The University of Sydney

School of Mathematics and Statistics

MATH1901/1906

Differential Calculus (Advanced)

June 2014 Lecturer: J Parkinson

Time Allowed: **One and a half hours**

Family Name: ....

Other Names: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Approved non-programmable calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Marker's use only	

Extended Answer Section

*There are **four** questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

1. (a) (*i*) Write the complex number $-\sqrt{2}+\sqrt{2}i$ in polar form.

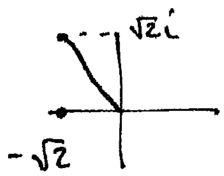
(*ii*) Calculate $(-\sqrt{2}+\sqrt{2}i)^{12}$, expressing your final answer in Cartesian form.

(*iii*) Find all solutions $w \in \mathbb{C}$ to the equation

$$w^3 = -\sqrt{2}+\sqrt{2}i,$$

expressing your answers in polar form.

Soln: (a) (i)



$|-\sqrt{2}+\sqrt{2}i| = \sqrt{2+2} = 2.$

$\arg = \frac{3\pi}{4}$

So $-\sqrt{2}+\sqrt{2}i = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(ii) $(-\sqrt{2}+\sqrt{2}i)^{12} = 2^{12}\,\mathrm{cis}\left(12 \times \frac{3\pi}{4}\right)$

$= 2^{12}\,\mathrm{cis}\left(\frac{9\pi}{\cancel{4}}\right) = 2^{12}\,\mathrm{cis}(\pi)$

$= -2^{12}.$

(iii) $w = r(\cos\theta + i\sin\theta)$

$r^3(\cos 3\theta + i\sin 3\theta) = 2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$r^3 = 2 \Rightarrow r = \sqrt[3]{2}$; $3\theta = \frac{3\pi}{4} + 2k\pi$

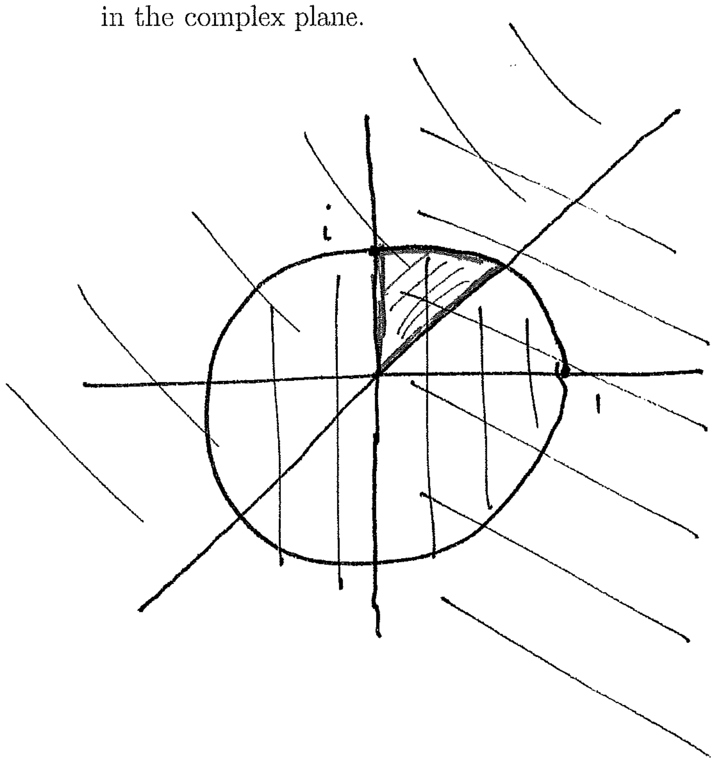
$\Rightarrow \quad w = \sqrt[3]{2}\,\mathrm{cis}\left(\frac{\frac{3\pi}{4}+2k\pi}{3}\right) \quad k = 0, 1, 2.$

(b) Shade the region

$$\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0 \text{ and } |z - i| \leq |z - 1| \text{ and } |z| \leq 1\}$$

in the complex plane.

(b)



$z = x + iy$

$|z - i| \leq |z - 1|$

$\Rightarrow x^2 + (y-1)^2 \leq (x-1)^2 + y^2$

$-2y + 1 \leqslant -2x + 1$

$y \geqslant x$.

(c) By writing $z = x + iy$, find all solutions $z \in \mathbb{C}$ of the equation

$$z^2 - 2\overline{z} + 1 = 0.$$

$(x+iy)^2 - 2(x-iy)+1 = 0$

$x^2 - y^2 + 2xyi - 2x + 2yi + 1 = 0$

$(x^2 - y^2 - 2x+1) + (2xy + 2y)i = 0 + 0i$

So $\quad x^2 - y^2 - 2x + 1 = 0 \qquad (1)$

$\quad 2y(x+1) = 0 \qquad (2)$

$(2) \Rightarrow \quad y = 0 \quad$ or $\quad x = -1$.

If $y=0$, $(1) \Rightarrow \quad x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 = 0 \Rightarrow \quad x = 1$.

If ~~y~~ $x=-1$, $(1) \Rightarrow \quad 1 - y^2 + 2 + 1 = 0 \Rightarrow y = \pm 2$.

Solns are: $\boxed{z = 1 + 0i,\ -1+2i,\ -1-2i.}$

(d) Find all roots of the polynomial $p(z) = z^4 + z^3 - 5z^2 + z - 6$ given that $z = i$ is a root.

We know $z = \bar{i} = -i$ is also a root, so the poly is divisible by $(z-i)(z+i) = z^2+1$

So $z^4 + z^3 - 5z^2 + z - 6 = (z^2+1)(z^2 + az + b)$

constant terms: $-6 = b \Rightarrow b = -6$

coeff. of z : $1 = a \Rightarrow a = 1.$

So $z^4 + z^3 - 5z^2 + z - 6 = (z^2+1)(z^2 + z - 6)$

$= (z^2+1)(z+3)(z-2)$

So roots are $\boxed{z = i, -i, -3, +2}$.

2. (a) Calculate the following limits, or show that they do not exist, showing all of your working. You may use any valid method.

(i) $\displaystyle\lim_{x\to\infty} \frac{x^3+5x^2+1}{5x^3+4}$ (ii) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

(iii) $\displaystyle\lim_{x\to 0^+} x^x$ (iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy(x+y)}{x^2+y^2}$

Soln (i) $\displaystyle\lim_{x\to\infty} \frac{x^3+5x^2+1}{5x^3+4} = \lim_{x\to\infty} \frac{1+5/x+1/x^3}{5+4/x^3} = \frac{1}{5}$.

(ii) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \lim_{x\to 0} \frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}$

$\displaystyle = \lim_{x\to 0} \frac{x+4-4}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{4}$

(or use L'Hôpital's Rule).

(iii) $\displaystyle\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x}$

Now: $\displaystyle\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{(1/x)} \underset{\text{(L'Hop)}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$

$\displaystyle = \lim_{x\to 0^+} (-x) = 0$.

So $\displaystyle\lim_{x\to 0^+} x^x = e^0 = 1$.

(iv) Write $(x,y)=(r\cos\theta, r\sin\theta)$. So $(x,y)\to(0,0) \Leftrightarrow r\to 0^+$

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy(x+y)}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^3\cos\theta\sin\theta(\cos\theta+\sin\theta)}{r^2}$

$\displaystyle = \lim_{r\to 0^+} r(\underline{\qquad}) = 0$.

(b) Let $f(x, y) = e^y - x\sin(x + y)$. The equation $f(x, y) = 1$ implicitly defines $y = g(x)$ as a function of x such that $g(\pi) = 0$. Find the derivative $g'(x)$, and hence find the equation of the tangent line to the level curve of $f(x, y) = 1$ at the point $(\pi, 0)$.

Soln: $f_y(x,y) = e^y - x\cos(x+y)$

so $f_y(\pi, 0) = 1 - \pi\cos\pi = 1+\pi \neq 0$.

Hence $f(x,y)=1$ implicitly defines y as a differentiable function of x around $(\pi, 0)$, by the Implicit Function Theorem. Moreover,

$$\boxed{g'(\pi) = -\frac{f_x(\pi,0)}{f_y(\pi,0)} = -\frac{\pi}{1+\pi}.}$$

$f_x(x,y) = -x\cos(x+y) - \sin(x+y)$

$\Rightarrow f_x(\pi, 0) = +\pi$

Actually: $\boxed{g'(x) = -\frac{f_x(x,y)}{f_y(x,y)} = -\frac{-x\cos(x+y) - \sin(x+y)}{e^y - x\cos(x+y)}}$

Tangent line: $y - y_1 = m(x - x_1)$

$y - 0 = \left(-\frac{\pi}{1+\pi}\right)(x - \pi)$

so $y = -\frac{\pi}{1+\pi}x + \frac{\pi^2}{1+\pi}$.

(c) An ant is standing on the kitchen floor. The floor has coordinates such that the x-axis points east, and the y-axis points north, and the temperature on the kitchen floor at the point (x, y) is given by the formula $T(x, y) = x^2 - 2y^2 + 4xy$. The ant is currently at the point $(x, y) = (2, 1)$ on the floor.

(i) In what direction should the ant walk to initially increase temperature most rapidly? What is the rate of change in temperature that the ant will experience if it walks in this direction?

(ii) What is the rate of change in temperature that the ant initially experiences if it walks in the direction $\mathbf{i} - \mathbf{j}$?

(i) Walk in direction $\nabla f(2,1)$ $\qquad (f = T)$

$$\nabla f(x,y) = (2x+4y)\underset{\sim}{i} + (-4y+4x)\underset{\sim}{j}$$

$$\nabla T(2,1) = (8)\underset{\sim}{i} + 4\underset{\sim}{j}$$

so direction $\boxed{8\underset{\sim}{i} + 4\underset{\sim}{j}}$

rate is $|\nabla T(2,1)| = \sqrt{8^2+4^2} = \sqrt{64+16} = \sqrt{80}$
$= 4\sqrt{5}$.

(ii) $D_{\underset{\sim}{i}-\underset{\sim}{j}} T(2,1)$ $\qquad \widehat{\underset{\sim}{i}-\underset{\sim}{j}} = \frac{1}{\sqrt{2}}\underset{\sim}{i} - \frac{1}{\sqrt{2}}\underset{\sim}{j}$

$$= \nabla T(2,1) \cdot (\widehat{\underset{\sim}{i}-\underset{\sim}{j}})$$

$$= (8\underset{\sim}{i}+4\underset{\sim}{j}) \cdot \left(\frac{1}{\sqrt{2}}\underset{\sim}{i} - \frac{1}{\sqrt{2}}\underset{\sim}{j}\right)$$

$$= \frac{8}{\sqrt{2}} - \frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

3. (a) Consider the equation

$$x^3 + x^2 + 3x - 2 = 0 \tag{1}$$

(i) Use the Intermediate Value Theorem to show that the equation (1) has at least one real solution.

(ii) Use Rolle's Theorem to show that the equation (1) has at most one real solution (and thus it has exactly 1 real solution).

(i) Let $f(x) = x^3 + x^2 + 3x - 2$. This is continuous and diff'ble everywhere. (Limit Laws).

$f(0) = -2 < 0$

$f(1) = 3 > 0$

So IVT $\Rightarrow$ there is $c \in (0,1)$ such that $f(c) = 0$. So at least one solution.

(ii) Suppose $c < d$ satisfy $f(c) = f(d) = 0$

By Rolle's Thm there is $e \in (c,d)$ such that

$f'(e) = 0$ (*)

But $f'(x) = 3x^2 + 2x + 3 = 0$

$b^2 - 4ac = 4 - 4 \times 3 \times 3 < 0$

so quadratic has no real solutions, a contradiction with (*).

Hence exactly one solution.

(b) (*i*) State the Mean Value Theorem.

(*ii*) Use the Mean Value Theorem to show that

$$\frac{1}{2\sqrt{a+1}} < \sqrt{a+1} - \sqrt{a} < \frac{1}{2\sqrt{a}} \quad \text{for all } a > 0.$$

See ~~assignment~~. lecture 14 .

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x,y) = \begin{cases} \dfrac{x^2\sin(x^2+y)}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ k & \text{if } (x,y) = (0,0). \end{cases}$$

Is there a choice of $k \in \mathbb{R}$ such that f is continuous at $(0,0)$? If so, find k and prove that with this choice of k the function f is continuous at $(0,0)$. If not, prove that no such k exists.

We need to decide if $\lim_{(x,y)\to(0,0)} \dfrac{x^2\sin(x^2+y)}{x^4+y^2}$ exists or not.

$$f(x,-x^2) = \frac{x^2\sin(0)}{2x^4} = 0 \to 0 \quad (\text{as } x \to 0)$$

$$f(x,0) = \frac{x^2\sin(x^2)}{x^4} = \frac{\sin(x^2)}{x^2} \to 1 \quad (\text{as } x \to 0)$$

So $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, so there is no k making f cts at $(0,0)$.

4. (a) Let $f(x) = \ln(1+x)$.

(i) Calculate the fourth order Taylor polynomial $T_4(x)$ for $f(x)$ centred at 0.

(ii) Use Taylor's Theorem to write down a formula for the fourth remainder term $R_4(x) = f(x) - T_4(x)$, and deduce that

$$\frac{x^5}{5(1+x)^5} \le f(x) - T_4(x) \le \frac{x^5}{5} \qquad \text{for all } x \ge 0.$$

(iii) Use the previous part to compute the limit

$$\lim_{x\to 0^+} \frac{\ln(1+x) - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4}{\sin^5 x}$$

Warning: Applying l'Hôpital's Rule will be very time consuming, and thus is not a good approach to this question.

(a)(i) $f(x) = \ln(1+x) \Rightarrow f(0) = 0$

$f'(x) = (1+x)^{-1} \Rightarrow f'(0) = 1$

$f^{(2)}(x) = -(1+x)^{-2} \Rightarrow f^{(2)}(0) = -1$

$f^{(3)}(x) = 2(1+x)^{-3} \Rightarrow f^{(3)}(0) = 2 = 2!$

$f^{(4)}(x) = -6(1+x)^{-4} \Rightarrow f^{(4)}(0) = -6 = -3!$

$f^{(5)}(x) = 4!.(1+x)^{-5}$.

So $\boxed{T_4(x) = 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4}$

(ii) $R_4(x) = \frac{f^{(5)}(c)}{5!}x^5$ for some c between 0 and x.

$= \frac{1}{5(1+c)^5}x^5$ $[0 \leqslant c \leqslant x$ as $x \geqslant 0]$

So $f(x) - T_4(x) = R_4(x)$; and

$$\frac{x^5}{5(1+x)^5} \leqslant R_4(x) \leqslant \frac{x^5}{5(1+0)^5} = \frac{x^5}{5}$$

(iii) The numerator is just $f(x) - T_4(x)$.

Since $\sin x > 0$ for small $x > 0$.

so the inequality in (ii) gives

$$\frac{x^5}{5(1+x)^5\sin^5 x} \leq \frac{f(x) - T_4(x)}{\sin^5 x} \leqslant \frac{x^5}{5\sin^5 x}$$

Now: $$\lim_{x\to 0^+} \frac{x^5}{5\sin^5 x} = \frac{1}{5}\lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)^5 = \frac{1}{5}$$

$$\lim_{x\to 0^+} \frac{x^5}{5(1+x)^5\sin^5 x} = \frac{1}{5}\lim_{x\to 0^+}\frac{1}{(1+x)^5}\cdot\left(\frac{x}{\sin x}\right)^5 = \frac{1}{5}$$

Hence $$\lim_{x\to 0^+} \frac{\ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}}{\sin^5 x} = \frac{1}{5}$$

by the Squeeze Law.

(b) Let $f : \mathbb{C}\backslash\{2\} \to \mathbb{C}$ be the function

$$f(z) = \frac{z+2}{z-2}$$

(*i*) Show that f is injective.

(*ii*) What is the range of f?

(*iii*) Find a formula for the inverse function $f^{-1} : B \to \mathbb{C}\backslash\{2\}$, where B is the range of f.

(*iv*) Show that the image of the unit circle $C = \{z \in \mathbb{C} \mid |z| = 1\}$ under f is a circle in the w plane, where $w = f(z)$. What is the centre and radius of the image circle?

(i) $f(z_1) = f(z_2) \Rightarrow \dfrac{z_1+2}{z_1-2} = \dfrac{z_2+2}{z_2-2}$

$\Rightarrow (z_1+2)(z_2-2) = (z_1-2)(z_2+2)$

$\Rightarrow z_1z_2 - 2z_1 + 2z_2 - 4 = z_1z_2 + 2z_1 - 2z_2 - 4$

$\Rightarrow 4z_2 = 4z_1$

$\Rightarrow z_1 = z_2.$

so injective.

(ii) $f(z) = \dfrac{z-2+4}{z-2} = 1 + \dfrac{4}{z-2}$

Since $\dfrac{4}{z-2}$ takes all values ~~on~~ in $\mathbb{C}$ except 0

The range is $\mathbb{C}\backslash\{1\}$

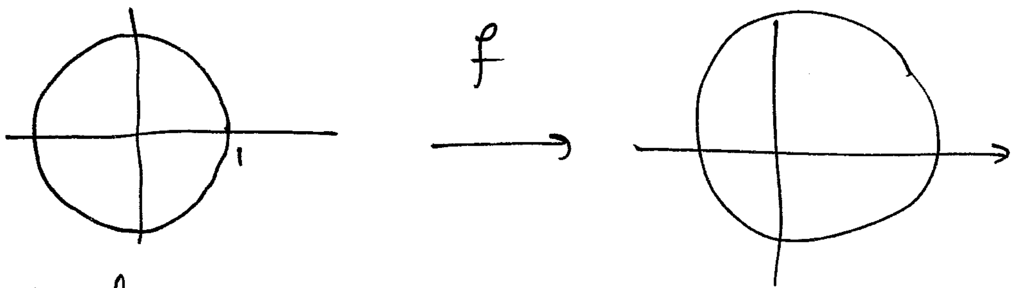
(iii) Write $w = \dfrac{z+2}{z-2}$ and solve for z.

$$w(z-2) = z+2$$
$$z(w-1) = 2(w+1)$$

So $z = \dfrac{2(w+1)}{w-1}$ since $w \neq 1$, as $w \in B$.

Hence $f^{-1}(w) = \dfrac{2(w+1)}{w-1}$; $f^{-1}: \mathbb{C}\setminus\{1\} \to \mathbb{C}\setminus\{2\}$

(iv)



z - plane

w - plane

Option: $z = x+iy$ $(x^2+y^2=1)$ and consider

$$w = \frac{z+2}{z-2} = \frac{(x+2)+iy}{(x-2)+iy} \cdots\cdots$$

$$= \frac{(x+2)+iy}{(x-2)+iy} \cdot \frac{(x-2)-iy}{(x-2)-iy} = \ldots \text{??}$$

Instead, use:

$$z = \frac{2(w+1)}{w-1}$$

So $$1 = 2\frac{|w+1|}{|w-1|} \qquad (\text{since } |z|=1)$$

$$|w-1| = 2|w+1|$$

Write $w = u + iv$ and so

$$(u-1)^2 + v^2 = 4\left((u+1)^2 + v^2\right)$$

$$u^2 - 2u + 1 + v^2 = 4u^2 + 8u + 4 + 4v^2$$

$$3u^2 + 3v^2 + 10u + 3 = 0$$

$$3\left(u^2 + \frac{10}{3}u + \left(\frac{5}{3}\right)^2\right) - 3\left(\frac{5}{3}\right)^2 + 3v^2 + 3 = 0$$

$$\cancel{3}\left(u + \frac{5}{3}\right)^2 + \cancel{3}v^2 + 1 - \frac{25}{9} = 0$$

$$\left(u + \frac{5}{3}\right)^2 + v^2 = \left(\frac{4}{3}\right)^2$$

Circle: centre $\left(-\frac{5}{3}, 0\right)$, radius $\frac{4}{3}$.

Standard Derivatives

The following derivatives can be quoted without proof unless a question specifically asks you to show details. These results can be combined with the standard rules of differentiation (not listed here) to differentiate more complicated functions. For example, $(d/dx)\sin(ax+b) = a\cos(ax+b)$. Natural domains common to both sides are assumed.

1. $\dfrac{d}{dx}\, x^k = kx^{k-1} \quad (k \in \mathbb{R})$
2. $\dfrac{d}{dx}\, e^x = e^x$
3. $\dfrac{d}{dx}\, \ln x = \dfrac{1}{x} \quad (x > 0)$
4. $\dfrac{d}{dx}\, \sin x = \cos x$
5. $\dfrac{d}{dx}\, \cos x = -\sin x$
6. $\dfrac{d}{dx}\, \tan x = \sec^2 x$
7. $\dfrac{d}{dx}\, \cot x = -\operatorname{cosec}^2 x$
8. $\dfrac{d}{dx}\, \sec x = \sec x \tan x$
9. $\dfrac{d}{dx}\, \operatorname{cosec} x = -\operatorname{cosec} x \cot x$
10. $\dfrac{d}{dx}\, \sinh x = \cosh x$
11. $\dfrac{d}{dx}\, \cosh x = \sinh x$
12. $\dfrac{d}{dx}\, \tanh x = \operatorname{sech}^2 x$
13. $\dfrac{d}{dx}\, \sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$
14. $\dfrac{d}{dx}\, \cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$
15. $\dfrac{d}{dx}\, \tan^{-1} x = \dfrac{1}{1+x^2}$
16. $\dfrac{d}{dx}\, \sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$
17. $\dfrac{d}{dx}\, \cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}} \quad (x > 1)$
18. $\dfrac{d}{dx}\, \tanh^{-1} x = \dfrac{1}{1-x^2} \quad (|x| < 1)$

End of Extended Answer Section

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