The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Exercises for Week 7** 2017

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) A *linear equation* in variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \;=\; b$$

where $a_1, a_2, \ldots, a_n, b$ are constants.

(ii) A *system* of linear equations has the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

with *augmented matrix*

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

The system is *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$.

(iii) Every system of linear equations has either no solutions, one solution, or infinitely many solutions. If it has no solutions then the system is called *inconsistent*. If it has at least one solution then the system is called *consistent*.

(iv) There are three types of *elementary row operations* performed on augmented matrices:

(a) interchanging the ith and jth rows (denoted by $R_i \leftrightarrow R_j$)

(b) multiplying the ith row through by a nonzero constant λ (denoted by $R_i \to \lambda R_i$)

(c) adding a multiple of the jth row to the ith row (denoted by $R_i \to R_i + \lambda R_j$)

(v) A matrix is in *row echelon form* if

(a) rows of zeros appear at the bottom,

(b) first nonzero (*leading*) entries of consecutive rows appear further to the right,

(c) leading entries of rows are equal to 1;

and in *reduced row echelon form* if, in addition,

(d) entries above (and below) leading entries are zero.

(vi) The process of *Gaussian elimination* applies elementary row operations (*row reduction*) to transform the augmented matrix of a system into row echelon form, after which the associated system is solved using *back substitution*:

(a) the *leading variables* corresponding to leading entries are evaluated one equation at a time from the bottom towards the top,

(b) parameters are assigned to each nonleading variable (if any).

(vii) A system is inconsistent if and only if at some stage in the process of row reduction a row of the following form is produced for some nonzero real number k:

$$0 \quad 0 \quad \cdots \quad 0 \quad | \quad k$$

(viii) The process of *Gauss-Jordan elimination* row reduces the augmented matrix to reduced row echelon form, after which the process of back substitution simplifies. However, Gauss-Jordan elimination is usually less efficient in terms of the overall number of arithmetic operations used than Gaussian elimination.

(ix) A given matrix can be row reduced to one and only one matrix in reduced row echelon form.

Preparatory Exercises:

1. Solve the following systems of equations:

(i)
$$\begin{aligned} x + y &= 6 \\ 2x - 3y &= 2 \end{aligned}$$

(ii)
$$\begin{aligned} x + 2y + 3z &= 6 \\ y - z &= 3 \\ 2z &= -4 \end{aligned}$$

2. Solve the following systems of equations by writing down the associated augmented matrix and row reducing, either to echelon form and using back substitution, or to reduced echelon form and reading off the answers:

(i)
$$\begin{aligned} x + y - z &= 0 \\ 2x - y + z &= 9 \\ x + z &= 10 \end{aligned}$$

(ii)
$$\begin{aligned} -3x + 2y + z &= 4 \\ 4x + y + 3z &= 9 \\ x - y - z &= -4 \end{aligned}$$

3. Solve the following systems of equations by assigning $z = t$ and expressing x and y in terms of the parameter t:

(i)
$$\begin{aligned} x - 2z &= 4 \\ y + z &= 2 \end{aligned}$$

(ii)
$$\begin{aligned} x + 2y + 3z &= 0 \\ y - 2z &= -1 \end{aligned}$$

4. Solve the following systems of equations:

(i)
$$\begin{aligned} 4x - 5y &= 7 \\ -3x + 8y &= -1 \end{aligned}$$

(ii)
$$\begin{aligned} x + 2y + z &= 1 \\ -x + y + 2z &= 2 \\ 2x + 3y + 2z &= 5 \end{aligned}$$

5. Find parametric scalar equations for the line of intersection of the two planes in each of the following cases:

(i)
$$\begin{aligned} x + y + z &= 2 \\ x - y + 3z &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} -3x + 2y + 7z &= 1 \\ 5x - 3y - 2z &= -2 \end{aligned}$$

Exercises:

16. For each of the following augmented matrices, decide whether the system of equations to which it corresponds has (a) no solution, (b) a unique solution, or (c) an infinite solution.

(i) $\left[\begin{array}{rrr|r} 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 5 \\ -1 & -4 & 0 & 1 \end{array}\right]$ (ii) $\left[\begin{array}{rrr|r} 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 5 \\ -1 & -4 & 0 & -1 \end{array}\right]$

(iii) $\left[\begin{array}{rrr|r} 1 & 4 & 2 & 3 \\ -1 & -2 & 3 & 6 \\ -1 & -3 & 0 & 1 \end{array}\right]$ (iv) $\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 3 & 1 & -1 \\ -2 & -2 & 4 & 6 & 2 & 0 \\ 0 & 0 & 0 & -3 & -1 & 4 \end{array}\right]$

17. By solving a system of four equations in x, y and z, find the point of intersection of the following two lines in space, expressed in Cartesian form:

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5} \qquad \text{and} \qquad \frac{x-8}{7} = y-4 = \frac{z-5}{3}$$

18.* Prove that the following system in x, y, z is inconsistent:

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} + \frac{3}{z} &= 0 \\ \frac{4}{x} + \frac{5}{y} + \frac{6}{z} &= 1 \\ \frac{7}{x} + \frac{10}{y} + \frac{9}{z} &= 2 \end{aligned}$$

19.* The combustion of petrol is described by the chemical reaction

$$C_8H_{18} + O_2 \longrightarrow CO_2 + H_2O\ .$$

Find the smallest positive integer values for x, y, z and w such that

$$x\,C_8H_{18} + y\,O_2 \longrightarrow z\,CO_2 + w\,H_2O$$

is balanced, in the sense that the number of atoms of each type agree on the left and right hand sides.

20.* Use row reduction on augmented matrices to show that the lines $ax+by=k$ and $cx+dy=\ell$ intersect in a single point if and only if $ad-bc \neq 0$.

21.* Let $\mathcal{S}$ denote the collection of systems of m equations in n variables $x_1, \ldots, x_n$ where m and n are positive integers. Let $\mathcal{A}$ denote the set of augmented matrices with m rows and $n+1$ columns. Let $f : \mathcal{S} \to \mathcal{A}$ where $f(\sigma)$ is the augmented matrix of a given system σ. Let ρ denote an elementary operation either on a system or on a matrix, read in context. Make sense of the following:

(i) f is one-one and onto (ii) $f(\rho\sigma) = \rho f(\sigma)$ for all ρ and σ

22.* Given a system of at least two linear equations, is the operation of simultaneously applying the elementary row operations $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_1$ reversible? If not, why not?

23.* Catalogue the possible reduced row echelon forms for coefficient matrices for the following homogeneous system in α, β, γ, δ:

$$\begin{array}{ccccccccc} u_1\alpha & + & v_1\beta & + & w_1\gamma & + & t_1\delta & = & 0 \\ u_2\alpha & + & v_2\beta & + & w_2\gamma & + & t_2\delta & = & 0 \\ u_3\alpha & + & v_3\beta & + & w_3\gamma & + & t_3\delta & = & 0 \end{array}$$

Find a simple connection that explains why any four geometric vectors in space must be linearly dependent.

24.** Prove that for any 2×3 matrix, the reduced row echelon form is unique, that is, no matter how one applies elementary row operations, it is possible to reach one and only one reduced row echelon matrix. (Now think about matrices of any size.)

25.** This exercise illustrates the fact that Gaussian elimination is more efficient than Gauss-Jordan elimination. In the following, all leading coefficients are assumed to be nonzero.

(i) Count the number $f(n)$ of arithmetic operations needed to find the values $x_1, x_2, \ldots, x_n$ by back substitution, given that the following equations hold:

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ & & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & & & \ddots & & \vdots & & \vdots \\ & & & & & & & & a_{nn}x_n & = & b_n \end{array}$$

(ii) Count the number $g(n)$ of arithmetic operations needed to bring the following matrix into reduced row echelon form, using row reduction from the top towards the bottom of the matrix:

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} & b_1 \\ 0 & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} & b_2 \\ 0 & 0 & a_{33} & \cdots & a_{3,n-1} & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} & b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} & b_n \end{array}\right]$$

(iii) Verify that $\lim_{n\to\infty} \dfrac{g(n)}{f(n)} = \infty$.

Short Answers to Selected Exercises:

1. (i) $x = 4\ ,\ \ y = 2$ (ii) $x = 10\ ,\ \ y = 1\ ,\ \ z = -2$

2. (i) $x = 3\ ,\ \ y = 4\ ,\ \ z = 7$ (ii) $x = -1\ ,\ \ y = -2\ ,\ \ z = 5$

3. (i) $x = 4 + 2t\ ,\ \ y = 2 - t\ ,\ \ z = t$ (ii) $x = 2 - 7t\ ,\ \ y = -1 + 2t\ ,\ \ z = t$

4. (i) $x = 3\ ,\ \ y = 1$ (ii) $x = 3\ ,\ \ y = -3\ ,\ \ z = 4$

5. (i) $x = 1 - 2t\ ,\ \ y = 1 + t\ ,\ \ z = t$ (ii) $x = -1 - 17t\ ,\ \ y = -1 - 29t\ ,\ \ z = t$

6. (i) $x = t\ ,\ \ y = -2t\ ,\ \ z = t$ (ii) $x = -\frac{2}{7}t\ ,\ \ y = \frac{8}{7}t\ ,\ \ z = -\frac{3}{7}t\ ,\ \ w = t$

7. $x_1 = -\frac{3}{2}s - t\ ,\ \ x_2 = s\ ,\ \ x_3 = -t\ ,\ \ x_4 = t\ ,\ \ x_5 = t$

8. Assigning each variable the value 0 yields at least one solution.

9. (i) inconsistent, no solution (ii) $x = 1 - 2s - t\ ,\ \ y = s\ ,\ \ z = 3 + 2t\ ,\ \ w = t$

10. $x^3 - 3x^2 + 3x - 3$

11. $A = 30\ ,\ \ B = 30\ ,\ \ C = 4\ ,\ \ D = 6$

12. $A = 1\ ,\ \ B = 3\ ,\ \ C = 3\ ,\ \ D = 1$

15. (i) $\lambda = -5$ (ii) $\lambda = 2$ (iii) $\lambda \neq 2, -5$

16. (i) (c) (ii) (a) (iii) (b) (iv) (a)

17. $x = 1\ ,\ \ y = 3\ ,\ \ z = 2$

19. $x = 2\ ,\ \ y = 25\ ,\ \ z = 16\ ,\ \ w = 18$

22. No, this operation is not one-one.