The University of Sydney
School of Mathematics and Statistics

Assignment 1

MATH1902: Linear Algebra (Advanced) — Semester 1, 2017

Web Page: http://sydney.edu.au/science/maths/u/UG/JM/MATH1902/
Lecturer: Holger Dullin and Alexander Kerschl

This assignment is due by **11:59pm Monday 20th March**, via Turnitin. A PDF copy of your answers must be uploaded in the Learning Management System (Blackboard) at https://elearning.sydney.edu.au. Please submit only a single PDF document (scan or convert other formats). It should include

- your name and SID,
- your tutorial time, day, room and Tutor's name.

Printed/typed solutions are acceptable. It is your responsibility to preview each page of your assignment after uploading to ensure each page is included in correct order and is legible (not sideways or upside down) before confirming your submission. The School of Mathematics and Statistics encourages some collaboration between students when working on problems, but students must write up and submit their own version of the solutions.

This assignment is worth 5% of your final assessment for this course.

Your answers should be well written, neat, thoughtful, mathematically concise, and a pleasure to read. Please cite any resources used and show all working. Present your arguments clearly using words of explanation and diagrams where relevant. After all, mathematics is about communicating your ideas. This is a worthwhile skill which takes time and effort to master.

The marker will give you feedback and allocate an overall letter grade and mark to your assignment using the following criteria:

Mark	Grade	Criterion
10	A+	Outstanding and scholarly work, answering all parts of all questions correctly, with clear accurate explanations and all relevant diagrams and working. There are at most only minor or trivial errors or omissions.
9	A	Very good work, making excellent progress on at least 1 question and good progress on the remaining question, but with one or two substantial errors, misunderstandings or omissions throughout the assignment.
7	B	Good work, making good progress on 1 question and some progress on the second, but making more than two distinct substantial errors, misunderstandings or omissions throughout the assignment.
6	C	A reasonable attempt, making substantial progress on only 1 question and some progress on another question.
4	D	Some attempt, with substantial progress made on only 1 question.
2	E	No substantial progress made on any of the 2 questions.
0	F	No credit awarded.

1. Given any vector space V we can define a subspace U. A non-empty subset U of V is called subspace, if for any two vectors $\mathbf{u}, \mathbf{v} \in U$ and any scalar λ we have $\mathbf{u} + \mathbf{v} \in U$ and $\lambda \mathbf{u} \in U$. So in words, a subset of vectors is a subspace, if the sum of any two vectors in the subset and the scalar multiple of any vector in the subset is also part of the subset. Now let $V = \mathbb{R}^3$ be the space of geometric vectors in three dimensions and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the standard unit vectors.

(a) For each of the following subsets U of V, either show that it is a subspace, or give a counterexample that shows that it is not a subspace.

(i) $U = \{\mathbf{i} - 3\mathbf{j}, \mathbf{i} - \mathbf{j} - \mathbf{k}, -2\mathbf{j} + \mathbf{k}, \mathbf{0}\}$, the set of the 4 given vectors

(ii) $U = \{\alpha \mathbf{j} \mid 0 \neq \alpha \in \mathbb{R}\}$

(iii) $U = \{\alpha(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \mid \alpha \in \mathbb{R}\}$

(iv) $U = \{\alpha \mathbf{i} + \beta \mathbf{k} \mid \alpha, \beta \in \mathbb{R}\}$

(v) $U = \{\mathbf{v} \mid |\mathbf{v}| \leq 1\}$, the set of vectors with length up to 1

(vi) $U = \{\mathbf{0}\}$, the set just containing the zero vector

(b) Consider U from part (iv). Write down two vectors $\mathbf{u}, \mathbf{v} \in U$ that are linearly independent. Show that if you take any third vector $\mathbf{w} \in U$, then the three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent. (This establishes that the dimension of U is 2.)

(c) Show that for any subspace U of V, the zero vector $\mathbf{0}$ is part of U.

2. Recall that the eight axioms of a vector space are

1)	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	commutative addition
2)	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	associative addition
3)	$\mathbf{v} + \mathbf{0} = \mathbf{v}$	existence of zero vector
4)	$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	existence of additive inverse
5)	$\lambda(\mu \mathbf{v}) = (\lambda \mu)\mathbf{v}$	associative scalar multiplication
6)	$\lambda(\mathbf{u} + \mathbf{v}) = \lambda \mathbf{u} + \lambda \mathbf{v}$	distributive I
7)	$(\lambda + \mu)\mathbf{v} = \lambda \mathbf{v} + \mu \mathbf{v}$	distributive II
8)	$1\mathbf{v} = \mathbf{v}$	scalar 1 is neutral element

where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are arbitrary vectors and λ, μ are scalars.

(a) Consider the set of polynomials $a + bx + cx^2$ in the variable x of up to degree 2. Here a, b, c are real numbers and the scalars are real numbers as well. Addition of 'vectors' is the usual addition of polynomials, and scalar multiplication is the usual multiplication of a polynomial by a real number. Show that there is a polynomial that acts like the zero vector and satisfies axiom 3. Show that for every polynomial $\mathbf{v}$ in the space there is a polynomial $\mathbf{u}$ such that $\mathbf{v} + \mathbf{u} = \mathbf{0}$, establishing axiom 4. (In fact all eight axioms hold, so this is a vector space.)

(b) Consider the set of ordered pairs of real numbers (a, b). The operation $+$ for ordered pairs is defined by $(a, b) + (c, d) = (a + c, b + d)$ and scalar multiplication is defined by $\lambda(a, b) = (\lambda a, \lambda b)$. Carefully prove that the axiom 1, 2, and 7 hold. (In fact all eight axioms hold, so this is a vector space.)

(c) Consider the set of ordered pairs of real numbers (a, b). The operation $+$ in this case is defined by $(a, b) + (c, d) = ((a + c)/2, (b + d)/2)$ and scalar multiplication is defined by $\lambda(a, b) = (\lambda a, \lambda b)$. Consider axioms 1, 2, 6, and 7. For each one, either prove that the axiom holds, or give an example that shows that it does not hold.