

THE UNIVERSITY OF SYDNEY
MATH1902 LINEAR ALGEBRA (ADVANCED)

Semester 1

Longer Solutions to Selected Exercises for Week 3

2017

7. (i) First observe that $\overrightarrow{QP} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{QR} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, yielding

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = 2 + 4 + 2 = 8 > 0,$$

so that $\angle PQR$ is acute (nonzero, since the vectors are clearly not parallel). Now observe that $\overrightarrow{RQ} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{RS} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, yielding

$$\overrightarrow{RQ} \cdot \overrightarrow{RS} = -2 - 4 - 2 = -8 < 0,$$

so that $\angle QRS$ is obtuse (not 180° , since the vectors are clearly not parallel).

- (ii) Observe that

$$\overrightarrow{PR} \cdot \overrightarrow{QS} = (-\mathbf{i} + \mathbf{k}) \cdot (3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) = -3 + 0 + 3 = 0,$$

so PR and QS are mutually perpendicular.

8. The vectors are orthogonal because

$$\begin{aligned} \mathbf{w} \cdot \left(\mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w} \right) &= \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w} \right) = \mathbf{w} \cdot \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} (\mathbf{w} \cdot \mathbf{w}) \\ &= \mathbf{w} \cdot \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} |\mathbf{w}|^2 = \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} \\ &= \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} = 0. \end{aligned}$$

10. Observe that

$$\begin{aligned} (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} - |\mathbf{b}|^2 \\ &= |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0, \end{aligned}$$

so that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are orthogonal.

11. (i) The cosine of the angle between a major diagonal and an edge is

$$\frac{|\mathbf{i} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})|}{|\mathbf{i}| |\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{1}{\sqrt{3}},$$

yielding an angle of approximately 55 degrees.

(ii) The cosine of the angle between a major diagonal and a face diagonal is

$$\frac{|(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})|}{|\mathbf{i} + \mathbf{j}| |\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{2}{\sqrt{6}},$$

yielding an angle of approximately 35 degrees.

(iii) The cosine of the angle between diagonals on adjacent faces is

$$\frac{|(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \mathbf{k})|}{|\mathbf{i} + \mathbf{j}| |\mathbf{i} + \mathbf{k}|} = \frac{1}{2},$$

yielding an angle of exactly 60 degrees.

(iv) The cosine of the angle between major diagonals is

$$\frac{|(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k})|}{|\mathbf{i} + \mathbf{j} + \mathbf{k}| |\mathbf{i} - \mathbf{j} + \mathbf{k}|} = \frac{1}{3},$$

yielding an angle of approximately 71 degrees.

12. (i) If $\mathbf{v} \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{y} = 0$ then

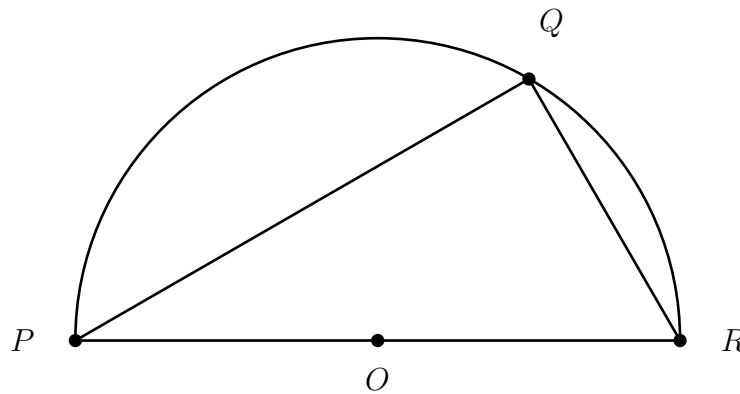
$$\mathbf{v} \cdot (a\mathbf{x} + b\mathbf{y}) = a(\mathbf{v} \cdot \mathbf{x}) + b(\mathbf{v} \cdot \mathbf{y}) = a0 + b0 = 0,$$

for any scalars a and b .

(ii) If $\mathbf{v} \cdot \mathbf{x} = \mathbf{v} \cdot (a\mathbf{x} + b\mathbf{y}) = 0$, where a and b are scalars such that $b \neq 0$, then

$$\begin{aligned} \mathbf{v} \cdot \mathbf{y} &= \mathbf{v} \cdot \left(-\frac{a}{b}\mathbf{x} + \frac{a}{b}\mathbf{x} + \mathbf{y} \right) = \mathbf{v} \cdot \left(-\frac{a}{b}\mathbf{x} + \frac{1}{b}(a\mathbf{x} + b\mathbf{y}) \right) \\ &= -\frac{a}{b}(\mathbf{v} \cdot \mathbf{x}) + \frac{1}{b}(\mathbf{v} \cdot (a\mathbf{x} + b\mathbf{y})) = -\frac{a}{b}0 + \frac{1}{b}0 = 0. \end{aligned}$$

13. Consider a semicircle and points O, P, Q, R as shown:



To show the angle at Q is a right-angle, it suffices to show $\overrightarrow{QP} \cdot \overrightarrow{QR} = 0$. But this follows from Exercise 10, since

$$\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP} \quad \text{and} \quad \overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR} = \overrightarrow{QO} - \overrightarrow{OP}$$

and the fact that $|\overrightarrow{QO}| = |\overrightarrow{OP}|$, the radius of the circle.

14. Note first, by commutativity, that also $\mathbf{j} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j} = 0$. Thus, if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{w} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$ then, by distributivity and compatibility of scalar multiplication,

$$\begin{aligned}
 \mathbf{v} \cdot \mathbf{w} &= (a\mathbf{i}) \cdot (d\mathbf{i}) + (a\mathbf{i}) \cdot (e\mathbf{j}) + (a\mathbf{i}) \cdot (f\mathbf{k}) \\
 &\quad + (b\mathbf{j}) \cdot (d\mathbf{i}) + (b\mathbf{j}) \cdot (e\mathbf{j}) + (b\mathbf{j}) \cdot (f\mathbf{k}) \\
 &\quad + (c\mathbf{k}) \cdot (d\mathbf{i}) + (c\mathbf{k}) \cdot (e\mathbf{j}) + (c\mathbf{k}) \cdot (f\mathbf{k}) \\
 &= (ad)(\mathbf{i} \cdot \mathbf{i}) + (ae)(\mathbf{i} \cdot \mathbf{j}) + (af)(\mathbf{i} \cdot \mathbf{k}) \\
 &\quad + (bd)(\mathbf{j} \cdot \mathbf{i}) + (be)(\mathbf{j} \cdot \mathbf{j}) + (bf)(\mathbf{j} \cdot \mathbf{k}) \\
 &\quad + (cd)(\mathbf{k} \cdot \mathbf{i}) + (ce)(\mathbf{k} \cdot \mathbf{j}) + (cf)(\mathbf{k} \cdot \mathbf{k}) \\
 &= (ad)(1) + (ae)(0) + (af)(0) \\
 &\quad + (bd)(0) + (be)(1) + (bf)(0) \\
 &\quad + (cd)(0) + (ce)(0) + (cf)(1) \\
 &= ad + be + cf.
 \end{aligned}$$

15. Given $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{w} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$ and the geometric definition, we have, applying the Cosine Rule and the length formula,

$$\begin{aligned}
 \mathbf{v} \cdot \mathbf{w} &= |\mathbf{v}||\mathbf{w}|\cos\theta = \frac{1}{2}(|\mathbf{v}|^2 + |\mathbf{w}|^2 - |\mathbf{v} - \mathbf{w}|^2) \\
 &= \frac{1}{2}\left(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - ((a-d)^2 + (b-e)^2 + (c-f)^2)\right) \\
 &= \frac{1}{2}(2ad + 2be + 2cf) = ad + be + cf.
 \end{aligned}$$

16. The projection of \mathbf{u} in the direction of \mathbf{v} is

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \frac{15 - 6 + 12}{9 + 36 + 4} \mathbf{v} = \frac{3}{7}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}).$$

The component of \mathbf{u} orthogonal to \mathbf{v} is

$$\mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \mathbf{u} - \frac{3}{7}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) = \frac{1}{7}(26\mathbf{i} + 25\mathbf{j} + 36\mathbf{k}).$$

Thus, the decomposition of \mathbf{u} as a sum of vectors, the first parallel to \mathbf{v} and the second perpendicular to \mathbf{v} , is

$$\mathbf{u} = \frac{3}{7}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) + \frac{1}{7}(26\mathbf{i} + 25\mathbf{j} + 36\mathbf{k}).$$

17. (i) The component of the force in the direction of $-\mathbf{i} + \mathbf{j}$ is

$$\frac{(15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j})}{|-\mathbf{i} + \mathbf{j}|^2} (-\mathbf{i} + \mathbf{j}) = \frac{5}{2}(-\mathbf{i} + \mathbf{j}) \text{ newtons},$$

and the component orthogonal to $-\mathbf{i} + \mathbf{j}$ is

$$15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} - \frac{5}{2}(-\mathbf{i} + \mathbf{j}) = \frac{1}{2}(35\mathbf{i} + 35\mathbf{j} + 12\mathbf{k}) \text{ newtons}.$$

(ii) The component of the force in the direction of $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is

$$\frac{(15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + \mathbf{k})}{|2\mathbf{i} - 3\mathbf{j} + \mathbf{k}|^2} (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = -\frac{12}{7}(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \text{ newtons ,}$$

and the component orthogonal to $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is

$$15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} + \frac{12}{7}(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = \frac{1}{7}(129\mathbf{i} + 104\mathbf{j} + 54\mathbf{k}) \text{ newtons .}$$

18. If \mathbf{a} and \mathbf{b} are mutually perpendicular then $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = 0$, so that

$$\begin{aligned} |\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 + 0 + 0 + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 . \end{aligned}$$

This is just the usual Theorem of Pythagoras where \mathbf{a} and \mathbf{b} label directed edges of a right-angled triangle.

19. Certainly \mathbf{a} and \mathbf{b} are perpendicular since $\mathbf{a} \cdot \mathbf{b} = 10 - 2 - 8 = 0$. Suppose $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is perpendicular to both \mathbf{a} and \mathbf{b} , so $\mathbf{v} \cdot \mathbf{a} = \mathbf{v} \cdot \mathbf{b} = 0$, yielding

$$2v_1 - v_2 + 4v_3 = 5v_1 + 2v_2 - 2v_3 = 0 .$$

Thus $2v_1 - v_2 = -4v_3$ and $5v_1 + 2v_2 = 2v_3$, which solving simultaneously yield

$$v_1 = -\frac{2}{3}v_3 \quad \text{and} \quad v_2 = \frac{8}{3}v_3 ,$$

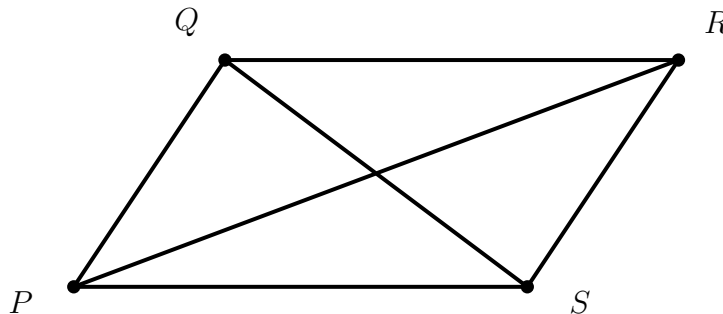
so that

$$\mathbf{v} = \frac{v_3}{3}(-2\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}) .$$

of length $|v_3|\sqrt{77}/3$. It follows quickly that the only unit vectors perpendicular to both \mathbf{a} and \mathbf{b} are

$$\pm \frac{1}{\sqrt{77}}(2\mathbf{i} - 8\mathbf{j} - 3\mathbf{k}) .$$

20. Consider the following parallelogram, and put $\mathbf{v} = \overrightarrow{PQ}$ and $\mathbf{w} = \overrightarrow{QR}$:



Then the sum of the squares of the lengths of the diagonals is

$$\begin{aligned}
|\overrightarrow{PR}|^2 + |\overrightarrow{QS}|^2 &= \overrightarrow{PR} \cdot \overrightarrow{PR} + \overrightarrow{QS} \cdot \overrightarrow{QS} \\
&= (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) + (\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) \\
&= \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v} \\
&= 2\mathbf{v} \cdot \mathbf{v} + 2\mathbf{w} \cdot \mathbf{w} = 2(|\mathbf{v}|^2 + |\mathbf{w}|^2),
\end{aligned}$$

which is the sum of the squares of the lengths of the sides.

21. The diagonals of the parallelogram of the previous exercise are perpendicular if and only if

$$\overrightarrow{QS} \cdot \overrightarrow{PR} = 0,$$

that is,

$$(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = |\mathbf{v}|^2 - |\mathbf{w}|^2 = 0,$$

that is,

$$|\mathbf{v}|^2 = |\mathbf{w}|^2,$$

that is,

$$|\mathbf{v}| = |\mathbf{w}|,$$

that is, the parallelogram is a rhombus.

22. By algebraic properties of the dot product,

$$\begin{aligned}
&(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{d} - \mathbf{c}) + (\mathbf{b} - \mathbf{c}) \cdot (\mathbf{d} - \mathbf{a}) + (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{b}) \\
&= \mathbf{a} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{d} + \mathbf{c} \cdot \mathbf{a} \\
&\quad + \mathbf{c} \cdot \mathbf{d} - \mathbf{c} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{d} + \mathbf{a} \cdot \mathbf{b} \\
&= \mathbf{a} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{d} + \mathbf{d} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{b} \\
&\quad - \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{d} + \mathbf{c} \cdot \mathbf{d} \\
&= \mathbf{a} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} \\
&\quad - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{d} + \mathbf{c} \cdot \mathbf{d} \\
&= 0 + 0 + 0 + 0 + 0 + 0 = 0.
\end{aligned}$$

We apply this identity where \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} are the position vectors of points A , B , C , D respectively, where ABC is a triangle and D is the point of intersection of the altitudes through A and B . Then

$$\overrightarrow{BA} = \mathbf{a} - \mathbf{b}, \quad \overrightarrow{AC} = \mathbf{c} - \mathbf{a}, \quad \overrightarrow{CB} = \mathbf{b} - \mathbf{c}.$$

The altitude through A is parallel to $\overrightarrow{AD} = \mathbf{d} - \mathbf{a}$ and the altitude through B is parallel to $\overrightarrow{BD} = \mathbf{d} - \mathbf{b}$, so that

$$(\mathbf{b} - \mathbf{c}) \cdot (\mathbf{d} - \mathbf{a}) = 0, \quad \text{and} \quad (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{b}) = 0.$$

The earlier identity immediately implies

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{d} - \mathbf{c}) = 0,$$

so that $\overrightarrow{DC} = \mathbf{d} - \mathbf{c}$ is perpendicular to \overrightarrow{AB} . Hence the altitude through C is parallel to \overrightarrow{DC} , which implies that D lies on it. This proves that the three altitudes intersect.

23. Suppose that PQR is a triangle and A, B, C are midpoints of QR, PR, PQ respectively. Let D be the intersection of the perpendicular bisectors of PR and QR , so

$$\overrightarrow{BD} \cdot \overrightarrow{PR} = \overrightarrow{AD} \cdot \overrightarrow{QR} = 0.$$

Then

$$\begin{aligned} \overrightarrow{CD} \cdot \overrightarrow{PQ} &= (\overrightarrow{CB} + \overrightarrow{BD}) \cdot \overrightarrow{PQ} = \overrightarrow{CB} \cdot \overrightarrow{PQ} + \overrightarrow{BD} \cdot \overrightarrow{PQ} \\ &= \frac{1}{2} \overrightarrow{QR} \cdot \overrightarrow{PQ} + \overrightarrow{BD} \cdot (\overrightarrow{PR} + \overrightarrow{RQ}) = \frac{1}{2} \overrightarrow{QR} \cdot \overrightarrow{PQ} + \overrightarrow{BD} \cdot \overrightarrow{PR} + \overrightarrow{BD} \cdot \overrightarrow{RQ} \\ &= \frac{1}{2} \overrightarrow{QR} \cdot \overrightarrow{PQ} + 0 + (\overrightarrow{BA} + \overrightarrow{AD}) \cdot \overrightarrow{RQ} = \frac{1}{2} \overrightarrow{QR} \cdot \overrightarrow{PQ} + \overrightarrow{BA} \cdot \overrightarrow{RQ} + \overrightarrow{AD} \cdot \overrightarrow{RQ} \\ &= \frac{1}{2} \overrightarrow{QR} \cdot \overrightarrow{PQ} + \frac{1}{2} \overrightarrow{PQ} \cdot \overrightarrow{RQ} + 0 = \frac{1}{2} \overrightarrow{QR} \cdot \overrightarrow{PQ} - \frac{1}{2} \overrightarrow{QR} \cdot \overrightarrow{PQ} = 0, \end{aligned}$$

so that the perpendicular bisector of PQ is parallel to \overrightarrow{CD} . This proves D lies on this perpendicular bisector, so that all three perpendicular bisectors intersect in the common point D .

24. By the Theorem of Pythagoras applied to the right-angled triangles PDB and RDB , and the fact that $|PB| = |BR|$, we have

$$|PD|^2 = |PB|^2 + |BD|^2 = |BR|^2 + |BD|^2 = |RD|^2,$$

so that $|PD| = |RD|$. Similarly $|PD| = |DQ|$, which proves D is equidistant from all three vertices of the triangle.

25. Suppose first that A, B, C, D lie on a plane. The triangle ABC is nondegenerate so we may form axes for the plane through AB and AC . Since D does not lie on either of these axes, it must lie on the interior of one of the quadrants formed by these axes, so that

$$\overrightarrow{AD} = \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}$$

for some nonzero scalars λ, μ . But then

$$\overrightarrow{AO} + \overrightarrow{OD} = \lambda(\overrightarrow{AO} + \overrightarrow{OB}) + \mu(\overrightarrow{AO} + \overrightarrow{OC})$$

so that

$$(\lambda + \mu - 1) \overrightarrow{OA} - \lambda \overrightarrow{OB} - \mu \overrightarrow{OC} + \overrightarrow{OD} = \mathbf{0},$$

giving

$$\alpha \overrightarrow{OA} + \beta \overrightarrow{OB} + \gamma \overrightarrow{OC} + \delta \overrightarrow{OD} = \mathbf{0}$$

where

$$\alpha = \lambda + \mu - 1, \quad \beta = -\lambda, \quad \gamma = -\mu, \quad \delta = 1.$$

Certainly $\alpha + \beta + \gamma + \delta = 0$ and β, γ, δ are nonzero. If $\alpha = 0$ then D lies on the line through BC , contradicting that B, C, D are not collinear. Hence α is nonzero also.

Suppose conversely that

$$\alpha \overrightarrow{OA} + \beta \overrightarrow{OB} + \gamma \overrightarrow{OC} + \delta \overrightarrow{OD} = \mathbf{0}$$

where $\alpha, \beta, \gamma, \delta$ are nonzero scalars such that $\alpha + \beta + \gamma + \delta = 0$. Then

$$\frac{-\beta - \gamma - \delta}{\delta} \overrightarrow{OA} + \frac{\beta}{\delta} \overrightarrow{OB} + \frac{\gamma}{\delta} \overrightarrow{OC} + \overrightarrow{OD} = \mathbf{0}$$

so that, after rearranging,

$$\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD} = \frac{\beta}{\delta}(\overrightarrow{OA} + \overrightarrow{BO}) + \frac{\gamma}{\delta}(\overrightarrow{OA} + \overrightarrow{CO}) = \frac{\beta}{\delta} \overrightarrow{BA} + \frac{\gamma}{\delta} \overrightarrow{CA}.$$

This proves that D lies in the plane determined by the triangle ABC , so all four points lie on a plane.

Suppose now that the lines through AD, BD, CD cut lines through BC, CA, AB in R, S, T respectively. We have

$$\alpha \overrightarrow{OA} + \beta \overrightarrow{OB} + \gamma \overrightarrow{OC} + \delta \overrightarrow{OD} = \mathbf{0}$$

for some nonzero scalars $\alpha, \beta, \gamma, \delta$ such that $\alpha + \beta + \gamma + \delta = 0$. Note that because AD intersects BC and A does not lie on the line through BC , the vectors \overrightarrow{AD} and \overrightarrow{BC} are not parallel. But the earlier calculation yields

$$\overrightarrow{AD} = \frac{\beta}{\delta} \overrightarrow{BA} + \frac{\gamma}{\delta} \overrightarrow{CA} = \frac{\beta}{\delta} \overrightarrow{BA} + \frac{\gamma}{\delta} (\overrightarrow{BA} + \overrightarrow{CB}) = \frac{\beta + \gamma}{\delta} \overrightarrow{BA} - \frac{\gamma}{\delta} \overrightarrow{BC}.$$

If $\beta + \gamma = 0$ then $\overrightarrow{AD} = -\frac{\gamma}{\delta} \overrightarrow{BC}$, contradicting that \overrightarrow{AD} and \overrightarrow{BC} are not parallel. Hence

$$\beta + \gamma = -\alpha - \delta \neq 0,$$

and so we may divide through and rearrange the earlier equation to get

$$\frac{\beta \overrightarrow{OB} + \gamma \overrightarrow{OC}}{\beta + \gamma} = \frac{\alpha \overrightarrow{OA} + \delta \overrightarrow{OD}}{\alpha + \delta}.$$

But these must represent the position vector of the point R of intersection of AD with BC , so that, from the left-hand side, R divides BC in the ratio $\gamma : \beta$. Similarly S divides CA in the ratio $\alpha : \gamma$ and T divides AB in the ratio $\beta : \alpha$, and the product of these ratios is

$$\frac{\gamma}{\beta} \frac{\alpha}{\gamma} \frac{\beta}{\alpha} = 1,$$

completing the proof of Ceva's Theorem.