University of Sydney

MATH 1906

SSP

# Assignment 2 - Fractals

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Monday 4pm

June 9, 2017

**1.** (a) Using the given initator and the generative steps to construct the fractal, we can construct the second, third and fourth iterations of the fractal, as seen below.

The following figure is Iteration 2.

IFS2_S.png

The following figure is Iteration 3.

IFS3_S.png

The following figure is Iteration 4, and is the final iteration we are to construct.

(b) The similarity dimension of a fractal is calculated using the following formula:

$$d_{ss} = \frac{\log N}{\log \frac{1}{r}}$$

where $d_{ss}$ is the self similarity dimension, $N$ is the number of copies of the initiator, and $r$ is the scaling factor. Applying this formula to our fractal, $N = 3$, and $r = \frac{1}{\sqrt{3}}$. Thus calculating the self similarity dimension of our fractal gives:

IFS4_S.png

$$\begin{aligned} d_{ss} &= \frac{\log N}{\log \frac{1}{r}} \\ &= \frac{\log 3}{\log \frac{1}{\frac{1}{\sqrt{3}}}} \\ &= \frac{\log 3}{\log \sqrt{3}} \\ &= \frac{\log 3}{\frac{1}{2} \log 3} \\ &= \frac{1}{\frac{1}{2}} \\ \therefore d_{ss} &= 2 \end{aligned}$$

(c) In order to calculate the IFS for the fractal given, we must examine the transformation of each of three copies from the initiator, as this process will be iteratively applied at every step. Examining the first line segment of the three that result in the first iteration of the fractal, we arrive at the following transformations.

Scaling by a factor of $\frac{1}{\sqrt{3}}$.

$$\begin{aligned} f(x,y) &\to f_1(x',y') \\ x &\to x' \\ &\to \frac{x}{\sqrt{3}} \\ \therefore x' &= \frac{x}{\sqrt{3}} \\ y &\to y' \\ &\to \frac{y}{\sqrt{3}} \\ \therefore y' &= \frac{y}{\sqrt{3}} \end{aligned}$$

Rotation by $\frac{\pi}{6}$.

$$f_1(x',y') \to f_1(x'',y'')$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

$$\begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{3}} \\ \frac{y}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{3}} \\ \frac{y}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{x}{2} - \frac{y}{2\sqrt{3}} \\ \frac{x}{2\sqrt{3}} + \frac{y}{2} \end{bmatrix}$$

$$\begin{aligned} \therefore x'' &= \frac{x}{2} - \frac{y}{2\sqrt{3}} \\ \therefore x'' &= \frac{\sqrt{3}x - y}{2\sqrt{3}} \\ \therefore y'' &= \frac{x}{2\sqrt{3}} + \frac{y}{2} \\ \therefore y'' &= \frac{x + \sqrt{3}y}{2\sqrt{3}} \end{aligned}$$

There is no Translation for the first segment. Thus the map for the first segment is given by the following map.

$$\begin{aligned} f(x,y) &\to f_1(x'',y'') \\ f_1(x'',y'') &= f_1(x_1,y_1) \\ \therefore f(x,y) &\to f_1(x_1,y_1) \\ \therefore f(x,y) &\to f_1\left(\frac{3x - \sqrt{3}y}{6}, \frac{\sqrt{3}x + 3y}{6}\right) \end{aligned}$$

Examining the second line segment of the three that result in the first iteration of the fractal, we arrive at the following transformations.

Scaling by a factor of $\frac{1}{\sqrt{3}}$.

$$\begin{aligned} f(x,y) &\to f_2(x',y') \\ x &\to x' \\ &\to \frac{x}{\sqrt{3}} \\ \therefore x' &= \frac{x}{\sqrt{3}} \\ y &\to y' \\ &\to \frac{y}{\sqrt{3}} \\ \therefore y' &= \frac{y}{\sqrt{3}} \end{aligned}$$

Rotation by $-\frac{\pi}{2}$.

$$f_2(x',y') \to f_2(x'',y'')$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

$$\begin{bmatrix} \cos\left(-\frac{\pi}{2}\right) & -\sin\left(-\frac{\pi}{2}\right) \\ \sin\left(-\frac{\pi}{2}\right) & \cos\left(-\frac{\pi}{2}\right) \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{3}} \\ \frac{y}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{3}} \\ \frac{y}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{y}{\sqrt{3}} \\ -\frac{x}{\sqrt{3}} \end{bmatrix}$$

$$\begin{aligned} \therefore x'' &= \frac{y}{\sqrt{3}} \\ \therefore y'' &= -\frac{x}{\sqrt{3}} \end{aligned}$$

Translation by $\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$.

$$
\begin{aligned}
f_2(x'', y'') &\to f_2(x''', y''') \\
x'' &\to x''' \\
&\to \frac{y}{\sqrt{3}} + \frac{1}{2} \\
\therefore x''' &= \frac{y}{\sqrt{3}} + \frac{1}{2} \\
\therefore x''' &= \frac{\sqrt{3} + 2y}{2\sqrt{3}} \\
y'' &\to y''' \\
&\to \frac{1}{2\sqrt{3}} - \frac{x}{\sqrt{3}} \\
\therefore y''' &= \frac{1}{2\sqrt{3}} - \frac{x}{\sqrt{3}} \\
\therefore y''' &= \frac{1 - 2x}{2\sqrt{3}}
\end{aligned}
$$

Thus the map for the second segment is given by the following map.

$$
\begin{aligned}
f(x, y) &\to f_2(x''', y''') \\
f_2(x''', y''') &= f_2(x_2, y_2) \\
\therefore f(x, y) &\to f_2(x_2, y_2) \\
\therefore f(x, y) &\to f_2\left(\frac{3 + 2\sqrt{3}y}{6}, \frac{\sqrt{3}(1 - 2x)}{6}\right)
\end{aligned}
$$

Examining the third line segment of the three that result in the first iteration of the fractal, we arrive at the following transformations.

Scaling by a factor of $\frac{1}{\sqrt{3}}$.

$$
\begin{aligned}
f(x, y) &\to f_3(x', y') \\
x &\to x' \\
&\to \frac{x}{\sqrt{3}} \\
\therefore x' &= \frac{x}{\sqrt{3}} \\
y &\to y' \\
&\to \frac{y}{\sqrt{3}} \\
\therefore y' &= \frac{y}{\sqrt{3}}
\end{aligned}
$$

Rotation by $\frac{\pi}{6}$.

$$f_3(x', y') \to f_3(x'', y'')$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

$$\begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{3}} \\ \frac{y}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{3}} \\ \frac{y}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{x}{2} - \frac{y}{2\sqrt{3}} \\ \frac{x}{2\sqrt{3}} + \frac{y}{2} \end{bmatrix}$$

$$\begin{aligned} \therefore x'' &= \frac{x}{2} - \frac{y}{2\sqrt{3}} \\ \therefore y'' &= \frac{x}{2\sqrt{3}} + \frac{y}{2} \end{aligned}$$

Translation by $\left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$.

$$\begin{aligned} f_3(x'', y'') &\to f_3(x''', y''') \\ x'' &\to x''' \\ &\to \frac{x}{2} - \frac{y}{2\sqrt{3}} + \frac{1}{2} \\ \therefore x''' &= \frac{x}{2} - \frac{y}{2\sqrt{3}} + \frac{1}{2} \\ \therefore x''' &= \frac{\sqrt{3}(x+1) - y}{2\sqrt{3}} \\ y'' &\to y''' \\ &\to \frac{x}{2\sqrt{3}} + \frac{y}{2} - \frac{1}{2\sqrt{3}} \\ \therefore y''' &= \frac{x}{2\sqrt{3}} + \frac{y}{2} - \frac{1}{2\sqrt{3}} \\ \therefore y''' &= \frac{x - 1 + \sqrt{3}y}{2\sqrt{3}} \end{aligned}$$

Thus the map for the third segment is given by the following map.

$$\begin{aligned} f(x, y) &\to f_3(x''', y''') \\ f_3(x''', y''') &= f_3(x_3, y_3) \\ \therefore f(x, y) &\to f_3(x_3, y_3) \\ \therefore f(x, y) &\to f_3\left(\frac{3(x+1) - \sqrt{3}y}{6}, \frac{\sqrt{3}(x-1) + 3y}{6}\right) \end{aligned}$$

Therefore the IFS for the fractal curve is $F = f_1 \cup f_2 \cup f_3$, using the results for $f_1$, $f_2$, and $f_3$ from above.

**2.** $\forall x \in [0,1]$, let $x = 0.a_{1x}a_{2x}a_{3x}\dots$ be a ternary representation of $x$, i.e., $x = \sum_{n=1}^{\infty} \frac{a_{nx}}{3^n}$, where $a_{nx} \in \{0,1,2\}$. Denote $N_x$ by the smallest $n$ with $a_{nx} = 1$ if it exists, and set $N_x = \infty$ if no such $a_{nx}$ exists. Define $G : [0,1] \to \mathbb{R}$ by $G(x) = \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1} \frac{a_{nx}}{2^n}$, $\forall x \in [0,1]$. $\forall n \in \mathbb{N}$, define $M_n := \int_0^1 x^n G(x)dx$.

(a) In order to prove that $G(x)$ is independent of the ternary expansion of $x$, we will use the summation definition of $x$, to write a digit of the expansion of $x$ in the form $\frac{a}{3^b}$, for some $b \in \mathbb{N}$. Thus we can write the term $\frac{a}{3^b}$ as either a single fraction or an infinite decimal sum using the ternary expansion of $x$. Now considering the cases for $G(x)$, we get the following results.

Case (1): $a = 0 \to G(x) = 0$, as it is the trivial case.

Case (2): $a = 1 \to \frac{1}{3^b}$ This can be written as either $\frac{1}{3^b}$, or as an infinite sum $\frac{0}{3^b} + \frac{2}{3^{b+1}} + \frac{2}{3^{b+2}} + \dots$ to infinity.

$$\begin{aligned}
G\left(\frac{1}{3^b}\right) &= \frac{1}{2^b} + \frac{1}{2}\sum_{n=1}^{b-1}\frac{a_{nx}}{2^n} \qquad \text{as } N_x = b \\
&= \frac{1}{2^b} + 0 \qquad \text{as } \forall n < b,\ \ a_{nx} = 0 \\
\therefore G\left(\frac{1}{3^b}\right) &= \frac{1}{2^b}
\end{aligned}$$

Considering the infinite sum method of writing the entry, we get the following result.

$$\begin{aligned}
G\left(\frac{0}{3^b} + \frac{2}{3^{b+1}} + \frac{2}{3^{b+2}} + \dots\right) &= \frac{1}{2^\infty} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{a_{nx}}{2^n} \qquad \text{as } N_x = \infty \\
&= 0 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{a_{nx}}{2^n} \\
&= \frac{1}{2}\sum_{n=1}^{b}\frac{a_{nx}}{2^n} + \frac{1}{2}\sum_{n=b+1}^{\infty}\frac{a_{nx}}{2^n} \qquad \text{as } \forall n \le b,\ \ a_{nx} = 0 \\
&= 0 + \frac{1}{2}\sum_{n=b+1}^{\infty}\frac{a_{nx}}{2^n} \\
&= \frac{1}{2}\sum_{n=b+1}^{\infty}\frac{2}{2^n} \qquad \text{as } \forall n > b,\ \ a_{nx} = 2 \\
&= \frac{1}{2}\sum_{n=b+1}^{\infty}\frac{1}{2^{n-1}} \\
&= \frac{1}{2}\left(\frac{1}{2^b}\right)\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) \\
&= \frac{1}{2}\left(\frac{1}{2^b}\right)\left(\frac{1}{1-\frac{1}{2}}\right) \qquad \text{Limiting Sum of a GP} \\
\therefore G\left(\frac{0}{3^b} + \frac{2}{3^{b+1}} + \frac{2}{3^{b+2}} + \dots\right) &= \frac{1}{2^b}
\end{aligned}$$

Case (3): $a = 2 \to \frac{2}{3^b}$ This can be written as either $\frac{2}{3^b}$, or as an infinite sum $\frac{1}{3^b} + \frac{2}{3^{b+1}} + \frac{2}{3^{b+2}} + \dots$ to infinity.

$$\begin{aligned}
G\left(\frac{2}{3^b}\right) &= \frac{1}{2^\infty} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{a_{nx}}{2^n} \qquad \text{as } N_x = \infty \\
&= 0 + \frac{1}{2}\left[\sum_{n=1}^{b-1}\frac{a_{nx}}{2^n} + \frac{2}{2^b} + \sum_{n=b+1}^{\infty}\frac{a_{nx}}{2^n}\right] \\
&\text{As } \forall n \leq b-1 \text{ , } a_{nx} = 0 \text{ , } n = b \text{ , } a_{nx} = 2 \text{ and } \forall n \geq b+1 \text{ , } a_{nx} = 0 \\
&= \frac{1}{2}\left[0 + \frac{2}{2^b} + 0\right] \\
\therefore G\left(\frac{2}{3^b}\right) &= \frac{1}{2^b}
\end{aligned}$$

Considering the infinite sum method of writing the entry, we get the following result.

$$\begin{aligned}
G\left(\frac{1}{3^b} + \frac{2}{3^{b+1}} + \frac{2}{3^{b+2}} + \dots\right) &= \frac{1}{2^b} + \frac{1}{2}\sum_{n=1}^{b-1}\frac{a_{nx}}{2^n} \qquad \text{as } N_x = b \\
&= \frac{1}{2^b} + 0 \qquad \text{as } \forall n < b, \;\; a_{nx} = 0 \\
\therefore G\left(\frac{1}{3^b} + \frac{2}{3^{b+1}} + \frac{2}{3^{b+2}} + \dots\right) &= \frac{1}{2^b}
\end{aligned}$$

Therefore regardless of the input from a ternary term of $x$, $G(x)$ returns the same value, a binary term, and is thus independent of the ternary term that is provided. As such, this can be generalised from a single term, to the input of a ternary representation to $G(x)$, simply returning a binary expanison independent of the value of $x$.

(b) To prove the continuity of $G(x)$, fix $\varepsilon > 0$ and let $c$ be a ternary representation with the same definition as $x$. Furthermore, $c \in [0, 1]$. Therefore, by $\varepsilon$-$\delta$ proofs, we get the following result for the continuity of the function $G(x)$. Set $\delta = \frac{1}{3^{N+1}}$. Therefore, $a_{nx} = a_{nc}$, $\forall n \leq N$, and thus $x = c$ to a sufficient accuracy.

$$\begin{aligned}
0 < |x - c| < \delta \implies |G(x) - G(c)| &< \varepsilon \\
\therefore |G(x) - G(c)| &= \left|\frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} - \left[\frac{1}{2^{N_c}} + \frac{1}{2}\sum_{n=1}^{N_c-1}\frac{a_{nc}}{2^n}\right]\right| \\
&= \left|\frac{1}{2^{N_x}} - \frac{1}{2^{N_c}} + \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} - \frac{1}{2}\sum_{n=1}^{N_c-1}\frac{a_{nc}}{2^n}\right| \dots\dots\dots (1) \\
&\leq \frac{1}{3^{N+1}} \\
&= \varepsilon \qquad \text{as } n \to N, \;\; \frac{1}{3^{N+1}} \to 0, \; \therefore \frac{1}{3^{N+1}} < \varepsilon \\
\therefore 0 < |x - c| < \delta \implies |G(x) - G(c)| &< \varepsilon
\end{aligned}$$

As a result, the limit exists at every point on the function $G(x)$, and thus by definition, is continuous over the interval $[0, 1]$.

Now in order to prove that the function $G(x)$ is increasing for all $x_1 < x_2$, we will introduce $x_1 < x_2$. As a result of the definition of $x$ as a ternary representation, for $x_1 < x_2$, $\therefore a_{nx_1} < a_{nx_2}$. As a result, if all $a_{nx_1} < a_{nx_2}$, $G(x_1) \leq G(x_2)$, for all $x_1 < x_2$.

(c) Each point in the Cantor set can be examined as a ternary representation of their location along the interval $[0, 1]$. Thus by construction, the location of a point in the Cantor set is $x = 0.a_{1x}a_{2x}a_{3x}\dots$, where $a_{nx} \neq 1$. If $a_{nx} = 1$, in other words, the location of the middle third interval, the point represented by $x$ is not in the Cantor set. Thus, as $a_{nx} = 1$, $N_x$ is fixed to the smallest value of $n$ that the $a_{nx} = 1$ holds for. Thus because $N_x$ is fixed, and the function $G(x) = \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1} \frac{a_{nx}}{2^n}$ is dependent on $N_x$, the function $G(x)$ will remain constant, no matter the continuation of $x$. This is due to the sum $\frac{1}{2}\sum_{n=1}^{N_x-1} \frac{a_{nx}}{2^n}$ becoming fixed as $N_x$ holds a value that is not $\infty$. As a consequnece, the function $G(x) = k$, for some constant $k$, for the intervals $[0, 1] \setminus \mathcal{C}$.

Now, $\forall x \in \left(\frac{1}{3}, \frac{2}{3}\right)$, we know this interval is not included in $\mathcal{C}$. Furthermore, by construction of our ternary representation of $\mathcal{C}$ and $x$, we know that $a_{1x} = 1$. As a result, $N_x = 1$, and thus we can compute $G(x)$, $\forall x \in \left(\frac{1}{3}, \frac{2}{3}\right)$.

$$
\begin{aligned}
G(x) &= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1} \frac{a_{nx}}{2^n} \\
&= \frac{1}{2^1} + \frac{1}{2}\sum_{n=1}^{1-1} \frac{1}{2^n} \\
&= \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{0} \frac{1}{2^n} \\
&= \frac{1}{2} + 0 \\
\therefore G(x) &= \frac{1}{2} \qquad \forall x \in \left(\frac{1}{3}, \frac{2}{3}\right)
\end{aligned}
$$

(d) As we have proven in part (a), $G(x)$ returns a value independent of the ternary representation of $x$, and thus returns some binary expansion. If $\forall n,\ a_{nx} \neq 1$, the function $G(x)$ returns some infinite binary expansion that maps one of the infinite points in the Cantor set to the interval $[0, 1]$. If, however, for some $n,\ a_{nx} = 1$, the function $G(x)$ is designed to remain constant, using the definition of $N_x$. This constant indicates that the interval is removed from the Cantor set. Thus the function $G(x)$ maps either an infinite binary sequence or a constant, the definition of the Cantor set.

We will examine the endpoints of the function to check that they are mapped to those points $[0, 1]$. Examining the endpoint $0$, $\forall a_{nx} = 0$, and $\therefore N_x = \infty$, we get the following result.

$$
\begin{aligned}
G(x) &= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1} \frac{a_{nx}}{2^n} \\
&= \frac{1}{2^\infty} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{a_{nx}}{2^n} \\
&= 0 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{0}{2^n} \\
\therefore G(x) &= 0 \qquad \forall a_{nx} = 0
\end{aligned}
$$

Examining the endpoint $1$, $\forall a_{nx} = 2$, and $\therefore N_x = \infty$, we get the following result.

$$
\begin{aligned}
G(x) &= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1} \frac{a_{nx}}{2^n} \\
&= \frac{1}{2^\infty} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{a_{nx}}{2^n} \\
&= 0 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{2}{2^n} \\
&= \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}}\right] \qquad \text{Using the formula for an infinite sum} \\
&= \frac{1}{2}\left[\frac{1}{\frac{1}{2}}\right] \\
&= \frac{1}{2}(2) \\
\therefore G(x) &= 1 \qquad \forall a_{nx} = 2
\end{aligned}
$$

Thus the function $G(x)$ maps the Cantor set onto the interval $[0, 1]$.

(e) We are required to prove that $G\left(\frac{x}{3}\right) = \frac{G(x)}{2}$, $\forall x \in [0, 1]$. In order to do so we must first examine $\frac{x}{3}$, and how it behaves as a ternary representation.

$$
\begin{aligned}
x &= 0.a_{1x}a_{2x}a_{3x}\dots \\
\therefore \frac{x}{3} &= 0.0a_{1x}a_{2x}a_{3x}\dots
\end{aligned}
$$

This occurs due to the consequence of dividing by 3 in base 3. The result is eqauivalent to a division by 10 in base 10 and is the reason as to why we get a leading zero after the decimal place. For the proof, we shall define $y := \frac{x}{3}$. Furthermore, we will define $y := 0.a_{1y}a_{2y}a_{3y}\dots$, which is a ternary representation of $y$. As a result, we get the following relationships:

$$
\begin{aligned}
a_{1y} &= 0 \\
a_{ny} &= a_{(n-1)x} \\
N_y &= N_x + 1
\end{aligned}
$$

Now examining the function $G(x)$ evaluated at $x = \frac{x}{3}$, or in other words, $x = y$, we get the following result.

$$\begin{aligned}
G(y) &= \frac{1}{2^{N_y}} + \frac{1}{2}\sum_{n=1}^{N_y-1}\frac{a_{ny}}{2^n} \\
&= \frac{1}{2^{N_y}} + \frac{1}{2}\sum_{n=2}^{N_y-1}\frac{a_{ny}}{2^n} \qquad \text{as } a_{1y} = 0 \\
\therefore G(y) &= \frac{1}{2^{N_x+1}} + \frac{1}{2}\sum_{n=2}^{N_x+1-1}\frac{a_{(n-1)x}}{2^n} \\
&= \frac{1}{2\times 2^{N_x}} + \frac{1}{2}\sum_{n=2}^{N_x}\frac{a_{(n-1)x}}{2^n} \\
&= \frac{1}{2\times 2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^{n+1}} \\
&= \frac{1}{2}\times\frac{1}{2^{N_x}} + \frac{1}{2}\times\frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= \frac{1}{2}\left[\frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n}\right] \\
\therefore G(y) &= \frac{1}{2}G(x) \\
\therefore G\left(\frac{x}{3}\right) &= \frac{G(x)}{2} \qquad \forall x \in [0,1]
\end{aligned}$$

We are now required to prove that $G(1-x) = 1 - G(x)$, $\forall x \in [0,1]$. In order to do so we must examine $G(1-x)$, and how it behaves as a ternary representation. When subtracting 1 in base 3, it has the effect of swapping $0$s and 2s. For this proof, we shall define $y := 1 - x$. Furthermore, we will define $y := 0.a_{1y}a_{2y}a_{3y}\ldots$, which is a ternary representation of $y$. As a result, we get the following relationships:

$$\begin{aligned}
a_{ny} &= 0 \text{ if } a_{nx} = 2 \\
a_{ny} &= 1 \text{ if } a_{nx} = 1 \\
a_{ny} &= 2 \text{ if } a_{nx} = 0 \\
\therefore a_{ny} &= 2 - a_{nx} \\
N_y &= N_x
\end{aligned}$$

Now examining the function $G(x)$ evaluated at $x = 1 - x$, or in other words, $x = y$, we get the following result.

$$
\begin{aligned}
G(y) &= \frac{1}{2^{N_y}} + \frac{1}{2}\sum_{n=1}^{N_y-1}\frac{a_{ny}}{2^n} \\
&= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{2-a_{nx}}{2^n} \\
&= \frac{1}{2^{N_x}} + \frac{1}{2}\left[\sum_{n=1}^{N_x-1}\frac{2}{2^n} - \sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n}\right] \\
&= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{2}{2^n} - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= \frac{1}{2^{N_x}} + \sum_{n=1}^{N_x-1}\frac{1}{2^n} - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= \frac{1}{2^{N_x}} + \frac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^{N_x-1} - 1\right)}{\frac{1}{2}-1} - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= \frac{1}{2^{N_x}} + \frac{\frac{1}{2}\left(\frac{1}{2^{N_x-1}} - 1\right)}{-\frac{1}{2}} - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= \frac{1}{2^{N_x}} - \left(\frac{1}{2^{N_x-1}} - 1\right) - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= \frac{1}{2^{N_x}} + 1 - \frac{1}{2^{N_x-1}} - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= 1 + \frac{1}{2^{N_x}} - \frac{1}{2^{N_x-1}} - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= 1 + \frac{1}{2^{N_x}} - \frac{2}{2^{N_x}} - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= 1 + \frac{1-2}{2^{N_x}} - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= 1 - \frac{1}{2^{N_x}} - \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} \\
&= 1 - \left[\frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n}\right] \\
\therefore G(y) &= 1 - G(x) \\
\therefore G(1-x) &= 1 - G(x) \qquad \forall x \in [0,1]
\end{aligned}
$$

(f) We are required to prove that $G(x) = G\left(x - \frac{2}{3}\right) + \frac{1}{2}$, $\forall x \in \left[\frac{2}{3}, 1\right]$. In order to do so, we must examine $x - \frac{2}{3}$, and how it behaves as a ternary representation. For the ternary representation of $x$, $\frac{2}{3} = 0.2$. Furthermore, as $x \in [2/3, 1]$, therefore $a_{1x} = 2$.

$$\begin{aligned} x &= 0.a_{1x}a_{2x}a_{3x}\dots \\ \therefore x &= 0.2a_{2x}a_{3x}\dots \qquad \forall x \in \left[\frac{2}{3}, 1\right] \\ \therefore x - \frac{2}{3} &= 0.2a_{2x}a_{3x}\dots - 0.2 \qquad \forall x \in \left[\frac{2}{3}, 1\right] \\ \therefore x - \frac{2}{3} &= 0.0a_{2x}a_{3x}\dots \qquad \forall x \in \left[\frac{2}{3}, 1\right] \end{aligned}$$

For the proof, we shall define $y \coloneqq x - \frac{2}{3}$. Furthermore, we will define $y \coloneqq 0.a_{1y}a_{2y}a_{3y}\dots$, which is a ternary representation of $y$. As a result, we get the following relationships:

$$\begin{aligned} a_{1y} &= 0 \\ a_{1y} &= a_{1x} - 2 \\ a_{ny} &= a_{nx} \\ N_y &= N_x \end{aligned}$$

Now examining the function $G(x)$ evaluated at $x = x - \frac{2}{3}$, or in other words, $x = y$, we get the following result.

$$\begin{aligned} G(y) &= \frac{1}{2^{N_y}} + \frac{1}{2}\sum_{n=1}^{N_y-1}\frac{a_{ny}}{2^n} \\ &= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=2}^{N_x-1}\frac{a_{nx}}{2^n} + \frac{1}{2}\left[\frac{a_{1x}}{2} - \frac{2}{2}\right] \\ &= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=2}^{N_x-1}\frac{a_{nx}}{2^n} + \frac{1}{2}\left[\frac{a_{1x}}{2}\right] - \frac{1}{2} \\ &= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1}\frac{a_{nx}}{2^n} - \frac{1}{2} \\ \therefore G(y) &= G(x) - \frac{1}{2} \\ \therefore G(x) &= G(y) + \frac{1}{2} \\ \therefore G(x) &= G\left(x - \frac{2}{3}\right) + \frac{1}{2} \qquad \forall x \in \left[\frac{2}{3}, 1\right] \end{aligned}$$

(g) For the following proofs, we define $M_n := \int_0^1 x^n G(x)dx \quad \forall x \in [0,1]$ and $\forall n \in \mathbb{N}$.

For the first proof, we are required to prove the $M_0 = \frac{1}{2}$. Using the definition of $M_n$, and the result from part (e), we get the following result.

$$\begin{aligned} M_n &= \int_0^1 x^n G(x)dx \\ \therefore M_0 &= \int_0^1 x^0 G(x)dx \\ &= \int_0^1 G(x)dx \end{aligned}$$

Let $x = 1-u$, and $\therefore dx = -du$. Furthermore, when $x = 1$, $u = 0$ and when $x = 0$, $u = 1$.

$$\begin{aligned} \therefore M_0 &= \int_1^0 G(1-u)\,(-du) \\ &= \int_0^1 G(1-u)du \\ &= \int_0^1 \big[1 - G(u)\big]du \qquad \text{Using part (e)} \\ &= \int_0^1 1du - \int_0^1 G(u)du \\ &= u\Big|_0^1 - M_0 \\ \therefore M_0 + M_0 &= 1 \\ \therefore M_0 &= \frac{1}{2} \end{aligned}$$

For the second proof, we are required to prove that $\forall n \geq 1$, it holds that $2M_n = \frac{1}{n+1} + \frac{1}{3^{n+1}-1}\sum_{k=0}^{n-1}\binom{n}{k}2^{n-k}M_k$. In order to complete the proof, we will split the integral $M_n$ into three separate integrals over the intervals $\big[0, \frac{1}{3}\big]$, $\big[\frac{1}{3}, \frac{2}{3}\big]$, and $\big[\frac{2}{3}, 1\big]$. We will also need to involve the results from parts (c),(e) and (f).

$$\begin{aligned} M_n &= \int_0^1 x^n G(x)dx \\ &= \int_0^{\frac{1}{3}} x^n G(x)dx + \int_{\frac{1}{3}}^{\frac{2}{3}} x^n G(x)dx + \int_{\frac{2}{3}}^1 x^n G(x)dx \end{aligned}$$

In order to complete the proof in the simplest and most logical method, we will define each integral in the expression above as $I_n$ for $n \in \{1, 2, 3\}$. Therefore we get the following results.

$$\begin{aligned}
I_1 &:= \int_0^{\frac{1}{3}} x^n G(x) dx \\
I_2 &:= \int_{\frac{1}{3}}^{\frac{2}{3}} x^n G(x) dx \\
I_3 &:= \int_{\frac{2}{3}}^{1} x^n G(x) dx \\
M_n &= I_1 + I_2 + I_3
\end{aligned}$$

Now computing each integral individually, we will concisely and logically arrive at the required expression for $2M_n$.

Examining $I_1$, we get the following result.

$$I_1 = \int_0^{\frac{1}{3}} x^n G(x) dx$$

Let $x = \frac{u}{3}$, and $\therefore dx = \frac{1}{3} du$. Furthermore, when $x = \frac{1}{3}$, $u = 1$ and when $x = 0$, $u = 0$.

$$\begin{aligned}
\therefore I_1 &= \int_0^1 \left(\frac{u}{3}\right)^n G\left(\frac{u}{3}\right) \frac{1}{3} du \\
&= \frac{1}{3^{n+1}} \int_0^1 u^n G\left(\frac{u}{3}\right) du \\
&= \frac{1}{3^{n+1}} \int_0^1 (u^n) \frac{G(u)}{2} du \qquad \text{Using part (e)} \\
&= \frac{1}{2 \times 3^{n+1}} \int_0^1 u^n G(u) \, du \\
&= \frac{1}{2 \times 3^{n+1}} \int_0^1 x^n G(x) \, dx \qquad \text{Using dummy variable switch u} \longrightarrow \text{x} \\
\therefore I_1 &= \frac{1}{2 \times 3^{n+1}} M_n
\end{aligned}$$

Examining $I_2$, we get the following result.

$$I_2 = \int_{\frac{1}{3}}^{\frac{2}{3}} x^n G(x) dx$$

Before we manipulate the integral $I_2$ using part (c), we must compute the value of $G(x)$, at $x = \frac{1}{3}$ and at $x = \frac{2}{3}$. First, examining $G(x)$ at $x = \frac{1}{3}$, we know that $x = 0.1$, the ternary representation of the location of the point $x = \frac{1}{3}$. Therefore, $a_{1x} = 1$, and $N_x = 1$. Thus $\forall n \in \mathbb{N}$, and $\forall n > 1$, $a_{nx} = 0$. As a consequence, we can compute the value of $G(x)$ at $x = \frac{1}{3}$.

$$\begin{aligned}
G(x) &= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1} \frac{a_{nx}}{2^n} \\
\therefore G\left(\frac{1}{3}\right) &= \frac{1}{2^1} + \frac{1}{2}\sum_{n=1}^{1-1} \frac{a_{nx}}{2^n} \\
&= \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{0} \frac{a_{nx}}{2^n} \\
&= \frac{1}{2} + 0 \\
\therefore G\left(\frac{1}{3}\right) &= \frac{1}{2}
\end{aligned}$$

Now, examining $G(x)$ at $x = \frac{2}{3}$, we know that $x = 0.2$, the ternary representation of the location of the point $x = \frac{2}{3}$. Therefore, $a_{1x} = 2$, and $N_x = \infty$. Thus $\forall n \in \mathbb{N}$, and $\forall n > 1$, $a_{nx} = 0$. As a consequence, we can compute the value of $G(x)$ at $x = \frac{2}{3}$.

$$\begin{aligned}
G(x) &= \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1} \frac{a_{nx}}{2^n} \\
\therefore G\left(\frac{2}{3}\right) &= \frac{1}{2^\infty} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{a_{nx}}{2^n} \\
&= 0 + \frac{1}{2}\left[\sum_{n=2}^{\infty} \frac{0}{2^n} + \frac{2}{2^1}\right] \\
&= \frac{1}{2}\left[0 + 1\right] \\
&= 0 + \frac{1}{2} \\
\therefore G\left(\frac{2}{3}\right) &= \frac{1}{2}
\end{aligned}$$

Combining these results with the result from part (c), we get the consequence that $\forall x \in \left[\frac{1}{3}, \frac{2}{3}\right]$, $G(x) = \frac{1}{2}$. Thus using this result to manipulate the integral $I_2$, we get the following result.

$$\begin{aligned}
I_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}} x^n G(x) dx \\
\therefore I_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}} x^n \left(\frac{1}{2}\right) dx \\
&= \left(\frac{1}{2}\right) \int_{\frac{1}{3}}^{\frac{2}{3}} x^n dx \\
&= \frac{1}{2(n+1)} x^{n+1} \Big|_{\frac{1}{3}}^{\frac{2}{3}} \\
&= \frac{1}{2(n+1)} \left(\frac{2}{3}\right)^{n+1} - \frac{1}{n+1} \left(\frac{1}{3}\right)^{n+1} \\
&= \frac{1}{2(n+1)3^{n+1}} (2^{n+1} - 1^{n+1}) \\
\therefore I_2 &= \frac{2^{n+1} - 1}{(n+1)2 \times 3^{n+1}}
\end{aligned}$$

Examining $I_3$, we get the following result.

$$I_3 = \int_{\frac{2}{3}}^{1} x^n G(x) dx$$

Let $x = u + \frac{2}{3}$, and $\therefore dx = du$. Furthermore, when $x = \frac{2}{3}$, $u = 0$ and when $x = 1$, $u = \frac{1}{3}$.

$$\begin{aligned}
\therefore I_3 &= \int_0^{\frac{1}{3}} \left(u + \frac{2}{3}\right)^n G\left(u + \frac{2}{3}\right) du \\
&= \int_0^{\frac{1}{3}} \left(u + \frac{2}{3}\right)^n \left[G(u) + \frac{1}{2}\right] du \quad \text{Using part (f)}
\end{aligned}$$

Now, let $u = \frac{v}{3}$, and $\therefore dx = \frac{1}{3}dv$. Furthermore, when $u = \frac{1}{3}$, $v = 1$ and when $u = 0$, $v = 0$.

$$
\begin{aligned}
\therefore I_3 &= \int_0^1 \left(\frac{v}{3} + \frac{2}{3}\right)^n \left[G\left(\frac{v}{3}\right) + \frac{1}{2}\right] \frac{1}{3} dv \\
&= \frac{1}{3} \int_0^1 \left(\frac{v+2}{3}\right)^n \left[\frac{G(v)}{2} + \frac{1}{2}\right] dv \qquad \text{Using part (e)} \\
&= \frac{1}{3} \int_0^1 \left(\frac{v+2}{3}\right)^n \left[\frac{G(v)+1}{2}\right] dv \\
&= \frac{1}{2 \times 3^{n+1}} \int_0^1 (v+2)^n \left[G(v)+1\right] dv \\
&= \frac{1}{2 \times 3^{n+1}} \left[\int_0^1 (v+2)^n G(v) dv + \int_0^1 (v+2)^n\right] \\
&= \frac{1}{2 \times 3^{n+1}} \left[\int_0^1 (v+2)^n G(v) dv + \frac{(v+2)^{n+1}}{n+1}\Bigg|_0^1\right] \\
&= \frac{1}{2 \times 3^{n+1}} \left[\int_0^1 (v+2)^n G(v) dv + \frac{1}{n+1}\left(3^{n+1}\right) - \frac{1}{n+1}\left(2^{n+1}\right)\right] \\
&= \frac{1}{2 \times 3^{n+1}} \left[\int_0^1 (v+2)^n G(v) dv + \frac{1}{n+1}\left(3^{n+1} - 2^{n+1}\right)\right] \\
&= \frac{1}{2 \times 3^{n+1}} \left[\int_0^1 (v+2)^n G(v) dv + \frac{\left(3^{n+1} - 2^{n+1}\right)}{n+1}\right] \\
\therefore I_3 &= \frac{1}{2 \times 3^{n+1}} \left[\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} M_k + \frac{\left(3^{n+1} - 2^{n+1}\right)}{n+1}\right]
\end{aligned}
$$

Using the above results for $I_1$, $I_2$, and $I_3$, we get the following result for $M_n$.

$$
\begin{aligned}
M_n &= I_1 + I_2 + I_3 \\
&= \frac{1}{2\times 3^{n+1}} M_n + \frac{2^{n+1}-1}{(n+1)2\times 3^{n+1}} + \frac{1}{2\times 3^{n+1}}\left[\sum_{k=0}^{n}\binom{n}{k}2^{n-k}M_k + \frac{\left(3^{n+1}-2^{n+1}\right)}{n+1}\right] \\
&= \frac{1}{2\times 3^{n+1}}\left[M_n + \frac{2^{n+1}-1}{n+1} + \sum_{k=0}^{n}\binom{n}{k}2^{n-k}M_k + \frac{3^{n+1}-2^{n+1}}{n+1}\right] \\
&= \frac{1}{2\times 3^{n+1}}\left[M_n + \frac{2^{n+1}}{n+1} - \frac{1}{n+1} + \frac{3^{n+1}}{n+1} - \frac{2^{n+1}}{n+1} + \sum_{k=0}^{n}\binom{n}{k}2^{n-k}M_k\right] \\
&= \frac{1}{2\times 3^{n+1}}\left[M_n + \frac{3^{n+1}-1}{n+1} + \sum_{k=0}^{n}\binom{n}{k}2^{n-k}M_k\right] \\
\therefore \left(2\times 3^{n+1}\right)(M_n) &= M_n + \frac{3^{n+1}-1}{n+1} + \sum_{k=0}^{n}\binom{n}{k}2^{n-k}M_k \\
2M_n\left(3^{n+1}\right) - M_n &= \frac{3^{n+1}-1}{n+1} + \sum_{k=0}^{n}\binom{n}{k}2^{n-k}M_k \\
&= \frac{3^{n+1}-1}{n+1} + \sum_{k=0}^{n-1}\binom{n}{k}2^{n-k}M_k + \binom{n}{n}2^{n-n}M_n \\
&= \frac{3^{n+1}-1}{n+1} + \sum_{k=0}^{n-1}\binom{n}{k}2^{n-k}M_k + M_n \\
2M_n\left(3^{n+1}\right) - 2M_n &= \frac{3^{n+1}-1}{n+1} + \sum_{k=0}^{n-1}\binom{n}{k}2^{n-k}M_k \\
2M_n\left(3^{n+1}-1\right) &= \frac{3^{n+1}-1}{n+1} + \sum_{k=0}^{n-1}\binom{n}{k}2^{n-k}M_k \\
\therefore 2M_n &= \frac{1}{n+1} + \frac{1}{3^{n+1}-1}\sum_{k=0}^{n-1}\binom{n}{k}2^{n-k}M_k \qquad \forall n \in \mathbb{N} \text{ and } \forall n \geq 1
\end{aligned}
$$

For the third proof, we are required to prove that $\forall n \geq 1$, it holds that $(1+(-1)^n)\, M_n = \dfrac{1}{n+1} + \displaystyle\sum_{k=0}^{n-1} (-1)^{k+1} \binom{n}{k} M_k$. In order to complete the proof, we will need to incorporate the results from parts (c), (e) and (f). In doing so, we get the following result.

$$M_n = \int_0^1 x^n G(x) dx$$

Let $x = 1 - u$, and $\therefore dx = -du$. Furthermore, when $x = 1$, $u = 0$ and when $x = 0$, $u = 1$.

$$\begin{aligned}
\therefore M_n &= \int_1^0 (1-u)^n\, G(1-u)\, (-du) \\
&= -\int_1^0 (1-u)^n\, G(1-u) du \\
&= \int_0^1 (1-u)^n \left[1 - G(u)\right] du \qquad \text{Using part (e)} \\
&= \int_0^1 (1-u)^n\, du - \int_0^1 (1-u)^n\, G(u) du \\
&= -\left[\frac{(1-u)^{n+1}}{n+1}\right]\Bigg|_0^1 - \int_0^1 (1-u)^n\, G(u) du \\
&= -\left[\frac{(1-u)^{n+1}}{n+1}\right]\Bigg|_0^1 - \sum_{k=0}^{n} (-1)^k \binom{n}{k} M_k \\
&= -\left[0 - \frac{1^{n+1}}{n+1}\right] + \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} M_k \\
&= \frac{1}{n+1} + \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n}{k} M_k + (-1)^{n+1} \binom{n}{n} M_n \\
&= \frac{1}{n+1} + \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n}{k} M_k - (-1)^n\, M_n \\
\therefore M_n + (-1)^n\, M_n &= \frac{1}{n+1} + \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n}{k} M_k \\
\therefore (1 + (-1)^n)\, M_n &= \frac{1}{n+1} + \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n}{k} M_k \qquad \forall n \in \mathbb{N} \text{ and } \forall n \geq 1
\end{aligned}$$