The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Longer Solutions to Selected Exercises for Week 13** 2012

1. Observe that

$$A\mathbf{v} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\mathbf{v}$$

and

$$A\mathbf{w} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3\mathbf{w}$$

so that $\mathbf{v}$ is an eigenvector corresponding to eigenvalue 5, and $\mathbf{w}$ an eigenvector corresponding to eigenvalue -3. Now

$$\begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 16 = \lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3)$$

with roots $\lambda = 5$ and -3, which correspond to the eigenvalues of A.

2. Observe that

$$B\mathbf{v}_1 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0\mathbf{v}_1 \,,$$

$$B\mathbf{v}_2 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1\mathbf{v}_2 \,,$$

$$B\mathbf{v}_3 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 3\mathbf{v}_3 \,,$$

so that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are eigenvectors corresponding to eigenvalues 0, 1, 3 respectively. Now

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & -1 \\ \lambda-1 & 3-\lambda & 1 \\ 0 & 1-\lambda & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 3-\lambda & 0 \\ 0 & 1-\lambda & -\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 3-\lambda & 0 \\ 1-\lambda & -\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)(-\lambda) = \lambda(\lambda-1)(3-\lambda)$$

with roots $\lambda = 0$, 1 and 3, which correspond to the eigenvalues of B.

3. (i) $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)$ with roots $\lambda = 1$, 2. To find the eigenspace corresponding to $\lambda = 1$:

$$M - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

corresponding to the system with one equation $y = 0$ with solution $x = t$, $y = 0$, yielding the eigenspace

$$\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to $\lambda = 2$:

$$M - \lambda I = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

with solution $x = 0$, $y = t$, yielding the eigenspace $\left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

(ii) $\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$ with roots $\lambda = 1, \ -1$. To find the eigenspace corresponding to $\lambda = 1$:

$$M - \lambda I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

corresponding to the system $x + y = 0$ with solution $x = -t$, $y = t$, yielding the eigenspace

$$\left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to $\lambda = -1$:

$$M - \lambda I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

with solution $x = t$, $y = t$, yielding the eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

(iii) $\begin{vmatrix} -1 - \lambda & 3 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$ with roots $\lambda = -3, \ 2$. To find the eigenspace corresponding to $\lambda = 2$:

$$M - \lambda I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

with solution $x = t$, $y = t$, yielding the eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$. To find the eigenspace corresponding to $\lambda = -3$:

$$M - \lambda I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$$

with solution $x = -3t$, $y = 2t$, yielding the eigenspace $\left\{ \begin{bmatrix} -3t \\ 2t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

4. (i) The only eigenvalue is $\lambda = 1$. To find its corresponding eigenspace:

$$M - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with solution $x = t$, $y = 0$, yielding the eigenspace $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$.

(ii) The eigenvalues are $\lambda = 2\,,\ -1\,$. To find the eigenspace corresponding to $\lambda = 2$:

$$M - \lambda I = \begin{bmatrix} 0 & 0 \\ -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

with solution $x = -3t\,,\ y = t\,$, yielding the eigenspace $\left\{ \begin{bmatrix} -3t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

To find the eigenspace corresponding to $\lambda = -1$:

$$M - \lambda I = \begin{bmatrix} 3 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

with solution $x = 0\,,\ y = t\,$, yielding the eigenspace $\left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(iii) The eigenvalues are $\lambda = 3\,,\ 5\,$. To find the eigenspace corresponding to $\lambda = 3$:

$$M - \lambda I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

with solution $x = t\,,\ y = 0\,,\ z = 0\,$, yielding the eigenspace $\left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

To find the eigenspace corresponding to $\lambda = 5$:

$$M - \lambda I = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

with solution $x = 3t\,,\ y = 2t\,,\ z = 4t$ and eigenspace $\left\{ \begin{bmatrix} 3t \\ 2t \\ 4t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

5. $$\begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ -4 & -4 & 7-\lambda \end{vmatrix} = \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ 0 & \lambda-3 & 3-\lambda \end{vmatrix} = \begin{vmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 3-\lambda \\ 0 & \lambda-3 & 0 \end{vmatrix}$$

$= (3-\lambda)\begin{vmatrix} -3-\lambda & 2 \\ -4 & 3-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2-1)$ with eigenvalues $\lambda = 3\,,\ 1$ and $-1\,$.

To find an eigenvector corresponding to $\lambda = 3$:

$$M - \lambda I = \begin{bmatrix} -6 & 0 & 2 \\ -4 & -4 & 4 \\ -4 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 6 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = t\,,\ y = 2t\,,\ z = 3t\,$. Thus an eigenvector is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

To find an eigenvector corresponding to $\lambda = 1$:

$$M - \lambda I = \begin{bmatrix} -4 & 0 & 2 \\ -4 & -2 & 4 \\ -4 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & -2 & 2 \\ 0 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = t$, $y = 2t$, $z = 2t$. Thus an eigenvector is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

To find an eigenvector corresponding to $\lambda = -1$:

$$M - \lambda I = \begin{bmatrix} -2 & 0 & 2 \\ -4 & 0 & 4 \\ -4 & -4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

with solution $x = t$, $y = t$, $z = t$. Thus an eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

6. (i) We may take $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

(ii) We have

$$\begin{aligned} B^n &= PD^nP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 4^n \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} -2^n & 4^n \\ 2^n & 4^n \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 2^{n-1} + 2(4^{n-1}) & -2^{n-1} + 2(4^{n-1}) \\ -2^{n-1} + 2(4^{n-1}) & 2^{n-1} + 2(4^{n-1}) \end{bmatrix}, \end{aligned}$$

so in particular $B^3 = \begin{bmatrix} 36 & 28 \\ 28 & 36 \end{bmatrix}$ and $B^4 = \begin{bmatrix} 136 & 120 \\ 120 & 136 \end{bmatrix}$.

7. (i) We may take $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(ii) By row reducing an augmented matrix we discover

$$P^{-1} = \frac{1}{3} \begin{bmatrix} -1 & -1 & 3 \\ 3 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix},$$

and so we have

$$\begin{aligned} C^n &= PD^nP^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} -1 & -1 & 3 \\ 3 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 + 3^{n-1} & 3^{n-1} & -1 \\ -1 + 2(3^{n-1}) & 2(3^{n-1}) & 1 \\ 3^{n-1} & 3^{n-1} & 0 \end{bmatrix}, \end{aligned}$$

so in particular $C^4 = \begin{bmatrix} 28 & 27 & -1 \\ 53 & 54 & 1 \\ 27 & 27 & 0 \end{bmatrix}$.

8. Suppose that $\mathbf{v}$ is an eigenvector for an invertible matrix A corresponding to the eigenvalue λ. If $\lambda = 0$ then

$$\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v} = A^{-1}0\mathbf{v} = \mathbf{0},$$

which contradicts that $\mathbf{v}$ is nonzero. Hence $\lambda \neq 0$. From $A\mathbf{v} = \lambda\mathbf{v}$ we deduce

$$A^{-1}\mathbf{v} = A^{-1}\lambda^{-1}\lambda\mathbf{v} = \lambda^{-1}A^{-1}A\mathbf{v} = \lambda^{-1}I\mathbf{v} = \lambda^{-1}\mathbf{v} ,$$

so that $\mathbf{v}$ is an eigenvector of A^{-1} corresponding to the eigenvalue λ^{-1}. If k is any positive integer then

$$A^k\mathbf{v} = A^{k-1}A\mathbf{v} = A^{k-1}\lambda\mathbf{v} = \lambda A^{k-1}\mathbf{v} = \lambda^2A^{k-2}\mathbf{v} = \ldots = \lambda^k\mathbf{v} ,$$

so that $\mathbf{v}$ is an eigenvector of A^k corresponding to the eigenvalue λ^k. If k is any negative integer, then the same argument, using A^{-1} in place of A, yields that $\mathbf{v}$ is an eigenvector of A^k corresponding to the eigenvalue λ^k. Finally, if $k = 0$ then certainly $1 = \lambda^0$ is an eigenvalue of $I = A^0$ with eigenvector $\mathbf{v}$.

9. We argue by contradiction. Suppose

$$\mathbf{v}_1 \;=\; \alpha\mathbf{v}_2$$

for some scalar α. Then $\alpha \neq 0$, since $\mathbf{v}_1$ is nonzero (being an eigenvector). Then

$$\lambda_2\mathbf{v}_2 \;=\; M\mathbf{v}_2 \;=\; M\alpha^{-1}\mathbf{v_1} \;=\; \alpha^{-1}\lambda_1\mathbf{v}_1 \;=\; \alpha^{-1}\lambda_1\alpha\mathbf{v_2} \;=\; \lambda_1\mathbf{v}_2 \,.$$

Hence

$$(\lambda_1 - \lambda_2)\mathbf{v}_2 \;=\; \mathbf{0} \,,$$

which implies $\lambda_1 - \lambda_2 = 0$ (now since $\mathbf{v}_2$ is nonzero), contradicting that λ_1 and λ_2 are different. Hence $\mathbf{v}_1$ is not a scalar mutliple of $\mathbf{v}_2$.

10. By the multiplicative property of the determinant and distributivity,

$$\begin{aligned}
\det(B^{-1}AB - \lambda I) \;&=\; \det(B^{-1}AB - \lambda B^{-1}B) \;=\; \det(B^{-1}AB - B^{-1}\lambda IB) \\
&=\; \det(B^{-1}(A - \lambda I)B) \;=\; \det B^{-1}\det(A - \lambda I)\det B \\
&=\; \det B^{-1}\det B\det(A - \lambda I) \;=\; \det(A - \lambda I) \,.
\end{aligned}$$

Since their characteristic polynomials are identical, the matrices A and $B^{-1}AB$ have the same eigenvalues.

11. Observe that

$$\begin{aligned}
\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} \;&=\; (\cos\theta - \lambda)^2 + \sin^2\theta \\
&=\; \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta \\
&= \lambda^2 - 2\lambda\cos\theta + 1 \,,
\end{aligned}$$

with roots

$$\lambda \;=\; \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \;=\; \cos\theta \pm i\sin\theta \;=\; \mathrm{cis}\ (\pm\theta) \,.$$

Thus the eigenvalues are real if and only if $\mathrm{cis}\ (\pm\theta)$ is real, which occurs precisely when $\mathrm{cis}\ (\pm\theta) = \pm 1$, that is, $\theta = 0$ or π. Geometrically, M is a rotation matrix in the plane, so that premultiplication of column vectors corresponds to anticlockwise rotation. Rotations send nonzero vectors to scalar multiples of themselves only in extreme circumstances, namely the zero rotation that fixes everything (eigenvalue 1) and the 180° rotation that flips things over (eigenvalue -1).

12. Let $\mathbf{v}$ be an eigenvector of A corresponding to λ.

(i) Suppose $A^2 = 0$. If $\lambda \neq 0$ then

$$\mathbf{v} \;=\; \lambda^{-2}\lambda^2\mathbf{v} \;=\; \lambda^{-2}A^2\mathbf{v} \;=\; \lambda^{-2}0\mathbf{v} \;=\; \mathbf{0}\,,$$

which contradicts that an eigenvector is nonzero. Hence $\lambda = 0$.

(ii) Suppose $A^2 = A$ and $\lambda \neq 0$. Then

$$\mathbf{v} \;=\; \lambda^{-1}\lambda\mathbf{v} \;=\; \lambda^{-1}A\mathbf{v} \;=\; \lambda^{-2}A^2\mathbf{v} \;=\; \lambda^{-1}\lambda^2\mathbf{v} \;=\; \lambda\mathbf{v}\,,$$

so that $(1-\lambda)\mathbf{v} = \mathbf{0}$. But $\mathbf{v} \neq \mathbf{0}$, so $1-\lambda = 0$, giving $\lambda = 1$.

(iii) Suppose $A^2 = I$. Then

$$\mathbf{v} \;=\; I\mathbf{v} \;=\; A^2\mathbf{v} \;=\; \lambda^2\mathbf{v}\,,$$

so that $(1-\lambda^2)\mathbf{v} = \mathbf{0}$. But $\mathbf{v} \neq \mathbf{0}$, so $1-\lambda^2 = 0$, giving $\lambda = 1$ or -1.

13. Suppose α, β and γ are scalars such that

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 \;=\; \mathbf{0}\,.$$

Mutliplying through by λ_1 gives

$$\lambda_1\alpha\mathbf{v}_1 + \lambda_1\beta\mathbf{v}_2 + \lambda_1\gamma\mathbf{v}_3 \;=\; \mathbf{0}\,.$$

Multiplying through by M, using the definitions of eigenvectors and eigenvalues, gives

$$\alpha\lambda_1\mathbf{v}_1 + \beta\lambda_2\mathbf{v}_2 + \gamma\lambda_3\mathbf{v}_3 \;=\; \mathbf{0}\,.$$

Subtracting gives

$$(\lambda_1-\lambda_2)\beta\mathbf{v}_2 + (\lambda_1-\lambda_3)\gamma\mathbf{v}_3 \;=\; \mathbf{0}\,.$$

Mutliplying through by λ_2 gives

$$\lambda_2(\lambda_1-\lambda_2)\beta\mathbf{v}_2 + \lambda_2(\lambda_1-\lambda_3)\gamma\mathbf{v}_3 \;=\; \mathbf{0}\,,$$

and by M gives

$$(\lambda_1-\lambda_2)\lambda_2\beta\mathbf{v}_2 + (\lambda_1-\lambda_3)\lambda_3\gamma\mathbf{v}_3 \;=\; \mathbf{0}\,.$$

Again subtracting gives

$$(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\gamma\mathbf{v}_3 \;=\; \mathbf{0}\,.$$

But $\lambda_1-\lambda_3 \neq 0$, $\lambda_2-\lambda_3 \neq 0$ and $\mathbf{v}_3 \neq \mathbf{0}$. Thus $\gamma = 0$ and so

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \;\mathbf{0}\,.$$

If $\alpha \neq 0$ or $\beta \neq 0$ then one of $\mathbf{v_1}$ or $\mathbf{v}_2$ is a scalar multiple of the other, contradicting the previous exercise. Hence

$$\alpha \;=\; \beta \;=\; \gamma \;=\; 0\,,$$

which proves that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

14. $\begin{vmatrix} 3-\lambda & 2 & 1 \\ -2 & -1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1 \\ \lambda-1 & -3-\lambda & 1 \\ 0 & 1+2\lambda & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & -3-\lambda & 2 \\ 0 & 1+2\lambda & -\lambda \end{vmatrix}$

$= (1-\lambda)\begin{vmatrix} -3-\lambda & 2 \\ 1+2\lambda & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-\lambda-2) = (1-\lambda)(\lambda-2)(\lambda+1)$ with roots 1, 2 and -1. But

$$M - I = \begin{bmatrix} 2 & 2 & 1 \\ -2 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector corresponding to the eigenvalue 1 is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$;

$$M - 2I = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector corresponding to the eigenvalue 2 is $\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$; and

$$M + I = \begin{bmatrix} 4 & 2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector corresponding to the eigenvalue -1 is $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

15. We may take $P = \begin{bmatrix} -1 & 5 & 1 \\ 1 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

16. By row reducing an augmented matrix we discover

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 9 & 12 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix},$$

and so we have

$$\begin{aligned} M^n &= PD^nP^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 5 & 1 \\ 1 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix}\begin{bmatrix} 3 & 9 & 12 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} -3+5(2^{n+1})-(-1)^n & -9+5(2^{n+1})-(-1)^n & -12+5(2^{n+1})+2(-1)^n \\ 3-6(2^n)+3(-1)^n & 9-6(2^n)+3(-1)^n & 12-6(2^n)-6(-1)^n \\ 2^{n+1}-2(-1)^n & 2^{n+1}-2(-1)^n & 2^{n+1}+4(-1)^n \end{bmatrix}, \end{aligned}$$

so in particular $M^4 = \begin{bmatrix} 26 & 25 & 25 \\ -15 & -14 & -15 \\ 5 & 5 & 6 \end{bmatrix}$.

17. The matrix is triangular so the eigenvalues are the diagonal entries. Quickly one discovers that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to 2, and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector corresponding to 1. Hence

$$\begin{aligned} M^n &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix} . \end{aligned}$$

18. The matrix is triangular so the eigenvalues are the diagonal entries. Quickly one discovers that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ corresponds to 1, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ corresponds to 2 and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ corresponds to 3. Hence

$$\begin{aligned} M^n &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^n \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2^n - 1 & 2^n - 1 \\ 0 & 2^n & 2^n - 3^n \\ 0 & 0 & 3^n \end{bmatrix} . \end{aligned}$$

19. We argue by contradiction. Suppose that M is diagonalisable, so $P^{-1}MP$ is diagonal for some invertible matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. But

$$P^{-1}MP = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} * & d^2 \\ -c^2 & * \end{bmatrix} ,$$

which implies $c = d = 0$, so that $ad - bc = 0$, contradicting that P is invertible.

20. Observe, by properties of transpose, that

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I^T) = \det(A^T - \lambda I) ,$$

so that A and A^T have identical characteristic polynomials and therefore the same eigenvalues.

21. Observe that

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc ,$$

and

$$\begin{aligned}
A^2 - &(a+d)A + (ad-bc)I \\
&= \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} - \begin{bmatrix} a^2+da & ab+db \\ ac+dc & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\
&= \begin{bmatrix} a^2+bc-a^2-da+ad-bc & ab+bd-ab-db \\ ca+dc-ac-dc & cb+d^2-ad-d^2+ad-bc \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0\,,
\end{aligned}$$

so that A is a root of its characteristic polynomial.

22. The characteristic polynomial is

$$\begin{vmatrix} -7-\lambda & -2 & 6 \\ -2 & 1-\lambda & 2 \\ -10 & -2 & 9-\lambda \end{vmatrix} = \begin{vmatrix} -7-\lambda & -2 & 6 \\ -2 & 1-\lambda & 2 \\ \lambda-3 & 0 & 3-\lambda \end{vmatrix} = \begin{vmatrix} -1-\lambda & -2 & 6 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$
$$= (-1-\lambda)(1-\lambda)(3-\lambda) = -\lambda^3 + 3\lambda^3 + \lambda - 3\,.$$

By the Cayley-Hamilton Theorem,

$$-M^3 + 3M^2 + M - 3I = 0\,,$$

so that

$$M(M^2 - 3M - I) = -3I\,,$$

and it follows quickly (suppressing some straightforward matrix calculations) that

$$M^{-1} = -\frac{1}{3}(M^2 - 3M - I) = \frac{1}{3}\begin{bmatrix} -13 & -6 & 10 \\ 2 & 3 & -2 \\ -14 & -6 & 11 \end{bmatrix}.$$

23. $\begin{vmatrix} 1/2-\lambda & 2/5 \\ 1/2 & 3/5-\lambda \end{vmatrix} = (\lambda-1/2)(\lambda-3/5)-1/5 = \lambda^2-11\lambda/10+1/10 = (\lambda-1)(\lambda-1/10)$ with roots 1 and $1/10$. But

$$M - I = \begin{bmatrix} -1/2 & 2/5 \\ 1/2 & -2/5 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/5 \\ 0 & 0 \end{bmatrix}$$

so the eigenspace corresponding to the eigenvalue 1 is $\left\{ \begin{bmatrix} 4t \\ 5t \end{bmatrix} \middle|\, t \in \mathbb{R} \right\}$. The element of this eigenspace whose entries add to 1 is

$$\mathbf{v} = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix},$$

which is the unique steady state vector of M. Also

$$M - \frac{1}{10}I = \begin{bmatrix} 2/5 & 2/5 \\ 1/2 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

so an eigenvector corresponding to the eigenvalue $1/10$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence

$$\begin{aligned} M^n &= \begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/10 \end{bmatrix}^n \begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{9} \begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1/10)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 4 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 4+5(1/10)^n & 4-4(1/10)^n \\ 5-5(1/10)^n & 5+4(1/10)^n \end{bmatrix}. \end{aligned}$$

But $(1/10)^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} M^n = \frac{1}{9} \begin{bmatrix} 4 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 4/9 & 4/9 \\ 5/9 & 5/9 \end{bmatrix} = \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix},$$

as the general theory predicted.

24. $\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$ with roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

But $M - \lambda_1 I \sim \begin{bmatrix} 1 & -\lambda_1 \\ 0 & 0 \end{bmatrix}$ so an eigenvector for λ_1 is $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$. Similarly an eigenvector for λ_2 is $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$. Hence

$$\begin{aligned} M^n &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1\lambda_2^{n+1} - \lambda_2\lambda_1^{n+1} \\ \lambda_1^n - \lambda_2^n & \lambda_1\lambda_2^n - \lambda_2\lambda_1^n \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = M^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n-1}(1-\lambda_2) - \lambda_2^{n-1}(1-\lambda_1) \\ \lambda_1^{n-2}(1-\lambda_2) - \lambda_2^{n-2}(1-\lambda_1) \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^n - \lambda_2^n \\ \lambda_1^{n-1} - \lambda_2^{n-1} \end{bmatrix} \end{aligned}$$

yielding finally the formula for the nth Fibonacci number:

$$x_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

25. Let M be a complex 2×2 matrix. If the eigenvalues are distinct then M is diagonalisable so M is similar to a diagonal matrix. Similarly if M has one eigenvalue whose eigenspace contains two nonparallel vectors, then again M is similar to a diagonal matrix. It remains to assume that M has just one eigenvalue λ such that the eigenspace is one dimensional (that is, consists precisely of all scalar multiples of some fixed nonzero vector). The characteristic equation is a perfect square, so, by the Cayley-Hamilton Theorem,

$$(M - \lambda I)^2 \;=\; 0\,.$$

Consider

$$\mathbf{w}_1 \;=\; (M-\lambda I)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 \;=\; (M-\lambda I)\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

the first and second columns of $M - \lambda I$ respectively. If both are zero then $M - \lambda I = 0$, so that $M = \lambda I$ is diagonal, contradicting that the eigenspace is one-dimensional. Without loss of generality, we may suppose $\mathbf{w}_1$ is nonzero. But

$$(M-\lambda I)\mathbf{w}_1 \;=\; (M-\lambda I)^2\begin{bmatrix} 1 \\ 0 \end{bmatrix} \;=\; 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} \;=\; 0\,,$$

so that $\mathbf{w}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ is an eigenvector. Certainly $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not an eigenvector, so $\mathbf{w}_1$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are not scalar multiples of each other. It follow quickly that $b \neq 0$ and so

$$P \;=\; \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$$

is invertible. Note that

$$M\mathbf{w}_1 \;=\; \lambda\mathbf{w}_1 \qquad \text{and} \qquad M\begin{bmatrix} 1 \\ 0 \end{bmatrix} \;=\; \mathbf{w}_1 + \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} a+\lambda \\ b \end{bmatrix},$$

so

$$MP \;=\; [M\mathbf{w}_1 \;\; \mathbf{w}_1] \;=\; \begin{bmatrix} \lambda a & a+\lambda \\ \lambda b & b \end{bmatrix} \;=\; \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \;=\; P\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

whence

$$M \;=\; P\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}P^{-1}\,.$$

Thus M is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. The real case is identical except when the eigenvalues form a complex conjugate pair $\lambda = r\,\mathrm{cis}\,(\pm\theta)$, in which case there is an invertible complex matrix P such that (suppressing some straightforward calculations)

$$M \;=\; P\begin{bmatrix} r\,\mathrm{cis}\,\theta & 0 \\ 0 & r\,\mathrm{cis}\,(-\theta) \end{bmatrix}P^{-1} \;=\; Q\begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}Q^{-1}$$

where $Q = P\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$, so that M is similar to a scalar multiple of a rotation matrix. In fact if we are careful to choose complex conjugate eigenvectors to form $P = \begin{bmatrix} \alpha & \overline{\alpha} \\ \beta & \overline{\beta} \end{bmatrix}$ (as we may since M is unchanged by conjugating all of its elements, and complex conjugation clearly preserves matrix multiplication) then $Q = \begin{bmatrix} \alpha+\overline{\alpha} & i\alpha - i\overline{\alpha} \\ \beta+\overline{\beta} & i\beta - i\overline{\beta} \end{bmatrix}$ will be real (as is readily checked because the entries are fixed by conjugation).