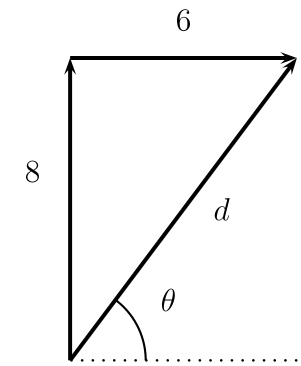
THE UNIVERSITY OF SYDNEY

MATH1902 LINEAR ALGEBRA (ADVANCED)

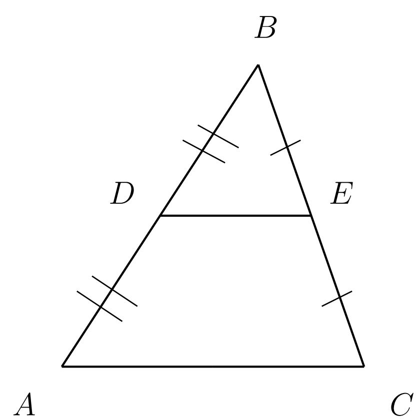
Semester 1 **Longer Solutions to Selected Exercises for Week 1** 2017

11.



By Pythagoras $d = \sqrt{8^2 + 6^2} = 10$. If θ is the angle to the horizontal then $\cos\theta = 6/10$, yielding an angle $\theta \approx 53°$. Thus the resultant force is 10 newtons in a direction 53° to the horizontal, towards the right.

12.



Observe that

$$\overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \frac{1}{2}\overrightarrow{AC}\,.$$

This tells us that the line segment joining D to E is parallel to and half the length of the line segment joining A to C.

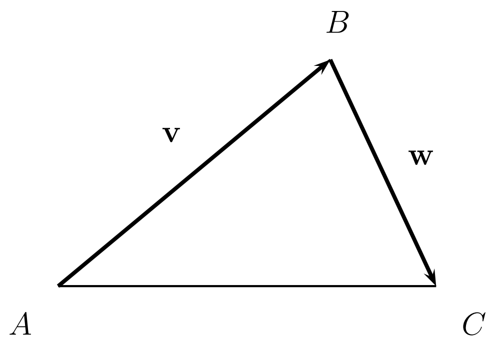
13. The associative law for addition of vectors says that, for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$,

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}\,.$$

To verify this, we suppose that the vectors have been lined up so that the point P is at the tail of $\mathbf{u}$, the point Q is both at the tip of $\mathbf{u}$ and at the tail of $\mathbf{v}$, the point R is both at the tip of $\mathbf{v}$ and at the tail of $\mathbf{w}$, and the point S is at the tip of $\mathbf{w}$. Then

$$\begin{aligned} \mathbf{u} + (\mathbf{v}+\mathbf{w}) &= \overrightarrow{PQ} + \left(\overrightarrow{QR} + \overrightarrow{RS}\right) = \overrightarrow{PQ} + \overrightarrow{QS} \\ &= \overrightarrow{PS} \\ &= \overrightarrow{PR} + \overrightarrow{RS} = \left(\overrightarrow{PQ} + \overrightarrow{QR}\right) + \overrightarrow{RS} = (\mathbf{u}+\mathbf{v}) + \mathbf{w} \, . \end{aligned}$$

To explain the Triangle Inequality, consider the following diagram:



The vectors $\mathbf{v}$ and $\mathbf{w}$ have been placed tip-to-tail so that they label two directed edges of a triangle ABC, so that

$$\mathbf{v} = \overrightarrow{AB} \, , \qquad \mathbf{w} = \overrightarrow{BC} \, .$$

Then $\mathbf{v}+\mathbf{w} = \overrightarrow{AC}$. The shortest distance between two points is a straight line, so that travelling from A to C via B is at least as far as travelling directly from A to C. Thus

$$|\mathbf{v}+\mathbf{w}| = \left|\overrightarrow{AC}\right| \le \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right| = |\mathbf{v}| + |\mathbf{w}| \, ,$$

which verifies the triangle inequality. This becomes equality precisely when B falls on the direct path joining A to C, so that the triangle becomes degenerate.

14. Later in the course we learn a general method for solving systems of equations, called Gaussian elimination. For this problem, we can find a quick ad hoc solution, by eliminating z, by taking multiples of the first two equations

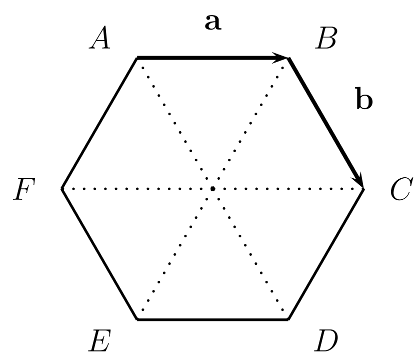
$$\begin{array}{rcrcrcr} 6\,x &+& 9\,y &+& 12\,z &=& -12 \\ 10\,x &+& 10\,y &+& 12\,z &=& -6 \end{array}$$

and subtracting one from the other to get $4x+y = 6$. By subtracting twice the original third equation from the first, we get $-4x + y = -2$. Solving these two equations in x and y quickly yields $x = 1$ and $y = 2$. From any of the original equations, we find $z = -3$. Thus the intersection point is $(1, 2, -3)$.

15. To find the reflected line is a simple three part process: (i) we first translate the line $ax + by = c$ vertically $-k$ units to get the line $ax + b(y + k) = c$, which becomes $ax + by = c - bk$; (ii) we then reflect this line in $y = x$, yielding $bx + ay = c - bk$; (iii) we finish the job by translating this last line vertically k units, to get the line $bx + a(y - k) = c - bk$, that is

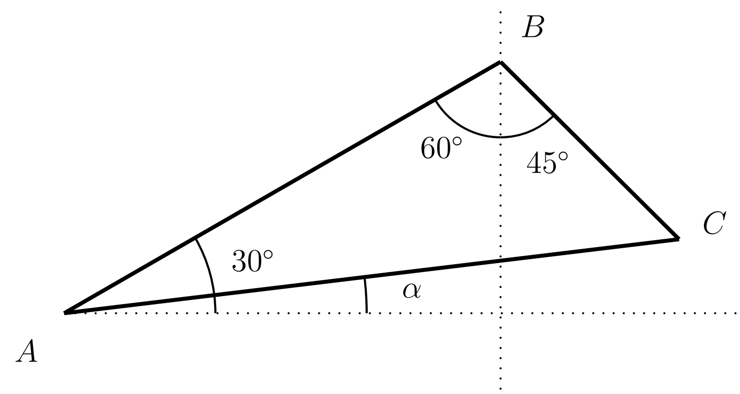
$$bx + ay = c + k(a - b) \, .$$

17.



$$\overrightarrow{CD} \;=\; \mathbf{b}-\mathbf{a}\,, \quad \overrightarrow{DE} \;=\; -\mathbf{a}\,, \quad \overrightarrow{EF} \;=\; -\mathbf{b}\,, \quad \overrightarrow{FA} \;=\; \mathbf{a}-\mathbf{b}\,.$$

18.



We have $|\overrightarrow{AB}| = 20$ and $|\overrightarrow{BC}| = 10$. By the Cosine Rule,

$$|\overrightarrow{AC}| \;=\; \sqrt{20^2+10^2-2(10)(20)\cos 105^\circ} \;\approx\; 25\,.$$

By the Sine Rule,

$$\sin(30^\circ-\alpha) \;=\; \frac{10\sin 105^\circ}{|\overrightarrow{AC}|}\,,$$

from which it follows that

$$30^\circ-\alpha \;\approx\; 23^\circ\,,$$

so that $\alpha \approx 7^\circ$. Hence the final distance and direction of the aircraft from the starting point are approximately 25 km and 7° north of east respectively.

19.

$$\overrightarrow{AD} \;=\; \overrightarrow{AB}+\overrightarrow{BD} \;=\; \overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC} \;=\; \overrightarrow{AB}+\frac{1}{2}\big(\overrightarrow{BA}+\overrightarrow{AC}\big)$$

$$\;=\; \overrightarrow{AB}+\frac{1}{2}\big(-\overrightarrow{AB}+\overrightarrow{AC}\big) \;=\; \frac{1}{2}\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AC} \;=\; \frac{1}{2}\big(\overrightarrow{AB}+\overrightarrow{AC}\big)\,.$$

20. Let $PQRS$ be a parallelogram and T the midpoint of the diagonal PR. Then

$$\overrightarrow{QT} \;=\; \overrightarrow{QP}+\overrightarrow{PT} \;=\; \overrightarrow{QP}+\frac{1}{2}\overrightarrow{PR} \;=\; \overrightarrow{QP}+\frac{1}{2}(\overrightarrow{PQ}+\overrightarrow{QR})$$

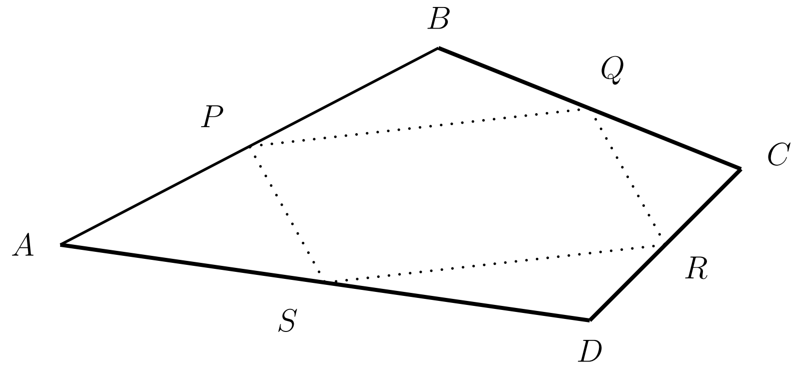
$$\;=\; \overrightarrow{QP}+\frac{1}{2}(-\overrightarrow{QP}+\overrightarrow{QR}) = \frac{1}{2}(\overrightarrow{QP}+\overrightarrow{QR}) \;=\; \frac{1}{2}(\overrightarrow{RS}+\overrightarrow{QR}) \;=\; \frac{1}{2}\overrightarrow{QS}\,,$$

which proves that T is the midpoint of the diagonal QS, so that the diagonals bisect each other.

21. Let P, Q, R, S be the respective midpoints of the edges AB, BC, CD, DA of the quadrilateral $ABCD$. Then, by two applications of Exercise 12, firstly to the triangle ABC, and then secondly to the triangle ADC,

$$\overrightarrow{PQ} \;=\; \frac{1}{2}\overrightarrow{AC} \;=\; \overrightarrow{SR}\,,$$

which is sufficient to prove that $PQRS$ is a parallelogram.



22. Consider a triangle ABC such that X is the midpoint of BC, Y the midpoint of AC and Z the midpoint of AB. The medians are represented by the vectors $\overrightarrow{AX}$, $\overrightarrow{BY}$ and $\overrightarrow{CZ}$. But, using Exercise 19, the vector sum is

$$\begin{aligned}
\overrightarrow{AX}+\overrightarrow{BY}+\overrightarrow{CZ} \;&=\; \frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC})+\frac{1}{2}(\overrightarrow{BA}+\overrightarrow{BC})+\frac{1}{2}(\overrightarrow{CA}+\overrightarrow{CB})\\
&=\; \frac{1}{2}(\overrightarrow{AB}+\overrightarrow{BA}+\overrightarrow{BC}+\overrightarrow{CB}+\overrightarrow{AC}+\overrightarrow{CA})\\
&=\; \frac{1}{2}(\mathbf{0}+\mathbf{0}+\mathbf{0}) \;=\; \mathbf{0}\,,
\end{aligned}$$

which proves that the medians can be shifted parallel to themselves to form another triangle.

23. To find the rotated line is a simple three part process: (i) we first translate the line $ax+by=c$ horizontally $-x_0$ units and vertically $-y_0$ units, to get the line

$$a(x+x_0)+b(y+y_0) \;=\; c\,,$$

which becomes

$$ax+by=c-ax_0-by_0\,;$$

(ii) we then rotate this line ninety degrees anticlockwise about the origin, yielding

$$-bx+ay=c-ax_0-by_0\,;$$

(iii) we finish the job by translating this last line horizontally x_0 units and vertically y_0 units, to get the line

$$-b(x-x_0)+a(y-y_0)=c-ax_0-by_0\,,$$

that is

$$-bx+ay \;=\; c+a(y_0-x_0)-b(x_0+y_0)\,.$$

24. A quick solution uses similar triangles: Let $r = |ST|$ and $s = |TR|$ so that $|PQ| = |SR| = r + s$. But $\triangle PQU$ is similar to $\triangle TRU$, so corresponding ratios of lengths of sides are equal:

$$\frac{|PU|}{|UR|} = \frac{|PU|}{|UR|} = \frac{r+s}{s} ,$$

so that the ratio in which U divides the diagonal is $r + s : s$.

An alternative solution using vectors does not require any geometric insight: suppose that α and β are scalars such that $\overrightarrow{PU} = \alpha\overrightarrow{PR}$ and $\overrightarrow{QU} = \beta\overrightarrow{QT}$. Then

$$\begin{aligned} \alpha(\overrightarrow{PS} + \overrightarrow{SR}) &= \alpha\overrightarrow{PR} = \overrightarrow{PU} = \overrightarrow{PQ} + \overrightarrow{QU} = \overrightarrow{PQ} + \beta\overrightarrow{QT} \\ &= \overrightarrow{PQ} + \beta(\overrightarrow{QP} + \overrightarrow{PS} + \overrightarrow{ST}) = \overrightarrow{PQ} + \beta(\overrightarrow{QP} + \overrightarrow{PS} + \frac{r}{r+s}\overrightarrow{SR}) , \end{aligned}$$

so that, rearranging, $(\alpha - \beta)\overrightarrow{PS} = (1 - \alpha - \beta + \beta\frac{r}{r+s})\overrightarrow{PQ}$. But $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ are not parallel so the scalar coefficients must be zero:

$$\alpha - \beta = (1 - \alpha - \beta + \beta\frac{r}{r+s}) = 0 ,$$

from which it follows quickly that $\alpha = \beta = \dfrac{r+s}{r+2s}$, so that the ratio in which U divides the diagonal is $r + s : s$.

25. The mindreader always produces the number 0 by the instructions. To see this let x be any positive integer, so that X becomes the integer part of $\sqrt{2x} + \frac{1}{2}$ and Y becomes the xth number along the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6$$

We want to prove that $X = Y$, so that when we take Y away from X we get zero. Observe that

$$1 + 2 + \ldots + (Y - 1) + 1 \leq x \leq 1 + 2 + \ldots + Y$$

giving

$$\frac{(Y-1)Y}{2} + 1 \leq x \leq \frac{Y(Y+1)}{2}$$

whence

$$Y^2 - Y + 2 \leq 2x \leq Y^2 + Y .$$

Thus

$$\left(Y - \frac{1}{2}\right)^2 + \frac{7}{4} \leq 2x < \left(Y + \frac{1}{2}\right)^2$$

so

$$Y - \frac{1}{2} \leq \sqrt{2x - \frac{7}{4}} < \sqrt{2x} < Y + \frac{1}{2} ,$$

whence

$$Y < \sqrt{2x} + \frac{1}{2} < Y + 1 .$$

Thus if we throw away everything to the right of the decimal point and the decimal point, in the decimal expansion of $\sqrt{2x} + \frac{1}{2}$, we must be left with the integer Y, that is, $X = Y$, voila!