THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Week 12** 2012

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Use an appropriate chain rule to find:

(a) $\dfrac{dz}{dt}$ where $z = x^3 + y^3$, $x = 3t$ and $y = 1 - t^2$;

(b) $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ where $z = xy$, $x = s + 2t$ and $y = s - 2t$.

Questions for the tutorial

2. Verify that $f_{xy}(x, y) = f_{yx}(x, y)$ for each function $f(x, y)$ given below.

(a) $x^2y^4 + 3x^2 + 5y^2$ (b) $\sin^2 x \cos y + 2$ (c) $xye^y + 3x + 5y$

3. For each function f given below, find all points (a, b) at which $f_x(x, y)$ and $f_y(x, y)$ are both zero. Determine whether $f(a, b)$ is a local maximum, a local minimum, or neither. (To do this, consider the sign of $f(a + h, b + k) - f(a, b)$.)

(a) $f(x, y) = x^2 + 2xy + 2y^2 - 6x + 8y$ (b) $f(x, y) = xy - 2x - 3y - 4$

4. Let f be the function given by $f(x, y) = \sqrt{20 - x^2 - 7y^2}$. Find the linear approximation to f at $(2, 1)$. Hence find an approximate value of $f(1.95, 1.08)$.

5. Estimate the volume of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter, if the metal in the top and base is 0.1 cm thick and the metal in the walls is 0.05 cm thick.

6. At time t, the temperature $u(x, t)$ at the point x of a long, thin insulated rod lying along the x axis satisfies the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad (k \text{ is a constant}).$$

Show that the function u given by the formula $u(x, t) = e^{-n^2kt} \sin nx$ satisfies the heat equation for any choice of the constant n.

7. A string is stretched along the x axis, fixed at each end, then set in vibration. The displacement $y(x, t)$ of the point at location x at time t satisfies the one-dimensional wave equation $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$, where the constant a depends upon the density and tension of the string. Show that each of the following functions satisfies the wave equation.

(a) $y = \cosh 3(x - at)$

(b) $y = \sin kx \cos kat$ (where k is a constant)

(c) $y = f(x - at) + g(x + at)$, where f and g are any twice-differentiable functions of one variable.

8. Car A is travelling north at 90 km/h and car B is travelling west at 80 km/h, both approaching the intersection of their highways. How fast is the distance between the cars changing when A is 0.3 km and B is 0.4 km from the intersection?

9. (a) Use two different methods to calculate $\dfrac{\partial u}{\partial t}$ if $u = \sqrt{x^2 + y^2}$, $x = e^{st}$ and $y = 1+s^2 \cos t$.

(b) Use two different methods to calculate $\dfrac{\partial z}{\partial x}$ if $z = \tan^{-1}\left(\dfrac{u}{v}\right)$, $u = 2x+y$ and $v = 3x-y$.

10. An object moves on the surface $z = (x-1)^2 + y^2$. The shadow of the object's path on the xy-plane is given by the parametric equations $x = 2\cos t$, $y = 2\sin t$ where $t \geq 0$ represents time. Use the chain rule to find the rate of change of height of the object above the xy plane. Hence find the maximum height achieved by the object.

Extra Question

11. Define a function f of two variables by

$$f(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$ at points $(x,y) \neq (0,0)$ and also at $(0,0)$. Observe that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Why does this not contradict the theorem mentioned in lectures?

Solution to Question 1

(a) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (3x^2)(3) + (3y^2)(-2t) = -6t + 81t^2 + 12t^3 - 6t^5.$

(b) $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = (y)(1)+(x)(1) = 2s,$ $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = (y)(2)+(x)(-2) = -8t.$