Math2221
Higher Theory and Applications of Differential Equations

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Contents

Part I

Linear ODEs

Introduction

In first year, you studied second-order linear ODEs with constant coefficients. We will see that the techniques you learned can be extended to handle higher-order linear ODEs with variable coefficients. A key idea, used repeatedly throughout the remainder of the course, is linear superposition.

Linear differential operators

In linear algebra, you have seen the compact notation $A\boldsymbol{x} = \boldsymbol{b}$ for a system of linear equations. A similarly notation when dealing with a linear ordinary differential equation

$$Lu = f.$$

Here, L is an operator (or transformation) that acts on a function u to create a new function Lu.

Notation

Given coefficients $a_0(x)$, $a_1(x)$, ..., $a_m(x)$ we define the linear differential operator L of order m,

$$
\begin{aligned}
Lu(x) &= \sum_{j=0}^{m} a_j(x) D^j u(x) \\
&= a_m D^m u + a_{m-1} D^{m-1} u + \cdots + a_0 u,
\end{aligned}
\tag{1}
$$

where $D^j u = d^j u/dx^j$ (with $D^0 u = u$).

We refer to a_m as the leading coefficient of L. For simplicity, we assume that each $a_j(x)$ is a smooth function of x.

The ODE $Lu = f$ is said to be singular with respect to an interval $[a, b]$ if the leading coefficient $a_m(x)$ vanishes for any $x \in [a, b]$. (Also say L is singular on $[a, b]$.)

Linearity

An operator of the form (1) is indeed linear: for any constants c_1 and c_2 and any (m-times differentiable) functions u_1 and u_2,

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2.$$

Example

$Lu = (x-3)u''' - (1+\cos x)u' + 6u$ is a linear differential operator of order 3, with leading coefficient $x-3$. Thus, L is singular on $[1,4]$, but not singular on $[0,2]$.

Example

$N(u) = u'' + u^2u' - u$ is a *non*linear differential operator of order 2.

Homogeneous or forced?

Ordinary differential equations of the form

$$Lu = 0$$

are known as homogeneous. Those of the form

$$Lu = f$$

are known as inhomogeneous or non-homogeneous. In physical systems the inhomogeneity is often described as a forcing term.

Example

Equation for an oscillating mass (m) subject to an external force (f):

$$m\,x'' + r\,x' + k\,x = f(t),$$

where x is position, r friction and k is restoring.

Homogeneous solutions form a vector space

Let u_1, u_2,...u_k be the solutions to the linear homogeneous differential equation $(Lu = 0)$. Because L is linear, the linear combination

$$u = c_1u_1 + c_2u_2 + ... + c_ku_k$$

is also a solution.

Hence, the set of solutions forms a vector space with the trivial solution $(u = 0)$ its zero vector.

Initial-value and boundary-value problems

When the solution to a differential equation (linear or non-linear, homogeneous or not) is prescribed at a particular point $x = x_0$, that is

$$u(x_0) = \nu_0, \quad u'(x_0) = \nu_1, \quad \ldots, \quad u^{(m-1)}(x_0) = \nu_{m-1},$$

we call it an initial value problem.

Where a differential equation is order 2 or greater, solutions at 2 or more locations can be prescribed. Such problems are called boundary value problems.

Unique solution to the linear initial problem

Consider a general mth-order linear differential operator

$$Lu = \sum_{j=0}^{m} a_j(x) D^j u.$$

Given $f(x)$ and m initial values ν_0, ν_1, ..., ν_{m-1} we seek $u = u(x)$ satisfying

$$Lu = f \quad \text{on } [a, b], \tag{2}$$

with

$$u(a) = \nu_0, \quad u'(a) = \nu_1, \quad \ldots, \quad u^{(m-1)}(a) = \nu_{m-1}. \tag{3}$$

Theorem

For an ODE $Lu = f$ which is not singular with respect to $[a, b]$, with f continuous on $[a, b]$, the IVP (2) and (3) has a unique solution.

See discussion of uniqueness in Technical Proofs.

Unique solution: special case

Consider the linear inhomogeneous initial value problem

$$\begin{aligned} a_2 u'' + a_1 u' + a_0 u &= f(x) \\ u(0) = \nu_0 \;\; ; \;\; u'(0) &= \nu_1 \end{aligned}$$

with a_2, a_1 and a_0 constants and f continuous for all x.

Assume two solutions u_a and u_b. Let $U = u_a - u_b$. Hence

$$\begin{gathered} U(0) = u_a(0) - u_b(0) = \nu_0 - \nu_0 = 0, \\ U'(0) = u_a'(0) - u_b'(0) = \nu_1 - \nu_1 = 0 \\ and \\ a_2 U'' + a_1 U' + a_0 U = f(x) - f(x) = 0. \end{gathered}$$

Unique solution: special case

Multiplying the homogeneous equation by U' and integrating over $0 \le x \le t$,

$$a_2 \int_0^t U''U' dx + a_1 \int_0^t (U')^2 dx + a_0 \int_0^t UU' dx = 0,$$

$$\frac{a_2}{2}[U'(t)^2 - U'(0)^2] + a_1 \int_0^t (U')^2 dx + a_0[U'(t)^2 - U'(0)^2] = 0.$$

Above, $U'(0) = 0 = U(0)$ and since each of the remaining terms is ≥ 0 we find that $U = 0$ is the only permitted solution. Therefore the two solutions to the inhomogeneous problem, u_a and u_b, are identical.

Solution to mth order problem has dimension m

Theorem

Assume that the linear, mth-order differential operator L is not singular on $[a, b]$. Then the set of all solutions to the homogeneous equation $Lu = 0$ on $[a, b]$ is a vector space of dimension m.

Proof.

Let $V = \{ u : Lu = 0 \text{ on } [a, b] \}$ and define the linear transformation $\Theta : V \to \mathbb{R}^m$ by

$$\Theta u = [u(a), u'(a), \ldots, u^{(m-1)}(a)]^T.$$

Uniqueness of solutions means that Θ is one-to-one, and existence means that Θ is onto. Hence, Θ is an isomorphism, and therefore the vector space V has dimension m. □

General solution

If $\{u_1, u_2, \ldots, u_m\}$ is any basis for the solution space of $Lu = 0$, then every solution can be written in a unique way as

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_m u_m(x) \quad \text{for } a \le x \le b. \tag{4}$$

We refer to (4) as the general solution of the homogeneous equation $Lu = 0$ on $[a, b]$.

Linear superposition refers to this technique of constructing a new solution out of a linear combination of old ones. Of course, this trick works only because L is linear.

Example

The general solution to $u'' - u' - 2u = 0$ is

$$u(x) = c_1 e^{-x} + c_2 e^{2x}.$$

Inhomogeneous problem

Consider the *in*homogeneous equation $Lu = f$ on $[a, b]$, and fix a particular solution u_{P}.

For *any* solution u, the difference $u - u_{\mathrm{P}}$ is a solution of the *homogeneous* equation because

$$L(u - u_{\mathrm{P}}) = Lu - Lu_{\mathrm{P}} = f - f = 0 \quad \text{on } [a, b].$$

Hence, $u(x) - u_{\mathrm{P}}(x) = c_1 u_1(x) + \cdots + c_m u_m(x)$ for some constants c_1, ..., c_m, and so

$$u(x) = u_{\mathrm{P}}(x) + \underbrace{c_1 u_1(x) + \cdots + c_m u_m(x)}_{u_{\mathrm{H}}(x)}, \qquad a \le x \le b,$$

is the general solution of the *in*homogeneous equation $Lu = f$.

Inhomogeneous problem

Example

The inhomogeneous ODE

$$u'' - u' + 2u = 2e^x$$

has the particular solution

$$u_{\mathrm{P}}(x) = e^x.$$

The general solution for its homogeneous counterpart is

$$u_{\mathrm{H}}(x) = c_1 e^{-x} + c_2 e^{2x}.$$

So the general solution of the inhomogeneous ODE is

$$u(x) = u_{\mathrm{P}}(x) + u_{\mathrm{H}}(x) = e^x + c_1 e^{-x} + c_2 e^{2x}.$$

Reduction of order

Theorem

For $u = u_1(x) \neq 0$, a solution to the ODE

$$u'' + p(x)u' + q(x)u = 0,$$

on some interval I, then a second solution is

$$u = u_1(x) \int \frac{1}{u_1^2 \exp(\int p\,dx)}\,dx$$

Substitute $u = u_1(x)v(x)$ into the ODE and rearrange to obtain

$$\underbrace{(u_1'' + pu_1' + qu_1)}_{=0}v + u_1 v'' + (2u_1' + pu_1)v' = 0.$$

This is just a *first*-order, linear ODE for the derivative of the unknown factor $v(x)$: put $w = v'$ then

$$u_1 w' + (2u_1' + pu_1)w = 0.$$

Reduction of order (continued)

Writing the ODE for w in the standard form

$$w' + (2u_1' u_1^{-1} + p)w = 0,$$

we seek an integrating factor
$A(x) = \exp\bigl(\int (2u_1' u_1^{-1} + p)\,dx\bigr) = u_1^2 \exp(\int p\,dx)$, so that

$$\frac{d}{dx}(Aw) = Aw' + A'w = A\bigl(w' + (2u_1' u_1^{-1} + p)w\bigr) = 0.$$

Then $Aw = C$ for some constant C, and so

$$v = \int \frac{C}{A(x)}\,dx.$$

Example

For the ODE $u'' - 6u' + 9u = 0$, take $u_1 = e^{3x}$ and find v.

Differential operators with constant coefficients

If L has constant coefficients, then the problem of solving $Lu = 0$ reduces to that of factorizing the polynomial having the same coefficients. Some complications occur if the polynomial has any repeated roots.

Characteristic polynomial

Suppose that a_j is constant for $0 \le j \le m$, with $a_m \ne 0$. We define the associated polynomial of degree m,

$$p(z) = \sum_{j=0}^{m} a_j z^j = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0,$$

so that, formally, $L = p(D)$.

Since $D^j e^{\lambda x} = \lambda^j e^{\lambda x}$ we have

$$p(D)e^{\lambda x} = \big(a_m \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0\big)e^{\lambda x} = p(\lambda)e^{\lambda x},$$

and so

$$p(D)e^{\lambda x} = 0 \iff p(\lambda) = 0.$$

Factorization

By the fundamental theorem of algebra,

$$p(z) = a_m(z - \lambda_1)^{k_1}(z - \lambda_2)^{k_2} \cdots (z - \lambda_r)^{k_r}$$

where λ_1, λ_2, ..., λ_r are the distinct roots of p, with corresponding multiplicities k_1, k_2, ..., k_r satisfying

$$k_1 + k_2 + \cdots + k_r = m.$$

Lemma

$(D - \lambda)x^j e^{\lambda x} = jx^{j-1}e^{\lambda x}$ *for* $j \geq 0$.

Lemma

$(D - \lambda)^k x^j e^{\lambda x} = 0$ *for* $j = 0, 1, \ldots, k - 1$.

Proof

An elementary calculation gives

$$(D-\lambda)x^j e^{\lambda x} = jx^{j-1}e^{\lambda x} + x^j\lambda e^{\lambda x} - \lambda x^j e^{\lambda x} = jx^{j-1}e^{\lambda x},$$

as claimed, and then

$$\begin{aligned}(D-\lambda)^2 x^j e^{\lambda x} &= (D-\lambda)jx^{j-1}e^{\lambda x} = j(j-1)x^{j-2}e^{\lambda x},\\ &\vdots\\ (D-\lambda)^j x^j e^{\lambda x} &= j!e^{\lambda x},\\ (D-\lambda)^{j+1}x^j e^{\lambda x} &= j!(D-\lambda)e^{\lambda x} = 0,\end{aligned}$$

so $(D-\lambda)^k x^j e^{\lambda x} = 0$ for all $k \geq j+1$, that is, for $j \leq k-1$.

Basic solutions

Lemma

If $(z-\lambda)^k$ *is a factor of* $p(z)$ *then the function* $u(x) = x^j e^{\lambda x}$ *is a solution of* $Lu = 0$ *for* $0 \le j \le k-1$*.*

Proof.

Write $p(z) = (z-\lambda)^k q(z)$, so that $q(z)$ is a polynomial of degree $m-k$. It follows that

$$p(D) = (D-\lambda)^k q(D) = q(D)(D-\lambda)^k$$

and so for $0 \le j \le k-1$,

$$p(D)x^j e^{\lambda x} = q(D)(D-\lambda)^k x^j e^{\lambda x} = q(D)0 = 0.$$

□

General solution

Theorem

For the constant-coefficient case, the general solution of the homogeneous equation $Lu = 0$ is

$$u(x) = \sum_{q=1}^{r} \sum_{l=0}^{k_q - 1} c_{ql} x^l e^{\lambda_q x},$$

where the c_{ql} are arbitrary constants.

Since $(z - \lambda_q)^{k_q}$ is a factor of $p(z)$,

$$Lu = \sum_{q=1}^{r} \sum_{l=0}^{k_q - 1} c_{ql} L x^l e^{\lambda_q x} = \sum_{q=1}^{r} \sum_{l=0}^{k_l - 1} c_{ql} \times 0 = 0.$$

Linear independence is shown in the Technical Proofs.

Distinct real roots

Example

From the factorization

$$D^4 - 2D^3 - 11D^2 + 12D = (D+3)D(D-1)(D-4)$$

we see that the general solution of

$$u'''' - 2u''' - 11u'' + 12u' = 0$$

is

$$u = c_1 e^{-3x} + c_2 + c_3 e^{x} + c_4 e^{4x}.$$

Repeated real root

Example

From the factorization

$$D^4 + 6D^3 + 9D^2 - 4D - 12 = (D-1)(D+2)^2(D+3)$$

we see that the general solution of

$$u'''' + 6u''' + 9u'' - 4u' - 12u = 0$$

is

$$u = c_1 e^{x} + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 e^{-3x}.$$

Complex root

Example

From the factorization

$$\begin{aligned} D^3 - 7D^2 + 17D - 15 &= (D^2 - 4D + 5)(D - 3) \\ &= (D - 2 - i)(D - 2 + i)(D - 3) \end{aligned}$$

we see that the general solution of

$$u''' - 7u'' + 17u' - 15u = 0$$

is

$$\begin{aligned} u(x) &= c_1 e^{(2+i)x} + c_2 e^{(2-i)x} + c_3 e^{3x} \\ &= c_4 e^{2x} \cos x + c_5 e^{2x} \sin x + c_3 e^{3x}. \end{aligned}$$

Simple oscillator

Second-order ODEs arise naturally in classical mechanics.

Consider a particle of mass m that moves along the x-axis with velocity $v = \dot{x} = dx/dt$ under the influence of

$$\begin{aligned} \text{an external applied force} &= f(t), \\ \text{a frictional resistance force} &= -r(v)v, \\ \text{a restoring force} &= -k(x)x. \end{aligned}$$

Newton's second law,

$$m\,\ddot{x} = m\,\frac{d^2x}{dt^2} = f(t) - r(v)v - k(x)x,$$

leads to a second-order differential equation

$$m\,\ddot{x} + r(\dot{x})\,\dot{x} + k(x)x = f(t).$$

Simplest case is when $r(v) = r_0 > 0$ and $k(x) = k_0 > 0$ are constant, giving a linear ODE with constant coefficients:

$$m\,\ddot{x} + r_0\,\dot{x} + k_0 x = f(t).$$

Typically interested in the case when the applied force is periodic with frequency ω; for example,

$$f(t) = F \sin \omega t.$$

The general solution $x(t) = x_{\mathrm{H}}(t) + x_{\mathrm{P}}(t)$.

We now show that $x_{\mathrm{H}}(t) \to 0$ as $t \to \infty$ and that $x_{\mathrm{P}}(t)$ exists. Since $x_{\mathrm{P}}(t+T) = x_{\mathrm{P}}(t)$ for $T = 2\pi/\omega$, it follows that the solution $x(t)$ always tends to a periodic function with frequency ω, regardless of the initial values $x(0)$ and $\dot{x}(0)$.

The characteristic equation is

$$m\lambda^2 + r_0\lambda + x_0 = m(\lambda - \lambda_+)(\lambda - \lambda_-),$$

and the roots are

$$\lambda_\pm = \frac{-r_0 \pm \sqrt{\Delta}}{2m} \quad \text{where} \quad \Delta = r_0^2 - 4mk_0.$$

If $\Delta > 0$, then $\lambda_- < \lambda_+ < 0$ so, for any constants A and B,

$$x_{\mathrm{H}}(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t} \to 0 \quad \text{as } t \to \infty.$$

If $\Delta = 0$ then $\lambda_- = \lambda_+ < 0$ so again $x_{\mathrm{H}}(t) \to 0$.

If $\Delta < 0$ then $\sqrt{\Delta} = i\sqrt{|\Delta|}$ so $\operatorname{Re}\lambda_+ = \operatorname{Re}\lambda_- < 0$, and again $x_{\mathrm{H}}(t) \to 0$.

Since $\operatorname{Re}\lambda_\pm \neq 0$ the particular solution has the form

$$x_{\mathrm{P}}(t) = C\cos\omega t + E\sin\omega t,$$

and we find that

$$\begin{aligned} m\ddot{x}_{\mathrm{P}} + r_0\dot{x}_{\mathrm{P}} + k_0 x_{\mathrm{P}} &= \bigl(-m\omega^2 C + r_0\omega E + k_0 C\bigr)\cos\omega t \\ &\qquad + \bigl(-m\omega^2 E - r_0\omega C + k_0 E\bigr)\sin\omega t, \end{aligned}$$

which equals $F\sin\omega t$ iff

$$\begin{aligned} (k_0 - m\omega^2)C + r_0\omega E &= 0, \\ -r_0\omega C + (k_0 - m\omega^2)E &= F. \end{aligned}$$

This 2×2 system has a unique solution since its determinant is

$$(k_0 - m\omega^2)^2 + (r_0\omega)^2 > 0.$$

Wronskians and linear independence

We introduce a function, called the Wronskian, that provides us with a way of testing whether a family of solutions to $Lu = 0$ is linearly independent. The Wronskian also turns out to have several other uses.

Linear independence

Let $u_1(x), u_2(x), \ldots, u_m(x)$ be functions defined on an interval $I \subset \mathbb{R}$. The functions $u_1, \ldots, u_m$ are called linearly dependent if there exist constants $a_1, a_2, \ldots, a_m$ not all zero such that

$$a_1 u_1(x) + a_2 u_2(x) + \cdots + a_m u_m(x) = 0 \quad \forall x \in I.$$

If the above equation only holds for

$$a_i = 0, \qquad i = 1, 2, \ldots, m$$

then the functions are linearly independent.

Example

$u_1 = sin\, 2x$ and $u_2 = sin\, x\, cos\, x$ are linearly dependent.
$u_1 = sin\, x$ and $u_2 = cos\, x$ are linearly independent.

Wronskian

The Wronskian of the functions u_1, u_2, ..., u_m is the $m \times m$ determinant

$$W(x) = W(x; u_1, u_2, \ldots, u_m) = \det[D^{i-1}u_j].$$

For instance, if $m = 2$ then

$$W(x) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_2 u_1',$$

and when $m = 3$,

$$W(x) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{vmatrix}.$$

Of course, $W(x)$ is defined only when the functions are differentiable $m - 1$ times.

Wronskian

Example

The Wronskian of the functions $u_1 = e^{2x}$, $u_2 = xe^{2x}$ and $u_3 = e^{-x}$ is

$$W = \begin{vmatrix} e^{2x} & xe^{2x} & e^{-x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} & -e^{-x} \\ 4e^{2x} & 4e^{2x} + 4xe^{2x} & e^{-x} \end{vmatrix} = 9e^{3x}.$$

Example

The Wronskian of the functions $u_1 = e^x \cos 3x$ and $u_2 = e^x \sin 3x$ is

$$W = \begin{vmatrix} e^x \cos 3x & e^x \sin 3x \\ e^x \cos 3x - 3e^x \sin 3x & e^x \sin 3x + 3e^x \cos 3x \end{vmatrix} = 3e^{2x}.$$

Linearly dependent functions

Lemma

If u_1, ..., u_m are linearly dependent over an interval $[a, b]$ then $W(x; u_1, \ldots, u_m) = 0$ for $a \leq x \leq b$.

Example

The functions $u_1 = \cosh x$, $u_2 = \sinh x$ and $u_3 = e^x$ are linearly dependent because

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x.$$

Their Wronskian is

$$W = \begin{vmatrix} \cosh x & \sinh x & e^x \\ \sinh x & \cosh x & e^x \\ \cosh x & \sinh x & e^x \end{vmatrix} = 0.$$

Proof (for $m = 3$)

Assume that u_1, u_2, u_3 are linearly dependent on the interval $[a, b]$, that is, there exist constants c_1, c_2, c_3, not all zero, such that

$$c_1 u_1(x) + c_2 u_2(x) + c_3 u_3(x) = 0 \quad \text{for } a \le x \le b.$$

Differentiating, it follows that

$$\begin{aligned} c_1 u_1'(x) + c_2 u_2'(x) + c_3 u_3'(x) &= 0, \\ c_1 u_1''(x) + c_2 u_2''(x) + c_3 u_3''(x) &= 0, \end{aligned}$$

so

$$\begin{bmatrix} u_1(x) & u_2(x) & u_3(x) \\ u_1'(x) & u_2'(x) & u_3'(x) \\ u_1''(x) & u_2''(x) & u_3''(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } a \le x \le b.$$

This 3×3 matrix must be singular and thus $W(x) = 0$.

Wronskian satisfies a first-order ODE

Lemma

If u_1, u_2, ..., u_m are solutions of $Lu = 0$ on the interval $[a, b]$ then their Wronskian satisfies

$$a_m(x)W'(x) + a_{m-1}(x)W(x) = 0, \qquad a \le x \le b.$$

Example

The second-order ODE

$$u'' + 3u' - 4u = 0$$

has solutions $u_1 = e^x$ and $u_2 = e^{-4x}$. Their Wronskian is

$$\begin{vmatrix} e^x & e^{-4x} \\ e^x & -4e^{-4x} \end{vmatrix} = -5e^{-3x},$$

and satisfies $W' + 3W = 0$.

Proof (for $m = 2$)

Differentiating $W = u_1 u_2' - u_1' u_2$ we have

$$W' = (u_1' u_2' + u_1 u_2'') - (u_1'' u_2 + u_1' u_2') = u_1 u_2'' - u_1'' u_2,$$

so

$$\begin{aligned}
a_2 W' + a_1 W &= a_2(u_1 u_2'' - u_1'' u_2) + a_1(u_1 u_2' - u_1' u_2) \\
&\quad + a_0 \underbrace{(u_1 u_2 - u_1 u_2)}_{=0} \\
&= u_1(a_2 u_2'' + a_1 u_2' + a_0 u_2) \\
&\quad - (a_2 u_1'' + a_1 u_1' + a_0 u_1) u_2 \\
&= u_1(L u_2) - (L u_1) u_2 = 0.
\end{aligned}$$

Linear independence of solutions

Theorem

Let u_1, u_2, ..., u_m be solutions of a non-singular, linear, homogeneous, mth-order ODE $Lu = 0$ on the interval $[a, b]$.
Either

> $W(x) = 0$ *for* $a \leq x \leq b$ *and the* m *solutions are linearly* ***dependent***,

or else

> $W(x) \neq 0$ *for* $a \leq x \leq b$ *and the* m *solutions are linearly* ***independent***.

Proof

The Wronskian satisfies

$$W' + pW = 0 \quad \text{for } a \le x \le b, \quad \text{where} \quad p = \frac{a_{m-1}}{a_m}.$$

Define an integrating factor

$$I(x) = \exp\left(\int p(x)\,dx\right) \neq 0,$$

so that $I' = Ip$ and hence

$$(IW)' = IW' + IpW = I(W' + pW) = 0.$$

Thus, $I(x)W(x) = C$ for some constant C.

Either $C = 0$ in which case $W(x) = 0$ for all $x \in [a, b]$, or else $C \neq 0$ in which case $W(x)$ is never zero for $x \in [a, b]$.

(Assume now that $m = 3$.) We already know that

$$u_1,\ u_2,\ u_3 \text{ linearly dependent} \implies W \equiv 0.$$

Hence, to complete the proof it suffices to show

$$W(a) = 0 \implies u_1,\ u_2,\ u_3 \text{ linearly dependent.}$$

If $W(a) = 0$, then there exist c_1, c_2, c_3, not all zero, such that

$$\begin{bmatrix} u_1(x) & u_2(x) & u_3(x) \\ u_1'(x) & u_2'(x) & u_3'(x) \\ u_1''(x) & u_2''(x) & u_3''(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{at } x = a,$$

so the function $u(x) = c_1u_1(x) + c_2u_2(x) + c_3u_3(x)$ satisfies

$$Lu = 0 \text{ for } a \le x \le b, \text{ with } u(a) = u'(a) = u''(a) = 0.$$

The solution of this initial-value problem is unique, so $u(x) \equiv 0$ and thus u_1, u_2, u_3 are linearly dependent.

Methods for inhomogeneous equations

In first year, you learned the method of undetermined coefficients for constructing a particular solution u_{P} to an inhomogeneous second-order linear ODE $Lu = f$ in some simple cases. We will study this method systematically for higher-order linear ODEs with constant coefficients.

We also discuss variation of parameters, a technique that applies for general L and f, but which requires the evaluation of possibly very difficult integrals.

Superposition of solutions

We now consider methods for finding a particular solution u_{P} satisfying $Lu_{\mathrm{P}} = f$.

First note that if

$$f(x) = c_1 f_1(x) + c_2 f_2(x)$$

and if we know $u_{\mathrm{P}1}$ and $u_{\mathrm{P}2}$ satisfying

$$Lu_{\mathrm{P}1} = f_1 \quad \text{and} \quad Lu_{\mathrm{P}2} = f_2,$$

then we can put

$$u_{\mathrm{P}}(x) = c_1 u_{\mathrm{P}1}(x) + c_2 u_{\mathrm{P}2}(x),$$

because by the linearity of L,

$$Lu_{\mathrm{P}} = c_1 Lu_{\mathrm{P}1} + c_2 Lu_{\mathrm{P}2} = c_1 f_1 + c_2 f_2 = f.$$

Polynomial solutions

Let $L = p(D)$ be a linear differential operator of order m with constant coefficients.

Theorem

Assume that $a_0 = p(0) \neq 0$. For any integer $r \geq 0$, there exists a unique polynomial u_{P} of degree r such that $Lu_{\mathrm{P}} = x^r$.

For simplicity, we prove the result only for the case $m = 2$. Thus,

$$Lu = a_2u'' + a_1u' + a_0u,$$

where a_0, a_1, a_2 are constants with $a_2 \neq 0$ and $a_0 \neq 0$.

Look for u_{P} in the form

$$u_{\mathrm{P}}(x) = \sum_{j=0}^{r} c_j \frac{x^j}{j!}.$$

We find that

$$
\begin{aligned}
Lu_{\mathrm{P}} &= a_2 \sum_{j=2}^{r} c_j \frac{x^{j-2}}{(j-2)!} + a_1 \sum_{j=1}^{r} c_j \frac{x^{j-1}}{(j-1)!} + a_0 \sum_{j=0}^{r} c_j \frac{x^j}{j!} \\
&= a_0 c_r \frac{x^r}{r!} + \left(a_0 c_{r-1} + a_1 c_r\right) \frac{x^{r-1}}{(r-1)!} \\
&\quad + \sum_{j=0}^{r-2} \left(a_2 c_{j+2} + a_1 c_{j+1} + a_0 c_j\right) \frac{x^j}{j!},
\end{aligned}
$$

which equals x^r if and only if

$$
\begin{aligned}
a_0 c_j + a_1 c_{j+1} + a_2 c_{j+2} &= 0, \quad 0 \le j \le r-2, \\
a_0 c_{r-1} + a_1 c_r &= 0, \\
a_0 c_r &= r!.
\end{aligned}
$$

This upper triangular system is uniquely solvable because $a_0 \neq 0$.

An example

Let $Lu = 3u'' - u' + 2u$ and suppose we want a particular solution to $Lu = 8x^3$. The theorem ensures that

$$u_{\mathrm{P}}(x) = C + Ex + Fx^2 + Gx^3$$

works for some C, E, F, G. In fact,

$$Lu_{\mathrm{P}} = (2C - E + 6F) + (2E - 2F + 18G)x + (2F - 3G)x^2 + 2Gx^3$$

so

$$\begin{aligned} 2C - E + 6F \qquad &= 0, \\ 2E - 2F + 18G &= 0, \\ 2F - 3G &= 0, \\ 2G &= 8 \end{aligned}$$

and back substitution gives $u_{\mathrm{P}} = -33 - 30x + 6x^2 + 4x^3$.

Exponential solutions

Theorem

Let $L = p(D)$ *and* $\mu \in \mathbb{C}$*. If* $p(\mu) \neq 0$*, then the function*

$$u_{\mathrm{P}}(x) = \frac{e^{\mu x}}{p(\mu)}$$

satisfies $Lu_{\mathrm{P}} = e^{\mu x}$*.*

Proof.

Follows at once because $p(D)e^{\mu x} = p(\mu)e^{\mu x}$. $\square$

Example

A particular solution of $u'' + 4u' - 3u = 3e^{2x}$ is $u_{\mathrm{P}} = e^{2x}/3$.

Product of polynomial and exponential

Theorem

Let $L = p(D)$ and assume that $p(\mu) \neq 0$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_{\mathrm{P}} = v(x)e^{\mu x}$ satisfies $Lu_{\mathrm{P}} = x^r e^{\mu x}$.

Proof.

Again, for simplicity, we prove the result only for $m = 2$.
Put $v = e^{-\mu x}u$ so that $u = ve^{\mu x}$, and observe that

$$\begin{aligned} Lu &= \bigl[a_2(v'' + 2\mu v' + \mu^2 v) + a_1(v' + \mu v) + a_0 v\bigr]e^{\mu x} \\ &= \bigl[a_2 v'' + (a_1 + 2a_2\mu)v' + (a_2\mu^2 + a_1\mu + a_0)v\bigr]e^{\mu x}. \end{aligned}$$

Thus, $Lu = e^{\mu x}q(D)v$ where $q(z) = a_2 z^2 + (a_1 + 2a_2\mu)z + p(\mu)$, and our earlier result yields the desired v satisfying $q(D)v = x^r$ because $q(D)1 = q(0) = p(\mu) \neq 0$. $\square$

An example

Consider

$$2u'' + u' - 3u = 9xe^{-2x}.$$

Here,

$$p(z) = 2z^2 + z - 3$$

so $p(-2) = 3 \neq 0$ and a particular solution $u_{\mathrm{P}} = (Cx + E)e^{-2x}$ exists. In fact, we find that

$$p(D)u_{\mathrm{P}} = (3Cx - 7C + 3E)e^{-2x}$$

so

$$3C = 9 \quad \text{and} \quad -7C + 3E = 0.$$

Thus, $C = 3$ and $E = 7$, giving

$$u_{\mathrm{P}} = (3x + 7)e^{-2x}.$$

Annihilator method

In the previous cases we proposed a solution $u = u_P$ and showed that it satisfied $Lu = f$. The following is a method to derive a particular solution given $Lu = f$. If f(x) is differentiable at least n times and

$$[a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D^1 + a_0] f(x) = 0$$

then $[a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D^1 + a_0]$ annihilates f.

Example

D^n annihilates x^{m-1} for $m \leq n$.

Example

$(D - \alpha)^n$ annihilates $x^{n-1} e^{\alpha x}$ for $m \leq n$.

Annihilator method: Three simple examples (1)

Given $Lu = f$ we can apply the appropriate annihilator to both sides and solve the resulting homogeneous DE.
Let $Lu = u' - u$ and suppose we want a solution such that $Lu = x^2$.
Annihilating both sides we have

$$D^3(u' - u) = u'''' - u''' = 0.$$

Setting $w = u'''$, clearly $w = Ce^x$ is the general solution. Integrating three times yields

$$u = Ce^x + Ex^2 + Fx + G.$$

Clearly $u_h = Ae^x$ and the form of the particular solution is $u_p = Ex^2 + Fx + G$ (a polynomial of degree 2, as expected). Substituting find $E = -1$, $F = 2$ and $G = 1$.

Annihilator method: Three simple examples (2)

Let $Lu = u'' - u'$ and suppose we want a solution such that $Lu = x^2$ (note that $p(0) \neq 0$ for $L = p(D)$ so we are not yet able to solve this). Annihilating both sides we have

$$D^3(u'' - u') = u^{(5)} - u^{(4)} = 0.$$

Setting $w = u^{(4)}$, $w = Ce^x$ is the general solution. Integrating four times yields

$$u = Ce^x + Ex^3 + Fx^2 + Gx + H.$$

Here $u_h = Ae^x + H$ is the homogeneous solution and the particular solution is $u_p = x(Ex^2 + Fx + G)$. Substituting find $E = -1/3$, $F = -1$, $G = -2$.

Annihilator method: Three simple examples (3)

Let $Lu = u' - u$ and suppose we want a solution such that $Lu = e^x$ (note that $\mu = 1$ and $p(\mu) \neq 0$ for $L = p(D)$ so we are not yet able to solve this). Annihilating both sides we have

$$(D - 1)(u' - u) = u'' - 2u' + u = 0.$$

The characteristic polynomial has a repeated root $u_h = Ae^x + Bxe^x$ the general solution. Here $u_h = Ae^x$ is the homogeneous solution and so the particular solution is $u_p = Bxe^x$. Substituting find $B = 1$.

Polynomial solutions: the remaining case

Theorem

Let $L = p(D)$ *and assume* $p(0) = p'(0) = \cdots = p^{(k-1)}(0) = 0$ *but* $p^{(k)}(0) \neq 0$ *where* $1 \leq k \leq m-1$*. For any integer* $r \geq 0$*, there exists a unique polynomial* v *of degree* r *such that* $u_p(x) = x^k v(x)$ *satisfies* $Lu_{\mathrm{P}} = x^r$*.*

Example

Let $Lu = u''' + 2u''$ and seek a particular solution to $Lu = 12x^2$. The theorem ensures that

$$u_\mathrm{P} = x^2(C + Ex + Fx^2) = Cx^2 + Ex^3 + Fx^4$$

works for some C, E, F. In fact,

$$Lu_\mathrm{P} = (4C + 6E) + (12E + 24F)x + 24Fx^2$$

so

$$\begin{aligned} 4C + 6\ \ E \qquad\quad &= 0, \\ 12E + 24F &= 0, \\ 24F &= 12 \end{aligned}$$

and back substitution gives

$$u_\mathrm{P} = \frac{x^2}{2}(3 - 2x + x^2).$$

Exponential times polynomial: remaining case

Lemma

If $u(x) = w(x)e^{\mu x}$ *then*

$$p(D)u = e^{\mu x}q(D)w \quad \textit{where} \quad q(z) = \sum_{j=0}^{m} p^{(j)}(\mu)\,\frac{z^j}{j!}.$$

Theorem

Let $L = p(D)$ *and assume* $p(\mu) = p'(\mu) = \cdots = p^{(k-1)}(\mu) = 0$ *but* $p^{(k)}(\mu) \neq 0$, *where* $1 \leq k \leq m-1$. *For any integer* $r \geq 0$, *there exists a unique polynomial* v *of degree* r *such that* $u_{\mathrm{P}}(x) = x^k v(x)e^{\mu x}$ *satisfies* $Lu_{\mathrm{P}} = x^r e^{\mu x}$.

Exponential times polynomial: remaining case

Proof.

Since $q^{(j)}(0) = p^{(j)}(\mu)$ for all j, there is a unique polynomial v of degree r such that $w(x) = x^k v(x)$ satisfies $q(D)w = x^r$ and hence

$$p(D)u_{\mathrm{P}} = e^{\mu x} q(D) w = e^{\mu x} x^r.$$

$\square$

Proof of Lemma

$$
\begin{aligned}
p(D)we^{\mu x} &= \sum_{k=0}^{m} a_k D^k \left(we^{\mu x}\right) = \sum_{k=0}^{m} a_k \sum_{j=0}^{k} \binom{k}{j} D^j w \, D^{k-j} e^{\mu x} \\
&= \sum_{k=0}^{m} a_k \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} D^j w \, D^{k-j} e^{\mu x} \\
&= \sum_{j=0}^{m} \frac{D^j w}{j!} \sum_{k=j}^{m} a_k k(k-1)(k-2)\cdots(k-j+1)\mu^{k-j} e^{\mu x} \\
&= \sum_{j=0}^{m} \frac{D^j w}{j!} \sum_{k=j}^{m} a_k k(k-1)(k-2)\cdots(k-j+1)\mu^{k-j} e^{\mu x} \\
&= \sum_{j=0}^{m} \frac{D^j w}{j!} p^{(j)}(\mu) e^{\mu x} = e^{\mu x} \sum_{j=0}^{m} p^{(j)}(\mu) \frac{D^j w}{j!} \\
&= e^{\mu x} q(D) w.
\end{aligned}
$$

An example

Consider the ODE

$$Lu = 12e^{2x} \quad \text{where} \quad Lu = u''' - 4u'' + 4u'.$$

Here, $L = p(D)$ for $p(z) = z^3 - 4z^2 + 4z = z(z-2)^2$, so $p(2) = p'(2) = 0$ but $p''(2) \neq 0$. Thus, we try

$$u_{\mathrm{P}} = Cx^2 e^{2x}$$

and find

$$u_{\mathrm{P}}' = C(2x + 2x^2)e^{2x}, \qquad u_{\mathrm{P}}'' = C(2 + 8x + 4x^2)e^{2x},$$
$$u_{\mathrm{P}}''' = C(12 + 24x + 8x^2)e^{2x},$$

so $Lu_{\mathrm{P}} = 4Ce^{2x}$ and we require $4C = 12$. Therefore, a particular solution is

$$u_{\mathrm{P}} = 3x^2 e^{2x}.$$

Complex conjugate roots

Consider

$$Lu \equiv p(D) \equiv u''' + u' + 10u = 13e^x \sin 2x.$$

Here, $p(z) = z^3 + z + 10 = [(z-1)^2 + 4](z+2)$ so $p(1 \pm 2i) = 0$, and

$$e^x \sin 2x = e^x \, \frac{e^{2ix} - e^{-2ix}}{2i}$$

is a linear combination of $e^{(1+2i)x}$ and $e^{(1-2i)x}$. Therefore put

$$u_{\mathrm{P}}(x) = Cxe^{(1+2i)x} + Exe^{(1-2i)x} = xe^x\big(F\cos 2x + G\sin 2x\big).$$

We find that if $F = -3/4$ and $G = -1/2$ then

$$Lu_{\mathrm{P}} = (-8F + 12G)e^x \cos 2x + (-12F - 8G)e^x \sin 2x = 13e^x \sin 2x,$$

so

$$u_{\mathrm{P}} = -\frac{xe^x}{4}\big(3\cos 2x + 2\sin 2x\big).$$

Variation of parameters

What if f is not a polynomial times an exponential, or if L does not have constant coefficients?

Consider a linear, second-order, inhomogeneous ODE with leading coefficient 1:

$$Lu = u''(x) + p(x)u'(x) + q(x)u(x) = f(x). \tag{5}$$

Let $u_1(x)$ and $u_2(x)$ be linearly independent solutions to the homogeneous equation and let $W(x) = W(x; u_1, u_2)$ denote their Wronskian. Thus,

$$Lu_1 = 0, \quad Lu_2 = 0, \quad W \neq 0.$$

We seek v_1 and v_2 such that

$$u(x) = v_1(x)u_1(x) + v_2(x)u_2(x)$$

is a (particular) solution to $Lu = f$.

Variation of parameters (continued)

To simplify the expression

$$u' = v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2'$$

we impose the condition $v_1' u_1 + v_2' u_2 = 0$, then (as if v_1 and v_2 were constant parameters)

$$u' = v_1 u_1' + v_2 u_2'.$$

A short calculation now shows

$$Lu = v_1 Lu_1 + v_2 Lu_2 + v_1' u_1' + v_2' u_2' = v_1' u_1' + v_2' u_2',$$

since by assumption $Lu_1 = 0 = Lu_2$.

Conclusion: $u = v_1 u_1 + v_2 u_2$ satisfies $Lu = f$ if

$$\begin{aligned} v_1' u_1 + v_2' u_2 &= 0, \\ v_1' u_1' + v_2' u_2' &= f. \end{aligned}$$

Thus, we have a pair of equations for the unknown v_1' and v_2'. In matrix form

$$\begin{bmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{bmatrix} \begin{bmatrix} v_1'(x) \\ v_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix},$$

so

$$\begin{bmatrix} v_1'(x) \\ v_2'(x) \end{bmatrix} = \frac{1}{W(x)} \begin{bmatrix} u_2'(x) & -u_2(x) \\ -u_1'(x) & u_1(x) \end{bmatrix} \begin{bmatrix} 0 \\ f(x) \end{bmatrix},$$

or in other words,

$$v_1'(x) = \frac{-u_2(x)f(x)}{W(x)} \quad \text{and} \quad v_2'(x) = \frac{u_1(x)f(x)}{W(x)}.$$

Example

Find the general solution to

$$u'' - 4u' + 4u = (x+1)e2x.$$

Solution via power series

If L has variable coefficients, then we cannot expect in general that the solution of $Lu = 0$ is expressible in terms of elementary functions like polynomials, trigonometric functions, exponentials etc. Power series provide a flexible way to represent u in this case.

Constructing a series solution

Consider the initial-value problem

$$Lu = (1 - x^2)u'' - 5xu' - 4u = 0, \quad u(0) = 1, \quad u'(0) = 2.$$

Look for a solution in the form of a power series

$$u(x) = \sum_{k=0}^{\infty} A_k x^k = A_0 + A_1 x + A_2 x^2 + \cdots .$$

Formal calculations show that

$$Lu = \sum_{k=0}^{\infty} (k+2)[(k+1)A_{k+2} - (k+2)A_k]x^k,$$

and the initial conditions imply $A_0 = 1$ and $A_1 = 2$.

Convergence?

Since Lu is identically zero iff the coefficient of x^k vanishes for every k, we obtain the recurrence relation

$$A_{k+2} = \frac{k+2}{k+1} A_k \quad \text{for } k = 0,\ 1,\ 2,\ \ldots.$$

Thus,

$$A_0 = 1, \quad A_1 = 2, \quad A_2 = 2, \quad A_3 = 3, \quad \ldots,$$

giving

$$u(x) = 1 + 2x + 2x^2 + 3x^3 + \cdots .$$

Since

$$\lim_{k\to\infty} \frac{A_{k+2}x^{k+2}}{A_k x^k} = \lim_{k\to\infty} \frac{k+2}{k+1} x^2 = x^2,$$

the ratio test shows that $\sum_{j=0}^{\infty} A_{2j}x^{2j}$ and $\sum_{j=0}^{\infty} A_{2j+1}x^{2j+1}$ converge for $x^2 < 1$ but diverge for $x^2 > 1$.

General case

Consider a general second-order, linear, homogeneous ODE

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = 0.$$

Equivalently,

$$u'' + p(x)u' + q(x)u = 0,$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)} \quad \text{and} \quad q(x) = \frac{a_0(x)}{a_2(x)}.$$

Assume that a_j is analytic at 0 for $0 \le j \le 2$, and that $a_2(0) \ne 0$. Then p and q are analytic at 0, that is, they admit power series expansions

$$p(z) = \sum_{k=0}^{\infty} p_k z^k \quad \text{and} \quad q(z) = \sum_{k=0}^{\infty} q_k z^k \quad \text{for } |z| < \rho,$$

for some $\rho > 0$.

Formal expansions

If

$$u(z) = \sum_{k=0}^{\infty} A_k z^k$$

then we find that

$$\begin{aligned} Lu(z) = (2A_2 + p_0 A_1 + q_0 A_0) \\ + (6A_3 + 2p_0 A_2 + p_1 A_1 + q_0 A_1 + q_1 A_0) z + \cdots , \end{aligned}$$

where, on the RHS, the coefficient of z^{n-1} for a general $n \geq 1$ is

$$(n+1)nA_{n+1} + \sum_{j=0}^{n-1} \big[(n-j) p_j A_{n-j} + q_j A_{n-1-j} \big].$$

Convergence theorem

Given $u(0)$ and $u'(0)$, we put

$$A_0 = u(0) \quad \text{and} \quad A_1 = u'(0),$$

and compute recursively

$$A_{n+1} = \frac{-1}{n(n+1)} \sum_{j=0}^{n-1} \big[(n-j)p_j A_{n-j} + q_j A_{n-1-j}\big], \quad n \geq 1.$$

Theorem

If the coefficients $p(z)$ and $q(z)$ are analytic for $|z| < \rho$, then the formal power series for the solution $u(z)$, constructed above, is also analytic for $|z| < \rho$.

Previous example

Earlier we considered

$$Lu = (1 - x^2)u'' - 5xu' - 4u = 0, \quad u(0) = 1, \quad u'(0) = 2.$$

In this case,

$$p(z) = \frac{-5z}{1 - z^2} = -5 \sum_{k=0}^{\infty} z^{2k+1}$$

and

$$q(z) = \frac{-4}{1 - z^2} = -4 \sum_{k=0}^{\infty} z^{2k}$$

are analytic for $|z| < 1$, so the theorem guarantees that $u(z)$, given by the formal power series, is also analytic for $|z| < 1$.

Expansion about a point other than 0

Suppose we want a power series expansion about a point $c \neq 0$, for instance because the initial conditions are given at $x = c$.

A simple change of the independent variable allows us to write

$$u = \sum_{k=0}^{\infty} A_k (z-c)^k = \sum_{k=0}^{\infty} A_k Z^k \quad \text{where } Z = z - c.$$

Since $du/dz = du/dZ$ and $d^2u/dz^2 = d^2u/dZ^2$, we obtain the translated equation

$$\frac{d^2u}{dZ^2} + p(Z+c)\,\frac{du}{dZ} + q(Z+c)u = 0.$$

Now compute the A_k using the series expansions of $p(Z+c)$ and $q(Z+c)$ in powers of Z.

Example

We construct a power series solution about $z = 1$ to Airy's equation:

$$u'' - zu = 0.$$

Put $Z = z - 1$ and find that $u'' - zu = u'' - (Z+1)u$ equals

$$\begin{aligned}
&\sum_{k=0}^{\infty} k(k-1)A_k Z^{k-2} - \sum_{k=0}^{\infty} A_k Z^{k+1} - \sum_{k=0}^{\infty} A_k Z^k \\
&\quad = \sum_{k=0}^{\infty}(k+2)(k+1)A_{k+2}Z^k - \sum_{k=1}^{\infty} A_{k-1}Z^k - \sum_{k=0}^{\infty} A_k Z^k \\
&\quad = (2A_2 - A_0) + \sum_{k=1}^{\infty}\big[(k+2)(k+1)A_{k+2} - A_{k-1} - A_k\big]Z^k.
\end{aligned}$$

Thus, the coefficients must satisfy $2A_2 - A_0 = 0$ and

$$(k+2)(k+1)A_{k+2} - A_{k-1} - A_k = 0 \quad \text{for all } k \geq 1,$$

so

$$A_2 = \frac{A_0}{2} \quad \text{and} \quad A_{k+2} = \frac{A_{k-1} + A_k}{(k+2)(k+1)} \quad \text{for } k \geq 1.$$

We find that

$$u(z) = A_0\left(1 + \frac{(z-1)^2}{2} + \frac{(z-1)^3}{6} + \frac{(z-1)^4}{24} + \cdots\right) \\ + A_1\left((z-1) + \frac{(z-1)^3}{6} + \frac{(z-1)^4}{12} + \cdots\right).$$

Singular ODEs

Recall that our basic existence and uniqueness theorem for $Lu = f$ assumes that L is not singular, that is, the leading coeffecient of L does not vanish on the interval of interest. However, some important applications lead to singular ODEs so we must now address this case.

Singular ODEs of second order

Consider

$$a_2(x)u'' + a_1(x)u' + a_0(x)u = 0 \quad \text{for } a \le x \le b,$$

and suppose that $a_2(x_0) = 0$ for some x_0 with $a < x_0 < b$, but $a_2(x) \neq 0$ if $x \neq x_0$. Put

$$b_j(y) = a_j(x) \quad \text{and} \quad v(y) = u(x) \quad \text{where } y = x - x_0,$$

so that, with $c = a - x_0 < 0 < d = b - x_0$,

$$b_2(y)v'' + b_1(y)v' + b_0(y)v = 0 \quad \text{for } c \le y \le d.$$

Since $y = 0$ when $x = x_0$, we have $b_2(0) = 0$.

In this way, it suffices to consider the case when the leading coefficient vanishes at the origin.

Cauchy–Euler ODE

A second-order Cauchy–Euler ODE has the form

$$Lu = ax^2u'' + bxu' + cu = f(x),$$

where a, b and c are constants, with $a \neq 0$. This ODE is singular at $x = 0$.

Noticing that

$$Lx^r = \big[ar(r-1) + br + c\big]x^r,$$

we see that $u = x^r$ is a solution of the homogeneous equation ($f = 0$) iff

$$ar(r-1) + br + c = 0.$$

Factorization

Suppose $ar(r-1)+br+c = a(r-r_1)(r-r_2)$. If $r_1 \neq r_2$ then the general solution of the homogeneous equation $Lu = 0$ is

$$u(x) = C_1 x^{r_1} + C_2 x^{r_2}, \qquad x > 0.$$

Lemma

If $r_1 = r_2$ then the general solution of the homogeneous Cauchy–Euler equation $Lu = 0$ is

$$u(x) = C_1 x^{r_1} + C_2 x^{r_1} \log x, \qquad x > 0.$$

Example

Solve $x^2 u'' - xu' + u = 0$.

Example

Solve $2(x-2)^2 u'' - 3(x-2)u' - 3u = 0$.

Proof of the lemma

Since $r_1 = r_2$ the function $F(x, r) = x^r$ satisfies

$$ax^2 F'' + bxF' + cF = a(r - r_1)^2 x^r,$$

where the dash means $\partial/\partial x$. Put

$$v(x) = \frac{\partial F}{\partial r}\bigg|_{r=r_1} = \frac{\partial}{\partial r}\, e^{r \log x}\bigg|_{r=r_1} = e^{r_1 \log x} \log x = x^{r_1} \log x$$

and observe that

$$\begin{aligned}
ax^2 v'' + bxv' + cv &= \left(ax^2 \frac{\partial^2}{\partial x^2} \frac{\partial F}{\partial r} + bx \frac{\partial}{\partial x} \frac{\partial F}{\partial r} + c \frac{\partial F}{\partial r}\right)\bigg|_{r=r_1} \\
&= \frac{\partial}{\partial r}\left(ax^2 \frac{\partial^2 F}{\partial x^2} + bx \frac{\partial F}{\partial x} + cF\right)\bigg|_{r=r_1} \\
&= \frac{\partial}{\partial r}\big(a(r - r_1)^2 x^r\big)\big|_{r=r_1} \\
&= \big(2a(r - r_1)x^r + (r - r_1)^2 x^r \log x\big)\big|_{r=r_1} = 0.
\end{aligned}$$

More general singular ODEs

A number of important applications lead to ODEs that can be written in the Frobenious normal form

$$z^2 u'' + zP(z)u' + Q(z)u = 0,$$

where $P(z)$ and $Q(z)$ are analytic at $z = 0$:

$$P(z) = \sum_{k=0}^{\infty} P_k z^k \quad \text{and} \quad Q(z) = \sum_{k=0}^{\infty} Q_k z^k, \qquad |z| < \rho. \tag{6}$$

Notice that $u'' + p(z)u' + q(z)u = 0$ but $p(z) = z^{-1}P(z)$ and $q(z) = z^{-2}Q(z)$ are not analytic at $z = 0$ unless $P(0) = 0$ and $Q(0) = Q'(0) = 0$.

So in general we cannot expect a solution $u(z)$ to be analytic at $z = 0$.

A clue

We can think of an ODE in Frobenius normal form as a Cauchy–Euler ODE with variable coefficients.

For z near 0 we have $P(z) \approx P_0$ and $Q(z) \approx Q_0$ so we might expect $u(z)$ to behave like a solution of

$$z^2u'' + P_0zu' + Q_0u = 0.$$

We therefore consider the indicial polynomial

$$I(r) = r(r-1) + P_0r + Q_0 = (r - r_1)(r - r_2).$$

If $r_1 \neq r_2$ then the approximating Cauchy–Euler ODE has the general solution $c_1z^{r_1} + c_2z^{r_2}$, so it is natural to seek a solution in the form

$$u(z) = z^r \sum_{k=0}^{\infty} A_k z^k = \sum_{k=0}^{\infty} A_k z^{k+r}, \quad |z| < \rho, \quad \text{with } A_0 \neq 0.$$

An example

Consider

$$Lu = 2z^2u'' + 7zu' - (z^2 + 3)u = 0.$$

Here, $P(z) = 7/2$ and $Q(z) = -(z^2 + 3)/2$ are trivially analytic at $z = 0$ (since they are polynomials).

The approximations $P(z) \approx P_0 = 7/2$ and $Q(z) \approx Q_0 = -3/2$ lead to the Cauchy–Euler equation $z^2u'' + (7/2)zu' - (3/2)u = 0$ or

$$2z^2u'' + 7zu' - 3u = 0.$$

Thus, the indicial polynomial is

$$2r(r-1) + 7r - 3 = 2r^2 + 5r - 3 = (2r-1)(r+3)$$

so $r_1 = 1/2$ and $r_2 = -3$.

Using

$$u = \sum_{k=0}^{\infty} A_k z^{k+r}, \qquad u' = \sum_{k=0}^{\infty} (k+r) A_k z^{k+r-1},$$

$$u'' = \sum_{k=0}^{\infty} (k+r)(k+r-1) A_k z^{k+r-2},$$

we find that

$$\begin{aligned} Lu &= (2z^2 u'' + 7zu' - 3u) - z^2 u \\ &= \sum_{k=0}^{\infty} \big[2(k+r)(k+r-1) + 7(k+r) - 3 \big] A_k z^{k+r} \\ &\qquad - \sum_{k=0}^{\infty} A_k z^{k+r+2}. \end{aligned}$$

Since

$$2(k+r)(k+r-1)+7(k+r)-3 = 2(k+r)^2+5(k+r)-3$$
$$= [2(k+r)-1][(k+r)+3]$$

and

$$\sum_{k=0}^{\infty} A_k z^{k+r+2} = \sum_{k=2}^{\infty} A_{k-2} z^{k+r}$$

it follows that

$$Lu = (2r-1)(r+3)A_0 z^r + (2r+1)(r+4)A_1 z^{r+1}$$
$$+ \sum_{k=2}^{\infty} \big[(2k+2r-1)(k+r+3)A_k - A_{k-2}\big] z^{k+r}.$$

Conclusion: u is a solution if $r \in \{1/2, -3\}$ with

$$A_1 = 0, \qquad A_k = \frac{A_{k-2}}{(2k+2r-1)(k+r+3)} \quad \text{for all } k \geq 2.$$

First solution: $r = 1/2$ with

$$A_1 = 0, \qquad A_k = \frac{A_{k-2}}{k(2k+7)} \quad \text{for all } k \geq 2,$$

so

$$A_2 = \frac{A_0}{22}, \quad A_3 = \frac{A_1}{39} = 0, \quad A_4 = \frac{A_2}{60} = \frac{A_0}{1320}, \quad \ldots$$

and

$$u(z) = A_0 z^{1/2}\left(1 + \frac{z^2}{22} + \frac{z^4}{1320} + \cdots\right).$$

Second solution: $r = -3$ with

$$A_1 = 0, \qquad A_k = \frac{A_{k-2}}{k(2k-7)} \quad \text{for all } k \geq 2,$$

so

$$A_2 = -\frac{A_0}{6}, \quad A_3 = -\frac{A_1}{3} = 0, \quad A_4 = -\frac{A_2}{4} = \frac{A_0}{24}, \quad \ldots$$

and

$$u(z) = A_0 z^{-3}\left(1 - \frac{z^2}{6} + \frac{z^4}{24} + \cdots\right).$$

General solution of $Lu = 0$:

$$u(z) = Az^{1/2}\left(1 + \frac{z^2}{22} + \frac{z^4}{1320} + \cdots\right) + Bz^{-3}\left(1 - \frac{z^2}{6} + \frac{z^4}{24} + \cdots\right).$$

General case

Now consider

$$z^2u'' + zP(z)u' + Q(z)u = 0$$

for $P(z)$ and $Q(z)$ satisfying (6). Formal manipulations show that $Lu(z)$ equals

$$I(r)A_0z^r + \sum_{k=1}^{\infty}\left(I(k+r)A_k + \sum_{j=0}^{k-1}[(j+r)P_{k-j} + Q_{k-j}]A_j\right)z^{k+r},$$

so we define $A_0(r) = 1$ and

$$A_k(r) = \frac{-1}{I(k+r)}\sum_{j=0}^{k-1}[(j+r)P_{k-j} + Q_{k-j}]A_j(r), \quad k \geq 1,$$

provided $I(k+r) \neq 0$ for all $k \geq 1$.

Choice of exponent

The preceding calculations show that the series

$$F(z;r) = \sum_{k=0}^{\infty} A_k(r) z^{k+r}$$

satisfies

$$z^2 F'' + zP(z)F' + Q(z)F = I(r)z^r,$$

with

$$I(r) = r(r-1) + P_0 r + Q_0 = (r-r_1)(r-r_2).$$

Assume, with no loss of generality, that $\operatorname{Re} r_1 \geq \operatorname{Re} r_2$. It follows that $I(k+r_1) \neq 0$ for all integers $k \geq 1$, and therefore $u_1(z) = F(z;r_1)$ is (formally) a solution.

If $r_1 - r_2$ is not a whole number, then a second, linearly independent solution is $u_2(z) = F(z;r_2)$.

Roots differing by an integer

Suppose that $r_1 = r_2$. In this case, $I(r) = (r - r_1)^2$ and so

$$z^2F'' + zP(z)F' + Q(z)F = (r - r_1)^2 z^r.$$

The function $v = \partial F/\partial r$ satisfies

$$z^2v'' + zP(z)v' + Q(z)v = 2(r - r_1)z^r + (r - r_1)^2 z^r \log z,$$

and the RHS is zero if $r = r_1$, so a second, linearly independent solution is

$$u_2(z) = \frac{\partial F}{\partial r}(z; r_1) = \sum_{k=0}^{\infty} A_k'(r_1)z^{k+r_1} + \underbrace{\sum_{k=0}^{\infty} A_k(r_1)z^{k+r_1}\log z}_{u_1(z)\log z}.$$

Even worse complications if $r_1 = r_2 + n$ for an integer $n \geq 1$.

Bessel and Legendre equations

We wrap up this part of the course with two particularly important examples of second-order, linear ODEs with variable coefficients. Both occur several times later in the course.

Bessel equation

The Bessel equation with parameter ν is

$$z^2u'' + zu' + (z^2 - \nu^2)u = 0.$$

This ODE is in Frobenius normal form, with indicial polynomial

$$I(r) = (r + \nu)(r - \nu),$$

and we seek a series solution

$$u(z) = \sum_{k=0}^{\infty} A_k z^{k+r}.$$

We assume $\operatorname{Re}\nu \geq 0$, so $r_1 = \nu$ and $r_2 = -\nu$.

Recurrence relation

We find that if

$$(r+1+\nu)(r+1-\nu)A_1 = 0,$$
$$(k+r+\nu)(k+r-\nu)A_k + A_{k-2} = 0, \quad k \geq 2.$$

then

$$z^2u'' + zu' + (z^2-\nu^2)u = (r+\nu)(r-\nu)A_0 z^r.$$

Taking $r = \nu$ gives

$$A_k = \frac{-A_{k-2}}{k(k+2\nu)} \quad \text{for } k \geq 2,$$

so with A_0 arbitrary and $A_1 = 0$ we obtain

$$u(z) = A_0 z^\nu \Bigg[1 - \frac{(z/2)^2}{1+\nu} + \frac{(z/2)^4}{2(2+\nu)(1+\nu)} - \frac{(z/2)^6}{3!(3+\nu)(2+\nu)(1+\nu)} + \cdots\Bigg].$$

Bessel function

With the normalisation

$$A_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$$

the series solution is called the Bessel function of order ν and is denoted

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} \left[1 - \frac{(z/2)^2}{1+\nu} + \frac{(z/2)^4}{2!(1+\nu)(2+\nu)} - \cdots \right].$$

From the functional equation $\Gamma(1+z) = z\Gamma(z)$ we see that

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} - \frac{(z/2)^{\nu+2}}{\Gamma(2+\nu)} + \frac{(z/2)^{\nu+4}}{2!\Gamma(3+\nu)} - \frac{(z/2)^{\nu+6}}{3!\Gamma(4+\nu)} + \cdots$$

and so

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\Gamma(k+1+\nu)}.$$

Bessel function of negative order

If ν is not an integer, then a second, linearly independent, solution is

$$J_{-\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k-\nu}}{k!\Gamma(k+1-\nu)}.$$

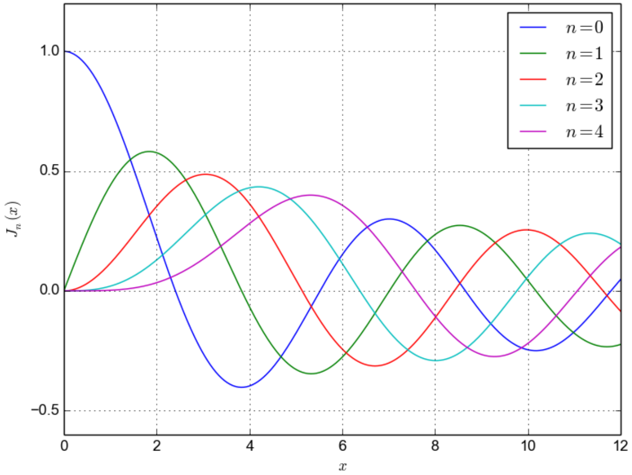
For an integer $\nu = n \in \mathbb{Z}$, since $\Gamma(n+1) = n!$ we have

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+n}}{k!(k+n)!}.$$

Also, since $1/\Gamma(z) = 0$ for $z = 0$, -1, -2, …, we find that J_n and J_{-n} are linearly dependent; in fact,

$$J_{-n}(z) = (-1)^n J_n(z).$$

Bessel functions of integer order



Neumann function

The Neumann function (or Bessel function of the second kind) is

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \quad \text{if } \nu \notin \mathbb{Z}.$$

For $n \in \mathbb{Z}$, L'Hospital's rule shows that if $\nu \to n$ then $Y_\nu(z)$ tends to a finite limit

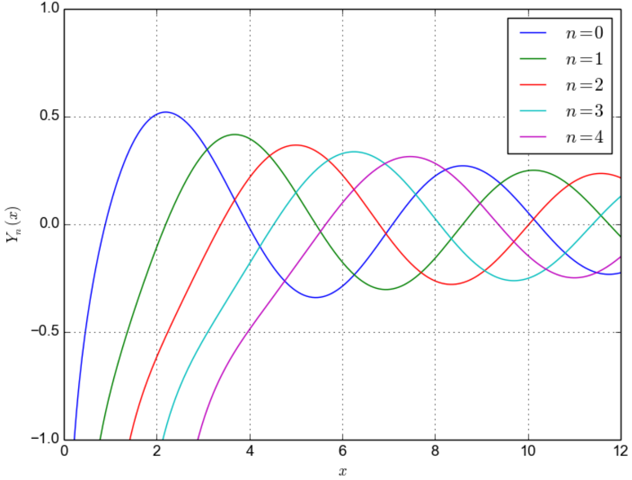
$$Y_n(z) = \lim_{\nu\to n} Y_\nu(z).$$

The functions J_ν and Y_ν are linearly independent solutions of Bessel's equation for all complex ν.
As $z \to 0$ with ν fixed,

$$J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)}, \quad \nu \notin \{-1,-2,-3,\ldots\},$$

$$Y_0(z) \sim \frac{2}{\pi}\log z, \qquad Y_\nu(z) \sim \frac{-\Gamma(\nu)}{\pi(z/2)^\nu}, \quad \operatorname{Re}\nu > 0.$$

Neumann functions of integer order



Legendre equation

The Legendre equation with parameter ν is

$$(1 - z^2)u'' - 2zu' + \nu(\nu + 1)u = 0.$$

This ODE is not singular at $z = 0$ so the solution has an ordinary Taylor series expansion

$$u = \sum_{k=0}^{\infty} A_k z^k.$$

The A_k must satisfy

$$(k + 1)(k + 2)A_{k+2} - [k(k + 1) - \nu(\nu + 1)]A_k = 0$$

for $k \geq 0$, and since

$$k(k + 1) - \nu(\nu + 1) = (k - \nu)(k + \nu + 1),$$

the recurrence relation is

$$A_{k+2} = \frac{(k - \nu)(k + \nu + 1)}{(k + 1)(k + 2)} A_k \quad \text{for } k \geq 0.$$

General solution

We have

$$u(z) = A_0 u_0(z) + A_1 u_1(z)$$

where

$$u_0(z) = 1 - \frac{\nu(\nu+1)}{2!} z^2 + \frac{(\nu-2)\nu(\nu+1)(\nu+3)}{4!} z^4 - \cdots$$

and

$$u_1(z) = z - \frac{(\nu-1)(\nu+2)}{3!} z^3 + \frac{(\nu-3)(\nu-1)(\nu+2)(\nu+4)}{5!} z^5 - \cdots .$$

Suppose now that $\nu = n$ is a non-negative integer. If n is even then the series for $u_0(z)$ terminates, whereas if n is odd then the series for $u_1(z)$ terminates.

Legendre polynomial

The terminating solution is called the Legendre polynomial of degree n and is denoted by $P_n(z)$ with the normalization

$$P_n(1) = 1.$$

The first few Legendre polynomials are

$$\begin{aligned} P_0(z) &= 1, & P_3(z) &= \tfrac{1}{2}(5z^3 - 3z), \\ P_1(z) &= z, & P_4(z) &= \tfrac{1}{8}(35z^4 - 30z^2 + 3), \\ P_2(z) &= \tfrac{1}{2}(3z^2 - 1), & P_5(z) &= \tfrac{1}{8}(63z^5 - 70z^3 + 15z). \end{aligned}$$

Notice that P_n is an even or odd function according to whether n is even or odd.

Behaviour of Legendre polynomials