(A)

MATH1903 Lecture 6 Fri 18/8/2017

Length of a curve: pp 64–69

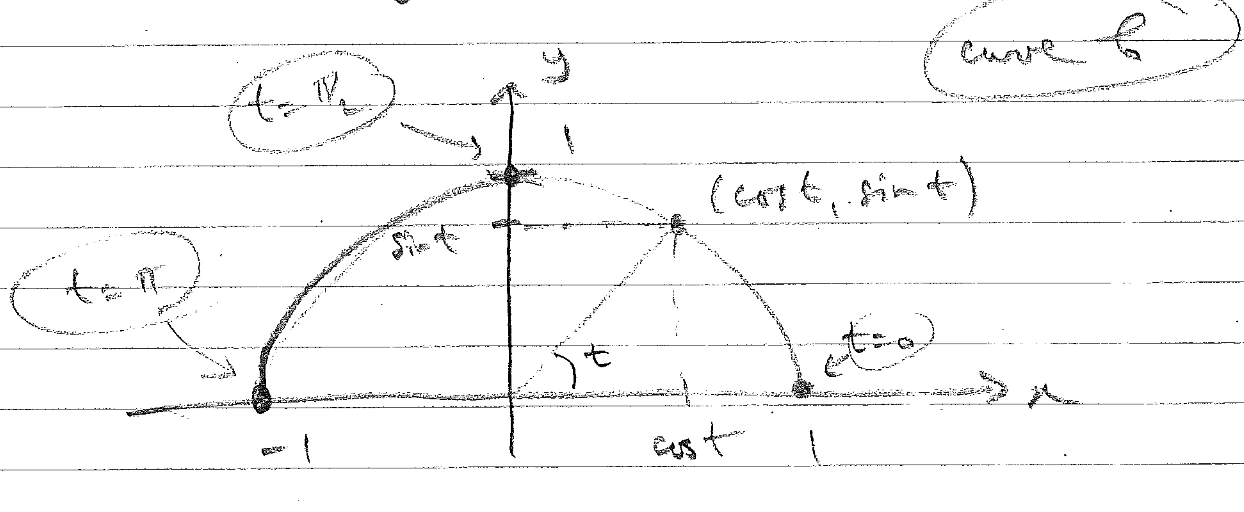
Let $\mathcal{C}$ be a curve in the plane described parametrically by differentiable functions

$$x = x(t), \quad y = y(t)$$

for $t \in [a, b]$. Then

$$\boxed{\text{length of } \mathcal{C} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt}$$

Example: $\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \quad 0 \le t \le \pi$



$$\text{length of } \mathcal{C} = \int_0^\pi \sqrt{\sin^2 t + \cos^2 t}\, dt$$

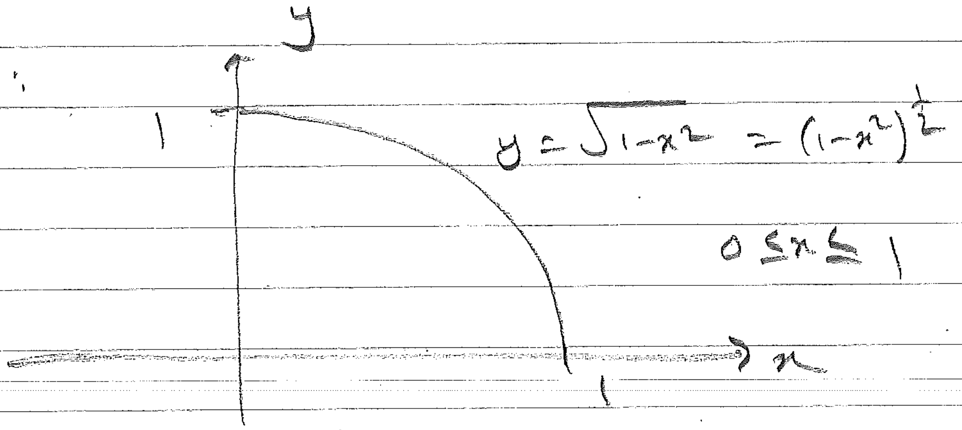
$$= \int_0^\pi 1\, dt$$

$$= [t]_0^\pi = \pi - 0 = \pi$$

✓ as expected.

Ⓑ

Example revisited:



$$y' = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$\text{length of } \tfrac{1}{4} \text{ circle} = \int_0^1 \sqrt{1+(y')^2}\,dx$$

$$= \int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx = \int_0^1 \sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx \underset{??}{\overset{\uparrow}{=}} \left[\sin^{-1} x\right]_0^1$$

$$= \sin^{-1} 1 - \sin^{-1} 0 = \pi/2 - 0 = \pi/2 \quad \checkmark$$

as expected

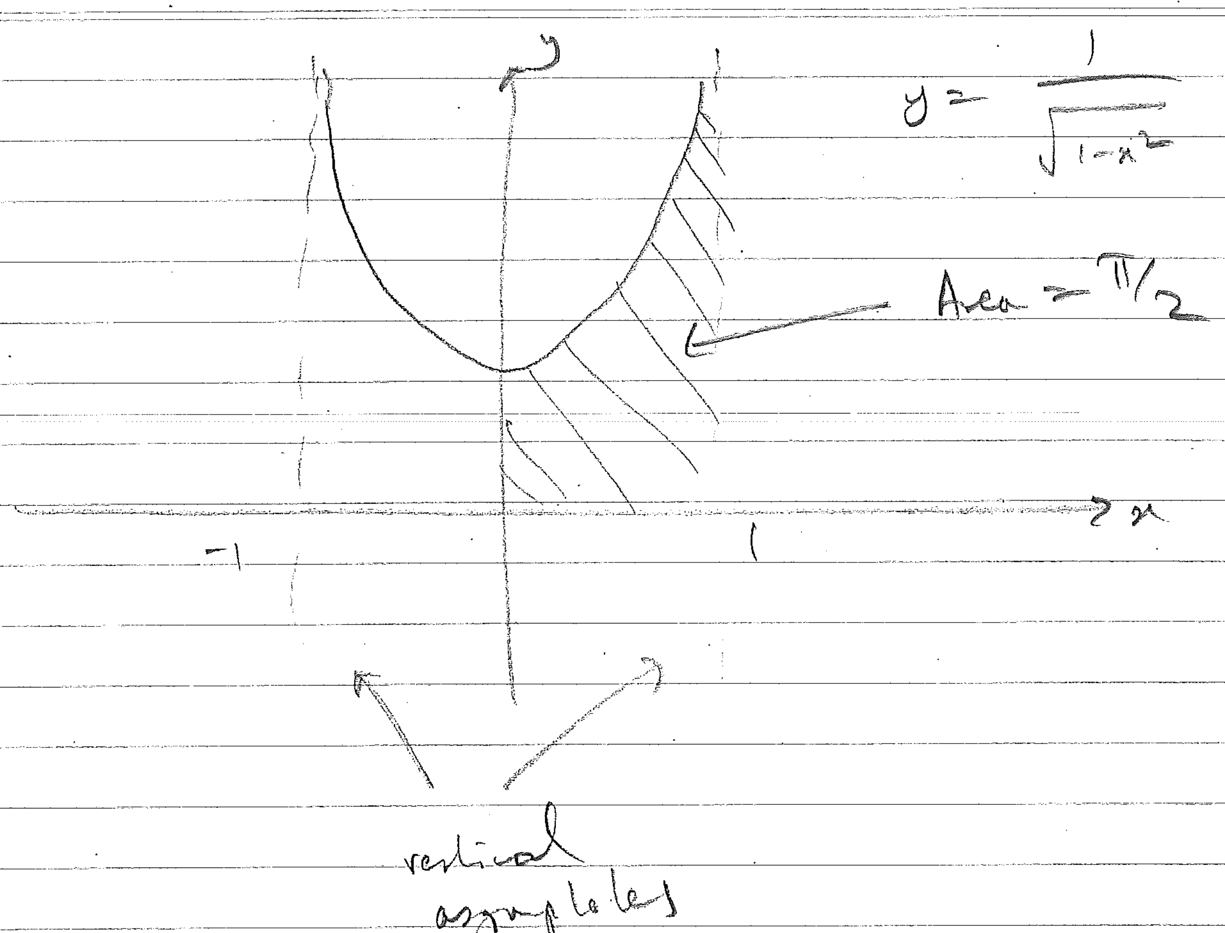
Corrected answer:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{b\to 1} \left[\sin^{-1} x\right]_0^b = \lim_{b\to 1} \left(\sin^{-1} b - \sin^{-1} 0\right)$$

$$= \pi/2 - 0 = \pi/2 \quad \checkmark\checkmark$$

(c)



$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ is an improper integral

discussed later

Logs and exponentials

- reference : pp 2.1 – 2.28 of Notes

pages 2.1, 2.2 existence of nth roots ?

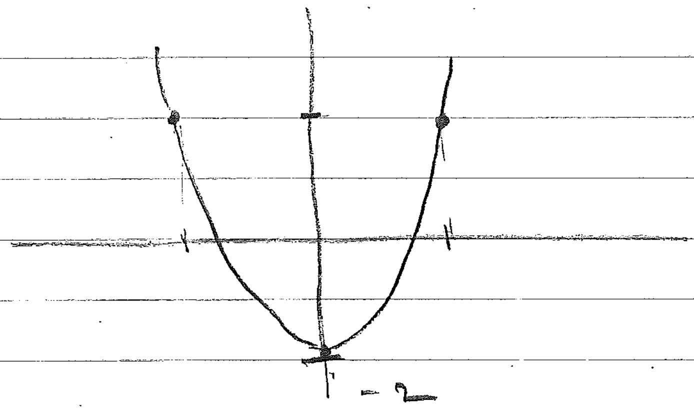
(nontrivial consequence of continuity theorems)

What about $2^{1/2} = \sqrt{2}$?

Proof $\sqrt{2}$ exists :

(i) (modern approach) :

Put $f(x) = x^2 - 2$, so f is continuous

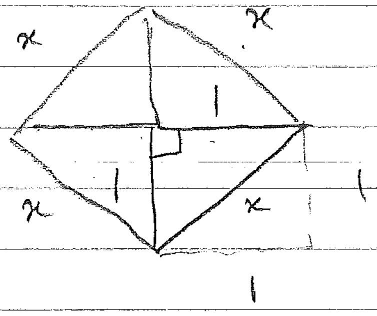


$f(0) = -2 < 0$, $f(2) = 2 > 0$, $f(0) < 0 < f(2)$.

By the Intermediate Value Theorem, $f(c) = 0$ for some $c \in (0,2)$, so $c^2 - 2 = 0$, so $c^2 = 2$,

so $c = \sqrt{2}$ exists. $\square$

(ii) (ancient approach) (Yale stone - Babylonian mathematics)



$$x^2 = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 2$$

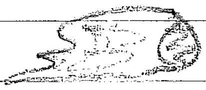
$x = \sqrt{2}$ exists (length of a line segment),

(isosceles case of Theorem of Pythagoras $\square$

However $\boxed{\sqrt{2} \notin \mathbb{Q} = \{\text{fractions}\}}$

Silly proof:

naturally decomposing log

Preliminaries 

Define for $x \in \mathbb{Z}^+$

$$\mathcal{O}(x) \overset{\text{def}}{=} \text{exponent of highest power of 2 dividing } x$$

e.g. $\mathcal{O}(24) = 3 = \mathcal{O}(8)$ $\quad 24 = 2^3 \times 3$

$\mathcal{O}(2) = 1$, $\mathcal{O}(1) = 0$.

$\boxed{\text{Simple fact: } \mathcal{O}(\alpha\beta) = \mathcal{O}(\alpha) + \mathcal{O}(\beta)}$

(F)

For $m, n \in \mathbb{Z}^+$, define

$$\mathcal{L}\left(\frac{m}{n}\right) \overset{\text{def}}{=} \mathcal{L}(m) - \mathcal{L}(n) \underbrace{\in \mathbb{Z}}$$

these logs give integers

Easy to check this is well-defined:

e.g. $\mathcal{L}\left(\frac{1}{2}\right) = \mathcal{L}(1) - \mathcal{L}(2) = 0 - 1 = -1$

$\mathcal{L}\left(\frac{16}{32}\right) = \mathcal{L}(16) - \mathcal{L}(32) = 4 - 5 = -1$.

$\boxed{\text{Simple fact continues to hold:}}$

"logs turn products into sums"

Proof that $\sqrt{2} \notin \mathbb{Q}$:

Suppose $\sqrt{2} = \alpha \in \mathbb{Q}$, so $\alpha^2 = 2$.

Then $\mathcal{L}(\alpha^2) = \mathcal{L}(2) = 1$

$$\mathcal{L}(\alpha^2) = \mathcal{L}(\alpha) + \mathcal{L}(\alpha) = 2\,\mathcal{L}(\alpha),$$

so 1 is even. (1 is odd).

Hence $\sqrt{2} \notin \mathbb{Q}$. $\square$

Exercise: modify this proof to show $\sqrt[3]{5} \notin \mathbb{Q}$.