THE UNIVERSITY OF SYDNEY

MATH1902 LINEAR ALGEBRA (ADVANCED)

Semester 1 **Board Tutorial for Week 8** 2017

Important Ideas and Useful Facts:

(v) The *identity matrix* is a square matrix with *diagonal* entries equal to 1 and all entries off the diagonal equal to 0. The identity matrix is denoted by I or I_n if it is $n \times n$ and the size needs to be emphasised.

(vi) If A is an $m \times n$ matrix and B is an $n \times p$ matrix then the *matrix product* AB is defined and is an $m \times p$ matrix. The (i,k)-entry of AB is the "dot product" of the ith row of A with the kth column of B, which can be expressed using sigma notation:

$$\sum_{j=1}^{n} a_{ij}b_{jk} \; = \; a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}$$

where a_{ij}, b_{jk} denote typical (i,j) and (j,k)-entries of A and B respectively.

(vii) If A, B, C are matrices of appropriate sizes for which the expressions make sense, and λ and μ are scalars, then the following properties hold:

$$A + B = B + A\,, \qquad (A+B)+C = A+(B+C)\,, \qquad A + 0 = 0 + A = A\,,$$

$$-(-A) = A\,, \qquad A + (-A) = A - A = 0\,, \qquad \lambda(\mu A) = (\lambda\mu)A\,,$$

$$\lambda(A+B) = \lambda A + \lambda B\,, \qquad (\lambda+\mu)A = \lambda A + \mu A\,, \qquad IA = AI = A\,,$$

$$(AB)C = A(BC)\,, \qquad A(B+C) = AB + AC\,, \qquad (A+B)C = AC + BC\,,$$

$$\lambda(BC) = (\lambda B)C = B(\lambda C)\,, \qquad 0A = 0 = A0\,.$$

(viii) Warning: Matrix multiplication is not in general commutative. Most of the time

$$AB \neq BA\,.$$

(ix) Transpose: The *transpose* of a matrix A is the matrix A^T obtained by interchanging rows and columns, that is, the (i,j)-entry of A becomes the (j,i)-entry of A^T. The following hold for matrices A and B of dimensions for which the expressions make sense:

$$(A^T)^T \; = \; A\,, \qquad (A+B)^T \; = \; A^T + B^T\,, \qquad (AB)^T \; = \; B^T A^T$$

Tutorial Exercises:

7. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$.

(i) Find AB, BA, CD and DC.

(ii) Simplify A^2B^2 and $C(DCDCD)^2C$ without any further matrix calculations.

8. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $Y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $Z = \begin{bmatrix} 5 & -2 \end{bmatrix}$, $W = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

Find AX, AY, ZA, WA, ZX, ZAX, ZY, ZAY, WX, WAX, WY, WAY.

9. Find a 2×2 matrix M such that $M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but every entry of M is nonzero.

10. Explain briefly why the associative law for matrix multiplication implies that every square matrix commutes with its square.

11. This is an exercise in interpreting notation. Suppose A is an $m \times n$ matrix where $m \neq n$. Explain why there is no paradox in the following assertions:

$$A0 \;=\; 0A \;=\; 0 \qquad \text{and} \qquad 0 \;\neq\; 0 \;\neq\; 0$$

12. Evaluate and simplify the product

$$\begin{bmatrix} r\cos\alpha & -r\sin\alpha \\ r\sin\alpha & r\cos\alpha \end{bmatrix} \begin{bmatrix} s\cos\beta & -s\sin\beta \\ s\sin\beta & s\cos\beta \end{bmatrix}$$

where α, β are any real numbers and r, s are nonnegative real numbers. Relate your answer to multiplication of complex numbers. Is there any connection with addition of complex numbers?

13. Which of the following do you know to be true or expect to be true for all square matrices A, B, C of the same size:

(i)	$(AB)C = A(BC)$	(ii)	$AB = BA$
(iii)	$(AB)^2 = A^2B^2$	(iv)	$A(B+C) = AB + AC$
(v)	$(-A)(-B) = AB$	(vi)	$A(B-C) = AB - AC$
(vii)	$(A+B)^2 = A^2 + 2AB + B^2$	(viii)	$(A+B)(A-B) = A^2 - B^2$
(ix)	$(A+I)^2 = A^2 + 2A + I$	(x)	$(A+I)(A-I) = A^2 - I$
(xi)	$(A^TB^T)^T = BA$		

Find a counterexample to each of these that you believe not to hold in general.

14.* Use sigma notation to verify the associative law for matrix multiplication.

15.* Consider the matrix

$$M = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}.$$

(i) Verify that $M^2 = 2M - I$.

(ii) Deduce that $M^3 = 3M - 2I$ and guess a general formula for powers of M. (You can use the technique of proof by induction to verify that your guess is correct.)

(iii) Evaluate M^5, M^{10} and M^{100}.