3 Continuous maps and classes of subsets

[M] – Maple/Gnuplot; [A] – additional/optional problems; [H] – harder problems.

3.1 Open and closed subsets

You are allowed to use Problem 50 to prove your answer to other problems in this subsection. Also, you are allowed to use the fact that an interval (and a disjoint union of intervals) in $\mathbb{R}^1$ is open (closed, respectively) if and only if it does not contain its end points (contains all of its end points, respectively).

49: Prove that, for any subsets $U, V \in \mathbb{R}^m$ and every map

$$\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m,$$

i) $f^{-1}(U^c) = \left(f^{-1}(U)\right)^c$;

ii) $f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V)$;

iii) $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.

[A] **50**: Let f be a function from $\mathbb{R}^m$ to $\mathbb{R}^n$.

i) Prove that f is continuous on $\mathbb{R}^m$ if and only if $f^{-1}(U)$ is open in $\mathbb{R}^m$ for all open sets U in $\mathbb{R}^n$.

ii) Use the part i) of Question 49 and deduce that f is continuous on $\mathbb{R}^m$ if and only if $f^{-1}(U)$ is closed in $\mathbb{R}^m$ for all closed sets U in $\mathbb{R}^n$.

51: Decide whether each of the following subsets is open or closed. Prove your answer.

i)
$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : \quad x^2 - y^2 < 1\right\}.$$

ii)
$$\left\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\right\}.$$

iii)
$$\left\{(x, y) \in \mathbb{R}^2 : \quad x^2 - y^2 < 1\right\}.$$

iv)
$$\left\{(x, y) \in \mathbb{R}^2 : \quad x^2 - y^2 \geq 1\right\}.$$

v)
$$\left\{(x, y, z) \in \mathbb{R}^3 : \quad \sqrt{x^2 + y^2} \leq z \leq 1\right\}.$$

52: For each set Ω below show that Ω is neither open nor closed. Do so, by finding $f^{-1}(\Omega)$ for the function f provided. Find suitable function f if no function is given.

i) $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, 0)$,
$$\left\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\right\}.$$

ii) $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, 1)$
$$\left\{(x, y) \in \mathbb{R}^2 : \quad 0 \leq x^2 - y^2 < 1\right\}.$$

iii)
$$\left\{(x, y, z) \in \mathbb{R}^3 : \quad \sqrt{x^2 + y^2} \leq z < 1\right\}.$$

3.2 Bounded and path-connected subsets

For the problems of this subsection you are allowed to use definitions only.

53: Decide whether the subsets of Question 51 are bounded or unbounded. Prove your answer.

54: Decide whether the subsets of Question 51 are path-connected or not path-connected. Prove your answer. *Hint:* The following "coordinates" may be useful in case you need to build a continuous path connecting two points:

i) hyperbolic:
$$x = r\cosh t,\quad y = r\sinh t$$

ii) polar:
$$x = r\cos t,\quad y = r\sin t$$

iii) cylindrical:
$$x = r\cos t,\quad y = r\sin t,\quad z = h$$

3.3 Continuous maps and subsets

[A] **55**: Let $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Prove that

i) $K \subset \Omega$ and K is compact $\Rightarrow \mathbf{f}(K)$ is compact.

ii) $B \subset \Omega$ and B is path connected $\Rightarrow \mathbf{f}(B)$ is path connected.

56: Consider

$$S_1 = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\right\}$$
$$S_2 = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\right\}$$

Is there a continuous function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

i) $\mathbf{f}(S_1) = S_2$?

ii) $\mathbf{f}(S_2) = S_1$?

iii) $\mathbf{f}(\mathbb{R}^2) = S_2$?

iv) $\mathbf{f}(\mathbb{R}^2) = S_1$?

v) $\mathbf{f}(S_2) = \mathbb{R}^2$?

vi) $\mathbf{f}(S_1) = \mathbb{R}^2$?

Hint: Consider using polar coordinates (see Question 53) to reduce the problem to one-dimension and map the radius component only.

[H] **57**: Do Question 56 again with the subset S_2 replaced by the subset

$$S_3 = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 - y^2 < 1\right\}.$$

Hint Consider using hyperbolic coordinates together with polar coordinates (see Question 53).

[H] **58**: Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = (x^2 - y^2, x + y + 1).$$

Find the images of the following sets under f.

i) $\{(x, y) \ : \ x \geq 0, \ y \geq 0\}$.

ii) $\{(x, y) \ : \ 0 \leq x \leq 1, \ 0 \leq y \leq 1\}$.

iii) $\{(x, y) \ : \ y \geq -x\}$.

Hint: Map boundary first. In case of part (iii), find the image of the straight lines $x + y = a$ with $a > 0$.

Answers to problems

A50: Part i) proved in these webnotes[10] **A52**: Argument for (i) is explained in these webnotes[11]; Also, if you are interested in the direct, based on definition only, argument for (i), look at these webnotes[12];

[10] http://web.maths.unsw.edu.au/~potapov/2111_2015/Continuity-via-preimage.html
[11] http://web.maths.unsw.edu.au/~potapov/2111_2015/Continuity-via-preimage-_002d_002d-Examples.html
[12] http://web.maths.unsw.edu.au/~potapov/2111_2015/Non_002dtrivial-example-of-open-subset.html