8027A SEMESTER 2 2009

THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

MATH1903/1907
INTEGRAL CALCULUS AND MODELLING (ADVANCED)

November 2009

LECTURERS: H Dullin, J Parkinson

TIME ALLOWED: **One and a half hours**

Family Name: ....

Other Names: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

MARKER'S USE ONLY	

Extended Answer Section

There are **four** *questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

MARKS

1. (a) Compute the area of the region in the first quadrant bounded by the curve $y = e^x$, the curve $y = x^2$, the y-axis, and the line $x = 1$. **2**

(b) Compute the volume of the solid obtained by rotating about the y-axis the region bounded by the curve $y = e^{-x^2}$, the y-axis, the x-axis, and the line $x = 1$. **2**

MARKS

(c) Find the length of the curve given by parametric equations 2

$$x(t) = e^t \cos t \qquad \text{and} \qquad y(t) = e^t \sin t$$

with $t \in [0, 2\pi]$.

(d) Compute the limit as $n \to \infty$ of the sequence 2

$$a_n = \frac{(2n)!2^{2n}}{(n!)^2(2n+1)5^{2n}}.$$

MARKS

(e) Compute the value of the improper integral $\int_0^\infty e^{-x} \sin x \, dx$ by taking the limit of an appropriate proper integral. 2

Question 2 begins on the next page

MARKS

2. (a) Write down the first and second order Taylor polynomials $T_1(x)$ and $T_2(x)$ for the function $f(x) = \ln(1+x)$ about $x = 0$, and use Taylor's Theorem to write down formulas for the first and second order remainder terms $R_1(x) = \ln(1+x) - T_1(x)$ and $R_2(x) = \ln(1+x) - T_2(x)$. **3**

MARKS

(b) Hence, or otherwise, prove that 2

$$x - \frac{1}{2}x^2 \leq \ln(1+x) \leq x \qquad \text{for all } x \geq 0.$$

(c) Show that 2

$$\ln(n+1) = \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) \qquad \text{for } n \geq 1.$$

MARKS

(d) Use parts (b) and (c) and an appropriate comparison test to deduce that the sequence **3**

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1)$$

converges as $n \to \infty$.

Question 3 begins on the next page

MARKS

3. (a) Use an integrating factor to find the general solution of the differential equation 2

$$\frac{dy}{dx} = \alpha + \frac{y}{x}.$$

(b) Solve the initial value problem 2

$$\frac{d^2y}{dx^2} + y = -2\sin x$$

where $y(0) = 0$ and $y'(0) = 1$.

MARKS

(c) Consider a differential equation of the form 3

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Introduce a new dependent variable z by $y = xz$ and show that the resulting differential equation for $z(x)$ can be written in separable form $dz/dx = G(x)/F(z)$. Use this method to again solve the differential equation from Question 3(a).

MARKS

(d) Let n be a positive integer and a, b, c, d real constants with $ad - bc \neq 0$. Find a transformation to a new dependent variable such that the differential equation 3

$$\frac{1}{x^{n-1}}\frac{dy}{dx} = \frac{ay + bx^n}{cy + dx^n}$$

becomes separable and derive the equation in separated form (do not solve the DE).

QUESTION 4 BEGINS ON THE NEXT PAGE

MARKS

4. A skydiver with mass M jumps out of a plane at $t = 0$ with initial vertical velocity $v(0) = 0$. Newton's second law states that mass times acceleration equals force. The force of gravity is $-Mg$, where g is the gravitational acceleration, and the resistive force due to turbulent drag is proportional to the velocity squared. (In this question the positive direction is "up").

   (a) Derive the differential equation for $v(t)$ and solve it for the given initial condition. Find the terminal speed $v_\infty = \left|\lim_{t\to\infty} v(t)\right|$. If the ratio of masses of two skydivers is $\rho = M_1/M_2$, what is the ratio of their terminal velocities? 4

MARKS

(b) Using the result of part (a), find the time T it takes to fall a height of h when initially $v(0) = 0$. **3**

(c) For small times $h(t) \approx gt^2/2$. Using the Taylor series of $h(t)$ about $t = 0$, find the first non-zero correction term to this expression due to air resistance. **3**

Table of Standard Integrals

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

3. $\displaystyle\int e^x\,dx = e^x + C$

4. $\displaystyle\int \sin x\,dx = -\cos x + C$

5. $\displaystyle\int \cos x\,dx = \sin x + C$

6. $\displaystyle\int \tan x\,dx = -\ln\big|\cos x\big| + C$

7. $\displaystyle\int \cot x\,dx = \ln\big|\sin x\big| + C$

8. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

9. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

10. $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

11. $\displaystyle\int \sec x\,dx = \ln\big|\sec x + \tan x\big| + C$

12. $\displaystyle\int \operatorname{cosec} x\,dx = \ln\big|\operatorname{cosec} x - \cot x\big| + C$

13. $\displaystyle\int \sinh x\,dx = \cosh x + C$

14. $\displaystyle\int \cosh x\,dx = \sinh x + C$

15. $\displaystyle\int \tanh x\,dx = \ln\cosh x + C$

16. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (|x| < a)$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C'$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C' \quad (x > a)$

Linearity: $\displaystyle\int \big(\lambda f(x) + \mu g(x)\big)\,dx = \lambda\int f(x)\,dx + \mu\int g(x)\,dx$

Integration by substitution: $\displaystyle\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$

Integration by parts: $\displaystyle\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.