The University of Sydney
School of Mathematics and Statistics

MATH1903/1907
Integral Calculus and Modelling (Advanced)

November 2011

Lecturers: D Daners, J Parkinson

Time Allowed: One and a half hours

Family Name: ....

Other Names: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Approved non-programmable calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Marker's use only

Extended Answer Section

*There are **four** questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

MARKS

1. (a) Calculate the volume of the solid obtained by revolving the region of $\mathbb{R}^2$ bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$ and $y = 0$ about the y-axis. **2**

   (b) Calculate the length of the curve in $\mathbb{R}^2$ with parametric equations **3**

$$x(t) = 3t^2 + 2, \qquad y(t) = 4 - t^3, \qquad \text{with } t \in [0, 1].$$

MARKS

(c) Calculate the value of the improper integral 3

$$\int_0^\infty \frac{1}{(x+1)(x+2)}\,dx.$$

(d) Find $\dfrac{d}{dx}\displaystyle\int_x^{e^x} \ln(1+t^2)\,dt.$ 3

QUESTION 2 BEGINS ON THE NEXT PAGE

MARKS

2. (a) (*i*) Let m, n be integers with $m < n$, and let $f(x)$ be a monotone decreasing continuous function with $f(x) \geq 0$ for all x. Use upper and lower Riemann sums on the interval $[m, n]$ to show that **3**

$$f(n) \leq \sum_{k=m}^{n} f(k) - \int_{m}^{n} f(x)\, dx \leq f(m).$$

(*ii*) Hence, or otherwise, show that the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges. **2**

QUESTION 2 CONTINUES ON THE NEXT PAGE

MARKS

(b) You are given that the equation

$$ye^y = x$$

implicitly defines a function $y = y(x)$ with domain $x \geq -e^{-1}$ and range $y \geq -1$, and that this function can be differentiated any number of times.

(*i*) Calculate the integral **3**

$$\int_0^e \frac{1}{1+y(x)}\,dx.$$

(*ii*) Find the second order Taylor polynomial for $y(x)$ about $x = 0$. **3**

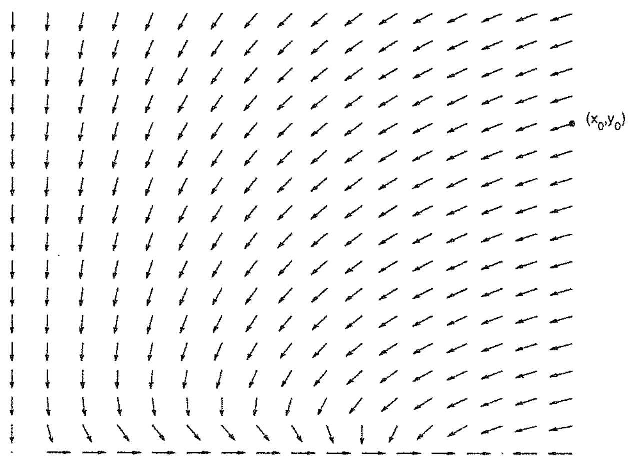
Question 3 begins on the next page

MARKS

3. (a) Find the general solution to the differential equation 2

$$y' \cos^2 x = y^2(1 - \sin x).$$

(b) The diagram below shows a vector field of a system of two differential equations. In that diagram, draw the trajectory of the solution starting at the point (x_0, y_0) marked in the diagram. 1



MARKS

(c) (*i*) Find the general solution of homogeneous second order differential equation **2**

$$\ddot{x} + \dot{x} - 6x = 0.$$

(*ii*) Find a particular solution of the inhomogeneous differential equation **3**

$$\ddot{x} + \dot{x} - 6x = e^{2t}.$$

Question 3 continues on the next page

MARKS

(d) Solve the initial value problem 3

$$u' = 2xu + x^3, \qquad u(0) = 2.$$

QUESTION 4 BEGINS ON THE NEXT PAGE

MARKS

4. (a) By infusion, the glucose concentration of blood is increased at a constant rate measured in mg/minute. At the same time, the glucose is converted and excreted from the blood at a rate proportional to the present concentration of the glucose.

   (i) Carefully define all dependent and independent variables needed to model the concentration of the glucose in the blood. **1**

   (ii) Derive a differential equation describing the concentration of the glucose as a function of time. Use the variables you introduced in (i). **2**

MARKS

(b) Consider the nonlinear differential equation

$$xy' = y + ax\sqrt{x^2 + y^2}, \qquad x > 0,$$

where $a > 0$ is a constant.

(*i*) Show that $v := yx^{-1}$ satisfies the separable differential equation **2**

$$v' = a\sqrt{1 + v^2}$$

(*ii*) Use the differential equation in part (*i*) to get the general solution to the original differential equation. (Note the table of standard integrals.) **2**

QUESTION 4 CONTINUES ON THE NEXT PAGE

MARKS

(c) Consider the system of differential equations

$$x' = 2x - y$$
$$y' = x + 2y$$

(*i*) Determine the stability of the zero solution $x = y = 0$. **1**

(*ii*) Find the solution of the system for the initial values $x(0) = 0$ and $y(0) = -1$. **3**

This page may be used if you need more space for your answers

THIS PAGE MAY BE USED IF YOU NEED MORE SPACE FOR YOUR ANSWERS

Table of Standard Integrals

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

3. $\displaystyle\int e^x\,dx = e^x + C$

4. $\displaystyle\int \sin x\,dx = -\cos x + C$

5. $\displaystyle\int \cos x\,dx = \sin x + C$

6. $\displaystyle\int \tan x\,dx = -\ln\big|\cos x\big| + C$

7. $\displaystyle\int \cot x\,dx = \ln\big|\sin x\big| + C$

8. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\Big(\frac{x}{a}\Big) + C$

9. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

10. $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

11. $\displaystyle\int \sec x\,dx = \ln\big|\sec x + \tan x\big| + C$

12. $\displaystyle\int \operatorname{cosec} x\,dx = \ln\big|\operatorname{cosec} x - \cot x\big| + C$

13. $\displaystyle\int \sinh x\,dx = \cosh x + C$

14. $\displaystyle\int \cosh x\,dx = \sinh x + C$

15. $\displaystyle\int \tanh x\,dx = \ln\cosh x + C$

16. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\Big(\frac{x}{a}\Big) + C \quad (|x| < a)$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\Big(\frac{x}{a}\Big) + C = \ln\Big(x + \sqrt{x^2+a^2}\Big) + C'$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\Big(\frac{x}{a}\Big) + C = \ln\Big(x + \sqrt{x^2-a^2}\Big) + C' \quad (x > a)$

Linearity: $\displaystyle\int \big(\lambda f(x) + \mu g(x)\big)\,dx = \lambda\int f(x)\,dx + \mu\int g(x)\,dx$

Integration by substitution: $\displaystyle\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$

Integration by parts: $\displaystyle\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.