THE UNIVERSITY OF SYDNEY

PHYS1902 – PHYSICS 1B (ADVANCED)

NOVEMBER 2009

Time allowed: THREE Hours

MARKS FOR QUESTIONS ARE AS INDICATED
TOTAL: 90 marks

INSTRUCTIONS

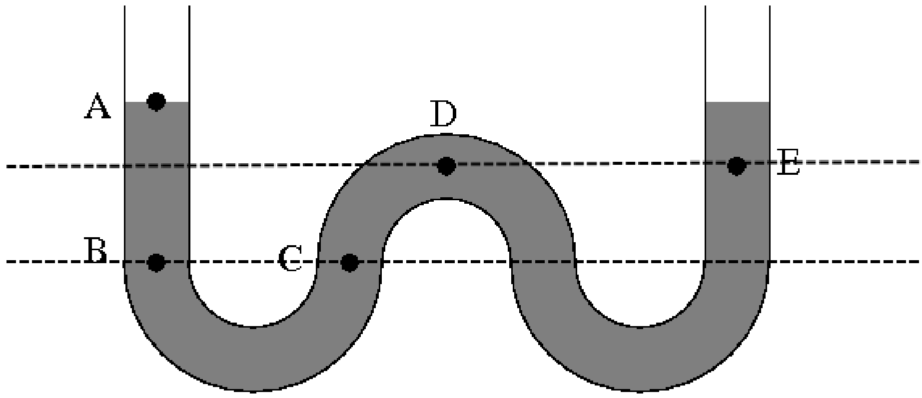
- **All questions are to be answered.**
- **Use a separate answer book for each section.**
- **All answers should include explanations in terms of physical principles.**

DATA

Density of water	ρ	=	1.00×10^{3} kg.m^{-3}
Density of air	ρ	=	1.20 kg.m^{-3}
Atmospheric pressure	1 atm	=	1.01×10^{5} Pa
Magnitude of local gravitational field	g	=	9.80 m.s^{-2}
Avogadro constant	N_A	=	6.022×10^{23} mol^{-1}
Permittivity of free space	ε_0	=	8.854×10^{-12} F.m^{-1}
Permeability of free space	μ_0	=	$4\pi \times 10^{-7}$ T.m.A^{-1}
Elementary charge	e	=	1.602×10^{-19} C
Speed of light in vacuum	c	=	2.998×10^{8} m.s^{-1}
Planck constant	h	=	6.626×10^{-34} J.s
Rest mass of an electron	m_e	=	9.110×10^{-31} kg
Rest mass of a neutron	m_n	=	1.675×10^{-27} kg
Rest mass of a proton	m_p	=	1.673×10^{-27} kg
Rest mass of a hydrogen atom	m_H	=	1.674×10^{-27} kg
Boltzmann constant	k	=	1.381×10^{-23} J.K^{-1}
Atomic mass unit	u	=	1.661×10^{-27} kg
Rydberg constant	R	=	1.097×10^{7} m^{-1}

ADV_Q01

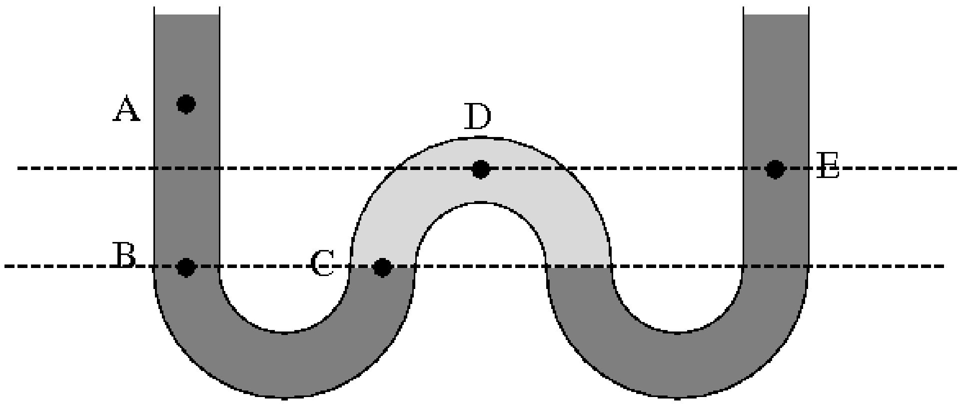
A W-shaped tube is partially filled with water. Both ends of the tube are open and exposed to the atmosphere. Points B and C are at the same height, as are points D and E.



(a) Using the diagram above, rank the points A–E according to pressure from largest to smallest. If the pressure at any two points is the same, state so explicitly. Explain your answers.

(b) Suppose that some oil, with a density less than that of water, was injected into the central portion of the tube, as shown in the diagram below. No water is removed, and no air is allowed inside the central portion of the tube. Would the pressure at point E increase, decrease, or stay the same? Explain your answer.

(c) Would the pressure at point D, in the diagram below, be greater than, less than, or equal to the pressure at point E? Explain your answer.



(5 marks)

Solution

(a) Since all the points are submerged in the same fluid, the pressure at each point is determined by the distance between each point and the surface of the water. The pressure increases with depth $P = P_0 + \rho g h$, and points at the same height will have the same pressure. Hence

$$P_B = P_C > P_D = P_E > P_A$$

(1 mark for correct answer, 1 mark for reasoning)

(b) There is still only water between point E and the surface, so the change in pressure at E is determined solely by the change in distance between point E and the surface. This distance has increased, so the pressure at E is *greater* than it was before the oil was added.

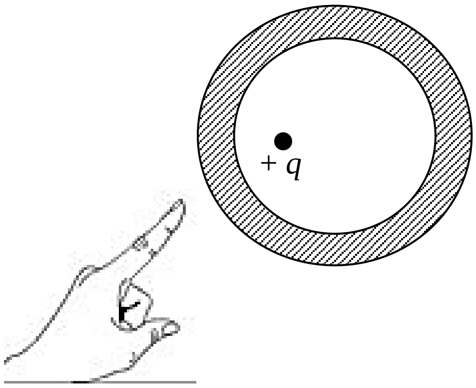
(1 mark)

(c) Take the reference point to be point C. Since point B is at the same level and in the same fluid, the pressure at point B is the same as the pressure at point C. The difference in pressure between a point at that level and a point at a different level is given by $\Delta P = \rho g \Delta h$. Points D and E are at the same height above the reference level, but the fluid between point D and the reference point is less dense than that between point E and the reference point. Hence the pressure difference between point D and the reference point is less than that between point E and the reference point. This means that the pressure at point D is greater than that at point E.

(1 mark for correct answer, 1 mark for reasoning)

ADV_Q02=TEC_Q02=ENV_Q03

The diagram below shows a cross-sectional view of a charge + q inside a cavity in a metal spherical shell. The shell carries no net charge.



(a) Sketch the distribution of any charges on the sphere.

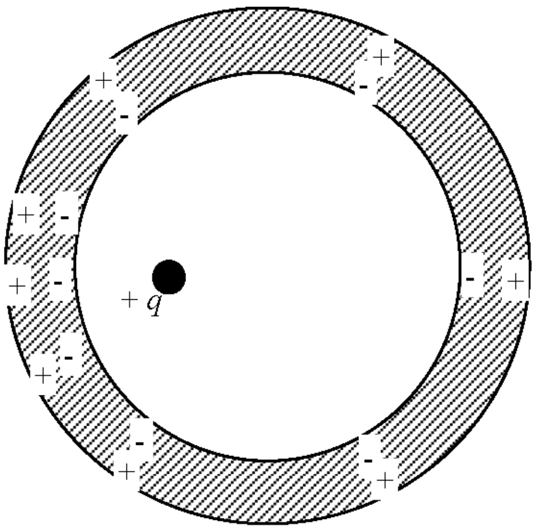
(b) On the same diagram, sketch some representative electric field lines in the various regions of the diagram, using them to indicate where the field is stronger or weaker (if anywhere).

(c) Someone now touches the outside surface of the spherical shell for a moment and then moves away. Draw a new diagram showing some representative electric field lines. Justify the differences (if any) between your two diagrams.

(5 marks)

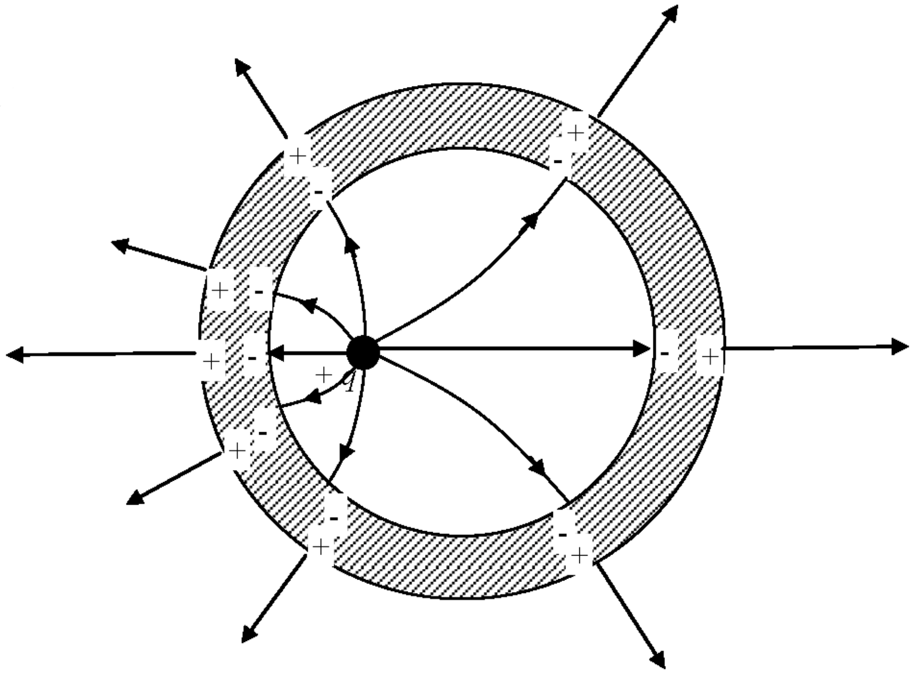
Solution

(a)



(1 mark)

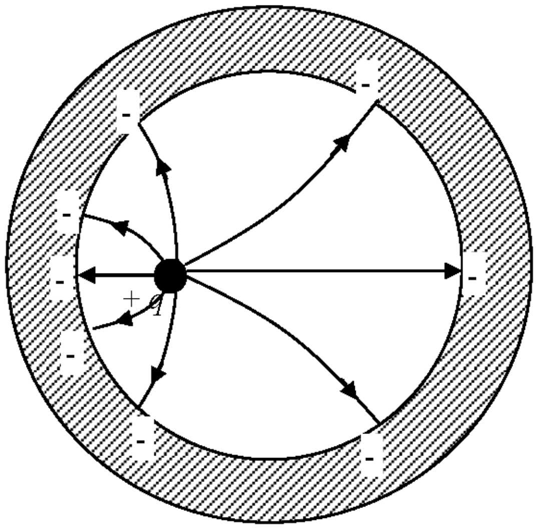
(b)



Note that the charges on the above diagram are located only on the surface of the shell.

(stronger field near the charge; field lines perpendicular to surface - 2 marks)

(c)



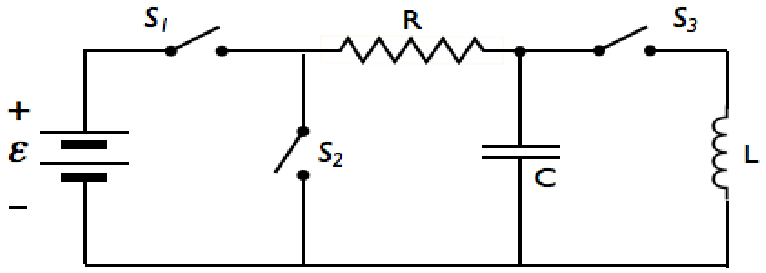
(1 mark)

Touching the outer surface of the shell allows charge to flow (electrons flow in to neutralise the outer positive changes), leaving a net negative charge on the shell. This leaves no net charge within a Gaussian surface surrounding the shell, and therefore no net flux and hence no field outside the shell.

(1 mark)

Total: 5 marks

ADV_Q03



(a) In the circuit shown above, initially all switches (S_1, S_2, S_3) are open and the capacitor is discharged. Switch S_1 is closed at a time $t = 0$. Sketch the current flowing through resistor *R* as a function of time indicating the initial value as well as a time scale.

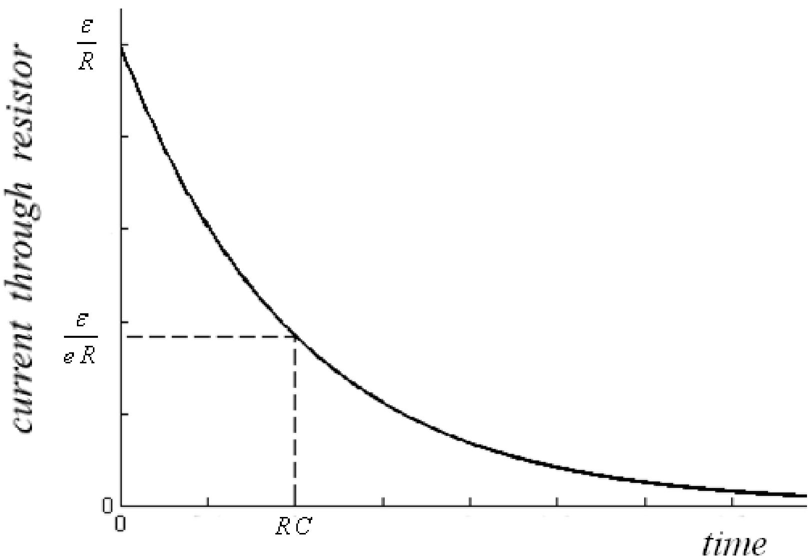
(b) After waiting for a long time with S_1 closed, switch S_1 is opened and switch S_3 is closed (but switch S_2 remains open). Sketch the charge on the capacitor as a function of time, clearly indicating the initial value and a time scale.

(c) With switch S_1 still open and switch S_3 still closed, switch S_2 is now closed. Briefly explain what happens to the energy in the circuit.

(5 marks)

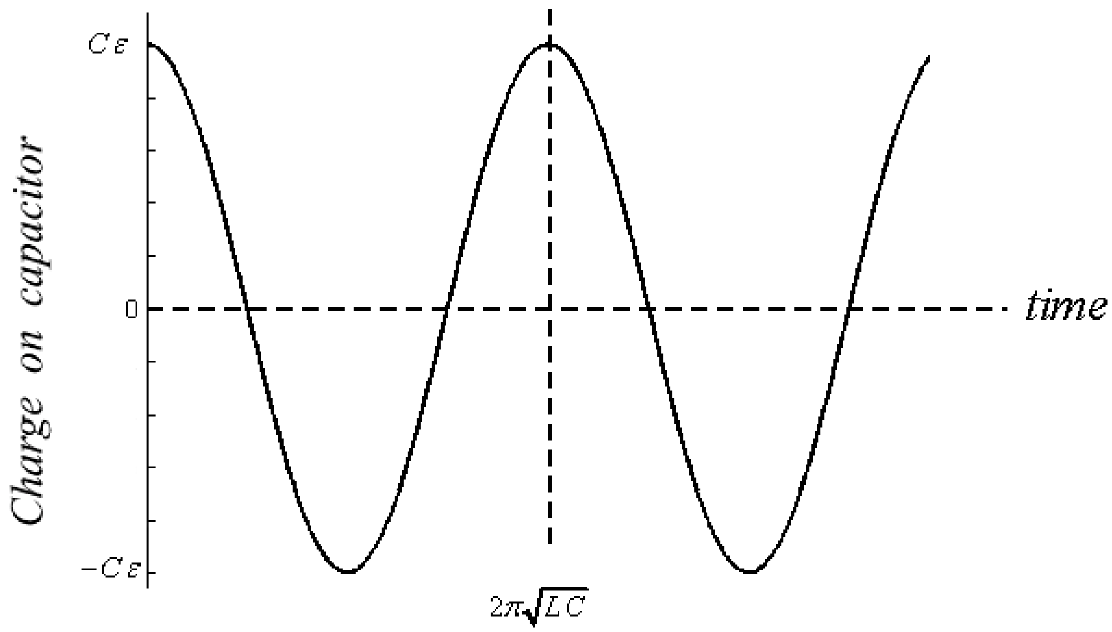
Solution

(a)



general shape (**1 mark**)
initial value of current (**½ mark**)
time scale (**½ mark**)

(b)



general shape **(1 mark)**
initial current **(½ mark)**
period **(½ mark)**

(c)
Energy will be dissipated in the resistor R. The oscillations in part (b) will be attenuated and stop. **(1 mark)**

ADV_Q04

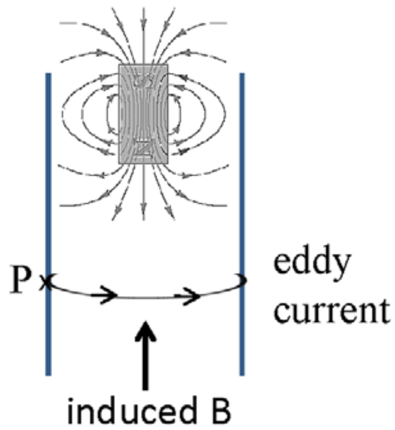
A magnet dropped through a hollow copper pipe is observed to fall very slowly. By considering what happens as the magnet moves past a fixed point P on the pipe, carefully explain this observation.

Assume the magnet falls without spinning, with its north pole downwards.

Your explanation should include a diagram and be given in terms of physical principles but without using equations. Your answer should be no more than half a page long, excluding the diagram.

(5 marks)

Solution



As the magnet approaches point P the magnetic flux through the pipe at the level of P increases. Faraday's law says that an emf is generated inducing eddy currents which in turn create a magnetic field (by Ampere's law).

Lenz's law tells us that the induced current opposes the *change* in the flux that caused it, so that the induced magnetic field when the north pole approaches is *increasing upwards* and acts to repel the falling magnet. This means that the magnet slows down.

After the south pole of the magnet has gone past point P the magnetic flux at the level of P decreases, the eddy currents are reversed in direction. The magnetic field induced by this diminishing flux is thus downwards and acts to attract the magnet. This means that the magnet falls more slowly.

Extra Note: The induced eddy currents around the tube do take many different paths other than the circumferential one shown in the diagram. The main point is that there should at least be a vertical component of the induced magnetic field.

(1 mark for sensible diagram)
(1 mark for emf generated by Faraday's law)
(1 mark for induced B field by Ampere's law)
(1 mark for Lenz's Law)
(1 mark for B to be upwards acting to repel magnet as it approaches)

ADV_Q05=TEC_Q05

Bremsstrahlung radiation is produced when electrons accelerated through a potential V strike a metal target. The radiation has two components: a continuum component and a line spectrum component.

(a) Briefly explain the origin of the continuum component.

Suppose the radiation is produced by a tungsten target which is then replaced by a nickel target.

(b) Briefly explain why the continuum component changes, or why not, if it does not change.

The continuum component exhibits a maximum frequency given by

$$f_{\max} = \frac{eV}{h}.$$

(c) Explain the origin of this maximum frequency, and hence derive the equation above.

(d) Explain why the existence of a maximum frequency is inconsistent with the classical (wave) picture of light.

(5 marks)

Solution

(a)

The continuum component is produced when the electrons are rapidly decelerated by close encounters with nuclei in the target material. They are decelerated and radiate.

(1 mark)

(b)

The continuum component is independent of the target material, so it will not change.

(1 mark)

(c)

The electrons incident on the metal can lose energy in multiple encounters with nuclei, or in a single encounter. If an electron gives up its energy in a single encounter it produces one photon, with energy equal to its kinetic energy. The electron is accelerated through a potential V and hence have a kinetic energy equal to eV. If all of this energy goes into one photon then the maximum possible energy for the photon is:

$$eV = h\,f_{\max}$$

$$\Rightarrow f_{\max} = \frac{eV}{h}.$$

(1 mark for explanation; 1 mark for derivation)

(d)

In the classical picture, the electron is braked continuously by a suitably close encounter with a (point) target nucleus. This leads to a continuous spectrum with no obvious reason for a maximum.

(1 mark)

ADV_Q06

The magnitude of the magnetic moment associated with a current loop with current I and area A is

$$\mu = I\,A.$$

(a) In the Bohr model for the Hydrogen atom the electron orbits the nucleus in a plane with radius *r* and speed *v*, and forms a small current loop. The magnitude of the orbital angular momentum *L* of the electron is assumed to have quantised values:

$$L = n\hbar,$$

with $n = 1, 2, ...$ Show that the magnitude of the magnetic moment of the ground state electron in the Bohr model is

$$\mu = \frac{e}{2m}\hbar.$$

(b) The quantum-mechanically correct quantisation of orbital angular momentum for the electron in the Hydrogen atom is that the magnitude of the angular momentum may have possible values

$$L = \sqrt{\ell(\ell+1)}\ \hbar,$$

where $\ell = 0, 1, 2, ..., n-1$ is the orbital quantum number and n is the principal quantum number. Any given component of the angular momentum, say the *z*–component, may have values

$$L_z = m_\ell \hbar,$$

where $m_\ell = 0, \pm 1, \pm 2, ..., \pm \ell$ is the magnetic quantum number.

For a given choice of n and ℓ, it follows that $|L_z| < L$. Briefly explain how this result is a requirement of the Heisenberg uncertainty principle.

(5 marks)

Solution

(a)

For the classical electron the current is:

$$I = \frac{e}{2\pi r / v} = \frac{e\,v}{2\pi r}.$$

Hence

$$\mu = I\,\pi r^2 = \frac{e\,v\,r}{2}.$$

The angular momentum is $L = m\,v\,r$ and hence we can write

$$\mu = \frac{e}{2m}L = \frac{e}{2m}\hbar$$

for the ground state of the Bohr atom for which $L = \hbar.$

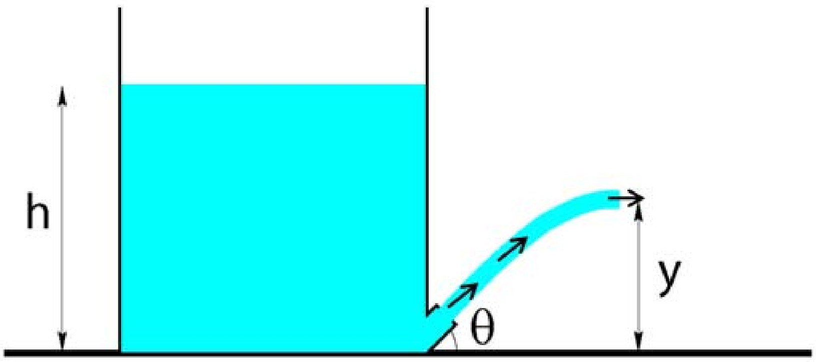
(3 marks)

(b)

We have $\vec{L} = \vec{r} \times \vec{p}$. If $L = |L_z|$ then $\vec{n}$ and $\vec{p}$ are in a plane perpendicular to $\hat{z}$. In that case $\Delta z = 0$ (motion is in a plane) and consequently the uncertainty in the z position of the electron is zero. The uncertainty principle then tells us that $\Delta p_z = \infty$ which means that the electron is not moving in a plane. This contradiction implies that it is not possible to have $L = |L_z|$ and hence $|L_z| < L$.

(2 marks)

ADV_Q07



A stream of (non-viscous) liquid emerges from a short outlet tube at the base of a large open tank, in which the depth of liquid is *h*. The tube opening has a cross-sectional area *A* and is at an angle θ to the ground (as shown in the diagram). You can ignore the height above ground of the end of the outlet tube since it is small compared to *h*.

(a) Obtain an expression for the speed of the liquid *v* as it exits the outlet tube.

(b) Show that the maximum height of the stream *y* can be expressed in terms of θ and *h*. by $y = h\sin^2\theta$.

(c) Show that this height *y* cannot be greater than *h*

(d) The cross-sectional area of the stream changes after it leaves the outlet tube. Where is this area maximum? Explain why this is so.

(e) Explain qualitatively how the maximum height of the stream would change if the liquid were replaced by a mildly viscous liquid.

(10 marks)

Solution

(a)

Use Bernoulli's equation between the surface of the tank and the exit of the outlet tube. These are both on the same streamline, so Bernoulli's equation applies. Assume that the tank is large enough that we can take the velocity of the surface of the water to be equal to zero.

Surface (point 1): $v_1 = 0;\ \ y_1 = h;\ \ p_1 = p_0$

Outlet (point 2): $v_2 = v;\ \ y_2 = 0;\ \ p_2 = p_0$

where p_0 is atmospheric pressure.

Bernoulli's equation reduces to:

$$\rho g h = \frac{1}{2}\rho v^2$$

$$\Rightarrow v = \sqrt{2gh}$$

(1 mark for correct approach (Bernoulli); 1 mark for correct setup; 1 mark for correct solution)

(b)
The initial *x*- and *y*- components of the velocity of the liquid as it exits the tube are:
$v_{x0} = v\cos\theta$
$v_{y0} = v\sin\theta$
The liquid will follow a ballistic trajectory so that we can use the equations from projectile motion so that the y-velocity at time t is

$$v_y(t) = v_{y0} - g\,t.$$

Maximum height occurs at time $t_{\max}$ when $v_y(t_{\max}) = 0,$ so

$$v_{y0} - g\,t_{\max} = 0$$

$$\Rightarrow t_{\max} = \frac{v_{y0}}{g}.$$

The height at time $t_{\max}$ is given by

$$\begin{aligned}
y_{\max} &= v_{y0} - \frac{1}{2} g\,t_{\max}{}^2 \\
&= v_{y0}\left(\frac{v_{y0}}{g}\right) - \frac{1}{g} g\left(\frac{v_{y0}}{g}\right)^2 \\
&= \frac{1}{2}\frac{v_{y0}{}^2}{g} \\
&= \frac{\left(v\sin\theta\right)^2}{2g}.
\end{aligned}$$

We can also use conservation of energy

$$m\,g\,y_{\max} = \frac{1}{2} m v_{y0}{}^2$$

$$\Rightarrow y_{\max} = \frac{v_{y0}{}^2}{2g}$$

$$= \frac{\left(v\sin\theta\right)^2}{2g}$$

In either case since $v = \sqrt{2\,g\,h}$ from part (a) then

$$y_{\max} = h^2 \sin\theta\,.$$

(1 mark for a correct approach, 1 mark for correct setup; 1mark for correct solution)

(c)
The maximum possible value for $\sin\theta$ is 1, so the maximum height is reached when $\sin\theta = 1$ in which case

$$y_{\max} = h\,.$$

(1 mark)

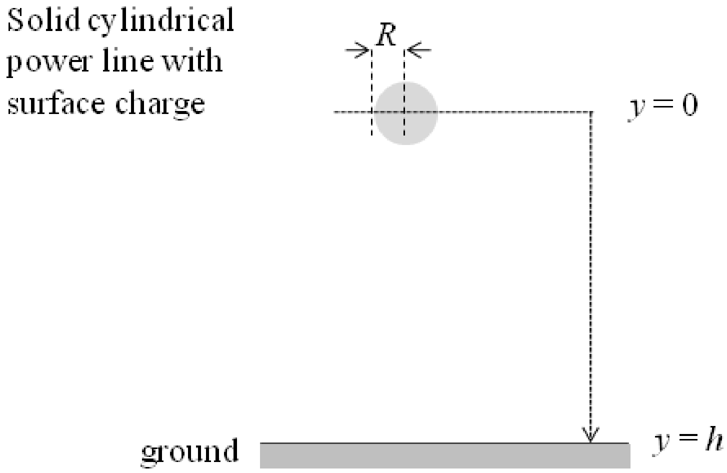
(d)
As the speed of the stream changes, the cross-sectional area changes to keep the flow rate *Av* constant, as given by the continuity equation: as the speed decreases, the area increases. The speed of the liquid drops as it rises, since the *x*-component of the velocity is constant but the *y*-component of the velocity reaches zero at its highest point. Thus the speed is *minimum* at the highest point, which is where the cross-sectional area is *maximum.*

(1 mark for continuity equation; 1 mark for correct solution)

(e)
If the liquid was viscous, then it would lose energy to internal friction, so the maximum height would be less than in the non-viscous case.

(1 mark)

ADV_Q08



A straight, cylindrical, infinitely long power transmission line is at potential V_0 with respect to the ground. The cable has a radius R and a surface charge density σ.

(a) Express the electric field (direction and magnitude) at any point outside the cable in terms of the surface charge density σ and the radius R.

(b) Using $V(h) = 0$, where h is the distance from the axis of the cable to the ground, show that σ can be expressed by

$$\sigma = \frac{\varepsilon_0 V_0}{R\ln\left(h/R\right)}.$$

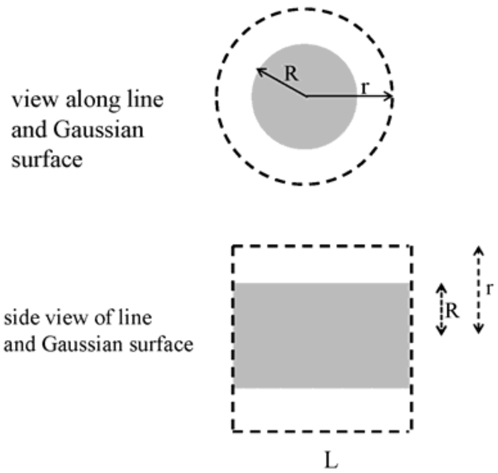
(c) Calculate the electric field just outside the line if $V_0 = 500\,\text{kV}$, the line is 30 m above the ground and is 0.03 m in radius.

(d) The dielectric strength of air is $3.0\times10^6\ \text{V.m}^{-1}$. Explain why it would not be a good idea to increase the operating voltage of the line to $750\,\text{kV}$.

(10 marks)

Solution

(a)



The circular/cylindrical symmetry means that the electric field $\vec{E}$ must be radial everywhere and only depend on r. This means that $\vec{E} = E(r)\,\hat{r}$.

Gauss's law using a cylinder of radius r and of length L centred on the cylinder is given by:

$$\oint \vec{E} \bullet d\vec{r} = \frac{Q_{enc}}{\varepsilon_0}$$

and can be written

$$E(r)(2\pi r L) = (2\pi R L)\frac{\sigma}{\varepsilon_0}.$$

This implies that

$$E(r) = \frac{\sigma}{\varepsilon_0}\frac{R}{r}$$

or

$$\vec{E}(r) = \frac{\sigma}{\varepsilon_0}\frac{R}{r}\hat{r}.$$

(1 mark for symmetry; 1 mark for correct Gaussian surface; 2 marks for answer)

(b)

$$V_0 - V(h) \;=\; \int_R^h \vec{E} \bullet d\vec{r} \;=\; \frac{\sigma}{\varepsilon_0} R \int_R^h \frac{1}{r}\,dr$$

$$\Rightarrow V_0 \;=\; \frac{\sigma}{\varepsilon_0} R\left[\ln(r)\right]_R^h \;=\; \frac{\sigma}{\varepsilon_0} R\,\ln\left(\frac{h}{R}\right)$$

$$\Rightarrow \sigma = \frac{\varepsilon_0 V_0}{R\,\ln\left(\frac{h}{R}\right)}.$$

(2 marks for approach)

(c)

$V_0 = 500\,\text{kV} = 5.00\times10^5\ \text{V}$

$h = 30\,\text{m}$

$R = 0.03\,\text{m}$

$$\sigma = \frac{\varepsilon_0 V_0}{R\ln(h/R)} = \frac{(8.85\times10^{-12})(5.00\times10^5)}{(0.03)\ln(30/0.03)}$$

$$= 2.135\times10^{-5}\ \text{C.m}^{-1}$$

$$E(R) \;=\; \frac{\sigma}{\varepsilon_0}\frac{R}{R} \;=\; \frac{2.135\times10^{-5}}{8.85\times10^{-12}} = \;2.41\times10^6\ \text{V.m}^{-1}$$

or more directly from

$$E(R) = \frac{\sigma}{\varepsilon_0} = \frac{V_0}{R\ln(h/R)} = \frac{5.00\times10^5}{(0.03)\ln(30/0.03)}$$

(1 mark for method; 1 mark for answer)

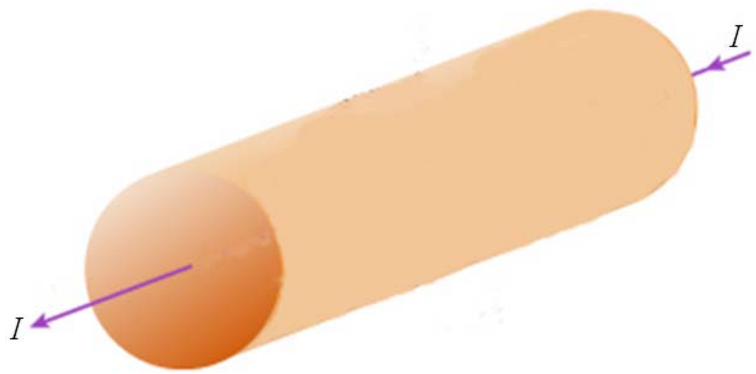
(d)
With
$V_0 = 750\,\mathrm{kV}$

$E(R) \approx 3.6\times10^6\ \mathrm{V.m^{-1}}$

This is greater than the dielectric strength, dielectric breakdown would occur and charge would be lost by corona discharge. Power would be lost from the line.

(1 mark for dielectric breakdown; 1 mark for power loss)

ADV_Q09



Consider a very long *solid* metal cylinder with radius *R*. A current *I* flows along the cylinder to the left with uniform current density.

(a) Using Ampere's Law, derive expressions for the magnitude of the magnetic field $B(r)$ in the range of $r > R$. State all the steps in the argument.

(b) Show that the magnitude of the magnetic field inside the cylinder ($r < R$) is

$$B(r) = \frac{\mu_0 I r}{2\pi R^2}.$$

A long hollow cylindrical shell of radius *2R* is now placed around this same cylinder, such that they share the same axis. A current *I* flows along this outer shell to the *right*, opposite in direction to the current in the inner cylinder.

(c) Calculate the magnetic field of this new configuration at a location outside of this shell ($r > 2R$).

(d) Has the field in the region $R < r < 2R$ changed from that given in part (a)? Justify your answer.

(e) Sketch the magnetic field lines (including their direction) in each region

(i) $r < R$

(ii) $R < r < 2R$

(iii) $r > 2R$.

(10 marks)

Solution

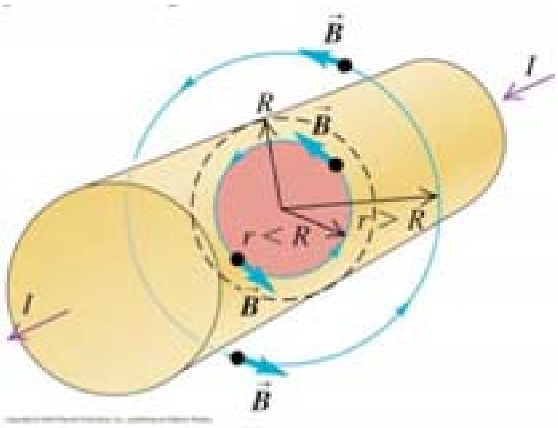
(a)

From symmetry arguments, the magnetic field is in circles.

(1 mark)

Apply Amperes Law using a circular path of radius r as illustrated

(1 mark)



We use the fact that the B field has constant magnitude along the path and is parallel to the path

(1 mark)

Thus Amperes Law gives

$$B(2\pi r) = \mu_0 I$$

and thus

$$B(r > R) = \frac{\mu_0 I}{2\pi r}$$

(1 mark)

(b)

Similar argument but with $I(r) = \frac{r^2}{R^2} I$ inside the cylinder.

(1 mark)

yields

$$B(r < R) = \frac{\mu_0 I r}{2\pi R^2}$$

(1 mark)

(c)

Again, Ampere's law is applied. For circular paths of radius *r>2R*, the two opposing currents cancel, given a net current of zero. The magnetic field in the region *r>2R* is zero.

(1 mark)

(d)

The magnetic field in the region $R < r < 2R$ is the same as given in part (a). The answer has not changed because the current in the outer shell does not contribute for circular paths of radius $r < 2R$, because it flows outside of the paths.

(1 mark)

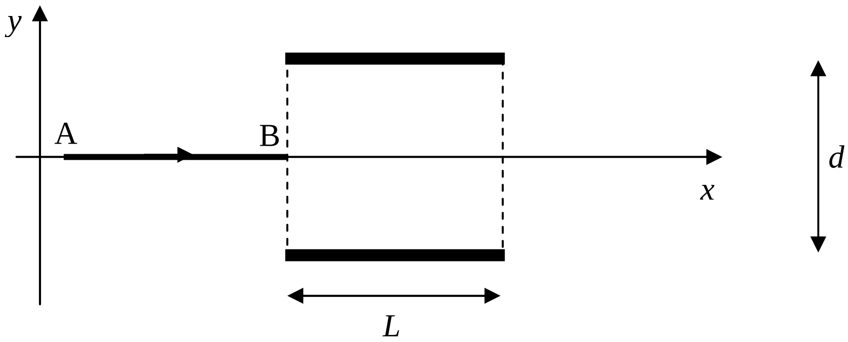
(e)

Inside outer shell: *B* circular – anticlockwise sense
Outside shell: *B* zero (no lines)

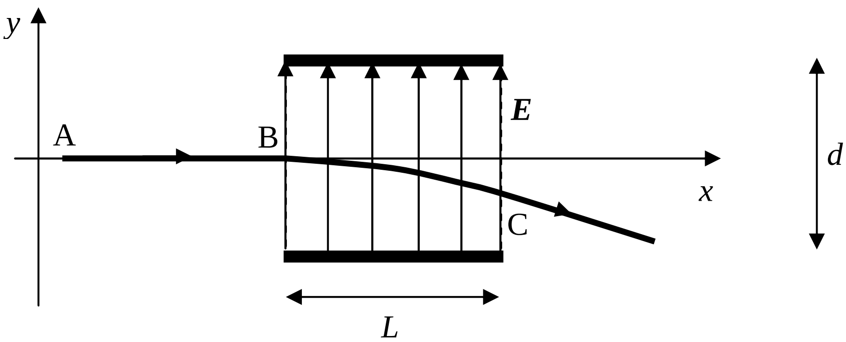
(2 marks)

ADV_Q10=TEC_Q10

(a) An electron at rest is accelerated from a point A at a potential V_A to a point B at potential V_B. Obtain an expression for the electron's velocity v_B when it reaches B expressed in terms of V_A, V_B, the electron charge e and the electron mass m.



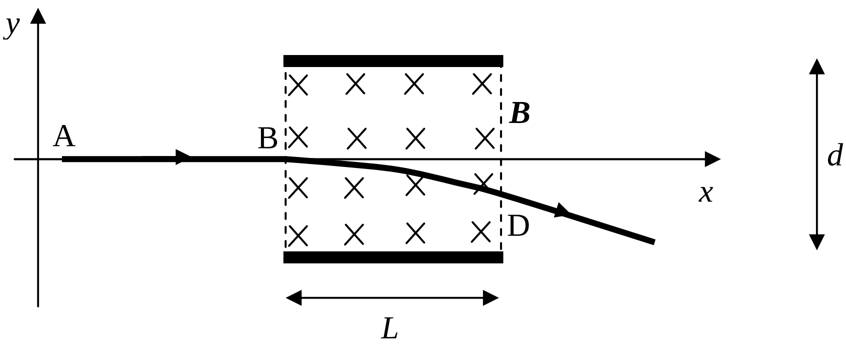
(b) The electron now traverses a uniform electric field $\boldsymbol{E}$ in the space between two charged metal plates. The electric field is zero outside the area of the plates. The plates have a length L and are separated by a distance d with a potential difference V between them.



The electron emerges from the area of the electric field at point C with a velocity v_C. Show that the deflection angle α (the angle between the vectors $\mathbf{v_B}$ and $\mathbf{v_C}$) is

$$\alpha = \arctan\left(\frac{-eVL}{m\,d\,v_B^2}\right)$$

(c) The electric field is replaced by a magnetic field so that the electron now traverses a uniform *magnetic* field $\boldsymbol{B}$ in exactly the same space between the plates. The magnetic field points into the page in the diagram below. The magnetic field is assumed to be zero outside the area of the plates.



The electron emerges from the area of the magnetic field at point D with a velocity v_D. Show that the deflection angle β (the angle between the vectors $\boldsymbol{v_B}$ and $\boldsymbol{v_D}$) is

$$\beta = \arcsin\left(\frac{-L\,e\,B}{m v_B}\right)$$

(d) Assuming the electrons at B have a small range of velocities about v_B, which of the two ways of deflecting the beam will produce the narrower beam of electrons emerging from the deflection region? Justify your answer.

(10 marks)

Solution

(a)

The work done on the electron by the field is given by

$$\begin{aligned} W_{A\to B} &= -\left(U_B - U_A\right) \\ &= qV_A - qV_B \\ &= -e\left(V_A - V_B\right) \end{aligned}$$

This equals the change in KE, therefore

$$\frac{1}{2} m v_B{}^2 - 0 = -e\left(V_A - V_B\right)$$

$$v_B = \sqrt{\frac{2e\left(V_B - V_A\right)}{m}}$$

(2 marks)

(b)

Electric field: $$E_y = \frac{V}{d}$$

Therefore, in the direction of the electric field:

- The force on the electron is $$F_y = \frac{-eV}{d}.$$

(1 mark)

- The acceleration of the electron is $$a_y = \frac{-eV}{m\,d}.$$

- The velocity of the electron is

$$v_y = \left(\frac{-eV}{m\,d}\right)t$$

where, t is the time the electron is in the field, which is set by the horizontal velocity

$$t = \frac{L}{v_B}.$$

So

$$v_y = \frac{-eV\,L}{m\,d\,v_B}$$

$$v_x = v_B$$

(1 mark)

The angle α between these vectors

$$\tan\alpha = \frac{v_y}{v_x} = \frac{-eV\,L}{m\,d\,v_B^2}$$

so

$$\alpha = \arctan\left(\frac{-eV\,L}{m\,d\,v_B^2}\right)$$

(1 mark)

(c)

The magnetic force on the electron is

$$F_y = -e\mathbf{v}\times\mathbf{B}$$

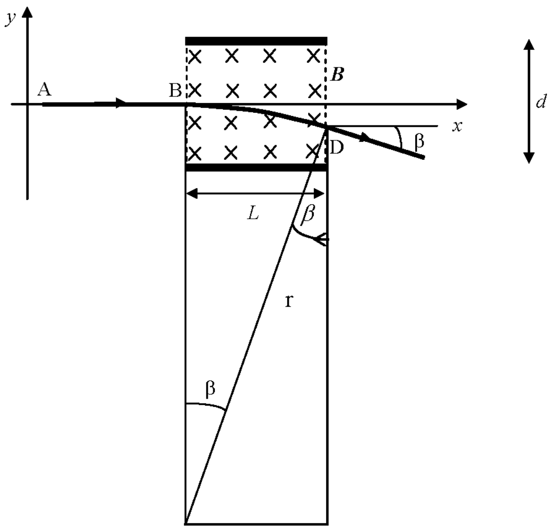
(1 mark)

The electron follows a circular path in which the centripetal force is the magnetic force – i.e.

$$\frac{m\,v_B^2}{r} = e\,v_B\,B$$

$$r = \frac{m\,v_B}{e\,B}$$

(1 mark)



L

Looking at the geometry

$$\sin\beta = \frac{-L}{r} = \frac{-L\,e\,B}{m\,v_B} \qquad (\beta \text{ clockwise as above})$$

So

$$\beta = \arcsin\left(\frac{-L\,e\,B}{m\,v_B}\right)$$

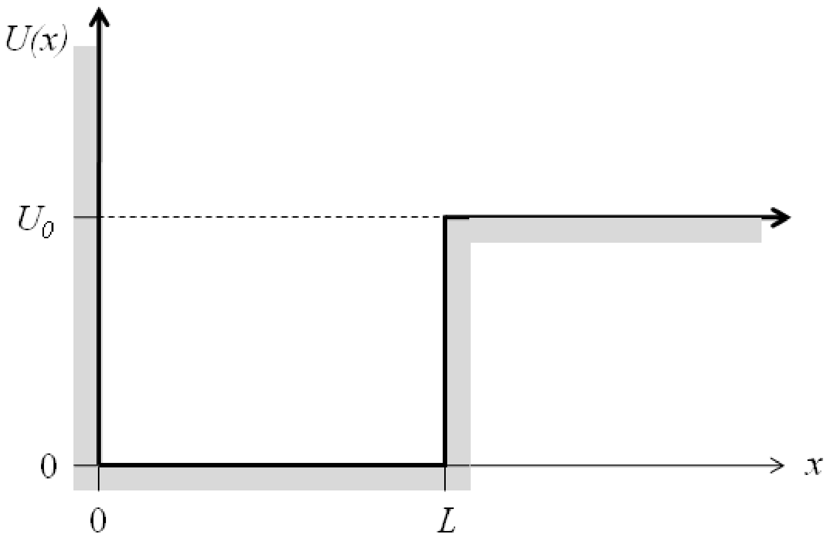
(1 mark)

(d)

The magnetic deflection has a v_B^{-1} dependence on electron velocity. The electric deflection has a v_B^{-2} dependence on electron velocity. Thus the electric deflection process is more sensitive to the electron velocity entering the deflection region and will therefore result in more angular spread of the beam.

(2 marks)

ADV_Q11



Consider the 1-D time-independent Schrödinger equation

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+U(x)\psi(x)=E\psi(x)$$

for an electron in a "half infinite square well", i.e.

$$U(x)=\begin{cases}\infty & \text{for } x\le 0\\ 0 & \text{for } 0<x<L\\ U_0 & \text{for } x\ge L\end{cases}$$

The electron is assumed to have an energy $E < U_0$.

(a) If the electron is considered to be a classical particle, where could it be found for this potential well? Where could it not be found? Briefly explain your answers.

(b) The wave function for the region $0 < x < L$ is

$$\psi_{\mathrm{I}}(x) = Ae^{ikx} + Be^{-ikx}.$$

Using the Schrödinger equation, determine the value of the wave number k.

(c) The wave function in the region $x \geq L$ is

$$\psi_{\mathrm{II}}(x) = Ce^{-\alpha x}.$$

Using the Schrödinger equation, determine the value of α.

(d) The boundary condition on the wave function at $x = 0$ is

$$\psi_I(0)=0,$$

and the boundary conditions at $x = L$ are

$$\psi_I(L)=\psi_{II}(L) \quad \text{and} \quad \psi'_I(L)=\psi'_{II}(L),$$

where a prime denotes the derivative. Applying these boundary conditions, show that the energy E of the electron is defined implicitly by the equation

$$\tan\theta = -\frac{\theta}{\sqrt{\theta_0^2 - \theta^2}},$$

where $\theta = \frac{\sqrt{2mE}}{\hbar}L$ and $\theta_0 = \frac{\sqrt{2mU_0}}{\hbar}L$.

(e) Sketch the curves

$$y = \tan\theta \quad \text{and} \quad y = -\frac{\theta}{\sqrt{\theta_0^2 - \theta^2}}$$

on the same plot. Hence, or otherwise, present an argument that this potential has no energy states at all if

$$\theta_0 < \frac{\pi}{2},$$

i.e., if the potential step satisfies

$$U_0 < \frac{\pi^2\hbar^2}{8mL^2}.$$

(10 marks)

Solution

(a)

For a classical particle the kinetic energy K is given by:

$$K = E - U = \frac{1}{2}mv^2.$$

The requirement that the kinetic energy be positive means that $E > U$. Hence a classical particle could be found in the region $0 < x < L$. It would not be found in the regions $x < 0$ or $x > L$.

(2 marks)

(b)

The wave function in $0 < x < L$ is

$$\psi_{\mathrm{I}}(x) = Ae^{ikx} + Be^{-ikx}.$$

Differentiating twice we have

$$\frac{d^2\psi_I}{dx^2} = -k^2\,\psi_I.$$

The Schrodinger equation is

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}E\psi$$

in the region $0 < x < L$ where $U = 0$.

Comparing these we have

$$k^2 = \frac{2mE}{\hbar^2}$$

$$\Rightarrow k = \frac{\sqrt{2mE}}{\hbar}.$$

(2 marks)

(c)

Similarly we have for $x \geq L$:

$$\frac{d^2\psi_{II}}{dx^2} = \alpha^2\, \psi_{II}\,.$$

The Schrodinger equation in this region is

$$\frac{d^2\psi_{II}}{dx^2} = \frac{2m}{\hbar^2}\left(U_0 - E\right)\psi_{II}\,.$$

And hence

$$\alpha^2 = \frac{2m\left(U_0 - E\right)}{\hbar^2}$$

$$\Rightarrow \alpha = \frac{\sqrt{2m\left(U_0 - E\right)}}{\hbar}.$$

(2 marks)

(d)

The wave function in $0 < x < L$ is

$$\psi_{\mathrm{I}}(x) = Ae^{ikx} + Be^{-ikx},$$

and the wave function for $x > L$ is

$$\psi_{II} = C\,e^{-\alpha x}.$$

Applying the boundary condition $\psi_I(0) = 0$ we have

$$A + B = 0 \quad \Rightarrow \quad B = -A$$

So we can write

$$\psi_I = A\,2i\sin kx \quad = D\sin kx,$$

where $D = 2i\,A.$

The derivatives are

$$\psi_I{}' = k\,D\cos kx$$

$$\psi_{II}{}' = -\alpha\, C\,e^{-\alpha x}.$$

Setting $\psi_I(L) = \psi_{II}(L)$ gives

$$D\sin kL = C\,e^{-\alpha L}$$

Setting $\psi_I{}'(L) = \psi_{II}{}'(L)$ gives

$$k\,D\cos kL = -\alpha\, C\,e^{-\alpha L}.$$

The ratio of these gives

$$\tan kL = -\frac{k}{\alpha}$$

or

$$\tan\left(\frac{\sqrt{2mE}}{\hbar}L\right) = -\sqrt{\frac{E}{U_0 - E}}\,.$$

Setting

$$\theta = \frac{\sqrt{2mE}}{\hbar}L$$

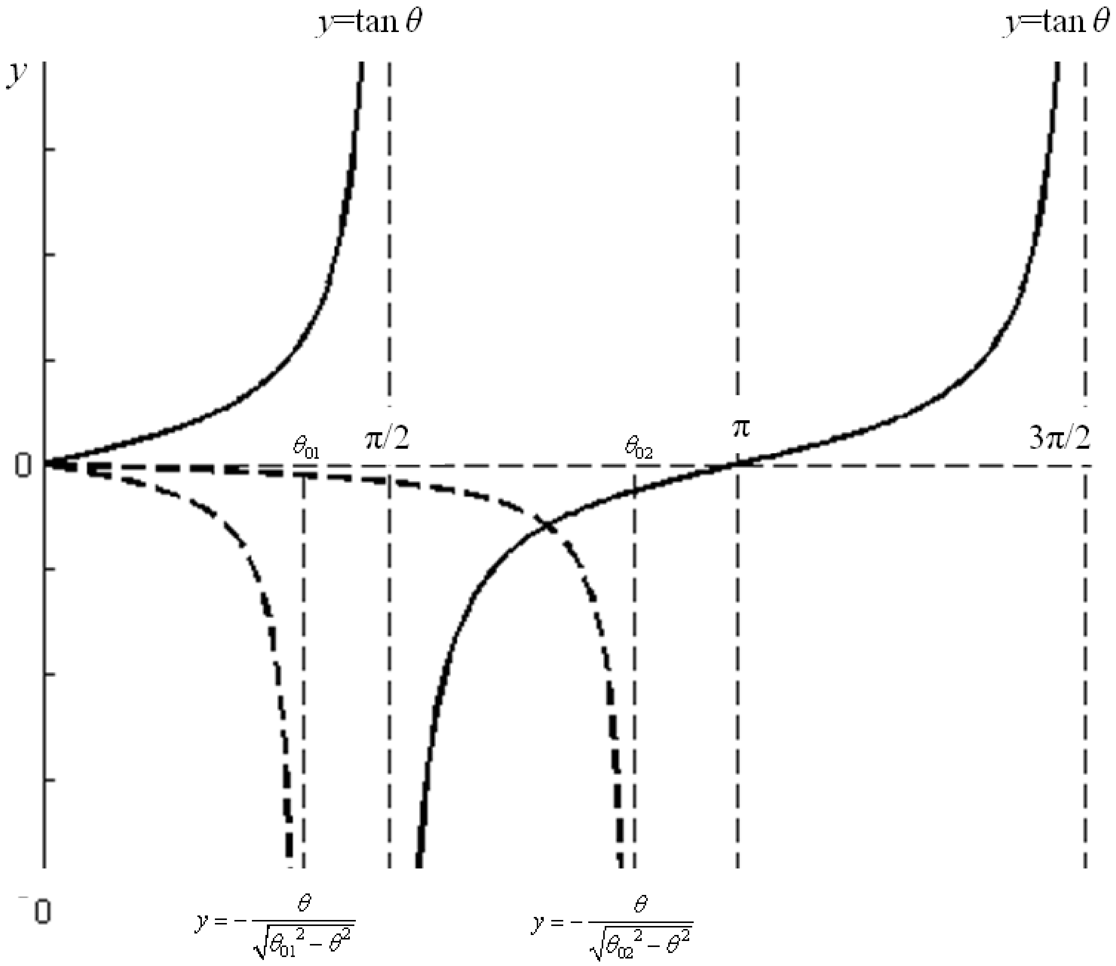
and

$$\theta_0 = \frac{\sqrt{2mU_0}}{\hbar}L$$

this can be written as

$$\tan\theta = -\frac{\theta}{\sqrt{\theta_0^{\,2} - \theta^2}}.$$

(2 marks)



The function

$$y = -\frac{\theta}{\sqrt{\theta_0^{\,2} - \theta^2}}$$

goes through the origin and has an asymptote at the value $\theta = \theta_0$. The diagram above shows two examples of this function, one with an asymptote at $\theta_0 = \theta_{01} < \pi/2$ and the other with an asymptote $\theta_0 = \theta_{02} > \pi/2$. From the diagram we can see that for $\theta_0 < \pi/2$ there is no solution to the equation

$$\tan\theta = -\frac{\theta}{\sqrt{\theta_0^{\,2} - \theta^2}}$$

and hence no states exist.

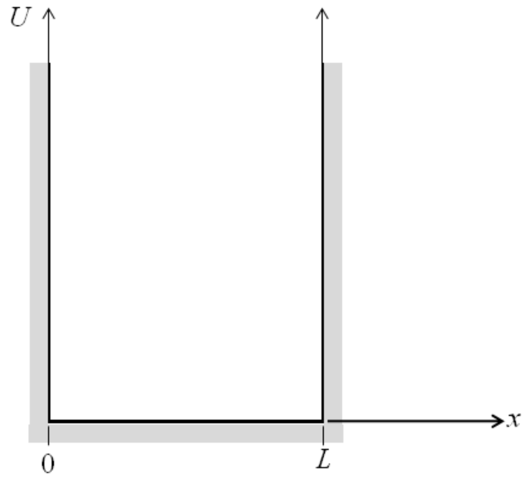
This corresponds to

$$\sqrt{\frac{2mU_0}{\hbar}}\,L < \frac{\pi}{2}$$

$$\Rightarrow U_0 = \frac{\pi^2\hbar^2}{8mL^2}.$$

(2 marks)

ADV_Q12



Consider an electron in an infinite square well which extends from $x = 0$ to $x = L$ (a "particle in a box"). The time-dependent wave function for the particle in the region of the well may be written

$$\Psi_n(x,t) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \exp\left(-\frac{iE_n t}{\hbar}\right),$$

where the possible energies E_n of the particle are

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

(a) These wave functions describe stationary (i.e., time-independent), states. Briefly explain what this means, given that the wave functions $\Psi_n(x, t)$ depend on time.

(b) Sketch the ground state ($n = 1$) wave function at time $t = 0$ for the region

$$-\frac{1}{2}L \leq x \leq \frac{3}{2}L.$$

(c) A reasonable estimate for the uncertainty in the position of the electron is

$$\Delta x = \frac{L}{2}.$$

(i) For the ground state ($n = 1$) write down an estimate for the uncertainty Δp_x in the momentum of the electron. Briefly justify your estimate.

(ii) Show that the estimates of Δx and Δp_x are consistent with the Heisenberg uncertainty principle.

(d) For the ground state ($n = 1$), show that the probability that the electron is located in the region $0 \leq x \leq \ell$, where $\ell \leq L$, is given by

$$p_\ell = \frac{1}{\pi}\left(\frac{\pi\ell}{L} - \frac{1}{2}\sin\frac{2\pi\ell}{L}\right).$$

(10 marks)

Solution
(a)
Although $\Psi(x,t)$ depends on time, we have

$$\left|\Psi(x,t)\right|^2 = \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)$$

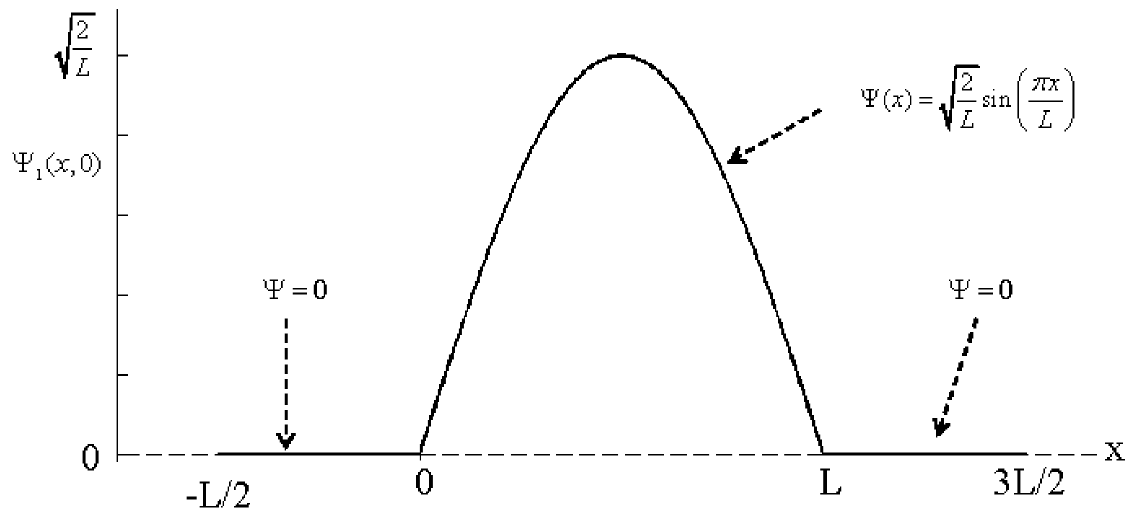
which is time-independent, and this is the observable quantity. Hence the state is stationary.

(2 marks)

(b)
At time $t = 0$ and for $n = 1$ we have

$$\Psi_1(x,0) = \sqrt{\frac{2}{L}}\ \sin\left(\frac{\pi x}{L}\right) \quad \text{(for } 0 < x < L\text{)},$$
$$\Psi_1(x,0) = 0 \quad \text{(for } x < 0\text{)},$$
$$\text{and } \Psi_1(x,0) = 0 \quad \text{(for } x > L\text{)}.$$

The wave function is as shown, for $-\dfrac{L}{2} \le x \le \dfrac{3L}{2}$.



(3 marks)

(c)(i)
The wave function in the well is built from two basic stationary states in the free particle case:

$$\psi = Ae^{ikx} + Be^{-ikx}.$$

These have momenta of $p = \hbar k$ and $p = -\hbar k$.

So an estimate of the uncertainty is:

$$\Delta p_x \approx \hbar k = \frac{2\hbar\pi}{L}$$

(c)(ii)
In this case we have

$$\Delta p_x\,\Delta x = \frac{2\hbar\pi}{L}\frac{1}{2}L = \pi\hbar > \hbar$$

(2 marks)

(d)
We have

$$p_\ell = \int_0^\ell \left|\Psi(x,t)\right|^2 dx$$
$$= \frac{2}{L}\int_0^\ell \sin^2\left(\frac{\pi x}{L}\right)dx$$

If we write

$$\theta = \frac{\pi\, x}{L} \quad \text{and} \quad d\theta = \frac{\pi}{L}dx$$

and use

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right),$$

Then

$$p_\ell = \frac{2}{L}\,\frac{L}{\pi}\int_0^{\pi\ell/L} \sin^2\theta\, d\theta$$
$$= \frac{2}{\pi}\,\frac{1}{2}\int_0^{\pi\ell/L}\left(1 - \cos 2\theta\right)d\theta$$
$$= \frac{1}{\pi}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi\ell/L}$$
$$= \frac{1}{\pi}\left(\frac{\pi\,\ell}{L} - \frac{1}{2}\sin\left(\frac{2\pi\,\ell}{L}\right)\right)$$

as required.

(3 marks)