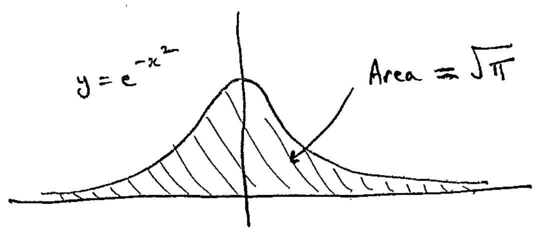
An improper integral of central importance to statistics

Theorem: $\boxed{\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$

existence by Comparison Test

evaluation using some tricks which are formalized in 2nd year



Existence: By symmetry

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx$$

so LHS converges precisely when

$$\int_0^{\infty} e^{-x^2}\,dx \quad \text{converges}.$$

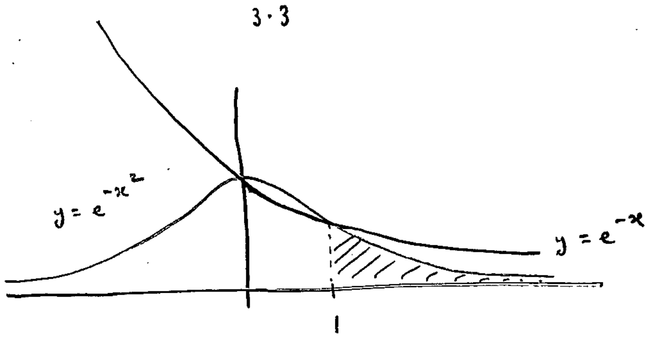
But

$$\int_0^{\infty} e^{-x^2}\,dx = \underbrace{\int_0^{1} e^{-x^2}\,dx}_{\text{proper}} + \underbrace{\int_1^{\infty} e^{-x^2}\,dx}$$

converges by comparison with

$$\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\int_1^{b} e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_1^{b} = \lim_{b\to\infty} -e^{-b}+e^{-1} = e^{-1}.$$



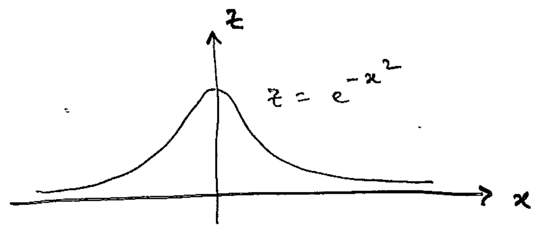
Calculation of

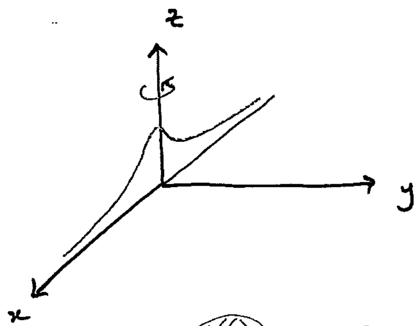
$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx :$$

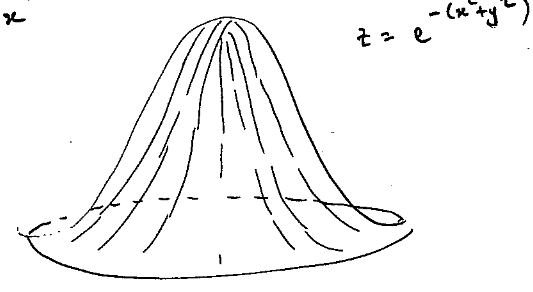
Some lateral thinking required !!

Put $\boxed{f(x,y) = e^{-(x^2+y^2)}}$

function of <u>two</u> variables obtained by revolving $z = e^{-x^2}$ about z-axis.



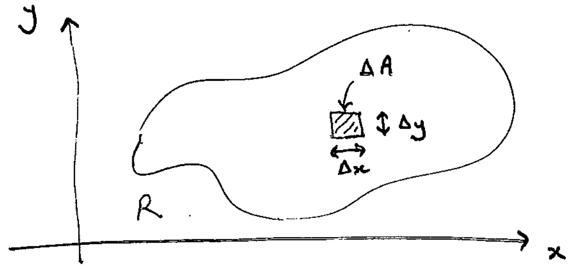




bell-shaped <u>surface</u>

3.5

Let R be a region in the xy-plane



and define

$$\iint_R f(x,y)\, dA \overset{\text{def}}{=} \text{volume under surface } z = f(x,y) \text{ over region } R$$

- called a <u>double integral</u>
- limit of Riemann sums by subdividing R into small squares

3.6

We prove the following

Claim:

$$\underbrace{I^2 = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA}_{} = \pi$$

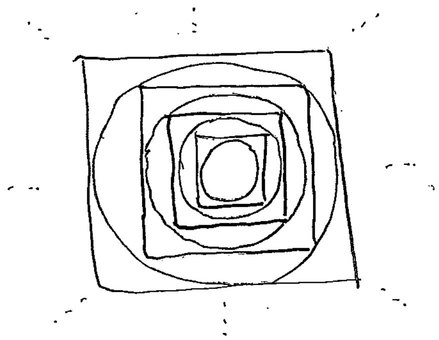
volume over the entire xy-plane!

We calculate this volume in two different ways, using limits over increasingly big

- square regions
- circular regions

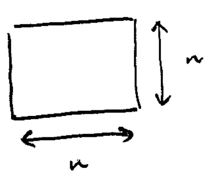
3.7

Circles and squares "interleave" and expand to cover the plane:



Let

S_n = $n \times n$ square



C_n = circle of radius n

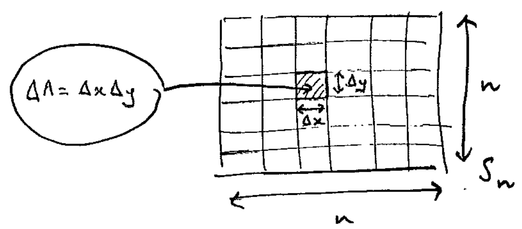
3.8

We can calculate the volume under the surface in two ways:

$$\iint_{\mathbb{R}^2} f(x,y)\, dA = \lim_{n\to\infty} \iint_{S_n} f(x,y)\, dA$$

$$= \lim_{n\to\infty} \iint_{C_n} f(x,y)\, dA$$

<u>Let's do the squares first</u>:



The approximating Riemann sums have the form

$$\sum f(x,y)\, \Delta A = \sum f(x,y)\, \Delta x\, \Delta y$$

3.9

We get

$$\iint_{S_n} e^{-(x^2+y^2)}\,dA = \lim \sum \left(e^{-(x^2+y^2)} \Delta x \Delta y\right)$$

$$= \lim \sum \left(e^{-x^2} \Delta x \; e^{-y^2} \Delta y\right)$$

$$= \lim \left(\sum e^{-x^2} \Delta x\right)\left(\sum e^{-y^2} \Delta y\right)$$

$$= \left(\lim \sum e^{-x^2} \Delta x\right)\left(\lim \sum e^{-y^2} \Delta y\right)$$

$$= \int_{-n}^{n} e^{-x^2}\,dx \int_{-n}^{n} e^{-y^2}\,dy$$

so

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$$

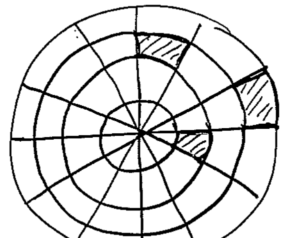
$$= \lim_{n\to\infty} \left(\int_{-n}^{n} e^{-x^2}\,dx \int_{-n}^{n} e^{-y^2}\,dy\right)$$

$$= \left(\lim_{n\to\infty} \int_{-n}^{n} e^{-x^2}\,dx\right)\left(\lim_{n\to\infty} \int_{-n}^{n} e^{-y^2}\,dy\right)$$

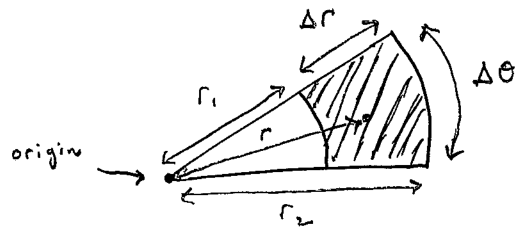
$$= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = I^2 .$$

3.10

Now we do the circles :



use polar rectangles !!



Area of polar rectangle

$$= \frac{\Delta\theta}{2} r_2^2 - \frac{\Delta\theta}{2} r_1^2$$

$$= \frac{\Delta\theta}{2}(r_2^2 - r_1^2) = \frac{\Delta\theta}{2}(r_2+r_1)(r_2-r_1)$$

$$= \frac{r_2+r_1}{2} \Delta r \Delta\theta \approx r \Delta r \Delta\theta$$

3.11

We get

$$\iint_{C_n} e^{-(x^2+y^2)}\,dA = \lim \sum e^{-r^2} r \Delta r \Delta\theta$$

since $x^2+y^2 = r^2$

$$= \lim \left(\sum e^{-r^2} r \Delta r\right)\left(\sum \Delta\theta\right)$$

$$= \left(\int_0^n r e^{-r^2}\,dr\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \left[-\frac{e^{-r^2}}{2}\right]_0^n \left[\theta\right]_0^{2\pi}$$

$$= \left(-\frac{e^{-n^2}}{2} + \frac{1}{2}\right)(2\pi)$$

$$= \pi\left(1 - e^{-n^2}\right)$$

3.12

Hence

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$$

$$= \lim_{n\to\infty} \pi\left(1 - e^{-n^2}\right) = \pi .$$

Thus $I^2 = \pi$

so $I = \sqrt{\pi}$

i.e.
$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

and the proof is complete.

$\square$