CHAPTER **FOUR**

Resisted Motion

4A Horizontal Resisted Motion

Exercise 4A

1. A certain drag-racing car of mass M kg is capable of a top speed of 288 km/h. After it reaches this top speed, two different retarding forces combine to bring it to rest. First there is a constant breaking force of magnitude $\frac{2}{3}M$ Newtons. Secondly there is a resistive force of magnitude $\dfrac{Mv^2}{180}$ Newtons, where v m/s is the speed of the car, acting against a parachute released from the rear-end of the vehicle. Let x metres be the distance of the car from the point at which the two retarding forces are activated.

(a) Show that $x = 90\ln\left(\dfrac{120+80^2}{120+v^2}\right)$.

(b) Hence calculate, to the nearest metre, the distance that the drag-racing car travels as it is brought from its top speed to rest.

2. A monorail of mass 10 000 kg is pulling out of a station S. Its motor provides a propelling force of magnitude 10 000 Newtons, and as it moves it experiences a resistive force of magnitude $100v^2$ Newtons, where v metres per second is its velocity.

(a) Show that the maximum speed the monorail can attain is 36 km/h.

(b) Show that $x = 50\ln\left(\dfrac{100}{100-v^2}\right)$, where x metres is the distance the monorail has travelled from S.

(c) What percentage (to the nearest per cent) of its maximum speed has the monorail reached when it has travelled 50 metres?

3. A particle of unit mass moves in a straight line against a resistance numerically equal to $v+v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q>0$.

(a) Show that v is related to the displacement x by the formula $x = \tan^{-1}\left(\dfrac{Q-v}{1+Qv}\right)$.

(b) Show that the time t which has elapsed when the particle is travelling with velocity v is given by $t = \frac{1}{2}\log_e\dfrac{Q^2(1+v^2)}{v^2(1+Q^2)}$.

(c) Show that $v^2 = \dfrac{Q^2}{(1+Q^2)e^{2t}-Q^2}$.

(d) What are the limiting values of v and x as $t\to\infty$?

4. When a jet aircraft touches down two different retarding forces combine to bring it to rest. If the aircraft has mass M kg and speed v m/s there is a constant frictional force of $\frac{1}{4}M$ Newtons and a force of $\frac{1}{108}Mv^2$ Newtons due to the reverse thrust of the engines. The reverse thrust does not take effect until 20 seconds after touchdown.

Let x be the distance in metres of the jet from its point of touchdown and let t be the time in seconds after touchdown.

(a) If the jet's speed at touchdown is 60 m/s, show that $v = 55$ and $x = 1150$ at the instant the reverse thrust of the engines takes effect.

(b) Show that when $t > 20$, $x = 1150 + 54\left(\ln(27 + 55^2) - \ln(27 + v^2)\right)$.

(c) How far from the point of touchdown, correct to the nearest metre, does the jet come to rest?

5. A particle of mass m kg experiences a resistance of kv^2 Newtons when moving along the x-axis, where k is a positive constant and v is the speed of the particle in metres per second. The maximum speed attainable by the particle is u metres per second under a variable propelling force of $\dfrac{P}{v}$ Newtons, where P is a positive constant.

(a) Show that $k = \dfrac{P}{u^3}$.

(b) Show that $\dfrac{dv}{dx} = \dfrac{P}{m}\left(\dfrac{1}{v^2} - \dfrac{v}{u^3}\right)$.

(c) Prove that the distance travelled as the speed changes from $\dfrac{u}{3}$ m/s to $\dfrac{2u}{3}$ m/s is $\dfrac{mu^3}{3P}\ln\frac{26}{19}$ metres.

(d) When the brakes are applied, the propelling force is no longer in operation. If the maximum force exerted by the brakes is B Newtons, prove that the minimum distance travelled in coming to rest from a speed of u m/s is $\dfrac{mu^3}{2P}\ln\left(1 + \dfrac{P}{Bu}\right)$ metres.

4B Vertical Resisted Motion

Exercise **4B**

1. An object of mass 5 kg is projected vertically upwards with velocity 40 m/s and experiences a resistive force in Newtons of magnitude $0{\cdot}2v^2$, where v is the velocity of the object at time t seconds. Assume that $g = 10\,\mathrm{m/s^2}$.

(a) Show that $\ddot{x} = \dfrac{-250 - v^2}{25}$.

(b) Find, correct to the nearest tenth of a second, the time that the object takes to reach its maximum height.

(c) Find the maximum height reached, correct to the nearest metre.

2. An object of mass $0{\cdot}5$ kg is projected upwards with velocity 40 m/s and experiences a resistive force in Newtons of magnitude $0{\cdot}2v$, where v is the velocity of the object at time t seconds. Assume that $g = 10\,\mathrm{m/s^2}$.

(a) Show that $\ddot{x} = \dfrac{-50 - 2v}{5}$.

(b) Show that the object takes $\frac{5}{2}\ln\frac{13}{5}$ seconds to reach its maximum height.

(c) Show that the maximum height reached, in metres, is $100 + \frac{125}{2}\ln\frac{5}{13}$.

3. An object of mass 100 kg is found to experience a resistive force, in Newtons, of one-tenth the square of its velocity in metres per second when it moves through the air. Suppose that the object falls from rest under gravity, and take $g = 9{\cdot}8\,\mathrm{m/s^2}$.

(a) Show that its terminal velocity is about 99 m/s.

(b) If the object reaches 80% of its terminal velocity before striking the ground, show that the point from which it was dropped was about 511 metres above the ground.

4. An object of mass 1 kg is projected vertically upwards from the ground at 20 m/s. The body is under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$, where v is the velocity at time t. (Take $g = 10\,\mathrm{m/s^2}$, and take upwards as the positive direction.)

(a) (i) Show that the greatest height reached by the object is $20\ln 2$ metres.

(ii) Show that the time taken to reach this greatest height is $\frac{\pi}{2}$ seconds.

(b) Having reached its greatest height the particle falls back to its starting point. The particle is still under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$.

(i) Write down the equation of motion of the object as it falls, this time taking downwards as the positive direction.

(ii) Find the speed of the object when it returns to its starting point.

5. A certain object, when projected vertically downwards with initial velocity V, experiences air resistance of magnitude mkv, where k is a positive constant. Take downwards as the positive direction.

(a) Show that $t = \frac{1}{k}\log_e\left(\frac{g-kV}{g-kv}\right)$.

(b) Hence show that $v = \frac{g}{k}(1 - e^{-kt}) + Ve^{-kt}$, and explain from this equation why the terminal velocity is $\frac{g}{k}$.

(c) Integrate again to show that $x = \frac{gt}{k} + \frac{kV-g}{k^2}\left(1 - e^{-kt}\right)$.

(d) Suppose that the terminal velocity of this object is 20 m/s, and that $g = 10\,\text{m/s}^2$. One of these objects is thrown vertically downwards from a lookout at the top of a cliff at precisely the terminal velocity, and, at the same instant, another of these objects is dropped. Show that the distance between the two falling objects after t seconds is, in metres, $40(1 - e^{-\frac{1}{2}t})$, and hence state the limiting distance between the two falling objects.

6. A particle of mass 10 kg is found to experience a resistive force, in Newtons, of one-tenth of the square of its velocity in metres per second, when it moves through the air. The particle is projected vertically upwards from a point O with a velocity of u metres per second, and the point A, vertically above O, is the highest point reached by the particle before it starts to fall to the ground again. Assuming that $g = 10\,\text{m/s}^2$,

(a) show that the particle takes $\sqrt{10}\tan^{-1}\frac{u}{10\sqrt{10}}$ seconds to reach A from O,

(b) show that the height OA is $50\log_e\frac{1000+u^2}{1000}$ metres,

(c) show that the particle's velocity w metres per second when it reaches O again is given by $w^2 = \dfrac{1000u^2}{1000+u^2}$.

7. (a) A particle of mass m falls from rest, from a point O, in a medium whose resistance is mkv, where k is a positive constant and v is the velocity at time t.

(i) Prove that the terminal velocity V is $V = \dfrac{g}{k}$.

(ii) Prove that the speed at time t is given by $\frac{g}{k}\left(1 - e^{-kt}\right)$.

(b) An identical particle is projected upwards from O with initial velocity U in the same medium. Suppose that this second particle is released simultaneously with the first.

(i) Prove that the second particle reaches its maximum height at $t = \frac{1}{k}\ln\frac{g+kU}{g}$.

(ii) Prove that the speed of the first particle when the second particle is at its maximum height is $\dfrac{UV}{U+V}$.

8. A particle P_1 of mass m kg is dropped from point A and falls towards point B, which is directly underneath A. At the instant when P_1 is dropped, a second particle P_2, also of mass m kg, is projected upwards from B towards A with an initial velocity equal to twice the terminal velocity of P_1. Each particle experiences a resistance of magnitude mkv as it moves, where $v\,\text{ms}^{-1}$ is the velocity and k is a constant.

(a) Show that the terminal velocity of P_1 is $\dfrac{g}{k}$, where g is acceleration due to gravity.

(b) For particle P_2, show that $t = \dfrac{1}{k}\ln\left(\dfrac{3g}{g+kv}\right)$, where $v\,\text{ms}^{-1}$ is the velocity after t seconds.

(c) Suppose that the particles collide at the instant when P_1 has reached 30% of its terminal velocity. Show that the velocity of P_2 when they collide is $\dfrac{11g}{10k}\,\text{ms}^{-1}$.

9. An object of mass 1 kg is dropped from a lookout on top of a high cliff.
Take the acceleration due to gravity to be $10\,\text{m/s}^2$.

(a) At first, air resistance causes a deceleration of magnitude $\dfrac{v}{10}$, where v m/s is the speed of the object t seconds after it is dropped.

(i) Taking downwards as positive, explain why its equation of motion is

$$\ddot{x} = 10 - \tfrac{v}{10},$$

where x is the distance that the object has fallen in the first t seconds.

(ii) Show that $\dfrac{dv}{dx} = \dfrac{100-v}{10v}$, and hence show that the speed V of the object when it is 40 metres below the lookout satisfies the equation

$$V + 100\log_e\left(1 - \tfrac{V}{100}\right) + 4 = 0.$$

(b) After the object has fallen 40 metres and reached this speed V, a very small parachute opens, and air resistance now causes a deceleration to its motion of magnitude $\dfrac{v^2}{10}$.

(i) Taking downwards as positive, write an expression for the new acceleration $\ddot{x}$ of the object, where x now is the distance that the object has fallen in the first t seconds after the parachute opens.

(ii) Show that $v^2 = 100 - (100 - V^2)e^{-\frac{1}{5}x}$, and hence find the terminal velocity of the object.

(iii) Show that t seconds after the parachute opens,

$$t = \tfrac{1}{2}\log_e\frac{(v+10)(V-10)}{(v-10)(V+10)}.$$

(iv) Given that the solution to the equation in part (ii) of part (a) is $V \doteqdot 25{\cdot}7\,\text{m/s}$, how long after the parachute opens does the object reach 105% of its terminal velocity?

10. A particle of mass 2 kg experiences a resistive force, in Newtons, of 10% of the square of its velocity v metres per second when it moves through the air. The particle is projected vertically upwards from a point A with velocity u metres per second. The highest point reached is B, directly above A. Assume that $g = 10\,\text{ms}^{-2}$, and take upwards as the positive direction.

(a) Show that the acceleration of the particle as it rises is given by

$$\ddot{x} = -\frac{v^2 + 200}{20}\,.$$

(b) Show that the distance x metres of the particle from A as it rises is given by

$$x = 10\log_e\left(\frac{200 + u^2}{200 + v^2}\right).$$

(c) Show that the time t seconds that the particle takes to reach a velocity of v metres per second is given by

$$t = \sqrt{2}\left(\tan^{-1}\frac{u}{10\sqrt{2}} - \tan^{-1}\frac{v}{10\sqrt{2}}\right).$$

(d) Now suppose that we take two of the 2 kg particles described above.
One of the particles is projected upwards from A with initial velocity $10\sqrt{2}\,\text{ms}^{-1}$, then, $\frac{3\sqrt{2}}{5}$ seconds later, the other particle is projected upwards from A with initial velocity $30\sqrt{2}\,\text{ms}^{-1}$. Will the second particle catch up to the first particle before the first particle reaches its maximum height? You must explain your reasoning and show your working.

11. (a) Consider the function

$$f(x) = x - \frac{g^2}{x} - 2g\ln\left(\frac{x}{g}\right), \quad \text{for } x \ge g\,.$$

(i) Evaluate $f(g)$.

(ii) Show that $f'(x) = \left(1 - \dfrac{g}{x}\right)^2$.

(iii) Explain why $f(x) > 0$ for $x > g$.

(b) A body is moving vertically through a resisting medium, with resistance proportional to its speed. The body is initially fired upwards from the origin with speed v_0. Let y metres be the height of the object above the origin at time t seconds, and let g be the constant acceleration due to gravity. Thus

$$\frac{d^2y}{dt^2} = -(g + kv) \quad \text{where} \quad k > 0\,.$$

(i) Find v as a function of t, and hence show that

$$k^2y = (g + kv_0)(1 - e^{-kt}) - gkt\,.$$

(ii) Find T, the time taken to reach the maximum height.

(iii) Show that when $t = 2T$,

$$k^2y = (g + kv_0) - \frac{g^2}{g + kv_0} - 2g\ln\left(\frac{g + kv_0}{g}\right).$$

(iv) Use this result and part (a) to show that the downwards journey takes longer.

Chapter Four

Exercise **4A** (Page 115)

1(b) 360 metres

2(c) 80%

3(d) 0 and $\tan^{-1} Q$

4(c) 1405 metres

Exercise **4B** (Page 117)

1(b) 1·9 seconds **(c)** 25 metres

4(b)(ii) $10\sqrt{2}\,\text{m/s}$

5(d) 40 metres

10(d) Yes.

11(b)(ii) $T = \frac{1}{k} \ln \left(\dfrac{g + kv_0}{g} \right)$