School of Mathematics University of New South Wales

MATH2701: Abstract Algebra and Fundamental Analysis

Short Assignment 2

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1. (a) As $f(x) = O(h(x))$, for some $M_1 > 0$, and some x_1, we have that for all $x > x_1$,

$$|f(x)| \leq M_1|h(x)|.$$

Similarly, as $g(x) = O(h(x))$, for some $M_2 > 0$, and some x_2, we have that for all $x > x_2$,

$$|g(x)| \leq M_2|h(x)|.$$

Now, select $M = M_1 + M_2$, and $x_0 = \max\{x_1, x_2\}$, such that for all $x > x_0$, we have

$$\begin{aligned} |f(x) + g(x)| &\leq |f(x)| + |g(x)| \\ &\leq M_1|h(x)| + M_2|h(x)| \\ &= (M_1 + M_2)|h(x)| \\ \therefore |f(x) + g(x)| &\leq M|h(x)|. \end{aligned}$$

Thus, $f(x) + g(x) = O(h(x))$.

(b) As $f(x) = O(g(x))$, for some $M_1 > 0$, and some x_1, we have that for all $x > x_1$,

$$|f(x)| \leq M_1|g(x)|.$$

Similarly, as $g(x) = O(h(x))$, for some $M_2 > 0$, and some x_2, we have that for all $x > x_2$,

$$|g(x)| \leq M_2|h(x)|.$$

Now, select $M = M_1M_2$, and $x_0 = \max\{x_1, x_2\}$, such that for all $x > x_0$, we have

$$\begin{aligned} |f(x)| &\leq M_1|g(x)| \\ &\leq M_1(M_2|h(x)|) \\ &= (M_1M_2)|h(x)| \\ \therefore |f(x)| &\leq M|h(x)|, \end{aligned}$$

Thus, $f(x) = O(h(x))$.

(c) We have $f(x) \sim g(x)$ as $x \to a$, so

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$

Also, $h(x) = o(g(x)$ as $x \to a$, so

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0.$$

Considering the above two limits, we have

$$\lim_{x\to a} \left(\frac{f(x) + h(x)}{g(x)} \right) = \lim_{x\to a} \left(\frac{f(x)}{g(x)} + \frac{h(x)}{g(x)} \right) = 1 + 0 = 0$$

By definition, $f(x) + h(x) \sim g(x)$.

(d) Consider $f(x) = x^3 + x^2 = O(x^4 + x)$ and $g(x) = x^3 = O(x^4)$. This gives $h(x) = x^4 + x$ and $k(x) = x^4$. Then $f(x) - g(x) = x^2$, but $h(x) - k(x) = x$. Clearly, $x^2 \neq O(x)$, and so by counter-example, the assertion is false.

2. (a) Consider the generalised AM-GM Inequality given in lectures,

$$(x_1 x_2 \dots x_n)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} x_k,$$

Noting that equality only occurs when $x_1 = x_2 = \dots = x_n$. Let $x_k = k$. This gives,

$$\sum_{k=1}^{n} x_k = \frac{n(n+1)}{2}, \qquad \text{and} \qquad x_1 x_2 \dots x_n = n!.$$

Using the generalised AM-GM Inequality given above, and the choice of x_k,

$$\begin{aligned} \therefore (n!)^{1/n} &\leq \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2} \\ \therefore (n!)^{1/n} &\leq \frac{n+1}{2} \\ \therefore n! &\leq \left(\frac{n+1}{2}\right)^n . \end{aligned}$$

Note that equality occurs when $x_1 = x_2 = \dots = x_n$. From our choice of $x_k = k$, the equality condition becomes $n = n - 1 = \dots = 1$, and thus $n = 1$ for equality to occur.

(b) Consider the generalised AM-GM Inequality given in lectures,

$$(x_1 x_2 \dots x_n)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} x_k.$$

Applying the inequality to the two factors of the LHS,

$$\begin{aligned} \left(\sum_{k=1}^{n} x_k\right) &\geq n(x_1 x_2 \dots x_n)^{1/n}, \\ \left(\sum_{k=1}^{n} \frac{1}{x_k}\right) &\geq n \left(\frac{1}{x_1 x_2 \dots x_n}\right)^{1/n}. \end{aligned}$$

Now, considering the LHS of the result we have to prove,

$$\begin{aligned} \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) &\geq \left(n(x_1 x_2 \dots x_n)^{1/n}\right)\left(n\left(\frac{1}{x_1 x_2 \dots x_n}\right)^{1/n}\right) \\ &= n^2 (x_1 x_2 \dots x_n)^{1/n} \left(\frac{1}{x_1 x_2 \dots x_n}\right)^{1/n} \\ &= n^2 \left(\frac{x_1 x_2 \dots x_n}{x_1 x_2 \dots x_n}\right)^{1/n} \\ &= n^2 \end{aligned}$$

$$\therefore \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \geq n^2.$$

3. (a) Consider first $e - e_n$, using their definitions,

$$\begin{aligned}
e - e_n &= \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} \\
&= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\
&= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots \\
&= \frac{1}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots \right).
\end{aligned}$$

For all $k > 1$, $n > 0$, we have $n + k > n + 1$, so

$$\frac{1}{n+k} < \frac{1}{n+1}.$$

Thus, using the result from above,

$$\begin{aligned}
e - e_n &= \frac{1}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots \right) \\
&< \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots \right) \\
&= \frac{1}{(n+1)!} \left(\frac{1}{1 - \frac{1}{n+1}} \right) \\
&= \frac{1}{(n+1)!} \left(\frac{1}{\frac{n+1-1}{n+1}} \right) \\
&= \frac{1}{(n+1)!} \left(\frac{n+1}{n} \right) \\
&= \frac{1}{n \cdot n!} \\
\therefore e - e_n &\le \frac{1}{n \cdot n!}.
\end{aligned}$$

Thus, let $M = 1$, and $n_0 = 1$, we have

$$\begin{aligned}
e - e_n &\le \frac{1}{n \cdot n!} \\
\therefore e - e_n &\le M \left| \frac{1}{n \cdot n!} \right|,
\end{aligned}$$

so $e - e_n = O\left(\dfrac{1}{n \cdot n!} \right)$.

(b) Clearly, $e - e_n > 0$. So, for all $n > 1$,

$$0 < e - e_n < \frac{1}{n \cdot n!}.$$

As $n > 1$, then $\frac{1}{n} < 1$, so,

$$\begin{aligned}
0 &< e - e_n < \frac{1}{n} \cdot \frac{1}{n!} \\
0 &< e - e_n < \frac{1}{n!} \\
0 &< n!(e - e_n) < 1.
\end{aligned}$$

(c) Assume $e \in \mathbb{Q}$, so there exists co-prime integers a and $b > 0$, such that $e = \frac{a}{b}$. Let $n = b$, so $n \in \mathbb{N}$. Using the result from the previous part,

$$\begin{aligned}
0 &< n!(e - e_n) < 1 \\
0 &< b!\left(\frac{a}{b} - e_b\right) < 1 \\
0 &< b!\frac{a}{b} - b!\sum_{k=0}^{b} \frac{1}{k!} < 1 \\
0 &< a(b-1)! - \sum_{k=0}^{b} b(b-1)\dots(k+1) < 1.
\end{aligned}$$

Clearly, $a(b-1)! \in \mathbb{Z}$, similarly, $\sum_{k=0}^{b} b(b-1)\dots(k+1) \in \mathbb{Z}$. As a result,

$$a(b-1)! - \sum_{k=0}^{b} b(b-1)\dots(k+1) \in \mathbb{Z}.$$

However, from the result above,

$$a(b-1)! - \sum_{k=0}^{b} b(b-1)\dots(k+1) \in (0,1).$$

This is a contradiction, and so $e \notin \mathbb{Q}$, and thus e is irrational.