LECTURE 0

Assumed Knowledge

This is a review of basic facts about complex numbers that ought to be familiar:

- the definition of complex numbers,
- their arithmetic,
- Cartesian and polar representations,
- the Argand diagram,
- de Moivre's theorem, and
- extracting nth roots of complex numbers.

Students who do not feel confident about this material need to do lots of exercises about these, such as those in the MATH1141 notes.

1. Complex numbers

DEFINITION 0.1. A *complex number* is an expression of the form $x + iy$, where x and y are real numbers. The *real part* of $x + iy$ is x and the *imaginary part* of $x + iy$ is y. We denote this by $\mathrm{Re}(x + iy) = x$ and $\mathrm{Im}(x + iy) = y$. The set of all complex numbers is denoted $\mathbb{C}$.

We often write $w = u + iv$ and $z = x + iy$, and work with w and z rather than $u + iv$ and $x + iy$. In this case, we write, for instance, $\mathrm{Re}(z) = x$ and $\mathrm{Im}(w) = v$. We abbreviate $x + i0$ and $0 + iy$ to x and iy, and $0 + i1$ to i.

DEFINITION 0.2. Suppose that $w = u + iv$ and $z = x + iy$, where $u, v, x, y \in \mathbb{R}$. Then we define the complex numbers $w + z$, $-z$, wz and, if $z \neq 0$, z^{-1} and w/z, as follows:

$$\begin{aligned}
w + z &= (u + x) + i(v + y)\\
-z &= -x + i(-y)\\
wz &= (ux - vy) + i(uy + vx)\\
z^{-1} &= (x^2 + y^2)^{-1}(x - iy)\\
w/z = wz^{-1} &= (x^2 + y^2)^{-1}[(ux + vy) + i(vx - uy)].
\end{aligned}$$

Then $i^2 = -1$ and $-z = (-1)z$.

We may combine these operations to make sense of more complicated expressions such as $w^m - z^n$, where m and n are integers.

PROPOSITION 0.3. *Complex numbers have the following properties:*

$$
\begin{aligned}
z_1 + z_2 &= z_2 + z_1 && \forall z_1, z_2 \in \mathbb{C} \\
(z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) && \forall z_1, z_2, z_3 \in \mathbb{C} \\
0 + z &= z && \forall z \in \mathbb{C} \\
(-z) + z &= 0 && \forall z \in \mathbb{C} \\
z_1 z_2 &= z_2 z_1 && \forall z_1, z_2 \in \mathbb{C} \\
(z_1 z_2) z_3 &= z_1 (z_2 z_3) && \forall z_1, z_2, z_3 \in \mathbb{C} \\
1z &= z && \forall z \in \mathbb{C} \\
z z^{-1} &= 1 && \forall z \in \mathbb{C} \setminus \{0\} \\
z_1(z_2 + z_3) &= z_1 z_2 + z_1 z_3 && \forall z_1, z_2, z_3 \in \mathbb{C}
\end{aligned}
$$

The symbol $\forall$ is read "for all". This proposition shows that the complex numbers form a *field*.

DEFINITION 0.4. If $z = x + iy$, then $\overline{z}$, the *(complex) conjugate* of z, and $|z|$, the *modulus* of z, are defined to be $x - iy$ and $(x^2 + y^2)^{1/2}$.

Some further properties of complex numbers relate conjugates and moduli.

PROPOSITION 0.5. *For all $z, z_1, z_2 \in \mathbb{C}$,*

(a)	$\operatorname{Re}(z) = \dfrac{z + \overline{z}}{2}$	(b)	$\operatorname{Im}(z) = \dfrac{z - \overline{z}}{2i}$
(c)	$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$	(d)	$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$
(e)	$\overline{-z} = -\overline{z}$	(f)	$\overline{z^{-1}} = (\overline{z})^{-1}$
(g)	$\lvert z_1 + z_2 \rvert \leq \lvert z_1 \rvert + \lvert z_2 \rvert$	(h)	$\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \, \lvert z_2 \rvert$
(i)	$z \overline{z} = \lvert z \rvert^2$	(j)	$z^{-1} = \lvert z \rvert^{-2} \overline{z}$.

Inequality (g) is called the *triangle inequality*. If $|z| = 1$, then $z^{-1} = \overline{z}$, from (j).

PROOF. We only prove the triangle inequality, because this is hardest.

First, here is an algebraic proof. Recall that $2ab \leq a^2 + b^2$ for real numbers a and b. Taking a and b to be $x_1 y_2$ and $x_2 y_1$, we deduce that

$$2 x_1 x_2 y_1 y_2 \leq x_1^2 y_2^2 + x_2^2 y_1^2,$$

and hence

$$x_1^2 x_2^2 + y_1^2 y_2^2 + 2 x_1 x_2 y_1 y_2 \leq x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2),$$

so, taking square roots,

$$x_1 x_2 + y_1 y_2 \leq |z_1| \, |z_2| \, .$$

Finally, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, and so

$$
\begin{aligned}
|z_1 + z_2|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 \\
&= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1 x_2 + y_1 y_2) \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1| \, |z_2| \\
&= \big(|z_1| + |z_2|\big)^2;
\end{aligned}
$$

the triangle inequality follows by taking square roots.

Alternatively, here is a geometric version. Consider the triangle whose vertices are w, z, and $w+z$, and the parallelogram with vertices 0, w, z, and $w+z$. The side of the parallelogram that joins w to $w+z$ is congruent to the side joining 0 to z, and the side of the parallelogram that joins z to $w+z$ is congruent to the side joining 0 to w. So we are just asserting the obvious fact that the length of one side of a triangle is less than the sum of the other two sides. $\square$

2. Euler's formula

The usual exponential function has a Taylor series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

so at least formally, for a real number θ,

$$\begin{aligned} e^{i\theta} &= 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\ &= \Big(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\Big) + i\Big(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\Big) \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

from the Taylor series for the cosine and sine functions.

Later we will make this rigorous and use power series very effectively. This observation leads us to make the following definition.

Definition 0.6. We define $e^{i\theta}$ to be $\cos\theta + i\sin\theta$, for any real number θ.

Suppose that $z = x + iy \neq 0$. Write r instead of $|z|$. Then $(x/r, y/r)$ lies on the unit circle in the Cartesian plane, so $(x/r, y/r) = (\cos\theta, \sin\theta)$ for an appropriate choice of θ, and hence

$$z = r\Big(\frac{x}{r} + i\frac{y}{r}\Big) = r\big(\cos\theta + i\sin\theta\big) = re^{i\theta}.$$

Definition 0.7. The *Cartesian form* of a complex number z is its representation in the form $x+iy$, where x and y are real. The *polar form* of a complex number z is its representation in the form $re^{i\theta}$, where $r \geq 0$ and $\theta \in \mathbb{R}$. The number θ is called the *argument* of z, and is written $\arg(z)$.

Lemma 0.8. *Suppose that $r, s \in \mathbb{R}^+$ and $\theta, \phi \in \mathbb{R}$. Then*

$$r(\cos\theta + i\sin\theta) = s(\cos\phi + i\sin\phi)$$

if and only if $r = s$ and $\theta - \phi = 2k\pi$ for some $k \in \mathbb{Z}$.

This lemma follows immediately from trigonometry. It tells us that the argument of a nonzero complex number z is ambiguous. The next definition is to avoid this ambiguity.

Definition 0.9. The *principal value* of the argument of a nonzero complex number z, written $\mathrm{Arg}(z)$, is the unique number θ such that $z = |z|\, e^{i\theta}$ and $-\pi < \theta \leq \pi$.

We do not define the argument of 0.

3. The Argand Diagram

Suppose that $w = u + iv$. Then to the complex number w we associate the point in the Cartesian plane with Cartesian coordinates (u, v). When we do this, we call the axes the real axis and the imaginary axis. See Figure 0.1.

Geometrically, $|w|$ is the length of the line joining w to O, and $\arg(w)$ is the angle between this line and the positive real axis (taking the anticlockwise direction to be positive). Further, $|w - z|$ is the length of the line joining w and z.

Adding two complex numbers corresponds to vector addition in the plane. Multiplying by r in $\mathbb{R}^+$ dilates by a factor of r, and multiplying by the complex number $e^{i\theta}$ (where $\theta \in \mathbb{R}$) rotates (anticlockwise) through the angle θ.

4. De Moivre's formula

THEOREM 0.10. *If $\theta, \phi \in \mathbb{R}$, then*

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta + \phi) + i\sin(\theta + \phi).$$

PROOF. For all $\theta, \phi \in \mathbb{R}$,

$$\begin{aligned}&(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\\ &= \cos(\theta + \phi) + i\sin(\theta + \phi),\end{aligned}$$

as required. □

COROLLARY 0.11 (de Moivre's formula). *If $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$, then*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{0.1}$$

PROOF. This is obviously true if $n = 0$ or 1. The result may be proved for $n \in \mathbb{Z}^+$ by induction. Suppose that $k \in \mathbb{Z}^+$ and formula (0.1) holds when $n = k$, i.e.,

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta.$$

Then by Theorem 0.10,

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k\\ &= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)\\ &= \cos(\theta + k\theta) + i\sin(\theta + k\theta)\\ &= \cos(k+1)\theta + i\sin(k+1)\theta,\end{aligned}$$

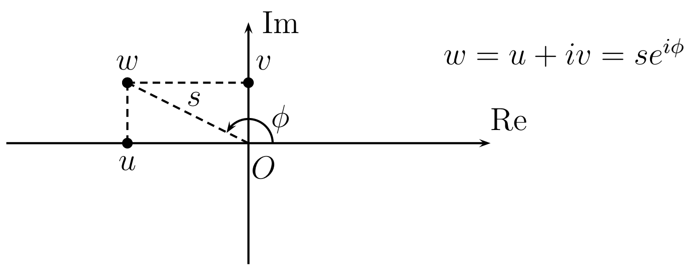


FIGURE 0.1. The complex plane

so the result holds when $n = k + 1$. By induction, the result holds for all $n \in \mathbb{Z}^+$.

To prove the result when $n \in \mathbb{Z}^-$, we use the fact that if $|z| = 1$, then $z\overline{z} = 1$, so $\overline{z} = z^{-1}$. That is,

$$(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta),$$

as required. □

In polar notation, Theorem 0.10 and Corollary 0.11 become

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi} \qquad \text{and} \qquad (e^{i\theta})^n = e^{in\theta}.$$

These are more "obvious" and easier to remember than the trigonometric formulae.

5. Roots of complex numbers

We use complex exponentials to find roots of complex numbers. Fix $w \in \mathbb{C}$ and $n \in \mathbb{Z}^+$, and suppose that $w = z^n$ for some $z \in \mathbb{C}$. Then z is called *an nth root of w*. Write z as $re^{i\theta}$ and w as $se^{i\phi}$. Then

$$se^{i\phi} = w = z^n = r^n e^{in\theta}.$$

From Lemma 0.8, $r = s^{1/n}$ and $n\theta = \phi + 2k\pi$ for some $k \in \mathbb{Z}$. Therefore $z = s^{1/n}e^{i\theta}$, where

$$\theta = \frac{\phi}{n} + \frac{2k\pi}{n}$$

for some $k \in \mathbb{Z}$. If $k = nl + r$, where $l \in \mathbb{Z}$ and $r = 0, 1, 2, \ldots, n-1$, then

$$e^{(\phi/n+2k\pi/n)} = e^{(\phi/n+2r\pi/n+2l\pi)} = e^{(\phi/n+2r\pi/n)},$$

so we get the same value of z by taking $k = nl + r$ as by taking $k = r$. Thus, to get all n possible values of z, it suffices to take the first n values of k, or any n consecutive values, or indeed any n values of k which give the n possible different remainders when divided by n.

The nth roots of any nonzero complex number w are uniformly spaced around the circle with centre 0 and radius $|w|^{1/n}$. For example, Figure 0.2 shows the seventh roots of unity. A symmetry argument shows that the sum of all these numbers is 0.

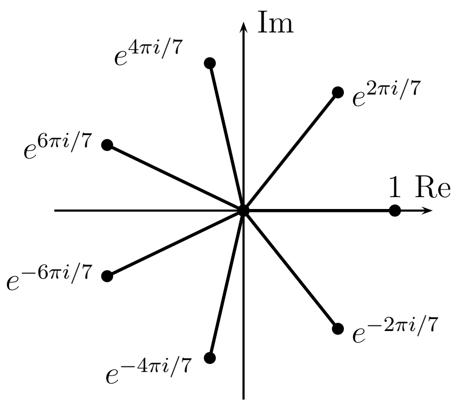


FIGURE 0.2. The seventh roots of unity

6. History†

Leonhard Euler found the formula the bears his name in 1748; he was one of the most prolific mathematicians ever. Some mathematicians did not believe that the geometric representation afforded by the Argand diagram was legitimate. The story of de Moivre's death is very curious. For more information, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/`
and then find Euler, Argand, and de Moivre.

LECTURE 1

Inequalities and Sets of Complex Numbers

In complex analysis, we consider functions whose domains or ranges or both are regions in the complex plane. So to be able to discuss functions, we need to be able to describe regions. Curves and regions in the complex plane are often described by equalities and inequalities involving $|\cdot|$, Arg, Re, Im, $\dots$.

In the first part of this lecture, we review some equalities and inequalities. Then we discuss different types of regions. Finally, we consider some examples.

1. Equalities and inequalities

We begin with a lemma that is related to the cosine rule. It implies that $\operatorname{Re}(w\bar{z})$ is the inner product of the vectors represented by the complex numbers w and z.

LEMMA 1.1. *For all complex numbers w and z,*

$$|w+z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2.$$

PROOF. Observe that

$$\begin{aligned} |w+z|^2 &= (w+z)(\bar{w}+\bar{z}) = w\bar{w} + w\bar{z} + \bar{w}z + z\bar{z} \\ &= |w|^2 + w\bar{z} + (w\bar{z})^{-} + |z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2, \end{aligned}$$

as required. □

The triangle inequality is one of the most useful results about complex numbers. It states:

$$|w+z| \le |w| + |z| \qquad \forall w, z \in \mathbb{C}.$$

Here are a variation on the triangle inequality, sometimes called the circle inequality.

LEMMA 1.2. *For all complex numbers w and z,*

$$\big||w| - |z|\big| \le |w-z|.$$

PROOF. Observe that $w = (w-z) + z$, so $|w| \le |w-z| + |z|$ by the triangle inequality, and hence

$$|w| - |z| \le |w-z|. \tag{1.1}$$

Interchanging the roles of w and z in (1.1),

$$|z| - |w| \le |z-w|.$$

Combining these inequalities and recalling that $|w-z| = |z-w|$, we see that

$$\big||w| - |z|\big| = \max\{|w|-|z|, |z|-|w|\} \le |w-z|,$$

as required.

Alternatively, consider points on circles of radii $|z|$ and $|w|$, and compare the distance between the points with the difference of the radii. □

Recall that if $z = x + iy$, then e^z is defined to be $e^x(\cos y + i \sin y)$. Then $e^w e^z = e^{w+z}$ for all complex numbers w and z. Here is another very useful result.

Lemma 1.3. *If $z \in \mathbb{C}$, then*

$$|e^z| = e^{\mathrm{Re}(z)}.$$

Proof. See Problem Sheet 1. □

Lemma 1.4. *For all real numbers θ,*

$$\left|e^{i\theta} - 1\right| \leq |\theta|.$$

Proof. See Problem Sheet 1. □

2. Properties of sets

Definition 1.5. The *open ball* with centre z_0 and radius ε, written $\mathrm{B}(z_0, \varepsilon)$, is the set $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$.

The *punctured open ball* with centre z_0 and radius ε, written $\mathrm{B}^\circ(z_0, \varepsilon)$, is the set $\{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$.

Sometimes these sets are called discs rather than balls.

Definition 1.6. Suppose that $S \subseteq \mathbb{C}$. For any point z_0 in $\mathbb{C}$, there are three mutually exclusive and exhaustive possibilities:

(1) When the positive real number ε is sufficiently small, $\mathrm{B}(z_0, \varepsilon)$ is a subset of S, that is, $\mathrm{B}(z_0, \varepsilon) \cap S = \mathrm{B}(z_0, \varepsilon)$. In this case, z_0 is an *interior point* of S.
(2) When the positive real number ε is sufficiently small, $\mathrm{B}(z_0, \varepsilon)$ does not meet S, that is, $\mathrm{B}(z_0, \varepsilon) \cap S = \emptyset$. In this case, z_0 is an *exterior point* of S.
(3) No matter how small the positive real number ε is, neither of the above holds, that is, $\emptyset \subset \mathrm{B}(z_0, \varepsilon) \cap S \subset \mathrm{B}(z_0, \varepsilon)$. In this case, z_0 is a *boundary point* of S.

These definitions are illustrated in Figure 1.1. We consider points z_1, z_2 and z_3.

If the radius of the ball centred at z_1 is small enough, then the ball lies inside the set S, and $\mathrm{B}(z_1, \varepsilon) \cap S = \mathrm{B}(z_1, \varepsilon)$. Thus z_1 is an interior point.

If the radius of the ball centred at z_2 is small enough, then the ball lies outside the set S, and $\mathrm{B}(z_2, \varepsilon) \cap S$ is empty. Thus z_2 is an exterior point.

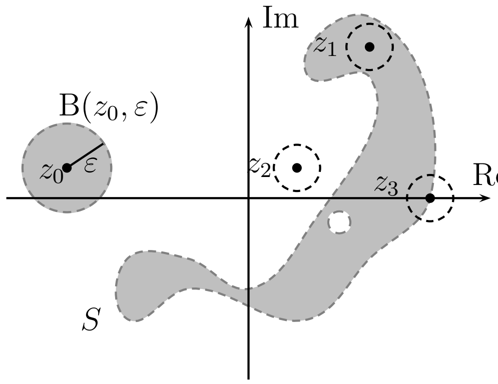


Figure 1.1. The ball with centre z_0 and radius ε, and interior, exterior and boundary points z_1, z_2 and z_3 of the set S

No matter how small the radius of the ball centred at z_3 is, part of the ball lies inside S, and part lies outside S, and $\mathrm{B}(z_3,\varepsilon)\cap S$ is neither empty nor all of $\mathrm{B}(z_3,\varepsilon)$. Thus z_3 is a boundary point.

DEFINITION 1.7. Suppose that $S \subseteq \mathbb{C}$.

(1) The set S is *open* if all its points are interior points.
(2) The set S is *closed* if it contains all of its boundary points, or equivalently, if its complement $\mathbb{C} \setminus S$ is open.
(3) The *closure* of S, written $\bar{S}$, is the set consisting of all the points of S together with all its boundary points.
(4) The set S is *bounded* if $S \subseteq \mathrm{B}(0, R)$ for some positive real number R.
(5) The set S is *compact* if it is both closed and bounded.
(6) The set S is a *region* if it is an open set together with none, some, or all of its boundary points.

For example, the dashed boundary lines of the set S in Figure 1.1 indicate that it does not contain any boundary points. Consequently, this set is open.

Note that open and closed are not exclusive nor exhaustive. There are sets that are open and closed, such as the whole plane, and sets that are neither open nor closed, such as $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) > 0\}$. In complex analysis, we focus on open sets. We often write Ω for an open set.

DEFINITION 1.8. A *polygonal path* is a finite sequence of finite line segments, where the end point of one line segment is the initial point of the next one. A *simple closed polygonal path* is a polygonal path that does not cross itself, but the final point of the last segment is the initial point of the first segment. The complement of a simple closed polygonal path is made up of two pieces: one, the *interior of the path*, is bounded, and the other, the *exterior*, is not.

DEFINITION 1.9. Suppose that $\Omega \subseteq \mathbb{C}$ and that Ω is open.

(1) The set Ω is *connected* if any two points of Ω can be joined by a polygonal path lying inside Ω.
(2) The set Ω is *simply connected* if the interior of every simple closed polygonal path in Ω lies in Ω, that is, if "Ω has no holes".
(3) The set Ω is *a domain* if it is connected as well as open.

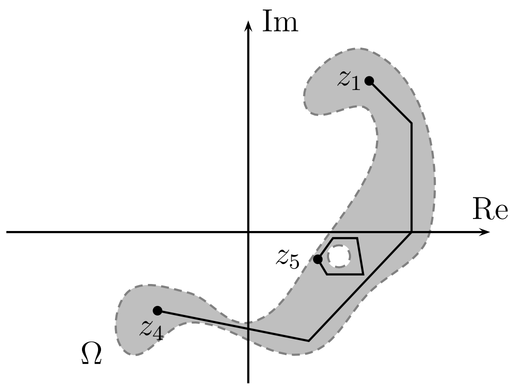


FIGURE 1.2. A connected, but not simply connected, set

Part (2) of this definition is not standard, and only agrees with the standard definition for open sets; see the exercise sheet for more on this. The set Ω in Figure 1.2 is connected, because any two points in Ω (such as z_1 and z_4) can be joined by a polygonal path. However, Ω is not simply connected, because part of the interior of the closed path shown going through z_5 is not in Ω.

3. Describing sets in the complex plane

EXERCISE 1.10. Suppose that $a, b, c, d \in \mathbb{C}$. Show that the set

$$\{z \in \mathbb{C} : |az + b| = |cz + d|\}$$

may be empty, a point, a line, a circle, or the whole complex plane, and all these possibilities occur for suitable values of the parameters a, b, c, d.

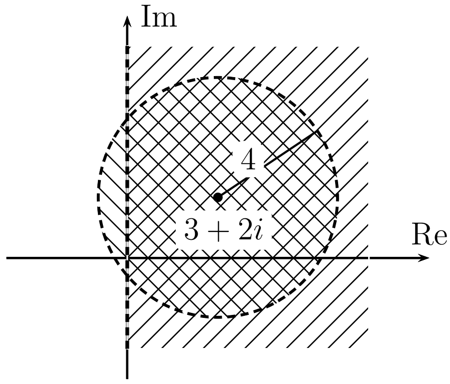
ANSWER.

EXERCISE 1.11. Sketch the set $\{z \in \mathbb{C} : |z - 3 - 2i| < 4,\ \mathrm{Re}(z) > 0\}$ in the complex plane. Is it open, closed, bounded, compact, connected, simply connected, a region, or a domain?

ANSWER.

$\triangle$

EXERCISE 1.12. Sketch the set $\{z \in \mathbb{C} : 4 \leq |z - 3 - 2i| \leq 5\}$ in the complex plane. Is it open, closed, bounded, compact, connected, simply connected, a region or a domain?



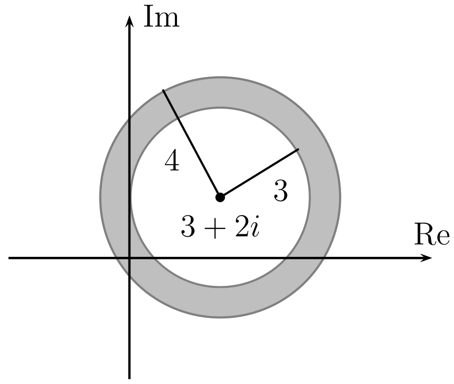


FIGURE 1.3. Two regions defined by inequalities

ANSWER.

Here is a more complicated example, related to conic sections.

EXERCISE 1.13. Sketch the set $\{z \in \mathbb{C} : |z+i| + |z-i| = 4\}$ in $\mathbb{C}$. Is it open or bounded? Describe the set $\{z \in \mathbb{C} : |z+i| + |z-i| < 4\}$. Is it open, closed, bounded, compact, connected, simply connected, a region or a domain?

ANSWER.

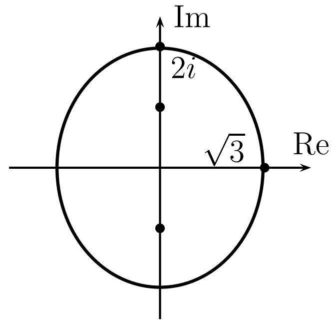


FIGURE 1.4. An ellipse

LECTURE 2

Functions of a complex variable

In this lecture, we introduce functions of a complex variable, and recall concepts such as domain and range. We examine some examples and consider the problem of estimating the size of a complex function.

1. Functions

In Mathematics, we often think of functions as machines: you give the machine a number, x say, press a button, and out comes $f(x)$. We sometimes write $x \mapsto f(x)$ to indicate that x is the input and $f(x)$ is the output.

- The *domain* of a function f, written $\mathrm{Domain}(f)$, is the set of all the numbers you are allowed to put in. Sometimes this is restricted in some way. If there is no explicit restriction, you should consider the *natural domain*, that is, the largest domain possible.
- A *codomain* is a set of numbers that includes all the numbers that you can get out, and perhaps more.
- The *range* or *image* of a function f, written $\mathrm{Range}(f)$, is the set of the numbers that you can get out, and no others.
- The *image* of a subset S of the domain of a function f, sometimes written $f(S)$, is the set of all possible values $f(s)$ as s varies over S.
- The *preimage* of a subset T of the codomain of a function f, sometimes written $f^{-1}(T)$, is the set of all x in $\mathrm{Domain}(f)$ such that $f(x) \in T$.

Definition 2.1. A *complex function* is one whose domain, or whose range, or both, is a subset of the complex plane $\mathbb{C}$ that is not a subset of the real line $\mathbb{R}$. To emphasize that the domain is complex, not real, the expression *function of a complex variable* may be used. To emphasize that the range is complex, not real, the expression *complex-valued function* may be used.

2. Examples of functions

Examples of functions of a complex variable include the real part function Re, the imaginary part function Im, the modulus function $z \mapsto |z|$, and the principal value of the argument Arg; these are all real-valued. Complex conjugation $z \mapsto \bar{z}$ is an example of a complex-valued function of a complex variable.

In this course, we are going to learn about a number of useful complex functions. Shortly we will define complex polynomials and then rational functions. In future lectures, we will define $\log z$, $\sin z$, and $\cosh z$ for a complex number z, and there are many other functions in the menagerie of complex functions.

Exercise 2.2. Suppose that $f(z) = 1/z$ for all $z \in \mathbb{C} \setminus \{0\}$, and that $g(z) = z$ for all $z \in \mathbb{C}$. Show that $f \circ f(z) = z$ for all $z \in \mathbb{C} \setminus \{0\}$. Is $f \circ f = g$?

ANSWER.

DEFINITION 2.3. A *complex polynomial* is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_d z^d + \cdots + a_1 z + a_0,$$

where $a_d, \ldots, a_1, a_0 \in \mathbb{C}$. If $a_d \neq 0$, we say that p is of *degree* d. A *rational function* is a quotient of polynomials.

Sums, differences, products and compositions of polynomials are polynomials.

THEOREM 2.4 (The fundamental theorem of algebra). *Every nonconstant complex polynomial p of degree d factorizes uniquely: there exist $\alpha_1, \alpha_2, \ldots, \alpha_d$ and c in $\mathbb{C}$ such that*

$$p(z) = c \prod_{j=1}^{d} (z - \alpha_j) \qquad \forall z \in \mathbb{C}.$$

Equivalently, every nonconstant complex polynomial has at least one root.

In the factorisation above, the roots α_j may occur more than once. Thus we could also write

$$p(z) = c \prod_{j=1}^{e} (z - \alpha_j)^{m_j},$$

where the α_j are all distinct, and the *multiplicities* m_j add to give the degree of p.

THEOREM 2.5 (Polynomial division and partial fractions). *Suppose that p and q are polynomials of degrees m and n. Then the rational function p/q may be written as a sum*

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)},$$

where r and s are polynomials, and the degree of r is strictly less than the degree of q. Further, if

$$q(z) = c \prod_{j=1}^{e} (z - \beta_j)^{m_j},$$

then we may decompose the term r/q into partial fractions:

$$\frac{r(z)}{q(z)} = \sum_{j=1}^{e} \sum_{k=1}^{m_j} \frac{a_{jk}}{(z - \beta_j)^k}.$$

At this stage, we do not prove these results, which should be familiar, though perhaps not in this generality; we will give proofs later.

The natural domain of any complex polynomial is $\mathbb{C}$. Sometimes we cannot determine the range of a real polynomial exactly, because we cannot find maxima or minima exactly. However, for complex polynomials, things are easier.

EXERCISE 2.6. Suppose that p is a nonconstant complex polynomial. Show that the range of p is $\mathbb{C}$.

ANSWER.

3. Real and imaginary parts

To a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$, we associate two *real-valued* functions u and v of *two real variables*:

$$f(x + iy) = u(x, y) + iv(x, y).$$

Then $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$. It is very useful and very important to be able to view a complex-valued function of a complex variable in this way.

EXERCISE 2.7. Suppose that $f(z) = z$ and that $g(z) = z^2$. Find the real and imaginary parts of f and g.

ANSWER.

EXERCISE 2.8. Suppose that $f(z) = z^3 + \overline{z} - 2$. Write the real and imaginary parts of this function as functions u and v of (x, y), where $z = x + iy$.

ANSWER.

EXERCISE 2.9. Suppose that $f(z) = 1/z$. Write the real and imaginary parts of this function as functions of x and y, where $z = x + iy$.

ANSWER.

EXERCISE 2.10. Write e^z in the form $u(x, y) + iv(x, y)$, where $z = x + iy$.

ANSWER.

Sometimes we view the complex number z in polar coordinates, that is, we write $z = re^{i\theta}$. In this case, we consider

$$f(z) = u(r, \theta) + iv(r, \theta).$$

EXERCISE 2.11. Write e^z in the form $u(r,\theta) + iv(r,\theta)$, where $z = re^{i\theta}$.

ANSWER.

4. The function $z \mapsto 1/z$

It is obvious that if $w = 1/z$, then $z = 1/w$, and the function $z \mapsto 1/z$ is one-to-one (injective). Further, the domain and the range of the function are both equal to $\mathbb{C} \setminus \{0\}$.

EXERCISE 2.12. Suppose that z varies on the line $x = 1$, and let $w = 1/z$. Show that w varies on the circle $|w - \frac{1}{2}| = \frac{1}{2}$.

ANSWER.

We can reverse the argument and show that every point on the circle except 0 arises in this way. Thus the image of the line is the circle with the point 0 removed.

5. Fractional linear transformations

The fractional linear transformations form an important family of complex functions. These are the functions of the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. We will study these functions in more detail later, but at the moment we just point out that if f is a fractional linear transformation and z varies on a line, then $f(z)$ varies on a line or on a circle. The same holds if z varies along a circle. Note that when $z \to -d/c$, then $cz + d \to 0$ and $f(z) \to \infty$ (we will define limits formally later). We can tell whether $f(z)$ varies on a line or on a circle as follows: if the points where z varies include $-d/c$, then the points where $f(z)$ varies will include ∞, and this means that $f(z)$ must vary on a line. Conversely, if the points where z varies do not include $-d/c$, then $f(z)$ will stay bounded, and this means that $f(z)$ must vary on a circle. Once we know whether $f(z)$ varies on a line or on a circle, we may find the equation of the line or the circle quite easily by finding a few values of $f(z)$.

EXAMPLE 2.13. Let $f(z) = 1/z$. As z varies on the line $x = 1$, its image $f(z)$ varies on a circle, because z stays away from 0 and so $1/z$ stays away from ∞. This circle passes through the points 1 and 0 (since $f(z) \to 0$ as $z \to \infty$), and is symmetric about the real axis, since $1/(1 - it) = (1/(1 + it))^{\bar{}}$. This must be the circle that we found above.

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6. Estimating the size of the values of a function

We will need to use what we know about inequalities to estimate how large the values of a complex function are.

EXERCISE 2.14. Suppose that $f(z) = \dfrac{1}{z^4 - 1}$ for all $z \in \mathbb{C} \setminus \{\pm 1, \pm i\}$. Show that

$$|f(z)| \leq \frac{1}{15}$$

if $|z| \geq 2$ (that is, if z lies on or outside the circle with centre 0 and radius 2).

ANSWER.

EXERCISE 2.15. Suppose that $p(z) = 10z^4 - 3z^3 + z - 10$. Show that when $|z|$ is large enough, $|p(z)| \leq 11|z|^4$.

ANSWER.

LECTURE 3

Sketching complex functions

We often use graphical methods to gain useful intuition about complex functions, and we spend some time investigating these. It is hard to represent complex functions, because there are up to four variables involved. Just as we often write $y = f(x)$ for a real function, it is common to consider a function in the form $w = f(z)$, and to use real variables x and y to describe the domain and u and v to describe the range. Typically, we draw "elementary" curves in the z plane, such as lines parallel to the axes, or concentric circles around and rays exiting from the origin, and then examine their images in the w plane, or we draw similar elementary curves in the w plane and then examine their preimages.

1. Linear and affine mappings

Suppose that $a \neq 0$, and consider the bijective linear map $z \mapsto az$. We write $a = c + id$ and $z = x + iy$; then

$$az = (c + id)(x + iy) = (cx - dy) + i(cy + dx).$$

In Cartesian coordinates, the map may be written

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $r = \sqrt{c^2 + d^2}$, while $\cos\theta = c/r$ and $\sin\theta = d/r$. Hence multiplying by the matrix

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

is the same as rotating through the angle θ and then dilating by the real number r. This corresponds to the representation of a in the form $re^{i\theta}$, where $r = |a|$ and $\theta = \operatorname{Arg}(a)$.

An affine map of the complex plane is a map of the form $z \mapsto az + b$; such mappings are also bijective (as long as $a \neq 0$). We may also represent this as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. We write $a = c + id$ and $b = e + if$. Then in Cartesian coordinates, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}.$$

It is an exercise in algebra to see that the image of a line under an affine mapping is a line, and the image of a circle under an affine mapping is a circle.

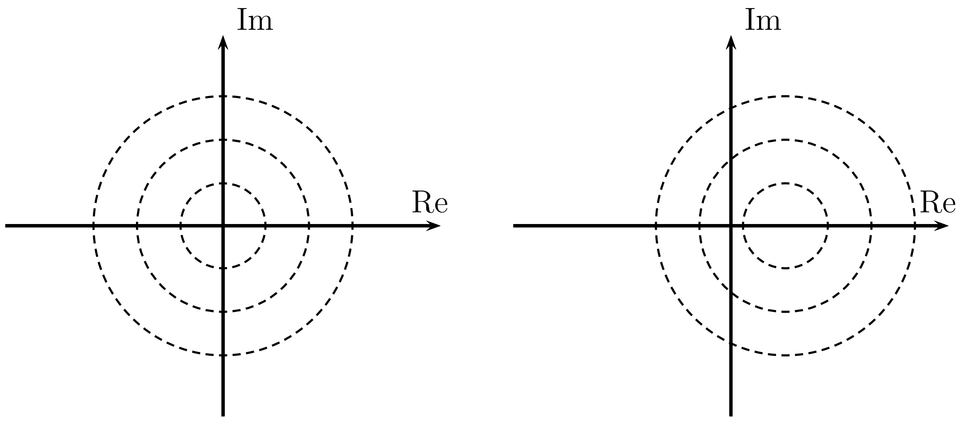

FIGURE 3.1. The translation $z \mapsto z+1$

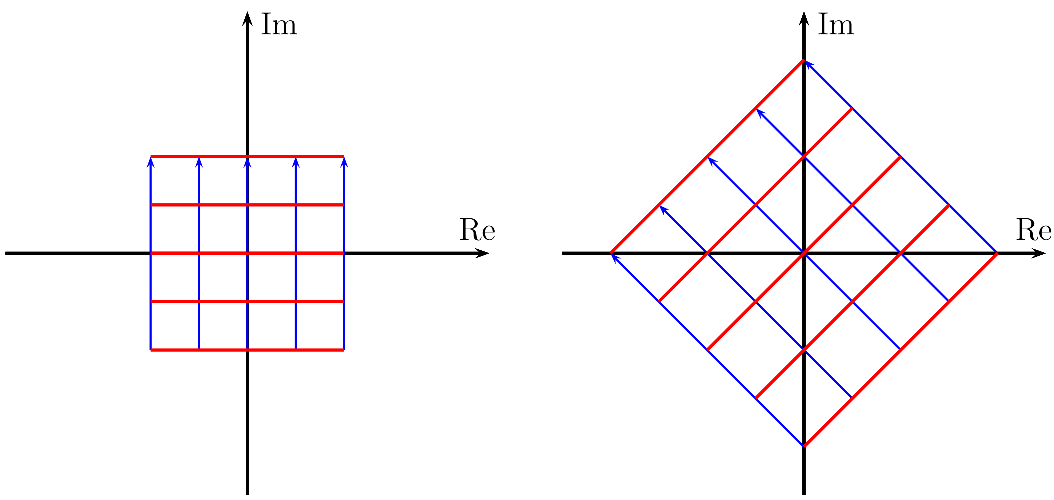

FIGURE 3.2. The multiplication $z \mapsto (1+i)z$

The inverse of an affine mapping is an affine mapping. It follows that the preimages of lines are lines and preimages of circles are circles; the preimage of a grid parallel to the axes is a rectangular grid, but not necessarily parallel to the axes.

In Figures 3.1 and 3.2, we illustrate the images of lines parallel to the axes and circles around the origin under affine mappings.

This is a good way to show how the function behaves, although a lot of space is used and care is needed to choose the points that are moved in a way that is not ambiguous. Indeed, it might be better to draw a very asymmetrical figure in the z plane and then its image in the w plane.

Sometimes we just draw the right hand figure of the two drawn above, labelling the curves with the corresponding curve in the domain of the function. In Figure 3.1, these are the circles $r = 1$, $r = 2$, and $r = 3$; in Figure 3.2, they are the horizontal lines $x = 0$, $x = \pm 1$, $x = \pm 2$, and the vertical lines $y = 0$, $y = \pm 1$, $y = \pm 2$.

On the other hand, we may look for curves in the xy plane whose images in the uv plane are the lines $u = c$ and $v = d$. So we are finding the level curves of the real and imaginary parts of the function. People who are used to map reading can build a picture in their mind of terrain, just knowing the contours that represent different heights. They imagine a surface above the page at the height indicated by the contour, and then fill in the gaps.

2. Quadratic functions

Now we consider the function $z \mapsto z^2$. Notice that this function is two-to-one in $\mathrm{B}^\circ(0,\infty)$.

On the one hand, we may represent the images in the uv plane of the curves in the xy plane given by $x = a$ and $y = b$, or by $r = a$ and $\theta = b$. For instance, if $x = a$ and $y = t$, where a is fixed and t varies, then

$$w = z^2 = (a + it)^2 = a^2 - t^2 + 2iat.$$

That is, $u = a^2 - t^2$ and $v = 2at$. We eliminate t to show that

$$u = a^2 - \frac{v^2}{4a^2}\,.$$

Alternatively, if $y = b$ and $x = t$, where b is fixed and t varies, then

$$w = z^2 = (t + ib)^2 = t^2 - b^2 + 2ibt.$$

That is, $u = t^2 - b^2$ and $v = 2bt$. We eliminate t to show that

$$u = \frac{v^2}{4b^2} - b^2.$$

See, for example, Figure 3.3.

EXERCISE† 3.1. Find the focus and the directrix of the parabola $u = v^2/4b^2 - b^2$ in the uv plane.

On the other hand, we may look for the values of x and y so that $\mathrm{Re}(z^2)$ or $\mathrm{Im}(z^2)$ takes a fixed value. For instance, if $\mathrm{Re}(z^2) = a$, then

$$x^2 - y^2 = a,$$

and this is a hyperbola opening to the left and right, or up and down, depending on the sign of a. Similarly, if $\mathrm{Im}(z^2) = b$, then

$$2xy = b,$$

and this is a right hyperbola in the first and third quadrants, or in the second and fourth quadrants, depending on the sign of b. See, for example, Figure 3.3.

EXERCISE 3.2. How would you “sketch the graph” of $w = (z - 1)^2 - 1$?

3. The function $w = 1/z$

As we saw in the last lecture, the function $w = 1/z$ is one-to-one. Further, it sends lines through the origin to lines through the origin. Actually, this is easiest to see using polar coordinates: as r varies in $\mathbb{R}^+$, the point $z = re^{i\theta}$ varies along a ray from the origin. Now $w = (1/r)e^{-i\theta}$, and $1/r$ also varies in $\mathbb{R}^+$ and this point too varies along a ray. However the new ray is the reflection of the old ray in the real axis. Since moreover diametrically opposed rays are mapped to diametrically opposed rays, lines through the origin do indeed go to lines through the origin.

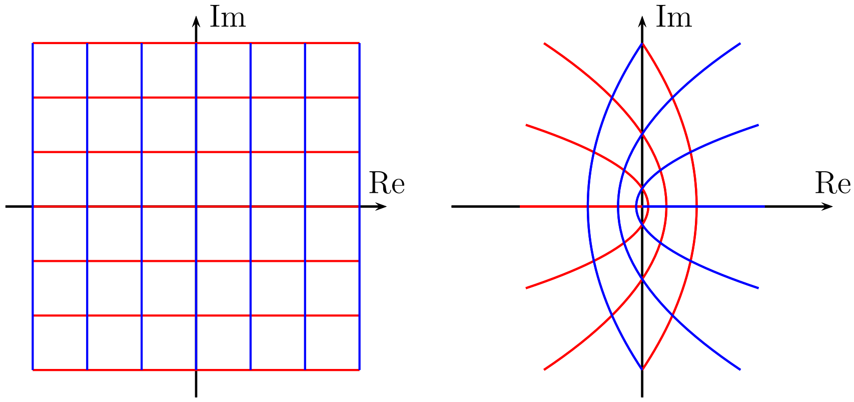

FIGURE 3.3. Images of lines $x = c$ and $y = d$ for $w = z^2$

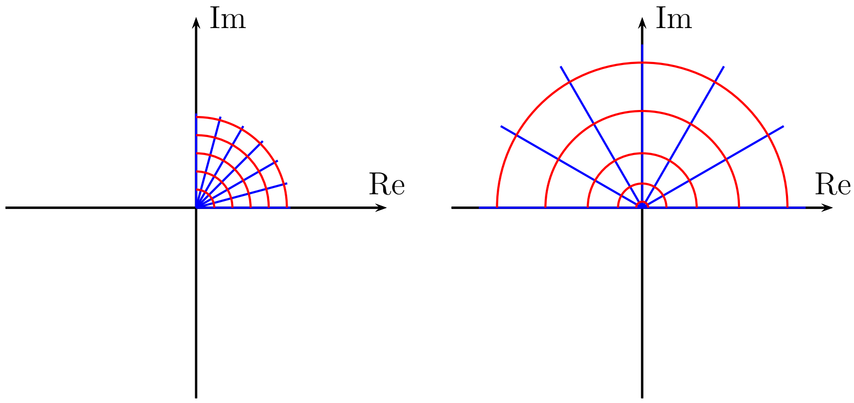

FIGURE 3.4. Images of curves $r = c$ and $\theta = d$ for $w = z^2$

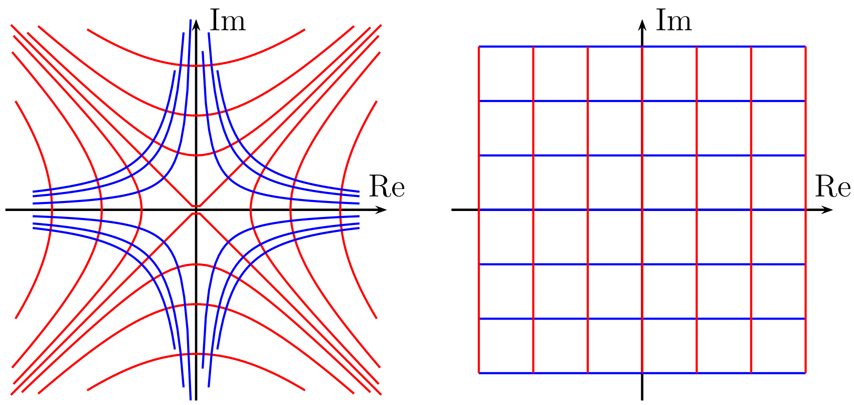

FIGURE 3.5. The level curves for $\mathrm{Re}(z^2)$ and $\mathrm{Im}(z^2)$

Similarly, as θ varies, the point $re^{i\theta}$ moves around a circle centred at the origin. Now $w = (1/r)e^{-i\theta}$, and this point moves around the same circle, but in the opposite direction.

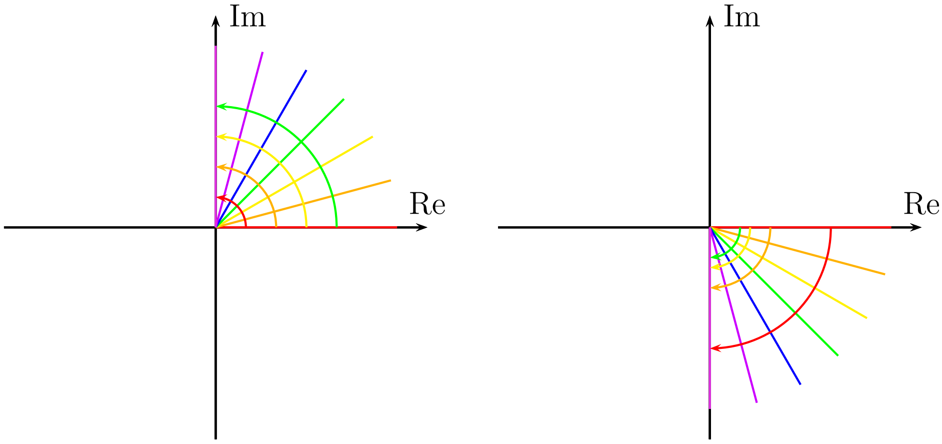

FIGURE 3.6. Images of curves $r = c$ and $\theta = d$ for $w = 1/z$

We represent this graphically in Figure 3.6.

For completeness, we now describe the images of lines and circles under the map $w = 1/z$ in more detail.

LEMMA 3.3. *Consider the mapping $w = 1/z$.*

(1) The image of a line through 0 (not including 0) is a line through 0 (not including 0).
(2) The image of a line that does not pass through 0 is a circle through 0, with 0 removed. If p is the closest point on the line to 0, then the line segment between 0 and $1/p$ is a diameter of the circle.
(3) The image of a circle that passes through 0 is a line. If q is the furthest point on the circle from 0, then the closest point on the line to 0 is $1/q$.
(4) The image of a circle that does not pass through 0 is a circle. If p and q are the points on the circle closest to and furthest from 0, then the points on the image circle closest to and furthest from 0 are $1/q$ and $1/p$.

PROOF. See the exercise sheet. □

EXERCISE 3.4. Suppose that z varies on the line $ax + by = c$, where $a, b, c \in \mathbb{R}$, and let $w = 1/z$. Show that w varies on a line when $c = 0$ and on a circle otherwise.

ANSWER.

4. The exponential function

The exponential function $w = e^z$ is ∞-to-1; that is, infinitely many different points in the xy plane are sent to the same point in the uv plane. See Figure 3.7 for a graphical representation.

5. More on graphical representations of complex functions

There are many good web-sites that explore different ways to represent complex functions.

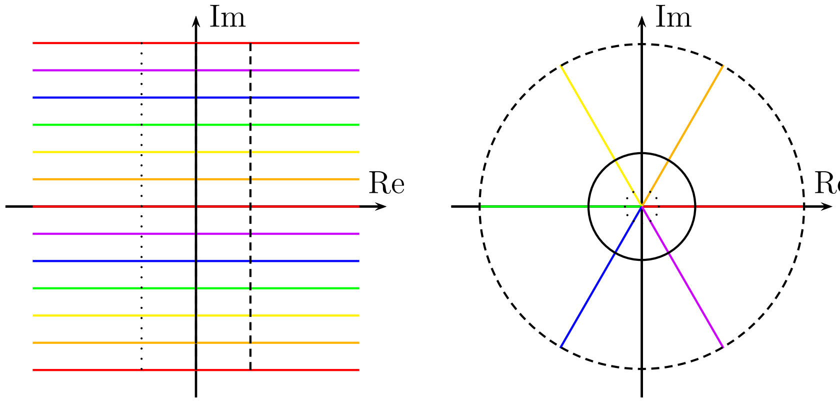

FIGURE 3.7. Images of curves $x = c$ and $y = d$ for $w = e^z$

LECTURE 4

Fractional linear transformations

In this lecture, we study a particular type of function: fractional linear transformations. These are easy to handle because we can use linear algebra to simplify computations.

1. Domains and ranges of fractional linear transformations

Let M be a 2×2 complex matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We write T_M for the associated *fractional linear transformation*:

$$T_M(z) = \frac{az+b}{cz+d}. \tag{4.1}$$

First, we need to assume, and will always do so, that $(c,d) \neq (0,0)$, otherwise the denominator is always 0. If $\det M = 0$, then $(a,b) = \lambda(c,d)$ for some $\lambda \in \mathbb{C}$, whence $az+b = \lambda(cz+d)$ and so $f(z) = \lambda$ for all $z \in \mathrm{Domain}(f)$. Thus fractional linear transformations associated to matrices with determinant 0 are essentially just constant functions, and we do not consider them further.

Observe now that $T_{\lambda M} = T_M$ if $\lambda \neq 0$, and recall that $\det(\lambda M) = \lambda^2 \det(M)$. This means that when $\det M \neq 0$, if we set $M' = (\det M)^{-1/2} M$, then $\det M' = 1$ and $T_M = T_{M'}$. Thus we may and shall henceforth restrict our attention to fractional linear transformations associated to matrices with determinant 1.

First we find the domain of T_M. If $c = 0$ and $d \neq 0$, then the denominator is never 0, so $T_M(z)$ is defined for all a, that is, $\mathrm{Domain}(T_M) = \mathbb{C}$; otherwise, the denominator is nonzero when $z \neq -d/c$, whence $\mathrm{Domain}(T_M) = \mathbb{C} \setminus \{-d/c\}$.

Now suppose that $w \in \mathrm{Range}(T_M)$. Then

$$w = \frac{az+b}{cz+d} \quad \Longrightarrow \quad wcz + wd = az + b \quad \Longrightarrow \quad z(wc-a) = b - wd,$$

and as long as $wc \neq a$, this in turn implies that

$$z = \frac{b-wd}{wc-a} = \frac{(-d)w+b}{cw+(-a)} = \frac{dw-b}{(-c)w+a} = T_{M'}(w)\,, \tag{4.2}$$

where

$$M' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then M' is exactly M^{-1}, since $\det M = 1$. We conclude that

$$\mathrm{Range}(T_M) = \begin{cases} \mathbb{C} \setminus \{a/c\} & \text{if } c \neq 0 \\ \mathbb{C} & \text{if } c = 0. \end{cases}$$

Our discussion of the domain and range had several cases; we can simplify the statements by enlarging the set of complex numbers by adding ∞. Indeed, we define

$$f(-d/c) = \lim_{z\to -d/c} \frac{az+b}{cz+d} = \infty \quad \text{and} \quad f(\infty) = \lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c}$$

(we will define limits formally in the next lecture) and now we can just write

$$T_M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}.$$

We often call $\mathbb{C} \cup \{\infty\}$ the Riemann sphere, and write it S: we imagine a unit sphere in three dimensions with centre at 0. We can define a function $\sigma : \mathbb{C} \to S$ geometrically by joining a point p in the plane to the north pole n of the sphere by a straight line. The line will cut the sphere at n and at one other point, which we call $\sigma(p)$. Then we may think of n as being $\sigma(\infty)$. The function σ is called *stereographic projection.*

2. Matrix products and composition of mappings

Exercise 4.1. Suppose that $M, N \in M_{2,2}(\mathbb{C})$. Show that $T_M T_N = T_{MN}$ (on the Riemann sphere). Deduce that the transformation T_M is bijective and that $(T_M)^{-1} = T_{M^{-1}}$.

Answer.

3. Factorisations of fractional linear transformations

Matrices may be factorised, and hence fractional linear transformations may be factorised too.

Theorem 4.2. *Every 2×2 complex matrix with determinant 1 may be written as a product of at most three matrices of the following special types:*

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. Consider the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We consider three cases, according to whether any of a or c is 0. since the determinant is 1, both cannot be 0.

If $c = 0$, the matrix itself is of the desired form.

Now suppose that $a = 0$. Then

$$\begin{aligned}
\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ 0 & b \end{pmatrix},
\end{aligned}$$

which is of the desired form.

Finally, if neither a nor c is 0, then we take $x = a/c$, and write

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - cx & b - dx \\ c & d \end{pmatrix};
\end{aligned}$$

since $a - cx = 0$, combining with the previous case shows that the factorisation holds in this case too. □

This factorisation simplifies a number of arguments; the next result is an example.

Theorem 4.3. *Let T_M be a fractional linear transformation. Then the image of a line under T_M is a line or a circle, and the image of a circle under T_M is also a line or a circle.*

Proof. If M is of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, then T_M is an affine transformation, and the theorem holds in this case. Otherwise, T_M is composed of some affine transformations and the inversion map $z \mapsto -1/z$; it therefore suffices to treat the inversion map.

We may write the equation of the circle with centre c and radius r in the form $|z - c|^2 = r^2$. Note that this is the same as $|z|^2 - 2\operatorname{Re}(z\bar{c}) + |c|^2 - |r|^2 = 0$.

If we set $w = -1/z$, then we find that

$$\left| \frac{-1}{w} - c \right| = r^2,$$

whence

$$|wc + 1| = r^2 |w|^2,$$

and

$$\left(|c|^2 - r^2\right) |w|^2 + 2\operatorname{Re}(wc) + 1 = 0,$$

which is the equation of a circle, unless $|c|^2 = r^2$, in which case we get

$$2\,\mathrm{Re}\,(wc) = -1,$$

which is the equation of a straight line.

The argument to show that the inversion mapping sends straight lines to straight lines or circles is similar, and we omit it; the starting point is that the equation of a straight line may be written as $|z-p| = |z-q|$. $\square$

EXERCISE 4.4. Show that the fractional linear transformation $T : z \mapsto \dfrac{z-i}{z+i}$ sends the upper half plane onto the unit disc $\{w \in \mathbb{C} : |w| < 1\}$.

Find the image under T of the sector $\{z \in \mathbb{C} : \varphi < \mathrm{Arg}(z) < \pi - \varphi\}$.

ANSWER.

4. Special classes of fractional linear transformations

EXERCISE 4.5. Suppose that $M \in M_{2,2}(\mathbb{R})$ and $\det(M) = 1$. Show that if $\mathrm{Im}(z) > 0$, then $\mathrm{Im}(T_M z) > 0$. Deduce from this and Exercise 4.1 that T_M maps the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ onto itself bijectively.

ANSWER.

LECTURE 5

Limits and continuity

In this lecture, we outline the key ideas and facts about limits and continuity, as a preliminary to defining differentiability.

1. Limits

We define limits for complex functions much as for real functions.

Recall that, given a set S, we define its *closure* $\bar{S}$ or S^{-} to be the set consisting of all points of S together with all boundary points.

DEFINITION 5.1. Suppose that f is a complex function, $\ell \in \mathbb{C}$, and z_0 is in $\mathrm{Domain}(f)^{-}$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = \ell,$$

if, for every $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \varepsilon$ provided that z is in $\mathrm{Domain}(f)$ and $0 < |z - z_0| < \delta$.

Suppose also S is a subset of $\mathrm{Domain}(f)$ and that $z_0 \in \bar{S}$. We say that $f(z)$ tends to ℓ as z tends to z_0 in S, or that ℓ is the limit of $f(z)$ as z tends to z_0 in S, and we write $f(z) \to \ell$ as $z \to z_0$ in S, or

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \ell,$$

if, for every $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \varepsilon$ provided that $z \in S$ and $0 < |z - z_0| < \delta$.

Informally, $f(z)$ tends to ℓ if we can make $f(z)$ arbitrarily close to ℓ by taking z close to, but not equal to, z_0.

Most of what follows about limits of the form $\lim_{z \to z_0} f(z)$ also applies to restricted limits, that is, limits of the form $\lim_{\substack{z \to z_0 \\ z \in S}} f(z)$.

We may rewrite the conditions $0 < |z - z_0| < \delta$ and $|f(z) - \ell| < \varepsilon$ as $z \in \mathrm{B}^{\circ}(z_0, \delta)$ and $f(z) \in \mathrm{B}(\ell, \varepsilon)$. We define limits involving infinity in a similar way by defining balls centred at infinity, and extending our previous definition slightly.

DEFINITION 5.2. Suppose that $\varepsilon > 0$. We define both $\mathrm{B}(\infty, \varepsilon)$ and $\mathrm{B}^{\circ}(\infty, \varepsilon)$ to be the set $\{z \in \mathbb{C} : |z| > 1/\varepsilon\}$.

DEFINITION 5.3. Suppose that f is a complex function, that $\ell \in \mathbb{C} \cup \{\infty\}$, and that either $z_0 \in \mathrm{Domain}(f)^{-}$ or $\mathrm{Domain}(f)$ is unbounded and $z_0 = \infty$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = \ell,$$

if for all $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $f(z) \in \mathrm{B}(\ell, \varepsilon)$ provided that $z \in \mathrm{Domain}(f) \cap \mathrm{B}^\circ(z_0, \delta)$.

With this definition, the following lemma holds; we omit the proof.

LEMMA 5.4 (Standard limits). *Suppose that $\alpha, c \in \mathbb{C}$. Then*

$$\lim_{z\to\alpha} c = c \qquad \lim_{z\to\infty} c = c$$

$$\lim_{z\to\alpha} z - c = \alpha - c \qquad \lim_{z\to\infty} z - \alpha = \infty$$

$$\lim_{z\to\alpha} \frac{1}{z-\alpha} = \infty \qquad \lim_{z\to\infty} \frac{1}{z-\alpha} = 0.$$

As the statement of these limits indicates, we are sometimes allowed to consider ∞ as a limit in this course.

The next results follows from the definition; we omit the proofs, which generalise arguments from calculus.

LEMMA 5.5. *Suppose that f is a complex function, that $T \subseteq S \subseteq \mathrm{Domain}(f)$, and that $z_0 \in \bar{T}$. If $\lim_{\substack{z\to z_0\\ z\in S}} f(z)$ exists, then so does $\lim_{\substack{z\to z_0\\ z\in T}} f(z)$, and these limits are equal.*

LEMMA 5.6. *Suppose that f is a complex function, and that $z_0 \in \mathrm{Domain}(f)^{-}$. If $\lim_{z\to z_0} f(z)$ exists, then it is unique.*

We may break complicated limits up into sums, products, and so on, of simpler limits.

THEOREM 5.7. *Suppose that f and g are complex functions and that $c \in \mathbb{C}$. Then*

$$\begin{aligned} \lim_{z\to z_0} cf(z) &= c \lim_{z\to z_0} f(z) \\ \lim_{z\to z_0} f(z) + g(z) &= \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z) \\ \lim_{z\to z_0} f(z)\, g(z) &= \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z) \\ \lim_{z\to z_0} \frac{f(z)}{g(z)} &= \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}, \end{aligned}$$

in the sense that if the right hand side exists, then so does the left hand side, and they are equal. In particular, for the quotient, we require that $\lim_{z\to z_0} g(z) \neq 0$.

We also omit the proof of this theorem, which is very similar to that of the corresponding theorem for limits of functions of a real variable.

Limits respect complex conjugation and related operations.

THEOREM 5.8. *Suppose that f is a complex function and that either $\mathrm{Domain}(f)$ is unbounded and $z_0 = \infty$ or $z_0 \in \mathrm{Domain}(f)^{-}$. Then*

$$\begin{aligned} \lim_{z\to z_0} \overline{f(z)} &= \overline{\lim_{z\to z_0} f(z)} \\ \lim_{z\to z_0} \mathrm{Re}(f(z)) &= \mathrm{Re} \lim_{z\to z_0} f(z) \\ \lim_{z\to z_0} \mathrm{Im}(f(z)) &= \mathrm{Im} \lim_{z\to z_0} f(z) \\ \lim_{z\to z_0} f(z) &= \lim_{z\to z_0} \mathrm{Re}(f(z)) + i \lim_{z\to z_0} \mathrm{Im}(f(z)), \end{aligned}$$

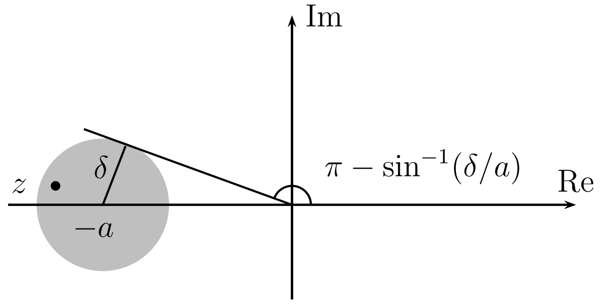

FIGURE 5.1. The trigonometry of the argument function

in the sense that if the right hand side exists, then so does the left hand side, and they are equal. In particular, $f(z)$ tends to ℓ as z tends to z_0 if and only if $\mathrm{Re}(f(z))$ *tends to* $\mathrm{Re}(\ell)$ *and* $\mathrm{Im}(f(z))$ *tends to* $\mathrm{Im}(\ell)$ *as z tends to z_0.*

PROOF. The proof of part (1) uses that fact that

$$\left|\overline{f(z)} - \bar{\ell}\right| = |f(z) - \ell| \,.$$

The rest follows from the first part and the first two parts of Theorem 5.7. □

2. Examples of limits

EXERCISE 5.9. Show from first principles that $\lim_{z\to z_0} z = z_0$.

ANSWER.

EXERCISE 5.10. Suppose that $f(z) = z^2 - \bar{z} + i$. Does $\lim_{z\to 2i} f(z)$ exist: if so find it, and if not, explain why not.

ANSWER.

We now consider an important example.

EXERCISE 5.11. Suppose that $a > 0$. Show that

$$\lim_{\substack{z\to -a\\ \mathrm{Im}(z)\geq 0}} \mathrm{Arg}(z) = \pi \qquad \text{and} \qquad \lim_{\substack{z\to -a\\ \mathrm{Im}(z)<0}} \mathrm{Arg}(z) = -\pi.$$

Does $\lim_{z\to -a} \mathrm{Arg}(z)$ exist, and if so, what is it?

ANSWER.

We may also show that

$$\lim_{\substack{z\to 0\\ \operatorname{Arg}(z)=\theta}} \operatorname{Arg}(z) = \theta.$$

Thus Arg is not continuous at any point of $(-\infty, 0]$. The function Arg is one of many important discontinuous complex functions.

EXERCISE 5.12. Suppose that $f(z) = \bar{z}/z$ and $g(z) = z^2/\bar{z}$. Does $\lim_{z\to 0} f(z)$ or $\lim_{z\to 0} g(z)$ exist? If so, find the limit; otherwise, explain why it does not exist.

ANSWER.

3. Stereographic projection and the Riemann sphere

To explain the role of ∞, we imagine a unit sphere S in xyz space, with center at 0, and we identity the xy plane with the complex plane. We may define a function $\sigma : \mathbb{C} \to S$ geometrically by joining a point p in the plane to the north pole n of the sphere by a straight line. The line will cut the sphere at n and at one other point, which we call $\sigma(p)$. Then we may think of n as being $\sigma(\infty)$.

The function σ is called *stereographic projection*, and the sphere is called the Riemann sphere. Then balls $\mathrm{B}(0, \varepsilon)$ correspond to spherical caps in the Riemann sphere, centred at $\sigma(0)$, while balls $\mathrm{B}(z_0, \varepsilon)$ correspond to spherical caps in the Riemann sphere containing the point $\sigma(z_0)$, and the "punctured balls" $\mathrm{B}^\circ(\infty, \varepsilon)$ correspond to "punctured" spherical caps in the Riemann sphere centred at $\sigma(\infty)$. These spherical caps shrink down towards the points $\sigma(0)$, to $\sigma(z)$ and to $\sigma(\infty)$ as ε tends to 0.

4. Continuity

DEFINITION 5.13. Suppose that f is a complex function. We say that f is continuous at a point z_0 if $f(z_0)$ is defined and $\lim_{z\to z_0} f(z) = f(z_0)$. We say that f is continuous in a set S if it is continuous at all points of S. We say that f is continuous if it is continuous at all points of its domain.

The functions $z \mapsto z$, $z \mapsto \overline{z}$, $z \mapsto |z|$, $z \mapsto \operatorname{Re}(z)$, and $z \mapsto \operatorname{Im}(z)$ are all continuous. The function Arg is continuous in the set $\{z \in \mathbb{C} \setminus \{0\} : \operatorname{Arg}(z) \neq \pi\}$.

Properties of limits lead to similar properties of continuous functions.

THEOREM 5.14. *Suppose that $c \in \mathbb{C}$, and that $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$. Then cf, $f + g$, $|f|$, $\overline{f}$, $\operatorname{Re} f$, $\operatorname{Im} f$ and fg are continuous in S, as is f/g provided that $g(z) \neq 0$ for any z in S.*

THEOREM 5.15. *Suppose that $f : S \to \mathbb{C}$ and $g : T \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$ and $T \subseteq \mathbb{C}$. Then $f \circ g$ is continuous where it is defined, that is, in $\{z \in T : g(z) \in S\}$.*

By using the theorems above, it follows that functions that are composed of the standard functions (except Arg), such as

$$z \mapsto \frac{\operatorname{Re}(z^2) + i \operatorname{Im}(z^3)}{|z| + 1 + \overline{z}},$$

are also continuous where they are defined (with this example, the tricky bit is finding the domain of definition; the natural domain is actually $\mathbb{C}$).

Generally speaking, any function that can be written down using the standard functions, and without choices in the definition, is continuous in its domain of definition, except when Arg is involved. Where there are choices in the definition, the difficulties usually lie where the different definitions match up.

Continuity is useful for two reasons. First, when functions are continuous, we do not have to worry about limits much. Next, continuous functions have some important properties.

THEOREM 5.16. *Suppose that the set $S \subseteq \mathbb{C}$ is compact (i.e., closed and bounded) and that f is a continuous complex function defined on S. Then there exists a point z_0 in S such that*

$$|f(z_0)| = \max\{|f(z)| : z \in S\}.$$

One says that the modulus of a continuous function *attains its maximum* in a compact set. As a consequence, if f is a continuous complex function defined in a compact set $S \subseteq \mathbb{C}$, then there is a number R such that

$$|f(z)| \leq R \qquad \forall z \in S.$$

Thus f is *bounded in S*.

Last but not least, continuous functions in compact sets are *uniformly continuous.* We will not explain this now.

5. Examples of continuous functions

EXERCISE 5.17. Show from first principles that $z \mapsto |z|$ is a continuous function in $\mathbb{C}$.

ANSWER.

EXERCISE 5.18. Show that $\operatorname{Arg}(z)$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$.

ANSWER.

DEFINITION 5.19. The function $\operatorname{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is defined by

$$\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z).$$

EXERCISE 5.20. Show that $\operatorname{Log}(z)$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$, and is not continuous at any point on $(-\infty, 0]$.

ANSWER.

LECTURE 6

Complex Differentiability

In this lecture, we investigate the differentiability of a function of a complex variable, defined much as for functions of a real variable. Many calculations are similar to the real-variable case; however, some functions that we might expect to be differentiable are not.

1. Definition

DEFINITION 6.1. Suppose that $S \subseteq \mathbb{C}$ and that $f : S \to \mathbb{C}$ is a complex function. Then we say that f is *differentiable at the point* z_0 in S if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{6.1}$$

exists and is finite. If it does, it is called the *derivative* of f at z_0, and is written $\dfrac{df(z_0)}{dz}$ or $f'(z_0)$.

We say that f is differentiable in a set S if it is differentiable at all points of S, and that f is differentiable if it is differentiable at all points of its domain.

REMARK 6.2. The limit (6.1) may also be written as

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}\,, \tag{6.2}$$

and the definition may be given with this limit instead.

2. Examples

EXERCISE 6.3. Suppose that $f_1(z) = z^2 + iz + 2$. Is f_1 differentiable at z_0 in $\mathbb{C}$? If so, find $f_1'(z_0)$?

ANSWER.

The computation above is almost identical to that to find the derivative of the real function $x^2 + x + 2$. Indeed, many formulae from the real case also hold in the complex case when x is replaced by z. So do many theorems.

3. More examples of differentiation of complex functions

The following examples show that there is a twist to the story.

EXERCISE 6.4. Suppose that $f_2(z) = \overline{z}$. Is f_2 differentiable at z_0 in $\mathbb{C}$? If so, find $f_2'(z_0)$?

ANSWER.

EXERCISE 6.5. Suppose that $f_3(z) = |z|^2$. Is f_3 differentiable at z_0 in $\mathbb{C}$? If so, find $f_3'(z_0)$?

ANSWER.

These examples show that a function may be differentiable everywhere, or nowhere, or at some points and not others. The nondifferentiable examples involved the complex conjugate, explicitly or implicitly.

4. The Cauchy–Riemann equations

We will now investigate differentiability using theoretical tools. If $\lim_{w\to 0} q(w)$ exists, then

$$\lim_{\substack{w\to 0\\ w\in\mathbb{R}}} q(w) = \lim_{\substack{w\to 0\\ w\in i\mathbb{R}}} q(w) = \lim_{w\to 0} q(w),$$

in the sense that the first two limits also exist, and are equal to the third. This allows us to relate the complex derivative to partial derivatives.

THEOREM 6.6. *Suppose that Ω is an open subset of $\mathbb{C}$, that f is a complex function defined in Ω, that $f(x+iy) = u(x,y) + iv(x,y)$, where u and v are real-valued functions of two real variables, and that f is differentiable at $z_0 \in \Omega$. Then the partial derivatives*

$$\frac{\partial u}{\partial x}(x_0,y_0), \quad \frac{\partial u}{\partial y}(x_0,y_0), \quad \frac{\partial v}{\partial x}(x_0,y_0) \quad \text{and} \quad \frac{\partial v}{\partial y}(x_0,y_0)$$

all exist, and

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)\,. \tag{6.3}$$

Further,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) = -i\Big(\frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)\Big). \tag{6.4}$$

REMARK 6.7. The pair of equations (6.3), which relate the partial derivatives of u and v, are known as the Cauchy–Riemann equations.

PROOF. If $f'(z_0)$ exists, then

$$\begin{aligned} f'(z_0) &= \lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{f(z_0+w)-f(z_0)}{w} \\ &= \lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{u(x_0+w,y_0)+iv(x_0+w,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{w} \\ &= \lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{u(x_0+w,y_0)-u(x_0,y_0)}{w} + i\lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{v(x_0+w,y_0)-v(x_0,y_0)}{w} \\ &= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0), \end{aligned}$$

because a limit exists if and only if its real and imaginary parts do. Thus

$$\frac{\partial u}{\partial x}(x_0,y_0) = \operatorname{Re}(f'(z_0)) \qquad \text{and} \qquad \frac{\partial v}{\partial x}(x_0,y_0) = \operatorname{Im}(f'(z_0)).$$

Since $f'(z_0) = \operatorname{Re}(f'(z_0)) + i\operatorname{Im}(f'(z_0))$, part of (6.4) follows.

Similarly, if $f'(z_0)$ exists, then

$$\begin{aligned} f'(z_0) &= \lim_{\substack{w\to 0\\ w\in i\mathbb{R}}} \frac{f(z_0+w)-f(z_0)}{w} \\ &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0,y_0+h)+iv(x_0,y_0+h)-u(x_0,y_0)-iv(x_0,y_0)}{ih} \\ &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{v(x_0,y_0+h)-v(x_0,y_0)}{h} - i\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0,y_0+h)-u(x_0,y_0)}{h} \\ &= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0) = -i\Big(\frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)\Big). \end{aligned}$$

Thus

$$\frac{\partial v}{\partial y}(x_0,y_0) = \operatorname{Re}(f'(z_0)) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x_0,y_0) = -\operatorname{Im}(f'(z_0)).$$

The Cauchy–Riemann equations follow by equating the two expressions for the real part of $f'(z_0)$ and the two expressions for the imaginary part of $f'(z_0)$, and the remaining part of (6.4) also follows.. □

One consequence of the previous theorem is that if f is differentiable at every point of an open set Ω in $\mathbb{C}$, then the Cauchy–Riemann equations hold at every point of Ω. Later on, we will see that in addition the four partial derivatives $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$ and $\partial v/\partial y$ are all continuous. For open sets Ω, the converse is true.

THEOREM 6.8. *If the four partial derivatives $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$ and $\partial v/\partial y$ are all continuous in an open set Ω, then f is complex differentiable at $z_0 \in \Omega$ if and only if the Cauchy–Riemann equations hold at z_0, and if so, then*

$$f'(x_0 + iy_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

We will justify this result later. When the partial derivatives are continuous in a set that is not open, the function might be differentiable, or it might not be.

5. Examples

We revisit our previous examples, and add another, using the Cauchy–Riemann equations.

EXAMPLES 6.9. (1) Suppose that $f_1(z) = z^2 + iz + 2$. Then

$$u(x, y) = x^2 - y^2 - y + 2 \quad \text{and} \quad v(x, y) = 2xy + x,$$

so

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = -2y - 1, \qquad \frac{\partial v}{\partial x} = 2y + 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = 2x,$$

and hence the Cauchy–Riemann equations hold for all (x, y) in $\mathbb{R}^2$. Since the partial derivatives are continuous and $\mathbb{C}$ is open, f_1 is differentiable in $\mathbb{C}$, and

$$f_1'(z) = 2x + i(2y + 1) = 2z + i.$$

(2) Suppose that $f_2(z) = \overline{z}$. Then

$$u(x, y) = x \quad \text{and} \quad v(x, y) = -y,$$

so

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = -1,$$

and hence the Cauchy–Riemann equations do not hold for any (x, y) in $\mathbb{R}^2$.

Hence f_2 is not differentiable at any point in $\mathbb{C}$.

(3) Suppose that $f_3(z) = |z|^2$. Then

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0,$$

so

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = 2y, \qquad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0,$$

and hence the Cauchy–Riemann equations hold if and only if $x = y = 0$. The partial derivatives are continuous in $\mathbb{C}$, which is open, and hence f_3 is differentiable at 0. Finally, f is not differentiable at any other point than 0, since the Cauchy–Riemann equations do not hold at any other point.

(4) Suppose that $f_4(z) = e^z$. Then

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y,$$

so

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial x} = e^x \sin y \quad \text{and} \quad \frac{\partial v}{\partial y} = e^x \cos y,$$

and hence the Cauchy–Riemann equations hold for all (x, y) in $\mathbb{R}^2$. Since the partial derivatives are continuous and $\mathbb{C}$ is open, f_4 is differentiable in $\mathbb{C}$, and

$$f_4'(z) = \frac{\partial u}{\partial x}(x, y) + \frac{\partial u}{\partial y}(x, y) = e^x(\cos y + i \sin y) = e^z.$$

Remark 6.10. Doing (4) using limits would be rather messy!

6. Properties of the derivative

Theorem 6.11. *Suppose that $z_0 \in \mathbb{C}$, that the complex functions f and g are differentiable at z_0, and that $c \in \mathbb{C}$. Then the functions cf, $f + g$ and $f\,g$ are differentiable at z_0, and*

$$\begin{aligned} (cf)'(z_0) &= c\,f'(z_0), \\ (f + g)'(z_0) &= f'(z_0) + g'(z_0), \\ (f\,g)'(z_0) &= f'(z_0)\,g(z_0) + f(z_0)\,g'(z_0). \end{aligned}$$

Further, if $g(z_0) \neq 0$, then the function f/g is differentiable at z_0, and

$$(f/g)'(z_0) = \frac{f'(z_0)\,g(z_0) - f(z_0)\,g'(z_0)}{g(z_0)^2}\,.$$

Theorem 6.12. *Suppose that $z_0 \in \mathbb{C}$, that the complex function f is differentiable at $g(z_0)$, and that the complex function g is differentiable at z_0. Then the function $f \circ g$ is differentiable at z_0, and*

$$(f \circ g)'(z_0) = f'(g(z_0))\,g'(z_0).$$

Theorem 6.13. *Suppose that f is a complex function and that $z_0 \in \mathrm{Domain}(f)$. If f is differentiable at z_0, then f is continuous at z_0.*

Proof. The proof of these results are very similar to those of the corresponding results for real functions, and we omit them. □

Theorem 6.14 (l'Hôpital's rule). *Suppose that $z_0 \in \mathbb{C} \cup \{\infty\}$ and that the complex functions f and g are differentiable at z_0. If $\lim_{z \to z_0} f(z)/g(z)$ is indeterminate, that is, of the form $0/0$ or ∞/∞, and if $\lim_{z \to z_0} f'(z)/g'(z)$ exists, then*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}\,. \tag{6.5}$$

We do not prove l'Hôpital's rule at this time. It follows from results that we prove later about Taylor series and Laurent series.