Solutions

THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Quiz 2 Sample

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2016

Full Name .... SID ....

Day .... Time .... Room ....

Tutor .... Signature ....

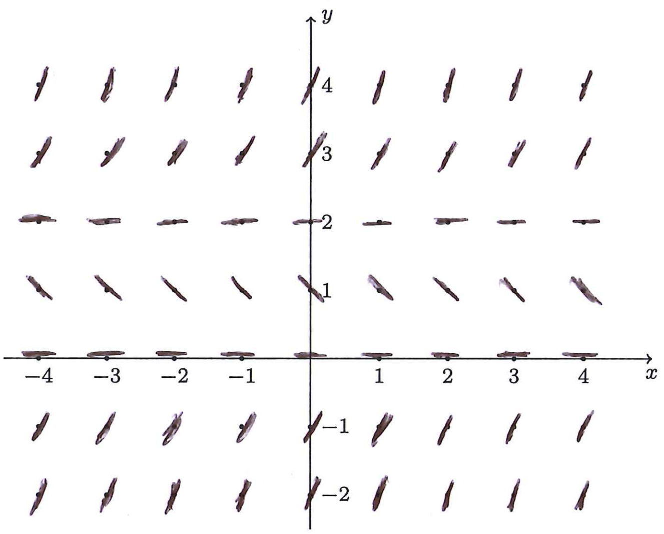
Time allowed: 40 minutes

1. Full marks will only be given if you obtain the correct answer **and** your working is sufficient to justify your answer.
2. Partial marks may be awarded for working.
3. Please write carefully and legibly.
4. All of your answers should be written using ink and **not** pencil, with your final answer placed in the answer box.
5. All working must be done on the quiz paper in the indicated space.
6. Each question is worth **2 marks**.
7. Only University of Sydney approved calculators may be used (must have a sticker).
8. All pages (including working) of the quiz paper must be handed in at the end of the quiz.

This quiz paper has seven pages and 10 questions.

1. Sketch the direction field of the differential equation $y' = y(y-2)$ in the region below.

Answer for Question 1.



2. Find the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{3y-1}{x}$.

Answer for Question 2.

$y = \frac{1}{3} + B x^3$, where $B \in \mathbb{R}$ is a constant

Working for Question 2.

The given DE is separable. Notice that $y = 1/3$ is a solution. If $y \neq 1/3$, then by separating the variables and integrating, we get that $\displaystyle\int \frac{dy}{3(y-\frac{1}{3})} = \int \frac{dx}{x}$. Hence,

$\ln|y - \frac{1}{3}| = 3\ln|x| + C$, for some constant $C \in \mathbb{R}$.

This gives that $|y - \frac{1}{3}| = e^C |x|^3$ and denoting $A = e^C > 0$, we get $y - \frac{1}{3} = \pm A x^3$. The general solution is

$y = \frac{1}{3} + B x^3$, where $B \in \mathbb{R} \setminus \{0\}$. By allowing $B = 0$, we recover the solution $y = \frac{1}{3}$.

3. Find the equilibrium solution of the equation $\frac{dy}{dx} = (y+1)\ln y$.

Answer for Question **3.**

$y = 1$

Working for Question **3.**

The equilibrium solutions are obtained by solving $\frac{dy}{dx} = 0$, that is $(y+1)\ln y = 0$. Since $\ln y$ is not well-defined for $y < 0$, we have only one solution $y = 1$.

4. Determine whether the equilibrium solution in the above question is stable or unstable. Justify your answer.

Answer for Question **4.**

$y = 1$ is unstable

Working for Question **4.**

For $y > 1$, we have $(y+1)\ln y > 0$ (the slope in the direction field would be positive), whereas for $0 < y < 1$, we have $(y+1)\ln y < 0$ in which case the slope is negative.

The equilibrium solution $y = 1$ is unstable as long as we can find a solution curve with the initial value $y(0)$ close to 1 (either $y(0) < 1$ or $y(0) > 1$) such that as $x \to \infty$, the solution goes away from $y = 1$.

In our case, a solution curve with $y(0) > 1$ (and $y(0)$ close to 1) would go away from the equilibrium sol. $y = 1$ as $x \to \infty$. (This also occurs for a sol. curve $y(x)$ with $y(0) < 1$ and $y(0)$ close to 1)

5. Determine whether the statement "The equation $\dfrac{dy}{dx} = x + y$ is both linear and separable" is true or false. Explain your answer.

Answer for Question **5.**
False

Working for Question **5.**

The equation $\dfrac{dy}{dx} = x+y$ is linear since it is of the form $a(x)\dfrac{dy}{dx} + b(x)y = f(x)$ with $a(x)=1$, $b(x)=-1$, $f(x)=x$

However, it is not separable since $x+y$ cannot be written in the form $f(x)g(y)$.

6. Find an integrating factor for the linear differential equation $e^x\left(y' + y\sin x\right) = 2$.

Answer for Question **6.**
$e^{-\cos x}$

Working for Question **6.**

To find an integrating factor, we first need to bring the DE to the standard form by dividing the equation by the coefficient of y'. Hence, we find the standard form to be

$$y' + y\sin x = 2e^{-x}.$$

An integrating factor is of the form $e^{\int \sin x\, dx}$ and we can take, for instance, $e^{-\cos x}$.

7. Find the particular solution of $\frac{dy}{dx} + y\cos x = 2xe^{-\sin x}$ with $y(\pi) = 0$.

Answer for Question 7.

$$y(x) = (x^2 - \pi^2)\, e^{-\sin x}$$

Working for Question 7.

The given DE is a first-order linear DE since the coefficients of the dependent variable y and its derivative depend only on the independent variable x and the remaining term in the DE depends only on x.

The equation is already in a standard form since the coefficient of y' is 1. An integrating factor $I(x)$ is given by $I(x) = e^{\int \cos x\, dx}$ and we choose $I(x) = e^{\sin x}$.

Multiply the given eq. (in standard form) by $I(x)$ to find

$\frac{d}{dx}(y\, I(x)) = 2x$, that is $\frac{d}{dx}(y e^{\sin x}) = 2x$.

Integrating with respect to x in both sides, we get

$$y e^{\sin x} = x^2 + C.$$

The general sol. is given by

$$y(x) = (x^2 + C)\, e^{-\sin x}.$$

By imposing the initial condition $y(\pi) = 0$, we obtain $C = -\pi^2$ and thus the particular sol. is

$$y(x) = (x^2 - \pi^2)\, e^{-\sin x}$$

8. Find the particular solution $y = y(t)$ of the differential equation $\dfrac{dt}{dy} = \dfrac{5t}{t^2+t-6}$ for $0 < t < 2$, which satisfies the condition $y(1) = \dfrac{1}{5}$.

Answer for Question 8.

$$y(t) = \frac{t^2}{10} + \frac{t}{5} - \frac{6}{5}\ln t - \frac{1}{10}.$$

Working for Question 8.

We rewrite the DE so that y becomes the dependent variable

Hence, $\dfrac{dy}{dt} = \dfrac{t^2+t-6}{5t} = \dfrac{t}{5} + \dfrac{1}{5} - \dfrac{6}{5t}.$

A direct integration gives the general sol.

$$y(t) = \frac{t^2}{10} + \frac{t}{5} - \frac{6}{5}\ln|t| + C$$

for some constant $C \in \mathbb{R}$. Since $y(1) = \frac{1}{5}$, we find $C = -\frac{1}{10}$

Given that $0 < t < 2$, we have $\ln|t| = \ln t$.

The particular solution is thus

$$y(t) = \frac{t^2}{10} + \frac{t}{5} - \frac{6}{5}\ln t - \frac{1}{10}$$

9. Suppose that $y(x)$ is a solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^2+xy}{xy+y^2}$. We make the transformation $v = \dfrac{y}{x}$. Find a differential equation for $v = v(x)$.

Answer for Question 9.

$$\frac{dv}{dx} = \frac{1}{x}\left(\frac{1}{v} - v\right)$$

Working for Question 9.

Using that $v = \frac{y}{x}$, the given DE becomes

$$\frac{dy}{dx} = \frac{x^2\left(1+\frac{y}{x}\right)}{x^2\left(\frac{y}{x}+\frac{y^2}{x^2}\right)} = \frac{1+v}{v+v^2} = \frac{1}{v}.$$

From $y = xv$, by differentiating w.r.t x, we get $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Thus $\dfrac{1}{v} - v = x\dfrac{dv}{dx}$, which implies that

$$\boxed{\frac{dv}{dx} = \frac{1}{x}\left(\frac{1}{v} - v\right)}$$

10. Find all functions f such that f' is continuous and

$$f(x) = \frac{1}{2} + \int_0^x \frac{[f(t)]^2}{1+t^2}\,dt \quad \text{for all real } x.$$

Answer for Question **10.**

$$f(x) = \frac{1}{2 - \tan^{-1}(x)}$$

Working for Question **10.**

Observe that $f(0) = \frac{1}{2}$ by taking $x=0$ in the equation

By differentiating the given equation, we find that

$$\frac{df}{dx} = \frac{(f(x))^2}{1+x^2}$$

which is a separable DE. Hence, by separating the variables, we have

$$\frac{df}{f^2} = \frac{dx}{1+x^2}$$

which by integration yields $\int \frac{df}{f^2} = \int \frac{dx}{1+x^2}$

that is, $-\frac{1}{f} = \tan^{-1}(x) + C$ for some $C \in \mathbb{R}$.

Thus, we have $f(x) = \frac{-1}{\tan^{-1}(x) + C}$

From $f(0) = \frac{1}{2}$, we find $C = -2$. Thus,

$$f(x) = \frac{-1}{\tan^{-1}(x) - 2} = \frac{1}{2 - \tan^{-1}(x)}$$