THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 6

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Limits (continued).
- ☐ Squeeze Law (see also last week's tutorial).
- ☐ Limits as $x \to \infty$, or $x \to -\infty$.
- ☐ Continuity, left continuity, right continuity.

Outcomes

After completing this tutorial you should

- ☐ work with limits;
- ☐ understand the definition of continuity, left and right continuity;
- ☐ be able to prove that certain functions are continuous, right continuous or left continuous.

Summary of essential material

Limits as $x \to \pm\infty$. We say that $\lim_{x\to\infty} f(x) = \ell$ if for every $\epsilon > 0$ there exists $M > 0$ such that

$$x > M \quad \Rightarrow \quad |f(x) - \ell| < \epsilon.$$

Improper limits. We say that $\lim_{x\to a} f(x) = \infty$ if for every $m \in \mathbb{R}$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad f(x) > m.$$

The latter is called an *improper limit* or *divergence to infinity*. There are more such concepts (limit to $-\infty$as $x \to a$, or as $x \to \infty$ etc.) We can also look at right and left hand limits.

Continuity. A function $f(x)$ is *continuous* at $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

We can also give an ϵ-δ definition of limit: $f(x)$ is continuous at $x = a$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

Note that we don't require $0 < |x - a| < \delta$, because if $x = a$ then $f(x) = f(a)$ is automatic.

Left and Right Continuity We say f is *right* or *left continuous* at $x = a$ if $\lim_{x\to a+} f(x) = f(a)$ or $\lim_{x\to a-} f(x) = f(a)$ respectively. A function is continuous at a if and only if it is left continuous and right continuous at a.

Continuity on Intervals. A function $f(x)$ is continuous on an open interval (a, b) if it is continuous at each point of (a, b). It is continuous on a closed interval $[a, b]$ if it is continuous on (a, b), right continuous at $x = a$, and left continuous at $x = b$.

How to show continuity of functions. As with limits, we use that elmentary functions are continuous such as x^α, $\sin x$, $\cos x$, e^x, $\ln x$, $\sin^{-1} x$, $\cos^{-1} x$ on their natural domains. From the limit laws, sums, products and quotients of continuous functions are continuous (denominator non-zero as always). By the composition/substitution law, compositions of continuous functions are continuous.

Questions to complete during the tutorial

1. Let $f(x) = \lfloor x \rfloor$, the largest integer less than or equal to x. Sketch the graph of f. At which points is f continuous? At which points is f right continuous, and at which points is it left continuous?

2. Provide a careful step-by-step argument to explain why $f(x)$ is continuous at $x = \pi$, where

$$f(x) = \sqrt{\ln(\cos x + \sin x + 2x) + e^x}.$$

3. Prove that if $f(x)$ is continuous at $x = a$, then the function $|f(x)|$ is continuous at $x = a$. (Use the reversed triangle inequality from a previous tutorial.) Is the converse true?

4. Determine whether the functions given by the following formulas are continuous the given x values.

(a) $h(x) = x^2 + \sqrt{7-x}$, at $x = 4$.

(b) $k(x) = \dfrac{x^2-1}{x+1}$, at $x = -1$.

(c) $F(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x > 0 \\ 1 - x & \text{if } x \leq 0 \end{cases}$, at $x = 0$.

(d) $K(x) = \begin{cases} \dfrac{x^2-1}{x+1} & \text{if } x \neq -1 \\ 6 & \text{if } x = -1 \end{cases}$, at $x = -1$.

5. Find a constant c so that g is continuous everywhere, where g is defined by:

(a) $g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4. \end{cases}$

(b) $g(x) = \begin{cases} -c + \sqrt{x-4} & \text{if } x \geq 4 \\ |x^2 - c^2| & \text{if } x < 4. \end{cases}$

6. Calculate the following limits using limit laws, the squeeze law, and/or the substitution law:

(a) $\displaystyle\lim_{x\to 0} x^2 \cos\frac{2}{x}$

(b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{3+2x}-\sqrt{3}}{x}$

(c) $\displaystyle\lim_{x\to\infty} \frac{x+\sin^3 x}{2x-1}$

(d) $\displaystyle\lim_{x\to\infty} \sqrt{\frac{3-x}{4-x}}$

(e) $\displaystyle\lim_{x\to\infty} \sqrt{\frac{3-x}{4-x^2}}$

(f) $\displaystyle\lim_{x\to\infty} (\sqrt{x} - \sqrt{x+1})$

7. (a) Suppose that f is a function such that $\displaystyle\lim_{x\to a} |f(x)| = \infty$. Use the definition of a limit to show that $\displaystyle\lim_{x\to a} \frac{1}{|f(x)|} = 0$, where a is either finite or $a = \infty$.

(b) Hence show that $\displaystyle\lim_{x\to\infty} e^{-x} = 0$ as $x \to \infty$.

Extra questions for further practice

8. (a) By comparing the areas of a suitable sector and triangle, show that $|\sin\theta| \le |\theta|$, where $\theta \in \mathbb{R}$ is measured in radians.

(b) Prove that $\sin x - \sin y = 2\sin\dfrac{x-y}{2}\cos\dfrac{x+y}{2}$ for all $x, y \in \mathbb{R}$.

(c) Hence, show that $|\sin x - \sin y| \le |x-y|$ for all $x, y \in \mathbb{R}$. Deduce that $\sin\colon \mathbb{R} \to \mathbb{R}$ is continuous.

(d) Using that the sine function is continuous, show that all other trigonometric functions are continuous. Use for instance that $\cos(x) = \sin(\pi/2 - x)$.

9. Compute the following limits using the limit laws and the substitution law.

(a) $\displaystyle\lim_{t\to 0}\frac{\tan t}{t}$.

(b) $\displaystyle\lim_{t\to 0}\frac{\sin(t^2)}{t}$.

(c) $\displaystyle\lim_{x\to\infty}\sqrt{x^2+1}\,\sin\frac{1}{x}$.

(d) $\displaystyle\lim_{x\to\infty}\Big[\cosh(x)\big(\cosh(x)-\sinh(x)\big)\Big]$.

(e) $\displaystyle\lim_{x\to 0}\frac{|3x+1|-|3x-1|}{x}$.

(f) $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{\sin(5x)}$.

10. Show that if $f(x)$ is continuous at $x = a$, and if $f(a) > 0$, then there is a number $\delta > 0$ such that $f(x) > 0$ whenever $|x-a| < \delta$.

Challenge questions (optional)

11. Consider the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } q > 0 \text{ and with } p \text{ and } q \text{ integers having no factors in common.} \end{cases}$$

For example $f(6/8) = 1/4$ since $6/8 = 3/4$. Prove that f is discontinuous at every rational number.