School of Mathematics University of New South Wales

**MATH2701: Abstract Algebra and Fundamental Analysis**
**Short Assignment 2**

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1. Firstly, we must generalise Young's Inequality, using Jensen's Inequality. Suppose that $f : R \to R$ is convex, that $x_1, \ldots, x_n \in \mathbb{R}$ and $a_1, \ldots, a_n > 0$. Then

$$f\left(\frac{\sum a_i x_i}{\sum a_j}\right) \leq \frac{\sum a_i f(x_i)}{\sum a_j}.$$

Let $\lambda_i = \dfrac{a_i}{\sum a_j}$, and consider $f(x) = -\ln x$, which is convex. Furthermore, from the question, we have

$$\sum_{l=1}^{k} \frac{1}{p_l} = 1.$$

Now, taking $\lambda_i = \dfrac{1}{p_i}$, and suppose $c_1, \ldots, c_n > 0$. Letting $x_i = c_i^{p_i}$, by Jensen's Inequality, we have

$$\begin{aligned} -\ln\left(\frac{c_1^{p_1}}{p_1} + \frac{c_2^{p_2}}{p_2} + \cdots + \frac{c_n^{p_n}}{p_n}\right) &\leq -\frac{1}{p_1}\ln c_1^{p_1} - \frac{1}{p_2}\ln c_2^{p_2} - \cdots - \frac{1}{p_n}\ln c_n^{p_n} \\ &= -\ln(c_1 c_2 \ldots c_n). \end{aligned}$$

Exponentiating gives the inequality

$$c_1 c_2 \ldots c_n \leq \frac{c_1^{p_1}}{p_1} + \frac{c_2^{p_2}}{p_2} + \cdots + \frac{c_n^{p_n}}{p_n} \ldots (1).$$

Using the triangle inequality,

$$\left|\sum_{i=1}^{n} x_{1,i} x_{2,i} \ldots x_{k,i}\right| \leq \sum_{i=1}^{n} |x_{1,i}||x_{2,i}| \ldots |x_{k,i}|,$$

and so it suffices to prove the result in the case that all the numbers are non-negative. We may assume that all the $x_{j,i}$ are strictly positive since if $x_{j,i} = 0$, then omitting the $i$-th terms of the sums doesn't change the left-hand side of the required result, and can only make the right-hand side of the required result smaller. Now, setting

$$A_1 = c_1^{p_1}, A_2 = c_2^{p_2}, \ldots, A_k = c_k^{p_k},$$

and considering the inequality given by $(1)$, we have

$$A_1^{1/p_1} A_2^{1/p_2} \ldots A_k^{1/p_k} \leq \frac{1}{p_1}A_1 + \frac{1}{p_2}A_2 + \cdots + \frac{1}{p_k}A_k \ldots (2).$$

For ease of notation, let

$$X_1 = \sum_{i=1}^{n} x_{1,i}^{p_1}, X_2 = \sum_{i=1}^{n} x_{2,i}^{p_2}, \ldots, X_k = \sum_{i=1}^{n} x_{k,i}^{p_k},$$

and furthermore, for $i = 1, \ldots, n$, let

$$A_{1,i} = \frac{x_{1,i}^{p_1}}{X_1}, A_{2,i} = \frac{x_{2,i}^{p_2}}{X_2}, \ldots, A_{k,i} = \frac{x_{k,i}^{p_k}}{X_k},$$

so that,

$$\sum_{i=1}^{n} A_{1,i} = \sum_{i=1}^{n} A_{2,i} = \cdots = \sum_{i=1}^{n} A_{k,i} = 1.$$

Using the above result $(2)$, we have

$$\begin{aligned}
\sum_{i=1}^{n} x_{1,i}x_{2,i}\dots x_{k,i} &= \sum_{i=1}^{n} X_1^{1/p_1} A_{1,i}^{1/p_1} X_2^{1/p_2} A_{2,i}^{1/p_2} \dots X_k^{1/p_k} A_{k,i}^{1/p_k} \\
&= X_1^{1/p_1} X_2^{1/p_2} \dots X_k^{1/p_k} \sum_{i=1}^{n} A_{1,i}^{1/p_1} A_{2,i}^{1/p_2} \dots A_{k,i}^{1/p_k} \\
&\le \prod_{l=1}^{k} X_l^{1/p_l} \left[\frac{1}{p_1}\sum_{i=1}^{n} A_{1,i} + \frac{1}{p_2}\sum_{i=1}^{n} A_{2,i} + \cdots + \frac{1}{p_k}\sum_{i=1}^{n} A_{k,i}\right] \\
&= \prod_{l=1}^{k} X_l^{1/p_l} \left[\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}\right] \\
&= \prod_{l=1}^{k} X_l^{1/p_l} \\
&= \prod_{l=1}^{k} \left(\sum_{i=1}^{n} x_{l,i}\right)^{1/p_l} \\
&= \prod_{l=1}^{k} \|\boldsymbol{x}_l\|_{p_l} \\
\therefore \sum_{i=1}^{n} x_{1,i}x_{2,i}\dots x_{k,i} &\le \prod_{l=1}^{k} \|\boldsymbol{x}_l\|_{p_l}
\end{aligned}$$

2. (a) Given orthonormal vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n$ and $d_1, d_2, \dots, d_n$ positive numbers, we have the set,

$$E = \left\{ \boldsymbol{x} = \sum_{k=1}^{n} c_k \boldsymbol{x}_k : \frac{c_1^2}{d_1^2} + \frac{c_2^2}{d_2^2} + \cdots + \frac{c_n^2}{d_n^2} \le 1 \right\}.$$

To prove that ellipsoids are convex bodies, we must prove that $E$ is a non-empty subset of $\mathbb{R}^n$, convex, centrally symmetric, closed, and bounded above and below. Firstly, for convexity of $E$, consider two vectors $\boldsymbol{x}, \boldsymbol{y} \in E$ where,

$$\boldsymbol{x} = \sum_{k=1}^{n} a_k \boldsymbol{x}_k \text{ and } \boldsymbol{y} = \sum_{k=1}^{n} b_k \boldsymbol{x}_k.$$

For later use, we will label the following vectors,

$$\boldsymbol{a} = \left(\frac{a_1}{d_1}, \frac{a_2}{d_2}, ..., \frac{a_n}{d_n}\right) \text{ and } \boldsymbol{b} = \left(\frac{b_1}{d_1}, \frac{b_2}{d_n}, ..., \frac{b_n}{d_n}\right).$$

Given $\boldsymbol{x}, \boldsymbol{y} \in E$, we have $\boldsymbol{a} \cdot \boldsymbol{a} = \|\boldsymbol{a}\|^2 \le 1$ and $\boldsymbol{b} \cdot \boldsymbol{b} = \|\boldsymbol{b}\|^2 \le 1$. Now, consider the vector

$\boldsymbol{w} = \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}$ for any $\lambda \in [0,1]$. As a result, we have,

$$\boldsymbol{w} = \sum_{k=1}^{n} (\lambda a_k + (1-\lambda)b_k)\boldsymbol{x_k}.$$

Let $v_k = (\lambda a_k + (1-\lambda)b_k)$. We can thus rewrite $\boldsymbol{w} = \sum_{k=1}^{n} v_k \boldsymbol{x_k}$. Consider the following,

$$\begin{aligned}
\frac{v_1^2}{d_1^2} + \frac{v_2^2}{d_2^2} + \ldots + \frac{v_n^2}{d_n^2} &= \sum_{k=1}^{n} \frac{(\lambda a_k + (1-\lambda)b_k)^2}{d_k^2} \\
&= \sum_{k=1}^{n} \left( \lambda^2 \frac{a_k^2}{d_k^2} + (1-\lambda)^2 \frac{b_k^2}{d_k^2} + 2\lambda(1-\lambda)\frac{a_k b_k}{d_k^2} \right) \\
&= \lambda^2(\boldsymbol{a}\cdot\boldsymbol{a}) + (1-\lambda)^2(\boldsymbol{b}\cdot\boldsymbol{b}) + 2\lambda(1-\lambda)(\boldsymbol{a}\cdot\boldsymbol{b}) \\
&\le \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\,\|\boldsymbol{a}\|\,\|\boldsymbol{b}\| \qquad \text{by Cauchy-Schwarz} \\
&\le \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) \\
&= 1
\end{aligned}$$

Hence $\boldsymbol{w} \in E$, and thus ellipsoids are convex. Now, for centrally symmetric, we consider $\boldsymbol{x} \in E$, such that

$$\boldsymbol{x} = \sum_{k=1}^{n} c_k \boldsymbol{x}_k \text{ and } \frac{c_1^2}{d_1^2} + \frac{c_2^2}{d_2^2} + \cdots + \frac{c_n^2}{d_n^2} \le 1.$$

Considering $-\boldsymbol{x}$,

$$\begin{aligned}
-\boldsymbol{x} &= \sum_{k=1}^{n} (-c_k)\boldsymbol{x}_k \\
\therefore \frac{(-c_1)^2}{d_1^2} + \frac{(-c_2)^2}{d_2^2} + \cdots + \frac{(-c_n)^2}{d_n^2} &= \frac{c_1^2}{d_1^2} + \frac{c_2^2}{d_2^2} + \cdots + \frac{c_n^2}{d_n^2} \le 1 \\
\therefore \frac{(-c_1)^2}{d_1^2} + \frac{(-c_2)^2}{d_2^2} + \cdots + \frac{(-c_n)^2}{d_n^2} &\le 1.
\end{aligned}$$

Clearly, $-\boldsymbol{x} \in E$, and so $E$ is centrally symmetric. We also know that $E$ contains its boundary. The boundary is given by,

$$\partial E = \left\{ \boldsymbol{x} = \sum_{k=1}^{n} c_k \boldsymbol{x_k} : \frac{c_1^2}{d_1^2} + \frac{c_2^2}{d_2^2} + \cdots + \frac{c_n^2}{d_n^2} = 1 \right\},$$

which is clearly a subset of $E$. Hence $E$ is closed. Finally we also know that $E$ is bounded above and below. Specifically it is bounded above by the ball $B_D$ where $D = \max\{d_k\}$. To check, consider $\boldsymbol{x} \in E$ where $\|\boldsymbol{x}\| > D$. Then we have

$$\begin{aligned}
c_1^2 + \cdots + c_n^2 &> D^2 \\
\frac{c_1^2}{D^2} + \cdots + \frac{c_n^2}{D^2} &> 1.
\end{aligned}$$

But we also have,

$$\frac{c_1^2}{D^2} + \cdots + \frac{c_n^2}{D^2} \leq \frac{c_1^2}{d_1^2} + \ldots + \frac{c_n^2}{d_n^2} \leq 1.$$

By contradiction, no such $\boldsymbol{x}$ exists and hence $E$ is bounded above by the ball $B_D$. Thus, all ellipsoids are convex bodies.

(b) Let $\boldsymbol{y} = \sum_{k=1}^{n} a_k \boldsymbol{x_k}$ and consider $\boldsymbol{x} \cdot \boldsymbol{y} \leq 1$ for all $\boldsymbol{x} = \sum_{k=1}^{n} c_k \boldsymbol{x_k} \in E$.

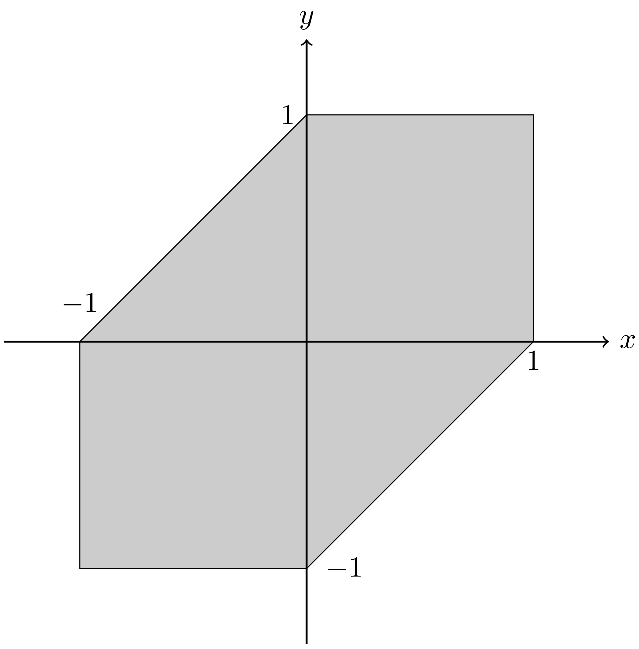
$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{y} &= \sum_{k=1}^{n} a_k c_k \\ &= \sum_{k=1}^{n} (d_k a_k) \left(\frac{c_k}{d_k}\right) \\ &\leq \left(\sum_{k=1}^{n} (d_k a_k)^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \left(\frac{c_k}{d_k}\right)^2\right)^{\frac{1}{2}} \qquad \text{by Cauchy-Schwarz} \\ &\leq \left(\sum_{k=1}^{n} d_k^2 a_k^2\right)^{\frac{1}{2}} \end{aligned}$$

Considering the condition for equality in Cauchy-Schwarz, we have,

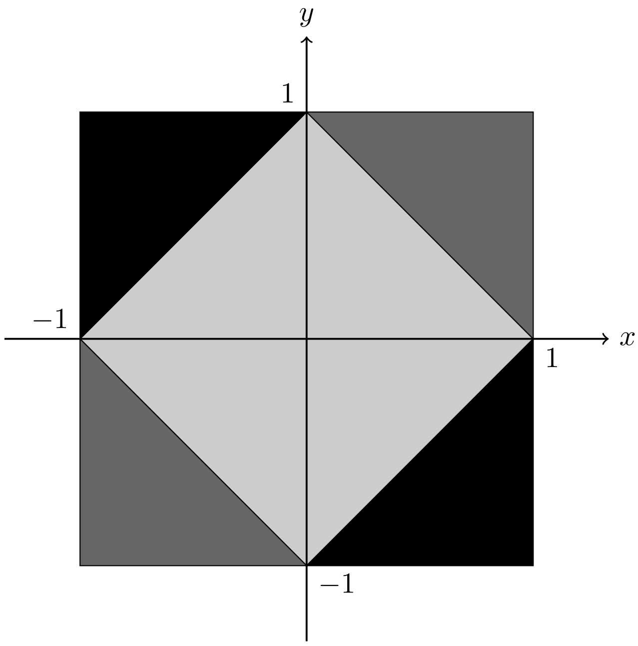
$$K^\circ = \left\{ \boldsymbol{y} = \sum_{k=1}^{n} a_k \boldsymbol{x_k} : \frac{a_1^2}{d_1^{-2}} + \frac{a_2^2}{d_2^{-2}} + \ldots + \frac{a_n^2}{d_n^{-2}} \leq 1 \right\},$$

which is an ellipsoid with corresponding set of positive numbers $d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1}$.

3. The set, $K = \{(x, y) \in \mathbb{R}^2 : \max\{|x - y|, |x|, |y|\} \leq 1\}$ is indicated in the following diagram.



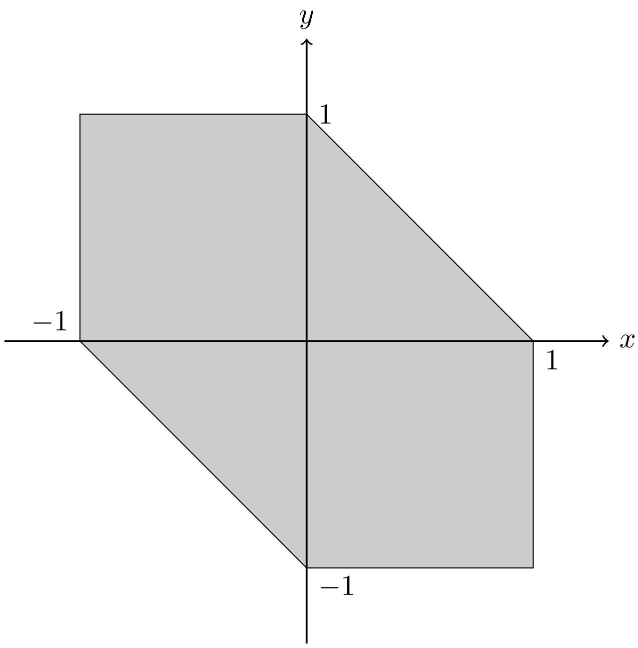
Consider $\boldsymbol{u} = (x_1, y_1) \in K$ and $\boldsymbol{v} = (x_2, y_2) \in \mathbb{R}^2$, where $\boldsymbol{u} \cdot \boldsymbol{v} \leq 1 \implies x_1 x_2 + y_1 y_2 \leq 1$. Considering the diagram below, it is obvious that for $\boldsymbol{u}, \boldsymbol{v}$ in the light-grey shaded region, $\boldsymbol{u} \cdot \boldsymbol{v} \leq 1$. However, as $\boldsymbol{u}$ can also reside in the dark-grey shaded regions, $\boldsymbol{v}$ cannot, as otherwise $\boldsymbol{u} \cdot \boldsymbol{v} > 1$. So, by symmetry, $\boldsymbol{v}$ can also reside in the black shaded regions, as $\boldsymbol{u}$ cannot.




Hence we can write,

$$K^\circ = \{(x, y) \in \mathbb{R}^2 : \max\{|x + y|, |x|, |y|\} \leq 1\}.$$

This is simply a rotation of $K$, with boundary as shown below.




Now we can compute the Mahler volume,

$$M(K) = \mathsf{vol}(K)\mathsf{vol}(K^\circ) = 3 \times 3 = 9.$$

4. (a) First note that if $M = \max\limits_{a \le x \le b} |f(x)|$, then we also have $M^p = \max\limits_{a \le x \le b} |f(x)|^p$. Therefore,

$$\begin{aligned} \|f\|_p &= \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \\ &\le \left( M^p \int_a^b dx \right)^{\frac{1}{p}} \\ &= M(b-a)^{\frac{1}{p}}. \end{aligned}$$

This gives us the right hand side of the inequality,

$$\|f\|_p \le (b-a)^{\frac{1}{p}} M.$$

(b) Taking the limit of the inequality in part (a),

$$\begin{aligned} \lim_{p\to\infty} c^{\frac{1}{p}}(M-\varepsilon) \le \lim_{p\to\infty} \|f\|_p &\le \lim_{p\to\infty} (b-a)^{\frac{1}{p}} M \\ M - \varepsilon \le \lim_{p\to\infty} \|f\|_p &\le M \\ -\varepsilon \le \lim_{p\to\infty} \|f\|_p - M &\le 0 \\ \left| \lim_{p\to\infty} \|f\|_p - M \right| &\le \varepsilon \end{aligned}$$

Suppose $\left| \lim\limits_{p\to\infty} \|f\|_p - M \right| = k \neq 0$. Then consider $\varepsilon = \dfrac{k}{2} < k$. This contradicts the inequality above, and hence by contradiction, $\left| \lim\limits_{p\to\infty} \|f\|_p - M \right| = 0$.

$$\therefore \lim_{p\to\infty} \|f\|_p = M.$$