The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 7

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2012

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Topics covered

In lectures last week:

- ☐ Taylor polynomials and the remainder term.
- ☐ Taylor series. Examples: $e^x, \cos x, \sin x, \cosh x, \sinh x, \ln(1+x), \tan^{-1} x, (1+x)^\alpha$.

Objectives

After completing this tutorial sheet you will be able to:

- ☐ Compute Taylor polynomials.
- ☐ Understand that error bounds are an essential part to any good approximation.
- ☐ Be able to use the remainder term to find polynomial bounds for a function.
- ☐ Show that certain Taylor series converge by showing that $R_n(x)$ tends to 0.
- ☐ Find Taylor series of complicated functions by using the Taylor series of the basic building blocks of the function.
- ☐ Approximate integrals using Taylor polynomials and series.

Preparation questions to do *before* class

1. (a) Compute the Taylor series for $\cos x$ about $x = 0$. Show that the Taylor series converges to $\cos x$ for all $x \in \mathbb{R}$.

(b) Write down the Taylor series for $\cos(x^3)$.

2. Approximate, with error bounds, the integral $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$.

Questions to attempt in class

3. (a) Compute the nth order Taylor polynomial of $f(x) = \ln(1+x)$ about $x = 0$.

(b) Use Taylor's Theorem to write down an expression for the remainder term.

(c) Deduce that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for all } x \in [0,1].$$

(This equation actually holds for $x \in (-1, 1]$).

4. Recall that the error function is $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt$. Assuming that any reasonable series manipulations are valid, derive a series formula for $\operatorname{erf}(x)$.

5. (a) Use Taylor's Theorem to show that for all $x \geq 0$

$$1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \leq \frac{1}{\sqrt{1+x}} \leq 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4.$$

(b) Hence give upper and lower bounds for the integral $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1+x^3}}\,dx$.

Questions for extra practice

6. Derive a series formula for $\displaystyle\int_0^x \frac{e^t - 1}{t}\,dt$.

7. (a) Compute the Taylor series of $f(x) = \sinh x$ about $x = 0$, and show that the series converges to $\sinh x$ for all $x \in \mathbb{R}$.

(b) Derive series formulas for $\displaystyle\int_0^1 \sinh(x^2)\,dx$ and $\displaystyle\int_0^1 \frac{\sinh x}{x}\,dx$.

8. The Taylor series for $\tan^{-1} x$ is hard to find directly; here's an indirect method.

(a) Show that $\displaystyle\frac{1}{1+t^2} = \sum_{k=0}^{n-1}(-1)^k t^{2k} + \frac{(-1)^n t^{2n}}{1+t^2}$ for all $t \in \mathbb{R}$, and deduce that

$$\tan^{-1} x = \sum_{k=0}^{n-1}(-1)^k \frac{x^{2k+1}}{2k+1} + E_n(x), \quad \text{where} \quad E_n(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt.$$

(b) Show that $|E_n(x)| \leq \int_0^{|x|} t^{2n}\,dt = \frac{|x|^{2n+1}}{2n+1}$, and conclude that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for all } -1 \leq x \leq 1.$$

Challenging problems

9. From Question 3 we have the formula

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Unfortunately this converges pathetically slowly - it turns out that you need 1565238 terms to get $\ln 2$ correct to 6 decimal places! We can do much better using the function

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)$$

and noticing that $f(1/3) = \ln 2$.

(a) Find the Taylor series of $f(x)$ about $x = 0$. *Hint:* $f(x) = \ln(1+x) - \ln(1-x)$.

(b) Use the Taylor polynomial $T_6(1/3)$ to approximate $\ln 2$. Estimate the size of the remainder term $R_6(1/3)$. Deduce that you have $\ln 2$ correct to 2 decimals.

10. Here we use Taylor's Theorem to justify the manipulations made in Question 4.

(a) Use Taylor's Theorem to show that

$$e^{-t^2} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} t^{2k} + E_n(t), \qquad \text{where} \qquad |E_n(t)| \leq \frac{t^{2n+2}}{(n+1)!}$$

(b) Hence show that $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cdot k!} x^{2k+1}$ for all $x \in \mathbb{R}$.

11. Give another proof of the Lagrange formula for the remainder term $R_n(x;a)$: Suppose that $f(x)$ is $(n+1)$-times differentiable, and (rather cleverly) let

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k - R_n(x;a)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}.$$

(a) Show that $g(a) = 0$ and $g(x) = 0$.

(b) Apply the Mean Value Theorem to $g(t)$ to show that there is a c strictly between a and x such that

$$R_n(x;a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

12. From Question 8 we have *Leibnitz's Formula*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

This series is essentially useless for the purpose of approximating π (try it!). But there is something clever we can do. Recall the identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \qquad \text{valid for } xy < 1.$$

(a) Show that $4\tan^{-1}(\frac{1}{5}) = \tan^{-1}(\frac{120}{119})$ and $\tan^{-1} 1 + \tan^{-1}(\frac{1}{239}) = \tan^{-1}(\frac{120}{119})$.

(b) Hence prove *Machin's formula*: $\pi = 16\tan^{-1}(1/5) - 4\tan^{-1}(1/239)$. Use the first five terms from the $\tan^{-1} x$ series from Question 8 to approximate π.

13. This question shows that we need to be careful when rearranging the terms of a series. Recall that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

Consider the rearrangement

$$S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \cdots$$

(a) Express the partial sums of S in terms of the harmonic numbers H_m.

(b) Calculate the value of the series (*Hint:* $H_n - \ln n \to \gamma$ *as* $n \to \infty$).

14. Use 'reasonable' series manipulations and the Euler series formula to show that

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \frac{\pi^2}{6}.$$