THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 13

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
Lecturer: Daniel Daners

Material covered

- ☐ Implicit Function Theorem.
- ☐ Tangents to level curves.
- ☐ Directional derivatives.
- ☐ The gradient vector.
- ☐ Properties of the gradient vector.

Outcomes

After completing this tutorial you should

- ☐ apply the implicit function theorem;
- ☐ calculate tangents to level curves;
- ☐ compute directional derivatives;
- ☐ understand the properties of the gradient vector.

Summary of essential material

Mixed Derivatives Theorem. Suppose $D \subseteq \mathbb{R}^2$ and $f: D \to \mathbb{R}$. If $(x_0, y_0) \in D$ that $f_{xy}(x, y)$ and $f_{yx}(x, y)$ both exist in a disc around (x_0, y_0) and are both continuous at (x_0, y_0), then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0),$$

that is, mixed partial derivatives can be computed in any order. For efficient computation choose the order that requires minimal effort.

Chain Rule. Let $z = f(x, y)$, and suppose that $x = x(t)$ and $y = y(t)$. If $x(t)$ and $y(t)$ are continuous at $t = t_0$, and if f_x and f_y are continuous at $(a, y_0) = (x(t_0), y(t_0))$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

The chain rule generalises to functions $x = x(u, v)$ and $y = y(u, v)$ of two variables (if we fix one of them we are back to the one variable case):

$$\frac{\partial z}{du} = \frac{\partial f}{\partial x}\frac{\partial x}{du} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \text{and} \qquad \frac{\partial z}{dv} = \frac{\partial f}{\partial x}\frac{\partial x}{dv} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Implicit Function Theorem. Suppose that (x_0, y_0) lies on the level curve C given by $f(x, y) = c$. If f has continuous partial derivatives f_x and f_y in a neighbourhood of (x_0, y_0), and if $f_y(x_0, y_0) \neq 0$, then there is a disc D around (x_0, y_0) such that the part of C inside D is the graph of a differentiable function $y = y(x)$, and the derivative of $y(x)$ at $(x, y) = (x_0, y_0)$ is

$$y'(a) = -\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)}.$$

Gradient Vectors. The *gradient vector* of $f(x, y)$ is

$$\nabla f(x, y) := f_x(x, y)\boldsymbol{i} + f_y(x, y)\boldsymbol{j}.$$

Properties of $\nabla f(x, y)$: Suppose that $f(x, y)$ has continuous partial derivatives at $(x, y) = (x_0, y_0)$. Then

(1) $\nabla f(x_0, y_0)$ points in the direction of the steepest increase of $z = f(x, y)$ at $(x, y) = (x_0, y_0)$.

(2) $|\nabla f(x_0, y_0)| = \sqrt{f_x(x_0, y_0)^2 + f_y(x_0, y_0)^2}$ is the magnitude of this steepest increase.

(3) If (x_0, y_0) lies on the level curve $f(x, y) = k$, then $\nabla f(x_0, y_0)$ is perpendicular to this level curve at (x_0, y_0).

Directional Derivatives. Let $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ be a *unit vector*. The *directional derivative* of a function $f(x, y)$ in the direction $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}} f(x, y) = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

if the limit exists. It is the slope of the tangent of the curve obtained by intersecting graph $z = f(x, y)$ with the plane through (x_0, y_0) parallel to the z axis and containing $\boldsymbol{u}$. If $f(x, y)$ has continuous partial derivatives, then

$$D_{\boldsymbol{u}} f(x, y) = \nabla f(x, y) \cdot \boldsymbol{u} = f_x(x, y)u_1 + f_y(x, y)u_2 \qquad \text{(the “}\cdot\text{” denotes the dot product).}$$

Special cases are the partial derivatives: they are the directional derivatives in the direction of $\boldsymbol{i}$ and $\boldsymbol{j}$.

Questions to complete during the tutorial

1. Find the equation of the tangent to the level curve

$$1 - 2x + 6y + \sinh(3 - 2x + y) = 3$$

at the point $(x, y) = (2, 1)$.

2. Let $f(x, y) = \sin(x^2 - y) + 4xy + 3$.

(a) Find the tangent plane to $z = f(x, y)$ at the point $(x, y) = (2, 4)$.

(b) What is the direction of the steepest slope to the graph $z = f(x, y)$ at the point $(x, y) = (2, 4)$? What is the magnitude of this slope?

(c) What is the slope of the graph $z = f(x, y)$ in the direction $\boldsymbol{i} + 3\boldsymbol{j}$ at the point $(x, y) = (2, 4)$?

3. Let $f(x, y) = e^y - x\sin(x + y)$. Show that the equation $f(x, y) = 1$ implicitly defines $y = y(x)$ as a function of x in a disc around $(x, y) = (\pi, 0)$. Compute $y'(x)$, and hence find the equation of the tangent line to the level curve $f(x, y) = 1$ at the point $(x, y) = (\pi, 0)$.

4. An ant is standing on the kitchen floor. The floor has coordinates such that the x-axis points east, and the y-axis points north. The temperature on the kitchen floor at the point (x, y) is given by the formula $T(x, y) = x^2 - 2y^2 + 4xy$. The ant is currently at the point $(x, y) = (2, 1)$ on the floor.

(a) In which direction should the ant walk to initially increase temperature most rapidly? What is the rate of change in temperature that the ant will experience if it walks in this direction?

(b) What is the rate of change in temperature that the ant initially experiences if it walks in the direction $\boldsymbol{i} - \boldsymbol{j}$?

5. (a) Use two different methods to calculate $\dfrac{\partial z}{\partial t}$ if $z = \sqrt{x^2 + y^2}$, $x = e^{st}$ and $y = 1 + s^2\cos t$.

(b) Use two different methods to calculate $\dfrac{\partial z}{\partial u}$ if $z = \tan^{-1}(x/y)$, $x = 2u + v$ and $y = 3u - v$.

6. An object moves on the surface $z = (x - 1)^2 + y^2$. The projection of the object's path onto the xy-plane is given by the parametric equations $x = 2\cos t$, $y = 2\sin t$ where $t \geq 0$ represents time. Use the chain rule to find the rate of change of height of the object above the xy plane. Hence find the maximum height achieved by the object.

7. Let $z = f(x, y)$, and suppose that $x = r\cos\theta$ and $y = r\sin\theta$. Show that

$$r^2\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right).$$

You may assume that $f(x, y)$ is nice enough so that the Chain Rule applies.

Extra questions for further practice

8. Let $f(x) = x^3 - 3x + 1$.

(a) Show that the function $f : [-1, 1] \to [-1, 3]$ is bijective.

(b) Let $f^{-1} : [-1, 3] \to [-1, 1]$ be the inverse function. Calculate the third order Taylor polynomial of $f^{-1}(x)$ centred at $x = 1$.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be function, and suppose that f satisfies

$$f(ta, tb) = t f(a, b) \qquad \text{for all} \quad (a, b) \in \mathbb{R}^2 \quad \text{and all} \quad t \in \mathbb{R}.$$

In this question you may assume that f is sufficiently smooth so that the chain rule, and any other theorems from lectures, apply.

(a) Use the chain rule to show that

$$f_x(a, b)a + f_y(a, b)b = f(a, b) \qquad \text{for all} \quad (a, b) \in \mathbb{R}^2.$$

(b) Show that the origin $(0, 0, 0) \in \mathbb{R}^3$ lies on every tangent plane to the surface $z = f(x, y)$.

(c) You are given that $f(2, 1) = 4$. Find the value of the directional derivative $D_{\boldsymbol{u}} f(6, 3)$ in the direction $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j}$.

10. Suppose that $f(x, y) = ye^{xy}$.

(a) Use the implicit function theorem to show that $f(x, y) = 1$ defines y implicitly as a function of x in a neighbourhood of the point $(0, 1)$.

(b) Use implicit differentiation to find the second order Taylor polynomial of the implicity defined function $y = y(x)$ centred at $x = 0$.