§2 Integers, Modular Arithmetic, and Relations

- Recall the commonly-used sets in our number system:
  - *Positive integers* $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.
  - *Natural numbers* $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
  - *Integers* $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
  - *Rational numbers* $\mathbb{Q} = \left\{\frac{p}{q} \,\middle|\, p, q \in \mathbb{Z},\ q \neq 0\right\}$.
  - *Real numbers* $\mathbb{R}$ (includes $\mathbb{Q}$ and *irrational numbers* such as $\sqrt{2}$, π, e).
  - *Complex numbers* $\mathbb{C} = \{x + i\,y \mid x, y \in \mathbb{R}\}$, with $i = \sqrt{-1}$.

  Note that $\mathbb{Z}^+ \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.
- *Number theory* focuses on $\mathbb{Z}$ and its subsets.
- We can add, multiply, subtract, and divide in $\mathbb{Q}$ and $\mathbb{R}$,
  but we *cannot always* divide in $\mathbb{Z}$; for instance, $\frac{2}{3} \notin \mathbb{Z}$.

- Let a and b be integers. If there is an integer m such that $b = am$, then
  - b is a *multiple* of a,
  - a is a *factor* of b,
  - a is a *divisor* of b,
  - a *divides* b,
  - b is *divisible* by a,

  and we write $a \mid b$.
- We write $a \nmid b$ if a does not divide b.
- If a and b are positive integers and $a \mid b$, then we must have $a \leq b$.
- $a \mid b$ ("a divides b") is a statement about divisibility that is either true or false.
  $\frac{a}{b}$ ("a divided by b") is a number that we get by carrying out division.
  The divisibility symbol $a \mid b$ and the division symbol a/b are not to be confused.
- Divisibility by zero is well-defined but mostly pointless,
  since $0 \mid b$ only holds when $b = 0$.

Exercise. Compare the following notations.

$12/48$	$\dfrac{12}{48}$	$12 \mid 48$	$12 \nmid 48$
$48/12$	$\dfrac{48}{12}$	$48 \mid 12$	$48 \nmid 12$

- Properties of divisibility: let a, b, and c be integers, then

  (i) $a \mid 0$, (Each integer is a factor of 0 and 0 is a multiple of every integer.)

  (ii) if $a \mid b$, then $a \mid bc$;

  (iii) if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$;

  (iv) if $a \mid b$ and $a \mid c$, then $a \mid (sb+tc)$ for all integers s and t; (Important!)

  (v) if $a \mid b$ and $b \mid c$, then $a \mid c$. (*Transitivity* of divisibility)

Proof. (v) Suppose that $a \mid b$ and $b \mid c$.
Then $b = am$ and $c = bn$ for some integers m and n. Thus, we have

$$c = bn = (am)n = a(mn) = ak,$$

where $k = mn$ is an integer. Hence, $a \mid c$.

Exercise. Prove (i)-(iv).

- Simple divisibility tests:

2	Last digit is 0, 2, 4, 6, or 8.
3	Sum of digits is divisible by 3.
4	Last two digits are divisible by 4.
5	Last digit is 0 or 5.
6	Divisible by 2 and 3.
7	Double the last digit and subtract it from the remaining leading truncated number. If the result is divisible by 7, then so was the original number. Apply this rule over and over again as necessary.
8	Last three digits is divisible by 8.
9	Sum of digits is divisible by 9.
10	Last digit is 0.
11	The difference between the sum of digits in the odd positions and the sum of digits in the even positions is divisible by 11.
$\vdots$	

Exercise.
Is 408254 a multiple of 3? Is 408254 divisible by 7? Does 11 divide 408254?

- An *even* number is an integer that is divisible by 2.
  - 0 is an even number.
  - Every even number n can be written as $n = 2k$ for some integer k.
- An *odd* number is an integer that is not an even number.
  - Every odd number n can be written as $n = 2k + 1$ for some integer k.

- A *prime* is an integer larger than 1 whose only positive factors are 1 and itself.
  - The first few primes are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$.
  - The smallest prime is 2, the only even prime.
  - There are infinitely many primes; this has been known for over 2000 years.
  - Primes of the form $2^p - 1$, where p is prime, are called *Mersenne primes*. The largest prime currently known is a Mersenne prime, $2^{82,589,933} - 1$, discovered in December 2018. It has 24,862,048 decimal digits.
  - Primes of the form $2^{2^n} + 1$ are known as *Fermat primes*. Only five Fermat primes are known: $3, 5, 17, 257, 65537$.
  - *Twin primes* are pairs of primes that differ by 2, such as 3 and 5, 5 and 7, 11 and 13, 17 and 19, and 1000000000061 and 1000000000063. There are thought to be infinitely many twin primes but no proof exists.
  - Yitang Zhang showed in 2013 that there are infinitely many pairs of primes that differ by at most some fixed finite number N. This inspired Terence Tao and others to show that N is smaller than 246. That is still far from 2 and infinitely many prime pairs though!

- A *composite* number is an integer that is not prime (and not -1, 0, or 1).
- 1 is neither prime nor composite.

- Prime numbers are the basic building blocks of all the integers. If a positive integer is not prime, then it can be factorized into a product of two smaller numbers. If these numbers are not prime, then they can be factorized further, and the process continues until all the factors are prime. This is called *prime factorization*.

Example. We can write

$$60 = 6 \times 10 = 2 \times 3 \times 2 \times 5 \qquad \text{or} \qquad 60 = 15 \times 4 = 3 \times 5 \times 2 \times 2.$$

It does not matter how we start; we always end up with the same prime factors, but maybe in different order.

- **The Fundamental Theorem of Arithmetic.**
  Every positive integer has a unique prime factorization,
  apart from the order of the prime factors.

Proof. The proof uses mathematical induction, to be covered in Topic 3. See also textbook [Epp, Exercise 41 of Section 10.4].

- We often standardize the prime factorization by writing the prime factors in increasing order and collecting the same prime into powers:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

  for primes $p_1 < p_2 < \cdots < p_k$ and exponents $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{Z}^+$.
- A prime number is a product of just one prime, namely itself.
- 1 is a product of no primes.

Example.

$$60 = 2^2 \times 3 \times 5\,;$$

$$1000 = 2^3 \times 5^3\,; \quad 1001 = 7 \times 11 \times 13\,; \quad 1002 = 2 \times 3 \times 167\,.$$

Exercise. Find the prime factorization of 345 and 567.

- How do we determine whether or not a given positive integer n is prime?
  - The obvious way to do this is to check whether n is a multiple of any of the numbers $2, 3, \ldots, n-1$. If none of these is a factor, then n is prime; if any of them is a factor, then n is composite.
    There are $n-2$ numbers to check.
  - It is enough to check only the primes among these $n-2$ numbers. (*Why*?)
  - It is enough to check only the primes up to $\sqrt{n}$. (See next page.)
  - There are methods to test primality much faster than prime factorization. (See MATH2400 and MATH3411.)

- **Theorem.** If n is composite, then n has a prime factor at most equal to $\sqrt{n}$.

  Equivalently...

- **Theorem.** If n has no prime factor less than or equal to $\sqrt{n}$, then it is prime.

Proof.

Exercise. Is 161 prime? Is 163 prime? Is 167 prime?

- Let a and b be non-zero integers.
  Integer d is a *common divisor* or *common factor* of a and b if $d|a$ and $d|b$.
  The largest such d is called the *greatest common divisor* $\gcd(a, b)$ of a and b.

- If $\gcd(a, b) = 1$, then a and b are *coprime* or *relatively prime* to each other.

- Integer m is a *common multiple* of a and b if $a \mid m$ and $b \mid m$.
  The smallest such $m > 0$ is the *least common multiple* $\operatorname{lcm}(a, b)$ of a and b.

- If a and b are positive integers, then $\gcd(a, b) \times \operatorname{lcm}(a, b) = ab$.

Example. The positive factors of 12 are $\{1, 2, 3, 4, 6, 12\}$.
The positive divisors of 42 are $\{1, 2, 3, 6, 7, 14, 21, 42\}$.
The common divisors of 12 and 42 are $\{1, 2, 3, 6\}$.
Thus, $\gcd(12, 42) = 6$.

The positive multiples of 12 are $\{12, 24, 36, 48, 60, 72, 84, \ldots\}$.
The positive multiples of 42 are $\{42, 84, 126, \ldots\}$.
Thus, $\operatorname{lcm}(12, 42) = 84$.

Example. We can find the gcd and lcm of two numbers through their prime factorizations. For example, consider

$$14175 = 3^4 \times 5^2 \times 7 \qquad \text{and} \qquad 16758 = 2 \times 3^2 \times 7^2 \times 19\,.$$

For the gcd, we multiply all the prime factors common to both:

$$\gcd(14175, 16758) = 3^2 \times 7 = 63\,.$$

For the lcm, take the smallest product that includes all factors of both numbers:

$$\operatorname{lcm}(14175, 16758) = 2 \times 3^4 \times 5^2 \times 7^2 \times 19 = 3770550\,.$$

Exercise. Find the gcd and lcm of $a = 2^3 \times 3 \times 5^2 \times 11$ and $b = 3 \times 5 \times 7$.

Exercise. If a is positive and is a factor of b, then what is $\gcd(a, b)$?

Exercise. If a is positive, then what is $\gcd(a, 0)$?

Exercise. If a is positive, then what is $\gcd(a, a)$?

Exercise. What happens if we try to compute $\gcd(0, 0)$?

Exercise. If a is any integer, then what happens if we try to compute $\operatorname{lcm}(a, 0)$?

The Division Algorithm. Let a be an integer and b be a positive integer. Then there is a unique pair of integers q and r (called *quotient* and *remainder*) such that

$$a = q\,b + r \qquad \text{and} \qquad 0 \leq r < b.$$

Proof. See textbook [Epp, Section 4.4 and Exercise 18 of Section 3.7].

Example. We can find the quotient and remainder by long division or by repeated subtraction. For example, we divide 92 and -92 by 7.

$$\begin{array}{rrrr}
\begin{array}{r} 13 \\ 7\ \overline{)92} \\ \underline{7\ } \\ 22 \\ \underline{21} \\ 1 \end{array} &
\begin{array}{r} 13 \\ 7\ \overline{)92} \\ \underline{91} \\ 1 \\ \\ \\ \end{array} &
\begin{array}{r} -13 \\ 7\ \overline{)\,-92} \\ \underline{-91} \\ -1 \\ \\ \\ \end{array} &
\begin{array}{r} -14 \\ 7\ \overline{)\,-92} \\ \underline{-98} \\ 6 \\ \\ \\ \end{array} \\
\textit{long division} & \textit{simplified} & \textit{incorrect} & \textit{correct}
\end{array}$$

We see that $92 = 13 \times 7 + 1$.
Thus, when 92 is divided by 7, the quotient is 13 and the remainder 1.

We have $-92 = (-13) \times 7 + (-1)$ and $-92 = (-14) \times 7 + 6$. Since the remainder should lie between 0 and 6, we conclude that when -92 is divided by 7, the quotient is -14 and the remainder 6.

Exercise. Find the quotient and remainder when 1001 is divided by 101.

Exercise. Find the quotient and remainder when 101 is divided by 1001.

Exercise. Find the quotient and remainder when -1001 is divided by 101.

Theorem. Let a, b, q, and r be integers such that $a = q\,b + r$, where a and b are not both zero. Then

$$\gcd(a,b) \;=\; \gcd(b,r).$$

Proof. Write $d_1 = \gcd(a,b)$ and $d_2 = \gcd(b,r)$.

Since $d_2 \mid b$ and $d_2 \mid r$, we have $d_2 \mid (q\,b + r)$ and thus $d_2 \mid a$.
We see that d_2 is a common divisor of a and b.
But since d_1 is the greatest common divisor of a and b, we must have $d_2 \leq d_1$.

Conversely, we can write $r = a - q\,b$.
Since $d_1 \mid a$ and $d_1 \mid b$, we have $d_1 \mid (a - q\,b)$ and thus $d_1 \mid r$.
This shows that d_1 is a common divisor of b and r, and hence $d_1 \leq d_2$.

For both $d_2 \leq d_1$ and $d_1 \leq d_2$ to be true we require that $d_1 = d_2$.

Euclidean Algorithm. Use the above theorem together with the Division Algorithm repeatedly to calculate the greatest common divisor of two numbers.

Example. We use the Euclidean Algorithm to compute the greatest common divisor of 16758 and 14175 as follows:

$$\begin{aligned}
\underline{16758} &= 1 \times \underline{14175} + \underline{2583}, && \text{so } \gcd(16758, 14175) = \gcd(14175, 2583).\\
\underline{14175} &= 5 \times \underline{2583} + \underline{1260}, && \text{so } \gcd(14175, 2583) = \gcd(2583, 1260).\\
\underline{2583} &= 2 \times \underline{1260} + \underline{63}, && \text{so } \gcd(2583, 1260) = \gcd(1260, 63).\\
\underline{1260} &= 20 \times \underline{63} + 0, && \text{thus } 63 \mid 1260 \text{ and so } \gcd(1260, 63) = 63.
\end{aligned}$$

Hence, $\gcd(16758, 14175) = 63$. Moreover, we have

$$\operatorname{lcm}(16758, 14175) \;=\; \frac{16758 \times 14175}{\gcd(16758, 14175)} \;=\; 3770550\,.$$

Exercise. Use the Euclidean Algorithm to find $\gcd(854, 651)$.

- We can use the Euclidean Algorithm to find an integer solution of x and y to the equation

$$ax + by = \gcd(a, b)\,.$$

This is done by working backward through the Euclidean Algorithm; this process is known as the *Extended Euclidean Algorithm*.

Example. We look for an integer solution of x and y to the equation

$$16758x + 14175y = 63\,.$$

Recall that we obtained $\gcd(16758, 14175) = 63$ by the Euclidean Algorithm

$$\begin{aligned}
\underline{16758} &= 1 \times \underline{14175} + \underline{2583} && (3)\\
\underline{14175} &= 5 \times \underline{2583} + \underline{1260} && (2)\\
\underline{2583} &= 2 \times \underline{1260} + \underline{63} && (1)\\
\underline{1260} &= 20 \times \underline{63} + 0\,.
\end{aligned}$$

We now begin by rearranging the second to last equation:

$$\begin{aligned}
\underline{63} &= \underline{2583} - 2 \times \underline{1260} && \text{by equation (1)}\\
&= \underline{2583} - 2\,(\underline{14175} - 5 \times \underline{2583}) && \text{by equation (2)}\\
&= 11 \times \underline{2583} - 2 \times \underline{14175} && \text{collect like terms}\\
&= 11\,(\underline{16758} - \underline{14175}) - 2 \times \underline{14175} && \text{by equation (3)}\\
&= 11 \times \underline{16758} - 13 \times \underline{14175} && \text{collect like terms}\,.
\end{aligned}$$

Thus,

$$16758 \times 11 \;+\; 14175 \times (-13) \;=\; 63\,.$$

Hence, $16758x + 14175y = 63$ has an integer solution $x = 11$ and $y = -13$.

Furthermore, we see that

- $16758x + 14175y = 126$ has an integer solution $x = 22$ and $y = -26$, since
$$16758 \times (11{\times}2) \;+\; 14175 \times (-13{\times}2) \;=\; 63{\times}2.$$
- $16758x + 14175y = 630$ has an integer solution $x = 110$ and $y = -130$, since
$$16758 \times (11{\times}10) \;+\; 14175 \times (-13{\times}10) \;=\; 63{\times}10.$$
- $16758x + 14175y = -189$ has an integer solution $x = -33$ and $y = 39$, since
$$16758 \times (11{\times}(-3)) \;+\; 14175 \times (-13{\times}(-3)) \;=\; 63{\times}(-3).$$
- $16758x + 14175y \;=\; 60$ has no integer solution, since $63 \nmid 60$.

The Bézout Property. Consider the equation

$$ax + by = c,$$

where a, b, and c are integers, with a and b not both zero. Then

(i) if $c = \gcd(a, b)$, then the equation has an integer solution;

(ii) if c is a multiple of $\gcd(a, b)$, then the equation has an integer solution;

(iii) if c is not a multiple of $\gcd(a,b)$, then the equation has no integer solution.

Proof. Let $d = \gcd(a, b)$.
The proof of (i) is a bit complicated; see Exercise 11 in Problem Set 2.
Assuming that (i) proved, then we have integers x_0 and y_0 such that

$$ax_0 + by_0 = d.$$

If c is a multiple of d, then $c = dm$ for some integer m, and we have

$$a(x_0m) + b(y_0m) = dm = c.$$

Thus, the equation $ax + by = c$ has an integer solution $x = x_0m$ and $y = y_0m$.

Suppose now that c is not a multiple of d. If x and y were integers, then we would have $d \mid (ax + by)$ and hence $d \mid c$, which contradicts the fact that c is not a multiple of d. Hence, in this case x and y cannot be integers.

Exercise. Use the Extended Euclidean Algorithm to find integer solutions to the equations

$$520x - 1001y = 13, \quad 520x - 1001y = -26, \quad \text{and} \quad 520x - 1001y = 1.$$

- Recall that the Division Algorithm states

  if a is an integer and m is a positive integer, then there exist unique integers q and r, called the quotient and the remainder, respectively, such that $a = q\,m + r$ and $0 \leq r < m$.

  We define $a \bmod m$ (reads "a *modulo* m") to be this remainder r.
  This is called *modular arithmetic*, and the number m is called the *modulus*.

Exercise. Evaluate

$11 \bmod 3$ $\qquad$ $5 \bmod 7$ $\qquad$ $-11 \bmod 3$ $\qquad$ $-5 \bmod 7$

- Let m be a positive integer.
  Integers a and b are *congruent modulo* m, denoted by $a \equiv b \pmod{m}$, if

  $$(a \bmod m) = (b \bmod m),$$

  that is, if a and b have the same remainder when divided by m.

- Equivalent definitions of congruence:

  (i) $a \equiv b \pmod{m}$,

  (ii) $(a \bmod m) = (b \bmod m)$,

  (iii) $m \mid (a - b)$,

  (iv) $a = b + km$ for some integer k.

Proof.
(i) and (ii) are equivalent by definition.

(iii) and (iv) are equivalent by definition.

Let us prove that (ii) implies (iii).
Suppose that $(a \bmod m) = (b \bmod m) = r$ for some integer $0 \leq r < m$.
Then $a = q_1 m + r$ and $b = q_2 m + r$ for some integers q_1 and q_2. Thus,

$$a - b = (q_1 m + r) - (q_2 m + r) = (q_1 - q_2)m = km\,,$$

where $k = q_1 - q_2$ is an integer. Hence, we have $m \mid (a - b)$.

Finally, let us prove that (iv) implies (ii). (*Why does this prove the result?*)

- Properties of congruence: if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then
  - (i) $a + c \equiv b + d \pmod{m}$;
  - (ii) $a - c \equiv b - d \pmod{m}$;
  - (iii) $ac \equiv bd \pmod{m}$;
  - (iv) $a^n \equiv b^n \pmod{m}$ for all $n \geq 0$;
  - (v) $\ell a \equiv \ell b \pmod{m}$ for all integers ℓ;
  - (vi) $a \equiv b \pmod{n}$ for all integers n satisfying $n \mid m$.

Proof.
Suppose that $a = b + k_1 m$ and $c = d + k_2 m$ for some integers k_1 and k_2.

(i) $a + c = (b + k_1 m) + (d + k_2 m) = (b + d) + (k_1 + k_2)m = (b + d) + km$, where $k = k_1 + k_2$ is an integer. Thus, $a + c \equiv b + c \pmod{m}$.

(ii) Similar to (i).

Example. The last two digits of the number 1234567 is the number 67. This can be formally expressed as

$$1234567 \bmod 100 = 67 \qquad \text{or} \qquad 1234567 \equiv 67 \pmod{100}.$$

Similarly, to find the last two digits of the number $7^{1234567}$, we need to evaluate $7^{1234567} \bmod 100$. We have

$$\begin{aligned}
7^1 &\equiv 7 \pmod{100};\\
7^2 &\equiv 49 \pmod{100};\\
7^3 &\equiv 49 \times 7 \equiv 343 \equiv 43 \pmod{100};\\
7^4 &\equiv 43 \times 7 \equiv 301 \equiv 1 \pmod{100}.
\end{aligned}$$

Then it is easy to obtain, for example,

$$\begin{aligned}
7^8 &\equiv (7^4)^2 \equiv 1^2 \equiv 1 \pmod{100};\\
7^{444} &\equiv (7^4)^{111} \equiv 1^{111} \equiv 1 \pmod{100};\\
7^{446} &\equiv (7^4)^{111} \times 7^2 \equiv 1^{111} \times 49 \equiv 49 \pmod{100};
\end{aligned}$$

and in particular, we have

$$7^{1234567} \equiv 7^{4\times 308641+3} \equiv (7^4)^{308641} \times 7^3 \equiv 1^{308641} \times 43 \equiv 43 \pmod{100}.$$

Exercise. Simplify $10^{123456789} \bmod 41$.

Example. We have seen that simplifying $a^n \bmod m$ becomes quite easy if there is a small number k such that $a^k \equiv 1 \pmod{m}$. In a similar way, it is also useful to have $a^k \equiv -1 \pmod{m}$. The trick is to try and keep the numbers between $-m/2$ and $m/2$.

For example, we will try to simplify $5^{115511} \bmod 29$. We have

$$\begin{aligned}
5^1 &\equiv 5 \pmod{29}\,;\\
5^2 &\equiv 25 \equiv -4 \pmod{29}\,;\\
5^3 &\equiv (-4)\times 5 \equiv -20 \equiv 9 \pmod{29}\,;\\
5^4 &\equiv 9 \times 5 \equiv 45 \equiv 16 \equiv -13 \pmod{29}\,;\\
5^5 &\equiv (-13)\times 5 \equiv -65 \equiv -7 \pmod{29}\,;\\
5^6 &\equiv (-7)\times 5 \equiv -35 \equiv -6 \pmod{29}\,;\\
5^7 &\equiv (-6)\times 5 \equiv -30 \equiv -1 \pmod{29}\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
5^{115511} \equiv 5^{7\times 16501+4} &\equiv (5^7)^{16501} \times 5^4\\
&\equiv (-1)^{16501} \times (-13) \equiv (-1)\times(-13) \equiv 13 \pmod{29}\,.
\end{aligned}$$

Example. Unfortunately, we cannot always find k with $a^k \equiv \pm 1 \pmod{m}$. We therefore need to keep an eye out for any "pattern" in the numbers.

For example, we now try to simplify $6^{54321} \bmod 100$. We have

$$\begin{aligned}
6^1 &\equiv 6 \pmod{100}\,;\\
6^2 &\equiv 36 \pmod{100}\,;\\
6^3 &\equiv 36 \times 6 \equiv 216 \equiv 16 \pmod{100}\,;\\
6^4 &\equiv 16 \times 6 \equiv 96 \equiv -4 \pmod{100}\,;\\
6^5 &\equiv (-4)\times 6 \equiv -24 \pmod{100}\,;\\
6^6 &\equiv (-24)\times 6 \equiv -144 \equiv -44 \pmod{100}\,;\\
6^7 &\equiv (-44)\times 6 \equiv -264 \equiv 36 \pmod{100}\,.
\end{aligned}$$

Since $6^7 \equiv 6^2 \pmod{100}$, the numbers repeat every 5 steps from here on. Thus,

$$6^{54321} \equiv 6^{54316} \equiv 6^{54311} \equiv \cdots \equiv 6^6 \equiv -44 \equiv 56 \pmod{100}.$$

Since $6^6 \not\equiv 6^1 \pmod{100}$, the pattern does *not* hold for smaller powers.

Recall that **real** numbers a, b are *inverses* if $ab = 1$.

- Let m be a positive integer and $a, b \in \mathbb{Z}$ be such that $ab \equiv 1 \pmod{m}$. Then
  - a, b are *inverses* modulo m.
  - b is an *inverse* of a modulo m.
- In this case, for any integer k,

$$a(b + km) \equiv ab \equiv 1 \pmod{m}$$

so $b + km$ is also an inverse of a modulo m.

Example. 3 and 4 are inverses modulo 11 since $3 \times 4 = 12 \equiv 1 \pmod{11}$.

Example. Find an inverse x of 40 modulo 77 by extended Euclidean algorithm.

1. Re-write the congruence equation $40x \equiv 1 \pmod{77}$ as the ordinary equation

$$40x - 1 = 77y \quad \text{for some } y \in \mathbb{Z}\,.$$

This is equivalent to $40x - 77y = 1$. (This has a solution since $\gcd(40, 77) = 1$.)

2. Now use the extended Euclidean algorithm to find x.

$$\begin{aligned}
\underline{77} &= 1 \times \underline{40} + \underline{37} \\
\underline{40} &= 1 \times \underline{37} + \underline{3} \\
\underline{37} &= 12 \times \underline{3} + \underline{1} \\
\underline{3} &= 3 \times \underline{1} + 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\underline{1} &= \underline{37} - 12 \times \underline{3} \\
&= \underline{37} - 12\,(\underline{40} - \underline{37}) \\
&= 13 \times \underline{37} - 12 \times \underline{40} \\
&= 13\,(\underline{77} - \underline{40}) - 12 \times \underline{40} \\
&= 13 \times \underline{77} - 25 \times \underline{40}
\end{aligned}$$

Thus, $40 \times (-25) - 77 \times (-13) = 1$.
This gives a solution $x = -25$ (we don't care about y).
This shows that we have the following inverses of 40 modulo 77:

$$x = \ldots, -25, -25 + 77, -25 + 2 \times 77, \ldots = \ldots, -25, 52, 129, \ldots$$

Exercise. Find an inverse n of 5 modulo 11.

Problem. For integers a, b and positive integer m, find all integers x so that

$$ax \equiv b \pmod{m}.$$

This is a problem of *solving linear congruence*.

There are several cases to consider in solving this congruence equation.

Theorem. If $\gcd(a, m) \nmid b$, then $ax \equiv b \pmod{m}$ has no solutions.

Proof. The congruence equation can be re-written as $ax + my = b$ where y is any integer. But this equation only has solutions if $\gcd(a, m)|b$.

Example. Does $6x \equiv 3 \pmod{8}$ have solutions?
Answer: No, because $\gcd(6, 8) = 2$ and $2 \nmid 3$.

Theorem. If $\gcd(a, m) = 1$, then $ax \equiv b \pmod{m}$ has the solution
$x \equiv cb \pmod{m}$ where c is an inverse of a modulo m.

Proof. If $\gcd(a, m) = 1$, then a has an inverse c modulo m.
Multiplying both sides of the congruence equation by c gives

$$cb \equiv cax \equiv 1x \equiv x \pmod{m}.$$

Conversely, if $x \equiv cb \pmod{m}$, then

$$ax \equiv acb \equiv 1b \equiv b \pmod{m}.$$

Example. Solve

$$79x \equiv 12 \pmod{45}.$$

1. Since $\gcd(79, 45) = 1$, we can find an inverse c of 79 modulo 45.

2. As before, we find c using the Extended Euclidean Algorithm:

$$\begin{aligned} \underline{79} &= 1 \times \underline{45} + \underline{34} \\ \underline{45} &= 1 \times \underline{34} + \underline{11} \\ \underline{34} &= 3 \times \underline{11} + \underline{1} \\ \underline{11} &= 11 \times \underline{1} + 0 \end{aligned}$$

$$\begin{aligned} \underline{1} &= \underline{34} - 3 \times \underline{11} \\ &= \underline{34} - 3\,(\underline{45} - \underline{34}) \\ &= 4 \times \underline{34} - 3 \times \underline{45} \\ &= 4\,(\underline{79} - \underline{45}) - 3 \times \underline{45} \\ &= 4 \times \underline{79} - 7 \times \underline{45} \end{aligned}$$

   Thus, $c = 4$ is an inverse of 79 modulo 45.

3. The solution to our linear congruence $79x \equiv 12 \pmod{45}$ is therefore

$$x \equiv 4 \times 12 = 48 \equiv 3 \pmod{45},$$

   i.e., $x = \ldots, -42, 3, 48, \ldots$.

Exercise. Solve $23x \equiv 11 \pmod{30}$.

Question What if $\gcd(a, m) \neq 1$?
We use the following trick.

Theorem. If $c \neq 0$, then the congruences

$$ax \equiv b \pmod{m} \qquad \text{and} \qquad cax \equiv cb \pmod{cm}$$

have the same solutions.

$\star$ We can cancel a factor from both sides of a congruence, provided that we cancel it from the modulus as well.

Proof.

$$\begin{aligned} & ax \equiv b \pmod{m} \\ \Longleftrightarrow\ & ax = b + my \quad \text{for some integer } y \\ \Longleftrightarrow\ & cax = cb + cmy \quad \text{for some integer } y \\ \Longleftrightarrow\ & cax \equiv cb \pmod{cm} \end{aligned}$$

Example. Solve the linear congruence

$$52x \equiv 8 \pmod{60}.$$

1. We divide by $\gcd(52, 60) = 4$ and use the theorem above to see that the congruence equation above has the same solutions as

$$13x \equiv 2 \pmod{15}.$$

2. Now, $\gcd(13, 15) = 1$ so we solve the new congruence equation by finding an inverse c of 13 modulo 15 using the Extended Euclidean Algorithm.

$$\begin{aligned} \underline{15} &= 1 \times \underline{13} + \underline{2} \\ \underline{13} &= 6 \times \underline{2} + \underline{1} \\ \underline{2} &= 2 \times \underline{1} + 0 \end{aligned} \qquad \begin{aligned} \underline{1} &= \underline{13} - 6 \times \underline{2} \\ &= \underline{13} - 6\,(\underline{15} - \underline{13}) \\ &= 7 \times \underline{13} - 6 \times \underline{15} \end{aligned}$$

We see that $c = 7$ is an inverse of 13 modulo 15.

3. Hence the solutions to $13x \equiv 2 \pmod{15}$ are $x \equiv 7 \times 2 = 14 \pmod{15}$, which is also the solution to the original congruence equation.

4. Equivalently, we can write the solution in terms of the original modulus

$$x \equiv 14, 29, 44, 59 \pmod{60}.$$

5. Note that there are now $4 = \gcd(52, 60)$ solutions modulo 60.

Exercise. Solve the congruence $9x \equiv -3 \pmod{24}$. Give your answer as a congruence to the smallest possible modulus, and as a congruence modulo 24.

To summarise:

Theorem. Consider the congruence $ax \equiv b \pmod{m}$.

(i) If $\gcd(a, m) = 1$, then the congruence has a unique solution modulo m.

(ii) If $\gcd(a, m)$ is not a factor of b, then the congruence has no solution.

(iii) If $d = \gcd(a, m)$ is a factor of b, then the congruence has

- one unique solution modulo m/d, and
- d different solutions modulo m.

Exercise. Without actually solving anything, determine how many solutions the following congruences have. Give your answers in terms of the original modulus, and in terms of a smaller modulus if appropriate.

(a) $15x \equiv 18 \pmod{21}$

(b) $16x \equiv 19 \pmod{22}$

(c) $17x \equiv 20 \pmod{23}$

Sometimes we can solve congruences simply by using the following fact.

Theorem. If $\gcd(c, m) = 1$, then

$$p \equiv q \pmod{m} \quad \text{if and only if} \quad cp \equiv cq \pmod{m}.$$

Example.

$$\begin{aligned}
& 52x \equiv 4 \pmod{60} && \\
\iff & 13x \equiv 1 \pmod{15} && \text{by earlier theorem on cancelling a factor} \\
\iff & -2x \equiv -14 \pmod{15} && 13 \equiv -2 \pmod{15}, 1 \equiv -14 \pmod{15} \\
\iff & x \equiv 7 \pmod{15} && \text{by above theorem with } c = -2
\end{aligned}$$

Exercise. Prove the divisibility by 7 test, namely that

$$10a + b \equiv 0 \pmod{7} \quad \text{if and only if} \quad a - 2b \equiv 0 \pmod{7}.$$

Example. Public Key Cryptography – the **RSA System** (1976):

- Find *two large primes* p and q (e.g., 200 digits each).
- Form the *modulus* $m = pq$.
- Find an *encryption exponent* α relatively prime to $(p-1)(q-1)$.
- Find the *decryption exponent* β satisfying $\alpha\beta \equiv 1 \pmod{(p-1)(q-1)}$.
- Publish the numbers α and m. Keep p and q secret.

To encrypt...

1. Convert plain text into a string of digits to form a large integer x.
2. Compute $y = (x^\alpha \bmod m)$.
3. Send y.

To decrypt...

1. Receive y.
2. Compute $x = (y^\beta \bmod m)$. *Need to know* β*.*
3. Convert x back to plain text.

Why is this secure?

To decrypt the message we must know β, which can be obtained if p and q are known. Recall that primality testing is much faster than prime factorization. Although it is easy to find two large primes p and q to form the product $m = pq$, it is close to impossible to factorize a large m to find the values of p and q.

- A *relation* R from a set A to a set B is a subset of $A \times B$.
  - If $(a,b) \in R$ we say that "a is related to b (by R)", and we write $a\,R\,b$.
  - If $(a,b) \notin R$ we write $a\not R\,b$.
- Representing a relation $R \subseteq A \times B$ on finite sets A and B:
  - *Arrow diagram*:
    List the elements of A and the elements of B,
    and then draw an arrow from a to b for each pair $(a,b) \in R$.
  - *Matrix* M_R:
    Arrange the elements of A and B in some order $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$,
    and then form a rectangular array of numbers where

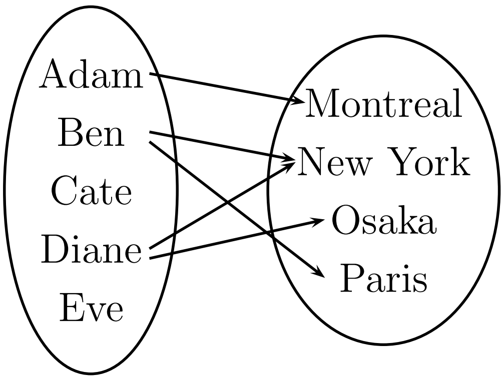
$$\text{the entry in the } i\text{th row and } j\text{th column } = m_{i,j} = \begin{cases} 1 & \text{if } a_i\,R\,b_j\,; \\ 0 & \text{if } a_i \not R\,b_j. \end{cases}$$

    - The matrix M_R has $|A|$ rows and $|B|$ columns.
    - The matrix changes if the elements are arranged in a different order.

Example. Five flatmates Adam, Ben, Cate, Diane, and Eve chatted about who had visited the four cities Montreal, New York, Osaka, and Paris.
Their travel experiences lead to a relation defined as follows:

$$\begin{aligned} A &= \{\text{Adam, Ben, Cate, Diane, Eve}\} \\ B &= \{\text{Montreal, New York, Osaka, Paris}\} \\ R &= \{(\text{Adam, Montreal}), (\text{Ben, New York}), (\text{Ben, Paris}), \\ &\qquad (\text{Diane, New York}), (\text{Diane, Osaka})\} \end{aligned}$$

The arrow diagram and matrix for this relation are



$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

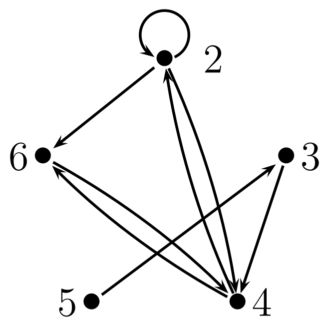
The matrix M_R is of size 5×4 (reads "5 by 4").
It is based on the alphabetical order of the names and cities.

- A *function* is a relation $R \subseteq A \times B$ with the special property that for every $a \in A$ there is exactly one $b \in B$ such that $a\,R\,b$.
- A relation over two sets as defined above is formally a *binary relation*. We can also define a *ternary relation* as a subset of the Cartesian product $A \times B \times C$ of three sets A, B, C, or in general, an *n-ary relation* as a subset of $A_1 \times A_2 \times \cdots \times A_n$ for sets $A_1, A_2, \ldots, A_n$.
- Here we shall consider mainly *binary relations on a set*, that is, a relation from a set to itself.
  - The arrow diagram in this case is essentially a *directed graph* (see Topic 5). We draw a dot for each element in the set and use an arrow or a loop to represent each ordered pair.
  - The corresponding matrix M_R is a *square* matrix; that is, there are as many rows as there are columns.

Example. We define a relation R on the set $A = \{2, 3, 4, 5, 6\}$ by

$$\begin{aligned} R &= \{(a, b) \in A \times A \mid a \text{ is a factor of } b + 2\} \\ &= \{(2,2), (2,4), (2,6), (3,4), (4,2), (4,6), (5,3), (6,4)\}\,. \end{aligned}$$

Then we can write, for example, $2\,R\,4$ and $3\,R\,4$, but $5 \not{R}\, 4$.
The arrow diagram and matrix are



$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Exercise. Let $R = \{(a, a), (a, b), (b, a), (b, b), (d, b)\}$ be a relation on the set $A = \{a, b, c, d\}$. Draw the arrow diagram of R and write the matrix of R.

- We say that a relation R on a set A is *reflexive* when for every $a \in A$,

$$a\,R\,a\,,$$

i.e., every element is related to itself.

- We say that a relation R on a set A is *symmetric* when for every $a, b \in A$,

$$a\,R\,b \quad \text{implies} \quad b\,R\,a\,,$$

i.e., if a is related to b, then b is related to a.

- We say that a relation R on a set A is *antisymmetric* when for every $a, b \in A$,

$$a\,R\,b \quad \text{and} \quad b\,R\,a \quad \text{implies} \quad a = b\,,$$

i.e., if a and b are related to each other, then they must be identical.

- We say that a relation R on a set A is *transitive* when for every $a, b, c \in A$,

$$a\,R\,b \quad \text{and} \quad b\,R\,c \quad \text{implies} \quad a\,R\,c\,,$$

i.e., if a is related to b and b is related to c, then a is related to c.

- In terms of arrow diagrams and matrices...

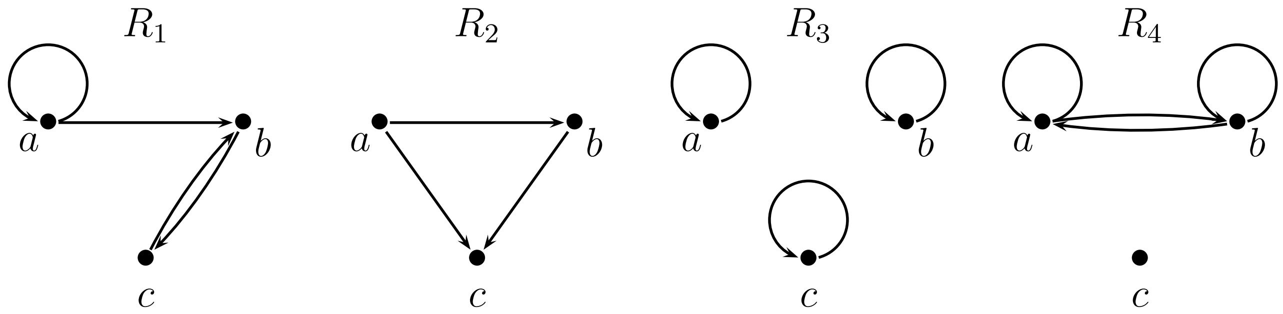
	arrow diagram	matrix
reflexive	we must have $\circlearrowleft\!\bullet$ at every dot	diagonal entries are all 1
symmetric	if we have $\bullet \longrightarrow \bullet$, then we must have $\bullet \rightleftarrows \bullet$	for $i \neq j$, $m_{i,j} = m_{j,i}$
antisymmetric	we cannot have $\bullet \rightleftarrows \bullet$	for $i \neq j$, $m_{i,j}$ and $m_{j,i}$ cannot both be 1
transitive	(i) if we have , then we must have , and (ii) if we have $\bullet \rightleftarrows \bullet$, then we must have $\circlearrowleft\!\bullet \rightleftarrows \bullet\!\circlearrowright$	for every nonzero entry in M^2 $(= M \times M)$, the corresponding entry in M must be 1

- Note that “antisymmetric” is not the opposite of “symmetric”.
A relation can be both symmetric and antisymmetric.

Exercise. Define the relations R_1, R_2, R_3, R_4 on the set $A = \{a, b, c\}$ by

$$R_1 = \{(a,a),(a,b),(b,c),(c,b)\}, \qquad R_2 = \{(a,b),(a,c),(b,c)\},$$
$$R_3 = \{(a,a),(b,b),(c,c)\}, \qquad R_4 = \{(a,a),(a,b),(b,a),(b,b)\}.$$

For each relation, determine whether it is reflexive, symmetric, antisymmetric, and/or transitive.



	R_1	R_2	R_3	R_4
reflexive				
symmetric				
antisymmetric				
transitive				

Exercise. For each relation R defined on the set of all human beings, determine whether or not it is reflexive, symmetric, antisymmetric, or transitive.

$(a,b) \in R$ if and only if	reflexive	symmetric	antisymmetric	transitive
a is the father of b				
a is a sibling of b				
a is taller than b				
a and b were born on the same day				
a and b speak a common language				
a likes b				

Exercise. For each relation R defined on the set of all integers, determine whether or not it is reflexive, symmetric, antisymmetric, or transitive.

	$(x, y) \in R$ if and only if	reflexive	symmetric	antisymmetric	transitive
(a)	$x = y$				
(b)	$x > y$				
(c)	$x \leq y$				
(d)	$x \neq y$				
(e)	$x \equiv y \pmod 7$				
(f)	x is a multiple of y				
(g)	$xy \geq 1$				
(h)	$x = y + 1$ or $x = y - 1$				
(i)	x and y are both negative or both nonnegative				
(j)	$x = y^2$				
(k)	$x \geq y^2$				

Give reasons for your answers.

- A reflexive, symmetric, and transitive relation is an *equivalence relation*.
- We often write $\sim$ to denote an equivalence relation:
  $a \sim b$ reads "a is equivalent to b".
- An equivalence relation tells us when two things are "of the same type".

Example. Let $\sim$ denote a relation on the set of real numbers defined by

$$x \sim y \quad \text{if and only if} \quad \cos x = \cos y\,.$$

(R) For all $x \in \mathbb{R}$, clearly $\cos x = \cos x$ and so $x \sim x$.
Thus, $\sim$ is reflexive.

(S) Suppose that $x \sim y$ for some $x, y \in \mathbb{R}$.
Then $\cos x = \cos y$, so $\cos y = \cos x$ which means $y \sim x$.
Thus, $\sim$ is symmetric.

(T) Suppose that $x \sim y$ and $y \sim z$ for some $x, y, z \in \mathbb{R}$.
Then $\cos x = \cos y$ and $\cos y = \cos z$.
Therefore, $\cos x = \cos z$, and so $x \sim z$.
Thus, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation.

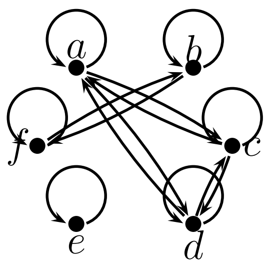
- Let $\sim$ be an equivalence relation on a set A. For any element $a \in A$, the *equivalence class* of a with respect to $\sim$, denoted by $[a]$, is the set

$$[a] \;=\; \{x \in A \,|\, x \sim a\}.$$

- Intuitively, an equivalence class collects all objects that are "of the same type".

Example. Let $A = \{a, b, c, d, e, f\}$ and

$$R \;=\; \{(a,a), (a,c), (a,d), (b,b), (b,f), (c,a), (c,c), (c,d), (d,a), (d,c), (d,d), (e,e), (f,b), (f,f)\}.$$



Since R is reflexive, symmetric, and transitive, it is an equivalence relation.
The equivalence classes are

$$[a] \;=\; \{a, c, d\} \qquad [b] \;=\; \{b, f\} \qquad [c] \;=\; \{a, c, d\}$$
$$[d] \;= \qquad\qquad [e] \;= \qquad\qquad [f] \;=$$

In particular, we have $[a] = [c] = [d]$ and $[b] = [f]$.

Exercise. Let $A = \{x \in \mathbb{N} \,|\, 2 \leq x \leq 12\}$ and define

$$x \sim y \quad \text{if and only if} \quad x \text{ and } y \text{ have exactly the same prime factors.}$$

List the equivalence classes with respect to $\sim$.

Theorem. Let $\sim$ be an equivalence relation on a set A. Then

(i) For all $a \in A$, $a \in [a]$.
- ⋆ Every element of A belongs to at least one equivalence class.
- ⋆ Every equivalence class contains at least one element.

(ii) For all $a, b \in A$, $a \sim b$ if and only if $[a] = [b]$.

(iii) For all $a, b \in A$, $a \not\sim b$ if and only if $[a] \cap [b] = \varnothing$.
- ⋆ The equivalence classes are either equal or disjoint.

Proof.

(i) Since $\sim$ is reflexive, $a \sim a$ and so $a \in [a]$.

(ii) Let $a \sim b$ and suppose that $x \in [a]$; then $a \sim x$.
Since $\sim$ is symmetric, $b \sim a$.
Since $\sim$ is transitive, $b \sim x$, so $x \in [b]$.
Hence, $[a] \subseteq [b]$. Similarly, we can show that $[b] \subseteq [a]$,
and we conclude that $a \sim b$ implies that $[a] = [b]$.

Now suppose that $[a] = [b]$. By i), we have $a \in [a]$ so $a \in [b]$. Then $a \sim b$.
Hence, $a \sim b$ if and only if $[a] = [b]$.

(iii) Suppose that $a \not\sim b$ and assume by contradiction that $x \in [a] \cap [b]$.
Then $x \sim a$ and $x \sim b$. Since $\sim$ is symmetric, $a \sim x$.
Since $\sim$ is transitive, $a \sim b$, a contradiction.
Hence, if $a \not\sim b$, then $[a] \cap [b] = \varnothing$.

Now suppose that $[a] \cap [b] = \varnothing$ and assume by contradiction that $a \sim b$.
Then $a \in [b]$. By (i), we also have $a \in [a]$, so $a \in [a] \cap [b]$, a contradiction.
Thus, if $[a] \cap [b] = \varnothing$, then $a \not\sim b$.

Hence, $a \not\sim b$ if and only if $[a] \cap [b] = \varnothing$.

- A *partition* of a set A is a collection of disjoint nonempty subsets of A whose union equals A. When this holds, we say that these sets *partition* A.

Example. Let $A = \{a, b, c, d, e, f\}$. The subsets

$$\{a, c, d\},\ \{b, f\},\ \{e\}$$

partition A.

- **Theorem.** Let A be a set.
  - (i) The equivalence classes of an equivalence relation on A partition A.
  - (ii) Any partition of A can be used to form an equivalence relation on A.

Proof.

(i) Since every element of A belongs to some equivalence class, the union of the equivalence classes equals A. Since the equivalence classes are either equal or disjoint, we conclude that the equivalence classes partition A.

(ii) Suppose that we have a partition of A, that is, we have a collection of disjoint nonempty subsets A_i of A whose union $\bigcup A_i$ equals A. Define a

relation $\sim$ on A by

$$a \sim b \quad \text{if and only if} \quad a \text{ and } b \text{ belong to the same subset.}$$

(R) Each element $a \in A$ belongs to one of the subsets A_i and, of course, a belongs to the same subset as itself, so $a \sim a$.
Thus, $\sim$ is reflexive.

(S) Suppose that $a \sim b$. Then a and b belong to the same subset A_i so, of course, b and a also belong to the same subset; hence, $b \sim a$.
Thus, $\sim$ is symmetric.

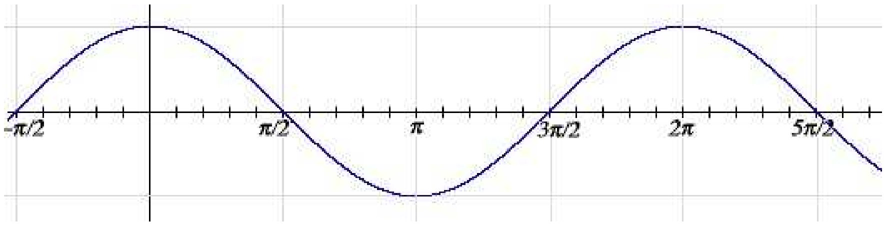
(T) Suppose that $a \sim b$ and $b \sim c$. Then a and b lie in the same subset A_i and b and c belong to the same subset A_j. Since A_i and A_j are either disjoint or identical and b lies in both, they must be identical.
Thus a, b, c all belong to the same subset; in particular, a and c belong to the same subset, so $a \sim c$.
Thus, $\sim$ is transitive.

Hence, $\sim$ is an equivalence relation on A.

Exercise. For the relation $\sim$ on $\mathbb{R}$ defined by

$$x \sim y \quad \text{if and only if} \quad \cos x = \cos y,$$

find $[0]$ and $[1]$; then find a general formula for the equivalence class $[a]$.



Exercise. List all equivalence relations on the set $A = \{1, 2, 3\}$.

Exercise. Let m be a positive integer.
Show that the relation *congruence modulo m* on the set of integers, that is,

$$a \sim b \quad \text{if and only if} \quad a \equiv b \pmod{m},$$

is an equivalence relation. List the equivalence classes for the case $m = 7$.

- A reflexive, antisymmetric, and transitive relation is a *partial order*.
- We often write $\preceq$ to denote a partial order: $a \preceq b$ reads "a precedes b".
- A partial ordering tells us which of two things "comes first" in some way.

Example. Consider the relation $\leq$ on the real numbers $\mathbb{R}$ and let $a, b, c \in \mathbb{R}$.

(R) We have $a \leq a$, so $\leq$ is reflexive.

(A) If $a \leq b$ and $b \leq a$, then $a = b$. Thus, $\leq$ is antisymmetric.

(T) If $a \leq b$ and $b \leq c$, then $a \leq c$. Thus, $\leq$ is transitive.

Since $\leq$ is reflexive, antisymmetric, and transitive, it is a partial order on $\mathbb{R}$. $a \leq b$ means that a comes before b if we list the numbers in increasing order.

Example. The relation $\geq$ is a partial ordering on the set of real numbers $\mathbb{R}$. $a \geq b$ means that a comes before b if we list the numbers in decreasing order.

Exercise. Prove that divisibility $\mid$ is a partial order on the positive integers $\mathbb{Z}^+$.

Exercise. For any set S, prove that the relation $\subseteq$ is a partial order on $P(S)$.

- A set A together with a partial order $\preceq$ is a *partially ordered set* or a *poset*. We denote this by $(A, \preceq)$.
- We say that two elements $a, b \in A$ are *comparable* with respect to a partial order $\preceq$ if and only if either $a \preceq b$ or $b \preceq a$ holds.
- A partial order in which every two elements are comparable is a *total order* or a *linear order*.

Example. $(\mathbb{R}, \leq)$ is a poset. Moreover, $\leq$ is a total order on $\mathbb{R}$. Similarly, $(\mathbb{R}, \geq)$ is a totally ordered set.

Example. $(\mathbb{Z}^+, |)$ is a poset but not a total order. For instance, $2 \nmid 7$ and $7 \nmid 2$, so 2 and 7 are not comparable in this poset.

Exercise. We have shown earlier that $(P(S), \subseteq)$ is a poset for any set S. Is the relation $\subseteq$ a total order on $P(S)$?

Exercise. On the set of complex numbers $\mathbb{C}$, we define

$$z \preceq w \quad \text{if and only if} \quad \mathrm{Re}(z) \leq \mathrm{Re}(w) \text{ and } \mathrm{Im}(z) \leq \mathrm{Im}(w)\,.$$

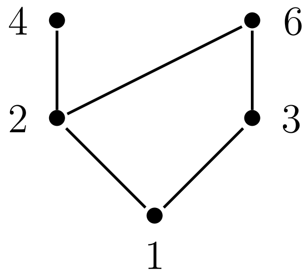
Prove that $\preceq$ is a partial order on $\mathbb{C}$. Is $\preceq$ a total order on $\mathbb{C}$?

- We represent a partial order $\preceq$ on a finite set by a *Hasse diagram*:
  - If $a \preceq b$ and $a \neq b$ (in which case, we often write $a \prec b$), then we draw a line between a and b, with a positioned lower than b in the diagram.
  - We do not draw any lines that can be deduced by the transitive property: $a \preceq b$ and $b \preceq c$ imply $a \preceq c$.
  - We do not draw any loops to indicate the reflexive property $a \preceq a$.

Example. $(\{1, 2, 3, 4, 6\}, |)$ is a poset. More precisely, the relation is

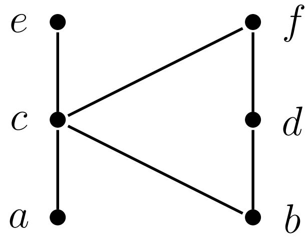
$$R = \{(1,1), (1,2), (1,3), (1,4), (1,6),\ (2,2), (2,4), (2,6),\ (3,3), (3,6),\ (4,4), (6,6)\}.$$

The corresponding Hasse diagram is



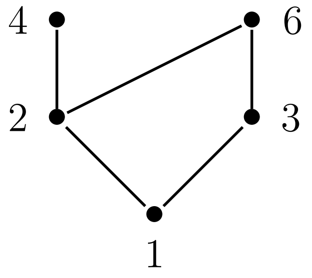
Exercise. Draw the Hasse diagram for the poset $(P(S), \subseteq)$ where $S = \{a, b, c\}$.

Exercise. Determine the poset represented by the following Hasse diagram.



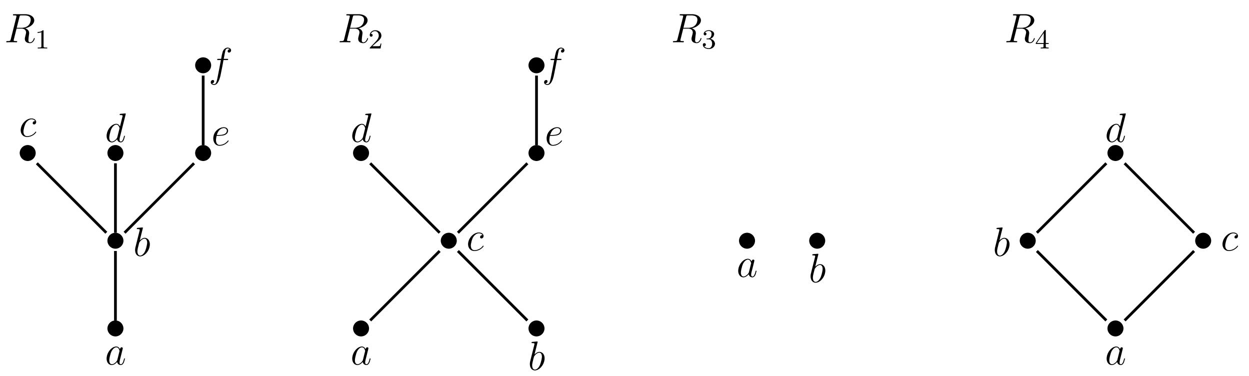
- Let $(A, \preceq)$ be a poset. An element $x \in A$ is called
  - a *maximal element* if there is no element $a \in A$ with $x \prec a$;
  - a *minimal element* if there is no element $a \in A$ with $a \prec x$;
  - the *greatest element* if $a \preceq x$ for all $a \in A$;
  - the *least element* if $x \preceq a$ for all $a \in A$.
    - ⋆ The greatest element in a poset is unique if it exists.
    - ⋆ The least element in a poset is unique if it exists.

Example. Consider the poset $(\{1, 2, 3, 4, 6\}, |)$:



The maximal elements are 4 and 6.
The minimal element is 1.
There is no greatest element.
The least element is 1.

Exercise. For the posets represented by the following Hasse diagrams, list the maximal, minimal, greatest and least elements if they exist.



	R_1	R_2	R_3	R_4
maximal elements				
minimal elements				
greatest element				
least element				

Exercise. Draw the Hasse diagram for the divisibility relation on the set

$$S = \{\text{positive factors of } 72\}.$$

Exercise. Let $S = \{1, 2, 3, 4, 5, 6\}$.
Draw the Hasse diagram for the following posets:

(a) $(S, \leq)$ (b) $(S, \geq)$ (c) $(S, |)$