THE UNIVERSITY OF SYDNEY
MATH1901/06 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Short answers to exam questions** **2010**

1. (a) Ellipse and its interior. Horizontal semiaxis $a = 5/2$, vertical semiaxis $b = 3/2$.

(b) Factorisation: $P(z) = (z^2 - 4z + 5)(z^2 + 2z + 2)$.

(c) Surjective because $z \mapsto z^3$ maps the 120° closed sector to the full complex plane. Not injective because points x and $xe^{2\pi i/3}$, $x > 0$, on the sides of the sector, map to the same image point x^3 in the codomain.

2. (a) (i). $f(x,y) = \tan^{-1}(x^2 + 3y^2)$, $\nabla f = \dfrac{2x\,\mathbf{i} + 6y\,\mathbf{j}}{1 + (x^2+3y^2)^2}$, $\nabla f(2,1) = \dfrac{2\,\mathbf{i} + 3\,\mathbf{j}}{25}$.

Directional derivative: $D_{\mathbf{u}} f(2,1) = \nabla f \cdot \hat{\mathbf{u}} = 1/(5\sqrt{17})$.

(ii). Tangent plane: $z = (2x + 3y - 7)/25 + \tan^{-1}(7)$.

(b) (i). Taylor series ($T_7(x)$ plus dots): $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots$, $\sinh x = x + x^3/3! + x^5/5! + x^7/7! + \dots$, divide both by x, then let $x \to \sqrt{x}$, $x \geq 0$, in first series, $x \to \sqrt{-x}$, $x < 0$, in second series.

Result: $g(x) = 1 - x/3! + x^2/5! - x^3/7! + \dots$ (two-sided).

(ii). $g'(0) = -1/6$, $g''(0) = 2!/5! = 1/60$, $g'''(0) = -3!/7! = -1/840$.

3. (a) (i). Limit is 5/6. (Cancel fraction to $\dfrac{x+3}{x+4}$ or use l'Hôpital's rule twice.)

(ii). Limit is 2. (L'Hôpital's rule or comparison with $(2\ln x)/(\ln 2x)$.)

(iii). No limit. (Let $y = x + \lambda x^2$ or $x + \lambda x^3$, limit depends on path.)

(iv). Limit is 0. (Use polar coordinates.)

(b) By MVT, $\dfrac{f(x) - f(a)}{x - a} = f'(x_1)$ for some x_1, $a < x_1 < x$. But $f'(x) \to L$ forces $f'(x_1) \to L$ as $x \to a^+$, and so $f'_+(a)$ exists and equals L.

4. (a) (i). Answer $f'(x) = -\sqrt{a^2 - x^2}/x$ given, use chain rule and simplify.

(ii). Slope of tangent: $-\sqrt{a^2 - c^2}/c$.

Equation of tangent: $y = a\cosh^{-1}(a/c) - x\sqrt{a^2 - c^2}/c$.

(iii). y-intercept of tangent is $a\cosh^{-1}(a/c)$. Distance to P is a, indep of c.

(b) Substituting $T_3(y)$ for $f(y)$ about $y = x$, leaving $f(x)$ as is, gives
$G(x,y) = f'(x) + f''(x)(y-x)/2 + f'''(x)(y-x)^2/6 + \dots$,
$G_y(x,y) = f''(x)/2 + f'''(x)(y-x)/3 + \dots$, and so $G_y(x,x) = f''(x)/2$.
Alternatively: use l'Hôpital on $(G(x,y) - G(x,x))/(y-x)$ or on $G_y(x,y)$.
Quick method (assuming answer exists): $G_x(x,x) + G_y(x,x) = f''(x)$ by chain rule, but symmetry $G_x(a,b) = G_y(b,a)$ implies $G_x(x,x) = G_y(x,x)$.