The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 12

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/
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Material covered

- ☐ Homogeneous linear second order differential equations with constant coefficients.
- ☐ Inhomogeneous linear second order differential equations with constant coefficients.

Outcomes

After completing this tutorial you should

- ☐ be confident in solving homogeneous second order homogeneous and inhomogeneous differential equations in various contexts.

Summary of essential material

Homogeneous linear second order equations with constant coefficients. Consider a differential equation of the form

$$ay'' + by' + c' = 0$$

with $a, b, c \in \mathbb{R}$ constants and $a \neq 0$. To find the general write down the *auxiliary equation*

$$a\lambda^2 + b\lambda + c = 0$$

and find its roots (real or complex). Depending on the nature of the roots apply the relevant case:

Case 1: The auxiliary equation has *two distinct real roots* $\lambda_1 \neq \lambda_2$. Then the general solution is

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

Case 2: The auxiliary equation has one (real) *double root* λ. Then the general solution is

$$y(t) = (A + Bt)e^{\lambda t}$$

Case 3: The auxiliary equation has a *pair of complex conjugate roots* $\lambda = \mu \pm i\omega$. Then the real form of the general solution is

$$y(t) = e^{\mu t}\bigl(A\cos(\omega t) + B\sin(\omega t)\bigr)$$

Inhomogeneous linear second order equations with constant coefficients. Consider a differential equations of the form

$$ay'' + by' + c' = f(t)$$

with $a, b, c \in \mathbb{R}$ constants and $a \neq 0$. The function f is called the *inhomogeneity*. The general solution is of the form

$$y(t) = y_h(t) + y_p(t),$$

where y_h is the general solution of the homogeneous problem $ay'' + by' + c' = 0$ and y_p an arbitrary solution of the inhomogeneous problem we call a *particular solution*. To find a particular solution we often find a solution that has a similar form to the inhomogeneity f. The idea is to determine the unknown parameters by subsitituion into the differential equations.

Inhomogeneity $f(t)$	Form of particular solution $y_p(t)$	$(C, D, E, \ldots$ to be determined)
$Ae^{\mu t}$	$Ce^{\mu t}$	
$A\cos(\omega t)$ or $B\cos(\omega t)$	$C\cos(\omega t) + D\sin(\omega t)$	(both terms unless there is symmetry)
At	$C + Dt$	
At^2	$Ct^2 + Dt + E$	(all terms unless there is symmetry)
polynomial of degree n	polynomial of degree n	(all terms, unless there is symmetry)
$f(t)$ solves the homogeneous equation	$Ctf(t)$	

Questions to do before the tutorial

1. Find the general solution of each of the following.

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 5y = 0.$ (b) $\dfrac{d^2y}{dt^2} + 9y = 0.$

2. Consider the second-order non-homogeneous differential equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2.$

(a) Find the general solution of the above differential equation.

(b) Find the particular solution of the above differential equation satisfying the initial conditions $y(0) = y'(0) = 4$.

Questions to complete during the tutorial

3. Find the general solution of each of the following differential equations.

(a) $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 9x = 0.$ (b) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0.$

4. Solve the following equations, giving the general solution and then the particular solution $y(x)$ satisfying the given boundary or initial conditions.

(a) $y'' + 4y' + 5y = 0, \quad y(0) = 2,\ y'(0) = 4$ (b) $y'' - 2y' + y = 0, \quad y(2) = 0,\ y'(2) = 1$

5. We considered the case of a second order differential equation where the auxiliary equation has a double root, say λ_0. Here we provide an argument why $te^{\lambda_0 t}$ is expected to be a solution. The differential equation in that case is

$$y'' - 2\lambda_0 y' + \lambda_0^2 y = 0.$$

The idea is to look at a perturbed equation that has two distinct real roots, then obtain the solution $te^{\lambda_0 t}$ as a limit of solutions of the perturbed equation.

(a) Check that $e^{\lambda_0 t}$ and $e^{(\lambda_0+h)t}$ are solutions to $y'' - (2\lambda_0 + h)y' + \lambda_0(\lambda_0 + h)y = 0$. Briefly explain why

$$\frac{e^{(\lambda_0+h)t} - e^{\lambda_0 t}}{h}$$

is a solution of the same perturbed equation.

(b) Let $h \to 0$ in the equation as well as the solution given in part (a) and relate it to the original unperturbed equation. Check that the limit of solutions as $h \to 0$ is a solution to the limit equation.

6. First find the general solution of each of the following non-homogeneous second-order differential equations, and then the particular solution for the given initial conditions.

(a) $y''+3y'+2y = 6e^t, \quad y(0) = 1,\ y'(0) = 0.$ (b) $y''+3y'+2y = 6e^{-t},\ y(0) = 2,\ y'(0) = 1.$

7. (a) For $\omega \neq 5$, find the general solution of the non-homogeneous differential equation,

$$\frac{d^2y}{dt^2} + 25y = 100\sin\omega t,$$

and the particular solution subject to the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$.

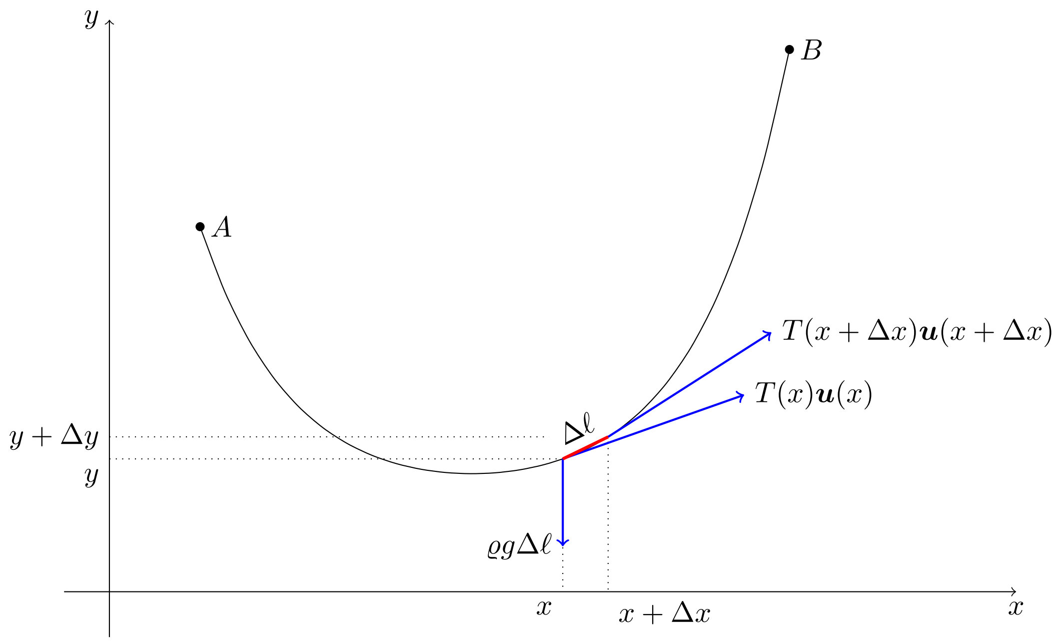
(b) For $\omega = 5$, find a particular solution of the differential equation. Then determine the particular solution with $y(0) = 0$ and $\dot{y}(0) = 0$.

(c) Find the corresponding particular solution of the differential equation for $\omega = 5$ by fixing t in the result of part (a) and taking the limit as ω approaches its special value.

Extra questions for further practice

8. A rope of length L is suspended at two points A and B and hangs freely in between. The rope has constant mass density ϱ per unit length, that is, a section of length ℓ has mass $\varrho\ell$. We assume that the rope is perfectly flexible, that is, there is no bending force.

The only forces acting on the rope are the tension force T tangent to the rope and the gravitational force in the downwards direction. Denote the unit tangent vector along the rope by $\boldsymbol{u}$. The height of the rope above ground is given by a function $y(x)$.



Consider a small section of rope of length $\Delta\ell$ between x and $x+\Delta x$. That section has mass $\varrho\Delta\ell$. We denote the unit vectors in the direction of the x-axis and the y-axis by $\boldsymbol{i}$ and $\boldsymbol{j}$, respectively.

(a) Using the fact that the sum of all forces on $\Delta\ell$ add up to zero, show that

$$\frac{d}{dx}\big(T(x)\boldsymbol{u}(x)\big) = \varrho g\sqrt{1+(y'(x))^2}\,\boldsymbol{j}.$$

(b) Show that the unit tangent vector $\boldsymbol{u}$ is given by

$$\boldsymbol{u}(x) = \frac{1}{\sqrt{1+(y'(x))^2}}\boldsymbol{i} + \frac{y'(x)}{\sqrt{1+(y'(x))^2}}\boldsymbol{j}.$$

(c) By considering the component of the differential equation from (a) in the x-direction, that is, the direction of $\boldsymbol{i}$, show that

$$T(x) = H\sqrt{1+(y'(x))^2}$$

for some constant H. Give a physical interpretation of H.

(d) By considering the component of the differential equation from (a) in the y-direction, that is, the direction of $\boldsymbol{j}$, show that

$$y''(x) = \frac{\varrho g}{H}\sqrt{1+(y'(x))^2}.$$

(e) Find the general solution of the differential equation in (d). Note that the differential equation is a first order differential equation for $z(x) = y'(x)$.

9. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0,$$

expressing your answer in real form. What is the particular solution satisfying $y(0) = 1$ and $y(\pi/4) = 2$?

10. Solve the following equations, giving the general solution and then the particular solution $y(x)$ satisfying the given boundary or initial conditions.

(a) $2y'' - 7y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 1$

(b) $y'' + 4y' + 3y = 0, \quad y(-2) = 1,\ y(2) = 1$

(c) $2y'' - 2y' + 5y = 0, \quad y(0) = 0,\ y(2) = 2$

(d) $y'' - 4y' + 4y = 0, \quad y(0) = -2,\ y(1) = 0$

11. Find the particular solution of the differential equation $y'' - 6y' + 9y = e^{3x}$ which satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.