THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education,
Engineering and Science

MATH1901/1906

Differential Calculus (Advanced)

June 2006

Lecturer: Jenny Henderson

Time Allowed: **One and a half hours**

Name: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 25% of the total examination;
there are 15 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 75% of the total examination;
there are 6 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

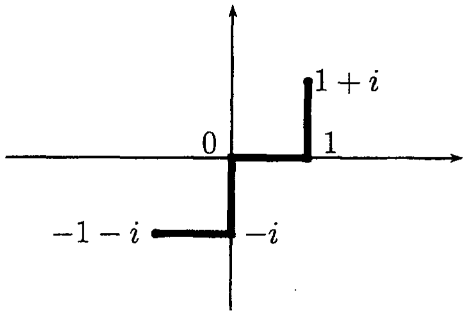
Extended Answer Section

Answer these questions in the answer book provided.
Ask for extra books if you need them.

1. (a) Let $z = 3 + 2i$ and $w = 2 - i$. Write $\dfrac{z}{w} + \overline{w}$ in cartesian form. (2 Marks)

   (b) Explain why the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$ is surjective but is not injective. (3 Marks)

   (c) Consider the set S of complex numbers forming a 'step' pattern between $-1 - i$ and $1 + i$, as shown.



   Find and sketch the image of S under the function $z \mapsto e^z$. (5 Marks)

2. (a) Calculate the Taylor polynomial $T_4(x)$ of order 4 for $f(x) = \ln(1 + x)$, about 0. (2 Marks)

   (b) By observing the pattern in the derivatives of $f(x)$ in the previous part, write down the Taylor polynomial $T_n(x)$ of order n for $f(x)$, where n is any positive integer. (1 Mark)

   (c) Write down the remainder term $R_n(x)$ for the Taylor polynomial in part (b). (2 Marks)

   (d) How large need n be taken to ensure that $T_n(x)$ gives a value of $\ln(1.3)$ which has an error of less than 0.0002? (3 Marks)

   (e) Using your answer to part (a), write down without any calculation the Taylor polynomial of order 8 for $\ln(1 + x^2)$, about 0. Explain briefly why this is valid. (2 Marks)

3. Find the following limits, showing your working.

   (a) $\displaystyle\lim_{x\to\infty} \left(\frac{x^2}{x+1} - \frac{x^2}{x-1}\right)$ (b) $\displaystyle\lim_{t\to 0} \frac{t}{\sqrt{4+t} - \sqrt{4-t}}$ (4 Marks)

   (c) $\displaystyle\lim_{x\to\infty} \left(1 + \sin\left(\frac{3}{x}\right)\right)^x$ (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ (6 Marks)

4. (a) (i) Apply the Mean Value Theorem to the function $f(x) = \tan^{-1} x$ to show that, for all positive $a < b$,

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}.$$

(3 Marks)

(ii) Use part (i) to show that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

(2 Marks)

(b) Suppose that $f(x, y)$ is a function of two variables with

$$f_x(0,2) = 2 \quad \text{and} \quad f_y(0,2) = -1.$$

Using the chain rule, compute the numerical value of $f_\theta(r\cos\theta, r\sin\theta)$ at $r = 2$, $\theta = \pi/2$. (5 Marks)

5. (a) Consider the surface given by the function F, where $F(x, y) = xe^{-(x^2+y^2)}$.

(i) You are standing on the surface at the point $(1, -1, e^{-2})$.
In which direction is the slope largest?
What is the value of this slope? (3 Marks)

(ii) Find all the points (a, b, c) on the surface where the tangent plane is horizontal. (3 Marks)

(b) The function g has domain $\mathbb{R}$, is continuous at the point 0 and satisfies the condition

$$g(x+y) = g(x) + g(y), \qquad \forall\, x,\, y \in \mathbb{R}.$$

Prove that g is continuous at the point a, for each $a \in \mathbb{R}$. (4 Marks)

6. The function f is defined as follows:

$$f(x) = \begin{cases} x + 2x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(a) Show, using the definition of derivative as a limit, that $f'(0) = 1$. (3 Marks)

(b) Show that f' is not continuous at 0. (3 Marks)

(c) Show that f is not increasing on any interval containing 0. (4 Marks)

End of Extended Answer Section