5. By the formula for 2×2 matrices, the inverse of $\begin{bmatrix} 5 & -3 \\ 7 & -4 \end{bmatrix}$ is $\begin{bmatrix} -4 & 3 \\ -7 & 5 \end{bmatrix}$, so that

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 11 & 4 \\ 15 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix}.$$

6. The matrix A must have some size, say $r \times s$, and B some size, say $t \times u$. Then $AB = I_n$ is both $r \times u$ and $n \times n$, whilst $BA = I_n$ is both $t \times s$ and $n \times n$, so

$$r = u = t = s = n\,,$$

which shows A and B are both $n \times n$.

7. (ii) The inverse of $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ does not exist because its determinant is $6(1) - 2(3) = 0$.

(iv) $$\left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

(v) $$\left[\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 7 & 11 & 6 & 0 & 1 & 0 \\ -6 & -6 & 12 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/2 & 0 & 0 \\ 0 & -3 & -15 & -7/2 & 1 & 0 \\ 0 & 6 & 30 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/2 & 0 & 0 \\ 0 & -3 & -15 & -7/2 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array}\right],$$

so the matrix is not invertible.

(vi) $$\left[\begin{array}{ccc|ccc} -4 & 3 & 3 & 1 & 0 & 0 \\ 8 & 7 & 3 & 0 & 1 & 0 \\ 4 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -4 & 3 & 3 & 1 & 0 & 0 \\ 0 & 13 & 9 & 2 & 1 & 0 \\ 0 & 6 & 6 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -3/4 & -3/4 & -1/4 & 0 & 0 \\ 0 & 1 & 1 & 1/6 & 0 & 1/6 \\ 0 & 0 & -4 & -1/6 & 1 & -13/6 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/8 & 0 & 1/8 \\ 0 & 1 & 0 & 1/8 & 1/4 & -3/8 \\ 0 & 0 & 1 & 1/24 & -1/4 & 13/24 \end{array}\right]$$

8. Suppose that A is $m \times m$ and D is $n \times n$. For $ABD = ACD$ to be sensibly defined, both B and C are $m \times n$. It does not matter if m and n are different: since $A^{-1}A = I_m$ and $DD^{-1} = I_n$, we have

$$B = I_m B I_n = A^{-1}ABDD^{-1} = A^{-1}ACDD^{-1} = I_m C I_n = C\,.$$

9. Observe that

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 3 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & -1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right],$$

so the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 3 \end{bmatrix}$ is $\begin{bmatrix} 6 & -1 & -1 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$. Observe also that

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

so that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 & -1 & -1 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -4 \end{bmatrix}.$$

10. Let A, B be square matrices of the same size. If A has a row of zeros then AB also has a row of zeros. If A has a column of zeros the BA also has a columns of zeros. In either case it is impossible to have $AB = BA = I$.

11. Suppose that E is an elementary matrix corresponding to the elementary row operation ρ. Let the inverse operation of ρ be called σ. In all possible cases, σ is itself an elementary row operation:

(i) if $\rho : R_i \leftrightarrow R_j$ then $\rho = \sigma$;

(ii) if $\rho : R_i \rightarrow \lambda R_i$ where $\lambda \neq 0$ then $\sigma : R_i \rightarrow \frac{1}{\lambda} R_i$;

(iii) if $\rho : R_i \rightarrow R_j + \lambda R_i$ then $\sigma : R_i \rightarrow R_j + (-\lambda) R_i$.

Denote by F the elementary matrix corresponding to σ. But E is the effect of ρ on I, so the effect of σ on E must be I, so $FE = I$. Hence $E^{-1} = F$ is elementary.

12. (i) This is false. For example take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$(ABC)^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

yet

$$A^{-1}B^{-1}C^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(ii) This is true, since $(ABA)^{-1} = A^{-1}(AB)^{-1} = A^{-1}B^{-1}A^{-1}$.

(iii) This is true. By uniqueness of inverses, since $A^{-1}A = AA^{-1} = I$, we have immediately that $(A^{-1})^{-1} = A$.

(iv) This is true. Observe that

$$(-A)(-A^{-1}) \;=\; (-1)(-1)AA^{-1} \;=\; AA^{-1} \;=\; I$$

and

$$(-A^{-1})(-A) \;=\; (-1)(-1)A^{-1}A \;=\; A^{-1}A \;=\; I\,,$$

so that, by uniqueness of inverses, $(-A)^{-1} = -A^{-1}$, yielding

$$-(-A)^{-1} = -(-A^{-1}) = A^{-1}\,.$$

(v) This is true, since $C^{-1}(ABC^{-1})^{-1}AB = C^{-1}(C^{-1})^{-1}B^{-1}A^{-1}AB = I$.

(vi) This is false even for 1×1 matrices, since $(A+B)^{-1}$ may not exist. For example, take $A = 1$ and $B = -1$, so that $A + B = 0$ has no inverse. Even when $(A+B)^{-1}$ exists, the statement is typically false. For example, take $A = B = 1$, so that $(A + B)^{-1} = 1/2 \neq 2 = A^{-1} + B^{-1}$.

(vii) This is true, since $A^{-1}(I + A)A = A^{-1}IA + A^{-1}AA = I + A = A + I$.

(viii) This is true, since $(A + I)(A^{-1} - I) = AA^{-1} - A + A^{-1} - I = A^{-1} - A$.

(ix) This is true, since

$$\begin{aligned} A^2 - 2A + I = 0 \quad &\Longrightarrow \quad 2A - A^2 = I \\ &\Longrightarrow \quad A(2I - A) = (2I - A)A = I \\ &\Longrightarrow \quad A^{-1} = 2I - A\,. \end{aligned}$$

(x) This is false. For example, take $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \neq I$, yet

$$A^2 - 2A + I \;=\; \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;=\; 0\,.$$

13. Observe that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \;\sim\; \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \;\sim\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

using elementary row operations ρ_1 and ρ_2, in that order, that correspond to elementary matrices

$$E_1 \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad E_2 \;=\; \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

respectively. We have

$$E_2E_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \;=\; E_1^{-1}E_2^{-1} \;=\; E_1E_2 \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Observe that

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

using elementary row operations ρ_1, ρ_2, ρ_3, ρ_4, in that order, that correspond to elementary matrices

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

respectively. We have

$$E_4E_3E_2E_1 \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$\begin{aligned} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}. \end{aligned}$$

14. Observe that

$$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 1 & 2 \\ -3 & -4 & \lambda \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -5 & 11 \\ 0 & -10 & \lambda+9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -5 & 11 \\ 0 & 0 & \lambda-13 \end{bmatrix},$$

so that the method of finding the inverse by row reduction fails if and only if $\lambda = 13$.

15. Observe first that, for the dimensions to match, B must be $n \times n$ and C must be $m \times m$. Consider $i \in \{1, \ldots, n\}$, and choose, as an instance of A, the case where all entries of A are 0 except for the $(1, i)$-entry which is 1. Then the first row of $A = AB$ becomes the ith row of B. Comparing entries shows that the (i, j)-entry of B is 1 if and only if $i = j$. Since i and j can be chosen arbitrarily, this proves $B = I_n$. The same argument using columns in place of rows proves $C = I_m$.

16. Observe that $(5M)^{-1} = \begin{bmatrix} 5 & 6 \\ 5 & 5 \end{bmatrix}$ so that $5M = \begin{bmatrix} 5 & 6 \\ 5 & 5 \end{bmatrix}^{-1} = -\frac{1}{5}\begin{bmatrix} 5 & -6 \\ -5 & 5 \end{bmatrix}$, yielding

$$M = -\frac{1}{25}\begin{bmatrix} 5 & -6 \\ -5 & 5 \end{bmatrix} = \begin{bmatrix} -1/5 & 6/25 \\ 1/5 & -1/5 \end{bmatrix}.$$

17. Observe that

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 18 & 7 & -5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/18 & 1/18 & 7/18 \\ 0 & 1 & 0 & 1/18 & 7/18 & -5/18 \\ 0 & 0 & 1 & 7/18 & -5/18 & 1/18 \end{array}\right]$$

so the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ is $\dfrac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$. Observe also that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

so that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{18}\begin{bmatrix} -5a+b+7c \\ a+7b-5c \\ 7a-5b+c \end{bmatrix}.$$

18. If any of the diagonal entries is zero, then the matrix has a row of zeros so is not invertible. If all of the diagonal entries are nonzero then

$$\left[\begin{array}{cccc|cccc} d_1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n & 0 & 0 & \cdots & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & d_1^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & d_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & d_n^{-1} \end{array}\right]$$

so that the inverse exists and is the diagonal matrix with reciprocals down the diagonal.

19. Observe that

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ -2 & -3 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ -2 & -3 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

using elementary row operations ρ_1, ρ_2, ρ_3, ρ_4 in that order, that correspond to elementary matrices

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

respectively. We have

$$F_4F_3F_2F_1\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ -2 & -3 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $A = E_1E_2E_3E_4B$ where

$$E_1 = F_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = F_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_3 = F_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$E_4 = F_4^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

20. If $n = 1$ then $I - J = 1 - 1 = 0$ which is not invertible. Suppose $n \geq 2$. Then $J^2 = nJ$, so that

$$(I-J)\left(I - \frac{1}{n-1}J\right) \;=\; I - \frac{1}{n-1}J - J + \frac{1}{n-1}J^2 \;=\; I - \frac{n}{n-1}J + \frac{n}{n-1}J \;=\; I\,,$$

and similarly $\left(I - \dfrac{1}{n-1}J\right)(I-J) = I$, so that $(I-J)^{-1} = I - \dfrac{1}{n-1}J$.

21. (i) Observe that $\begin{bmatrix} 2-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if and only if $\lambda \neq 2$ and $\lambda \neq -3$,

so that $A - \lambda I$ is not invertible if and only if $\lambda = 2$ or $\lambda = -3$.

(ii) Observe that

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} \sim \begin{bmatrix} 1 & \lambda-4 \\ 1-\lambda & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \lambda-4 \\ 0 & \lambda^2-5\lambda+6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if and only if $\lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) \neq 0$. Hence $A - \lambda I$ is not invertible if and only if $(\lambda-2)(\lambda-3) = 0$, that is, $\lambda = 2$ or $\lambda = 3$.

(iii) Observe that

$$\begin{bmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ -4 & -4 & 7-\lambda \end{bmatrix} \sim \begin{bmatrix} -4 & -4 & 7-\lambda \\ 0 & 3-\lambda & \lambda-3 \\ -3-\lambda & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & (\lambda-7)/4 \\ 0 & 3-\lambda & \lambda-3 \\ 0 & \lambda+3 & (\lambda^2-4\lambda-13)/4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & (\lambda-7)/4 \\ 0 & 3-\lambda & \lambda-3 \\ 0 & 6 & (\lambda^2-25)/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & (\lambda-7)/4 \\ 0 & 1 & (\lambda^2-25)/24 \\ 0 & 3-\lambda & \lambda-3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & (\lambda-7)/4 \\ 0 & 1 & (\lambda^2-25)/24 \\ 0 & 0 & (\lambda-3)(\lambda-1)(\lambda+1)/24 \end{bmatrix},$$

which can be row reduced to the identity matrix if and only if

$$(\lambda-3)(\lambda-1)(\lambda+1) \neq 0\,.$$

Hence $A - \lambda I$ is not invertible if and only if

$$(\lambda-3)(\lambda-1)(\lambda+1) = 0\,,$$

that is, $\lambda = 3$, 1 or -1.

22. If $A = A^T$ and A^{-1} exists then

$$I \;=\; I^T \;=\; (AA^{-1})^T \;=\; \left(A^{-1}\right)^T A^T \;=\; \left(A^{-1}\right)^T A\,,$$

so that $A^{-1} = \left(A^{-1}\right)^T$. If $A = -A^T$ and A^{-1} exists then

$$I \;=\; I^T \;=\; (AA^{-1})^T \;=\; \left(A^{-1}\right)^T A^T \;=\; \left(A^{-1}\right)^T(-A) \;=\; \left(-\left(A^{-1}\right)^T\right)A\,,$$

so that $A^{-1} = -\left(A^{-1}\right)^T$.

23. Let M be any $n \times n$ matrix. Then M^T is also $n \times n$, so we can form $A = M + M^T$ and $B = M - M^T$. But

$$A^T = (M + M^T)^T = M^T + \left(M^T\right)^T = M^T + M = M + M^T = A\,,$$

and

$$B^T = (M - M^T)^T = M^T - \left(M^T\right)^T = M^T - M = -(M - M^T) = -B\,.$$

Certainly then $\frac{1}{2}A$ is symmetric and $\frac{1}{2}B$ is skew-symmetric, and

$$\frac{1}{2}A + \frac{1}{2}B = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T) = M\,,$$

which shows that M is the sum of a symmetric matrix and a skew-symmetric matrix.

24. No, it is impossible. We argue by contradiction. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} s & t \\ u & v \end{bmatrix}$ are matrices such that

$$A \begin{bmatrix} x & y \\ z & w \end{bmatrix} B = \begin{bmatrix} y & w \\ x & z \end{bmatrix}$$

for all real numbers x, y, z, w. In particular,

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s & t \\ u & v \end{bmatrix} = \begin{bmatrix} as & at \\ cs & ct \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s & t \\ u & v \end{bmatrix} = \begin{bmatrix} au & av \\ cu & cv \end{bmatrix}.$$

In particular, $as = 0$, $cs = 1$ and $au = 1$. The second equation implies that $s \neq 0$ and the third that $a \neq 0$, so that $as \neq 0$, contradicting the first equation.

25. Yes, we can find infinite arrays A and B such that $AB = I$ and $BA \neq I$. For example, let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

the result of adding a column of zeros to the front of I, moving all the other columns along one space, and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

the result of adding a row of zeros to the top of I, moving all the other rows down one space. Then $AB = I$, but

$$BA = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \neq I \ .$$