CHAPTER **TEN**

Inequalities

10A Proving Inequalities by Algebra

Exercise 10A

1. Suppose that a and b are real numbers.
Prove that $\dfrac{a^2+b^2}{2} \geq \left(\dfrac{a+b}{2}\right)^2$.

2. If $a > b$, prove that $a^3 - b^3 \geq a^2b - ab^2$.

3. (a) Given that x and y are non-negative, prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$.

(b) Hence prove that $(x+y)(x+z)(y+z) \geq 8xyz$.

4. Suppose that p, q and r are real and distinct.

(a) Prove that $p^2 + q^2 > 2pq$.

(b) Hence prove that $p^2 + q^2 + r^2 > pq + qr + rp$.

(c) Given that $p + q + r = 1$, prove that $pq + qr + rp < \frac{1}{3}$.

5. Suppose that a, b and c are real numbers.

(a) Prove that $a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$.

(b) Hence show that $a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + b^2ac + c^2ab$.

(c) Deduce that if $a + b + c = d$, then $a^4 + b^4 + c^4 \geq abcd$.

6. Suppose that a, b and c are positive.

(a) Prove that $a^2 + b^2 \geq 2ab$.

(b) Hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

(c) Given that $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$,
prove that $a^3 + b^3 + c^3 \geq 3abc$.

(d) If x, y and z are positive, show that $x + y + z \geq 3\,(xyz)^{\frac{1}{3}}$.

(e) Suppose that $(1+x)(1+y)(1+z) = 8$. Prove that $xyz \leq 1$.

7. (a) Show that $a^2 + b^2 \geq 2ab$ for all real numbers a and b.
(b) Hence deduce that for all real numbers a, b and c:
(i) $(a+b+c)^2 \geq 3(ab+bc+ca)$
(ii) $ab(a+b)+bc(b+c)+ca(c+a) \geq 6abc$
(c) Suppose that a, b and c are the side lengths of a triangle.
(i) Explain why $(b-c)^2 \leq a^2$.
(ii) Deduce that $(a+b+c)^2 \leq 4(ab+bc+ca)$.

8. Suppose that a, b and c are positive.
(a) Prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$.
(b) Hence show that $(a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right) \geq 9$.
(c) (i) Prove that $a^3+b^3 \geq \left(\dfrac{a}{c}+\dfrac{b}{c}\right)abc$, and write down similar inequalities for b^3+c^3 and c^3+a^3.
(ii) Hence prove that $a^3+b^3+c^3 \geq 3abc$.
(iii) Deduce that $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a} \geq 3$.

9. Suppose that a, b, c and d are positive.
Use the fact that $\dfrac{a^2+b^2}{2} \geq ab$ to show that $\dfrac{a^2+b^2+c^2+d^2}{4} \geq \sqrt{abcd}$.

10. Suppose that x and y are positive. Prove that:
(a) $\dfrac{1}{x}+\dfrac{1}{y} \geq \dfrac{4}{x+y}$
(b) $\dfrac{1}{x^2}+\dfrac{1}{y^2} \geq \dfrac{8}{(x+y)^2}$

11. (a) Prove by induction that $2^n > n$, for all positive integers n.
(b) Hence show that $1 < \sqrt[n]{n} < 2$, if n is a positive integer greater than 1.
(c) Suppose that a and n are positive integers. It is known that if $\sqrt[n]{a}$ is a rational number, then it is an integer. Explain why $\sqrt[n]{n}$, where n is a positive integer greater than 1, is never a rational number.

12. (a) (i) Prove by induction that $(1+c)^n > 1+cn$, for all integers $n \geq 2$, where c is a nonzero constant greater than -1.
(ii) Hence show that $(1-\frac{1}{2n})^n > \frac{1}{2}$, for all integers $n \geq 2$.
(b) (i) Solve the inequation $x^2 > 2x+1$.
(ii) Hence prove by induction that $2^n > n^2$, for all integers $n \geq 5$.
(c) Suppose that $a > 0$, $b > 0$, and n is a positive integer.
(i) Divide the expression $a^{n+1} - a^n b + b^{n+1} - b^n a$ by $a-b$, and hence show that $a^{n+1}+b^{n+1} \geq a^n b + b^n a$.
(ii) Hence prove by induction that $\left(\dfrac{a+b}{2}\right)^n \leq \dfrac{a^n+b^n}{2}$.

13. (a) Given that $\sin x > \dfrac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, show that:

(i) $e^{-\sin x} < e^{-\frac{2x}{\pi}}$ for $0 < x < \frac{\pi}{2}$,

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\,dx$.

(b) Use the substitution $u = \pi - x$ to show that

$$\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx = \int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx\,.$$

(c) Hence show that $\displaystyle\int_0^{\pi} e^{-\sin x}\,dx < \frac{\pi}{e}(e-1)$.

14. For $n = 0,\ 1,\ 2,\ \ldots$ let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n\theta\,d\theta$.

(a) Show that $I_1 = \frac{1}{2}\ln 2$.

(b) Show that, for $n \ge 2$, $I_n + I_{n-2} = \dfrac{1}{n-1}$.

(c) For $n \ge 2$, explain why $I_n < I_{n-2}$, and deduce that

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}\,.$$

(d) Use the reduction formula in part (b) to find I_5, and hence deduce that

$$\tfrac{2}{3} < \ln 2 < \tfrac{3}{4}\,.$$

15. Let $I_n = \displaystyle\int_0^1 \frac{x^{n-1}}{(x+1)^n}\,dx$, for $n = 1,\ 2,\ 3,\ \ldots$

(a) Show that $I_1 = \ln 2$.

(b) Use integration by parts to show that $I_{n+1} = I_n - \dfrac{1}{n\,2^n}$.

(c) The maximum value of $\dfrac{x}{x+1}$, for $0 \le x \le 1$, is $\frac{1}{2}$.
Use this fact to show that $I_{n+1} < \frac{1}{2}I_n$.

(d) Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}$.

(e) Use the reduction formula in part (b) and the inequality in part (d) to show that $\frac{2}{3} < \ln 2 < \frac{17}{24}$.

10B Proving Inequalities by Geometry

Exercise 10B

1. (a) A regular hexagon is drawn inside a circle of radius 1 cm and centre O so that its vertices lie on the circumference, as shown in the first diagram.

(i) Show that $\triangle OAB$ is equilateral and hence find its area.

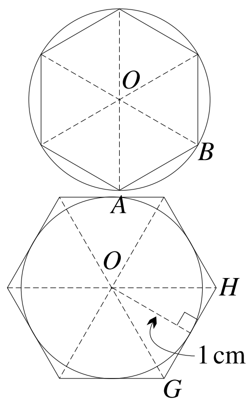
(ii) Hence find the exact area of this hexagon.

(b) A second regular hexagon is drawn so that each side is tangent to the circle, as shown.

(i) Find the area of $\triangle OGH$.

(ii) Hence find the exact area of the outer hexagon.

(c) By considering the results in parts (a) and (b), show that $\dfrac{3\sqrt{3}}{2} < \pi < 2\sqrt{3}$.



2. The diagram shows the points $A(0, 1)$ and $B(1, e^{-1})$ on the curve $y = e^{-x}$.

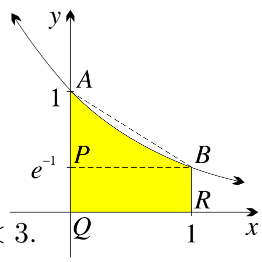
(a) Show that the exact area of the region bounded by the curve, the x-axis and the vertical lines $x = 0$ and $x = 1$ is $(1 - e^{-1})$ square units.

(b) Find the area of:

(i) rectangle $PBRQ$,

(ii) trapezium $ABRQ$.

(c) Use the areas found in the previous parts to show that $2 < e < 3$.



3. The diagram shows the curve $y = \sin x$ for $0 \le x \le \frac{\pi}{2}$. The points $P(\frac{\pi}{2}, 1)$ and $Q(\frac{\pi}{6}, \frac{1}{2})$ lie on the curve.

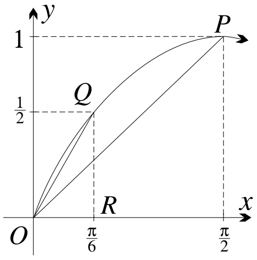
(a) Find the equation of the tangent at O.

(b) Find the equation of the chord OP, and hence show that $\frac{2x}{\pi} < \sin x < x$, for $0 < x < \frac{\pi}{2}$.

(c) Find the equation of the chord OQ, and hence show that $\frac{3x}{\pi} < \sin x < x$, for $0 < x < \frac{\pi}{6}$.

(d) By integrating $\sin x$ from 0 to $\frac{\pi}{6}$ and comparing this to the area of $\triangle ORQ$, show that

$$\pi < 12(2 - \sqrt{3}) \doteqdot 3{\cdot}2\,.$$



4. The points A, P and B on the curve $y = \dfrac{1}{x}$ have x-coordinates 1, $1\frac{1}{2}$ and 2 respectively. The points C and D are the feet of the perpendiculars drawn from A and B to the x-axis. The tangent to the curve at P cuts AC and BD at M and N respectively.

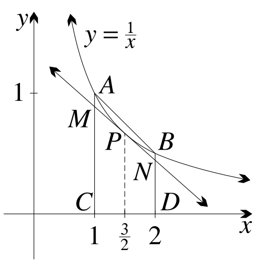
(a) Show that the tangent at P has equation

$$4x + 9y = 12.$$

(b) Find the coordinates of M and N.

(c) Find the areas of trapezia $ABDC$ and $MNDC$.

(d) Hence show that $\frac{2}{3} < \ln 2 < \frac{3}{4}$.



5. (a) Show, using calculus, that the graph of $y = \ln x$ is concave down throughout its domain.
(b) Sketch the graph of $y = \ln x$, and mark two points $A(a, \ln a)$ and $B(b, \ln b)$ on the curve, where $0 < a < b$.
(c) Find the coordinates of the point P that divides the interval AB in the ratio $2 : 1$.
(d) Using parts (b) and (c), deduce that $\frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln(\frac{1}{3}a + \frac{2}{3}b)$.

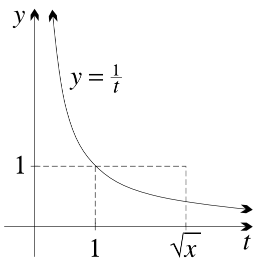
6. Let $f(x) = x^n\, e^{-x}$, where $n > 1$.
(a) Show that $f'(x) = x^{n-1}e^{-x}(n - x)$.
(b) Show that $(n, n^n e^{-n})$ is a maximum turning point of the graph of $f(x)$, and hence sketch the graph for $x \geq 0$. (Don't attempt to find points of inflexion.)
(c) Explain why $x^n e^{-x} < n^n e^{-n}$ for $x > n$. Begin by considering the graph of $f(x)$ for $x > n$,
(d) Deduce from part (c) that $(1 + \frac{1}{n})^n < e$.

7. The function $f(x)$ is defined by $f(x) = x - \log_e(1 + x^2)$.
(a) Show that $f'(x)$ is never negative.
(b) Explain why the graph of $y = f(x)$ lies completely above the x-axis for $x > 0$.
(c) Hence prove that $e^x > 1 + x^2$, for all positive values of x.

8. Consider the function $y = e^x\left(1 - \dfrac{x}{10}\right)^{10}$.
(a) Find the two turning points of the graph of the function.
(b) Discuss the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.
(c) Sketch the graph of the function.
(d) From your graph, deduce that $e^x \leq \left(1 - \dfrac{x}{10}\right)^{-10}$, for $x < 10$.
(e) Hence show that $\left(\dfrac{11}{10}\right)^{10} \leq e \leq \left(\dfrac{10}{9}\right)^{10}$.

9. Let $A(1, 1)$ and $B(k, \frac{1}{k})$, where $k > 1$, be points on the hyperbola $y = \dfrac{1}{x}$.
(a) Show that the tangents to the hyperbola at A and B intersect at $T\left(\frac{2k}{k+1}, \frac{2}{k+1}\right)$.
(b) Suppose that A', B' and T' are the feet of the perpendiculars drawn from A, B and T to the x-axis.
(i) Show that the sum of the areas of the two trapezia $AA'T'T$ and $TT'B'B$ is $\dfrac{2(k-1)}{k+1}$ square units.
(ii) Hence prove that $\dfrac{2u}{u+2} < \log(u + 1) < u$, for all $u > 0$.

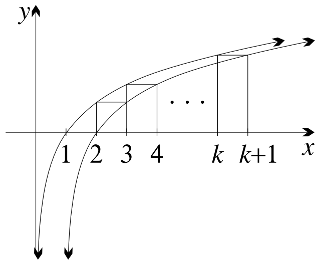
10. The diagram shows the curve $y = \dfrac{1}{t}$, for $t > 0$.
(a) If $x > 1$, show that $\displaystyle\int_1^{\sqrt{x}} \frac{1}{t}\, dt = \tfrac{1}{2}\log x$.
(b) Explain why $0 < \frac{1}{2}\log x < \sqrt{x}$, for all $x > 1$.
(c) Hence show that $\displaystyle\lim_{x\to\infty}\left(\frac{\log x}{x}\right) = 0$.



11. The diagram shows the curves

$$y = \log x \qquad \text{and} \qquad y = \log(x-1)\,,$$

and $k-1$ rectangles constructed between $x=2$ and $x=k+1$, where $k \ge 2$.



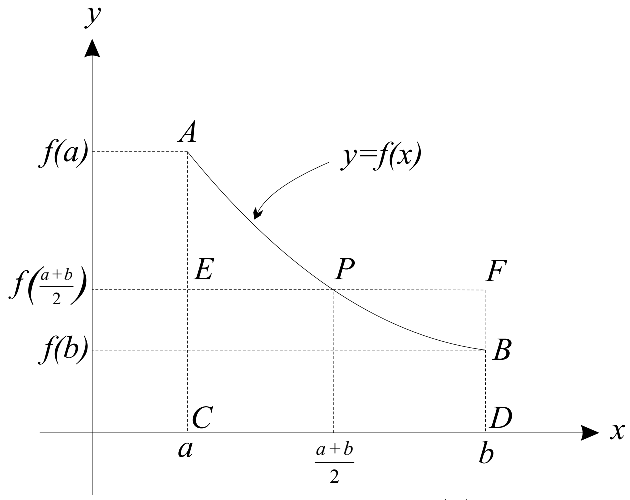
(a) Show that:

(i) $\displaystyle\int_2^{k+1} \log(x-1)\,dx = k\log k - k + 1$

(ii) $\displaystyle\int_2^{k+1} \log x\,dx = (k+1)\log(k+1) - \log 4 - k + 1$

(b) Deduce that $k^k < k!\,e^{k-1} < \frac{1}{4}(k+1)^{k+1}$, for all $k \ge 2$.

12.



The diagram above shows the curve $y = f(x)$ for $a \le x \le b$. Note that $f''(x)$ is positive for $a \le x \le b$.

(a) Use areas to explain briefly why

$$(b-a)\,f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a)\,\frac{f(a)+f(b)}{2}\,.$$

(b) Hence show that, for $n = 2, 3, 4, \ldots$,

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right).$$

(c) Deduce that

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

(d) Show that

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\,.$$

(e) Hence show that $\displaystyle\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}\,.$

13. (a) Show that $\displaystyle\int_1^n \ln x\,dx = n\ln n - n + 1$.

(b) Use the trapezoidal rule on the intervals with endpoints 1, 2, 3, ..., n to show that

$$\int_1^n \ln x\,dx \doteqdot \tfrac{1}{2}\ln n + \ln(n-1)!$$

(c) Hence show that $n! < n^{n+\frac{1}{2}}\,e^{1-n}$. NOTE: This is a preparatory lemma in the proof of Stirling's formula $n! \doteqdot \sqrt{2\pi}\,n^{n+\frac{1}{2}}\,e^{-n}$, which gives an approximation for $n!$ whose percentage error converges to 0 for large integers n.

14. (a) Prove that $\log_e x \le x - 1$ for $x > 0$.

(b) Suppose that p_1, p_2, p_3, ..., p_n are positive real numbers whose sum is 1.
Prove that $\displaystyle\sum_{r=1}^{n} \log_e(np_r) \le 0$.

(c) Let x_1, x_2, x_3, ..., x_n be positive real numbers.
Prove that $\dfrac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \ge (x_1x_2x_3\ldots x_n)^{\frac{1}{n}}$.

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Chapter Ten

Exercise **10A** (Page 1)

12(b)(i) $x > 1 + \sqrt{2}$ or $x < 1 - \sqrt{2}$
14(d) $I_5 = \frac{1}{4}(2 \ln 2 - 1)$

Exercise **10B** (Page 4)

1(a)(i) $\frac{\sqrt{3}}{4}$ sq. units **(ii)** $\frac{3\sqrt{3}}{2}$ sq. units **(b)(i)** $\frac{1}{\sqrt{3}}$ sq. units **(ii)** $2\sqrt{3}$ sq. units
2(b)(i) e^{-1} **(ii)** $\frac{1}{2}(1 + e^{-1})$
3(a) $y = x$ **(b)** $y = \frac{2x}{\pi}$ **(c)** $y = \frac{3x}{\pi}$
4(b) $M = (1, \frac{8}{9}),\ N = (2, \frac{4}{9})$ **(c)** $\frac{3}{4}$ and $\frac{2}{3}$ square units.
5(c) $\left(\frac{a+2b}{3}, \frac{\ln a + 2\ln b}{3}\right)$
8(a) $(0, 1)$ is a maximum turning point, $(10, 0)$ is a minimum turning point.
(b) $y \to \infty$ as $x \to \infty$, and $y \to 0$ as $x \to -\infty$.

