THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION, ENGINEERING AND SCIENCE

MATH1902

LINEAR ALGEBRA (ADVANCED)

June 1999

TIME ALLOWED: Two Hours

LECTURERS: WG Gibson
RB Howlett

This Examination has 3 Printed Components.

1. AN EXTENDED ANSWER QUESTION PAPER (THIS BOOKLET, GREEN 80/16): 3 PAGES NUMBERED 1 TO 3; 7 QUESTIONS NUMBERED 1 TO 7.
2. A MULTIPLE CHOICE QUESTION PAPER (YELLOW 80/16A): 3 PAGES NUMBERED 1 TO 3; 15 QUESTIONS NUMBERED 1 TO 15.
3. A MULTIPLE CHOICE ANSWER SHEET (WHITE 80/16B): 1 PAGE.

Components 2 and 3 MUST NOT be removed from the examination room.

This Examination has 2 Sections: **Extended Answer** *and* **Multiple Choice**.

The **Extended Answer Section** *is worth 70% of the total marks for the paper: all questions may be attempted; questions are of equal value; working must be shown.*

The **Multiple Choice Section** *is worth 30% of the total marks for the paper: all questions may be attempted; questions are of equal value; answers must be coded onto the* **Multiple Choice Answer Sheet**.

Calculators will be supplied; no other electronic calculators are permitted.

1. (*i*) Let $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OQ}$ be the position vectors of the points P and Q. Show that the midpoint of the line PQ has position vector $(\mathbf{p} + \mathbf{q})/2$.

   (*ii*) Use vector methods to prove that the figure formed by joining the midpoints of a plane quadrilateral is a parallelogram.

2. (*i*) Find the fifth roots of unity; that is, find all the solutions of the equation $w^5 = 1$ where w is a complex number.

   (*ii*) Show that the equation $(1 - z)^5 = z^5$ has solutions $z = \frac{1}{2}$, $\frac{1}{2}(1 \pm i \tan \frac{\pi}{5})$, $\frac{1}{2}(1 \pm i \tan \frac{2\pi}{5})$.

3. Given the planes:

$$p_1 : \quad 3x + 2y - z = 1,$$
$$p_2 : \quad \mathbf{r} \cdot (\mathbf{i} - \mathbf{j} - \mathbf{k}) = 3,$$

   find

   (*i*) a vector $\mathbf{v}_1$ perpendicular to p_1;

   (*ii*) a vector $\mathbf{v}_2$ perpendicular to p_2;

   (*iii*) a point P lying on both p_1 and p_2;

   (*iv*) the equation of the line of intersection of p_1 and p_2.

4. (*i*) What does it mean to say that a matrix B is a *reduced row-echelon* matrix?

   (*ii*) Using the fact (which you are not asked to prove) that any matrix can be transformed into a reduced row-echelon matrix by a sequence of elementary row operations, explain why for every $r \times n$ matrix A there is an invertible $r \times r$ matrix M such that MA is a reduced row-echelon matrix.

   (*iii*) Let A be an $r \times n$ matrix, B an $n \times p$ matrix, and C a $p \times q$ matrix. Prove that $(AB)C = A(BC)$.

5. (*i*) Describe how a permutation of $\{1, 2, \ldots, n\}$ may be associated with a diagram, from which one may determine whether the permutation is even or odd. Use the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$$

to illustrate your answer.

(*ii*) Using the elementary row operations method, compute the determinant of the following matrix A:

$$A = \begin{bmatrix} 2 & 2 & 4 & -2 \\ 3 & 3 & 6 & 5 \\ 4 & 4 & 11 & 1 \\ 1 & 2 & 2 & -1 \end{bmatrix}.$$

(*iii*) Suppose that A, B, C and D are $n \times n$ matrices, all of which are invertible, and suppose that

$$B(C + AD) = B + C.$$

Find a formula that expresses A in terms of B, C and D. (That is, make A the subject of the formula.)

6. (*i*) (a) Let A be an $n \times n$ matrix. What does it mean to say that a matrix B is an *inverse* of the matrix A?

(b) Suppose that B and C are both inverses of the matrix A. Prove that $B = C$.

(*ii*) Let $M = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 4 \\ -2 & -5 & 0 \end{bmatrix}$.

(a) Calculate the inverse of M.

(b) Find a matrix X such that $XM = \begin{bmatrix} -1 & 3 & 4 \\ 1 & -1 & 5 \end{bmatrix}$.

7. Let A be a 7×7 matrix whose entries are real numbers.

(*i*) By considering the characteristic equation of A, show that A has at least one real eigenvalue.

(*ii*) Suppose that A satisfies the equation $A^2 + A = cI$, where I is the 7×7 identity matrix, and c is some fixed real number. Show that every eigenvalue λ of A satisfies $\lambda^2 + \lambda = c$.

(*iii*) Prove that the number c in Part (*ii*) must satisfy $c \geq -1/4$.

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