# Technical Proofs

Math2221

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## 1 Local existence and uniqueness

We consider an initial-value problem for a nonlinear system of ODEs,

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{F}(\boldsymbol{x}, t) \quad \text{for all } t, \quad \text{with } \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{1}$$

Since $\boldsymbol{x}(t)$ is a solution iff it satisfies the nonlinear Volterra equation

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{F}\big(\boldsymbol{x}(s), s\big)\, ds,$$

we attempt to define the sequence of *Picard iterates*

$$\begin{aligned} \boldsymbol{x}_1(t) &= \boldsymbol{x}_0, \\ \boldsymbol{x}_{k+1}(t) &= \boldsymbol{x}_0 + \int_0^t \boldsymbol{F}\big(\boldsymbol{x}_k(s), s\big)\, ds, \quad \text{for } k = 1,\, 2,\, 3,\, \ldots. \end{aligned}$$

Fix $r > 0$ and $\tau > 0$, and let

$$S = \{\, (\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : |\boldsymbol{x} - \boldsymbol{x}_0| \le r \text{ and } |t| \le \tau \,\}.$$

We assume that there exist positive constants $M$ and $L$ such that

$$|\boldsymbol{F}(\boldsymbol{x}, t)| \le M \quad \text{for } (\boldsymbol{x}, t) \in S, \tag{2}$$

and

$$\big|\boldsymbol{F}(\boldsymbol{x}_1, t) - \boldsymbol{F}(\boldsymbol{x}_2, t)\big| \le L|\boldsymbol{x}_1 - \boldsymbol{x}_2| \quad \text{for } (\boldsymbol{x}_1, t),\, (\boldsymbol{x}_2, t) \in S. \tag{3}$$

These conditions are satisfied if, for example, $\boldsymbol{F}$ and $\partial F_i/\partial x_j$ are continuous on $S$ for $i$, $j \in \{1, 2, \ldots, n\}$.

**Lemma 1.** *With* $\tau_1 = \min(\tau, r/M)$*, the Picard iterates* $\boldsymbol{x}_1$*,* $\boldsymbol{x}_1$*,* $\boldsymbol{x}_2$*, ... exist and satisfy*

$$|\boldsymbol{x}_k(t) - \boldsymbol{x}_0| \leq r \quad \textit{for } |t| \leq \tau_1 \textit{ and } k \geq 0.$$

*Proof.* We will use induction on $k$. The claim is trivial for $k = 0$. Let $k \geq 0$ and assume that $\boldsymbol{x}_k$ satisfies the estimate, then $\bigl(\boldsymbol{x}_k(t), t\bigr) \in S$ for $|t| \leq \tau_1$, so

$$|\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_0| \leq \int_0^t \bigl|\boldsymbol{F}\bigl(\boldsymbol{x}_k(s), s\bigr)\bigr| \, ds \leq Mt \leq r \quad \text{for } 0 \leq t \leq \tau_1.$$

Similarly, if $-\tau_1 \leq t \leq 0$ then

$$|\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_0| \leq \int_t^0 \bigl|\boldsymbol{F}\bigl(\boldsymbol{x}_k(s), s\bigr)\bigr| \, ds \leq M|t| \leq r \quad \text{for } -\tau_1 \leq t \leq 0.$$

□

**Lemma 2.** *The Picard iterates satisfy*

$$|\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t)| \leq \frac{M}{L} \frac{(Lt)^{k+1}}{(k+1)!} \quad \textit{for } |t| \leq \tau_1 \textit{ and } k \geq 1.$$

*Proof.* If $k = 1$, we have

$$|\boldsymbol{x}_1(t) - \boldsymbol{x}_0(t)| \leq \int_0^t \bigl|\boldsymbol{F}(\boldsymbol{x}_0, s)\bigr| \, ds \leq Mt \quad \text{for } 0 \leq t \leq \tau_1,$$

and similarly for $-\tau_1 \leq t \leq 0$. Proceeding by induction on $k$, we let $k \geq 2$ and assume that

$$|\boldsymbol{x}_k(t) - \boldsymbol{x}_{k-1}(t)| \leq \frac{M}{L} \frac{(Lt)^k}{k!} \quad \text{for } |t| \leq \tau_1,$$

then

$$\begin{aligned}
|\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t)| &= \left| \int_0^t \bigl[\boldsymbol{F}\bigl(\boldsymbol{x}_k(s), s\bigr) - \boldsymbol{F}\bigl(\boldsymbol{x}_{k-1}(s), s\bigr)\bigr] \, ds \right| \\
&\leq L \int_0^t \bigl|\boldsymbol{x}_k(s) - \boldsymbol{x}_{k-1}(s)\bigr| \, ds \leq L \int_0^t \frac{M}{L} \frac{(Ls)^k}{k!} \, ds = \frac{M}{L} \frac{(Lt)^{k+1}}{(k+1)!}
\end{aligned}$$

for $0 \leq t \leq \tau_1$, and similarly for $-\tau_1 \leq t \leq 0$. □

Thus, if $|t| \le \tau_1$, then

$$\sum_{k=1}^{\infty} \bigl|\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t)\bigr| \le \frac{M}{L} \sum_{k=0}^{\infty} \frac{(Lt)^{k+1}}{(k+1)!} = \frac{M}{L}\bigl(e^{Lt} - 1\bigr) < \infty,$$

and therefore the limit

$$\boldsymbol{x}(t) = \lim_{k\to\infty} \boldsymbol{x}_k(t) = \lim_{k\to\infty} \left( \boldsymbol{x}_0 + \sum_{j=1}^{k-1} \bigl[\boldsymbol{x}_{j+1}(t) - \boldsymbol{x}_j(t)\bigr] \right)$$

exists; equivalently, we can write

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \sum_{k=1}^{\infty} \bigl[\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t)\bigr]. \tag{4}$$

Moreover, the convergence is uniform for $|t| \le \tau_1$, so $\boldsymbol{x}(t)$ is continuous for $|t| \le \tau_1$. We conclude from the definition of the sequence $\boldsymbol{x}_k(t)$ that this limit satisfies

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{F}\bigl(\boldsymbol{x}(s), s\bigr)\, ds \quad \text{for } |t| \le \tau_1,$$

and hence that $\boldsymbol{x}(t)$ is a solution of the initial-value problem (1).

**Lemma 3.** *The solution $\boldsymbol{x}(t)$ defined above satisfies*

$$|\boldsymbol{x}(t) - \boldsymbol{x}_0| \le M|t| \le r \quad \textit{for } |t| \le \tau_1.$$

*Proof.* By sending $k \to \infty$ in Lemma 1, we see that $\|\boldsymbol{x}(t) - \boldsymbol{x}_0\| \le r$ for $|t| \le \tau_1$. It then follows that

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_0\| = \left\| \int_0^t \boldsymbol{F}\bigl(\boldsymbol{x}(s), s\bigr)\, ds \right\| \le \int_0^{|t|} M\, ds = M|t|.$$

for $|t| \le \tau_1 \le r/M$. □

The next result will allow us to show that the solution $\boldsymbol{x}(t)$ is unique.

**Lemma 4.** *Let $g : [a-\delta, a+\delta] \to \mathbb{R}$. If $g$ is continuous on the closed interval $[a-\delta, a+\delta]$ and differentiable on the open interval $(a-\delta, a+\delta)$, and if there is a constant $K$ such that*

$$|g'(t)| \le K g(t) \quad \textit{for } a-\delta < t < a+\delta,$$

*then*

$$0 \le g(t) \le g(a) e^{K|t-a|} \quad \textit{for } a-\delta \le t \le a+\delta.$$

*Proof.* Since $-Kg(t) \le g'(t) \le Kg(t)$, it follows that

$$\frac{d}{dt}\big[\pm e^{\pm Kt} g(t)\big] = \pm e^{\pm Kt}\big[g'(t) \pm Kg(t)\big] \ge 0 \quad \text{for } a-\delta < t < a+\delta.$$

Hence, the Mean Value Theorem implies that

$$e^{-Kt}g(t) - e^{-Ka}g(a) \le 0 \quad \text{for } a \le t \le a+\delta,$$

and

$$e^{Kt}g(t) - e^{Ka}g(a) \le 0 \quad \text{for } a-\delta \le t \le a.$$

In the first case $g(t) \le g(a)e^{K(t-a)}$, whereas in the second case $g(t) \le g(a)e^{K(a-t)}$, so in both cases $g(t) \le g(a)e^{K|t-a|}$. □

Our main result now follows.

**Theorem 5.** *If $\boldsymbol{F}$ satisfies* (2) *and* (3)*, then the initial-value problem* (1) *has a unique solution for $|t| \le \tau_1 = \min(\tau, r/M)$.*

*Proof.* We have already shown the existence of a solution, via Picard iteration. To prove uniqueness, suppose that $\boldsymbol{y}$ is a second solution:

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{F}(\boldsymbol{y}, t) \quad \text{for } |t| \le \tau_1, \quad \text{with } \boldsymbol{y}(0) = \boldsymbol{x}_0.$$

Let $0 < \epsilon < \tau_1$ and consider the set $E = \{\, t : |t| \le \tau_1 - \epsilon \text{ and } \boldsymbol{x}(t) = \boldsymbol{y}(t)\,\}$. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are continuous, $E$ must be closed. Also, $E \ne \emptyset$ because $0 \in E$. Thus, if we show that $E$ is relatively open in the interval $I_\epsilon = [-\tau_1+\epsilon, \tau_1-\epsilon]$ then it will follow that $E = I_\epsilon$.

Let $a \in E$. By the triangle inequality and Lemma 3,

$$\begin{aligned}\|\boldsymbol{y}(t) - \boldsymbol{x}_0\| &\le \|\boldsymbol{y}(t) - \boldsymbol{y}(a)\| + \|\boldsymbol{y}(a) - \boldsymbol{x}_0\| = \|\boldsymbol{y}(t) - \boldsymbol{y}(a)\| + \|\boldsymbol{x}(a) - \boldsymbol{x}_0\| \\ &\le \|\boldsymbol{y}(t) - \boldsymbol{y}(a)\| + M|a|,\end{aligned}$$

and since $M|a| \le M(\tau_1 - \epsilon) < r$ and $\boldsymbol{y}(t)$ is continuous, there exists $\delta > 0$ such that

$$\|\boldsymbol{y}(t) - \boldsymbol{x}_0\| \le r \quad \text{for } t \in (a-\delta, a+\delta) \cap I_\epsilon.$$

The function $g(t) = |\boldsymbol{x}(t) - \boldsymbol{y}(t)|^2$ satisfies

$$g'(t) = 2\big[\boldsymbol{x}(t) - \boldsymbol{y}(t)\big] \cdot \big[\boldsymbol{F}(\boldsymbol{x}, t) - \boldsymbol{F}(\boldsymbol{y}, t)\big],$$

and thus, by (3),

$$|g'(t)| \le 2|\boldsymbol{x}(t) - \boldsymbol{y}(t)|L|\boldsymbol{x}(t) - \boldsymbol{y}(t)| = 2Lg(t) \quad \text{for } t \in (a-\delta, a+\delta) \cap I_\epsilon.$$

Applying Lemma 4, we see that if $t \in (a-\delta, a+\delta) \cap I_\epsilon$ then $g(t) \le e^{2L|t-a|}g(a) = 0$ and hence $t \in E$, showing that $E$ is relatively open in $I_\epsilon$, as claimed.

Since $\epsilon$ can be arbitrarily small, it follows that $\boldsymbol{x}(t) = \boldsymbol{y}(t)$ for $|t| < \tau_1$. Finally, by continuity, $\boldsymbol{x}(\pm\tau_1) = \boldsymbol{y}(\pm\tau_1)$. □

# 2 General solution of a linear ODE with constant coefficients

We saw in lectures that if $L = p(D)$ where

$$p(z) = \sum_{j=0}^{m} a_j z^j = a_m(z-\lambda_1)^{k_1}(z-\lambda_2)^{k_2}\cdots(z-\lambda_r)^{k_r},$$

then the function

$$u(x) = \sum_{q=1}^{r}\sum_{l=0}^{k_q-1} c_{ql}x^l e^{\lambda_q x}$$

satisfies $Lu = 0$, for any choice of the constants $c_{ql}$. To prove that this $u$ is the general solution, it remains to show that the functions $x^l e^{\lambda_q x}$ are linearly independent. Thus, suppose that

$$\sum_{q=1}^{r}\sum_{l=0}^{k_q-1} c_{ql}x^l e^{\lambda_q x} = 0 \quad \text{for all } x \in [a,b]; \tag{5}$$

we will use the following lemmas to show that $c_{ql} = 0$ for all $q$ and $l$.

**Lemma 6.** $(D-\lambda)x^j e^{\mu x} = (\mu-\lambda)x^j e^{\mu x} + jx^{j-1}e^{\mu x}$.

*Proof.* A direct calculation shows that

$$\begin{aligned}(D-\lambda)x^j e^{\mu x} &= jx^{j-1}e^{\mu x} + x^j\mu e^{\mu x} - \lambda x^j e^{\mu x}\\ &= (\mu-\lambda)x^j e^{\mu x} + jx^{j-1}e^{\mu x}.\end{aligned}$$

□

**Lemma 7.** *There exist constants $c_{kjl}$ such that*

$$(D-\lambda)^k x^j e^{\mu x} = \left((\mu-\lambda)^k x^j + \sum_{q=0}^{k-1}\sum_{l=0}^{j-1} c_{kjql}(\mu-\lambda)^q x^l\right)e^{\mu x}.$$

*Proof.* We use induction on $k$. If $k = 0$, then the formula reduces to $x^j e^{\mu x} = x^j e^{\mu x}$, which is obviously true.

Assume now that the formula holds for some $k \geq 0$. Using Lemma **??**,

$$\begin{aligned}(D-\lambda)^{k+1}x^j e^{\mu x} &= (D-\lambda)\left((\mu-\lambda)^k x^j + \sum_{q=0}^{k-1}\sum_{l=0}^{j-1} c_{kjql}(\mu-\lambda)^q x^l\right)e^{\mu x}\\ &= (\mu-\lambda)^{k+1}x^j e^{\mu x} + (\mu-\lambda)^k jx^{j-1}e^{\mu x}\\ &\quad + \sum_{q=0}^{k-1}\sum_{l=0}^{j-1} c_{kjql}(\mu-\lambda)^q\big[(\mu-\lambda)x^l + lx^{l-1}\big]e^{\mu_q x},\end{aligned}$$

and therefore

$$(D-\lambda)^{k+1}x^j e^{\mu x} = \left((\mu-\lambda)^{k+1}x^j e^{\mu x} + \sum_{q=0}^{k}\sum_{l=0}^{j-1} c_{k+1,jql}(\mu-\lambda)^q x^l\right)e^{\mu x},$$

for appropriately chosen $c_{k+1,jql}$. $\square$

Let $p_1(z) = p(z)/(z-\lambda_1)^{k_1}$, so that from the results in lectures,

$$p_1(D)x^l e^{\lambda_q x} = 0 \quad \text{for } 2 \le q \le r \text{ and } 0 \le l \le k_q - 1.$$

However, by Lemma **??**, in the case $q = 1$,

$$p_1(D)x^l e^{\lambda_1 x} = \big[p_1(\lambda_1)x^l + \phi_l(x)\big]e^{\lambda_1 x},$$

where $\phi_l$ is a polynomial of degree at most $l-1$. Thus, applying $p_1(D)$ to (**??**) gives

$$\sum_{l=0}^{k_1-1} c_{1l}\big[p_1(\lambda_1)x^l + \phi_l(x)\big] = 0 \quad \text{for all } x \in [a,b].$$

Here, the coefficient of $x^{k_1-1}$ is $c_{1,k_1-1}p_1(\lambda_1)$, and since $p_1(\lambda_1) \ne 0$ we conclude that $c_{1,k_1-1} = 0$. In the same way, $c_{1l} = 0$ for $l = k_1-2$, $k_1-2$, …, 1.

Defining $p_2(z) = p(z)/(z-\lambda_2)^{k_2}$ and arguing in the same way, we see that $c_{2l} = 0$ for $0 \le l \le k_2 - 1$. Continuing in this fashion, we may conclude that $c_{ql} = 0$ for all $q$ and $l$, and hence that the functions $x^l e^{\lambda_q x}$ are linearly independent, as required.

# 3 Norm of a Bessel function

Recall the following theorem from lectures.

**Theorem 8.** *Assume that $c_0c_1 \ge 0$ and let $\phi_j(x) = J_\nu(k_j x)$ denote the $j$th Bessel eigenfunction. The $\phi_j$ have the orthogonality property*

$$\int_0^\ell \phi_i(x)\phi_j(x)x\,dx = 0 \quad \textit{if } i \ne j. \tag{6}$$

*Moreover, if $c_1 \ne 0$, then*

$$\int_0^\ell \phi_j(x)^2 x\,dx = \frac{1}{2k_j^2}\left[\left(\frac{\ell c_0}{c_1}\right)^2 + (k_j\ell)^2 - \nu^2\right]J_\nu(k_j\ell)^2 \quad \textit{for } j \ge 1,$$

*whereas if $c_1 = 0$, so that $\phi_j(\ell) = 0$ by (**??**), then*

$$\int_0^\ell \phi_j(x)^2 x\,dx = \frac{\ell^2}{2}\,J_{\nu+1}(k_j\ell)^2 \quad \textit{for } j \geq 1.$$

*In the case $c_0 = 0 = \nu$, we have $\int_0^\ell \phi_0(x)^2 x\,dx = \ell^2/2$.*

*Proof.* We proved the orthogonality property (**??**) in lectures. Recall also that $u = \phi_j(x)$ and $\lambda = \lambda_j = k_j^2$ satisfy

$$(xu')' + (\lambda x - \nu^2 x^{-1} u = 0 \quad \text{for } 0 \leq x \leq \ell, \tag{7}$$

and

$$\begin{gathered} u(x) \text{ bounded and } xu'(x) \to 0 \text{ as } x \to 0^+, \\ c_1 u' + c_0 u = 0 \text{ at } x = \ell. \end{gathered} \tag{8}$$

Multiply both sides of (**??**) by $xu'$ to obtain

$$\begin{gathered} (xu')'(xu') + (\lambda x^2 - \nu^2)uu' = 0, \\ \frac{d}{dx}\frac{(xu')^2}{2} + (\lambda x^2 - \nu^2)\,\frac{d}{dx}\,\frac{u^2}{2} = 0, \end{gathered}$$

and then multiply by 2 and integrate:

$$\big[(xu')^2\big]_0^\ell + \lambda \int_0^\ell x^2(u^2)'\,dx - \nu^2\big[u^2\big]_0^\ell = 0.$$

Integration by parts gives

$$\int_0^\ell x^2(u^2)'\,dx = \big[x^2u^2\big]_0^\ell - \int_0^\ell 2xu^2\,dx,$$

so

$$2\lambda \int_0^\ell u^2 x\,dx = \big[(xu')^2 + \lambda(xu)^2 - (\nu u)^2\big]_0^\ell.$$

Since the boundary conditions (**??**) imply that $xu' \to 0$ and $xu \to 0$ as $x \to 0$, and

$$\nu u(0) = \nu J_\nu(0) = 0 \quad \text{for all } \nu \geq 0,$$

we conclude that

$$2\lambda \int_0^\ell u^2 x\,dx = \big(\ell u'(\ell)\big)^2 + (\lambda \ell^2 - \nu^2)u(\ell)^2.$$

If $c_1 \neq 0$ then $u'(\ell) = -c_0 u(\ell)/c_1$ implying the formula for $\int_0^\ell \phi_j(x)^2 x\,dx$.

If $c_1 = 0$ then $u(\ell) = 0$ and hence $u = J_\nu(k_j x)$ satisfies

$$2k_j^2 \int_0^\ell u^2 x\, dx = \big(\ell u'(\ell)\big)^2 = \big(k_j \ell J_\nu'(k_j \ell)\big)^2.$$

We saw in one of the tutorial problems that $(x^{-\nu} J_\nu)' = -x^{-\nu} J_{\nu+1}$, so

$$\begin{aligned} -\nu x^{-\nu-1} J_\nu(x) + x^{-\nu} J_\nu'(x) &= -x^{-\nu} J_{\nu+1}(x), \\ -\nu J_\nu(x) + x J_\nu'(x) &= -x J_{\nu+1}(x), \end{aligned}$$

and $J_\nu(k_j \ell) = u(\ell) = 0$, so putting $x = k_j \ell$ gives $J_\nu'(k_j \ell) = -J_{\nu+1}(k_j \ell)$, giving the desired formula in this case also.

Finally, when $c_0 = 0 = \nu$ and $\phi_0(x) = 1$,

$$\int_0^\ell \phi_0(x) x\, dx = [x^2/2]_0^\ell = \ell^2/2.$$

$\square$

# 4 Reference

These notes are based on parts of


Garrett Birkhoff and Gian-Carlo Rota, *Ordinary Differential Equations*, Blaisdell Publishing Company, 1969. **PX517.382/11N**