8025A Semester 1 2009

The University of Sydney

Faculties of Arts, Economics, Education, Engineering and Science

MATH1901/1906

Differential Calculus (Advanced)

June 2009

Lecturer: C M Cosgrove

Time Allowed: One and a half hours

Name: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

MARKS

1. (a) In the complex plane, sketch the set $\{z \in \mathbb{C} \mid 1 \leq |z+1-i| \leq 2\}$. **3**

   (b) Find all real and complex solutions of the equation,
$$z^4 - 4z^3 + 9z^2 - 16z + 20 = 0,$$
   given that $2+i$ is a root. **5**

   (c) Show that the function, $g : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^4$, is surjective but not injective. (Please keep your answer short.) **4**

2. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto \ln(x^2 + 4y^2)$, and let P be the point $(3, 1)$ in the xy-plane.

   (*i*) Calculate the directional derivative $D_{\mathbf{u}}f$ at P in the direction of the vector $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$. **2**

   (*ii*) Find the equation of the tangent plane to the graph of $z = f(x, y)$ at the point on the graph vertically above P. Express your answer in the form $z = ax + by + c$. **2**

   (b) Let f denote the function $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$
   Calculate the Taylor polynomial of order 6 of the function $f(x)$ about $x = 0$ and deduce the values of the even-order derivatives $f''(0)$, $f^{(4)}(0)$ and $f^{(6)}(0)$. (*Hint.* Use the standard Taylor polynomial of $\sin x$ to a suitable order. Do not try to calculate derivatives using the quotient rule or l'Hôpital's rule, for example.) **5**

   (c) Evaluate the limit,
$$\lim_{x \to 0} \frac{\sin 3x - 3\sin x}{x^3},$$
   by using Taylor polynomials of suitable order. **3**

MARKS

3. (a) Find the following limits, showing the steps of your working clearly, or show that the limit does not exist. (You may use any valid method. Allow $+\infty$ and $-\infty$ as values that a limit can take.)

(*i*) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+2ax} - \sqrt{x^2-2bx}$, a and b real constants. **3**

(*ii*) $\displaystyle\lim_{(x,y)\to(0,0)} \ln\big(\sin(x^2+y^2)\big)$. **3**

(*iii*) $\displaystyle\lim_{x\to 1} \frac{(\ln x)^2}{1+\cos \pi x}$. **3**

(b) Prove that the graph of $y = x^{3/5}$ has a vertical tangent at the origin. **3**

4. (a) A cardioid is a closed plane curve having the parametric equations:

$$x = R(2\cos\theta - \cos 2\theta), \qquad y = R(2\sin\theta - \sin 2\theta),$$

$0 \le \theta \le 2\pi$, R positive constant. Find the equation of the tangent line to the cardioid at the point $(x, y) = (R, 2R)$. **4**

(b) The function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by the rule

$$f(x,y) = \begin{cases} \dfrac{x^5 y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(*i*) Evaluate the partial derivatives,

$$f_x(0,y), \qquad f_x(0,0), \qquad f_y(x,0), \qquad f_y(0,0).$$

(The definition of partial derivative as a limit is the recommended method.) **4**

(*ii*) Evaluate the mixed second derivatives,

$$f_{xy}(0,0), \qquad f_{yx}(0,0).$$

(You will notice that they are not equal to each other.) **4**

End of Extended Answer Section