THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Assignment 3

MATH1906: Mathematics Special Studies Program A — Semester 1, 2017

Web Page: http://sydney.edu.au/science/maths/u/UG/JM/MATH1906/
Lecturer: Eduardo G. Altmann

This assignment is due by **23:59 on Friday 9th June**. A scanned or typeset copy of your answers must be upload in the Learning Management System (Blackboard) at https://elearning.sydney.edu.au, where it will be passed through the text matching service Turnitin. It should include your name and SID.

The School of Mathematics and Statistics encourages some collaboration between students when working on problems, but students must write up and submit their own version of the solutions.

The two questions below should be answered using analytical calculations, drawings, and (possibly) numerical evaluations using standard calculators. Show and justify every step of your calculations. The use of computer simulations is not mandatory, but it is encouraged as a consistency *test* of the answers.

Consider the dynamical system

$$x_{t+1} = f(x_t) = a - x_t^2, \tag{1}$$

where $t \in \mathbb{Z}$ is the discrete time, $a \in [0, 2]$ is a control parameter, and $x_t \in [-2, 2]$ is the variable of interest.

1. We say that the point x belongs to a periodic orbit of period n if $x = f^{(n)}(x)$, where $f^{(n)}$ is the composition of f with itself n times $(f \circ f \circ \ldots \circ f)$. A periodic orbit of period n is composed by n such points, $x, f(x), f^{(2)}(x), \ldots, f^{(n-1)}(x)$. *5 Marks*

(a) Compute, as a function of a, the points belonging to orbits of period 1 of system (1).

(b) Compute, as a function of a, the points belonging to orbits of period 2 of system (1).

(c) Compute the linear stability of all periodic orbits of period 1 and 2, indicating the intervals in a for which they are stable and unstable. Denote by a_1 and a_2 the values of a for which the periodic orbits of period 1 and 2 loose stability.

(d) Draw the graph x_t vs. x_{t+1} for a parameter a of your choice in the interval $a_1 < a < a_2$. Choose two different initial conditions (x_0 and x'_0) in the interval $(0, 1)$ and draw (using the graphical method) in the graph the two trajectories ($\{x_0, x_1, x_2, \ldots\}$ and $\{x'_0, x'_1, x'_2, \ldots\}$). You may use your calculator to iterate x_0 and x'_0 using Eq. (1).

2. Cascades of period-doubling bifurcations, such as the one happening in system (1), are described by universal constants known as the Feigenbaum constants (in honour of Mitchell J. Feigenbaum, who obtained these results in the late 1970's). Feigenbaum's first constant is defined as *5 Marks*

$$\lim_{k\to\infty} \delta_k = \delta = 4.669201..., \tag{2}$$

with

$$\delta_k = \frac{a_k - a_{k-1}}{a_{k+1} - a_k}, \tag{3}$$

where a_k is the $k-th$ bifurcation of the cascade (in the case discussed in question 1, this corresponds to the bifurcation of the orbit with period $n = 2^{k-1}$). By approximating $\delta_k \approx \delta$ the value of the Feigenbaum constant δ given in (2) can be used to estimate other bifurcation parameters a_k.

(a) Draw the bifurcation diagram a vs. x indicating: (i) the solutions and stability of the orbits computed in question 1; (ii) the bifurcation parameters a_1 and a_2, indicating their values; and (iii) a sketch of the diagram for $a > a_2$, indicating in the plot a_3, and a_∞ (values are unknown).

(b) Use the values of a_1, a_2 (computed in question 1), and δ to estimate a_3.

(c) Use the values of a_1, a_2 (computed in question 1), and δ to estimate a_∞, i.e., the parameter a for which the bifurcation cascade accumulates (edge of chaos).
Hint 1: By defining $\Delta_k \equiv a_{k+1} - a_k$, write Δ_k as a function of Δ_0 and δ.
Hint 2: Use the following identity

$$\sum_{k=1}^{\infty} \frac{1}{y^k} = \frac{y}{y-1}, \text{ for any } y > 1.$$

Can you show it?
Hint 3: The value of a_∞ estimated here is $\approx 1\%$ smaller than the true value (relevant if you estimate a_∞ using computer simulations).