The University of Sydney
School of Mathematics and Statistics

MATH1901/1906
Differential Calculus (Advanced)

June 2015

Lecturer: J Parkinson

Time Allowed: One and a half hours

Family Name: ....

Other Names: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Approved non-programmable calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Marker's use only	

Extended Answer Section

*There are **four** questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

1. (a) (*i*) Write the complex number $2 + 2\sqrt{3}\, i$ in polar form.

(*ii*) Find all solutions $z \in \mathbb{C}$ to the equation

$$z^4 = 2 + 2\sqrt{3}\, i,$$

expressing your final answers in polar form.

(b) Find all solutions $z \in \mathbb{C}$ of the equation

$$e^{2z} - 1 = i,$$

expressing your final answers in Cartesian form.

(c) Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = iz^2 + 3z$ and let $A = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = 1\}$. Sketch the image of A under f in the complex plane.

(d) Use the ϵ, δ definition of limits to show that

$$\lim_{x \to 2}(2x - 3) = 1.$$

QUESTION 2 BEGINS ON THE NEXT PAGE

2. (a) Calculate the following limits, or show that they do not exist, showing all of your working. You may use any valid method.

(*i*) $\displaystyle\lim_{x\to 0} \frac{\ln(1+3x)}{x(2+x^2)}$

(*ii*) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy+|y|}{x^2+|y|}$

(*iii*) $\displaystyle\lim_{x\to 2} \frac{\sqrt{x+2}-2\sqrt{x}}{x-2}$

(*iv*) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 \sin y}{x^4+y^4}$

(b) Let $f(x, y) = 1 - 2x + 6y + \sinh(3 - 2x + y)$. Find the equation of the tangent to the level curve $f(x, y) = 3$ at the point $(x, y) = (2, 1)$.

(c) Let $g(x, y) = \sin(x^2 - y) + 4xy + 3$.

(i) Find the tangent plane to the graph $z = g(x, y)$ at the point $(x, y) = (2, 4)$.

(ii) What is the direction of the steepest slope of the graph $z = g(x, y)$ at the point $(x, y) = (2, 4)$.

(iii) What is the slope of the graph $z = g(x, y)$ in the direction $\mathbf{i} + 3\mathbf{j}$ at the point $(x, y) = (2, 4)$?

QUESTION 3 BEGINS ON THE NEXT PAGE

3. (a) (i) State Rolle's Theorem.

(ii) Use Rolle's Theorem to show that if $h : [a, b] \to \mathbb{R}$ is continuous with $h'(x) \neq 0$ for all $x \in (a, b)$ then the function h is injective.

(b) Suppose that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable everywhere, and that $f'(x) = g'(x)$ for all $x \in \mathbb{R}$. Use the Mean Value Theorem to prove that $f(x) = g(x) + k$ for all $x \in \mathbb{R}$, where k is a constant.

(c) Let $f(x) = \sqrt{1+x}$.

(i) Calculate the second order Taylor polynomial $T_2(x)$ for $f(x)$ about $x = 0$.

(ii) Write down a formula for the remainder term $R_2(x) = f(x) - T_2(x)$.

(iii) Hence show that

$$1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 \le \sqrt{1+x^4} \le 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12}$$

for all $x \in \mathbb{R}$.

(iv) Hence, or otherwise, compute the limit

$$\lim_{x\to 0} \frac{2\sqrt{1+\sin^4 x} - 2 - \sin^4 x}{\sin^8 x}.$$

Question 4 begins on the next page

4. (a) Let $f(x) = x^3 - 3x + 1$.

(i) Show that the function $f : [-1, 1] \to [-1, 3]$ is bijective.

(ii) Let $f^{-1} : [-1, 3] \to [-1, 1]$ be the inverse of the function $f : [-1, 1] \to [-1, 3]$. Calculate the third order Taylor polynomial of $f^{-1}(x)$ centred at $x = 1$.

(b) You are given that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(a+b) = \frac{f(a)f(b)}{2} \quad \text{for all } a, b \in \mathbb{R},$$

and that f is differentiable at $x = 0$ with $f'(0) = 5$.

(i) Calculate $f(0)$.

(ii) Show that f is differentiable everywhere.

(iii) Find an explicit formula for the function $f(x)$.

Standard Derivatives

The following derivatives can be quoted without proof unless a question specifically asks you to show details. These results can be combined with the standard rules of differentiation (not listed here) to differentiate more complicated functions. For example, $(d/dx)\sin(ax+b) = a\cos(ax+b)$. Natural domains common to both sides are assumed.

1. $\dfrac{d}{dx}\, x^k \;=\; kx^{k-1} \quad (k \in \mathbb{R})$

2. $\dfrac{d}{dx}\, e^x \;=\; e^x$

3. $\dfrac{d}{dx}\, \ln x \;=\; \dfrac{1}{x} \quad (x > 0)$

4. $\dfrac{d}{dx}\, \sin x \;=\; \cos x$

5. $\dfrac{d}{dx}\, \cos x \;=\; -\sin x$

6. $\dfrac{d}{dx}\, \tan x \;=\; \sec^2 x$

7. $\dfrac{d}{dx}\, \cot x \;=\; -\operatorname{cosec}^2 x$

8. $\dfrac{d}{dx}\, \sec x \;=\; \sec x \tan x$

9. $\dfrac{d}{dx}\, \operatorname{cosec} x \;=\; -\operatorname{cosec} x \cot x$

10. $\dfrac{d}{dx}\, \sinh x \;=\; \cosh x$

11. $\dfrac{d}{dx}\, \cosh x \;=\; \sinh x$

12. $\dfrac{d}{dx}\, \tanh x \;=\; \operatorname{sech}^2 x$

13. $\dfrac{d}{dx}\, \sin^{-1} x \;=\; \dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$

14. $\dfrac{d}{dx}\, \cos^{-1} x \;=\; -\dfrac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$

15. $\dfrac{d}{dx}\, \tan^{-1} x \;=\; \dfrac{1}{1+x^2}$

16. $\dfrac{d}{dx}\, \sinh^{-1} x \;=\; \dfrac{1}{\sqrt{1+x^2}}$

17. $\dfrac{d}{dx}\, \cosh^{-1} x \;=\; \dfrac{1}{\sqrt{x^2-1}} \quad (x > 1)$

18. $\dfrac{d}{dx}\, \tanh^{-1} x \;=\; \dfrac{1}{1-x^2} \quad (|x| < 1)$

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.