A

MATH1903 Lecture 3 Thurs 10/8/2017

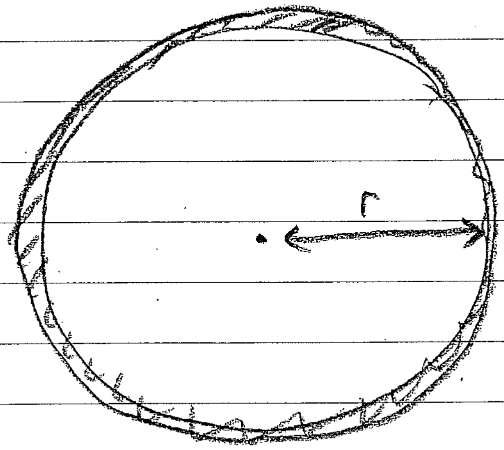
The Fundamental Theorem of Calculus

- connection between areas & derivatives
- recall area of a circle $A = \pi r^2$

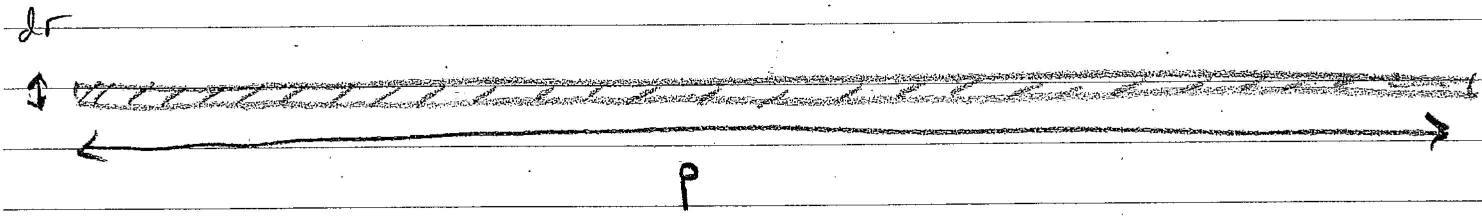
and $P = 2\pi r = \frac{dA}{dr}$, so

$$\boxed{dA = P\,dr}$$

nice heuristics



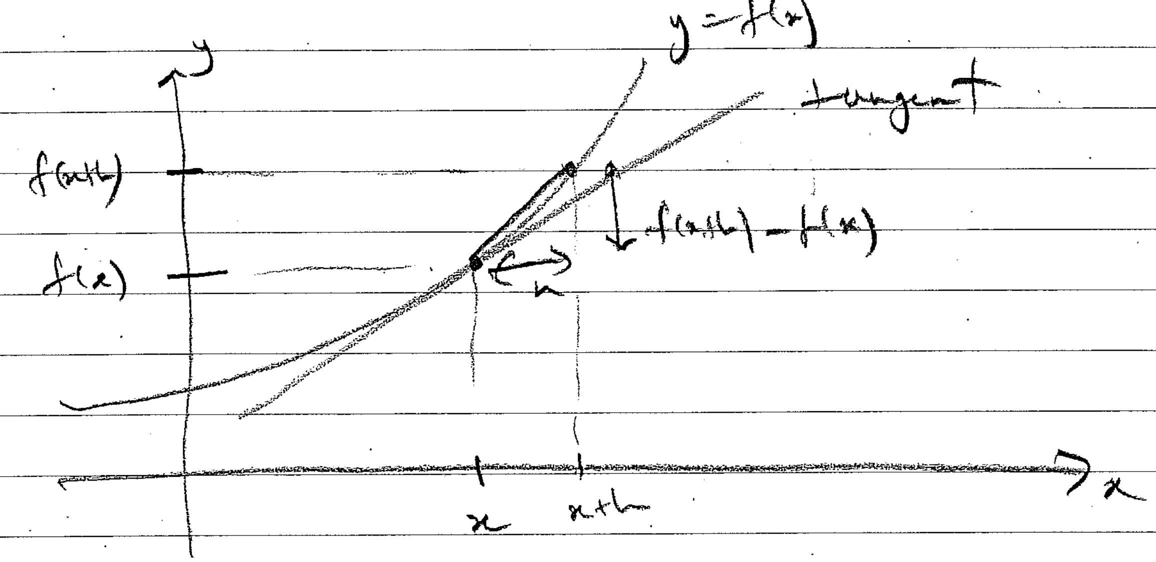
"infinitely" thin change in area produced by "infinitely" thin change in radius



pp 38-42 of Notes

- Fundamental Theorem of Calculus Part 1.

Recall definition of derivative



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- We interpret this definition for the area function.

Note: the drawings on pp 38, 39 assume the function is increasing, but the argument can be adapted to the general case

Practice: Find $g'(x)$ and $h'(x)$ where

$$g(x) = \int_0^x \cos(t^2)\,dt$$

$$h(x) = \int_{\sin x}^{3x^2} \cos(t^4)\,dt.$$

(c)

<u>Solution</u> : $g'(x) = \frac{d}{dx}\left(\int_0^x \cos(t^2)\,dt\right) = \cos(x^2)$.

$$h(x) = \int_{\sin x}^{0} \cos(t^2)\,dt + \int_0^{3x^2} \cos(t^2)\,dt$$

$$= -\int_0^{\sin x} \cos(t^2)\,dt + \int_0^{3x^2} \cos(t^2)\,dt$$

$$= -\int_0^{u} \cos(t^2)\,dt + \int_0^{v} \cos(t^2)\,dt$$

where $u = \sin x$, so $\frac{du}{dx} = \cos x$,

and $v = 3x^2$, so $\frac{dv}{dx} = 6x$.

Hence

$$h'(x) = -\frac{d}{dx}\left(\int_0^{\sin x} \cos(t^2)\,dt\right) + \frac{d}{dx}\left(\int_0^{3x^2} \cos(t^2)\,dt\right)$$

$$= -\frac{d}{du}\left(\int_0^{u} \cos(t^2)\,dt\right)\frac{du}{dx} + \frac{d}{dv}\left(\int_0^{v} \cos(t^2)\,dt\right)\frac{dv}{dx}$$

$$= -\cos(u^2)(\cos x) + \cos(v^2)(6x)$$

$$= 6x\cos(9x^4) - (\cos x)\cos(\sin^2 x).$$

D

pp 43 - 47 of Notes

- Fundamental Theorem of Calculus Part 2.

<u>Exercise (using the Mean Value Theorem)</u>: If f is differentiable on the interval (a,b) and $f'(x) = 0$ for all $x \in (a,b)$, then there exists a constant C such that

$$f(x) = C \quad \text{for all } x \in (a,b).$$

<u>Corollary</u>: If f and g are differentiable on the interval (a,b) and $f'(x) = g'(x)$ for all $x \in (a,b)$, then there exists a constant C such that

$$f(x) = g(x) + C \quad \text{for all } x \in (a,b).$$

<u>Proof</u>: Define a function h by the rule

$$h(x) \equiv f(x) - g(x) \quad \text{for } x \in (a,b).$$

Then $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a,b)$,

so there exists a constant k such that

$$f(x) - g(x) = h(x) = C,$$

so that $f(x) = g(x) + C$ for all $x \in (a,b)$. $\square$

E

p48 – indefinite integrals

e.g. $\frac{d}{dx}(\sin x) = \cos x$,

$\frac{d}{dx}(\cos x) = -\sin x$

carry the same information as

$$\int \cos x \, dx = \sin x + C,$$

$$\int \sin x \, dx = -\cos x + C$$

respectively.

pp 49-50 – integration by substitution.

$\boxed{\text{Example: Find } \int \frac{dx}{1+e^{-x}}}$

Solution ①: $\int \frac{dx}{1+e^{-x}} = \int \frac{1}{1+e^{-x}} dx$

$= \int \frac{e^x}{e^x+1} dx = \int \frac{du}{u}$

$= \ln|u| + C$

$= \ln(e^x+1) + C$

Put $u = e^x+1$,

so $\frac{du}{dx} = e^x$

so $du = e^x du$

F

Solution (2) : Put $u = 1 + e^{-x}$, so

$$\frac{du}{dx} = -e^{-x} = -(1+e^{-x}) + 1 = 1 - u,$$

so $\boxed{\dfrac{du}{1-u} = dx}$.

Hence
$$\int \frac{dx}{1+e^{-x}} = \int \frac{1}{1+e^{-x}}\,dx$$
$$= \int \left(\frac{1}{u}\right)\left(\frac{1}{1-u}\right)\,du$$
$$= \int \frac{du}{u(1-u)}.$$

But
$$\frac{1}{u(1-u)} = \frac{1}{u} + \frac{1}{1-u}$$

(see method of partial fractions, later)

so
$$\int \frac{du}{u(1-u)} = \int \frac{du}{u} + \int \frac{du}{1-u}$$
$$= \ln|u| - \ln|1-u| + C$$
$$= \ln\left|\frac{u}{1-u}\right| + C$$
$$= \ln\left(\frac{1+e^{-x}}{e^{-x}}\right) + C$$
$$= \ln(e^{x}+1) + C$$

, as before