Solutions

1. Put $u = c - kx^2$, so $du = -2kx\,dx$,

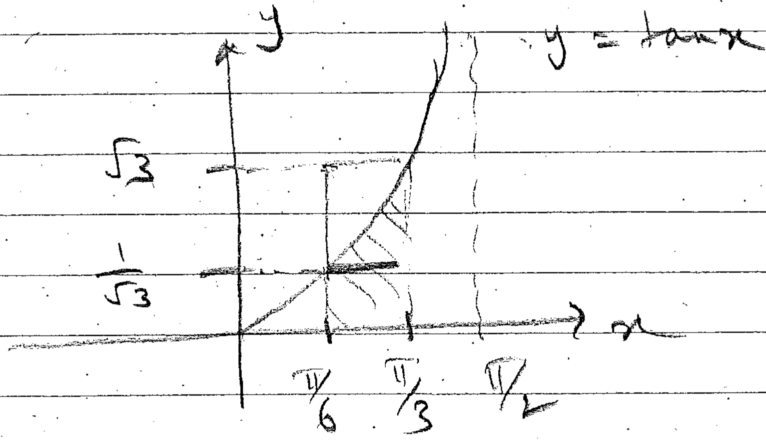
so $x\,dx = \dfrac{-du}{2k}$, giving

$$\int_a^b x\, f(c-kx^2)\,dx = \int_a^b f(c-kx^2)\, x\,dx$$

$$= \int_{c-ka^2}^{c-kb^2} f(u)\left(-\frac{1}{2k}\right) du = -\frac{1}{2k}\int_{c-ka^2}^{c-kb^2} f(u)\,du$$

$$= \frac{1}{2k}\int_{c-kb^2}^{c-ka^2} f(x)\,dx,$$ as required.

2.



area of lower rectangle $= \left(\pi/3 - \pi/6\right)\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6\sqrt{3}}$,

area of upper rectangle $= \left(\pi/3 - \pi/6\right)\sqrt{3} = \dfrac{\sqrt{3}\pi}{6} = \dfrac{\pi}{2\sqrt{3}}$.

Hence $$\frac{\pi}{6\sqrt{3}} \le \int_{\pi/6}^{\pi/3} \tan x\,dx \le \frac{\pi}{2\sqrt{3}}.$$

3. $\int_{-2}^{2} (2-g(x))\,(f(x)+\sin x+1)\,dx$

$= \int_{-2}^{2} (2-g(x))(f(x)+\sin x) + 2-g(x)\,dx$

$= \int_{-2}^{2} \underbrace{\underbrace{(2-g(x))}_{\text{even}}\,\underbrace{(f(x)+\sin x)}_{\text{odd}}}_{\text{odd}}\,dx + \int_{-2}^{2} 2\,dx - \int_{-2}^{2} g(x)\,dx$

$= 0 + 2(2-(-2)) - 2\int_{0}^{2} g(x)\,dx$

$= 2(4) - 2(-3) = 8+6 = 14.$

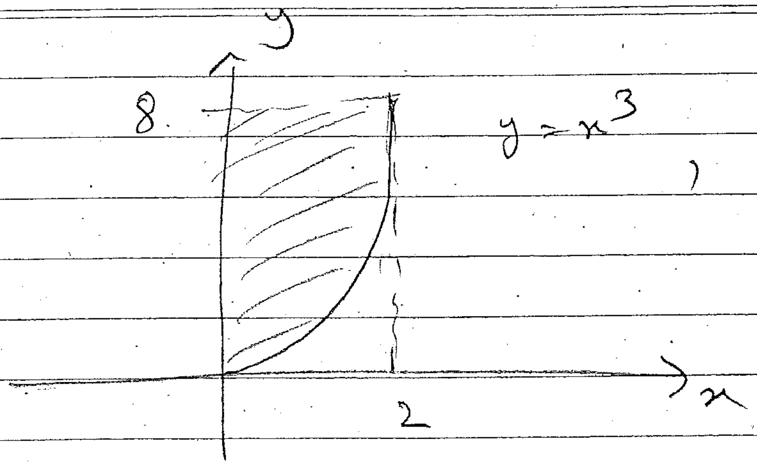
4. $y = \frac{1}{x} = x^{-1}$, $y' = -x^{-2}$, $(y')^2 = x^{-4}$.

Hence length of curve for $1/100 \le x \le 100$ is

$= \int_{1/100}^{100} \sqrt{1+(y')^2}\,dx$

$= \int_{1/100}^{100} \sqrt{1+x^{-4}}\,dx = \int_{1/100}^{100} \sqrt{\frac{x^4+1}{x^4}}\,dx$

$= \int_{1/100}^{100} \frac{\sqrt{x^4+1}}{x^2}\,dx,$ as required.

5. 

$y = x^3$, $x = y^{1/3}$

$$\text{Volume} = \int_0^8 \pi x^2 \, dy \qquad (\text{disc method})$$

$$= \pi \int_0^8 y^{2/3} \, dy = \pi \left[\frac{3}{5} y^{5/3} \right]_0^8$$

$$= \frac{3\pi}{5} \left(8^{5/3} - 0 \right) = \frac{3\pi}{5} (2^5) = \frac{96\pi}{5}.$$

6. By the shell method,

$$\text{Volume} = \int_0^2 2\pi x (8 - y) \, dx = 2\pi \int_0^2 x(8 - x^3) \, dx$$

$$= 2\pi \int_0^2 8x - x^4 \, dx = 2\pi \left[4x^2 - \frac{x^5}{5} \right]_0^2$$

$$= 2\pi \left(16 - \frac{32}{5} - 0 \right)$$

$$= 2\pi \left(\frac{80 - 32}{5} \right) = 2\pi \left(\frac{48}{5} \right)$$

$$= \frac{96\pi}{5},$$ as expected from Q5.

7. $$\int_0^{\pi/2} \sin^5\theta \cos^5\theta \, d\theta = \int_0^{\pi/2} (1-\cos^2\theta)^2 \cos^5\theta \sin\theta \, d\theta$$

$$= -\int_1^0 (1-u^2)^2 u^5 \, du$$

Put $u = \cos\theta$

so $du = -\sin\theta \, d\theta$

$$= \int_0^1 (1-2u^2+u^4) u^5 \, du$$

$$= \int_0^1 u^5 - 2u^7 + u^9 \, du = \left[\frac{u^6}{6} - \frac{u^8}{4} + \frac{u^{10}}{10}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{10-15+6}{60} = \frac{1}{60}.$$

Alternatively, $$\int_0^{\pi/2} \sin^5\theta \cos^5\theta \, d\theta = \int_0^{\pi/2} \sin^5\theta (1-\sin^2\theta)^2 \cos\theta \, d\theta$$

$$= \int_0^1 u^5 (1-u^2)^2 \, du$$

Put $u = \sin\theta$

so $du = \cos\theta \, d\theta$

$$= \int_0^1 u^5 (1-2u^2+u^4) \, du$$

$$= \int_0^1 u^5 - 2u^7 + u^9 \, du$$

$$= \ldots = \frac{1}{60}$$, as before.

8. $$F(x) = \int_{\sin x}^{\sqrt{x}} \cos(t^3)\, dt$$

$$= \int_0^{\sqrt{x}} \cos(t^3)\, dt - \int_0^{\sin x} \cos(t^3)\, dt,$$

so $$F'(x) = \frac{1}{2} x^{-1/2} \cos\left((\sqrt{x})^3\right) - \cos x \cos\left((\sin x)^3\right)$$

$$= \frac{\cos(x^{3/2})}{2\sqrt{x}} - \cos x \cos(\sin^3 x).$$

9. $$\frac{\sin(\pi x)}{x} = \int_0^{x^2} f(t)\, dt,$$

so, differentiating both sides gives

$$-\frac{\sin(\pi x)}{x^2} + \frac{\pi \cos(\pi x)}{x} = 2x\, f(x^2).$$

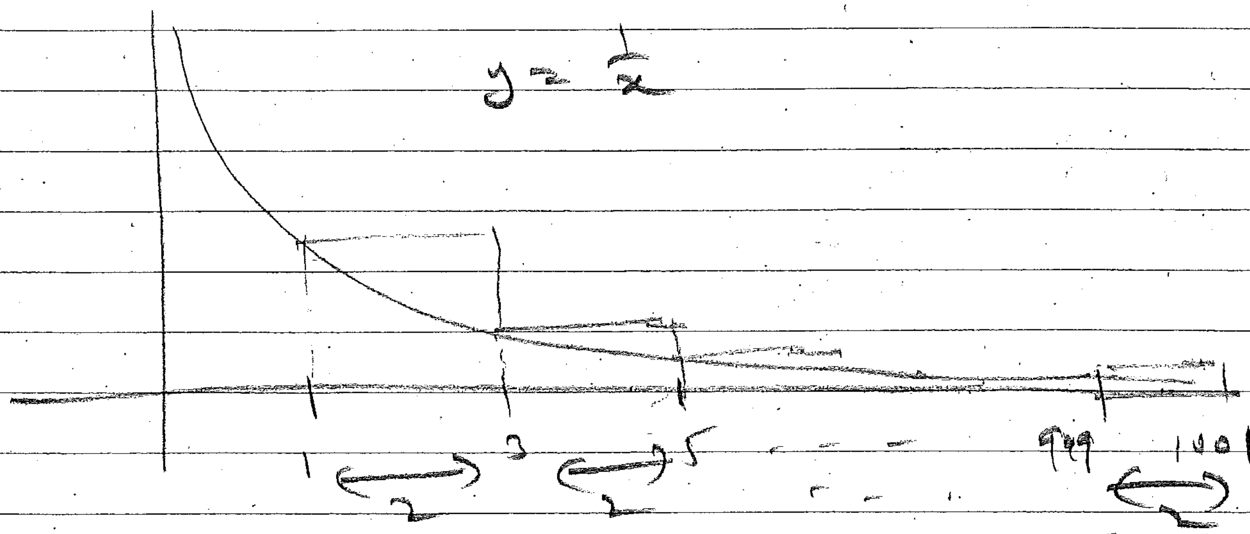
Putting $x = 5$ gives

$$-\frac{\sin(5\pi)}{25} + \frac{\pi \cos(5\pi)}{5} = 10\, f(25),$$

i.e. $$0 - \frac{\pi}{5} = 10\, f(25),$$

so $$f(25) = -\frac{\pi}{50}.$$

10.



Using upper rectangles of width 2 we get the Riemann sum over the interval $[1, 1001]$ for $y = \frac{1}{x}$

$$2\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{999}\right)$$

which must be at least the area

$$\int_1^{1001} \frac{1}{x}\,dx = \Big[\ln x\Big]_1^{1001}$$

$$= \ln 1001 - \ln 1$$

$$= \ln 1001,$$

so

$$1 + \frac{1}{3} + \cdots + \frac{1}{999} \geq \frac{\ln 1001}{2},$$

as required.