MATH1903 Lectures

Semester 2, 2012

Week 12

Daniel Daners

Summary:

General solution to $ay'' + by' + cy = f(x)$
is of the form

$$y(x) = y_p(x) + y_h(x)$$

where
- $y_p(x)$ is a particular solution
- $y_h(x)$ is the general solution of $ay'' + by' + cy = 0$

Often $y_p(x)$ has a similar form as $f(x)$:

$f(x)$	$y_p(x)$
• polynomial of degree n e.g. $f(x) = 2 + 3x^2$	polynomial of degree n try $A + Bx + Cx^2$
• $Ce^{\lambda x}$	try $Ae^{\lambda x}$
• trig functions $\cos \omega t, \sin \omega t$	try $A\cos\omega x + B\sin\omega x$
If $f(x)$ is a solution to homogeneous equation	try $Axf(x)$ multiply by x

Simple harmonic motion with a forcing term

$$y'' + \omega_0^2 y = \cos\omega t$$

forcing term

For a particular solution try

$$y_p(t) = A\cos\omega t + B\sin\omega t$$

Since $y'' + \omega_0^2 y$ is even if y is even we try

$$y_p(t) = A\cos\omega t$$ (which is even as the forcing term)

Substitute into equation:

$$-A\omega^2\cos\omega t + \omega_0^2 A\cos\omega t = \cos\omega t$$

$$A(\omega_0^2 - \omega^2)\cos\omega t = \cos\omega t$$

Hence $A(\omega_0^2-\omega^2) = 1$, so $A = \dfrac{1}{\omega_0^2-\omega^2}$ if $\omega \neq \omega_0$

Particular solution

$$y_p(t) = \frac{\cos\omega t}{\omega_0^2 - \omega^2} \quad \text{if } \omega \neq \omega_0$$

What if $\omega = \omega_0$?

We could try $y_p(t) = t(A\cos\omega_0 t + B\sin\omega_0 t)$

As $t\sin\omega_0 t$ is even we only need $y_p = tB\sin\omega_0 t$

Alternative: take limit as $\omega \to \omega_0$ of solution with $\omega \neq \omega_0$.

Adding a solution of the homogeneous equation

$$y'' + \omega^2 y = 0$$

we can get a different solution: $y_h = A\cos\omega t + B\sin\omega t$

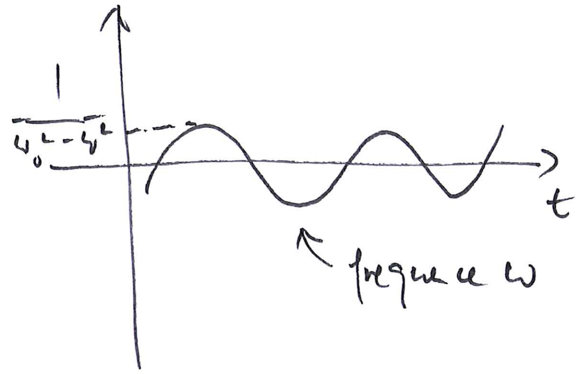
Hence

$$y_p(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2} = -\frac{1}{\omega_0 + \omega}\,\frac{\cos\omega t - \cos\omega_0 t}{\omega - \omega_0}$$

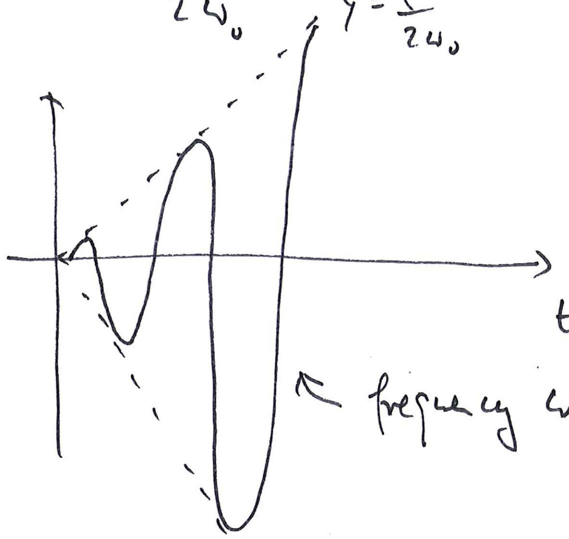
$$\xrightarrow[\omega \to \omega_0]{} -\frac{1}{2\omega_0}\frac{d}{d\omega}\cos\omega t\Big|_{\omega=\omega_0} = \frac{t\sin\omega_0 t}{2\omega_0}$$

Graph of $y_p(t)$ if $\omega \neq \omega_0$ | Graph of $y_p(t)$ if $\omega = \omega_0$

$$y_p(t) = \frac{\cos \omega t}{\omega_0^2 - \omega^2}$$



$$y_p(t) = \frac{t \sin \omega_0 t}{2\omega_0}$$



case of <u>resonance</u>

Systems of first order differential equations

Consider two equations for two unknown functions:

$$x'(t) = f(t, x(t), y(t))$$
$$y'(t) = g(t, x(t), y(t))$$

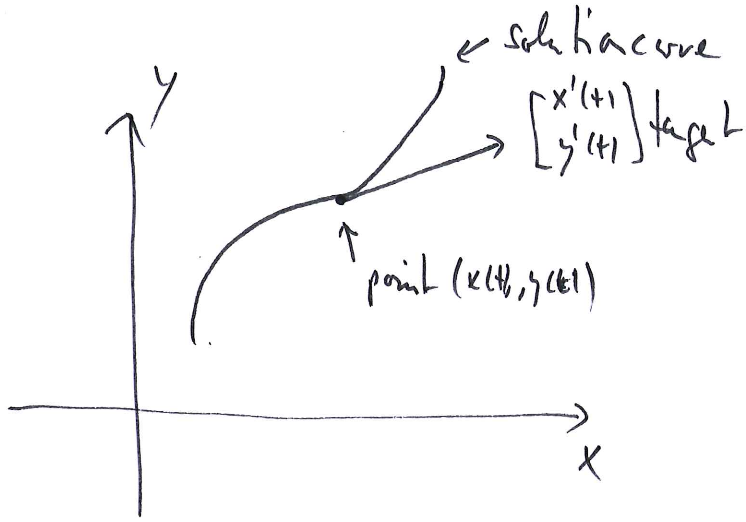
non-autonomous if f, g explicitly depend on t

We only look at autonomous equations:

$$x'(t) = f(x(t), y(t))$$
$$y'(t) = g(x(t), y(t))$$

We can view a solution as a curve $(x(t), y(t))$ in the plane. The tangent to that curve is the vector

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

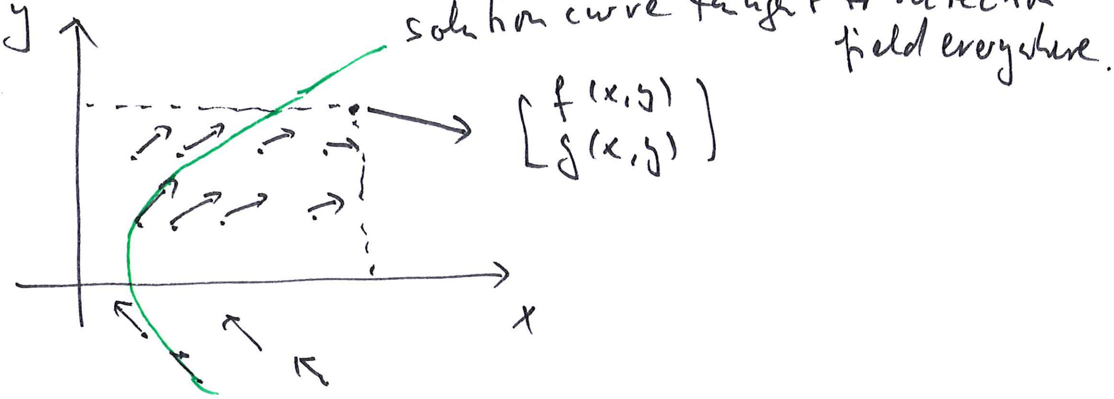


From the differential equation, the direction of the tangent is

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} f(x(t), y(t)) \\ g(x(t), y(t)) \end{bmatrix}$$

Hence we can find the direction of the solution at any point (x,y) without solving the equation.

If we plot this direction at any point we get the direction field of the system



Example Simple predator-prey system

$x(t)$ population of prey at time t (rabits)

$y(t)$ population of predator at time t (foxes)

Equation for $x(t)$

$$\frac{dx}{dt} = k(y)\, x$$

Equation of $y(t)$

$$\frac{dy}{dt} = h(x)\, y$$

Determine $k(y)$

$$k(y) = a - by$$

reduced rate due to predator.

unrestricted growth

Determine $h(x)$:

$$h(x) = -d + cx$$

extinction if no prey

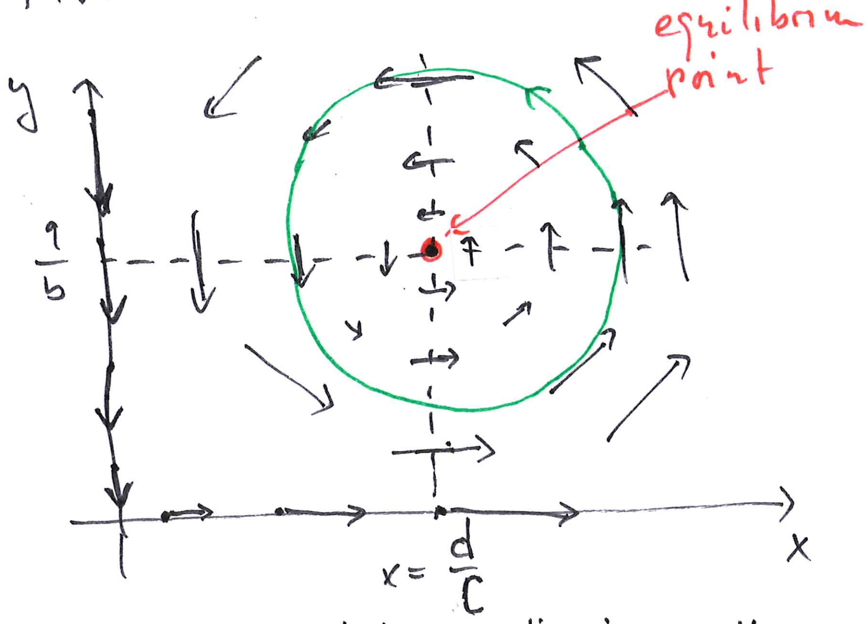
growth depending on size of x

Hence $x(t), y(t)$ satisfy the system of DEs:

$$\frac{dx}{dt} = (a - by)x \qquad (a, b, c, d > 0 \text{ const})$$

$$\frac{dy}{dt} = (cx - d)y$$

Plot the direction field:



How to draw dir. field?

- what happens on x, y-axis? $y=0$, $x=0$
- Find points where field is horizontal $\frac{dy}{dt} = 0 = (cx-d)y = 0$ $y=0$ or $cx-d=0$
- Find points where field is vertical $\frac{dx}{dt} = 0 = (a-by)x$ $x=0$ or $a-by=0$

Unless we are at the equilibrium, the populations oscillate with time.

Linear systems of differential equations

$$x' = ax + by$$
$$y' = cx + dy$$

a, b, c, d const.

One way to solve:

- differentiate one equation
- eliminate one of x or y
- solve a second order equation
- then recover the other variable.

Example:

$x' = 2x + y$ ①

$y' = 4x - y$ ②

Differentiate ①

$x'' = 2x' + y'$

$x'' = 2x' + 4x - y$ (use ②)

$x'' = 2x' + 4x - (x' - 2x)$ (use ①)

$= x' + 6x$

Equation for x: $x'' - x' - 6x = 0$

auxiliary equation $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$

general solution: $x(t) = Ae^{3t} + Be^{-2t}$

Compute y from ①:

$y = x' - 2x = 3Ae^{3t} - 2Be^{-2t} - 2Ae^{3t} - 2Be^{-2t}$

$= Ae^{3t} - 4Be^{-2t}$

In vector form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{3t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-2t}$

Different approach: write system in vector form:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a\,x(t) + b\,y(t) \\ c\,x(t) + d\,y(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\shortparallel$$

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Hence the system takes the form

$$\frac{d}{dt} \underset{\sim}{u}(t) = A \underset{\sim}{u}(t), \quad \text{where } \underset{\sim}{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The previous example suggests that there are solutions of the form

$$\underset{\sim}{u}(t) = e^{\lambda t} \underset{\sim}{v} \qquad (\underset{\sim}{v} \text{ a vector})$$

(λ, $\underset{\sim}{v}$ to be determined)

Substitute into equation:

$$\frac{d}{dt}\left(e^{\lambda t}\underset{\sim}{v}\right) = \lambda e^{\lambda t}\underset{\sim}{v} \underset{\text{want}}{=} A\left(e^{\lambda t}\underset{\sim}{v}\right) = e^{\lambda t}A\underset{\sim}{v}$$

Divide equation by $e^{\lambda t} \neq 0$:

$$A\underset{\sim}{v} = \lambda \underset{\sim}{v}$$

Hence $e^{\lambda t}\underset{\sim}{v}$ is a solution of the DE if and only if $\underset{\sim}{v}$ is an eigenvector of A corresponding to the eigenvalue λ.

Example revisited:

$$\begin{aligned} x' &= 2x+y \\ y' &= 4x-y \end{aligned} \quad \rightsquigarrow \text{ vector form } \quad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

characteristic polynomial of matrix

$$\det\begin{bmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{bmatrix} = +(\lambda-2)(1+\lambda) - 4$$

$$= \lambda^2 - \lambda - 2 - 4 = \lambda^2 - \lambda - 6 = 0$$

(same as aux. eq from before!)

We can write a 2nd order equation as a 1st order system.

$$ax'' + bx' + cx = 0$$

$$x := x$$
$$y := x'$$

Then $y' = x'' = -\frac{b}{a}x' - \frac{c}{a}x = -\frac{b}{a}y - \frac{c}{a}x$

We get the system

$$x' = y$$
$$y' = -\frac{c}{a}x - \frac{b}{a}y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Characteristic polynomial:

$$\det\begin{bmatrix} \lambda & -1 \\ \frac{c}{a} & \lambda + \frac{b}{a} \end{bmatrix} = \lambda\left(\lambda + \frac{b}{a}\right) + \frac{c}{a}$$

$$= \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} \rightsquigarrow a\lambda^2 + b\lambda + c = 0$$

aux. eq of $ax'' + bx' + cx = 0$