MATH 1906 - Differential Calculus SSP

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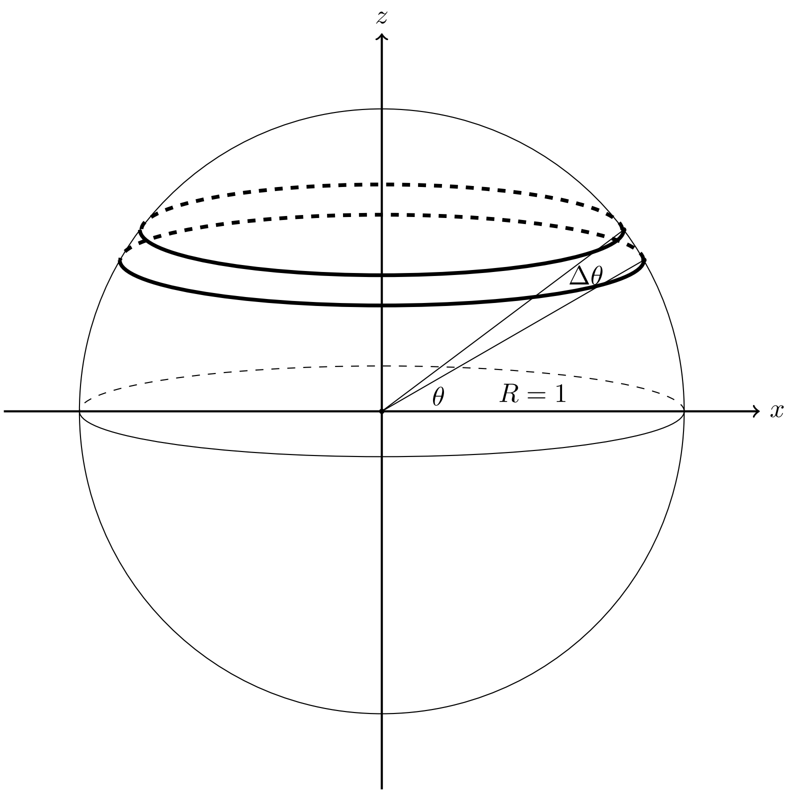
Assignment 1 - Maps

(a) Using an equal area map projection, we are required to prove that $b^2 cos\theta = 2v\frac{dv}{d\theta}$. In order to do this we need to examine the area on the sphere and secondly on the map projection, to derive the above relationship.

Now we know that the gradient of the border lines of the map projection is $\pm\frac{b}{a}$. Furthermore we are given that the total area of the map projection is equal to the surface area of the sphere.

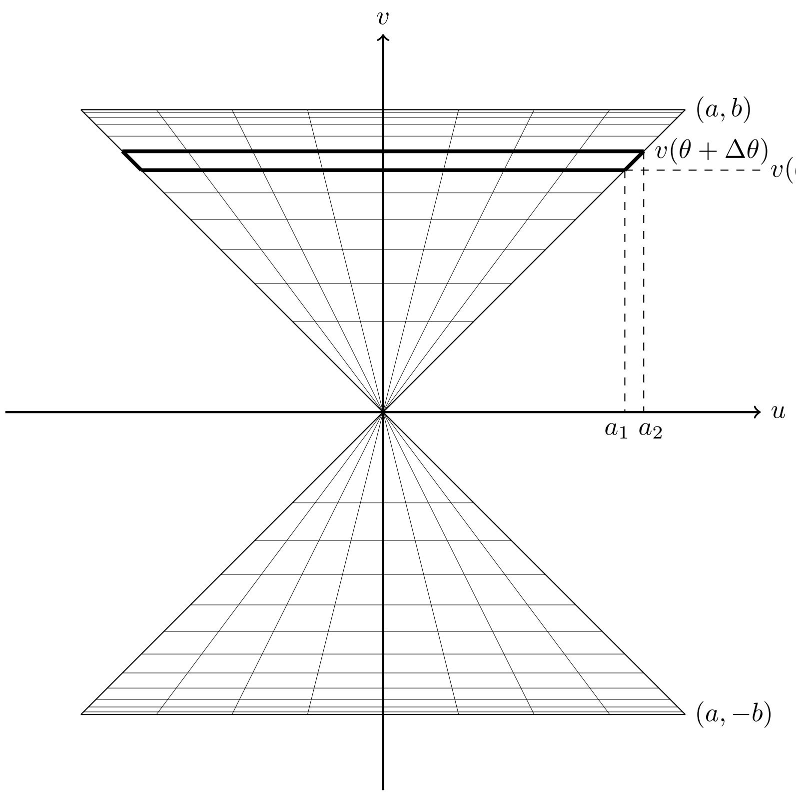
$$\begin{aligned} 4\pi R^2 &= \frac{1}{2}(2a)(b) + \frac{1}{2}(2a)(b) \\ &= ab + ab \\ &= 2ab \\ \therefore 4\pi &= 2ab \\ \therefore 2\pi &= ab \end{aligned}$$

Examining the sphere, with radius 1 and centered on the origin, we will need to examine the area of a strip that is created by the difference in the latitudes θ and $\theta + \Delta\theta$. The area of the strip is given by integrating the circumference from θ to $\theta + \Delta\theta$. The radius of the latitude θ is given by $\cos\theta$.



$$
\begin{aligned}
A_{strip_{sphere}} &= \int_{\theta}^{\theta+\Delta\theta} 2\pi\cos\theta d\theta \\
&= 2\pi\int_{\theta}^{\theta+\Delta\theta} \cos\theta d\theta \\
&= 2\pi\Big[\sin(\theta+\Delta\theta) - \sin(\theta)\Big] \\
&= ab\Big[\sin(\theta+\Delta\theta) - \sin(\theta)\Big]
\end{aligned}
$$

Now considering a strip on the map projection, we will find the area using the area of the triangles that bound the strip on the map projection. Considering the triangles that lie to the right of the y-axis, we will label the base length of the larger triangle as a_2, and the base length of the smaller triangle as a_1. Thus the area of each triangle is given by:



$$
\begin{aligned}
A_1 &= \frac{1}{2}a_1 v(\theta) \\
A_2 &= \frac{1}{2}a_2 v(\theta+\Delta\theta)
\end{aligned}
$$

Thus the area of the strip is given by subtracting the total area of the smaller triangle from the larger triangle. The total area is given by twice the above area for each triangle as we only considered half the total area originally.

$$
\begin{aligned}
A_{1_T} &= a_1 v(\theta) \\
A_{2_T} &= a_2 v(\theta+\Delta\theta)
\end{aligned}
$$

Now considering the equation of the lines that bound the map projection we can determine the values of a_1 and a_2.

$$\begin{aligned} y &= mx + b \\ \therefore y &= \pm\frac{b}{a}x + 0 \end{aligned}$$

For the values of a_1 and a_2, we will consider the positive value of the gradient.

$$\begin{aligned} y &= \frac{b}{a}x \\ \therefore v(\theta) &= \frac{b}{a}a_1 \\ \therefore a_1 &= \frac{a}{b}v(\theta) \\ \therefore v(\theta+\Delta\theta) &= \frac{b}{a}a_2 \\ \therefore a_2 &= \frac{a}{b}v(\theta+\Delta\theta) \\ \therefore A_{1_T} &= a_1 v(\theta) \\ &= \frac{a}{b}v(\theta)v(\theta) \\ \therefore A_{2_T} &= a_2 v(\theta+\Delta\theta) \\ &= \frac{a}{b}v(\theta+\Delta\theta)v(\theta+\Delta\theta) \end{aligned}$$

Now in order to determine the area of the strip we will subtract the area of the smaller triangle from the area of the larger triangle.

$$\begin{aligned} A_{strip_{map}} &= A_{2_T} - A_{1_T} \\ &= \frac{a}{b}v(\theta+\Delta\theta)v(\theta+\Delta\theta) - \frac{a}{b}v(\theta)v(\theta) \\ &= \frac{a}{b}\Big[\Big[v(\theta+\Delta\theta)\Big]^2 - \Big[v(\theta)\Big]^2\Big] \end{aligned}$$

As the map projection is by definition an equal area map, we can equate the area on the globe to the area on the map projection. Thus the following relationship can be derived:

$$\begin{aligned} A_{strip_{sphere}} &= A_{strip_{map}} \\ \therefore ab\Big[\sin(\theta+\Delta\theta) - \sin(\theta)\Big] &= \frac{a}{b}\Big[\Big[v(\theta+\Delta\theta)\Big]^2 - \Big[v(\theta)\Big]^2\Big] \\ \therefore b^2\Big[\sin(\theta+\Delta\theta) - \sin(\theta)\Big] &= \Big[v(\theta+\Delta\theta)\Big]^2 - \Big[v(\theta)\Big]^2 \\ \therefore b^2\frac{\Big[\sin(\theta+\Delta\theta) - \sin(\theta)\Big]}{\Delta\theta} &= \frac{\Big[v(\theta+\Delta\theta)\Big]^2 - \Big[v(\theta)\Big]^2}{\Delta\theta} \end{aligned}$$

Now taking the limit as $\Delta\theta \to 0$, we can derive the required result, as we are using the definition of differentiation from first principles.

$$
\begin{aligned}
\lim_{\Delta\theta \to 0} b^2 \frac{\Big[\sin(\theta + \Delta\theta) - \sin(\theta)\Big]}{\Delta\theta} &= \lim_{\Delta\theta \to 0} \frac{\Big[v(\theta + \Delta\theta)\Big]^2 - \Big[v(\theta)\Big]^2}{\Delta\theta} \\
\therefore b^2 \frac{d}{d\theta}\Big[\sin(\theta)\Big] &= \frac{d}{d\theta}\Big[\Big[v(\theta)\Big]^2\Big] \\
\therefore b^2 \cos\theta &= 2v(\theta)\frac{d}{d\theta}\Big[v(\theta)\Big] \\
\therefore b^2 \cos\theta &= 2v\frac{dv}{d\theta}
\end{aligned}
$$

Thus we have proved the required result.

(b) From part (a), we have a differential equation, which we can use to solve for v as a funtion of θ.

$$
\begin{aligned}
b^2 \cos\theta &= 2v\frac{dv}{d\theta} \\
\therefore b^2 \cos\theta d\theta &= 2v dv \\
\therefore \int b^2 \cos\theta d\theta &= \int 2v dv \\
\therefore b^2 \int \cos\theta d\theta &= 2\int v dv \\
\therefore b^2 \sin\theta + c_1 &= v^2 + c_2 \\
\therefore b^2 \sin\theta &= v^2 + C
\end{aligned}
$$

Now in order to determine the value of C, we will use the value of $\theta = 0$ and thus $v = 0$.

$$
\begin{aligned}
b^2 \sin\theta &= v^2 + C \\
b^2 \sin 0 &= 0^2 + C \\
\therefore C &= 0 \\
\therefore b^2 \sin\theta &= v^2 \\
\therefore v &= \pm b\sqrt{|\sin\theta|}
\end{aligned}
$$

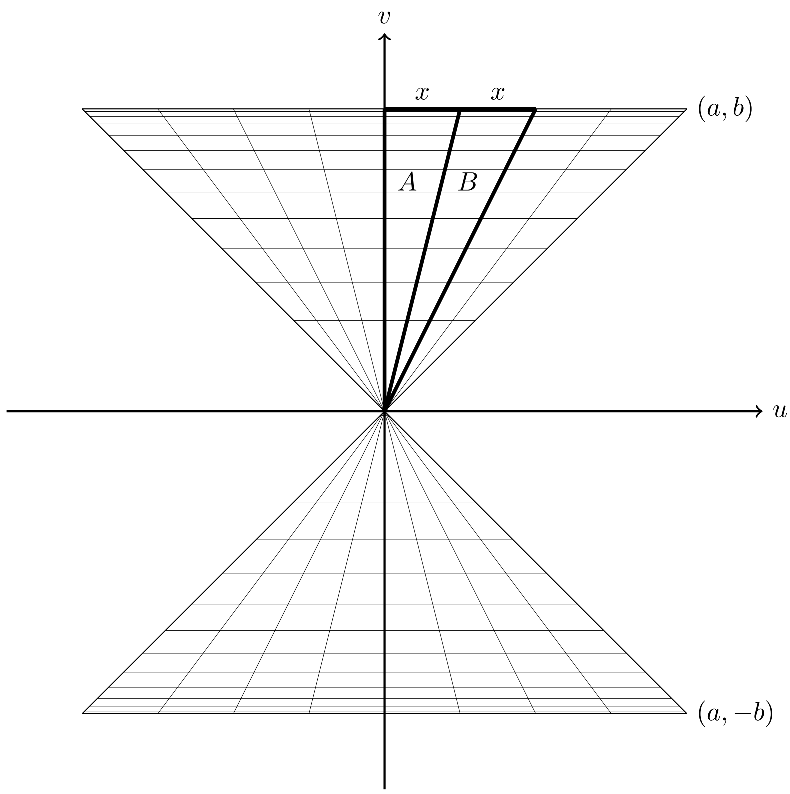
As θ can take negative values, the use of the absolute value function ensures this solution works for both positve and negative values of θ. Furthermore, this function of v in terms of θ is defined properly as a piecewise function, as $v(\theta)$ is negative on the map projection for negative values of θ.

$$
v = \begin{cases} b\sqrt{|\sin\theta|} & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\ -b\sqrt{|\sin\theta|} & \text{for } -\frac{\pi}{2} \leq \theta < 0 \end{cases}
$$

$$
-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}
$$

This range for θ gives all potential values for the map projection, however it is clear that our map is restricted.

(c) We are now required to find the variables in terms of the latitude and longitude. We are required to find $u = u(\varphi, \theta)$ and $v = v(\theta)$.



Using the solution to the differential equation we solved in part (b), we have the variable v in terms of θ.

$$v(\theta) = \begin{cases} b\sqrt{|\sin\theta|} & \text{for } 0 \le \theta \le \frac{\pi}{2} \\ -b\sqrt{|\sin\theta|} & \text{for } -\frac{\pi}{2} \le \theta < 0 \end{cases}$$
$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

See the following page for the coordinate $u(\varphi, \theta)$.

Furthermore, this map projection is equal area, so we know that the areas A and B are equal. As they have the same height, the intervals, x, that each longitudinal area subtends must be equivalent to preserve area. Therefore, $x = \frac{\varphi}{\pi}$, and thus $u(\varphi, \theta)$ is dependent upon $v(\theta)$, and φ. Hence the derivation of $u(\varphi, \theta)$ is from the area of the triangle A or B.

$$
\begin{aligned}
Area_A &= \frac{1}{2}bh \\
&= \frac{1}{2} \times \frac{\varphi}{\pi} v(\theta) \\
&= \frac{v\varphi}{2\pi} \\
\therefore u(\varphi, \theta) &= \frac{v\varphi}{2\pi} \\
&= \frac{\varphi}{2\pi} b\sqrt{|\sin\theta|} \\
&= \frac{\varphi}{ab} b\sqrt{|\sin\theta|} \\
&= \frac{\varphi}{a}\sqrt{|\sin\theta|}
\end{aligned}
$$

$$
\begin{aligned}
\therefore u(\varphi, \theta) &= \frac{\varphi}{a}\sqrt{|\sin\theta|} \\
-\pi < \varphi &\leq \pi
\end{aligned}
$$

The range of φ incorporates the full rotation around the sphere, and its sign determines the sign of $u(\varphi, \theta)$, and thus we do not need to consider the piecewise branches of $v(\theta)$.