MATH1903/1907 Lectures

Week 11, Semester 2, 2017

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General solution of

$$a y'' + b y' + c y = f(t) \quad (a, b, c \text{ const})$$

is of the form

$$y(t) = y_h(t) + y_p(t),$$

where

- y_h is general solution of the homogeneous equation $a y'' + b y' + c y = 0$
- y_p is a particular solution of the inhomogeneous eq.

<u>Example:</u>

$$y'' - 5y' + 4y = t^2$$

Since derivatives of polynomials are polynomials we try

$$y_p(t) = A + Bt + Ct^2 \quad (A, B, C \text{ to be determined})$$

Substitute into equation:

$$\underbrace{2C}_{y_p''} - 5(\underbrace{B + 2Ct}_{y_p'}) + 4(\underbrace{A + Bt + Ct^2}_{y_p}) = t^2$$

Rearrange:

$$(2C - 5B + 4A) + (-10C + 4B)t + 4t^2 \underset{\uparrow \text{ want}}{=} t^2$$

Example

$$y'' - 3y' - 10y = 2e^{3t}$$

Since derivatives of a function of the form Ae^{3t} is of the same form we try

$$y_p = Ae^{3t} \quad (A \text{ to be determined})$$

Substitute into the equation:

$$9Ae^{3t} - 3(3e^{3t}) - 10Ae^{3t} \underset{\substack{\uparrow \\ \text{want.}}}{=} 2e^{3t}$$

Divide by e^{3t} $(\neq 0)$

$$\cancel{9A} - \cancel{9A} - 10A = 2, \quad \text{so } A = -\frac{2}{10} = -\frac{1}{5}$$

Hence $y_p(t) = -\frac{1}{5}e^{3t}$ is a particular solution.

Auxiliary equation of the homogeneous problem:

$$\lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5) = 0, \quad \text{so } \lambda = -2, 5$$

$$\boxed{\text{General solution: } y(t) = ae^{-2t} + be^{5t} - \frac{1}{5}e^{3t}.}$$

Example:

$$y'' - 3y' + 5y = \cos 3t$$

Note: up to a constant, the derivatives of $\cos 3t$ are $\cos 3t$, $\sin 3t$

For that reason we try a solution of the form

$$y_p(t) = A\cos 3t + B\sin 3t$$

Substitute into equation:

$$\underbrace{-9A\cos 3t - 9B\sin 3t}_{y_p''} - 3\underbrace{(-3A\sin 3t + 3B\cos 3t)}_{y_p'}$$

$$+5(A\cos 3t + B\sin 3t) \underset{\text{want}}{=} \cos 3t$$

collect coefficients of $\cos 3t$, $\sin 3t$:

$\cos 3t$: $-9A - 9B + 5A = 1$

$-4A - 9B = 1$

$\sin 3t$: $-9B + 9A + 5B = 0$

$-4B + 9A = 0$

This leads to a system of equations:

$$-4A - 9B = 1$$
$$9A - 4B = 0$$

Solving this system will give a particular solution .....

Example:

$$y'' + y' - 2y = e^{-2t}$$

As before, try $y_p(t) = A e^{-2t}$

Substitute into equation:

$$\underbrace{4Ae^{-2t}}_{y_p''} \underbrace{-2Ae^{-2t}}_{y_p'} \underbrace{-2Ae^{-2t}}_{y_p} = 0 = e^{-2t}$$

want

Problem: e^{-2t} solves the homogeneous equation

If that is the case try $y_p(t) = Ate^{-2t}$

multiply by t

Substitute into equation:

$$y_p'(t) = Ae^{-2t} - 2Ate^{-2t}$$

$$y_p''(t) = -2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t}$$

$$= 4A(t-1)e^{-2t}$$

$$y_p'' + y_p' - 2y_p$$

$$= 4A(t-1)e^{-2t} + Ae^{-2t} - 2Ate^{-2t} - 2Ate^{-2t} = e^{-2t}$$

want

Cancel $e^{-2t} \neq 0$ and collect coeff of $1, t$:

$$(\underbrace{4A - 2A - 2A}_{=0})t + (-4A + A) = 1$$

so $A = -\frac{1}{3}$.

$$\boxed{\text{particular solution: } y_p = -\frac{t}{3}e^{-2t}}$$

Summary:

General solution for $ay'' + by' + cy = f(t)$ is of the form

$$y(t) = y_p(t) + y_h(t),$$

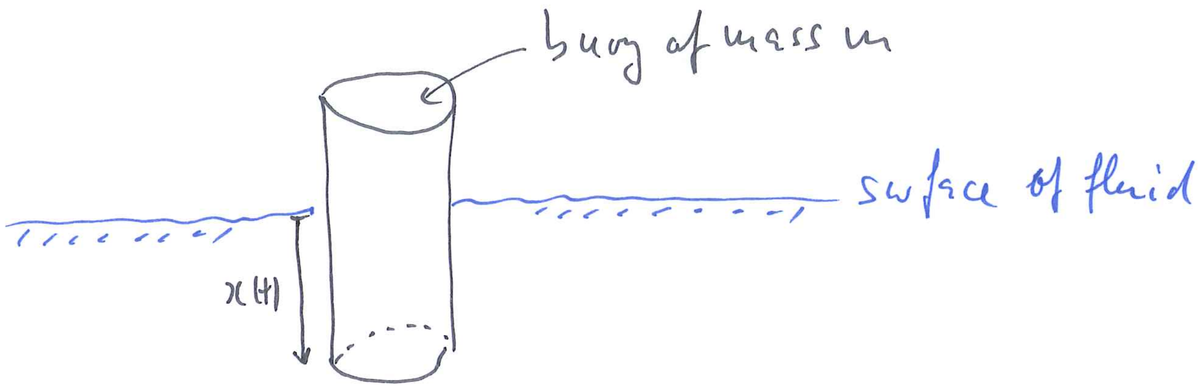
where

- $y_p(t)$ is a particular solution
- $y_h(t)$ is the general solution of the homogeneous equation $ay'' + by' + cy = 0$

Often $y_p(t)$ has a similar form as $f(t)$:

$f(t)$	$y_p(t)$
• polynomial of degree n e.g. $f(t) = 2 + 3t^2$	• try polynomial of degree n e.g. $y_p(t) = A + Bt + Ct^2$
• $c e^{\lambda t}$	• try $y_p(t) = A e^{\lambda t}$
• trig: $\cos\omega t$, $\sin\omega t$	• try $y_p(t) = A\cos\omega t + B\sin\omega t$
• If $f(t)$ solves the homogeneous eq	• try $y_p(t) = A t f(t)$

Application: Buoy in a fluid



Forces acting on buoy:

- gravitational force
- force of buoyancy
- resistive force

$x(t)$ = distance of the bottom of the buoy from surface of the fluid at time t

Motion is determined by Newton's second law.

gravitational force: mg

force of buoyancy: Archimedes law of buoyancy:
force is proportional to the volume $V(x)$ of the part of the buoy submerged i the fluid.

resistive force: $R(x, \dot{x})$.

$$m\ddot{x} = \underbrace{mg}_{\text{gravitational force}} - \underbrace{\alpha V(x)}_{\text{force of buoyancy}} - \underbrace{R(x, \dot{x})}_{\text{resistive force.}}$$

Assumptions

- Buoy has cylindrical shape with cross-section of area A. Hence

$$V(x) = Ax$$

- resistive force is proportional to velocity:

$$R(x, \dot{x}) = \beta \dot{x} \quad (\beta > 0)$$

Substitute into equation:

$$m\ddot{x} = mg - \alpha A x - \beta \dot{x}$$

Transformation (make sure equilibrium is at zero)

Equilibrium: $\ddot{x} = \dot{x} = 0$

$$0 = mg - \alpha A x - 0, \quad \text{so } x = \frac{mg}{\alpha A}$$

Set $y = x - \frac{mg}{\alpha A}$ and substitute: $\dot{x} = \dot{y}, \ddot{x} = \ddot{y}$

Then

$$m\ddot{y} = -\alpha A y - \beta \dot{y}$$

Hence

$$\boxed{m\ddot{y} + \beta\dot{y} - \alpha A y = 0}$$

Homogeneous linear equation

Auxiliary equation:

$$m\lambda^2 + \beta\lambda + \alpha A = 0$$

The solutions are

$$\lambda = \frac{1}{2m}\left(-\beta \pm \sqrt{\beta^2 - 4m\alpha A}\right)$$

Consider cases depending on discriminant $\beta^2 - 4m\alpha A$

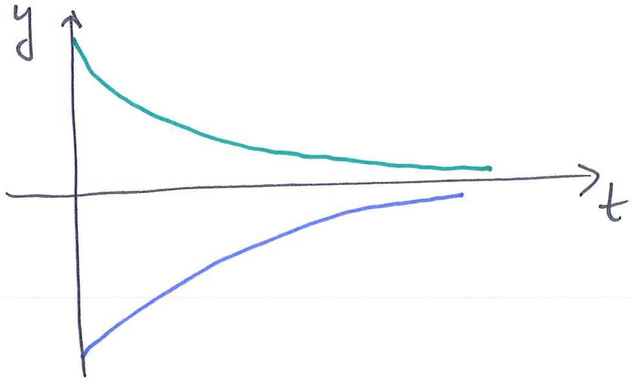
Case 1: There are two distinct real roots:

$\beta^2 - 4m\alpha A > 0$

Note: $\sqrt{\beta^2 - 4m\alpha A} < \sqrt{\beta^2} = \beta$

Hence $\lambda = \frac{1}{2m}\left(-\beta \pm \underbrace{\sqrt{\beta^2 - 4m\alpha A}}_{<\beta}\right) < 0$

Hence $y(t) = Ce^{\lambda_1 t} + De^{\lambda_2 t}$ is decaying as $t \to \infty$ since $\lambda_1, \lambda_2 < 0$.



This is the case of a very viscous fluid (buoy in a honey pot)

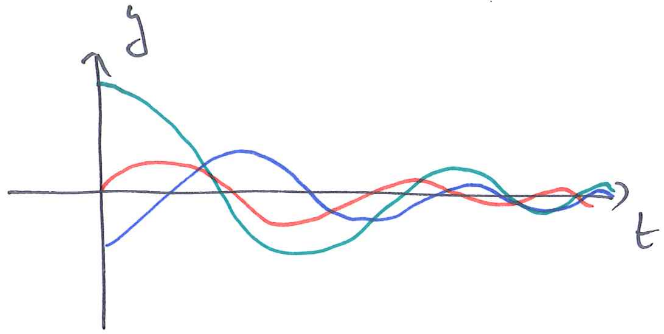
Case 2: $\beta^2 - 4m\alpha A < 0$ pair of complex conjugate roots $\lambda = \mu \pm i\omega$

$$\mu = -\frac{\beta}{2m}, \quad \omega = \frac{1}{2m}\sqrt{4m\alpha A - \beta^2}$$

General solution:

$$y(t) = e^{-\frac{\beta}{2m}t}\left(C\cos\omega t + D\sin\omega t\right)$$

is a damped oscillation

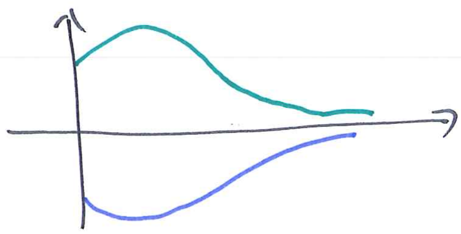


Case 3: one single real root $\beta^2 - 4m\alpha A = 0$

$$\lambda = -\frac{\beta}{2m}$$

General solution:

$$y(t) = C e^{-\frac{\beta}{2m}t} + D t e^{-\frac{\beta}{2m}t} \longrightarrow 0 \text{ as } t \to \infty$$



Limit case between decay and oscillation "critical damping"

Tries to oscillate, but cannot.