Semester 2 **Solutions to Exercises for Week 7** 2017

1. To say that $\lim_{n\to\infty} a_n = L$ means

$$(\forall \epsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N) \quad |a_n - L| < \epsilon \,.$$

Suppose this holds and take any $\epsilon > 0$. Then there exists $N \in \mathbb{Z}^+$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. But if $n \geq N$ then certainly $n + k \geq N$ so that

$$|b_n - L| \;=\; |a_{n+k} - L| \;<\; \epsilon \,.$$

This verifies that $\lim_{n\to\infty} b_n = L$ also.

2. Taylor's Theorem tells us that if f is infinitely differentiable and $n \geq 0$, then

$$f(x) \;=\; T_n(x) + R_n(x)$$

where $T_n(x)$ is the Taylor polynomial of degree n of f about $x = a$ and

$$R_n(x) \;=\; \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

represents the remainder term for some c between a and x. If $P(x)$ denotes the Taylor series expansion of f about $x = a$ then, when convergence takes place,

$$P(x) \;=\; \lim_{n\to\infty} T_n(x) \,.$$

3. The definite integral $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ is technically improper only because the integrand is undefined at $x = 0$ (not because of any unbounded behaviour). If we define a function $f : \mathbb{R} \to \mathbb{R}$ by the rule

$$f(x) \;=\; \begin{cases} 1 & \text{if } x = 0 \\ \frac{\sin x}{x} & \text{if } x \neq 0 \end{cases}$$

then f is continuous and $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 f(x)\,dx$ becomes proper.

4. The Taylor polynomial of degree 6 for f about $x = 0$ is $T_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ (technically a polynomial of degree 5) and the remainder term is

$$R_6(x) \;=\; \frac{f^{(7)}(c)}{7!}x^7 \;=\; \frac{-\cos c}{7!}x^7$$

for some c between 0 and x. But $|\cos c| \leq 1$ so that $|R_6(x)| \leq \frac{|x^7|}{7!}$. Hence, for $0 < x \leq 1$,

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \;\leq\; \sin x \;\leq\; x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \,,$$

so that

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \;\leq\; \frac{\sin x}{x} \;\leq\; 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} \,.$$

5. The previous inquality also holds for $x = 0$ so that, integrating over the interval $[0, 1]$ gives

$$1 - \frac{1}{3(3!)} + \frac{1}{5(5!)} - \frac{1}{7(7!)} \;\leq\; \int_0^1 \frac{\sin x}{x}\, dx \;\leq\; 1 - \frac{1}{3(3!)} + \frac{1}{5(5!)} + \frac{1}{7(7!)} ,$$

which simplifies to the following (quoting at least 5 decimal places):

$$0.94608\ldots \;\leq\; \int_0^1 \frac{\sin x}{x}\, dx \;\leq\; 0.94613\ldots$$

Hence $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx = 0.946$ to three decimal places.

6. If $\displaystyle\sum_{n=0}^{\infty} a_n = L < \infty$ then

$$\lim_{n\to\infty} a_n \;=\; \lim_{n\to\infty} \left(\sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k \right) \;=\; \left(\lim_{n\to\infty} \sum_{k=0}^{n} a_k \right) - \left(\lim_{n\to\infty} \sum_{k=0}^{n-1} a_k \right) \;=\; L - L \;=\; 0 .$$

The converse is false: for example, the harmonic series diverges but the terms tend to zero.

7. (i) We have that $\displaystyle\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} \;=\; \lim_{n\to\infty} \frac{2}{n+1} \;=\; 0 \;<\; 1$, so the series converges by the ratio test.

(ii) We have that $\displaystyle\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^3} \frac{n^3}{2^n} \;=\; \lim_{n\to\infty} \frac{2}{(1+\frac{1}{n})^3} \;=\; 2 \;>\; 1$, so the series diverges by the ratio test.

(iii) We have that $\displaystyle\lim_{n\to\infty} \frac{3(n+1)+1}{2^{n+1}} \frac{2^n}{3n+1} \;=\; \lim_{n\to\infty} \frac{3+\frac{4}{n}}{2(3+\frac{1}{n})} \;=\; \frac{1}{2} \;<\; 1$, so the series converges by the ratio test.

8. We have

$$\begin{aligned} f(x) \;=\; \frac{1}{x} \;=\; \frac{1}{1-(1-x)} &\;=\; 1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots \\ &\;=\; 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots , \end{aligned}$$

which must be the Taylor series about $x = 1$, by uniqueness of power series expansions.

9. Observe first that $\displaystyle\frac{1}{1+x^2} \;=\; \frac{1}{1-(-x^2)} \;=\; 1 - x^2 + x^4 - x^6 + \cdots$ so, antidifferentiating,

$$\tan^{-1} x \;=\; C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

for some constant C. But $\tan^{-1}(0) = 0$, so $C = 0$, giving finally the Maclaurin series

$$\tan^{-1} x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

10. (i) Expanding and collecting like powers of x yields

$$\begin{aligned} e^{-x^2}\sinh x &= \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= x - x^3 + \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^5}{3!} + \frac{x^5}{2!} + \cdots \\ &= x - \frac{5}{6}x^3 + \frac{41}{120}x^5 + \cdots \end{aligned}$$

(ii) Observe first that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ,$$

so, after antidifferentiating,

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots ,$$

yielding

$$\begin{aligned} e^{-x}\ln(1-x) &= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) \\ &= -x + x^2 - \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^3}{2} - \frac{x^3}{3} + \cdots \\ &= -x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots \end{aligned}$$

11. The following calculation is justified by the addition limit law, which holds for complex numbers. For any real number θ we have, using the usual Maclaurin series for real sin and cos,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \cdots + \frac{i^n\theta^n}{n!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\ &= \cos\theta + i\sin\theta . \end{aligned}$$

12. The integral $f(x) = \displaystyle\int_0^x \frac{e^t - 1}{t}\,dt$ is technically improper because the integrand is not defined at $t = 0$. However, by L'Hopital's Rule, $\displaystyle\lim_{t\to 0}\frac{e^t-1}{t} = \lim_{t\to 0}\frac{e^t}{1} = 1$, so that the function

$$g(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{e^t-1}{t} & \text{if } t \neq 0 \end{cases}$$

is continuous and $f(x) = \displaystyle\int_0^x \frac{e^t-1}{t}\,dt = \int_0^x g(t)\,dt$ becomes proper. Using the series expansion

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots$$

we get

$$\frac{e^t - 1}{t} = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \dots ,$$

noting that the right-hand side also represents the entire rule for g. Since f is an antiderivative of g it suffices to antidifferentiate this series, replacing t with x to get

$$f(x) = x + \frac{x^2}{2(2!)} + \frac{x^3}{3(3!)} + \frac{x^4}{4(4!)} + \dots ,$$

noting that the constant term must be $f(0) = 0$.

13. It is easy to see that $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$ so that $f^{(n)}(0) = (-1)^{n-1}(n-1)!$. It follows that the Taylor polynomial of degree n for f about $x = 0$ is

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^n}{n} .$$

By Taylor's Theorem, the remainder term has the form

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{(-1)^n n!(1+c)^{-n-1}}{(n+1)!} x^{n+1} = \frac{(-1)^n}{(n+1)(1+c)^{n+1}} x^{n+1}$$

for some c between 0 and x. In particular (taking $x = 1$),

$$|R_n(1)| = \frac{1}{(n+1)(1+c)^{n+1}} \leq \frac{1}{n+1} \to 0$$

as $n \to \infty$, so that $\lim\limits_{n\to\infty} R_n(1) = 0$. Hence

$$\ln 2 = f(1) = \lim_{n\to\infty} T_n(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ,$$

the alternating harmonic series.

14. For each positive integer n, put

$$\begin{aligned} A_n &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n} , \\ S_n &= A_{2n} , \\ T_n &= A_{2n+1} = S_n + \frac{1}{2n+1} . \end{aligned}$$

Then

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) ,$$

so the S_n form a monotonically increasing sequence, since $\frac{1}{2n-1} - \frac{1}{2n} > 0$ for each n. Further, this sequence is bounded above by 1 since

$$S_n = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2n-2} - \frac{1}{2n-1}\right) - \frac{1}{2n} .$$

By the Monotone Convergence Theorem, $\lim\limits_{n\to\infty} S_n = L$ for some L. Hence also

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} S_n + \frac{1}{2n+1} = L + 0 = L .$$

Let $\epsilon > 0$. Then there exist N_1, N_2 such that

$$|S_n - L| \;<\; \epsilon \quad \text{for all } n \geq N_1 \;, \qquad \text{and} \quad |T_n - L| \;<\; \epsilon \quad \text{for all } n \geq N_2 \;.$$

Put $N = \max\{N_1, N_2\}$, so

$$|S_n - L| < \epsilon \quad \text{and} \quad |T_n - L| < \epsilon \quad \text{for all } n \geq N.$$

Let $m \geq 2N$. If m is even then $m/2 \geq N$ and

$$|A_m - L| \;=\; |S_{m/2} - L| \;<\; \epsilon \;.$$

If m is odd then $\frac{m-1}{2} \geq N$ and

$$|A_m - L| \;=\; |T_{\frac{m-1}{2}} - L| \;<\; \epsilon \;.$$

Thus $|A_m - L| < \epsilon$ for all $m \geq N$, which shows $\lim\limits_{m\to\infty} A_m = L$. Thus the alternating harmonic series converges.

15. (i) Observe that

$$\lim_{n\to\infty} \frac{\frac{1}{3^n-2}}{\frac{1}{3^n}} \;=\; \lim_{n\to\infty} \frac{3^n}{3^n - 2} \;=\; 1 \;,$$

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 2}$ converges by limit comparison with the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$.

(ii) Observe that

$$\lim_{n\to\infty} \frac{\frac{1}{3n-2}}{\frac{1}{n}} \;=\; \lim_{n\to\infty} \frac{n}{3n - 2} \;=\; \frac{1}{3} \;,$$

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n - 2}$ diverges by limit comparison with the harmonic series.

(iii) Observe that

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} \;=\; \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \;=\; 1 \;,$$

so $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$ diverges by limit comparison with the harmonic series.

16. Suppose $\displaystyle\sum_{n=0}^{\infty} a_n$ and $\displaystyle\sum_{n=0}^{\infty} b_n$ are series with positive terms and $0 < \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L < \infty$.

Suppose $\displaystyle\sum_{n=0}^{\infty} b_n$ converges, say $\displaystyle\sum_{n=0}^{\infty} b_n = K < \infty$. Then there exist positive integers N_1, N_2 such that

$$(\forall n \geq N_1) \qquad \left|\frac{a_n}{b_n} - L\right| < L \;, \quad \text{so} \quad a_n < 2Lb_n$$

and

$$(\forall n \geq N_2) \qquad \left|\sum_{k=0}^{n} b_n - K\right| < K \;, \quad \text{so} \quad \sum_{k=0}^{n} b_n < 2K \;.$$

Put $N = \max\{N_1, N_2\}$. If $n > N$ then

$$\begin{aligned}
\sum_{k=0}^{n} a_k &= \sum_{k=0}^{N} a_k + \sum_{k=N+1}^{n} a_k \\
&< \sum_{k=0}^{N} a_k + \sum_{k=N+1}^{n} 2Lb_k \\
&\leq \sum_{k=0}^{N} a_k + 2L\sum_{k=0}^{n} b_k \\
&< \sum_{k=0}^{N} a_k + 4LK \ .
\end{aligned}$$

By the Monotone Convergence Theorem, $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} a_k$ exists, that is, $\sum\limits_{n=0}^{\infty} a_n$ converges.

Since $0 < \lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = \dfrac{1}{L} < \infty$, the same argument shows that if $\sum\limits_{n=0}^{\infty} a_n$ converges then so does $\sum\limits_{n=0}^{\infty} b_n$.

This proves both directions of the limit comparison test.

17. Define the following sequence recursively: $a_0 = \sqrt{2}$, $a_n = \sqrt{2 + a_{n-1}}$ for $n \geq 1$. We claim that $a_0, a_1, a_2, \ldots$ is increasing and bounded above by 2. To see this we prove

$$a_{n-1} < a_n < 2 \qquad \text{for each } n \geq 1.$$

Clearly, $a_0 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_1 < 2$, which starts an induction. If $i \geq 1$ and $a_{i-1} < a_i < 2$ then $2 + a_{i-1} < 2 + a_i < 4$, so

$$a_i = \sqrt{2 + a_{i-1}} < a_{i+1} = \sqrt{2 + a_i} < \sqrt{4} = 2 \ ,$$

and the result follows by induction. Hence $\lim\limits_{n\to\infty} a_n = L$ exists by the Monotone Convergence Theorem. Thus

$$L = \lim_{n\to\infty} \sqrt{2 + a_{n-1}} = \sqrt{2 + \lim_{n\to\infty} a_{n-1}} = \sqrt{2 + L} \ ,$$

so $L^2 = 2 + L$, so $L^2 - L - 2 = (L-2)(L+1) = 0$, so $L = 2$ or $L = -1$. But $L > 0$, so $L = 2$, that is,

$$2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} \ .$$

Now define the following sequence recursively: $b_0 = \sqrt{2}$, $b_n = \sqrt{2 \times b_{n-1}}$ for $n \geq 1$. We claim that $b_0, b_1, b_2, \ldots$ is increasing and bounded above by 2. To see this we prove

$$b_{n-1} < b_n < 2 \qquad \text{for each } n \geq 1.$$

Clearly, $b_0 = \sqrt{2} < \sqrt{2 \times \sqrt{2}} = b_1 < 2$, which starts an induction. If $i \geq 1$ and $b_{i-1} < b_i < 2$ then $2 \times b_{i-1} < 2 \times b_i < 4$, so

$$b_i = \sqrt{2 \times b_{i-1}} < b_{i+1} = \sqrt{2 \times b_i} < \sqrt{4} = 2 ,$$

and the result follows by induction. Hence $\lim_{n\to\infty} b_n = K$ exists by the Monotone Convergence Theorem. Thus

$$K = \lim_{n\to\infty} \sqrt{2 \times b_{n-1}} = \sqrt{2 \times \lim_{n\to\infty} b_{n-1}} = \sqrt{2 \times K} ,$$

so $K^2 = 2K$, so $K^2 - 2K = (K-2)K = 0$, so $K = 2$ or $K = 0$. But $K > 0$, so $K = 2$, that is,

$$2 = \sqrt{2 \times \sqrt{2 \times \sqrt{2 \times \sqrt{2 \times \cdots}}}} .$$