THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 5

MATH1901: Differential Calculus (Advanced) Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- [ ] Intuitive concept of a limit.
- [ ] Formal definition of limits in terms of ε and δ.
- [ ] The limit laws.
- [ ] The squeeze law.

Outcomes

After completing this tutorial you should

- [ ] understand the intuitive notion of limits;
- [ ] be able to work with the ε-δ definition of a limit in concrete and theoretical contexts;
- [ ] be able to prove the limit laws;
- [ ] calculate complicated limits using limit laws and the squeeze law;
- [ ] use the squeeze law to prove theoretical results;

Summary of essential material

Definition of a limit. We say that $\lim_{x\to a} f(x) = \ell$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon.$$

Note that

$$\lim_{x\to a} f(x) = \ell \qquad \iff \qquad \lim_{x\to a} |f(x) - \ell| = 0.$$

We often use the latter in conjuction with the squeeze law. We can also consider limits from the right: We say that $\lim_{x\to a+} f(x) = \ell$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < x-a < \delta \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon.$$

Similarly we consider limits from the left: replace $x \to a+$ by $x \to a-$ and $0 < x - a < \delta$ by $0 < a - x < \delta$.

- A limit exists if and only if right and left hand limits exist and are equal.

Note on computing limits. The ε-δ definition of a limit cannot be used to compute a limit! It can be used to prove some number is the limit. If you do that you must make sure that the argument you provide works for *every* choice of $\varepsilon > 0$! The *limit laws* and the *squeeze law*, in conjunction with a number of elementary limits, are the main tools to compute limits!

Limit Laws. If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$, then

(1) $\lim_{x\to a}(kf(x)) = k\ell$ for all $k \in \mathbb{R}$.

(2) $\lim_{x\to a}(f(x) + g(x)) = \ell + m$.

(3) $\lim_{x\to a}(f(x)g(x)) = \ell m$.

(4) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{m}$ provided $m \neq 0$.

Squeeze Law. Suppose that

$$f(x) \le g(x) \le h(x) \quad \text{for all } x \text{ near } a \text{ (but not necessarily at } x = a).$$

If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = \ell$ then $\lim_{x\to a} g(x) = \ell$.

Questions to complete during the tutorial

Questions marked by * are harder questions.

1. Calculate the following limits using one or more of the limit laws and squeeze law. It is very tedious to write this down, please at least explain your group exactly how to apply the laws.

(a) $\lim\limits_{x\to 3} \dfrac{x^2+3x+2}{4x^2-x+1}$

Solution: Using the Addition, Product and Quotient Laws, we have

$$\lim_{x\to 3} \frac{x^2+3x+2}{4x^2-x+1} = \frac{(\lim_{x\to 3} x)^2 + 3(\lim_{x\to 3} x) + 2}{4(\lim_{x\to 3} x)^2 - \lim_{x\to 3} x + 1} = \frac{3^2+3(3)+2}{4(3)^2-3+1} = \frac{10}{17}.$$

(b) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

Solution: For all $x \neq 1$,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1.$$

Using the Addition Law, we see that $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 1+1 = 2$.

(c) $\lim\limits_{x\to 1} \dfrac{x^2-3x+2}{x^3-1}$

Solution: Observe that for all $x \neq 1$,

$$\frac{x^2-3x+2}{x^3-1} = \frac{(x-1)(x-2)}{(x-1)(x^2+x+1)} = \frac{(x-2)}{(x^2+x+1)}.$$

Then, using the Addition, Product and Quotient Laws, we have

$$\lim_{x\to 1} \frac{x^2-3x+2}{x^3-1} = \lim_{x\to 1} \frac{x-2}{x^2+x+1} = \frac{1-2}{1+1+1} = -\tfrac{1}{3}.$$

(d) $\lim\limits_{x\to 0} \dfrac{\sqrt{3+2x}-\sqrt{3}}{x}$

Solution: We can't use the limit laws with the expression in its present form, so we manipulate it first to make sure the denominator does not converge to zero.

$$\begin{aligned}\frac{\sqrt{3+2x}-\sqrt{3}}{x} &= \frac{(\sqrt{3+2x}-\sqrt{3})(\sqrt{3+2x}+\sqrt{3})}{x(\sqrt{3+2x}+\sqrt{3})} \\ &= \frac{3+2x-3}{x(\sqrt{3+2x}+\sqrt{3})} \\ &= \frac{2}{\sqrt{3+2x}+\sqrt{3}}.\end{aligned}$$

The above is like "rationalising the numerator" (you learnt that at school do this for the denominator). Hence

$$\lim_{x\to 0} \frac{\sqrt{3+2x}-\sqrt{3}}{x} = \lim_{x\to 0} \frac{2}{\sqrt{3+2x}+\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Technically there is a gap here and we need to show that $\sqrt{x} \to \sqrt{a}$ as $x \to a$ if $a > 0$. See Question 4 for a justification.

2. Use the Squeeze Law to calculate the limit $\lim\limits_{x\to 0} x^2 \sin \dfrac{1}{x}$.

Solution: Since $-1 \leq \sin \dfrac{1}{x} \leq 1$ we have that $-x^2 \leq x^2 \sin \dfrac{1}{x} \leq x^2$. As $\lim\limits_{x\to 0} \pm x^2 = 0$, we conclude from the squeeze law that $\lim\limits_{x\to 0} x^2 \sin \dfrac{2}{x} = 0$.

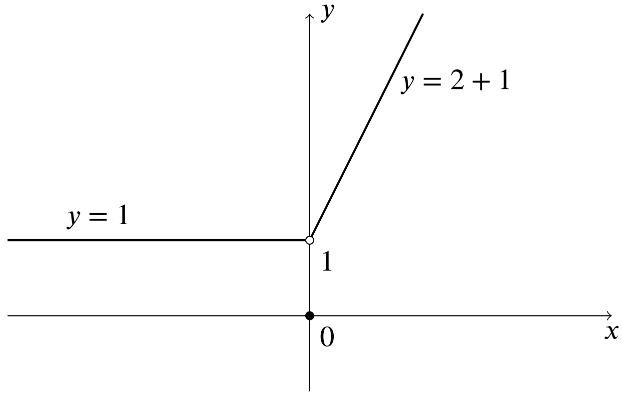
3. Sketch the function with formula

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 2x + 1 & \text{if } x > 0. \end{cases}$$

Find suitable values of δ such that whenever $0 < |x| < \delta$, we have:

(a) $|f(x) - 1| < 0.01$

Solution: The sketch of the graph is



We want to find δ such if $0 < |x-0| = |x| < \delta$, then $|f(x)-1| < 0.01$; that is, $0.99 < f(x) < 1.01$. First observe that if $x < 0$, we have $f(x) = 1$ and so the condition $0.99 < f(x) < 1.01$ is automatically satisfied. If $x > 0$, then $0.99 < f(x) < 1.01$ if and only if $2x + 1 < 1.01$, that is, $x < 0.005$. Therefore if we take $\delta = 0.005$ (or any smaller positive number), then $0 < |x| < \delta$ implies that $|f(x) - 1| < 0.01$.

(b) $|f(x) - 1| < 0.001$

Solution: This time, when $x > 0$ we require $0.999 < f(x) < 1.001$, that is, $2x+1 < 1.001$. If we take $\delta = 0.0005$ (or any smaller positive number), then whenever $0 < |x| < \delta$, we are guaranteed that $|f(x) - 1| < 0.001$.

(c) $|f(x) - 1| < \varepsilon$.

Solution: When $x > 0$ we require $2x + 1 < 1 + \varepsilon$, that is, $0 < x < \frac{\varepsilon}{2}$. We can take δ to be $\frac{\varepsilon}{2}$ (or any smaller positive number).

4. (a) Let $a > 0$. Use the squeeze law to show that $\lim_{x \to a} \sqrt{x} = \sqrt{a}$.

Hint: Rewrite $|\sqrt{x} - \sqrt{a}|$ so that $|x - a|$ appears in the expression.

Solution: Let $\varepsilon > 0$ be given. If $x \geq 0$ with $0 < |x - a| < \delta$ then

$$|\sqrt{x} - \sqrt{a}| = \left| \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}} \right| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\delta}{\sqrt{x} + \sqrt{a}} \leq \frac{\delta}{\sqrt{a}}.$$

Thus, choosing $\delta = \sqrt{a}\varepsilon$ we have that $0 < |x - a| < \delta$ implies that $|\sqrt{x} - \sqrt{a}| < \varepsilon$, and hence the result.

(b) Use the ε-δ definition of limits to show that $\lim_{x \to 0+} \sqrt{x} = 0$.

Solution: Given $\varepsilon > 0$ we set $\delta := \varepsilon^2$. If we do that, then

$$0 < x < \delta (= \varepsilon^2) \quad \Rightarrow \quad 0 < \sqrt{x} < \varepsilon.$$

5. The function f is defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

Use the squeeze law to show that $\lim_{x\to 0} f(x) = 0$. Also write down an ε-δ proof.

Solution: We observe that $0 \le f(x) \le x^2$ for all x, and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} x^2$. So, by the Squeeze Law, $\lim_{x\to 0} f(x) = 0$.

To write down an ε, δ proof: Let $\varepsilon > 0$ be given. Let $\delta = \sqrt{\varepsilon}$. If $0 < |x - 0| < \delta$ then

$$|f(x) - 0| = |f(x)| \le |x|^2 < \delta^2 = \varepsilon.$$

Hence the result.

6. Suppose that $\lim\limits_{x\to a} f(x) = \ell$ and $\lim\limits_{x\to a} g(x) = m$. Use the ε-δ definition of limits to prove the following limit laws:

(a) $\lim\limits_{x\to a} (kf(x)) = k\ell$, $k \in \mathbb{R}$ is a constant.

Solution: If $k = 0$ then both sides are zero. So suppose that $k \neq 0$. Let $\varepsilon > 0$ be given. Since $\lim_{x\to a} f(x) = \ell$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \qquad \Rightarrow \qquad |f(x) - \ell| < \frac{\varepsilon}{|k|}.$$

Thus

$$0 < |x - a| < \delta \qquad \Rightarrow \qquad |kf(x) - k\ell| = |k||f(x) - \ell| < |k|\frac{\varepsilon}{|k|} = \varepsilon,$$

and so $\lim\limits_{x\to a} (kf(x)) = k\ell$ by the definition of limits.

(b) $\lim\limits_{x\to a} (f(x) + g(x)) = \ell + m$.

Solution: Let $\varepsilon > 0$ be given.

(1) Since $\lim_{x\to a} f(x) = \ell$, there is $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon/2.$$

(2) Since $\lim_{x\to a} g(x) = m$, there is $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \quad \Rightarrow \quad |g(x) - m| < \varepsilon/2.$$

Let δ be the smallest of δ_1 and δ_2. Then if $0 < |x - a| < \delta$ both (1) and (2) hold, and so

$$\begin{aligned} |(f(x) + g(x)) - (\ell + m)| &= |(f(x) - \ell) + (g(x) - m)| \\ &\le |f(x) - \ell| + |g(x) - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As the above argument works for every choice of $\varepsilon > 0$ we conclude that $\lim_{x\to a}(f(x) + g(x)) = \ell + m$ by the definition of limits.

7. (a) Assume that $\lim\limits_{x\to a} f(x) = \ell$. Let M_1, M_2 such that $M_1 < \ell < M_2$. Use the ε-δ definition of a limit to show that there exists a $\delta > 0$ such that $M_1 < f(x) < M_2$ for all $x \in (a - \delta, a + \delta)$, $x \neq a$.

Solution: We just $\varepsilon := \min\{\ell - M_1, M_2 - \ell\}$. Then, by the ε-δ definition of a limit there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \qquad \Rightarrow \qquad |f(x) - \ell| < \varepsilon.$$

Now $|f(x) - \ell| < \varepsilon$ is equivalent to $-\varepsilon < f(x) - \ell < 1\varepsilon$ or $\ell - \varepsilon < f(x) < \ell + \varepsilon$. By choice of ε we have $\ell - \varepsilon \ge M_1$ and $\ell + \varepsilon \le M_2$. Hence,

$$m < f(x) < M \qquad \text{for all } x \in (a - \delta, a + \delta),\ x \neq a$$

as claimed.

*(b) If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$, show that $\lim_{x\to a}\big(f(x)g(x)\big) = \ell m$ (limit law of multiplication).
Hint: Write $f(x)g(x) - \ell m = f(x)\big(g(x) - m\big) + m\big(f(x) - \ell\big)$ and use the ε-δ definition of limits.

Solution: From the identity in the hint and the triangle inequality

$$\Big|f(x)g(x) - \ell m\Big| \le |f(x)|\Big|g(x) - m\Big| + |m|\Big|f(x) - \ell\Big| \tag{1}$$

By definition of a limit, if we choose $\varepsilon = 1$, there exists $\delta_0 > 0$ such that

$$0 < |x - a| < \delta_0 \qquad \Rightarrow \qquad 0 < |f(x) - \ell| < 1.$$

Hence in particular

$$0 < |x - a| < \delta_0 \qquad \Rightarrow \qquad |f(x)| = |f(x) - \ell + \ell| \le |f(x) - \ell| + |\ell| < |\ell| + 1.$$

Hence, by (1) we conclude that

$$\Big|f(x)g(x) - \ell m\Big| \le (|\ell| + 1)\Big|g(x) - m\Big| + |m|\Big|f(x) - \ell\Big| \tag{2}$$

whenever $0 < |x - a| < \delta_0$. We now fix $\varepsilon > 0$. As $f(x) \to \ell$ there exists $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \qquad \Rightarrow \qquad 0 < |f(x) - \ell| < \frac{\varepsilon}{2(|\ell| + 1)}.$$

Similarly, as $g(x) \to m$ there exists $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \qquad \Rightarrow \qquad 0 < |g(x) - m| < \frac{\varepsilon}{2|m|}.$$

If we choose δ to be the minumum of δ_0, δ_1 and δ_2, then we deduce from (2) that

$$0 < |x - a| < \delta_2 \quad \Rightarrow \quad \Big|f(x)g(x) - \ell m\Big| \le (|\ell| + 1)\Big|g(x) - m\Big| + |m|\Big|f(x) - \ell\Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As the above argument works for every choice of $\varepsilon > 0$ we conclude that $\lim_{x\to a}(f(x)g(x)) = \ell m$ by the definition of limits.

8. Prove or disprove the following statements:

(a) If $\lim_{x\to 0} f(x)^2$ exists then $\lim_{x\to 0} f(x)$ exists.

Solution: We disprove this, by exhibiting a counterexample: Let $f(x) = 1$ if $x \ge 0$ and $f(x) = -1$ if $x < 0$. Then $\big(f(x)\big)^2 = 1$ for all x and so $\lim_{x\to 0}\big(f(x)\big)^2$ exists and equals 1, whereas $\lim_{x\to 0} f(x)$ does not exist. (Side comment: if $\lim_{x\to 0} f(x)^2 = 0$ then it does follow that $\lim_{x\to 0} f(x) = 0$, because the relevant inequality $|f(x)| < \varepsilon$ is equivalent to $|f(x)^2| < \varepsilon^2$.)

(b) If $\lim_{x\to 0} f(x^2)$ exists then $\lim_{x\to 0} f(x)$ exists.

Solution: We disprove this, by exhibiting a counterexample: Again, we can let $f(x) = 1$ if $x \ge 0$ and $f(x) = -1$ for $x < 0$. Then $\lim_{x\to 0} f(x)$ does not exist but $\lim_{x\to 0} f(x^2)$ exists and equals 1, as $f(x^2) = 1$ for all x.

*(c) If $\lim_{x\to a} g(x) = m$ and $\lim_{y\to m} f(y) = \ell$ then $\lim_{x\to a} f(g(x)) = \ell$. Here f and g both have domain $\mathbb{R}$.

Solution: Perhaps surprisingly this is not a true statement in general. For example, consider the functions $g(x) = 0$ for all $x \in \mathbb{R}$ and

$$f(x) = \begin{cases} 1 & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Certainly we have $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} f(x) = 1$, yet since

$$f(g(x)) = f(0) = 0 \quad \text{for all } x \in \mathbb{R}$$

we have $\lim_{x\to 0} f(g(x)) = 0$ (not 1). We will return to this problem in next week's tutorial – the reason this counter example works is because $f(x)$ is not continuous at $x = 0$.

Extra questions for further practice

9. Prove that $\lim_{x\to a} f(x) = \ell$ if and only if $\lim_{x\to a} |f(x) - \ell| = 0$.

Solution: Suppose that $\lim_{x\to a} f(x) = \ell$. Let $\epsilon > 0$. By the definition of limits there is $\delta > 0$ such that $0 < |x - a| < \delta$ implies that $|f(x) - \ell| < \epsilon$. This in turn says that $\Big||f(x) - \ell| - 0\Big| < \epsilon$ whenever $0 < |x - a| < \delta$, which shows that $\lim_{x\to a} |f(x) - \ell| = 0$.

Now suppose that $\lim_{x\to a} |f(x) - \ell| = 0$. Then

$$-|f(x) - \ell| \leq f(x) - \ell \leq |f(x) - \ell| \quad \text{for all } x.$$

The squeeze law now implies that $\lim_{x\to a}(f(x) - \ell) = 0$, and by the limit laws

$$\lim_{x\to a} f(x) = \lim_{x\to a}(f(x) - \ell + \ell) = \lim_{x\to a}(f(x) - \ell) + \lim_{x\to a} \ell = 0 + \ell = \ell.$$

10. (a) For $x, y \in \mathbb{R}$ we obviously have $|x| = \Big|(x - y) + y\Big|$ and $|y| = \Big|(y - x) + x\Big|$. Use the triangle inequality to show that $\Big||x| - |y|\Big| \leq |x - y|$. This is called the *reversed triangle inequalty*.

Solution: By the triangle inequality

$$|x| = \Big|(x - y) + y\Big| \leq |x - y| + |y| \qquad \text{so} \quad |x| - |y| \leq |x - y|$$

and similarly

$$|y| = \Big|(y - x) + x\Big| \leq |y - x| + |x| \qquad \text{so} \quad |y| - |x| \leq |y - x| = |x - y|.$$

Combining the two inequality, using the definition of absolute value, we obtain

$$\Big||x| - |y|\Big| \leq |x - y|$$

for all $x, y \in \mathbb{R}$ (this works the same way in $\mathbb{C}$).

(b) Hence, use the squeeze law to show $\lim_{x\to a} f(x) = \ell$ implies that $\lim_{x\to a} |f(x)| = |\ell|$.

Solution: Using the reversed triangle inequality from the previous part we have that

$$\Big||f(x)| - |\ell|\Big| \leq |f(x) - \ell| \to 0$$

as $x \to a$ and hence $|f(x)| \to |\ell|$ as $x \to a$ by the squeeze law.

(c) Is the converse of the statement in part (b) true? Give a proof or a counter example.

Solution: The claim is not true, consider $f(x) := \dfrac{x}{|x|}$ for $x \neq 0$. Then the left hand limit as $x \to 0$ is -1 and the right hand limit as $x \to 0$ is $+1$. As they are distinct the limit does not exist. However $|f(x)| = 1$ for all $x \neq 0$, so $\lim_{x\to 0} |f(x)| = 1$ exists.

11. Prove the following results using the ϵ, δ definition:

(a) $\lim_{x\to 4} f(x) = -3$, where $f(x) = \begin{cases} 5 - 2x & \text{if } x \neq 4, \\ 100 & \text{if } x = 4. \end{cases}$

Solution: Observe that to find $\lim_{x\to 4} f(x)$, the value of the function at 4 is irrelevant. So for this proof we will need to use the formula $f(x) = 5 - 2x$. The first step is to investigate $|f(x) - (-3)|$ and if possible to get some idea of which values of x will make this expression less than any given $\epsilon > 0$.

Now $|f(x) - (-3)| = |(5 - 2x) - (-3)| = |8 - 2x| = 2|4 - x|$. This tells us that the difference between $f(x)$ and -3 can be made as small as we like by making the difference between x and 4 small enough. In particular, to guarantee that $|f(x) - (-3)| < \epsilon$ (which is the same as saying that $2|4 - x| < \epsilon$), we need only ensure that the x values satisfy $|x - 4| < \epsilon/2$.

$$|f(x) - (-3)| = |(5 - 2x) - (-3)| = |8 - 2x| = 2|4 - x|.$$

Thus $|f(x) - (-3)| < \epsilon$ whenever $0 < |x - 4| < \delta$, where $\delta = \epsilon/2$.

(b) $\lim_{x \to 0} g(x) = 0$, where $g(x) = \begin{cases} 3x & \text{if } x \text{ is rational,} \\ 7x & \text{if } x \text{ is irrational.} \end{cases}$

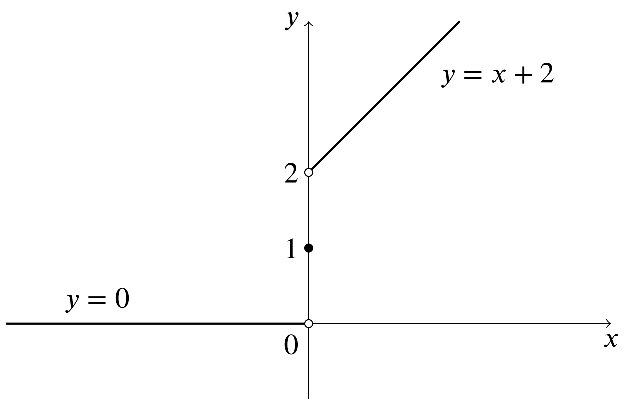
Solution: We must prove that for each $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$, we have $|g(x)| < \epsilon$. But clearly $|g(x)| \leq 7|x|$, so $\delta = \epsilon/7$ has the required property.

12. Sketch the graph of the function:

$$f(x) = \begin{cases} 0 & x < 0, \\ 1 & x = 0, \\ x + 2 & x > 0. \end{cases}$$

Find $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ (no need for formal proofs). Does $\lim_{x \to 0} f(x)$ exist?

Solution: The graph is



The right hand limit is 2 and the left hand limit is 0 as $x \to 0$.

13. Using the limit laws, show that the limit of a function exists as $x \to a$, then the limit is unique. That is, prove that if $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} f(x) = m$, then $\ell = m$.
Hint: Write $\ell - m = (f(x) - m) - (f(x) - \ell)$.

Solution: By the hint and the triangle inequality

$$0 \leq |\ell - m| \leq |f(x) - m| - |f(x) - \ell|.$$

By assumption the right hand side goes to zero as $x \to a$. As $|\ell - m|$ is a constant it must be zero, that is, $\ell = m$.

Challenge questions (optional)

14. Prove that $\lim\limits_{x \to 0} f(x)$ does not exist, where $f(x) = \begin{cases} 0 & \text{if } x \text{ rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$

Solution: The idea is that any interval (no matter how small) surrounding 0 contains rational and irrational numbers, and so in this interval, f takes the values 1 and 0. Hence $f(x)$ cannot approach a limit as $x \to 0$.

Here is a formal proof by contradiction. Suppose that $\lim_{x\to 0} f(x) = \ell$. Given $\epsilon = \frac{1}{2}$, there exists a $\delta > 0$ such that

$$0 < |x| < \delta \Rightarrow |f(x) - \ell| < \tfrac{1}{2}.$$

Take any positive rational number r with $0 < r < \delta$. Then $f(r) = 0$, so $|0 - \ell| < \frac{1}{2}$, that is, $-\frac{1}{2} < \ell < \frac{1}{2}$.

Now take any positive irrational number s with $0 < s < \delta$. Then $f(s) = 1$, so $|1 - \ell| < \frac{1}{2}$, that is, $\frac{1}{2} < \ell < \frac{3}{2}$. This contradicts $-\frac{1}{2} < \ell < \frac{1}{2}$, so $\lim_{x\to 0} f(x)$ does not exist.

Such rational and irrational numbers exist by Question 20 in the Problem Sheet for Week 2.

*__15.__ Prove the quotient limit law: If $\lim\limits_{x\to a} g(x) = m$ and $m \neq 0$, then $\lim\limits_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{m}$.

Solution: We have

$$\left|\frac{1}{g(x)} - \frac{1}{m}\right| = \left|\frac{m - g(x)}{g(x)m}\right| = \frac{|g(x) - m|}{|g(x)||m|}.$$

Since $\lim_{x\to a} g(x) = m$, and since $m \neq 0$, taking $\epsilon = \dfrac{|m|}{2} > 0$ there is $\delta > 0$ such that

$$|g(x) - m| \le \frac{|m|}{2} \quad \text{for all } 0 < |x - a| < \delta.$$

Thus $|m| - |g(x)| \le \dfrac{|m|}{2}$, and so $|g(x)| \ge \dfrac{|m|}{2}$ whenever $0 < |x - a| < \delta$. This gives $\dfrac{1}{|g(x)|} \le \dfrac{2}{|m|}$, and so

$$\left|\frac{1}{g(x)} - \frac{1}{m}\right| = \frac{|g(x) - m|}{|g(x)||m|} \le \frac{2}{|m|^2}|g(x) - m|$$

whenever $0 < |x - a| < \delta$. Now the result follows from the squeeze law since $\dfrac{2}{|m|^2}$ is a constant and $|g(x) - m| \to 0$.