula: Now we can use the shifting procedures of Chapter Two to generalise the result above to generate a formula for the perpendicular distance p of the line ℓ: $ax + by + c = 0$ from any given point $P(x_1, y_1)$. The perpendicular distance remains the same if we shift both line and point x_1 units to the left and y_1 units down. This shift moves the point P to the origin, and moves the line ℓ to the new line ℓ' with equation



$$a(x + x_1) + b(y + y_1) + c = 0$$
$$ax + by + (ax_1 + by_1 + c) = 0.$$

Then, using the formula previously established for the distance from the origin, we obtain:

18 PERPENDICULAR DISTANCE FORMULA: $p = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

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WORKED EXERCISE: Find the perpendicular distance of $P(-2, 5)$ from $y = 2x - 1$.

SOLUTION: The line in general form is $2x - y - 1 = 0$,

$$\begin{aligned} \text{so distance} &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} \\ &= \frac{|2 \times (-2) - 1 \times 5 - 1|}{\sqrt{2^2 + (-1)^2}} \\ &= \frac{|-10|}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \\ &= \frac{10\sqrt{5}}{5} \\ &= 2\sqrt{5}\,. \end{aligned}$$

Circles and the Perpendicular Distance Formula: A line is a tangent to a circle when its perpendicular distance from the centre is equal to the radius. Lines closer to the centre are secants, and lines more distant miss the circle entirely.

WORKED EXERCISE: Solve these using the perpendicular distance formula.

(a) Show that ℓ: $3x+4y-20 = 0$ is a tangent to the circle $\mathcal{C}$: $(x-7)^2+(y-6)^2 = 25$.

(b) Find the length of the chord of $\mathcal{C}$ cut off by the line m: $3x + 4y - 60 = 0$.

SOLUTION: The circle has centre $(7, 6)$ and radius 5.

(a) The distance p_ℓ from the line ℓ to the centre is

$$\begin{aligned} p_\ell &= \frac{|21 + 24 - 20|}{\sqrt{3^2+4^2}} \\ &= \frac{25}{5} \\ &= 5, \end{aligned}$$

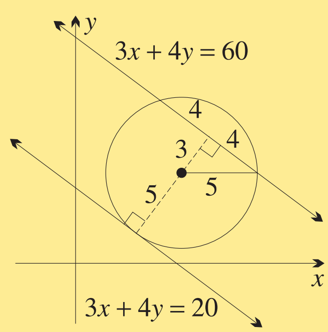
so ℓ is a tangent to the circle.

(b) The distance p_m from the line m to the centre is

$$\begin{aligned} p_m &= \frac{|21 + 24 - 60|}{\sqrt{3^2+4^2}} \\ &= \frac{|-15|}{5} \\ &= 3. \end{aligned}$$

Using Pythagoras' theorem in the circle on the right,

$$\begin{aligned} \text{chord length} &= 2 \times 4 \\ &= 8 \text{ units.} \end{aligned}$$



WORKED EXERCISE: For what values of k will the line $5x - 12y + k = 0$ never intersect the circle with centre $P(-3, 1)$ and radius 6?

SOLUTION: The condition is

$$\begin{aligned} \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} &> 6 \\ \frac{|-15 - 12 + k|}{\sqrt{5^2+12^2}} &> 6 \\ \frac{|-27 + k|}{13} &> 6 \\ |k - 27| &> 78 \\ k < -51 \;\text{ or }\; k &> 105. \end{aligned}$$

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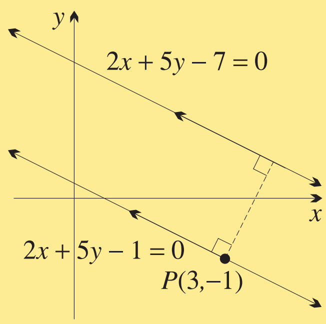
Distance between Parallel Lines: The distance between two parallel lines can be found by choosing any point on one line and finding its perpendicular distance from the second line.

WORKED EXERCISE: Find the perpendicular distance between the two parallel lines $2x + 5y - 1 = 0$ and $2x + 5y - 7 = 0$.

SOLUTION: Choose any point on the first line, say $P(3, -1)$.
The distance between the lines is the perpendicular distance from P to the second line:

$$\begin{aligned}\text{distance} &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}\\ &= \frac{|6 - 5 - 7|}{\sqrt{2^2 + 5^2}}\\ &= \frac{|-6|}{\sqrt{4 + 25}}\\ &= \frac{6}{\sqrt{29}}.\end{aligned}$$



Exercise 5E

1. Find the perpendicular distance from each line to the origin:
(a) $x + 3y + 5 = 0$ (b) $2x - y + 4 = 0$ (c) $2x + 4y - 5 = 0$

2. Find the perpendicular distance between each point and line:
(a) $(2, 0)$ and $3x + 4y - 1 = 0$
(b) $(-2, 1)$ and $12x - 5y + 3 = 0$
(c) $(-3, 2)$ and $4x - y - 3 = 0$
(d) $(-3, -2)$ and $x + 3y + 4 = 0$
(e) $(3, -1)$ and $x + 2y - 1 = 0$
(f) $(1, 3)$ and $2x + 4y + 1 = 0$

3. Which of the given points is: (a) closest to, (b) furthest from, the line $6x - 8y - 9 = 0$?
$A(1, -1)$ $B(3, 2)$ $C(-4, 1)$ $D(-3, -3)$

4. Which of the given lines is: (a) closest to, (b) furthest from, the point $(-1, 5)$?
ℓ_1: $2x + 3y + 4 = 0$ ℓ_2: $x - 4y + 7 = 0$ ℓ_3: $3x + y - 8 = 0$

DEVELOPMENT

5. (a) The line $y - 2x + \mu = 0$ is $2\sqrt{5}$ units from the point $(1, -3)$. Find the possible values of μ.
(b) The line $3x - 4y + 2 = 0$ is $\frac{3}{5}$ units from the point $(-1, \lambda)$. Find the possible values of λ.

6. (a) The line $y - x + h = 0$ is more than $\frac{1}{\sqrt{2}}$ units from the point $(2, 7)$. What range of values may h take?
(b) The line $x + 2y - 5 = 0$ is at most $\sqrt{5}$ units from the point $(k, 3)$. What range of values may k take?

7. Use the perpendicular distance formula to determine how many times each line intersects the given circle:
(a) $3x - 5y + 16 = 0$, $x^2 + y^2 = 5$
(b) $7x + y - 10 = 0$, $x^2 + y^2 = 2$
(c) $3x - y - 8 = 0$, $(x - 1)^2 + (y - 5)^2 = 10$
(d) $x + 2y + 3 = 0$, $(x + 2)^2 + (y - 1)^2 = 6$

8. Use a point on the first line to find the distance between each pair of parallel lines:
(a) $x - 3y + 5 = 0$, $x - 3y - 2 = 0$ (b) $4x + y - 2 = 0$, $4x + y + 8 = 0$

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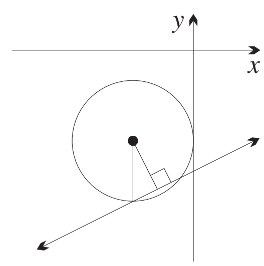
9. The vertices of a triangle are $A(-3,-2)$, $B(3,1)$ and $C(-1,4)$.
(a) Find the equation of the side AB in general form. (b) How far is C from this line?
(c) Find the length of AB and hence find the area of this triangle.
(d) Similarly find the area of the triangle with vertices $P(1,-1)$, $Q(-1,5)$ and $R(-3,1)$.

10. Draw on a number plane the triangle ABC with vertices $A(5,0)$, $B(8,4)$ and $C(0,10)$.
(a) Show that the line AB has equation $3y = 4x - 20$.
(b) Show that the gradient of BC is $-\frac{3}{4}$.
(c) Hence show that AB and BC are perpendicular.
(d) Show that AB is 5 units.
(e) Show that triangles AOC and ABC are congruent.
(f) Find the area of quadrilateral $OABC$.
(g) Find the distance from the point $D(8,0)$ to the line AB.



11. (a) Write down the centre and radius of the circle with equation $(x+2)^2+(y+3)^2 = 4$. Then find the distance from the line $2y - x + 8 = 0$ to the centre.
(b) Hence determine the length of the chord cut off from the line by the circle.

12. Choose two points in the first quadrant on the line $3x-5y+4 = 0$ and find their distances from the line $4x - 5y - 3 = 0$. What can be concluded about the two lines?

13. The point $P(x,y)$ is equidistant from the lines $2x + y - 3 = 0$ and $x - 2y + 1 = 0$, which intersect at A.
(a) Use the distance formula to show that $|2x + y - 3| = |x - 2y + 1|$.
(b) Hence find the equations of the lines that bisect the angles at A.

14. (a) Write down the equation of a line through the origin with gradient m.
(b) Write down the distance from this line to the point $(3,1)$.
(c) If the line is tangent to the circle $(x - 3)^2 + (y - 1)^2 = 4$, show that m satisfies the equation $5m^2 - 6m - 3 = 0$.
(d) Find the possible values of m and hence the equations of the two tangents.

EXTENSION

15. Use the perpendicular distance formula to prove that the distance between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.

16. (a) Find the vertex of $y = x^2 - 2x + 3$, and show that the minimum value of y is 2. (b) The point $Q(q,q^2)$ lies on the parabola $\mathcal{P}$: $y = x^2$. Write down the distance from Q to the line ℓ: $2x - y - 3 = 0$. (c) Hence find the minimum distance from the line ℓ to the parabola $\mathcal{P}$.

17. (a) If the centre of the circle $(x - 4)^2 + (y - 1)^2 = 25$ is moved 3 right and 2 down, what is the equation of the new circle? (b) Write down the distance from the centre of the second circle to the line $y = mx$. (c) Find the values of m if $y = mx$ is tangent to this circle. (d) Hence find the equations of the two tangents from the point $(-3,2)$ to the circle $(x - 4)^2 + (y - 1)^2 = 25$.

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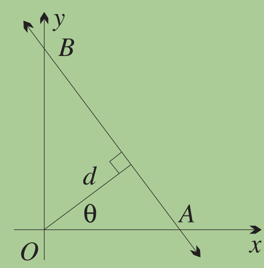
18. [Perpendicular distance — a derivation by trigonometry] Consider the straight line $ax + by = c$ where, for the sake of convenience, the values a, b and c are positive. The perpendicular from the origin to the line has length d and makes and angle θ with the positive x-axis.



(a) Show that $\angle OBA = \theta$.

(b) Show that $\cos\theta = \dfrac{ad}{c}$ and $\sin\theta = \dfrac{bd}{c}$.

(c) Use the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ in order to show that $d = \dfrac{c}{\sqrt{a^2 + b^2}}$.

19. [Perpendicular distance — a quadratic derivation] Consider the circle $x^2 + y^2 = d^2$ and the line $px + qy + r = 0$.

(a) Show that the coordinates of any point of intersection of the line and the circle must satisfy the quadratic equation $(p^2 + q^2)x^2 + 2prx + r^2 - q^2d^2 = 0$.

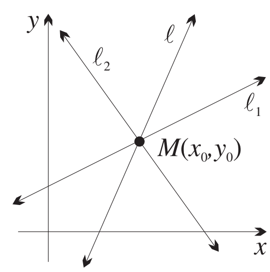
(b) If the line is tangent to the circle, then this equation has only one solution and in the quadratic formula $b^2 - 4ac = 0$. Use this result to find the distance to the origin.

5 F Lines Through the Intersection of Two Given Lines

This section develops an ingenious way of finding the equations of particular lines through the intersection of two given lines without actually solving the two given lines simultaneously to find their point of intersection. The method is another situation where the general form of the equation of the line is used.

The General Form of Such a Line: When two lines intersect, the set of all the lines through the point M of intersection forms a family of lines through M. The first task is to write the equations of all the lines in the family in the one form.

Suppose that ℓ_1: $a_1x + b_1y + c_1 = 0$ and ℓ_2: $a_2x + b_2y + c_2 = 0$ are two lines intersecting at a point $M(x_0, y_0)$. Let ℓ be any line of the form



$$\ell\colon (a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0, \qquad (*)$$

where k is a constant. We are going to prove that ℓ also passes through M.

First, ℓ_1 passes through M, and so, substituting the coordinates of M into the equation of ℓ_1,

$$a_1x_0 + b_1y_0 + c_1 = 0. \qquad (1)$$

Similarly, ℓ_2 passes through M, and so

$$a_2x_0 + b_2y_0 + c_2 = 0. \qquad (2)$$

To prove that ℓ passes through M, substitute $M(x_0, y_0)$ into $(*)$:

$$\begin{aligned} \text{LHS} &= (a_1x_0 + b_1y_0 + c_1) + k(a_2x_0 + b_2y_0 + c_2) \\ &= 0 + 0, \quad \text{by (1) and (2)} \\ &= \text{RHS}, \end{aligned}$$

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as required. Hence the equation of every line through the intersection of ℓ_1 and ℓ_2 has the following form:

19 LINE THROUGH THE INTERSECTION OF TWO GIVEN LINES:

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0, \text{ where } k \text{ is a constant.}$$

NOTE: Careful readers will realise that there is one line through M which is not included in this standard form, namely the second line ℓ_2. When this case occurs, it can be recognised because k becomes infinite. To overcome this problem, it is not good enough to put the k in front of the first expression instead, because then the line ℓ_1 would not be included. The way through is a little more complicated; one must use a *homogeneous form* such as

$$h(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0,$$

in which every line through M corresponds to a unique ratio of h and k or, when the line is ℓ_1 or ℓ_2, to one of k and h becoming zero.

This same situation arose with gradient. When we replaced the two parameters rise and run by their ratio, a single parameter, the case of vertical gradient was excluded, because when the run is zero the ratio is undefined. There too it would be more correct, though also more complicated, to define gradient as the ordered pair $(\text{rise}, \text{run})$, with the qualification that two ordered pairs like $(3, 1)$ and $(6, 2)$ in which one pair is a multiple of another represent the same gradient. This is the easiest way to generalise gradient to three or more dimensions.

WORKED EXERCISE: Write down the equation of a line through the intersection M of the lines

$$\ell_1\colon\ x + 2y - 6 = 0 \qquad \text{and} \qquad \ell_2\colon 3x - 2y - 6 = 0.$$

Hence, without finding the point of intersection, find the line through M that:

(a) passes through $P(2, -1)$,

(b) has gradient 5,

(c) is horizontal,

(d) is vertical.

SOLUTION: The general form of a line through M is

$$(x + 2y - 6) + k(3x - 2y - 6) = 0, \quad \text{for some constant } k. \qquad (1)$$

(a) Substituting $P(2, -1)$ into (1),

$$\begin{aligned} (2 - 2 - 6) + k(6 + 2 - 6) &= 0 \\ 2k &= 6 \\ k &= 3, \end{aligned}$$

and substituting into (1), the required line is

$$\begin{aligned} (x + 2y - 6) + 3(3x - 2y - 6) &= 0 \\ 10x - 4y - 24 &= 0 \\ 5x - 2y - 12 &= 0. \end{aligned}$$

(b) Rearranging (1) gives $(1 + 3k)x + (2 - 2k)y + (-6 - 6k) = 0$, (2)

which has gradient $\dfrac{1 + 3k}{2k - 2}$. Hence

$$\begin{aligned} \frac{1 + 3k}{2k - 2} &= 5 \\ 1 + 3k &= 10k - 10 \\ k &= \tfrac{11}{7}, \end{aligned}$$

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so substituting into (1), the required line is

$$\begin{aligned}(x+2y-6)+\tfrac{11}{7}(3x-2y-6)&=0\\ 7(x+2y-6)+11(3x-2y-6)&=0\\ 40x-8y-108&=0\\ 10x-2y-27&=0.\end{aligned}$$

(c) The gradient is zero, so the coefficient of x in (2) is zero, $1+3k=0$

$$k=-\tfrac{1}{3}.$$

Substituting into (1), the required line is

$$\begin{aligned}(x+2y-6)-\tfrac{1}{3}(3x-2y-6)&=0\\ 3(x+2y-6)-(3x-2y-6)&=0\\ 8y-12&=0\\ y&=1\tfrac{1}{2}.\end{aligned}$$

(d) The line is vertical, so the coefficient of y in (2) is zero, $2-2k=0$

$$k=1.$$

Substituting into (1), the required line is

$$\begin{aligned}(x+2y-6)+(3x-2y-6)&=0\\ 4x-12&=0\\ x&=3.\end{aligned}$$

NOTE: Parts (c) and (d) together actually do tell us that the two given lines intersect at $(3, 1\frac{1}{2})$.

Exercise 5F

1. (a) Graph the lines $x-y=0$ and $x+y-2=0$ on grid or graph paper and label them (they intersect at $(1,1)$).

(b) Simplify the equation $(x-y)+k(x+y-2)=0$ for $k=2$, 1, $\frac{1}{2}$ and 0. Add these lines to your graph, and label each line with its value of k. Observe that each line passes through $(1,1)$.

(c) Repeat this process for $k=-\frac{1}{2}$, -1 and -2, adding these lines to your graph.

2. The lines $x+2y+9=0$ and $2x-y+3=0$ intersect at B.

(a) Write down the general equation of a line through B.

(b) Hence find the equation of the line ℓ through B and the origin O.

3. Find the equation of the line through the intersection of the lines $x-y-3=0$ and $y+3x-5=0$ and the given point, without finding the point of intersection of the lines:

(a) $(0,-2)$ (b) $(-1,5)$ (c) $(3,0)$

4. The lines $2x+y-5=0$ and $x-y+2=0$ intersect at A.

(a) Write down the general equation of a line through A, and show that it can be written in the form $x(2+k)+y(1-k)+(2k-5)=0$.

(b) Find the value of k that makes the coefficient of x zero, and hence find the equation of the horizontal line through A.

(c) Find the value of k that makes the coefficient of y zero, and hence find the equation of the vertical line through A.

(d) Hence write down the coordinates of A.

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5. (a) Find the point P of intersection of $x + y - 2 = 0$ and $2x - y - 1 = 0$.
(b) Show that P satisfies the equation $x + y - 2 + k(2x - y - 1) = 0$.
(c) Find the equation of the line through P and $Q(-2, 2)$:
(i) using the coordinates of both P and Q,
(ii) without using the coordinates of P. Your answers should be the same.

DEVELOPMENT

6. (a) Write down the general form of a line through the point T of intersection of the two lines $2x - 3y + 6 = 0$ and $x + 3y - 15 = 0$.
(b) Hence find the equation of the line through T and:
(i) $(3, 8)$ (ii) $(6, 0)$ (iii) $(-3, 3)$ (iv) $(0, 0)$

7. (a) The general form of a line through the intersection M of $x-2y+5 = 0$ and $x+y+2 = 0$ is ℓ: $(x - 2y + 5) + k(x + y + 2) = 0$. Show that the gradient of ℓ is $\dfrac{1+k}{2-k}$.
(b) Hence find the equation of the lines through M:
(i) parallel to $3x + 4y = 5$,
(ii) perpendicular to $2x - 3y = 6$,
(iii) perpendicular to $5y - 2x = 4$,
(iv) parallel to $x - y - 7 = 0$.

8. (a) Show that every line of the family $(2x-y-7)+k(x+y-5) = 0$ passes through a fixed point by finding the coordinates of that point. [HINT: Use the method of question 4 to find the point.]
(b) Similarly, show that the lines of each family pass through a fixed point:
(i) $(x + 2y - 8) + k(x - y + 4) = 0$ (ii) $(3x + y + 2) + k(5x + 2y + 1) = 0$

9. Show that the three lines ℓ_1: $2x - 3y + 13 = 0$, ℓ_2: $x + y - 1 = 0$ and ℓ_3: $4x + 3y - 1 = 0$ are concurrent by the following method:
(a) Without finding any points of intersection, find the equation of the line through the intersection of ℓ_1 and ℓ_2 parallel to ℓ_3. (b) Show that this line is the same line as ℓ_3.

10. (a) Use the perpendicular distance formula to show that $A(-2, 3)$ is equidistant from the two lines $x - 3y + 1 = 0$ and $3x + y - 7 = 0$.
(b) Hence find the equation of the line through A that bisects the angle between the two lines, without finding their point of intersection.

11. (a) It is known that ℓ: $x + 2y + 10 = 0$ is tangent to the circle $\mathcal{C}$: $x^2 + y^2 = 20$ at T. Write down the equation of the radius of the circle at T (it will be perpendicular to ℓ).
(b) Use part (a) to find the equation of the line through $S(1, -3)$ and the point of contact without actually finding the point of contact.

12. The lines $3x - y + 2 = 0$ and $x - 4y - 3 = 0$ form two sides PQ and QR of parallelogram $PQRS$. Given that S has coordinates $(4, 3)$, find the equations of the following lines without finding the coordinates of any other point: (a) RS (b) PS (c) QS (d) PR. [HINT: PR is the line through the intersection of PQ and PS that is parallel to the line through the intersection of QR and RS and has the same y-intercept.]

EXTENSION

13. (a) Show that every circle that passes through the intersections of the circle $x^2 + y^2 = 2$ and the straight line $y = x$ can be written in the form $(x - \mu)^2 + (y + \mu)^2 = 2(1 + \mu^2)$.
(b) Hence find the equation of such a circle which also passes through $(4, -1)$.

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14. [A form of the circle of Apollonius] Consider the expressions $C_1 = (x-2)^2 + y^2 - 5$ and $C_2 = (x+2)^2 + y^2 - 5$. (a) Show that the equations $C_1 = 0$ and $C_2 = 0$ represent circles of radius $\sqrt{5}$ which intersect at 1 and -1 on the y-axis. (b) Show that the equation $C_1 + \lambda C_2 = 0$ can be rewritten as $x^2 + 4x\dfrac{\lambda - 1}{\lambda + 1} + y^2 = 1$, provided $\lambda \neq -1$.
(c) Draw graphs of the circles $C_1 + \lambda C_2 = 0$ for $\lambda = 0, \frac{1}{3}, 1, 3$ and note the y-intercepts.
(d) Investigate the equation $C_1 + \lambda C_2 = 0$ for negative values of λ. [HINT: Begin with the values $-\frac{1}{3}, -1, -3$.] (e) Find the equation of the circle through $(0,1)$, $(0,-1)$ and $(5,1)$.

15. (a) Show that the parabola through the intersection of the parabolas $y = x^2 - 1$ and $y = 1 - x^2$ can be written in the form $y = \dfrac{1-k}{1+k}(x^2 - 1)$, provided $k \neq -1$.
(b) Hence find the parabola that also passes through $(2, 12)$.
(c) Show that there is no parabola that passes through the intersection of the parabolas $y = 2x - x^2$ and $y = x^2 - 4x$, and the point $(1, -1)$.

5 G Coordinate Methods in Geometry

When the French mathematician and philosopher René Descartes introduced coordinate geometry in the 17th century, he intended it to be a system in which all the theorems of Euclidean geometry could be proven in an alternative way. Here are two well-known theorems about centres of triangles proven by this alternative coordinate method. In the first proof, it is convenient to place the figure carefully with important points on the axes or at the origin so that the algebra is simplified. In the second proof, however, using general coordinates for all three vertices displays the symmetry of the situation. Many questions use the words *altitude* and *median*, which should be known.

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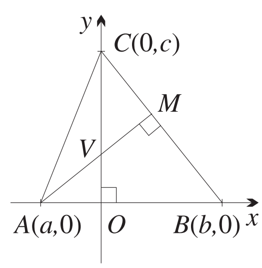
MEDIAN: A *median* of a triangle is the interval from a vertex of the triangle to the midpoint of the opposite side.
ALTITUDE: An *altitude* of a triangle is the perpendicular from a vertex of the triangle to the opposite side (produced if necessary).

Example — The Three Altitudes of a Triangle are Concurrent: This theorem asserts the concurrence of the three altitudes of a triangle, and is most easily proven by placing one side on the x-axis and the opposite vertex on the y-axis.

THEOREM: The three altitudes of a triangle are concurrent. (Their point of intersection is called the *orthocentre* of the triangle.)

PROOF: Let the side AB of the triangle ABC lie on the x-axis, and the vertex C lie on the positive side of the y-axis. Let the coordinates of the vertices be $A(a,0)$, $B(b,0)$ and $C(0,c)$, as in the diagram.



The altitude through C is the interval CO on the y-axis.
Let M be the foot of the altitude through A.
Since BC has gradient $-c/b$, AM has gradient b/c,

so the equation of AM is $y - 0 = \dfrac{b}{c}(x - a)$

$$y = \frac{bx}{c} - \frac{ab}{c}$$

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and substituting $x = 0$, the altitude AM meets CO at $V(0, -ab/c)$.
Exchanging a and b, the third altitude BN through B similarly has equation

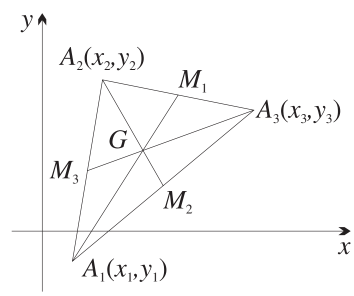
$$y = \frac{ax}{c} - \frac{ab}{c}$$

and again substituting $x = 0$, BN meets CO at the same point $V(0, -ab/c)$.
Hence the three altitudes are concurrent.

Example — The Three Medians of a Triangle are Concurrent: The three medians of any triangle are also concurrent. In contrast to the previous proof, the concurrence of the medians is most clearly proven by allowing the vertices to have general coordinates.

THEOREM: The medians of a triangle are concurrent, and their point of intersection (called the *centroid* of the triangle) divides each median in the ratio 2 : 1.

PROOF: Let the triangle $A_1A_2A_3$ have as coordinates $A_1(x_1, y_1)$, $A_2(x_2, y_2)$ and $A_3(x_3, y_3)$. Let M_1, M_2 and M_3 be the midpoints of A_2A_3, A_3A_1 and A_1A_2 respectively.



Let G be the point dividing A_1M_1 in the ratio 2 : 1.
By the midpoint formula, the coordinates of M_1 are

$$M_1 = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right),$$

and so, using the ratio division formula,

$$G = \left(\frac{1 \times x_1 + 2 \times \frac{1}{2}(x_2 + x_3)}{1 + 2}, \frac{1 \times y_1 + 2 \times \frac{1}{2}(y_2 + y_3)}{1 + 2}\right)$$

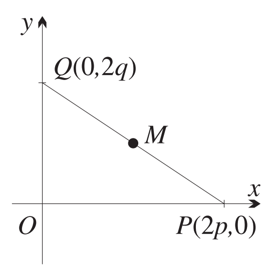
$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

Because this result is symmetric, G must divide A_2M_2 and A_3M_3 in the ratio 2 : 1.
So each median passes through G, which divides it in the ratio 2 : 1.

Exercise 5G

NOTE: Diagrams should be drawn wherever possible.

1. (a) The points O, $P(8, 0)$ and $Q(0, 10)$ form a right-angled triangle, and M is the midpoint of PQ. (i) Find the coordinates of M. (ii) Then find the distance OM, PM and QM, and show that M is equidistant from each of the vertices. (iii) Explain why a circle with centre M can be drawn through the three vertices O, P and Q



(b) It is true in general that the midpoint of the hypotenuse of a right triangle is the centre of a circle through all three vertices. Prove that this result is true for any right triangle by placing its vertices at $O(0, 0)$, $P(2p, 0)$ and $Q(0, 2q)$, and repeating the procedures of part (a).

2. (a) $PQRS$ is a quadrilateral with vertices on the axes at $P(1, 0)$, $Q(0, 2)$, $R(-3, 0)$ and $S(0, -4)$. Show that $PQ^2 + RS^2 = PS^2 + QR^2$.

(b) It is true in general that if the diagonals of a quadrilateral are perpendicular, then the two sums of squares of opposite sides are equal. Prove that this result is true for any quadrilateral by placing the vertices on the axes, giving them coordinates $P(p, 0)$, $Q(0, q)$, $R(-r, 0)$ and $S(0, -s)$, and proceeding as in part (a).

3. (a) A triangle has vertices at $A(1, -3)$, $B(3, 3)$ and $C(-3, 1)$.

(i) Find the coordinates of P and Q, the midpoints of AB and BC respectively.

(ii) Show that PQ is parallel to AC and that $PQ = \frac{1}{2}AC$.

(b) It is true in general that the line joining the midpoints of two sides of a triangle is parallel to the base and half its length. Prove that this is true for any triangle by placing its vertices at $A(2a, 0)$, $B(2b, 2c)$ and $C(0, 0)$, where $a > 0$, and proceeding as in part (a).

4. Triangle OBA has vertices at the origin O, $A(3, 0)$ and $B(0, 4)$. C is a point on AB such that OC is perpendicular to AB.

(a) Find the equations of AB and OC and hence find the coordinates of C.

(b) Find the lengths OA, AB, OC, BC and AC.

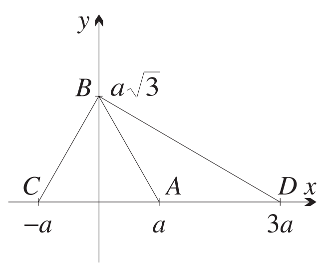
(c) Thus confirm these important corollaries for a right-angled triangle:

(i) $OC^2 = AC \times BC$ (ii) $OA^2 = AC \times AB$

5. $A(a, 0)$ and $Q(q, 0)$ are points on the positive x-axis, and $B(0, b)$ and $P(0, p)$ are points on the positive y-axis. Show that $AB^2 - AP^2 = QB^2 - QP^2$.

DEVELOPMENT

6. The diagram opposite shows the points A, B, C and D on the number plane.



(a) Show that $\triangle ABC$ is equilateral.

(b) Show that $\triangle ABD$ is isosceles, with $AB = AD$.

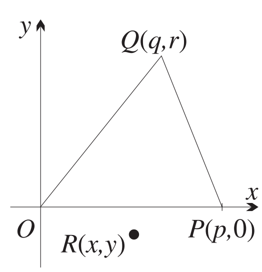
(c) Show that $AB^2 = \frac{1}{3}BD^2$.

7. (a) Suppose that D is the midpoint of AC in triangle ABC. Prove that $AB^2 + BC^2 = 2(CD^2 + BD^2)$. Begin by placing the vertices at $A(a, 0)$, $B(b, c)$ and $C(-a, 0)$. Then find the coordinates of D, then find the squares AB^2, BC^2, BD^2 and CD^2.

(b) The result proven in part (a) is *Apollonius' theorem*. Give a geometric statement of the result (use the word *median* for the interval joining any vertex to the midpoint of the opposite side).

8. Prove that the diagonals of a parallelogram bisect each other. Begin by showing that the quadrilateral with vertices $W(a, 0)$, $X(b, c)$, $Y(-a, 0)$ and $Z(-b, -c)$, where the constants a, b and c are all positive, is a parallelogram. Then show that the midpoints of both diagonals coincide.

9. [A condition for a point to lie on an altitude of a triangle] In the diagram, $\triangle OPQ$ has its vertices at the origin, $P(p, 0)$ and $Q(q, r)$. Another point $R(x, y)$ satisfies the condition $PQ^2 - OQ^2 = PR^2 - OR^2$. Substitute the coordinates of O, P, Q and R into this condition, and show that R lies on the altitude through Q.



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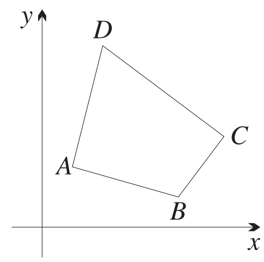
10. The points $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$ and $D(d_1, d_2)$ are the vertices of a quadrilateral such as the one in the diagram.

(a) Find the midpoints P, Q, R and S of AB, BC, CD and DA respectively.

(b) Show that the diagonals of $PQRS$ bisect each other, by showing that the midpoints of PR and QS coincide.

(c) What is the most general type of quadrilateral $PQRS$ can be?



11. The previous question proves the general result that the midpoints of any quadrilateral form a parallelogram. Prove this result in an alternative way by finding the gradients of the four sides of $PQRS$ and showing that opposite sides are parallel.

12. The points $A(1, -2)$, $B(5, 6)$ and $C(-3, 2)$ are the vertices of a triangle, and P, Q and R are the midpoints of BC, AC and AB respectively.

(a) Find the equations of the three medians BQ, CR and AP.

(b) Find the intersection of BQ and CR, and show that it lies on the third median AP.

13. [The three medians of a triangle are concurrent — an alternative proof] The previous question was a special case of the general result that the medians of a triangle are concurrent (their point of intersection is called the *centroid*). Prove that the result is true for any triangle by choosing as vertices $A(6a, 6b)$, $B(-6a, -6b)$ and $C(0, 6c)$, and following these steps:

(a) Find the midpoints P, Q and R of BC, CA and AB respectively. Show that the median through C is $x = 0$, and find the equations of the other two medians.

(b) Find where the median through C meets another median, and show that the point lies on the third median.

14. It is true in general that the perpendicular bisectors of the sides of a triangle are concurrent, and that the point of intersection (called the *circumcentre*) is the centre of a circle through all three vertices (called the *circumcircle*). Prove this result in general by placing the vertices at $A(2a, 0)$, $B(-2a, 0)$ and $C(2b, 2c)$, and proceeding as follows:

(a) Find the gradients of AB, BC and CA, and hence find the equations of the three perpendicular bisectors.

(b) Find the intersection M of any two perpendicular bisectors, and show that it lies on the third.

(c) Explain why the circumcentre must be equidistant from each vertex.

EXTENSION

15. Triangle ABC is right-angled at A. P is the midpoint of AB and Q is the midpoint of BC. Choose suitable coordinates in order to prove that $BQ^2 - PC^2 = 3(PB^2 - QC^2)$.

16. Prove the corollaries in question 4 for a general right-angled triangle using the vertices $A(a, 0)$, $B(0, b)$ and the origin.

17. The points $P(cp, c/p)$, $Q(cq, c/q)$, $R(cr, c/r)$ and $S(cs, c/s)$ lie on the curve $xy = c^2$.

(a) If $PQ \parallel RS$, show that $pq = rs$.

(b) Show that $PQ \perp RS$ if and only if $pqrs = -1$.

(c) Conclude from part (b) that if a triangle is drawn with its vertices on a rectangular hyperbola, then the orthocentre (intersection of the altitudes) will lie on the hyperbola.

GO Online Multiple Choice Quiz

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