THE UNIVERSITY OF SYDNEY
FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH1903/1907
INTEGRAL CALCULUS AND MODELLING (ADVANCED)

November 2008

LECTURERS: C M Cosgrove, H Dullin

TIME ALLOWED: **One and a half hours**

Name: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.
There is a table of integrals after the last question in this booklet.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

MARKS

1. (a) Evaluate the definite integral, **3**

$$\int_0^{\pi/4} \cos^2(4\theta)\, d\theta.$$

(b) Evaluate the indefinite integral, **4**

$$\int \frac{dx}{(x-3)(x^2+1)}.$$

(c) Let S_n denote the finite sum $1 + 2^{2/3} + 3^{2/3} + \ldots + n^{2/3}$.

(*i*) Calculate the integral $J = \int_0^{1000} x^{2/3}\, dx$, expressing your answer as a positive integer. **1**

(*ii*) Use suitable upper and lower Riemann sums for the function $f(x) = x^{2/3}$ on the interval $[0, 1000]$ to prove that $S_{999} < J < S_{1000}$. **2**

(*iii*) Hence, or otherwise, find integer lower and upper bounds, no more than 100 units apart, for S_{1000}. **2**

2. (a) The area under the graph of $y = (4 - x^2)^{-1/4}$ from $x = 0$ to $x = 1$ is rotated about the x-axis to form a solid of revolution. Calculate the volume V of this solid. **3**

(b) A cardioid is a closed plane curve having the parametric equations:

$$x = R(2\cos\theta - \cos 2\theta), \qquad y = R(2\sin\theta - \sin 2\theta),$$

$0 \le \theta \le 2\pi$, $R > 0$. Calculate the arc length of the cardioid. **3**

(c) Use the first two or three terms (whichever is needed) of the standard Taylor series expansions of $\sin x$ and $\cos x$ about the origin to evaluate the following l'Hôpital-type limit: **3**

$$\lim_{x \to 0} \frac{(ax - \sin ax)^2}{(1 - \cos bx)^3}, \qquad b \neq 0.$$

(d) Use a suitable comparison test to determine the convergence or divergence of the following improper integral: **3**

$$\int_{2\pi}^{\infty} \frac{\sin^3 x}{x^{3/2}}\, dx.$$

MARKS

3. (a) Find the general solution of the differential equation,

$$(\cos\theta)\frac{dw}{d\theta} + (\sin\theta - \cos\theta)w = e^{\theta}.$$

Express your answer in the form $w = f(\theta)$. **4**

(b) Use the change of variable $y = xw$ to find the general solution of the differential equation,

$$\frac{dy}{dx} = \frac{x^3 + 2y^3}{xy^2}.$$

Express your answer in the form $y = g(x)$. **3**

(c) Determine the general solution of the differential equation,

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 2e^{-2t},$$

and find the particular solution that satisfies the initial conditions $x(0) = 3$ and $x'(0) = 6$. **5**

4. (a) Water is leaking from a hole in the base of an upright cylindrical tank. According to Torricelli's Law, the rate at which the depth of the water drops is proportional to the square root of the depth. In this particular tank, the depth of the water has dropped from 3 metres to 2 metres in 24 hours. Define suitable variables and write down a differential equation governing the depth of the water as a function of time. Calculate the time taken for the tank to drain completely from the moment when the depth was 3 metres. **6**

(b) A hot object at a temperature of 400°C is placed in a well-insulated room at a temperature of 10°C. The object heats the air in the room as it cools down. One hour later, the object has cooled to 200°C and the air in the room has heated to 15°C. Let x and y denote the temperatures in degrees Celsius of the hot object and of the room, respectively, as functions of time t, where t is measured in hours since the hot object was placed in the room. Newton's Law of Cooling gives the pair of differential equations,

$$\frac{dx}{dt} = -\alpha(x - y), \qquad \frac{dy}{dt} = -\beta(y - x),$$

for positive constants α and β. Use the information given to determine the limiting temperature of the room for large t. **6**

Table of Standard Integrals

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

3. $\displaystyle\int e^x\,dx = e^x + C$

4. $\displaystyle\int \sin x\,dx = -\cos x + C$

5. $\displaystyle\int \cos x\,dx = \sin x + C$

6. $\displaystyle\int \tan x\,dx = -\ln\left|\cos x\right| + C$

7. $\displaystyle\int \cot x\,dx = \ln\left|\sin x\right| + C$

8. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

9. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

10. $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

11. $\displaystyle\int \sec x\,dx = \ln\left|\sec x + \tan x\right| + C$

12. $\displaystyle\int \operatorname{cosec} x\,dx = \ln\left|\operatorname{cosec} x - \cot x\right| + C$

13. $\displaystyle\int \sinh x\,dx = \cosh x + C$

14. $\displaystyle\int \cosh x\,dx = \sinh x + C$

15. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (|x| < a)$

16. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C'$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C' \quad (x > a)$

Linearity: $\displaystyle\int \left(\lambda f(x) + \mu g(x)\right) dx = \lambda \int f(x)\,dx + \mu \int g(x)\,dx$

Integration by substitution: $\displaystyle\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$

Integration by parts: $\displaystyle\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.