CHAPTER **EIGHT**

The Quadratic Function

The previous chapter on differentiation established that the derivative of any quadratic function is a linear function, for example,

$$\frac{d}{dx}(x^2 - 5x + 6) = 2x - 5.$$

In this sense, quadratics are the next most elementary functions to study after the linear functions of Chapter Five. This relationship between linear and quadratic functions is the underlying reason why quadratics arise in so many applications of mathematics.

Study Notes: Sections 8A–8D review the known theory of quadratics — factoring, completing the square, and the formulae for the roots and the axis of symmetry — presenting them in the more general context of functions and their graphs, and leading to maximisation problems in Section 8E. From this basis, Sections 8F and 8G develop a classification of quadratics based on the discriminant. The final Sections 8H and 8I on the sum and product of roots and quadratic identities will be generalised later to polynomials of higher degree.

8 A Factorisation and the Graph

A *quadratic function* is a function that can be written in the form

$$f(x) = ax^2 + bx + c, \text{ where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

A *quadratic equation* is an equation that can be written in the form

$$ax^2 + bx + c = 0, \text{ where } a, b \text{ and } c \text{ are constants, and } a \neq 0,$$

that is, in a form where the LHS is a quadratic function. The requirement that $a \neq 0$ means that the term in x^2 cannot vanish, so that linear functions and equations are not to be regarded as special cases of quadratics. The word ‘quadratic’ comes from the Latin root *quadrat*, meaning ‘square’, and reminds us that quadratics tend to arise as the area of a plane shape, or more generally as the product of two linear functions. In the same way, the terms ‘square of x’ and ‘cube of x’ are used for x^2 and x^3 because they are the area and volume respectively of a square and cube of side length x.

Zeroes and Roots: One usually speaks of the solutions of a quadratic equation as *roots* of the equation, and of the x-intercepts of a quadratic function as *zeroes* of the function. However, the distinction between the words ‘roots’ and ‘zeroes’ is not strictly observed, and questions about quadratic functions and their graphs are closely related to questions about quadratic equations.

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The Four Questions about the Graph of a Quadratic: Our first task is to review the sketching of the graph of a quadratic function $f(x) = ax^2 + bx + c$. The graph is a parabola, and before attempting any sketch, there are four questions that need to be to asked.

1 FOUR QUESTIONS ABOUT THE GRAPH OF A PARABOLA:
1. Which way up is the curve? Answer: Look at the sign of a.
2. What is the y-intercept? Answer: Put $x = 0$, and then $y = c$.
3. Where are the axis of symmetry and the vertex?
4. Where are the x-intercepts, if there are any?

The first two questions are very straightforward to answer, but the second two questions need close attention. This section and the next two will review in succession the three standard approaches to them: factorisation, completing the square, and using the formulae generated by completing the square.

Factorisation and the Zeroes: Most quadratics cannot easily be factored, but when factorisation is possible, this is usually the quickest approach to sketching the curve. The zeroes are found using the following principle:

2 FACTORISATION AND THE ZEROES: If $AB = 0$, then $A = 0$ or $B = 0$, so we find the zeroes by putting each factor equal to zero.

For example, if $y = (2x - 3)(2x - 5)$, then the zeroes are given by

$$2x - 3 = 0 \qquad \text{or} \qquad 2x - 5 = 0,$$

so they are $x = 1\frac{1}{2}$ and $x = 2\frac{1}{2}$.

Finding the Axis of Symmetry and Vertex from the Zeroes: The axis of symmetry is always midway between the x-intercepts, so it can be found by taking the average of the zeroes.

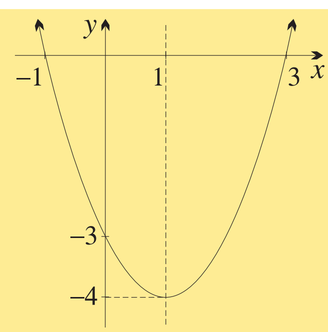
3 ZEROES AND THE AXIS OF SYMMETRY AND VERTEX:
1. If a quadratic has zeroes α and β, its axis is the line $x = \frac{1}{2}(\alpha + \beta)$.
2. Substitution into the quadratic gives the y-coordinate of the vertex.

For example, we saw that $y = (2x - 3)(2x - 5)$ has zeroes $x = 1\frac{1}{2}$ and $x = 2\frac{1}{2}$.
Averaging these zeroes, the axis of symmetry is the line $x = 2$.
Substituting $x = 2$ gives $y = -1$, so the vertex is $(2, -1)$.

WORKED EXERCISE: Sketch the curve $y = x^2 - 2x - 3$.

SOLUTION: Since $a > 0$, the curve is concave up.

When $x = 0$, $y = -3$.
Factoring, $y = (x + 1)(x - 3)$.
When $y = 0$, $x + 1 = 0$ or $x - 3 = 0$
$x = -1$ or $x = 3$.
Then the axis of symmetry is $x = \frac{1}{2}(-1 + 3)$
$x = 1$.
When $x = 1$, $y = -4$, so the vertex is $(1, -4)$.



WORKED EXERCISE: Sketch the graph of the function $f(x) = -2x^2 + 9x - 7$.

SOLUTION: Since $a < 0$, the curve is concave down.

When $x = 0$, $\quad f(0) = -7$.

Factoring, $\quad f(x) = -(2x - 7)(x - 1)$.

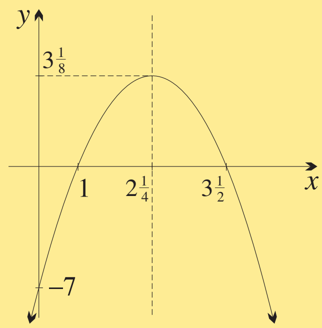
When $f(x) = 0$, $\quad 2x - 7 = 0$ or $x - 1 = 0$

$x = 3\frac{1}{2}$ or $x = 1$.

Then the axis of symmetry is $\quad x = \frac{1}{2}(3\frac{1}{2} + 1)$

$x = 2\frac{1}{4}$.

Since $f(2\frac{1}{4}) = 3\frac{1}{8}$, the vertex is $(2\frac{1}{4}, 3\frac{1}{8})$.



Quadratic Inequations: As discussed in Chapter Three, a quadratic inequation is best solved from a sketch of the quadratic function.

WORKED EXERCISE: From the graphs above, solve:

(a) $x^2 - 2x - 3 \le 0$ (b) $2x^2 + 7 > 9x$

SOLUTION:

(a) $x^2 - 2x - 3 \le 0$

From the first graph,

$-1 \le x \le 3$.

(b) $2x^2 + 7 > 9x$

$-2x^2 + 9x - 7 < 0$

From the second graph,

$x < 1$ or $x > 3\frac{1}{2}$.

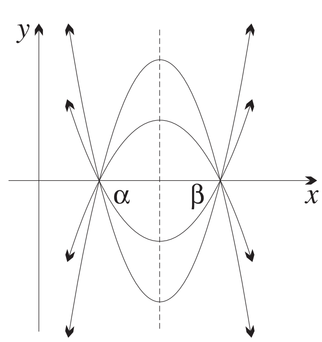
Domain and Range of Quadratic Functions: The natural domain of a quadratic function is the set $\mathbf{R}$ of real numbers, and the graphs above show that its range is clear once the vertex and concavity are established. When the domain is restricted, the range can be read off the graph, taking account of the vertex and endpoints.

WORKED EXERCISE: From the graph of $y = x^2 - 2x - 3$ on the previous page, find the range of the function: (a) with unrestricted domain, (b) with domain $x \ge 4$,
(c) with domain $0 \le x \le 4$.

SOLUTION:

(a) With no restriction on the domain, the range is $y \ge -4$.
(b) When $x = 4$, $y = 5$, so the range is $y \ge 5$.
(c) When $x = 0$, $y = -3$, so the range is $-4 \le y \le 5$.

Quadratics with Given Zeroes: If it is known that a quadratic $f(x)$ has zeroes α and β, then the quadratic must have the form $f(x) = a(x - \alpha)(x - \beta)$, where a is the coefficient of x^2. By taking different values of a, this equation forms a family of quadratics all with the same x-intercepts, as sketched opposite. In general:



4 QUADRATICS WITH GIVEN ZEROES:
The quadratics whose zeroes are α and β form a family with equation $y = a(x - \alpha)(x - \beta)$.

WORKED EXERCISE: Write down the family of quadratics with zeroes -2 and 4, then find the equation of such a quadratic:

(a) with y-intercept 6, (b) with vertex $(1, 21)$.

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SOLUTION: The family of quadratics with zeroes -2 and 4 is $y = a(x+2)(x-4)$.

(a) When $x = 0$, $y = -8a$, so $\quad -8a = 6$,
so $a = -\frac{3}{4}$, and the quadratic is $\quad y = -\frac{3}{4}(x+2)(x-4)$.

(b) [Taking the average of the zeroes, the axis of symmetry is indeed $x = 1$.]
When $x = 1$, $y = -9a$, so $\quad -9a = 21$,
so $a = -\frac{7}{3}$, and the quadratic is $\quad y = -\frac{7}{3}(x+2)(x-4)$.

Monic Quadratics: Factorisation in quadratics is a little easier to handle when the coefficient of x^2 is 1. Such quadratics are called *monic*.

5 DEFINITION: A quadratic is called *monic* if the coefficient of x^2 is 1.

Then $(x-\alpha)(x-\beta)$ is the only monic quadratic whose zeroes are α and β.

Exercise 8A

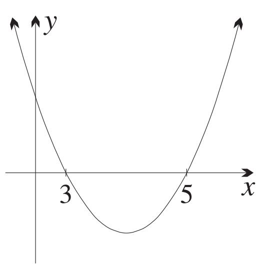
1. Use factorisation where necessary to find the zeroes of these quadratic functions. Use the fact that the axis of symmetry is halfway between the zeroes to find the equation of the axis, then find the vertex. Hence sketch a graph showing all intercepts and the vertex.

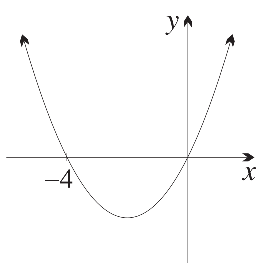
(a) $y = (x-1)(x+3)$
(b) $y = x(x-3)$
(c) $y = (5-x)(x+1)$
(d) $y = (2x-1)(2x+5)$
(e) $y = x^2 - 9$
(f) $y = x^2 - 5x + 6$
(g) $y = x^2 + 4x + 3$
(h) $y = 3 + 2x - x^2$

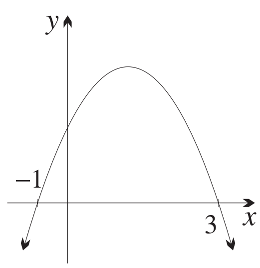
2. Use the graphs sketched in the question above to solve the following inequations:

(a) $(x-1)(x+3) > 0$
(b) $x(x-3) \le 0$
(c) $5 + 4x - x^2 \ge 0$
(d) $4x^2 + 8x - 5 < 0$
(e) $x^2 \ge 9$
(f) $x^2 < 5x - 6$
(g) $x^2 + 4x \le -3$
(h) $3 + 2x > x^2$

3. Give a possible equation of each quadratic function sketched below:

(a)


(b)


(c)


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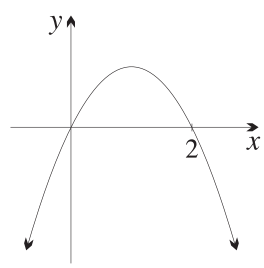

4. State the axis of symmetry and equation of the monic quadratic with zeroes:

(a) 4 and 6
(b) 3 and 8
(c) -3 and 5
(d) -6 and -1

5. Use factorisation to sketch each quadratic, showing the intercepts and vertex:

(a) $y = 2x^2 - 9x - 5$
(b) $y = 3x^2 - 10x - 8$
(c) $y = -3x^2 - 5x + 2$
(d) $y = 7x - 3 - 4x^2$

DEVELOPMENT

6. The general form of a quadratic with zeroes 2 and 8 is $y = a(x-2)(x-8)$. By evaluating a, find the equation if such a quadratic:

(a) has y-intercept -16,
(b) passes through $(3, 10)$,
(c) has constant term -3,
(d) has coefficient of x^2 3,
(e) has vertex $(5, -12)$,
(f) passes through $(1, -20)$.

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7. Write down the general form of a monic quadratic for which one of the zeroes is $x = 1$. Then find the equation of such a quadratic in which:

(a) the curve passes through the origin,
(b) there are no other zeroes,
(c) the axis of symmetry is $x = -7$,
(d) the curve passes through $(3, 9)$.

8. Factor and sketch each quadratic. Hence find the range of each function (i) with unrestricted domain, (ii) with domain $x \geq 5$, (iii) with domain $0 \leq x \leq 5$:

(a) $y = x^2 - 6x + 8$
(b) $y = x^2 - 12x + 27$
(c) $y = -x^2 + 4x - 3$

9. Sketch each of the following regions on a number plane:

(a) $y \geq x^2$
(b) $y \leq -x^2$
(c) $-x^2 \leq y \leq x^2$
(d) $x^2 - 4 \leq y \leq 4 - x^2$
(e) $x^2 - 1 \leq y \leq 9 - x^2$
(f) $x^2 + 2x - 3 \leq y \leq 4x - x^2$

10. Explain why $y = ax(x - \alpha)$ is the general form of a quadratic whose graph passes through the origin. Hence find the equation of such a quadratic:

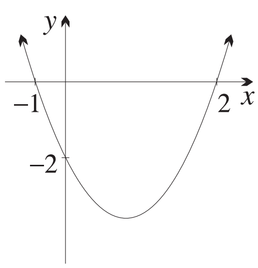
(a) with another x-intercept at $x = -3$, and monic,
(b) with vertex $(1, 4)$,
(c) with no other x-intercepts, and passing through $(2, 6)$,
(d) with no other x-intercepts, and having gradient 1 at $x = -2$,
(e) with another x-intercept at $x = 5$, and having gradient 2 at the origin,
(f) with axis of symmetry $x = -3$, and with -12 as coefficient of x.

11. The general form of a quadratic with zeroes α and β is $y = a(x - \alpha)(x - \beta)$. Find a in terms of α and β if:

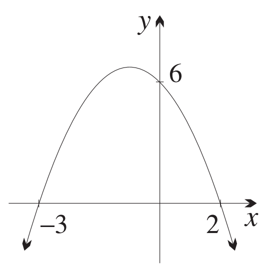
(a) the y-intercept is c, (b) the coefficient of x is b, (c) the curve passes through $(1, 2)$.

12. Find the equations of each quadratic function sketched below:

(a)



(b)



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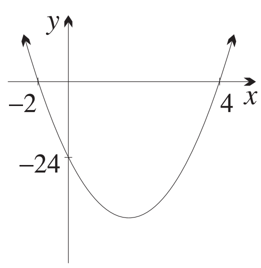
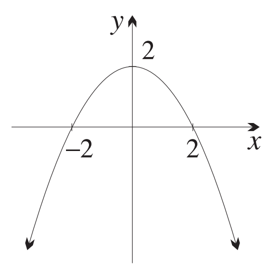


(d)



13. Use factorisation to find the x-intercepts of each graph, and hence find its axis:

(a) $y = x^2 + bx + cx + bc$
(b) $y = x^2 + (1 - a^2)x - a^2$
(c) $y = ax^2 - bx - (a + b)$
(d) $y = x^2 - 2cx - 1 + c^2$

14. (a) (i) Sketch the graph of $f(x) = (x - 3)^2$. (ii) Find $f'(x)$, show that $f(3) = f'(3) = 0$, and explain the geometrical significance of this result.

(b) Show that the derivative of $f(x) = p(x - q)^2$ is $f'(x) = 2p(x - q)$. Hence show that $f(q) = f'(q) = 0$, and explain the geometrical significance of this result.

15. Use the product rule to show that the derivative of $y = a(x-\alpha)(x-\beta)$ is $y' = a(2x-\alpha-\beta)$. Hence show that the vertex is at $x = \frac{1}{2}(\alpha + \beta)$, and that the gradients at the x-intercepts are opposites of each other. Show more generally that the gradients at $x = \frac{1}{2}(\alpha + \beta) + h$ and $x = \frac{1}{2}(\alpha + \beta) - h$ are opposites of each other.

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EXTENSION

16. Show that $y = x^4 - 13x^2 + 36$ is an even function, then sketch its graph.

17. (a) If $f(x) = (x-3)(x-1)(x+4)(x+6)$, show that $f(-1) = f(-2)$ and $f(2) = f(-5)$.
(b) Show that $f(-\frac{3}{2} + a) = f(-\frac{3}{2} - a)$.
(c) Show that $f'(-\frac{3}{2} - a) = -f'(-\frac{3}{2} + a)$.
(d) On the same set of axes sketch $f(x) = (x-3)(x-1)(x+4)(x+6)$ and the line $x = -\frac{3}{2}$.
(e) Sketch a graph of the function $f(x) = (x-a)(x-b)(x-c)(x-d)$ where $b - a = d - c$.

18. Suppose that a quadratic has equation $f(x) = k(x-\alpha)(x-\beta)$. Prove the following identities, and explain how each establishes that $x = \frac{1}{2}(\alpha + \beta)$ is the axis of symmetry.
(a) $f\big(\frac{1}{2}(\alpha+\beta) + h\big) = f\big(\frac{1}{2}(\alpha+\beta) - h\big)$ (b) $f(\alpha + \beta - x) = f(x)$

8 B Completing the Square and the Graph

Completing the square is the fundamental method of approach to quadratics. It works in every case, in contrast with factoring, which really only works in exceptional cases. Although important formulae for the zeroes and vertex can be developed by completing the square in a general quadratic, the method remains necessary in many situations and needs to be learnt well.

The Method of Completing the Square: For *monic quadratics*, where the coefficient of x^2 is 1, the goal is to express the quadratic $y = x^2 + bx + c$ in the form

$$y = (x-h)^2 + k, \text{ which expands to } y = x^2 - 2hx + h^2 + k.$$

Since then $h = -\frac{1}{2}b$, the method is usually expressed rather concisely as:

6 COMPLETING THE SQUARE: Halve the coefficient of x, then add and subtract its square.

For *non-monic quadratics*, where the coefficient of x^2 is not 1, the coefficient should be removed by bracketing before the calculation begins.

WORKED EXERCISE: Complete the square in each of the following quadratics:
(a) $y = x^2 - 4x - 5$
(b) $y = 2x^2 - 12x + 16$
(c) $y = x^2 + x + 1$
(d) $y = -3x^2 - 4x + 2$

SOLUTION:

(a) $$\begin{aligned} y &= x^2 - 4x - 5 \\ &= (x^2 - 4x + 4) - 4 - 5 \\ &= (x-2)^2 - 9 \end{aligned}$$

(b) $$\begin{aligned} y &= 2x^2 - 12x + 16 \\ &= 2(x^2 - 6x + 8) \\ &= 2\Big((x^2 - 6x + 9) - 9 + 8\Big) \\ &= 2(x-3)^2 - 2 \end{aligned}$$

(c) $$\begin{aligned} y &= x^2 + x + 1 \\ &= (x^2 + x + \tfrac{1}{4}) - \tfrac{1}{4} + 1 \\ &= (x + \tfrac{1}{2})^2 + \tfrac{3}{4} \end{aligned}$$

(d) $$\begin{aligned} y &= -3x^2 - 4x + 2 \\ &= -3(x^2 + \tfrac{4}{3}x - \tfrac{2}{3}) \\ &= -3\Big((x^2 + \tfrac{4}{3}x + \tfrac{4}{9}) - \tfrac{4}{9} - \tfrac{2}{3}\Big) \\ &= -3(x + \tfrac{2}{3})^2 + \tfrac{10}{3} \end{aligned}$$

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Sketching the Graph from the Completed Square: The work in Chapter Two on transformations of graphs tells us that $y = a(x-h)^2 + k$ is just $y = ax^2$ shifted h units to the right and k units upwards. Hence its vertex must be at (h, k).

7 THE COMPLETED SQUARE AND TRANSLATION: The curve $y = a(x-h)^2 + k$ is the translation of $y = ax^2$ to a parabola with vertex at (h, k).

From the completed square form, the zeroes can then be calculated by the usual method of setting $y = 0$. If the zeroes exist, the quadratic can then be written in factored form.

WORKED EXERCISE: Use the previous completed squares to sketch the graphs of:

(a) $y = x^2 - 4x - 5$

(b) $y = 2x^2 - 12x + 16$

(c) $y = x^2 + x + 1$

(d) $y = -3x^2 - 4x + 2$

If possible, express each quadratic in the form $y = a(x-\alpha)(x-\beta)$.

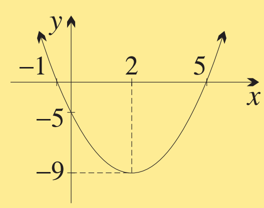
NOTE: When the square is completed, the axis of symmetry and vertex can be read off directly — the zeroes of the quadratic can then be found with a subsequent calculation. This contrasts with factoring, where the zeroes are found first and the vertex then follows.

SOLUTION:

(a) $y = x^2 - 4x - 5$ is concave up with y-intercept -5.
Since $y = (x-2)^2 - 9$, the vertex is $(2, -9)$.
Put $y = 0$, then $(x-2)^2 = 9$

$$x - 2 = 3 \text{ or } x - 2 = -3$$
$$x = 5 \text{ or } -1.$$

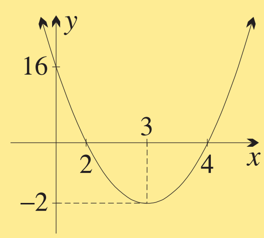
Hence also $y = (x-5)(x+1)$.



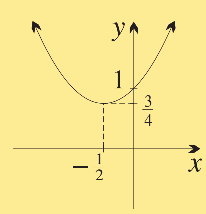
(b) $y = 2x^2 - 12x + 16$ is concave up with y-intercept 16.
Since $y = 2(x-3)^2 - 2$, the vertex is $(3, -2)$.
Put $y = 0$, then $2(x-3)^2 = 2$

$$x - 3 = 1 \text{ or } x - 3 = -1$$
$$x = 4 \text{ or } 2\,.$$

Hence also $y = 2(x-4)(x-2)$.



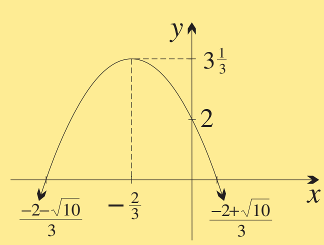
(c) $y = x^2 + x + 1$ is concave up with y-intercept 1.
Since $y = (x+\frac{1}{2})^2 + \frac{3}{4}$, the vertex is $(-\frac{1}{2}, \frac{3}{4})$.
Put $y = 0$, then $(x+\frac{1}{2})^2 = -\frac{3}{4}$,
and since this has no solutions, there are no x-intercepts.



(d) $y = -3x^2 - 4x + 2$ is concave down with y-intercept 2.
Since $y = -3(x+\frac{2}{3})^2 + \frac{10}{3}$, the vertex is $(-\frac{2}{3}, 3\frac{1}{3})$.
Put $y = 0$, then $3(x+\frac{2}{3})^2 = \frac{10}{3}$

$$(x+\tfrac{2}{3})^2 = \tfrac{10}{9}$$
$$x + \tfrac{2}{3} = \tfrac{1}{3}\sqrt{10} \text{ or } x + \tfrac{2}{3} = -\tfrac{1}{3}\sqrt{10}$$
$$x = -\tfrac{2}{3} + \tfrac{1}{3}\sqrt{10} \text{ or } -\tfrac{2}{3} - \tfrac{1}{3}\sqrt{10}\,.$$

Hence also $y = -3\left(x + \frac{2}{3} - \frac{1}{3}\sqrt{10}\right)\left(x + \frac{2}{3} + \frac{1}{3}\sqrt{10}\right)$.



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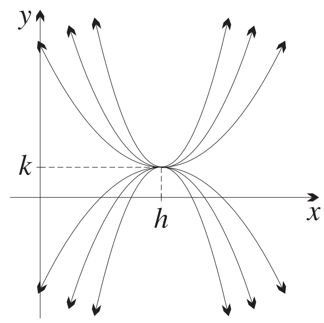
The Family of Quadratics with a Common Vertex: If a quadratic is known to have its vertex at (h, k), then by the theory above, its equation must have the form $y = a(x - h)^2 + k$, for some value of a. This equation gives a family of quadratics with vertex (h, k), as different values of a are taken, as in the sketch opposite. The general case is:



8 QUADRATICS WITH A COMMON VERTEX: The quadratics with vertex (h, k) form a family of curves with equation $y = a(x - h)^2 + k$.

WORKED EXERCISE: Write down the family of quadratics with vertex $(-3, 2)$, then find the equation of such a quadratic:

(a) if $x = 5$ is one of its zeroes, (b) if the coefficient of x is equal to 1.

SOLUTION: The family of quadratics with vertex $(-3, 2)$ is

$$y = a(x + 3)^2 + 2.$$

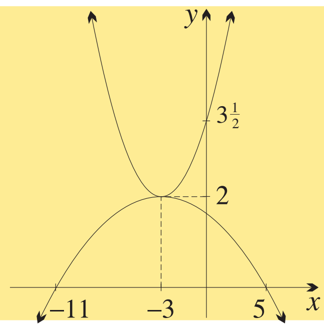
(a) Substituting $(5, 0)$ gives $0 = a \times 64 + 2$,

so $a = -\frac{1}{32}$, and the quadratic is $y = -\frac{1}{32}(x + 3)^2 + 2$.

(b) Expanding, $y = ax^2 + 6ax + (9a + 2)$,

so $6a = 1$,

so $a = \frac{1}{6}$ and the quadratic is $y = \frac{1}{6}(x + 3)^2 + 2$.



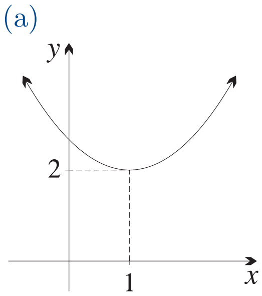
Exercise 8B

1. Complete the square where necessary in each quadratic, expressing it in the form $y = (x - h)^2 + k$. Hence find the axis of symmetry, vertex and any intercepts, giving irrational zeroes in exact form. Sketch their graphs, showing vertex and intercepts.

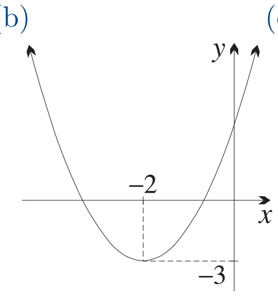
(a) $y = (x - 3)^2 - 9$ (d) $y = (x - 5)^2 - 2$ (g) $y = x^2 - 2x + 5$
(b) $y = (x + 2)^2 - 1$ (e) $y = x^2 - 2x$ (h) $y = x^2 + x + 1$
(c) $y = (x + 1)^2 + 3$ (f) $y = x^2 - 4x + 3$ (i) $y = x^2 - 3x + 1$

2. Give a possible equation for each of the quadratic functions sketched below:

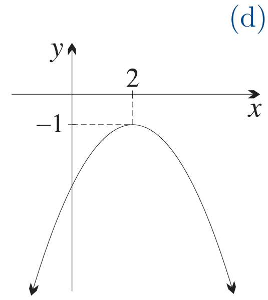
(a)



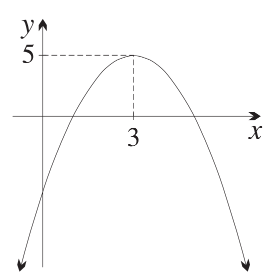
(b)



(c)



(d)



3. Write down the equation of the monic quadratics with the following vertices. Also find the axis of symmetry and y-intercept of each.

(a) $(2, 5)$ (b) $(0, -3)$ (c) $(-1, 7)$ (d) $(3, -11)$

4. Explain why any quadratic with vertex $(0, 1)$ has equation $y = ax^2 + 1$, for some value of a. Hence find the equation of such a quadratic passing through:

(a) $(1, 3)$ (b) $(-2, -11)$ (c) $(9, 28)$ (d) $(\frac{1}{2}, \frac{15}{16})$

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5. Explain why $y = a(x+4)^2 + 2$ is the general form of a quadratic with vertex $(-4, 2)$. Then find the equation of such a quadratic for which:

(a) the quadratic is monic,
(b) the coefficient of x^2 is 3,
(c) one of the zeroes is $x = 3$,
(d) the y-intercept is 16,
(e) the curve passes through the origin,
(f) the curve passes through $(1, 20)$.

DEVELOPMENT

6. Express each quadratic function in the form $y = a(x-h)^2 + k$ (notice that $a \neq 1$ in each example). Find any x-intercepts, the y-intercept and the vertex. Write down the equation of the axis of symmetry. Then sketch the curve, showing this information.

(a) $y = -x^2 - 2x$
(b) $y = -x^2 + 4x + 1$
(c) $y = -x^2 + 5x - 6$
(d) $y = 2x^2 - 4x + 3$
(e) $y = 4x^2 - 16x + 13$
(f) $y = -3x^2 + 6x + 3$
(g) $y = -5x^2 - 20x - 23$
(h) $y = 2x^2 + 5x - 12$
(i) $y = 3x^2 + 2x - 8$

7. Sketch the graph of each function, showing the intercepts and vertex. From the graph, find the range of each function with: (i) unrestricted domain, (ii) domain $x \leq -1$, (iii) domain $-1 \leq x \leq 2$.

(a) $y = (x-1)^2 + 2$
(b) $y = 2(x-3)^2 + 1$
(c) $y = -(x+2)^2 + 5$

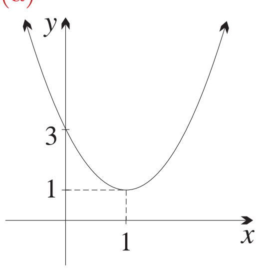
8. Complete the square to find the vertex of the function $y = x^2 - 6x + c$. Hence find the values of c for which the graph of the function: (a) touches the x-axis, (b) cuts the x-axis in two places, (c) does not intersect the x-axis.

9. Write down the general form of a monic quadratic whose axis of symmetry is $x = -2$. Hence find the equation of such a quadratic:

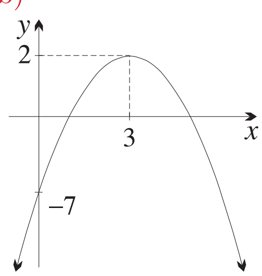
(a) passing through the origin,
(b) passing through $(5, 1)$,
(c) with $x = 1$ as one zero,
(d) with y-intercept -6,
(e) touching the line $y = -2$,
(f) with range $y \geq 7$,
(g) whose tangent at $x = 1$ passes through $(0, 0)$,
(h) which is tangent to $y = -x^2$.

10. Find the equations of the quadratic functions sketched below.

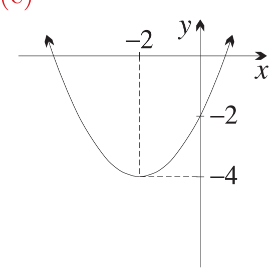
(a)



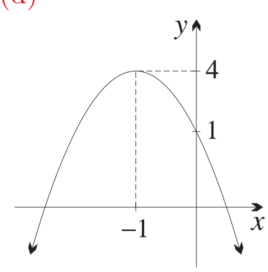
(b)



(c)



(d)



11. Complete the square to find the roots α and β, and show that $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

(a) $x^2 - x - 6 = 0$ (b) $x^2 - 4x + 1 = 0$ (c) $2x^2 + 3x - 5 = 0$ (d) $5x^2 - 15x + 11 = 0$

12. (a) Complete the square to find the vertex of each quadratic function. Then sketch all five functions on the same number plane. (i) $y = x^2 - 4x + 4$ (ii) $y = x^2 - 2x + 4$ (iii) $y = x^2 + 4$ (iv) $y = x^2 + 2x + 4$ (v) $y = x^2 + 4x + 4$

(b) What is the effect of varying b on the graph of $y = x^2 + bx + 4$?

13. Complete the square to find the vertex and x-intercepts of the function $y = x^2 + ax + b$. Then sketch a possible graph of the function if:

(a) $a > 0$ and $a^2 > 4b$,
(b) $a > 0$ and $a^2 < 4b$,
(c) $a > 0$ and $a^2 = 4b$,
(d) $a < 0$ and $a^2 > 4b$,
(e) $a < 0$ and $a^2 < 4b$,
(f) $a < 0$ and $a^2 = 4b$.

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14. Complete the square in $y = ax^2 + bx + c$. Hence write down the vertex and find the zeroes.

15. Expand $y = a(x - \alpha)(x - \beta)$, complete the square, and hence find the vertex.

16. If $f(x) = (x - h)^2 + k$, show that $f'(h + r) = -f'(h - r)$. Give a geometric interpetation.

17. Write down the general form of a quadratic with vertex (h, k). Find an expression for the coefficient of x^2 if:

(a) the y-intercept is c,

(b) the curve passes through $(1, 2)$,

(c) the coefficient of x is b,

(d) one of the zeroes is α.

18. (a) Find the zeroes of the monic quadratic $y = (x + d)^2 - e$, where $e > 0$.

(b) Find an expression for the difference between the two zeroes.

(c) Hence find the condition for the difference between the two zeroes to be 2, and describe geometrically the family of quadratics with this property.

19. The monic quadratics $y = (x - h_1)^2 + k_1$ and $y = (x - h_2)^2 + k_2$ do not intersect at all. Find the corresponding condition on the constants h_1, h_2, k_1 and k_2, and describe geometrically the relationship between the two curves.

EXTENSION

20. (a) Complete the square to find the vertex and x-intercepts of $y = x^2 + 2x - 3$, and sketch the curve. (b) Hence sketch $y = \left|x^2 + 2x - 3\right|$. (c) Complete the square to solve the equations $x^2 + 2x - 3 = 1$ and $x^2 + 2x - 3 = -1$. (d) Hence sketch $y = \dfrac{1}{x^2 + 2x - 3}$.

21. Consider the quadratic $f(x) = a(x - h)^2 + k$ with vertex (h, k). Prove the following identities and hence establish that $x = h$ is the axis of symmetry.

(a) $f(h + t) = f(h - t)$

(b) $f(2h - x) = f(x)$.

22. Show that the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, cannot have more than two distinct roots. [HINT: Assume that the equation can have three distinct roots α, β and γ. Substitute α, β and γ into the equation and conclude that $a = b = c = 0$.]

8 C The Quadratic Formulae and the Graph

Completing the square in a general quadratic yields formulae for the axis of symmetry and for the zeroes of a quadratic function. These formulae are extremely useful, and will allow the theory of quadratics to be advanced considerably. The previous exercise asked for these formulae to be generated, but in view of their importance, they are derived again here.

Completing the Square in the General Quadratic: Here then is the completion of the square in the general quadratic $y = ax^2 + bx + c$:

$$\begin{aligned} y &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right), \quad \text{since half the coefficient of } x \text{ is } \frac{b}{2a}, \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}. \end{aligned}$$

Hence the axis of symmetry is $x = -\dfrac{b}{2a}$, and the vertex is $\left(-\dfrac{b}{2a}, -\dfrac{b^2 - 4ac}{4a}\right)$.

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Remember the formula for the axis, and find the y-coordinate of the vertex by substitution.

9 THE AXIS OF SYMMETRY: The axis of symmetry is the line $x = -\dfrac{b}{2a}$.

To find the formula for the zeroes, put $y = 0$ into the completed square:

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ is called the *discriminant* and is given the symbol Δ (Greek capital delta). When calculating the zeroes, the discriminant should always be found first, so the formula for the zeroes should be remembered in the form:

10 THE ZEROES: $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ or $\dfrac{-b - \sqrt{\Delta}}{2a}$, where $\Delta = b^2 - 4ac$.

The discriminant $\Delta = b^2 - 4ac$ will become important theoretically as the chapter develops. For now it will be enough to notice two things. First, if the discriminant is negative, then the quadratic has no zeroes, because negative numbers don't have square roots. Secondly, if the discriminant is zero, then the quadratic has only one zero, because the only square root of zero is zero itself.

NOTE: The vertex (found above) can be written in the form $\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right)$. Some people prefer memorising this formula for the coordinates of the vertex rather than substituting the axis of symmetry to find the y-coordinate.

WORKED EXERCISE: Use the quadratic formulae to sketch the following quadratics. Give any irrational zeroes first in simplified surd form, then approximated to four significant figures. If possible, write each quadratic in factored form.

(a) $y = -x^2 + 6x + 1$ (b) $y = 3x^2 - 6x + 4$

SOLUTION:

(a) The curve $y = -x^2 + 6x + 1$ is concave down with y-intercept 1.

The formulae are applied with $a = -1$, $b = 6$ and $c = 1$.

The axis is $x = -\dfrac{b}{2a}$

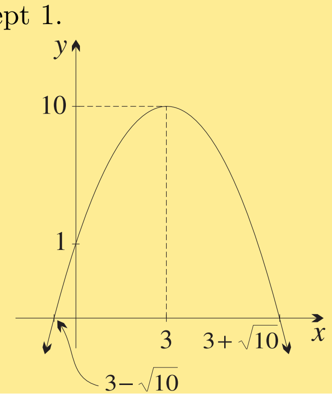
$x = 3.$

When $x = 3$, $y = 10$, so the vertex is $(3, 10)$.

Also $\Delta = b^2 - 4ac$

$= 40 = 4 \times 10,$

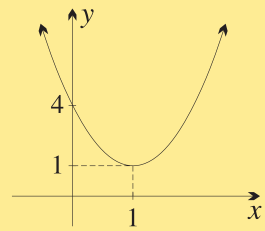
so $y = 0$ when $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ or $\dfrac{-b - \sqrt{\Delta}}{2a}$



$= 3 - \sqrt{10}$ or $3 + \sqrt{10}$
$\doteqdot -0{\cdot}1623$ or $6{\cdot}162$.

Hence also $y = -\left(x - 3 + \sqrt{10}\right)\left(x - 3 - \sqrt{10}\right)$.

(b) The curve $y = 3x^2 - 6x + 4$ is concave up, and its y-intercept is 4.
Using the formulae with $a = 3$, $b = -6$ and $c = 4$,
the axis is $x = 1$,
and substituting $x = 1$, the vertex is $(1, 1)$.
Also $\Delta = 36 - 48$
which is negative, so there are no zeroes.



Exercise 8C

1. Find the discriminant $\Delta = b^2 - 4ac$, and hence the zeroes, of these quadratics. Give irrational zeroes in surd form, then approximated to four significant figures.

(a) $y = x^2 + 6x + 5$
(b) $y = x^2 + 4x + 4$
(c) $y = -x^2 + 2x + 24$
(d) $y = -x^2 + 2x + 1$
(e) $y = x^2 + 4x - 1$
(f) $y = 2x^2 + 2x - 1$
(g) $y = -5x^2 + 7x + 3$
(h) $y = 4x^2 - 3x - 3$
(i) $y = 4x^2 - 9$

2. For each quadratic in the previous question, find the equation of the axis of symmetry using the formula $x = -b/2a$. Substitute into the function to find the vertex, then sketch the curve, showing the vertex and all intercepts.

3. Use the graphs in parts (a)–(d) of the previous question to solve:

(a) $x^2 + 6x + 5 < 0$ (b) $x^2 + 4x > -4$ (c) $2x + 24 \le x^2$ (d) $x^2 \le 2x + 1$

4. By substituting the axis of symmetry $x = -b/2a$ into the equation of the general quadratic $y = ax^2 + bx + c$, show that the vertex has y-coordinate $-\Delta/4a$. Use this formula to check the vertices that you obtained in question 2 above.

5. Evaluate the discriminant $\Delta = b^2 - 4ac$ for each quadratic, and hence establish how many times each function will intersect the x-axis:

(a) $y = x^2 + 2x - 3$ (b) $y = x^2 + 3x + 1$ (c) $y = 9x^2 - 6x + 1$ (d) $y = -2x^2 + 5x - 7$

6. Use the quadratic formula to find the roots α and β of each quadratic, and show that $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

(a) $3x^2 - 10x - 8 = 0$ (b) $x^2 - 2x - 4 = 0$ (c) $x^2 - 6x + 1 = 0$ (d) $-3x^2 + 5x + 2 = 0$

DEVELOPMENT

7. Use the quadratic formula to find the zeroes of each of the following quadratic functions. Hence write each function in the form $y = a(x - \alpha)(x - \beta)$.

(a) $y = x^2 - 6x + 4$ (b) $y = 3x^2 + 6x + 2$ (c) $y = -x^2 + 3x + 1$ (d) $y = -2x^2 - x + 1$

8. Solve the following pairs of equations simultaneously. Hence state how many times the parabola and the line intersect.

(a) $y = x^2 - 4x + 3$ and $y = x + 3$,
(b) $y = 2x^2 + 7x - 4$ and $y = 3x - 6$,
(c) $y = -x^2 + x - 3$ and $y = 2x + 1$,
(d) $y = -2x^2 + 5x - 1$ and $y = 3 - x$.

9. The interval PQ has length p, and the point A lies between the points P and Q. Find PA when $PQ \times QA = PA^2$.

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10. Find the derivative $f'(x)$ of the general quadratic $f(x) = ax^2 + bx + c$, and hence show that the derivative is zero when $x = -b/2a$. Explain how this relates to the axis of symmetry.

11. (a) Expand $f(x) = (x-h)^2 + k$ and show that $\Delta = -4k$. Then use the quadratic formulae to show that the axis is $x = h$ (as expected) and to find an expression for the zeroes.

(b) Expand $f(x) = (x-\alpha)(x-\beta)$ and show that $\Delta = (\alpha-\beta)^2$. Hence show that the zeroes are $x = \alpha$ and $x = \beta$ (as expected) and that the vertex is $\left(\dfrac{\alpha+\beta}{2}, -\left(\dfrac{\alpha-\beta}{2}\right)^2\right)$.

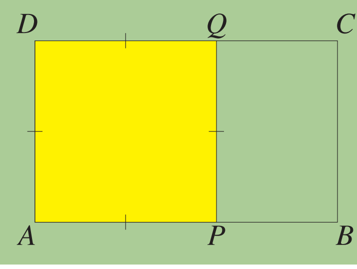
12. (a) Find the axis of symmetry and the vertex of $y = x^2 + bx + c$. (b) Find the zeroes, and then find the difference between them. (c) What condition on the constants b and c must be satisfied for the difference to be exactly 1? (d) Hence show that the family of such quadratics is the family of parabolas whose vertices are on the line $y = -\frac{1}{4}$.

EXTENSION

13. [The golden mean] Sketch $y = x^2 - x - 1$, showing the vertex and all intercepts.

(a) If $\alpha = \frac{1}{2}(\sqrt{5}+1)$, show that: (i) $\alpha^2 = \alpha + 1$

(ii) $\dfrac{1}{\alpha} = \alpha - 1$ (iii) $\alpha^6 = 8\alpha + 5$



(b) $ABCD$ is a rectangle whose length and breadth are in the ratio $\alpha : 1$. It is divided into a square $APQD$ and a second rectangle $PBCQ$, as shown. Show that the length and breadth of rectangle $PBCQ$ are also in the ratio $\alpha : 1$.

8 D Equations Reducible to Quadratics

There are many equations, including many trigonometric equations, which can be solved by using substitutions that reduce them to quadratic equations. For example, the degree 4 equation $x^4 - 13x^2 + 36 = 0$ becomes a quadratic equation with the substitution $u = x^2$. Substitution can also help to determine the graph if the function is reducible to a quadratic.

WORKED EXERCISE: By making substitutions that will reduce them to quadratic equations, solve:

(a) $x^4 - 13x^2 + 36 = 0$ (b) $2\cos^2 x - 3\cos x + 1 = 0$, for $0° \le x \le 360°$

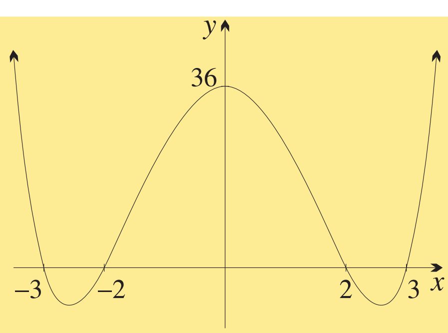
SOLUTION:

(a) Let $u = x^2$.
Then $u^2 - 13u + 36 = 0$
$(u-9)(u-4) = 0$
$u = 9$ or $u = 4$.
So $x^2 = 9$ or $x^2 = 4$
$x = 3, -3, 2$ or -2.

(b) Let $u = \cos x$.
Then $2u^2 - 3u + 1 = 0$
$(2u-1)(u-1) = 0$
$u = \frac{1}{2}$ or $u = 1$.
So $\cos x = \frac{1}{2}$ or $\cos x = 1$
$x = 0°, 60°, 300°$ or $360°$.

WORKED EXERCISE: Using only factorisation, sketch $y = x^4 - 13x^2 + 36$.

SOLUTION: From the previous example, the zeroes are 3, −3, 2 and −2, so $y = (x-3)(x+3)(x-2)(x+2)$. The y-intercept is 36.



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Exercise 8D

1. Solve the following equations for real values of x by reducing them to quadratic equations:

(a) $x^4 - 10x^2 + 9 = 0$

(b) $x^4 + 100 = 29x^2$

(c) $3x^4 - 10x^2 + 8 = 0$

(d) $x^6 - 9x^3 + 8 = 0$

(e) $x^3 + \dfrac{27}{x^3} = 28$

(f) $16x^2 + 16x^{-2} = 257$

(g) $(x^2 - x)^2 - 18(x^2 - x) + 72 = 0$

(h) $(x^2 - 4x) + 8 = \dfrac{48}{x^2 - 4x}$

(i) $3^{2x} - 12 \times 3^x + 27 = 0$

(j) $4^x - 12 \times 2^x + 32 = 0$

DEVELOPMENT

2. By making suitable substitutions, solve the following for $0^\circ \le x \le 360^\circ$:

(a) $2\sin^2 x - 3\sin x + 1 = 0$

(b) $2\sin^2 x = 3(\cos x + 1)$

(c) $\sec^2 x + 2\tan x = 0$

(d) $\cot^2 x = \operatorname{cosec} x + 1$

3. Solve the following simultaneous equations:

(a) $x^2 + y^2 = 10$ and $x + 2y = 7$

(b) $x^2 + y^2 - 2y = 7$ and $x - y = 3$

(c) $x^2 + y^2 - 2x + 6y - 35 = 0$ and $2x + 3y = 5$

4. Solve the following equations for real values of x by reducing them to quadratic equations:

(a) $2\left(x + \dfrac{1}{x}\right)^2 + \left(x + \dfrac{1}{x}\right) - 15 = 0$

(b) $x(x + 1)(x + 2)(x + 3) = 35$
[HINT: Expand $(x + 1)(x + 2)$ and $x(x + 3)$ and let $u = x^2 + 3x$.]

(c) $(x + 1)(x + 2)(x + 3)(x + 4) = 18 + 5(x^2 + 5x)$

5. Solve for x. Each solution must be checked in the original equation.

(a) $3x - \sqrt{x} = 2$

(b) $x + 2\sqrt{x + 1} = 7$

(c) $\sqrt{5 + x} + \sqrt{x} = 5$

(d) $\sqrt{x + 5} + \sqrt{x - 2} = \sqrt{5x - 6}$

6. Solve for x (the second will need the change of base law):

(a) $\dfrac{x+5}{x-5} - \dfrac{x-6}{x+6} = \dfrac{x+4}{x-4} - \dfrac{x-7}{x+7}$

(b) $2\log_5 x - 9\log_x 5 = 3$

7. (a) Solve for x and y simultaneously $\dfrac{p^2}{x} + \dfrac{q^2}{y} = \dfrac{(p+q)^2}{r}$ and $x + y = r$.

(b) Hence solve simultaneously $\dfrac{9}{x} + \dfrac{16}{y} = \dfrac{49}{2}$ and $x + y = 2$.

8. (a) Sketch the following functions, clearly indicating all x- and y-intercepts:
(i) $y = x^4 - 10x^2 + 9$ (ii) $y = 2x^4 - 11x^2 + 12$ (iii) $y = (x^2 - 4x)^2 - (x^2 - 4x) - 6$

(b) Use the graphs drawn in part (a) to solve the following inequations:
(i) $x^4 - 10x^2 + 9 \ge 0$ (ii) $2(x^4 + 6) \le 11x^2$ (iii) $x^4 - 8x^3 + 15x^2 + 4x - 6 > 0$

EXTENSION

9. Quartic equations of the form $Ax^4 + Bx^3 + Cx^2 + Bx + A = 0$, $A \ne 0$, are also reducible to quadratics using the substitution $u = x + 1/x$ and grouping terms appropriately.

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(a) Copy and complete:
$$\begin{aligned} \text{'}\, x^4 - 5x^3 + 8x^2 - 5x + 1 &= 0 \\ x^2\left(x^2 - 5x + 8 - \frac{5}{x} + \frac{1}{x^2}\right) &= 0 \\ x^2\left(x^2 + 2 + \frac{1}{x^2} - 5x - \frac{5}{x} + 6\right) &= 0. \end{aligned}$$
Let $u = x + \dfrac{1}{x}$, then $\quad x^2(u^2 - 5u + 6) = 0.$'

(b) Solve for x: (i) $x^4 + 3x^3 - 8x^2 + 3x + 1 = 0$ (ii) $3x^4 - 10x^3 + 13x^2 - 10x + 3 = 0$

8 E Problems on Maximisation and Minimisation

We come now to an entirely new type of problem, which involves finding the maximum or minimum value of a function, and the value of x for which it occurs. This section will only be able to deal with quadratic functions, but in the next chapter, the calculus will be used to deal with far more general functions.

There are, as usual, three approaches to maximising a quadratic — completing the square, using the formula for the axis of symmetry, and factorisation. While completing the square may sometimes seem a little complicated, it is worth repeating that this approach is the real foundation of work on quadratics, and will repay study.

Finding the Maximum or Minimum By Completing the Square: A square like $(x-6)^2$ can never be negative, and it reaches its minimum value of zero when $x = 6$. This is the key observation that allows us to deal with any quadratic whose square has been completed. Consider, for example,

$$(x-6)^2 + 5 \qquad \text{and} \qquad -(x-6)^2 + 7.$$

The first has a minimum of 5 when $x = 6$, and the second has a maximum of 7 when $x = 6$. Hence the general method of approaching the maximum or minimum values of a quadratic is:

11 MAXIMISATION AND MINIMISATION BY COMPLETING THE SQUARE: Complete the square, then use the fact that a square can never be negative to read off the maximum or minimum and the value of x for which it occurs.

WORKED EXERCISE: Find the maximum or minimum values of these quadratic functions, and the values of x for which they occur:

(a) $y = x^2 - 4x + 7$ (b) $y = 3 - 8x - x^2$

SOLUTION:

(a) Completing the square,
$$\begin{aligned} y &= x^2 - 4x + 7 \\ y &= (x^2 - 4x + 4) - 4 + 7 \\ y &= (x-2)^2 + 3. \end{aligned}$$
Now $(x-2)^2$ can never be negative, and $(x-2)^2$ is zero when $x = 2$, so y has a minimum of 3 when $x = 2$.

(b) Completing the square,
$$\begin{aligned} y &= 3 - 8x - x^2 \\ y &= -\left((x^2 + 8x + 16) - 16 - 3\right) \end{aligned}$$

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$$y = -(x+4)^2 + 19.$$

Now $-(x+4)^2$ can never be positive, and $-(x+4)^2$ is zero when $x = -4$, so y has a maximum of 19 when $x = -4$.

Maximisation and Minimisation Using the Axis of Symmetry: The maximum or minimum of a quadratic must occur at the vertex. When $a > 0$, the graph is concave up and so must have a minimum, while if $a < 0$, it is concave down and so must have a maximum. This gives an alternative approach using the formula for the axis of symmetry.

12 MAXIMISATION AND MINIMISATION USING THE AXIS OF SYMMETRY:

1. Find the axis of symmetry and substitute it to find the vertex.
2. The sign of a distinguishes between maximum and minimum.

WORKED EXERCISE: Repeat the previous worked example using the formula for the axis of symmetry.

SOLUTION:

(a) For $y = x^2 - 4x + 7$, the axis of symmetry is $x = 2$.
When $x = 2$, $y = 4 - 8 + 7 = 3$.
Since $a > 0$, the curve is concave up,
so there is a minimum of 3 when $x = 2$.

(b) For $y = 3 - 8x - x^2$, the axis of symmetry is $x = -4$.
When $x = -4$, $y = 3 + 32 - 16 = 19$.
Since $a < 0$, the curve is concave down,
so there is a maximum of 19 when $x = -4$.

Maximisation and Minimisation Using Factorisation: The axis of symmetry is the arithmetic mean of the zeroes, so if the quadratic can be factored (or is already factored), the axis of symmetry is easily found and substituted. As before, the sign of a will distinguish between maximum and minimum. An example of this approach is given in the problem below.

Solving Problems on Maxima and Minima: When a maximisation problem is presented in words rather than symbols, great care needs to be taken when setting up the function to be maximised. Two variables will need to be introduced — one variable (usually called y) will be the quantity to be maximised, the other (usually called x) will be the quantity that can be changed.

13 PROBLEMS ON MAXIMA AND MINIMA: After drawing a picture:

1. If no variables have been named, introduce two variables:
   ‘Let y (or whatever) be the variable to be maximised or minimised.
   Let x (or whatever) be the variable than can be changed.’
2. Express y as a function of x.
3. Use an acceptable method to find the maximum or minimum value of y, and the value of x for which it occurs.
4. Write a careful conclusion.

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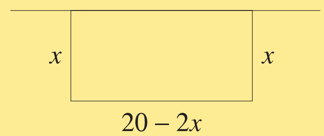
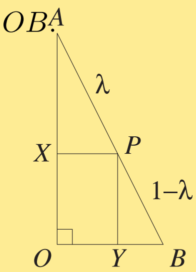
WORKED EXERCISE: Farmer Brown builds a rectangular chookyard using an existing wall as one fence. If she has 20 metres of fencing, find the maximum area of the chookyard and the length of the fence parallel to the wall.

SOLUTION: Let x be the length in metres perpendicular to the wall.
Let A be the area of the chookyard.
The length parallel to the wall is $20 - 2x$ metres,
so $A = x(20 - 2x)$.
Since the zeroes are 0 and 10, the axis is $x = 5$,
and when $x = 5$, $A = 50$.
Hence the maximum area is 50 square metres,
and occurs when the fence parallel to the wall is 10 metres long.



WORKED EXERCISE: [A subtle choice of variables to prove a significant result]
The point P lies on the hypotenuse AB of a right triangle $\triangle OAB$. The points X and Y are the feet of the perpendiculars from P to the sides OA and OB respectively. Show that the rectangle $OXPY$ has a maximum area equal to half the area of the triangle OAB when P is the midpoint of the hypotenuse AB.

SOLUTION: Let P divide AB in the ratio $\lambda : (1 - \lambda)$. Let $a = OA$ and $b = OB$.
Then by similar triangles, X divides AO in the ratio $\lambda : (1 - \lambda)$,
and Y divides OB in the ratio $\lambda : (1 - \lambda)$,
so $XO = (1 - \lambda)a$ and $OY = \lambda b$, and area of $OXPY = ab\lambda(1 - \lambda)$.
The zeroes of this quadratic are $\lambda = 0$ and $\lambda = 1$, and it is upside down,
so the area is maximum when $\lambda = \frac{1}{2}$ and P is the midpoint of AB,
and then $OXPY$ has area $\frac{1}{4}ab$, which is half the area of $\triangle OAB$.



Exercise 8E

1. (a) Complete the square to find the minimum value of each quadratic function:
(i) $y = x^2 - 4$ (ii) $y = x^2 - 10x + 16$ (iii) $y = x^2 - 5x + 6$ (iv) $y = 2x^2 + 5x - 3$

(b) Using the formula for the axis of symmetry, $x = -\dfrac{b}{2a}$, find the minimum value of each of the following quadratic functions:
(i) $y = x^2 - 2x + 5$ (ii) $y = 2x^2 + 4x + 5$ (iii) $y = x^2 - 6x + 7$ (iv) $y = 4x^2 - 2x + 3$

(c) Factor each of the following quadratic functions in order to find its zeroes. Sketch a graph of each function, clearly indicating its minimum value.
(i) $y = x^2 - 2x - 35$ (ii) $y = 6x^2 - 13x + 6$

2. (a) Complete the square in each function to find the maximum value:
(i) $y = 9 - x^2$ (ii) $y = 6 + x - x^2$ (iii) $y = 8 - 2x - x^2$ (iv) $y = 5x - 2x^2 - 3$

(b) Using the formula for the axis of symmetry, $x = -\dfrac{b}{2a}$, find the maximum value of each of the following quadratic functions:
(i) $y = -x^2 - 4x - 5$ (ii) $y = -3x^2 + 3x - 2$ (iii) $y = 4 + x - x^2$ (iv) $y = 3x - 2x^2 + 1$

(c) Factor each of the following quadratic functions in order to find its zeroes. Sketch a graph of each function, clearly indicating its maximum value.
(i) $y = 3 + 2x - x^2$ (ii) $y = 13x - 10x^2 - 4$

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3. Two numbers have a sum of 3.
(a) Let the numbers be x and $3-x$, and show that their product is $P = 3x - x^2$.
(b) Find the value of x for which P will be a maximum, and hence find the maximum value of P.

4. Two numbers have a sum of 30. Using the method of the previous question, find the numbers if their product is a maximum.

5. Two numbers have a sum of 6.
(a) Let the two numbers be x and $6-x$, and show that the sum of the squares of the two numbers is $S = 2x^2 - 12x + 36$.
(b) Find the value of x for which S is a minimum, and hence find the least value of S.

6. A rectangle has a perimeter of 16 metres. Let x be the length of one side, and find a formula for the area A in terms of x. Hence find the maximum value of A.

7. A stone is thrown upwards so that at any time t seconds after throwing, the height of the stone is $h = 100 + 10t - 5t^2$ metres. Find the maximum height reached.

8. A manufacturer finds that the cost $C(x)$, in thousands of dollars, of manufacturing his product is given by $C(x) = 2x^2 - 8x + 15$, where x is the number of machines operating. Find how many machines he should operate in order to minimise the cost of production, and hence the minimum cost of production.

DEVELOPMENT

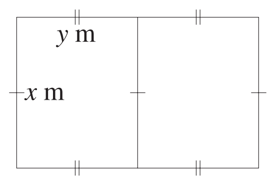
9. (a) A rectangle has a perimeter of 64 cm. If the length of the rectangle is x and its width is y, find an expression for the square of the length of the diagonal in terms of x.
(b) Find the dimensions of the rectangle if the square of the length of the diagonal is a minimum.

10. $PQRS$ is a square of side length 5 cm. A and B are points on the sides PQ and SP of the square respectively such that $PA = BP = x$.
(a) Show that the area of the quadrilateral $BARS$ is given by $\frac{1}{2}(25 + 5x - x^2)$.
(b) Hence find the maximum area of the quadrilateral $BARS$.

11. A dairy farmer has 4 km of fencing to enclose a rectangular paddock. There is to be a gate of length 15 metres on each of the shortest sides of the paddock. The gates require no fencing.
(a) If she uses x metres of fencing on each of the longer sides, and y metres of fencing on each of the shorter sides, find an expression for the area enclosed in terms of x only.
(b) Hence find the maximum area that the dairy farmer can enclose.

12. A piece of wire of length 80 cm is to be cut into two sections. One section is to be bent into a square, and the other into a rectangle 4 times as long as it is wide.
(a) Let x be the side length of the square and y be the width of the rectangle. Write a formula connecting x and y and show that if A is the sum of the areas of the square and rectangle, then $A = \frac{41}{4}y^2 - 100y + 400$.
(b) Find the lengths of both sections of wire if A is to be a minimum.

13. 1600 metres of fencing is to be used to enclose a rectangular area and to divide it into two equal areas as shown.
(a) Using the pronumerals given, show that the combined enclosed area A is given by $A = 800x - \frac{3}{2}x^2$.
(b) Hence find the values of x and y for which the area enclosed is greatest.



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14. A Tasmanian orchardist notices that an apple tree will produce 300 apples per year if 16 trees are planted in every standard-sized field. For every additional tree planted in the standard-sized field, she finds that the yield per tree decreases by 10 apples per year.

(a) If she plants an additional x trees in every standard-sized field, show that the total number of apples produced will be $N = -10x^2 + 140x + 4800$.

(b) How many trees should be planted in each field in order to maximise the number of apples that are produced?

15. A string 72 cm long is to be cut into two pieces. One piece is used to form a circle and the other a square. What should be the perimeter of the square in order to minimise the sum of the two areas.

16. A farmer with m dollars to spend is constructing a rectangular paddock $PQRS$. The side PQ runs along a river and costs n dollars per metre to fence. The remaining three sides of the paddock cost r dollars per metre to fence. Find in terms of m, n, and r the lengths of the sides of the paddock in order to maximise its area.

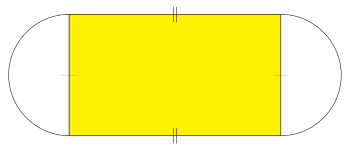
17. The total cost of producing x items per day is $\frac{1}{3}x^2 + 45x + 27$ dollars, and the price per item at which each may be sold is $60 - \frac{1}{2}x$ dollars. Find an expression for the daily profit, and hence find the maximum possible profit.

18. Suppose that the cost of producing x items per hour is given by $C(x)$ where $C(x) = x^2 + 10$, and the number of items sold per hour at a price of p dollars per item is $x = 16 - p$.
(a) Find in terms of x the revenue gained from the sales. (b) Hence show that the profit achieved per hour is given by $-2x^2 + 16x - 10$. (c) Find the number of items that should be produced each hour in order to maximise the profit. (d) Find the maximum profit.

19. (a) Find where the graphs of the functions $y = x(x-4)$ and $y = x(5-x)$ intersect, and carefully draw graphs of both functions on the same number plane.

(b) P is a point on the the graph of the function $y = x(x-4)$ and Q is a point on the graph of the function $y = x(5-x)$. P and Q have the same x-coordinate, where $0 \le x \le \frac{9}{2}$. Find an expression for the length of PQ and hence the maximum length of PQ.

20. A running track is 1000 metres long. It is designed using two sides of a rectangle and the perimeter of two semicircles as shown. The shaded rectangular section is to be used for field events. Find the dimensions of this section so as to maximise its area.



21. Highway A and Highway B intersect at right angles. A car on Highway A is presently 80 km from the intersection and is travelling towards the intersection at 50 km per hour. A car on Highway B is presently 70 km from the intersection and is travelling towards the intersection at 45 km per hour. (a) Find an expression for the square of the distance between the two cars if they continue in this manner for h hours. (b) If the cars can continue through the intersection and remain on the same highways, in how many minutes will the distance between them be a minimum?

22. The point $P(x, y)$ lies on the curve $y = 3x^2$. Find the coordinates of P so that the distance from P to the line $y = 2x - 1$ is a minimum.

23. A piece of string of length ℓ is bent to form the sector of a circle of radius r. Show that the area of the sector is maximised when $r = \frac{1}{4}\ell$.

24. Prove that the rectangle of greatest area that can be inscribed in a circle is a square. [HINT: Recall that the maximum of A occurs when the maximum of A^2 occurs.]

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25. The sum of the radii of two circles remains constant. Prove that the sum of the areas of the circles is least when the circles are congruent. [HINT: Let the radii be r and $k - r$, where k is a constant.]

26. OAB is a triangle in which $OA \perp OB$. OA and OB have lengths of 60 cm and 80 cm respectively. A rectangle is inscribed inside the triangle so that one of its sides lies along the base OA of the triangle. (a) By using similar triangles find the size of the rectangle of maximum area that may be inscribed in the triangle. (b) Repeat the question using a method similar to that in the second worked example.

27. A rectangle is inscribed in an isosceles triangle with one of the sides of the rectangle on the base of the triangle. Prove that the rectangle of greatest area occupies half the area of the triangle.

EXTENSION

28. Give a complete proof that the largest triangle that can be inscribed in a circle is an equilateral triangle.

8 F The Theory of the Discriminant

In Section 8C, we established that the zeroes of the general quadratic function $y = ax^2 + bx + c$ are

$$x = \frac{-b + \sqrt{\Delta}}{2a} \text{ or } \frac{-b - \sqrt{\Delta}}{2a}, \quad \text{where } \Delta = b^2 - 4ac.$$

In this section, we develop the theory of the discriminant Δ a little further, because it is one of the keys to understanding the behaviour of a quadratic function.

The Discriminant Discriminates: At first glance, one would expect the formula above to mean that every quadratic has two zeroes. The square root in the formula, however, makes the situation more complicated, because negative numbers have no square roots, zero has just one square root, and only positive numbers have two square roots. This means that the number of zeroes depends on the sign of the discriminant.

14 THE DISCRIMINANT AND THE NUMBER OF ZEROES:

If $\Delta > 0$, there are two zeroes, $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ and $x = \dfrac{-b - \sqrt{\Delta}}{2a}$.

If $\Delta = 0$, there is only one zero, $x = -\dfrac{b}{2a}$.

If $\Delta < 0$, there are no zeroes.

Unreal Zeroes and Double Zeroes: When $\Delta < 0$, we will sometimes say that *there are two unreal zeroes* (meaning two zeroes whose values are not real numbers), rather than saying that there are no zeroes. When $\Delta = 0$, it's often appropriate to think of the situation as two zeroes coinciding, and we say that *there are two equal zeroes*, or that the zero is a *double zero*. This adjustment of the language allows us to say that every quadratic has two zeroes, and the question then is whether those zeroes are real or unreal, and whether they are equal or distinct.

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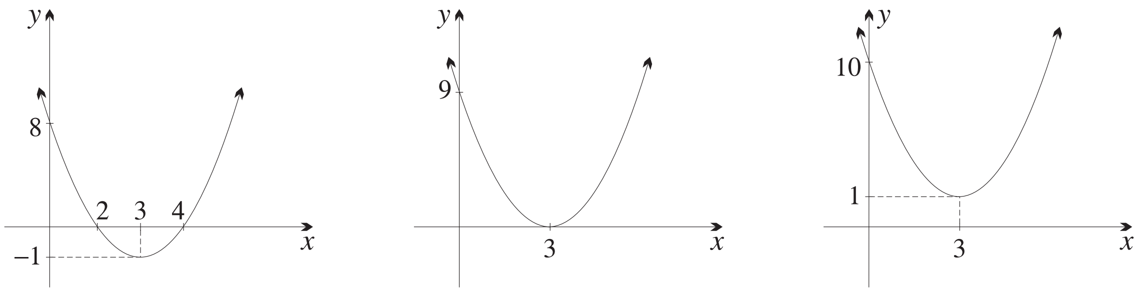
15 THE LANGUAGE OF DOUBLE ZEROES AND UNREAL ZEROES:

If $\Delta > 0$, there are two distinct real zeroes.

If $\Delta = 0$, there is one real double zero (or two equal zeroes).

If $\Delta < 0$, there are no zeroes (or two distinct unreal zeroes).



$$y = x^2 - 6x + 8 = (x-3)^2 - 1, \quad \Delta = 6^2 - 4 \times 8 = 4$$

$$y = x^2 - 6x + 9 = (x-3)^2, \quad \Delta = 6^2 - 4 \times 9 = 0$$

$$y = x^2 - 6x + 10 = (x-3)^2 + 1, \quad \Delta = 6^2 - 4 \times 10 = -4$$

The three graphs above all have axis of symmetry $x = 3$ and differ only in their constant terms. The first has two real and distinct zeroes, 2 and 4. In the second, the parabola has risen so that the two zeroes coincide to give the one double zero, $x = 3$. In the third, the parabola has risen further so that there are no longer any zeroes (or as we shall sometimes say, there are two unreal zeroes).

Quadratics that are Perfect Squares: The middle graph above is an example of a quadratic that is a *perfect square*:

$$x^2 - 6x + 9 = (x-3)^2 \qquad \text{and} \qquad \Delta = 6^2 - 4 \times 9 = 0.$$

In general, when $\Delta = 0$ and the two zeroes coincide, the quadratic meets the x-axis in only one point, where it is tangent, and the quadratic can be expressed as a multiple of a perfect square.

16 THE DISCRIMINANT AND PERFECT SQUARES:

When $\Delta = 0$, the quadratic is a multiple of a perfect square, $y = a(x - \alpha)^2$, and the x-axis is a tangent to the parabola at the double zero.

Are the Zeroes Rational or Irrational: Suppose now that all the three coefficients in $y = ax^2 + bx + c$ are rational numbers. Then because we need to take the square root of Δ, the zeroes will also be rational numbers if Δ is square, otherwise the zeroes will involve a surd and be irrational. So the discriminant allows another distinction to be made about the zeroes:

17 THE DISCRIMINANT AND RATIONAL ZEROES: Suppose that a, b and c are rational.

If Δ is a square, then the zeroes are rational.

If Δ is positive but not a square, then the zeroes are irrational.

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WORKED EXERCISE: Use the discriminant to describe the zeroes of:

(a) $y = 5x^2 - 2x - 3$ (b) $y = 3x^2 - 12x + 12$ (c) $y = 8 + 3x - 2x^2$

If the quadratic is a multiple of a perfect square, express it in this form.

SOLUTION:

(a) For $y = 5x^2 - 2x - 3$, $\Delta = 4 + 4 \times 15$
$= 64$,
so there are two real zeroes, and they are rational.

(b) For $y = 3x^2 - 12x + 12$, $\Delta = 144 - 4 \times 36$
$= 0$,
so there is one rational zero.
Also $y = 3(x-2)^2$.

(c) For $y = 8 + 3x - 2x^2$, $\Delta = 9 + 4 \times 16$
$= 73$,
so there are two real zeroes, and they are irrational.

WORKED EXERCISE: For what values of λ does $x^2 - (\lambda + 5)x + 9 = 0$ have:

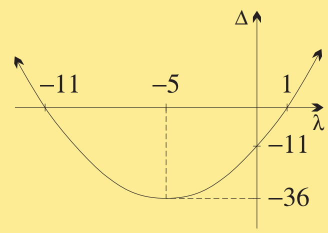
(a) equal roots, (b) no roots?

SOLUTION: Here $\Delta = (\lambda + 5)^2 - 36$.

(a) $\Delta = 0$ when $(\lambda + 5)^2 = 36$

$$\lambda + 5 = 6 \text{ or } \lambda + 5 = -6$$
$$\lambda = 1 \text{ or } -11,$$

so there are equal roots when $\lambda = 1$ and when $\lambda = -11$.



(b) There are no roots when Δ is negative,
so from the graph of Δ as a function of λ, there are no roots for $-11 < \lambda < 1$.

WORKED EXERCISE: [A harder example] Use the discriminant to find the equations of the lines through $A(3, -3)$ which are tangent to the rectangular hyperbola $y = 3/x$.

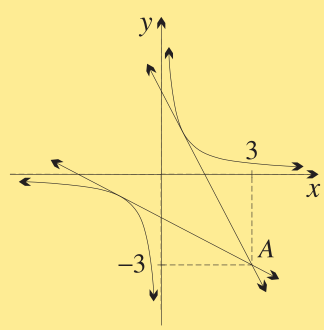
SOLUTION: The family of lines through $A(3, -3)$ is

$y + 3 = m(x - 3)$, where m is the gradient,

$y = mx - (3m + 3)$.

Solving this line simultaneously with the hyperbola,

$$mx - (3m + 3) = \frac{3}{x}$$
$$mx^2 - 3(m + 1)x - 3 = 0.$$



For the line to be a tangent, there must be a double zero.

So putting $\Delta = 0$, $9(m + 1)^2 + 12m = 0$

$\boxed{\div 3}$ $3m^2 + 6m + 3 + 4m = 0$

$$3m^2 + 10m + 3 = 0$$
$$(m + 3)(3m + 1) = 0$$
$$m = -3 \text{ or } -\tfrac{1}{3}.$$

So the lines are $y = -3x + 6$ and $y = -\frac{1}{3}x - 2$.

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Exercise 8F

1. Describe the roots of quadratic equations with rational coefficients that have the following discriminants. If the roots are real, state whether they are equal or unequal, rational or irrational.

(a) $\Delta = 7$
(b) $\Delta = -9$
(c) $\Delta = 0$
(d) $\Delta = 64$
(e) $\Delta = \frac{4}{9}$
(f) $\Delta = -0{\cdot}3$

2. Find the discriminant Δ of each equation. Hence state how many roots there are, and whether or not they are rational.

(a) $x^2 - 4x + 3 = 0$
(b) $2x^2 - 3x + 5 = 0$
(c) $x^2 - 6x + 9 = 0$
(d) $x^2 + 2x - 7 = 0$
(e) $6x^2 + 11x - 10 = 0$
(f) $9x^2 - 1 = 0$

NOTE: In questions 3–7, first find the discriminant Δ, then answer the question. If it is necessary to solve a quadratic inequality, this should be done by sketching a graph.

3. Find g, if the following quadratic functions have exactly one distinct zero:

(a) $y = x^2 + 10x + g$
(b) $y = gx^2 - 4x + 1$
(c) $y = 2x^2 - 3x + (g + 1)$
(d) $y = (g - 2)x^2 + 6x + 1$
(e) $y = gx^2 - gx + 1$
(f) $y = gx^2 + 7x + g$
(g) $y = 4x^2 + 4gx + (6g + 7)$
(h) $y = 9x^2 - 2(g + 1)x + 1$

4. Find the values of k for which the roots of the following equations are real numbers:

(a) $x^2 + 2x + k = 0$
(b) $kx^2 - 8x + 2 = 0$
(c) $3x^2 - 4x + (k + 1) = 0$
(d) $(2k - 1)x^2 - 5x + 2 = 0$
(e) $x^2 + kx + 4 = 0$
(f) $x^2 - 3kx + 9 = 0$
(g) $4x^2 - (6 + k)x + 1 = 0$
(h) $9x^2 + (k - 6)x + 1 = 0$

5. Find the values of ℓ for which the following quadratic functions have no real zeroes:

(a) $y = x^2 + \ell x + 4$
(b) $y = \ell x^2 + 6x + \ell$
(c) $y = x^2 + (\ell + 1)x + 4$
(d) $y = 9x^2 - 4(\ell - 1)x - \ell$
(e) $y = \ell x^2 - 4\ell x - (\ell - 5)$
(f) $y = (\ell - 3)x^2 + 2\ell x + (\ell + 2)$

6. (a) Show that the x-coordinates of the points of intersection of the circle $x^2 + y^2 = 4$ and the line $y = x + 1$ satisfy the equation $2x^2 + 2x - 3 = 0$.
(b) Evaluate the discriminant Δ and explain why this shows that there are two points of intersection.

7. Using the method outlined in the previous question, determine how many times the line and circle intersect in each case:

(a) $x^2 + y^2 = 9$, $y = 2 - x$
(b) $x^2 + y^2 = 1$, $y = x + 2$
(c) $x^2 + y^2 = 5$, $y = -2x + 5$
(d) $(x - 3)^2 + y^2 = 4$, $y = x - 4$

DEVELOPMENT

8. Find Δ for each equation. By writing Δ as a perfect square, show that each equation has rational roots for all rational values of m and n:

(a) $4x^2 + (m - 4)x - m = 0$
(b) $(m - 1)x^2 + mx + 1 = 0$
(c) $mx^2 + (2m + n)x + 2n = 0$
(d) $2mx^2 - (4m + 1)x + 2 = 0$
(e) $2(m - 2)x^2 + (6 - 7m)x + 6m = 0$
(f) $(4m + 1)x^2 - 2(m + 1)x + (1 - 2m) = 0$

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9. Prove that the roots of the following equations are real and distinct for all real values of λ. [HINT: Find Δ and write it in such a way that it is obviously positive.]

(a) $x^2 + \lambda x - 1 = 0$
(b) $3x^2 + 2\lambda x - 4 = 0$
(c) $\lambda x^2 - (\lambda + 4)x + 2 = 0$
(d) $x^2 + (\lambda + 1)x + (\lambda - 2) = 0$

NOTE: In the following questions you may need to rearrange the quadratic equation into the form $ax^2 + bx + c = 0$ before finding Δ.

10. Find the values of m for which the roots of the quadratic equation:

(a) $1 - 3x - mx^2 = 0$ are real and distinct,
(b) $2x^2 + 4x + 5 = 3x + m$ are real and equal,
(c) $x(x - 2m) = m - 2x - 3$ are unreal,
(d) $12m(x^2 - 2x) + 12(2x^2 + x) = 38m + 11$ are real.

11. Show that the roots of $(x - a)(x - b) = c^2$ are always real, where a, b and c are real.

12. (a) For what values of b is the line $y = x + b$ a tangent to the curve $y = 2x^2 - 7x + 4$?
(b) The line $2x + y + b = 0$ is a tangent to $y = 2x^2 + 3x + 1$. Find the value of b.
(c) The line $y = mx + 4$ is a tangent to $y = 3x^2 + 5x + 7$. Find the value of m.

13. Find the equation of the tangent to the parabola $y = x^2 - 5x - 3$ that is parallel to the line $3x - y - 7 = 0$.

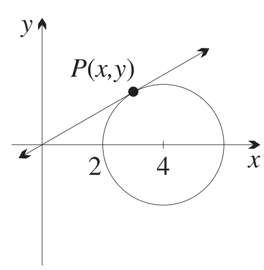
14. Find the gradients of the lines that pass through the point $(1, 7)$ and are tangent to the parabola $y = (2 - x)(1 + 3x)$.

15. The line $y = 4x - 7$ is tangent to a parabola that has a y-intercept of -3 and the line $x = \frac{1}{2}$ as its axis of symmetry. Find the equation of the parabola.

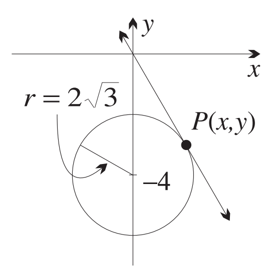
16. How many horizontal tangents may be drawn to each of the following cubic functions? [HINT: You will need to differentiate and set the derivative equal to zero, then use the discriminant to find how many solutions this equation has.]

(a) $y = x^3 + 5x^2 - 8x + 7$ (b) $y = 3x^3 - 3x^2 + x - 1$ (c) $y = \frac{1}{3}x^3 + x^2 + 5x + 11$

17. If in $y = ax^2 + bx + c$ we find $ac < 0$, explain why the graph of the parabola must have two x-intercepts.



18. (a) Write down the equation of the circle in the diagram.
(b) Write down the equation of the line through the origin with gradient m.
(c) By solving the circle and the line simultaneously, show that the x-coordinate of the point P in the diagram satisfies the equation $(m^2 + 1)x^2 - 8x + 12 = 0$.
(d) Use the theory of the discriminant to find the value of m.
(e) Hence or otherwise find the coordinates of P.



19. Use the method outlined in the previous question to find the gradient of the line in the diagram and hence the coordinates of the point P.

20. Use the discriminant to find the gradients of the lines that pass through the point $(7, 1)$ and are tangent to the circle $x^2 + y^2 = 25$.

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21. (a) Find a if $y = 3(a+2)x^2 + 6ax + (4-3a)$ has no zeroes.
(b) Find b if $y = (2b-3)x^2 + (5b-1)x + (3b+2)$ has two distinct zeroes.
(c) Find g if $y = (g+1)x^2 - (3-5g)x - (g-12)$ has one zero.
(d) Find k if $(3k-2)x^2 + 2(k+6)x + (k-4) = 0$ has two distinct roots.

22. (a) Show that the quadratic equation $(a^2+b^2)x^2 + 2b(a+c)x + (b^2+c^2) = 0$, where a, b and c are real constants, has real roots when $(b^2-ac)^2 \le 0$.
(b) State this condition in a simpler form.

EXTENSION

23. Show that $\dfrac{x^2-mn}{2x-m-n}$ takes no real values between m and n. [HINT: Put $\dfrac{x^2-mn}{2x-m-n} = \lambda$ and find the discriminant.]

24. Show that the equation $(x-m)(x-n) + (x-n)(x-\ell) + (x-\ell)(x-m) = 0$ cannot have equal roots unless $m = n = \ell$.

8 G Definite and Indefinite Quadratics

The last section showed how the discriminant discriminated between quadratics with two, one or no zeroes. The distinction between quadratics with no zeroes and those with one zero or two is sufficiently important for special words to be used to describe them.

Definition: Let $f(x) = ax^2 + bx + c$ be a quadratic.

18

POSITIVE DEFINITE, NEGATIVE DEFINITE AND INDEFINITE QUADRATICS:
$f(x)$ is called *positive definite* if $f(x)$ is positive for all values of x,
and *negative definite* if $f(x)$ is negative for all values of x.
$f(x)$ is called *definite* if it positive definite or negative definite.
$f(x)$ is called *indefinite* if it is not definite.

These definitions may be clearer when expressed in terms of zeroes:

19

DEFINITE AND INDEFINITE AND ZEROES:
A quadratic is *definite* if it has no zeroes,
being *positive definite* if it is always positive,
and *negative definite* if it is always negative.
A quadratic is *indefinite* if it has at least one zero.

The word ‘definite’ means ‘we can be definite about the sign of $f(x)$ whatever the value of x’. An ‘indefinite’ quadratic takes different signs (or is zero at least once) for different values of x.

The Six Cases: There are three possibilities for Δ — negative, zero and positive — and two possibilities for a — positive and negative. This makes six possible cases altogether, and these cases are graphed below.

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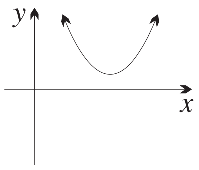
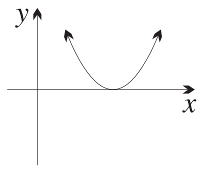
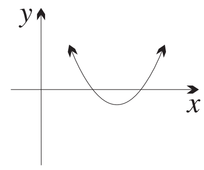


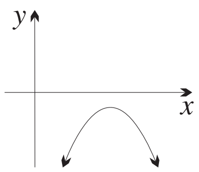
$\Delta < 0$ and $a > 0$,
positive definite



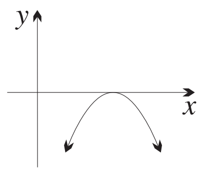
$\Delta = 0$ and $a > 0$,
indefinite



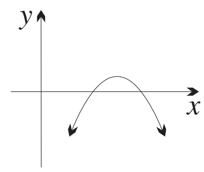
$\Delta > 0$ and $a > 0$,
indefinite



$\Delta < 0$ and $a < 0$,
negative definite



$\Delta = 0$ and $a < 0$,
indefinite



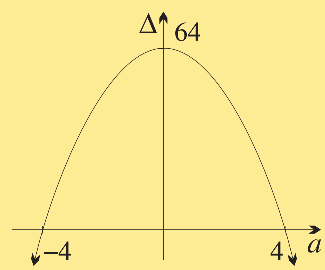
$\Delta > 0$ and $a < 0$,
indefinite

Definite Quadratics and Factorisation: If a quadratic is indefinite, then it can be factored, either as $a(x-\alpha)(x-\beta)$ if it has two zeroes α and β, or as $a(x-\alpha)^2$ if it has one double zero α. A definite quadratic, however, cannot be factored, because otherwise it would have zeroes.

20 INDEFINITE QUADRATICS AND FACTORING: A quadratic can be factored into real factors if and only if it is indefinite.

WORKED EXERCISE: For what values of a is $f(x) = ax^2 + 8x + a$ positive definite, negative definite and indefinite?

SOLUTION: Here $\Delta = 64 - 4a^2$
$= 4(16 - a^2)$,
so $\Delta \ge 0$ when $-4 \le a \le 4$,
and $\Delta < 0$ when $a < -4$ and when $a > 4$.
Hence $f(x)$ is indefinite for $-4 \le a \le 4$
(but $a \ne 0$, because when $a = 0$ it is not a quadratic),
and $f(x)$ is positive definite for $a > 4$,
and $f(x)$ is negative definite for $a < -4$.



Exercise 8G

1. Use a graph to solve the following quadratic inequations:

(a) $x^2 \ge x$
(b) $7 - x^2 > 0$
(c) $x^2 + 9 > 6x$
(d) $(x+1)^2 \le 34$
(e) $3x^2 + 5x - 2 \le 0$
(f) $-2x^2 + 13x \ge 15$
(g) $-5 > 4x(2 - x)$
(h) $x^2 + 4x + 5 \le 0$
(i) $-2x^2 - 3x - 3 < 0$

2. Evaluate the discriminant and look carefully at the coefficient of x^2 to determine whether the following functions are positive definite, negative definite or indefinite:

(a) $y = 2x^2 - 5x + 7$
(b) $y = x^2 - 4x + 4$
(c) $y = 5x - x^2 - 9$
(d) $y = -x^2 + 7x - 3$
(e) $y = 25 - 20x + 4x^2$
(f) $y = 3x + 2x^2 + 11$

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3. Find the discriminant as a function of k, and hence find the values of k for which the following expressions are: (i) positive definite, (ii) indefinite.

(a) $2x^2 - 5x + 4k$
(b) $2x^2 - kx + 8$
(c) $3x^2 + (12 - k)x + 12$
(d) $x^2 - 2(k - 3)x + (k - 1)$

4. Find the discriminant as a function of m, and hence find the values of m for which these expressions are: (iii) negative definite, (iv) indefinite.

(a) $-x^2 + mx - 4$
(b) $-2x^2 + 3x - m$
(c) $-x^2 + (m - 2)x - 25$
(d) $-4x^2 + 4(m + 1)x - (4m + 1)$

5. Find the values of ℓ that will make each quadratic below a perfect square:

(a) $x^2 - 2\ell x + 16$
(b) $2\ell x^2 + 2\ell x + 1$
(c) $(5\ell - 1)x^2 - 4x + (2\ell - 1)$
(d) $(4\ell + 1)x^2 - 6\ell x + 4$

DEVELOPMENT

6. Show that the discriminant of $(3k - 5)x^2 + 2(4 - k)x + 4 = 0$ is $\Delta = 4(k^2 - 20k + 36)$, and hence find the values of k for which $(3k - 5)x^2 + 2(4 - k)x + 4 = 0$ is:

(a) positive definite, (b) negative definite, (c) indefinite.

7. (a) Sketch a graph of the function $y = \dfrac{(x - 1)(x + 2)}{x(x + 4)}$.

(b) Prove that the roots of the equation $kx(x + 4) = (x - 1)(x + 2)$ are always real.

(c) How can you establish this result from the graph you have drawn?

8. Find the domain of each of the following expressions:

(a) $\sqrt{x^2 - 5x + 1}$
(b) $\sqrt{2x - 3 - x^2}$
(c) $\sqrt{2x^2 - 9x + 4}$
(d) $\dfrac{x + 2}{\sqrt{2x - x^2}}$
(e) $\sqrt{6 + 5x - 4x^2}$
(f) $\dfrac{(x - 1)(2x + 3)}{\sqrt{x^2 - 9}}$

9. State in terms of a, b and c the conditions necessary for $ax^2 + 2bx + 3c$ to be:

(a) positive definite, (b) negative definite, (c) indefinite.

10. Sketch a possible graph of the quadratic function $y = ax^2 + bx + c$ if:

(a) $a > 0$, $b > 0$, $c > 0$ and $b^2 - 4ac > 0$
(b) $a < 0$, $c < 0$ and $b = 0$
(c) $a > 0$, $b < 0$, $c > 0$ and $b^2 = 4ac$
(d) $a > 0$, $b < 0$ and $b^2 - 4ac < 0$

11. State in terms of b and c the condition for the roots of $x^2 + 2bx + 3c = 0$ to be:

(a) equal,
(b) real and distinct,
(c) unreal,
(d) opposite in sign,
(e) distinct and positive,
(f) distinct and negative.

12. The expression $x^2 - xy - 2y^2 + x + 7y - 5$ can be treated as a function in x with y as an arbitrary constant. Show that it is positive definite when $1 < y < \frac{7}{3}$.

13. Find the range of values of x for which the equation in y, $2x^2 - 3xy + y^2 - 5x + 11 = 0$ will have real roots. Find also for what values of y the equation in x will have real roots.

14. The expression $3x^2 + 2xy - 8y^2 - 8x + 14y - 3$ can be treated as a function in x with y as an arbitrary constant or as a function in y with x as an arbitrary constant. Show that in either case the expression is indefinite. Factor the expression.

15. Find the values of λ for which $4a^2 - 10ab + 10b^2 + \lambda(3a^2 - 10ab + 3b^2)$ is a perfect square.

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EXTENSION

16. The equation $2x^2 + ax + (b+3) = 0$ has real roots. Find the minimum value of $a^2 + b^2$.

17. (a) Show that $f(x) = (x-5)^2 + (x+2)^2$ is positive definite, first by expanding and finding the discriminant and secondly by explaining directly why $f(x)$ must be positive for all values of x.

(b) Express $2x^2 + 4x + 10$ in the form $(x-r)^2 + (x-s)^2$.

(c) Find the discriminant of $f(x) = 2x^2 + 2bx + c$. Hence show that $f(x)$ can be expressed as a sum $(x-r)^2 + (x-s)^2$ of two distinct squares (that is, with $r \neq s$) if and only if $f(x)$ is positive definite.

8 H Sum and Product of Roots

Many problems on quadratics depend on the sum and product of the roots rather than on the roots themelves. For example, the axis of symmetry is found by taking the average of the zeroes, which is half the sum of the zeroes. The formulae for the sum and product of the roots are very straightforward, and do not involve the surds that often appear in the roots themselves.

Forming a Quadratic Equation with Given Roots: Suppose that we are asked to form a quadratic equation with roots α and β. The simplest such equation is

$$(x-\alpha)(x-\beta) = 0.$$

Expanding this out, $x^2 - (\alpha+\beta)x + \alpha\beta = 0$.

21 A QUADRATIC WITH GIVEN ROOTS α AND β: $(x-\alpha)(x-\beta) = 0$
OR $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$.

WORKED EXERCISE: Form quadratic equations with integer coefficients and roots:

(a) $3\frac{1}{2}$ and $-2\frac{1}{3}$, (b) $2+\sqrt{7}$ and $2-\sqrt{7}$.

SOLUTION:

(a) Such an equation is

$$\left(x - 3\tfrac{1}{2}\right)\left(x + 2\tfrac{1}{3}\right) = 0$$

$\boxed{\times 6}$ $(2x-7)(3x+7) = 0$

$$6x^2 - 7x - 49 = 0.$$

OR

Since $\alpha + \beta = \frac{7}{6}$ and $\alpha\beta = -\frac{49}{6}$, such an equation is

$$x^2 - \tfrac{7}{6}x - \tfrac{49}{6} = 0$$

$\boxed{\times 6}$ $6x^2 - 7x - 49 = 0.$

(b) Taking the sum, $\alpha + \beta = 4$,
and taking the product, $\alpha\beta = 4 - 7 = -3$ (difference of squares),
so such an equation is $x^2 - 4x - 3 = 0$.

Formulae for the Sum and Product of Roots: Let $ax^2 + bx + c = 0$ have roots α and β. Dividing through by a gives $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$, so by the previous result:

22 SUM AND PRODUCT OF ROOTS: $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.

These formulae can also be proven directly using the general equation for the roots — see the first question in the second group of the exercise below.

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WORKED EXERCISE: If α and β are the roots of the equation $2x^2 - 6x - 1 = 0$, find:
(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2\beta + \alpha\beta^2$ (d) $1/\alpha + 1/\beta$ (e) $\alpha^2 + \beta^2$

SOLUTION: Since $a = 2$, $b = -6$ and $c = -1$:

(a) $\alpha + \beta = 3$

(b) $\alpha\beta = -\frac{1}{2}$

(c) $$\begin{aligned}\alpha^2\beta + \alpha\beta^2 &= \alpha\beta(\alpha + \beta)\\ &= -1\tfrac{1}{2}\end{aligned}$$

(d) $$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} &= \frac{\beta + \alpha}{\alpha\beta}\\ &= -6\end{aligned}$$

(e) $$\begin{aligned}\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta\\ &= 9 + 1\\ &= 10\end{aligned}$$

Expressions Symmetric in α and β: Expressions in α and β like those in the worked exercise above are called *symmetric* in α and β, because if α and β are exchanged, the expression remains the same. With enough ingenuity, it should be possible to evaluate any expression symmetric in α and β by the sort of methods used here.

WORKED EXERCISE: Prove that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$. Then use this identity to find the difference $|\alpha - \beta|$ of the roots of the equation $x^2 - 9x + 2 = 0$.

SOLUTION: $$\begin{aligned}(\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2\\ &= \alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta\\ &= (\alpha + \beta)^2 - 4\alpha\beta\end{aligned}$$

In the given equation, $\alpha + \beta = 9$ and $\alpha\beta = 2$, so $(\alpha - \beta)^2 = 9^2 - 4 \times 2$
$= 73$,
so the difference of the roots is $|\alpha - \beta| = \sqrt{73}$.

WORKED EXERCISE: [A problem where a relation between the roots is known]
Find m, given that one of the roots of $x^2 + mx + 18 = 0$ is twice the other.

SOLUTION: Let the roots be α and 2α. [NOTE: This is the essential step here.]
Then using the product of roots, $\alpha \times 2\alpha = 18$
$\alpha = 3$ or -3.
Now using the sum of roots, $\alpha + 2\alpha = -m$
$m = -3\alpha$
$m = 9$ or -9.

Unreal Roots: A quadratic equation like $x^2 - 4x + 8 = 0$ has no roots, because its discriminant is $\Delta = -16$ which has no square roots. Nevertheless, the formulae for the sum and product of the roots give answers as usual:

$$\alpha + \beta = 4 \quad \text{and} \quad \alpha\beta = 8,$$

and the question is, what meaning do these answers have? Now blind use of the formula for the roots of a quadratic would give the following expressions for them:

$$\alpha = 2 + \sqrt{-4} \quad \text{and} \quad \beta = 2 - \sqrt{-4}\,.$$

If we calculate $\alpha + \beta$ and $\alpha\beta$ ignoring the fact that these expressions are meaningless:

$$\alpha + \beta = 2 + 2 = 4 \quad \text{and} \quad \alpha\beta = 2^2 - (-4) = 8 \quad \text{(difference of squares)}$$

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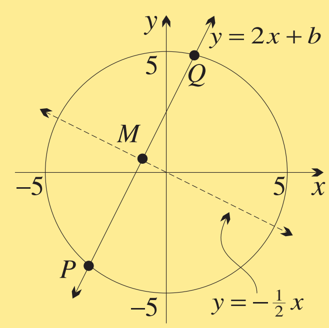
which are the values we obtained above. Considerations like these lead mathematicians to take seriously objects such as $\sqrt{-4}$. Since $\sqrt{-4}$ can't be a real number, such arithmetic requires an extension of the real number system. Square roots of negative numbers are called *imaginary numbers*, and sums like $2+\sqrt{-4}$ of real and imaginary numbers are called *complex numbers*.

Arithmetic and Geometric Means: If α and β are the roots of a given quadratic equation, then their arithmetic and geometric means are easily found, because they are simply half the sum of the roots and the square root of the product of the roots. This means that various problems in Euclidean and coordinate geometry can be solved far more easily.

WORKED EXERCISE: The line $y = 2x + b$ intersects the circle $x^2 + y^2 = 25$ at P and Q. Use the sum and product of roots to find the coordinates of the midpoint $M(X, Y)$ of PQ. Hence find the locus of M as b varies, and describe it geometrically.

SOLUTION: Solving the line and the circle simultaneously,

$$x^2 + (4x^2 + 4bx + b^2) = 25$$
$$5x^2 + 4bx + (b^2 - 25) = 0.$$



The x-coordinate of M is the arithmetic mean of the roots:

$$\begin{aligned} X &= \tfrac{1}{2}(\alpha+\beta) \\ &= \tfrac{1}{2} \times (-\tfrac{4}{5}b) \\ &= -\tfrac{2}{5}b, \end{aligned}$$

and substituting into the line, $M = \left(-\tfrac{2}{5}b, \tfrac{1}{5}b\right)$.

This point lies on the line $y = -\tfrac{1}{2}x$,

which is the diameter perpendicular to the family of lines $y = 2x + b$.

Exercise 8H

1. Use the formulae $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$ to write down the sum $\alpha + \beta$ and the product $\alpha\beta$ of the roots of $x^2 + 7x + 10 = 0$. Then solve $x^2 + 7x + 10 = 0$ by factoring, and check your results.

2. Repeat the previous question for these equations. In parts (b) and (c), use the formula to find the roots, and the difference of squares identity to find their product.
   (a) $3x^2 - 10x + 3 = 0$ (b) $x^2 + 4x + 1 = 0$ (c) $x^2 - x - 1 = 0$

3. If α and β are the roots of the following quadratic equations, write down the values of $\alpha + \beta$ and $\alpha\beta$ without solving the equations.
   (a) $x^2 - 2x + 5 = 0$
   (b) $x^2 + x - 6 = 0$
   (c) $x^2 + x = 0$
   (d) $2x^2 + 3x - 1 = 0$
   (e) $4 + 5x^2 = -5x$
   (f) $3x^2 + 2x = 4(x + 1)$
   (g) $x^2 - mx + n = 0$
   (h) $px^2 + qx - 3r = 0$
   (i) $ax(x - 1) = 3 - 4x$

4. A quadratic equation with roots α and β has the form $x^2 - (\alpha+\beta)x + \alpha\beta = 0$. Form a quadratic equation, with integral coefficients, whose roots are:
   (a) 1 and 3
   (b) -2 and 6
   (c) -1 and -4
   (d) $\tfrac{1}{2}$ and $\tfrac{3}{2}$
   (e) $2+\sqrt{3}$ and $2-\sqrt{3}$
   (f) $-1-\sqrt{5}$ and $-1+\sqrt{5}$

5. If α and β are the zeroes of $y = x^2 - 3x + 2$, without finding the zeroes, find the values of:
   (a) $\alpha + \beta$
   (b) $\alpha\beta$
   (c) $7\alpha + 7\beta$
   (d) $\alpha^2\beta + \alpha\beta^2$
   (e) $(\alpha+3)(\beta+3)$
   (f) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$
   (g) $\alpha^2 + \beta^2$
   (h) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

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6. If α and β are the roots of $2x^2 - 5x + 1 = 0$, without solving the equation, find the values of:

(a) $\alpha + \beta$
(b) $\alpha\beta$
(c) $(\alpha - 1)(\beta - 1)$
(d) $\alpha^{-1} + \beta^{-1}$
(e) $\alpha^3\beta^2 + \alpha^2\beta^3$
(f) $\dfrac{2}{\alpha} + \dfrac{2}{\beta}$
(g) $\alpha^2 + \beta^2$
(h) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

7. (a) Show that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$.

(b) If α and β are the roots of the following equations, without solving the equation, find the values of $\alpha + \beta$, $\alpha\beta$ and hence $(\alpha - \beta)^2$.

(i) $x^2 - 3x + 1 = 0$
(ii) $x^2 + 5x - 7 = 0$
(iii) $3x^2 - 7x + 2 = 0$

(c) Hence find the difference $|\alpha - \beta|$ for each equation.

8. If α and β are the roots of the equation $ax^2 + bx + c = 0$, show that $(\alpha - \beta)^2 = \dfrac{\Delta}{a^2}$.

DEVELOPMENT

9. Write down the formulae for the two zeroes α and β of $f(x) = ax^2 + bx + c$, assuming that $\alpha < \beta$. Hence prove directly that $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$ (you will need to use the difference of squares identity).

10. Without solving, find the arithmetic and geometric means of the roots of $3x^2 - 5x + 4 = 0$.

11. If α and β are the roots of the equation $3x^2 + 2x + 7 = 0$, find $\alpha + \beta$ and $\alpha\beta$. Hence form the equation with integer coefficients having roots:

(a) 2α and 2β
(b) $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$
(c) $\alpha + 2\beta$ and $\beta + 2\alpha$
(d) α^2 and β^2

12. Find the values of g for which the function $y = 2x^2 - (3g - 1)x + (2g - 5)$ has:

(a) one zero equal to 0 (let the zeroes be 0 and α),
(b) the sum of the zeroes equal to their product (put $\alpha + \beta = \alpha\beta$),
(c) the zeroes as reciprocals of one another (let the zeroes be α and $1/\alpha$),
(d) the zeroes equal in magnitude but opposite in sign (let the zeroes be α and $-\alpha$).

13. Given that α and β are the roots of $(2m - 1)x^2 + (1 + m)x + 1 = 0$, find m if:

(a) $\alpha = -\beta$
(b) $\alpha = \dfrac{1}{\beta}$
(c) $\alpha = 2$
(d) $\alpha + \beta = 2\alpha\beta$

14. Given that the roots of the quadratic equation $ax^2 + bx + c = 0$ are real, explain by inspection of the coefficients how one can determine:

(a) whether the roots have opposite signs,
(b) the sign of the roots if they both have the same sign,
(c) the sign of the numerically greater root if they have opposite signs.

15. If α and β are the zeroes of the function $y = 3x^2 - 5x - 4$, find the value of $\alpha^3 + \beta^3$.

16. Find the value of m if one root of the equation $x^2 + 6x + m = 0$ is double the other.

17. Find the value of ℓ if one zero of the function $y = x^2 - 2\ell x + (\ell + 3)$ is three times the other.

18. If the roots of the quadratic equation $x^2 - mx + n = 0$ differ by 1, without solving the equation, prove that $m^2 = 4n + 1$.

19. If one zero of the function $y = x^2 + mx + n$ is the square of the other, without finding the zeroes, prove that $m^3 = n(3m - n - 1)$.

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20. The line $y = 6x + 9$ crosses the parabola $y = x^2 + 1$ at A and B.
(a) Show that the x-coordinates of A and B satisfy the equation $x^2 - 6x - 8 = 0$.
(b) Without solving the equation, find the sum of the roots of the quadratic.
(c) Hence find the coordinates of the midpoint M of AB.

21. Using a method similar to that in the previous question, find the midpoints of the chords formed by:
(a) the line $x + y - 1 = 0$ and the circle $x^2 + y^2 = 13$,
(b) the line $y = x + 3$ and the parabola $y = (x - 2)^2$,
(c) the line $x + y + 4 = 0$ and the rectangular hyperbola $y = \dfrac{1}{x}$.

22. The line $3x - y + b = 0$ cuts the circle $x^2 + y^2 = 16$ at two points P and Q. Find the coordinates $M(X, Y)$ of the midpoint of PQ. Hence find the locus of M as b varies, and describe it geometrically.

23. (a) Expand $(\sqrt{\alpha} + \sqrt{\beta})^2$. (b) If α and β are the zeroes of the function $y = x^2 - 6x + 16$, without finding the zeroes, evaluate $\sqrt{\alpha} + \sqrt{\beta}$.

24. The line $x + y - 1 = 0$ intersects the circle $x^2 + y^2 = 13$ at $A(\alpha_1, \alpha_2)$ and $B(\beta_1, \beta_2)$. Without finding the coordinates of A and B, find the length of the chord AB. [HINT: Form a quadratic equation in x and evaluate $|\alpha_1 - \beta_1|$, and similarly find $|\alpha_2 - \beta_2|$.]

25. For what values of m are the roots of $x^2 + 2x + 3 = m(2x + 1)$ real and positive?

EXTENSION

26. If the equations $mx^2 + 2x + 1 = 0$ and $x^2 + 2x + m = 0$ have a common root, find the possible values of m and the value of the common root in each case.

27. If α and β are the roots of $x^2 = 5x - 8$, find $\sqrt[3]{\alpha} + \sqrt[3]{\beta}$ without finding the roots.

28. The distinct positive numbers α and β are roots of the quadratic equation $ax^2 + bx + c = 0$. Use the fact that the discriminant is positive to prove that their arithmetic mean is greater than their geometric mean.

8 I Quadratic Identities

This final section has three purposes. The first purpose is to prove some theorems about the coefficients of quadratics that have been tacitly assumed throughout the chapter. The second purpose is to develop some elegant methods of proving quadratic identities and finding coefficients in quadratics. The third purpose is to establish some very general geometrical ideas lying behind the algebra of quadratics.

Theorem — Three Values Determine a Quadratic: We shall prove that if two quadratics agree for at least three distinct values of x, then they agree for all values of x and their coefficients are equal.

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THEOREM: Suppose that the two expressions

$$f(x) = ax^2 + bx + c \qquad \text{and} \qquad g(x) = a_1x^2 + b_1x + c_1$$

take the same value for at least three distinct values of x. Then $f(x)$ and $g(x)$ are equal for all values of x, and the coefficients a_1, b_1 and c_1 are respectively equal to the coefficients a, b and c.

PROOF:

A. First we prove the following particular case:

'Suppose that the expression $Ax^2 + Bx + C$ is zero for the three distinct values $x = \alpha$, $x = \beta$ and $x = \gamma$. Then $A = B = C = 0$.'

Substituting,
$$A\alpha^2 + B\alpha + C = 0 \qquad (1)$$
$$A\beta^2 + B\beta + C = 0 \qquad (2)$$
$$A\gamma^2 + B\gamma + C = 0. \qquad (3)$$

Subtracting (2) from (1), $A(\alpha^2 - \beta^2) + B(\alpha - \beta) = 0$,

and since $\alpha \neq \beta$, we can divide through by $\alpha - \beta$ so that
$$A(\alpha + \beta) + B = 0. \qquad (4)$$

Similarly from (2) and (3), $A(\beta + \gamma) + B = 0. \qquad (5)$

Now subtracting (5) from (4), $A(\alpha - \gamma) = 0$,

and since $\alpha \neq \gamma$, $A = 0$.

Then from (4), $B = 0$ and from (1), $C = 0$.

B. To prove the main theorem, let
$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= (a - a_1)x^2 + (b - b_1)x + (c - c_1). \end{aligned}$$

Then since $f(x)$ and $g(x)$ agree at three distinct values of x,
it follows that $h(x)$ is zero for these three values of x,
so, by the result in part A, $a - a_1 = b - b_1 = c - c_1 = 0$, as required.

NOTE: We have of course been equating coefficients of two identically equal quadratic functions since the first section of this chapter, but the theorem now justifies that practice. The theorem is a special case of the result, to be proven later, that if two polynomials of degree n are equal for $n + 1$ values of x, then they are equal for all values of x and their coefficients are the same.

A Notation for Identically Equal: If two functions $f(x)$ and $g(x)$ are equal for all values of x, they are called *identically equal*. The notation for this is $f(x) \equiv g(x)$. There is an essential distinction between

$$f(x) = g(x) \qquad \text{and} \qquad f(x) \equiv g(x),$$

in that the first is an *equation*, which will be true for some set of values of x, and the second is an *identity*, which is true for all values of x.

Application of the Theorem to Identities: An identity involving quadratics can now be proven by proving it true for just three values of x, conveniently chosen to simplify the calculations.

WORKED EXERCISE: Given that a, b and c are distinct constants, prove that

$$(x - a)(x - b) + (x - b)(x - c) \equiv (x - b)(2x - a - c).$$

SOLUTION: It will be sufficient to prove the result when $x = a$, $x = b$ and $x = c$.
When $x = a$, LHS $= 0 + (a - b)(a - c)$ and RHS $= (a - b)(a - c)$,
when $x = b$, LHS $= 0 + 0$ and RHS $= 0$,
when $x = c$, LHS $= (c - a)(c - b)$ and RHS $= (c - b)(c - a)$,
so, being true for three distinct values of x, the identity holds for all x.

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Application of the Theorem to Finding Coefficients: If two quadratic expressions are identically equal, there are two ways of generating equations for finding unknown coefficients.

24 TWO METHODS OF GENERATING EQUATIONS FOR FINDING COEFFICIENTS:
1. Equate coefficients of like terms.
2. Substitute carefully chosen values of x.

WORKED EXERCISE: Express n^2 as a quadratic in $n-3$.

SOLUTION: Let $n^2 \equiv a(n-3)^2 + b(n-3) + c$.
Equating coefficients of n^2, $1 = a$.
Put $n = 3$, then since $n - 3 = 0$, $9 = c$.
Put $n = 0$, then $0 = 9a - 3b + c$,
and since $a = 1$ and $c = 9$, $b = 6$.
So $n^2 \equiv (n-3)^2 + 6(n-3) + 9$.

Geometrical Implications of the Theorem: Here are some of the geometrical versions of the theorem, given in the language of coordinate geometry.

25 GEOMETRICAL IMPLICATIONS:
1. The graph of a quadratic function is completely determined by any three points on the curve.
2. The graphs of two distinct quadratic functions cannot intersect in more than two points.
3. A line cannot intersect a parabola in more than two points.

Algebraically, a quadratic is said to have *degree 2* because it has only a term in x^2, a term in x and a constant term. But geometrically, a parabola is said to have *degree 2* because some lines intersect it in two points, but no line intersects it in more than two points. The third statement above shows that these two constrasting ideas of degree coincide. Hence the theorem at the start of this section provides one of the fundamental links between the algebra of the quadratic function and the geometry of the parabola. This theorem will later be generalised to polyomial functions of any degree, and so will link the algebra of polynomials to the geometry of their graphs.

Exercise 8I

1. Show that the following quadratic identity holds when $x = a$, $x = b$ and $x = 0$, where a and b are distinct and nonzero. Explain why it follows that it is true for all values of x.

$$x(x-b) + x(x-a) \equiv x(2x - a - b)$$

2. Show that $x = 0$, $x = a$ and $x = b$ are all solutions of the quadratic equation

$$\frac{x(x-a)}{b-a} + \frac{x(x-b)}{a-b} = x,$$

where a and b are distinct and nonzero. For what values of x does it now follow that this equation is true?

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3. Similarly prove the following identities in x, where a, b and c are distinct:

(a) $\dfrac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \dfrac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \dfrac{c^2(x-a)(x-b)}{(c-a)(c-b)} \equiv x^2$

(b) $(b-c)(x-a) + (c-a)(x-b) + (a-b)(x-c) \equiv 0$

NOTE: The last identity is only linear, and so only two values of x need be tested.

4. (a) If $x^2 + x + 1 = a(x-1)^2 + b(x-1) + c$ for all values of x, form three equations by substituting $x = 0$ and $x = 1$, and equating coefficients of x^2. Hence find a, b and c.

(b) Find a, b and c if $n^2 - n \equiv a(n-4)^2 + b(n-4) + c$. [HINT: Substitute $n = 4$, substitute $n = 0$, and equate coefficients of n^2.]

5. (a) Express $2x^2 + 3x - 6$ in the form $a(x+1)^2 + b(x+1) + c$.

(b) Find a, b and c if $2x^2 + 4x + 5 \equiv a(x-3)^2 + b(x-3) + c$.

(c) Express $2x^2 - 5x + 3$ in the form $a(x-2)^2 + b(x-2) + c$.

6. (a) Express x^2 in the form $a(x+1)^2 + b(x+1) + c$.

(b) Express n^2 as a quadratic in $(n-4)$.

(c) Express x^2 as a quadratic in $(x+2)$.

DEVELOPMENT

7. Prove the following identities, where p, q and r are distinct:

(a) $(p+q+x)(pq+px+qx) - pqx \equiv (p+q)(p+x)(q+x)$

(b) $\dfrac{p(x-q)(x-r)}{(p-q)(p-r)} + \dfrac{q(x-p)(x-r)}{(q-r)(q-p)} + \dfrac{r(x-p)(x-q)}{(r-p)(r-q)} \equiv x$

(c) $\dfrac{(p-x)(q-x)}{(p-r)(q-r)} + \dfrac{(q-x)(r-x)}{(q-p)(r-p)} + \dfrac{(r-x)(p-x)}{(r-q)(p-q)} \equiv 1$

8. (a) If $2x^2 - x - 1 = a(x-b)(x-c)$ for all real values of x, find a, b and c.

(b) Express m^2 in the form $a(m-1)^2 + b(m-2)^2 + c(m-3)^2$.

9. (a) Express $2x - 5$ in the form $A(2x+1) + B(x+1)$.

(b) Express $\dfrac{2x+3}{(x+1)(x+2)}$ in the form $\dfrac{A}{x+1} + \dfrac{B}{x+2}$. [HINT: You will need to multiply both sides of $\dfrac{2x+3}{(x+1)(x+2)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+2}$ by $(x+1)(x+2)$.]

(c) Express $\dfrac{x+1}{x(x+2)}$ in the form $\dfrac{A}{x} + \dfrac{B}{x+2}$.

10. (a) Find a, b and c if the graph of the quadratic function $f(x) = ax^2 + bx + c$ passes through $O(0,0)$, $A(4,0)$ and $B(5,5)$. Then check by substitution whether the point $D(-2,10)$ lies on the curve.

(b) Use a similar method to find whether the points $P(0,6)$, $Q(2,0)$, $R(4,2)$ and $S(6,12)$ lie on the graph of a quadratic function.

11. (a) Show that $m^2 - (m-1)^2 = 2m - 1$.

(b) Hence find the sum of the series $1 + 3 + 5 + \cdots + 61$.

(c) Check your answer using the formula for the sum of an arithmetic series.

(d) Find the sum of the series $1 + 3 + 5 + \cdots$ to n terms, using each of these two methods.

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EXTENSION

12. (a) Express $\dfrac{x^2}{(x-m)(x-n)}$ in the form $A + \dfrac{B}{x-m} + \dfrac{C}{x-n}$. Hence or otherwise, find the value of $\dfrac{m^2}{(m-n)(m-r)} + \dfrac{n^2}{(n-m)(n-r)} + \dfrac{r^2}{(r-m)(r-n)}$, where m, n and r are distinct.

(b) Similarly, given that the theorems proven in this section may be extended to cubics, find the value of $\dfrac{m^3}{(m-n)(m-r)(m-p)} + \dfrac{n^3}{(n-m)(n-r)(n-p)} + \dfrac{r^3}{(r-m)(r-n)(r-p)} + \dfrac{p^3}{(p-m)(p-n)(p-r)}$, where m, n, r and p are distinct.

13. Show that the curve $ax^2 + by^2 + cxy + dx + ey + f = 0$ has degree at most 2 in the geometric sense, by showing that no line can intersect it in more than two points. [HINT: Substitute the general form for the equation of a straight line into the curve.]

GO Online Multiple Choice Quiz

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