The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Longer Solutions to Selected Exercises for Week 4** 2017

5. We have

$$\overrightarrow{QP} \times \overrightarrow{QR} \;=\; \big(2\mathbf{i}+\mathbf{j}+3\mathbf{k}\big) \times \big(\mathbf{i}+2\mathbf{j}-\mathbf{k}\big) \;=\; -7\mathbf{i}+5\mathbf{j}+3\mathbf{k}\,.$$

Hence the area of the triangle $\triangle PQR$ is

$$\frac{\big|7\mathbf{i}-5\mathbf{j}-3\mathbf{k}\big|}{2} \;=\; \frac{\sqrt{83}}{2}\,.$$

6. (i) Observe that, suppressing some of the detail,

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) \;=\; (\mathbf{i}+\mathbf{j})\cdot(-\mathbf{i}+2\mathbf{j}-\mathbf{k}) \;=\; 1$$

and, on the other hand,

$$(\mathbf{v}\times\mathbf{u})\cdot\mathbf{w} \;=\;=\; (-2\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k}) \;=\; -1\,.$$

(ii) The volume of the parallelopiped is just $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |1| = 1$.

7. (i) $\mathbf{a}\times\mathbf{b} \;=\; -\mathbf{i}-2\mathbf{j}+3\mathbf{k}\,.$ (ii) $\mathbf{a}\times\mathbf{c} \;=\; -\mathbf{i}-2\mathbf{j}-2\mathbf{k}\,.$

(iii) $\mathbf{b}\times\mathbf{c} \;=\; \mathbf{i}-3\mathbf{j}+2\mathbf{k}\,.$ (iv) $\mathbf{a}\times\big(\mathbf{b}\times\mathbf{c}\big) \;=\; -2\mathbf{i}-4\mathbf{j}-5\mathbf{k}\,.$

(v) $\big(\mathbf{a}\times\mathbf{b}\big)\times\mathbf{c} \;=\; -2\mathbf{i}-5\mathbf{j}-4\mathbf{k}\,.$ (vi) $\mathbf{a}\times\big(\mathbf{a}\times\mathbf{c}\big) \;=\; 2\mathbf{i}+4\mathbf{j}-5\mathbf{k}\,.$

(vii) $\mathbf{a}\times\big(\mathbf{a}+\mathbf{c}\big) \;=\; \mathbf{a}\times\mathbf{c} \;=\; -\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ (viii) $\big(\mathbf{a}\times\mathbf{a}\big)\times\mathbf{c} \;=\; \mathbf{0}\,.$

(ix) $\mathbf{a}\times\big(\mathbf{b}-2\mathbf{c}\big) \;=\; \mathbf{a}\times\mathbf{b}-2(\mathbf{a}\times\mathbf{c}) \;=\; \mathbf{i}+2\mathbf{j}+7\mathbf{k}\,.$

(x) the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$ equals $\dfrac{\big|\mathbf{a}\times\mathbf{b}\big|}{|\mathbf{a}|\,|\mathbf{b}|} \;=\; \dfrac{\sqrt{14}}{\sqrt{15}}\,.$

(xi) the area of the parallelogram inscribed by $\mathbf{a}$ and $\mathbf{c}$ equals $\big|\mathbf{a}\times\mathbf{c}\big| \;=\; 3\,.$

(xii) the area of the triangle inscribed by $\mathbf{b}$ and $\mathbf{c}$ equals $\dfrac{\big|\mathbf{b}\times\mathbf{c}\big|}{2} \;=\; \dfrac{\sqrt{14}}{2}\,.$

(xiii) the volume of the parallelopiped inscribed by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ equals $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 5\,.$

8. (i) $\mathbf{w}\times\mathbf{v} \;=\; -\big(\mathbf{v}\times\mathbf{w}\big) \;=\; -2\mathbf{i}+\mathbf{j}-3\mathbf{k}\,.$

(ii) $\big(\mathbf{v}+3\mathbf{w}\big)\times\big(2\mathbf{w}-\mathbf{v}\big) \;=\; 2\big(\mathbf{v}\times\mathbf{w}\big)-3\big(\mathbf{w}\times\mathbf{v}\big) \;=\; 5\big(\mathbf{v}\times\mathbf{w}\big) \;=\; 10\mathbf{i}-5\mathbf{j}+15\mathbf{k}\,.$

9. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$ measured between 0 and 180 degrees. Then

$$\cos\theta \;=\; \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \;=\; \frac{-21}{28} \;=\; -\frac{3}{4}\,,$$

so $\sin\theta \;=\; \sqrt{1-\dfrac{9}{16}} \;=\; \dfrac{\sqrt{7}}{4}$. Hence

$$|\mathbf{a}\times\mathbf{b}| \;=\; |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta \;=\; 7\sqrt{7}\,.$$

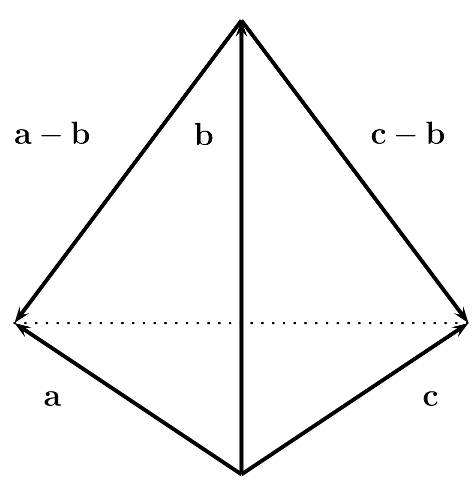
10. Write $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$. Then

$$\begin{aligned}
(\mathbf{v}\times\mathbf{w})\cdot\mathbf{v} &= \Big((v_2w_3 - v_3w_2)\,\mathbf{i} + (v_3w_1 - v_1w_3)\,\mathbf{j} + (v_1w_2 - v_2w_1)\,\mathbf{k}\Big)\cdot\mathbf{v} \\
&= (v_2w_3 - v_3w_2)v_1 + (v_3w_1 - v_1w_3)v_2 + (v_1w_2 - v_2w_1)v_3 \\
&= v_2w_3v_1 - v_3w_2v_1 + v_3w_1v_2 - v_1w_3v_2 + v_1w_2v_3 - v_2w_1v_3 \\
&= v_1v_2w_3 - v_1v_2w_3 + v_1w_2v_3 - v_1w_2v_3 + w_1v_2v_3 - w_1v_2v_3 \\
&= 0 + 0 + 0 \;=\; 0\,, \\
(\mathbf{v}\times\mathbf{w})\cdot\mathbf{w} &= \Big((v_2w_3 - v_3w_2)\,\mathbf{i} + (v_3w_1 - v_1w_3)\,\mathbf{j} + (v_1w_2 - v_2w_1)\,\mathbf{k}\Big)\cdot\mathbf{w} \\
&= (v_2w_3 - v_3w_2)w_1 + (v_3w_1 - v_1w_3)w_2 + (v_1w_2 - v_2w_1)w_3 \\
&= v_2w_3w_1 - v_3w_2w_1 + v_3w_1w_2 - v_1w_3w_2 + v_1w_2w_3 - v_2w_1w_3 \\
&= w_1v_2w_3 - w_1v_2w_3 + w_1w_2v_3 - w_1w_2v_3 + v_1w_2w_3 - v_1w_2w_3 \\
&= 0 + 0 + 0 \;=\; 0\,, \\
\mathbf{v}\times\mathbf{w} &= (v_2w_3 - v_3w_2)\,\mathbf{i} + (v_3w_1 - v_1w_3)\,\mathbf{j} + (v_1w_2 - v_2w_1)\,\mathbf{k} \\
&= -(w_2v_3 - w_3v_2)\,\mathbf{i} - (w_3v_1 - w_1v_3)\,\mathbf{j} - (w_1v_2 - w_2v_1)\,\mathbf{k} \\
&= -\Big((w_2v_3 - w_3v_2)\,\mathbf{i} + (w_3v_1 - w_1v_3)\,\mathbf{j} + (w_1v_2 - w_2v_1)\,\mathbf{k}\Big) \\
&= -\big(\mathbf{w}\times\mathbf{v}\big)\,, \\
\mathbf{v}\times\mathbf{v} &= (v_2v_3 - v_3v_2)\,\mathbf{i} + (v_3v_1 - v_1v_3)\,\mathbf{j} + (v_1v_2 - v_2v_1)\,\mathbf{k} \\
&= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} \;=\; \mathbf{0}\,.
\end{aligned}$$

11. The cross product $\mathbf{u}\times\mathbf{v}$ is the zero vector if and only if its magnitude is zero, and this occurs if and only if the parallelogram inscribed by $\mathbf{u}$ and $\mathbf{v}$ has zero area, that is, $\mathbf{u}$ and $\mathbf{v}$ are parallel.

13. The expression $\mathbf{u}\times\mathbf{v}\times\mathbf{w}$ is ambiguous (and hence not sensible) because it can be interpreted as either $\big(\mathbf{u}\times\mathbf{v}\big)\times\mathbf{w}$ or $\mathbf{u}\times\big(\mathbf{v}\times\mathbf{w}\big)$, which need not be equal, since the cross product is not in general associative. The equation $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w}$ does not imply $\mathbf{v} = \mathbf{w}$ whenever $\mathbf{u} \neq \mathbf{0}$, and a simple counterexample provided by taking $\mathbf{u} = \mathbf{v} = \mathbf{i}$ and $\mathbf{w} = \mathbf{0}$.

14. Consider a tetrahedron with directed edges labelled as follows



Using the Right-Hand Rule for successive faces, we may take

$$\mathbf{v}_1 = \frac{1}{2}\big(\mathbf{b}\times\mathbf{a}\big)\ ,\quad \mathbf{v}_2 = \frac{1}{2}\big(\mathbf{c}\times\mathbf{b}\big)\ ,\quad \mathbf{v}_3 = \frac{1}{2}\big(\mathbf{a}\times\mathbf{c}\big)\ ,\quad \mathbf{v}_4 = \frac{1}{2}\Big[\big(\mathbf{c}-\mathbf{b}\big)\times\big(\mathbf{a}-\mathbf{b}\big)\Big]\ .$$

Hence

$$\begin{aligned} 2\big(\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4\big) &= \mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\big(\mathbf{c}-\mathbf{b}\big)\times\big(\mathbf{a}-\mathbf{b}\big) \\ &= \mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}-\mathbf{c}\times\mathbf{b}-\mathbf{b}\times\mathbf{a} \\ &= \mathbf{a}\times\mathbf{c}-\mathbf{a}\times\mathbf{c} = \mathbf{0}\ . \end{aligned}$$

This shows $\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4 = \mathbf{0}$.

15. Put $\mathbf{a} = a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}$, $\mathbf{b} = b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}$ and $\mathbf{c} = c_1\,\mathbf{i}+c_2\,\mathbf{j}+c_3\,\mathbf{k}$. Then, by the algebraic formulae for dot and cross products,

$$\begin{aligned} \mathbf{a}\cdot\big(\mathbf{b}\times\mathbf{c}\big) &= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1) \\ &= a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1 \\ &= (a_2b_3-a_3b_2)c_1+(a_3b_1-a_1b_3)c_2+(a_1b_2-a_2b_1)c_3 \\ &= \big(\mathbf{a}\times\mathbf{b}\big)\cdot\mathbf{c}\ . \end{aligned}$$

For a quick geometric verification, observe that

$$\left|\mathbf{a}\cdot\big(\mathbf{b}\times\mathbf{c}\big)\right| = \left|\mathbf{c}\cdot\big(\mathbf{a}\times\mathbf{b}\big)\right| = \left|\big(\mathbf{a}\times\mathbf{b}\big)\cdot\mathbf{c}\right|$$

commonly measure the volume of the parallelopiped inscribed by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. But $\mathbf{a}\cdot\big(\mathbf{b}\times\mathbf{c}\big)$ and $\big(\mathbf{a}\times\mathbf{b}\big)\cdot\mathbf{c}$ have the same sign, since the triples $(\mathbf{a},\mathbf{b},\mathbf{c})$ and $(\mathbf{c},\mathbf{a},\mathbf{b})$ are right or left-handed together, so that

$$\mathbf{a}\cdot\big(\mathbf{b}\times\mathbf{c}\big) = \big(\mathbf{a}\times\mathbf{b}\big)\cdot\mathbf{c}\ .$$

Finally, by anticommutativity,

$$\mathbf{a}\cdot\big(\mathbf{b}\times\mathbf{c}\big) = \big(\mathbf{a}\times\mathbf{b}\big)\cdot\mathbf{c} = \big(-(\mathbf{b}\times\mathbf{a})\big)\cdot\mathbf{c} = -\big(\mathbf{b}\times\mathbf{a}\big)\cdot\mathbf{c}\ .$$

16. Observe that

$$\mathbf{v}\times\mathbf{w} = \big(\mathbf{i}+2\mathbf{j}-7\mathbf{k}\big)\times\big(5\mathbf{i}+\mathbf{j}+\mathbf{k}\big) = 9\mathbf{i}-36\mathbf{j}-9\mathbf{k} = 9\big(\mathbf{i}-4\mathbf{j}-\mathbf{k}\big)\ ,$$

which has length $9\sqrt{1+16+1} = 9\sqrt{18} = 27\sqrt{2}$. Hence two unit vectors perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ are

$$\pm\frac{\sqrt{2}}{6}\big(\mathbf{i}-4\mathbf{j}-\mathbf{k}\big)\ .$$

17. (i) Area equals $\dfrac{\big|(2\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})\big|}{2} = \dfrac{\big|-7\mathbf{j}-14\mathbf{k}\big|}{2} = \dfrac{7\sqrt{5}}{2}$.

(ii) Area equals $\dfrac{\big|(-2\mathbf{i}-5\mathbf{k})\times(\mathbf{i}-2\mathbf{j}-\mathbf{k})\big|}{2} = \dfrac{\big|-10\mathbf{i}-7\mathbf{j}+4\mathbf{k}\big|}{2} = \dfrac{\sqrt{165}}{2}$.

18. (i) The area of triangle PQR is

$$\frac{|\overrightarrow{PQ}\times\overrightarrow{PR}|}{2} = \frac{\big|(-2\mathbf{i}-2\mathbf{j}-\mathbf{k})\times(-\mathbf{i}+\mathbf{k})\big|}{2} = \frac{\big|-2\mathbf{i}+3\mathbf{j}-2\mathbf{k}\big|}{2} = \frac{\sqrt{17}}{2},$$

and the area of triangle QRS is

$$\frac{|\overrightarrow{QR}\times\overrightarrow{QS}|}{2} = \frac{\big|(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}+4\mathbf{j}+3\mathbf{k})\big|}{2} = \frac{\big|-2\mathbf{i}+3\mathbf{j}-2\mathbf{k}\big|}{2} = \frac{\sqrt{17}}{2}.$$

This is not suprising because the figure $PQRS$ is a rhombus (as determined in an earlier exercise).

(ii) Observe that

$$d_1 = |\overrightarrow{PR}| = |-\mathbf{i}+\mathbf{k}| = \sqrt{2}$$

and

$$d_2 = |\overrightarrow{QS}| = |3\mathbf{i}+4\mathbf{j}+3\mathbf{k}| = \sqrt{34},$$

so that

$$\frac{d_1 d_2}{4} = \frac{\sqrt{2}\sqrt{34}}{4} = \frac{\sqrt{17}}{2}.$$

This is not surprising because the diagonals of a rhombus are mutually perpendicular (as determined in an earlier exercise), so that the product of their lengths should be four times the area of either of the triangles PQR or QRS.

20. (i) Observe that

$$(-\mathbf{i}+2\mathbf{j})\times(\mathbf{j}+3\mathbf{k}) = 6\mathbf{i}+3\mathbf{j}-\mathbf{k},$$

so a unit vector perpendicular to both $-\mathbf{i}+2\mathbf{j}$ and $\mathbf{j}+3\mathbf{k}$ is

$$\pm\frac{1}{\sqrt{46}}\big(6\mathbf{i}+3\mathbf{j}-\mathbf{k}\big).$$

(ii)* We want a unit vector pointing in the direction of

$$\overrightarrow{BC}\times\overrightarrow{BA} = (-\mathbf{j}-3\mathbf{k})\times(-\mathbf{i}+2\mathbf{j}) = (-\mathbf{i}+2\mathbf{j})\times(\mathbf{j}+3\mathbf{k}),$$

which is $\dfrac{1}{\sqrt{46}}\big(6\mathbf{i}+3\mathbf{j}-\mathbf{k}\big)$.

21. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Using the geometric formulae for the dot and cross product, we get

$$\sqrt{|\mathbf{a}\cdot\mathbf{b}|^2+|\mathbf{a}\times\mathbf{b}|^2} = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta+|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta}$$

$$= \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2\big(\cos^2+\sin^2\theta\big)} = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2} = |\mathbf{a}|\;|\mathbf{b}|.$$

22. Write $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$, $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$. Then

$$\begin{aligned}
(\mathbf{u}+\mathbf{v}) \times \mathbf{w} &= \Big((u_1+v_1)\,\mathbf{i} + (u_2+v_2)\,\mathbf{j} + (u_3+v_3)\,\mathbf{k}\Big) \times \mathbf{w} \\
&= \big((u_2+v_2)w_3 - (u_3+v_3)w_2\big)\,\mathbf{i} + \big((u_3+v_3)w_1 - (u_1+v_1)w_3\big)\,\mathbf{j} \\
&\qquad + \big((u_1+v_1)w_2 - (u_2+v_2)w_1\big)\mathbf{k} \\
&= \big(u_2w_3 + v_2w_3 - u_3w_2 - v_3w_2\big)\,\mathbf{i} + \big(u_3w_1 + v_3w_1 - u_1w_3 - v_1w_3\big)\,\mathbf{j} \\
&\qquad + \big(u_1w_2 + v_1w_2 - u_2w_1 + v_2w_1\big)\,\mathbf{k} \\
&= \big(u_2w_3 - u_3w_2 + v_2w_3 - v_3w_2\big)\,\mathbf{i} + \big(u_3w_1 - u_1w_3 + v_3w_1 - v_1w_3\big)\,\mathbf{j} \\
&\qquad + \big(u_1w_2 - u_2w_1 + v_1w_2 + v_2w_1\big)\,\mathbf{k} \\
&= \big(u_2w_3 - u_3w_2\big)\,\mathbf{i} + \big(v_2w_3 - v_3w_2\big)\,\mathbf{i} + \big(u_3w_1 - u_1w_3\big)\,\mathbf{j} + \big(v_3w_1 - v_1w_3\big)\,\mathbf{j} \\
&\qquad + \big(u_1w_2 - u_2w_1\big)\,\mathbf{k} + \big(v_1w_2 + v_2w_1\big)\,\mathbf{k} \\
&= \big(u_2w_3 - u_3w_2\big)\,\mathbf{i} + \big(u_3w_1 - u_1w_3\big)\,\mathbf{j} + \big(u_1w_2 - u_2w_1\big)\,\mathbf{k} \\
&\qquad + \big(v_2w_3 - v_3w_2\big)\,\mathbf{i} + \big(v_3w_1 - v_1w_3\big)\,\mathbf{j} + \big(v_1w_2 + v_2w_1\big)\,\mathbf{k} \\
&= \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}\,.
\end{aligned}$$

By anticommutativity, and what we have just proved,

$$\begin{aligned}
\mathbf{w} \times (\mathbf{u} + \mathbf{v}) &= -\Big((\mathbf{u} + \mathbf{v}) \times \mathbf{w}\Big) = -\Big(\mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}\Big) \\
&= -(\mathbf{u} \times \mathbf{w}) - (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \times \mathbf{u} + \mathbf{w} \times \mathbf{v}\,,
\end{aligned}$$

which verifies distributivity on the other side.

23. Put $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ and $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$. Then, by the algebraic formulae for dot and cross products,

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= \big[(a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}\big] \times \mathbf{c} \\[1ex]
&= \big[(a_3b_1 - a_1b_3)c_3 - (a_1b_2 - a_2b_1)c_2\big]\mathbf{i} + \big[(a_1b_2 - a_2b_1)c_1 - (a_2b_3 - a_3b_2)c_3\big]\mathbf{j} \\
&\qquad + \big[(a_2b_3 - a_3b_2)c_2 - (a_3b_1 - a_1b_3)c_1\big]\mathbf{k} \\[1ex]
&= (a_2c_2 + a_3c_3)b_1\,\mathbf{i} + (a_1c_1 + a_3c_3)b_2\,\mathbf{j} + (a_1c_1 + a_2c_2)b_3\,\mathbf{k} \\
&\qquad - (b_2c_2 + b_3c_3)a_1\,\mathbf{i} - (b_1c_1 + b_3c_3)a_2\,\mathbf{j} - (b_1c_1 + b_2c_2)a_3\,\mathbf{k} \\[1ex]
&= (a_1c_1 + a_2c_2 + a_3c_3)b_1\,\mathbf{i} + (a_1c_1 + a_2c_2 + a_3c_3)b_2\,\mathbf{j} + (a_1c_1 + a_2c_2 + a_3c_3)b_3\,\mathbf{k} \\
&\qquad - (b_1c_1 + b_2c_2 + b_3c_3)a_1\,\mathbf{i} - (b_1c_1 + b_2c_2 + b_3c_3)a_2\,\mathbf{j} - (b_1c_1 + b_2c_2 + b_3c_3)a_3\,\mathbf{k} \\[1ex]
&= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}\,.
\end{aligned}$$

By anti-commutativity of the cross product, commutativity of the dot product, and what we have just proved,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\big[(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}\big] = -\big[(\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}\big] = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,.$$

24. From the previous exercise,

$$
\begin{aligned}
(\mathbf{u}\times\mathbf{v}) \times \mathbf{w} \;+\; &(\mathbf{v}\times\mathbf{w})\times\mathbf{u} \;+\; (\mathbf{w}\times\mathbf{u})\times\mathbf{v} \\
&= (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u} + (\mathbf{v}\cdot\mathbf{u})\mathbf{w} - (\mathbf{w}\cdot\mathbf{u})\mathbf{v} + (\mathbf{w}\cdot\mathbf{v})\mathbf{u} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} \\
&= \big(\mathbf{u}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{u}\big)\mathbf{v} + \big(\mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v}\big)\mathbf{w} + \big(\mathbf{w}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w}\big)\mathbf{u} \\
&= 0\,\mathbf{v} + 0\,\mathbf{w} + 0\,\mathbf{u} \;= \mathbf{0}+\mathbf{0}+\mathbf{0} \;=\; \mathbf{0}\,,
\end{aligned}
$$

which verifies the Jacobi identity.

25. By Exercise 23,

$$\big(\mathbf{u}\times\mathbf{v}\big)\times\mathbf{w} \;=\; \mathbf{u}\times\big(\mathbf{v}\times\mathbf{w}\big)$$

if and only if

$$\big(\mathbf{u}\cdot\mathbf{w}\big)\mathbf{v} - \big(\mathbf{v}\cdot\mathbf{w}\big)\mathbf{u} \;=\; \big(\mathbf{u}\cdot\mathbf{w}\big)\mathbf{v} - \big(\mathbf{u}\cdot\mathbf{v}\big)\mathbf{w}\,,$$

if and only if

$$\big(\mathbf{v}\cdot\mathbf{w}\big)\mathbf{u} \;=\; \big(\mathbf{u}\cdot\mathbf{v}\big)\mathbf{w}\,.$$

This implies that either both scalars $\mathbf{v}\cdot\mathbf{w}$ and $\mathbf{u}\cdot\mathbf{v}$ are zero, so that $\mathbf{v}$ is perpendicular to both $\mathbf{w}$ and $\mathbf{u}$, or at least one of these scalars is nonzero, so that $\mathbf{u}$ and $\mathbf{w}$ are parallel. Conversely, suppose either that $\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{w}$ or that $\mathbf{u}$ and $\mathbf{w}$ are parallel. In the first case, $\mathbf{v}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{u} = 0$, and the above equation holds trivially. In the second case, either $\mathbf{u} = \lambda\mathbf{w}$ or $\mathbf{w} = \lambda\mathbf{u}$ for some (possibly zero) scalar λ, and then either

$$\big(\mathbf{v}\cdot\mathbf{w}\big)\mathbf{u} \;=\; \big(\mathbf{v}\cdot\mathbf{w}\big)(\lambda\mathbf{w}) \;=\; \big(\mathbf{v}\cdot(\lambda\,\mathbf{w})\big)\,\mathbf{w} \;=\; \big(\mathbf{v}\cdot\mathbf{u}\big)\,\mathbf{w} \;=\; \big(\mathbf{u}\cdot\mathbf{v}\big)\mathbf{w}\,,$$

or

$$\big(\mathbf{v}\cdot\mathbf{w}\big)\mathbf{u} \;=\; \big(\mathbf{v}\cdot(\lambda\,\mathbf{u})\big)\mathbf{u} \;=\; \big(\mathbf{v}\cdot\mathbf{u}\big)(\lambda\,\mathbf{u}) \;=\; \big(\mathbf{u}\cdot\mathbf{v}\big)\mathbf{w}\,,$$

and again the above equation holds. This completes the proof.