(A)

MATH1902 1997 exam solutions 11/6/2010

Q1/ (i) $\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

(a) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 2 + 1 - 2 = 1$

(b) $\cos\theta = \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \dfrac{1}{\sqrt{1+1+4}\sqrt{4+1+1}} = \dfrac{1}{6}$

(c) $\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -1 & 2 \\ 2 & -1 & -1 \end{vmatrix} = 3\underset{\sim}{i} + 5\underset{\sim}{j} + \underset{\sim}{k}$

(d) $\widehat{\underset{\sim}{u}\times\underset{\sim}{v}} = \dfrac{1}{\sqrt{9+25+1}}\, \underset{\sim}{u}\times\underset{\sim}{v} = \dfrac{1}{\sqrt{35}}(3\underset{\sim}{i} + 5\underset{\sim}{j} + \underset{\sim}{k})$

(e) $(3\underset{\sim}{u} - 2\underset{\sim}{v}) \times (\underset{\sim}{u} + 5\underset{\sim}{v}) = 15(\underset{\sim}{u}\times\underset{\sim}{v}) - 2(\underset{\sim}{v}\times\underset{\sim}{u}) = 17(\underset{\sim}{u}\times\underset{\sim}{v})$

$= \dfrac{17}{\sqrt{35}}(3\underset{\sim}{i} + 5\underset{\sim}{j} + \underset{\sim}{k})$

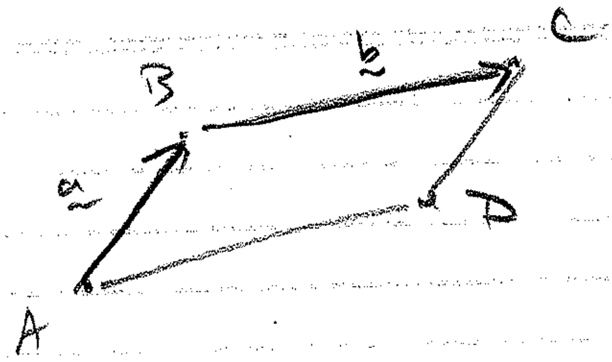
(ii) (a) Plane contains $A(2,-1,-1)$ and has normal $\overrightarrow{OA} = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$, so has equation

$$2x - y - z = 4 + 1 + 1 = 6.$$

(b) The line ℓ given by $\dfrac{x-3}{2} = \dfrac{y-4}{3} = \dfrac{z-5}{4}$ has direction vector $\underset{\sim}{v} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$, and the plane $\mathcal{P}$ given by $4x + 4y - 5z = 14$ has normal $\underset{\sim}{n} = 4\underset{\sim}{i} + 4\underset{\sim}{j} - 5\underset{\sim}{k}$ and $\underset{\sim}{v}\cdot\underset{\sim}{n} = 8 + 12 - 20 = 0$, so ℓ is parallel to $\mathcal{P}$.

(B)

Q2/ (i)



$\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$, $\overrightarrow{BD} = \underset{\sim}{b} - \underset{\sim}{a}$, so

$$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = |\underset{\sim}{a}+\underset{\sim}{b}|^2 + |\underset{\sim}{b}-\underset{\sim}{a}|^2$$

$$= (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b}) + (\underset{\sim}{b}-\underset{\sim}{a})\cdot(\underset{\sim}{b}-\underset{\sim}{a})$$

$$= \underset{\sim}{a}\cdot\underset{\sim}{a} + \cancel{2\underset{\sim}{a}\cdot\underset{\sim}{b}} + \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} - \cancel{2\underset{\sim}{b}\cdot\underset{\sim}{a}} + \underset{\sim}{a}\cdot\underset{\sim}{a}$$

$$= 2|\underset{\sim}{a}|^2 + 2|\underset{\sim}{b}|^2$$

= sum of squares of four sides, as required.

(ii) l passes through $B(0,0,0)$ and $C(-1,0,2)$

and $A = (1,2,-1)$.

(a) l has direction vector $\overrightarrow{BC} = -\underset{\sim}{i} + 2\underset{\sim}{k}$,

so has vector equation $\boxed{\underset{\sim}{r} = t(-\underset{\sim}{i} + 2\underset{\sim}{k})}$

(b) l has parametric equations

$$\boxed{\left.\begin{aligned} x &= -t \\ y &= 0 \\ z &= 2t \end{aligned}\right\} \quad t \in \mathbb{R}}$$

Q2/ (ii) (b) (Cont.)

so $N = (-t, 0, 2t)$ for $t \in \mathbb{R}$

(c) Want $\overrightarrow{AN} = (-t-1)\underset{\sim}{i} - 2\underset{\sim}{j} + (2t+1)\underset{\sim}{k}$ to be perpendicular to ℓ, so $\overrightarrow{AN} \cdot \overrightarrow{BC} = 0$,

i.e. $\left((-t-1)\underset{\sim}{i} - 2\underset{\sim}{j} + (2t+1)\underset{\sim}{k}\right) \cdot (-\underset{\sim}{i} + 2\underset{\sim}{k}) = 0$,

i.e. $t+1+4t+2 = 0$,

i.e. $5t = -3$, $t = -3/5$

so $N = (3/5, 0, -6/5)$.

(d) $|\overrightarrow{AN}| = |-2/5\,\underset{\sim}{i} - 2\underset{\sim}{j} - 11/5\,\underset{\sim}{k}|$

$$= \sqrt{\frac{4}{25} + 4 + \frac{121}{25}}$$

$$= \sqrt{\frac{4+100+121}{25}} = \sqrt{\frac{225}{25}}$$

$$= \sqrt{9} = 3.$$

Q4/ (i)
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 4 & -7 \\ 7 & 7 & 5 & 11 & -8 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 2 & -11 \\ 0 & 0 & -2 & 4 & -22 \end{array}\right]$$

Q4/ (i) (cont.)

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & -9 \\ 0 & 0 & 1 & -2 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x + y \quad + 3w &= -9 \\ z - 2w &= 11 \end{aligned}$$

Put $y = s$, $w = t$, so $z = 11 + 2t$, $x = -9 - s - 3t$,

so solution is

$$\boxed{(x, y, z, w) = (-9 - s - 3t,\ s,\ 11 + 2t,\ t)} \quad t \in \mathbb{R}$$

(ii) $$\begin{vmatrix} 1 & 5 & -1 & 4 \\ -1 & -3 & -1 & -4 \\ 2 & 10 & -1 & 9 \\ 0 & 1 & -1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -1 & 4 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 6 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -12 \\ 0 & 1 & 1 \\ 1 & -1 & 6 \end{vmatrix} = \begin{vmatrix} 0 & -12 \\ 1 & 1 \end{vmatrix} = 0 - (-12) = 12,$$

Q5/ (i) $$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

so $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Q5/ (ii) $X \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\Rightarrow X = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -2 & -3 \\ -2 & 1 & 1 \end{bmatrix}.$$

(iii) Given $BA = AB = CA = AC = I$, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

(iv) $A = \begin{bmatrix} \boxed{\quad\quad} \\ 0\ 0 \cdots 0 \\ \boxed{\quad\quad} \end{bmatrix}$ then $AB = \underset{A}{\begin{bmatrix} \boxed{\quad\quad} \\ 0\ 0 \cdots 0 \\ \boxed{\quad\quad} \end{bmatrix}} \underset{B}{\begin{bmatrix} \boxed{\,} & \boxed{\,} & \boxed{\,} & \boxed{\,} \end{bmatrix}}$

$$= \begin{bmatrix} \boxed{\quad\quad} \\ 0\ 0 \cdots 0 \\ \boxed{\quad\quad} \end{bmatrix}$$

So AB has a row of zeros (in the same position).

If A^{-1} exists then $I = AA^{-1}$ has a row of zeros (taking $B = A^{-1}$), which is nonsense. Hence A^{-1} does not exist.