THE UNIVERSITY OF SYDNEY

MATH1902 LINEAR ALGEBRA (ADVANCED)

Semester 1 **Exercises for Week 3** 2017

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

**Important Ideas and Useful Facts:**

(i) Geometric definition of dot product: If $\mathbf{v}$ and $\mathbf{w}$ are vectors and $\theta$ is the angle between them, then

$$\mathbf{v} \cdot \mathbf{w} \;=\; |\mathbf{v}||\mathbf{w}| \cos\theta \;,$$

so that, in the case both vectors are nonzero,

$$\cos\theta \;=\; \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|} \;.$$

(ii) Algebraic definition of dot product: If $\mathbf{v} = (v_k)_{1 \leq k \leq n}$ and $\mathbf{w} = (w_k)_{1 \leq k \leq n}$, then

$$\mathbf{v} \cdot \mathbf{w} \;=\; \sum_{k=1}^{n} v_k w_k \;.$$

(iii) The angle between two vectors is zero or acute if their dot product is positive. The angle is obtuse or $180°$ if the dot product is negative. Two vectors are mutually perpendicular (orthogonal) if the dot product is zero.

(iv) Cauchy-Schwarz Inequality: $|\mathbf{v} \cdot \mathbf{w}| \;\leq\; |\mathbf{v}||\mathbf{w}|$ .

(v) Commutativity of dot product: $\mathbf{v} \cdot \mathbf{w} \;=\; \mathbf{w} \cdot \mathbf{v}$ .

(vi) Distributivity of dot over plus: $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} \;=\; \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ .

(vii) If $\mathbf{v}$ is any vector then $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ , so $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ .

(viii) If $\mathbf{v}$ and $\mathbf{w}$ are vectors and $\lambda$ is a scalar then $(\lambda\mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (\lambda\mathbf{w})$ .

(ix) The *vector projection* of $\mathbf{v}$ in the direction of $\mathbf{w}$ is $\dfrac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$ , which is the best approximation of $\mathbf{v}$ using a scalar multiple of $\mathbf{w}$.

(x) The *scalar component* of $\mathbf{v}$ in the direction of $\mathbf{w}$ is $\dfrac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}$ , which is plus or minus the magnitude of the vector projection (minus in the case that the angle is obtuse or $180°$).

(xi) The *vector component of* $\mathbf{v}$ *orthogonal to* $\mathbf{w}$ is the difference between $\mathbf{v}$ and its vector projection, which is

$$\mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} \;.$$

**Preparatory Exercises:**

**1.** Use the Theorem of Pythagoras to verify the Cosine Rule.

**2.** Given that

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}\,, \qquad \mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\,, \qquad \mathbf{w} = 3\mathbf{i} - \mathbf{k}\,,$$

find

(i) $\mathbf{u}\cdot\mathbf{v}$ (ii) $\mathbf{u}\cdot\mathbf{w}$ (iii) $\mathbf{v}\cdot\mathbf{w}$ (iv) $\mathbf{u}\cdot\mathbf{u}$ (v) $\mathbf{v}\cdot\mathbf{v}$ (vi) $\mathbf{w}\cdot\mathbf{w}$
(vii) $|\mathbf{u}|$ (viii) $|\mathbf{v}|$ (ix) $|\mathbf{w}|$ (x) $\mathbf{u}\cdot\big(\mathbf{v}+\mathbf{w}\big)$ (xi) $\mathbf{u}\cdot\big(\mathbf{v}-\mathbf{w}\big)$

**3.** Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be as in the previous exercise. Let $\alpha$ be the angle between $\mathbf{u}$ and $\mathbf{v}$, $\beta$ be the angle between $\mathbf{u}$ and $\mathbf{w}$, and $\gamma$ the angle between $\mathbf{v}$ and $\mathbf{w}$. Find

(i) $\cos\alpha$ (ii) $\cos\beta$ (iii) $\cos\gamma$

**4.** Given that

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\,, \qquad \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}\,, \qquad \mathbf{c} = 3\mathbf{i} + 6\mathbf{j}\,,$$

determine whether the following are true or false:

(i) The angle between $\mathbf{a}$ and $\mathbf{b}$ is acute. (ii) The angle between $\mathbf{b}$ and $\mathbf{c}$ is acute.
(iii) The vectors $\mathbf{a}$ and $\mathbf{c}$ are mutually perpendicular.
(iv) The angle between the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}-\mathbf{c}$ is obtuse.

**5.** Given that $P = (8, 4, -1)$, $Q = (6, 3, -4)$ and $R = (7, 5, -5)$, find

$$\overrightarrow{QP}\,, \qquad \big|\overrightarrow{QP}\big|\,, \qquad \overrightarrow{QR}\,, \qquad \big|\overrightarrow{QR}\big|\,, \qquad \overrightarrow{QP}\cdot\overrightarrow{QR}\,,$$

and the cosine of $\angle PQR$.

**6.** Given that $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$, find

(i) $\mathbf{u}\cdot\mathbf{v}$ (ii) $\widehat{\mathbf{u}}$ (iii) $\widehat{\mathbf{v}}$ (iv) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}$ (v) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$ (vi) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$
(vii) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\,\mathbf{u}$ (viii) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}$ (ix) $\mathbf{v} - \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\,\mathbf{u}$ (x) $\mathbf{u} - \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}$
(xi) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$
(xii) the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$
(xiii) the scalar component of $\mathbf{v}$ in the direction of $\mathbf{u}$
(xiv) the vector projection of $\mathbf{u}$ in the direction of $\mathbf{v}$
(xv) the vector projection of $\mathbf{v}$ in the direction of $\mathbf{u}$
(xvi) the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$
(xvii) the vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$

**Exercises:**

**16.** Resolve the vector $\mathbf{u} = 5\mathbf{i} + \mathbf{j} + 6\mathbf{k}$ into a sum of two vectors, one of which is parallel and the other perpendicular to $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$.

**17.** Find the (vector) components of the force $15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}$ newtons in the direction of and orthogonal to

(i) $-\mathbf{i} + \mathbf{j}$ (ii) $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

**18.** Use the dot product to verify that if $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular vectors then

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 .$$

Interpret this result in terms of a well-known fact about triangles.

**19.*** Verify that $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ are perpendicular. Find two vectors of unit length that are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. (This will become easy after next week, using cross products.)

**20.*** Verify that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.

**21.*** Prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus (that is, has all sides of equal length).

**22.*** Verify the following identity for all geometric vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and use it to deduce that the three altitudes of a triangle intersect in a common point:

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{d} - \mathbf{c}) + (\mathbf{b} - \mathbf{c}) \cdot (\mathbf{d} - \mathbf{a}) + (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{b}) = \mathbf{0}$$

**23.*** Prove that the perpendicular bisectors of the sides of a triangle intersect in a common point (known as the *circumcentre*).

**24.** Given that it exists, verify that the circumcentre of a triangle is the same distance from each vertex (which explains its name).

**25.**** Suppose that $A$, $B$, $C$ and $D$ are distinct points in space such that no three are collinear. Verify that these points lie on a plane if and only if there are four nonzero scalars, $\alpha$, $\beta$, $\gamma$ and $\delta$ such that $\alpha + \beta + \gamma + \delta = 0$ and

$$\alpha\, \overrightarrow{OA} + \beta\, \overrightarrow{OB} + \gamma\, \overrightarrow{OC} + \delta\, \overrightarrow{OD} = \mathbf{0} .$$

Deduce Ceva's Theorem, that says if $D$ is a point in the plane of the triangle $ABC$, and the lines through $AD$, $BD$, $CD$ cut $BC$, $CA$, $AB$ in $R$, $S$, $T$ respectively, then the product of the ratios in which $R$, $S$, $T$ divide $BC$, $CA$, $AB$ respectively is 1.

**Short Answers to Selected Exercises:**

**1.** Drop a perpendicular to create right angled triangles.

**2.** (i) 6 (ii) 5 (iii) 1 (iv) 6 (v) 9 (vi) 10 (vii) $\sqrt{6}$ (viii) 3

(ix) $\sqrt{10}$ (x) 11 (xi) 1

**3.** (i) $\dfrac{\sqrt{6}}{3}$ (ii) $\dfrac{\sqrt{15}}{6}$ (iii) $\dfrac{\sqrt{10}}{30}$

**4.** (i) false (ii) true (iii) true (iv) true

**5.** $2\mathbf{i}+\mathbf{j}+3\mathbf{k}\,,\ \sqrt{14}\,,\ \mathbf{i}+2\mathbf{j}-\mathbf{k}\,,\ \sqrt{6}\,,\ 1\,,\ \dfrac{1}{2\sqrt{21}}$

**6.** (i) $-4$ (ii) $\frac{1}{\sqrt{5}}(\mathbf{i}-2\mathbf{j})$ (iii) $\frac{1}{\sqrt{5}}(-2\mathbf{i}+\mathbf{j})$ (iv) $-\frac{4}{\sqrt{5}}$ (v) $-\frac{4}{\sqrt{5}}$ (vi) $-\frac{4}{5}$

(vii) $-\frac{4}{5}(\mathbf{i}-2\mathbf{j})$ (viii) $\frac{4}{5}(2\mathbf{i}-\mathbf{j})$ (ix) $-\frac{3}{5}(2\mathbf{i}+\mathbf{j})$ (x) $-\frac{3}{5}(\mathbf{i}+2\mathbf{j})$ (xi) $-\frac{4}{5}$ (xii) $-\frac{4}{\sqrt{5}}$

(xiii) $-\frac{4}{\sqrt{5}}$ (xiv) $\frac{4}{5}(2\mathbf{i}-\mathbf{j})$ (xv) $-\frac{4}{5}(\mathbf{i}-2\mathbf{j})$ (xvi) $-\frac{3}{5}(\mathbf{i}+2\mathbf{j})$ (xvii) $-\frac{3}{5}(2\mathbf{i}+\mathbf{j})$

**9.** (i) $\frac{2}{3\sqrt{33}}$ (ii) $\frac{2}{\sqrt{33}}$ (iii) $\frac{2}{3}$ (iv) $\frac{2}{33}(-4\mathbf{i}+4\mathbf{j}-\mathbf{k})$ (v) $\frac{2}{9}(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$

(vi) $\frac{1}{33}(41\mathbf{i}+58\mathbf{j}+68\mathbf{k})$ (vii) $-\frac{1}{9}(38\mathbf{i}-32\mathbf{j}+13\mathbf{k})$ .

**11.** (i) $55^\circ$ (ii) $35^\circ$ (iii) $60^\circ$ (iv) $71^\circ$

**16.** $\mathbf{u} \ = \ \dfrac{3}{7}\big(3\mathbf{i}-6\mathbf{j}+2\mathbf{k}\big)+\dfrac{1}{7}\big(26\mathbf{i}+25\mathbf{j}+36\mathbf{k}\big)$

**17.** (i) $\dfrac{5}{2}\,(-\mathbf{i}+\mathbf{j})$ newtons , $\dfrac{1}{2}\big(35\mathbf{i}+35\mathbf{j}+12\mathbf{k}\big)$ newtons

(ii) $-\dfrac{12}{7}\big(2\mathbf{i}-3\mathbf{j}+\mathbf{k}\big)$ newtons , $\dfrac{1}{7}\big(129\mathbf{i}+104\mathbf{j}+54\mathbf{k}\big)$ newtons

**19.** $\pm\frac{1}{\sqrt{77}}(2\mathbf{i}-8\mathbf{j}-3\mathbf{k})$