MATH1902 2009 exam solutions 8/6/2010

Q1/ (a) (i) $\underset{\sim}{u}_1 = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{u}_2 = 2\underset{\sim}{i} - \underset{\sim}{j} + 8\underset{\sim}{k}$.

If P_1 is parallel to P_2 then $\underset{\sim}{u}_1 = \lambda \underset{\sim}{u}_2$ for some scalar λ, so that

$$\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} = \lambda(2\underset{\sim}{i} - \underset{\sim}{j} + 8\underset{\sim}{k}),$$

so that $2\lambda = 1$, $-\lambda = 2$, $8\lambda = -1$, impossible.

Hence P_1 is not parallel to P_2.

(ii) $\underset{\sim}{u}_1 \times \underset{\sim}{u}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & -1 \\ 2 & -1 & 8 \end{vmatrix} = 15\underset{\sim}{i} - 10\underset{\sim}{j} - 5\underset{\sim}{k} = 5(3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k})$

so we may take $\underset{\sim}{v} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$ in the direction of $\mathcal{L}$.

Putting $z = 0$, the equations for P_1 and P_2 become

$$\begin{aligned} x + 2y &= 3 \\ 2x - y &= 1 \\ 4x - 2y &= 2 \end{aligned}$$

giving $5x = 5$, so $x = 1$, $y = 1$, so that $(1,1,0)$ is a point on $\mathcal{L}$. Hence a parametric vector equation for $\mathcal{L}$ is

$$\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + t(3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}).$$

Q1/ (a) (iii) P_3 has equation $3x-2y-z=5$, so has normal vector $3\underline{i}-2\underline{j}-\underline{k}$, so that P_3 is perpendicular to $\mathcal{L}$, so must have an intersection point.

(b) (i) The area of the parallelogram determined by OA and OB is

$$|\vec{OA}\times\vec{OB}| = \left|\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 0 \\ -2 & 1 & 0 \end{vmatrix}\right| = |8\underline{k}| = 8.$$

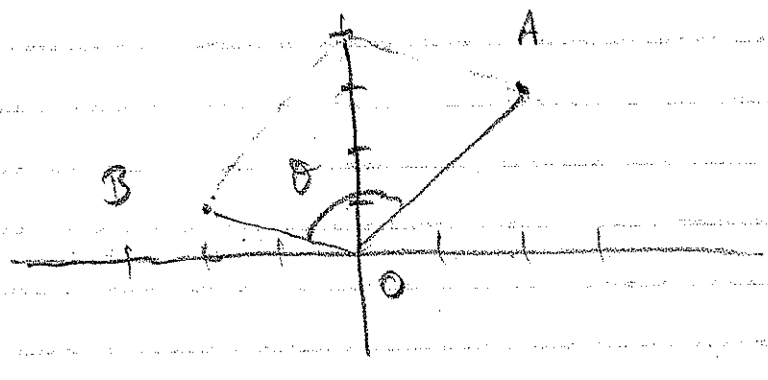
(ii) If θ = angle AOB then

$$\cos\theta = \frac{\vec{OA}\cdot\vec{OB}}{|\vec{OA}||\vec{OB}|} = \frac{(2\underline{i}+3\underline{j})\cdot(-2\underline{i}+\underline{j})}{|2\underline{i}+3\underline{j}|\,|-2\underline{i}+\underline{j}|}$$

$$= \frac{-4+3}{\sqrt{4+9}\sqrt{4+1}} = \frac{-1}{\sqrt{13}\sqrt{5}}$$

so

$$\theta = \arccos\left(\frac{-1}{\sqrt{13}\sqrt{5}}\right).$$



Q2/ (a) $\quad a\underset{\sim}{u} + b\underset{\sim}{v} = c\underset{\sim}{u} + d\underset{\sim}{v}$

$$\Rightarrow \quad (a\underset{\sim}{u} + b\underset{\sim}{v}) \cdot \underset{\sim}{u} = (c\underset{\sim}{u} + d\underset{\sim}{v}) \cdot \underset{\sim}{u}$$

$$\text{and} \quad (a\underset{\sim}{u} + b\underset{\sim}{v}) \cdot \underset{\sim}{v} = (c\underset{\sim}{u} + d\underset{\sim}{v}) \cdot \underset{\sim}{v}$$

$$\Rightarrow \quad a|\underset{\sim}{u}|^2 + b\,\underset{\sim}{v} \cdot \underset{\sim}{u} = c|\underset{\sim}{u}|^2 + d\,\underset{\sim}{v} \cdot \underset{\sim}{u}$$

$$\text{and} \quad a\,\underset{\sim}{u} \cdot \underset{\sim}{v} + b|\underset{\sim}{v}|^2 = c\,\underset{\sim}{u} \cdot \underset{\sim}{v} + d|\underset{\sim}{v}|^2$$

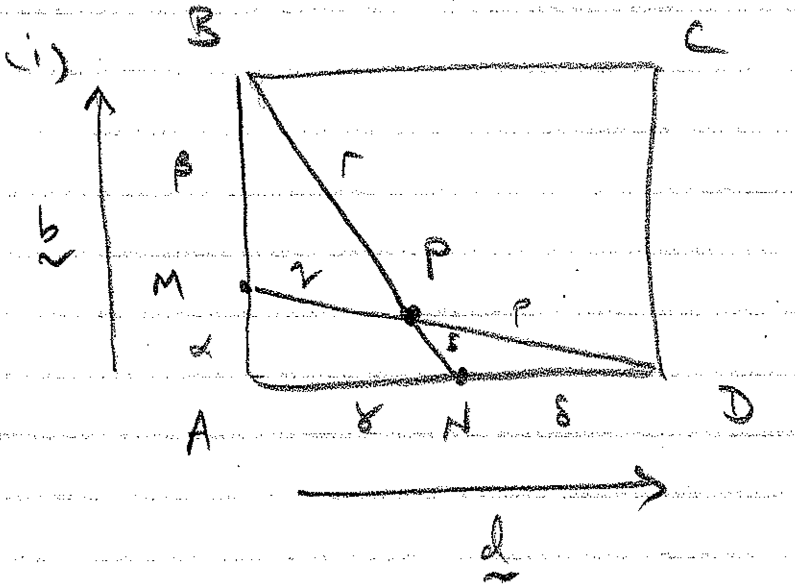
$$\Rightarrow \quad a|\underset{\sim}{u}|^2 = c|\underset{\sim}{u}|^2 \quad \text{and} \quad b|\underset{\sim}{v}|^2 = d|\underset{\sim}{v}|^2$$

$$(\text{since } \underset{\sim}{u} \cdot \underset{\sim}{v} = 0)$$

$$\Rightarrow \quad a = c \quad \text{and} \quad b = d$$

$$(\text{since } |\underset{\sim}{u}|^2, |\underset{\sim}{v}|^2 \neq 0).$$

(b) (i)



$\alpha + \beta = \gamma + \delta = 1$

(ii) We have $p + q = r + s = 1$

Q2/(b) (cont.)

$$\overrightarrow{AP} = \overrightarrow{AM} + \overrightarrow{MP} = \alpha \overrightarrow{AB} + q\overrightarrow{MD}$$

$$= \alpha \underset{\sim}{b} + q(\overrightarrow{MA} + \overrightarrow{AD})$$

$$= \alpha \underset{\sim}{b} + q(-\alpha \underset{\sim}{b} + \underset{\sim}{d})$$

$$= \alpha(1-q)\underset{\sim}{b} + q\underset{\sim}{d}$$

and also

$$\overrightarrow{AP} = \overrightarrow{AN} + \overrightarrow{NP} = \gamma \overrightarrow{AD} + s\overrightarrow{NB}$$

$$= \gamma \underset{\sim}{d} + s(\overrightarrow{NA} + \overrightarrow{AB})$$

$$= \gamma \underset{\sim}{d} + s(-\gamma \underset{\sim}{d} + \underset{\sim}{b})$$

$$= s\underset{\sim}{b} + \gamma(1-s)\underset{\sim}{d}.$$

By (a), $\alpha(1-q) = s$ and $q = \gamma(1-s)$

so that $q = \gamma(1-\alpha(1-q)) = \gamma - \gamma\alpha + \gamma\alpha q$,

so $q(1-\gamma\alpha) = \gamma(1-\alpha)$,

so $q = \dfrac{\gamma(1-\alpha)}{1-\gamma\alpha}$, $p = 1-q = \dfrac{1-\gamma\alpha-\gamma+\gamma\alpha}{1-\gamma\alpha}$

$$= \frac{1-\gamma}{1-\gamma\alpha}$$

and $s = \dfrac{\alpha(1-\gamma)}{1-\gamma\alpha}$,

$$r = 1-s = \frac{1-\gamma\alpha-\alpha+\alpha\gamma}{1-\gamma\alpha} = \frac{1-\alpha}{1-\gamma\alpha}.$$

Q4/ $$C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) $$\det(C - xI) = \begin{vmatrix} 1-x & 1 & 0 \\ 1 & 2-x & 1 \\ 0 & 1 & 1-x \end{vmatrix} = \begin{vmatrix} 1-x & 1 & 0 \\ 0 & 2-x & 1 \\ x-1 & 1 & 1-x \end{vmatrix}$$

$$= (1-x) \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2-x & 1 \\ -1 & 1 & 1-x \end{vmatrix} = (1-x) \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2-x & 1 \\ 0 & 2 & 1-x \end{vmatrix}$$

$$= (1-x) \begin{vmatrix} 2-x & 1 \\ 2 & 1-x \end{vmatrix} = (1-x)\left[(2-x)(1-x) - 2\right]$$

$$= (1-x)(x^2 - 3x + 2 - 2) = (1-x)\,x\,(x-3).$$

(b) eigenvalues are roots of $\det(C - xI)$,

that is, $x = 1, 0, 3$.

(c) eigenvalues of C^k are $1^k = 1$, $0^k = 0$, 3^k.

<u>Reason</u>: If $C\underline{v} = \lambda \underline{v}$ then

$$C^k \underline{v} = C^{k-1} C \underline{v} = C^{k-1} \lambda \underline{v} = \lambda C^{k-1} \underline{v}$$

$$= \dots = \lambda^k \underline{v}.$$

(d) characteristic polynomial of C^5 is

$$(1-x)\,x\,(x - 3^5).$$