A

MATH1903 Lecture 9 Thurs 31/8/2017

Improper integrals

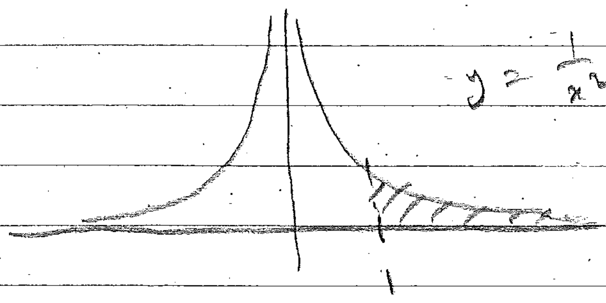
reference pp 2.53 – 2.62

p2.53 : areas over infinite intervals

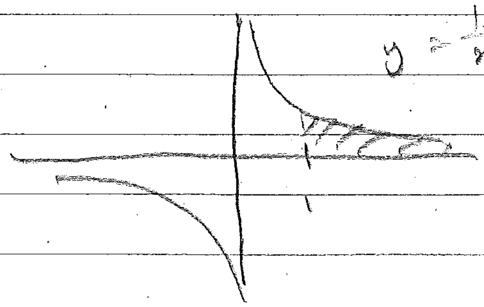
p2.54 : areas where integrand becomes unbounded
(sometimes "look like" proper integrals)

p2.55 : convergence when integral exists,
divergence otherwise

p2.56 : $\int_1^\infty \frac{dx}{x^2} = \dots = 1$.

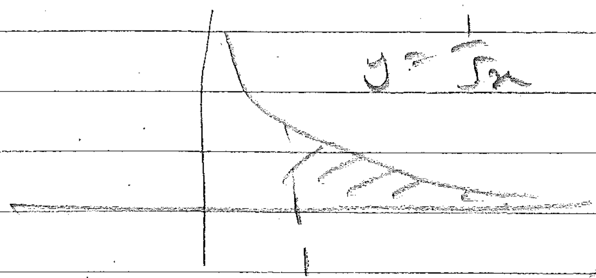


p2.57 : $\int_1^\infty \frac{dx}{x} = \dots = \infty$.



(B)

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \infty \quad \text{by } \underline{\text{comparison}}$$



$$\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} \qquad \text{since } \frac{1}{\sqrt{x}} \geq \frac{1}{x}$$

for $x \geq 1$

More generally, if $p > 0$, $p \neq 1$,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b \to \infty} \int_1^b x^{-p}\, dx$$

$$= \lim_{b \to \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b = \lim_{b \to \infty} \left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p} \right)$$

$$= \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \quad (\text{so } 1-p < 0) \\ \infty & \text{if } 0 < p < 1. \end{cases}$$

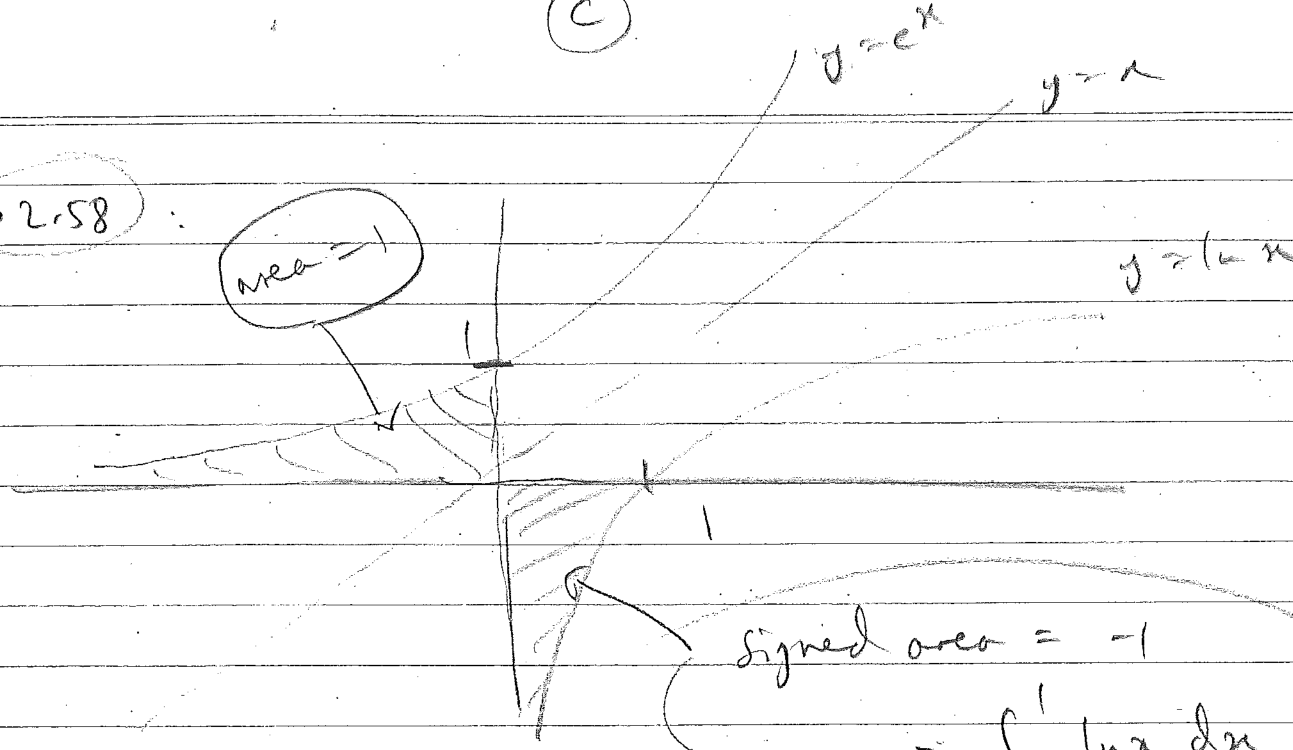
e.g. $\int_1^\infty \frac{dx}{x^2} = \frac{1}{2-1} = 1$ ✓ as before

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \int_1^\infty \frac{dx}{x^{1/2}} = \infty = \int_1^\infty \frac{dx}{x^{1/3}}$$

$$\int_1^\infty \frac{dx}{x^3} = \frac{1}{3-1} = \frac{1}{2}, \quad \int_1^\infty \frac{dx}{x^4} = \frac{1}{4-1} = \frac{1}{3}, \text{ etc.}$$

C

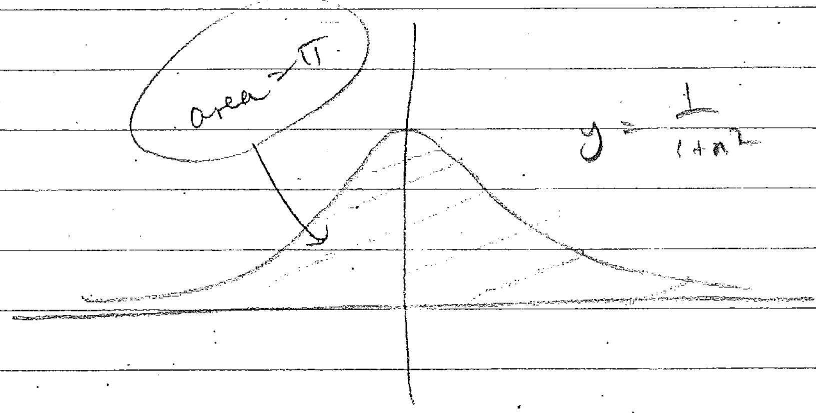
p. 258 :



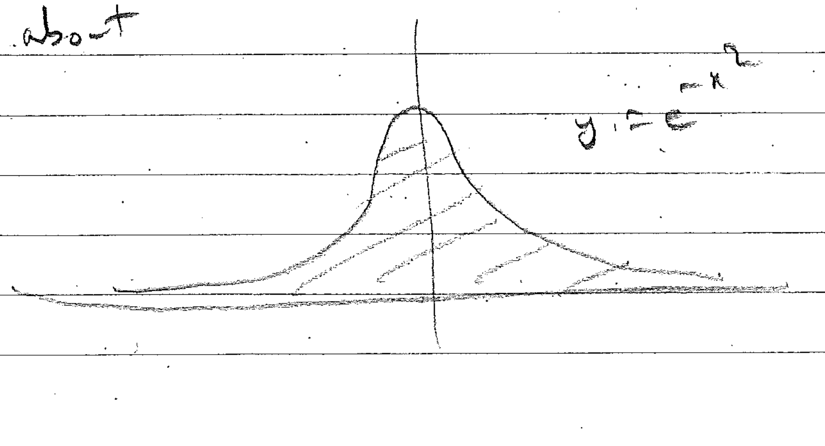
Signed area $= -1$
$= \int_0^1 \ln x \, dx$

Exercise: check this directly

p. 259 : witch of Maria Agnesi (1718 – 1799)



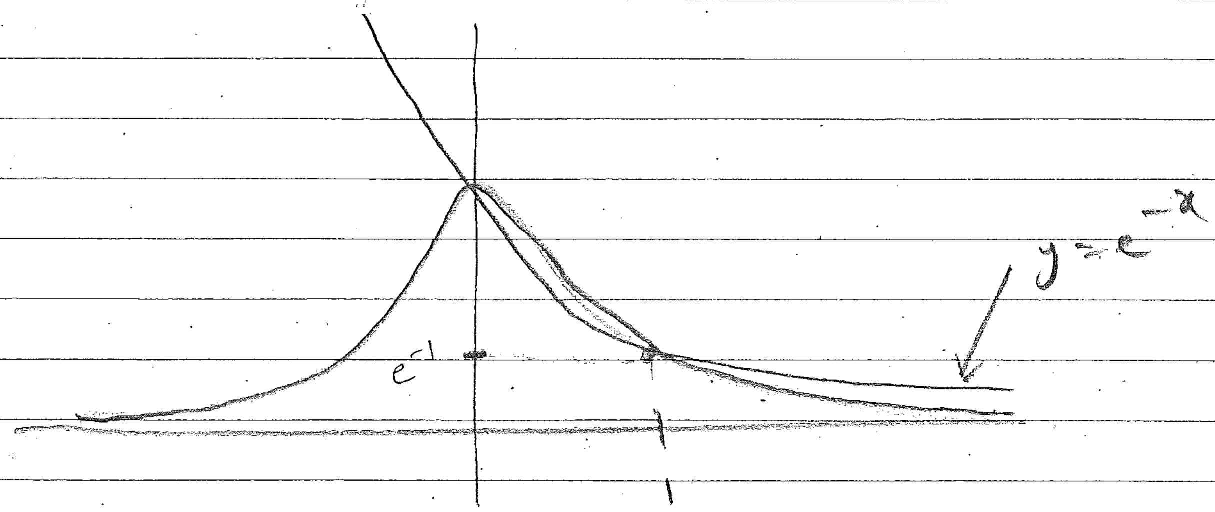
What about



(Gaussian bell-shaped curve)

?

Compare with $y = e^{-x}$:



Observe $e^{-x^2} \leq e^{-x}$ for $x \geqslant 1$ since $x^2 \geqslant x$.

By comparison (formalised shortly),

$$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx < 1$$

$$\left(1 = \int_0^\infty e^{-x}\,dx\right)$$

so $\int_1^\infty e^{-x^2}\,dx$ converges.

Hence $\int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx$ (using evenness)

$$= 2\left(\int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx\right)$$

$< \infty$, so converges

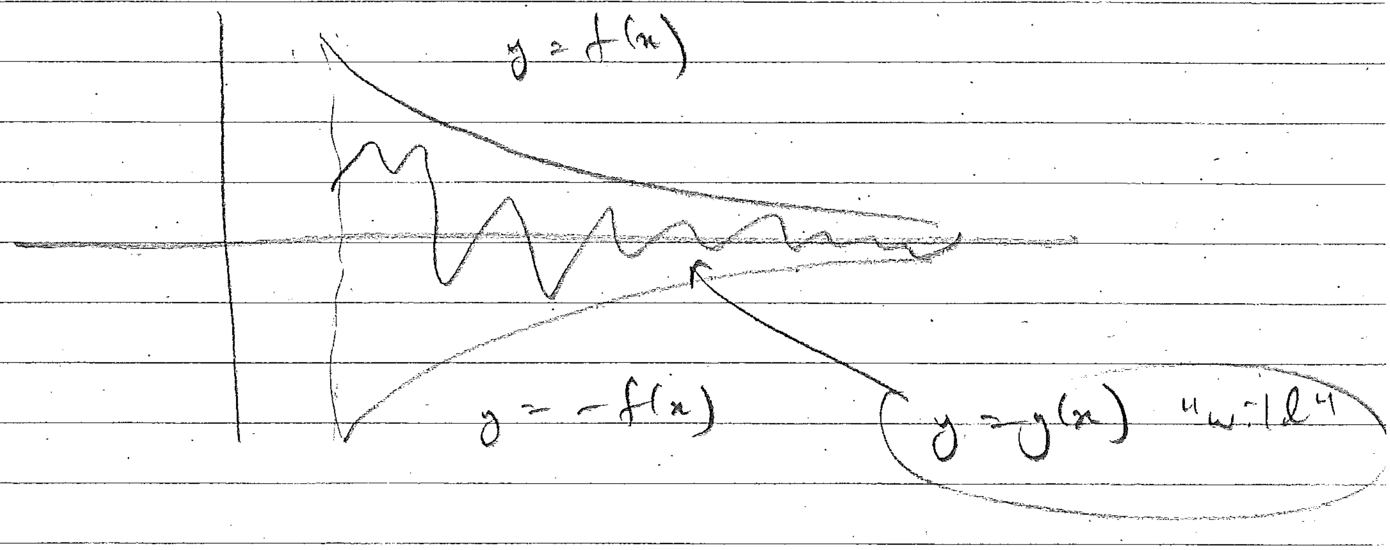
In fact : $\boxed{\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}}$ (proof difficult)

Comparison Test: If f, g are continuous

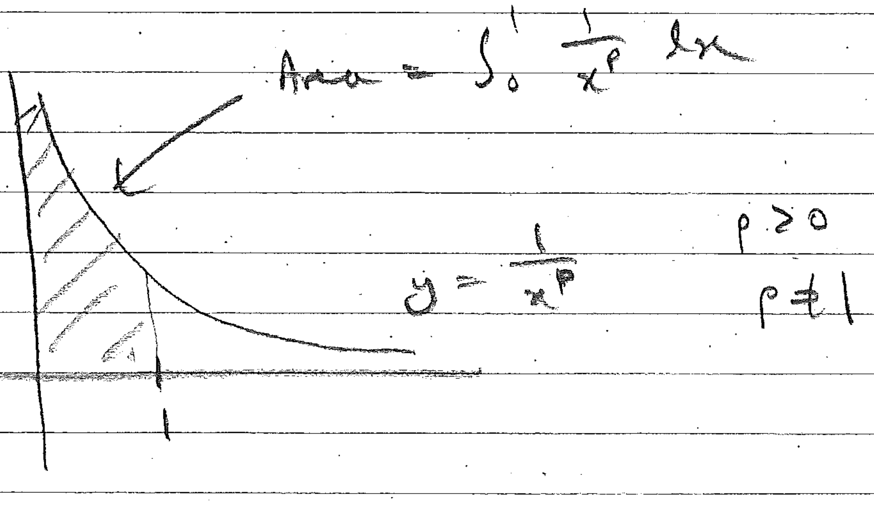
and $f(x) \geq |g(x)|$ over $[a, \infty)$

and $\int_a^\infty f(x) < \infty$

then $\int_a^\infty g(x) < \infty$.



The other half of a p-integral?



F

<u>Direct calculation</u>:

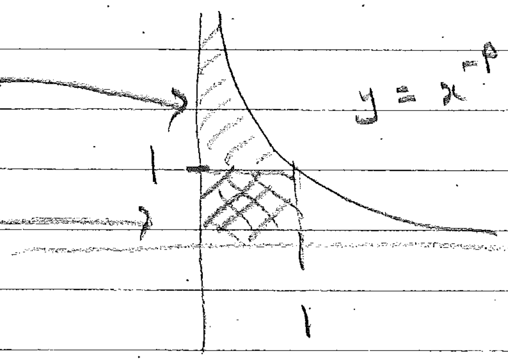
$$\int_0^1 \frac{dx}{x^p} = \lim_{a\to 0^+} \int_a^1 x^{-p}\,dx$$

$$= \lim_{a\to 0^+} \left[\frac{x^{1-p}}{1-p}\right]_a^1$$

$$= \lim_{a\to 0^+} \left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right) = \begin{cases} \frac{1}{1-p} & \text{if } 0<p<1 \\ \infty & \text{if } p>1. \end{cases}$$

<u>Indirectly, using our other formula</u>:

$$\int_0^1 \frac{dx}{x^p} = 1 + \int_1^\infty \frac{dy}{y^{1/p}}$$



$$= 1 + \begin{cases} \frac{1}{\frac{1}{p}-1} & \text{if } \frac{1}{p}>1 \\ \infty & \text{if } 0<\frac{1}{p}<1 \end{cases}$$

$$= \begin{cases} 1+\frac{p}{1-p} = \frac{1}{1-p} & \text{if } 0<p<1 \\ \infty & \text{if } p>1 \end{cases}$$

✓ as before