The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Exercises for Week 4 (beginning 26 March)** 2012

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Geometric definition of dot product: If $\mathbf{v}$ and $\mathbf{w}$ are vectors and θ is the angle between them, then

$$\mathbf{v}\cdot\mathbf{w} \;=\; |\mathbf{v}||\mathbf{w}|\cos\theta\ ,$$

so that, in the case both vectors are nonzero,

$$\cos\theta \;=\; \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\ .$$

(ii) Algebraic definition of dot product: If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and $\mathbf{w} = d\,\mathbf{i} + e\,\mathbf{j} + f\,\mathbf{k}$ then

$$\mathbf{v}\cdot\mathbf{w} \;=\; ad + be + cf\ .$$

(iii) The angle between two vectors is zero or acute if their dot product is positive. The angle is obtuse or 180° if the dot product is negative. Two vectors are mutually perpendicular (orthogonal) if the dot product is zero.

(iv) Cauchy-Schwarz Inequality: $|\mathbf{v}\cdot\mathbf{w}| \;\leq\; |\mathbf{v}||\mathbf{w}|$.

(v) Commutativity of dot product: $\mathbf{v}\cdot\mathbf{w} \;=\; \mathbf{w}\cdot\mathbf{v}$.

(vi) Distributivity of dot over plus: $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} \;=\; \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$.

(vii) If $\mathbf{v}$ is any vector then $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$, so $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$.

(viii) If $\mathbf{v}$ and $\mathbf{w}$ are vectors and λ is a scalar then $(\lambda\mathbf{v})\cdot\mathbf{w} = \lambda(\mathbf{v}\cdot\mathbf{w}) = \mathbf{v}\cdot(\lambda\mathbf{w})$.

(ix) The *vector projection* of $\mathbf{v}$ in the direction of $\mathbf{w}$ is $\dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$, which is the best approximation of $\mathbf{v}$ using a scalar multiple of $\mathbf{w}$.

(x) The *scalar component* of $\mathbf{v}$ in the direction of $\mathbf{w}$ is $\dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|}$, which is plus or minus the magnitude of the vector projection (minus in the case that the angle is obtuse or 180°).

(xi) The *vector component of* $\mathbf{v}$ *orthogonal to* $\mathbf{w}$ is the difference between $\mathbf{v}$ and its vector projection, which is

$$\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\ .$$

Preparatory Exercises:

1. Use the Theorem of Pythagoras to verify the Cosine Rule.

2. Given that

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}\,, \qquad \mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\,, \qquad \mathbf{w} = 3\mathbf{i} - \mathbf{k}\,,$$

find

(i) $\mathbf{u}\cdot\mathbf{v}$ (ii) $\mathbf{u}\cdot\mathbf{w}$ (iii) $\mathbf{v}\cdot\mathbf{w}$ (iv) $\mathbf{u}\cdot\mathbf{u}$ (v) $\mathbf{v}\cdot\mathbf{v}$ (vi) $\mathbf{w}\cdot\mathbf{w}$

(vii) $|\mathbf{u}|$ (viii) $|\mathbf{v}|$ (ix) $|\mathbf{w}|$ (x) $\mathbf{u}\cdot\big(\mathbf{v}+\mathbf{w}\big)$ (xi) $\mathbf{u}\cdot\big(\mathbf{v}-\mathbf{w}\big)$

3. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be as in the previous exercise. Let α be the angle between $\mathbf{u}$ and $\mathbf{v}$, β be the angle between $\mathbf{u}$ and $\mathbf{w}$, and γ the angle between $\mathbf{v}$ and $\mathbf{w}$. Find

(i) $\cos\alpha$ (ii) $\cos\beta$ (iii) $\cos\gamma$

4. Given that

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\,, \qquad \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}\,, \qquad \mathbf{c} = 3\mathbf{i} + 6\mathbf{j}\,,$$

determine whether the following are true or false:

(i) The angle between $\mathbf{a}$ and $\mathbf{b}$ is acute. (ii) The angle between $\mathbf{b}$ and $\mathbf{c}$ is acute.

(iii) The vectors $\mathbf{a}$ and $\mathbf{c}$ are mutually perpendicular.

(iv) The angle between the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}-\mathbf{c}$ is obtuse.

5. Given that $P = (8, 4, -1)$, $Q = (6, 3, -4)$ and $R = (7, 5, -5)$, find

$$\overrightarrow{QP}\,, \qquad |\overrightarrow{QP}|\,, \qquad \overrightarrow{QR}\,, \qquad |\overrightarrow{QR}|\,, \qquad \overrightarrow{QP}\cdot\overrightarrow{QR}\,,$$

and the cosine of $\angle PQR$.

6. Given that $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$, find

(i) $\mathbf{u}\cdot\mathbf{v}$ (ii) $\widehat{\mathbf{u}}$ (iii) $\widehat{\mathbf{v}}$ (iv) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}$ (v) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$ (vi) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

(vii) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\,\mathbf{u}$ (viii) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}$ (ix) $\mathbf{v} - \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\,\mathbf{u}$ (x) $\mathbf{u} - \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}$

(xi) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

(xii) the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$

(xiii) the scalar component of $\mathbf{v}$ in the direction of $\mathbf{u}$

(xiv) the vector projection of $\mathbf{u}$ in the direction of $\mathbf{v}$

(xv) the vector projection of $\mathbf{v}$ in the direction of $\mathbf{u}$

(xvi) the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$

(xvii) the vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$

Tutorial Exercises:

7. Consider the following points in space:

$$P(1,1,1)\,,\quad Q(-1,-1,0)\,,\quad R(0,1,2)\,,\quad S(2,3,3)$$

(i) Recall from an earlier exercise that $PQRS$ is a rhombus that is not a square. Determine whether the angles

$$\angle PQR \quad \text{and} \quad \angle QRS$$

are acute or obtuse (which will be further confirmation that this rhombus is not a square).

(ii) Use a dot product to verify that the diagonals PR and QS are mutually perpendicular.

8. Use the dot product to verify that if $\mathbf{v}$ and $\mathbf{w}$ are any vectors and $\mathbf{w}$ is nonzero, then

$$\mathbf{w} \qquad \text{and} \qquad \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\,\mathbf{w}$$

are mutually perpendicular.

9. Given that $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, find

(i) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

(ii) the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$

(iii) the scalar component of $\mathbf{v}$ in the direction of $\mathbf{u}$

(iv) the vector projection of $\mathbf{u}$ in the direction of $\mathbf{v}$

(v) the vector projection of $\mathbf{v}$ in the direction of $\mathbf{u}$

(vi) the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$

(vii) the vector component of $\mathbf{v}$ orthogonal to $\mathbf{u}$

10. Verify that if $\mathbf{a}$ and $\mathbf{b}$ are vectors of the same length then

$$\mathbf{a}+\mathbf{b} \qquad \text{and} \qquad \mathbf{a}-\mathbf{b}$$

are mutually perpendicular.

11. (suitable for group discussion) Use vectors to find the following angles in a cube:

(i) between a major diagonal (between opposite vertices) and an edge,

(ii) between a major diagonal and a face diagonal,

(iii) between diagonals on adjacent faces,

(iv) between major diagonals.

12. (suitable for group discussion) Use dot products to show that

(i) if $\mathbf{v}$ is orthogonal to $\mathbf{x}$ and $\mathbf{y}$, then $\mathbf{v}$ is orthogonal to any linear combination $a\mathbf{x} + b\mathbf{y}$ where a and b are scalars;

(ii) conversely, if $\mathbf{v}$ is orthogonal to $\mathbf{x}$ and $a\mathbf{x}+b\mathbf{y}$ where a and b are scalars such that b is nonzero, then $\mathbf{v}$ is orthogonal to $\mathbf{y}$.

13.* Use vectors to show that any angle inscribed in a semicircle is a right angle.

14.* (uniqueness of the dot product) Suppose we have a dot operation on geometric vectors that is commutative and distributes over vector addition, such that

$$\mathbf{i}\cdot\mathbf{j} \;=\; \mathbf{i}\cdot\mathbf{k} \;=\; \mathbf{j}\cdot\mathbf{k} \;=\; 0 \quad \text{and} \quad \mathbf{i}\cdot\mathbf{i} \;=\; \mathbf{j}\cdot\mathbf{j} \;=\; \mathbf{k}\cdot\mathbf{k} \;=\; 1\,.$$

Suppose further that the dot operation is compatible with scalar multiplication in the sense of Useful Fact (viii). Deduce the usual algebraic rule for dot products.

15.* Imagine that you don't know the algebraic formula for the dot product. Starting with the geometric definition for the dot product, apply the Cosine Rule to 'discover' the algebraic formula.

Further Exercises:

16. Resolve the vector $\mathbf{u} = 5\mathbf{i} + \mathbf{j} + 6\mathbf{k}$ into a sum of two vectors, one of which is parallel and the other perpendicular to $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$.

17. Find the (vector) components of the force $15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}$ newtons in the direction of and orthogonal to

(i) $-\mathbf{i} + \mathbf{j}$ (ii) $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

18. Use the dot product to verify that if $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular vectors then

$$|\mathbf{a} + \mathbf{b}|^2 \;=\; |\mathbf{a}|^2 + |\mathbf{b}|^2\,.$$

Interpret this result in terms of a well-known fact about triangles.

19.* Verify that $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ are perpendicular. Find two vectors of unit length that are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. (This will become easy after next week, using cross products.)

20.* Verify that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.

21.* Prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus (that is, has all sides of equal length).

22.* Verify the following identity for all geometric vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and use it to deduce that the three altitudes of a triangle intersect in a common point:

$$(\mathbf{a}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{c})+(\mathbf{b}-\mathbf{c})\cdot(\mathbf{d}-\mathbf{a})+(\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b}) \;=\; \mathbf{0}$$

23.* Prove that the perpendicular bisectors of the sides of a triangle intersect in a common point (known as the *circumcentre*).

24. Given that it exists, verify that the circumcentre of a triangle is the same distance from each vertex (which explains its name).

25.** Suppose that A, B, C and D are distinct points in space such that no three are collinear. Verify that these points lie on a plane if and only if there are four nonzero scalars, α, β, γ and δ such that $\alpha+\beta+\gamma+\delta=0$ and

$$\alpha\,\overrightarrow{OA}+\beta\,\overrightarrow{OB}+\gamma\,\overrightarrow{OC}+\delta\,\overrightarrow{OD} \;=\; \mathbf{0}\,.$$

Deduce Ceva's Theorem, that says if D is a point in the plane of the triangle ABC, and the lines through AD, BD, CD cut BC, CA, AB in R, S, T respectively, then the product of the ratios in which R, S, T divide BC, CA, AB respectively is 1.

Short Answers to Selected Exercises:

1. Drop a perpendicular to create right angled triangles.

2. (i) 6 (ii) 5 (iii) 1 (iv) 6 (v) 9 (vi) 10 (vii) $\sqrt{6}$ (viii) 3 (ix) $\sqrt{10}$ (x) 11 (xi) 1

3. (i) $\dfrac{\sqrt{6}}{3}$ (ii) $\dfrac{\sqrt{15}}{6}$ (iii) $\dfrac{\sqrt{10}}{30}$

4. (i) false (ii) true (iii) true (iv) true

5. $2\mathbf{i}+\mathbf{j}+3\mathbf{k}\,,\ \sqrt{14}\,,\ \mathbf{i}+2\mathbf{j}-\mathbf{k}\,,\ \sqrt{6}\,,\ 1\,,\ \dfrac{1}{2\sqrt{21}}$

6. (i) -4 (ii) $\frac{1}{\sqrt5}(\mathbf{i}-2\mathbf{j})$ (iii) $\frac{1}{\sqrt5}(-2\mathbf{i}+\mathbf{j})$ (iv) $-\frac{4}{\sqrt5}$ (v) $-\frac{4}{\sqrt5}$ (vi) $-\frac45$ (vii) $-\frac45(\mathbf{i}-2\mathbf{j})$ (viii) $\frac45(2\mathbf{i}-\mathbf{j})$ (ix) $-\frac35(2\mathbf{i}+\mathbf{j})$ (x) $-\frac35(\mathbf{i}+2\mathbf{j})$ (xi) $-\frac45$ (xii) $-\frac{4}{\sqrt5}$ (xiii) $-\frac{4}{\sqrt5}$ (xiv) $\frac45(2\mathbf{i}-\mathbf{j})$ (xv) $-\frac45(\mathbf{i}-2\mathbf{j})$ (xvi) $-\frac35(\mathbf{i}+2\mathbf{j})$ (xvii) $-\frac35(2\mathbf{i}+\mathbf{j})$

9. (i) $\frac{2}{3\sqrt{33}}$ (ii) $\frac{2}{\sqrt{33}}$ (iii) $\frac23$ (iv) $\frac{2}{33}(-4\mathbf{i}+4\mathbf{j}-\mathbf{k})$ (v) $\frac29(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$ (vi) $\frac{1}{33}(41\mathbf{i}+58\mathbf{j}+68\mathbf{k})$ (vii) $-\frac19(38\mathbf{i}-32\mathbf{j}+13\mathbf{k})$.

11. (i) 54° (ii) 35° (iii) 60° (iv) 71°

16. $\mathbf{u} \;=\; \dfrac{3}{7}\big(3\mathbf{i}-6\mathbf{j}+2\mathbf{k}\big)+\dfrac{1}{7}\big(26\mathbf{i}+25\mathbf{j}+36\mathbf{k}\big)$

17. (i) $\dfrac{5}{2}\,(-\mathbf{i}+\mathbf{j})$ newtons, $\dfrac{1}{2}\big(35\mathbf{i}+35\mathbf{j}+12\mathbf{k}\big)$ newtons

(ii) $-\dfrac{12}{7}\big(2\mathbf{i}-3\mathbf{j}+\mathbf{k}\big)$ newtons, $\dfrac{1}{7}\big(129\mathbf{i}+104\mathbf{j}+54\mathbf{k}\big)$ newtons

19. $\pm\frac{1}{\sqrt{77}}(2\mathbf{i}-8\mathbf{j}-3\mathbf{k})$