(A)

MATH1902 2002 exam solutions 11/6/2010

Q1/ (i) ℓ passes through $P(2,3,-1)$ and $Q(3,2,-3)$

so has direction vector $\overrightarrow{PQ} = \underline{i} - \underline{j} - 2\underline{k}$, so has vector

equation $\boxed{\underline{r} = 2\underline{i} + 3\underline{j} - \underline{k} + t(\underline{i} - \underline{j} - 2\underline{k})}$.

(ii) This becomes $\begin{cases} x = 2 + t \\ y = 3 - t \\ z = -1 - 2t \end{cases}$

yielding Cartesian equations $\boxed{x - 2 = \frac{y-3}{-1} = \frac{z+1}{2}}$

(iii) $A = (1,2,1)$, $B = (4,3,2)$, $C = (3,-2,4)$ so

$\overrightarrow{AB} = 3\underline{i} + \underline{j} + \underline{k}$ and $\overrightarrow{AC} = 2\underline{i} - 4\underline{j} + 3\underline{k}$, so

$$\underline{u} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & 1 \\ 2 & -4 & 3 \end{vmatrix} = 7\underline{i} - 7\underline{j} - 14\underline{k}$$
$$= 7(\underline{i} - \underline{j} - 2\underline{k})$$

(iv) Plane containing A, B, C has equation

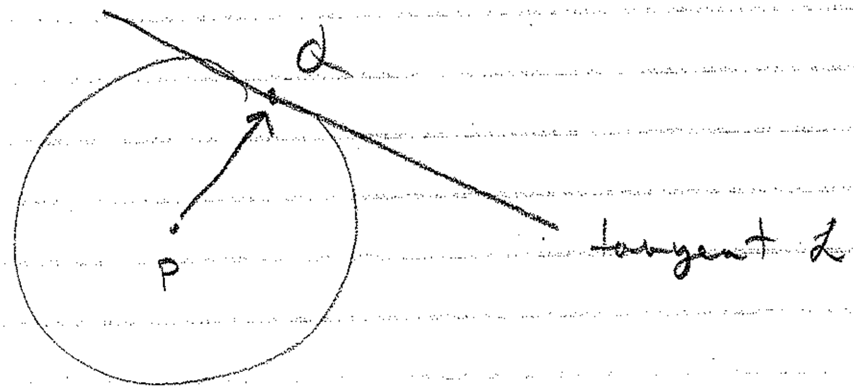
$$x - y - 2z = 1 - 2 - 2 = -3.$$

(v) ℓ has direction vector $\overrightarrow{PQ} = \frac{1}{7}\underline{u}$, normal to

the plane, so the line is perpendicular to the plane.

B

Q2/ (i) $P = (2,-3)$, $Q = (5,1)$

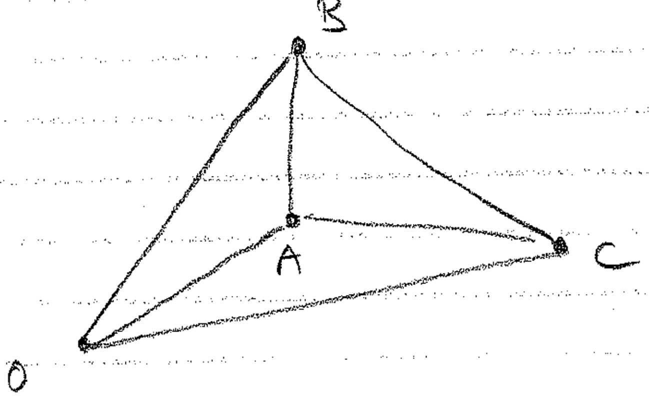


$\overrightarrow{PQ} = 3i + 4j$ so direction vector for $\mathcal{L}$ is

perpendicular to $\overrightarrow{PQ}$, say $v = 4i - 3j$,

so $\mathcal{L}$ has equation $\boxed{r = 5i + j + t(4i - 3j)}$.

(ii)



(a) $k = \overrightarrow{OK} = \frac{\alpha}{\alpha+\beta} a$, $l = \overrightarrow{OL} = \frac{\alpha}{\alpha+\beta} b$,

$m = \overrightarrow{OM} = \overrightarrow{OC} + \frac{\alpha}{\alpha+\beta}\overrightarrow{CB} = c + \frac{\alpha}{\alpha+\beta}(b - c)$, $n = \overrightarrow{ON} = \overrightarrow{OC} + \frac{\alpha}{\alpha+\beta}\overrightarrow{CA} = c + \frac{\alpha}{\alpha+\beta}(a - c)$

(b) Let S be the midpt of KM. Then

$$\overrightarrow{LS} = -l + k + \overrightarrow{KS} = k - l + \tfrac{1}{2}\overrightarrow{KM} = k - l + \tfrac{1}{2}(m - k)$$

$$= \frac{\alpha}{\alpha+\beta} a - \frac{\alpha}{\alpha+\beta} b + \frac{1}{2}\left(c + \frac{\alpha}{\alpha+\beta}(b - c) - \frac{\alpha}{\alpha+\beta} a\right)$$

2/ (ii) (b) (cont.)

$$= \frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\underset{\sim}{a} - \frac{\alpha}{\alpha+\beta}\underset{\sim}{b} - \frac{\alpha}{\alpha+\beta}\underset{\sim}{c} + \underset{\sim}{c}\right)$$

$$= \frac{1}{2}\left(-\frac{\alpha}{\alpha+\beta}\underset{\sim}{b} + \underset{\sim}{c} + \frac{\alpha}{\alpha+\beta}(\underset{\sim}{a}-\underset{\sim}{c})\right)$$

$$= \frac{1}{2}(-\underset{\sim}{l} + \underset{\sim}{n})$$

$$= \frac{1}{2}\overrightarrow{LN},$$

so that S is also the midpoint of LN.

This shows KM and LN intersect at their midpoints.

Q3/ (i) (a) $\left\{\begin{array}{l} x = 0 \\ y = 0 \\ x+y = 0 \end{array}\right\}$ has a unique solution.

(b) $\left\{\begin{array}{l} x+y = 0 \\ x+y = 0 \\ x+y = 0 \end{array}\right\}$ has infinitely many solutions.

(ii) $\left.\begin{array}{l} x + 2y + a^2 z = 2 \\ x + 3y + 4z = 1 \\ 2x + ay + 8z = 6 \end{array}\right\}$

$$\left[\begin{array}{ccc|c} 1 & 2 & a^2 & 2 \\ 1 & 3 & 4 & 1 \\ 2 & a & 8 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & a^2 & 2 \\ 0 & 1 & 4-a^2 & -1 \\ 0 & a-4 & 8-2a^2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & a^2 & 2 \\ 0 & 1 & 4-a^2 & -1 \\ 0 & 0 & & a-2 \end{array}\right]$$

$$8-2a^2-(a-4)(4-a^2)$$
$$= 8-2a^2-(-a^3+4a^2+4a-16)$$
$$= a^3-6a^2-4a+24$$

$$2+a-4 = a-2$$

D

Q3/ (b) (iii) The system is inconsistent iff

$p(a) = a^3 - 6a^2 - 4a + 24 = 0$ and $a - 2 \neq 0$.

Certainly $p(-2) = -8 - 24 + 8 + 24 = 0$

and $-2 - 2 = -4 \neq 0$, so $a = -2$ yields an

inconsistent system. Observe that

$$\begin{array}{r|l} & a^2 - 8a + 12 \\ a+2 & a^3 - 6a^2 - 4a + 24 \\ & a^3 + 2a^2 \\ \hline & -8a^2 - 4a + 24 \\ & -8a^2 - 16a \\ \hline & 12a + 24 \\ & 12a + 24 \\ \hline & 0 \end{array}$$

so that $p(a) = (a+2)(a^2 - 8a + 12)$

$= (a+2)(a-6)(a-2)$

so $\boxed{a = 6}$ also yields an inconsistent system

($a = 2$ yields an infinite solution.)

Q4/ (i) $\begin{bmatrix} 0 & n \\ \frac{1}{n} & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & n \\ \frac{1}{n} & 0 \end{bmatrix} \begin{bmatrix} 0 & n \\ \frac{1}{n} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

so $A = \begin{bmatrix} 0 & n \\ \frac{1}{n} & 0 \end{bmatrix}$ for $n \neq 0$ provides infinitely many

solutions to $A^2 = I$.

E

Q4/ (ii)
$$\begin{vmatrix} a+b & a & a & a \\ a & a+b & a & a \\ a & a & a+b & a \\ a & a & a & a+b \end{vmatrix} = \begin{vmatrix} 2a+b & 2a+b & 2a & 2a \\ a & a+b & a & a \\ a & a & a+b & a \\ a & a & a & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 3a+b & 3a+b & 3a+b & 3a \\ a & a+b & a & a \\ a & a & a+b & a \\ a & a & a & a+b \end{vmatrix} = \begin{vmatrix} 4a+b & 4a+b & 4a+b & 4a+b \\ a & a+b & a & a \\ a & a & a+b & a \\ a & a & a & a+b \end{vmatrix}$$

(iii) continuing (ii) :
$$= (4a+b) \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & a+b & a & a \\ a & a & a+b & a \\ a & a & a & a+b \end{vmatrix}$$

$$= (4a+b) \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & b & 0 & 0 \\ a & 0 & b & 0 \\ a & 0 & 0 & b \end{vmatrix} = (4a+b)\, b^3 .$$

(iv) $\det(J-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = (4-\lambda)(-\lambda)^3$

($a=1$, $b=-\lambda$ in (iii))

So eigenvalues are $\lambda = 4, 0$.

$$J = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ so $x+y+z+w=0$

Q4 (iv) (cont.)

giving eigenspace $\left\{ \begin{bmatrix} -s-t-u \\ s \\ t \\ u \end{bmatrix} \middle| \; s,t,u \in \mathbb{R} \right\}$

for $\lambda = 0$.

$$J-4I = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & -8 & 4 & 4 \\ 0 & 4 & -4 & 0 \\ 0 & 4 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left\{ \begin{array}{l} x - w = 0 \\ y - w = 0 \\ z - w = 0 \end{array} \right.$$

giving eigenspace $\left\{ \begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}$.

Q5/ (i) If $A\underset{\sim}{v} = \lambda \underset{\sim}{v}$ and $\underset{\sim}{v} \neq \underset{\sim}{0}$ then, for any scalar a,

$$(A + aI)\underset{\sim}{v} = A\underset{\sim}{v} + aI\underset{\sim}{v} = \lambda\underset{\sim}{v} + a\underset{\sim}{v} = (\lambda + a)\underset{\sim}{v},$$

which shows $\underset{\sim}{v}$ is an eigenvector for $A + aI$ with respect to eigenvalue $\lambda + a$.

Q5/ (ii) If $A\underset{\sim}{v} = \lambda \underset{\sim}{v}$ and $\underset{\sim}{v} \neq \underset{\sim}{0}$ then, for any scalar b,

$(bA)\underset{\sim}{v} = b(A\underset{\sim}{v}) = b\lambda\underset{\sim}{v} = (b\lambda)\underset{\sim}{v}$, which shows $\underset{\sim}{v}$ is an eigenvector for bA with respect to eigenvalue $b\lambda$.

(iii)
$$A\underset{\sim}{v} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & 1 & 0 & 1 \\ 0 & \cdots & & & 1 & 0 \end{bmatrix} \begin{bmatrix} \sin\theta \\ \sin 2\theta \\ \sin 3\theta \\ \vdots \\ \sin n\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin 2\theta \\ \sin\theta + \sin 3\theta \\ \sin 2\theta + \sin 4\theta \\ \vdots \\ \sin(n-2)\theta + \sin n\theta \\ \sin(n-1)\theta \end{bmatrix} = \begin{bmatrix} 2\cos\theta\sin\theta \\ 2\cos\theta\sin 2\theta \\ 2\cos\theta\sin 3\theta \\ \vdots \\ 2\cos\theta\sin(n-1)\theta \\ 2\cos\theta\sin n\theta \end{bmatrix}$$

noting $\sin(n+1)\theta = \sin\left(\frac{n+1}{n+1}\right)k\pi = 0$

so $\sin(n-1)\theta = \sin(n-1)\theta + \sin(n+1)\theta$
$= 2\cos\theta\sin n\theta$

$$= 2\cos\theta \begin{bmatrix} \sin\theta \\ \sin 2\theta \\ \vdots \\ \sin n\theta \end{bmatrix} = 2\cos\theta\, \underset{\sim}{v}$$

so $\underset{\sim}{v}$ is an eigenvector with respect to eigenvalue $2\cos\theta$.

Q5/(iv) By (i) and (ii), $\underline{v} = \begin{bmatrix} \sin\theta \\ \vdots \\ \sin n\theta \end{bmatrix}$

is an eigenvector for $bA + aI$ with corresponding eigenvalue $b\cos\theta + a$ where $\theta = \frac{k\pi}{n+1}$ for integers k such that $1 \leq k \leq n$.

These are all of the eigenvalues because there are n of them, all distinct.

because $\cos\frac{\pi}{n+1}, \cos\frac{2\pi}{n+1}, \ldots, \cos\frac{n\pi}{n+1}$ are all distinct