THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education, Engineering and Science

MATH1902

Linear Algebra (Advanced)

June 2005

Time Allowed: One and a half hours

Lecturers: R Howlett, N Joshi

This examination has three printed components:

(1) An Extended Answer Question Paper (this booklet, Green 8016A), 4 pages numbered 1 to 4, 5 questions numbered 1 to 5;

(2) A Multiple Choice Question Paper (Yellow 8016B), 3 pages numbered 1 to 3, 15 questions numbered 1 to 15;

(3) A Multiple Choice Answer Sheet (White 8016C), 1 page.

Components 2 and 3 must not be removed from the examination room.

The Extended Answer Section is worth 75% of the total examination; the five questions are of equal value; working must be shown.

The Multiple Choice Section is worth 25% of the total examination; the fifteen questions are of equal value.

Answers to the multiple choice questions must be coded onto the Multiple Choice Answer Sheet.

Calculators will be supplied; no other calculators are permitted.

1. (*i*) **(8 marks)**. Let $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.
   (a) Find $\mathbf{u} \cdot \mathbf{v}$.
   (b) Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$;
   (c) Find $\mathbf{u} \times \mathbf{v}$.
   (d) Find a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.
   (e) Find $(3\mathbf{u} + 5\mathbf{v}) \times (2\mathbf{u} + 4\mathbf{v})$.

   (*ii*) **(7 marks)**. Let $\mathbf{v}$ be a unit vector and c a real number, and let $\mathcal{P}$ be the plane whose equation is $\mathbf{r} \cdot \mathbf{v} = c$. Suppose also that $\mathbf{r}_0$ is a vector, and A the point whose position vector relative to the origin is $\mathbf{r}_0$.
   (a) Find, in parametric vector form, the equation of the line ℓ that is perpendicular to $\mathcal{P}$ and passes through A.
   (b) Let B be the point of intersection of the plane $\mathcal{P}$ and the line ℓ in Part (a). Find a formula for the position vector of B relative to the origin, in terms of $\mathbf{r}_0$, $\mathbf{v}$ and c.
   (c) Use Part (b) to find a formula for the distance from A to $\mathcal{P}$.
   (d) Find the distance from the point $(1, 1, 1)$ to the plane $3x + 2y - 6z = 21$.

2. (*i*) **(8 marks)**. Let A, B and C be the points $(2, 1, -1)$, $(1, 2, 2)$ and $(3, -1, -1)$ respectively, and let O be the origin. Let $\mathcal{H}$ be the parallelepiped that has OA, OB and OC as three of its sides.
   (a) Find the volume of $\mathcal{H}$.
   (b) Find the coordinates of the points A', B' and C' such that $COBA'$, $BOAC'$ and $AOCB'$ are faces of $\mathcal{H}$.
   (c) Find the area of $AOBC'$.
   (d) Let M, N and P be the midpoints of OA', OB' and OC' respectively. Find the coordinates of a point that lies on all three of the lines AM, BN and CP.

   (*ii*) **(7 marks)**. Let $ABCD$ be a plane quadrilateral. Suppose that the diagonals AC and BD intersect at the point P and the sides AB and CD (extended) meet at the point Q. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be the position vectors of A, B, C and D relative to some origin O.
   (a) Show that if P divides AC in the ratio $\alpha : (1 - \alpha)$ and BD in the ratio $\beta : (1 - \beta)$ then $(1 - \alpha)\mathbf{a} + \alpha\mathbf{c} = (1 - \beta)\mathbf{b} + \beta\mathbf{d}$.
   (b) Suppose that the scalars α and β in Part (a) are both positive and not equal to each other. Show that $\frac{1-\alpha}{\beta-\alpha}\mathbf{a} - \frac{1-\beta}{\beta-\alpha}\mathbf{b} = -\frac{\alpha}{\beta-\alpha}\mathbf{c} + \frac{\beta}{\beta-\alpha}\mathbf{d}$, and hence determine the ratios in which Q divides AB and CD (externally).
   (c) Show that if P divides AC in the ratio $7 : 1$ and BD in the ratio $5 : 3$ then Q divides AB in the ratio $-3 : 1$ and CD in the ratio $5 : -7$.

3. (*i*) **(4 marks).** Find the general solution of the system of linear equations

$$\begin{aligned} x + 2y - z + w &= 2 \\ 2x + 4y - z \phantom{{}+w} &= -7 \\ -3x - 6y + z + w &= 16. \end{aligned}$$

(*ii*) **(6 marks).** Describe the three types of elementary row operations, and say what effect each has on determinants of square matrices.

(*iii*) **(5 marks).** Use elementary row operations to calculate the determinant of

$$\begin{pmatrix} 1 & 6 & 1 & -2 \\ -1 & -3 & -4 & 2 \\ 2 & 12 & 3 & -3 \\ 0 & 1 & -1 & 7 \end{pmatrix}$$

4. (*i*) **(7 marks).** Let σ and ρ be the permutations of $\{1, 2, 3, 4\}$ given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \qquad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

(a) Draw diagrams representing σ and ρ, and hence determine $\operatorname{sgn}(\sigma)$ and $\operatorname{sgn}(\rho)$.

(b) Calculate the composite $\sigma \circ \rho$.

(*ii*) **(8 marks).** Suppose that A is a 4×3 matrix.

(a) Explain why there must exist an invertible 4×4 matrix T such that TA has a zero row.

(b) Explain why the matrix T in Part (a) cannot have a zero row.

(c) Let T be as in Part (a), and let B be any matrix with 3 rows. Explain why $T(AB)$ must have a zero row.

(d) Use the preceding parts to show that A cannot have a right inverse.

5. (*i*) **(8 marks)**. Let $\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}$ and let α_n and β_n be defined for $n = 1, 2, \ldots$ by the recurrence relation

$$\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix} = A\begin{bmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{bmatrix},$$

where A is the 2×2 matrix

$$A = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}.$$

(a) Find the eigenvalues λ_1 and λ_2 of A, and also find eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to λ_1 and λ_2.

(b) Find scalars s and t such that

$$\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2.$$

and hence find a general formula for $\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}$.

(c) By using Part (b), show that α_n/β_n approaches $\sqrt{2}$ as $n \to \infty$.

(*ii*) **(7 marks)**. Suppose that x is a given number and that $a_0, a_1, a_2, \ldots$ is a given infinite sequence of numbers. For each positive integer n, define $s_n = \det A_n$ and $t_n = \det B_n$, where A_n and B_n are the following $n \times n$ matrices:

$$A_n = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n-1}\\ a_1 & a_2 & \ldots & a_n\\ \vdots & \vdots & & \vdots\\ a_{n-1} & a_n & \ldots & a_{2n-2}\end{bmatrix}, \quad B_n = \begin{bmatrix} a_0 & a_1x & \ldots & a_{n-1}x^{n-1}\\ a_1x & a_2x^2 & \ldots & a_nx^n\\ \vdots & \vdots & & \vdots\\ a_{n-1}x^{n-1} & a_nx^n & \ldots & a_{2n-2}x^{2n-2}\end{bmatrix}.$$

(a) Show that $t_2 = x^2 s_2$.

(b) Show that $t_n = x^{n(n-1)}s_n$, for every positive integer n.

(c) Suppose that $a_i = x^{-i}$ for all integers $i \geq 0$. Show that in this case $s_n = 0$ whenever $n \geq 1$.