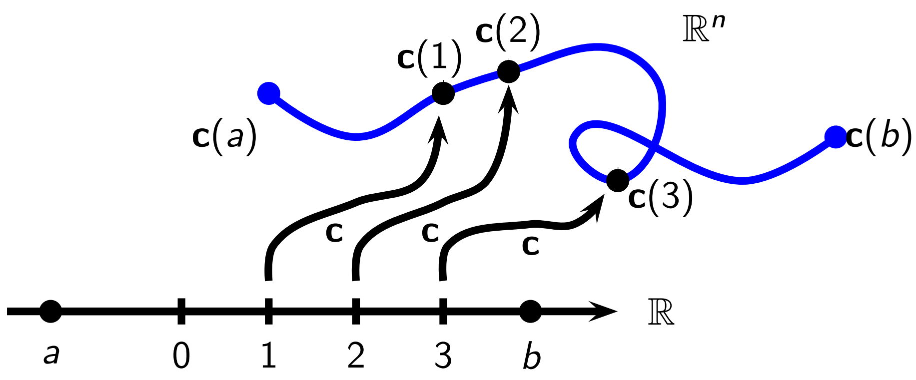
Curves

Definition

A curve in $\mathbb{R}^n$ is a vector valued function

$$\mathbf{c} : I \to \mathbb{R}^n$$

where I is an interval on $\mathbb{R}$.



Often we think of the image of I under $\mathbf{c}$ as the curve, but this is not the definition.

The function $\mathbf{c}$ is also called a parameterisation.

Curves

Example: A curve (or parameterisation) $\mathbf{r} : [-1, 3] \to \mathbb{R}^2$ is given by

$$\mathbf{r}(t) = \left(1 + t, \tfrac{4}{3}t^2\right).$$

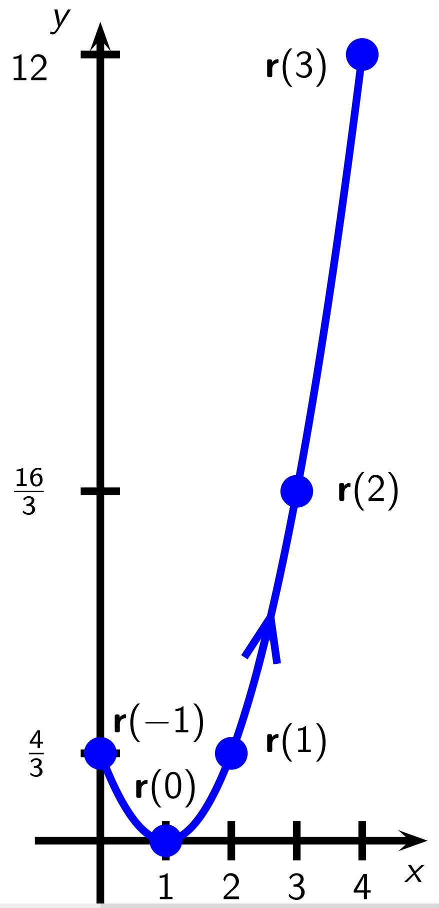
The image of $[-1, 3]$ is

$$\left\{(x, y) : x = 1 + t, y = \tfrac{4}{3}t^2, -1 \le t \le 3\right\}.$$

Plot and label the points $\mathbf{r}(-1)$, $\mathbf{r}(0)$, $\mathbf{r}(1)$, $\mathbf{r}(2)$ and $\mathbf{r}(3)$. Find a Cartesian equation for the image of $[-1, 3]$ and sketch the curve. Indicate the direction of increasing parameter.

A Cartesian equation can be obtained by eliminating the parameter.

$$x = 1 + t, y = \frac{4}{3}t^2 \qquad \Rightarrow \qquad y = \frac{4}{3}(x-1)^2.$$

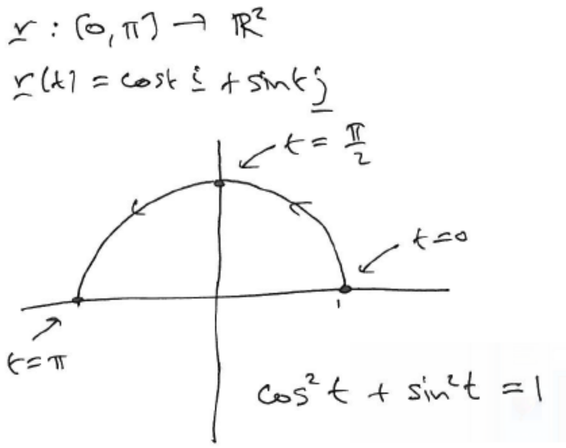


Curves

Examples: Sketch the following curves.

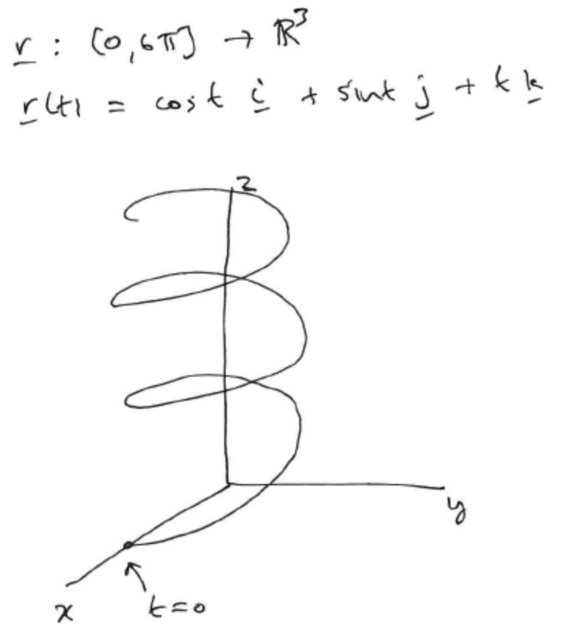
$\mathbf{r} : [0, \pi] \to \mathbb{R}^2$ given by

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}.$$



$\mathbf{r} : [0, 6\pi] \to \mathbb{R}^3$ given by

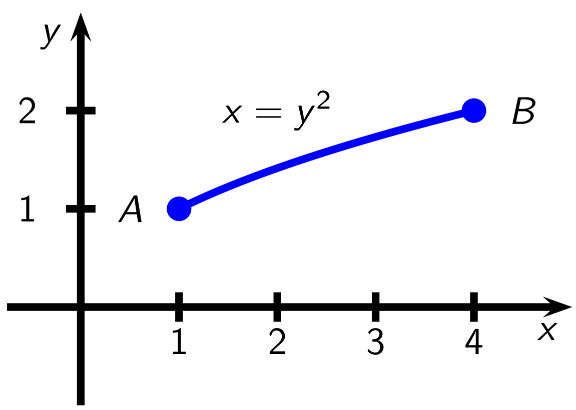
$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}.$$



Curves

Example

Find two different curves with the image drawn below. For each curve, describe the direction of increasing parameter.



Give a parameterisation that traverses from B to A and another that traverses from A to B and then back to A again.

Each value of x corresponds to only one point in the image. So we can use x as a parameter.

$$\mathbf{r_1} : [1, 4] \to \mathbb{R}^2,$$

$$\mathbf{r_1}(t) = (t, \sqrt{t}).$$

The parameter increases from A to B.

We could also use y as a parameter.

$$\mathbf{r_2} : [1, 2] \to \mathbb{R}^2,$$

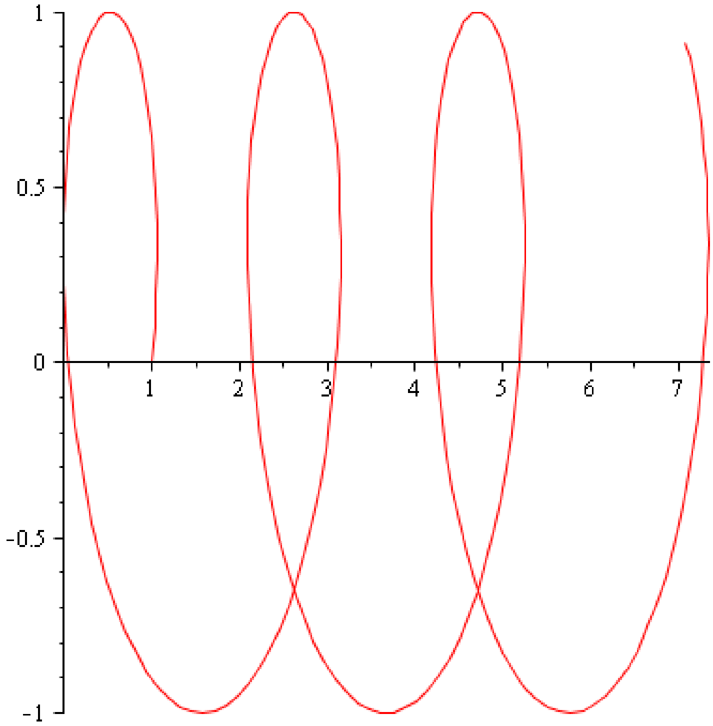
$$\mathbf{r_2}(t) = (t^2, t).$$

The parameter increases from A to B.

Curves

Example: Sketch $\mathbf{r} : [0, 20] \to \mathbb{R}^2$ given by $\mathbf{r}(t) = \left(\cos t + \frac{1}{3}t\right)\mathbf{i} + \sin t\mathbf{j}$.

```
> plot([cos(t)+t/3,sin(t),t=0..20]);
```

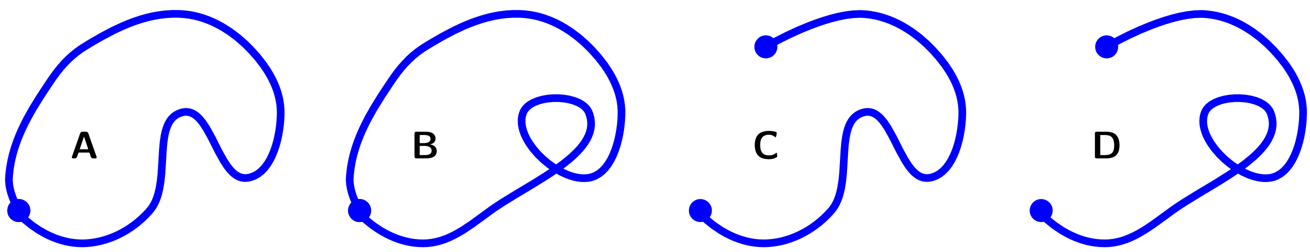


Curves

Definition

- A multiple point is a point through which the curve passes more than once.
- For a curve $\mathbf{c} : [a, b] \to \mathbb{R}^n$, $\mathbf{c}(a)$ and $\mathbf{c}(b)$ are called end points.
- A curve is closed if its end points are the same point.

Which of the following are the image of a closed curve? How many multiple points (other than end points) does each curve have?



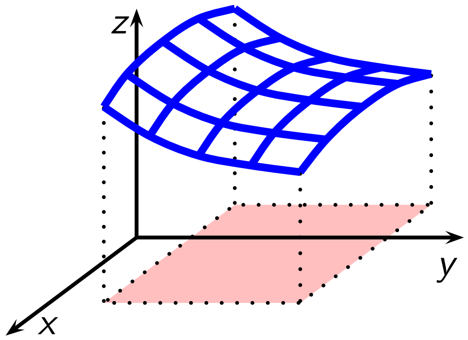
A and **B** are closed. **B** and **D** have one multiple point each.

What assumption has been made in the above answers?

Surfaces

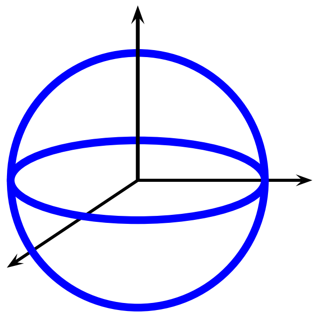
You have seen surfaces in $\mathbb{R}^3$ described in 3 ways.

Graph of a function



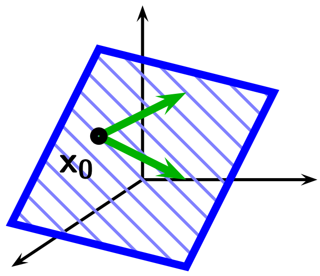
Eg, $z = f(x, y)$

Implicitly



Eg, a sphere given by $x^2 + y^2 + z^2 = 1$.

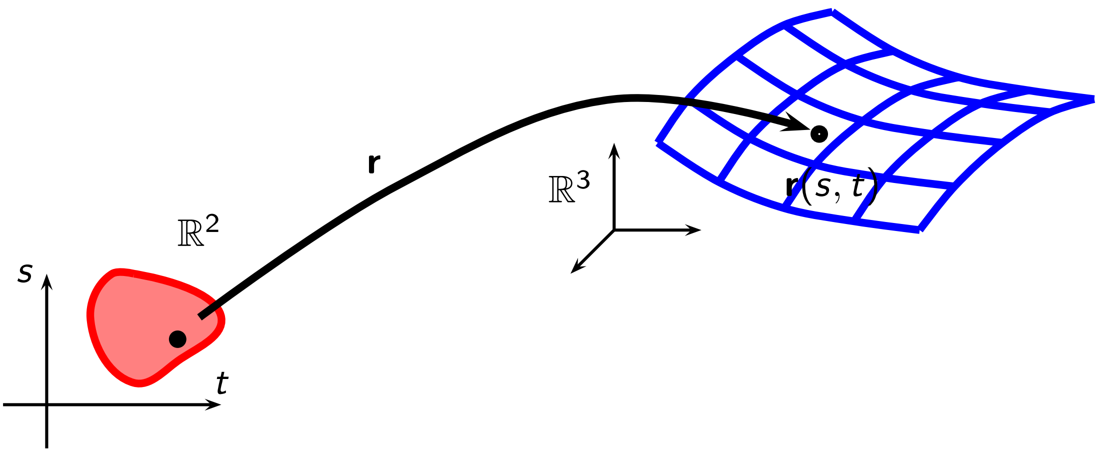
Parametrically



Eg, a plane given by $\mathbf{x} = \mathbf{x_0} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$.

Parameterisation defined surface

For $D \subset \mathbb{R}^2$, the image of D under $\mathbf{r} : D \to \mathbb{R}^3$ is a surface in $\mathbb{R}^3$. Note that unlike for curves, a surface is the image of the parameterisation.



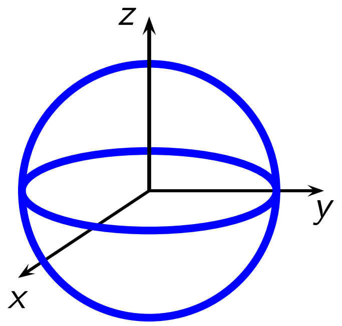
Eg, $\mathbf{r} : D \to \mathbb{R}^3$ where $D = \{(x, y) : x^2 + y^2 \leq 1\}$ and

$$\mathbf{r}(s, t) = (s, t, \sqrt{1 - s^2 - t^2})$$

is a parameterisation of the upper unit hemisphere.

Implicitly defined surface

We can define a surface in $\mathbb{R}^3$ as the set of points satisfying an equation. Eg, a sphere given by $x^2 + y^2 + z^2 = 1$.



Later in the course we will study a theorem that tells you when parts of this surface are the graph of a function of some of the variables — the Implicit Function Theorem.

Some other implicitly defined surfaces will be discussed in tutorial 1.

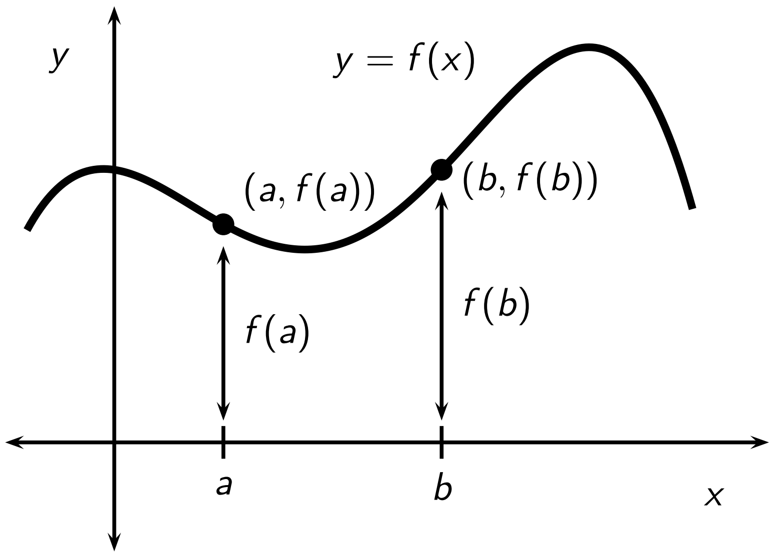
Graphs of functions of one variable

The graph of

$$f : \mathbb{R} \to \mathbb{R}$$

is the set of points

$$\{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$



On the graph of f, input values are represented by distance across the page and output values by distance up the page.

Graphs of functions of two variables

The graph of

$$f : D \to \mathbb{R}$$

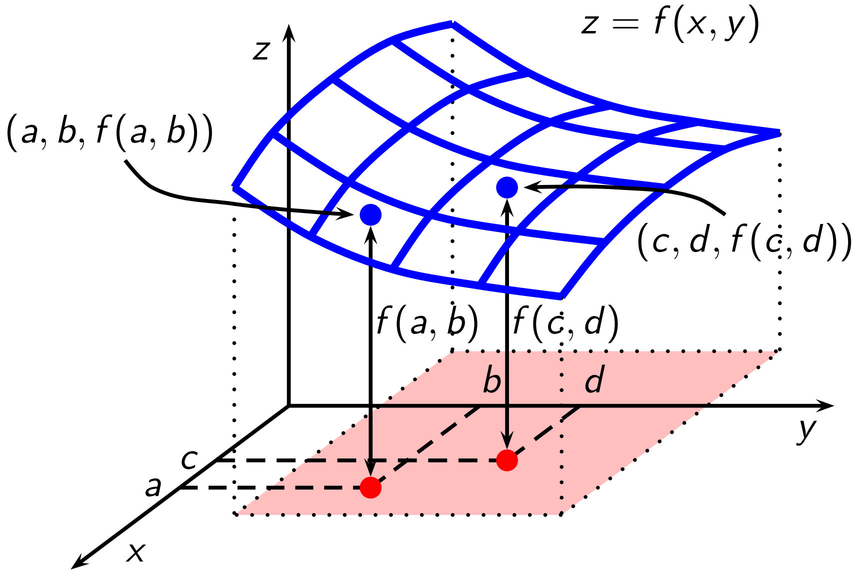
is the set of points

$$\{(x, y, z) : z = f(x, y)$$

$$\text{for all } (x, y) \in D\}.$$

In this example the domain is the subset of $\mathbb{R}^2$ shaded pink in the diagram.

In other examples, it could be all of $\mathbb{R}^2$ or any other subset of $\mathbb{R}^2$.



Note the orientation of the axes. If you sat on top of z-axis and looked down, you would see the usual orientation for the x and y axes.

Graphs of functions of two variables

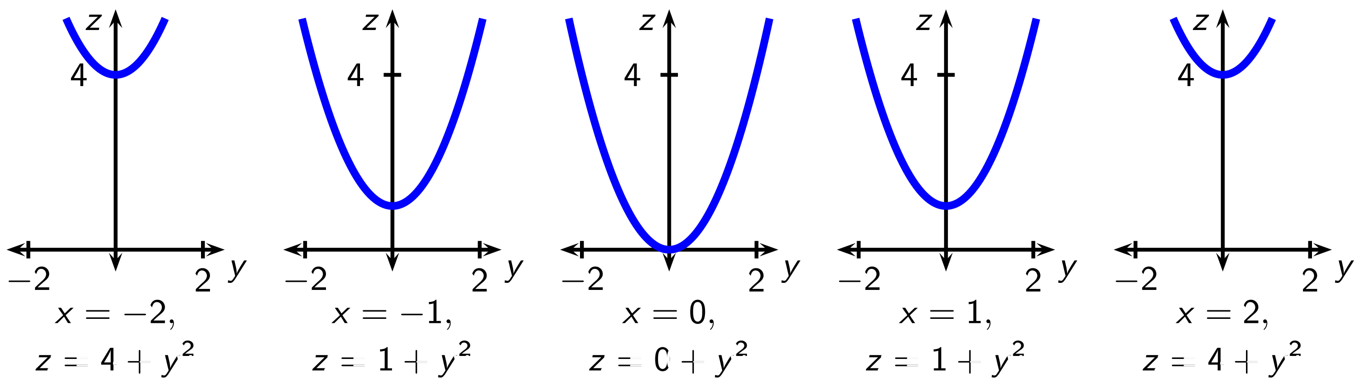
Given a function of two variables, how can we visualise it?

For example, what does the graph of

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x^2 + y^2$$

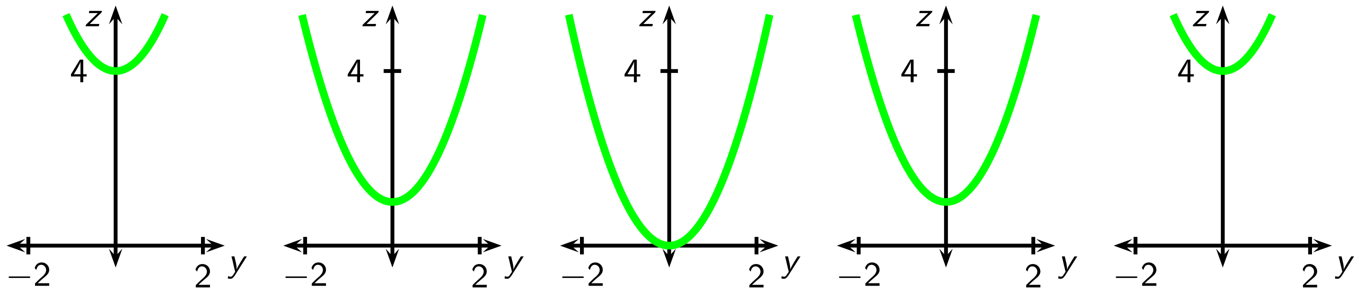
look like. That is, we want to sketch the set of points in $\mathbb{R}^3$ satisfying $z = f(x, y)$.

Let's start by looking at some vertical slices with constant x.

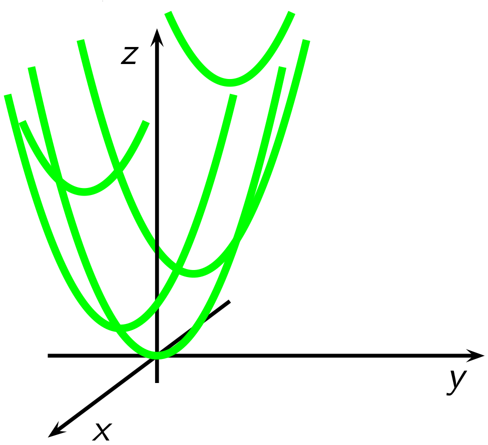


Next put these together.

Graphs of functions of two variables



$x=-2,$	$x=-1,$	$x=0,$	$x=1,$	$x=2,$
$z=4+y^2$	$z=1+y^2$	$z=0+y^2$	$z=1+y^2$	$z=4+y^2$

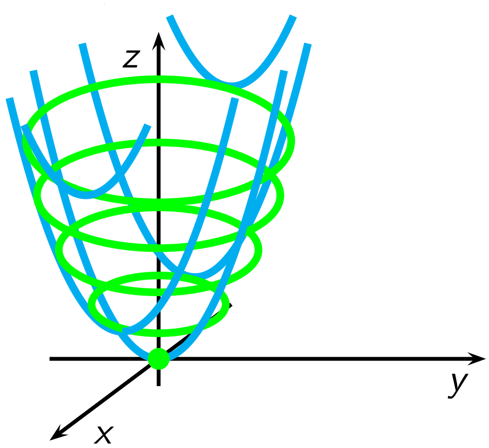


We could also take slices of constant y.
Try plotting these yourself.

Horizontal slices

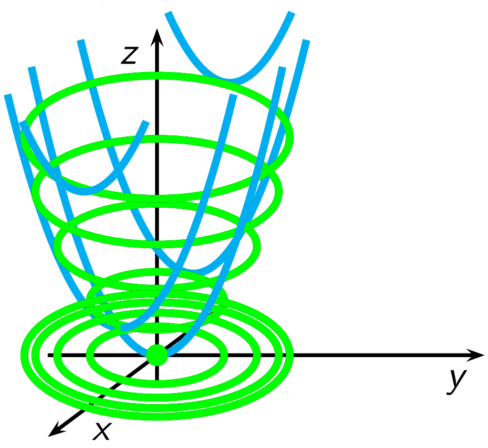
We could also take horizontal slices, that is, slices of constant z.

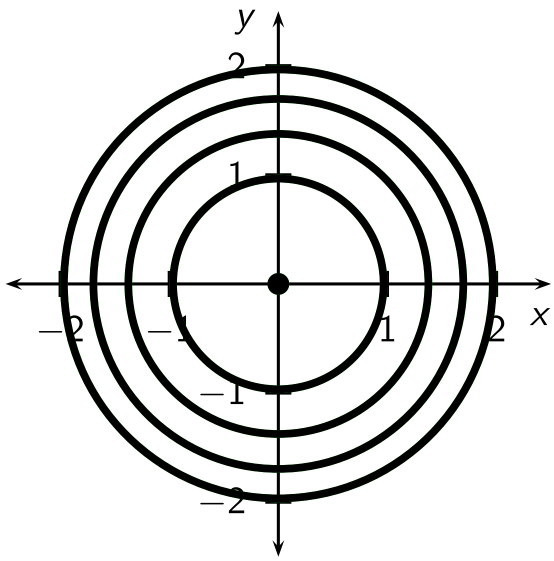
$$\begin{array}{lllll} z & = & -1 & : & \text{no solution} \\ z & = & 0 & : & (x,y)=(0,0) \quad \text{a single point} \\ z & = & 1 & : & x^2+y^2=1 \quad \text{a circle of radius } 1 \\ z & = & 2 & : & x^2+y^2=2 \quad \text{a circle of radius } \sqrt{2} \\ & \text{etc} & & & \end{array}$$



Horizontal slices

If we plot the horizontal slices in the xy-plane, we have a contour map.



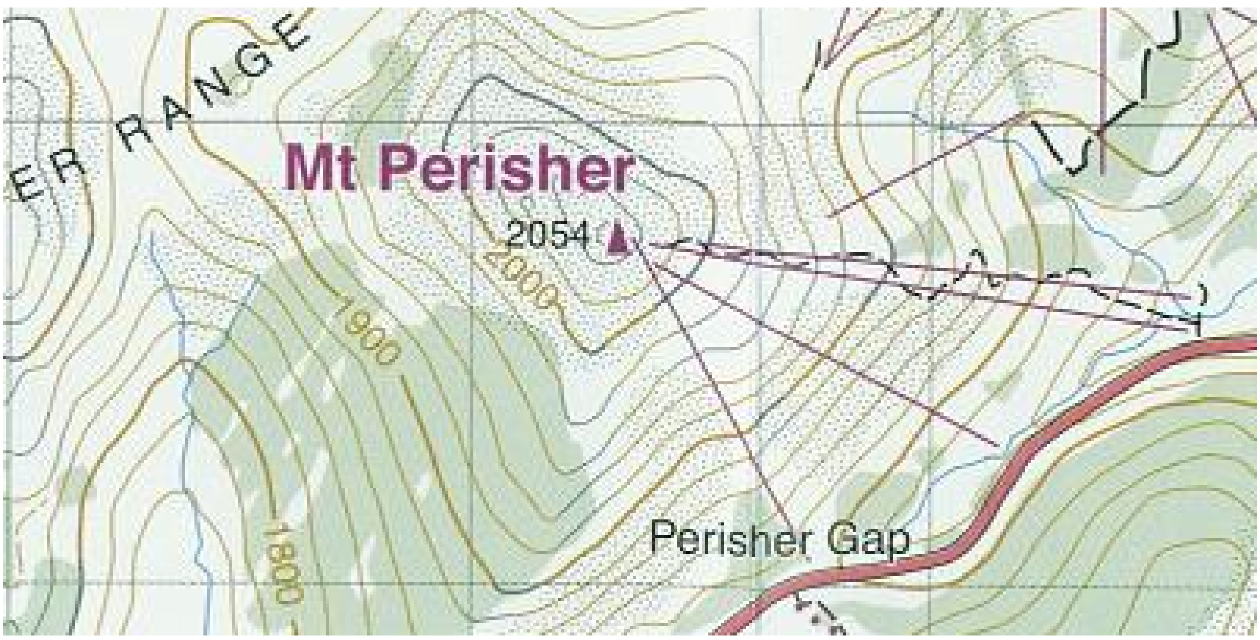


We have plotted some level curves or contours of f.

Contours or other slices are a good way of visualising a surface.

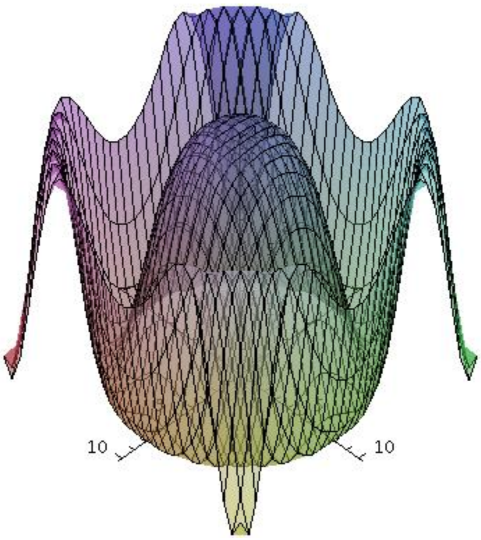
Level curves - examples

Contours on topographical maps are used to describe a surface. Maps Downunder have some sample maps on their website.
http://www.mapsdownunder.com.au/cgi-bin/mapshop/ABC-MTPKT.html

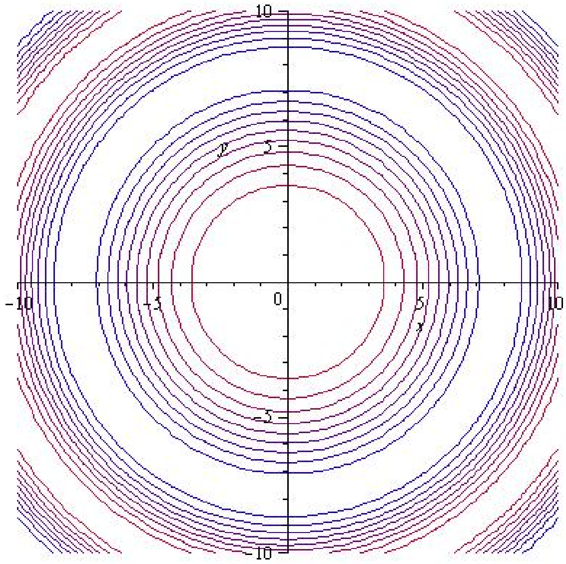


Level curves - examples

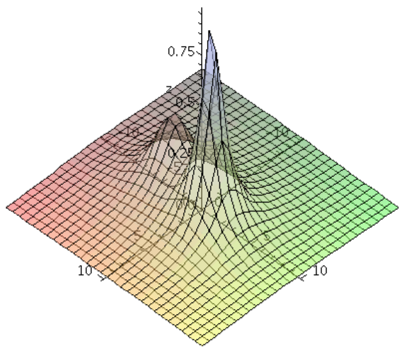
```
> f1 := 1-sin((x^2+y^2)/40)^2:
> with(plots):
  plot3d(f1,x=-10..10,y=-10..10,
  axes=normal,transparency=0.5,
  labels=[x,y,z],grid=[30,30]);
```

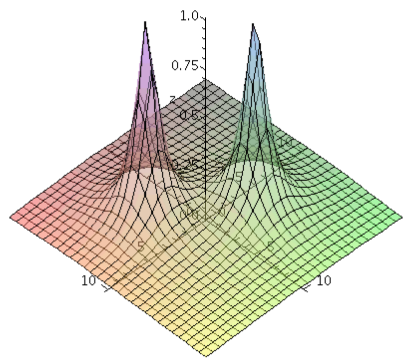


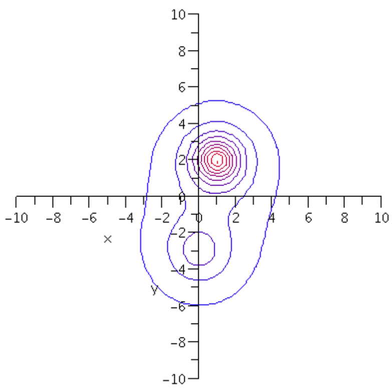

```
> contourplot(f1,x=-10..10,y=-10..10,
  grid=[50,50],view=[-10..10,-10..10],
  contours=[0,0.1,0.2,0.3,0.4,0.5,0.6,
  0.7,0.8,0.9,1],coloring=[blue,red]);
```

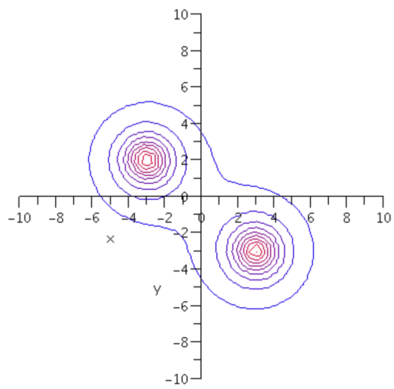


Level curves - examples

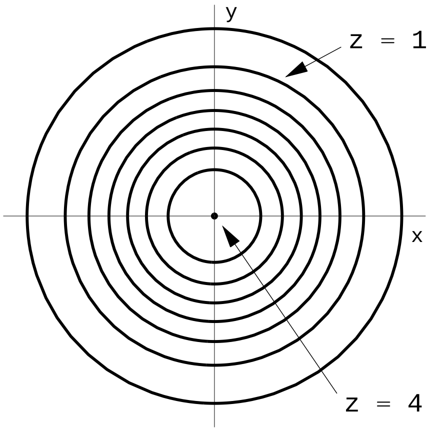






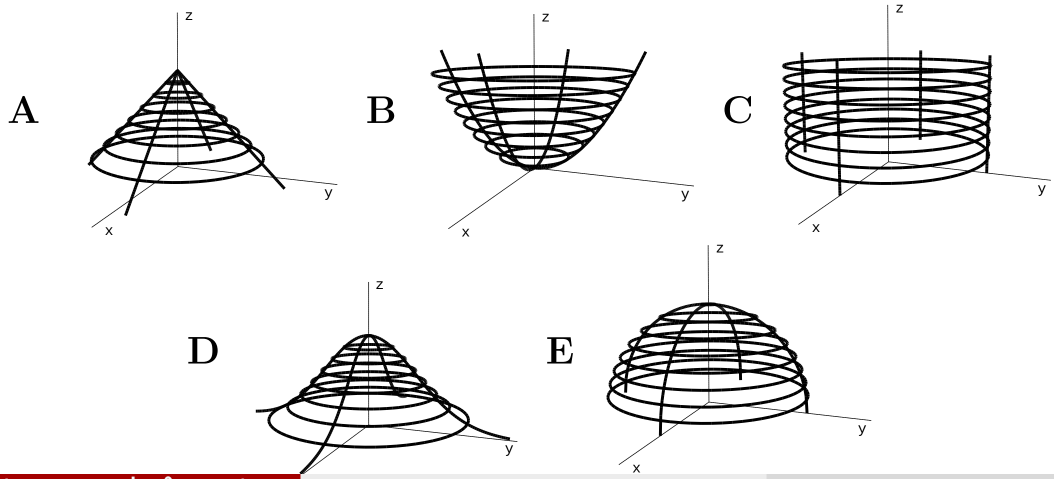


Level curves - example



Let f be a function of two variables. The $f(x, y) = 0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0,\ 3.5,\ 4.0$ level curves are draw on the left.

Which of the surfaces below could be the graph $z = f(x, y)$? Give reasons for your choice



Surfaces - an example

Sketch the level curves of

$$f(x, y) = 4 - \sqrt{x^2 + y^2}$$

and describe the surface $z = f(x, y)$.

$z = f(x,y) = 4 - \sqrt{x^2+y^2}$

level curves are $c = 4 - \sqrt{x^2+y^2}$

$\Rightarrow \quad c - 4 = -\sqrt{x^2+y^2}$ (Note $c-4 \leq 0$ $\Rightarrow c \leq 4$)

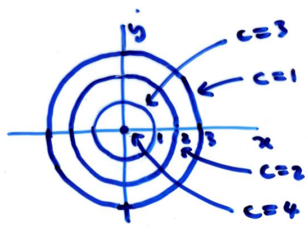
$c=4$: $(x,y) = (0,0)$

$c=3$: $x^2+y^2=1$ (circle of radius 1)

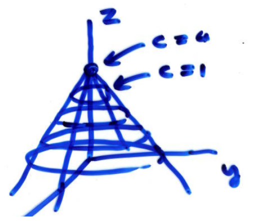
$c=2$: $x^2+y^2=4$ (" " " 2)

$c=1$: $x^2+y^2=9$ (" " " 3)

$\vdots$



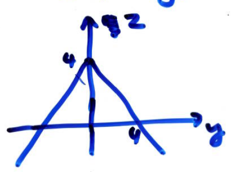
This is a cone with vertex at $(0,0,4)$



$x=0$ slice:

$z = 4 - \sqrt{0+y^2}$

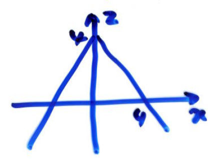
$= 4 - |y|$



$y=0$ slice:

$z = 4 - \sqrt{x^2+0}$

$= 4 - |x|$



Surfaces - an example

Sketch the level curves of

$$f(x, y) = \sqrt{1 - x^2 - 3y^2}$$

and describe the surface $z = f(x, y)$.

$z = f(x,y) = \sqrt{1-x^2-3y^2}$

level curves are $c = \sqrt{1-x^2-3y^2}$

$\Rightarrow x^2+3y^2 = 1-c^2 \quad (c \geq 0)$

$c=1 \; : \; (x,y) = (0,0)$

$$\left.\begin{array}{ll} c=\frac{3}{4} : & x^2+3y^2 = \frac{7}{16} \\ c=\frac{1}{2} : & x^2+3y^2 = \frac{3}{4} \\ c=\frac{1}{4} : & x^2+3y^2 = \frac{15}{16} \\ c=0 : & x^2+3y^2 = 1 \end{array}\right\} \text{ellipses}$$