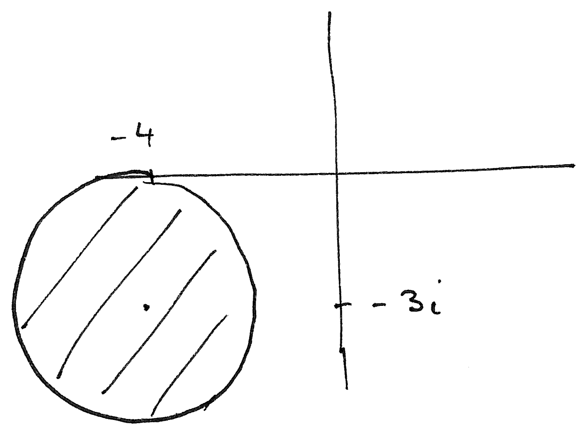
MATH 1901: 2012 Exam Solutions

Question 1:

(a) Circle, radius 3, centre $-4-3i$



(b) $\overline{2+3i} = 2-3i$ is also a root, so $P(z)$ divisible

by $(z-(2+3i))(z-(2-3i)) = (z-2)^2+9$
$= z^2-4z+13.$

Thus
$z^4-3z^3+10z^2+9z+13 = (z^2-4z+13)(z^2+az+b)$
for some a,b. To find a and b, expand RHS and compare coefficients:

coeff of $z^0 \Rightarrow \quad 13 = 13b \Rightarrow b=1$

coeff of $z^3 \Rightarrow \quad -3 = a-4 \Rightarrow a=1.$

So $P(z) = (z^2-4z+13)(z^2+z+1)$
(neither quadratic has real solns, so no further factorisation is possible)

(c) In polar form, $-8 = 8 \operatorname{cis} \pi = 8e^{i\pi}$

The cube roots are the numbers $z = re^{i\theta}$ satisfying

$$z^3 = -8$$

So $$r^3 e^{3i\theta} = 8e^{i\pi}$$

So $r^3 = 8$ and $3\theta = \pi + 2k\pi \quad (k \in \mathbb{Z})$

So $r = 2$ and $\theta = \dfrac{\pi + 2k\pi}{3} \quad (k = 0, 1, 2)$.

$k=0$: $z = 2e^{\frac{i\pi}{3}} = 1 + i\sqrt{3}$

$k=1$: $z = 2e^{i\pi} = -2$

$k=2$: $z = 2e^{i\frac{5\pi}{3}} = 1 - i\sqrt{3}$

(d) $$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \times \frac{1}{1+\cos x}$$

$$= 1^2 \times \tfrac{1}{2} \quad \text{(limit laws)}$$

$$= \tfrac{1}{2}.$$

Question 2.

(a) $f(x,y) = \ln(x^2+3y^2)$

(i) $\hat{\underset{\sim}{u}} = \frac{1}{|\underset{\sim}{u}|}\underset{\sim}{u} = \frac{1}{\sqrt{4^2+1}}\underset{\sim}{u} = \frac{4}{\sqrt{17}}\underset{\sim}{i} - \frac{1}{\sqrt{17}}\underset{\sim}{j}$

$$D_{\hat{u}}f(x,y) = f_x(x,y)\frac{4}{\sqrt{17}} + f_y(x,y)\left(-\frac{1}{\sqrt{17}}\right)$$

$$= \frac{2x}{x^2+3y^2}\,\frac{4}{\sqrt{17}} - \frac{6y}{x^2+3y^2}\,\frac{1}{\sqrt{17}}$$

$$D_{\hat{u}}f(2,1) = \frac{4}{7}\,\frac{4}{\sqrt{17}} - \frac{6}{7}\cdot\frac{1}{\sqrt{17}}$$

$$= \frac{10}{7\sqrt{17}}.$$

(ii) Direction is

$$\underset{\sim}{v} = \nabla f(2,1) = f_x(2,1)\underset{\sim}{i} + f_y(2,1)\underset{\sim}{j}$$

$$= \frac{4}{7}\underset{\sim}{i} + \frac{6}{7}\underset{\sim}{j}$$

So $\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{\frac{4^2}{7^2}+\frac{6^2}{7^2}}}\left(\frac{4}{7}\underset{\sim}{i} + \frac{6}{7}\underset{\sim}{j}\right) = \frac{7}{\sqrt{52}}\left(\frac{4}{7}\underset{\sim}{i} + \frac{6}{7}\underset{\sim}{j}\right)$

$$= \frac{4}{\sqrt{52}}\underset{\sim}{i} + \frac{6}{\sqrt{52}}\underset{\sim}{j} = \frac{2}{\sqrt{13}}\underset{\sim}{i} + \frac{3}{\sqrt{13}}\underset{\sim}{j}$$

The magnitude is

$$|\nabla f(2,1)| = \frac{2\sqrt{13}}{7}$$

(iii) $z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$$= \ln 7 + \frac{4}{7}(x-2) + \frac{6}{7}(y-1)$$

So $\boxed{z = \frac{4}{7}x + \frac{6}{7}y + (\ln 7 - 2)}$

(b) $f(x) = e^{2x}\cos x \Rightarrow f(0) = 1$

$$f'(x) = 2e^{2x}\cos x - e^{2x}\sin x$$

$$= e^{2x}(2\cos x - \sin x) \Rightarrow f'(0) = 2$$

$$f''(x) = 2e^{2x}(2\cos x - \sin x) + e^{2x}(-2\sin x + \cos x)$$

$$= e^{2x}(3\cos x - 4\sin x) \Rightarrow f''(0) = 3$$

$$f'''(x) = 2e^{2x}(3\cos x - 4\sin x) + e^{2x}(-3\sin x - 4\cos x)$$

$$= e^{2x}(2\cos x - 11\sin x) \Rightarrow f'''(0) = 2.$$

So $T_3(x) = 1 + \frac{2}{1!}x + \frac{3}{2!}x^2 + \frac{2}{3!}x^3$

$$\boxed{T_3(x) = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3}$$

Question 3.

(a) (i) $$\lim_{x\to 2} \frac{x^3+5x^2-32x+36}{x^3-12x+16} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 2} \frac{3x^2+10x-32}{3x^2-12} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 2} \frac{6x+10}{6x} = \boxed{\frac{11}{6}}.$$

(ii) $$\lim_{x\to 0} (\cos x)^{\cot^2 x} = \lim_{x\to 0} e^{\cot^2 x \ln(\cos x)}$$

$$= e^{\lim_{x\to 0} \frac{\ln(\cos x)}{\tan^2 x}}$$ } we now see that this is $0/0$ type

$$\lim_{x\to 0} \frac{\ln(\cos x)}{\tan^2 x} = \lim_{x\to 0} \frac{\left(-\frac{\sin x}{\cos x}\right)}{2\tan x \sec^2 x}$$

$$= \lim_{x\to 0} \frac{-\tan x}{2\tan x \sec^2 x}$$

$$= \lim_{x\to 0} \frac{-1}{2\sec^2 x} = -\frac{1}{2}.$$

So limit is $\boxed{e^{-1/2}}.$

(iii) Along $y=0$,

$f(x,0) = 0 \to 0$ as $x \to 0$

Along $y=x$,

$$f(x,x) = \frac{3x^4}{(2x^2)^2} = \frac{3}{4} \to \frac{3}{4}$$

So limit does not exist.

(b) Show that $\sinh x = 2x$ has exactly one solution on interval $[2, 2.5]$, and find interval length 0.1 containing this root.

Let $g(x) = \sinh x - 2x$.

This function is cts and diff'ble everywhere.

$g(2) = \sinh 2 - 4 = -0.373\ldots < 0$

$g(2.5) = \sinh(2.5) - 5 = \cancel{6.05}\ 1.050\ldots > 0$.

So by IVT there is a number $a \in (2, 2.5)$ with $g(a) = 0$.

Suppose that there are two solns;

$$2 < a < b < 2.5.$$

with $g(a) = g(b) = 0$. By Rolle's Thm there is a number $c \in (a,b)$ with

$$g'(c) = 0.$$

But $g'(x) = \cosh x - 2$

$\geqslant \cosh 2 - 2 = 1.7622$ for $x \geqslant 2$

(we use fact that cosh is increasing).

Thus there is at most one solution in the interval $[2, 2.5]$.

To find the subinterval:

$$g(2.0) = -0.37\ldots < 0$$

$$\left.\begin{aligned} g(2.1) &= -0.178\ldots < 0 \\ g(2.2) &= 0.0571\ldots > 0 \end{aligned}\right\} \text{so solution lies in } [2.1, 2.2].$$

Question 4.

$f(x) = \sin x \qquad g(x) = \sin(x^3)$.

(a) $T_4(x) = x - \frac{x^3}{3!} + 0x^4 = x - \frac{x^3}{3!}$

(Here T_4 is taylor poly of f)

So the Taylor poly for $g(x)$ is

$$T_{14}(x) = (x^3) - \frac{(x^3)^3}{3!}$$

$$= x^3 - \frac{x^9}{3!}$$ (the $x^{10}, x^{11}, x^{12}, x^{13}, x^{14}$ terms are zero)

(b) The remainder for $f(x)$ is

$$R_4(x) = \frac{f^{(5)}(c)}{5!} x^5$$ for some c between 0 & x

$$= \frac{\cos c}{120} x^5$$

If $0 < x \leq \frac{\pi}{2}$ then $0 < c < \frac{\pi}{2}$, so

$$0 < \cos c < 1$$

Thus $\cancel{R_4(x) = \frac{\cos c}{120} x^5 < \frac{x^5}{120}}$

$$0 < R_4(x) = \frac{\cos c}{120} x^5 < \frac{x^5}{120}.$$

Replacing x by x^3,

$$0 < R_4(x^3) < \frac{x^{15}}{120}$$

But $R_4(x^3) = R_{14}(x) \leftarrow$ remainder for $g(x)$

because
$$\begin{aligned} g(x) &= T_4(x^3) + R_4(x^3) \\ &= T_{14}(x) + R_4(x^3) \end{aligned}$$

gives $R_4(x^3) = g(x) - T_{14}(x)$

(which equals $R_{14}(x)$ by defn).

So, for $0 < x < (\frac{\pi}{2})^{1/3}$,

$$T_{14}(x) < g(x) = T_{14}(x) + R_{14}(x) < T_{14}(x) + \frac{x^{15}}{120}.$$

Thus
$$T_{14}(x) < \sin(x^3) < \underbrace{T_{14}(x) + \frac{x^{15}}{120}}_{= T_{15}(x)}.$$

(c) from (b),

$$\int_0^{\frac{1}{2}} T_{14}(x)\,dx < \int_0^{\frac{1}{2}} \sin(x^3)\,dx < \int_0^{\frac{1}{2}} T_{15}(x)\,dx$$

$$\int_0^{\frac{1}{2}} T_{14}(x)\,dx = \int_0^{\frac{1}{2}} \left(x^3 - \frac{x^9}{6}\right) dx = \frac{1}{64} - \frac{1}{61440}$$

$$\int_0^{\frac{1}{2}} T_{15}(x)\,dx = \int_0^{\frac{1}{2}} \left(x^3 - \frac{x^9}{6} + \frac{x^{15}}{120}\right) dx$$

$$= \frac{1}{64} - \frac{1}{61440} + \frac{1}{125829120}$$

Thus

$$0.0156087 2396 < \int_0^{\frac{1}{2}} \sin(x^3)\,dx < 0.0156087 3191$$