THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

MATH1903/1907

INTEGRAL CALCULUS AND MODELLING (ADVANCED)

November 2010

LECTURERS: H Dullin, J Parkinson

TIME ALLOWED: **One and a half hours**

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination; there are 20 questions; the questions are of equal value; all questions may be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination; there are 4 questions; the questions are of equal value; all questions may be attempted; working must be shown.

Approved non-programmable calculators may be used.
There is a table of integrals after the last question in this booklet.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

MARKER'S USE ONLY	

Extended Answer Section

There are **four** *questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

MARKS

1. (a) Let D be the region of the plane bounded by the x-axis, the y-axis, the line $x = 1$, and the curve $y = \cosh x$.

(i) Compute the area of D. **2**

(ii) Compute the volume of the solid obtained by rotating D about the y-axis **2**

MARKS

(b) Let $I(x) = \int_0^x \sqrt{1+t^3}\,dt$. Calculate the integral 2

$$\int_0^1 xI(x)\,dx.$$

Note: The constant $I(1)$ will appear in your answer.

MARKS

(c) Let $s_n = \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}$.

(*i*) Let P be the partition of $[0, n]$ into n subintervals of length 1. Use the corresponding upper and lower Riemann sums for the integral $\displaystyle\int_0^n \sqrt{x}\, dx$ to find upper and lower bounds for s_n, such that the bounds differ by at most $\sqrt{n}$. **3**

(*ii*) Hence, or otherwise, calculate the limit $\displaystyle\lim_{n\to\infty} \frac{s_n}{n^{3/2}}$. **1**

Question 2 begins on the next page

MARKS

2. (a) (i) Use a comparison test to show that $\displaystyle\int_0^\infty \frac{e^x}{7+2\cosh(2x)}\,dx$ converges. **2**

(ii) Using an appropriate substitution, or otherwise, calculate the integral **2**

$$\int_0^1 \frac{xe^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}\,dx.$$

MARKS

(b) (i) For integers $m, n \geq 0$ let $I_{m,n} = \int_0^1 x^m (\ln x)^n \, dx$. Show that for $n \geq 1$, **3**

$$I_{m,n} = -\frac{n}{m+1} I_{m,n-1},$$

and hence compute $I_{m,n}$.

[You may use the fact that $\lim_{x \to 0^+} x^\alpha (\ln x)^\beta = 0$ for all $\alpha > 0$ and $\beta \geq 0$.]

MARKS

(*ii*) Hence show that 3

$$\int_0^1 x^{-x}\,dx = \sum_{k=1}^{\infty} n^{-n}.$$

You may assume that any reasonable series manipulations are valid.

QUESTION 3 BEGINS ON THE NEXT PAGE

MARKS

3. (a) Find the general solution of the differential equation 4

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3e^{-2x}$$

MARKS

(b) Find the general solution of 3

$$\frac{dy}{dx} = \frac{2x+1}{x^2+x+1}(1-y),$$

and show that every solution converges to the equilibrium solution $y = 1$ for $x \to \infty$.

MARKS

(c) Consider the differential equation of the form **3**

$$\frac{dy}{dx} - e^{-x-y} + 1 = 0\,.$$

Introduce a new dependent variable u given by $u = x + y$, and hence find the general solution of the original equation.

Question 4 begins on the next page

MARKS

4. (a) A spherical raindrop evaporates at a rate proportional to its surface area, retaining the spherical shape. Derive a differential equation for the radius $r(t)$ of the raindrop and solve it for a raindrop with initial radius r_0 to show that **2**

$$r(t) = r_0 - \alpha t$$

for a constant $\alpha > 0$.

[Note that the volume of a sphere of radius r is $V = 4\pi r^3/3$, and that the surface area is $A = 4\pi r^2$, and assume that the density of water is 1.]

MARKS

(b) The evaporating raindrop is falling towards the ground. For this type of problem with time-dependent mass the appropriate form of Newton's second law states that the rate of change of the product of mass m with velocity v is equal to the force. The force is given by mg (with positive direction down), where g is the constant gravitational acceleration, with an additional air friction force proportional to the area πr^2 times the velocity. The friction force opposes the velocity. Show that the differential equation for the velocity v of the falling raindrop can be written as **3**

$$\frac{dv}{dt} - \frac{k\alpha}{r(t)}v = g$$

for some constant k.

MARKS

(c) Find the particular solution of the differential equation for the falling raindrop for which initially the raindrop is at rest. Assume that $k \neq -1$. **3**

MARKS

(d) Assume that $k = -2$. Compute the distance the drop falls from rest until it is completely evaporated. **2**

THERE IS MORE SPACE AVAILABLE ON THE NEXT PAGE

Table of Standard Integrals

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

3. $\displaystyle\int e^x\,dx = e^x + C$

4. $\displaystyle\int \sin x\,dx = -\cos x + C$

5. $\displaystyle\int \cos x\,dx = \sin x + C$

6. $\displaystyle\int \tan x\,dx = -\ln\left|\cos x\right| + C$

7. $\displaystyle\int \cot x\,dx = \ln\left|\sin x\right| + C$

8. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

9. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

10. $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

11. $\displaystyle\int \sec x\,dx = \ln\left|\sec x + \tan x\right| + C$

12. $\displaystyle\int \operatorname{cosec} x\,dx = \ln\left|\operatorname{cosec} x - \cot x\right| + C$

13. $\displaystyle\int \sinh x\,dx = \cosh x + C$

14. $\displaystyle\int \cosh x\,dx = \sinh x + C$

15. $\displaystyle\int \tanh x\,dx = \ln\cosh x + C$

16. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (|x| < a)$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C'$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C' \quad (x > a)$

19. $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \quad (|x| \neq a)$

End of Extended Answer Section