MATH3611: Higher Analysis
Assignment 1

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1. **Claim:** The set S of all eventually constant sequences of natural numbers is countable.

   Proof: Define an eventually constant sequence of natural numbers as $\{x_k\}_{k=0}^{\infty}$, where $x_k \in \mathbb{N}$, and $x_k = x_{k+1}\ \forall k \geq K \in \mathbb{N}$. Now, let $S_n \subset S$ be the set of all eventually constant sequences of natural numbers that become constant once $K = n$. Thus, $S_n = \{\{x_k\}_{k=0}^{\infty} \mid x_k = x_{k+1}\ \forall k \geq n\}$. Consider the sequence $s \in S_n$ where $s = (s_0, s_1, s_2, \ldots, s_{n-1}, s_n, s_n, \ldots)$, and consider the function $f : S_n \to \mathbb{N}^{n+1}$ defined by

   $$f(s) = (s_0, s_1, s_2, \ldots, s_{n-1}, s_n).$$

   To see that f is injective, consider $s, t \in S$ such that $s = (s_0, s_1, s_2, \ldots, s_{l-1}, s_l, s_l, \ldots)$, and $t = (t_0, t_1, t_2, \ldots, t_{m-1}, t_m, t_m, \ldots)$. Suppose that $f(s) = f(t)$. Clearly, the sequences must become eventually constant at $l = m = n$. Furthermore, $s_i = t_i$, $\forall i \in \mathbb{N}$ and $i \leq n$. Thus $s = t$, and so $f : S_n \hookrightarrow \mathbb{N}^{n+1}$.

   In lectures it was shown inductively that the set $\mathbb{N}^{n+1}$ was countable. Thus, $|S_n| \leq |\mathbb{N}^{n+1}| \leq |\mathbb{N}|$. The set S is then simply the union of all countable sets S_n, where $n \in \mathbb{N}$, that is

   $$S = \bigcup_{n \in \mathbb{N}} S_n.$$

   From lectures, a countable union of countable sets is countable, and so S is countable.

2. **Claim:** The set T of all sequences of rational numbers which converge to 3 is uncountable.

   Proof: Define the set B of infinite binary strings, $B = \{b_1 b_2 \ldots b_i \ldots \mid b_i \in \{0,1\}\}$. Furthermore, define $T_b = \left\{ \left\{3 + \frac{b_i}{i}\right\}_{i=1}^{\infty} \,\middle|\, b_i \in \{0,1\}, i \in \mathbb{Z}^+ \right\}$, which is a proper subset of T as $\lim\limits_{i \to \infty} \left(3 + \frac{b_i}{i}\right) = 3$, and is right hand convergence. Thus $T_b \subset T$. Consider the infinite binary string $b \in B$ where $b = b_1 b_2 \ldots b_i \ldots$, and consider the function $f : B \to T_b$ defined by

   $$f(b) = \left\{3 + \frac{b_i}{i}\right\}_{i=1}^{\infty}.$$

   To see that f is injective, consider $c, d \in T_b$ such that $c = c_1 c_2 \ldots c_i \ldots$, and $d = d_1 d_2 \ldots d_i \ldots$. Suppose that $f(c) = f(d)$. Thus $3 + {c_i}/{i} = 3 + {d_i}/{i}$ and so $c_i = d_i\ \forall i \in \mathbb{N}$. Thus $c = d$, and so $f : B \hookrightarrow T_b$.

   In lectures, we proved that there exists a bijection $g : B \to \mathbb{P}(\mathbb{N})$. Thus, $|B| = |\mathbb{P}(\mathbb{N})|$. Furthermore, we also proved that $|\mathbb{P}(\mathbb{N})| > |\mathbb{N}|$, and thus B is uncountable. As we have the injection $f : B \hookrightarrow T_b$, clearly $|B| \leq |T_b|$. Finally, as $T_b \subset T$, we have $|T_b| < |T|$. Hence, $|T| > |\mathbb{N}|$ and so the set T is uncountable.