THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial for Week 5

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

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Topics covered

In lectures last week:

- ☐ Review of integration techniques.
- ☐ Integrating functions with discontinuities: Improper integrals.
- ☐ Integrating over an unbounded domain: Improper integrals.
- ☐ The p-integrals $\int_0^1 \frac{1}{x^p}\,dx$ and $\int_1^\infty \frac{1}{x^p}\,dx$.
- ☐ The Comparison Test for integrals.

Objectives

After completing this tutorial sheet you will be able to:

- ☐ Determine if an improper integral exists by taking a limit of proper integrals.
- ☐ Obtain an intuition for which integrals exist/do not exist.
- ☐ Apply the comparison test to show that an integral exists of does not exist.
- ☐ Work with inequalities to find suitable bounds.
- ☐ Understand that integration by parts can be used to study convergence.
- ☐ Use integration by parts and limit laws to prove the elegant *Wallis product.*

Preparation questions to do *before* class

1. Determine whether the following improper integrals exist by evaluating an appropriate limit of a proper integral. If the integral exists, compute its value.

(a) $\displaystyle\int_0^\infty xe^{-x}\,dx$

Solution: By definition, the improper integral is

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx.$$

Integrating by parts gives

$$\int_0^b xe^{-x}\,dx = -be^{-b} + \int_0^b e^{-x}\,dx = -be^{-b} - e^{-b} + 1.$$

Therefore

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\left(-be^{-b} - e^{-b} + 1\right) = 1,$$

and so the improper integral exists, and equals 1.

(b) $\displaystyle\int_0^1 \frac{\ln x}{x}\,dx$

Solution: The problem here is at the 0 endpoint of the integration range. By definition the improper integral is

$$\int_0^1 \frac{\ln x}{x}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{\ln x}{x}\,dx.$$

Using the change of variable $u = \ln x$ gives

$$\int_a^1 \frac{\ln x}{x}\,dx = \int_{\ln a}^0 u\,du = -\frac{1}{2}(\ln a)^2.$$

Therefore

$$\int_0^1 \frac{\ln x}{x}\,dx = -\frac{1}{2}\lim_{a\to 0^+}(\ln a)^2 = -\infty,$$

and so the integral diverges.

2. Use the Comparison Test to determine if the following improper integrals exist.

(a) $\displaystyle\int_0^1 \frac{e^{-x}}{x}\,dx$

Solution: Because $e^{-x}/x \sim 1/x$ for x near 0, we expect to get divergence by comparison with the integral of $1/x$ on $[0,1]$. Indeed

$$\frac{e^{-x}}{x} \geq \frac{e^{-1}}{x} \quad \text{for } 0 \leq x \leq 1,$$

and since $\displaystyle\int_0^1 \frac{e^{-1}}{x}\,dx$ diverges (p-integral), so does $\displaystyle\int_0^1 \frac{e^{-x}}{x}\,dx$ by the Comparison Test.

(b) $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$

Solution: We guess that this improper integral exists, because the dominant behaviour comes from the $\frac{1}{x^2}$, and $\int_1^\infty \frac{1}{x^2}\,dx$ exists. Indeed,

$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}\,,$$

and since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ exists (p-integral), so does the original integral by the Comparison Test.

Questions to attempt in class

3. Decide if the following improper integrals exist or not (either use the Comparison Test, or make a direct limit calculation). If they exist, try to compute their value (this is not always possible!).

(a) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

Solution: Integrating by parts gives

$$\int_1^b \frac{\ln x}{x^2}\,dx = -\frac{\ln}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx = -\frac{\ln b}{b} + 1 - b^{-1}.$$

Since $\lim_{b\to\infty} \frac{\ln b}{b} = 0$ we see that $\lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx = 1$. Therefore the improper integral exists, and equals 1.

(b) $\displaystyle\int_1^\infty \sin(\pi x)\,dx$

Solution: We compute

$$\int_1^b \sin(\pi x)\,dx = -\frac{\cos b\pi}{\pi} - \frac{1}{\pi}.$$

Since $\lim_{b\to\infty} \cos b\pi$ does not exist we see that the improper integral does not exist. Note that it does not diverge to ∞; rather $\int_1^b \cos(\pi x)\,dx$ oscillates, taking values between 0 and $-2/\pi$.

(c) $\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$

Solution: $0 \le e^{-x}/\sqrt{x} \le e^{-x}$ for all $x \ge 1$. Also,

$$\int_1^b e^{-x}\,dx = \Big[-e^{-x}\Big]_1^b = e^{-1} - e^{-b} \to e^{-1} \quad \text{as } b \to \infty.$$

So $\int_1^\infty e^{-x} dx$ converges, so $\int_1^\infty \frac{e^{-x}}{\sqrt{x}} dx$ converges too by the Comparison Test.

(d) $\displaystyle\int_0^\infty \frac{\cosh x}{x^2+1}\,dx$

Solution: Since $\cosh x = \frac{1}{2}(e^x + e^{-x}) \approx \frac{1}{2}e^x$ for large x, we expect that the integral does not exist – the integrand blows up as $x \to \infty$. Indeed, since

$$\lim_{x\to\infty} \frac{\cosh x}{x^2+1} = \infty,$$

there is a number X such that $\frac{\cosh x}{x^2+1} \ge 1$ for all $x > X$. Since

$$\int_X^\infty 1\,dx$$

does not exist, we conclude that the given integral does not exist by the Comparison Test.

(e) $\displaystyle\int_{\pi/4}^{\pi/2} \sec^2 x\,dx$

Solution: Since $\sec^2 x = 1/\cos^2 x \to \infty$ as $x \to \pi/2$, the integrand is unbounded, and the integral is improper.

$$\int_{\pi/4}^{\pi/2} \sec^2 x\,dx = \lim_{b\to(\pi/2)^-} \int_{\pi/4}^{b} \sec^2 x\,dx = \lim_{b\to(\pi/2)^-} (\tan(b) - \tan(\pi/4)) = \infty.$$

So the improper integral does not exist.

(f) $\displaystyle\int_{-\infty}^{0} e^x \cos x \, dx$

Solution: The dominant behaviour comes from the e^x, and as $x \to -\infty$ this decays very quickly. So we guess that the integral exists. Indeed,

$$0 \leq |e^x \cos x| \leq e^x,$$

and since $\int_{-\infty}^{0} e^x \, dx = 1$ exists, we conclude that the given integral exists by the Comparison Test.

It is possible to compute the value of the integral. Integrating by parts we have

$$\begin{aligned}\int_a^0 e^x \cos x \, dx &= -e^a \sin a - \int_a^0 e^x \sin x \, dx \\ &= -e^a \sin a + 1 - e^a \cos a - \int_a^0 e^x \cos x \, dx.\end{aligned}$$

Therefore

$$\int_a^0 e^x \cos x \, dx = \frac{1}{2} - \frac{1}{2} e^a \sin a - \frac{1}{2} e^a \cos a \to \frac{1}{2} \qquad \text{as } a \to -\infty.$$

So the improper integral equals $1/2$.

(g) $\displaystyle\int_0^1 \sin\left(\frac{1}{x}\right) dx$

Solution: The integrand has a (rather nasty) discontinuity at $x = 0$, and so we need to consider

$$\lim_{a \to 0^+} \int_a^1 \sin\left(\frac{1}{x}\right) dx.$$

It is easier to see what is happening after making the change of variable $y = \frac{1}{x}$. Then

$$\int_a^1 \sin\left(\frac{1}{x}\right) dx = \int_1^{a^{-1}} \frac{\sin y}{y^2} \, dy,$$

and our improper integral is

$$\int_0^1 \sin\left(\frac{1}{x}\right) dx = \lim_{b \to \infty} \int_1^b \frac{\sin y}{y^2} \, dy.$$

This integral exists by comparison with $\int_1^\infty \frac{1}{x^2} \, dx$. It is not easy to compute the exact value of the integral.

(h) $\displaystyle\int_0^\infty \frac{\cos x}{x^2 + 1} \, dx$

Solution: This integral exists by comparison to $\int_1^\infty \frac{1}{x^2} \, dx$. It is not easy to compute the exact value of this integral using elementary methods. But some "complex analysis" shows that the integral equals $\pi e/2$.

4. Does the improper integral
$$\int_1^\infty \frac{\sin x}{x}\,dx$$
exist? *Hint: Integration by parts might help.*

Solution:
$$\int_1^b \frac{\sin x}{x}\,dx = \int_1^b \frac{1}{x}\frac{d}{dx}(-\cos x)\,dx = \Big[-\frac{\cos x}{x}\Big]_1^b - \int_1^b (-\cos x)\Big(-\frac{1}{x^2}\Big)\,dx$$
$$= \cos(1) - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\,dx.$$

Now use the comparison $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ and the fact that $\int_1^\infty \frac{1}{x^2}\,dx$ converges. Thus the Comparison Test shows that $\int_1^\infty \frac{\cos x}{x^2}\,dx$ exists, and hence
$$\int_1^\infty \frac{\sin x}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{\sin x}{x}\,dx = \cos(1) - \int_1^\infty \frac{\cos x}{x^2}\,dx$$
converges

The point of using integration by parts is that the Comparison Test is not immediately applicable to the function $(\sin x)/x$ on $[1,\infty)$. Indeed, one might initially guess that the integral diverges because of the $1/x$, but the oscillation of the $\sin x$, and the subsequence cancellations that occur, turn out to be just enough to make the integral converge.

5. (a) Find a reduction formula for the integral $\displaystyle\int (\ln x)^n\,dx$ $(n \ge 0)$.

Solution: Let $u = (\ln x)^n$ and $\frac{dv}{dx} = 1$. Then
$$I_n = \int (\ln x)^n\,dx \;=\; x(\ln x)^n \;-\; n\int (\ln x)^{n-1}\,dx = x(\ln x)^n - nI_{n-1}.$$

(b) Hence evaluate the improper integral $\displaystyle\int_0^1 (\ln x)^n\,dx$ $(n \ge 1)$.

Solution: let $J_n = \int_0^1 (\ln x)^n\,dx = \lim_{\epsilon\to 0^+}\int_\epsilon^1 (\ln x)^n\,dx$. Then, for $n \ge 1$, the reduction formula in the previous question gives
$$\begin{aligned} J_n &= \int_0^1 (\ln x)^n\,dx \\ &= \lim_{a\to 0^+}\Big[x(\ln x)^n\Big]_a^1 - n\int_0^1 (\ln x)^{n-1}\,dx \\ &= -nJ_{n-1}\,. \end{aligned}$$

The limit can be proved by applying l'Hôpital's rule n times to the ratio $(\ln x)^n/(x^{-1})$. Repeated application of the reduction formula for J_n gives
$$J_n = -nJ_{n-1} = n(n-1)J_{n-2} = -n(n-1)(n-2)J_{n-3} = \cdots = (-1)^n n! J_0.$$

A trivial integration gives $J_0 = 1$. Hence,
$$J_n \;=\; (-1)^n n!\,.$$

Discussion question

6. *Gabriel's horn* is the solid given by rotating $y = \frac{1}{x}$ ($x \geq 1$) about the x-axis.

(a) Show that the volume of Gabriel's horn is finite.

Solution: Using the disc method, the volume of Gabriel's horn is

$$V = \pi \int_1^\infty \frac{1}{x^2}\, dx = \pi \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx = \pi \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = \pi.$$

(b) Show that the surface area of Gabriel's horn is infinite.

Solution: Using the formula for surface areas of revolution we have

$$A = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx.$$

But

$$\frac{\sqrt{x^4+1}}{x^3} \geq \frac{\sqrt{x^4}}{x^3} = \frac{1}{x},$$

and since $\int_1^\infty \frac{1}{x}\, dx$ diverges the Comparison Test tells us that the integral for the surface area also diverges. [As a challenging exercise, try to compute the integral $\int x^{-3}\sqrt{x^4+1}\, dx$ by making a substitution].

(c) Interesting: Part (a) seems to say that Gabriel's horn can be filled with a finite volume of paint (and in doing so you will paint the 'inside' of the horn), but part (b) seems to say that it takes an infinite amount of paint to paint the outside of Gabriel's horn. What's going on here?

Solution: This feels paradoxical at first. But consider the question: Given π litres of paint (that is, enough paint to fill Gabriel's horn), what is the maximum area that could be painted with it? In an ideal world the answer is ∞ meters squared, for we could just spread the paint thinner and thinner. (But in reality the atomic graininess would prevent us from doing this). So there is not really a paradox here. I guess it comes down to the fact that volume and area have different units of measurement.

Questions for extra practice

7. Decide if the following improper integrals exist or not. If they exist, try to compute their value (this is not always possible!).

(a) $\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\, dx$

Solution: $0 \leq e^{-x}/\sqrt{x} \leq e^{-x}$ for all $x \geq 1$. Also,

$$\int_1^b e^{-x}\, dx = \Big[-e^{-x}\Big]_1^b = e^{-1} - e^{-b} \to e^{-1} \quad \text{as } b \to \infty.$$

So $\int_1^\infty e^{-x} dx$ converges, so $\int_1^\infty \frac{e^{-x}}{\sqrt{x}} dx$ converges too by the Comparison Test.

(b) $\displaystyle\int_0^\infty x^3 e^{-x}\,dx$

Solution: Integrating by parts we have

$$\int_0^b x^3 e^{-x}\,dx = 6 - (b^3 + 3b^2 + 6b + 6)e^{-b}.$$

Therefore

$$\int_0^\infty x^3 e^{-x}\,dx = 6 - \lim_{b\to\infty}(b^3 + 3b^2 + 6b + 6)e^{-b} = 6.$$

In particular, the integral exists.

Alternatively, we could prove more 'abstractly' that the integral exists (without calculating its value). Note first that $x^3/e^x \to 0$ as $x \to \infty$, as you can see using L'Hôpital's Rule, for example. So there is a number M such that $x^3/e^x \le 1$ once $x \ge M$. Replacing x by $x/2$, we see that $(x/2)^3/e^{x/2} \le 1$ once $x/2 \ge M$. That is, $x^3 \le 8e^{x/2}$ once $x \ge 2M$. So the integrand x^3e^{-x} may be estimated as follows:

$$x^3 e^{-x} \;\le\; 8e^{x/2}e^{-x} \;=\; 8e^{-x/2} \quad \text{once } x \ge 2M.$$

Now $\int_{2M}^\infty 8e^{-x/2}\,dx$ converges, by an easy calculation. So $\int_{2M}^\infty x^3e^{-x}\,dx$ converges by the Comparison Test. For $0 \le x \le 2M$, x^3e^{-x} is continuous, and so $\int_0^{2M} x^3e^{-x}\,dx$ exists. Hence

$$\int_0^\infty x^3 e^{-x}\,dx \;=\; \int_0^{2M} x^3 e^{-x}\,dx \;+\; \int_{2M}^\infty x^3 e^{-x}\,dx$$

exists. Using integration by parts, it is easy to calculate its value exactly: it equals 6.

If you prefer to avoid breaking the integral up into two parts as above, you could instead argue as follows: By Calculus, we find that $x^3e^{-x/2}$ takes its maximum value of $C = 216e^{-3}$ at $x = 6$. Thus $x^3 \le Ce^{x/2}$ for all $x \ge 0$. Hence $x^3e^{-x} \le Ce^{-x/2}$ for all $x \ge 0$. Since $\int_0^\infty Ce^{-x/2}\,dx$ converges by an easy direct calculation, so does $\int_0^\infty x^3e^{-x}\,dx$, by the Comparison Test.

(c) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

Solution: We have

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\tan^{-1}(b) = \frac{\pi}{2}.$$

So the integral exists, and equals $\frac{\pi}{2}$.

(d) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Solution: The integrand is unbounded at $x = 1$ and $x = -1$. We haven't really looked at these types of things in class, but it seems sensible to define

$$\begin{aligned}\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{a\to1}\int_{-a}^a \frac{1}{\sqrt{1-x^2}}\,dx = 2\lim_{a\to1}\int_0^a \frac{1}{\sqrt{1-x^2}}\,dx \\ &= 2\lim_{a\to1}\sin^{-1}(a) = \pi.\end{aligned}$$

(e) $\displaystyle\int_1^\infty \frac{e^{-x^2}}{\sqrt{x-1}}\,dx$

Solution: We need to worry about both integration limits here. Write

$$\int_1^\infty \frac{e^{-x^2}}{\sqrt{x-1}}\,dx = \int_1^2 \frac{e^{-x^2}}{\sqrt{x-1}}\,dx + \int_2^\infty \frac{e^{-x^2}}{\sqrt{x-1}}\,dx,$$

and treat each integral separately. The second integral exists by a similar argument to that in (a). The first integral exists by comparison with

$$e^{-1}\int_1^2 \frac{1}{\sqrt{x-1}}\,dx.$$

Therefore the given improper integral exists.

(f) $\displaystyle\int_0^1 \sin\left(\frac{1}{x^2}\right)\,dx$

Solution: A change of variable helps here: Let $y = x^{-2}$. Then

$$\int_a^1 \sin\left(\frac{1}{x^2}\right)\,dx = \int_1^{a^{-2}} \frac{\sin y}{y^{3/2}}\,dy.$$

Since $\int_0^\infty \frac{\sin y}{y^{3/2}}\,dy$ exists (by comparison with $\int_1^\infty \frac{1}{y^{3/2}}\,dy$) we see that

$$\int_0^1 \sin\left(\frac{1}{x^2}\right)\,dx = \lim_{a\to 0^+}\int_1^{a^{-2}} \frac{\sin y}{y^{3/2}}\,dy = \int_1^\infty \frac{\sin y}{y^{3/2}}\,dy,$$

and so the improper integral exists. It is not easy to give the value of the integral.

(g) $\displaystyle\int_0^\infty \operatorname{erf}(x)\,dx$

Solution: Since $\operatorname{erf}(x) \to 1$ as $x \to \infty$ we see that this integral will diverge. For example, for sufficiently large x we have

$$\operatorname{erf}(x) \geq \frac{1}{2},$$

and hence the integral $\displaystyle\int_0^\infty \operatorname{erf}(x)\,dx$ will diverge by comparison with $\displaystyle\int_0^\infty \frac{1}{2}\,dx$.

(h) $\displaystyle\int_0^\infty \cosh(3x)e^{-4x}\,dx$

Solution: Using the definition of $\cosh(3x)$ we have

$$\int_0^b \cosh(3x)e^{-4x}\,dx = \frac{1}{2}\int_0^b \left(e^{-x} + e^{-7x}\right)\,dx = \frac{1}{2}\left(1 - e^{-b}\right) + \frac{1}{14}\left(1 - e^{-7b}\right).$$

Taking the limit as $b \to \infty$ we see that the improper integral exists, and equals $\frac{1}{2} + \frac{1}{14} = \frac{4}{7}$.

(i) $\displaystyle\int_1^2 \frac{1}{\ln x}\,dx$

Solution: The integrand has a discontinuity at $x = 1$. Notice that

$$\ln x \le x - 1 \qquad \text{for all } x > 0.$$

In particular this is true for all $x \ge 1$. (To see this you can use some calculus: Consider the function $f(x) = \ln x - x + 1$. It has $f'(x) < 0$ for all $x \ge 1$, and $f(1) = 0$, so $f(x) \le 0$ for all $x \ge 1$). Therefore

$$\frac{1}{\ln x} \ge \frac{1}{x-1} \qquad \text{for all } x > 1.$$

The integral $\int_1^2 \frac{1}{x-1}\,dx$ does not exist, because

$$\int_1^2 \frac{1}{x-1}\,dx = \lim_{a\to 1^+} \int_a^2 \frac{1}{x-1}\,dx = -\lim_{a\to 1^+} \ln(a-1) = \infty.$$

Therefore the given improper integral also does not exist, by the Comparison Test.

(j) $\displaystyle\int_2^\infty \frac{\mathrm{Li}(x)}{x^2}\,dx$

Solution: Integrating by parts gives

$$\int_2^b \frac{\mathrm{Li}(x)}{x^2}\,dx = -\frac{\mathrm{Li}(b)}{b} + \int_2^b \frac{1}{x\ln x}\,dx = -\frac{\mathrm{Li}(b)}{b} + \ln(\ln b) - \ln(\ln 2).$$

By L'Hôpital's Rule and the Fundamental Theorem of Calculus we have

$$\lim_{b\to\infty} \frac{\mathrm{Li}(b)}{b} = \lim_{b\to\infty} \frac{\frac{1}{\ln b}}{1} = 0.$$

But since $\ln(\ln b) \to \infty$ as $b \to \infty$ we see that the given improper integral does not exist. But note that it diverges pathetically slowly. For example,

$$\int_2^{10000000000} \frac{\mathrm{Li}(x)}{x^2}\,dx \le 4.$$

(k) $\displaystyle\int_0^1 \frac{1}{1-x^2}\,dx$

Solution: The integrand has a discontinuity at $x = 1$. Therefore the integral is an improper integral, and by definition

$$\int_0^1 \frac{1}{1-x^2}\,dx = \lim_{b\to 1} \int_0^b \frac{1}{1-x^2}\,dx.$$

Using partial fractions, we have

$$\int_0^b \frac{1}{1-x^2}\,dx = \frac{1}{2}\int_0^b \frac{1}{1+x}\,dx + \frac{1}{2}\int_0^b \frac{1}{1-x}\,dx = \frac{1}{2}\ln(1+b) - \frac{1}{2}\ln(1-b).$$

As $b \to 1$ we have $\ln(1+b) \to \ln 2$ and $\ln(1-b) \to -\infty$, and so the integral does not exist (it diverges to ∞).

(l) $\displaystyle\int_0^1 \frac{\ln x}{x^{1/3}}\,dx$

Solution: As $x \to 0^+$ we see that $(\ln x)/x^{1/3} \to -\infty$. So the integral is improper. We compute

$$\begin{aligned}\int_a^1 \frac{\ln x}{x^{1/3}}\,dx &= \Big[\ln x\,\frac{3}{2}x^{2/3}\Big]_{x=a}^{x=1} - \frac{3}{2}\int_a^1 x^{2/3}\,\frac{1}{x}\,dx \quad \text{(integrating by parts)}\\ &= -\frac{3}{2}a^{2/3}\ln a - \frac{9}{4}\big(1 - a^{2/3}\big).\end{aligned}$$

As $a \to 0^+$, by L'Hôpital's Rule $a^c \ln a \to 0$ for any fixed $c > 0$. Therefore

$$\int_0^1 \frac{\ln x}{x^{1/3}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{\ln x}{x^{1/3}}\,dx = -\frac{9}{4} \quad \text{as } a \to 0.$$

So the improper integral exists and equals $-9/4$.

Some questions involving reduction formulae

8. Let $x \in \mathbb{R}$, and let $n \geq 0$ be an integer.

(a) Use a reduction formula to prove that

$$\int_0^x (x-t)^n e^t\,dt = n!\left(e^x - \sum_{k=0}^{n} \frac{x^k}{k!}\right).$$

Solution: Let $u = (x-t)^n$ and $\frac{dv}{dx} = e^t$. Then

$$I_n = \int_0^x (x-t)^n e^t\,dt = -x^n + nI_{n-1}.$$

The stated formula now follows by induction (using the reduction formula for the induction step).

(b) Set $x = 1$ in (a) and deduce that e is irrational.

Hint: If e is rational then the integral is an integer for sufficiently large n.

Solution: Suppose that $e = \frac{a}{b}$ is rational. Then

$$n!\left(\frac{a}{b} - \sum_{k=0}^{n}\frac{1}{k!}\right) = \int_0^1 (1-t)^n e^t\,dt.$$

The left hand side is an integer whenever $n \geq b$, and therefore the integral is an integer for all sufficiently large n. But

$$0 < \int_0^1 (1-t)^n e^t\,dt \leq e\int_0^1 (1-t)^n\,dt < \frac{3}{n+1},$$

giving a contradiction.

(c) Use (a) to show that

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!}=e^x \qquad \text{for all } x\in\mathbb{R}.$$

Solution: If $x\geq 0$ then

$$0\leq\int_0^x (x-t)^n e^t\,dt\leq e^x\int_0^x (x-t)^n\,dt=\frac{x^{n+1}e^x}{n+1}.$$

If $x<0$ then

$$\left|\int_0^x (x-t)^n e^t\,dt\right|=\left|-(-1)^n\int_x^0 (t-x)^n e^t\,dt\right|=\int_x^0 (t-x)^n e^t\,dt.$$

Therefore in this case,

$$0\leq\left|\int_0^x (x-t)^n e^t\,dt\right|\leq\int_x^0 (t-x)^n\,dt=\frac{(-x)^{n+1}}{n+1}=\frac{|x|^{n+1}}{n+1}.$$

Therefore for all $x\in\mathbb{R}$ we have

$$0\leq\left|e^x-\sum_{k=0}^{n}\frac{x^k}{k!}\right|\leq\frac{|x|^{n+1}}{(n+1)!}\max\{1,e^x\}.$$

For each *fixed* x the right hand side tends to 0 as $n\to\infty$, and so by the squeeze law

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!}=e^x \qquad \text{for all } x\in\mathbb{R}.$$

Remark: In this question you actually showed that the Taylor series for e^x converges to e^x for all $x\in\mathbb{R}$. We will be discussing Taylor series later in the course, and will reprove this result using different methods.

9. For $n \geq 0$ let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$.

(a) Derive a reduction formula for I_n, and use it to deduce that

$$I_{2n} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2} \qquad \text{and} \qquad I_{2n+1} = \frac{(2n)!!}{(2n+1)!!},$$

where $(2n)!! = 2 \cdot 4 \cdots (2n)$ and $(2n+1)!! = 1 \cdot 3 \cdots (2n+1)$.

Solution: We have

$$I_n = \int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\sin^{n-2}\theta \, d\theta = I_{n-2} - \int_0^{\frac{\pi}{2}} \cos^2\theta \sin^{n-2}\theta \, d\theta.$$

Using integration by parts (with $u = \cos\theta$ and $\frac{dv}{d\theta} = \cos\theta\sin^{n-2}\theta$) gives

$$I_n = I_{n-2} - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = I_{n-2} - \frac{1}{n-1} I_n.$$

Solving for I_n gives the reduction formula

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{for } n \geq 1.$$

Therefore

$$I_{2n} = \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{(2n)(2n-2)\cdots 4 \cdot 2} I_0 = \frac{(2n-1)!!}{(2n)!!}\int_0^{\frac{\pi}{2}} 1 \, d\theta = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$

and

$$I_{2n+1} = \frac{(2n)(2n-2)\cdots 4 \cdot 2}{(2n+1)(2n-1)\cdots 5 \cdot 3} I_1 = \frac{(2n)!!}{(2n+1)!!}\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = \frac{(2n)!!}{(2n+1)!!}.$$

(b) Show that $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$, and deduce that that

$$\frac{2n}{2n+1} \leq \frac{1 \cdot 3 \cdot 3 \cdots (2n-1)(2n-1)}{2 \cdot 2 \cdot 4 \cdots (2n-2)(2n)}\frac{\pi}{2} \leq 1\,.$$

Hence prove the *Wallis Product Formula* for π:

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7} \cdots = \lim_{n\to\infty} \frac{2^{4n} n!^4}{2n(2n)!^2}\,.$$

Solution: Since $0 \leq \sin^{2n+1}\theta \leq \sin^{2n}\theta \leq \sin^{2n-1}\theta$ for all $0 \leq \theta \leq \frac{\pi}{2}$ we have $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$. Therefore by part (a) we have

$$\frac{(2n)!!}{(2n+1)!!} \leq \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2} \leq \frac{(2n-2)!!}{(2n-1)!!}.$$

Rearranging gives

$$\frac{2n}{2n+1} \leq \frac{1 \cdot 3 \cdot 3 \cdots (2n-1)(2n-1)}{2 \cdot 2 \cdot 4 \cdots (2n-2)(2n)}\frac{\pi}{2} \leq 1,$$

and it follows from the Squeeze Theorem that

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1} \right).$$

Since

$$(2n)!! = 2^n n! \quad \text{and} \quad (2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}$$

we can rewrite the above limit as

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{(2n)!!^2}{2n(2n-1)!!^2} = \lim_{n\to\infty} \frac{2^{4n} n!^4}{2n(2n)!^2}.$$