THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

MATH1902
LINEAR ALGEBRA (ADVANCED)

June 2011 LECTURER: D. Easdown

TIME ALLOWED: **One and a half hours**

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Approved non-programmable calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

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Extended Answer Section

There are **four** *questions in this section, each with a number of parts. Write your answers in the answer book(s) provided. Ask for extra books if you need them.*

1. (a) Find A^{-1} where $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 4 & 1 & -1 \end{bmatrix}$.

(b) Use the result of part (a) to solve the following system of equations:

$$\begin{array}{rcrcrcl} r & & & + & 4t & = & 2 \\ & & s & + & t & = & 1 \\ 4r & + & s & - & t & = & 1 \end{array}$$

(c) Consider points $P_0(1,1,2)$ and $Q_0(3,0,1)$ and vectors $\mathbf{v} = \mathbf{i} - 4\mathbf{k}$ and $\mathbf{w} = \mathbf{j} + \mathbf{k}$.

(*i*) Let $r, s, t \in \mathbb{R}$ be parameters such that

$$\overrightarrow{PQ} = t(\mathbf{v} \times \mathbf{w})$$

where $P = (1+r, 1, 2-4r)$ and $Q = (3, s, 1+s)$. Verify that this constraint on r, s and t is equivalent to the system in part (b).

(*ii*) Find the closest points on the skew lines through P_0 and Q_0 parallel to $\mathbf{v}$ and $\mathbf{w}$ respectively, and find the shortest distance between these skew lines.

[5+3+4+3=15 marks]

2. (a) Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are three nonzero geometric vectors.

(*i*) Give a simple counterexample to demonstrate that the equation $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ does not imply $\mathbf{v} = \mathbf{w}$.

(*ii*) Explain briefly why the equation $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ implies that $\mathbf{v} - \mathbf{w}$ is parallel to $\mathbf{u}$.

(*iii*) Prove that the equations $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ together imply that $\mathbf{v} = \mathbf{w}$.

(b) Diagonalise $M = \begin{bmatrix} \frac{7}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$ and show that

$$\lim_{n \to \infty} M^n = \begin{bmatrix} \frac{4}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix}.$$

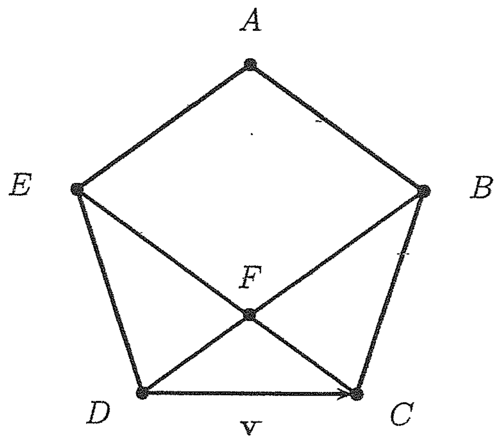
[1+2+3+9=15 marks]

3. (a) Let P, Q, R and S be any points in space.

(i) Use vector arithmetic to verify that
$$\overrightarrow{PQ} + \overrightarrow{RS} = \overrightarrow{RQ} + \overrightarrow{PS}\,.$$

(ii) Suppose now that P, Q and R are not collinear, that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{SR}$ and that $\overrightarrow{QR}$ is parallel to $\overrightarrow{PS}$. Prove that $\overrightarrow{PQ} = \overrightarrow{SR}$. (Thus $PQRS$ is a nondegenerate parallelogram.)

(b) Let the figure $ABCDE$ be the regular pentagon depicted below (so that all the sides lengths are equal and all of the internal angles between adjacent sides are equal). Let F be the intersection point of the line segments BD and EC, and put $\mathbf{v} = \overrightarrow{DC}$.



We have $\mathbf{v} = \lambda\overrightarrow{EB}$ for some positive scalar λ (since DC is parallel to EB by reflectional symmetry of the regular pentagon, and you may take this as granted).

(i) Explain briefly why $ABFE$ is a rhombus. [Hint: use (a)(ii).]

(ii) Prove that $\lambda\mathbf{v} = \overrightarrow{CB} + \overrightarrow{ED}$.

(iii) Deduce that $\lambda\mathbf{v} = \overrightarrow{EB} + \overrightarrow{CD}$ and that $\lambda^2 = 1 - \lambda$.

(iv) Deduce that F divides the line segment BD in the ratio $(\sqrt{5} - 1) : (3 - \sqrt{5})$.

[2+4+2+2+2+3=15 marks]

4. (a) Suppose that M is a 2×2 matrix such that $10M^2 - 11M + I = 0$.

   (i) Verify that M is invertible.

   (ii) Prove that if M is not a scalar multiple of the identity matrix then the eigenvalues of M are 1 and $\frac{1}{10}$. [Hint: apply the Cayley-Hamilton Theorem.]

   (b) Let $a_0, a_1, \ldots, a_{n-1}$ be (complex) numbers and consider the $n \times n$ matrix

$$M_n = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}$$

consisting of zeros in all positions except for the last row as indicated and 1's just above the diagonal. We interpret M_1 as the 1×1 matrix $[-a_0]$ and M_2 as the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$.

   (i) Find $\det(\lambda I - M_1)$ and $\det(\lambda I - M_2)$ as polynomials in λ.

   (ii) Conjecture and prove a formula for the polynomial $\det(\lambda I - M_n)$ for any positive integer n.

   (iii) It is a theorem that all square matrices with complex entries have at least one eigenvalue. Use this theorem to deduce the Fundamental Theorem of Algebra. (The converse implication of course holds, so the result of this exercise is that these two theorems are logically equivalent.)

[2+4+2+5+2=15 marks]

End of Extended Answer Section

THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

MATH1015
BIOSTATISTICS

June 2011

LECTURERS: Dr J. Chan, Dr D. Marchev

TIME ALLOWED: **90 Minutes**

Family Name: ....

Other Names: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 50% of the total examination;
there are 25 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 50% of the total examination;
there are 3 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Use your own (NSW Board of Studies approved) Calculator. A formula sheet and Statistical tables for use in this examination are printed after the last question in this booklet.

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