The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2 **Solutions to Exercises for Week 6** 2014

1. Put $S_n = a + ar + \ldots + ar^n$, so that $rS_n = ar + \ldots + ar^n + ar^{n+1}$, and

$$(1-r)S_n \;=\; S_n - rS_n \;=\; a - ar^{n+1} \;=\; a(1-r^{n+1})\,,$$

whence $S_n = \dfrac{a(1-r^{n+1})}{1-r}$.

2. To keep the problem nontrivial, assume $a \neq 0$. Using the notation from the previous solution,

$$a + ar + \ldots + ar^n + \ldots \;=\; \lim_{n\to\infty} S_n \;=\; \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} \;=\; \frac{a}{1-r}\,,$$

provided $|r| < 1$, using the fact that $\lim_{n\to\infty} r^{n+1} = 0$ at the last step. However, if $r \geq 1$ then $\lim_{n\to\infty} S_n$ is ∞ if a is positive and $-\infty$ if a is negative. If $r = -1$ then S_n oscillates infinitely often between 0 and a. If $r < -1$ then S_n oscillates infinitely often between increasingly large positive and negative numbers that grow in size without bound.

3. Take $a = 0.9$ and $r = 0.1$, in the previous exercise, so that $0.\dot{9}$ may be regarded as an abbreviation for

$$\lim_{n\to\infty} S_n \;=\; \frac{a}{1-r} \;=\; \frac{0.9}{0.9} \;=\; 1\,.$$

4. (i) Take $a = 0.009$ and $r = 0.1$ in the definition of the infinite geometric series, and then $0.009999\ldots$ becomes an abbreviation for

$$\lim_{n\to\infty} S_n \;=\; \frac{a}{1-r} \;=\; \frac{0.009}{0.9} \;=\; 0.01 \;=\; \frac{1}{100}\,.$$

(ii) Take $a = 0.11$ and $r = 0.001$ in the definition of the infinite geometric series, and then $0.1101101\ldots$ becomes an abbreviation for

$$\lim_{n\to\infty} S_n \;=\; \frac{a}{1-r} \;=\; \frac{0.11}{0.999} \;=\; \frac{110}{999}\,.$$

(iii) Observe that $0.0102102\ldots$ is a geometric series with $a = 0.0102$ and $r = 0.001$, so that

$$0.1102102\ldots \;=\; 0.1 + \frac{0.0102}{1-0.001} \;=\; \frac{1}{10} + \frac{102}{9990} \;=\; \frac{367}{3330}\,.$$

5. (i) Take $a = 1$ and $r = \frac{5}{11}$, and we have $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{11}\right)^n \;=\; \frac{1}{1-\frac{5}{11}} \;=\; \frac{11}{6}\,.$

(ii) Take $a = \frac{11}{5}$ and $r = \frac{1}{5}$, and we have $\displaystyle\sum_{n=1}^{\infty}\left(\frac{11}{5^n}\right) \;=\; \frac{\frac{11}{5}}{1-\frac{1}{5}} \;=\; \frac{11}{4}\,.$

(iii) Take $a = -\frac{3}{8}$ and $r = -\frac{3}{8}$, and we have $\displaystyle\sum_{n=2}^{\infty}\left(\frac{-3}{8}\right)^{n-1} \;=\; \frac{-\frac{3}{8}}{1+\frac{3}{8}} \;=\; -\frac{3}{11}\,.$

6. Observe that 21 years equals 664, 070, 400 seconds which is fewer than 7×10^8 seconds, so the final partial sum calculated after 21 years is certainly

$$\begin{aligned} &< 1 + \frac{1}{2} + \cdots + \frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{7 \times 10^8} \\ &< 3.02 + \int_{11}^{7\times 10^8} \frac{dt}{t} \;=\; 3.02 + \ln\left(\frac{7 \times 10^8}{11}\right) \\ &= 20.99 \;<\; 21\,. \end{aligned}$$

7. (i) Observe that $1 + 2x + 4x^2 + 8x^3 + \cdots \;=\; \dfrac{1}{1-2x}$ converges when $|2x| < 1$, that is, when $-\frac{1}{2} < x < \frac{1}{2}$.

(ii) Observe that $1 - 2x + 4x^2 - 8x^3 + \cdots \;=\; \dfrac{1}{1+2x}$ converges when $|2x| < 1$, that is, when $-\frac{1}{2} < x < \frac{1}{2}$.

8. (i) Observe that $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n} \;=\; \frac{1}{1-\frac{x}{2}} \;=\; \frac{2}{2-x}$ converges when $|x/2| < 1$, that is, when $-2 < x < 2$.

(ii) Observe that $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n} \;=\; \frac{\frac{x+2}{3}}{1-\frac{x+2}{3}} \;=\; \frac{x+2}{1-x}$ converges when $|\frac{x+2}{3}| < 1$, that is, when $-5 < x < 1$.

(iii) Observe that $\displaystyle\sum_{n=0}^{\infty} \tanh^{2n} x \;=\; \frac{1}{1-\tanh^2 x} \;=\; \frac{1}{\operatorname{sech}^2 x} \;=\; \cosh^2 x$ converges when $\tanh^2 x < 1$, that is, for all x.

9. (i) We have, putting $u = 3x + 1$,

$$\int_1^{\infty} \frac{dx}{(3x+1)^2} \;=\; \lim_{b\to\infty} \int_1^b \frac{dx}{(3x+1)^2} \;=\; \lim_{b\to\infty} \frac{1}{3}\int_4^{3b+1} \frac{du}{u^2}$$

$$=\; \frac{1}{3}\lim_{b\to\infty} \big[-u^{-1}\big]_4^{3b+1} \;=\; \frac{1}{12}\,.$$

(ii) We have, noting the vertical asymptote, and putting $u = x - 2$,

$$\int_2^5 \frac{dx}{\sqrt{x-2}} \;=\; \lim_{a\to 2^+} \int_a^5 \frac{dx}{\sqrt{x-2}} \;=\; \lim_{a\to 2^+} \int_{a-2}^3 \frac{du}{\sqrt{u}}$$

$$=\; \lim_{a\to 2^+} \big[2u^{1/2}\big]_{a-2}^3 \;=\; 2\sqrt{3}\,.$$

(iii) We have, noting the vertical asymptote,

$$\int_{\pi/4}^{\pi/2} \sec^2 x\; dx \;=\; \lim_{b\to\pi/2^-} \int_{\pi/4}^b \sec^2 x\; dx \;=\; \lim_{b\to\pi/2^-} \big[\tan x\big]_{\pi/4}^b$$

$$=\; \lim_{b\to\pi/2^-} (\tan b - 1) \;=\; \infty\,.$$

(iv) We have, using integration by parts, and noting the vertical asymptote,

$$\int_0^1 \ln x \, dx \;=\; \lim_{a\to 0^+} \int_a^1 \ln x \, dx \;=\; \lim_{a\to 0^+} \left(\big[x \ln x\big]_a^1 - \int_a^1 1 \, dx \right)$$

$$= \lim_{a\to 0^+} \left(-a \ln a - \big[x\big]_a^1 \right) \;=\; \lim_{a\to 0^+} \left(-\frac{\ln a}{1/a} - (1-a) \right)$$

$$= -1 + \lim_{a\to 0^+} \frac{-1/a}{-1/a^2} \;=\; -1 + \lim_{a\to 0^+} a \;=\; -1 \, .$$

(v) We have, using integration by parts,

$$\int_1^\infty \frac{\ln x}{x^2} \, dx \;=\; \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2} \, dx \;=\; \lim_{b\to\infty} \left[-\frac{\ln x}{x} \right]_1^b + \int_1^b \frac{dx}{x^2}$$

$$= \lim_{b\to\infty} -\frac{\ln b}{b} + \left[-\frac{1}{x} \right]_1^b \;=\; \lim_{b\to\infty} \frac{-1/b}{1} - \frac{1}{b} + 1 \;=\; 1 \, .$$

(vi) We have, using integration by parts,and noting the vertical asymptote,

$$\int_0^1 \frac{\ln x}{x^2} \, dx \;=\; \lim_{a\to 0^+} \int_a^1 \frac{\ln x}{x^2} \, dx \;=\; \lim_{a\to 0^+} \left[-\frac{\ln x}{x} \right]_a^1 + \int_a^1 \frac{dx}{x^2}$$

$$= \lim_{a\to 0^+} \frac{\ln a}{a} + \left[-\frac{1}{x} \right]_a^1 \;=\; \lim_{a\to 0^+} \frac{\ln a}{a} - 1 + \frac{1}{a}$$

$$= \lim_{a\to 0^+} \frac{\ln a + 1}{a} - 1 \;=\; -\infty \, .$$

10. (i) Here $\dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2}$ and

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} \left[-\frac{1}{x} \right]_1^b = 1$$

so $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2} \, dx$ converges by the Comparison Test.

(ii) Here $\dfrac{|\cos x|}{1+x^2} \le \dfrac{1}{1+x^2}$ and

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \Big[\tan^{-1} x \Big]_1^b = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

so $\displaystyle\int_1^\infty \frac{\cos x}{1+x^2} \, dx$ converges by the Comparison Test.

(iii) For $x \geq 1$, $e^{-x} < x$ so $\dfrac{1}{x+e^{-x}} > \dfrac{1}{2x}$ and

$$\int_1^\infty \frac{dx}{2x} = \frac{1}{2}\lim_{b\to\infty}\int_1^b \frac{dx}{x} = \frac{1}{2}\lim_{b\to\infty}\big[\ln x\big]_1^b = \infty$$

so $\displaystyle\int_1^\infty \frac{dx}{x+e^{-x}}$ diverges by the Comparison Test.

(iv) For $x \geq 1$, $\dfrac{e^{-x}}{x} \leq e^{-x}$ and

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\big[-e^{-x}\big]_1^b = e^{-1}$$

so $\displaystyle\int_1^\infty \frac{e^{-x}}{x}\,dx$ converges by the Comparison Test.

(v) For $0 < x \leq 1$, $\dfrac{e^{-x}}{x} \geq \dfrac{e^{-1}}{x}$ and

$$\int_0^1 \frac{e^{-1}}{x}\,dx = e^{-1}\lim_{a\to 0^+}\int_a^1 \frac{dx}{x} = e^{-1}\lim_{a\to 0^+}\big[\ln x\big]_a^1 = \infty$$

so $\displaystyle\int_0^1 \frac{e^{-x}}{x}\,dx$ diverges by the Comparison Test.

(vi) First note that $\displaystyle\int_0^1 e^{-x^2}\,dx$ exists since $y = e^{-x^2}$ is a continuous function on $[0,1]$. Now $e^{-x^2} \leq e^{-x}$ for $x \geq 1$ and

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\big[-e^{-x}\big]_1^b = e^{-1}$$

so $\displaystyle\int_1^\infty e^{-x^2}\,dx$ converges by the Comparison Test. Hence

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

converges.

11. The volume occupied by Gabriel's horn is, by the disc method,

$$\int_1^\infty \pi x^{-2}\,dx = \pi\lim_{b\to\infty}\int_1^b x^{-2}\,dx = \pi\lim_{b\to\infty}\big[-x^{-1}\big]_1^b = \pi\lim_{b\to\infty}(-b^{-1}+1) = \pi\,,$$

which is finite. The surface area however is

$$\int_1^\infty 2\pi x^{-1}\sqrt{1+\frac{1}{x^4}}\,dx \geq \int_1^\infty x^{-1}\,dx = \infty\,,$$

which is therefore infinite, by the Comparison Test.

12. Note first that

$$\lim_{b\to\infty} be^{-b^2} \;=\; \lim_{b\to\infty} \frac{b}{e^{b^2}} \;=\; \lim_{b\to\infty} \frac{1}{2be^{b^2}} \;=\; 0\,.$$

Hence

$$\begin{aligned}
\int_0^\infty x^2e^{-x^2}\,dx \;&=\; \lim_{b\to\infty}\int_0^b x^2e^{-x^2}\,dx \;=\; \lim_{b\to\infty}\left(\left[\frac{-xe^{-x^2}}{2}\right]_0^b + \int_0^b \frac{e^{-x^2}}{2}\,dx\right) \\
&=\; -\frac{1}{2}\lim_{b\to\infty} be^{-b^2} + \frac{1}{2}\lim_{b\to\infty}\int_0^b e^{-x^2}\,dx \;=\; 0 + \frac{1}{2}\lim_{b\to\infty}\int_0^b e^{-x^2}\,dx \\
&=\; \frac{1}{2}\int_0^\infty e^{-x^2}\,dx\,.
\end{aligned}$$

13. Observe first that

$$\lim_{a\to 0^+} a\ln a \;=\; \lim_{a\to 0^+}\frac{\ln a}{1/a} \;=\; \lim_{a\to 0^+}\frac{1/a}{-1/a^2} \;=\; -\lim_{a\to 0^+} a \;=\; 0$$

which starts an induction, and, for $n \geq 2$,

$$\lim_{a\to 0^+} a(\ln a)^n \;=\; \lim_{a\to 0^+}\frac{(\ln a)^n}{1/a} \;=\; \lim_{a\to 0^+}\frac{n(\ln a)^{n-1}1/a}{-1/a^2} \;=\; -n\lim_{a\to 0^+} a(\ln a)^{n-1} \;=\; n0 \;=\; 0$$

by an inductive hypothesis. Now we tackle the question. Observe by **9**(iv) that

$$\int_0^1 \ln x\,dx \;=\; -1 \;=\; (-1)1!\,,$$

which starts a new induction. For $n \geq 2$,

$$\begin{aligned}
\int_0^1 (\ln x)^n\,dx \;&=\; \lim_{a\to 0^+}\int_a^1 (\ln x)^n\,dx \;=\; \lim_{a\to 0^+}\left(\big[x(\ln x)^n\big]_a^1 - \int_a^1 n(\ln x)^{n-1}\frac{x}{x}\,dx\right) \\
&=\; \lim_{a\to 0^+}\left(-a(\ln a)^n - n\int_a^1 (\ln x)^{n-1}\,dx\right) \\
&=\; -\lim_{a\to 0^+} a(\ln a)^n \;-\; n\lim_{a\to 0^+}\int_a^1 (\ln x)^{n-1}\,dx \\
&=\; 0 \;-\; n\int_0^1 (\ln x)^{n-1}\,dx \\
&=\; -n(-1)^{n-1}(n-1)! \;=\; (-1)^n n!
\end{aligned}$$

by our previous result and by a new inductive hypothesis.

14. Observe that

$$\int_1^b \sin(\pi x)\,dx \;=\; \left[\frac{-\cos(\pi x)}{\pi}\right]_1^b \;=\; -\frac{\cos(b\pi)}{\pi} - \frac{1}{\pi} \;=\; \begin{cases} -2/\pi & \text{if } b \text{ is an even integer} \\ 0 & \text{if } b \text{ is an odd integer} \end{cases}$$

which oscillates infinitely often between 0 and $-2/\pi$ as $b \to \infty$, so that

$$\int_1^\infty \sin(\pi x)\,dx \;=\; \lim_{b\to\infty}\int_1^b \sin(\pi x)\,dx$$

does not exist.

15. Observe first that $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ converges, by the Comparison Test, since $\displaystyle\frac{|\cos x|}{x^2} \leq \frac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges. Observe also that $\displaystyle\lim_{b\to\infty}\frac{\cos b}{b} = 0$ by the Squeeze Law. Hence

$$\begin{aligned}
\int_1^\infty \frac{\sin x}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\sin x}{x}\,dx \\
&= \lim_{b\to\infty}\left(\left[\frac{1}{x}(-\cos x)\right]_1^b + \int_1^b \frac{-\cos x}{x^2}\,dx\right) \\
&= \lim_{b\to\infty}\left(\frac{-\cos b}{b} + \frac{\cos 1}{1} - \int_1^b \frac{\cos x}{x^2}\,dx\right) \\
&= \frac{\cos 1}{1} - \lim_{b\to\infty}\int_1^b \frac{\cos x}{x^2}\,dx \\
&= \frac{\cos 1}{1} - \int_1^\infty \frac{\cos x}{x^2}\,dx
\end{aligned}$$

converges.

16. Suppose $f(x) \geq |g(x)|$ for all $x \geq 1$ and $\int_1^\infty f(x)\,dx$ converges. Our task is to show $\int_1^\infty g(x)\,dx$ converges.

Special case (i): Suppose first that $f(x) \geq g(x) \geq 0$ for all $x \geq 1$. Then, for each $b \geq 1$,

$$\int_1^b g(x)\,dx \;\leq\; \int_1^b f(x)\,dx \;\leq\; \int_1^\infty f(x)\,dx \;<\infty\,,$$

so the set

$$X \;=\; \left\{\; \int_1^b g(x)\,dx \;\middle|\; b \geq 1 \right\}$$

is nonempty and bounded above. By completeness of $\mathbb{R}$, X has a least upper bound L. Note that if $1 \leq b_1 \leq b_2$ then

$$\int_1^{b_1} g(x)\,dx \;\leq\; \int_1^{b_1} g(x)\,dx + \int_{b_1}^{b_2} g(x)\,dx \;=\; \int_1^{b_2} g(x)\,dx \;\leq\; L\,.$$

Let $\epsilon > 0$. If $\int_1^b g(x)\,dx \leq L - \epsilon$ for all $b \geq 1$ then L would not be the least upper bound of X. Hence

$$\int_1^B g(x)\,dx \;>\; L - \epsilon \qquad \text{for some } B\,,$$

so, for $b \geq B$,

$$L - \epsilon \;<\; \int_1^B g(x)\,dx \;\leq\; \int_1^b g(x)\,dx \leq L\,.$$

Hence

$$\left|L - \int_1^b g(x)\,dx\right| \;<\; \epsilon \qquad \text{for all } b > B.$$

This proves $\lim_{b\to\infty}\int_1^b g(x)\,dx \;=\; L$, that is, $\int_1^\infty g(x)\,dx$ converges.

General case (ii): Now suppose $f(x) \geq |g(x)|$ for all $x \geq 1$, so $0 \leq f(x) \pm g(x) \leq 2f(x)$. But

$$\int_1^\infty 2f(x)\,dx = 2\int_1^\infty f(x)\,dx$$

converges. By Case (i), $\int_1^\infty f(x) \pm g(x)\,dx$ converges. Hence

$$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{f(x)+g(x)}{2}\,dx - \int_1^\infty \frac{f(x)-g(x)}{2}\,dx$$

converges.

17. (i) First oberve $\frac{1}{x-1} = \frac{(x+1)(x^2+1)}{x^4-1}$, $\frac{1}{x+1} = \frac{(x-1)(x^2+1)}{x^4-1}$, $\frac{1}{x^2+1} = \frac{x^2-1}{x^4-1}$, $\frac{x}{x^2+1} = \frac{x(x^2-1)}{x^4-1}$ are all members of V. Suppose now $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ and

$$\frac{\lambda_1}{x-1} + \frac{\lambda_2}{x+1} + \frac{\lambda_3}{x^2+1} + \frac{\lambda_4 x}{x^2+1} = 0$$

for all $x \neq \pm 1$. Then

$$\lambda_1(x+1)(x^2+1) + \lambda_2(x-1)(x^2+1) + \lambda_3(x^2-1) + \lambda_4 x(x^2-1) = 0$$

for all x (by continuity of polynomials). But $x = 1$ gives $4\lambda_1 = 0$, so $\lambda_1 = 0$; $x = -1$ gives $-4\lambda_2 = 0$, so $\lambda_2 = 0$; $x = 0$ gives $\lambda_1 - \lambda_2 - \lambda_3 = 0$, so $\lambda_3 = 0$; $x = 2$ gives $6\lambda_4 = 0$, so $\lambda_4 = 0$. This verifies linear independence.

(ii) First observe that $\frac{1}{(x-a)^k} = \frac{(x-a)^{n-k}}{(x-a)^n}$ is an element of W for $1 \leq k \leq n$. Suppose now $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and

$$\frac{\lambda_1}{x-a} + \frac{\lambda_2}{(x-a)^2} + \ldots + \frac{\lambda_n}{(x-a)^n} = 0$$

for all $x \neq a$. Then

$$\lambda_1(x-a)^{n-1} + \lambda_2(x-a)^{n-2} + \ldots + \lambda_n = 0$$

for infinitely many x. Nonzero polynomials have finitely many roots, so $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$, verifying linear independence.

(iii) The relevance in part (i) is that $\frac{1}{x-1}, \frac{1}{x+1}, \frac{1}{x^2+1}, \frac{x}{x^2+1}$ span V, so that if $p(x)$ is a polynomial of degree < 4 then

$$\frac{p(x)}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x^2+1} + \frac{Dx}{x^2+1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Dx+C}{x^2+1}$$

for some constants A, B, C, D, which is precisely existence of the partial fraction decomposition.

The relevance in part (ii) is that $\frac{1}{x-a}, \frac{1}{(x-a)^2}, \ldots, \frac{1}{(x-1)^n}$ span W, so that if $p(x)$ is a polynomial of degree $< n$ then

$$\frac{p(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_n}{(x-a)^n}$$

for some constants $A_1, \ldots, A_n$, which is again existence of the partial fraction decomposition.