CHAPTER **FOURTEEN**

The Trigonometric Functions

This chapter will extend the calculus to the sine and cosine functions and the other trigonometric functions. The sine and cosine functions are extremely important because their graphs are waves. Mathematically, they are the simplest wave forms, and every other wavy graph can be constructed from combinations of them. They are therefore essential in the modelling of all the many wave-like phenomena such as sound waves, light and radio waves, vibrating strings, tides and economic cycles.

In the last two chapters, we saw how e was the right number to use for the base of the exponential and logarithmic functions in calculus. In this chapter, a new measure of angles based on the number π will turn out to be appropriate for the calculus of the trigonometric functions. This is the reason why the irrational numbers e and π are so important in calculus.

Study Notes: This chapter has two parts. The first part (Sections 14A–14F) leads towards the proof that the derivative of $\sin x$ is $\cos x$. On this basis, Sections 14G–14J then develop the differentiation and integration of the trigonometric functions, applying the derivative and the integral in the usual ways. Establishing that the derivative of $\sin x$ is $\cos x$ requires a connected body of theory to be developed. In Sections 14A and 14B, radian measure is introduced and applied to the mensuration of circles. This allows the six trigonometric functions to be sketched in their true form in Section 14C. In Section 14D the formulae for the trigonometric functions of compound angles are developed, and are applied in Section 14E to the angle between two lines. In Section 14F the fundamental limit $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$ is proven using an appeal to geometry and the mensuration of the circle. Combining all this material finally allows the derivative of $\sin x$ to be established in Section 14G. In all study of the trigonometric functions, sketches of the graphs should be made whenever possible. Computers and other machines may help here in generating the graphs of a number of functions quickly, so that the effect of changing the formulae or the parameters can be discovered by experimentation.

14 A Radian Measure of Angle Size

The use of degrees to measure angle size is based on astronomy, not on mathematics. There are 360 days in the year, to the nearest convenient number, so 1° is the angle through which our sun moves against the fixed stars each day, or (after Copernicus) the angle swept out by the Earth each day in its orbit around the sun. Mathematics is far too general a discipline to be tied to the particularities

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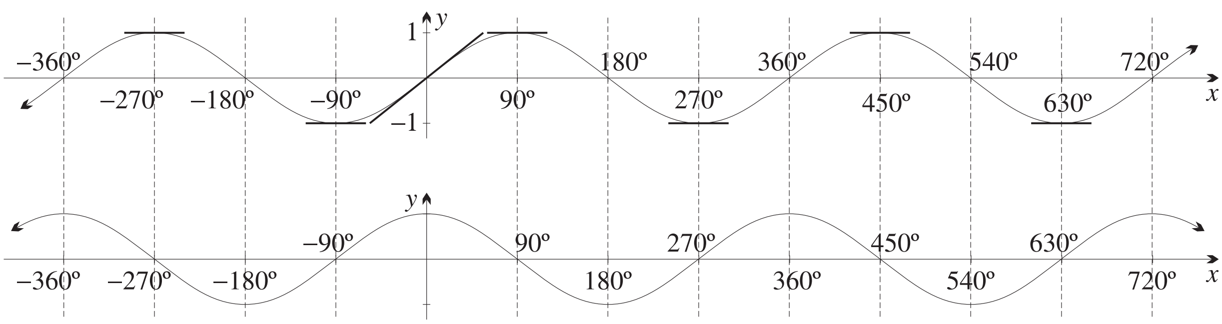
of our solar system, so it is quite natural that we should pass to a new system for measuring angles once the mathematics becomes a little more sophisticated.

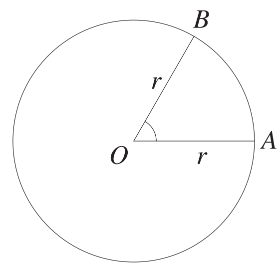
The Problem of How to Measure Angles: The need for a new system for measuring angles arises when one attempts to differentiate the trigonometric functions. The upper graph in the sketch below is $y = \sin x^\circ$. The lower graph is a rough sketch of the derivative of $y = \sin x^\circ$, paying attention to where the gradient of $y = \sin x^\circ$ is zero, maximum and minimum.

The lower graph seems unmistakably to be some sort of cosine graph, but it is not at all clear what the scale on its vertical axis should be. Most importantly, the y-intercept of the lower graph is the gradient of the upper graph at the origin. But gradients are not properly defined on the upper graph yet, — there are numbers on the vertical axis, but degrees on the horizontal axis, and there is no obvious way to set up the relationship between the scales on the two axes. If 1 unit on the x-axis were chosen to be 1 degree, then 90° would be placed about a metre off the page, making the upper graph very flat indeed. If one unit on the x-axis were chosen to be 90°, then the gradient of $\sin x$ at the origin would be somewhat steeper than 1.

We shall make the choice of units on the horizontal axis so that the gradient of $y = \sin x$ at the origin is exactly 1. The result will be the most convenient situation, because the derivative of $\sin x$ will turn out to be exactly $\cos x$, giving the simple formula $\dfrac{d}{dx}\sin x = \cos x$. A bit of experimentation stretching the graph horizontally shows that we need 90° to be about $1\frac{1}{2}$ units — the precise value will turn out to be $\frac{\pi}{2}$. This is the real purpose of the new units for measuring angles introduced in the next paragraph, and then developed over the next six sections in Sections 14A–14F.



Radian Measure: The new units for measuring angles are called *radians*. Their definition is purely mathematical, being the ratio of two lengths, and is thus very similar to the definitions of the trigonometric functions. Given an angle with vertex O, construct a circle with centre O meeting the two arms of the angle at A and B. Then:



1 RADIAN MEASURE: Size of $\angle AOB = \dfrac{\text{arc length } AB}{\text{radius } OA}$

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Since the whole circumference is $2\pi r$, there must be 2π radians in a revolution, π radians in a straight angle, and $\frac{\pi}{2}$ radians in a right angle.

2 CONVERSIONS: $360° = 2\pi$, $180° = \pi$, $90° = \frac{\pi}{2}$

One radian is then the angle subtended at the centre of a circle by an arc of length equal to the radius. The sector OAB in the diagram must then be almost an equilateral triangle, and so 1 radian is about $60°$. More precisely:

3 MORE CONVERSIONS: $1 \text{ radian} = \dfrac{180°}{\pi} \doteqdot 57°18'$, $1 \text{ degree} = \dfrac{\pi}{180} \doteqdot 0{\cdot}017\,453$

NOTE: The measure of an angle in radians is a ratio of lengths, and so is a dimensionless real number. The units are therefore normally omitted. For example, 'an angle of size 1·3' means an angle of 1·3 radians. Calculators set to the wrong mode routinely cause havoc at this point!

WORKED EXERCISE:

(a) Express $60°$, $495°$ and $37°$ in radians.

(b) Express the angles $\frac{\pi}{6}$, $\frac{3\pi}{4}$ and $0{\cdot}3$ in degrees.

(c) Evaluate $\cos\frac{\pi}{6}$, $\sec\frac{\pi}{4}$ and $\sin 1$ (to four decimal places).

SOLUTION:

(a)
$$\begin{aligned} 60° &= 60 \times \tfrac{\pi}{180} \\ &= \tfrac{\pi}{3} \\ 495° &= 495 \times \tfrac{\pi}{180} \\ &= \tfrac{11\pi}{4} \\ 37° &= 37 \times \tfrac{\pi}{180} \\ &= \tfrac{37\pi}{180} \end{aligned}$$

(b)
$$\begin{aligned} \tfrac{\pi}{6} &= \tfrac{\pi}{6} \times \tfrac{180}{\pi}^{\circ} \\ &= 30° \\ \tfrac{3\pi}{4} &= \tfrac{3\pi}{4} \times \tfrac{180}{\pi}^{\circ} \\ &= 135° \\ 0{\cdot}3 &= 0{\cdot}3 \times \tfrac{180}{\pi}^{\circ} \\ &= \tfrac{54}{\pi}^{\circ} \end{aligned}$$

(c) $\cos\frac{\pi}{6} = \frac{1}{2}\sqrt{3}$
(notice $\frac{\pi}{6} = 30°$)
$\sec\frac{\pi}{4} = \sqrt{2}$
(notice $\frac{\pi}{4} = 45°$)
$\sin 1 \doteqdot 0{\cdot}8415$
(set calculator to radians)

Solving Trigonometric Equations in Radians: The steps for giving the solution in radians to a trigonometric equation are unchanged. First, establish the quadrants in which the angle can lie. Secondly, find the related angle, but use radian measure, not degrees.

WORKED EXERCISE:

(a) Solve $\cos x = -\frac{1}{2}$, for $0 \le x \le 2\pi$.

(b) Solve $\operatorname{cosec} x = -3$ correct to five significant figures, for $0 \le x \le 2\pi$.

SOLUTION:

(a) Since $\cos x = -\frac{1}{2}$,
x is in the second or third quadrant, and the related angle is $\frac{\pi}{3}$.
Hence $x = \pi - \frac{\pi}{3}$ or $\pi + \frac{\pi}{3}$
$= \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

(b) Since $\operatorname{cosec} x = -3$
$\sin x = -\frac{1}{3}$,
so x is in the third or fourth quadrant, and the related angle is $\sin^{-1}\frac{1}{3}$.
Hence $x = \pi + \sin^{-1}\frac{1}{3}$ or $2\pi - \sin^{-1}\frac{1}{3}$
$\doteqdot 3{\cdot}4814$ or $5{\cdot}9433$.

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Exercise 14A

1. Express the following angles in radians as multiples of π:

(a) $90°$ (b) $45°$ (c) $30°$ (d) $60°$ (e) $120°$ (f) $150°$ (g) $135°$ (h) $225°$ (i) $360°$ (j) $300°$ (k) $270°$ (l) $210°$

2. Express the following angles in degrees:

(a) π (b) 2π (c) 4π (d) $\dfrac{\pi}{2}$ (e) $\dfrac{\pi}{3}$ (f) $\dfrac{\pi}{4}$ (g) $\dfrac{2\pi}{3}$ (h) $\dfrac{5\pi}{6}$ (i) $\dfrac{3\pi}{4}$ (j) $\dfrac{3\pi}{2}$ (k) $\dfrac{4\pi}{3}$ (l) $\dfrac{7\pi}{4}$ (m) $\dfrac{11\pi}{6}$

3. Use your calculator to express in radians correct to three decimal places:

(a) $73°$ (b) $14°$ (c) $168°$ (d) $21°36'$ (e) $95°17'$ (f) $211°12'$

4. Use your calculator to express in degrees and minutes:

(a) 2 radians (b) 0·3 radians (c) 1·44 radians (d) 0·123 radians (e) 3·1985 radians (f) 5·64792 radians

5. Use your calculator to evaluate correct to two decimal places:

(a) $\sin 2$ (b) $\cos 2{\cdot}5$ (c) $\tan 3{\cdot}21$ (d) $\operatorname{cosec} 0{\cdot}7$ (e) $\sec 1{\cdot}23$ (f) $\cot 5{\cdot}482$

6. Using the two special triangles and your knowledge of angles of any magnitude, find the exact value of:

(a) $\sin \frac{\pi}{3}$ (b) $\sin \frac{\pi}{4}$ (c) $\cos \frac{5\pi}{6}$ (d) $\tan \frac{4\pi}{3}$ (e) $\tan \frac{3\pi}{4}$ (f) $\cos \frac{5\pi}{3}$ (g) $\sin \frac{5\pi}{4}$ (h) $\tan \frac{7\pi}{6}$

7. Solve for x over the domain $0 \le x \le 2\pi$:

(a) $\sin x = \frac{1}{2}$ (b) $\cos x = -\frac{1}{2}$ (c) $\tan x = -1$ (d) $\sin x = 1$ (e) $2\cos x = \sqrt{3}$ (f) $\sqrt{3}\tan x = 1$ (g) $\cos x + 1 = 0$ (h) $\sqrt{2}\sin x + 1 = 0$ (i) $\cot x = 1$

DEVELOPMENT

8. Express in radians as multiples of π:

(a) $20°$ (b) $22{\cdot}5°$ (c) $36°$ (d) $100°$ (e) $112{\cdot}5°$ (f) $252°$ (g) $32°30'$ (h) $65°45'$

9. Express in degrees:

(a) $\dfrac{\pi}{12}$ (b) $\dfrac{2\pi}{5}$ (c) $\dfrac{20\pi}{9}$ (d) $\dfrac{11\pi}{8}$ (e) $\dfrac{17\pi}{10}$ (f) $\dfrac{23\pi}{15}$

10. Find: (a) the complement of $\frac{\pi}{6}$, (b) the supplement of $\frac{\pi}{6}$.

11. Two angles of a triangle are $\frac{\pi}{3}$ and $\frac{2\pi}{9}$. Find, in radians, the third angle.

12. Find, correct to three decimal places, the angle in radians through which:

(a) the second hand of a clock turns in 7 seconds,

(b) the hour hand of a clock turns between 6 am and 6·40 am.

13. If $f(x) = \sin x$, $g(x) = \cos 2x$ and $h(x) = \tan 3x$, find, correct to three significant figures:

(a) $f(1) + g(1) + h(1)$ (b) $f(g(h(1)))$

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14. (a) Copy and complete the table, giving values correct to three decimal places.

x	1	1·5	2
$\sin x$			

(b) Hence use Simpson's rule with three function values to find $\int_1^2 \sin x\, dx$ (give your answer to one decimal place).

15. Simplify:

(a) $\sin(\pi - x)$ (c) $\cos(\pi - x)$ (e) $\tan(\pi + x)$ (g) $\sec(\pi - x)$
(b) $\sin(\frac{\pi}{2} - x)$ (d) $\cos(\pi + x)$ (f) $\tan(2\pi - x)$ (h) $\operatorname{cosec}(\frac{\pi}{2} - x)$

16. Find the exact value of:

(a) $\cos(-\frac{\pi}{3})$ (c) $\tan\frac{29\pi}{6}$ (e) $\operatorname{cosec}\frac{3\pi}{2}$
(b) $\sin\frac{11\pi}{4}$ (d) $\cot\frac{2\pi}{3}$ (f) $\sec\frac{5\pi}{4}$

17. Solve each of these equations for $0 \le x \le 2\pi$:

(a) $\tan x = \sqrt{2} - 1$ (c) $\cos^2 x + \cos x = 0$ (e) $3\sin^2 x = \cos^2 x$
(b) $2\sin x \cos x = \cos x$ (d) $2\cos^2 x = \cos x + 1$ (f) $\tan^2 x + \tan x = \sqrt{3}\tan x + \sqrt{3}$

18. Solve for $-\pi \le x \le \pi$:

(a) $\sin 2x = \frac{1}{2}$ (c) $\tan(x - \frac{\pi}{6}) = \sqrt{3}$ (e) $\operatorname{cosec}(x - \frac{3\pi}{4}) = 1$
(b) $\cos 3x = -1$ (d) $\sec(x + \frac{\pi}{4}) = -\sqrt{2}$ (f) $\cot(x + \frac{5\pi}{6}) = \sqrt{3}$

19. Express $\frac{11\pi}{320}$ radians in degrees, minutes and seconds.

20. Express $25°21'$ in radians in terms of π.

21. The angles of a certain pentagon are in arithmetic progression, and the largest angle is double the smallest. Find, in radians, the size of each angle of the pentagon.

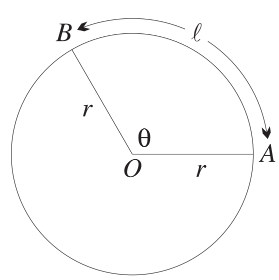
EXTENSION

22. (a) Explain why $\sin n \ne 0$, for all integers n. (b) Use your calculator to find the first positive integer n for which $|\sin n| < 0{\cdot}01$, and explain your result.

23. Given that $0 \le \theta \le 2\pi$, solve the equation $4\cos^2\theta + 2\sin\theta = 3$ (give your solutions in terms of π).

14 B Mensuration of Arcs, Sectors and Segments

Calculations of the lengths of arcs and the areas of sectors and segments are already possible, but radian measure allows the three formulae to be expressed in more elegant forms.



Arc Length: In the diagram on the right, AB is an *arc* of length ℓ subtending an angle θ at the centre O of a circle with radius r. The definition of angle size is $\theta = \dfrac{\ell}{r}$, which means that $\ell = r\theta$.

4 ARC LENGTH: $\ell = r\theta$

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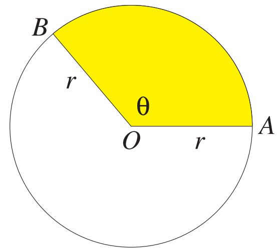
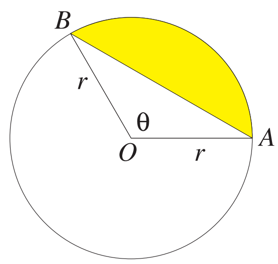
Area of a Sector: In the next diagram, the *sector* AOB is the shaded area bounded by the arc AB and the two radii OA and OB. Its area is a proportion of the total area:



$$\begin{aligned}\text{area of sector} &= \frac{\theta}{2\pi} \times (\text{area of circle})\\ &= \frac{\theta}{2\pi} \times \pi r^2\\ &= \tfrac{1}{2}r^2\theta.\end{aligned}$$

5 AREA OF SECTOR: $\text{Area} = \tfrac{1}{2}r^2\theta$

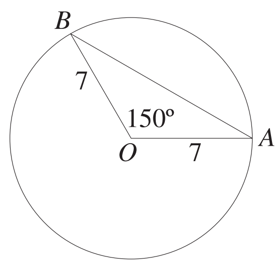
Area of a Segment: In the third diagram, the *segment* is the shaded area between the arc AB and the chord AB. Its area is the area of the sector OAB minus the area of the isosceles triangle $\triangle OAB$, whose area is given by the usual area formula for a triangle:



$$\begin{aligned}\text{area of segment} &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\\ &= \tfrac{1}{2}r^2(\theta - \sin\theta).\end{aligned}$$

6 AREA OF SEGMENT: $\text{Area} = \tfrac{1}{2}r^2(\theta - \sin\theta)$

WORKED EXERCISE:



(a) Find the lengths of the minor and major arcs formed by two radii of a circle of radius 7 metres meeting at 150°.
(b) Find the areas of the minor and major sectors.
(c) Find the length of the chord AB
(d) Find the areas of the major and minor segments.

NOTE: An arc, sector or segment is called *minor* if the angle subtended at the centre is less than 180°, and *major* if the angle is a reflex angle. Obviously the angles subtended at the centre by a minor arc and the corresponding major arc add to 360°.

SOLUTION:

(a) The minor arc subtends 150° at the centre, which in radians is $\frac{5\pi}{6}$, and the major arc subtends an angle of 210°, which in radians is $\frac{7\pi}{6}$,

so
$$\begin{aligned}\text{minor arc} &= r\theta\\ &= 7 \times \tfrac{5\pi}{6}\\ &= \tfrac{35\pi}{6}\text{ metres.}\end{aligned}$$
and
$$\begin{aligned}\text{major arc} &= r\theta\\ &= 7 \times \tfrac{7\pi}{6}\\ &= \tfrac{49\pi}{6}\text{ metres.}\end{aligned}$$

(b)
$$\begin{aligned}\text{Minor sector} &= \tfrac{1}{2}r^2\theta\\ &= \tfrac{1}{2} \times 7^2 \times \tfrac{5\pi}{6}\\ &= \tfrac{245\pi}{12}\text{ m}^2.\end{aligned}$$
$$\begin{aligned}\text{Major sector} &= \tfrac{1}{2}r^2\theta\\ &= \tfrac{1}{2} \times 7^2 \times \tfrac{7\pi}{6}\\ &= \tfrac{343\pi}{12}\text{ m}^2.\end{aligned}$$

(c) Using the cosine rule,
$$\begin{aligned}AB^2 &= 7^2 + 7^2 - 2 \times 7 \times 7 \times \cos\tfrac{5\pi}{6}\\ &= 98(1 + \tfrac{1}{2}\sqrt{3}\,)\\ &= 49(2 + \sqrt{3}\,),\\ AB &= 7\sqrt{2 + \sqrt{3}}\text{ metres.}\end{aligned}$$

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(d) Minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$
$= \frac{1}{2} \times 49(\frac{5\pi}{6} - \sin\frac{5\pi}{6})$
$= \frac{49}{2}(\frac{5\pi}{6} - \frac{1}{2})$
$= \frac{49}{12}(5\pi - 3)\,\text{m}^2.$

Major segment $= \frac{1}{2}r^2(\theta - \sin\theta)$
$= \frac{1}{2} \times 49(\frac{7\pi}{6} - \sin\frac{7\pi}{6})$
$= \frac{49}{2}(\frac{7\pi}{6} + \frac{1}{2})$
$= \frac{49}{12}(7\pi + 3)\,\text{m}^2.$

WORKED EXERCISE: Find, to the nearest millimetre, the radius of a circle in which:
(a) a sector, (b) a segment,
subtending an angle of 90° at the centre has area 1 square metre.

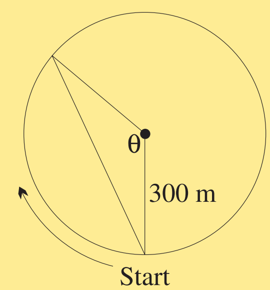
SOLUTION:

(a) Area of sector $= \frac{1}{2}r^2\theta$
$1 = \frac{1}{2} \times r^2 \times \frac{\pi}{2}$
$r^2 = \dfrac{4}{\pi}$
$r \doteqdot 1{\cdot}128$ metres.

(b) Area of segment $= \frac{1}{2}r^2(\theta - \sin\theta)$
$1 = \frac{1}{2} \times r^2 \times (\frac{\pi}{2} - 1)$
$r^2 = \dfrac{4}{\pi - 2}$
$r \doteqdot 1{\cdot}872$ metres.

WORKED EXERCISE: An athlete runs at a steady 4 m/s around a circular track of radius 300 metres. She runs clockwise, starting at the southernmost point on the track. (a) How far has she run after 10 minutes? (b) What total angle does her complete path subtend at the centre? (c) How far, in a direct line across the field, is she from her start? (d) What is then her bearing from the centre?

SOLUTION:



(a) Distance run $= 4 \times 10 \times 60$
$= 2400$ metres.

(b) Substituting into $\ell = r\theta$,
$2400 = 300\,\theta$
$\theta = 8$ (which is about $458°22'$).

(c) $(\text{Distance from start})^2 = 300^2 + 300^2 - 2 \times 300^2 \times \cos 8$
$= 300^2(2 - 2\cos 8),$
distance $= 300\sqrt{2 - 2\cos 8}$
$\doteqdot 454{\cdot}08$ metres.

(d) The original bearing was 180°T,
so final bearing $\doteqdot 180° + 458°22'$
$\doteqdot 278°22'$T.

Exercise 14B

1. A circle has radius 6 cm. Find the length of an arc of this circle that subtends an angle at the centre of: (a) 2 radians (b) $\frac{\pi}{3}$ radians

2. A circle has radius 8 cm. Find the area of a sector of this circle that subtends an angle at the centre of: (a) 1 radian (b) $\frac{3\pi}{8}$ radians

3. What is the radius of the circle in which an arc of length 10 cm subtends an angle of 2·5 radians at the centre?

4. If the area of a sector of a circle of radius 4 cm is 12 cm^2, find the angle at the centre in radians.

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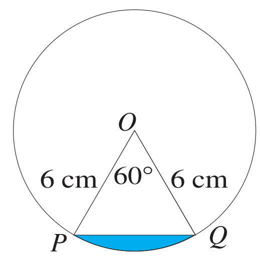
5. A circle has radius $3{\cdot}4$ cm. Find, correct to the nearest millimetre, the length of an arc of this circle that subtends an angle at the centre of: (a) $40°$ (b) $73°38'$

6. Find, correct to the nearest square metre, the area of a sector of a circle of radius 100 metres if the angle at the centre is $100°$.

7. A circle has radius 12 cm. Find, in exact form:
(a) the length of an arc that subtends an angle of $120°$ at the centre,
(b) the area of a sector in which the angle at the centre is $40°$.

8. An arc of a circle of radius $7{\cdot}2$ cm is $10{\cdot}6$ cm in length. Find the angle subtended at the centre by this arc, correct to the nearest degree.

9. A sector of a circle has area $52\,\text{cm}^2$ and contains an angle of $44°16'$. Find, in cm correct to one decimal place, the radius of the circle.

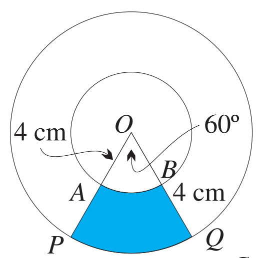


10. In the diagram on the right:
(a) Find the exact area of sector OPQ.
(b) Find the exact area of $\triangle OPQ$.
(c) Hence find the exact area of the shaded minor segment.

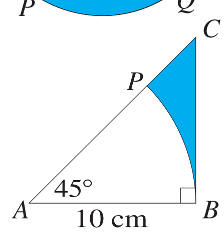
11. A chord of a circle of radius 4 cm subtends an angle of $150°$ at the centre. Use the formula $A = \frac{1}{2}r^2(\theta - \sin\theta)$ to find the areas of the minor and major segments cut off by the chord.

12. A circle has centre C and radius 5 cm, and an arc AB of this circle has length 6 cm. Find the area of the sector ACB.

DEVELOPMENT

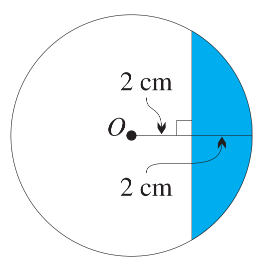


13. The diagram on the right shows two concentric circles with common centre O.
(a) Find the exact perimeter of the region $APQB$.
(b) Find the exact area of the region $APQB$.



14. In the diagram on the right, $\triangle ABC$ is a triangle that is right-angled at B, $AB = 10$ cm and $\angle A = 45°$. The circular arc BP has centre A and radius AB. It meets the hypotenuse AC at P.
(a) Find the exact area of sector ABP.
(b) Hence find the exact area of the shaded portion BCP.

15. (a) Through how many radians does the minute hand of a watch turn between 7:10 am and 7:50 am?
(b) If the minute hand is $1{\cdot}2$ cm long, find, correct to the nearest centimetre, the distance travelled by its tip in that time.



16. (a) A wheel on a fixed axle is rotating at 200 revolutions per minute. Convert this angular velocity into radians per second, correct to the nearest whole number.
(b) If the radius of the wheel is $0{\cdot}3$ metres, how far, correct to the nearest metre, will a point on the outside edge of the wheel travel in one second?

17. Find the exact area of the shaded region.

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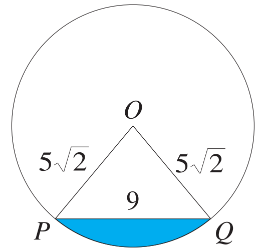
18. In a circle with centre O and radius $5\sqrt{2}$ cm, a chord of length 9 cm is drawn.

(a) Use the cosine rule to find $\angle POQ$ in radians, correct to two decimal places.

(b) Hence find, correct to the nearest cm^2, the area of the minor segment cut off by the chord.



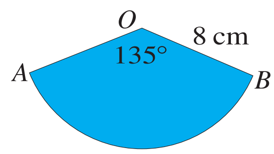
19. A piece of paper is in the shape of a sector of a circle. The radius is 8 cm and the angle at the centre is 135°. The straight edges of the sector are placed together so that a cone is formed.

(a) Show that the base of the cone has radius 3 cm.

(b) Show that the cone has perpendicular height $\sqrt{55}$ cm.

(c) Hence find, in exact form, the volume of the cone.

(d) Find the curved surface area of the cone.



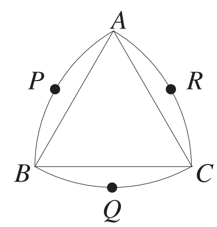
20. $\triangle ABC$ is equilateral and its side length is 2 cm. AB, BC and CA are circular arcs with centres C, A and B respectively.

(a) Find the length of the arc AB.

(b) Find the area of the sector $CAPBC$.

(c) Find the length of the perimeter $APBQCRA$.

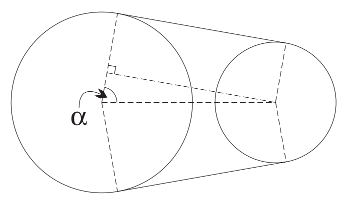
(d) Find the area of $\triangle ABC$ and hence find the area enclosed by the perimeter $APBQCRA$. (Give all answers in exact form.)



21. The diameters of two circular pulleys are 6 cm and 12 cm and their centres are 10 cm apart.

(a) Calculate the angle α in radians, correct to four decimal places.

(b) Hence find, in centimetres correct to one decimal place, the length of a taut belt that goes round the two pulleys.

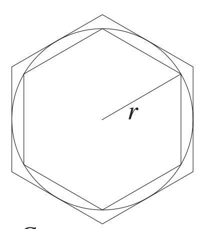


22. OP and OQ are radii of length r centimetres of a circle centred at O. The arc PQ of the circle subtends an angle of θ radians at O and the perimeter of the sector OPQ is 12 cm.

(a) Show that the area $A\,\text{cm}^2$ of the sector is given by $A = \dfrac{72\theta}{(2+\theta)^2}$.

(b) Hence find the maximum area of the sector.

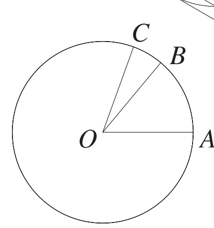
23. Consider two regular hexagons drawn inside and outside a circle of radius r, as shown in the diagram opposite. By considering the perimeters of the circle and the two hexagons, show that $3 < \pi < 2\sqrt{3}$.



24. In the diagram to the right, O is the centre of the circle and the arc AB is equal in length to the radius of the circle. C is a point on the circle such that $\triangle OAC$ is equilateral.

(a) By definition, what is the size of $\angle AOB$?

(b) Explain why B must lie on the minor arc AC, and hence show geometrically that one radian is less than 60°.



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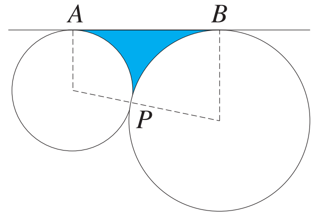
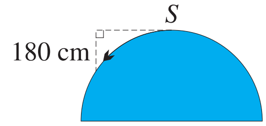


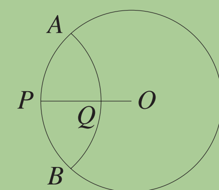
25. Two circles of radii 2 cm and 3 cm touch externally at P. AB is a common tangent. Calculate, in cm^2 correct to two decimal places, the area of the region bounded by the tangent and the arcs AP and BP.



26. A certain hill is represented by a hemisphere of radius 1 km. A man 180 cm tall walks down the hill from the summit S at 6 km/h. How long (correct to the nearest second) will it be before he is invisible to a person lying on the ground at S?

EXTENSION

27. A right circular cone is to be constructed by cutting a major sector from a circular sheet of paper and joining the two straight edges. Show that the volume of the cone is maximised when the central angle of the sector is $\dfrac{2\sqrt{2}\pi}{\sqrt{3}}$ radians.



28. In the diagram, P is a point on a circle with centre O. A circular arc AQB, with centre P, is drawn so as to divide the area of the given circle in the ratio $1:2$. If $\angle POA = \theta$, show that $\sin\theta + (\pi - \theta)\cos\theta = \frac{2\pi}{3}$. (You may need to use the result $\sin 2\theta = 2\sin\theta\cos\theta$.)

14 C Graphs of the Trigonometric Functions in Radians

Now that angle size has been defined as a ratio, that is, as a pure number, the trigonometric functions can be drawn in their true shape. On the next full page, the graphs of the six functions have been drawn using the same scale on the x-axis and y-axis. This means that the gradient of the tangent at each point now equals the true value of the derivative there, and the areas under the graphs faithfully represent the appropriate definite integrals.

Place a ruler on the graph of $y = \sin x$ to represent the gradient of the tangent at the origin. The ruler should lie along the line $y = x$, meaning that the gradient of the tangent is 1. As discussed earlier, this is necessary if the derivative of $y = \sin x$ is to be exactly $\cos x$. But all this needs to be proven properly, which will be done in later sections.

Amplitude of the Sine and Cosine Functions: The *amplitude* of a wave is the maximum height of the wave above the mean position. As we have seen, $y = \sin x$ and $y = \cos x$ have a maximum value of 1, a minimum value of -1, and a mean value of 0, so both have amplitude 1.

More generally, for positive constants a and b, the functions $y = a\sin bx$ and $y = a\cos bx$ have maximum value a and minimum value $-a$, so their amplitude is a.

7 AMPLITUDE: The functions $y = \sin x$ and $y = \cos x$ have amplitude 1.
The functions $y = a\sin bx$ and $y = a\cos bx$ have amplitude a.

The other four trigonometric functions increase without bound near their asymptotes, and so the idea of amplitude makes no sense. The vertical scale of $y = a\tan bx$, however, can conveniently be tied down using the fact that $\tan\frac{\pi}{4} = 1$.

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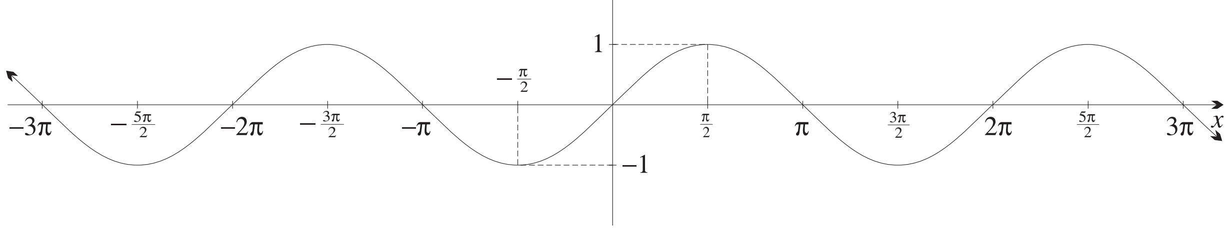
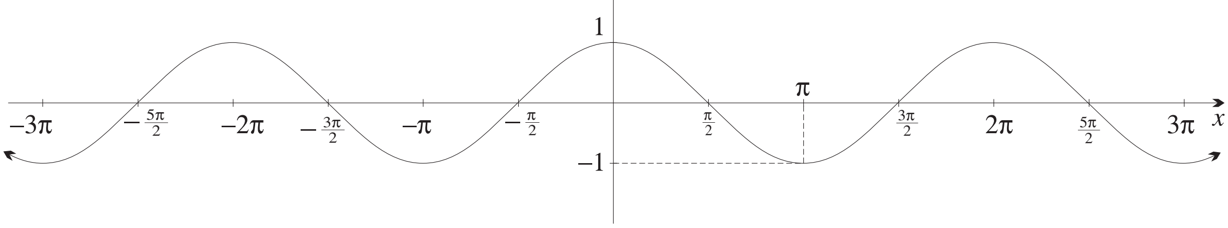
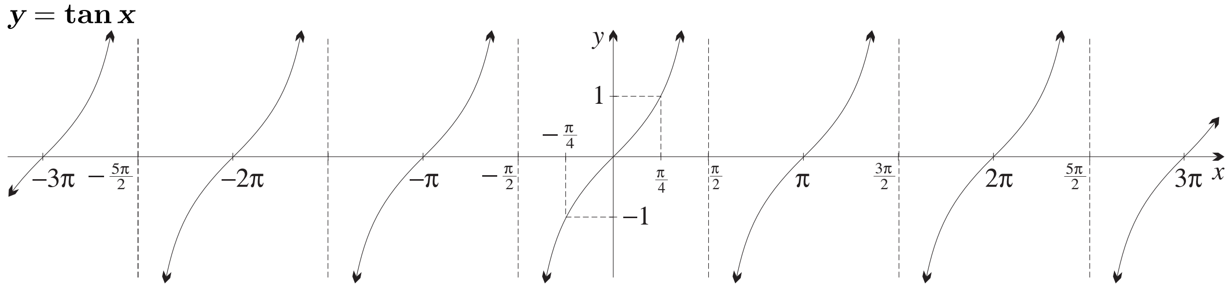
$y = \sin x$



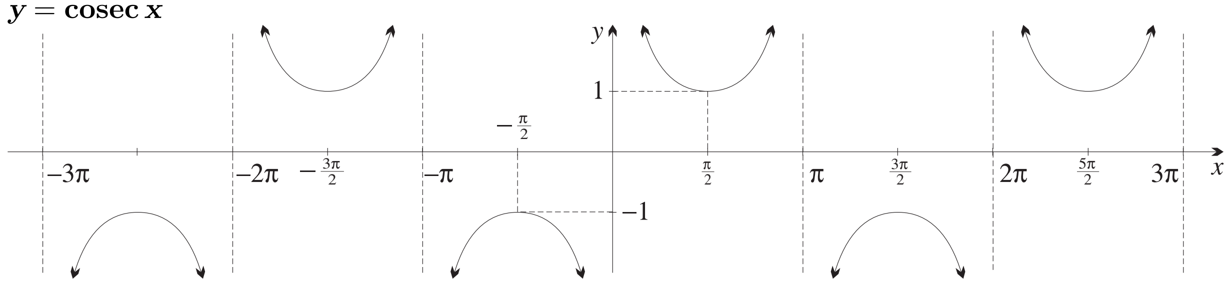
$y = \cos x$



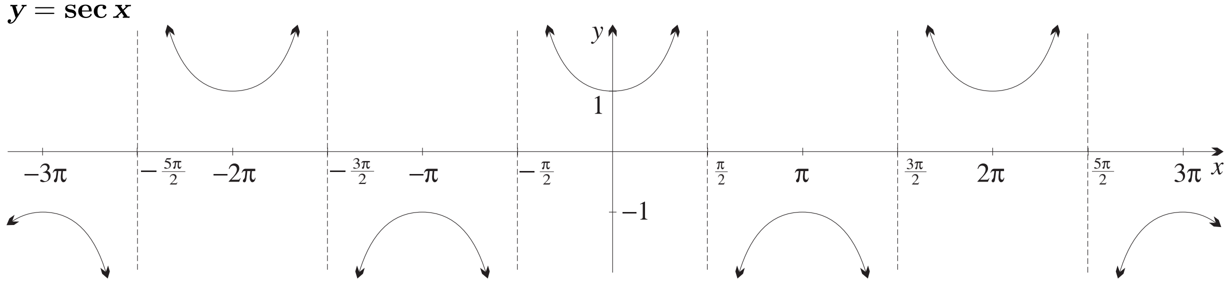
$y = \tan x$



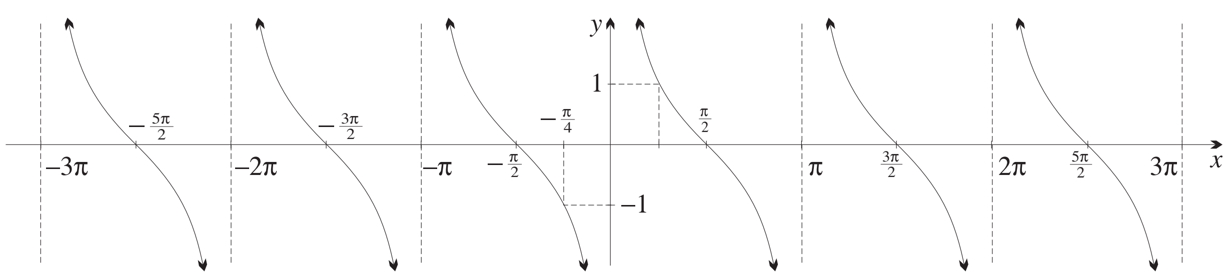
$y = \operatorname{cosec} x$



$y = \sec x$



$y = \cot x$



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The Periods of the Trigonometric Functions: The six trigonometric functions are called *periodic functions* because each of them repeats itself exactly over and over again. The *period* of such a function is the length of the smallest repeating unit. The sine and cosine curves on the previous page are waves that repeat every revolution, so they have period 2π. Their reciprocal functions secant and cosecant also have period 2π. The tangent and cotangent functions, on the other hand, repeat every half revolution, so they have period π.

For positive constants a and b, the function $y = a\sin bx$ is also periodic. As explained in Section 2I, replacing x by bx stretches the graph horizontally by a factor of $\frac{1}{b}$, so $y = a\sin bx$ must have period $\frac{2\pi}{b}$. The same applies to the other five trigonometric functions, and in summary:

8 THE PERIODS OF THE TRIGONOMETRIC FUNCTIONS:

A. $y = \sin x$, $y = \cos x$, $y = \sec x$ and $y = \operatorname{cosec} x$ each have period 2π.

$y = a\sin bx$, $y = a\cos bx$, $y = a\sec bx$ and $y = a\operatorname{cosec} bx$ have period $\dfrac{2\pi}{b}$.

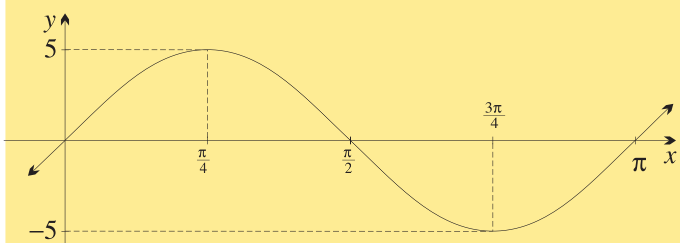
B. $y = \tan x$ and $y = \cot x$ each have period π.

$y = a\tan bx$ and $y = a\cot bx$ each have period $\dfrac{\pi}{b}$.

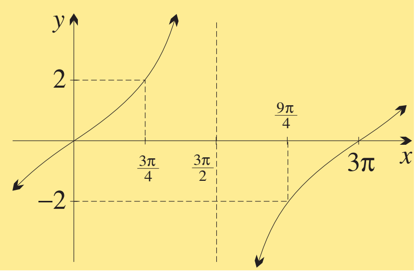
WORKED EXERCISE: Sketch one period of: (a) $y = 5\sin 2x$, (b) $y = 2\tan\frac{1}{3}x$, showing all intercepts, turning points and asymptotes.

SOLUTION: 7 5

(a) $y = 5\sin 2x$ has an amplitude of 5, and a period of $\dfrac{2\pi}{2} = \pi$.

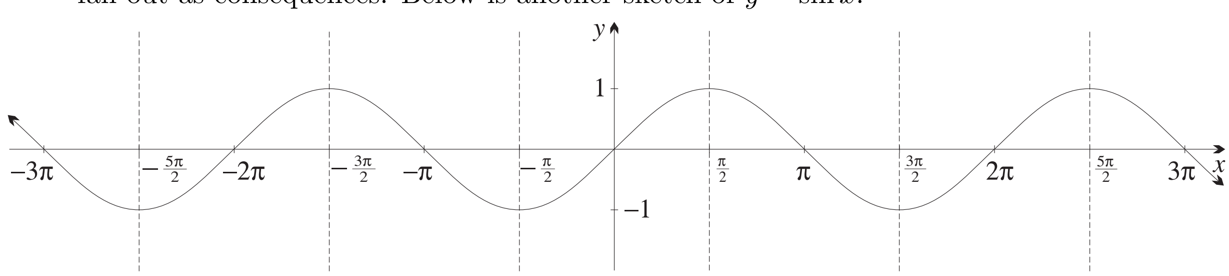


(b) $y = 2\tan\frac{1}{3}x$ has period $\dfrac{\pi}{1/3} = 3\pi$, and when $x = \frac{3\pi}{4}$, $y = 2$.



The Symmetries of the Sine Wave: The wavy graph of $y = \sin x$ has a great number of symmetries. Essentially one can build the whole curve by taking many copies of the rising part from $x = 0$ to $x = \frac{\pi}{2}$, and joining them all together in different orientations.

What follows is an account of these various transformations of the curve. When the symmetries are expressed algebraically, properties including oddness, the period, and the various identities known from the **A**ll **S**tations **T**o **C**entral diagram fall out as consequences. Below is another sketch of $y = \sin x$.



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Rotations of the Sine Curve: The sine curve has rotational symmetry about the origin, meaning that a rotation of 180° about the origin maps the curve onto itself. That is, $y = \sin x$ is odd, which can be stated algebraically as

$$\sin(-x) = -\sin x.$$

The curve is also mapped onto itself under rotations of 180° about each of its x-intercepts.

Translations of the Sine Curve: The period of the sine curve is 2π, and the curve is stable under translations to the right or left by 2π or by integer multiples of 2π. The stability under translation to the left by 2π can be written algebraically as

$$\sin(x + 2\pi) = \sin x.$$

If the curve is translated to the left by only π, it is the same as reflecting the graph in the x-axis (taking opposites). This is represented algebraically as

$$\sin(x + \pi) = -\sin x.$$

Translations to the right or left by odd multiples of π will all change $y = \sin x$ to $y = -\sin x$.

Reflections of the Sine Curve: The sine curve is mapped onto itself under reflections in every one of the dotted lines in the diagram above, that is, in the vertical lines through any turning point of the wave. The symmetry in the vertical line $x = \frac{\pi}{2}$ is expressed algebraically by

$$\sin(\pi - x) = \sin x,$$

because any point x on the horizontal axis is reflected to $\pi - x$ by reflection in $x = \frac{\pi}{2}$. On the other hand, reflection in the vertical line through any x-intercept has the same effect as reflecting the curve in the x-axis (taking opposites). In particular, reflection in the vertical line $x = \pi$ is represented algebraically by

$$\sin(2\pi - x) = -\sin x.$$

Transforming the Sine Wave into the Cosine Wave: The sine and cosine waves are sketched below on one set of axes. Clearly they have exactly the same shape and size, or in geometric language are *congruent*. The most obvious way of moving one wave onto the other is by translation:

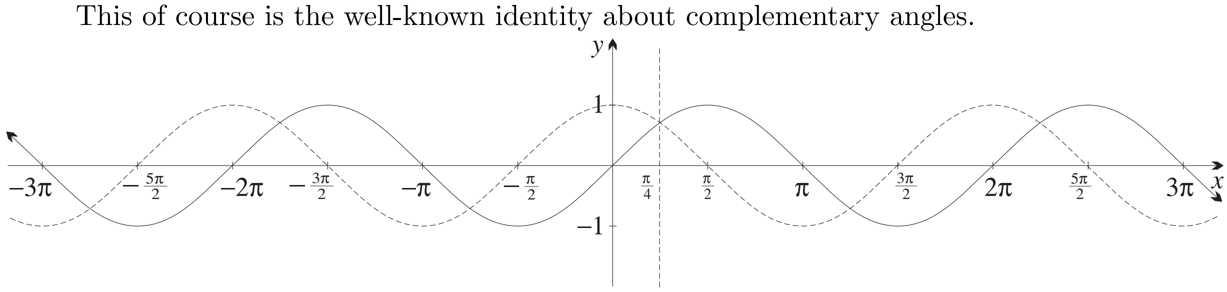
$\cos(x - \frac{\pi}{2}) = \sin x$ translates the cosine graph to the right by $\frac{\pi}{2}$,

$\sin(x + \frac{\pi}{2}) = \cos x$ translates the sine graph to the left by $\frac{\pi}{2}$.

The sine and cosine curves can also be reflected into each other by reflection in the vertical line $x = \frac{\pi}{4}$, as is clear in the diagram below:

$$\sin(\tfrac{\pi}{2} - x) = \cos x \qquad \text{or} \qquad \cos(\tfrac{\pi}{2} - x) = \sin x.$$

This of course is the well-known identity about complementary angles.



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Symmetries of the Other Trigonometric Functions: This rich variety of symmetry can be seen with each of the other five trigonometric functions, and the same treatment could well be given to each of them. One should not try to memorise these results. The intention is that they can easily be written down using knowledge of the various symmetries. Of particular interest are their evenness or oddness:

9

$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$
$\operatorname{cosec}(-x) = -\operatorname{cosec} x$	$\sec(-x) = \sec x$	$\cot(-x) = -\cot x$

their periods, associated with translations to the left by 2π or π:

10

$\sin(x + 2\pi) = \sin x$	$\cos(x + 2\pi) = \cos x$	$\tan(x + \pi) = \tan x$
$\operatorname{cosec}(x + 2\pi) = \operatorname{cosec} x$	$\sec(x + 2\pi) = \sec x$	$\cot(x + \pi) = \cot x$

the identities associated with the **A**ll **S**tations **T**o **C**entral diagram:

11

$\sin(\pi - x) = \sin x$	$\cos(\pi - x) = -\cos x$	$\tan(\pi - x) = -\tan x$
$\sin(\pi + x) = -\sin x$	$\cos(\pi + x) = -\cos x$	$\tan(\pi + x) = \tan x$
$\sin(2\pi - x) = -\sin x$	$\cos(2\pi - x) = \cos x$	$\tan(2\pi - x) = -\tan x$
$\operatorname{cosec}(\pi - x) = \operatorname{cosec} x$	$\sec(\pi - x) = -\sec x$	$\cot(\pi - x) = -\cot x$
$\operatorname{cosec}(\pi + x) = -\operatorname{cosec} x$	$\sec(\pi + x) = -\sec x$	$\cot(\pi + x) = \cot x$
$\operatorname{cosec}(2\pi - x) = -\operatorname{cosec} x$	$\sec(2\pi - x) = \sec x$	$\cot(2\pi - x) = -\cot x$

and the three identities associated with reflection in the vertical line $x = \frac{\pi}{4}$, identities that involve exchanging an angle with its **co**mplement, and are the origin of the names of the three **co**-functions:

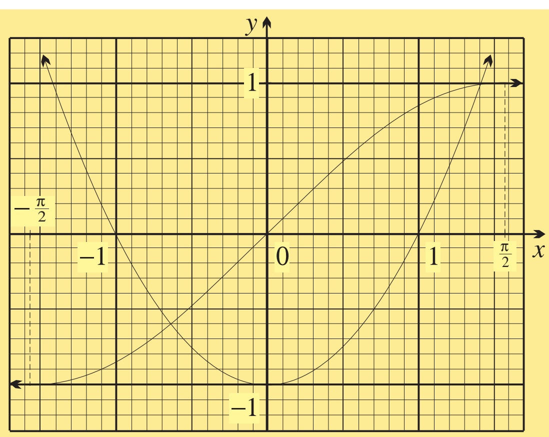
12

$\sin(\frac{\pi}{2} - x) = \cos x$	$\tan(\frac{\pi}{2} - x) = \cot x$	$\sec(\frac{\pi}{2} - x) = \operatorname{cosec} x$

Graphical Solutions of Trigonometric Equations: Many trigonometric equations cannot be solved by algebraic methods. As usual, a graph-paper sketch will show where the solutions are, and their approximate values.

WORKED EXERCISE: Find, by drawing a graph, the number of solutions to $\sin x = x^2 - 1$. Then use the graph to find approximations correct to one decimal place.

SOLUTION: Here are $y = \sin x$ and $y = x^2 - 1$.
Clearly the equation has two solutions.
The positive solution is $x \doteqdot 1{\cdot}4$,
and the negative solution is $x \doteqdot -0{\cdot}6$.



NOTE: Graphics calculators and computer packages are particularly useful here. They allow sketches to be drawn quickly, and many programs will give the approximate coordinates of the intersections.

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Exercise 14C

NOTE: Machine sketching would allow experience of many more similar examples, and may help elucidate the importance of period and amplitude.

1. Sketch on separate diagrams the graphs of the following functions for $-2\pi \le x \le 2\pi$. State the period in each case.
(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

2. On the same diagram, sketch each of the following functions for $0 \le \theta \le 2\pi$:
(a) $y = \sin\theta$ (b) $y = 2\sin\theta$ (c) $y = 4\sin\theta$

3. On the same diagram, sketch each of the following functions for $0 \le \alpha \le 2\pi$:
(a) $y = \cos\alpha$ (b) $y = \cos 2\alpha$ (c) $y = \cos 4\alpha$

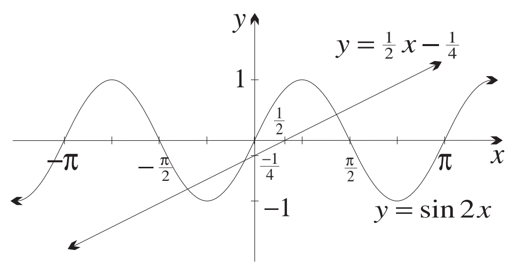
4. On the same diagram, sketch each of the following functions for $0 \le t \le 2\pi$:
(a) $y = \cos t$ (b) $y = \cos(t - \pi)$ (c) $y = \cos(t - \frac{\pi}{4})$

5. State the period and amplitude, then sketch on separate diagrams for $0 \le x \le 2\pi$:
(a) $y = \sin 2x$ (b) $y = 2\cos 2x$ (c) $y = 4\sin 3x$ (d) $y = 3\cos\frac{1}{2}x$ (e) $y = \tan 2x$

6. Sketch $y = \sin 2\pi x$ for $-1 \le x \le 2$.

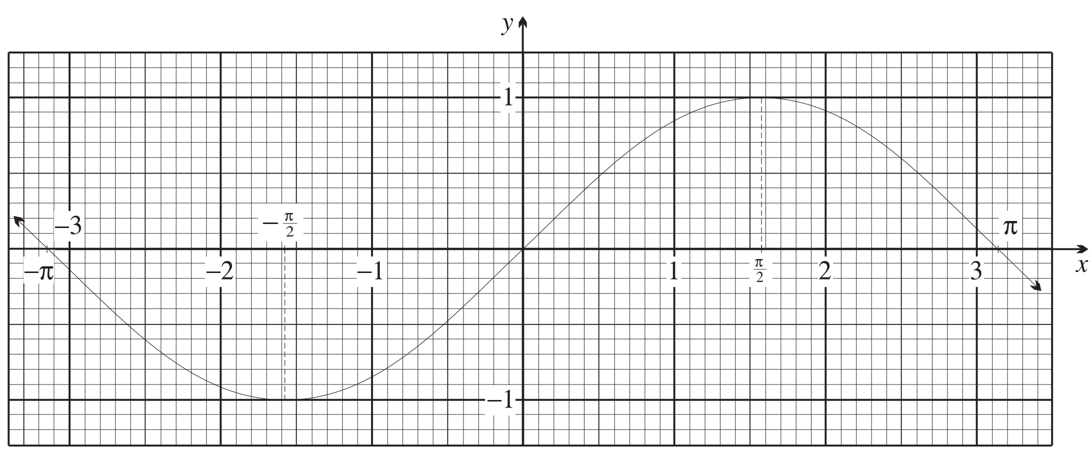
7. (a) Sketch $y = \sin x$ for $-\pi \le x \le \pi$. (b) On the same diagram sketch $y = \sin(x + \frac{\pi}{2})$ for $-\pi \le x \le \pi$. (c) Hence simplify $\sin(x + \frac{\pi}{2})$.

8. In the given diagram, the curve $y = \sin 2x$ is graphed for $-\pi \le x \le \pi$ and the line $y = \frac{1}{2}x - \frac{1}{4}$ is graphed.



(a) In how many points does the line $y = \frac{1}{2}x - \frac{1}{4}$ meet the curve $y = \sin 2x$?
(b) State the number of solutions of the equation $\sin 2x = \frac{1}{2}x - \frac{1}{4}$. How many of these solutions are positive?
(c) Briefly explain why the line $y = \frac{1}{2}x - \frac{1}{4}$ will not meet the curve $y = \sin 2x$ outside the domain $-\pi \le x \le \pi$.

9.



Photocopy the above graph of $y = \sin x$ for $-\pi \le x \le \pi$, and on it graph the line $y = \frac{1}{2}x$. Hence find the three solutions of the equation $\sin x = \frac{1}{2}x$, giving answers to one decimal place where necessary.

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DEVELOPMENT

10. (a) Sketch the curve $y = \cos 3x$ for $-\pi \le x \le \pi$.
(b) Hence sketch, on the same diagram, $y = 4\cos 3(x - \frac{\pi}{6})$ for $-\pi \le x \le \pi$.

11. Sketch the curve $y = 3 - \cos 2x$ for $0 \le x \le 2\pi$.

12. Sketch on separate diagrams for $0 \le \alpha \le 2\pi$:
(a) $y = \operatorname{cosec} 2\alpha$ (b) $y = 3\sec \frac{1}{2}\alpha$ (c) $y = \cot 2\alpha$

13. (a) Carefully sketch the curve $y = \sin^2 x$ for $0 \le x \le 2\pi$ after completing the table.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y					

(b) Explain why $y = \sin^2 x$ has range $0 \le y \le 1$.
(c) Write down the period and amplitude of $y = \sin^2 x$.

14. (a) Sketch the graph of $y = 2\cos x$ for $-2\pi \le x \le 2\pi$.
(b) On the same diagram, carefully sketch the line $y = 1 - \frac{1}{2}x$, showing its x- and y-intercepts.
(c) How many solutions does the equation $2\cos x = 1 - \frac{1}{2}x$ have?
(d) Mark with the letter P the point on the diagram from which the negative solution of the equation in (c) is obtained.
(e) Prove algebraically that if n is a solution of the equation in (c), then $-2 \le n \le 6$.

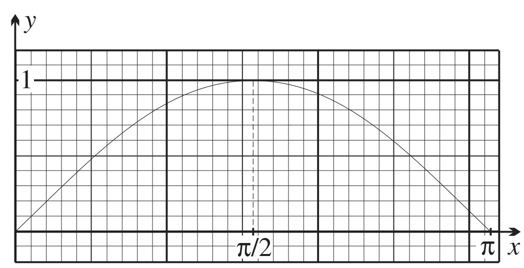
15. (a) What is the period of the function $y = \sin \frac{\pi}{2}x$?
(b) Sketch the curve $y = 1 + \sin \frac{\pi}{2}x$ for $0 \le x \le 4$.
(c) Through what fixed point does the line $y = mx$ always pass for varying values of m?
(d) By considering possible points of intersection of the graphs of $y = 1 + \sin \frac{\pi}{2}x$ and $y = mx$, find the range of values of m for which the equation $\sin \frac{\pi}{2}x = mx - 1$ has exactly one real solution in the domain $0 \le x \le 4$.

16. (a) Sketch the curve $y = 2\cos 2x$ for $0 \le x \le 2\pi$.
(b) Sketch the line $y = 1$ on the same diagram.
(c) How many solutions does the equation $\cos 2x = \frac{1}{2}$ have in the domain $0 \le x \le 2\pi$?
(d) What is the first positive solution to $\cos 2x = \frac{1}{2}$?
(e) Use your diagram to help you find the values of x in the domain $0 \le x \le 2\pi$ for which $\cos 2x < \frac{1}{2}$.

17. (a) On the same diagram, sketch $y = \sin x$ and $y = \cos x$, for $0 \le x \le 2\pi$.
(b) Hence, on the same diagram, carefully sketch $y = \sin x + \cos x$ for $0 \le x \le 2\pi$.
(c) What is the period of $y = \sin x + \cos x$?
(d) Estimate the amplitude of $y = \sin x + \cos x$ to one decimal place.

18. (a) Sketch $y = 3\sin 2x$ and $y = 4\cos 2x$ on the same diagram for $-\pi \le x \le \pi$.
(b) Hence sketch the graph of $y = 3\sin 2x - 4\cos 2x$ on the same diagram, for $-\pi \le x \le \pi$.
(c) Estimate the amplitude of the graph in (b).

19. (a) (i) Photocopy this graph of $y = \sin x$ for $0 \le x \le \pi$, and on it graph the line $y = \frac{3}{4}x$.
(ii) Measure the gradient of $y = \sin x$ at the origin. (Later, in Section 14G, you will prove that the exact value is 1.)



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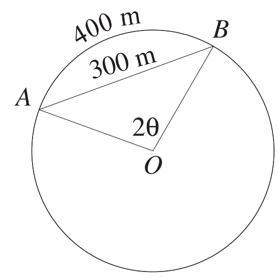
(iii) For what values of k does $\sin x = kx$ have a solution for $0 < x < \pi$?

(b) The diagram shows points A and B on a circle with centre O. $\angle AOB = 2\theta$, chord AB is of length 300 metres and the minor arc AB is of length 400 metres.

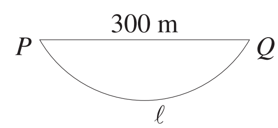


(i) Show that $\sin\theta = \frac{3}{4}\theta$.

(ii) Use the graph from (a)(i) to determine θ correct to one decimal place.

(iii) Hence find $\angle AOB$ in radians, correct to one decimal place, and show that the radius of the circle is about 154 metres.

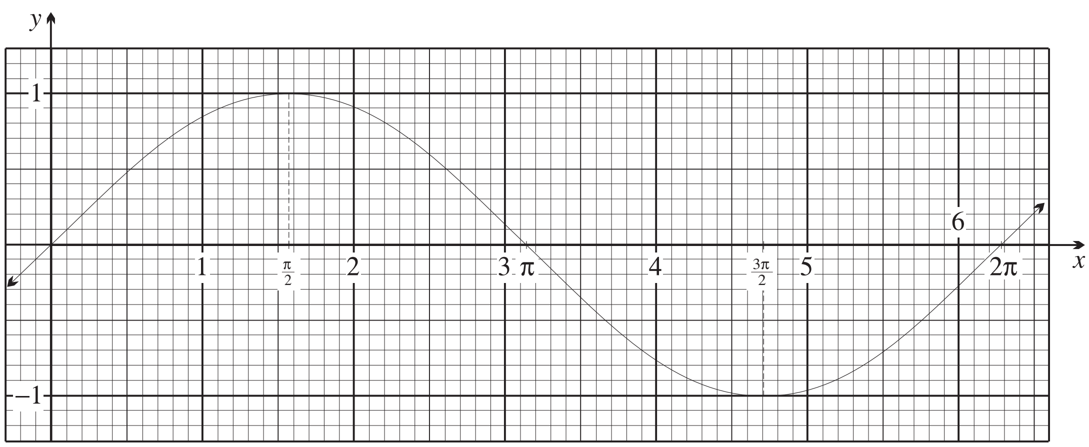
(c) P and Q are two points 300 metres apart. The circular arc PQ is of length ℓ metres.



(i) If C is the centre of the arc and $\angle PCQ = 2\alpha$, show that $\sin\alpha = \dfrac{300\alpha}{\ell}$.

(ii) Use your answer to (a)(iii) to find the possible range of values of ℓ.

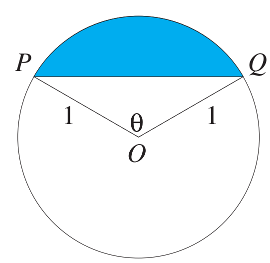
20.



(a) (i) Photocopy the above graph of $y = \sin x$ for $0 \le x \le 2\pi$, and on it carefully graph the line $y = x - 2$.

(ii) Use your graph to estimate, to two decimal places, the value of x for which $\sin x = x - 2$.

(b) The diagram shows points P and Q on a circle with centre O whose radius is one unit. $\angle POQ = \theta$. If the area of the shaded segment is one square unit, use part (a) to find θ, correct to the nearest degree.



(c) Suppose instead that the area of the segment is 2 square units.

(i) Show that $\sin\theta = \theta - 4$.

(ii) By drawing a suitable line on the graph in part (a), find θ to the nearest degree.

21. Show graphically that $x^2 - 2x + 4 > 3\sin x$ for all real values of x.

22. Sketch $y = \cos x$ and then answer these questions:

(a) Give the equations of all axes of symmetry. ($x = -2\pi$ to 2π will do.)

(b) Around which points does the graph have rotational symmetry?

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(c) What translations will leave the graph unchanged?

(d) Describe two translations that will move the graph of $y = \cos x$ to $y = -\cos x$.

(e) Describe two translations that will move the graph of $y = \cos x$ to the graph of $y = \sin x$.

(f) Name two vertical lines such that reflection of $y = \cos x$ in either of these lines will reflect the graph into the graph of $y = \sin x$.

23. Sketch $y = \tan x$ and then answer these questions:

(a) Explain whether $y = \tan x$ has any axes of symmetry.

(b) Name the points about which $y = \tan x$ has rotational symmetry.

(c) What translations will leave the graph unchanged?

(d) Name two vertical lines such that reflection of $y = \tan x$ in either of these lines will reflect the graph into the graph of $y = \cot x$.

EXTENSION

24. (a) Use a graphical approach to determine the number of positive solutions of $\sin x = \dfrac{x}{200}$.

(b) Find a positive integer value of n such that $\sin x = \dfrac{x}{n}$ has 69 positive solutions.

14 D Trigonometric Functions of Compound Angles

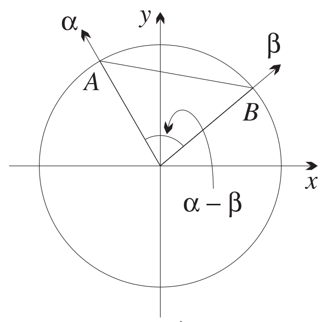
Before the differentiation of $\sin x$ at the origin can be extended to differentiation at all points on the curve, we will need to develop formulae for expanding objects like $\sin(x + h)$, $\tan(x - y)$, $\cos 2x$ and so forth. These trigonometric identities are most important for all sorts of other reasons as well, and must be thoroughly memorised.

As with all fundamental results in trigonometry, these formulae require an appeal to geometry, in this case Pythagoras' theorem in the form of the distance formula, the cosine rule, and the Pythagorean trigonometric identity. The approach given here begins with the expansion of $\cos(\alpha - \beta)$ and uses that result to derive the other expansions. There are many alternative approaches.

The Expansion of cos($\alpha - \beta$): We shall prove that for all real numbers α and β,

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

PROOF: Let the rays corresponding to the angles α and β intersect the circle $x^2 + y^2 = r^2$ at the points A and B respectively. Then by the definitions of the trigonometric functions for general angles,



$$A = (r\cos\alpha\,,\ r\sin\alpha) \quad \text{and} \quad B = (r\cos\beta\,,\ r\sin\beta).$$

Now we can use the distance formula to find AB^2:

$$\begin{aligned} AB^2 &= r^2(\cos\alpha - \cos\beta)^2 + r^2(\sin\alpha - \sin\beta)^2 \\ &= r^2(\cos^2\alpha + \cos^2\beta + \sin^2\alpha + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta) \\ &= 2r^2(1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta), \ \text{ since } \sin^2\theta + \cos^2\theta = 1. \end{aligned}$$

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But the angle $\angle AOB$ is $\alpha - \beta$, and so by the cosine rule,

$$\begin{aligned} AB^2 &= r^2 + r^2 - 2r^2\cos(\alpha-\beta) \\ &= 2r^2\big(1-\cos(\alpha-\beta)\big). \end{aligned}$$

Equating these two expressions for AB^2,

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

NOTE: It was claimed in the proof that $\angle AOB = \alpha - \beta$. This is not necessarily the case, because it's also possible that $\angle AOB = \beta - \alpha$, or that $\angle AOB$ differs from either of these two values by a multiple of 2π. But the cosine function is even, and it is periodic with period 2π. So it will still follow in every case that $\cos\angle AOB = \cos(\alpha-\beta)$, which is all that is required in the proof.

The Six Compound-Angle Formulae: Here are the compound-angle formulae for the sine, cosine and tangent functions, followed by the remaining five proofs.

13

THE COMPOUND-ANGLE FORMULAE:

A. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

B. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

C. $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

D. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

E. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

F. $\tan(\alpha-\beta) = \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

PROOF: We proceed from formula E, which has already been proven.

B. Replacing β by $-\beta$ in E, which is the expansion of $\cos(\alpha-\beta)$,

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\big(\alpha-(-\beta)\big) \\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \text{since cosine is even and sine is odd.} \end{aligned}$$

A. Using the identity $\sin\theta = \cos(\frac{\pi}{2}-\theta)$,

$$\begin{aligned} \sin(\alpha+\beta) &= \cos\big(\tfrac{\pi}{2}-(\alpha+\beta)\big) \\ &= \cos\big((\tfrac{\pi}{2}-\alpha)-\beta\big) \\ &= \cos(\tfrac{\pi}{2}-\alpha)\cos\beta + \sin(\tfrac{\pi}{2}-\alpha)\sin\beta \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta. \end{aligned}$$

D. Replacing β by $-\beta$, and noting that cosine is even and sine is odd,

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

C. Since $\tan\theta$ is the ratio of $\sin\theta$ and $\cos\theta$,

$$\begin{aligned} \tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}, \quad \text{dividing top and bottom by } \cos\alpha\cos\beta. \end{aligned}$$

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F. Replacing β by $-\beta$, and noting that the tangent function is odd,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

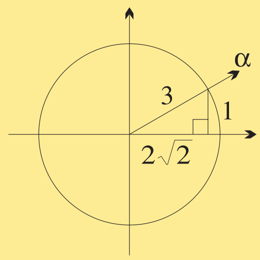
WORKED EXERCISE: Express $\sin(x + \frac{2\pi}{3})$ in the form $a\cos x + b\sin x$.

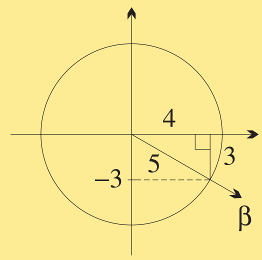
SOLUTION:
$$\begin{aligned}\sin(x + \tfrac{2\pi}{3}) &= \sin x\cos\tfrac{2\pi}{3} + \cos x\sin\tfrac{2\pi}{3}\\ &= -\tfrac{1}{2}\sin x + \tfrac{1}{2}\sqrt{3}\,\cos x\end{aligned}$$

WORKED EXERCISE: Given that $\sin\alpha = \frac{1}{3}$ and $\cos\beta = \frac{4}{5}$, where α is acute and $-\frac{\pi}{2} < \beta < 0$, find $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$.

SOLUTION: First, using the diagrams on the right,
$\cos\alpha = \frac{2}{3}\sqrt{2}$ and $\sin\beta = -\frac{3}{5}$.

So
$$\begin{aligned}\sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\\ &= \tfrac{4}{15} + \tfrac{6}{15}\sqrt{2}\\ &= \tfrac{2}{15}(2 + 3\sqrt{2}),\end{aligned}$$
and
$$\begin{aligned}\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \tfrac{8}{15}\sqrt{2} + \tfrac{3}{15}\\ &= \tfrac{1}{15}(8\sqrt{2} + 3).\end{aligned}$$





The Double-Angle Formulae: Replacing both α and β by θ in the compound-angle formula $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ gives

$$\sin 2\theta = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta.$$

The same process gives expansions of $\cos 2\theta$ and $\tan 2\theta$.

14 THE DOUBLE-ANGLE FORMULAE:
$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ \cos 2\theta &= \cos^2\theta - \sin^2\theta\\ \tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

The expansion of $\cos 2\theta$ can then be combined with the Pythagorean identity to give two other forms of the expansion, first by using $\sin^2\theta = 1 - \cos^2\theta$, then by using $\cos^2\theta = 1 - \sin^2\theta$.

15 THE $\cos 2\theta$ FORMULAE:
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\end{aligned}$$

WORKED EXERCISE: Prove that $(\sin x + \cos x)^2 = 1 + \sin 2x$.

SOLUTION:
$$\begin{aligned}\text{LHS} &= (\sin x + \cos x)^2\\ &= \sin^2 x + \cos^2 x + 2\sin x\cos x\\ &= 1 + \sin 2x\\ &= \text{RHS}.\end{aligned}$$

WORKED EXERCISE: Express $\cot 2x$ in terms of $\cot x$.

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SOLUTION: $\cot 2x = \dfrac{1}{\tan 2x}$

$= \dfrac{1 - \tan^2 x}{2 \tan x}$

$= \dfrac{\cot^2 x - 1}{2 \cot x}$, after division of top and bottom by $\tan^2 x$.

Further Exact Values of Trigonometric Functions: The various compound-angle formulae can be used to find expressions in surds for trigonometric functions of many angles other than ones whose related angles are the standard 30°, 45° and 60°.

WORKED EXERCISE: Find exact values of: (a) $\sin 75°$ (b) $\cos 75°$ (c) $\tan 22\frac{1}{2}°$

SOLUTION: [There are many alternative methods.]

(a) $\sin 75° = \sin(30° + 45°)$
$= \sin 30° \cos 45° + \cos 30° \sin 45°$
$= \frac{1}{2} \times \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3} \times \frac{1}{2}\sqrt{2}$
$= \frac{1}{4}(\sqrt{2} + \sqrt{6}\,)$

(b) $\cos 75° = \cos(135° - 60°)$
$= \cos 135° \cos 60° + \sin 135° \sin 60°$
$= -\frac{1}{2}\sqrt{2} \times \frac{1}{2} + \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3}$
$= \frac{1}{4}(\sqrt{6} - \sqrt{2}\,)$

(c) $\tan 45° = \dfrac{2 \tan 22\frac{1}{2}°}{1 - \tan^2 22\frac{1}{2}°}$
$1 \times (1 - \tan^2 22\frac{1}{2}°) = 2 \tan 22\frac{1}{2}°$
$\tan^2 22\frac{1}{2}° + 2 \tan 22\frac{1}{2}° - 1 = 0.$
$\Delta = 4 + 4$
$= 4 \times 2$
$\tan 22\frac{1}{2}° = \dfrac{-2 + 2\sqrt{2}}{2}$ or $\dfrac{-2 - 2\sqrt{2}}{2}$
$= \sqrt{2} - 1$, since $\tan 22\frac{1}{2}° > 0$.

Exercise 14D

1. Expand:
(a) $\sin(x - y)$
(b) $\cos(2A + 3B)$
(c) $\sin(3\alpha + 5\beta)$
(d) $\cos(\theta - \frac{\phi}{2})$
(e) $\tan(A + 2B)$
(f) $\tan(3\alpha - 4\beta)$

2. Express as a single trigonometric function:
(a) $\cos x \cos y - \sin x \sin y$
(b) $\sin 3\alpha \cos 2\beta + \cos 3\alpha \sin 2\beta$
(c) $\dfrac{\tan 40° - \tan 20°}{1 + \tan 40° \tan 20°}$
(d) $\sin 5A \cos 2A - \cos 5A \sin 2A$
(e) $\cos 70° \cos 20° + \sin 70° \sin 20°$
(f) $\dfrac{\tan \alpha + \tan 10°}{1 - \tan \alpha \tan 10°}$

3. Use the double-angle formulae to simplify:
(a) $2 \sin x \cos x$
(b) $\cos^2 \theta - \sin^2 \theta$
(c) $\dfrac{2 \tan \alpha}{1 - \tan^2 \alpha}$
(d) $2 \sin 20° \cos 20°$
(e) $2 \cos^2 50° - 1$
(f) $\dfrac{2 \tan 70°}{1 - \tan^2 70°}$
(g) $2 \sin 3\theta \cos 3\theta$
(h) $1 - 2 \sin^2 2A$
(i) $\dfrac{2 \tan 4x}{1 - \tan^2 4x}$

4. Use the compound-angle formulae to prove:
(a) $\sin(90° + A) = \cos A$
(b) $\cos(90° - A) = \sin A$
(c) $\tan(360° - A) = -\tan A$
(d) $\tan(180° + A) = \tan A$
(e) $\cos(270° - A) = -\sin A$
(f) $\sin(360° - A) = -\sin A$

5. Prove the identities:
(a) $\sin(A + 45°) = \dfrac{1}{\sqrt{2}}(\sin A + \cos A)$
(b) $2 \cos(\theta + \frac{\pi}{3}) = \cos \theta - \sqrt{3} \sin \theta$
(c) $\tan(\frac{\pi}{4} - A) = \dfrac{1 - \tan A}{1 + \tan A}$
(d) $\sin(A - 30°) = \frac{1}{2}(\sqrt{3} \sin A - \cos A)$

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6. Prove:

(a) $(\cos A - \sin A)(\cos A + \sin A) = \cos 2A$

(b) $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$

(c) $\sin 2\theta = 2\sin\theta\sin(\frac{\pi}{2} - \theta)$

(d) $\dfrac{1}{1-\tan\theta} - \dfrac{1}{1+\tan\theta} = \tan 2\theta$

7. (a) By expressing 15°as (45° − 30°), show that:

(i) $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (iii) $\tan 15° = 2 - \sqrt{3}$

(b) Hence find surd expressions for: (i) $\sin 75°$ (ii) $\cos 75°$ (iii) $\tan 75°$

8. (a) If $\cos\alpha = \frac{4}{5}$, find $\cos 2\alpha$.

(b) If $\sin x = \frac{2}{3}$, find $\cos 2x$.

(c) If $\sin\theta = \frac{5}{13}$ and θ is acute, find $\sin 2\theta$.

(d) If $\tan A = \frac{1}{2}$, find $\tan 2A$.

9. (a) If $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$, find $\tan(\alpha+\beta)$.

(b) If $\cos A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$, where A and B are both acute, find $\sin(A+B)$.

(c) If $\sin\theta = \frac{2}{3}$ and $\cos\phi = \frac{1}{4}$, where θ and ϕ are both acute, find $\cos(\theta - \phi)$.

10. Prove the identities:

(a) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

(b) $\cos(x-y) - \cos(x+y) = 2\sin x\sin y$

(c) $\sin(x+y) + \cos(x-y) = (\sin x + \cos x)(\sin y + \cos y)$

DEVELOPMENT

11. If $\sin x = \frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of $\sin 2x$.

12. If $\sin A = \frac{2}{3}$, $\frac{\pi}{2} < A < \pi$, and $\tan B = \frac{2}{3}$, $\pi < B < \frac{3\pi}{2}$, show that $\cos(A+B) = \dfrac{3\sqrt{5}+4}{3\sqrt{13}}$.

13. Find the exact value of: (a) $\cos 105°$ (b) $\sin\frac{13\pi}{12}$ (c) $\cot 285°$

14. Show that: (a) $\sin(\frac{\pi}{4} - \theta)\cos(\frac{\pi}{4} + \theta) + \cos(\frac{\pi}{4} - \theta)\sin(\frac{\pi}{4} + \theta) = 1$

(b) $\tan 35° + \tan 10° + \tan 35°\tan 10° = 1$

15. Show that:

(a) $\cos^4\alpha - \sin^4\alpha = \cos 2\alpha$

(b) $\cos 2x + \cos x = (\cos x + 1)(2\cos x - 1)$

(c) $\dfrac{\sin 2A}{1 - \cos 2A} = \cot A$

(d) $\cos\theta - \sin\theta\sin 2\theta = \cos\theta\cos 2\theta$

(e) $\tan(\frac{\pi}{4} + \alpha) - \tan(\frac{\pi}{4} - \alpha) = 2\tan 2\alpha$

(f) $\dfrac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta + 1} = \tan\theta$

16. Prove: (a) $\dfrac{2\sin(x-y)}{\cos(x+y) - \cos(x-y)} = \cot x - \cot y$

(b) $\dfrac{\sin(\theta+\phi)}{\cos(\theta-\phi)} = \dfrac{\tan\theta + \tan\phi}{1 + \tan\theta\tan\phi}$

(c) $\dfrac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} + \dfrac{\sin(\beta-\gamma)}{\sin\beta\sin\gamma} + \dfrac{\sin(\gamma-\alpha)}{\sin\gamma\sin\alpha} = 0$

17. (a) Show that $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

(b) Hence simplify $\sin^2 75° - \sin^2 15°$.

18. Show that: (a) $2\sin\frac{4\pi}{5}\cos\frac{\pi}{5} = \sin\frac{2\pi}{5}$ (b) $\cos^2\frac{4\pi}{7} - \sin^2\frac{3\pi}{7} = \cos\frac{6\pi}{7}$

19. (a) Find the value of $\tan 135°$.

(b) Let $\tan 67\frac{1}{2}° = t$. Use the double-angle formula for $\tan 2\theta$ to show that $t^2 - 2t - 1 = 0$.

(c) Hence find the exact value of $\tan 67\frac{1}{2}°$.

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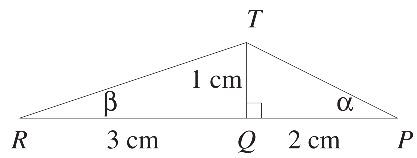
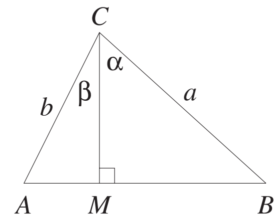
20. (a) Show that $\dfrac{1-\cos 2x}{1+\cos 2x} = \tan^2 x$.

(b) Hence find the exact value of $\tan \frac{\pi}{8}$.



21. (a) Write down the formula for $\sin(\alpha+\beta)$.

(b) Hence show that the angles α and β in the diagram to the right have a sum of $45°$.



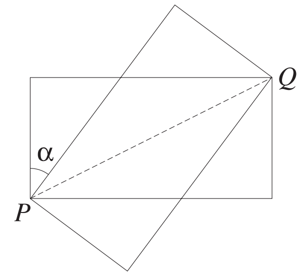
22. [The expansion of $\sin(\alpha+\beta)$ using areas] The diagram shows the altitude CM of $\triangle ABC$, with $\angle MCB = \alpha$ and $\angle MCA = \beta$.

(a) Show that $CM = b\cos\beta = a\cos\alpha$.

(b) Find the areas of $\triangle MCB$, $\triangle MCA$ and $\triangle CAB$ in terms of a, b, α and β.

(c) Hence prove the compound-angle formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$



23. The diagram shows two rectangles. Each rectangle is 6 cm long and 3 cm wide, and they share a common diagonal PQ. Show that $\tan\alpha = \frac{3}{4}$.

EXTENSION

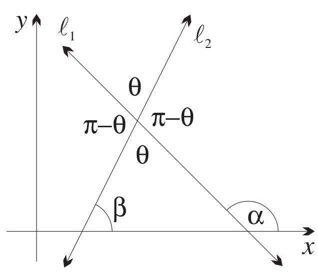
24. (a) Prove the trigonometric identity

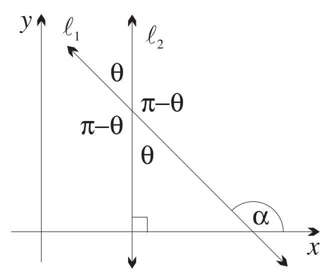
$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1-\tan A\tan B - \tan B\tan C - \tan C\tan A}.$$

(b) Hence show that in any triangle ABC, $\tan A + \tan B + \tan C = \tan A\tan B\tan C$.

14 E The Angle Between Two Lines

Suppose that the lines ℓ_1 and ℓ_2 are neither parallel nor perpendicular, as in the diagrams below. When the lines meet, they form four angles adding to 2π, with vertically opposite angles equal and adjacent angles supplementary. Let one of the acute angles be θ, then the four angles are θ, $\pi-\theta$, θ and $\pi-\theta$. Our task is to use trigonometry to find an exact expression for θ in terms of the gradients of the lines.





A Formula for the Angle Between Two Lines: Suppose first that neither line is vertical, as in the diagram on the left. Let the angles of inclination of ℓ_1 and ℓ_2 be α and β respectively, and let the gradients be m_1 and m_2 respectively. Then $m_1 = \tan\alpha$ and $m_2 = \tan\beta$. On the diagram given here, $\theta = \alpha - \beta$, but depending on which angle is larger, and whether the difference is acute or obtuse, θ could be

$$\alpha-\beta \quad \text{or} \quad \beta-\alpha \quad \text{or} \quad \pi-(\alpha-\beta) \quad \text{or} \quad \pi-(\beta-\alpha).$$

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The tangents of these four angles are the same apart from negative signs. Since θ is acute, $\tan\theta$ must be positive, so we can deal with all cases at once by using absolute value notation:

$$\tan\theta = |\tan(\alpha-\beta)| = \left|\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right| = \left|\frac{m_1-m_2}{1+m_1m_2}\right|.$$

Now suppose that the second line is vertical and the first line has gradient m, as in the diagram on the right. The acute angle θ is either $\frac{\pi}{2}-\alpha$ or $\alpha-\frac{\pi}{2}$, so in both cases

$$\tan\theta = \left|\tan(\tfrac{\pi}{2}-\alpha)\right| = |\cot\alpha| = \left|\frac{1}{\tan\alpha}\right| = \left|\frac{1}{m}\right|.$$

16 **ANGLES BETWEEN TWO LINES:** Suppose that two lines are neither parallel nor perpendicular, and let θ be the acute angle between them.

1. If neither line is vertical, then
$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|,\quad \text{where } m_1 \text{ and } m_2 \text{ are the respective gradients.}$$
2. If one line is vertical, then
$$\tan\theta = \left|\frac{1}{m}\right|,\quad \text{where } m \text{ is the gradient of the other line.}$$

NOTE: If substitution into the first formula gives the answer zero, then $m_1 = m_2$ and the lines are parallel, so the lines either coincide or don't meet at all. If substitution into that formula gives a zero denominator, then $m_1m_2 = -1$ and the lines are perpendicular.

WORKED EXERCISE: Find the tangents of the angles of the triangle formed by the three lines $\ell_1 : 2x + y = 1$, $\ell_2 : 2x - y = 5$ and $\ell_3 : x - 2y = 8$, then approximate the angles to the nearest minute.

SOLUTION: The gradients of the three lines are $m_1 = -2$, $m_2 = 2$ and $m_3 = \frac{1}{2}$.
First, $m_1 \times m_3 = -1$, so the lines ℓ_1 and ℓ_3 are perpendicular.
Let θ be the acute angle between ℓ_1 and ℓ_2, and ϕ the acute angle between ℓ_2 and ℓ_3.

Then
$$\begin{aligned}\tan\theta &= \left|\frac{m_2-m_1}{1+m_2m_1}\right| \\ &= \left|\frac{2+2}{1+2\times(-2)}\right| \\ &= \tfrac{4}{3},\end{aligned}$$
so $\theta \doteqdot 53^\circ 8'$.

Also,
$$\begin{aligned}\tan\phi &= \left|\frac{m_3-m_2}{1+m_3m_2}\right| \\ &= \left|\frac{\frac{1}{2}-2}{1+\frac{1}{2}\times 2}\right| \\ &= \tfrac{3}{4},\end{aligned}$$
so $\phi \doteqdot 36^\circ 52'$.

Notice that $\theta + \phi = 90^\circ$, as expected.

The Angle Between Two Curves: The angle between two curves can be defined clearly using tangents.

17 **ANGLE BETWEEN CURVES:** The angle between two curves at a point of intersection is the angle between the two tangents at this point.

WORKED EXERCISE: Find the points of intersection of $y = x^3 - x$ and $y = x - x^2$, and the exact values of the acute angles between the curves at these points.

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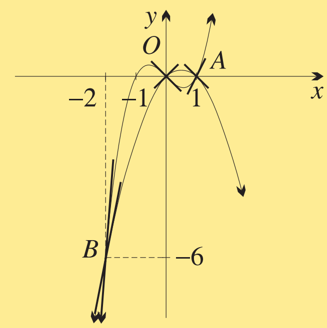

SOLUTION: Solving simultaneously, $x^3 - x = x - x^2$

$$x^3 + x^2 - 2x = 0$$
$$x(x+2)(x-1) = 0,$$

so they intersect at $O(0,0)$, $A(1,0)$ and $B(-2,-6)$.

The two derivatives are $\dfrac{dy}{dx} = 3x^2 - 1$ and $\dfrac{dy}{dx} = 1 - 2x$,

so the gradients of the tangents at the three points are:

point	$O(0,0)$	$A(1,0)$	$B(-2,-6)$
gradient of cubic	-1	2	11
gradient of quadratic	1	-1	5

The curves meet at right angles at O.

Let θ and ϕ be the acute angles between the curves at A and B respectively.

At A, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{2-(-1)}{1+2\times(-1)}\right| = 3,$

and so $\theta = \tan^{-1} 3$ [$\doteqdot 71^\circ 34'$].

At B, $\tan\phi = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{11-5}{1+11\times 5}\right| = \frac{3}{28},$

and so $\phi = \tan^{-1}\frac{3}{28}$ [$\doteqdot 6^\circ 7'$].

WORKED EXERCISE: [Questions like this are a little harder because they result in an absolute value equation.] Find the gradients of the lines making an angle of 60° with $3x - y + 4 = 0$.

SOLUTION: Let the gradient be m, then since the given line has gradient 3,

$$\tan 60^\circ = \left|\frac{m-3}{1+3m}\right|$$
$$|1+3m|\sqrt{3} = |m-3|.$$

Squaring both sides, $27m^2 + 18m + 3 = m^2 - 6m + 9$

$$13m^2 + 12m - 3 = 0.$$

Using the quadratic formula, $\Delta = 144 + 156 = 300,$

$$m = \tfrac{1}{26}(-12 + 10\sqrt{3}) \text{ or } \tfrac{1}{26}(-12 - 10\sqrt{3})$$
$$= \tfrac{1}{13}(-6 + 5\sqrt{3}) \text{ or } \tfrac{1}{13}(-6 - 5\sqrt{3}).$$

Exercise 14E

NOTE: In each question that asks for an angle, first find and simplify the tangent of the angle, which is known exactly, before finding an approximation to the angle.

1. Find, to the nearest degree, the acute angle between two lines whose gradients are:

(a) $m_1 = 2$ and $m_2 = \frac{1}{2}$ (b) $m_1 = 4$ and $m_2 = -2$ (c) $m_1 = 1\frac{1}{4}$ and $m_2 = \frac{1}{3}$

2. Find the acute angle between each pair of lines:

(a) $y = 3x - 1$ and $y = -2x + 5$

(b) $x - 2y + 1 = 0$ and $x + 3y + 2 = 0$

(c) $4x - 3y + 2 = 0$ and $7x + y - 1 = 0$

(d) $4x + 3y + 5 = 0$ and $3x - 4y - 2 = 0$

(e) $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$

(f) $y + 4x - 7 = 0$ and $y = 5 - 4x$

3. Find, to the nearest minute, the acute angle between the lines $3x + 4y = 8$ and $2x + 3y = 5$.

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4. Use the formula $\tan\theta = \left|\dfrac{1}{m}\right|$ to find, correct to the nearest minute, the acute angle between the vertical line $x = -1$ and the line $4x - 3y + 5 = 0$.

5. (a) Find the point of intersection of the parabolas $y = x^2$ and $y = (x-2)^2$.
(b) Sketch the two curves, showing their point of intersection.
(c) Find the acute angle, to the nearest degree, at which the curves intersect.

6. (a) Show that the curves $y = e^x$ and $y = x^2 - 3x + 1$ have a common point at $(0, 1)$.
(b) Find, to the nearest degree, the acute angle between the tangents to the curves at $(0, 1)$.

DEVELOPMENT

7. Find the acute angle and the obtuse angle between the lines:
(a) $2x - y + 1 = 0$ and $11x - (8 + 5\sqrt{3})y + 5\sqrt{3} = 0$
(b) $y = mx + b$ and $(m-1)x - (m+1)y = (m^2 - 1)b$

8. A, B and C are the points $(-2, 1)$, $(1, 7)$ and $(5, 0)$ respectively. Calculate the three interior angles of $\triangle ABC$ correct to the nearest minute and check their sum. Explain why the sum is not exactly 180°.

9. (a) The line $y = mx$ is inclined at 45° to $y = 2x - 4$. Find the two possible values of m.
(b) Explain geometrically why these two values of m represent perpendicular gradients.

10. (a) Show that the line $y - 1 = m(x - 1)$ always passes through the point $(1, 1)$.
(b) Find, in general form, the equations of the two lines through $(1, 1)$ that are inclined at 45° to the line $y = 5x + 6$.

11. (a) Find, in general form, the equations of the lines that pass through the point $(1, 2)$ and are inclined at an angle of 60° to the line $x + y = 3$.
(b) Find, in general form, the equations of the lines that pass through $(2, 3)$ and are inclined at an angle whose tangent ratio is $\frac{1}{3}$ to the line $y = 2x + 4$.

12. (a) Sketch the parabolas $y = 3x^2$ and $y = 4x - x^2$, showing their points of intersection.
(b) Find, to the nearest minute, the acute angles between these curves at their points of intersection.

13. (a) Show that the curves $y = \log_e(x - 1)$ and $y = \dfrac{2 - x}{x}$ meet at $(2, 0)$.
(b) Find $\tan\alpha$, where α is the acute angle between the tangents at $(2, 0)$.

14. Find the points of intersection of $y = x^2$ with the curve defined parametrically by $x = t^3$ and $y = t^2$. Find the angle between the curves at these points. Sketch the situation.

EXTENSION

15. Find the equations of the lines that pass through the origin and are inclined at 75° to the line $x + y + (y - x)\sqrt{3} = a$.

16. If ϕ is the acute angle between the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$, prove that

$$\cos\phi = \frac{|A_1A_2 + B_1B_2|}{\sqrt{({A_1}^2 + {B_1}^2)({A_2}^2 + {B_2}^2)}}\,.$$

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14 F The Behaviour of $\sin x$ Near the Origin

The next step in proving that $\sin x$ has derivative $\cos x$ is to establish that the curve $y = \sin x$ has gradient exactly 1 at the origin. Geometrically, this means that that the line $y = x$ is the tangent to $y = \sin x$ at the origin.

A Fundamental Inequality: First, we make an appeal to geometry and use areas to establish the following inequality concerning x, $\sin x$ and $\tan x$.

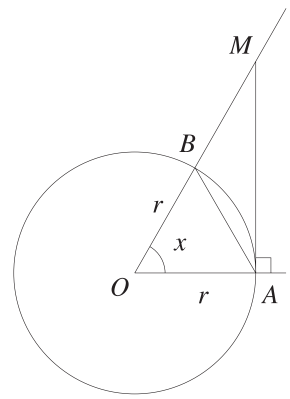
18 AN INEQUALITY CONCERNING $\sin x$ AND $\tan x$ NEAR THE ORIGIN:
A. $\sin x < x < \tan x$, for $0 < x < \frac{\pi}{2}$ (that is, x acute).
B. $\sin x > x > \tan x$, for $-\frac{\pi}{2} < x < 0$.

PROOF:

A. Suppose that x is an acute angle.
Construct a circle with centre O and any positive radius r, and a sector AOB subtending the angle x at the centre O.
Let the tangent at A meet the radius OB at M (OB will need to be produced), and join the chord AB.
Then area $\triangle OAB <$ area sector $OAB <$ area $\triangle OAM$

$$\tfrac{1}{2}r^2 \sin x < \tfrac{1}{2}r^2 x < \tfrac{1}{2}r^2 \tan x$$

$\boxed{\div \tfrac{1}{2}r^2}$ $\qquad \sin x < x < \tan x.$



B. Since x, $\sin x$ and $\tan x$ are all odd functions,
$\sin x > x > \tan x$, for $-\frac{\pi}{2} < x < 0$.

The Main Theorem: This inequality now allows the fundamental limits to be proven:

19 THE FUNDAMENTAL LIMITS: $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\displaystyle\lim_{x \to 0} \frac{\tan x}{x} = 1$

PROOF: Suppose first that x is acute, so that $\sin x < x < \tan x$.

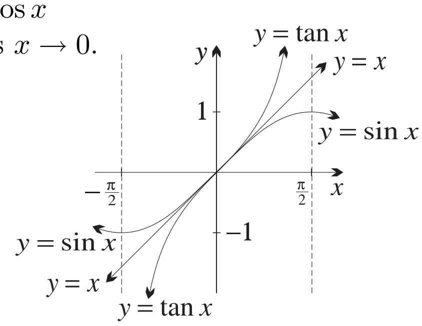
Dividing through by $\sin x$ gives $\qquad 1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$.

But $\cos x \to 1$ as $x \to 0^+$, and so $\qquad \dfrac{x}{\sin x} \to 1$ as $x \to 0^+$, as required.

Since $\dfrac{x}{\sin x}$ is even, it follows also that $\dfrac{x}{\sin x} \to 1$ as $x \to 0^-$.

Finally,
$$\frac{\tan x}{x} = \frac{\sin x}{x} \times \frac{1}{\cos x}$$
$$\to 1 \times 1, \text{ as } x \to 0.$$

The diagram on the right shows what has been proven about the graphs of $y = x$, $y = \sin x$ and $y = \tan x$ near the origin. The line $y = x$ is a common tangent at the origin to both $y = \sin x$ and $y = \tan x$. On both sides of the origin, $y = \sin x$ curves away from the tangent towards the x-axis, and $y = \tan x$ curves away from the tangent in the opposite direction.



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20 THE BEHAVIOUR OF $\sin x$ AND $\tan x$ NEAR THE ORIGIN: The line $y = x$ is a tangent to both $y = \sin x$ and $y = \tan x$ at the origin.
When $x = 0$, the derivatives of both $\sin x$ and $\tan x$ are exactly 1.

WORKED EXERCISE: Find: (a) $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{3x}$ (b) $\displaystyle\lim_{h\to 0}\frac{2h}{\sin\frac{1}{2}h}$ (c) $\displaystyle\lim_{x\to 0}\frac{\tan 5x}{\sin\frac{1}{3}x}$

SOLUTION:

(a) $$\lim_{x\to 0}\frac{\sin 5x}{3x} = \frac{5}{3}\times\lim_{x\to 0}\frac{\sin 5x}{5x} = \tfrac{5}{3}\times 1 = \tfrac{5}{3}$$

(b) $$\lim_{h\to 0}\frac{2h}{\sin\frac{1}{2}h} = 4\times\lim_{h\to 0}\frac{\frac{1}{2}h}{\sin\frac{1}{2}h} = 4\times 1 = 4$$

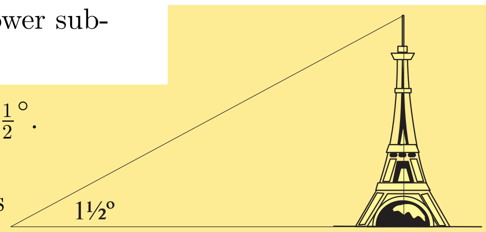
(c) $$\lim_{x\to 0}\frac{\tan 5x}{\sin\frac{1}{3}x} = 15\lim_{x\to 0}\left(\frac{\tan 5x}{5x}\times\frac{\frac{1}{3}x}{\sin\frac{1}{3}x}\right) = 15\times 1\times 1 = 15$$

Approximations to the Trigonometric Functions for Small Angles: For small angles, positive or negative, the limits above yield good approximations for the three trigonometric functions (the angle must of course be expressed in radians).

21 SMALL ANGLE APPROXIMATIONS:
$$\sin x \doteqdot x$$
$$\cos x \doteqdot 1$$
$$\tan x \doteqdot x$$

Question 1 in the following exercise asks for tables of values for $\sin x$, $\cos x$ and $\tan x$ for progressively smaller angles. This will give some idea about how good the approximations are.

WORKED EXERCISE: Approximately how high is a tower subtending an angle of $1\frac{1}{2}^\circ$ if it is 20 km away?

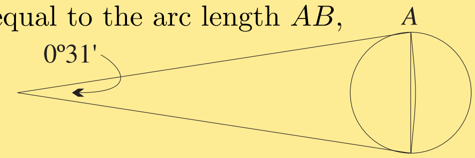


SOLUTION: From the diagram, height $= 20\,000\tan 1\frac{1}{2}^\circ$.
But $1\frac{1}{2}^\circ$ is $\frac{\pi}{120}$, so $\tan 1\frac{1}{2}^\circ \doteqdot \frac{\pi}{120}$.
Hence, approximately, height $\doteqdot \frac{500\pi}{3}$ metres
$\doteqdot 524$ metres.

WORKED EXERCISE: The sun subtends an angle of $0°31'$ at the Earth, which is 150 000 000 km away. What is the sun's approximate diameter?

NOTE: This problem can be done similarly to the previous problem, but like most small-angle problems, it can also be done by approximating the diameter to an arc of the circle.

SOLUTION: Since the diameter AB is approximately equal to the arc length AB,



$$\text{diameter} \doteqdot r\theta$$
$$\doteqdot 150\,000\,000\times\frac{31}{60}\times\frac{\pi}{180}$$
$$\doteqdot 1\,353\,000 \text{ km.}$$

Exercise 14F

1. (a) Copy and complete the following table of values, giving entries to four significant figures. For each column, hold x in the calculator's memory until the column is complete:

angle size in degrees	60°	30°	10°	5°	2°	1°	$20'$	$5'$	$1'$	$30''$	$10''$
angle size x in radians											
$\sin x$											
$\dfrac{\sin x}{x}$											
$\tan x$											
$\dfrac{\tan x}{x}$											
$\cos x$											

(b) Write x, $\sin x$ and $\tan x$ in ascending order, for acute angles x.

(c) Although $\sin x \to 0$ and $\tan x \to 0$ as $x \to 0$, what are the limits, as $x \to 0$, of $\dfrac{\sin x}{x}$ and $\dfrac{\tan x}{x}$?

(d) Experiment with your calculator to find how small x must be for $\dfrac{\sin x}{x} > 0{\cdot}999$.

2. Find the following limits:

(a) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$ (c) $\displaystyle\lim_{x\to 0}\frac{\sin x}{2x}$ (e) $\displaystyle\lim_{x\to 0}\frac{5x}{\sin 3x}$

(b) $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{x}$ (d) $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{2x}$ (f) $\displaystyle\lim_{x\to 0}\frac{\sin 3x + \sin 5x}{x}$

3. Find the following limits:

(a) $\displaystyle\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$ (b) $\displaystyle\lim_{\theta\to 0}\frac{\tan 7\theta}{5\theta}$ (c) $\displaystyle\lim_{\theta\to 0}\frac{7\theta}{3\tan 5\theta}$

DEVELOPMENT

4. Find: (a) $\displaystyle\lim_{x\to 0}\frac{\sin x}{\frac{x}{2}}$ (b) $\displaystyle\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}$ (c) $\displaystyle\lim_{x\to 0}\frac{\sin\frac{x}{3}}{2x}$

5. Find: (a) $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{\sin x}$ (b) $\displaystyle\lim_{y\to 0}\frac{\tan 3y}{\tan 2y}$ (c) $\displaystyle\lim_{t\to 0}\frac{\sin 5t}{\tan 7t}$

6. Find: (a) $\displaystyle\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta^2}$ (b) $\displaystyle\lim_{\theta\to 0}\frac{\tan a\theta}{\sin b\theta}$ (c) $\displaystyle\lim_{\theta\to 0}\frac{\theta^2}{1-\sin^2\theta}$

7. Show that: (a) $\displaystyle\lim_{n\to 0}\frac{\sin n^\circ}{n} = \frac{\pi}{180}$ (b) $\displaystyle\lim_{n\to\infty} n\sin\tfrac{\pi}{n} = \pi$

8. (a) Write down the compound-angle formula for $\cos(A+B)$.

(b) By replacing both A and B with x, show that $\cos 2x = 1 - 2\sin^2 x$.

(c) Hence find $\displaystyle\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$.

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9. (a) Express 2° in radians.

(b) Explain why $\sin 2^\circ \doteqdot \dfrac{\pi}{90}$.

(c) Taking π as $3{\cdot}142$, find $\sin 2^\circ$ correct to four decimal places *without* using a calculator.

10. A car travels 1 km up a road which is inclined at 5° to the horizontal. Through what vertical distance has the car climbed? (Use the fact that $\sin x \doteqdot x$ for small angles, and give your answer correct to the nearest metre.)

11. A tower is 30 metres high. What angle, correct to the nearest minute, does it subtend at a point 4 km away? (Use the fact that when x is small, $\tan x \doteqdot x$.)

12. The moon subtends an angle of $31'$ at an observation point on Earth 400 000 km away. Use the fact that the diameter of the moon is approximately equal to an arc of a circle whose centre is the point of observation to show that the diameter of the moon is approximately 3600 km.

13. A regular polygon of 300 sides is inscribed in a circle of radius 60 cm. Show that each side is approximately $1{\cdot}26$ cm.

14. (a) Show that $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \tfrac{1}{2}$ by multiplying top and bottom by $1 + \cos x$.

(b) Explain why this means that $\cos x \doteqdot 1 - \frac{1}{2}x^2$ for small angles.

(c) Use the calculator to approximate the difference between $\cos x$ and $1 - \frac{1}{2}x^2$ for small angles, and find the smallest angle, in degrees, for which this difference exceeds $0{\cdot}001$.

(d) A post 5 metres long leans at 10° to the vertical. Find how high the top is above the ground using the approximation $\cos x \doteqdot 1 - \frac{1}{2}x^2$. Give your answer correct to four significant figures, and then check using the calculator's cosine function.

EXTENSION

15. (a) Write down the compound-angle formula for $\sin(A - B)$.

(b) Hence show that, for small x, $\sin(\theta - x) \doteqdot \sin\theta - x\cos\theta$.

(c) Use the result in (b) to show that $\sin 29^\circ 57' \doteqdot \dfrac{3600 - \sqrt{3}\pi}{7200}$.

(d) To how many decimal places is the approximation in (c) accurate?

(e) Use similar methods to obtain approximations to $\sin 29^\circ$, $\cos 31^\circ$, $\tan 61^\circ$, $\cot 59^\circ$ and $\sin 46^\circ$, checking the accuracy of your approximations using the calculator.

16. The chord AB of a circle of radius r subtends an angle x at the centre O.

(a) Find AB^2 by the cosine rule, and find the length of the arc AB.

(b) By equating arc and chord, show that for small angles, $\cos x \doteqdot 1 - \dfrac{x^2}{2}$. Explain whether the approximation is bigger or smaller than $\cos x$.

NOTE: The approximation $\cos x = 1 - \frac{1}{2}x^2$ is an excellent small-angle approximation for $\cos x$. It can be shown further that $\cos x$ is exactly the limit of the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

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14 G The Derivatives of the Trigonometric Functions

We can now establish the derivative of $\sin x$, which is the central theorem of the chapter. From this, the derivatives of $\cos x$ and $\tan x$ can quickly be calculated.

The Derivative of sin x: The following proof that $\dfrac{d}{dx}\sin x = \cos x$ uses the compound-angle formulae to bring everything back to the fundamental limit proven earlier, $\lim\limits_{u\to 0}\dfrac{\sin u}{u} = 1$. The first part of the proof develops a further identity involving compound angles. This identity is part of a group of 'sums-to-products' identities, which are part of the 4 Unit course, but play no further role in the 3 Unit course.

PROOF:

A. We know that $\sin(A+B) = \sin A\cos B + \cos A\sin B$,
and $\sin(A-B) = \sin A\cos B - \cos A\sin B$.
Subtracting these, $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$.
Now let $S = A+B$
and $T = A-B$.
Then adding and halving, $A = \frac{1}{2}(S+T)$
subtracting and halving, $B = \frac{1}{2}(S-T)$,
giving the identity, $\sin S - \sin T = 2\cos\frac{1}{2}(S+T)\sin\frac{1}{2}(S-T)$.

B. Using the first-principles definition of the derivative,

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \lim_{u\to x}\frac{\sin u - \sin x}{u-x}\\ &= \lim_{u\to x}\frac{2\,\cos\frac{1}{2}(u+x)\,\sin\frac{1}{2}(u-x)}{u-x}\\ &= \cos x \times \lim_{u\to x}\frac{\sin\frac{1}{2}(u-x)}{\frac{1}{2}(u-x)}\\ &= \cos x,\ \text{ because the limit is 1.}\end{aligned}$$

OR

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}\\ &= \lim_{h\to 0}\frac{2\,\cos(x+\frac{1}{2}h)\,\sin\frac{1}{2}h}{h}\\ &= \cos x \times \lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\\ &= \cos x,\ \text{ because the limit is 1.}\end{aligned}$$

The Derivatives of cos x and tan x: Once the derivative of $\sin x$ is proven, it is straightforward to differentiate the other trigonometric functions.

A. Let $y = \cos x$
$= \sin(\frac{\pi}{2} - x)$.
Applying the chain rule,
let $u = \frac{\pi}{2} - x$.
Then $y = \sin u$.
Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}\times\frac{dy}{du}\\ &= \cos u \times (-1)\\ &= -\cos(\tfrac{\pi}{2} - x)\\ &= -\sin x.\end{aligned}$$

B. Let $y = \tan x$
$= \dfrac{\sin x}{\cos x}$.
Let $u = \sin x$ and $v = \cos x$.
Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \qquad \text{(quotient rule)}\\ &= \frac{\cos x\cos x + \sin x \sin x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} \qquad \text{(Pythagoras)}\\ &= \sec^2 x.\end{aligned}$$

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22 STANDARD DERIVATIVES:
$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$

WORKED EXERCISE: Use the chain rule to differentiate $y = \sin(ax + b)$.

SOLUTION: Let $u = ax + b$, so that $y = \sin u$.

Then applying the chain rule, $\dfrac{dy}{dx} = \dfrac{du}{dx} \times \dfrac{dy}{du}$
$$= a \times \cos u$$
$$= a\cos(ax + b).$$

WORKED EXERCISE: Use the chain rule to differentiate $y = \tan^2 x$.

SOLUTION: Let $u = \tan x$, so that $y = u^2$.

Then applying the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$$= 2u \times \sec^2 x$$
$$= 2\tan x \sec^2 x.$$

WORKED EXERCISE: Use the product and quotient rules to differentiate:

(a) $y = e^x \cos x$ (b) $y = \dfrac{\sin x}{x}$

SOLUTION:

(a) Applying the product rule,

let $u = e^x$ and $v = \cos x$.

Then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$
$$= e^x \cos x - e^x \sin x$$
$$= e^x(\cos x - \sin x).$$

(b) Applying the quotient rule,

let $u = \sin x$ and $v = x$.

Then $\dfrac{dy}{dx} = \dfrac{vu' - uv'}{v^2}$
$$= \frac{x\cos x - \sin x}{x^2}.$$

Replacing x by $ax+b$: The first result in the previous worked exercise can be extended to the other trigonometric functions, giving the following standard forms.

23 FUNCTIONS OF $ax + b$:
$$\frac{d}{dx}\sin(ax + b) = a\cos(ax + b)$$
$$\frac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$$
$$\frac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$$

WORKED EXERCISE: Differentiate the following functions:

(a) $y = 4\sin(3x - \frac{\pi}{3})$ (b) $y = \frac{3}{2}\tan\frac{3}{2}x$ (c) $y = 5\cos 2x\,\cos\frac{1}{2}x$

SOLUTION:

(a) $\dfrac{dy}{dx} = 12\cos(3x - \frac{\pi}{3})$

(b) $\dfrac{dy}{dx} = \frac{9}{4}\sec^2\frac{3}{2}x$

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(c) Applying the product rule, let $u = 5\cos 2x$ and $v = \cos \frac{1}{2}x$.

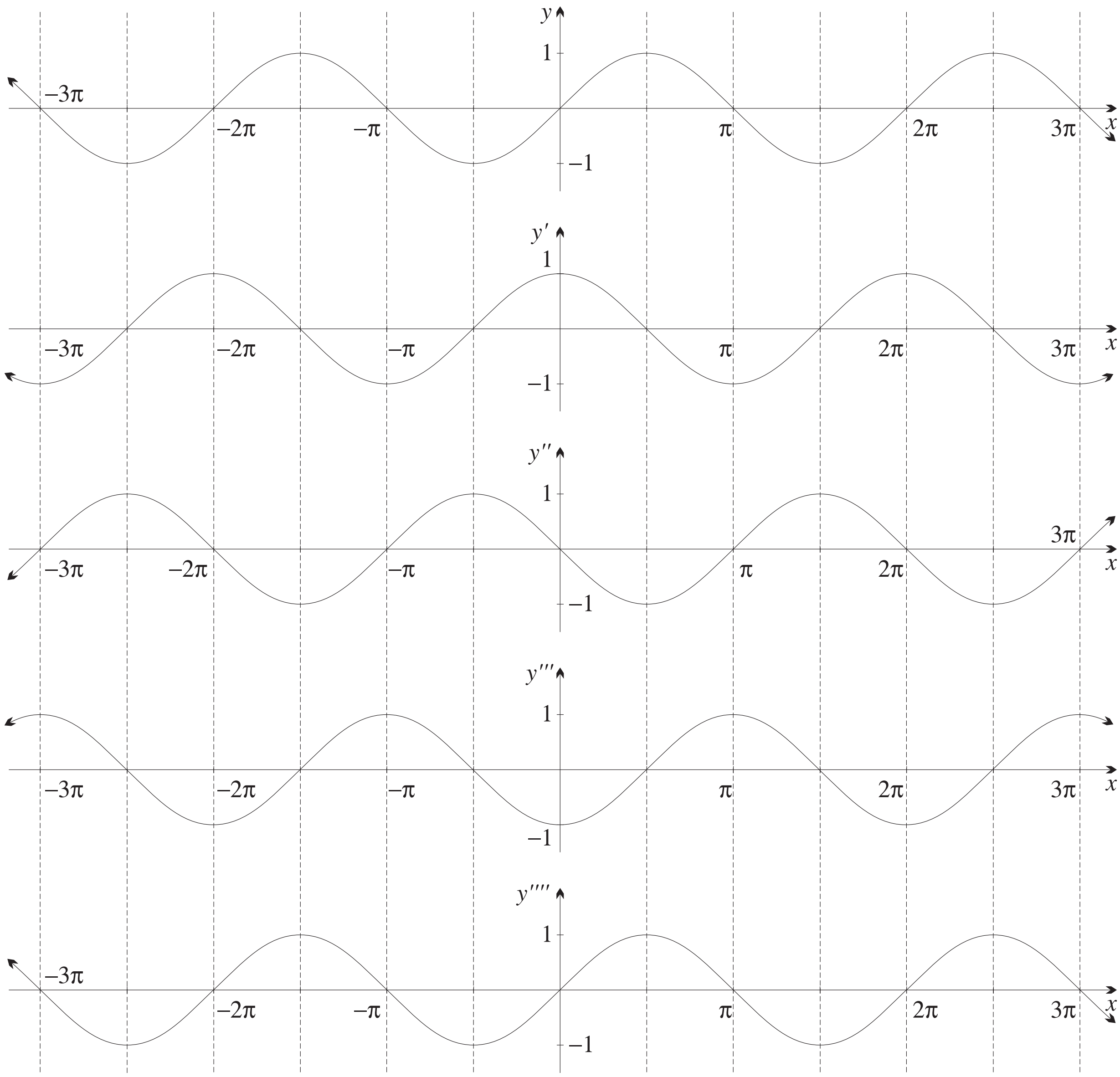
Then $\dfrac{dy}{dx} = v\,\dfrac{du}{dx} + u\,\dfrac{dv}{dx}$

$= -10\cos \frac{1}{2}x\,\sin 2x - \frac{5}{2}\cos 2x \sin \frac{1}{2}x.$

Successive Differentiation of Sine and Cosine: Differentiating $y = \sin x$ repeatedly,

$$\frac{dy}{dx} = \cos x, \qquad \frac{d^2y}{dx^2} = -\sin x, \qquad \frac{d^3y}{dx^3} = -\cos x, \qquad \frac{d^4y}{dx^4} = \sin x.$$

So differentiation is an order 4 operation on the sine function, which means that when differentiation is applied four times, the original function returns. Below are the graphs of $y = \sin x$ and its first four derivatives.



Each application of the differentiation operator shifts the wave backwards a quarter revolution, so four applications shift it backwards one revolution, where it coincides with itself again. Thus differentiation is behaving like a rotation of 90° anticlockwise, and of course a rotation of 90° has order 4.

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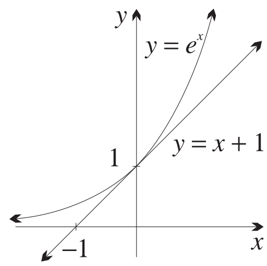
Notice that double differentiation exchanges $y = \sin x$ with its opposite function $y = -\sin x$, with each graph being the reflection of the other in the x-axis. It has a similar effect on the cosine function. So both $y = \sin x$ and $y = \cos x$ satisfy the equation $y'' = -y$. This idea will later be central to simple harmonic motion.

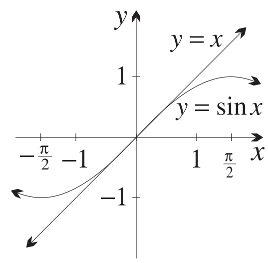
The exponential function should be mentioned in this context. The first derivative of $y = e^x$ is $y' = e^x$, and the second derivative of $y = e^{-x}$ is $y'' = e^{-x}$. This means that we now have four functions whose fourth derivatives are equal to themselves:

$$y = \sin x, \qquad y = \cos x, \qquad y = e^x, \qquad y = e^{-x}.$$

This is one clue amongst many others in our course that the trigonometric functions and the exponential functions are very closely related.



Some Analogies between π and e: In the previous chapter, we found that by choosing the base of our exponential function to be the special number e, then the derivative of $y = e^x$ was exactly $y' = e^x$. In contrast, the derivative of any other exponential function $y = a^x$ is $y = a^x \log a$, which is only a multiple of itself. In particular, the tangent of $y = e^x$ at the y-intercept has gradient exactly 1.

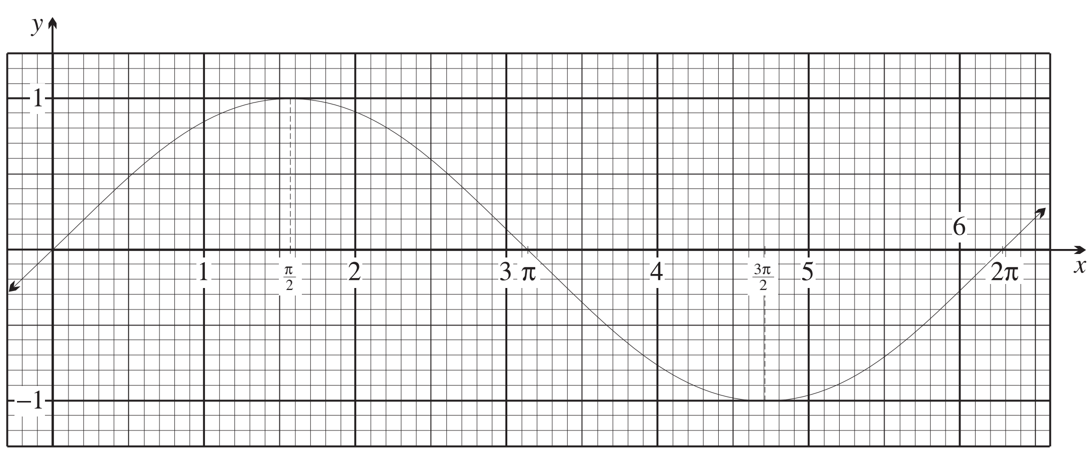


The choice of radian measure, based on the special number π, was motivated exactly the same way — as we have just proven, the derivative of $y = \sin x$ using radian measure is exactly $y' = \cos x$, but would only be a multiple of $\cos x$ were another system of measuring angles used. In particular, the gradient of $y = \sin x$ at the origin is exactly 1.

Both numbers $\pi \doteqdot 3{\cdot}1416$ and $e \doteqdot 2{\cdot}7183$ are irrational (although this is not easy to prove), with π defined using areas of circles, and e defined using areas associated with the rectangular hyperbola. All these things are further hints of deeper connections between trigonometric and exponential functions that lie beyond this course.

Exercise 14G

1.



Photocopy the sketch above of $f(x) = \sin x$. Carefully draw tangents at the points where $x = 0,\ 0{\cdot}5,\ 1,\ 1{\cdot}5,\ \ldots,\ 3$, and also at $x = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$. Measure the gradient of each tangent to two decimal places, and copy and complete the following table.

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x	0	0·5	1	1·5	$\frac{\pi}{2}$	2	2·5	3	π	3·5	4	4·5	$\frac{3\pi}{2}$	5	5·5	6	2π
$f'(x)$																	

Then use these values to plot the graph of $y = f'(x)$. What is the equation of this graph?

2. Differentiate with respect to x:

(a) $\sin x$
(b) $\cos x$
(c) $\tan x$
(d) $\sin 2x$
(e) $2\cos x$
(f) $2\tan 2x$
(g) $\sin 2\pi x$
(h) $\tan \frac{\pi}{2}x$
(i) $3\sin x + \cos 5x$
(j) $4\sin \pi x + 3\cos \pi x$
(k) $\cos(5x+4)$
(l) $7\sin(2-3x)$
(m) $10\tan(10-x)$
(n) $\frac{1}{\pi}\tan(\frac{\pi}{2} - 2\pi x)$
(o) $6\sin\left(\frac{x+1}{2}\right)$
(p) $15\cos\left(\frac{3-2x}{5}\right)$

3. Differentiate the following functions:

(a) $y = \sin(x^2)$
(b) $y = \cos(x^3+1)$
(c) $y = \sin\left(\dfrac{1}{x}\right)$
(d) $y = \tan\sqrt{x}$
(e) $y = x\sin x$
(f) $y = x^2\cos 2x$
(g) $y = \cos^2 x$
(h) $y = \sin^3 x$
(i) $y = \dfrac{1}{1+\sin x}$
(j) $y = \dfrac{\sin x}{1+\cos x}$
(k) $y = \dfrac{1-\sin x}{\cos x}$
(l) $y = \dfrac{\cos x}{\cos x + \sin x}$
(m) $y = \log_e(\cos x)$
(n) $y = e^{\tan x}$

4. Sketch $y = \cos x$ for $-3\pi \le x \le 3\pi$. Find y', y'', y''' and y'''', and sketch them underneath the first graph. What geometrical interpretations can be given of the facts that:

(a) $y'' = -y$?
(b) $y'''' = y$?

DEVELOPMENT

5. Find $f'(x)$, given that:

(a) $f(x) = e^{\sin 2x}$
(b) $f(x) = \sin(e^{2x})$
(c) $f(x) = \sqrt{\sin 4x}$
(d) $f(x) = \log_e(\sin 4x)$
(e) $f(x) = (1-\cos 3x)^3$
(f) $f(x) = \sin 2x \sin 4x$
(g) $f(x) = \cos^5 3x$
(h) $f(x) = \dfrac{1}{3+4\cos 5x}$
(i) $f(x) = \tan^3(5x-4)$
(j) $f(x) = \sin\sqrt{x^2+1}$

6. Given that $y = e^{-x}(\cos 2x + \sin 2x)$, show that $y'' + 2y' + 5y = 0$.

7. (a) Express $x°$ (that is, x degrees) in radians.

(b) Hence find: (i) $\dfrac{d}{dx}(\sin x°)$ (ii) $\dfrac{d}{dx}(\tan(x° + 45°))$ (iii) $\dfrac{d}{dx}(\cos 2x°)$

8. (a) By writing $\sec x$ as $(\cos x)^{-1}$, show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

(b) Similarly, show that: (i) $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ (ii) $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

9. Given that $y = \frac{1}{3}\tan^3 x - \tan x + x$, show that $\dfrac{dy}{dx} = \tan^4 x$.

10. (a) Copy and complete: $\log_b(\frac{P}{Q}) = \ldots$.

(b) If $f(x) = \log_e\left(\dfrac{1+\sin x}{\cos x}\right)$, show that $f'(x) = \sec x$.

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11. (a) Show that $\frac{1}{2}\left(\sin(m+n)x + \sin(m-n)x\right) = \sin mx \cos nx$.
(b) Hence, without using the product rule, find the derivative of $\sin mx \cos nx$.
(c) Simplify $\frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right)$, and hence differentiate $\cos mx \cos nx$.

12. (a) If $y = \sin x$, prove: (i) $\dfrac{dy}{dx} = \sin(\frac{\pi}{2}+x)$ (ii) $\dfrac{d^2y}{dx^2} = \sin(\pi+x)$ (iii) $\dfrac{d^3y}{dx^3} = \sin(\frac{3\pi}{2}+x)$
(b) Deduce an expression for $\dfrac{d^n y}{dx^n}$.

13. (a) If $y = \ln(\tan 2x)$, show that $\dfrac{dy}{dx} = 2\sec 2x \operatorname{cosec} 2x$.
(b) Show that $\dfrac{d}{dx}(\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x) = \sin^4 x \cos^3 x$.
(c) Show that $\dfrac{d}{dx}\left(\dfrac{\cos x + \sin x}{\cos x - \sin x}\right) = \sec^2(\frac{\pi}{4} + x)$.
(d) (i) Show that $\cos 2x = \cos^2 x - \sin^2 x$ by using the formula for $\cos(A+B)$.
(ii) Hence show that $\dfrac{d}{dx}\left(\dfrac{1+\cos 2x}{1-\cos 2x}\right) = -2\cot x \operatorname{cosec}^2 x$.
(e) Show that $\dfrac{d}{dx}\left(\ln \dfrac{\sqrt{2} - \cos x}{\sqrt{2} + \cos x}\right) = \dfrac{2\sqrt{2}\sin x}{1 + \sin^2 x}$.

14. Given the parametric equations $x = e^t \cos t$, $y = e^t \sin t$, show that $\dfrac{dy}{dx} = \tan(t + \frac{\pi}{4})$.

15. [First-principles differentiation of $\sin x$, $\cos x$ and $\tan x$]
(a) Use the definition $\dfrac{d}{dx}\tan x = \lim\limits_{h\to 0} \dfrac{\tan(x+h) - \tan x}{h}$ and the usual expansion of $\tan(x+h)$ to prove that the derivative of $\tan x$ is $\sec^2 x$.
(b) (i) Show that $\lim\limits_{h\to 0} \dfrac{1-\cos h}{h} = 0$ by multiplying top and bottom by $1 + \cos h$.
(ii) Use the $h \to 0$ definition of the derivative to show that

$$\frac{d}{dx}\sin x = \cos x \times \lim_{h\to 0}\frac{\sin h}{h} - \sin x \times \lim_{h\to 0}\frac{1-\cos h}{h},$$

and then use part (i) to show that the derivative of $\sin x$ is $\cos x$.
(c) Use methods similar to those in part (b) to prove, using the $h \to 0$ definition of the derivative, that the derivative of $\cos x$ is $-\sin x$.

EXTENSION

16. Find $\dfrac{dy}{dx}$ in terms of x and y by differentiating implicitly:
(a) $\sin x + \cos y = 1$ (b) $x \sin y + y \sin x = 1$ (c) $\sin(x+y) = \cos(x-y)$

17. (a) Use the expansions of $\cos(A+B)$ and $\cos(A-B)$ to prove that

$$\cos S - \cos T = -2\sin\tfrac{1}{2}(S+T)\sin\tfrac{1}{2}(S-T).$$

(b) Hence differentiate $\cos x$ from first principles, using $f'(x) = \lim\limits_{u\to x}\dfrac{f(u)-f(x)}{u-x}$.

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18. (a) It can be shown that $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$. Making the assumption that the series can be differentiated term by term (which is true here, but not for all series), show that $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$. Show also that the fourth derivative of each series is the original series.

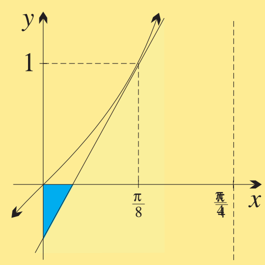
(b) Using the first few terms, find $\sin 1$ and $\cos 1$ to four significant figures.

14 H Applications of Differentiation

We have now reached an important point in the course where the differentiation of the trigonometric functions can be applied to the analysis of a number of very significant functions. With trigonometric functions, it is usually easier to determine the nature of stationary points by using the second derivative rather than from a table of values of the derivative.

WORKED EXERCISE: Find the x-intercept and y-intercept of the tangent to $y = \tan 2x$ at the point on the curve where $x = \frac{\pi}{8}$, and find the area of the triangle formed by this tangent and the coordinate axes.

SOLUTION: Since $y = \tan 2x$,
$y' = 2\sec^2 2x$.
When $x = \frac{\pi}{8}$, $y = 1$
and $y' = 4$,
so the tangent is $y - 1 = 4(x - \frac{\pi}{8})$.
When $x = 0$, $y - 1 = -\frac{\pi}{2}$
$y = \frac{1}{2}(2 - \pi)$,
and when $y = 0$, $-1 = 4(x - \frac{\pi}{8})$
$x = \frac{1}{8}(\pi - 2)$,
so area of triangle $= \frac{1}{2} \times \frac{1}{2}(\pi - 2) \times \frac{1}{8}(\pi - 2)$
$= \frac{1}{32}(\pi - 2)^2$ square units.

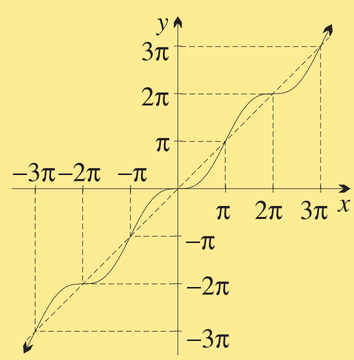


WORKED EXERCISE: Use the standard curve sketching menu to sketch $y = x - \sin x$.

NOTE: This function is essentially the function describing the area of a segment, if the radius in the formula $A = \frac{1}{2}r^2(x - \sin x)$ is held constant while the angle x at the centre varies.

SOLUTION:

1. The domain is the set of all real numbers.
2. $f(x)$ is odd, since both $\sin x$ and x are odd.
3. The function is zero at $x = 0$ and nowhere else, since $\sin x < x$ for $x > 0$, and $\sin x > x$ for $x < 0$.
4. $\sin x$ always remains between -1 and 1, so $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.
5. $f'(x) = 1 - \cos x$,
   so $f'(x)$ has zeroes whenever $\cos x = 1$,
   that is, for $x = \ldots, -2\pi, 0, 2\pi, 4\pi, \ldots$.
   But $f'(x)$ is never negative, since $\cos x$ is always between -1 and 1,



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so each stationary point is a stationary inflexion,
and these points are $\ldots, (-2\pi, -2\pi), (0, 0), (2\pi, 2\pi), (4\pi, 4\pi), \ldots$.

6. $f''(x) = \sin x$,
which is zero for $x = \ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$.
We know that $\sin x$ changes sign around each of these points,
so $\ldots, (-\pi, -\pi), (\pi, \pi), (3\pi, 3\pi), \ldots$ are also inflexions.
Since $f'(\pi) = 1 - (-1) = 2$, the gradient at these other inflexions is 2.

WORKED EXERCISE: [A harder example] Find, as multiples of π, the x-coordinates of the turning points of $y = e^{-x} \sin 4x$, for $0 \le x \le 2\pi$. Then sketch the curve, ignoring inflexions.

NOTE: Various forms of this important function describe *damped oscillations*, like the decaying vibrations of a plucked guitar string, or a swing gradually slowing down. The $\sin x$ factor supplies the oscillations, and the e^{-x} factor provides the natural decay.

SOLUTION: For $y = e^{-x} \sin 4x$, using the product rule,

$$y' = -e^{-x} \sin 4x + 4e^{-x} \cos 4x$$
$$= e^{-x}(-\sin 4x + 4\cos 4x).$$

This has no discontinuities, and has zeroes when

$$\sin 4x = 4\cos 4x,$$

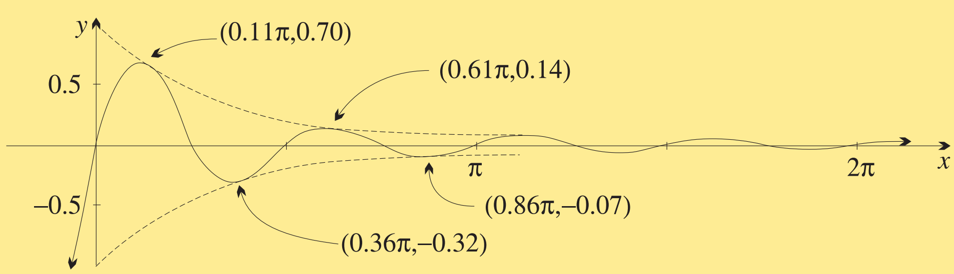
that is, $\tan 4x = 4$.
The related angle here is about $0{\cdot}42\pi$, so

$$4x \doteqdot 0{\cdot}42\pi, 1{\cdot}42\pi, 2{\cdot}42\pi, 3{\cdot}42\pi, 4{\cdot}42\pi, 5{\cdot}42\pi, 6{\cdot}42\pi \text{ or } 7{\cdot}42\pi,$$
$$x \doteqdot 0{\cdot}11\pi, 0{\cdot}36\pi, 0{\cdot}61\pi, 0{\cdot}86\pi, 1{\cdot}11\pi, 1{\cdot}36\pi, 1{\cdot}61\pi \text{ or } 1{\cdot}86\pi.$$

Using the product rule again,

$$y'' = -e^{-x}(-\sin 4x + 4\cos 4x) + e^{-x}(-4\cos 4x - 16\sin 4x)$$
$$= -e^{-x}(15\sin 4x + 8\cos 4x)$$
$$= -e^{-x} \cos 4x(15\tan 4x + 8)$$

so when $\tan 4x = 4$, the sign of y'' is opposite to the sign of $\cos 4x$,
and the stationary points are alternately maxima and minima.
Thus $(0{\cdot}11\pi, 0{\cdot}6965)$, $(0{\cdot}61\pi, 0{\cdot}1448)$, $(1{\cdot}11\pi, 0{\cdot}0301)$, $(1{\cdot}61\pi, 0{\cdot}0063)$ are maxima,
and $(0{\cdot}36\pi, -0{\cdot}3175)$, $(0{\cdot}86\pi, 0{\cdot}0660)$, $(1{\cdot}36\pi, -0{\cdot}0137)$, $(1{\cdot}86\pi, -0{\cdot}0029)$ are minima.

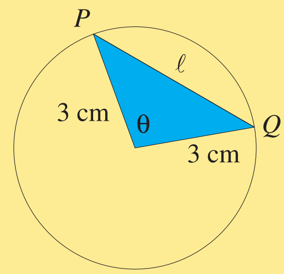


WORKED EXERCISE: Two radii OP and OQ of a circle of radius 3 cm meet at an angle θ at the centre O. The angle $\theta = \angle POQ$ between them is increasing at $0{\cdot}2$ radians per minute. Find, when $\theta = \frac{\pi}{3}$, the rate of increase of:

(a) the area of $\triangle OPQ$, (b) the length of the chord PQ.

SOLUTION: Let A be the area of $\triangle OPQ$, and ℓ be the length of the chord PQ.

(a) Since $A = \frac{1}{2}r^2 \sin\theta = \frac{9}{2}\sin\theta$,
$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt} = \frac{9}{2}\cos\theta \times \frac{d\theta}{dt},$$
and substituting $\frac{d\theta}{dt} = \frac{1}{5}$ and $\theta = \frac{\pi}{3}$,
$$\frac{dA}{dt} = \frac{9}{2} \times \frac{1}{2} \times \frac{1}{5} = \frac{9}{20}\ \text{cm}^2/\text{min}.$$



(b) By the cosine rule,
$$\ell^2 = 9 + 9 - 18\cos\theta$$
$$\ell = 3\sqrt{2 - 2\cos\theta},$$
so by the chain rule,
$$\frac{d\ell}{dt} = \frac{3\sin\theta}{\sqrt{2 - 2\cos\theta}} \times \frac{d\theta}{dt}$$
and substituting $\frac{d\theta}{dt} = \frac{1}{5}$ and $\theta = \frac{\pi}{3}$,
$$\frac{d\ell}{dt} = \frac{\frac{3}{2}\sqrt{3}}{\sqrt{2-1}} \times \frac{1}{5} = \frac{3}{10}\sqrt{3}\ \text{cm/min}.$$

Exercise 14H

NOTE: The large number of sketches should allow the alternative of machine sketching of some graphs, followed by algebraic explanation of the features.

1. Find the gradient of the tangent to each of the following curves at the point indicated:

(a) $y = \cos x$ at $x = \frac{\pi}{6}$
(b) $y = \sin x$ at $x = \frac{\pi}{4}$
(c) $y = \tan x$ at $x = 0$
(d) $y = \cos 2x$ at $x = \frac{\pi}{4}$
(e) $y = \sin\frac{x}{2}$ at $x = \frac{2\pi}{3}$
(f) $y = \tan 2x$ at $x = \frac{\pi}{6}$

2. Find the equation of the tangent at the given point on each of the following curves:

(a) $y = \cos 2x$ at $(\frac{\pi}{4}, 0)$
(b) $y = \sin 2x$ at $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$
(c) $y = x\sin x$ at $(\pi, 0)$

3. (a) Find the equations of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.

(b) If the tangent meets the x-axis at P and the normal meets the y-axis at Q, find the area of $\triangle OPQ$, where O is the origin.

4. Find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on each of the following curves where the gradient of the tangent is zero.

(a) $y = 2\sin x$
(b) $y = 2\cos x + x$
(c) $y = 2\sin x + \sqrt{3}x$
(d) $y = e^{\sin x}$

5. Find the equations of the tangent and normal to the curve $y = 2\sin x - \cos 2x$ at $(\frac{\pi}{6}, \frac{1}{2})$.

6. (a) Find the first and second derivatives of $y = \cos x + \sqrt{3}\sin x$.

(b) Find the stationary points in the domain $0 \le x \le 2\pi$ and use the second derivative to determine their nature.

(c) Find the points of inflexion.

(d) Hence sketch the curve for $0 \le x \le 2\pi$.

7. Repeat the previous question for $y = \cos x - \sin x$. Verify your results by sketching $y = \cos x$ and $y = -\sin x$ on the same diagram and then sketching $y = \cos x - \sin x$ by addition of heights. (Notice that the final graph is a sine wave shifted sideways. This situation will be discussed more fully in the Year 12 volume.)

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8. (a) Find the first and second derivatives of $y = x + \sin x$.

(b) Find the stationary points in the domain $-2\pi < x < 2\pi$ and determine their nature.

(c) Find the points of inflexion for $-2\pi < x < 2\pi$.

(d) Sketch the curve for $-2\pi \le x \le 2\pi$.

9. Repeat the previous question for $y = x - \cos x$.

DEVELOPMENT

10. (a) Find the first and second derivatives of $y = 2\sin x + \cos 2x$.

(b) Show that $y' = 0$ when $\cos x = 0$ or $\sin x = \frac{1}{2}$. (You will need to use the formula $\sin 2x = 2\sin x \cos x$.)

(c) Hence find the stationary points in the interval $-\pi \le x \le \pi$ and determine their nature.

(d) Sketch the curve for $-\pi \le x \le \pi$ using this information.

11. Repeat the previous question for $y = 2\cos x + \sin 2x$. (You will need to use the formula $\cos 2x = 1 - 2\sin^2 x$ to find the stationary points.)

12. (a) Find the first and second derivatives of $y = e^{-x}\cos x$. (Note that this function models damped oscillations.)

(b) Find the stationary points for $-\pi \le x \le \pi$ and determine their nature.

(c) Find the points of inflexion for $-\pi \le x \le \pi$.

(d) Hence sketch the curve for $-\pi \le x \le \pi$.

13. Repeat the previous question for $y = e^{x}\sin x$. (Note that this function models oscillations such as feedback loops which grow exponentially.)

14. The angle θ between two radii OP and OQ of a circle of radius 6 cm is increasing at the rate of 0·1 radians per minute.

(a) Show that the area of sector OPQ is increasing at the rate of $1{\cdot}8\,\text{cm}^2/\text{min}$.

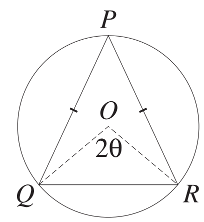
(b) Find the rate at which the area of $\triangle OPQ$ is increasing at the instant when $\theta = \frac{\pi}{4}$.

(c) Find the value of θ for which the rate of increase of the area of the segment cut off by the chord PQ is at its maximum.

15. An isosceles triangle has equal sides of length 10 cm. The angle θ between these equal sides is increasing at the rate of $3°$ per minute. Show that the area of the triangle is increasing at $\frac{5\sqrt{3}\pi}{12}\,\text{cm}^2$ per minute at the instant when $\theta = 30°$.

16. A rotating light L is situated at sea 180 metres from the nearest point P on a straight shoreline. The light rotates through one revolution every 10 seconds. Show that the rate at which a ray of light moves along the shore at a point 300 metres from P is 136π m/s.

17. PQR is an isosceles triangle inscribed in a circle with centre O of radius one unit, as shown in the diagram. Let $\angle QOR = 2\theta$, where θ is acute.



(a) Show that the area A of $\triangle PQR$ is given by $A = \sin\theta(\cos\theta + 1)$.

(b) Hence show that, as θ varies, $\triangle PQR$ has its maximum possible area when it is equilateral.

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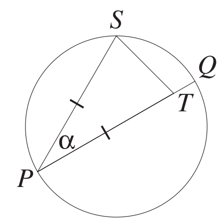

18. PQ is a diameter of the given circle and S is a point on the circumference. T is the point on PQ such that $PS = PT$. Let $\angle SPT = \alpha$.
(a) Show that the area A of $\triangle SPT$ is $A = \frac{1}{2}d^2\cos^2\alpha\sin\alpha$, where d is the diameter of the circle.
(b) Hence show that the maximum area of $\triangle SPT$ as S varies on the circle is $\frac{1}{9}d^2\sqrt{3}$ units^2.

19. A straight line passes through the point $(2, 1)$ and has positive x- and y-intercepts at P and Q respectively. Let $\angle OPQ = \alpha$, where O is the origin.
(a) Explain why the line has gradient $-\tan\alpha$.
(b) Find the x- and y-intercepts in terms of α.
(c) Show that the area of $\triangle OPQ$ is given by $A = \dfrac{(2\tan\alpha + 1)^2}{2\tan\alpha}$.
(d) Hence show that this area is maximised when $\tan\alpha = \frac{1}{2}$.

20. Find the maximum and minimum values of the expression $\dfrac{2 - \sin\theta}{\cos\theta}$ for $0 \le \theta \le \frac{\pi}{4}$, and state the values of θ for which they occur.

21. Find any stationary points and inflexions, then sketch each curve for $0 \le x \le 2\pi$:
(a) $y = 2\sin x + x$ (b) $y = 2x - \tan x$

22. Find any stationary points and any other important features, then sketch for $0 \le x \le 2\pi$:
(a) $y = \sin^2 x + \cos x$ (b) $y = \sin^3 x\cos x$ (c) $y = \tan^2 x - 2\tan x$

EXTENSION

23. Find the equation of the tangent to the curve $y = \pi\tan\frac{y}{x} + x - 4$ at the point $(4, \pi)$.

24. (a) Show that the line $y = x$ is the tangent to the curve $y = \tan x$ at $(0, 0)$.
(b) Show that $\tan x > x$ for $0 < x < \frac{\pi}{2}$.
(c) Let $f(x) = \dfrac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$. Find $f'(x)$ and show that $f'(x) < 0$ in the given domain.
(d) Sketch the graph of $f(x)$ over the given domain and explain why $\sin x > \dfrac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$.

25. (a) Find the derivative of $y = e^{\lambda x}\sin nx$, where $\lambda \ne 0$ and $n > 0$.
(b) Show that in each interval of length $\dfrac{2\pi}{n}$ there are two stationary points, and explain why one is a maximum and the other is a minimum.
(c) What happens to the x values of the stationary points as: (i) $\lambda \to \infty$ (ii) $\lambda \to 0$?

26. Let $f(x) = \dfrac{\sin x}{x}$. Remember that we proved in Section 14F that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$.
(a) Write down the domain of $f(x)$, show that $f(x)$ is even, find the zeroes of $f(x)$, and determine $\lim\limits_{x\to\infty} f(x)$.
(b) Differentiate $f(x)$, and hence show that $f(x)$ has stationary points when $\tan x = x$.
(c) Sketch $y = \tan x$ and $y = x$ on one set of axes, and hence use your calculator to estimate the turning points for $0 \le x \le 4\pi$. Give the x-coordinates in the form $\lambda\pi$, with λ to no more than two decimal places.
(d) Using this information, sketch $y = f(x)$.

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14 I Integration of the Trigonometric Functions

The Standard Forms for Integrating the Trigonometric Functions: Since the derivative of $\sin x$ is $\cos x$, the primitive of $\cos x$ is $\sin x$. Each of the standard forms for differentiation can be reversed in this way, giving the following three standard integrals.

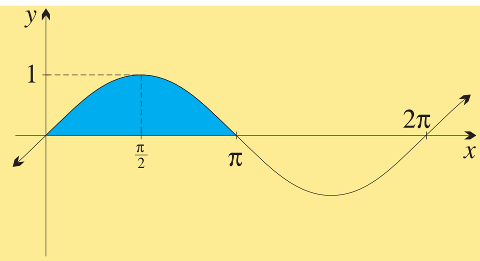
24 STANDARD INTEGRALS:

$$\int \cos x\,dx = \sin x$$

$$\int \sin x\,dx = -\cos x$$

$$\int \sec^2 x\,dx = \tan x$$

WORKED EXERCISE: Show that the shaded area of the first arch of $y = \sin x$ is 2 square units.



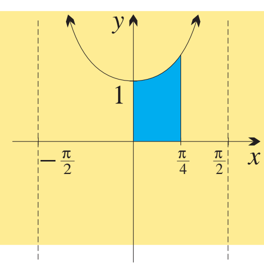
SOLUTION:

$$\begin{aligned}\text{Area} &= \int_0^{\pi} \sin x\,dx \\ &= \Big[-\cos x\Big]_0^{\pi} \\ &= -\cos \pi + \cos 0 \\ &= -(-1) + 1 \\ &= 2 \text{ square units.}\end{aligned}$$

NOTE: The fact that this area is such a simple number is another confirmation that radians are the correct angle unit to use for the calculus of the trigonometric functions. Comparable simple results were obtained earlier when e was used as the base for logarithms and powers, for example,

$$\int_1^e \log x\,dx = 1 \qquad \text{and} \qquad \lim_{N\to\infty} \int_0^N e^{-x}\,dx = 1.$$

WORKED EXERCISE: Find the volume generated when the shaded area under $y = \sec x$ between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x-axis.



SOLUTION:

$$\begin{aligned}\text{Volume} &= \int_0^{\frac{\pi}{4}} \pi \sec^2 x\,dx \\ &= \pi\Big[\tan x\Big]_0^{\frac{\pi}{4}} \\ &= \pi(\tan \tfrac{\pi}{4} - \tan 0) \\ &= \pi \text{ cubic units.}\end{aligned}$$

Replacing x by $ax + b$: Reversing the standard forms for derivatives:

25 FUNCTIONS OF $ax + b$:

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)$$

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WORKED EXERCISE: Evaluate: (a) $\displaystyle\int_0^{\frac{3\pi}{2}} \cos \tfrac{1}{3}x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x+\pi)\,dx$

SOLUTION:

(a) $$\int_0^{\frac{3\pi}{2}} \cos \tfrac{1}{3}x\,dx = 3\Big[\sin \tfrac{1}{3}x\Big]_0^{\frac{3\pi}{2}} = 3(\sin \tfrac{\pi}{2} - 3\sin 0) = 3$$

(b) $$\int_0^{\frac{\pi}{8}} \sec^2(2x+\pi)\,dx = \tfrac{1}{2}\Big[\tan(2x+\pi)\Big]_0^{\frac{\pi}{8}} = \tfrac{1}{2}(\tan \tfrac{5\pi}{4} - \tan\pi) = \tfrac{1}{2}$$

Given a Derivative, Find an Integral: As always, the results of chain-rule and product-rule differentiations can be reversed to give a primitive.

WORKED EXERCISE:

(a) Use the product rule to differentiate $x\sin x$.

(b) Hence find $\displaystyle\int x\cos x\,dx$.

SOLUTION:

(a) Let $y = x\sin x$.

By the product rule, $y' = vu' + uv'$
$= \sin x + x\cos x$.

Let $u = x$
and $v = \sin x$.
Then $u' = 1$
and $v' = \cos x$.

(b) Reversing this result,

$$\int(\sin x + x\cos x)\,dx = x\sin x$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + C, \text{ for some constant } C.$$

WORKED EXERCISE:

(a) Use the chain rule to differentiate $\cos^5 x$.

(b) Hence find $\displaystyle\int_0^{\pi} \sin x\cos^4 x\,dx$.

SOLUTION:

(a) Let $y = \cos^5 x$.

By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$= -5\sin x\cos^4 x$.

Let $u = \cos x$,
then $y = u^5$.
So $\dfrac{du}{dx} = -\sin x$
and $\dfrac{dy}{du} = 5u^4$.

(b) Reversing this result,

$$\int_0^{\pi}(-5\sin x\cos^4 x)\,dx = \Big[\cos^5 x\Big]_0^{\pi}$$

$$\int_0^{\pi} \sin x\cos^4 x\,dx = -\tfrac{1}{5}\Big[\cos^5 x\Big]_0^{\pi} = -\tfrac{1}{5}(-1-1) = \tfrac{2}{5}.$$

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Using the Reverse Chain Rule: The integral in the previous worked exercise could have been evaluated directly using the reverse chain rule.

WORKED EXERCISE: Use the reverse chain rule to evaluate:

(a) $\int_0^{\pi} \sin x \cos^4 x \, dx$ (b) $\int_0^{\frac{\pi}{3}} \tan^7 x \sec^2 x \, dx$

SOLUTION:

(a) $\int_0^{\pi} \sin x \cos^4 x \, dx = -\int_0^{\pi} (-\sin x) \cos^4 x \, dx$
$= -\frac{1}{5}\Big[\cos^5 x\Big]_0^{\pi}$
$= \frac{2}{5}$

Let $u = \cos x$.
Then $\frac{du}{dx} = -\sin x$.
$\int u^4 \frac{du}{dx}\, dx = \frac{1}{5}u^5$

(b) $\int_0^{\frac{\pi}{3}} \tan^7 x \sec^2 x \, dx = \frac{1}{8}\Big[\tan^8 x\Big]_0^{\frac{\pi}{3}}$.
$= \frac{1}{8}(\tan^8 \frac{\pi}{3} - \tan^8 0)$
$= \frac{1}{8} \times (\sqrt{3})^8$
$= \frac{81}{8}$

Let $u = \tan x$.
Then $\frac{du}{dx} = \sec^2 x$.
$\int u^7 \frac{du}{dx}\, dx = \frac{1}{8}u^8$

The Primitives of $\tan x$ and $\cot x$: The primitives of $\tan x$ and $\cot x$ can be found by using the ratio formulae $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$, followed by the reverse chain rule.

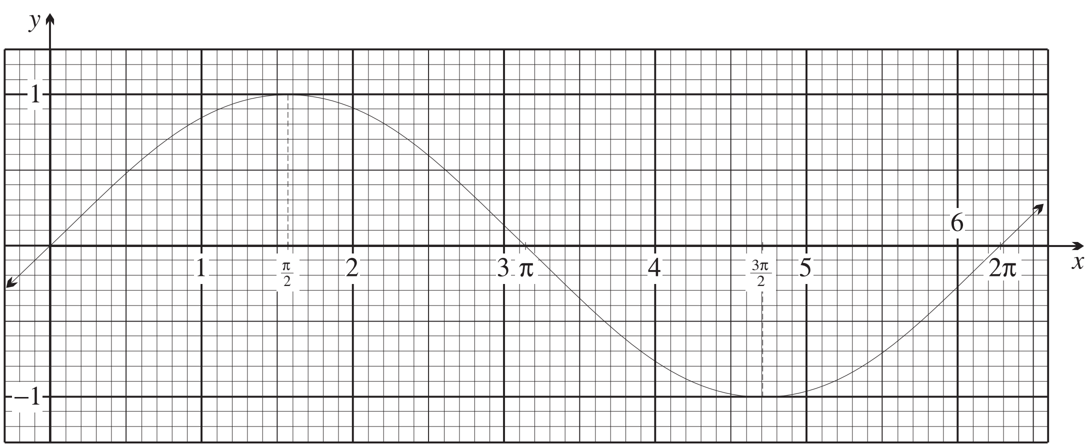
WORKED EXERCISE: Find the primitive of $\tan x$.

SOLUTION: $\int \tan x \, dx = \int \frac{\sin x}{\cos x}\, dx$
$= -\int \frac{-\sin x}{\cos x}\, dx$
$= -\log(\cos x) + C$

Let $u = \cos x$.
Then $\frac{du}{dx} = -\sin x$.
$\int \frac{1}{u}\frac{du}{dx}\, dx = \log u$

Exercise 14I

1.



(a) The first worked exercise in the theory for this section proved that $\int_0^{\pi} \sin x \, dx = 2$. Count squares on the graph of $y = \sin x$ drawn above to confirm this result.

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(b) Count squares and use symmetry to find:

(i) $\int_0^{\frac{\pi}{4}} \sin x\,dx$ (iii) $\int_0^{\frac{3\pi}{4}} \sin x\,dx$ (v) $\int_0^{\frac{3\pi}{2}} \sin x\,dx$

(ii) $\int_0^{\frac{\pi}{2}} \sin x\,dx$ (iv) $\int_0^{\frac{5\pi}{4}} \sin x\,dx$ (vi) $\int_0^{\frac{7\pi}{4}} \sin x\,dx$

(c) Evaluate these integrals using the fact that $-\cos x$ is a primitive of $\sin x$ and confirm the results of (b).

2. Find:

(a) $\int \sec^2 x\,dx$ (c) $\int \sin 2x\,dx$ (e) $\int \cos(3x-2)\,dx$ (g) $\int \sec^2(4-x)\,dx$

(b) $\int \cos(x+2)\,dx$ (d) $\int \sec^2(\frac{1}{3}x)\,dx$ (f) $\int \sin(7-5x)\,dx$ (h) $\int \sec^2\left(\frac{1-x}{3}\right)\,dx$

3. Find the value of:

(a) $\int_0^{\frac{\pi}{6}} \cos x\,dx$ (c) $\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$ (e) $\int_0^{\frac{\pi}{3}} \sin 2x\,dx$ (g) $\int_{\frac{\pi}{3}}^{\pi} \cos(\frac{1}{2}x)\,dx$

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx$ (d) $\int_0^{\frac{\pi}{4}} 2\cos 2x\,dx$ (f) $\int_0^{\frac{\pi}{2}} \sec^2(\frac{1}{2}x)\,dx$ (h) $\int_0^{\pi} (2\sin x - \sin 2x)\,dx$

4. Find: (a) $\int (6\cos 3x - 4\sin\frac{1}{2}x)\,dx$ (b) $\int (8\sec^2 2x - 10\cos\frac{1}{4}x + 12\sin\frac{1}{3}x)\,dx$

DEVELOPMENT

5. Find the following indefinite integrals, where a, b, u and v are constants:

(a) $\int a\sin(ax+b)\,dx$ (b) $\int \pi^2\cos\pi x\,dx$ (c) $\int \frac{1}{u}\sec^2(v+ux)\,dx$ (d) $\int \frac{a}{\cos^2 ax}\,dx$

6. (a) Copy and complete $1+\tan^2 x = \ldots$, and hence find $\int \tan^2 x\,dx$.

(b) Simplify $1-\sin^2 x$, and hence find the value of $\int_0^{\frac{\pi}{3}} \frac{2}{1-\sin^2 x}\,dx$.

7. Comment on the validity and the result of the following computation:

$$\int_0^{\pi} \sec^2 x\,dx = \Big[\tan x\Big]_0^{\pi} = \tan\pi - \tan 0 = 0.$$

8. (a) Copy and complete $\int \frac{f'(x)}{f(x)}\,dx = \ldots$, and hence show that $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}\,dx \doteqdot 0{\cdot}4$.

(b) Write $\tan x$ in terms of $\sin x$ and $\cos x$, and hence show that $\int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{1}{2}\ln 2$.

(c) Differentiate $e^{\sin x}$, and hence find the value of $\int_0^{\frac{\pi}{2}} \cos x\,e^{\sin x}\,dx$.

(d) Copy and complete $\int f'(x)e^{f(x)}\,dx = \ldots$, and hence find $\int_0^{\frac{\pi}{4}} \sec^2 x\,e^{\tan x}\,dx$.

9. (a) Find $\frac{d}{dx}(\sin x^2)$, and hence find $\int 2x\cos x^2\,dx$.

(b) Find $\frac{d}{dx}(\cos x^3)$, and hence find $\int x^2\sin x^3\,dx$.

(c) Find $\frac{d}{dx}(\tan\sqrt{x})$, and hence find $\int \frac{1}{\sqrt{x}}\sec^2\sqrt{x}\,dx$.

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10. (a) Show that $\dfrac{d}{dx}(\sin x - x\cos x) = x\sin x$, and hence find $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\,dx$.

(b) Show that $\dfrac{d}{dx}(\tfrac{1}{3}\cos^3 x - \cos x) = \sin^3 x$, and hence find $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x\,dx$.

11. (a) Find $\dfrac{d}{dx}(\sin^5 x)$, and hence find $\displaystyle\int \sin^4 x\cos x\,dx$.

(b) Find $\dfrac{d}{dx}(\tan x)^{-3}$, and hence find $\displaystyle\int \frac{\sec^2 x}{\tan^4 x}\,dx$.

12. Use the rule $\displaystyle\int u^n\frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1}$ or $\displaystyle\int f'(x)f(x)^n\,dx = \frac{(f(x))^{n+1}}{n+1}$, to evaluate:

(a) $\displaystyle\int_0^{\pi} \sin x\cos^8 x\,dx$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\cos^n x\,dx$

(c) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\sin^7 x\,dx$

(d) $\displaystyle\int_0^{\frac{\pi}{6}} \cos x\sin^n x\,dx$

(e) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\tan^7 x\,dx$

(f) $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\tan^n x\,dx$

13. (a) Use the compound-angle formula for $\sin(A+B)$ to show that $\sin 2x = 2\sin x\cos x$.

(b) Hence find: (i) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\cos x\,dx$ (ii) $\displaystyle\int_0^{\frac{\pi}{4}} \sin 2x\cos 2x\,dx$ (iii) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 2x\cos 2x\,dx$

14. (a) Use the compound-angle formula for $\cos(A+B)$ to show that $\cos 2x = \cos^2 x - \sin^2 x$.

(b) Hence show that: (i) $\cos 2x = 1 - 2\sin^2 x$ (ii) $\cos 2x = 2\cos^2 x - 1$

(c) Hence show that: (i) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ (ii) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

(d) Hence find: (i) $\displaystyle\int \sin^2 x\,dx$ (ii) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

(e) Use part (c)(i) to write $\sin^2 2x$ in terms of $\cos 4x$, and hence find $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 2x\,dx$.

(f) Find: (i) $\displaystyle\int_0^{\frac{\pi}{3}} \cos^2 \tfrac{x}{2}\,dx$ (ii) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 \tfrac{x}{2}\,dx$

15. Find:

(a) $\displaystyle\int e^{2x}\cos e^{2x}\,dx$

(b) $\displaystyle\int \frac{\sin e^{-2x}}{e^{2x}}\,dx$

(c) $\displaystyle\int \frac{\sec^2 x}{3\tan x + 1}\,dx$

(d) $\displaystyle\int \frac{3\sin x}{4 + 5\cos x}\,dx$

(e) $\displaystyle\int \frac{1 - \cos^3 x}{1 - \sin^2 x}\,dx$

(f) $\displaystyle\int_0^{\pi} \sin x\cos^2 x\,dx$

16. Find $\dfrac{d}{dx}(x\sin 2x)$, and hence find $\displaystyle\int_0^{\frac{\pi}{4}} x\cos 2x\,dx$.

17. (a) Show that $\dfrac{d}{dx}(\tan^3 x) = 3(\sec^4 x - \sec^2 x)$. (b) Hence find $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x\,dx$.

18. (a) Show that $\tan^3 x = \tan x\sec^2 x - \tan x$.

(b) Hence find: (i) $\displaystyle\int \tan^3 x\,dx$ (ii) $\displaystyle\int \tan^5 x\,dx$

19. (a) Show that $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

(b) Hence find: (i) $\displaystyle\int_0^{\frac{\pi}{2}} 2\sin 3x\cos 2x\,dx$ (ii) $\displaystyle\int_0^{\pi} \sin 3x\cos 4x\,dx$

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(c) Show that $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$ for positive integers m and n:

(i) using the primitive, (ii) using symmetry arguments.

20. (a) Show that $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$.

(b) Hence find: (i) $\int_0^{\frac{\pi}{2}} 2\cos 3x \cos 2x \, dx$ (ii) $\int_0^{\pi} \cos 3x \cos 4x \, dx$

(c) Find $\int \cos mx \cos nx \, dx$, where m and n are:

(i) distinct positive integers, (ii) equal positive integers.

(d) Show that for positive integers m and n, $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & \text{when } m \neq n, \\ 2\pi, & \text{when } m = n. \end{cases}$

21. (a) Show, by finding the integral in two different ways, that for constants C and D,

$$\int \sin x \cos x \, dx = \tfrac{1}{2}\sin^2 x + C = -\tfrac{1}{4}\cos 2x + D.$$

(b) How may the two answers be reconciled?

EXTENSION

22. (a) Find the values of A and B in the identity

$$A(2\sin x + \cos x) + B(2\cos x - \sin x) = 7\sin x + 11\cos x.$$

(b) Hence show that $\int_0^{\frac{\pi}{2}} \dfrac{7\sin x + 11\cos x}{2\sin x + \cos x} \, dx = \frac{5\pi}{2} + \ln 8$.

23. (a) If $y = Ae^{\lambda x}\sin x + Be^{\lambda x}\cos x$, show that $y'' - 2\lambda y' + (\lambda^2 + 1)y = 0$.

(b) A function satisfies $y'' - 2\lambda y' + (\lambda^2 + 1)y = 0$, and $y(0) = y'(0) = 1$. Evaluate A and B, and hence find the equation of the function.

24. [The power series for $\sin x$ and $\cos x$] Here is the outline of a proof of the two power series for $\sin x$ and $\cos x$, introduced informally in the Extension group of Exercise 14G:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{and} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

(a) We know that $\cos t \leq 1$, for t positive, since $\cos t$ was defined as the ratio of a semichord of a circle over the radius. Integrate this inequality over the interval $0 \leq t \leq x$, where x is positive, and hence show that $\sin x \leq x$.

(b) Change the variable to t, integrate the inequality $\sin t \leq t$ over $0 \leq t \leq x$, and hence show that $\cos x \geq 1 - \dfrac{x^2}{2!}$.

(c) Do it twice more, and show that: (i) $\sin x \geq x - \dfrac{x^3}{3!}$ (ii) $\cos x \leq 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

(d) Now use induction (informally) to show that for all positive integers n,

$$\sin x \;\leq\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4n+1}}{(4n+1)!} \;\leq\; \sin x + \frac{x^{4n+3}}{(4n+3)!},$$

and use this inequality to conclude that $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$ converges, with limit $\sin x$.

(e) Proceeding similarly, prove that $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$ converges, with limit $\cos x$.

(f) Use evenness and oddness to extend the results of (d) and (e) to negative values of x.

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14 J Applications of Integration

The integrals of the trigonometric functions can now be applied to finding areas and volumes, and to finding functions whose derivatives are known.

WORKED EXERCISE: If $f'(x) = \cos x + \sin 2x$ and $f(\pi) = 0$, find $f(\frac{\pi}{4})$.

SOLUTION: $f(x) = \sin x - \frac{1}{2}\cos 2x + C$, for some constant C.

Substituting $f(\pi) = 0$, $0 = 0 - \frac{1}{2} + C$,

so $C = \frac{1}{2}$ and $f(x) = \sin x - \frac{1}{2}\cos 2x + \frac{1}{2}$.

Substituting $x = \frac{\pi}{4}$, $f(\frac{\pi}{4}) = \frac{1}{2}\sqrt{2} - 0 + \frac{1}{2}$

$= \frac{1}{2}(1 + \sqrt{2})$.

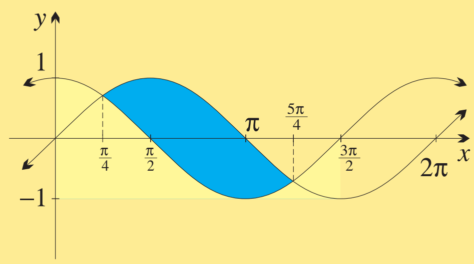
WORKED EXERCISE: Find where the curves $y = \sin x$ and $y = \cos x$ intersect in the interval $0 \le x \le 2\pi$, sketch the curves, and find the area contained between them.

SOLUTION: Put $\sin x = \cos x$,

then $\tan x = 1$

$x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$,

so they intersect at $(\frac{\pi}{4}, \frac{1}{2}\sqrt{2})$ and $(\frac{5\pi}{4}, -\frac{1}{2}\sqrt{2})$.



$$\text{Area between} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$$

$$= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -\cos\tfrac{5\pi}{4} - \sin\tfrac{5\pi}{4} + \cos\tfrac{\pi}{4} + \sin\tfrac{\pi}{4}$$

$$= 4 \times \tfrac{1}{2}\sqrt{2}$$

$$= 2\sqrt{2} \text{ square units.}$$

WORKED EXERCISE:

(a) Differentiate $y = \log(\cos x)$, and hence find $\int \tan x\,dx$.

(b) Sketch $y = 1 + \tan x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$, then, using the integral in part (a), find the volume generated when the area under the curve from $x = 0$ to $x = \frac{\pi}{4}$ is rotated about the x-axis.

SOLUTION:

(a) By the chain rule, $\dfrac{d}{dx}\log(\cos x) = \dfrac{1}{\cos x} \times (-\sin x)$,

$= -\tan x$.

Reversing this, $\int \tan x\,dx = -\log(\cos x) + C$, for some constant C.

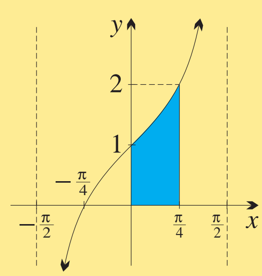
(b) Volume $= \displaystyle\int_0^{\frac{\pi}{4}} \pi(1 + \tan x)^2\,dx$

$= \pi \displaystyle\int_0^{\frac{\pi}{4}} (1 + 2\tan x + \tan^2 x)\,dx.$

Using the Pythagorean identity $1 + \tan^2 x = \sec^2 x$,

volume $= \pi \displaystyle\int_0^{\frac{\pi}{4}} (\sec^2 x + 2\tan x)\,dx$

$= \pi \Big[\tan x - 2\log(\cos x)\Big]_0^{\frac{\pi}{4}}$



$$= \pi\left(\tan\tfrac{\pi}{4} - 2\log\frac{1}{\sqrt{2}} - \tan 0 + 2\log 1\right)$$
$$= \pi(1 + 2 \times \tfrac{1}{2}\log 2 - 0 + 0)$$
$$= \pi(1 + \log 2) \text{ cubic units.}$$

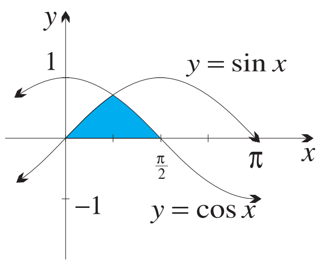
Exercise 14J

1. Find, using a diagram, the area bounded by one arch of each curve and the x-axis:

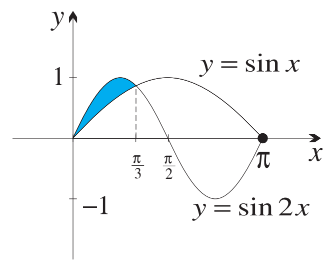
(a) $y = \sin x$ (b) $y = \cos 2x$

2. Calculate the area of the shaded region in each diagram below:

(a)



(b)



3. Sketch the area enclosed between each curve and the x-axis over the specified domain, and then find the exact value of the area. (Make use of symmetry wherever possible.)

(a) $y = \cos x$ from $x = 0$ to $x = \pi$

(b) $y = \sin x$ from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$

(c) $y = \sec^2 x$ from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$

(d) $y = \cos 2x$ from $x = 0$ to $x = \pi$

(e) $y = \sin 2x$ from $x = \frac{\pi}{3}$ to $x = \frac{2\pi}{3}$

(f) $y = \sec^2 \frac{1}{2}x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

(g) $y = \sin x$ from $x = -\frac{5\pi}{6}$ to $x = \frac{7\pi}{6}$

(h) $y = \cos 3x$ from $x = \frac{\pi}{6}$ to $x = \frac{2\pi}{3}$

4. (a) The gradient function of a certain curve is given by $y' = \cos x - 2\sin 2x$. If the curve passes through the origin, find its equation.

(b) If $f'(x) = \cos \pi x$ and $f(0) = \frac{1}{2\pi}$, find $f(\frac{1}{6})$.

(c) If $f'(x) = \pi\cos \pi x$ and $f(0) = 0$, find $f(\frac{1}{3})$.

(d) If $f''(x) = 18\cos 3x$ and $f'(0) = f(\frac{\pi}{2}) = 1$, find $f(x)$.

5. Find the exact volume of the solid of revolution formed when each region described below is rotated about the x-axis:

(a) the region bounded by the curve $y = \sec x$ and the x-axis from $x = 0$ to $x = \frac{\pi}{4}$,

(b) the region bounded by the curve $y = \sqrt{\cos 4x}$ and the x-axis from $x = 0$ to $x = \frac{\pi}{8}$,

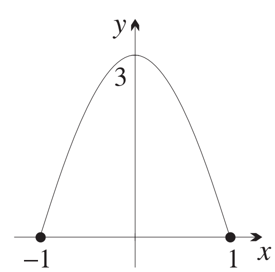
(c) the region bounded by the curve $y = \sqrt{1 + \sin 2x}$ and the x-axis from $x = 0$ to $x = \frac{\pi}{4}$.

DEVELOPMENT

6. (a) Sketch the curve $y = 2\cos \pi x$ for $-1 \le x \le 1$, clearly marking the two x-intercepts.

(b) Find the exact area bounded by the curve $y = 2\cos \pi x$ and the x-axis, between the two x-intercepts.

7. An arch window 3 metres high and 2 metres wide is made in the shape of the curve $y = 3\cos(\frac{\pi}{2}x)$. Find the area of the window in square metres, correct to one decimal place.



8. (a) Show that $\displaystyle\int_0^{2\pi} \sin nx\,dx = \int_0^{2\pi} \cos nx\,dx = 0$, for all positive integers n.

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(b) Sketch each graph, and then find the area from $x = 0$ to $x = 2\pi$ between the x-axis and: (i) $y = \sin x$ (ii) $y = \sin 2x$ (iii) $y = \sin 3x$ (iv) $y = \sin nx$ (v) $y = \cos nx$

9. (a) Show that $\displaystyle\int_0^1 \sin \pi x\, dx = \frac{2}{\pi}$. (b) Use Simpson's rule with five function values to approximate $\displaystyle\int_0^1 \sin \pi x\, dx$. (c) Hence show that $\pi(1 + 2\sqrt{2}\,) \doteqdot 12$.

10. The graphs of $y = x - \sin x$ and $y = x$ are sketched together in the notes to Section 14H. Find the total area enclosed between these graphs from $x = 0$ to $x = 2\pi$.

11. The region R is bounded by the curve $y = \tan x$, the x-axis and the vertical line $x = \frac{\pi}{3}$.
(a) Sketch R and then find its area.
(b) Find the volume generated when R is rotated about the x-axis.

12. (a) Express $\cot x$ as the ratio of $\cos x$ and $\sin x$, and hence find a primitive of $\cot x$.
(b) Find the area between the curve $y = \cot x$ and the x-axis, from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$.
(c) Comment on the validity and the results of the following calculations:

(i) $$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cot x\, dx = \Big[\log \sin x\Big]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \log(\tfrac{1}{2}\sqrt{2}\,) - \log(\tfrac{1}{2}\sqrt{2}\,) = 0$$

(ii) $$\int_{\frac{3\pi}{4}}^{\frac{9\pi}{4}} \cot x\, dx = \Big[\log \sin x\Big]_{\frac{3\pi}{4}}^{\frac{9\pi}{4}} = \log(\tfrac{1}{2}\sqrt{2}\,) - \log(\tfrac{1}{2}\sqrt{2}\,) = 0$$

13. The region R is bounded by the curve $y = \sin x$, the x-axis and the vertical line $x = \frac{\pi}{2}$.
(a) Sketch R and then find its area.
(b) Find the exact volume of the solid generated when R is rotated through one complete revolution about the x-axis.

14. The region R is bounded by the curve $y = \cos 2x$, the x-axis and the lines $x = \frac{\pi}{6}$ and $x = -\frac{\pi}{6}$. (a) Sketch R and then find its area.
(b) Find the exact volume generated when the region R is rotated about the x-axis.

15. Show that for $0 < b < \frac{\pi}{2}$, the volume generated by rotating $y = \tan x$ from $x = 0$ to $x = b$ about the x-axis is πb less than the volume generated by rotating $y = \sec x$.

16. (a) Sketch the region bounded by the graphs of $y = \sin x$, $y = \cos x$, $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.
(b) Find the area of the region in part (a).

17. (a) Show that the curves $y = \sin x$ and $y = \cos 2x$ meet at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.
(b) Sketch the curves $y = \sin x$ and $y = \cos 2x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{6}$.
(c) Hence find the area of the region bounded by the two curves.

18. (a) Show that $\displaystyle\int_0^n (1 + \sin 2\pi x)\, dx = n$, for all positive integers n.
(b) Sketch $y = 1 + \sin 2\pi x$, and interpret the result geometrically.

19. (a) Find $\displaystyle\lim_{n\to\infty} \int_0^a \sin nx\, dx$, where a is any real number. (b) Explain geometrically what is happening in part (a).

20. (a) Show that $\sqrt{2}\,\sin(x + \frac{\pi}{4}) = \sin x + \cos x$. (b) Hence, or otherwise, find:
(i) the exact area under one arch of the curve $y = \sin x + \cos x$,
(ii) the exact volume generated when the arch is rotated about the x-axis.

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21. (a) Sketch $y = 1 - \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and shade the region R bounded by the curve and the coordinate axes. (b) Find:
(i) the area of R, (ii) the volume generated when R is rotated about the x-axis.

22. A champagne glass is designed by rotating the curve $y = 4 + 4\sin\frac{x}{4}$ from $x = 4\pi$ to $x = 6\pi$ about the x-axis through $360°$. Find, in millilitres correct to two decimal places, the capacity of the glass, if 1 unit = 1 cm.

23. Sketch $y = |\sin x|$ for $0 \le x \le 6\pi$, and hence evaluate $\displaystyle\int_0^{6\pi} |\sin x|\,dx$.

24. (a) Explain why $x^2\sin x < x^3 < x^2\tan x$ for $0 < x < \frac{\pi}{2}$.

(b) Hence show that $\displaystyle\int_0^{\frac{\pi}{4}} x^2\sin x\,dx < \frac{\pi^4}{4^5} < \int_0^{\frac{\pi}{4}} x^2\tan x\,dx$.

25. (a) Given that $y = \dfrac{1}{1+\sin x}$, show that $y' = -\dfrac{\cos x}{(1+\sin x)^2}$.

(b) Hence explain why the function $y = \dfrac{1}{1+\sin x}$ is decreasing for $0 < x < \frac{\pi}{2}$.

(c) Sketch the curve for $0 \le x \le \frac{\pi}{2}$, and hence show that $\displaystyle\frac{\pi}{4} < \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx < \frac{\pi}{2}$.

26. Use symmetry arguments to help evaluate:

(a) $\displaystyle\int_{-4\pi}^{4\pi} \sin 3x\,dx$

(b) $\displaystyle\int_{-2\pi}^{2\pi} \cos^2 x\sin^3 x\,dx$

(c) $\displaystyle\int_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}} \cos x\,dx$

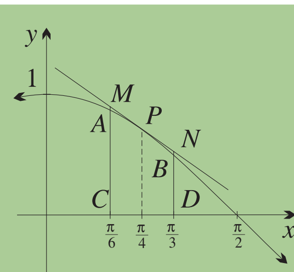
(d) $\displaystyle\int_{-\pi}^{\pi} \sec^2 \tfrac{1}{3}x\,dx$

(e) $\displaystyle\int_{-\pi}^{\pi} (3 + 2x + \sin x)\,dx$

(f) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin 2x + \cos 3x + 3x^2)\,dx$

EXTENSION

27. In the diagram, P is the point where $x = \frac{\pi}{4}$ on the curve $y = \cos x$. The vertical lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ meet the curve at A and B respectively, and they meet the tangent at P at M and N respectively.



(a) Show that the tangent at P is $x + \sqrt{2}\,y = 1 + \frac{\pi}{4}$.

(b) Show that M has y-coordinate $\frac{1}{24}\sqrt{2}\,(12+\pi)$ and find the y-coordinate of N.

(c) Find the areas of the trapezia $ABDC$ and $MNDC$.

(d) Hence show, without using a calculator, that $3\sqrt{2}\,(\sqrt{3}-1) < \pi < 6(\sqrt{3}-1)^2$.

28. (a) Show that $\dfrac{d}{dx}\left(-\frac{1}{2}\,e^{-x}(\sin x + \cos x)\right) = e^{-x}\sin x$.

(b) Find $\displaystyle\int_0^N e^{-x}\sin x\,dx$, and show that $\displaystyle\int_0^\infty e^{-x}\sin x\,dx$ converges to $\frac{1}{2}$.

(c) Find $\displaystyle\int_0^\pi e^{-x}\sin x\,dx$, $\displaystyle\int_{2\pi}^{3\pi} e^{-x}\sin x\,dx$, ..., and show that the areas of the arches above the x-axis form a GP with limiting sum $\dfrac{e^\pi}{2(e^\pi-1)}$.

(d) Show that the areas of the arches below the x-axis also form a GP, and hence show that the total area contained between the curve and the x-axis, to the right of the y-axis, is $\dfrac{e^\pi+1}{2(e^\pi-1)}$. Also confirm by subtraction the result of part (b).

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