University of Sydney

MATH 1903

Integral Calculus and Modelling Advanced

Assignment 1

Author: Keegan Gyoery
SID: 470413467

Tutor: Dr Daniel Daners
Tutorial: Carslaw 361 Tuesday 1pm

August 17, 2017

1. Let $f \in \mathbb{R}^{\Delta}$ and define $g, h \in \mathbb{R}^{\Delta}$ by the rules

$$\begin{aligned} g(x) &= \frac{f(x) + f(-x)}{2} \\ h(x) &= \frac{f(x) - f(-x)}{2} \end{aligned}$$

for all $x \in \Delta$.

We are now asked to prove which function out of g and h is even and which is odd. The definition of an even function is $f(-x) = f(x)$, and the definition of an odd function $f(-x) = -f(x)$. Both of these defintions will be used to determine the nature of the functions g and h.

Firstly, examining the function $g(x)$, we get the following results.

$$\begin{aligned} g(x) &= \frac{f(x) + f(-x)}{2} \\ \therefore g(-x) &= \frac{f(-x) + f(-(-x))}{2} && \text{substituting } -x \\ &= \frac{f(-x) + f(x)}{2} \\ &= \frac{f(x) + f(-x)}{2} \\ \therefore g(-x) &= g(x) \end{aligned}$$

Therefore, the function $g(x)$ is an even function as it satisfies the above defintion of an even function.

Secondly, examining the function $h(x)$, we get the following results.

$$\begin{aligned} h(x) &= \frac{f(x) - f(-x)}{2} \\ \therefore h(-x) &= \frac{f(-x) - f(-(-x))}{2} && \text{substituting } -x \\ &= \frac{f(-x) - f(x)}{2} \\ &= \frac{-f(x) + f(-x)}{2} \\ &= \frac{-\Big[f(x) - f(-x)\Big]}{2} \\ \therefore h(-x) &= -h(x) \end{aligned}$$

Therefore, the function $h(x)$ is an odd function as it satisfies the above defintion of an odd function.

2. Let $f \in \mathbb{R}^{\Delta}$. We are required to prove that

$$f = f_{even} + f_{odd}$$

for some unique functions, $f_{even}, f_{odd} \in \mathbb{R}^{\Delta}$ such that f_{even} is even and f_{odd} is odd.

Using the functions from the previous question $g(x)$ and $h(x)$, we can define two functions, f_{even} and f_{odd}, that are both unique and will satisfy the above condition. If we use the following definitions, the required result will become obvious.

$$\begin{aligned}
f_{even} &\coloneqq \frac{f(x) + f(-x)}{2} \\
f_{odd} &\coloneqq \frac{f(x) - f(-x)}{2} \\
\therefore f_{even} + f_{odd} &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\
&= \frac{f(x)}{2} + \frac{f(-x)}{2} + \frac{f(x)}{2} - \frac{f(-x)}{2} \\
&= \frac{f(x)}{2} + \frac{f(x)}{2} \\
&= 2\frac{f(x)}{2} \\
&= f(x) \\
\therefore f_{even} + f_{odd} &= f(x) \\
\therefore f &= f_{even} + f_{odd}
\end{aligned}$$

3. In order to find simplified expressions for $f_{even}(x)$ and $f_{odd}(x)$, in each of the following cases, we will use both the results from parts 1 and 2, in order to construct unique expressions for the $f(x)$ given in the question.

(a) For the first case we have $\Delta = \mathbb{R}$, and $f(x) = e^x$ for all $x \in \Delta$.

$$\begin{aligned}
f(x) &= e^x \\
f_{even}(x) &= \frac{f(x) + f(-x)}{2} \\
\therefore f_{even}(x) &= \frac{e^x + e^{-x}}{2} \\
f_{odd}(x) &= \frac{f(x) - f(-x)}{2} \\
\therefore f_{odd}(x) &= \frac{e^x - e^{-x}}{2}
\end{aligned}$$

(b) For the second case we have $\Delta = \mathbb{R} \setminus \{\pm 1\}$, and $f(x) = \dfrac{1}{1-x}$ for all $x \in \Delta$.

$$
\begin{aligned}
f(x) &= \frac{1}{1-x} \\
f_{even}(x) &= \frac{f(x) + f(-x)}{2} \\
&= \frac{\frac{1}{1-x} + \frac{1}{1+x}}{2} \\
&= \frac{\frac{1+x+1-x}{(1-x)(1+x)}}{2} \\
&= \frac{\frac{2}{1-x^2}}{2} \\
&= \frac{1}{1-x^2} \\
\therefore f_{even}(x) &= \frac{1}{1-x^2} \\
f_{odd}(x) &= \frac{f(x) - f(-x)}{2} \\
&= \frac{\frac{1}{1-x} - \frac{1}{1+x}}{2} \\
&= \frac{\frac{1+x-1+x}{(1-x)(1+x)}}{2} \\
&= \frac{\frac{2x}{1-x^2}}{2} \\
&= \frac{x}{1-x^2} \\
\therefore f_{odd}(x) &= \frac{x}{1-x^2}
\end{aligned}
$$

(c) For the third case we have $\Delta = \mathbb{R} \setminus \mathbb{Z}$, and $f(x) = \lfloor x \rfloor$ for all $x \in \Delta$. Due to the nature of the floor function and the defintion of the set in which Δ lies, we have the following facts for x and $\lfloor x \rfloor$. Firstly, due to the definition of the set in which x lies, that is $x \in \Delta$, where $\Delta \in \mathbb{R} \setminus \mathbb{Z}$, we have the following result.

$$
\begin{aligned}
n < x < n+1 &\quad \text{for some } n \in \mathbb{Z} \\
\therefore \lfloor x \rfloor &= n \\
\therefore -n > -x &> -(n+1) \\
\therefore -(n+1) < -x &< -n \\
\therefore \lfloor -x \rfloor &= -(n+1) \\
\therefore \lfloor x \rfloor + \lfloor -x \rfloor &= n - (n+1) \\
&= n - n - 1 \\
&= -1 \\
\therefore \lfloor x \rfloor + \lfloor -x \rfloor &= -1 \ldots\ldots (*)
\end{aligned}
$$

From this result we can find the necessary functions $f_{even}(x)$ and $f_{odd}(x)$, which are as follows.

$$
\begin{aligned}
f(x) &= \lfloor x \rfloor \\
f_{even}(x) &= \frac{f(x) + f(-x)}{2} \\
&= \frac{\lfloor x \rfloor + \lfloor -x \rfloor}{2} \\
&= -\frac{1}{2} \quad \text{using } (*) \\
\therefore f_{even}(x) &= -\frac{1}{2} \\
f_{odd}(x) &= \frac{f(x) - f(-x)}{2} \\
&= \frac{\lfloor x \rfloor - \lfloor -x \rfloor}{2} \\
&= \frac{\lfloor x \rfloor - \Big[-1 - \lfloor x \rfloor \Big]}{2} \quad \text{using } (*) \\
&= \frac{\lfloor x \rfloor + 1 + \lfloor x \rfloor}{2} \\
&= \frac{2\lfloor x \rfloor + 1}{2} \\
&= \lfloor x \rfloor + \frac{1}{2} \\
\therefore f_{odd}(x) &= \lfloor x \rfloor + \frac{1}{2}
\end{aligned}
$$

4. For the following proofs, we define $f \in \mathbb{R}^{\mathbb{R}}$, and f continuous such that all definite integrals $\displaystyle\int_a^b f(x)dx$ exist and are defined for all $a, b \in \mathbb{R}$.

Firstly, we shall determine the substitution $t = -u$, for use in the proofs that follow in $4a$, and $4b$.

$$\begin{aligned} t &= -u \\ t = x &\implies u = -x \\ t = -x &\implies u = x \\ t = a &\implies u = -a \\ t = -a &\implies u = a \\ t = 0 &\implies u = 0 \\ \frac{dt}{du} &= -1 \\ \therefore dt &= -du \end{aligned}$$

(a) Using the properties of integrals, we will construct the following two proofs to show that, for all $a \in \mathbb{R}$,

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\displaystyle\int_0^a f(x)dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd} \end{cases}$$

For the first case, f is even, and so we construct the following proof.

$$\begin{aligned} \int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx \\ &= \int_a^0 f(-t)(-dt) + \int_0^a f(x)dx \quad \text{substituting } x = -t \\ &= -\int_a^0 f(t)dt + \int_0^a f(x)dx \quad \text{as } f(x) \text{ is even} \\ &= \int_0^a f(t)dt + \int_0^a f(x)dx \\ &= \int_0^a f(x)dx + \int_0^a f(x)dx \quad \text{dummy variable switch} \\ \therefore \int_{-a}^{a} f(x)dx &= 2\int_0^a f(x)dx \quad \text{if } f \text{ is even} \end{aligned}$$

And thus the first result is proven.

Now for the second case, f is odd, and so we construct the following proof.

$$
\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= \int_{a}^{0} f(-t)(-dt) + \int_{0}^{a} f(x)\,dx \qquad \text{substituting } x = -t \\
&= -\int_{a}^{0} \big[-f(t)\big]\,dt + \int_{0}^{a} f(x)\,dx \qquad \text{as } f(x) \text{ is odd} \\
&= \int_{a}^{0} f(t)\,dt + \int_{0}^{a} f(x)\,dx \\
&= -\int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(x)\,dx \\
&= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{dummy variable switch} \\
\therefore \int_{-a}^{a} f(x)\,dx &= 0 \qquad \text{if } f \text{ is odd}
\end{aligned}
$$

And thus the second and final result is proven.

(b) For this question, we fix $a \in \mathbb{R}$ and define the area function $A \in \mathbb{R}^{\mathbb{R}}$ by the rule

$$A(x) = \int_a^x f(t)\,dt$$

for all $x \in \mathbb{R}$.

i. We are now required to prove that A is even if and only if f is odd. In order to complete this proof, we must prove the result from both sides of the implication. For the first part of the proof, we are required to prove f is odd if A is even.

$$\begin{aligned}
A(x) &= A(-x) \\
\therefore A(x) - A(-x) &= 0 \\
\therefore \text{LHS} &= \int_a^x f(t)\,dt - \int_a^{-x} f(t)\,dt \\
&= \int_a^0 f(t)\,dt + \int_0^x f(t)\,dt - \int_a^0 f(t)\,dt - \int_0^{-x} f(t)\,dt \\
&= \int_0^x f(t)\,dt - \int_0^{-x} f(t)\,dt \\
&= \int_0^x f(t)\,dt - \int_0^x f(-u)(-du) \quad \text{substituting } t = -u \\
&= \int_0^x f(t)\,dt + \int_0^x f(-u)\,du \\
&= \int_0^x f(t)\,dt + \int_0^x f(-t)\,dt \quad \text{dummy variable switch} \\
&= \int_0^x \Big[f(t) + f(-t)\Big]\,dt \\
\therefore \int_0^x \Big[f(t) + f(-t)\Big]\,dt &= 0 \\
\therefore \frac{d}{dx}\left[\int_0^x \Big[f(t) + f(-t)\Big]\,dt\right] &= \frac{d}{dx}[0] \\
\therefore f(x) + f(-x) &= 0 \quad \text{by the Fundamental Theorem of Calculus} \\
\therefore f(-x) &= -f(x)
\end{aligned}$$

Therefore f is odd if A is even.

The second part requires the proof that A is even if f is odd, which is as follows.

$$\begin{aligned}
\int_{-x}^{x} f(t)dt &= 0 \quad \text{as f is odd} \\
\therefore \text{LHS} &= \int_{0}^{x} f(t)dt + \int_{-x}^{0} f(t)dt \\
&= \int_{0}^{x} f(t)dt - \int_{0}^{-x} f(t)dt \\
&= \int_{0}^{x} f(t)dt + \int_{a}^{0} f(t)dt - \int_{a}^{0} f(t)dt - \int_{0}^{-x} f(t)dt \\
&= \int_{a}^{x} f(t)dt - \int_{a}^{-x} f(t)dt \\
\therefore \int_{a}^{x} f(t)dt - \int_{a}^{-x} f(t)dt &= 0 \\
\therefore \int_{a}^{x} f(t)dt &= \int_{a}^{-x} f(t)dt \\
\therefore A(x) &= A(-x)
\end{aligned}$$

Therefore A is even if f is odd. Thus A is even if and only if f is odd.

ii. We are now required to prove that A is odd if and only if f is even and $A(0) = 0$. In order to complete this proof, we must prove the result from both sides of the implication. For the first part of the proof, we are required to prove f is even, and $A(0) = 0$ if A is odd.

Firstly, we shall prove that $A(0) = 0$ if A is odd.

$$\begin{aligned}
A(-x) &= -A(x) \\
\therefore A(0) &= -A(0) \\
\therefore 2A(0) &= 0 \\
\therefore A(0) &= 0
\end{aligned}$$

Now we shall complete the remainder of the first proof.

$$\begin{aligned}
A(x) &= -A(-x) \\
\therefore A(x) + A(-x) &= 0 \\
\therefore \text{LHS} &= \int_a^x f(t)dt + \int_a^{-x} f(t)dt \\
&= \int_a^0 f(t)dt + \int_0^x f(t)dt + \int_a^0 f(t)dt + \int_0^{-x} f(t)dt \\
&= 2\int_a^0 f(t)dt + \int_0^x f(t)dt + \int_0^{-x} f(t)dt \\
&= 2\int_a^0 f(t)dt + \int_0^x f(t)dt + \int_0^x f(-u)(-du) \quad \text{substituting } t = -u \\
&= 2\int_a^0 f(t)dt + \int_0^x f(t)dt - \int_0^x f(-u)du \\
&= 2\int_a^0 f(t)dt + \int_0^x f(t)dt - \int_0^x f(-t)dt \quad \text{dummy variable switch} \\
&= 2A(0) + \int_0^x f(t)dt - \int_0^x f(-t)dt \\
&= \int_0^x f(t)dt - \int_0^x f(-t)dt \\
\therefore \int_0^x f(t)dt - \int_0^x f(-t)dt &= 0 \\
\therefore \int_0^x f(t)dt &= \int_0^x f(-t)dt \\
\therefore \frac{d}{dx}\left[\int_0^x f(t)dt\right] &= \frac{d}{dx}\left[\int_0^x f(-t)dt\right] \\
\therefore f(x) &= f(-x) \quad \text{by the Fundamental Theorem of Calculus}
\end{aligned}$$

Therefore f is even if A is odd and $A(0) = 0$.

The second part requires the proof that A is odd if f is even and $A(0) = 0$, which is as follows.

$$\begin{aligned}
\int_{-x}^{x} f(t)dt &= 2\int_{0}^{x} f(t)dt \qquad \text{as f is even} \\
\therefore \int_{-x}^{x} f(t)dt - 2\int_{0}^{x} f(t)dt &= 0 \\
\therefore \text{LHS} &= \int_{-x}^{x} f(t)dt - 2\int_{0}^{x} f(t)dt \\
&= \int_{-x}^{x} f(t)dt - 2\int_{0}^{x} f(t)dt - 2A(0) \qquad \text{as } A(0) = 0 \\
&= \int_{0}^{x} f(t)dt + \int_{-x}^{0} f(t)dt - 2\int_{0}^{x} f(t)dt - 2A(0) \\
&= -\int_{0}^{x} f(t)dt + \int_{-x}^{0} f(t)dt - 2\int_{a}^{0} f(t)dt \\
&= -\int_{0}^{x} f(t)dt - \int_{a}^{0} f(t)dt - \int_{0}^{-x} f(t)dt - \int_{a}^{0} f(t)dt \\
&= -\int_{a}^{x} f(t)dt - \int_{a}^{-x} f(t)dt \\
\therefore -\int_{a}^{x} f(t)dt - \int_{a}^{-x} f(t)dt &= 0 \\
\therefore -\int_{a}^{x} f(t)dt &= \int_{a}^{-x} f(t)dt \\
\therefore -A(x) &= A(-x)
\end{aligned}$$

Therefore A is odd if f is even and $A(0) = 0$. Therefore A is odd if and only if f is even and $A(0) = 0$.