The University of Sydney

MATH1903 Integral Calculus and Modelling (Advanced)

Semester 2 **Solutions to Exercises for Week 2** 2017

1. Let $f : A \to \mathbb{R}$ be a function where A is a subset of $\mathbb{R}$ closed under taking negatives. Call f *even* if $f(-x) = f(x)$ for all $x \in A$. Call f *odd* if $f(-x) = -f(x)$ for all $x \in A$.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is both even and odd. Then, for all $x \in \mathbb{R}$,

$$f(x) \;=\; f(-x) \;=\; -f(x)\ ,$$

so that $2f(x) = 0$, whence $f(x) = 0$. This proves that f is the zero function.

2. If $y = \sin x$ then $dy/dx = \cos x$. If $y = \cos x$ then $dy/dx = -\sin x$. If $y = \tan x$ then $dy/dx = \sec^2 x$. If $y = \tan^{-1} x$ then $dy/dx = \frac{1}{1+x^2}$.

If $y = \frac{1}{x^2+1} = (1+x^2)^{-1}$ then, by the Chain Rule,

$$dy/dx \;=\; -(x^2+1)^{-2}(2x) \;=\; \frac{-2x}{(x^2+1)^2}\ .$$

If $y = \frac{x}{x^2+1}$ then, by the Quotient Rule,

$$dy/dx \;=\; \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} \;=\; \frac{1-x^2}{(x^2+1)^2}\ .$$

3. The rules $y = \cos x$ and $y = \frac{1}{x^2+1}$ define even functions, and their derivatives are odd. All of the other rules in the previous exercise define odd functions, and their derivatives are even.

We conjecture that if f is a differentiable even or odd function then the derivative f' is odd or even respectively. The proof of this conjecture is not given as it may relate to the First Assignment.

4. The Mean Value Theorem says that if f is continuous on $[a, b]$ and differentiable on (a, b) then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Suppose $f : [0, \infty) \to \mathbb{R}$ has positive derivative $f'(x)$ for all $x > 0$. Assume $0 < x < y$. By the Mean Value Theorem, there exists $c \in (x, y)$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$, so that $f(y) - f(x) = f'(c)(y - x) > 0$, whence $f(x) < f(y)$. This proves that f is strictly increasing on $(0, \infty)$. If $f(x) \le f(0)$ for some $x > 0$, then by the Mean Value Theorem again, there must be some c such that $c \in (0, x)$ and $f'(c) = \frac{f(x)-f(0)}{x} \le 0$, contradicting that $f'(c) > 0$. This proves f is strictly increasing on $[0, \infty)$.

5. If $y = \sqrt{1 + x^3}$ then $y' = \frac{3}{2}x^2(1+x^3)^{-\frac{1}{2}} > 0$ for $x > 0$, so the curve is strictly increasing on $[0, \infty)$, by the previous exercise.

Alternatively, one may observe that $y = \sqrt{1 + x^3}$ is a composite of very simple strictly increasing functions, so must be increasing. (It is easy to see that if f and g are strictly increasing then $g \circ f$ is strictly increasing, for if $x < y$ then $f(x) < f(y)$, so $g \circ f(x) = g(f(x)) < g(f(y)) = g \circ f(y)$.)

6. We have

$$\begin{aligned}
1.97 &\approx 0.2\big(\sqrt{1+1^3}+\sqrt{1+1.2^3}+\sqrt{1+1.4^3}+\sqrt{1+1.6^3}+\sqrt{1+1.8^3}\big)\\
&\le \int_1^2 \sqrt{1+x^3}\,dx\\
&\le 0.2\big(\sqrt{1+1.2^3}+\sqrt{1+1.4^3}+\sqrt{1+1.6^3}+\sqrt{1+1.8^3}+\sqrt{1+2^3}\big)\\
&\approx 2.29\ .
\end{aligned}$$

7. (i) The lower sum is $4(0+8+19+29+36)=368$ and the upper sum is $4(8+19+29+36+40)=528$, so the boat travels between 368 and 528 metres.

(ii) If measurements are taken every t seconds then

$$\text{upper sum} \ - \text{lower sum} \ = \ 40t\ ,$$

so to have $40t < 10$ we require $t < \frac{10}{40} = 0.25$, that is, measurements are taken every $1/4$ second, or 4 times per second.

8. (i) $\int_2^3 f(x)\,dx = \int_0^3 f(x)\,dx - \int_0^2 f(x)\,dx = 7-10=-3.$

(ii) $\int_0^3 \dfrac{f(x)-3g(x)}{2}\,dx = \frac{1}{2}\int_0^3 f(x)\,dx - \frac{3}{2}\int_0^3 g(x)\,dx = \frac{7}{2}-\frac{3}{2}(-2) = \frac{13}{2}.$

(iii) $\int_{-3}^3 f(x)\,dx = 0$ since f is odd.

(iv) $\int_3^{-3} g(x)\,dx = -\int_{-3}^3 g(x)\,dx = -2\int_0^3 g(x)\,dx = 4$, using the fact g is even.

(v) Put $h(x) = (x-f(x)g(x))^{\frac{1}{3}}$ and observe that

$$h(-x) \ = \ (-x-f(-x)g(-x))^{\frac{1}{3}} \ = \ -(x-f(x)g(x))^{\frac{1}{3}} \ = \ -h(x)\ ,$$

which verifies that h is odd. Hence $\int_{-3}^3 h(x)\,dx = 0$.

9. Using the following telescoping sum

$$\begin{aligned}
n^4 &= \sum_{i=1}^n (i^4-(i-1)^4) \ = \ \sum_{i=1}^n i^4-(i^4-4i^3+6i^2-4i+1)\\
&= 4\sum_{i=1}^n i^3 \ - \ 6\sum_{i=1}^n i^2 \ + \ 4\sum_{i=1}^n i \ - \ n\\
&= 4\sum_{i=1}^n i^3 \ - \ n(2n+1)(n+1) \ + \ 2n(n+1) \ - \ n\ ,
\end{aligned}$$

we get

$$\begin{aligned}
\sum_{i=1}^n i^3 &= \frac{n^4+n(n+1)(2n-1)+n}{4} = \frac{n(n^3+1)+n(n+1)(2n-1)}{4}\\
&= \frac{n(n+1)(n^2-n+1)+n(n+1)(2n-1)}{4} = \frac{n(n+1)(n^2+n)}{4}\\
&= \left(\frac{n(n+1)}{2}\right)^2 .
\end{aligned}$$

10. We have

$$\begin{aligned}\int_0^1 x^3\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3 \\ &= \lim_{n\to\infty}\frac{n^2(n+1)^2}{4n^4} = \lim_{n\to\infty}\frac{(1+\frac{1}{n})^2}{4} = \frac{1}{4}\,.\end{aligned}$$

11. For each n, inscribe $\mathcal{C}_1$ with n congruent triangles fanning out from the centre, each of which has two side lengths r_1 and the third side say ℓ_n, so that

$$P_1 = \lim_{n\to\infty} n\ell_n\,.$$

Inscribe $\mathcal{C}_2$ with similar triangles, each of which has two side lengths r_2 and the third side say m_n. But ratios of corresponding sides of similar triangles are equal, so $\frac{m_n}{\ell_n} = \frac{r_2}{r_1}$, giving $m_n = \ell_n\frac{r_2}{r_1}$. Thus

$$P_2 = \lim_{n\to\infty} nm_n = \frac{r_2}{r_1}\lim_{n\to\infty} n\ell_n = \frac{r_2}{r_1}P_1\,,$$

from which it is immediate that $\dfrac{P_2}{2r_2} = \dfrac{P_1}{2r_1}$, so indeed π is well-defined.

12. (i) The overestimate is $12+8.4+5.9+4.1+2.9+2.0 = 35.3$ and the underestimate is $8.4+5.9+4.1+2.9+2.0+1.4 = 24.7$.

(ii) If measurements are taken every t seconds then the difference between the estimates must be $(12-1.4)t = 10.6t$, so for $10.6t < 1$ we require measurements taken about 11 times per second.

13. Suppose $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ are two functions such that $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist. Then, for any Riemann sums $\sum_{i=1}^{n} f(x_i)\Delta x_i$ and $\sum_{i=1}^{n} g(x_i)\Delta x_i$ for f and g respectively which use common partitions and choices of x_i, we have

$$\sum_{i=1}^{n}\big(f(x_i)+g(x_i)\big)\Delta x_i$$

is also a Riemann sum for $f+g$. By the addition limit law,

$$\begin{aligned}\int_a^b f(x)+g(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\big(f(x_i)+g(x_i)\big)\Delta x_i \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x_i + \sum_{i=1}^{n} g(x_i)\Delta x_i\right) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i + \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\Delta x_i \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\,.\end{aligned}$$

14. For each i,

$$t_{i-1} \;=\; \sqrt{t_{i-1}^2} \;<\; \sqrt{t_{i-1}t_i} \;=\; x_i \;<\; \sqrt{t_i^2} \;=\; t_i \;,$$

so $x_i \in [t_{i-1}, t_i]$, and

$$f(x_i)\Delta x_i \;=\; \frac{1}{t_{i-1}t_i}(t_i - t_{i-1}) \;=\; \frac{1}{t_{i-1}} - \frac{1}{t_i} \;.$$

Hence

$$\sum_{i=1}^{n} f(x_i)\Delta x_i \;=\; \left(\frac{1}{t_0} - \frac{1}{t_1}\right) + \left(\frac{1}{t_1} - \frac{1}{t_2}\right) + \cdots + \left(\frac{1}{t_{n-1}} - \frac{1}{t_n}\right) \;=\; \frac{1}{a} - \frac{1}{b} \;.$$

Thus

$$\int_a^b \frac{1}{x^2}\,dx \;=\; \frac{1}{a} - \frac{1}{b} \;.$$

15. Let a and b be two real numbers with $a < b$. Choose a positive integer n larger than $\dfrac{1}{b-a}$, so that

$$\frac{1}{n} \;<\; b - a \;.$$

Let $\dfrac{m}{n}$ be as large as possible without exceeding a, where m is a positive integer. Put $q = \dfrac{m+1}{n}$, Then $a < q$, by the choice of m. If $b \leq q$ then

$$\frac{1}{n} \;=\; \frac{m+1-m}{n} \;=\; \frac{m+1}{n} - \frac{m}{n} \;=\; q - \frac{m}{n} \;\geq b - a \;,$$

contradicting our choice of n. Hence $q < b$, so q is a rational number between a and b. This result applied to $a + \sqrt{2}$ and $b + \sqrt{2}$ guarantees the existence of a rational r such that

$$a + \sqrt{2} \;<\; r < b + \sqrt{2} \;,$$

so that $a < r - \sqrt{2} < b$. But $r - \sqrt{2}$ is irrational (for otherwise we could rearrange the information to yield that $\sqrt{2}$ is rational). Hence $r - \sqrt{2}$ is an irrational number between a and b.

16. Observe that, for all h,

$$\left|\frac{f(0+h)-f(0)}{h}\right| \;=\; \left|\frac{f(h)}{h}\right| \;\leq\; \left|\frac{h^2}{h}\right| \;=\; |h|\;,$$

and so, by the squeeze law, $\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=0$. Therefore f is differentiable at $x=0$, and $f'(0)=0$. In particular, f is continuous at $x=0$.

Suppose $c\neq 0$. By the previous exercise, there exists a sequence $(x_n)_{n=1}^{\infty}$ of rational numbers and also a sequence $(y_n)_{n=1}^{\infty}$ of irrational numbers, such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c\,.$$

If f is continuous at $x=c$ then

$$0 \;=\; \lim_{n\to\infty} f(x_n) \;=\; f(c) \;=\; \lim_{n\to\infty} f(y_n) \;=\; \lim_{n\to\infty} y_n^2 \;=\; c^2\;,$$

which is impossible. Hence f is not continuous (so also not differentiable) at $x=c$.

Choose a sequence of partitions of $[0,1]$ where lengths of subintervals tend to zero. On the one hand, choose representatives of subintervals that are rational, always possible by the previous exercise, in which case the Riemann sums approach $\int_0^1 0\,dx = 0$. On the other hand, choose representatives of subintervals that are irrational, again always possible by the previous exercise, in which case the Riemann sums approach $\int_0^1 x^2 dx = \frac{1}{3}$. Since these limits are different, f is not Riemann integrable on $[0,1]$.

17. Suppose $\int_a^b f(x)\,dx = 0$. Suppose, by way of contradiction, that $f(c)\neq 0$ for some $c\in(a,b)$. By continuity, there exists $\delta>0$, such that $f(x)\geq\frac{f(c)}{2}$ for all $x\in(c-\delta,c+\delta)$. Without loss of generality, we may assume $a<c-\delta<c+\delta<b$. Then, since f is nonnegative,

$$\begin{aligned} 0 \;&=\; \int_a^b f(x)\,dx \;=\; \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^{b} f(x)\,dx \\ &\geq\; 0 \;+\; \int_{c-\delta}^{c+\delta}\frac{f(c)}{2}\,dx \;+\; 0 \;=\; \delta f(c) \;>\; 0\,, \end{aligned}$$

which is impossible. Hence $f(c)=0$ for all $c\in(a,b)$, so, by continuity, $f(c)=0$ for all $c\in[a,b]$.

The conclusion may fail if the continuity assumption is dropped. For example, if f is the function defined by the rule

$$f(x)=\begin{cases}0 & \text{if } x\neq\frac{1}{2}\\ 1 & \text{if } x=\frac{1}{2},\end{cases}$$

then f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)\,dx=0$.