The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 9

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/

Lecturers: Daniel Daners and David Easdown

Material covered

- ☐ Solution of first order differential equations by separation of variables
- ☐ Particular solutions of differential equations
- ☐ Applications of differential equations in various contexts

Outcomes

After completing this tutorial you should

- ☐ confident in solving separable first order differential equations with or without initial conditions
- ☐ be able to deal with differential equations arising from a variety of models and be able to interpret the solutions.

Questions to do before the tutorial

1. Find the general solutions of the following differential equations.

(a) $(1+x^2)\dfrac{dy}{dx} + xy = 0,$

Solution: Separating variables we get

$$\frac{dy}{y} = -\frac{x}{1+x^2}dx$$

Integrating we obtain

$$\int \frac{dy}{y} = -\int \frac{x}{1+x^2}\,dx$$

(where the constant is part of the indefinite integrals). Hence

$$\ln|y| = -\frac{1}{2}\ln(1+x^2) + C.$$

With a new constant of integration $A = \pm e^C$, we get the general solution,

$$y(x) = \frac{A}{\sqrt{1+x^2}}.$$

(b) $x\dfrac{dy}{dx} = y^2 - 1.$

Solution: Separating variables we get

$$\frac{dy}{y^2-1} = \frac{dx}{x}$$

Integrating by partial fractions we obtain

$$\int \frac{dx}{x} = \int \frac{dy}{y^2-1} = \frac{1}{2}\int \frac{1}{y-1} - \frac{1}{y+1}\,dy.$$

Hence

$$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \ln|x| + C.$$

With a new constant $A = \pm e^{2C}$, exponentiation gives

$$\frac{y-1}{y+1} = Ax^2,$$

which can be rearranged to give the general solution,

$$y = \frac{1+Ax^2}{1-Ax^2}.$$

Questions to complete during the tutorial

2. Find the general solution of the following differential equations.

(a) $(x^2y^2 + x^2 + y^2 + 1)\dfrac{dy}{dx} = xy + x,$

Solution: To be able to separate variables we first factorise the left hand side, so that

$$(x^2+1)(y^2+1)\frac{dy}{dx} = x(y+1).$$

We then separate variables and integrate:

$$\int \frac{xdx}{x^2+1} = \int \frac{y^2+1}{y+1}\,dy = \int y - 1 + \frac{2}{y+1}\,dy.$$

Carrying out the integration we get

$$\frac{1}{2}\ln(x^2+1) = \frac{1}{2}y^2 - y + 2\ln|y+1| + C.$$

Exponentiating and renaming the constant, we get the general solution,

$$(1+y)^4e^{y^2-2y} = A(1+x^2).$$

This relation defines $y(x)$ implicitly. There is no explicit expression for $y(x)$. (If desired, we could solve for x in terms of y.)

(b) $ye^x\dfrac{dy}{dx} = y^2 + y - 2.$

Solution: We first separate variables and factorise $y^2 + y - 2$ to get

$$e^{-x}\,dx = \frac{y}{(y-1)(y+2)}\,dy = \frac{1}{3}\Big(\frac{1}{y-1} + \frac{2}{y+2}\Big)\,dy$$

Integrating both sides we get

$$-e^{-x} + C = \frac{1}{3}\ln\big|(y-1)(y+2)^2\big|.$$

Exponentiating and renaming the constant, we get the general solution,

$$(y-1)(y+2)^2 = A\exp\big\{-3e^{-x}\big\}.$$

In this case, there is an explicit expression for $y(x)$, but it is messy. Leave the answer as a cubic equation for y as shown.

3. Find the particular solutions of the following differential equations.

(a) $\dfrac{dy}{dx} = xe^{y-x^2}, \quad y(0) = 0,$

Solution: Separating variables and integrating we get

$$\int e^{-y}dy = \int xe^{-x^2}dx,$$

so

$$-e^{-y} = -\frac{1}{2}e^{-x^2} + C_1$$

for some constant C_1. Therefore

$$y = -\ln\Big(\frac{1}{2}e^{-x^2} + C\Big)$$

if we set $C = -C_1$. Substituting $x = 0$ and $y = 0$ into this equation gives

$$0 = -\ln\Big(\frac{1}{2} + C\Big)$$

and hence $C = 1/2$. So the particular solution is

$$y = -\ln\Big(\frac{1}{2}(e^{-x^2} + 1)\Big) = \ln 2 - \ln(e^{-x^2} + 1).$$

(b) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}, \quad y(0) = a$, a a constant.

Solution: We first compute the general solution by separation of variables:

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\tan^{-1} y = \tan^{-1} x + C.$$

To determine the constant C we substitute $x = 0$ and $y = a$ into this equation. This gives $C = \tan^{-1} a$. Hence,

$$y = \tan\big(\tan^{-1} x + \tan^{-1} a\big) = \frac{\tan(\tan^{-1} x) + \tan(\tan^{-1} a)}{1 - \tan(\tan^{-1} x)\tan(\tan^{-1} a)} = \frac{a+x}{1-ax}$$

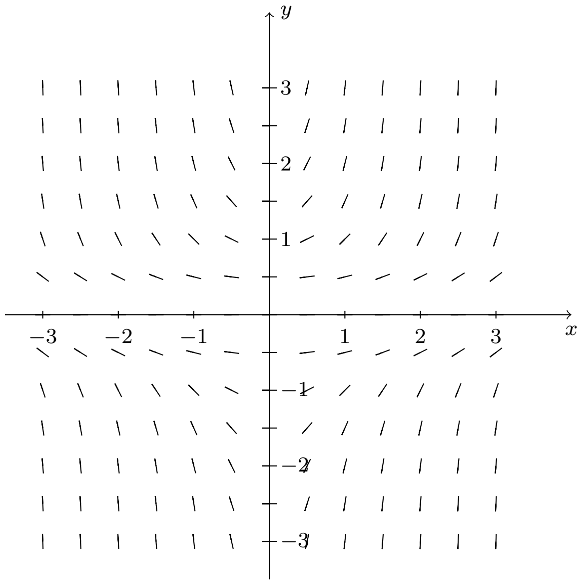
if we use the formula for $\tan(A+B)$ from an earlier tutorial.

Remark. This differential equation has an interesting feature. Both the equation and its solution involve only rational functions while the method of obtaining the solution took a detour through the realm of transcendental functions. A similar example involving algebraic functions (left as an exercise) is $dy/dx = \sqrt{ay^2+b}/\sqrt{ax^2+b}$, whose general solution is $y = x\sqrt{1+aa^2} + a\sqrt{ax^2+b}$. It is remarkable that a similar construction works when the quadratic function under the square root is replaced by a cubic or quartic function, even though the intermediate transcendental functions are not elementary.

4. Consider the differential equation $y' = xy^2$.

(a) Sketch the direction field for the given differential equation for $-3 \le x, y \le 3$.

Solution: The direction field is



(b) Solve the differential equation with the initial condition $y(1) = -2$.

Solution: The differential equation is separable. Separating and integrating we get

$$\int_{-2}^{y} \frac{1}{z^2}\, dz = \int_{1}^{x} \xi \, d\xi.$$

Evaluating the integrals we deduce that

$$-\frac{1}{z}\Big|_{-2}^{y} = -\frac{1}{y} - \frac{1}{2} = \frac{\xi^2}{2}\Big|_{1}^{x} = \frac{x^2}{2} - \frac{1}{2}.$$

Solving for y we conclude that

$$y = -\frac{2}{x^2}.$$

The solution is always a single curve containing the initial value. Since $y(x) \to -\infty$ as $x \to 0$ from the right, the existence interval for the solution is $(0, \infty)$.

Alternatively we could compute the general solution and then use the initial condition to determine the constant.

(c) The equilibrium solution $y = 0$ is stable if any solution starting near zero stays near zero for all $x > 0$. Is the zero solution $y = 0$ for the given differential equation stable? Briefly justify your answer.

Solution: The solution is unstable. From the direction field it is evident that for every solution $y(x)$ starting with an initial condition $y(x_0) > 0$ we have $y(x) \to \infty$ as $x \to \infty$.

5. Many special functions including the exponential function can be defined as a particular solution of a differential equation. This is in particular true for the exponential function: It is the unique function that is its own derivative and has value one at zero. More formally, we are interested in a function $u\colon \mathbb{R} \to \mathbb{R}$ satisfying the differential equation

$$u'(x) = u(x) \qquad \text{and} \qquad u(0) = 1.$$

This question shows the existence and uniqueness of such a function and its properties.

(a) Use the fundamental theorem of calculus to show that u solves the above initial value problem if and only if

$$u(x) = 1 + \int_0^x u(t)\,dt$$

for all $x \in \mathbb{R}$.

Solution: Suppose that u is a solution of the differential equation. Then by the fundamental theorem of calculus and assumption that $u' = u$

$$u(x) - u(0) = \int_0^x u'(t)\,dt = \int_0^x u(t)\,dt$$

Using that $u(0) = 1$ the claim follows. To show the converse we again use the Fundamental Theorem of Calculus. We see that

$$\frac{d}{dx}u(x) = \frac{d1}{dx} + \frac{d}{dx}\int_0^x u(t)\,dt = 0 + u(x)$$

and hence $u' = u$. Also $u(0) = 1 + \int_0^0 u(t)\,dt = 1$.

(b) We approximate solutions by an iterative process. We let

$$u_0(x) = 1 \qquad \text{and} \qquad u_{n+1}(x) = 1 + \int_0^x u_n(t)\,dt \quad \text{for } n \geq 0.$$

Using mathematical induction on n prove that

$$u_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

(This is a special case of "Picard-Lindelöf iteration" to prove the existence and uniqueness of solutions to differential equations.)

Solution: For $n = 0$ the identity

$$u_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

reduces to $u_0 = 1$ which is given. Assuming the formula holds for n we compute

$$\begin{aligned}
u_{n+1} &= 1 + \int_0^x u_n(t)\,dt = 1 + \int_0^x 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!}\,dt \\
&= 1 + \left[t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots + \frac{t^{n+1}}{(n+1)!}\right]_0^x \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{(n+1)!} - 0 = u_{n+1}(x)
\end{aligned}$$

as required.

(c) Using the ratio test show that $u(x) := \lim_{n\to\infty} u_n(x)$ exists. Assuming that you can take a limit inside an integral, show that u is a solution of the differential equation under consideration (This shows the existence of a solution.)

Solution: For any given $x \in \mathbb{R}$ We have

$$\frac{|x|^{n+1}}{(n+1)!}\frac{n!}{|x|^n} = \frac{|x|}{n+1} \to 0$$

as $n \to \infty$. Hence the ratio test implies convergence of $u_n(x)$ to some $u(x)$. Now, assuming that we can interchange limit and integral we have

$$\begin{array}{ccc} u_{n+1}(x) & = & 1 + \displaystyle\int_0^x u_n(t)\,dt \\ \Big\downarrow n\to\infty & & \Big\downarrow n\to\infty \\ u(x) & = & 1 + \displaystyle\int_0^x u(t)\,dt. \end{array}$$

Hence u satisfies the integral equation from part (b) and hence is a solution of $u' = u$ as required.

(d) Assume that u and v are solutions of the differential equation. Show that $\dfrac{u(x)}{v(x)} = 1$ for all $x \in \mathbb{R}$ and hence that the solution to the differential equation is unique.

Solution: Differentiating the quotient and using that $u' = u$ and $v' = v$ we have

$$\frac{d}{dx}\frac{u(x)}{v(x)} = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{u(x)v(x) - u(x)v(x)}{v(x)^2} = 0.$$

Hence, $\dfrac{u(x)}{v(x)} = c$ for some constant c. As $\dfrac{u(0)}{v(0)} = \dfrac{1}{1} = 1$ we deduce that $c = 1$.

The next few parts establish fundamental properties of the solution to link it to the exponential function.

(e) Let u be a solution of the differential equation. Show that $u(x)u(-x) = 1$ for all $x \in \mathbb{R}$ and hence deduce that $u(x) \neq 0$.

Solution: Differentiating $u(x)u(-x)$ and using that u is a solution to the differential equation we obtain

$$\big(u(x)u(-x)\big)' = u'(x)u(-x) - u(x)u'(-x) = u(x)u(-x) - u(x)u(-x) = 0$$

for all $x \in \mathbb{R}$. Hence the product is constant. As $u(0) = 1$ we conclude that $u(x)u(-x) = u(0)u(0) = 1$ as claimed. In particular, the product can only be non-zero if both factors are non-zero, so $u(x) \neq 0$ for all $x \in \mathbb{R}$.

(f) Let u be a solution of $u' = u$. Fix $y \in \mathbb{R}$ and show that $v(x) := u(x+y)/u(y)$ is a solution of the differential equation. Hence show that $u(x+y) = u(x)u(y)$ for all $x, y \in \mathbb{R}$.

Solution: By the previous part $u(y) \neq 0$, so we can divide by $u(y)$. By differentiation and using the differential equation we see that

$$v'(x) = \Big(\frac{u(x+y)}{u(y)}\Big)' = \frac{u'(x+y)}{u(y)} = \frac{u(x+y)}{u(y)} = v(x).$$

Moreover $v(0) = u(0+y)/u(y) = 1$. Hence by the uniqueness of solutions we see that $v = u$ and thus $u(x) = u(x+y)/u(y)$, that is, $u(x+y) = u(x)u(y)$.

(g) Set $e := u(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$. Show that $u(n) = e^n$ and $u(1/n) = \sqrt[n]{e}$. This justifies the definition $e^x := u(x)$ for all $x \in \mathbb{R}$.

Solution: Applying the previous part repeatedly we have

$$u(n) = u(n-1+1) = u(n-1)u(1) = \cdots = (u(1))^n = e^n$$

for all $n \in \mathbb{N}$. Similarly we have

$$e = u(1) = u(n(1/n)) = u(1/n + 1/n + \cdots + 1/n) = (u(1/n))^n,$$

so $u(1/n) = \sqrt[n]{e}$.

*(h) Let $M > 0$ be given. Show that for $|x| \le M$ and $n + 1 \ge 2M$ we have that

$$|u_n(x) - u(x)| \le 2\frac{M^n}{n!}.$$

Deduce that for $|x| \le M$

$$\lim_{n\to\infty} \int_0^x u_n(t)\,dt = \int_0^x u(t)\,dt.$$

(This completes part (c), justifying the interchange of the integral and the limit.)

Solution: Assume now $k \ge n + 1 \ge 2M \ge |x|$. As

$$\begin{aligned} k! &= (1 \cdot 2 \cdots n) \cdot (n+1) \cdot (n+2) \cdots (k-1)k \\ &= n!(n+1) \cdot (n+2) \cdots (k-1)k \ge n!\big(2M\big)^{k-n} \end{aligned}$$

we deduce that

$$\frac{|x|^k}{k!} \le \frac{M^k}{k!} \le \frac{M^k}{n!\big(2M\big)^{k-n}} = \frac{M^n}{n!}\frac{1}{2^{k-n}}$$

Hence, if $n + 1 \ge 2M \ge |x|$, then

$$|u_n(x) - u(x)| = \Big|\sum_{k=n+1}^{\infty} \frac{x^k}{k!}\Big| \le \frac{M^n}{n!} \sum_{k=n+1}^{\infty} \frac{1}{2^{k-n}} = \frac{M^n}{n!}$$

where in the last step we used the sum of a geometric series. This means that given $M > 0$ and $|x| \le M$ we have

$$\begin{aligned} \Big|\int_0^x u(t)\,dt - \int_0^x u_n(t)\,dt\Big| &= \Big|\int_0^x u(t) - u_n(t)\,dt\Big| \\ &\le \int_{-|x|}^{|x|} |u(t) - u_n(t)|\,dt \le \int_{-M}^{M} \frac{M^n}{n!}\,dt = 2M\frac{M^n}{n!} \end{aligned}$$

for all $n + 1 \ge M$. As $M^n/n! \to 0$ as $n \to \infty$ we deduce that

$$\lim_{n\to\infty} \int_0^x u_n(t)\,dt = \int_0^x u(t)\,dt$$

as claimed.

Extra questions for further practice

6. Einstein's Theory of Relativity predicts the existence of black holes: regions in space from which nothing can escape, due to strong gravitational forces. The theory predicts that black holes will be formed when large stars collapse.

However, Einstein's theory did not take into account quantum mechanical effects. In 1975, Stephen Hawking used quantum theory to show that a black hole should glow slightly; that is, it should radiate energy and particles in the same way that a heated object does. Assuming that nothing else falls into the black hole, this causes its mass M to decrease at the rate governed by the differential equation,

$$\frac{dM}{dt} = -\frac{\alpha}{M^2},$$

where t denotes time and α is a constant whose value is not yet known precisely.

(a) Find the general solution $M(t)$ of this differential equation.

Solution: Rewriting the equation as $\alpha dt/dM = -M^2$ and integrating with respect to M gives

$$\alpha t = -\frac{1}{3}M^3 + C,$$

with C an arbitrary constant of integration. Thus

$$M(t) = (3C - 3\alpha t)^{1/3}.$$

(b) Find the particular solution which satisfies the condition that the mass is M_0 when $t = 0$.

Solution: With $t = 0$, the solution above gives $M_0 = (3C)^{1/3}$. Hence $C = M_0^3/3$ and so the particular solution is

$$M(t) = \left(M_0^3 - 3\alpha t\right)^{1/3}.$$

(c) How long does it take for a black hole which initially has mass M_0 to lose half its mass? How long does it take for it to evaporate completely?

Solution: We want to find $t_{1/2}$ such that $M(t_{1/2}) = \frac{1}{2}M_0$. Thus $\left(M_0^3 - 3\alpha t_{1/2}\right)^{1/3} = \frac{1}{2}M_0$, which gives $M_0^3 - 3\alpha t_{1/2} = \frac{1}{8}M_0^3$ and hence $3\alpha t_{1/2} = \frac{7}{8}M_0^3$. Thus the time taken for the black hole to evaporate half its mass is

$$t_{1/2} = \frac{7M_0^3}{24\alpha}.$$

We also want to find t_1 such that $M(t_1) = 0$. Thus $\left(M_0^3 - 3\alpha t_1\right)^{1/3} = 0$, which gives $3\alpha t_1 = M_0^3$. Hence the time taken to evaporate completely is

$$t_1 = \frac{M_0^3}{3\alpha}.$$

The rate of evaporation increases towards the end. In particular, $t_1 = (8/7)t_{1/2}$.

7. Find the general solutions of

(a) $\dfrac{dy}{dx} = \dfrac{x + \sin x}{3y^2}$,

Solution: Separating variables

$$\int 3y^2 dy = \int x + \sin x \, dx$$

and so

$$y^3 = \frac{1}{2}x^2 - \cos x + C.$$

Solving for y

$$y = \left(\frac{1}{2}x^2 - \cos x + C\right)^{1/3}.$$

(b) $\dfrac{dx}{dt} = 1 + t - x - tx$,

Solution: The right-hand side factorises as $(1+t)(1-x)$. Hence, the differential equation is separable. Separating and integrating gives

$$\int \frac{dx}{1-x} = \int (1+t)dt$$

and so

$$-\ln|x-1| = t + \frac{1}{2}t^2 + C$$

Solving for x we get

$$x = Ae^{-t-t^2/2} + 1.$$

(c) $\dfrac{dy}{dx} = \dfrac{\ln x}{xy + xy^3}$.

Solution: By separation of variables and integration

$$\int (y + y^3)dy = \int \frac{\ln x}{x} dx = \int u du,$$

where $u = \ln x$. Integrating gives

$$\frac{1}{2}y^2 + \frac{1}{4}y^4 = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

Completing the square gives $(y^2+1)^2 = 2(\ln x)^2 + D$, where $D = 4C+1$. It follows that

$$y^2 + 1 = \pm\sqrt{2(\ln x)^2 + D},$$

and so (ignoring the imaginary solution that results if we use the negative root),

$$y = \pm\sqrt{\sqrt{2(\ln x)^2 + D} - 1}.$$

8. Find particular solutions satisfying the given conditions.

(a) $\dfrac{dy}{dx} = \dfrac{1+x}{xy} \quad (x > 0), \quad y(1) = -4;$

Solution: We first compute the general solution by separating variales:

$$\int y \, dy = \int \frac{1}{x} + 1 \, dx$$

and so

$$\frac{1}{2}y^2 = \ln|x| + x + C$$

Hence the general solution is $y(x) = \pm\sqrt{2x + 2\ln|x| + 2C}$. At $x = 1$ we then get $y(1) = \pm\sqrt{2 + 2C}$, and so the initial condition $y(1) = -4$ implies that we must choose the negative sign and therefore $C = 7$. Hence the required particular solution is

$$y(x) = -\sqrt{2x + 2\ln x + 14}$$

valid for $x > 0$.

(b) $\dfrac{dy}{dt} = \dfrac{ty + 3t}{t^2 + 1}, \quad y(2) = 2.$

Solution: Separating variables and integrating we get

$$\int \frac{1}{y+3}\,dy = \int \frac{t}{t^2+1}\,dt$$

and therefore

$$\ln|y + 3| = \frac{1}{2}\ln(t^2 + 1) + C$$

The general solution therefore is $y(t) = A\sqrt{t^2+1} - 3$. We then get $y(2) = A\sqrt{5} - 3$, and so the initial condition $y(2) = 2$ implies that we must choose $A = \sqrt{5}$. Hence the particular solution is

$$y(t) = \sqrt{5(t^2+1)} - 3.$$

9. Find a function $g(x)$ such that $g'(x) = g(x)\big(1 + g(x)\big)$ and $g(0) = 1$.

Solution: The equation $dg/dx = g(1 + g)$ is separable. Separation and integration gives

$$\int \frac{dg}{g(1+g)} = \int dx,$$

which gives $\ln\big|g/(g+1)\big| = x + C$. Hence $(g + 1)/g = Ae^{-x}$ and

$$g(x) = \frac{1}{Ae^{-x} - 1}.$$

This is the general solution. The initial condition $g(0) = 1$ implies that $A = 2$. So the required particular solution is

$$g(x) = \frac{1}{2e^{-x} - 1}.$$

10. A molecule of substance A can combine with a molecule of substance B to form a molecule of substance X, in a reaction which is denoted $A + B \to X$. According to the Law of Mass Action, the rate of formation of X is proportional to the product of the amounts of A and B present. A test-tube initially contains amounts a and b of substances A and B, respectively, (measured in moles), but none of substance X.

(a) Let $x(t)$ denote the amount of substance X (in moles) produced within the first t seconds. Write down a differential equation for $x(t)$.

Solution: The amounts of substances A and B left after time t are, respectively, $a - x(t)$ and $b - x(t)$. We are told that the rate of increase of x is proportional to the product of these quantities, and so

$$\frac{dx}{dt} = k(a - x)(b - x),$$

where k is a positive constant which characterises the reaction $A + B \to X$.

(b) Assuming that $a \neq b$, solve this equation to obtain an expression for $x(t)$.

Solution: Separating and integrating by using partial fractions we get

$$\int \frac{dx}{(a-x)(b-x)} = \int k dt \quad \text{and hence} \quad \frac{1}{b-a} \ln \left| \frac{b-x}{a-x} \right| = kt + C,$$

where C is an undetermined constant of integration. Putting $A = \pm e^{(b-a)C}$ we then have

$$\frac{b - x(t)}{a - x(t)} = Ae^{(b-a)kt},$$

and, since $x(0) = 0$, we deduce that $A = b/a$. Hence,

$$\frac{b-x}{a-x} = \frac{b}{a} e^{(b-a)kt}. \tag{1}$$

After rearrangement we see that

$$x(t) = \frac{ab\big(e^{(b-a)kt} - 1\big)}{be^{(b-a)kt} - a}.$$

(c) Suppose that initially there are two molecules of B for every molecule of A, and that after 10 seconds there are six molecules of B for every molecule of A. What is the ratio after 30 seconds?

Solution: Equation (1) tells us how many molecules of B there are for each molecule of A after t seconds. But we know that $b = 2a$ (since there is initially twice as much of B as of A), and so this expression reduces to

$$\frac{2a - x}{a - x} = \frac{\text{number of molecules of } B \text{ after } t \text{ seconds}}{\text{number of molecules of } A \text{ after } t \text{ seconds}} = 2e^{akt}.$$

We are also told that this ratio is equal to 6 when $t = 10$, so $6 = 2e^{10ak}$, and hence $e^{10ak} = 3$. Thus, after 30 seconds, the number of molecules of B per molecule of A must be

$$2e^{30ak} = 2\big[e^{10ak}\big]^3 = 2 \times 3^3 = 54.$$

(d) The experiment is repeated, but with the initial amount of substance B halved so as to equal the initial amount a of substance A. (As before, substance X is absent initially.) What fraction of A molecules remain after 30 seconds?

Solution: The amount of substance A present initially is again a, and this time the amount of B present initially is also a. Thus, after t seconds, the test-tube will contain amounts $a - x(t)$ of both substances. Hence, the rate of formation of X is given by the differential equation,

$$\frac{dx}{dt} = k(a - x)^2,$$

where the constant k has the same value as in the previous experiment, where we found that $e^{10ak} = 3$ and hence $k = (\ln 3)/(10a)$. Separating and integrating this equation, we get

$$\frac{1}{a - x(t)} = kt + C_2,$$

where C_2 is a new constant of integration. Recalling that $x(0) = 0$, we deduce that $C_2 = 1/a$, and so the amount of A remaining at time t is

$$a - x(t) = \frac{a}{1 + akt} = \frac{a}{1 + (\ln 3)t/10}.$$

Dividing by a (the amount of A present initially), and taking $t = 30$, we see that the proportion of the original A molecules which remain after 30 seconds is

$$\frac{a - x(t)}{a} = \frac{1}{1 + (\ln 3) \times 30/10} = \frac{1}{1 + 3\ln 3} = \frac{1}{4.2958} = 0.23278 \cdots .$$

Thus, about 23%.