THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1	**Tutorial Week 13**	2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Find the directional derivative of $f(x,y) = x^2 + 2e^{x+y}$ in the direction of $\mathbf{v} = \mathbf{i} - \mathbf{j}$ at the point $(1,2)$.

Questions for the tutorial

2. Use the formula $\dfrac{dy}{dx} = -\dfrac{f_x(x,y)}{f_y(x,y)}$ to find an expression for $\dfrac{dy}{dx}$ where y is defined implicitly as a function of x by the equation $x^3 + y^3 = 3xy$. Hence evaluate the slope of the tangent to the curve $x^3 + y^3 = 3xy$ at the point $(2/3, 4/3)$.

3. Let $f(x,y) = 1 + 2x\sqrt{y}$ and $g(x,y) = e^{-x}\sin y$.

(a) Find $\nabla f(x,y)$, $\nabla f(3,4)$, $\nabla g(x,y)$, $\nabla g(2,0)$.

(b) Let $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$. Determine the unit vector $\widehat{\mathbf{v}}$. Hence find $D_{\widehat{\mathbf{v}}} f(x,y)$ and also the special case $D_{\widehat{\mathbf{v}}} f(3,4)$. Similarly, if $\mathbf{w} = 3\mathbf{i} + 2\mathbf{j}$, find $D_{\widehat{\mathbf{w}}} g(x,y)$ and $D_{\widehat{\mathbf{w}}} g(2,0)$.

4. Instead of the one-sided limit used in the definition of the directional derivative in this course, many texts use the following two-sided limit:

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector and h may be either positive or negative.

(a) Let $f(x,y) = \sqrt{xy}$ and let $\mathbf{u}$ be a unit vector. Prove that $D_{\mathbf{u}} f(0,0)$, defined using the two-sided limit above, exists if and only if $\mathbf{u} = \mathbf{i}$, $-\mathbf{i}$, $\mathbf{j}$ or $-\mathbf{j}$.

(b) Now use our one-sided definition for the limit and find all directions for which $D_{\mathbf{u}} f(0,0)$ exists.

5. Find the directions in which the directional derivative of $f(x,y) = x^2 + \sin(xy)$ at $(1,0)$ has value 1.

6. Find the greatest slope and the (two) directions one could begin to move to stay level if one is standing at the point

(a) $(3,4,13)$ on the surface $z = 1 + 2x\sqrt{y}$;

(b) $(2,0,0)$ on the surface $z = e^{-x}\sin y$.

7. Suppose you are climbing a hill whose shape is given by the equation

$$z = 1000 - 0.01x^2 - 0.02y^2,$$

where x, y, z are measured in metres, and you are standing at a point with coordinates $(50, 80, 847)$. The positive x axis points east and the positive y axis points north.

(a) If you walk due south, will you start to ascend or descend?

(b) If you walk northwest, will you start to ascend or descend?

(c) In which direction is the slope largest? What is the value of this slope? At what angle above the horizontal does the path in that direction begin?

(d) In which horizontal direction should you move to maintain a height of 847 metres?

8. Let $f(x, y) = x - y^2$. Find $\nabla f(3, -1)$, and use it to find the parametric equation of the normal (perpendicular) line to the level curve $f(x, y) = 2$ at $(3, -1)$.

Extra Question

9. A function f of two variables is called *homogeneous of degree* $n \geq 1$ if

$$f(tx, ty) = t^n f(x, y)$$

for all t, x, y. Assume that all functions are well-behaved so that the chain rule applies.

(a) Verify that $g(x, y) = x^3 + xy^2 + y^3$ and $h(x, y) = (x^4 + y^4)^{3/2}$ are homogeneous of degrees 3 and 6 respectively.

(b) Suppose f is homogeneous of degree n and let $x = ta$, $y = tb$ where a and b are constants and t is a parameter. Put $F(t) = f(ta, tb)$. Differentiate $F(t)$ in two different ways (one using the chain rule) to conclude

$$nt^{n-1} f(a, b) = a\frac{\partial f}{\partial x}(ta, tb) + b\frac{\partial f}{\partial y}(ta, tb).$$

Set $t = 1$ and replace a by x and b by y to deduce *Euler's Theorem*:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

Solution to Question 1

First calculate $\nabla f(x, y) = (2x + 2e^{x+y})\mathbf{i} + 2e^{x+y}\mathbf{j}$. A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$, and

$$D_{\mathbf{u}} f(x, y) = (\tfrac{1}{\sqrt{2}}\mathbf{i} - \tfrac{1}{\sqrt{2}}\mathbf{j}) \cdot ((2x + 2e^{x+y})\mathbf{i} + 2e^{x+y}\mathbf{j}) = \sqrt{2}x.$$

So the directional derivative at $(1, 2)$ is $\sqrt{2}$.