THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Solutions Week 12** 2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Use an appropriate chain rule to find:

(a) $\dfrac{dz}{dt}$ where $z = x^3 + y^3$, $x = 3t$ and $y = 1 - t^2$;

(b) $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ where $z = xy$, $x = s + 2t$ and $y = s - 2t$.

Questions for the tutorial

2. Verify that $f_{xy}(x, y) = f_{yx}(x, y)$ for each function $f(x, y)$ given below.

(a) $x^2y^4 + 3x^2 + 5y^2$ (b) $\sin^2 x \cos y + 2$ (c) $xye^y + 3x + 5y$

Solution

(a) $f_{xy}(x, y) = \dfrac{\partial}{\partial y}(\dfrac{\partial f}{\partial x}) = \dfrac{\partial}{\partial y}(2xy^4 + 6x) = 8xy^3.$

$f_{yx}(x, y) = \dfrac{\partial}{\partial x}(\dfrac{\partial f}{\partial y}) = \dfrac{\partial}{\partial x}(4x^2y^3 + 10y) = 8xy^3.$

(b) $f_{xy}(x, y) = f_{yx} = -2 \sin x \cos x \sin y.$

(c) $f_{xy}(x, y) = f_{yx} = e^y(1 + y).$

3. For each function f given below, find all points (a, b) at which $f_x(x, y)$ and $f_y(x, y)$ are both zero. Determine whether $f(a, b)$ is a local maximum, a local minimum, or neither. (To do this, consider the sign of $f(a + h, b + k) - f(a, b)$.)

(a) $f(x, y) = x^2 + 2xy + 2y^2 - 6x + 8y$ (b) $f(x, y) = xy - 2x - 3y - 4$

Solution

(a) We have $f_x(x, y) = 2x + 2y - 6$ and $f_y(x, y) = 2x + 4y + 8$. Setting these equal to zero and solving simultaneously gives a single solution $x = 10$, $y = -7$. For these values of x and y, $f(x, y) = -58$. We now investigate whether these values give a local maximum, a local minimum or neither. Calculating $f(10 + h, -7 + k) - (-58)$ gives

$$(10 + h)^2 + 2(10 + h)(-7 + k) + 2(-7 + k)^2 - 6(10 + h) + 8(-7 + k) + 58.$$

This simplifies to $f(10 + h, -7 + k) - (-58) = h^2 + 2hk + 2k^2 = (h + k)^2 + k^2 \geq 0$. Therefore there is a local minimum at $x = 10$, $y = -7$.

(b) We have $f_x(x, y) = y - 2$ and $f_y(x, y) = x - 3$. Setting these equal to zero gives a single solution $x = 3$, $y = 2$. For these values of x and y, $f(x, y) = -10$. Calculating $f(3 + h, 2 + k) - (-10)$ gives an answer of hk. If h and k have opposite sign, this is negative. If they are the same sign, it is positive. Hence there is neither a local minimum nor a local maximum at $x = 3$, $y = 2$.

4. Let f be the function given by $f(x,y)=\sqrt{20-x^2-7y^2}$. Find the linear approximation to f at $(2,1)$. Hence find an approximate value of $f(1.95,1.08)$.

Solution

We have $f(2,1)=3$, $f_x(x,y)=\dfrac{-x}{\sqrt{20-x^2-7y^2}}$ and $f_y(x,y)=\dfrac{-7y}{\sqrt{20-x^2-7y^2}}$.

So $f_x(2,1)=-\frac{2}{3}$ and $f_y(2,1)=-\frac{7}{3}$. Thus the linear approximation to $f(x,y)$ at $(2,1)$ is given by

$$\begin{aligned} f(x,y) &\approx f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1) \\ &= 3-\tfrac{2}{3}(x-2)-\tfrac{7}{3}(y-1). \end{aligned}$$

Thus $f(1.95,1.08)\approx 3-\frac{2}{3}(-0.05)-\frac{7}{3}(0.08)\approx 2.85$.

5. Estimate the volume of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter, if the metal in the top and base is 0.1 cm thick and the metal in the walls is 0.05 cm thick.

Solution

The volume of a can of radius r cm and height h cm is $V=\pi r^2 h$ cm^3. If we change the radius by Δr and change the height by Δh then the change in volume is given by

$$\Delta V = V(r+\Delta r, h+\Delta h)-V(r,h)\approx \frac{\partial V}{\partial r}\Delta r+\frac{\partial V}{\partial h}\Delta h = 2\pi rh\,\Delta r+\pi r^2\,\Delta h.$$

This gives an estimate for the volume of metal. Using the values $r=2$, $h=10$, $\Delta r=-0.05$ and $\Delta h=-0.20$,

$$\Delta V = 2\pi(2)(10)(-0.05)+\pi(2^2)(-0.20)=-2.8\pi\approx -8.8.$$

Thus the volume of metal in the can is estimated to be approximately 8.8 cm^3.

6. At time t, the temperature $u(x,t)$ at the point x of a long, thin insulated rod lying along the x axis satisfies the one-dimensional heat equation,

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2} \qquad (k \text{ is a constant}).$$

Show that the function u given by the formula $u(x,t)=e^{-n^2kt}\sin nx$ satisfies the heat equation for any choice of the constant n.

Solution

We have $\dfrac{\partial u}{\partial t}=-n^2ke^{-n^2kt}\sin nx$.

Also,

$$k\frac{\partial^2 u}{\partial x^2}=k\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=k\frac{\partial}{\partial x}(e^{-n^2kt}n\cos nx)=-n^2ke^{-n^2kt}\sin nx.$$

Therefore the given function satisfies the heat equation.

7. A string is stretched along the x axis, fixed at each end, then set in vibration. The displacement $y(x,t)$ of the point at location x at time t satisfies the one-dimensional wave equation $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$, where the constant a depends upon the density and tension of the string. Show that each of the following functions satisfies the wave equation.

(a) $y = \cosh 3(x - at)$

(b) $y = \sin kx \cos kat$ (where k is a constant)

(c) $y = f(x - at) + g(x + at)$, where f and g are any twice-differentiable functions of one variable.

Solution

(a) As $y = \cosh 3(x - at)$, we have $\dfrac{\partial y}{\partial t} = -3a \sinh 3(x - at)$ and $\dfrac{\partial y}{\partial x} = 3 \sinh 3(x - at)$. Hence $\dfrac{\partial^2 y}{\partial t^2} = 9a^2 \cosh 3(x-at)$ and $a^2 \dfrac{\partial^2 y}{\partial x^2} = 9a^2 \cosh 3(x-at)$. That is, $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$.

(b) As $y = \sin kx \cos kat$, we have $\dfrac{\partial y}{\partial t} = -ka \sin kx \sin kat$ and $\dfrac{\partial y}{\partial x} = k \cos kx \cos kat$. Hence $\dfrac{\partial^2 y}{\partial t^2} = -k^2 a^2 \sin kx \cos kat$ and $a^2 \dfrac{\partial^2 y}{\partial x^2} = a^2(-k^2 \sin kx \cos kat)$.

That is, $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$.

(c) We have $\dfrac{\partial y}{\partial t} = -af'(x - at) + ag'(x + at)$ and so

$$\frac{\partial^2 y}{\partial t^2} = a^2 f''(x - at) + a^2 g''(x + at).$$

Also, $\dfrac{\partial y}{\partial x} = f'(x - at) + g'(x - at)$, so

$$\frac{\partial^2 y}{\partial x^2} = f''(x - at) + g''(x + at).$$

It is now evident that $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$.

8. Car A is travelling north at 90 km/h and car B is travelling west at 80 km/h, both approaching the intersection of their highways. How fast is the distance between the cars changing when A is 0.3 km and B is 0.4 km from the intersection?

Solution

Let $s_1 = s_1(t)$ and $s_2 = s_2(t)$ be the distances from the intersection to car A and car B respectively at time t, so $z = \sqrt{s_1^2 + s_2^2}$ is the distance between them. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial s_1}\frac{ds_1}{dt} + \frac{\partial z}{\partial s_2}\frac{ds_2}{dt} = \frac{s_1}{\sqrt{s_1^2 + s_2^2}}(-90) + \frac{s_2}{\sqrt{s_1^2 + s_2^2}}(-80).$$

When $s_1 = 0.3$ and $s_2 = 0.4$, this becomes

$$\frac{dz}{dt} = \frac{0.3}{0.5}(-90) + \frac{0.4}{0.5}(-80) = -118,$$

that is, the distance between the cars is decreasing at the rate 118 km/h.

9. (a) Use two different methods to calculate $\dfrac{\partial u}{\partial t}$ if $u = \sqrt{x^2 + y^2}$, $x = e^{st}$ and $y = 1+s^2 \cos t$.

(b) Use two different methods to calculate $\dfrac{\partial z}{\partial x}$ if $z = \tan^{-1}\left(\dfrac{u}{v}\right)$, $u = 2x+y$ and $v = 3x-y$.

Solution

(a) First substitute for x and y into the formula for u, to obtain u as a function of s and t directly. We obtain

$$u = \sqrt{e^{2st} + (1 + s^2 \cos t)^2},$$

and hence

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\frac{\partial}{\partial t}\left(e^{2st} + (1 + s^2\cos t)^2\right)}{2\sqrt{e^{2st} + (1 + s^2 \cos t)^2}} \\ &= \frac{2se^{2st} + 2(1 + s^2\cos t)(-s^2 \sin t)}{2\sqrt{e^{2st} + (1 + s^2 \cos t)^2}} \\ &= \frac{se^{2st} - s^2 \sin t(1 + s^2 \cos t)}{\sqrt{e^{2st} + (1 + s^2 \cos t)^2}}\end{aligned}$$

Alternatively, use the chain rule:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \\ &= \frac{x}{\sqrt{x^2+y^2}} \times se^{st} + \frac{y}{\sqrt{x^2+y^2}} \times -s^2 \sin t \\ &= \frac{e^{st}se^{st} + (1 + s^2\cos t)(-s^2 \sin t)}{\sqrt{x^2+y^2}} \\ &= \frac{se^{2st} - s^2 \sin t(1 + s^2 \cos t)}{\sqrt{e^{2st} + (1 + s^2 \cos t)^2}}.\end{aligned}$$

(b) As in the previous part, we start with $z = \tan^{-1}\left(\dfrac{2x+y}{3x-y}\right)$ to obtain

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{2(3x-y) - 3(2x+y)}{(3x-y)^2[1 + \left(\frac{2x+y}{3x-y}\right)^2]} \\ &= \frac{-5y}{(3x-y)^2 + (2x+y)^2}.\end{aligned}$$

Using the chain rule,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\ &= \frac{1/v}{1 + \frac{u^2}{v^2}} \times 2 + \frac{-u/v^2}{1 + \frac{u^2}{v^2}} \times 3 \\ &= \frac{2v - 3u}{v^2 + u^2} \\ &= \frac{-5y}{(3x-y)^2 + (2x+y)^2}.\end{aligned}$$

10. An object moves on the surface $z = (x-1)^2 + y^2$. The shadow of the object's path on the xy-plane is given by the parametric equations $x = 2\cos t$, $y = 2\sin t$ where $t \geq 0$ represents time. Use the chain rule to find the rate of change of height of the object above the xy plane. Hence find the maximum height achieved by the object.

Solution

By the chain rule

$$\frac{dz}{dt} = 2(x-1)(-2\sin t) + 2y(2\cos t) = 2(2\cos t - 1)(-2\sin t) + 2(2\sin t)(2\cos t),$$

which simplifies to $4\sin t$. Thus $\dfrac{dz}{dt} = 0$ iff t is a multiple of π , corresponding to the points $(2,0,1)$ and $(-2,0,9)$ on the surface. Thus the maximum height equals 9, which occurs whenever t is an odd multiple of π .

Extra Question

11. Define a function f of two variables by

$$f(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$ at points $(x,y) \neq (0,0)$ and also at $(0,0)$. Observe that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Why does this not contradict the theorem mentioned in lectures?

Solution

For points $(x,y) \neq (0,0)$, we find (after some calculation) that

$$f_x(x,y) = \frac{8x^2y^3 + 2x^4y - 2y^5}{(x^2+y^2)^2},$$

$$f_y(x,y) = \frac{2x^5 - 8x^3y^2 - 2xy^4}{(x^2+y^2)^2},$$

and

$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{2x^6 - 2y^6 + 18x^4y^2 - 18x^2y^4}{(x^2+y^2)^3}.$$

At the point $(0,0)$ we find

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0/h^2}{h} = 0,$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0/k^2}{k} = 0.$$

Also, at $(0,0)$ we have

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{-2k^5/k^4}{k} = -2,$$

and

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{2h^5/h^4}{h} = 2.$$

We observe that $f_{xy}(0,0) \neq f_{yx}(0,0)$. This does not contradict the theorem mentioned in lectures because the function f doesn not satisfy the conditions of the theorem. It is true that f is continuous at $(0,0)$, since (using polar coordinates for x and y)

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0} \frac{2r^4\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta)}{r^2} \\
&= \lim_{r\to 0} r^2 \sin 2\theta \cos 2\theta \\
&= \lim_{r\to 0} \tfrac{1}{2} r^2 \sin 4\theta \\
&= 0 = f(0,0),
\end{aligned}$$

using the squeeze law. However, the function f_{xy} $(= f_{yx})$ is not continuous at $(0,0)$, since along the x axis (where $y = 0$),

$$\lim_{(x,y)\to(0,0)} f_{xy}(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x^6 - 2y^6 + 18x^4y^2 - 18x^2y^4}{(x^2+y^2)^3} = \lim_{x\to 0} \frac{2x^6}{x^6} = 2,$$

and along the y axis (where $x = 0$),

$$\lim_{(x,y)\to(0,0)} f_{xy}(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x^6 - 2y^6 + 18x^4y^2 - 18x^2y^4}{(x^2+y^2)^3} = \lim_{y\to 0} \frac{-2y^6}{y^6} = -2.$$

This shows that $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ doesn't exist, and so certainly f_{xy} (and therefore f_{yx}) is not continuous at $(0,0)$.

Solution to Question 1

(a) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (3x^2)(3) + (3y^2)(-2t) = -6t + 81t^2 + 12t^3 - 6t^5.$

(b) $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = (y)(1) + (x)(1) = 2s$, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = (y)(2) + (x)(-2) = -8t.$