THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 9

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ L'Hôpital's Rule;
- ☐ Taylor Polynomials;
- ☐ Differentiability.

Outcomes

After completing this tutorial you should

- ☐ use L'Hôpital's Rule to compute limits;
- ☐ construct Taylor polynomials of various functions;
- ☐ understand practical and theoretical properties of derivatives.

Summary of essential material

L'Hôpital's Rule: Suppose that f and g are differentiable in a neighbourhood of a but not necessarily at $x = a$. Further assume that either $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or $f(x) \to \pm\infty$ and $g(x) \to \infty$ as $x \to a$. (We say $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of is of type $0/0$ or $\pm\infty/\infty$.) If $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists (or is $\pm\infty$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

If $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ is still of type $0/0$, then we can apply L'Hôpital's rule again: If $\lim_{x\to a} \dfrac{f''(x)}{g''(x)}$ exists (or is $\pm\infty$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}.$$

More applications are possible if necessary. The given limits are not always in the form of a ratio, but need to be brought into that form. Commonly used methods:

- $fg = \dfrac{f}{1/g}$
- $f(x)^x = e^{x \ln f(x)}$, then compute the limit of the exponent $x \ln f(x) = \frac{\ln f(x)}{1/x}$ and use the continuity of the exponential function. This method can also be used for limits of the form $f(x)^{g(x)}$.

Taylor Polynomials: Let $f(x)\colon (a, b) \to \mathbb{R}$ be a function differentiable at least n times at $x = x_0$. The *n-th order Taylor polynomial* of $f(x)$ centred at $x = x_0$ is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

where, by convention, $f^{(0)}(x) = f(x)$ and $0! = 1$.

Note: The n-th order Taylor polynomial provides the best approximation of the function f near x_0 by a polynomial of order n. In particular, it is uniquely determined by the condition

$$f^{(k)}(x_0) = T_n^{(k)}(x_0) \qquad \text{for } k = 0, 1, \dots, n.$$

(All derivatives up to order n coincide with those of f.)

Questions to complete during the tutorial

1. Find the following limits. Some need L'Hôpital's rule, others can be done without.

(a) $\displaystyle\lim_{x\to -1} \frac{x^6+x^4-2}{x^4-1}$

Solution: The limit is of the type $0/0$. Using L'Hôpital's Rule,

$$\lim_{x\to -1} \frac{x^6+x^4-2}{x^4-1} = \lim_{x\to -1} \frac{6x^5+4x^3}{4x^3} = \lim_{x\to -1} \frac{6x^2+4}{4} = \frac{10}{4} = \frac{5}{2}.$$

(b) $\displaystyle\lim_{x\to \pi} \frac{\tan x}{x-\pi}$

Solution: The limit is of the form $0/0$. By L'Hôpital's Rule,

$$\lim_{x\to \pi} \frac{\tan x}{x-\pi} = \lim_{x\to \pi} \frac{\sec^2 x}{1} = 1.$$

(c) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\ln(\ln x)}$

Solution: The limit is of the form ∞/∞. By L'Hôpital's Rule,

$$\lim_{x\to \infty} \frac{\ln x}{\ln(\ln x)} = \lim_{x\to \infty} \frac{1/x}{1/(x\ln x)} = \lim_{x\to \infty} \ln x = \infty.$$

(d) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$

Solution: Using l'Hôpital's rule,

$$\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to \infty} \frac{(1/x)}{(1/2\sqrt{x})} = \lim_{x\to \infty} \frac{2}{\sqrt{x}} = 0.$$

(e) $\displaystyle\lim_{x\to \infty} \frac{x}{\sqrt{x^2+1}}$

Solution: Using l'Hôpital's rule twice we see that we end up with the same limit, so that the rule cannot be applied successfully. Fortunately, we can evaluate this limit directly, as

$$\lim_{x\to \infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to \infty} \sqrt{\frac{x^2}{x^2+1}} = \lim_{x\to \infty} \sqrt{\frac{1}{1+\frac{1}{x^2}}} = 1.$$

(f) $\displaystyle\lim_{x\to \infty} x^{1/x}$

Solution: Note first that

$$\lim_{x\to \infty} x^{1/x} = \lim_{x\to \infty} e^{\frac{\ln x}{x}}.$$

By L'Hôpital's Rule,

$$\lim_{x\to \infty} \frac{\ln x}{x} = \lim_{x\to \infty} \frac{1}{x} = 0,$$

and so by continuity of the exponential function,

$$\lim_{x\to \infty} x^{1/x} = e^0 = 1.$$

2. Find the Taylor polynomial $T_5(x)$ of order five about $x = 0$ for each of the following functions.

(a) $f(x) = \cosh x$

Solution: Note that $\frac{d}{dx}\cosh x = \sinh x$, and $\frac{d}{dx}\sinh x = \cosh x$. So

$$f^{(n)}(x) = \begin{cases} \sinh x, & \text{if } n \text{ is odd.} \\ \cosh x, & \text{if } n \text{ is even.} \end{cases}$$

Hence $f^{(n)}(0) = \sinh 0 = 0$ for $n = 1, 3, 5$, and $f^{(n)}(0) = \cosh 0 = 1$ for $n = 0, 2, 4$. The Taylor polynomial of order five about $x = 0$ is therefore the quartic polynomial

$$T_5(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}.$$

By spotting the pattern we can write down the general formula for the Taylor polynomial of $\cosh x$ centred at $x = 0$:

$$T_{2n+1}(x) = T_{2n}(x) = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}.$$

The reason why $T_{2n+1}(x) = T_{2n}(x)$ is because the odd terms are all zero.

(b) $f(x) = \ln(1 + x)$

Solution: The derivatives of $f(x)$ are:

$$\begin{aligned} f(x) &= \ln(1+x) \\ f^{(1)}(x) &= (1+x)^{-1} \\ f^{(2)}(x) &= -1!(1+x)^{-2} \\ f^{(3)}(x) &= 2!(1+x)^{-3} \\ f^{(4)}(x) &= -3!(1+x)^{-4} \\ f^{(5)}(x) &= 4!(1+x)^{-5} \end{aligned} \quad \Rightarrow \quad \begin{aligned} f(0) &= 0 \\ f^{(1)}(0) &= 1 \\ f^{(2)}(0) &= -1! \\ f^{(3)}(0) &= 2! \\ f^{(4)}(0) &= -3! \\ f^{(5)}(0) &= 4! \end{aligned}$$

Therefore the 5th order Taylor polynomial is

$$T_5(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.$$

Again, it is easy to spot the pattern and write down the general formula:

$$T_n(x) = \sum_{k=1}^{n} (-1)^k \frac{x^k}{k}.$$

(c) $f(x) = \sqrt{1 + x}$

Solution: Computing the derivatives of $f(x)$ we find that

$$\begin{aligned} f(x) &= \sqrt{1+x} = (1+x)^{1/2}, & f(0) &= 1 \\ f'(x) &= \tfrac{1}{2}(1+x)^{-1/2}, & f'(0) &= \tfrac{1}{2} \\ f''(x) &= -\tfrac{1}{2}\cdot\tfrac{1}{2}(1+x)^{-3/2}, & f''(0) &= -\tfrac{1}{2^2} \\ f^{(3)}(x) &= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}(1+x)^{-5/2}, & f^{(3)}(0) &= \tfrac{3}{2^3} \\ f^{(4)}(x) &= -\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}(1+x)^{-7/2}, & f^{(4)}(0) &= -\tfrac{3\times 5}{2^4} \\ f^{(5)}(x) &= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot\tfrac{7}{2}(1+x)^{-9/2}, & f^{(5)}(0) &= \tfrac{3\cdot 5\cdot 7}{2^5} \end{aligned}$$

So the Taylor polynomial of $f(x)$ of order 5 about $x = 0$ is

$$T_5(x) = 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{3x^3}{2^3 \cdot 3!} - \frac{15x^4}{2^4 \cdot 4!} + \frac{105x^5}{2^5 \cdot 5!}.$$

It is tougher to spot the general pattern this time, and we leave that for later.

3. Let $\alpha > 0$. Show that $\lim_{x \to 0+} x^\alpha \ln x = 0$.

Solution: We write

$$x^\alpha \ln x = \frac{\ln x}{x^{-\alpha}}$$

and apply L'Hôpital's rule. We can do that since $x^{-\alpha} \to \infty$ and $\ln x \to -\infty$ as $x \to 0+$. As $\alpha > 0$ we have

$$\frac{(\ln x)'}{(x^{-\alpha})'} = \frac{x^{-1}}{-\alpha x^{-\alpha-1}} = -\frac{x^\alpha}{\alpha} \to 0$$

as $x \to 0^+$. Hence $\lim_{x \to 0+} x^\alpha \ln x = 0$.

4. Find the n-th order Taylor polynomial of $f(x) = \dfrac{1}{1-x}$ about $x = 0$.

Solution: We compute the derivatives of $f(x)$ at zero:

$$\begin{aligned}
f(x) &= \frac{1}{1-x} & f(0) &= 1 \\
f'(x) &= \frac{1}{(1-x)^2} & f'(0) &= 1 \\
f''(x) &= \frac{2}{(1-x)^3} & f''(0) &= 2 \\
f'''(x) &= \frac{3!}{(1-x)^4} & f'''(0) &= 3! \\
&\vdots & &\vdots \\
f(n)(x) &= \frac{n!}{(1-x)^{n+1}} & f'''(0) &= n!
\end{aligned}$$

Hence the n-th order Taylor polynomial is

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = 1 + x + x^2 + x^3 + \cdots + x^n.$$

5. We know that the n-th order Taylor polynomial of a function f centred at x_0 is the unique polynomial T_n such that $f^{(k)}(x_0) = T_n^{(k)}(x_0)$ for $k = 0, 1, \dots, n$. Use this characterisation to derive the following facts.

(a) Suppose that T_n is the n-th order Taylor polynomial of f centred at x_0. Let $g := f'$. Show that T_n' is the Taylor polynomial g of order $(n-1)$ centred at x_0.

Solution: The derivative of a polynomial of degree n is a polynomial of at most degree $n - 1$. Clearly we have

$$g^{(k)}(x_0) = f^{(k+1)}(x_0) = T_n^{(k+1)}(x_0) = (T_n')^{(k)}(x_0)$$

for $k = 0, 1, 2, \dots, n - 1$. As the k-th derivatives of g and T_n' coincide for $k = 0, 1, 2, \dots, n - 1$ it follows from the characterisation mentioned abouve that T_n' is the Taylor polynomial of g centred at x_0.

(b) How can you find the Taylor polynomial of f if you have the one for $g = f'$?

Solution: We can apply the result from the previous part and take a primitive (anti-derivative) of the Taylor polynomial of g. More precisely, if

$$g(0) + g'(0)(x - x_0) + \frac{g''(0)}{2}(x - x_0)^2 + \frac{g'''(0)}{3!}(x - x_0)^3 + \cdots + \frac{g^{(n)}(0)}{n!}(x - x_0)^n$$

is the n-th Taylor polynomial of g, then the primitive

$$f(0)+g(0)(x-x_0)+\frac{g'(0)}{2}(x-x_0)^2+\frac{g''(0)}{3!}(x-x_0)^3+\frac{g'''(0)}{4!}(x-x_0)^4+\cdots+\frac{g^{(n)}(0)}{n!}(x-x_0)^{n+1}$$

is the $(n + 1)$-th order Taylor polynomial of f.

(c) Suppose that T_n is the n-th order Taylor polynomial of f centred at 0. Let $g(x) := f(ax^2)$ with $a \in \mathbb{R}$. Show that $T_n(ax^2)$ is the $2n$-th order Taylor polynomial of g centred at 0.

Solution: First note that $x \mapsto g(ax^2)$ is an even function. Hence all derivatives of odd order must be zero at $x = 0$. Similarly, $x \mapsto T_n(ax^2)$ is an even polynomial. Hence all coefficients of odd powers of x are zero. This means that all derivatives of odd order must be zero at $x = 0$.
By assumption $g(0) := f(a0^2) = f(0) = T_n(0)$. Looking at the first derivative we have $g'(x) = 2axf'(ax^2)$. The second derivative is

$$2af'(ax^2) + 4a^2x^2f''(ax^2)$$

At $x = 0$ only the term involving f'. The same is the case when differentiating T_n, so the derivatives are the same. The third derivative is

$$2a(2ax)f''(ax^2) + 8a^2xf''(ax^2) + 8a^3x^3f'''(ax^2) = 12a^2xf''(ax^2) + 8a^3x^3f'''(ax^2)$$

The forth derivative is

$$12a^2f''(ax^2) + 48a^3x^2f'''(ax^2) + 16a^4x^4f^{(4)}(ax^2)$$

At $x = 0$ only the term involving f''. The same is the case when differentiating T_n, so the derivative is the same. This pattern continues, with every second derivative involving a term not explicitly multiplied by x.

(d) Use the above facts to find the Taylor polynomials of order n centred at 0 for the following functions. In each case think about why it is easier than a direct computation.

(i) e^{-x^2} using the Taylor polynomial of e^x.

Solution: Since $(e^x)' = e^x$ the Taylor polynomial of e^x is $1+x+x^2/2+x^3/3!+\cdots+x^n/n!$. Substituting $-x^2$ we see that the Taylor polynomial of e^{-x^2} of order $2n$ is

$$1 - x^2 + x^4/2 - x^6/3! + \cdots + (-1)^n x^{2n}/n!.$$

(ii) $\ln(1 - x)$ using the Taylor polynomial of the derivative.

Solution: The derivative of $\ln(1 - x)$ is $-\frac{1}{1-x}$. According to Question 4 its Taylor polynomial is

$$-1 - x - x^2 - \cdots - x^n$$

The primitive of this polynomial is

$$c - x - x^2/2 - x^3/3 - \cdots - x^{n+1}/(n + 1)$$

for some constant c. As $\ln(1+0) = \ln 0 = 0$ we have $c = 0$ and thus the Taylor polynomial of $\ln(1 - x)$ of order $(n + 1)$ is

$$-x - x^2/2 - x^3/3 - \cdots - x^{n+1}/(n + 1).$$

(iii) $\dfrac{1}{1+x^2}$ using the Taylor polynomial of $\dfrac{1}{1-x}$

Solution: According to Question 4 its Taylor polynomial is

$$1 + x + x^2 + \cdots + x^n.$$

Substituting $-x^2$ shows that the Taylor polynomial for $\dfrac{1}{1+x^2}$ is

$$1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n}.$$

(iv) $\tan^{-1}(x)$ using the Taylor polynomial of the derivative.

Solution: We know that $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$. Hence, the Taylor polynomial of $\tan^{-1} x$ is a primitive of the one in part (iii), that is,

$$c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{n+1}$$

as $\tan^{-1} 0 = 0$ we have $c = 0$.

(v) $\cos x$ using the Taylor polynomial of $\sin x$.

Solution: From lectures (or a short calculation) the Taylor polynomial of $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

We know that $(\sin x)' = \cos x$, so the Taylor polynomial of $\cos x$ is the derivative of that of $\sin x$, that is,

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{2n}.$$

6. Define a function f by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable everywhere and that f' is not continuous at 0. Thus we cannot compute $f'(0)$ by using the formula $x^2 \sin \frac{1}{x}$ to calculate $f'(x)$ for $x \neq 0$ and then taking a limit.

Solution: As $f(0) = 0$ we have

$$f(x) = f(0) + \left(x \sin \frac{1}{x}\right)(x - 0),$$

so the derivative at zero is

$$f'(0) = \lim_{x\to 0} x \sin \frac{1}{x} = 0,$$

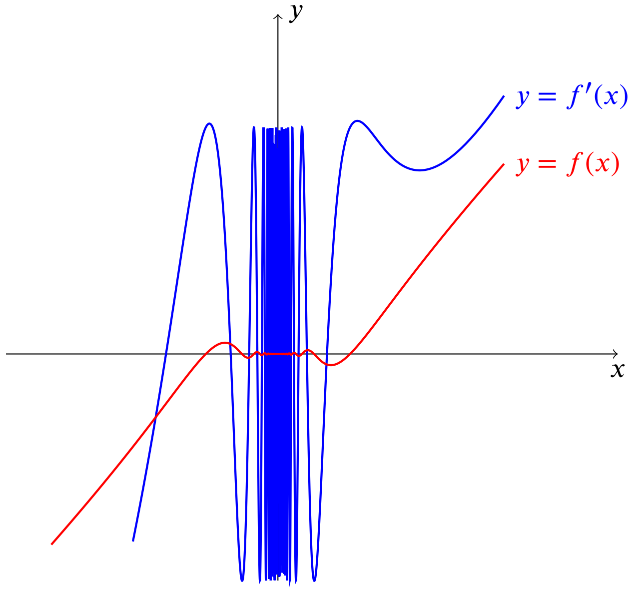
by the squeeze law. At points other than 0, we can simply differentiate f using the product and chain rules:

$$f'(x) = 2x \sin \frac{1}{x} + x^2\left(-\frac{1}{x^2} \cos \frac{1}{x}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \text{ for } x \neq 0.$$

So f is differentiable everywhere. However, f' is not continuous at 0 because $\lim_{x\to 0} f'(x)$ does not exist. To see this, suppose for a contradiction that $\lim_{x\to 0} f'(x) = \ell$. Then

$$\lim_{x\to 0} \cos \frac{1}{x} = \lim_{x\to 0} 2x \sin \frac{1}{x} - f'(x) = 0 - \ell = -\ell,$$

which is impossible for the same reason as in the proof that $\lim_{x\to 0} \sin \dfrac{1}{x}$ does not exist.



7. The derivative of a function does not need to be continuous as the example in Question 6 shows. However, the nature of such a discontinuity must be quite complicated as the following facts show.

(a) Assume that $f : (a, b) \to \mathbb{R}$ is differentiable and that $\lim_{x\to x_0} f'(x) = L$ exists. Use L'Hôpital's rule to prove that f' is continuous at x_0. (Such a statement is certainly not true for arbitrary functions!)

Solution: We assumed that the function is differentiable, hence it is continuous. In particular it is continuous at x_0, so the limit

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

is of type "0/0". Hence we can apply L'Hôpital's rule. By assumption

$$\frac{(f(x) - f(x_0))'}{(x - x_0)'} = \frac{f'(x)}{1} = f'(x) \to L$$

as $x \to x_0$. Hence L'Hôpital's rule implies that

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L$$

as well, so that $f'(x_0) = L$. Therefore, f' is continuous at x_0.

(b) Hence show that the function given by $f(x) := 1$ for $x \neq 0$ and $f(0) := -1$ on $\mathbb{R}$ cannot be the derivative of any function.

Solution: By definition of f we have $f(x) \to 1$ as $x \to 0$. If $f = F'$ for some function, then this implies that $F'(0) = 0 \neq f(0) = -1$. Hence, even though derivatives can be discontinuous, they cannot have "arbitrary" discontinuities.

Extra questions for further practice

8. Compute the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

Solution: Using l'Hôpital's rule twice,

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

(b) $\lim_{x\to\frac{\pi}{2}^-}(\tan x)^{\cos x}$

Solution: We write $(\tan x)^{\cos x}$ as $e^{\cos x\ln(\tan x)}$. Now

$$\lim_{x\to\frac{\pi}{2}^-}\cos x\ln(\tan x)=\lim_{x\to\frac{\pi}{2}^-}\sin x\,\frac{\ln(\tan x)}{\tan x}.$$

But by L'Hôpital's Rule,

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\tan x)}{\tan x}=\lim_{x\to\frac{\pi}{2}^-}\frac{\sec^2 x/\tan x}{\sec^2 x}=\lim_{x\to\frac{\pi}{2}^-}\frac{1}{\tan x}=0.$$

Since $\lim_{x\to\frac{\pi}{2}^-}\sin x=1$, we see, by limit laws, that $\lim_{x\to\frac{\pi}{2}^-}\cos x\ln(\tan x)=0$. Thus

$$\lim_{x\to\frac{\pi}{2}^-}(\tan x)^{\cos x}=e^0=1.$$

(c) $\lim_{x\to 0^+}(\sinh\frac{4}{x})^x$

Solution: Write $(\sinh\frac{4}{x})^x=e^{x\ln\sinh\frac{4}{x}}$. Then

$$\begin{aligned}\lim_{x\to 0^+}x\ln\sinh\frac{4}{x}&=\lim_{x\to 0^+}\frac{\ln(\sinh\frac{4}{x})}{1/x}\\&=\lim_{x\to 0^+}\frac{\frac{-\frac{4}{x^2}\cosh\frac{4}{x}}{\sinh\frac{4}{x}}}{\frac{-1}{x^2}}\\&=4\lim_{x\to 0^+}\frac{\cosh\frac{4}{x}}{\sinh\frac{4}{x}}\\&=4\lim_{x\to 0^+}\left(\frac{e^{4/x}+e^{-4/x}}{e^{4/x}-e^{-4/x}}\right)\\&=4\lim_{x\to 0^+}\left(\frac{1+e^{-8/x}}{1-e^{-8/x}}\right)\\&=4.\end{aligned}$$

Hence $\lim_{x\to 0^+}(\sinh\frac{4}{x})^x=e^4$.

(d) $\lim_{x\to 0}\frac{2^x-1}{x}$

Solution: The limit is of the form $0/0$. Since $\frac{d}{dx}2^x=\frac{d}{dx}e^{2\ln 2}=\ln 2e^{x\ln 2}=(\ln 2)2^x$ we have

$$\lim_{x\to 0}\frac{2^x-1}{x}=\lim_{x\to 0}\frac{(\ln 2)2^x}{1}=\ln 2.$$

(e) $\lim_{x\to\infty}(1+e^{-x})^x$

Solution: We have

$$\lim_{x\to\infty}(1+e^{-x})^x=\lim_{x\to\infty}e^{x\ln(1+e^{-x})}.$$

Now,

$$\begin{aligned}
\lim_{x\to\infty} x\ln(1+e^{-x}) &= \lim_{x\to\infty} \frac{\ln(1+e^{-x})}{1/x} \\
&= \lim_{x\to\infty} \frac{x^2e^{-x}}{1+e^{-x}} \\
&= \lim_{x\to\infty} \frac{x^2}{e^x+1} \\
&= \lim_{x\to\infty} \frac{2x}{e^x} \\
&= \lim_{x\to\infty} \frac{2}{e^x} \\
&= 0,
\end{aligned}$$

where we have used L'Hôpital's Rule three times. Thus

$$\lim_{x\to\infty}(1+e^{-x})^x = e^0 = 1.$$

(f) $\displaystyle\lim_{x\to\infty} \frac{x^{-1/2}+x^{-3/2}}{x^{-1/2}-x^{-3/2}}$

Solution: This limit looks easiest to compute directly by multiplying through by $x^{1/2}$:

$$\lim_{x\to\infty} \frac{x^{-1/2}+x^{-3/2}}{x^{-1/2}-x^{-3/2}} = \lim_{x\to\infty} \frac{1+x^{-1}}{1-x^{-1}} = 1.$$

However since the limit is of type $0/0$ you might also try L'Hôpital's Rule. If you do so you get:

$$\lim_{x\to\infty} \frac{x^{-1/2}+x^{-3/2}}{x^{-1/2}-x^{-3/2}} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-3/2}+\frac{3}{2}x^{-5/2}}{\frac{1}{2}x^{-3/2}-\frac{3}{2}x^{-5/2}} = \lim_{x\to\infty} \frac{\frac{3}{4}x^{-5/2}+\frac{15}{4}x^{-7/2}}{\frac{3}{4}x^{-5/2}-\frac{15}{4}x^{-7/2}} = \cdots$$

and you don't get anywhere!

9. Use induction on n and L'Hôpital's rule to prove that $\lim_{x\to 0^+} x(\ln x)^n = 0$ for $n \in \mathbb{N}$.

Solution: For $n=0$, $\lim_{x\to 0^+} x(\ln x)^n = \lim_{x\to 0^+} x = 0$.

Assume that $\lim_{x\to 0^+} x(\ln x)^n = 0$ (induction hypothesis). Now

$$\begin{aligned}
\lim_{x\to 0^+} x(\ln x)^{n+1} &= \lim_{x\to 0^+} \frac{(\ln x)^{n+1}}{1/x} \\
&= \lim_{x\to 0^+} \frac{(n+1)(\ln x)^n(1/x)}{-1/x^2} && \text{by L'Hôpital's Rule} \\
&= -(n+1)\lim_{x\to 0^+} x(\ln x)^n,
\end{aligned}$$

and this is equal to 0 by the induction hypothesis. So the result is true by induction.

10. Using the 5th order Taylor polynomial of $f(x)=\ln(1+x)$ (see Question 2) to approximate $\ln 2$ we get

$$\ln 2 \approx 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5} = 0.78\dot{3}.$$

This is not so impressive, because $\ln 2 = 0.693147\ldots$ In fact it turns out that you need to use the Taylor polynomial of order 1565237 to get $\ln 2$ correct to only 6 decimal places! We can do much better using the function

$$f(x)=\ln\left(\frac{1+x}{1-x}\right)$$

and noticing that $f(1/3)=\ln 2$.

(a) Find the general formula of the Taylor polynomial of $f(x)$ about $x = 0$.
Hint: $f(x) = \ln(1 + x) - \ln(1 - x)$.

Solution: Writing $f(x) = \ln(1 + x) - \ln(1 - x)$ makes it clear that

$$f^{(n)}(x) = \frac{d^n}{dx^n}\ln(1+x) - \frac{d^n}{dx^n}\ln(1-x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} + \frac{(n-1)!}{(1-x)^n}.$$

Evaluating at $x = 0$, we get

$$f^{(n)}(0) = (n-1)!\big((-1)^{n-1} + 1\big),$$

and so

$$\frac{f^{(n)}(0)}{n!} = \frac{1 + (-1)^{n-1}}{n} = \begin{cases} 2/n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Hence the Taylor polynomial of order $2n + 1$ is

$$T_{2n+1}(x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1}\right).$$

(b) Use the Taylor polynomial $T_5(1/3)$ to approximate $\ln 2$.

Solution: The Taylor polynomial $T_5(x)$ is

$$T_5(x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right).$$

Therefore $T_5(1/3) = 0.693004115\ldots$ This is much better!

11. Let f and g be differentiable at $x = a$, with $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. A proposed "converse" to L'Hôpital's Rule reads as follows:

$$\text{“If } \lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ does not exist, then } \lim_{x\to a}\frac{f(x)}{g(x)} \text{ does not exist.”}$$

By considering $f(x) = x^2\sin(1/x)$ and $g(x) = x$, sh that the above statement is false.

Solution: The limit $\lim_{x\to 0}\frac{f(x)}{g(x)}$ is of the form $0/0$. For $x \neq 0$ we have

$$\begin{aligned} f'(x) &= 2x\sin(1/x) - \cos(1/x) \\ g'(x) &= 1, \end{aligned}$$

and therefore

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\left(2x\sin(1/x) - \cos(1/x)\right).$$

We have $\lim_{x\to 0} 2x\sin(1/x) = 0$ by the squeeze law, however $\lim_{x\to 0}\cos(1/x)$ does not exist (it oscillates like mad between -1 and 1 as $x \to 0$, with the frequency becoming larger and larger). Therefore

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} \quad \text{does not exist.}$$

However

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0} x\sin(1/x) = 0,$$

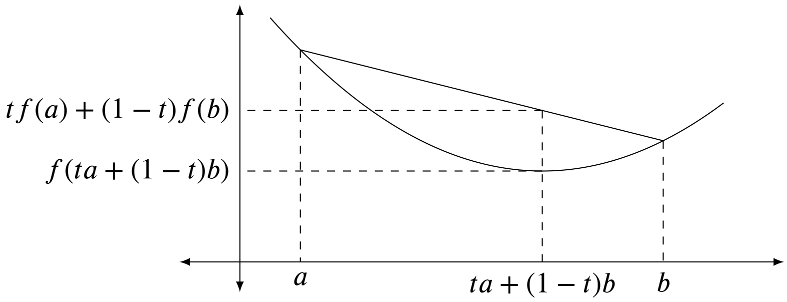
by the Squeeze law. So the proposed converse to L'Hôpital's rule is false.

Challenge questions (optional)

12. (Very challenging!) Use the Mean Value Theorem to show that if $f''(x) \geq 0$ for all $x \in [a, b]$ then

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \qquad \text{for all } t \in [0,1].$$

Geometrically this says that f is concave up on $[a, b]$:



Hint: Let $p_t = ta + (1-t)b$. Apply MVT twice – once on $[a, p_t]$, and also on $[p_t, b]$.

Solution: If $t = 0$ or $t = 1$ then the proposed inequality is an equality. So assume that $t \in (0, 1)$. Let $p_t = ta + (1-t)b$. Note that $p_t \in (a, b)$, because $p_t > ta + (1-t)a = a$ and $p_t < tb + (1-t)b = b$.

Applying the Mean Value Theorem on $[a, p_t]$ and $[p_t, b]$ gives

$$f'(c_1) = \frac{1}{1-t}\frac{f(p_t) - f(a)}{b-a} \qquad \text{and} \qquad f'(c_2) = \frac{1}{t}\frac{f(b) - f(p_t)}{b-a} \tag{1}$$

for some $c_1 \in (a, p_t)$ and $c_2 \in (p_t, b)$. Now apply the Mean Value Theorem on the interval $[c_1, c_2]$ to the function $f'(x)$. Therefore there is $c \in (c_1, c_2)$ such that

$$f''(c) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1}.$$

Then using (1) we have

$$f''(c) = \frac{(1-t)(f(b) - f(p_t)) - t(f(p_t) - f(a))}{t(1-t)(b-a)(c_2 - c_1)} = \frac{tf(a) + (1-t)f(b) - f(p_t)}{t(1-t)(b-a)(c_2 - c_1)}.$$

Since $f''(c) \geq 0$ this implies that $f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$.