The University of Sydney

MATH1901/06 Differential Calculus (Advanced)

Semester 1 **Short answers to exam questions** **2011**

1. (a) Circle and its interior, centre $3 - 2i$, radius 2, touching positive real axis.

(b) Factorisation: $P(z) = (z^2 - 6z + 10)(z^2 + z + 1)$.

(c) $f'(x) = (x\cos x - \sin x)/x^2$. Negative on $(0, \pi/2]$ according to the given inequality. Negative on $[\pi/2, \pi]$ because both terms are negative or zero.

2. (a) (i). $f(x,y) = \ln(4x^2 + y^2)$, $\nabla f = \dfrac{8x\,\mathbf{i} + 2y\,\mathbf{j}}{4x^2 + y^2}$, $\nabla f(1,2) = \mathbf{i} + \frac{1}{2}\mathbf{j}$.

Directional derivative at P: $D_{\mathbf{u}} f(1,2) = \nabla f \cdot \hat{\mathbf{u}} = 2/\sqrt{13}$.

(ii). $\hat{\mathbf{v}} = \nabla f / |\nabla f|\Big|_P = (2\mathbf{i} + \mathbf{j})/\sqrt{5}$, $\partial f/\partial n = |\nabla f| = \sqrt{5}/2$.

(iii). Tangent plane: $z = x + \frac{1}{2}y + 3\ln 2 - 2$.

(b) Put $x \to 2x$ in standard exp series: $e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \ldots$.
Put $x \to 3x$ in standard cosine series: $\cos 3x = 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \ldots$.
Multiply and stop at x^4. $T_4(x)$ for $e^{2x}\cos 3x$ is $1 + 2x - \frac{5}{2}x^2 - \frac{23}{3}x^3 - \frac{119}{24}x^4$.

3. (a) $g(1) = \ln 2 - \ln(1 + \sqrt{2}) < 0$, $g(10) = \ln 20 - \ln(1 + \sqrt{101}) > 0$. Sign change and continuity imply at least one zero on $[1, 10]$ by IVT. Zero is unique because $g'(x) = 1/x - x/(\sqrt{1 + x^2} + 1 + x^2) > 1/x - x/(1 + x^2) > 0$.
Alternatively, solve $g(x) = 0$ directly and get $x = 4/3$ (unique).

(b) (i). Limit is 10/9. (Cancel fraction to $\dfrac{x+7}{x+6}$ or use l'Hôpital's rule twice.)

(ii). Limit is $e^{1/6}$. $\big(\lim_{x\to 0}(1 + x^2/6 + \ldots)^{1/x^2} = \lim_{n\to\infty}(1 + 1/(6n))^n = e^{1/6}.\big)$

(iii). No limit. (Let $y = 0$ and $y = x$, limit depends on path.)

4. (a) $T_3(x) = x - x^3/6$, $R_3(x) = (\sin c)x^4/24 > 0$ because $0 < c < \pi$ when $x \in (0, \pi]$.

(b) $T_3(3) = -3/2$ and $T_3(x)$ decreasing for $x \geq 3$ imply $\sin x \geq -1 > T_3(x)$ on $[3, \infty)$. Part (a) covers $(0, 3]$. $x - \sin x$ is increasing, so positive on $(0, \infty)$.

(c) $x - \alpha x^3 < \sin x < x - x^3/6 + x^5/120$ implies $\alpha > 1/6 - x^2/120$. Small x forces $\alpha \geq 1/6$.

(d) $\left|\dfrac{\sin x}{x} - 1\right| = 1 - \dfrac{\sin x}{x} < 1 - \dfrac{T_3(x)}{x} = \dfrac{x^2}{6}$. The given inequality holds on $(0, \delta)$ when $\delta^2/6 \leq \epsilon$. In particular, it holds when $\delta = \sqrt{6\epsilon}$. The coefficient $A = \sqrt{6}$ is largest possible by (c). If the power of ϵ could be lowered, then A in $A\epsilon^{1/2}$ could be raised indefinitely. So $A = \sqrt{6}$ and $b = 1/2$. (Of course, $\sqrt{6\epsilon}$ is not the largest possible δ, just the largest of the form $A\epsilon^b$.)