THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 6

MATH1901: Differential Calculus (Advanced) Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
Lecturer: Daniel Daners

Material covered

- ☐ Limits (continued).
- ☐ Squeeze Law (see also last week's tutorial).
- ☐ Limits as $x \to \infty$, or $x \to -\infty$.
- ☐ Continuity, left continuity, right continuity.

Outcomes

After completing this tutorial you should

- ☐ work with limits;
- ☐ understand the definition of continuity, left and right continuity;
- ☐ be able to prove that certain functions are continuous, right continuous or left continuous.

Summary of essential material

Limits as $x \to \pm\infty$. We say that $\lim_{x\to\infty} f(x) = \ell$ if for every $\epsilon > 0$ there exists $M > 0$ such that

$$x > M \quad \Rightarrow \quad |f(x) - \ell| < \epsilon.$$

Improper limits. We say that $\lim_{x\to a} f(x) = \infty$ if for every $m \in \mathbb{R}$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad f(x) > m.$$

The latter is called an *improper limit* or *divergence to infinity*. There are more such concepts (limit to $-\infty$as $x \to a$, or as $x \to \infty$ etc.) We can also look at right and left hand limits.

Continuity. A function $f(x)$ is *continuous* at $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

We can also give an ϵ-δ definition of limit: $f(x)$ is continuous at $x = a$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

Note that we don't require $0 < |x - a| < \delta$, because if $x = a$ then $f(x) = f(a)$ is automatic.

Left and Right Continuity We say f is *right* or *left continuous* at $x = a$ if $\lim_{x\to a+} f(x) = f(a)$ or $\lim_{x\to a-} f(x) = f(a)$ respectively. A function is continuous at a if and only if it is left continuous and right continuous at a.

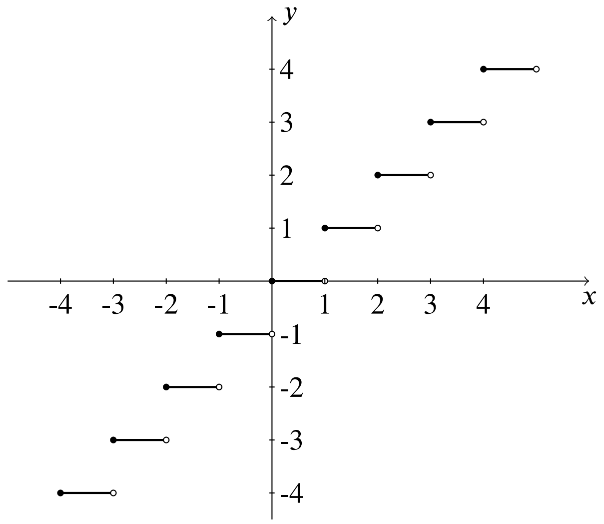
Continuity on Intervals. A function $f(x)$ is continuous on an open interval (a, b) if it is continuous at each point of (a, b). It is continuous on a closed interval $[a, b]$ if it is continuous on (a, b), right continuous at $x = a$, and left continuous at $x = b$.

How to show continuity of functions. As with limits, we use that elmentary functions are continuous such as x^α, $\sin x$, $\cos x$, e^x, $\ln x$, $\sin^{-1} x$, $\cos^{-1} x$ on their natural domains. From the limit laws, sums, products and quotients of continuous functions are continuous (denominator non-zero as always). By the composition/substitution law, compositions of continuous functions are continuous.

Questions to complete during the tutorial

1. Let $f(x) = \lfloor x \rfloor$, the largest integer less than or equal to x. Sketch the graph of f. At which points is f continuous? At which points is f right continuous, and at which points is it left continuous?

Solution: The graph is shown below:



This function is continuous at every non-integer value. It is left continuous only at non-integer values. It is right continuous everywhere.

2. Provide a careful step-by-step argument to explain why $f(x)$ is continuous at $x = \pi$, where

$$f(x) = \sqrt{\ln(\cos x + \sin x + 2x) + e^x}.$$

Solution: We break the function down into pieces:

- The functions $\cos x$, $\sin x$, and $2x$ are all continuous at $x = \pi$.
- Thus by limit laws the function $f_1(x) = \cos x + \sin x + 2x$ is continuous at $x = \pi$.
- The function $f_2(x) = \ln x$ is continuous at $x = \cos\pi + \sin\pi + 2\pi = 2\pi - 1$ (note that $2\pi - 1 > 0$ is in the domain of ln).
- Hence by the Composition Law $f_3(x) = f_2 \circ f_1(x) = \ln(\cos x + \sin x + 2x)$ is continuous at $x = \pi$.
- The function $f_4(x) = e^x$ is continuous at $x = \pi$.
- Hence by the limit laws the function $f_5(x) = f_3(x) + f_4(x) = \ln(\cos x + \sin x + 2x) + e^x$ is continuous at $x = \pi$.
- The function $f_6(x) = \sqrt{x}$ is continuous at $x = f_5(\pi) = \ln(2\pi - 1) + e^\pi$ (note that this number is positive).
- Hence by the Composition Law our function $f(x) = (f_6 \circ f_5)(x)$ is continuous at $x = \pi$.

3. Prove that if $f(x)$ is continuous at $x = a$, then the function $|f(x)|$ is continuous at $x = a$. (Use the reversed triangle inequality from a previous tutorial.) Is the converse true?

Solution: Note that for any real numbers r, s it is true that $\big||r| - |s|\big| \leq |r - s|$ (reversed triangle inequality, see last week's tutorial). This shows that

$$\big||f(x)| - |f(a)|\big| \leq |f(x) - f(a)|.$$

As f is continuous at a we have that $|f(x) - f(a)| \to 0$ as $s \to a$. Thus by the squeeze law also $\big||f(x)| - |f(a)|\big| \to 0$ as $x \to a$.

The converse assertion – that if $|f|$ is continuous at a, then so is f – is false. For instance, let f be the function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Then $|f|$ is the constant function with value 1, so it is continuous at 0, but f is not continuous at 0.

4. Determine whether the functions given by the following formulas are continuous the given x values.

(a) $h(x) = x^2 + \sqrt{7-x}$, at $x = 4$.

Solution: The function $x \mapsto x^2$ is continuous everywhere, and the square root function $x \mapsto \sqrt{x}$ is continuous everywhere in its domain $[0, \infty)$, so $h(x)$ is continuous everywhere in its domain $(-\infty, 7]$. In particular, it is continuous at 4.

(b) $k(x) = \dfrac{x^2 - 1}{x+1}$, at $x = -1$.

Solution: The domain of k does not include -1. Thus the function $k(x)$ is not continuous at $x = -1$.

(c) $F(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x > 0 \\ 1 - x & \text{if } x \leq 0 \end{cases}$, at $x = 0$.

Solution: As $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \dfrac{\sin x}{x} = 1$, $\lim_{x\to 0^-} F(x) = \lim_{x\to 0^-} 1 - x = 1$, and $F(0) = 1$, we see that F is continuous at 0.

(d) $K(x) = \begin{cases} \dfrac{x^2 - 1}{x+1} & \text{if } x \neq -1 \\ 6 & \text{if } x = -1 \end{cases}$, at $x = -1$.

Solution: Since $K(x) = \frac{x^2-1}{x+1} = x - 1$ for $x \neq -1$, we have

$$\lim_{x\to -1} K(x) = \lim_{x\to -1} x - 1 = -2.$$

However, $K(-1) = 6$, so $\lim_{x\to -1} K(x) \neq K(-1)$. Therefore K is discontinuous at -1.

5. Find a constant c so that g is continuous everywhere, where g is defined by:

(a) $g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4. \end{cases}$

Solution: The functions $x^2 - c^2$ and $cx + 20$, considered on the intervals $(-\infty, 4)$ and $[4, \infty)$ respectively, are continuous for any value of c. Thus the only possible discontinuity is at $x = 4$. For g to be continuous at 4, we require $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^+} g(x) = g(4)$, that is,

$$\lim_{x\to 4^-} (x^2 - c^2) = \lim_{x\to 4^+} (cx + 20) = g(4).$$

Hence $16 - c^2 = 4c + 20$, giving $c = -2$.

(b) $g(x) = \begin{cases} -c + \sqrt{x-4} & \text{if } x \geq 4 \\ |x^2 - c^2| & \text{if } x < 4. \end{cases}$

Solution: As in part (a), for g to be continuous at 4, we require $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^+} g(x) = g(4)$, that is,

$$\lim_{x\to 4^-} |x^2 - c^2| = \lim_{x\to 4^+} (-c + \sqrt{x-4}) = g(4).$$

Hence we require $|16 - c^2| = -c$. From this we see that $c \leq 0$. If $c \leq -4$ then we require $c^2 - 16 = -c$, that is, $c = \frac{-1-\sqrt{65}}{2}$. If $-4 < c \leq 0$ then we require $16 - c^2 = -c$, that is, $c = \frac{1-\sqrt{65}}{2}$. Hence the given function is continuous at 4 for two values of c, namely $c = \frac{-\sqrt{65}\pm 1}{2}$.

6. Calculate the following limits using limit laws, the squeeze law, and/or the substitution law:

(a) $\lim_{x\to 0} x^2 \cos \frac{2}{x}$

Solution: We use the Squeeze Law. Since $-x^2 \leq x^2 \cos \frac{2}{x} \leq x^2$ and $\lim_{x\to 0} \pm x^2 = 0$, we have $\lim_{x\to 0} x^2 \cos \frac{2}{x} = 0$.

(b) $\lim_{x\to 0} \frac{\sqrt{3+2x} - \sqrt{3}}{x}$

Solution: We can't use the limit laws with the expression in its present form, so we manipulate it first.

$$\begin{aligned}
\frac{\sqrt{3+2x} - \sqrt{3}}{x} &= \frac{(\sqrt{3+2x} - \sqrt{3})(\sqrt{3+2x} + \sqrt{3})}{x(\sqrt{3+2x} + \sqrt{3})} \\
&= \frac{3 + 2x - 3}{x(\sqrt{3+2x} + \sqrt{3})} \\
&= \frac{2}{\sqrt{3+2x} + \sqrt{3}}.
\end{aligned}$$

Hence

$$\lim_{x\to 0} \frac{\sqrt{3+2x} - \sqrt{3}}{x} = \lim_{x\to 0} \frac{2}{\sqrt{3+2x} + \sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Note that the last step used the substitution law to evaluate the limit of the denominator.

(c) $\lim_{x\to\infty} \frac{x + \sin^3 x}{2x - 1}$

Solution: $\lim_{x\to\infty} \frac{x + \sin^3 x}{2x - 1} = \lim_{x\to\infty} \frac{1 + \frac{\sin^3 x}{x}}{2 - \frac{1}{x}} = \frac{1}{2}$.

Note: we have used the fact that $\lim_{x\to\infty} \frac{\sin^3 x}{x} = 0$, which follows from an application of the Squeeze Law. Since $-1 \leq \sin^3 x \leq 1$, we have (for $x > 0$)

$$-\frac{1}{x} \leq \frac{\sin^3 x}{x} \leq \frac{1}{x} \quad \text{and} \quad \lim_{x\to\infty} \pm\frac{1}{x} = 0.$$

(d) $\lim_{x\to\infty} \sqrt{\frac{3-x}{4-x}}$

Solution: Divide top and bottom inside the square root sign by $-x$. We obtain

$$\sqrt{\frac{3-x}{4-x}} = \sqrt{\frac{-\frac{3}{x} + 1}{-\frac{4}{x} + 1}}.$$

Now as $x \to \infty$, $-\frac{3}{x} + 1 \to 1$ and $-\frac{4}{x} + 1 \to 1$. By the substitution law, as the square root function is continuous, we see that $\lim_{x\to 0} \sqrt{\frac{3-x}{4-x}} = \sqrt{\frac{1}{1}} = 1$.

(e) $\lim_{x\to\infty} \sqrt{\dfrac{3-x}{4-x^2}}$

Solution: This time we divide top and bottom by $-x^2$. We obtain

$$\sqrt{\frac{3-x}{4-x^2}} = \sqrt{\frac{-\frac{3}{x^2}+\frac{1}{x}}{-\frac{4}{x^2}+1}}.$$

Now as $x \to \infty$, $-\frac{3}{x^2}+\frac{1}{x} \to 0$ and $-\frac{4}{x^2}+1 \to 1$. By the substitution law, as the square root function is continuous, we see that $\lim_{x\to\infty} \sqrt{\frac{3-x}{4-x^2}} = \sqrt{\frac{0}{1}} = 0$.

(f) $\lim_{x\to\infty} (\sqrt{x} - \sqrt{x+1})$

Solution: $\lim_{x\to\infty}(\sqrt{x}-\sqrt{x+1}) = \lim_{x\to\infty} \frac{(\sqrt{x}-\sqrt{x+1})(\sqrt{x}+\sqrt{x+1})}{\sqrt{x}+\sqrt{x+1}} = \lim_{x\to\infty} \frac{-1}{\sqrt{x}+\sqrt{x+1}} = 0.$

7. (a) Suppose that f is a function such that $\lim_{x\to a} |f(x)| = \infty$. Use the definition of a limit to show that $\lim_{x\to a} \dfrac{1}{|f(x)|} = 0$, where a is either finite or $a = \infty$.

Solution: Let a be finite and fix $\varepsilon > 0$. Then clearly

$$\frac{1}{|f(x)|} < \varepsilon \quad \Longleftrightarrow \quad |f(x)| > \frac{1}{\varepsilon}$$

As $\lim_{x\to a} |f(x)| = \infty$ there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Rightarrow \quad |f(x)| > \frac{1}{\varepsilon}$$

Putting the two conditions together we see that

$$0 < |x-a| < \delta \quad \Rightarrow \quad \left|\frac{1}{|f(x)|} - 0\right| = \frac{1}{|f(x)|} < \varepsilon.$$

As the above argument works for any choice of $\varepsilon > 0$ we conclude that $\lim_{x\to a} \dfrac{1}{|f(x)|} = 0$.
We proceed similarly if $a = \infty$. Given $m \in \mathbb{R}$ we have

$$\frac{1}{|f(x)|} < \varepsilon \quad \Longleftrightarrow \quad |f(x)| > \frac{1}{\varepsilon}$$

As $\lim_{x\to\infty} |f(x)| = \infty$ there exists $m \in \mathbb{R}$ such that

$$x > m \quad \Rightarrow \quad |f(x)| > \frac{1}{\varepsilon}$$

Putting the two conditions together we see that

$$x > m \quad \Rightarrow \quad \left|\frac{1}{|f(x)|} - 0\right| = \frac{1}{|f(x)|} < \varepsilon.$$

As the above argument works for any choice of ε we conclude that $\lim_{x\to\infty} \dfrac{1}{|f(x)|} = 0$.

(b) Hence show that $\lim_{x\to\infty} e^{-x} = 0$ as $x \to \infty$.

Solution: We know that $e^x \to \infty$ as $x \to \infty$. Hence from part (a) we conclude that

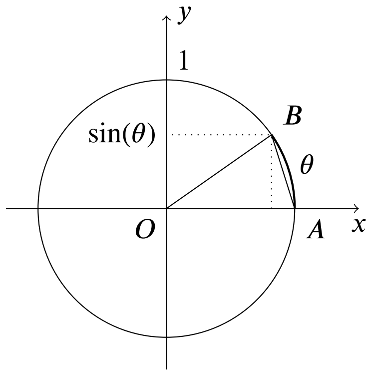
$$e^{-x} = \frac{1}{e^x} \to 0$$

as $x \to \infty$.

Extra questions for further practice

8. (a) By comparing the areas of a suitable sector and triangle, show that $|\sin\theta| \le |\theta|$, where $\theta \in \mathbb{R}$ is measured in radians.

Solution: Consider the diagram, where the circle is the unit circle:



To begin with, suppose that $0 < \theta < \pi/2$. The area of $\triangle OAB$ is less than the area of the sector OAB, which gives

$$0 \le \frac{1}{2} \times 1 \times \sin\theta \le \frac{\theta}{2\pi} \times \pi,$$

and so

$$0 \le \sin x \le x \quad \text{for all } 0 < \theta < \pi/2.$$

Multiplying by -1 this gives $0 \ge -\sin\theta \ge -\theta$ for all $\theta \in (0, \pi/2)$, and thus since $-\sin\theta = \sin(-\theta)$ we have $0 \ge \sin\theta \ge \theta$ for all $\theta \in (-\pi/2, 0)$. It follows that

$$|\sin x| \le |x| \quad \text{for all } 0 < |x| < \pi/2,$$

and this is clearly true also for $\theta = 0$, and also for $|\theta| \ge \pi/2$, because in this case $|\sin\theta| \le 1 < \pi/2 \le |x|$. Therefore $|\sin\theta| \le |\theta|$ for all $\theta \in \mathbb{R}$.

(b) Prove that $\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$ for all $x, y \in \mathbb{R}$.

Solution: You could either use various double angle formulae, or argue as follows. Recall from class that $\sin x$ and $\cos x$ can be written in terms of the complex exponential function as

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \quad \text{and} \quad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \quad \text{for all } x \in \mathbb{R}.$$

We have

$$\begin{aligned} 2\sin\frac{x-y}{2}\cos\frac{x+y}{2} &= \frac{1}{2i}\left(e^{i(x-y)/2} - e^{-i(x-y)/2}\right)\left(e^{i(x+y)/2} + e^{-i(x+y)/2}\right) \\ &= \frac{1}{2i}\left(e^{ix} + e^{-iy} - e^{iy} - e^{-ix}\right) \\ &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) - \frac{1}{2i}\left(e^{iy} - e^{-iy}\right) \\ &= \sin x - \sin y. \end{aligned}$$

(c) Hence, show that $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbb{R}$. Deduce that $\sin : \mathbb{R} \to \mathbb{R}$ is continuous.

Solution: Using the previous parts, and also the facts that $|\sin t| \le 1$ and $|\cos t| \le 1$ for all real numbers t, we have

$$|\sin x - \sin y| = 2\left|\sin\frac{x-y}{2}\right|\left|\cos\frac{x+y}{2}\right| \le 2\left|\frac{x-y}{2}\right| \times 1 = |x-y|.$$

(d) Using that the sine function is continuous, show that all other trigonometric functions are continuous. Use for instance that $\cos(x) = \sin(\pi/2 - x)$.

Solution: As $\cos(x) = \sin(\pi/2 - x)$ for all $x \in \mathbb{R}$ the subsitution law implies continuity of the cosine. Alternatively we can see this using the inequality from the previous part:

$$|\cos x - \cos y| = \left|\sin\left(\frac{\pi}{2} - x\right) - \sin\left(\frac{\pi}{2} - y\right)\right| \leq \left|\left(\frac{\pi}{2} - x\right) - \left(\frac{\pi}{2} - y\right)\right| = |y - x| = |x - y|$$

for all $x, y \in \mathbb{R}$.

Next, $\tan x = \dfrac{\sin x}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$ and $\operatorname{cosec} x = \dfrac{1}{\sin x}$ are continuous by the quotient law.

9. Compute the following limits using the limit laws and the substitution law.

(a) $\displaystyle\lim_{t\to 0} \frac{\tan t}{t}$.

Solution: We have $\dfrac{\tan t}{t} = \dfrac{\sin t}{t}\dfrac{1}{\cos t} \to 1\frac{1}{1} = 1$ as $t \to 0$ by using the elementary limit $\dfrac{\sin t}{t} \to 1$ and $\cos t \to 1$ as well as the product law.

(b) $\displaystyle\lim_{t\to 0} \frac{\sin(t^2)}{t}$.

Solution: We have $\dfrac{\sin(t^2)}{t} = t\dfrac{\sin(t^2)}{t^2} \to 0 \times 1 = 0$ as $t \to 0$ by using the elementary limit $\dfrac{\sin x}{x} \to 1$ (since $x = t^2 \to 0$) as well as the product law.

(c) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+1}\sin\frac{1}{x}$.

Solution: As $\sin\theta \leq \theta$ for all $\theta \geq 0$ we have that

$$\sqrt{2x^2+1}\sin\frac{1}{x} = x\sin\frac{1}{x}\sqrt{2+\frac{1}{x^2}} = \frac{\sin\frac{1}{x}}{\frac{1}{x}}\sqrt{1+\frac{1}{x^2}} \to \sqrt{2}\times 1 = \sqrt{2}$$

as $x \to \infty$, using that $1/x \to 0$ and the elementary limit $\dfrac{\sin\theta}{\theta} \to 1$ substituting $\theta = 1/x$.

(d) $\displaystyle\lim_{x\to\infty} \Big[\cosh(x)\big(\cosh(x) - \sinh(x)\big)\Big]$.

Solution: By definition of the hyperbolic functions

$$\cosh x - \sinh x = \frac{1}{2}\big((e^x + e^{-x}) - (e^x - e^{-x})\big) = \frac{1}{2}\big((e^x + e^{-x}) - (e^x - e^{-x})\big) = e^{-x}.$$

Hence,

$$\cosh(x)\big(\cosh(x) - \sinh(x)\big) = e^{-x}\frac{e^x + e^{-x}}{2} = \frac{2 + e^{-2x}}{2} \to \frac{1+0}{2} = \frac{1}{2}$$

as $x \to \infty$, using the limit laws.

(e) $\displaystyle\lim_{x\to 0} \frac{|3x+1| - |3x-1|}{x}$.

Solution: We have

$$\begin{aligned}\frac{|3x+1| - |3x-1|}{x} &= \frac{\big(|3x+1| - |3x-1|\big)\big(|3x+1| + |3x-1|\big)}{x\big(|3x+1| + |3x-1|\big)} \\ &= \frac{(3x+1)^2 - (3x-1)^2}{x\big(|3x+1| + |3x-1|\big)} = \frac{(9x^2+6x+1) - (9x^2-6x+1)}{|x|\big(|3x+1| + |3x-1|\big)} \\ &= \frac{12x}{x\big(|3x+1| + |3x-1|\big)} = \frac{12}{|3x+1| + |3x-1|}\end{aligned}$$

The limit of the denominator in the last expression as $x \to 0$ is 2, so $\dfrac{|3x+1| - |3x-1|}{x} \to \dfrac{12}{2} = 6$ as $x \to 0$.

(f) $\lim_{x\to 0} \dfrac{\sin(2x)}{\sin(5x)}$.

Solution: We can rewrite the expression in the form

$$\frac{\sin(2x)}{\sin(5x)} = \frac{2}{5}\frac{\sin(2x)}{2x}\left(\frac{\sin(5x)}{5x}\right)^{-1} \to \frac{2}{5}\times 1 \times \frac{1}{1} = \frac{2}{5}$$

as $x \to 0$.

10. Show that if $f(x)$ is continuous at $x = a$, and if $f(a) > 0$, then there is a number $\delta > 0$ such that $f(x) > 0$ whenever $|x - a| < \delta$.

Solution: By continuity of $f(x)$ at $x = a$, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

In particular, taking $\epsilon = f(a) > 0$ there is $\delta > 0$ such that

$$\begin{aligned} |x - a| < \delta &\Rightarrow |f(x) - f(a)| < f(a) \\ &\Rightarrow f(a) - f(x) < f(a) \\ &\Rightarrow f(x) > 0. \end{aligned}$$

Challenge questions (optional)

11. Consider the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } q > 0 \text{ and with } p \text{ and } q \text{ integers having no factors in common.} \end{cases}$$

For example $f(6/8) = 1/4$ since $6/8 = 3/4$. Prove that f is discontinuous at every rational number.

Solution:

Let a be any rational number, and suppose (for a contradiction) that f is continuous at $x = a$. We have $f(a) > 0$, so by the previous question there is $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x - a| < \delta$. However there is an irrational number y with $0 < y < \delta$ (see Tutorial 1). Then $x = a + y$ is also irrational, and $|x - a| < \delta$. But $f(x) = 0$, contradicting $f(x) > 0$. This is the desired contradiction, proving that f is discontinuous at every rational number (in particular, it has infinitely many discontinuities).

Remark: Rather remarkably, it turns out that $\lim_{x\to a} f(x) = 0$ for all $a \in \mathbb{R}$. Thus $f(x)$ is actually continuous at every irrational number!