Lecture 2 on Fractals: The Cantor Set

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1 Introduction

Georg Cantor (1845–1918) introduced in 1883 the Cantor set, which is by far the most important in the category of early fractals. The Cantor set starts with a line segment of length one. Its generator is the same line segment from which the middle third is removed. The rule is to keep replacing the ever-shorter lines with ever more porous generators. If repeating this ad infinitum, the result is totally unexpected. No solid bit of line is left anywhere. All that is left is an irregular collection of individual points.

The Cantor set has remarkable and deep properties, which helped Cantor and others lay the foundations of set theory. Nature appears to like Cantor's construction. On earth, researchers have found that the spectra of some organic chemicals resemble the Cantor set. The complex groove structure in the rings of Saturn is amazingly similar to the Cantor set.

1.1 Construction of the Cantor set

Start with the closed interval $\mathcal{I}_0 = [0, 1]$.

Step 1: Divide $\mathcal{I}_0 = [0, 1]$ into three equal parts and remove the middle open interval. This leaves two closed intervals $[0, 1/3]$ and $[2/3, 1]$, whose union is denoted by $\mathcal{I}_1$. Hence

$$\mathcal{I}_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Step 2: Take each of the closed subintervals generated in Step 1 (that is $[0, 1/3]$ and $[2/3, 1]$) and remove their middle thirds. This yields $4 = 2^2$ subintervals, each of length $1/3^2$. The union of these four subintervals is denoted by $\mathcal{I}_2$, namely

$$\mathcal{I}_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Continue in this way. After the n-th Step, this construction generates a sequence of 2^n closed intervals of length $1/3^n$, whose union is denoted by $\mathcal{I}_n$. Since $\mathcal{I}_n$ is a *finite* union of closed intervals, we obtain that $\mathcal{I}_n$ is a *closed* set for each $n \geq 1$.

The following figure illustrates the first four steps in the above construction.

The Cantor set, say $\mathcal{C}$, is defined as the set of points which remain if we carry out the removal steps infinitely often. Mathematically, this means that

$$\mathcal{C} = \cap_{n=0}^{\infty} \mathcal{I}_n. \tag{1}$$

From the construction of the Cantor set, we infer that:

1. The Cantor set is *non-empty*. We notice that the end points of the closed intervals created in the steps will remain no matter how often we carry out the removal steps.

2. The Cantor set is *closed* since it is defined in (1) as the intersection of closed sets.

Question: *How many points do we have in the Cantor set? Can we count them?*

To answer this question, we need to introduce the concept of *countable* and *uncountable* sets. We next discuss about the cardinality of a set in connection with a topological game.

2 Cardinality and an Infinite Real Number Game

2.1 The Game

Alice and Bob alternate playing an infinite game on the real number line. A subset S of the unit interval $[0, 1]$ is fixed. Alice moves first by choosing a real number a_1 that is strictly between 0 and 1. Then Bob chooses any real number b_1 strictly between a_1 and 1. On each subsequent turn, the players must choose a point strictly between the previous two choices.

Equivalently, if we set $a_0 = 0$ and $b_0 = 1$, then for $n \geq 1$, Alice chooses a number a_n so that

$$a_{n-1} < a_n < b_{n-1}, \tag{2}$$

then Bob chooses a real number b_n with the property that

$$a_n < b_n < b_{n-1} < 1. \tag{3}$$

The sequence $(a_n)_{n\geq 1}$ is strictly increasing and bounded above. Hence, $(a_n)_{n\geq 1}$ converges as $n \to \infty$, meaning that $\lim_{n\to\infty} a_n = \alpha$ is a well-defined real number (between 0 and 1). If $\alpha \in S$, then Alice wins the game. If $\alpha \notin S$, then Bob wins the game.

2.2 Countable and Uncountable Sets

Definition. A non-empty set X is called *countable* if it is possible to list the elements of X in a (possibly repeating) infinite sequence $x_1, x_2, x_3, \ldots$ Equivalently, $X \neq \emptyset$ is countable if there exists an *injective* function $f : X \to \mathbf{N}$ from the set X to the set $\mathbf{N}$ of natural numbers.

Examples of countable sets:

(a) Every finite set is countable.

(b) The set $\mathbf{N}$, $\mathbf{Z}$ (the set of integers) and $\mathbf{Q}$ (the set of rational numbers).

But there are sets which are *not* countable, such as the interval $[0, 1]$. A set that is not countable is called *uncountable*. Cantor proved that the interval $[0, 1]$ is uncountable using a *diagonalization argument*. A different proof can be obtained using the above game.

Proposition 1. *If S is countable, then Bob has a winning strategy.*

Proof. If $S = \emptyset$, then Bob clearly wins. Otherwise, since S is countable, we can enumerate its elements as $s_1, s_2, s_3, \ldots$ Notice that

$$a_n < \alpha < b_n \quad \text{for every } n \geq 1. \tag{4}$$

Indeed, α being the limit of the strictly increasing sequence $(a_n)_{n\geq 1}$, we have $a_n < \alpha$ for every $n \geq 1$. Similarly, the sequence $(b_n)_{n\geq 1}$ being *strictly decreasing* from (3) and bounded below, it must converge to a number β (i.e., $\beta = \lim_{n\to\infty} b_n$) and $\beta < b_n$ for every $n \geq 1$. Passing to the limit $n \to \infty$ in (3), we find $\alpha \leq \beta$. Hence $\alpha < b_n$ for every $n \geq 1$.

From (4) we see that Bob can win if for each move $n \geq 1$ either $s_n \leq a_n$ or $s_n \geq b_n$. So, on each move $n \geq 1$, he compares s_n with a_n (Alice's choice on move n) and with his previous choice b_{n-1}. There are three cases:

$$\textbf{Case I}: s_n \leq a_n, \qquad \textbf{Case II}: s_n \geq b_{n-1}, \qquad \textbf{Case III}: a_n < s_n < b_{n-1}.$$

In Case I and II, he can choose any allowable number b_n (strictly between a_n and b_{n-1}). In the third case, if he chooses $b_n = s_n$ (which is a legal move), then $\alpha < s_n$ from (4). With this strategy, Bob always wins.

Corollary. *The interval $[0, 1]$ is uncountable.*
If $S = [0, 1]$, then Alice always wins. By Proposition 1, we thus find that $[0, 1]$ is uncountable.

Are there other instances (besides $S = [0, 1]$) for which Alice has a winning strategy? (If there are such instances for S, then S must be uncountable.)

Proposition 2. *If S is the Cantor set, then Alice has a winning strategy.*

Proof. Let S be the Cantor set. Alice chooses a_n to be always a *left end point* of one of the closed intervals in the construction of the Cantor set. Any such end point remains in the Cantor set, which being closed must contain $\lim_{n\to\infty} a_n$. Hence Alice wins.

Now the details. Alice can start with $a_1 = 2/3$. This is the *left end point* of the interval $[a_1, a_1 + 1/3^j]$, which is followed by $[a_1 + 2/3^j, a_1 + 3/3^j]$ in Step j (with $j \geq 2$) of the construction of the Cantor set. Note that $a_1 + 2/3^j$ remains in the Cantor set S.

Bob selects b_1 such that $a_1 < b_1 < 1$. Next, Alice chooses $a_2 = a_1 + 2/3^j$ with $j \geq 1$ large enough such that $a_2 < b_1$. This is always possible since $2/3^j \to 0$ as $j \to \infty$. Then Bob chooses b_2 in the interval (a_2, b_1). At each move n, Alice chooses

$$a_n = a_{n-1} + \frac{2}{3^j},$$

where $j \geq 2$ is large enough such that $a_n < b_{n-1}$. Note that a_n belongs to the Cantor set S for each $n \geq 1$. Hence $\alpha = \lim_{n\to\infty} a_n$ is in S and Alice wins.

For the Game in Section 2 and other topological games, see [1].

3 Properties of the Cantor set

Property 1. The Cantor set is *uncountable*. This follows from Propositions 1 and 2.

Property 2. The *interior* of the Cantor set is *empty*[1].

Proof. We show that the lengths of all the middle intervals that are removed in the construction of the Cantor set will add up to 1 (the length of the unit interval we start up with). In Step 1 the open interval $(1/3, 2/3)$ is removed, whose length is $1/3$. In Step 2, two open intervals of length $1/3^2$ are removed. More generally, in Step n we remove 2^{n-1} intervals of length $1/3^n$. Summing up all these lengths for all $n \geq 1$, we find

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum_{k=0}^{\infty} \frac{2^k}{3^k} = \frac{1}{3} \frac{1}{(1 - \frac{2}{3})} = 1.$$

Hence the Cantor set cannot contain any interval of non-zero length.

Property 3. The Cantor set is *self-similar*, consisting of two copies of itself, each scaled by a factor $r = 1/3$. Hence the *self-similarity dimension* of the Cantor set is given by $\frac{\log 2}{\log 3}$.

Remark. A self-similar set is a fractal if the self-similarity dimension exceeds the topological dimension of the set. Not every self-similar set is a fractal. Notice that lines, squares and cubes are examples of self-similar sets which are not fractals.

Property 4. The Cantor set has *topological dimension* zero and *fractal* dimension $\frac{\log 2}{\log 3}$, which is greater than zero. Hence, the Cantor set is a *fractal*.

We now recall several useful concepts. Let X be a subset of the interval $[0, 1]$.

Definition. A point x in $[0, 1]$ is called a *limit point* of X if for every $\epsilon > 0$, the open interval $(x - \epsilon, x + \epsilon)$ contains an element of X other than x.

Example. If $X = \{1/n : \ n \geq 1 \text{ is a natural number}\}$, then 0 is a limit point of X. Notice that 0 does *not* belong to X. In the definition of the limit point of X, one does not require the limit point to be in X, but rather in a larger space (in our case $[0, 1]$) that contains X.

Notation. Let $L(X)$ denote the set of limit points of X.

What is $L(X)$ for the set X in the above example?

Definition. A non-empty set X is *closed* if and only if X contains all its limit points (that is, $L(X) \subseteq X$). Furthermore, if $X = L(X)$, then X is called a *perfect set*.

The Cantor set emerged as an example of certain exceptional sets[2]. More precisely,

Property 5. The Cantor set $\mathcal{C}$ equals its set of limit points (i.e., $\mathcal{C}$ is a *perfect set*).

Proof. Since $\mathcal{C}$ is closed, we have $L(\mathcal{C}) \subseteq \mathcal{C}$. We now show that $\mathcal{C} \subseteq L(\mathcal{C})$. Let $p \in \mathcal{C}$ be an arbitrary point in the Cantor set. We want to show that $p \in L(\mathcal{C})$, that is

$$\text{for every } \epsilon > 0 \ \text{ the open interval } (p - \epsilon, p + \epsilon) \ \text{ contains a point } q \text{ in } \mathcal{C} \text{ other than } p. \quad (5)$$

[1] This means that the Cantor set $\mathcal{C}$ is *nowhere dense*.

[2] Cantor set is an example of a set which is *perfect* and *nowhere dense*.

From (1), the point p belongs to each interval $\mathcal{I}_n$ with $n \geq 1$. Hence, p must belong to one of the 2^n closed intervals of length $1/3^n$ that make up $\mathcal{I}_n$. Let q be one of the end points of the interval of length $1/3^n$ containing p such that $q \neq p$. We know that the end points of all the closed intervals in the construction of the Cantor set will remain in the Cantor set. Hence $q \in \mathcal{C}$ and the distance between p and q is smaller than $1/3^n$. By choosing $n \geq 1$ sufficiently large such that $1/3^n < \epsilon$, we ensure that $q \in (p - \epsilon, p + \epsilon)$. This proves (5). Since $p \in \mathcal{C}$ is arbitrary, we have shown that $\mathcal{C} \subseteq L(\mathcal{C})$. Hence $\mathcal{C} = L(\mathcal{C})$, that is $\mathcal{C}$ is a perfect set.

Remark. Proposition 2 is a particular case of Proposition 2 in [1], which proves that if S is a non-empty perfect set, then Alice has a winning strategy.

References

[1] MATTHEW BAKER, *Uncountable sets and in infinite real number game*, Mathematics Magazine, Vol. 80. No. 5, December 2007, 377–380.