CHAPTER **TWO**

Integration

Chapter Overview: The art of integration is a skill that all mathematicians must possess, as integrals arise in all areas of mathematics. For example, in the seemingly unrelated topic of prime numbers the integral $\displaystyle\int \frac{dx}{\log x}$ appears.

As integration is an art form, it requires plenty of practice to become proficient. Thus students are encouraged to attempt as many of the exercise questions as possible in the time they have available.

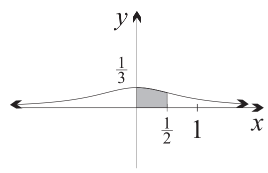
The work in this chapter builds on the content of the Mathematics Extension 1 course. A methodical approach is needed to study the material. In particular, it is important to be able to recognise the different forms of integrals, and to quickly determine which method is appropriate to apply.

The first four sections are relatively straightforward, being based on algebraic manipulation. In Section 2E the new method of integration by parts is introduced, which is based on the product rule for differentiation. Section 2F covers various types of harder Trignometric integrals. Section 2G introduces the concept of integrals that can be referenced by an index, and the corresponding reduction formulae. The chapter concludes with Section 2I which deals with theorems about integrals that can be used to simplify certain problems.

2A Algebraic Manipulation

Standard integrals: Students will know that each examination is accompanied by a table of Standard Integrals, located at the end of each paper. A copy of a table is included in the appendix to this chapter. Most of the results in that table will have already been encountered in the Mathematics Extension 1 course. The ability to make simple manipulations to the integrals in this table is expected.

Worked Exercise: Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{3+4x^2}$.



Solution: Take out a factor of $\frac{1}{4}$ to get:

$$\int_0^{\frac{1}{2}} \frac{dx}{3+4x^2} = \frac{1}{4}\int_0^{\frac{1}{2}} \frac{dx}{(\frac{\sqrt{3}}{2})^2 + x^2}$$

$$= \frac{1}{4} \times \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right]_0^{\frac{1}{2}} \qquad \text{(table of Standard Integrals)}$$

$$\begin{aligned}&= \frac{1}{2\sqrt{3}}\tan^{-1}\tfrac{1}{\sqrt{3}}\\ &= \frac{\pi}{12\sqrt{3}}\,.\end{aligned}$$

Algebraic Manipulation: Many of the integrals encountered contain fractions which require some sort of rearrangement before proceeding. In the first worked exercise the numerator is almost identical to the denominator.

WORKED EXERCISE: Determine $\displaystyle\int \frac{x^2-1}{x^2+1}\,dx$.

SOLUTION: Noting that $x^2 - 1 = (x^2+1) - 2$ we write:

$$\begin{aligned}\int \frac{x^2-1}{x^2+1}\,dx &= \int \frac{x^2+1}{x^2+1} - \frac{2}{x^2+1}\,dx\\ &= \int 1 - \frac{2}{x^2+1}\,dx\\ &= x - 2\tan^{-1}x + C\,.\end{aligned}$$

In harder problems long division is required, though in some cases the numerator is almost a multiple of the denominator, as in the next worked exercise.

WORKED EXERCISE: Find $\displaystyle\int \frac{4x^3-2x^2+1}{2x-1}\,dx$.

SOLUTION:

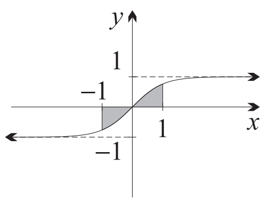
$$\begin{aligned}\int \frac{4x^3-2x^2+1}{2x-1}\,dx &= \int \frac{2x^2(2x-1)+1}{2x-1}\,dx\\ &= \int 2x^2 + \frac{1}{2x-1}\,dx\\ &= \tfrac{2}{3}x^3 + \tfrac{1}{2}\log(2x-1) + C\,.\end{aligned}$$

A Hard Example: The final worked exercise demonstrates a fraction which first requires multiplication or division by a common factor. The result is a numerator which is the dervative of the denominator.

WORKED EXERCISE: Evaluate $\displaystyle\int_{-1}^{1} \frac{e^{2x}-1}{e^{2x}+1}\,dx$.

SOLUTION: Divide numerator and denominator by e^x to get:

$$\begin{aligned}\int_{-1}^{1} \frac{e^{2x}-1}{e^{2x}+1}\,dx &= \int_{-1}^{1} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx\\ &= \big[\log(e^x+e^{-x})\big]_{-1}^{1}\\ &= \log(e+e^{-1}) - \log(e^{-1}+e)\\ &= 0\,.\end{aligned}$$



Two New Integrals: The final two integrals in the standard table will be new to most readers. Here, the result for the last integral is proven using a similar approach to the previous worked exercise.

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \int \frac{(x+\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}\,(x+\sqrt{x^2+a^2})}\,dx$$

$$= \int \frac{(\frac{x}{\sqrt{x^2+a^2}} + 1)}{(x + \sqrt{x^2 + a^2})}\, dx$$
$$= \int \frac{(1 + \frac{x}{\sqrt{x^2+a^2}})}{(x + \sqrt{x^2 + a^2})}\, dx\,.$$

Looking carfeully at the last line, notice that the numerator is the derivative of the denominator and hence we have the desired result:

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log(x + \sqrt{x^2 + a^2}) + C\,.$$

The other new integral in the list may be done in a similar way and is one of the questions in the Exercise.

Exercise **2A**

1. Use a table of Standard Integrals to determine the following. A copy of a table of Standard Integrals may be found in the appendix to this chapter.

(a) $\displaystyle\int \cos 2x\, dx$ (c) $\displaystyle\int \frac{1}{25 + x^2}\, dx$ (e) $\displaystyle\int \frac{1}{\sqrt{x^2 + 3}}\, dx$

(b) $\displaystyle\int \sec^2 \tfrac{x}{3}\, dx$ (d) $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx$ (f) $\displaystyle\int \frac{1}{\sqrt{x^2 - 5}}\, dx$

2. Evaluate the following with the aid of a table of Standard Integrals. A copy of a table of Standard Integrals may be found in the appendix to this chapter.

(a) $\displaystyle\int_0^4 e^{\frac{x}{2}}\, dx$ (c) $\displaystyle\int_{-4}^4 \frac{1}{16 + x^2}\, dx$ (e) $\displaystyle\int_{\sqrt{5}}^3 \frac{1}{\sqrt{x^2 - 4}}\, dx$

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x\, dx$ (d) $\displaystyle\int_0^1 \frac{1}{\sqrt{2 - x^2}}\, dx$ (f) $\displaystyle\int_{-4}^4 \frac{1}{\sqrt{x^2 + 9}}\, dx$

3. Determine these logarithmic integrals.

(a) $\displaystyle\int \frac{x}{1 - x^2}\, dx$ (b) $\displaystyle\int \frac{1 + \sec^2 x}{x + \tan x}\, dx$ (c) $\displaystyle\int \frac{\cos 3x}{1 + \sin 3x}\, dx$

4. Evaluate:

(a) $\displaystyle\int_0^1 \frac{x^2}{1 + x^3}\, dx$ (b) $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x} + 1}\, dx$ (c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x}\, dx$

DEVELOPMENT

5. Evaluate the following with the aid of a table of Standard Integrals.

(a) $\displaystyle\int_0^1 \frac{dx}{1 + 3x^2}$ (b) $\displaystyle\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{4 - 9x^2}}$ (c) $\displaystyle\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{4x^2 + 9}}$ (d) $\displaystyle\int_1^{\frac{7}{\sqrt{5}}} \frac{dx}{\sqrt{5x^2 - 4}}$

6. Determine the following by rewriting the numerator in terms of the denominator.

(a) $\displaystyle\int \frac{x}{x - 1}\, dx$ (b) $\displaystyle\int \frac{x - 1}{x + 1}\, dx$ (c) $\displaystyle\int \frac{x + 1}{x - 1}\, dx$

7. Evaluate the following.

(a) $\displaystyle\int_0^1 \frac{x - 1}{x + 1}\, dx$ (b) $\displaystyle\int_0^2 \frac{x}{2x + 1}\, dx$ (c) $\displaystyle\int_0^1 \frac{3 - x^2}{1 + x^2}\, dx$

8. Evaluate the following. In each case, begin by rewriting the given fraction as two fractions by separating the terms in the numerator.

(a) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{1 - x}{\sqrt{1 - x^2}}\, dx$ (b) $\displaystyle\int_0^1 \frac{2x + 1}{1 + x^2}\, dx$ (c) $\displaystyle\int_0^1 \frac{1 - x}{1 + x^2}\, dx$ (d) $\displaystyle\int_0^2 \frac{1 + x}{4 + x^2}\, dx$

9. Use a similar approach to that shown in the text to prove that
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left(x+\sqrt{x^2-a^2}\right)+C\,.$$

10. (a) Given that $x^3 = (x^3+1)-1$, determine $\displaystyle\int \frac{x^3}{x+1}\,dx$.

(b) Given that $x^3 = (x^3+x)-x$, determine $\displaystyle\int \frac{x^3}{x^2+1}\,dx$.

(c) Use similar approaches to those shown in parts (a) and (b) to determine the following.

(i) $\displaystyle\int \frac{x^3}{x-1}\,dx$ (iii) $\displaystyle\int \frac{1}{1+e^x}\,dx$ (v) $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$

(ii) $\displaystyle\int \frac{x^4}{x^2+1}\,dx$ (iv) $\displaystyle\int \frac{x}{\sqrt{2+x}}\,dx$ (vi) $\displaystyle\int \frac{x^3}{x^2+4}\,dx$

11. Evaluate these by first muliplying or dividing by an appropriate factor.

(a) $\displaystyle\int_1^2 \frac{e^{2x}+1}{e^{2x}-1}\,dx$ (b) $\displaystyle\int_0^1 \frac{e^x}{e^x+e^{-x}}\,dx$ (c) $\displaystyle\int_1^{\sqrt{3}} \frac{2+\frac{1}{x}}{x+\frac{1}{x}}\,dx$

12. By using long division or otherwise, determine:

(a) $\displaystyle\int \frac{x^2+x+1}{x+1}\,dx$ (b) $\displaystyle\int \frac{x^3-2x^2+3}{x-2}\,dx$ (c) $\displaystyle\int \frac{(x+1)^2}{1+x^2}\,dx$

EXTENSION

13. Divide numerator and denominator by an appropriate factor to help determine
$$\int \frac{1}{x+\sqrt{x}}\,dx\,.$$

2B Substitution

Many of the techniques used in integration are derived from differentiation. This is not so surprising since the two processes are essentially mutually inverse. One particularly useful technique is substitution which is the integration equivalent of the chain rule for differentiation, and is sometimes called the reverse chain rule.

The Chain Rule: Suppose that F is a function of u, which is in turn a function of x. Further suppose that $F(u)$ is a primitive of $f(u)$. Differentiating F with respect to x and treating the derivative like a fraction we get:
$$\frac{d}{dx}\,F(u) = \frac{dF}{du}\times\frac{du}{dx}$$
so
$$\frac{d}{dx}\,F(u) = f(u)\times u'\,.$$

Integrating both sides of this result
$$\int\left(\frac{d}{dx}\,F(u)\right)\,dx = \int f(u)\times u'\,dx$$
or
$$F(u)+C = \int f(u)\times u'\,dx\,.$$
It is this last result which proves most useful for integration. Thus if an integrand can be expressed as a product, where one factor is a chain of functions $f(u)$ and the other factor is u' then we can immediately write down the primitive.

Substitution: In the simplest examples, the primitive can be determined mentally. For example, a standard integral in the exponential function topic is

$$\int 2x\, e^{x^2}\, dx = e^{x^2} + C\,.$$

In harder examples a formal procedure should be followed.

WORKED EXERCISE: Determine $\displaystyle\int \frac{x^2}{\sqrt{x^3+1}}\, dx$ by using a suitable substitution.

SOLUTION: Let $I = \displaystyle\int \frac{x^2}{\sqrt{x^3+1}}\, dx$ and put $u = x^3 + 1$, then

$$\frac{du}{dx} = 3x^2$$

or $\quad \tfrac{1}{3}du = x^2\, dx \qquad$ (treating the derivative like a fraction.)

Thus $\quad I = \displaystyle\int \frac{1}{3\sqrt{u}}\, du$

$\quad = \tfrac{2}{3}\sqrt{u} + C\,.$

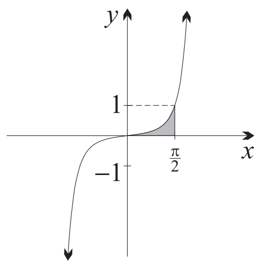
Hence $\quad I = \tfrac{2}{3}\sqrt{x^3+1} + C\,.$

Notice that the final step of the solution is a back substitution to get the integral I in terms of x. It is important to remember to do this.

It is equally important to follow this formal procedure when definite integrals are involved, paying particular attention to the limits of integration.

WORKED EXERCISE: Use a suitable substitution to find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)^3}\, dx\,.$

SOLUTION: Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)^3}\, dx$ and put $u = 1 + \cos x$ to get



$$\frac{du}{dx} = -\sin x$$

so $\quad -du = \sin x\, dx\,.$

When $\quad x = 0\,, \qquad u = 2\,,$

and when $\quad x = \tfrac{\pi}{2}\,, \qquad u = 1\,,$

thus
$$\begin{aligned} I &= \int_2^1 \frac{-1}{u^3}\, du \\ &= \left[\frac{1}{2u^2}\right]_2^1 \\ &= \tfrac{1}{2} - \tfrac{1}{8} \\ &= \tfrac{3}{8}\,. \end{aligned}$$

The step where the limits are expressed in terms of the substitute variable is important. Had this step not been done then the wrong answer is obtained since

$$\int_0^{\frac{\pi}{2}} \frac{-1}{u^3}\, du = \left[\frac{1}{2u^2}\right]_0^{\frac{\pi}{2}}$$

which is undefined at the lower limit. Again notice that the derivative is treated like a fraction in the third line of the solution.

In simple examples like those above, candidates are expected to determine the appropriate substitution for themselves. In harder problems the substitution will be given in the question. Implicit differentiation may also be convenient.

WORKED EXERCISE: Use the substitution $u = \sqrt{x}$ to determine $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$.

SOLUTION: Let $I = \displaystyle\int \frac{dx}{x+\sqrt{x}}$ and note that $u^2 = x$, so:

$$2u\,\frac{du}{dx} = 1$$

or $\quad 2u\;du = dx\,.$

Hence
$$\begin{aligned} I &= \int \frac{2u\;du}{u^2+u} \\ &= \int \frac{2\,du}{u+1} \\ &= 2\log(u+1) + C \\ &= 2\log(\sqrt{x}+1) + C\,. \end{aligned}$$

Take Care with Some Substitutions: There are many integrals which require a careful choice of substitution so as to avoid subsequent difficulties. For example, the correct choice of substitution in the previous worked exercise is $u = \sqrt{x}$.

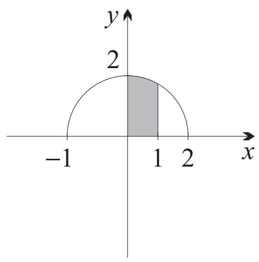
On first inspection, it would seem to make no difference to make the alternate substitution $u^2 = x$, however observe what happens in the denominator.

$$x + \sqrt{x} = u^2 + \sqrt{u^2} = u^2 + |u|\,.$$

Thus in this case we have accidentally introduced a new complication, namely the absolute value function. Here is another example where a careful choice of substitution must be made.

WORKED EXERCISE: Evaluate $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$ by applying a suitable substitution.

SOLUTION: Let $I = \displaystyle\int_0^1 \sqrt{4-x^2}\;dx$ and put $\theta = \sin^{-1}(\frac{x}{2})$ so that $\cos\theta \ge 0$.



Rearranging $\quad x = 2\sin\theta$

so $\quad dx = 2\cos\theta\;d\theta\,.$

When $\quad x = 0\,,\quad \theta = 0\,,$

and when $\quad x = 1\,,\quad \theta = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}\,.$

Thus
$$\begin{aligned} I &= \int_0^{\frac{\pi}{6}} 2\cos\theta\,\sqrt{4-4\sin^2\theta}\;d\theta \\ &= \int_0^{\frac{\pi}{6}} 4\cos\theta\,\sqrt{\cos^2\theta}\;d\theta \qquad \text{(by the Pythagorean identity)} \\ &= \int_0^{\frac{\pi}{6}} 4\cos^2\theta\;d\theta \qquad \text{(since } \cos\theta \ge 0\text{)} \\ &= \int_0^{\frac{\pi}{6}} 2(1+\cos 2\theta)\,d\theta \qquad \text{(by the double-angle formula)} \\ &= \Big[2\theta + \sin 2\theta\Big]_0^{\frac{\pi}{6}} \\ &= \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}\,. \end{aligned}$$

On first inspection, the alternate substitution $x = 2\sin\theta$ would seem to make no difference. However in this case the limits of integration are indeterminate. For example when $x = 1$, there are multiple solutions, namely $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$, and the problem then is to find the correct choice of limits.

Two Guidelines for Substitutions: The infinite variety of integrals that may be encountered make it impractical to give a specific recipe for making the correct substitution. However the following two guidelines may help, and can be observed in practice in the previous worked exercises.

- Try to replace the part of the integral which causes difficulty, such as the innermost function in a chain of functions. In particular, if the integral involves square-roots of sums or differences of squares then a trigonometric substitution is likely to work.
- It is better to use a substitution which is a function $u = f(x)$ rather than a relation $x = g(u)$. Substituting a relation such as $x = u^2$ can lead to problems later in the calculations, as demonstrated above.

Exercise **2B**

1. (a) Use the result $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log\big(f(x)\big) + C$ to help determine these indefinite integrals.

(i) $\displaystyle\int \frac{x}{1-x^2}\,dx$ (ii) $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$ (iii) $\displaystyle\int \frac{1}{x\log x}\,dx$

(b) Do likewise for these definite integrals.

(i) $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$ (ii) $\displaystyle\int_0^1 \frac{x^2}{1+x^3}\,dx$ (iii) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx$

2. (a) Use the result $\displaystyle\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$ to help determine these indefinite integrals.

(i) $\displaystyle\int 6x^2 e^{x^3}\,dx$ (ii) $\displaystyle\int \sec^2 x\, e^{\tan x}\,dx$ (iii) $\displaystyle\int \tfrac{1}{x^2} e^{\frac{1}{x}}\,dx$

(b) Do likewise for these definite integrals.

(i) $\displaystyle\int_0^1 x\,e^{1-x^2}\,dx$ (ii) $\displaystyle\int_0^{\frac{\pi}{2}} \cos x\, e^{\sin x}\,dx$ (iii) $\displaystyle\int_1^4 \tfrac{1}{\sqrt{x}} e^{\sqrt{x}}\,dx$

3. Try to find these integrals mentally, otherwise use a suitable substitution.

(a) $\displaystyle\int 2x(x^2+1)^4\,dx$ (c) $\displaystyle\int \frac{6x^2}{(1+x^3)^2}\,dx$ (e) $\displaystyle\int \frac{x}{\sqrt{x^2-2}}\,dx$

(b) $\displaystyle\int 3x^2(1+x^3)^6\,dx$ (d) $\displaystyle\int \frac{4x}{(3-x^2)^5}\,dx$ (f) $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

DEVELOPMENT

4. Use a suitable substitution where necessary to find:

(a) $\displaystyle\int \frac{\cos x}{\sin^3 x}\,dx$ (c) $\displaystyle\int \frac{(\log x)^2}{x}\,dx$ (e) $\displaystyle\int \frac{x}{1+x^4}\,dx$

(b) $\displaystyle\int \frac{\sec^2 x}{(1+\tan x)^2}\,dx$ (d) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ (f) $\displaystyle\int \frac{x^2}{\sqrt{1-x^6}}\,dx$

5. Use a suitable substitution where necessary to evaluate:

(a) $\displaystyle\int_0^1 x^3(1+3x^4)^2\,dx$ (c) $\displaystyle\int_3^4 \frac{x+1}{\sqrt{x^2+2x+3}}\,dx$ (e) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\sec^2 x\,dx$

(b) $\displaystyle\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$ (d) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x\cos x\,dx$ (f) $\displaystyle\int_1^{e^2} \frac{\log x}{x}\,dx$

6. (a) Use a suitable substitution to help evaluate $\displaystyle\int_0^1 x(x-1)^5\ dx$.

(b) How could this integral have been evaluated using just algebraic manipultaion?

7. Use the given substitution to find:

(a) $\displaystyle\int x\sqrt{x+1}\,dx$ [put $u=\sqrt{x+1}$] (c) $\displaystyle\int \frac{1}{1+x^{\frac{1}{4}}}\,dx$ [put $u=x^{\frac{1}{4}}$]

(b) $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$ [put $u=1+\sqrt{x}$] (d) $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$ [put $u=\sqrt{e^{2x}-1}$]

8. In each case, use the given substitution to evaluate the integral.

(a) $\displaystyle\int_0^1 \frac{2-x}{(2+x)^3}\,dx$ [put $u=2+x$] (c) $\displaystyle\int_0^4 \frac{1}{5+\sqrt{x}}\,dx$ [put $u=\sqrt{x}$]

(b) $\displaystyle\int_0^4 x\sqrt{4-x}\,dx$ [put $u=\sqrt{4-x}$] (d) $\displaystyle\int_4^{12} \frac{1}{(4+x)\sqrt{x}}\,dx$ [put $u=\sqrt{x}$]

9. In each case, use the given substitution to determine the primitive.

(a) $\displaystyle\int \frac{1}{(1-x)\sqrt{x}}\,dx$ [put $u=\sqrt{x}$] (b) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$ [put $u=\sqrt{x+1}$]

10. In each case use the given trigonometric substitution to evaluate the integral. You may assume that $0\le\theta<\frac{\pi}{2}$.

(a) $\displaystyle\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$ [put $x=\tan\theta$] (c) $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ [put $x=5\sin\theta$]

(b) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ [put $x=2\sin\theta$] (d) $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$ [put $x=\tan\theta$]

11. (a) Use a suitable substitution to help evaluate $\displaystyle\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\,dx$.

(b) How could this integral have been evaluated using just algebraic manipultaion?

12. (a) Use a trigonometric substitution to show that $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx=\frac{\pi}{12}-\frac{\sqrt{3}}{8}$.

(b) How could this integral have been evaluated using just algebraic manipultaion?

13. (a) Use a suitable substitution to show that $\displaystyle\int_1^2 \sqrt{4-x^2}\,dx=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$.

(b) Redo this problem by geometric means.

EXTENSION

14. Consider the indefinite integral $I=\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$. Clearly the domain of the integrand is disjoint, being $x>1$ or $x<-1$. Thus it seems appropriate to use a different substitution in each part of the domain.

(a) Find I for $x>1$ by using the substitution $u=\sqrt{x^2-1}$.

(b) Find I for $x<-1$ by using the substitution $u=-\sqrt{x^2-1}$.

15. (a) Use a suitable substitution to determine $\displaystyle\int_{2+\epsilon}^{4} \frac{dx}{x^2\sqrt{x^2-4}}$, where $\epsilon > 0$.

(b) Take the limit of this result as $\epsilon \to 0^+$ and hence find $\displaystyle\int_{2}^{4} \frac{dx}{x^2\sqrt{x^2-4}}$.

2C Partial Fractions

In arithmetic, when given the sum of two fractions, the normal procedure is to combine them into a single fraction using the lowest common denominator. Thus

$$\frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

Unfortunately when the fractions are functions and integration is involved, this is exactly the wrong thing to do. Whilst it is true that

$$\frac{3}{x+2} + \frac{2}{x-1} = \frac{5x+1}{x^2+x-2},$$

when considering the corresponding integrals,

$$\int \frac{3}{x+2} + \frac{2}{x-1}\, dx = \int \frac{5x+1}{x^2+x-2}\, dx,$$

it should be clear that the left hand side is far simpler to determine than the right hand integral. So:

$$\begin{aligned}\int \frac{5x+1}{x^2+x-2}\, dx &= \int \frac{3}{x+2} + \frac{2}{x-1}\, dx \\ &= 3\log(x+2) + 2\log(x-1) + C\,.\end{aligned}$$

This example is typical of integrals of rational functions. It is easiest to first split the fraction into its simpler components. In mathematical terminology, the fraction is *decomposed into partial fractions*.

A Theorem About Partial Fractions: Consider the rational function

$$\frac{P(x)}{A(x) \times B(x)},$$

where P, A and B are polynomials, with no common factors between any pair, and where $\deg P < \deg A + \deg B$. It can be shown that it is always possible to write

$$\frac{P(x)}{A(x) \times B(x)} = \frac{R_A(x)}{A(x)} + \frac{R_B(x)}{B(x)},$$

where R_A and R_B are polynomials with $\deg R_A < \deg A$ and $\deg R_B < \deg B$. The proof is beyond the scope of this course.

Linear Factors: In the simplest examples, $A(x)$ and $B(x)$ are linear. Since the degrees of R_A and R_B are less, they must be constants, yet to be found.

WORKED EXERCISE: (a) Decompose $\dfrac{x+1}{(x-1)(x+3)}$ into its partial fractions.

(b) Hence evaluate $\displaystyle\int_{2}^{6} \frac{x+1}{(x-1)(x+3)}\, dx$.

SOLUTION: (a) Let $\dfrac{x+1}{(x-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+3}$, where A and B are unknown constants. Multiply this equation by $(x-1)(x+3)$ to get:

$$x+1 = A(x+3) + B(x-1)$$

or
$$x+1 = (A+B)x + (3A-B)\,.$$

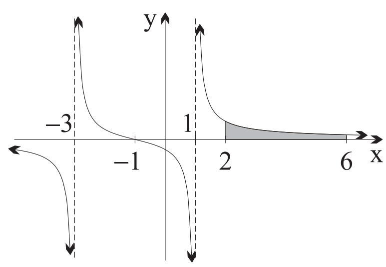
Equating coefficients of like powers of x yields the simultaneous equations

$$A+B = 1$$
$$3A - B = 1\,.$$

These can be solved mentally to get $A = \frac{1}{2}$ and $B = \frac{1}{2}$. Thus

$$\frac{x+1}{(x-1)(x+3)} = \frac{(\frac{1}{2})}{x-1} + \frac{(\frac{1}{2})}{x+3}\,.$$

(b) Hence
$$\begin{aligned}\int_2^6 \frac{x+1}{(x-1)(x+3)}\,dx &= \tfrac{1}{2}\int_2^6 \frac{1}{x-1} + \frac{1}{x+3}\,dx \\ &= \tfrac{1}{2}\Big[\log(x-1) + \log(x+3)\Big]_2^6 \\ &= \tfrac{1}{2}\Big((\log 9 + \log 5) - (\log 5 + \log 1)\Big) \\ &= \log 3\,.\end{aligned}$$



This method of equating coefficients of like powers of x is usually only convenient in straight forward examples like this one.

Finding the Constants by Substitution: A more generalised method of finding the unknown constants in partial fractions uses substitution. In many cases it is also a quicker method.

WORKED EXERCISE: Decompose $\dfrac{3x-5}{(x-3)(x+1)}$ into partial fractions.

SOLUTION: Let $\dfrac{3x-5}{(x-3)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x+1}$, where A and B are unkown constants. Multiply this equation by $(x-3)(x+1)$ to get:

$$3x - 5 = A(x+1) + B(x-3)\,.$$

When $x = 3$, $\quad 4 = 4A$

so $\quad A = 1\,.$

When $x = -1$, $\quad -8 = -4B$

so $\quad B = 2\,.$

Thus $\quad \dfrac{3x-5}{(x-3)(x+1)} = \dfrac{1}{x-3} + \dfrac{2}{x+1}\,.$

The careful reader will have noticed a point of contention with the solution. The fraction is undefined when $x = 3$ and when $x = -1$, yet these values were used in the substitution steps. How can this be valid? The answer is that some of the detail of the solution has been omitted. Here is a more complete explanation.

Since $\quad \dfrac{3x-5}{(x-3)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x+1}$ where $x \neq -1, 3$

we have $\quad 3x - 5 = A(x+1) + B(x-3)$ where $x \neq -1, 3\,.$

Now this last equation is true whenever $x \neq -1, 3$. That is, it is a linear equation which is true for at least two other values of x. Hence, by the work done in Year 11 on identities, it is true for all x, including $x = -1$ and $x = 3$. Thus these values can be substituted to determine A and B. It is not necessary to give this complete explanation as part of a solution, but students should be aware of it.

Numerators with Higher Degree: In slightly harder problems, the degree of the numerator is greater than or equal to the degree of the denominator. In such cases, the fraction should be expressed as a sum of a polynomial and the partial fractions. Long division may be used at this step, but it is often easier to use a polynomial with unkown coefficients, as in the following worked exercise.

WORKED EXERCISE: Determine $\displaystyle\int \frac{x^3 + x - 3}{x^2 - 3x + 2}\,dx\,.$

SOLUTION: First note that $\dfrac{x^3 + x - 3}{x^2 - 3x + 2} = \dfrac{x^3 + x - 3}{(x-2)(x-1)}\,,$

so let $\qquad \dfrac{x^3 + x - 3}{(x-2)(x-1)} = Ax + B + \dfrac{C}{x-2} + \dfrac{D}{x-1}\,,$

thus $\qquad x^3 + x - 3 = (Ax + B)(x-2)(x-1) + C(x-1) + D(x-2)\,.$

Equating the coefficients of x^3, $A = 1$.

At $x = 1$ $\qquad -1 = -D \quad$ so $\quad D = 1\,.$

At $x = 2$ $\qquad 7 = C\,.$

At $x = 0$ $\qquad -3 = 2B - 7 - 2$

so $\qquad B = 3\,.$

Finally
$$\begin{aligned}\int \frac{x^3 + x - 3}{x^2 - 3x + 2}\,dx &= \int x + 3 + \frac{7}{x-2} + \frac{1}{x-1}\,dx \\ &= \tfrac{1}{2}x^2 + 3x + 7\log(x-2) + \log(x-1) + C\,.\end{aligned}$$

A Special Case: There is an even quicker method to find the unknown constants of the partial fractions, provided that the original denominator is a product of distinct linear factors, and provided that the degree of the numerator is less than the degree of the denominator. The trick is to multiply by just one linear factor at a time.

WORKED EXERCISE: Express $\dfrac{7 - 5x}{(x+1)(x-2)(x-3)}$ as a sum of partial fractions.

SOLUTION: Let $\qquad \dfrac{7 - 5x}{(x+1)(x-2)(x-3)} = \dfrac{C_1}{x+1} + \dfrac{C_2}{x-2} + \dfrac{C_3}{x-3}\,. \qquad (*)$

$(*) \times (x+1)$ gives $\qquad \dfrac{7 - 5x}{(x-2)(x-3)} = C_1 + \dfrac{C_2(x+1)}{x-2} + \dfrac{C_3(x+1)}{x-3}$

so at $x = -1$ $\qquad C_1 = \dfrac{12}{(-3)(-4)} = 1\,.$

$(*) \times (x-2)$ gives $\qquad \dfrac{7 - 5x}{(x+1)(x-3)} = \dfrac{C_1(x-2)}{x+1} + C_2 + \dfrac{C_3(x-2)}{x-3}$

so at $x = 2$ $\qquad C_2 = \dfrac{-3}{3 \times (-1)} = 1\,.$

Finally $\qquad \dfrac{7 - 5x}{(x+1)(x-2)} = \dfrac{C_1(x-3)}{x+1} + \dfrac{C_2(x-3)}{x-2} + C_3$

so at $x = 3$ $$C_3 = \frac{-8}{4 \times 1} = -2\,.$$

Hence $$\frac{7 - 5x}{(x+1)(x-2)(x-3)} = \frac{1}{x+1} + \frac{1}{x-2} - \frac{2}{x-3}\,.$$

This method of finding the constants is sometimes called the *cover up rule*. Look carefully at how the three constants are determined. For each constant, the matching linear factor is effectively omitted, or "covered up". Thus for C_1, $(x+1)$ is left out of the original fraction. For C_2, $(x-2)$ is excluded, and for C_3, $(x-3)$ is omitted from the original fraction. In each case, the resulting rational function is then evaluated at the corresponding value of x. With practice, most students should be able to determine the constants mentally using this method. Here is a proof for the general case.

PROOF: Consider the rational function $\dfrac{P(x)}{Q(x)}$ where $\deg P < \deg Q$, and where $Q(x)$ is a product of distinct linear factors, that is
$$\begin{aligned} Q(x) &= C \times (x - a_1) \times (x - a_2) \times \ldots \times (x - a_n) \\ &= C \prod_{i=1}^{n} (x - a_i) \qquad \text{(note the use of product notation, } \textstyle\prod\text{.)} \end{aligned}$$

Let $$\frac{P(x)}{Q(x)} = \frac{C_1}{x - a_1} + \frac{C_2}{x - a_2} + \ldots + \frac{C_k}{x - a_k} + \ldots + \frac{C_n}{x - a_n}$$

Multiply this last equation by $(x - a_k)$ to get
$$\frac{P(x)(x - a_k)}{Q(x)} = \frac{C_1(x - a_k)}{x - a_1} + \frac{C_2(x - a_k)}{x - a_2} + \ldots + C_k + \ldots + \frac{C_n(x - a_k)}{x - a_n}\,.$$

Now take the limit as $x \to a_k$. All terms except C_k on the right hand side are zero and we get:
$$\begin{aligned} C_k &= \lim_{x \to a_k} \frac{P(x)(x - a_k)}{Q(x)} \\ &= \lim_{x \to a_k} \frac{P(x)}{C \displaystyle\prod_{\substack{i=1 \\ i \neq k}}^{n} (x - a_i)} \qquad \text{(that is, cancel the } k\text{th linear factor)} \end{aligned}$$

hence $$C_k = \frac{P(a_k)}{C \displaystyle\prod_{\substack{i=1 \\ i \neq k}}^{n} (a_k - a_i)}\,.$$

The mathematical notation may seem difficult, but the result is exactly as before. To get the kth coefficient C_k, omit the kth linear factor from the denominator and evaluate the rest of the fraction at $x = a_k$.

Quadratic Factors: In certain instances, the denominator of the rational function being considered will have a quadratic factor with no real zero. For example, in
$$\frac{3x + 10}{(x-2)(x^2+4)}$$
the quadratic factor $(x^2 + 4)$ has no real zero. Thus the denominator of the rational function cannot be expressed as a product of real linear factors.

Nevertheless, the method for finding the partial fraction decomposition remains essentially the same. And since the only requirement is that the degree of the numerator is less than the degree of the denominator, it follows that for any quadratic factor the numerator can be a linear polynomial.

WORKED EXERCISE: (a) Rewrite $\dfrac{3x+10}{(x-2)(x^2+4)}$ in its partial fractions.

(b) Hence determine $\displaystyle\int \frac{3x+10}{(x-2)(x^2+4)}\,dx$.

SOLUTION: (a) Let $\dfrac{3x+10}{(x-2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4}$, where A, B and C are unknown constants. Then

$$3x+10 = A(x^2+4) + (Bx+C)(x-2)$$

At $x = 2$ $\qquad 16 = 8A \quad$ so $\quad A = 2$.

Equating coefficients of x^2 yields

$$0 = 2 + B \quad \text{so} \quad B = -2\,.$$

At $x = 0$ $\qquad 10 = 8 - 2C$

so $\qquad C = -1$.

Thus $\qquad \dfrac{4x+10}{(x-2)(x^2+4)} = \dfrac{2}{x-2} - \dfrac{2x+1}{x^2+4}$.

(b) Hence
$$\int \frac{4x+10}{(x-2)(x^2+4)}\,dx = \int \frac{2}{x-2} - \frac{2x}{x^2+4} - \frac{1}{x^2+4}\,dx$$
$$= 2\log(x-2) - \log(x^2+4) - \tfrac{1}{2}\tan^{-1}(\tfrac{x}{2}) + C\,.$$

Repeated Factors: In a polynomial, a factor which has degree greater than one is called a repeated factor. For example in the denominator of the fraction

$$\frac{8-x}{(x-2)^2(x+1)}\,,$$

the factor $(x-2)^2$ is a repeated factor since its index is two. When a partial fraction question involves repeated factors, normally the initial decomposition is given in the question and it is simply a matter of finding the values of the unknown constants.

WORKED EXERCISE: (a) Given that $\dfrac{8-x}{(x-2)^2(x+1)}$ can be written as

$$\frac{8-x}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}\,,$$

where A, B and C are real numbers, find A, B and C.

(b) Hence evaluate $\displaystyle\int_0^1 \frac{8-x}{(x-2)^2(x+1)}\,dx$.

SOLUTION:

(a) Now $\qquad 8 - x = A(x-2)(x+1) + B(x+1) + C(x-2)^2$.

At $x = -1$ $\qquad 9 = 9C \quad$ so $\quad C = 1$.

At $x = 2$ $\qquad 6 = 3B \quad$ so $\quad B = 2$.

At $x = 3$ $\qquad 5 = 4A + 8 + 1$

so $\qquad A = -1$.

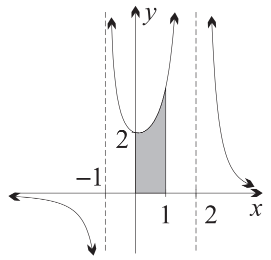
(b) Hence $\displaystyle\int_0^1 \frac{8-x}{(x-2)^2(x+1)}\,dx$

$$= \int_0^1 \frac{1}{x+1} + \frac{2}{(x-2)^2} - \frac{1}{x-2}\,dx$$

$$= \left[\log(x+1) - \frac{2}{x-2} - \log|x-2|\right]_0^1$$

$$= (\log 2 + 2 - \log 1) - (\log 1 + 1 - \log 2)$$

$$= 1 + 2\log 2\,.$$



Exercise **2C**

1. Decompose the following fractions into partial fractions.

(a) $\dfrac{2}{(x-1)(x+1)}$ (c) $\dfrac{4x}{x^2-9}$ (e) $\dfrac{x-1}{x^2+x-6}$

(b) $\dfrac{1}{(x-4)(x-1)}$ (d) $\dfrac{x}{x^2-3x+2}$ (f) $\dfrac{3x+1}{(x-1)(x^2+3)}$

2. Find:

(a) $\displaystyle\int \frac{2}{(x-4)(x-2)}\,dx$ (c) $\displaystyle\int \frac{3x-2}{(x-1)(x-2)}\,dx$ (e) $\displaystyle\int \frac{4x+5}{(2x+3)(x+1)}\,dx$

(b) $\displaystyle\int \frac{4}{x^2+4x+3}\,dx$ (d) $\displaystyle\int \frac{2x+10}{x^2+2x-3}\,dx$ (f) $\displaystyle\int \frac{10x}{2x^2-x-3}\,dx$

3. Evaluate:

(a) $\displaystyle\int_4^6 \frac{1}{x^2-4}\,dx$ (c) $\displaystyle\int_2^5 \frac{11}{2x^2+5x-12}\,dx$

(b) $\displaystyle\int_2^4 \frac{3}{x^2+x-2}\,dx$ (d) $\displaystyle\int_{-1}^0 \frac{1}{3x^2-4x+1}\,dx$

4. Determine:

(a) $\displaystyle\int \frac{x^2-2x+5}{(x-2)(x^2+1)}\,dx$ (b) $\displaystyle\int \frac{6-x}{(2x+1)(x^2+3)}\,dx$ (c) $\displaystyle\int \frac{x^2+x+3}{x^3+x}\,dx$

5. Find the value of:

(a) $\displaystyle\int_0^{\frac{1}{2}} \frac{1+2x-4x^2}{(x+1)(4x^2+1)}\,dx$ (b) $\displaystyle\int_{-1}^1 \frac{7-x}{(x+3)(x^2+1)}\,dx$ (c) $\displaystyle\int_1^{\sqrt{2}} \frac{x^2-4}{x^3+2x}\,dx$

DEVELOPMENT

6. Find:

(a) $\displaystyle\int \frac{2x+3}{(x-1)(x-2)(2x-3)}\,dx$ (b) $\displaystyle\int \frac{4x+12}{x^3-6x^2+8x}\,dx$

7. Evalute:

(a) $\displaystyle\int_2^7 \frac{3x+5}{(x-1)(x+2)(x+1)}\,dx$ (b) $\displaystyle\int_1^2 \frac{13x+6}{x^3-x^2-6x}\,dx$

8. (a) (i) Let $\dfrac{2x^2+1}{(x-1)(x+2)} = A + \dfrac{B}{x-1} + \dfrac{C}{x+2}$. Find the values of A, B and C.

(ii) Hence find $\displaystyle\int \frac{2x^2+1}{(x-1)(x+2)}\,dx$

(b) Use a similar technique to part (a) in order to find:

(i) $\displaystyle\int \frac{x^2-2x+3}{(x+1)(x-2)}\,dx$ (ii) $\displaystyle\int \frac{3x^2-66}{(x+4)(x-5)}\,dx$

9. (a) (i) Find the values of A and B such that

$$\frac{3x^2-10}{x^2-4x+4}=3+\frac{A}{x-2}+\frac{B}{(x-2)^2}.$$

(ii) Hence find $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx$.

(b) (i) Find the integers A, B, C and D such that

$$\frac{3x+7}{(x-1)^2(x-2)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2}.$$

(ii) Hence find $\displaystyle\int \frac{3x+7}{(x-1)^2(x-2)^2}\,dx$.

10. (a) (i) Find the values of A, B, C and D such that

$$\frac{x^3-3x^2-4}{(x+1)(x-3)}=Ax+B+\frac{C}{x+1}+\frac{D}{x-3}.$$

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{x^3-3x^2-4}{(x+1)(x-3)}\,dx$.

(b) Use a similar method to evaluate $\displaystyle\int_2^4 \frac{x^3+4x^2+x-3}{(x+2)(x-1)}\,dx$.

11. Show that:

(a) $\displaystyle\int_4^6 \frac{x^2-8}{x^3+4x}\,dx=\tfrac{3}{2}\log 2-2\log\tfrac{3}{2}$.

(b) $\displaystyle\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}\,dx=\tfrac{1}{2}\log 20$.

12. (a) Let $\displaystyle\frac{x^2-1}{x^4+x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$. Find A, B, C and D.

(b) Hence show that $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{x^2-1}{x^4+x^2}\,dx=\tfrac{1}{3}(\pi-2\sqrt{3})$.

13. Use appropriate methods to find:

(a) $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx$

(b) $\displaystyle\int \frac{x^2+1}{x^2-x}\,dx$

(c) $\displaystyle\int \frac{x^3+1}{x^3+x}\,dx$

(d) $\displaystyle\int \frac{x^2}{x^2-5x+6}\,dx$

(e) $\displaystyle\int \frac{x^3+5}{x^2+x}\,dx$

(f) $\displaystyle\int \frac{x^4}{x^2-3x+2}\,dx$

EXTENSION

14. Use a similar approach to Question 10 for repeated factors to show that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5x-x^2}{(x+1)^2(x-1)}\,dx=4-3\log 3.$$

15. (a) In the notation of the text, if $Q(x)$ is a product of distinct linear factors then:

$$C_k=\lim_{x\to a_k}\frac{P(x)(x-a_k)}{Q(x)}.$$

Use this result to prove that

$$C_k=\frac{P(a_k)}{Q'(a_k)}.$$

[HINT: What is the value of $Q(a_k)$?]

(b) Use this formula to redo Questions 6(b) and 7(b).

2D Quadratics in the Denominator

Many practical applications yield integrals with a quadratic in the denominator. In the simplest cases it is a matter of applying the four standard integral results:

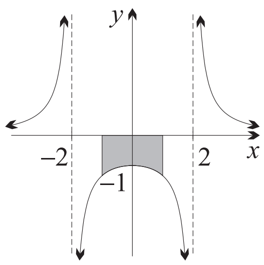
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Another common integral is $\displaystyle\int \frac{dx}{x^2-a^2}$. Although a formula exists for this, it is not part of the course. It is expected that candidates determine the primitive by use of partial fractions whenever this type of integral is encountered.

WORKED EXERCISE: Evaluate $\displaystyle\int_{-1}^{1} \frac{4}{x^2-4}\,dx$.

SOLUTION: Let $\dfrac{4}{x^2-4} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$, then



$$4 = A(x+2) + B(x-2)$$

At $x=2$ $\quad 4 = 4A$, so $A = 1$.

At $x=-2$ $\quad 4 = -4B$, so $B = -1$.

Hence
$$\begin{aligned}\int_{-1}^{1} \frac{4}{x^2-4}\,dx &= \int_{-1}^{1} \frac{1}{x-2} - \frac{1}{x+2}\,dx \\ &= \Big[\log|x-2| - \log(x+2)\Big]_{-1}^{1} \\ &= (\log 1 - \log 3) - (\log 3 - \log 1) \\ &= -2\log 3\,.\end{aligned}$$

Quadratics with Linear Terms: Frequently the quadratic will have a linear term, such as in $3+2x-x^2$. In these instances the method is to complete the square to obtain either the sum of two squares or the difference of two squares.

WORKED EXERCISE: Find $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$.

SOLUTION: Completing the square in the denominator:

$$\begin{aligned}\int \frac{1}{\sqrt{3+2x-x^2}}\,dx &= \int \frac{1}{\sqrt{4-(x-1)^2}}\,dx \\ &= \int \frac{1}{\sqrt{4-u^2}}\,du \qquad \text{where } u = x-1 \\ &= \sin^{-1}\tfrac{u}{2} + C \\ &= \sin^{-1}\tfrac{x-1}{2} + C\,.\end{aligned}$$

Notice that the solution uses a substitution. This step may be omitted by using a result from the Mathematics Extension 1 course. Recall that if $F(x)$ is a primitive of $f(x)$ then

$$\int f(ax+b)\,dx = \tfrac{1}{a}F(ax+b) + C\,.$$

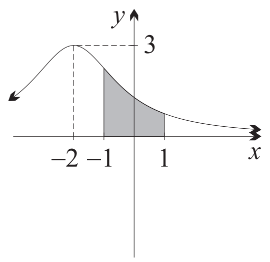
In this particular instance, $f(x) = \dfrac{1}{\sqrt{4-x^2}}$, the primitive is $F(x) = \sin^{-1}\frac{x}{2}$ with $a = 1$ and $b = 1$. Thus it is permissible to write

$$\int \frac{1}{\sqrt{4-(x-1)^2}}\,dx = \sin^{-1}\tfrac{x-1}{2} + C\,,$$

without showing any working. Here is a similar example.

WORKED EXERCISE: Find the value of $\displaystyle\int_{-1}^{1} \frac{9}{7+4x+x^2}\,dx$.

SOLUTION: Completing the square in the denominator:



$$\begin{aligned}\int_{-1}^{1} \frac{9}{7+4x+x^2}\,dx &= \int_{-1}^{1} \frac{9}{3+(4+4x+x^2)}\,dx \\ &= \int_{-1}^{1} \frac{9}{3+(2+x)^2}\,dx \\ &= \tfrac{9}{\sqrt{3}}\Big[\tan^{-1}\tfrac{x+2}{\sqrt{3}}\Big]_{-1}^{1} \\ &= 3\sqrt{3}\left(\tfrac{\pi}{3}-\tfrac{\pi}{6}\right) \\ &= \tfrac{\pi\sqrt{3}}{2}\,.\end{aligned}$$

1 QUADRATICS WITH LINEAR TERMS: Complete the square, then use the result

$$\int f(ax+b)\,dx = \tfrac{1}{a}F(ax+b) + C\,,$$

where $F(x)$ is the primitive of $f(x)$.

Linear Numerators: So far in all the worked exercises the numerator has been a constant. When the numerator is linear it is best to carefully split it into two parts. The first term should be a multiple of the derivative of the quadratic in the denominator. The second term will then be a constant.

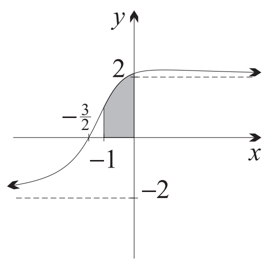
WORKED EXERCISE: Determine $\displaystyle\int \frac{4x+3}{x^2+9}\,dx$.

SOLUTION:

$$\begin{aligned}\int \frac{4x+3}{x^2+9}\,dx &= 2\int \frac{2x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx \\ &= 2\log(x^2+9) + \tan^{-1}\tfrac{x}{3} + C\,.\end{aligned}$$

In harder examples the quadratic will also contain a linear term.

WORKED EXERCISE: Evaluate $\displaystyle\int_{-1}^{0} \frac{2x+3}{\sqrt{x^2+2x+2}}\,dx$.



SOLUTION:

$$\begin{aligned}&\int_{-1}^{0} \frac{2x+3}{\sqrt{x^2+2x+2}}\,dx \\ &= \int_{-1}^{0} \frac{2x+2}{\sqrt{x^2+2x+2}}\,dx + \int_{-1}^{0} \frac{1}{\sqrt{(x+1)^2+1}}\,dx\end{aligned}$$

$$= \Big[2\sqrt{x^2+2x+2}\,\Big]_{-1}^{0} + \Big[\log\big((x+1)+\sqrt{(x+1)^2+1}\,\big)\Big]_{-1}^{0}$$
$$= 2\sqrt{2} - 2 + \log(1+\sqrt{2}) - \log 1$$
$$= 2(\sqrt{2}-1) + \log(1+\sqrt{2})\,.$$

2 LINEAR NUMERATORS: When the numerator is linear it is best to split it into a multiple of the derivative of the quadratic in the denominator plus a constant.

Rationalising the Numerator: In much previous work it has been convenient to rationalise the denominator when a surd appears. In contrast, when calculus is involved it is often convenient to rationalise the numerator instead.

WORKED EXERCISE: Find $\displaystyle\int \sqrt{\frac{x+1}{x+7}}\,dx\,.$

SOLUTION: Rationalising the numerator

$$\int \sqrt{\frac{x+1}{x+7}}\,dx = \int \frac{x+1}{\sqrt{x^2+8x+7}}\,dx$$
$$= \int \frac{x+4}{\sqrt{x^2+8x+7}}\,dx - \int \frac{3}{\sqrt{x^2+8x+7}}\,dx$$
$$= \int \frac{x+4}{\sqrt{x^2+8x+7}}\,dx - \int \frac{3}{\sqrt{(x+4)^2-3^2}}\,dx$$
$$= \sqrt{x^2+8x+7} - 3\log\Big((x+4)+\sqrt{(x+4)^2-3^2}\Big) + C\,.$$

Notice that in the first line of working, by rationalising, the numerator has become linear. This is typical of the questions done in this section.

Care is needed when applying this technique to definite integrals. For example whilst $\displaystyle\int_{-1}^{1} \sqrt{\frac{x+1}{x+7}}\,dx$ is well defined, the resulting integral $\displaystyle\int_{-1}^{1} \frac{x+1}{\sqrt{x^2+8x+7}}\,dx$ is not, since the denominator is zero at the lower limit. Definite integrals of this type are dealt with in the last section of this chapter.

3 RATIONALISING THE NUMERATOR: When calculus is involved it is often convenient to rationalise the numerator.

Exercise **2D**

1. Find:

(a) $\displaystyle\int \frac{1}{9+x^2}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$ (e) $\displaystyle\int \frac{1}{x^2-9}\,dx$

(b) $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx$ (d) $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$ (f) $\displaystyle\int \frac{1}{9-x^2}\,dx$

2. Determine:

(a) $\displaystyle\int \frac{1}{x^2+4x+3}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{x^2-6x+13}}\,dx$ (e) $\displaystyle\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$

(b) $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$ (d) $\displaystyle\int \frac{1}{\sqrt{x^2+8x+12}}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt{4x^2+8x+6}}\,dx$

3. Evaluate:

(a) $\displaystyle\int_1^3 \frac{1}{x^2-2x+5}\,dx$ (c) $\displaystyle\int_{-1}^0 \frac{1}{\sqrt{3-2x-x^2}}\,dx$ (e) $\displaystyle\int_{-1}^3 \frac{1}{\sqrt{x^2+2x+10}}\,dx$

(b) $\displaystyle\int_1^5 \frac{4}{x^2-6x+13}\,dx$ (d) $\displaystyle\int_0^1 \frac{3}{\sqrt{3+4x-4x^2}}\,dx$ (f) $\displaystyle\int_{\frac{1}{2}}^1 \frac{2}{\sqrt{x^2-x+1}}\,dx$

DEVELOPMENT

4. Find:

(a) $\displaystyle\int \frac{2x+1}{x^2+2x+2}\,dx$ (c) $\displaystyle\int \frac{x}{\sqrt{x^2+2x+10}}\,dx$ (e) $\displaystyle\int \frac{x}{\sqrt{6x-x^2}}\,dx$

(b) $\displaystyle\int \frac{x}{x^2+2x+10}\,dx$ (d) $\displaystyle\int \frac{x+3}{\sqrt{x^2-2x-4}}\,dx$ (f) $\displaystyle\int \frac{x+3}{\sqrt{4-2x-x^2}}\,dx$

5. Find the value of:

(a) $\displaystyle\int_0^2 \frac{x+1}{x^2+4}\,dx$ (c) $\displaystyle\int_1^2 \frac{2x-3}{x^2-2x+2}\,dx$ (e) $\displaystyle\int_{-1}^3 \frac{1-2x}{\sqrt{x^2+2x+3}}\,dx$

(b) $\displaystyle\int_1^2 \frac{x+1}{x^2-4x+5}\,dx$ (d) $\displaystyle\int_{-1}^0 \frac{x}{\sqrt{3-2x-x^2}}\,dx$ (f) $\displaystyle\int_0^1 \frac{x+3}{\sqrt{x^2+4x+1}}\,dx$

6. Determine each primitive.

(a) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,dx$ (b) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$ (c) $\displaystyle\int \sqrt{\frac{3-x}{2+x}}\,dx$

7. Evaluate:

(a) $\displaystyle\int_{-1}^0 \sqrt{\frac{1-x}{x+3}}\,dx$ (b) $\displaystyle\int_{-1}^0 \sqrt{\frac{x+2}{1-x}}\,dx$ (c) $\displaystyle\int_0^1 \sqrt{\frac{x+1}{x+3}}\,dx$

EXTENSION

8. (a) Why is it not valid to evalute $\displaystyle\int_0^2 \sqrt{\frac{x}{4-x}}\,dx$ using the techniques of this section?

(b) Nevertheless, show that its value is $\displaystyle\lim_{\epsilon\to 0^+}\int_\epsilon^2 \sqrt{\frac{x}{4-x}}\,dx = \pi - 2\,.$

9. (a) Show that $x^3+3x^2+5x+1=(x+1)(x^2+2x+2)+(x-1)\,.$

(b) Hence or otherwise show that

$$\int_{-1}^0 \frac{x^3+3x^2+5x+1}{\sqrt{x^2+2x+2}}\,dx = \tfrac{1}{3}(5\sqrt{2}-4) - 2\log(1+\sqrt{2}\,)\,.$$

2E Integration by Parts

Whilst there are well known and relatively simple formulae for the derivatives of products and quotients of functions, there are no such general formulae for the integrals of products and quotients. Nevertheless, as was found in the previous two sections, certain quotients can be integrated relatively easily. In this section, a method of integration is developed that can be applied to certain types of products. It begins with the product rule for differentiation.

Now $$\frac{d}{dx}(uv) = u'v + u\,v'\,.$$

Swapping sides and integrating yields
$$\int u'v\,dx + \int u\,v'\,dx = uv\,,$$
hence $$\int u\,v'\,dx = uv - \int u'v\,dx\,.$$

This last equation provides a way to rearrange an integral of one product into an integral of a different product. The formula is applied with the aim that the new integral is in some way simpler. The process is called *integration by parts*.

WORKED EXERCISE: Use integration by parts to find $\displaystyle\int xe^x\,dx$.

SOLUTION:

Let $$\begin{aligned} I &= \int xe^x\,dx \\ &= \int u\,v'\,dx\,, \end{aligned}$$
where $u = x$ and $v' = e^x$
so $u' = 1$ and $v = e^x$.

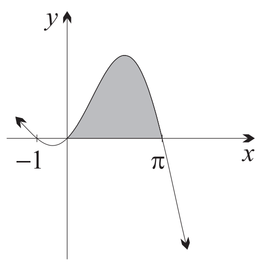
Hence $$\begin{aligned} I &= uv - \int u'v\,dx \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C \end{aligned}$$
or $$I = e^x(x-1) + C\,.$$

Notice the lack of any constant of integration until the process is finished.

4 INTEGRATION BY PARTS: The integral of the product $u\,v'$ can be rearranged using the integration by parts formula:
$$\int u\,v'\,dx = uv - \int u'v\,dx\,.$$

Reducing Polynomials: When one of the factors of the integrand is a polynomial, it is common to let u be that polynomial. In that way the new integral, which depends on u', will contain a polynomial of lesser degree. That is, the aim is to reduce the degree of the polynomial.

WORKED EXERCISE: Evaluate $\displaystyle\int_0^{\pi} (x+1)\sin x\,dx$.



SOLUTION:

Let $$\begin{aligned} I &= \int_0^{\pi} (x+1)\sin x\,dx \\ &= \int_0^{\pi} u\,v'\,dx\,, \end{aligned}$$

where $\quad u = (x+1) \quad$ and $\quad v' = \sin x$
so $\quad u' = 1 \quad$ and $\quad v = -\cos x\,.$

Thus $I = \Big[uv\Big]_0^{\pi} - \displaystyle\int_0^{\pi} u'v\,dx$

$= \Big[-(x+1)\cos x\Big]_0^{\pi} + \displaystyle\int_0^{\pi} \cos x\,dx$

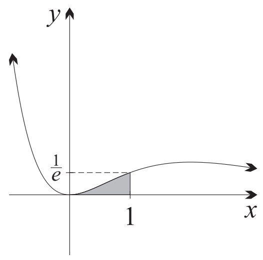
$= (\pi+1) + 1 + \Big[\sin x\Big]_0^{\pi},$

hence $I = \pi + 2\,.$

Repeated Applications: It may be necessary to apply integration by parts more than once in order to complete the process of integration. In simpler examples it may be possible to do some of the steps mentally.

WORKED EXERCISE: Evaluate $\displaystyle\int_0^1 x^2 e^{-x}\,dx\,.$

SOLUTION:



Let $\quad I = \displaystyle\int_0^1 x^2 e^{-x}\,dx$

and put $\quad u = x^2 \quad$ and $\quad v' = e^{-x}$
so $\quad u' = 2x \quad$ and $\quad v = -e^{-x}\,.$

Then $I = \Big[-x^2 e^{-x}\Big]_0^1 + \displaystyle\int_0^1 2xe^{-x}\,dx \qquad$ (by parts.)

Now put $\quad u = 2x \quad$ and $\quad v' = e^{-x}$
so $\quad u' = 2 \quad$ and $\quad v = -e^{-x}\,.$

Thus $I = -e^{-1} + \left(\Big[-2xe^{-x}\Big]_0^1 + \displaystyle\int_0^1 2e^{-x}\,dx\right) \qquad$ (by parts again)

$= -e^{-1} - 2e^{-1} - \Big[2e^{-x}\Big]_0^1$

$= 2 - 5e^{-1}\,.$

Exceptions with Polynomials: Although it is common to reduce the degree of a polynomial using integration by parts, there are many exceptions. In this course these exceptions typically involve the logarithm function.

WORKED EXERCISE: Determine $\displaystyle\int x\log x\,dx\,.$

SOLUTION:

Let $\quad I = \displaystyle\int x\log x\,dx$

and put $\quad u = \log x \quad$ and $\quad v' = x$
so $\quad u' = \frac{1}{x} \quad$ and $\quad v = \frac{1}{2}x^2\,.$

Thus $I = \frac{1}{2}x^2\log x - \displaystyle\int \tfrac{1}{2}x^2 \times \tfrac{1}{x}\,dx \qquad$ (by parts)

$= \frac{1}{2}x^2\log x - \displaystyle\int \tfrac{1}{2}x\,dx$

$= \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C$

or $\quad I = \frac{1}{4}x^2(2\log x - 1) + C\,.$

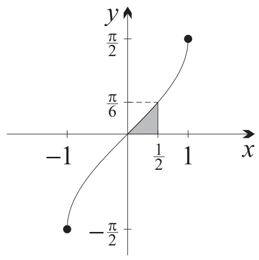
Integrands where $v' = 1$: The prime number 5 has only two distinct factors, namely 1×5. In the same way we may treat a function like a prime and write:

$$\sin^{-1} x = 1 \times \sin^{-1} x\,.$$

This somewhat artificial form of factoring is applied to facilitate integration by parts. It is then usual to put u equal to the function and $v' = 1$.

WORKED EXERCISE: Find the value of $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$.

SOLUTION:



$$\begin{aligned}
\text{Let}\quad I &= \int_0^{\frac{1}{2}} \sin^{-1} x\, dx \\
&= \int_0^{\frac{1}{2}} 1 \times \sin^{-1} x\, dx\,.
\end{aligned}$$

Put $\quad u = \sin^{-1} x \quad$ and $\quad v' = 1$

so $\quad u' = \dfrac{1}{\sqrt{1-x^2}} \quad$ and $\quad v = x\,.$

$$\begin{aligned}
\text{Thus } I &= \Big[x \sin^{-1} x\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\, dx \qquad \text{(by parts)} \\
&= \Big[x \sin^{-1} x + \sqrt{1-x^2}\,\Big]_0^{\frac{1}{2}} \\
&= \left(\tfrac{1}{2} \times \tfrac{\pi}{6} + \sqrt{\tfrac{3}{4}}\right) - (0+1) \\
&= \tfrac{\pi}{12} + \tfrac{\sqrt{3}}{2} - 1\,.
\end{aligned}$$

A Recurrence of the Integral: Integration by parts may lead to a recurrence of the original integral. It is then simply a matter of collecting like terms.

WORKED EXERCISE: Find a primitive of $e^x \sin x$.

SOLUTION:

Let $\quad I = \displaystyle\int e^x \sin x\, dx$

and put $\quad u = \sin x \quad$ and $\quad v' = e^x$

so $\quad u' = \cos x \quad$ and $\quad v = e^x\,.$

Then $\quad I = e^x \sin x - \displaystyle\int e^x \cos x\, dx \qquad$ (by parts)

Now put $\quad u = \cos x \quad$ and $\quad v' = e^x$

so $\quad u' = -\sin x \quad$ and $\quad v = e^x\,.$

$$\begin{aligned}
\text{Thus}\quad I &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x\, dx\right) \qquad \text{(by parts again)} \\
&= e^x(\sin x - \cos x) - I
\end{aligned}$$

or $\quad 2I = e^x(\sin x - \cos x)$

hence $\quad I = \frac{1}{2}e^x(\sin x - \cos x) + C \quad$ is the general primitive.

In this example it was important to apply the method consistently. Notice that u was always the trigonometric function and v' was always the exponential function. As an exercise to highlight the significance of these choices, repeat the worked exercise but put $u = e^x$ and $v' = \cos x$ at the second integration by parts.

Exercise **2E**

1. Find:

(a) $\displaystyle\int xe^x\,dx$ (c) $\displaystyle\int (x+1)e^{3x}\,dx$ (e) $\displaystyle\int (x-1)\sin 2x\,dx$

(b) $\displaystyle\int xe^{-x}\,dx$ (d) $\displaystyle\int x\cos x\,dx$ (f) $\displaystyle\int (2x-3)\sec^2 x\,dx$

2. Evaluate:

(a) $\displaystyle\int_0^{\pi} x\sin x\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$ (e) $\displaystyle\int_0^1 (1-x)e^{-x}\,dx$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx$ (d) $\displaystyle\int_0^1 xe^{2x}\,dx$ (f) $\displaystyle\int_{-2}^0 (x+2)e^x\,dx$

3. In these questions put $v' = 1$.

(a) $\displaystyle\int \log x\,dx$ (b) $\displaystyle\int \log(x^2)\,dx$ (c) $\displaystyle\int \cos^{-1} x\,dx$

4. Find the value of:

(a) $\displaystyle\int_0^1 \tan^{-1} x\,dx$ (b) $\displaystyle\int_1^e \log x\,dx$ (c) $\displaystyle\int_1^e \log\sqrt{x}\,dx$

5. In each case use integration by parts to increase the power of x.

(a) $\displaystyle\int x\log x\,dx$ (b) $\displaystyle\int x^2\log x\,dx$ (c) $\displaystyle\int \frac{\log x}{x^2}\,dx$

DEVELOPMENT

6. Use repeated applications of integration by parts in order to find:

(a) $\displaystyle\int x^2e^x\,dx$ (b) $\displaystyle\int x^2\cos x\,dx$ (c) $\displaystyle\int (\log x)^2\,dx$

7. These integrals are more naturally done by substitution. Nevertheless they can also be done by parts. Use integration by parts here and then compare your answers with similar questions in Exercise 2B.

(a) $\displaystyle\int_0^1 x(x-1)^5\,dx$ (b) $\displaystyle\int_0^1 x\sqrt{x+1}\,dx$ (c) $\displaystyle\int_0^4 x\sqrt{4-x}\,dx$

8. Determine: (a) $\displaystyle\int e^x\cos x\,dx$ (b) $\displaystyle\int e^{-x}\sin x\,dx$

9. Evaluate: (a) $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} e^x\sin 2x\,dx$

10. Use integration by parts to evaluate:

(a) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \sin^{-1} x\,dx$ (b) $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \cos^{-1} x\,dx$ (c) $\displaystyle\int_0^1 4x\tan^{-1} x\,dx$

11. Show that:

(a) $\displaystyle\int_0^{\pi} x^2\cos 2x\,dx = \frac{\pi}{2}$ (c) $\displaystyle\int_1^e \sin(\log x)\,dx = \tfrac{1}{2}e(\sin 1 - \cos 1) + \tfrac{1}{2}$

(b) $\displaystyle\int_0^{\pi} x^2\sin\tfrac{1}{2}x\,dx = 8\pi - 16$ (d) $\displaystyle\int_1^e \cos(\log x)\,dx = \tfrac{1}{2}e(\sin 1 + \cos 1) - \tfrac{1}{2}$

12. Determine formulae for the following:

(a) $\displaystyle\int \sqrt{a^2-x^2}\,dx$ (b) $\displaystyle\int \log(x+\sqrt{x^2+a^2}\,)\,dx$ (c) $\displaystyle\int \log(x+\sqrt{x^2-a^2}\,)\,dx$

13. (a) Determine $\int x \log x\, dx$. (b) Hence find $\int x(\log x)^2\, dx$.

14. Use trigonometric identities and then integration by parts to show that:

(a) $\int_0^{\frac{\pi}{2}} x \sin x \cos x\, dx = \frac{\pi}{8}$

(b) $\int_0^{\frac{\pi}{2}} x \sin^2 x\, dx = \frac{1}{16}(\pi^2 + 4)$

(c) $\int_0^{\frac{\pi}{4}} x \tan^2 x\, dx = \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\log 2$

(d) $\int_0^{\pi} x^2(\cos^2 x - \sin^2 x)\, dx = \frac{\pi}{2}$

EXTENSION

15. Determine:

(a) $\int x \sin x \cos 3x\, dx$ (b) $\int x \cos 2x \cos x\, dx$ (c) $\int e^x \sin 2x \cos x\, dx$

16. Determine: (a) $\int_0^{\frac{1}{2}} x \sin^{-1} x\, dx$ (b) $\int_0^1 x^2 \tan^{-1} x\, dx$

17. Let s be a positive constant. Show that $\lim_{N\to\infty} \int_0^N te^{-st}\, dt = \dfrac{1}{s^2}$.

2F Trigonometric Integrals

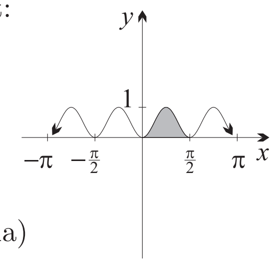
Powers of Cosine and Sine: There are two methods for the integral

$$\int \cos^m x \sin^n x\, dx$$

depending on whether the constants m and n are odd or even. If both are even then it is best to use the double angle identities.

WORKED EXERCISE: Evaluate $\int_0^{\frac{\pi}{2}} 4\cos^2 x \sin^2 x\, dx$

SOLUTION: Apply the double angle formula for sine to get:



$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} 4\cos^2 x \sin^2 x\, dx \\
&= \int_0^{\frac{\pi}{2}} \sin^2 2x\, dx \\
&= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos 4x\, dx \qquad \text{(cosine double angle formula)} \\
&= \tfrac{1}{2}\Big[x - \tfrac{1}{4}\sin 4x\Big]_0^{\frac{\pi}{2}} \\
&= \tfrac{\pi}{4}.
\end{aligned}$$

In the second method one or both of m and n is odd. Work with cosine if m is odd, otherwise work with sine. The odd index of the chosen trigonometric function can be reduced to 1 via the Pythagorean identity, $\cos^2 x + \sin^2 x = 1$. It is then a matter of making a substitution for the other trigonometric function. The result is a polynomial integral.

Worked Exercise: Determine $\displaystyle\int \cos^3 x \sin^2 x\, dx$.

Solution:

$$\begin{aligned}
\text{Let}\quad I &= \int \cos^3 x \sin^2 x\, dx \\
&= \int \cos x(1-\sin^2 x)\sin^2 x\, dx \qquad \text{(by Pythagoras.)}
\end{aligned}$$

Put $u = \sin x$,

$$\begin{aligned}
\text{then } I &= \int (1-u^2)u^2\, du \\
&= \int u^2 - u^4\, dx \\
&= \tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 + C \\
&= \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C\,.
\end{aligned}$$

5 Powers of cosine and sine: Given an integral of the form $\displaystyle\int \cos^m x \sin^n x\, dx$:

- if m and n are both even then use the double angle formulae,
- if either m or n is odd then use the Pythagorean identity and a substitution.

Powers of Secant and Tangent: There are three general methods for the integral

$$\int \sec^m x \tan^n x\, dx\,,$$

again depending on whether the constants m and n are odd or even. There are also two special cases which should be dealt with first.

When $m = 0$ and $n = 1$ the situation is trivial, viz:

$$\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\
&= -\log(\cos x) + C\,.
\end{aligned}$$

A very clever trick is required for the other special case when $m = 1$ and $n = 0$.

$$\begin{aligned}
\int \sec x\, dx &= \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\, dx \\
&= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\, dx \\
&= \log(\sec x + \tan x) + C\,.
\end{aligned}$$

Notice that in both special cases the result is a logarithmic function since the numerator of the integrand can be written as the derivative of the denominator.

6 The integrals of the tangent and secant functions:

$$\int \tan x\, dx = -\log(\cos x) + C$$

$$\int \sec x\, dx = \log(\sec x + \tan x) + C$$

Now for the general cases. If m and n are both even then separate out a factor of $\sec^2 x$ and substitute $u = \tan x$. The Pythagorean identity $1 + \tan^2 x = \sec^2 x$ may be required, particularly when $m = 0$.

WORKED EXERCISE: Find $\displaystyle\int \tan^4 x\,dx$.

SOLUTION:

$$\begin{aligned}
\int \tan^4 x\,dx &= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx && \text{(by Pythagoras)}\\
&= \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x - 1\,dx && \text{(by Pythagoras again)}\\
&= \int u^2\,du - \int \sec^2 x\,dx + \int 1\,dx && \text{where } u = \tan x\\
&= \tfrac{1}{3}u^3 - \tan x + x + C\\
&= \tfrac{1}{3}\tan^3 x - \tan x + x + C\,.
\end{aligned}$$

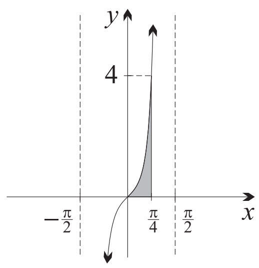
WORKED EXERCISE: Show that $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan^2 x\,dx = \tfrac{8}{15}$.

SOLUTION: Let $\displaystyle I = \int_0^{\frac{\pi}{4}} \sec^4 x \tan^2 x\,dx$

so $\displaystyle I = \int_0^{\frac{\pi}{4}} \sec^2 x\,(\tan^2 x + 1)\tan^2 x\,dx$ (by Pythagoras.)

Put $u = \tan x$,

then
$$\begin{aligned}
I &= \int_0^1 (u^2+1)u^2\,du\\
&= \int_0^1 u^4 + u^2\,du\\
&= \Big[\tfrac{1}{5}u^5 + \tfrac{1}{3}u^3\Big]_0^1\\
&= \tfrac{8}{15}\,.
\end{aligned}$$



If n is odd then factor out the term $\sec x \tan x$ and substitute $u = \sec x$. The Pythagorean identity may be required.

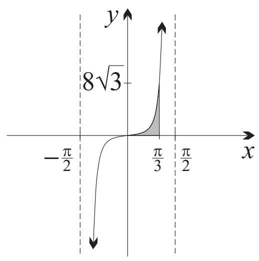
WORKED EXERCISE: Determine the value of $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan x\,dx$.

SOLUTION: Let $\displaystyle I = \int_0^{\frac{\pi}{3}} \sec^3 x \tan x\,dx$,

so $\displaystyle I = \int_0^{\frac{\pi}{3}} \sec^2 x \times \sec x \tan x\,dx$.

Put $u = \sec x$,

then
$$\begin{aligned}
I &= \int_1^2 u^2\,du\\
&= \Big[\tfrac{1}{3}u^3\Big]_1^2\\
&= \tfrac{7}{3}\,.
\end{aligned}$$



Whenever m is odd and n is even it is best to apply integrate by parts. Once again the Pythagorean identity may again be required.

WORKED EXERCISE: Find $\displaystyle\int \sec^3\, dx$.

SOLUTION:

$$
\begin{aligned}
\text{Let}\quad I &= \int \sec^3\, dx \\
&= \int \sec^2 x \times \sec x\, dx\,.
\end{aligned}
$$

Put $\quad u = \sec x \qquad$ and $\qquad v' = \sec^2 x$

so $\quad u' = \sec x \tan x \qquad$ and $\qquad v = \tan x$.

$$
\begin{aligned}
\text{Thus}\quad I &= \sec x \tan x - \int \sec x \tan^2 x\, dx \qquad \text{(by parts)} \\
&= \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx \qquad \text{(by Pythagoras)} \\
&= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx\,.
\end{aligned}
$$

$$
\begin{aligned}
\text{So}\quad I &= \sec x \tan x - I + \log(\sec x + \tan x) \qquad \text{(from the special case)} \\
\text{or}\quad 2I &= \sec x \tan x + \log(\sec x + \tan x)\,, \\
\text{hence}\quad I &= \tfrac{1}{2}\big(\sec x \tan x + \log(\sec x + \tan x)\big) + C\,.
\end{aligned}
$$

7 POWERS OF SECANT AND TANGENT: Given an integral of the form $\displaystyle\int \sec^m x \tan^n x\, dx$:

- if m and n are both even then factor out $\sec^2 x$ and substitute $u = \tan x$
- if n is odd then factor out the term $\sec x \tan x$ and substitute $u = \sec x$
- if m is odd and n is even then use integration by parts

Products to Sums: There are three standard formulae for converting products of trigonometric functions to sums. These will be familiar to some readers and are easily proved by expanding each right hand side.

8 PRODUCTS TO SUMS:

$$
\begin{aligned}
\sin A \cos B &= \tfrac{1}{2}\big(\sin(A - B) + \sin(A + B)\big) \\
\cos A \cos B &= \tfrac{1}{2}\big(\cos(A - B) + \cos(A + B)\big) \\
\sin A \sin B &= \tfrac{1}{2}\big(\cos(A - B) - \cos(A + B)\big)
\end{aligned}
$$

These formulae can be applied to simplify an integral, as in the following example.

WORKED EXERCISE: Find $\displaystyle\int \cos 3x \cos 2x\, dx$.

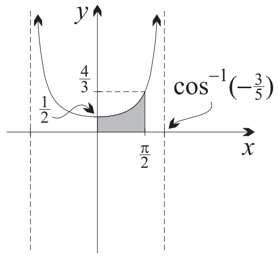
SOLUTION:

$$
\begin{aligned}
\int \cos 3x \cos 2x\, dx &= \tfrac{1}{2}\int \cos x + \cos 5x\, dx \qquad \text{(products to sums)} \\
&= \tfrac{1}{2}\sin x + \tfrac{1}{10}\sin 5x + C\,.
\end{aligned}
$$

The t-substitution: The t-substitution, namely $t = \tan\frac{x}{2}$, should be well known to readers, being part of the Mathematics Extension 1 course. Here it is applied to some harder integral problems.

WORKED EXERCISE: Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{4}{3+5\cos x}\,dx = \log 3$.

SOLUTION: Let $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{4}{3+5\cos x}\,dx$.



Put $t = \tan\frac{x}{2}$

so that $dx = \dfrac{2\,dt}{1+t^2}$

and $\cos x = \dfrac{1-t^2}{1+t^2}$,

then
$$I = \int_0^1 \frac{4}{3+5\,\frac{1-t^2}{1+t^2}} \times \frac{2\,dt}{1+t^2}$$
$$= \int_0^1 \frac{8}{8-2t^2}\,dt$$
$$= \int_0^1 \frac{4}{4-t^2}\,dt\,.$$

Let $\dfrac{4}{(2-t)(2+t)} = \dfrac{A}{2-t} + \dfrac{B}{2+t}$ (partial fractions)

then $A = 1$ and $B = 1$ (by the cover up method.)

Thus
$$I = \int_0^1 \frac{1}{2-t} + \frac{1}{2+t}\,dt$$
$$= \Big[\log(2+t) - \log(2-t)\Big]_0^1$$
$$= \log 3\,.$$

As a final note, take care with this method if the limits of integration include odd multiples of π since $\tan\frac{x}{2}$ is undefined there. Definite integrals of this type are dealt with in the last section of this chapter.

Exercise **2F**

1. Find:

(a) $\displaystyle\int \cos x\,dx$ (b) $\displaystyle\int \sin x\,dx$ (c) $\displaystyle\int \tan x\,dx$ (d) $\displaystyle\int \cot x\,dx$

2. Each of the following can be found with a substitution; either $u = \sin x$ or $u = \cos x$. You may also need to apply the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

(a) $\displaystyle\int \cos x \sin^2 x\,dx$ (c) $\displaystyle\int \sin^3 x\,dx$ (e) $\displaystyle\int \cos^5 x\,dx$

(b) $\displaystyle\int \cos^2 x \sin x\,dx$ (d) $\displaystyle\int \cos^3 x\,dx$ (f) $\displaystyle\int \sin^3 x \cos^3 x\,dx$

3. Use the double angle formulae to evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ (b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x\,dx$ (c) $\displaystyle\int_0^{\pi} \sin^2 x \cos^2 x\,dx$

4. Use the substitution $u = \tan x$ to find the following. You may also need to apply the Pythagorean identity $1 + \tan^2 x = \sec^2 x$.

(a) $\displaystyle\int \sec^2 x\,dx$ (b) $\displaystyle\int \tan^2 x\,dx$ (c) $\displaystyle\int \sec^4 x\,dx$ (d) $\displaystyle\int \tan^4 x\,dx$

5. Use the substitution $u = \sec x$ to help evaluate the following. You may also need to apply the Pythagorean identity $1 + \tan^2 x = \sec^2 x$.

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3 x\,dx$ (e) $\displaystyle\int_0^{\frac{\pi}{4}} \sec x \tan^3 x\,dx$

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \sec^3 x \tan x\,dx$ (d) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan x\,dx$ (f) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan^3 x\,dx$

DEVELOPMENT

6. Evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x \sin x\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 \cos x\,dx$ (e) $\displaystyle\int_0^{\pi} \sin^3 x \cos^2 x\,dx$

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \cos^3 x\,dx$ (d) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^5 x\,dx$ (f) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x \cos^3 x\,dx$

7. Determine:

(a) $\displaystyle\int \cos^4 x\,dx$ (b) $\displaystyle\int \sin^4 x\,dx$ (c) $\displaystyle\int \sin^4 x \cos^4 x\,dx$

8. Show that:

(a) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x \tan^2 x\,dx = \sqrt{3}$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan x\,dx = \tfrac{3}{4}$

(b) $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan^3 x\,dx = 2\tfrac{2}{9}$ (d) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^5 x\,dx = \tfrac{1}{4}(2\log 2 - 1)$

9. Use the t-substitution to help evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x}\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4 + 5\cos x}\,dx$ (c) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5 + 3\sin x}\,dx$

10. In each case use a suitable trigonometric substitution to evaluate the integral.

(a) $\displaystyle\int_0^1 \sqrt{1 - x^2}\,dx$ (b) $\displaystyle\int_0^1 x^3\sqrt{1 + x^2}\,dx$ (c) $\displaystyle\int_0^1 x^2\sqrt{1 - x^2}\,dx$

11. Let $I = \displaystyle\int \sin x \cos x\,dx\,.$

(a) Find I using a suitable substitution. (b) Find I by the double angle formulae.

(c) Show that the answers to parts (a) and (b) are equivalent.

12. (a) Use the t-substitution to show that $\displaystyle\int \sec x\,dx = \log\left(\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right) + C$

(b) Show that this answer and the result in Box 6 are equivalent.

13. Use integration by parts to find the following. You may also need to apply the Pythagorean identity $1 + \tan^2 x = \sec^2 x$.

(a) $\displaystyle\int \sec x \tan^2 x\,dx\,.$ (b) $\displaystyle\int \sec^3 x\,dx$ (c) $\displaystyle\int 8\sec^5 x\,dx$

14. Evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x \sec^2 x \, dx$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3 x + \tan x \, dx$

(b) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x \sec^4 x \, dx$ (d) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x - \cos^3 x \, dx$

15. Find these integrals by first converting the products to sums:

(a) $\displaystyle\int \sin 3x \cos x \, dx$ (b) $\displaystyle\int \cos 3x \sin x \, dx$ (c) $\displaystyle\int \cos 6x \cos 2x \, dx$

16. Evaluate these integrals by first converting the products to sums.

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \sin 3x \sin x \, dx$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos 4x \cos 2x \, dx$ (c) $\displaystyle\int_0^{\frac{\pi}{3}} \sin 4x \cos 2x \, dx$

17. Use the substitution $t = \tan \frac{x}{2}$ to determine:

(a) $\displaystyle\int \frac{1}{1 + \cos x} \, dx$ (b) $\displaystyle\int \frac{1}{1 + \sin x - \cos x} \, dx$ (c) $\displaystyle\int \frac{1}{3 \sin x + 4 \cos x} \, dx$

EXTENSION

18. Find $\displaystyle\int x \sec x \tan x \, dx \, .$

19. Repeat Question 16 for the integral $\displaystyle\int \sin^3 x \cos^3 x \, dx \, .$

20. In the chapter on complex numbers it was shown that $(\operatorname{cis}\theta)^3 = \operatorname{cis} 3\theta$. Use this result to help determine $\displaystyle\int \cos^3 \theta \, d\theta$.

21. Show that $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x \sec^3 x \, dx = \tfrac{1}{8}\big(3\sqrt{2} - \log(\sqrt{2} + 1)\big) \, .$

2G Reduction Formulae

The reader should already be familiar with sequences and series, such as the sequence of odd numbers,

$$1,\ 3,\ 5,\ 7,\ \ldots \qquad \text{or} \qquad u_n = 2n - 1 \, ,$$

or the powers of 2,

$$1,\ 2,\ 4,\ 8,\ \ldots \qquad \text{or} \qquad u_n = 2^{n-1} \, .$$

In this section we deal with sequences of integrals such as

$$\int_0^{\frac{\pi}{2}} \sin x \, dx \, , \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \, , \int_0^{\frac{\pi}{2}} \sin^3 x \, dx \, , \ \ldots \qquad \text{or} \qquad I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx \, .$$

Of particular interest are the equations which relate the terms of a sequence of integrals.

Thus, for example, if

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx \, ,$$

it can be shown that

$$I_n = \tfrac{n-1}{n} \times I_{n-2} \, .$$

Such equations are called *reduction formulae*, because they enable the index to be reduced, in this case from n to $n-2$. In practical terms, this means that if one of the integrals in the squence is known then other terms can be simply calculated from it without the need for further integration. Returning to the above example, since

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x\,dx = 1\,,$$

it follows that $I_3 = \frac{2}{3} I_1 = \frac{2}{3}$,

and $I_5 = \frac{4}{5} I_3 = \frac{8}{15}$.

This is obviously a significant saving of effort since it was not necessary to find the primitives of $\sin^3 x$ and $\sin^5 x$ in order to evaluate I_3 and I_5. It should now be clear that reduction formulae are of particular importance.

Identities: In a few cases the reduction formula can be generated by use of an identity.

WORKED EXERCISE: Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,dx$.

(a) Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$. (b) Evaluate I_1 and hence find I_5.

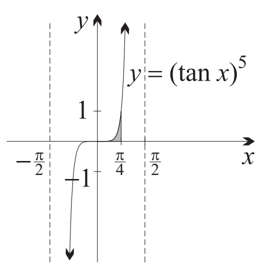
SOLUTION:

(a)
$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,(\sec^2 x - 1)\,dx \qquad \text{(by Pythagoras)} \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,\sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx \\ &= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2} \\ &= \frac{1}{n-1} - I_{n-2}\,. \end{aligned}$$

(b)
$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{4}} \tan x\,dx \\ &= \Big[-\log(\cos x)\Big]_0^{\frac{\pi}{4}} \\ &= \tfrac{1}{2}\log 2\,. \end{aligned}$$

Thus
$$\begin{aligned} I_3 &= \tfrac{1}{2} - I_1 \\ &= \tfrac{1}{2} - \tfrac{1}{2}\log 2\,, \end{aligned}$$

and
$$\begin{aligned} I_5 &= \tfrac{1}{4} - I_3 \\ &= \tfrac{1}{2}\log 2 - \tfrac{1}{4}\,. \end{aligned}$$



By Parts: Many examples of reduction formulae use integration by parts.

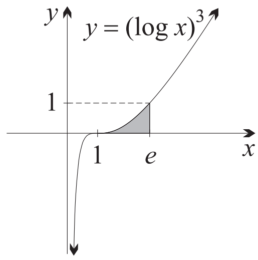
WORKED EXERCISE: (a) Let $I_n = \displaystyle\int_1^e (\log x)^n\,dx$ and show that $I_n = e - nI_{n-1}$.

(b) Find I_1 and hence show that $I_3 = 6 - 2e$.

SOLUTION:

(a)
$$\begin{aligned} I_n &= \int_1^e 1 \times (\log x)^n\, dx \\ &= \Big[x(\log x)^n\Big]_1^e - \int_1^e x \times \tfrac{n}{x}(\log x)^{n-1}\, dx \qquad \text{(by parts)} \\ &= (e-0) - n\int_1^e (\log x)^{n-1}\, dx \\ &= e - nI_{n-1}\,. \end{aligned}$$

(b)
$$\begin{aligned} I_1 &= \int_1^e \log x\, dx \\ &= \Big[x\log x - x\Big]_1^e \\ &= 0-(-1) \\ &= 1\,. \end{aligned}$$



Thus $\quad I_2 = e - 2I_1$
$\qquad\quad = e-2\,,$

and $\quad I_3 = e - 3I_2$
$\qquad\quad = e - 3(e-2)$
$\qquad\quad = 6 - 2e\,.$

By Parts with an Identity: In harder examples, integration by parts is used along with an identity.

WORKED EXERCISE: Let $I_n = \displaystyle\int_0^1 x^2(1-x^2)^n\, dx\,.$

(a) (i) Show that $x^2 = 1-(1-x^2)$. (ii) Hence show that $I_n = \frac{2n}{2n+3}I_{n-1}\,.$

(b) Evaluate I_0 and hence find $I_3\,.$

SOLUTION:

(a) (i) $RHS = 1 - 1 + x^2$
$\qquad\qquad = x^2 \quad$ as required.

(ii) Apply integration by parts to get:

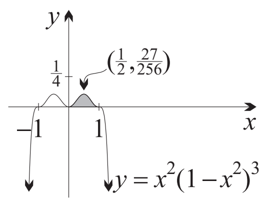
$$\begin{aligned} I_n &= \Big[\tfrac{1}{3}x^3(1-x^2)^n\Big]_0^1 - \int_0^1 \tfrac{1}{3}x^3 \times (-2nx)(1-x^2)^{n-1}\, dx \\ &= 0 + \tfrac{2n}{3}\int_0^1 x^2 \times x^2(1-x^2)^{n-1}\, dx \\ &= \tfrac{2n}{3}\int_0^1 x^2(1-x^2)^{n-1} - x^2(1-x^2)^n\, dx \qquad \text{by part (i)} \end{aligned}$$

so $\qquad I_n = \frac{2n}{3}I_{n-1} - \frac{2n}{3}I_n\,.$

thus $\quad \frac{2n+3}{3}I_n = \frac{2n}{3}I_{n-1}$

or $\qquad I_n = \frac{2n}{2n+3}I_{n-1}\,.$

(b) $\qquad I_0 = \displaystyle\int_0^1 x^2\, dx = \tfrac{1}{3}\,.$



Thus $\quad I_1 = \frac{2}{5}I_0 = \frac{2}{15}\,,$

$\qquad\quad I_2 = \frac{4}{7}I_1 = \frac{8}{105}\,,$

and $\quad I_3 = \frac{6}{9}I_2 = \frac{16}{315}\,.$

Exercise **2G**

1. (a) Given that $I_n = \displaystyle\int \tan^n x\, dx$, prove that $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}\,.$

(b) Hence show that $I_6 = \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$

2. (a) If $I_n = \displaystyle\int x^n e^x\, dx$, show that $I_n = x^n e^x - nI_{n-1}\,.$

(b) Hence show that $\displaystyle\int x^3 e^x\, dx = (x^3 - 3x^2 + 6x - 6)e^x + C\,.$

3. (a) If $I_n = \displaystyle\int_1^e x(\log x)^n\, dx$, show that $I_n = \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}.$

(b) Find I_0 and hence show that $I_4 = \frac{1}{4}(e^2 - 3)\,.$

4. Let $u_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n x\, dx\,.$

(a) Use integration by parts and the Pythagorean identity to prove that $u_n = \frac{n-1}{n}u_{n-2}\,.$

(b) Hence evaluate u_5.

DEVELOPMENT

5. Let $T_n = \displaystyle\int_0^{\frac{\pi}{4}} \sec^n x\, dx\,.$

(a) Show that $T_n = \dfrac{(\sqrt{2})^{n-2}}{n-1} + \dfrac{n-2}{n-1}T_{n-2}\,.$ (b) Deduce that $T_6 = \frac{28}{15}.$

6. Let $C_n = \displaystyle\int_0^{\frac{\pi}{2}} x^n \cos x\, dx$, where $n \ge 0\,.$

(a) Prove that $C_n = (\frac{\pi}{2})^n - n(n-1)C_{n-2}$, for $n \ge 2\,.$ (b) Hence evaluate C_6.

7. (a) If $I_n = \displaystyle\int_0^1 (1 - x^2)^n\, dx$, show that $I_n = \dfrac{2n}{2n+1}\, I_{n-1}$ for $n \ge 1$.

(b) Evaluate I_0 and hence find I_4.

8. (a) If $u_n = \displaystyle\int_0^1 x(1 - x^3)^n\, dx$, show that $u_n = \dfrac{3n}{3n+2}\, u_{n-1}.$

(b) Show that $u_0 = \frac{1}{2}$ and hence evaluate u_4.

9. Suppose that $J_n = \displaystyle\int \frac{x^n}{\sqrt{1-x^2}}\, dx\,.$

(a) Show that $J_n = \frac{1}{n}\left((n-1)J_{n-2} - x^{n-1}\sqrt{1-x^2}\right).$

[HINT: Do by parts with $u = x^{n-1}$ and $v' = \dfrac{x}{\sqrt{1-x^2}}$.]

(b) Hence determine $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\, dx\,.$

10. Let $u_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x\, dx.$

(a) Show that $u_n = \left(\frac{n-1}{n+2}\right) u_{n-2}$, for $n \ge 2\,.$

[HINT: Do by parts with $u = \sin^{n-1} x$ and $v' = \sin x \cos^2 x$.]

(b) Hence show that $u_4 = \frac{\pi}{32}\,.$

11. Let $T_n = \displaystyle\int_0^1 x^n \sqrt{1-x}\, dx$.

(a) Deduce the reduction formula $T_n = \dfrac{2n}{2n+3}\, T_{n-1}$. (b) Show that $T_3 = \frac{32}{315}$.

(c) Use the reduction formula to help prove by induction that $T_n = \dfrac{n!(n+1)!}{(2n+3)!}\, 4^{n+1}$.

12. Consider the integral $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x}}\, dx$.

(a) Show that $I_0 = 2\sqrt{2} - 2$.

(b) Show that $I_{n-1} + I_n = \displaystyle\int_0^1 x^{n-1}\sqrt{1+x}\, dx$.

(c) Use integration by parts to show that $I_n = \dfrac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$.

(d) Hence evaluate I_2.

13. (a) Show that $(1+t^2)^{n-1} + t^2(1+t^2)^{n-1} = (1+t^2)^n$.

(b) Put $P_n = \displaystyle\int_0^x \left(1+t^2\right)^n dt$. Use integration by parts and the result in part (a) to show that $P_n = \frac{1}{2n+1}\Big(\left(1+x^2\right)^n x + 2nP_{n-1}\Big)$.

(c) Hence determine P_4:

(i) by the reduction formula, (ii) by using the binomial theorem.

(d) Hence write $1 + \frac{4}{3}x^2 + \frac{6}{5}x^4 + \frac{4}{7}x^6 + \frac{1}{9}x^8$ in powers of $(1+x^2)$.

EXTENSION

14. Let $I_n = \displaystyle\int_0^1 (1-x^2)^n\, dx$ and $J_n = \displaystyle\int_0^1 x^2(1-x^2)^n\, dx$.

(a) Apply integration by parts to I_n to show that $I_n = 2n\, J_{n-1}$.

(b) Hence show that $I_n = \dfrac{2n}{2n+1}\, I_{n-1}$.

(c) Show that $J_n = I_n - I_{n+1}$, and hence deduce that $J_n = \dfrac{1}{2n+3}\, I_n$.

(d) Hence write down a reduction formula for J_n in terms of J_{n-1}.

15. For $n = 0, 1, 2, \ldots$ let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n\theta\, d\theta$.

(a) Show that $I_1 = \frac{1}{2}\ln 2$.

(b) Show that, for $n \ge 2$, $I_n + I_{n-2} = \dfrac{1}{n-1}$.

(c) For $n \ge 2$, explain why $I_n < I_{n-2}$, and deduce that $\dfrac{1}{2(n+1)} < I_n < \dfrac{1}{2(n-1)}$.

(d) Use the reduction formula in part (b) to find I_5, and hence deduce that $\frac{2}{3} < \ln 2 < \frac{3}{4}$.

2H Miscellaneous Integrals

As was stated in the chapter overview, integration is an art form and requires much practice. In particular, it is important to be able to recognise the different forms of integrals, and to quickly determine which method is appropriate to apply. To that end, this section has been included. The exercise contains a mixture of all integral types encountered so far. Some questions can be done using several of the techniques. It is up to the reader to determine which method is most efficient.

Exercise **2H**

1. Evaluate:

(a) $\displaystyle\int_{-1}^{1} \frac{x^2}{(5+x^3)^2}\,dx$ (c) $\displaystyle\int_{2}^{3} \frac{2x+2}{(x+3)(x-1)}\,dx$ (e) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{3\cos x}{\sin^4 x}\,dx$

(b) $\displaystyle\int_{0}^{\pi} x\sin x\,dx$ (d) $\displaystyle\int_{0}^{2} \frac{x-1}{x+1}\,dx$ (f) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{4x^2+1}}\,dx$

2. Find:

(a) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$ (d) $\displaystyle\int \frac{1}{2x^2+3x+1}\,dx$ (g) $\displaystyle\int \frac{1}{x^2+6x+25}\,dx$

(b) $\displaystyle\int \frac{1+x}{1+x^2}\,dx$ (e) $\displaystyle\int x^3\log x\,dx$ (h) $\displaystyle\int 3x\cos 3x\,dx$

(c) $\displaystyle\int \sin x\cos^4 x\,dx$ (f) $\displaystyle\int \sin^3 2x\,dx$ (i) $\displaystyle\int \frac{x}{\sqrt{4+x}}\,dx$

3. Show that:

(a) $\displaystyle\int_{0}^{1} x^2e^{-x}\,dx = 2-\tfrac{5}{e}$ (f) $\displaystyle\int_{2}^{4} \frac{x}{\sqrt{6x-8-x^2}}\,dx = 3\pi$

(b) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^3 x\cos^5 x\,dx = \tfrac{1}{24}$ (g) $\displaystyle\int_{0}^{1} \frac{\sqrt{x}}{1+x}\,dx = \tfrac{1}{2}(4-\pi)$

(c) $\displaystyle\int_{0}^{1} \frac{x}{(x+1)(x^2+1)}\,dx = \tfrac{1}{8}(\pi-2\log 2)$ (h) $\displaystyle\int_{0}^{\frac{\pi}{3}} \sec x\,dx = \log(2+\sqrt{3})$

(d) $\displaystyle\int_{0}^{\frac{1}{2}} (1-x^2)^{-\frac{3}{2}}\,dx = \tfrac{1}{\sqrt{3}}$ (i) $\displaystyle\int_{0}^{\frac{\pi}{4}} \sin 2x\cos 3x\,dx = \tfrac{1}{10}(3\sqrt{2}-4)$

(e) $\displaystyle\int_{0}^{1} \frac{1-x^2}{1+x^2}\,dx = \tfrac{\pi}{2}-1$ (j) $\displaystyle\int_{0}^{\pi} e^{-x}\cos x\,dx = \tfrac{1}{2}(1+e^{-\pi})$

DEVELOPMENT

4. (a) Find the rational numbers A, B and C such that

$$\frac{x-1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\,.$$

(b) Hence show that $\displaystyle\int_{0}^{1} \frac{x^3+x}{x^3+1}\,dx = 1-\tfrac{2}{3}\log 2.$

5. Use integration by parts to show that $\displaystyle\int x^3e^{-x^2}\,dx = -\tfrac{1}{2}e^{-x^2}(1+x^2)+C\,.$

6. Use integration by parts to evaluate $I = \displaystyle\int_{0}^{\frac{\pi}{4}} \sec^3 x\,dx\,.$

7. In each case let $t = \tan \frac{x}{2}$ in order to show that:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{3 + 5\cos x}\, dx = \tfrac{1}{4}\log 3$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\cos x - 2\sin x + 3}\, dx = \tfrac{\pi}{4}$

8. (a) Find the values of A, B, C, and D such that

$$\frac{4t}{(1+t)^2(1+t^2)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{Ct+D}{1+t^2}\,.$$

(b) Hence use the t-substitution to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\, dx\,.$

9. Use the substitution $u = \sqrt[6]{x}$ to show that $\displaystyle\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx = 11 - 6\log\tfrac{3}{2}\,.$

10. Find $\displaystyle\int \sqrt{a^2 - x^2}\, dx$ using:

(a) the substitution $\theta = \sin^{-1}\frac{x}{a}$, (b) integration by parts.

11. (a) Show that $\displaystyle\int_0^1 \frac{5 - 5x^2}{(1+2x)(1+x^2)}\, dx = \tfrac{1}{2}(\pi + \log\tfrac{27}{16})\,.$

(b) Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \cos x + 2\sin x}\, dx$ using the substitution $t = \tan\frac{x}{2}$.

12. (a) Find integers P and Q such that

$$8\sin x + \cos x - 2 = P(3\sin x + 2\cos x - 1) + Q(3\cos x - 2\sin x)\,.$$

(b) Hence find $\displaystyle\int \frac{8\sin x + \cos x - 2}{3\sin x + 2\cos x - 1}\, dx\,.$

13. (a) If $T_n = \displaystyle\int_0^{\pi} \sin^n x\, dx$, show that $T_n = \frac{n-1}{n}T_{n-2}$. (b) Hence show that $T_5 T_6 = \frac{\pi}{3}$.

14. (a) Let $I_n = \displaystyle\int_1^e (\log x)^n\, dx$ and show that $I_n = e - nI_{n-1}$. (b) Hence evaluate I_3.

EXTENSION

15. Let $I_n = \displaystyle\int_0^1 \frac{x^{n-1}}{(x+1)^n}\, dx\,,$ for $n = 1, 2, 3, \ldots$

(a) Show that $I_1 = \ln 2\,.$

(b) Use integration by parts to show that $I_{n+1} = I_n - \dfrac{1}{n\,2^n}\,.$

(c) The maximum value of $\dfrac{x}{x+1}$, for $0 \le x \le 1$, is $\frac{1}{2}$.
Use this fact to show that $I_{n+1} < \frac{1}{2}I_n\,.$

(d) Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}\,.$

(e) Use the reduction formula in part (b) and the inequality in part (d) to show that

$\frac{2}{3} < \ln 2 < \frac{17}{24}\,.$

16. Given that $\displaystyle\int_0^{\pi} \frac{1}{5 + 3\cos x}\, dx = \tfrac{\pi}{4}$, show that $\displaystyle\int_0^{\pi} \frac{\cos x + 2\sin x}{5 + 3\cos x}\, dx = \tfrac{1}{12}(16\log 2 - \pi)\,.$

17. (a) Use the substitution $u = t - t^{-1}$ to show that $\displaystyle\int \frac{1+t^2}{1+t^4}\,dt = \tfrac{1}{\sqrt{2}}\tan^{-1}\tfrac{\sqrt{2}(t^2-1)}{2t} + C$.

(b) Alternatively, use the result $(1+t^4) = (1+t^2)^2 - (\sqrt{2}\,t)^2$ and partial fractions to show that $\displaystyle\int \frac{1+t^2}{1+t^4}\,dt = \tfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\,t+1) + \tfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\,t-1) + C$.

18. Consider the two new functions $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$.
Show that $\displaystyle\int_0^{\log 2} \frac{1}{5\cosh x - 3\sinh x}\,dx = \tfrac{1}{2}\tan^{-1}\tfrac{1}{3}$.

2I Further Integration

Dummy Variables: In the case of definite integrals it does not matter what variable is used in the integrand, provided that the variable chosen is used consistently during the calculations. Thus for example, the three integrals below all have the same value, despite using different variables in the calculations.

$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3} \qquad \int_0^1 t^2\,dt = \left[\tfrac{1}{3}t^3\right]_0^1 = \tfrac{1}{3} \qquad \int_0^1 \theta^2\,d\theta = \left[\tfrac{1}{3}\theta^3\right]_0^1 = \tfrac{1}{3}$$

To be particularly absurd we could choose anything we like for the variable, such as a picture of an elephant from behind, . Thus:

$\int_0^1$  $^2\,d$  $= \left[\tfrac{1}{3}\right.$  $\left.^3\right]_0^1 = \tfrac{1}{3}$

which still gives the same value. In such cases as these, we say that the variable used is a *dummy variable*, since it is only seen in intermediate calculations and does not appear in the final answer.

In itself, the notion of dummy variables is not a particularly exciting result. However it is a feature that can be used to help prove some useful theorems about definite integrals. These theorems can then be used to help evaluate more complicated integrals.

Odd and Even Symmetry: If $f(x)$ exhibits odd or even symmetry then we may use the following to quickly simplify an integral:

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd,} \\ 2\displaystyle\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even.} \end{cases}$$

Clearly in the case where $f(x)$ is odd the result is immediate. In the case where the integrand is even let $F(x)$ be a primitive of $f(x)$. If the constant of integration is omitted then $F(0) = 0$. Hence

$$\begin{aligned} \int_{-a}^{a} f(x)\,dx &= 2\int_0^a f(x)\,dx \\ &= 2\big(F(a) - F(0)\big) \\ &= 2F(a)\,. \end{aligned}$$

Thus it is only necessary to evaluate the primitive at the upper limit.

WORKED EXERCISE: Evaluate $\displaystyle\int_{-1}^{1} \frac{x^2}{1+x^2}\,dx\,.$

SOLUTION: Clearly the integrand is even.

$$\text{Now}\qquad \int \frac{x^2}{1+x^2}\,dx = \int 1 - \frac{1}{1+x^2}\,dx$$
$$= x - \tan^{-1} x \qquad \text{(omitting the constant.)}$$

$$\text{Hence}\qquad \int_{-1}^{1} \frac{x^2}{1+x^2}\,dx = 2(1-\tan^{-1} 1)$$
$$= 2 - \tfrac{\pi}{2}\,.$$

WORKED EXERCISE: Evaluate $\displaystyle\int_{-1}^{1} f(x)\,dx$, where $f(x) = \dfrac{x^2}{\sqrt{2+x}} - \dfrac{x^2}{\sqrt{2-x}}\,.$

SOLUTION:
$$f(-x) = \frac{(-x)^2}{\sqrt{2+(-x)}} - \frac{(-x)^2}{\sqrt{2-(-x)}}$$
$$= \frac{x^2}{\sqrt{2-x}} - \frac{x^2}{\sqrt{2+x}}$$
$$= -f(x)\,.$$

Hence $f(x)$ is odd and thus $\displaystyle\int_{-1}^{1} f(x)\,dx = 0.$

9 ODD AND EVEN SYMMETRY: Let $F(x)$ be a primitive of $f(x)$ without constant, then:

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd,} \\ 2F(a) & \text{if } f(x) \text{ is even.} \end{cases}$$

Here is a proof of the case where $f(x)$ is even. The odd case is left as an exercise.

PROOF: Let $f(x)$ be even with primitive $F(x)$, then

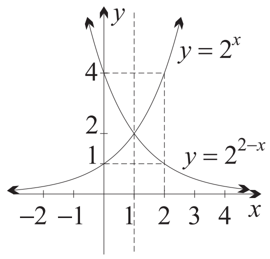
$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$
$$= -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)\,dx \quad \text{where } t = -x\,.$$

$$\text{Thus}\ \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-t)\,dt + \int_{0}^{a} f(x)\,dx \qquad \text{(reversing the limits)}$$
$$= \int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(x)\,dx \qquad \text{(since } f \text{ is even)}$$
$$= 2\int_{0}^{a} f(x)\,dx \qquad \text{(since } x \text{ and } t \text{ are dummy variables)}$$
$$= 2F(a)\,.$$

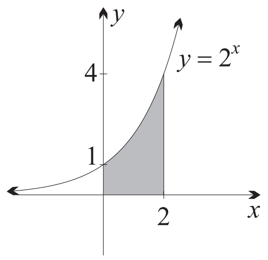
Reflection in the Line $x = a$: Integrals of the form $\displaystyle\int_{0}^{2a} f(x)\,dx$ can often be simplified by a reflection in the vertical line $x = a$. This is achieved by replacing x with $(2a - x)$. Such reflections are dealt with in more detail in the chapter on Graphs. The following example demonstrates the situation.

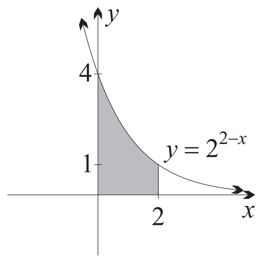
The graphs of $y = 2^x$ and $y = 2^{2-x}$ are to the right of the table of values.

x	-2	-1	0	1	2	3	4
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
2^{2-x}	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



It should be clear that the third line of the table of values is just the reverse of the second line. That is, there is symmetry about the middle value $x = 1$. The graphs also make it clear that $y = 2^{2-x}$ is obtained by reflecting $y = 2^x$ in the line $x = 1$.





The second pair of graphs should further make it clear that since a reflection is involved, the areas under the exponential curves between $x = 0$ and $x = 2$ are the same. That is:

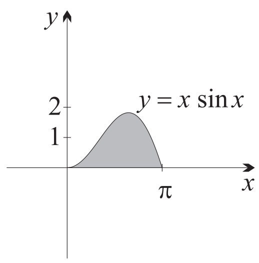
$$\int_0^2 2^x \, dx = \int_0^2 2^{2-x} \, dx \, .$$

Notice that in the integrand x has been replaced with $2a - x = 2 - x$, since $a = 1$.

WORKED EXERCISE: Determine $\displaystyle\int_0^\pi x \sin x \, dx$ by a suitable reflection.

SOLUTION: Let $I = \displaystyle\int_0^\pi x \sin x \, dx$. Reflect in the line $x = \frac{\pi}{2}$.

Thus replace x with $(\pi - x)$ to get:



$$\begin{aligned} I &= \int_0^\pi (\pi - x) \sin(\pi - x) \, dx \\ &= \int_0^\pi (\pi - x) \sin x \, dx \quad \text{(expanding } \sin(\pi - x)) \\ &= \int_0^\pi \pi \sin x \, dx - \int_0^\pi x \sin x \, dx \\ &= \int_0^\pi \pi \sin x \, dx - I \, . \end{aligned}$$

Hence $2I = \displaystyle\int_0^\pi \pi \sin x \, dx$

thus
$$\begin{aligned} I &= \tfrac{\pi}{2} \int_0^\pi \sin x \, dx \\ &= \tfrac{\pi}{2} \times 2 \\ &= \pi \, . \end{aligned}$$

The integral can also be done using integration by parts. The method of reflection in $x = \frac{\pi}{2}$ provides a geometric alternative which in some ways is simpler.

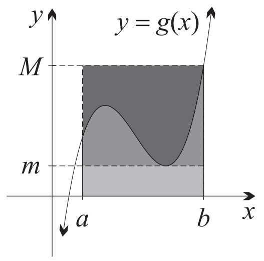
10	Reflection in a vertical line: The integral of a function $f(x)$ between $x = 0$ and $x = 2a$ is unchanged by a reflection in the line $x = a$, thus: $$\int_0^{2a} f(x)\,dx = \int_0^{2a} f(2a - x)\,dx\,.$$

The proof is straight forward, and again makes use of dummy variables.

Proof: Put $x = 2a - t$ so that $dx = (-1)\,dt$.
When $x = 0$, $t = 2a$, and when $x = 2a$, $t = 0$.

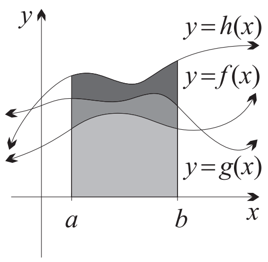
$$\begin{aligned}\text{Thus}\quad \int_0^{2a} f(x)\,dx &= \int_{2a}^{0} f(2a - t) \times (-1)\,dt \\ &= \int_0^{2a} f(2a - t)\,dt \quad \text{(reversing the limits)} \\ &= \int_0^{2a} f(2a - x)\,dx \quad \text{(since } x \text{ and } t \text{ are dummy variables)}\end{aligned}$$

Bounding: There are times when it is not necessary to know the exact value of an integral, just that it lies within certain bounds. For example, in the interval $a \le x \le b$ in the graph on the right the function $y = g(x)$ lies between its minimum value $y = m$ and its maximum value $y = M$. It should be clear then that the area under $y = g(x)$ in this interval is bigger than the lower rectangle and less than the upper rectangle, hence:



$$m(b - a) \le \int_a^b g(x)\,dx \le M(b - a)\,.$$

Again, by comparing areas, it should be clear in general that if $f(x) \le g(x) \le h(x)$ whenever $a \le x \le b$ then



$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \le \int_a^b h(x)\,dx\,.$$

Worked Exercise: (a) Prove that $\dfrac{1}{x+1} \le \dfrac{1}{x + \cos^2 x} \le \dfrac{1}{x}$ for $x > 0$.

(b) Hence show that $\log \frac{3}{2} \le \displaystyle\int_1^2 \frac{1}{x + \cos^2 x}\,dx \le \log 2$.

Solution: (a) Now $0 \le \cos^2 x \le 1$, so:

$$x \le x + \cos^2 x \le x + 1 \qquad \text{for all } x\,,$$

$$\text{hence}\qquad \frac{1}{x+1} \le \frac{1}{x + \cos^2 x} \le \frac{1}{x} \qquad \text{for } x > 0\,.$$

(b) Integrating all three parts:

$$\int_1^2 \frac{1}{x+1}\,dx \le \int_1^2 \frac{1}{x + \cos^2 x}\,dx \le \int_1^2 \frac{1}{x}\,dx$$

$$\text{so}\qquad \Big[\log(x+1)\Big]_1^2 \le \int_1^2 \frac{1}{x + \cos^2 x}\,dx \le \Big[\log x\Big]_1^2$$

$$\text{hence}\qquad \log \tfrac{3}{2} \le \int_1^2 \frac{1}{x + \cos^2 x}\,dx \le \log 2\,.$$

Improper Integrals and Limits: A definite integral is called an *improper integral* if the integrand is undefined at some point in the interval or if the interval of integration is unbounded. Thus

$$\int_1^2 \frac{1}{x-1}\,dx$$

is an improper integral since $\dfrac{1}{x-1}$ is undefined at $x=1$. The integral

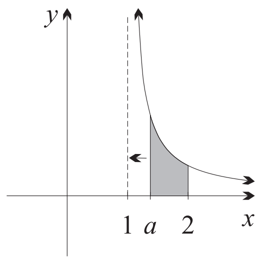
$$\int_0^\infty e^{-x}\,dx$$

is also an improper integral, in this case since the interval, $0 \le x < \infty$, is unbounded on the right hand side.

The value of an improper integral, if it exists, is found by taking the limit of a related integral.

WORKED EXERCISE: Find the value of $\displaystyle\int_1^2 \frac{1}{x-1}\,dx$, if it exists.

SOLUTION:



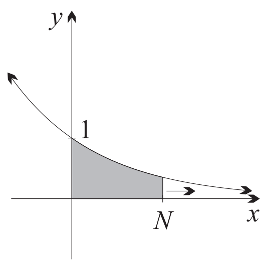
$$\begin{aligned}\text{Let}\quad I(a) &= \int_a^2 \frac{1}{x-1}\,dx\\ &= \Big[\log(x-1)\Big]_a^2\\ &= \log 1 - \log(a-1)\\ &= -\log(a-1)\,.\end{aligned}$$

$$\begin{aligned}\text{Thus}\quad \int_1^2 \frac{1}{x-1}\,dx &= \lim_{a\to1^+} I(a) \qquad \text{(if the limit exists)}\\ &= \lim_{a\to1^+} -\log(a-1)\end{aligned}$$

which is undefined. Hence $\displaystyle\int_1^2 \frac{1}{x-1}\,dx$ is undefined.

WORKED EXERCISE: Determine $\displaystyle\int_0^\infty e^{-x}\,dx$.

SOLUTION:



$$\begin{aligned}\text{Let}\quad I(N) &= \int_0^N e^{-x}\,dx\\ &= \Big[e^{-x}\Big]_0^N\\ &= 1 - e^{-N}\,.\end{aligned}$$

$$\begin{aligned}\text{Thus}\quad \int_0^\infty e^{-x}\,dx &= \lim_{N\to\infty} I(N) \qquad \text{(if the limit exists)}\\ &= \lim_{N\to\infty} 1 - e^{-N}\\ &= 1\,.\end{aligned}$$

Exercise **2I**

1. (a) Prove that $f(x) = \sqrt{2+x} - \sqrt{2-x}$ is odd and hence evaluate $\displaystyle\int_{-2}^{2} \sqrt{2+x} - \sqrt{2-x}\, dx$.

(b) Prove that $g(x) = e^x - e^{-x}$ is odd and hence evaluate $\displaystyle\int_{-1}^{1} e^x - e^{-x}\, dx$.

2. Use the formula $\displaystyle\int_0^{2a} f(x)\, dx = \int_0^{2a} f(2a-x)\, dx$ to help evaluate:

(a) $\displaystyle\int_0^1 x(1-x)^{10}\, dx$ (b) $\displaystyle\int_0^1 x^2\sqrt{1-x}\, dx$ (c) $\displaystyle\int_0^{\pi} x\sin^2 x\, dx$

3. (a) Use a graph to show that $\dfrac{1}{2x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$ for $\frac{\pi}{6} \le x \le \frac{\pi}{2}$.

(b) Hence show that $\frac{1}{2}\log 3 < \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx < \log 3$.

4. (a) Use a graph to show that $\dfrac{1}{x} \le \dfrac{\tan x}{x} \le \dfrac{\sqrt{3}}{x}$ for $\frac{\pi}{4} \le x \le \frac{\pi}{3}$.

(b) Hence show that $\log\frac{4}{3} < \displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x}\, dx < \sqrt{3}\log\frac{4}{3}$.

5. (a) Use a graph to show that $\tan x \le \sqrt{\tan x} \le 1$ for $0 \le x \le \frac{\pi}{4}$.

(b) Hence show that $\frac{1}{2}\log 2 < \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\, dx < \frac{\pi}{4}$.

6. (a) Explain why $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$ is an improper integral.

(b) Find $I(a) = \displaystyle\int_0^a \frac{dx}{\sqrt{1-x}}$, where $a < 1$.

(c) Determine $\displaystyle\lim_{a\to 1^-} I(a)$ and hence state the value of $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$.

7. (a) Explain why $\displaystyle\int_0^{\infty} \frac{dx}{4+x^2}$ is an improper integral.

(b) Find $I(N) = \displaystyle\int_0^N \frac{dx}{4+x^2}$.

(c) Determine $\displaystyle\lim_{N\to\infty} I(N)$ and hence state the value of $\displaystyle\int_0^{\infty} \frac{dx}{4+x^2}$.

DEVELOPMENT

8. (a) Use the substitution $u = -x$ to prove that $\displaystyle\int_{-a}^{0} f(x)\, dx = \int_0^a f(-x)\, dx$.

(b) Hence prove that $\displaystyle\int_{-a}^{a} f(x)\, dx = \int_0^a \big(f(x) + f(-x)\big)\, dx$.

(c) Use the theorem in part (b) to show that:

(i) $\displaystyle\int_{-a}^{a} f(x)\, dx = 0$ if $f(x)$ is odd

(ii) $\displaystyle\int_{-1}^{1} \frac{1}{1+e^{-x}}\, dx = 1$

(iii) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}\, dx = 2$

(iv) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x\sin^2 x}{1+e^x}\, dx = \frac{\pi}{4}$

9. Use the result of Box 10 with a suitable choice of a to evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$ (b) $\displaystyle\int_0^{\pi} \frac{\cos x}{3 + \sin^2 x}\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \sin 2x}\,dx$

10. (a) Show that $\displaystyle\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\,dx = \tfrac{\pi}{2}$.

(b) Use the substitution $x = \pi - u$ to find $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx$.

11. Evaluate the following improper integrals by applying an appropriate limit.

(a) $\displaystyle\int_0^{2} \frac{dx}{\sqrt{2 - x}}$ (c) $\displaystyle\int_0^{1} \frac{dx}{\sqrt{1 - x^2}}$ (e) $\displaystyle\int_0^{1} \sqrt{\frac{1 + x}{1 - x}}\,dx$

(b) $\displaystyle\int_0^{4} \frac{dx}{\sqrt{x}}$ (d) $\displaystyle\int_0^{1} \frac{dx}{\sqrt{2x - x^2}}$ (f) $\displaystyle\int_0^{e} (\log x)^2 dx$

12. Evaluate the following improper integrals by applying an appropriate limit.

(a) $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$ (c) $\displaystyle\int_1^{\infty} \frac{dx}{x^2 - 4x + 5}$ (e) $\displaystyle\int_{-\infty}^{0} e^x\,dx$

(b) $\displaystyle\int_0^{\infty} \frac{dx}{1 + x^2}$ (d) $\displaystyle\int_0^{\infty} x e^{-x^2}\,dx$ (f) $\displaystyle\int_0^{\infty} \frac{2\,dx}{e^x + e^{-x}}$

13. (a) Given that $0 < t < 1$, show that $\tfrac{1}{2} < \dfrac{1}{1 + t} < 1$.

(b) Hence, for $0 < x < 1$, show that $\tfrac{1}{2}x < \log(1 + x) < x$.

14. (a) Prove that $y = \tfrac{1}{2}(x + 1)$ is the tangent to $y = \sqrt{x}$ at $x = 1$.

(b) Hence explain why $\tfrac{1}{2}(x + 1) \geq \sqrt{x}$ for $x \geq 0$.

(c) Hence prove that $x + \sqrt{x} + 1 \leq \tfrac{3}{2}(x + 1)$.

(d) Hence show that $\displaystyle\int_1^{2} \frac{1}{x + \sqrt{x} + 1}\,dx \geq \tfrac{2}{3} \log \tfrac{3}{2}$.

15. (a) Given that $n > 2$ and $0 < x < 1$, show that $0 < x^n < x^2$.

(b) Hence, for $n > 2$ and $0 < x < 1$, show that $1 < \dfrac{1}{\sqrt{1 - x^n}} < \dfrac{1}{\sqrt{1 - x^2}}$.

(c) Deduce that $\tfrac{1}{2} < \displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^n}}\,dx < \tfrac{\pi}{6}$.

16. (a) Given that $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, explain why $\displaystyle\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\,dx$.

(b) Use the substitution $u = \pi - x$ to show that $\displaystyle\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx = \int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx$.

(c) Deduce that $\displaystyle\int_0^{\pi} e^{-\sin x}\,dx < \tfrac{\pi}{e}(e - 1)$.

17. (a) Show that $\displaystyle\int_0^{1} x^2(1 - x)^2\,dx = \tfrac{1}{30}$ and that $\displaystyle\int_0^{1} \frac{x^2(1 - x)^2}{x + 2}\,dx = 36 \ln \tfrac{3}{2} - \tfrac{175}{12}$.

(b) Explain why $\tfrac{1}{3}x^2(1 - x)^2 < \dfrac{x^2(1 - x)^2}{x + 2} < \tfrac{1}{2}x^2(1 - x)^2$, for $0 < x < 1$.

(c) Hence show that $\frac{2627}{6480} < \ln \frac{3}{2} < \frac{2628}{6480}$.

18. (a) Show that $\int_0^1 x^4(1-x)^4\,dx = \frac{1}{630}$ and that $\int_0^1 \dfrac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$.

(b) Explain why $\frac{1}{2}x^4(1-x)^4 < \dfrac{x^4(1-x)^4}{1+x^2} < x^4(1-x)^4$, for $0 < x < 1$.

(c) Hence show that $\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$.

19. Explain why $0 \le \left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$. A diagram may help.

20. (a) Let $I_n = \int x(\log x)^n\,dx$. Show that $I_n = \frac{1}{2}x^2(\log x)^n - \frac{1}{2}nI_{n-1}$.

(b) Given that $\lim_{x\to 0} x^n \log x = 0$ for $n > 0$, deduce a similar reduction formula for the improper integral $u_n = \int_0^1 x(\log x)^n\,dx$.

(c) Hence evaluate u_4.

21. (a) Use a suitable substitution to show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

(b) A function $g(x)$ has the property that $g(x) + g(a-x) = g(a)$. Use part (a) to prove that $\int_0^a g(x)\,dx = \frac{a}{2}g(a)$.

EXTENSION

22. Let $I_n(x) = \int_0^x t^n e^{-t}\,dt$, where n is a positive integer.

(a) Prove by induction that $I_n(x) = n!\left(1 - e^{-x}\sum_{j=0}^{n} \dfrac{x^j}{j!}\right)$, where $0! = 1$.

(b) Show that $0 \le \int_0^1 t^n e^{-t}\,dt \le \dfrac{1}{n+1}$.

(c) Hence show that $0 \le 1 - e^{-1}\sum_{j=0}^{n} \dfrac{1}{j!} \le \dfrac{1}{(n+1)!}$.

(d) Hence find $\lim_{n\to\infty} \left(\sum_{j=0}^{n} \dfrac{1}{j!}\right)$.

23. (a) Given that $e < 3$, show that $\int_0^1 x^n e^x\,dx < \dfrac{3}{n+1}$.

(b) Show by induction that for $n = 0, 1, 2, \ldots$ there exist integers a_n and b_n such that

$$\int_0^1 x^n e^x\,dx = a_n + b_n e.$$

(c) Let r be a positive rational number so that $r = \dfrac{p}{q}$, where p and q are positive integers. Show that for all integers a and b, either $|a + br| = 0$ or $|a + br| \ge \dfrac{1}{q}$.

(d) Prove that e is irrational.

24. Show that $\int_0^1 \dfrac{\log(1+x)}{1+x^2}\,dx = \frac{1}{8}\pi \log 2$.

Appendix — Table of Standard Integrals

Here is a table of standard integrals. A similar table is supplied on the last page of each examination paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Chapter Two

Exercise 2A (Page 53)

1(a) $\frac{1}{2}\sin 2x + C$ **(b)** $3\tan\frac{x}{3} + C$
(c) $\frac{1}{5}\tan^{-1}(\frac{x}{5}) + C$ **(d)** $\sin^{-1}(\frac{x}{2}) + C$
(e) $\log\left(x + \sqrt{x^2+3}\right) + C$
(f) $\log\left(x + \sqrt{x^2-5}\right) + C$
2(a) $2(e^2 - 1)$ **(b)** $\frac{1}{2}$ **(c)** $\frac{\pi}{8}$ **(d)** $\frac{\pi}{4}$
(e) $\log\left(\frac{3+\sqrt{5}}{1+\sqrt{5}}\right) = \log\left(\frac{1+\sqrt{5}}{2}\right)$ **(f)** $2\log 3$
3(a) $-\frac{1}{2}\log(1-x^2) + C$ **(b)** $\log(x + \tan x) + C$
(c) $\frac{1}{3}\log(1 + \sin 3x) + C$
4(a) $\frac{1}{3}\log 2$ **(b)** $\frac{1}{2}\log(\frac{e^2+1}{2})$ **(c)** $\log 2$
5(a) $\frac{\pi}{3\sqrt{3}}$ **(b)** $\frac{\pi}{18}$ **(c)** $\frac{1}{2}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \log(\sqrt{2}+1)$
(d) $\frac{1}{\sqrt{5}}\log\left(\frac{15+7\sqrt{5}}{5+\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\log(2+\sqrt{5})$
6(a) $x + \log(x-1) + C$ **(b)** $x - 2\log(x+1) + C$
(c) $x + 2\log(x-1) + C$
7(a) $1 - \log 4$ **(b)** $1 - \frac{1}{4}\log 5$ **(c)** $\pi - 1$
8(a) $\frac{\pi}{3} - \frac{1}{2}$ **(b)** $\frac{\pi}{4} + \log 2$ **(c)** $\frac{1}{4}(\pi - \log 4)$
(d) $\frac{\pi}{8} + \frac{1}{2}\log 2$
10(a) $\frac{x^3}{3} - \frac{x^2}{2} + x - \log(x+1) + C$
(b) $\frac{1}{2}\left(x^2 - \log(x^2+1)\right) + C$
(c)(i) $\frac{x^3}{3} + \frac{x^2}{2} + x + \log(x-1) + C$
(ii) $\frac{x^3}{3} - x + \tan^{-1}x + C$ **(iii)** $x - \log(1+e^x) + C$
(iv) $\frac{1}{3}(2x-8)\sqrt{2+x}+C$ **(v)** $-\frac{2}{3}(2+x)\sqrt{1-x}+C$
(vi) $\frac{1}{2}x^2 - 2\log(x^2+4) + C$
11(a) $\log(e+e^{-1})$ **(b)** $\frac{1}{2}\log\left(\frac{e^2+1}{2}\right)$ **(c)** $\frac{\pi}{12} + \log 2$
12(a) $\frac{1}{2}x^2 + \log(x+1) + C$ **(b)** $\frac{1}{3}x^3 + 3\log(x-2) + C$
(c) $x + \log(1+x^2) + C$
13 $2\log(1+\sqrt{x}) + C$

Exercise 2B (Page 57)

1(a)(i) $-\frac{1}{2}\log(1-x^2) + C$ **(ii)** $\log(1+\sin x) + C$
(iii) $\log(\log x) + C$ **(b)(i)** $\frac{1}{2}\left(\log(e^2+1) - \log 2\right)$
(ii) $\frac{1}{3}\log 2$ **(iii)** $\frac{1}{2}\log 3$
2(a)(i) $2e^{x^3} + C$ **(ii)** $e^{\tan x} + C$ **(iii)** $-e^{\frac{1}{x}} + C$
(b)(i) $\frac{1}{2}(e-1)$ **(ii)** $e - 1$ **(iii)** $2e(e-1)$
3(a) $\frac{1}{5}(x^2+1)^5 + C$ **(b)** $\frac{1}{7}(1+x^3)^7 + C$
(c) $-\frac{2}{1+x^3} + C$ **(d)** $\frac{1}{2(x^2-3)^4} + C$
(e) $\sqrt{x^2-2} + C$ **(f)** $\frac{1}{2}\sqrt{1+x^4} + C$
4(a) $\frac{-1}{2\sin^2 x} + C$ **(b)** $\frac{-1}{1+\tan x} + C$ **(c)** $\frac{1}{3}(\log x)^3 + C$
(d) $2\sin\sqrt{x} + C$ **(e)** $\frac{1}{2}\tan^{-1}x^2 + C$
(f) $\sin^{-1}x^3 + C$
5(a) $\frac{7}{4}$ **(b)** $2 - \sqrt{3}$ **(c)** $3(\sqrt{3} - \sqrt{2})$
(d) $\frac{1}{5}$ **(e)** $\frac{1}{3}$ **(f)** 2
6(a) $-\frac{1}{42}$ **(b)** Begin by writing $x = (x-1) + 1$.
7(a) $\frac{2}{15}(3x-2)(1+x)\sqrt{1+x} + C$
(b) $2\left(\sqrt{x} - \log(1+\sqrt{x})\right) + C$
(c) $4\left(x^{\frac{1}{4}} - \frac{1}{2}\sqrt{x} + \frac{1}{3}x^{\frac{3}{4}} - \log(1 + x^{\frac{1}{4}})\right) + C$
(d) $\tan^{-1}\sqrt{e^{2x}-1} + C$
8(a) $\frac{1}{9}$ **(b)** $\frac{128}{15}$ **(c)** $4 + 10\log\frac{5}{7}$ **(d)** $\frac{\pi}{12}$
9(a) $\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + C$ **(b)** $\frac{2}{3}(x-2)\sqrt{x+1} + C$
10(a) $\frac{x}{\sqrt{1+x^2}} + C$ **(b)** $\frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$
(c) $-\frac{\sqrt{25-x^2}}{25x} + C$ **(d)** $-\frac{1}{x}\sqrt{1+x^2} + C$
11(a) $\frac{2}{3}$ **(b)** Begin by writing $x^3 = x(x^2+1) - x$.
12(b) Begin by writing $x^2 = 1 - (1-x^2)$.
13(b) The region is half a segment.
14(a) $\tan^{-1}\sqrt{x^2-1} + C_1$ **(b)** $\tan^{-1}\sqrt{x^2-1} + C_2$
15(a) $\frac{\sqrt{3}}{8} - \frac{\sqrt{\epsilon(4+\epsilon)}}{4(2+\epsilon)}$ **(b)** $\frac{\sqrt{3}}{8}$

Exercise 2C (Page 64)

1(a) $\frac{1}{x-1} - \frac{1}{x+1}$ **(b)** $\frac{1}{3(x-4)} - \frac{1}{3(x-1)}$ **(c)** $\frac{2}{x-3} + \frac{2}{x+3}$
(d) $\frac{2}{x-2} - \frac{1}{x-1}$ **(e)** $\frac{1}{5(x-2)} + \frac{4}{5(x+3)}$ **(f)** $\frac{1}{x-1} + \frac{2-x}{x^2+3}$
2(a) $\ln(x-4) - \ln(x-2) + C$
(b) $2\ln(x+1) - 2\ln(x+3) + C$
(c) $4\log(x-2) - \log(x-1) + C$
(d) $3\log(x-1) - \log(x+3) + C$
(e) $\log(x+1) + \log(2x+3) + C$
(f) $2\log(x+1) + 3\log(2x-3) + C$
3(a) $\frac{1}{4}\log\frac{3}{2}$ **(b)** $\log 2$ **(c)** $\log\frac{14}{3}$ **(d)** $\frac{1}{2}\log 2$
4(a) $\log(x-2) - 2\tan^{-1}x + C$
(b) $\log(2x+1) - \frac{1}{2}\log(x^2+3) + C$
(c) $\tan^{-1}x + 3\log x - \log(x^2+1) + C$
5(a) $\frac{\pi}{4} - \log\frac{3}{2}$ **(b)** $\pi + \log 2$ **(c)** $\log 4 - \frac{3}{2}\log 3$
6(a) $5\log(x-1) + 7\log(x-2) - 12\log(2x-3) + C$
(b) $\frac{3}{2}\log(x) - 5\log(x-2) + \frac{7}{2}\log(x-4) + C$
7(a) $\frac{5}{3}\log 3 - \log 2$ **(b)** $2\log 3 - 8\log 2$
8(a)(i) $A = 2,\ B = 1,\ C = -3$
(ii) $2x + \log(x-1) - 3\log(x+2) + C$
(b)(i) $x + \log(x-2) - 2\log(x+1) + C$
(ii) $3x + 2\log(x+4) + \log(x-5) + C$
9(a)(i) $A = 12,\ B = 2$
(ii) $3x + 12\log(x-2) - \frac{2}{x-2} + C$
(b)(i) $A = 23,\ B = 10,\ C = -23,\ D = 13$
(ii) $23\log\left(\frac{x-1}{x-2}\right) - \frac{10}{x-1} - \frac{13}{x-2} + C$
10(a)(i) $A = 1,\ B = -1,\ C = 2,\ D = -1$
(ii) $\log 3 + \log 2 - \frac{1}{2}$ **(b)** $12 + \log 2$
12(a) $A = 0,\ B = 1,\ C = 0,\ D = 2$
13(a) $x + \log(x-1) - \log(x+1) + C$
(b) $x + 2\log(x-1) - \log x + C$
(c) $x - \tan^{-1}x + \log x - \frac{1}{2}\log(x^2+1) + C$
(d) $x + 9\log(x-3) - 4\log(x-2) + C$

(e) $\frac{1}{2}x^2 - x + 5\log(x) - 4\log(x+1) + C$
(f) $\frac{1}{3}x^3 + \frac{3}{2}x^2 + 7x + 16\log(x-2) - \log(x-1) + C$

Exercise **2D** (Page 68)

1(a) $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$ **(b)** $\log(x + \sqrt{9+x^2}) + C$
(c) $\sin^{-1}\frac{x}{3} + C$ **(d)** $\log(x + \sqrt{x^2-9}) + C$
(e) $\frac{1}{6}\Big(\log(x-3) - \log(x+3)\Big) + C$
(f) $\frac{1}{6}\Big(\log(3+x) - \log(3-x)\Big) + C$
2(a) $\frac{1}{2}\Big(\log(x+1) - \log(x+3)\Big) + C$
(b) $\tan^{-1}(x+2) + C$
(c) $\log(x - 3 + \sqrt{x^2-6x+13}\,) + C$
(d) $\log(x + 4 + \sqrt{x^2+8x+12}\,) + C$
(e) $\sin^{-1}\frac{x-4}{5} + C$
(f) $\frac{1}{2}\log\left(x + 1 + \sqrt{x^2+2x+\frac{3}{2}}\right) + C$
3(a) $\frac{\pi}{8}$ **(b)** π **(c)** $\frac{\pi}{6}$ **(d)** $\frac{\pi}{2}$ **(e)** $\log 3$ **(f)** $\log 3$
4(a) $\log(x^2+2x+2) - \tan^{-1}(x+1) + C$
(b) $\frac{1}{2}\log(x^2+2x+10) - \frac{1}{3}\tan^{-1}\frac{x+1}{3} + C$
(c) $\sqrt{(x+1)^2+9} - \log\left(x+1+\sqrt{(x+1)^2+9}\right)$
(d) $\sqrt{x^2-2x-4} + 4\log\left(x-1+\sqrt{x^2-2x-4}\right)$
(e) $-\sqrt{6x-x^2} + 3\sin^{-1}\frac{x-3}{3} + C$
(f) $-\sqrt{4-2x-x^2} + 2\sin^{-1}\frac{x+1}{\sqrt{5}} + C$
5(a) $\frac{1}{2}\log 2 + \frac{\pi}{8}$ **(b)** $\frac{1}{4}(3\pi - \log 4)$ **(c)** $\log 2 - \frac{\pi}{4}$
(d) $2 - \sqrt{3} - \frac{\pi}{6}$ **(e)** $3\log(3+2\sqrt{2}\,) - 4\sqrt{2}$
(f) $\log\left(1 + \sqrt{\frac{2}{3}}\right) + \sqrt{6} - 1$
6(a) $\sqrt{x^2-1} - \log\left(x + \sqrt{x^2-1}\right) + C$
(b) $\sin^{-1}x - \sqrt{1-x^2} + C$
(c) $\sqrt{6+x-x^2} + \frac{5}{2}\sin^{-1}\frac{2x-1}{5} + C$
7(a) $\frac{\pi}{3} + \sqrt{3} - 2$ **(b)** $3\sin^{-1}\frac{1}{3}$
(c) $2\sqrt{2} - \sqrt{3} + \log\left(\frac{2+\sqrt{3}}{3+2\sqrt{2}}\right)$
8(a) $\frac{x}{\sqrt{4x-x^2}}$ is undefined at $x = 0$.

Exercise **2E** (Page 72)

1(a) $e^x(x-1) + C$ **(b)** $-e^{-x}(x+1) + C$
(c) $\frac{1}{9}e^{3x}(3x+2) + C$ **(d)** $x\sin x + \cos x + C$
(e) $-\frac{1}{2}(x-1)\cos 2x + \frac{1}{4}\sin 2x + C$
(f) $(2x-3)\tan x + 2\log(\cos x) + C$
2(a) π **(b)** $\frac{\pi}{2} - 1$ **(c)** $\frac{\pi}{4} - \frac{1}{2}\log 2$ **(d)** $\frac{1}{4}(e^2+1)$
(e) e^{-1} **(f)** $1 + e^{-2}$
3(a) $x(\log x - 1) + C$ **(b)** $2x(\log x - 1) + C$
(c) $x\cos^{-1}x - \sqrt{1-x^2} + C$
4(a) $\frac{\pi}{4} - \frac{1}{2}\log 2$ **(b)** 1 **(c)** $\frac{1}{2}$
5(a) $\frac{1}{4}x^2(2\log x - 1) + C$ **(b)** $\frac{1}{9}x^3(3\log x - 1) + C$
(c) $-\frac{1}{x}(\log x + 1) + C$
6(a) $(2 - 2x + x^2)e^x + C$
(b) $x^2\sin x + 2x\cos x - 2\sin x + C$
(c) $x(\log x)^2 - 2x\log x + 2x + C$
7(a) $-\frac{1}{42}$ **(b)** $\frac{4}{15}(1+\sqrt{2}\,)$ **(c)** $\frac{128}{15}$
8(a) $\frac{1}{2}e^x(\cos x + \sin x) + C$
(b) $-\frac{1}{2}e^{-x}(\cos x + \sin x) + C$
9(a) $\frac{1}{5}(e^\pi - 2)$ **(b)** $\frac{1}{5}(e^{\frac{\pi}{4}} + 2)$
10(a) $\frac{1}{2\sqrt{3}}(\pi - \sqrt{3})$ **(b)** $\frac{\sqrt{3}\pi}{2}$ **(c)** $\pi - 2$
12(a) $\frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}(\frac{x}{a})\right) + C$
(b) $x\log(x + \sqrt{x^2+a^2}\,) - \sqrt{x^2+a^2} + C$
(c) $x\log(x + \sqrt{x^2-a^2}\,) - \sqrt{x^2-a^2} + C$
13(a) $\frac{1}{4}x^2(2\log x - 1) + C$
(b) $\frac{1}{4}x^2\left(2(\log x)^2 - 2\log x + 1\right) + C$
15(a) $\frac{1}{32}(\sin 4x - 4x\cos 4x + 8x\cos 2x - 4\sin 2x) + C$
(b) $\frac{1}{18}(3x\sin 3x + \cos 3x + 9x\sin x + 9\cos x) + C$
(c) $\frac{1}{4}e^x(\sin 3x - 3\cos 3x + 5\sin x - 5\cos x) + C$
16(a) $\frac{1}{48}(3\sqrt{3} - \pi)$ **(b)** $\frac{1}{12}(\pi + 2\log 2 - 2)$

Exercise **2F** (Page 78)

1(a) $\sin x + C$ **(b)** $-\cos x + C$ **(c)** $-\log(\cos x) + C$
(d) $\log(\sin x) + C$
2(a) $\frac{1}{3}\sin^3 x + C$ **(b)** $-\frac{1}{3}\cos^3 x + C$
(c) $\frac{1}{3}\cos^3 x - \cos x + C$ **(d)** $\sin x - \frac{1}{3}\sin^3 x + C$
(e) $\frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + \sin x + C$
(f) $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$
3(a) $\frac{\pi}{4}$ **(b)** $\frac{\pi}{12}$ **(c)** $\frac{\pi}{8}$
4(a) $\tan x + C$ **(b)** $\tan x - x + C$
(c) $\frac{1}{3}\tan^3 x + \tan x + C$ **(d)** $\frac{1}{3}\tan^3 x - \tan x + x + C$
5(a) $\sqrt{2} - 1$ **(b)** $\frac{1}{27}(8\sqrt{3} - 9)$ **(c)** $\frac{1}{2}(1 - \log 2)$
(d) $\frac{4}{3}$ **(e)** $\frac{1}{3}(2 - \sqrt{2}\,)$ **(f)** $\frac{58}{15}$
6(a) $\frac{1}{4}$ **(b)** $\frac{11}{24}$ **(c)** $\frac{9}{64}$ **(d)** $\frac{53}{480}$ **(e)** $\frac{4}{15}$ **(f)** $\frac{7}{60\sqrt{2}}$
7(a) $\frac{1}{32}(\sin 4x + 8\sin 2x + 12x) + C$
(b) $\frac{1}{32}(\sin 4x - 8\sin 2x + 12x) + C$
(c) $\frac{1}{1024}(24x - 8\sin 4x + \sin 8x) + C$
9(a) 1 **(b)** $\frac{1}{3}\log 2$ **(c)** $\frac{\pi}{4}$
10(a) $\frac{\pi}{4}$ **(b)** $\frac{2}{15}(1 + sqrt2)$ **(c)** $\frac{\pi}{16}$
11(a) $\frac{1}{2}\sin^2 x + C_1$ **(b)** $-\frac{1}{4}\cos 2x + C_2$
13(a) $\frac{1}{2}\Big(\sec x\tan x - \log(\sec x + \tan x)\Big) + C$
(b) $\frac{1}{2}\Big(\sec x\tan x + \log(\sec x + \tan x)\Big) + C$
(c) $\sec x\tan x(2\sec^2 x + 3) + 3\log(\sec x + \tan x) + C$
14(a) $\frac{1}{2}$ **(b)** $\frac{4}{3}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{\sqrt{3}}{4}$
15(a) $-\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$
(b) $-\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + C$
(c) $\frac{1}{16}\sin 8x + \frac{1}{8}\sin 4x + C$
16(a) $\frac{1}{4}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{3}{8}$
17(a) $\tan\frac{x}{2} + C$ **(b)** $\log\left(\frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right) + C$

(c) $\frac{1}{5}\log\left(\frac{1+2\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right)+C$
18 $x\sec x-\log(\sec x+\tan x)+C$
20 $\frac{1}{3}\sin 3\theta+\sin^3\theta+C$

Exercise **2G** (Page 82)

3(b) $\frac{1}{2}(e^2-1)$
4(b) $\frac{8}{15}$
6(b) $(\frac{\pi}{2})^6-30(\frac{\pi}{2})^4+360(\frac{\pi}{2})^2-720$
7(b) $I_0=1,\ I_4=\frac{128}{315}$
8(b) $u_4=\frac{243}{1540}$
9(b) $J_2=-\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x+C$
12(d) $\frac{1}{15}(14\sqrt{2}-16)$
13(d) $\frac{1}{9}\Big((1+x^2)^4+\frac{8}{7}(1+x^2)^3+\frac{48}{35}(1+x^2)^2$
$+\frac{192}{105}(1+x^2)+\frac{384}{105}\Big)$
14(d) $J_n=\dfrac{2n}{2n+3}\,J_{n-1}$
15(d) $I_5=\frac{1}{4}\,(2\ln 2-1)$

Exercise **2H** (Page 85)

1(a) $\frac{1}{36}$ **(b)** π **(c)** $\log\frac{12}{5}$ **(d)** $2-2\log 3$ **(e)** $2\sqrt{2}-1$
(f) $\frac{1}{2}\log(2+\sqrt{5}\,)$
2(a) $\sqrt{1+x^2}+C$ **(b)** $\tan^{-1}x+\frac{1}{2}\ln(1+x^2)+C$
(c) $-\frac{1}{5}\cos^5 x+C$ **(d)** $\log\left(\frac{2x+1}{x+1}\right)+C$
(e) $\frac{1}{4}x^4\log x-\frac{1}{16}x^4+C$ **(f)** $\frac{1}{6}\cos^3 2x-\frac{1}{2}\cos 2x+C$
(g) $\frac{1}{4}\tan^{-1}\frac{x+3}{4}+C$ **(h)** $x\sin 3x+\frac{1}{3}\cos 3x+C$
(i) $\frac{2}{3}(x-8)\sqrt{4+x}+C$
4(a) $A=-\frac{2}{3}\,,\ B=\frac{2}{3}\,,\ C=-\frac{1}{3}$
6 $\frac{1}{\sqrt{2}}+\frac{1}{2}\log(1+\sqrt{2})$
8(a) $A=0,\ B=-2,\ C=0,\ D=2$ **(b)** $\frac{\pi}{2}-1$
10 $\frac{1}{2}a^2\sin^{-1}\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$
11(b) $\frac{1}{10}(\pi+\log\frac{27}{16})$
12(a) $P=2,\ Q=-1$
(b) $2x-\log(3\sin x+2\cos x-1)+C$
14(b) $6-2e$

Exercise **2I** (Page 92)

1(a) 0 **(b)** 0
2(a) $\frac{1}{132}$ **(b)** $\frac{16}{105}$ **(c)** $\frac{\pi^2}{4}$
6(a) The integrand is undefined at $x=1$.
(b) $2(1-\sqrt{1-a})$ **(c)** 2
7(a) The interval is unbounded.
(b) $\frac{1}{2}\tan^{-1}\left(\frac{N}{2}\right)$ **(c)** $\frac{\pi}{4}$
9(a) $\frac{\pi}{4}$ **(b)** 0 **(c)** $1-\frac{\pi}{4}$
10(b) $\frac{\pi^2}{4}$
11(a) $2\sqrt{2}$ **(b)** 4 **(c)** $\frac{\pi}{2}$ **(d)** $\frac{\pi}{2}$ **(e)** $1+\frac{\pi}{2}$ **(f)** e
12(a) 1 **(b)** $\frac{\pi}{2}$ **(c)** $\frac{3\pi}{4}$ **(d)** $\frac{1}{2}$ **(e)** 1 **(f)** $\frac{\pi}{2}$

20(b) $u_n=-\frac{n}{2}u_{n-1}$ **(c)** $\frac{3}{4}$
22(d) e

