CHAPTER SIX

Further Calculus

This chapter deals with four related topics which complete the treatment of calculus at 3 Unit level, apart from the extra work on rates of change in the next chapter. First, the systematic differentiation and integration of trigonometric functions is completed. Secondly, the reverse chain rule is extended to a more general method of integration called *integration by substitution*. Thirdly, the derivative is used to develop a very effective method, called *Newton's method*, of finding approximate solutions of equations. Fourthly, some of the earlier material on limits and inequalities is reviewed and summarised.

Study Notes: The reciprocal trigonometric functions $\sec x$, $\operatorname{cosec} x$ and $\cot x$ are not central to the course, but Sections 6A and 6B are intended to teach sufficient familiarity with them, as well as reviewing and summarising the previous approaches to the calculus of trigonometric functions. Later questions in Exercises 6A and 6B become difficult, and many students may not want to pursue the exercises very far. Sections 6C and 6D develop integration by substitution, extending the reverse chain rule presented first in Chapter Ten of the Year 11 volume, and they should also provide a good summary of the previous methods of integration. It is suggested that 4 Unit students study Sections 6A–6D before or while they embark on the systematic integration of the 4 Unit course.

Section 6E develops two methods of finding approximate solutions of equations called *halving the interval* and *Newton's method*. The final Section 6F is very demanding. It is intended for 4 Unit students and for the more ambitious 3 Unit students, and could well be left until final revision. The section reviews and develops previous approaches to inequalities and limits, and involves arguments based on the derivative, on bounding a carefully chosen integral, on geometry, and on algebra.

6 A Differentiation of the Six Trigonometric Functions

So far, the derivatives of $\sin x$, $\cos x$ and $\tan x$ have been established. While the derivatives of the other three trigonometric functions can be calculated when needed, the patterns become clearer when all six derivatives are listed, and it is recommended that they all be memorised.

Differentiating the Three Reciprocal Functions: The differentiation of the three functions $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$ depends on the formula for differentiating the reciprocal of a function:

$$\text{if } y = \frac{1}{u}, \text{ then by the chain rule, } \frac{dy}{dx} = -\frac{du/dx}{u^2}\,.$$

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A. Let $y = \sec x$
$$= \frac{1}{\cos x}.$$
Then $y' = -\dfrac{-\sin x}{\cos^2 x}$
$$= \sec x \tan x.$$

B. Let $y = \operatorname{cosec} x$
$$= \frac{1}{\sin x}.$$
Then $y' = -\dfrac{\cos x}{\sin^2 x}$
$$= -\operatorname{cosec} x \cot x.$$

C. Let $y = \cot x$
$$= \frac{1}{\tan x}.$$
Then $y' = -\dfrac{\sec^2 x}{\tan^2 x}$
$$= -\operatorname{cosec}^2 x.$$

Here then is the list of all six derivatives.

1 THE DERIVATIVES OF THE SIX TRIGONOMETRIC FUNCTIONS:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\cot x = -\operatorname{cosec}^2 x \qquad \frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$$

The extensions of these standard forms to trigonometric functions of linear functions of x now follow easily. For example,

$$\frac{d}{dx}\sec(ax+b) = a\sec(ax+b)\tan(ax+b).$$

Remarks on these Derivatives: There are two patterns here that will help in memorising the results. These patterns should be studied in comparison with the graphs of all six trigonometric functions, reproduced again on the next full page.

First, the derivatives of the three co-functions — cosine, cotangent and cosecant — all begin with a negative sign. This is because the three co-functions all have negative gradient in the first quadrant, as can be seen from their graphs on the next page.

Secondly, the derivative of each co-function is obtained by adding the prefix 'co-', as well as adding the minus sign. For example,

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \text{and} \qquad \frac{d}{dx}\cot x = -\operatorname{cosec}^2 x.$$

WORKED EXERCISE:

(a) If $y = \tan x$, show that $y'' - 2y^3 - 2y = 0$.

(b) If $y = \sec x$, show that $y'' - 2y^3 + y = 0$.

SOLUTION:

(a) If $y = \tan x$,
then $y' = \sec^2 x$.
Using the chain rule,
$$\begin{aligned} y'' &= 2\sec x \times \sec x \tan x \\ &= 2\sec^2 x \tan x \\ &= 2(\tan^2 x + 1)\tan x. \end{aligned}$$
Hence $y'' = 2y^3 + 2y$.

(b) If $y = \sec x$,
then $y' = \sec x \tan x$.
Using the product rule,
$$\begin{aligned} y'' &= (\sec x \tan x)\tan x + \sec x \sec^2 x \\ &= \sec x(\sec^2 x - 1) + \sec^3 x \\ &= 2\sec^3 x - \sec x. \end{aligned}$$
Hence $y'' = 2y^3 - y$.

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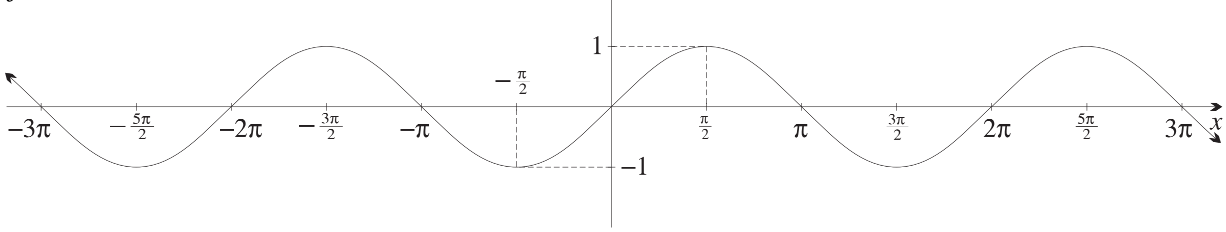
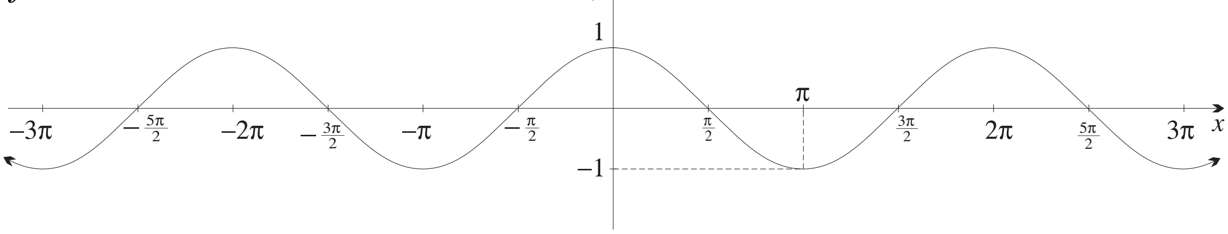
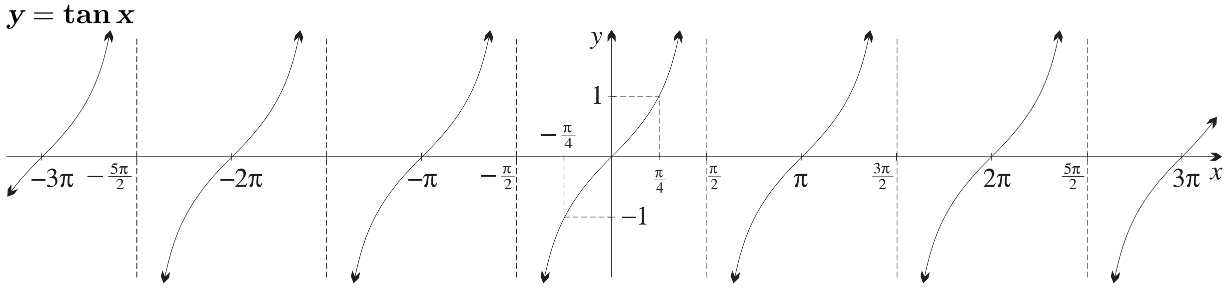
$y = \sin x$



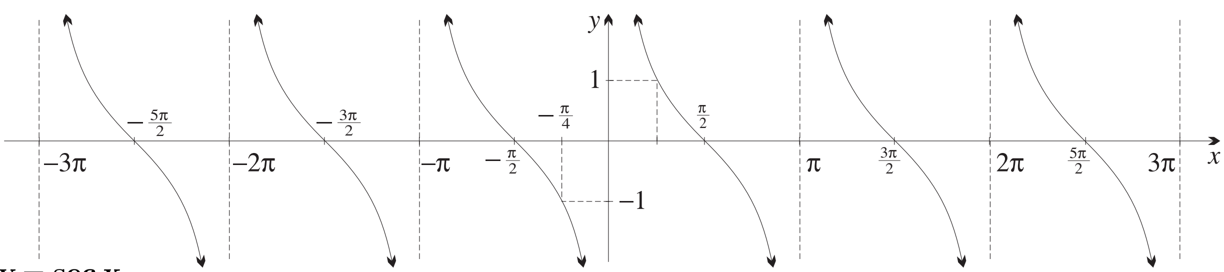
$y = \cos x$



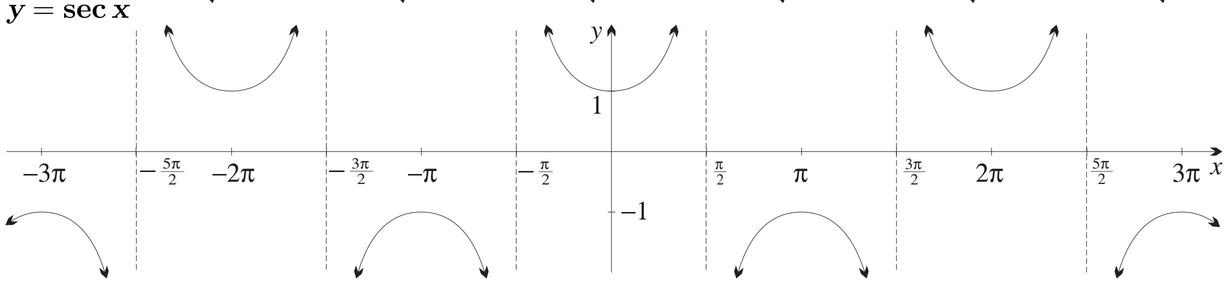
$y = \tan x$



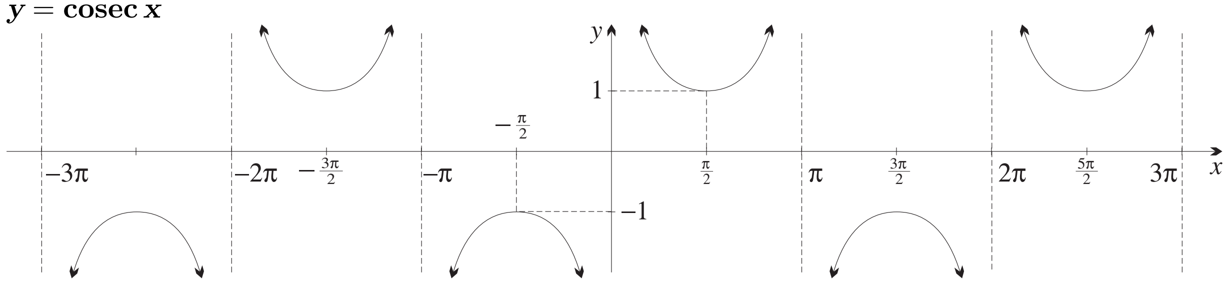
$y = \cot x$



$y = \sec x$



$y = \operatorname{cosec} x$



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WORKED EXERCISE: Find any points on $y = \sec x$, for $0 \le x \le 2\pi$, where the tangent has gradient $\sqrt{2}$.

SOLUTION: Differentiating, $y' = \sec x \tan x$.

Put $y' = \sqrt{2}$, then $\sec x \tan x = \sqrt{2}$

$\boxed{\times \cos^2 x}$ $\sin x = \sqrt{2}\cos^2 x$.

Since $\cos^2 x = 1 - \sin^2 x$, $\sqrt{2}\sin^2 x + \sin x - \sqrt{2} = 0$.

Since $\Delta = 1 + 4 \times 2 = 9$, $\sin x = \dfrac{-1+3}{2\sqrt{2}}$ or $\dfrac{-1-3}{2\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}}$ or $-\sqrt{2}$.

The second value is less than -1 and so gives no solutions.

Hence $x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, and the points are $(\frac{\pi}{4}, \sqrt{2})$ and $(\frac{3\pi}{4}, -\sqrt{2})$.

Exercise 6A

1. Differentiate with respect to x:

(a) $\sec x$
(b) $\operatorname{cosec} x$
(c) $\cot x$
(d) $\operatorname{cosec} 3x$
(e) $\cot(1-x)$
(f) $\sec(5x-2)$

2. Find the gradient of each curve at the point on it where $x = \frac{\pi}{6}$:

(a) $y = \sec 2x$
(b) $y = \cot 2x$

3. Find the equation of the tangent to each curve at the point indicated:

(a) $y = \cot 3x$ at $x = \frac{\pi}{12}$
(b) $y = \operatorname{cosec} x$ at $x = \frac{\pi}{4}$
(c) $y = \cos x + \sec x$ at $x = \frac{\pi}{3}$
(d) $y = \sec 5x$ at $x = \frac{\pi}{5}$

4. Differentiate with respect to x:

(a) $e^{\cot x}$
(b) $\log_e(\sec x)$
(c) $x \operatorname{cosec} x$
(d) $\cot^2 x$
(e) $\sec^4 x$
(f) $\log(\cot x)$
(g) $e^{2x}\sec 2x$
(h) $\dfrac{\operatorname{cosec}^2 x}{x^2}$

5. Consider the curve $y = \tan x + \cot x$, for $-\pi < x < \pi$.

(a) For which values of x in the given domain is y undefined?
(b) Is the function even or odd or neither?
(c) Show that the curve has no x-intercepts, and examine its sign in the four quadrants.
(d) Show that $y' = 0$ when $\tan^2 x = 1$.
(e) Find the stationary points in the given domain and determine their nature.
(f) Sketch the curve over the given domain.
(g) Show that the equation of the curve can be written as $y = 2\operatorname{cosec} 2x$.

DEVELOPMENT

6. Show that:

(a) $\dfrac{d}{dx}\left(x\sec^2 x - \tan x\right) = 2x\sec^2 x \tan x$
(b) $\dfrac{d}{dx}\ln(\sec x + \tan x) = \sec x$
(c) $\dfrac{d}{dx}\left(\dfrac{1+\tan x}{\sec x}\right) = \dfrac{1-\tan x}{\sec x}$
(d) $\dfrac{d}{dx}\tan^{-1}(\operatorname{cosec} x + \cot x) = -\frac{1}{2}$

7. If $y = \operatorname{cosec} x$, show that $y'' = 2y^3 - y$.

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8. (a) Show that $\dfrac{d}{dx}(\sec x \tan x) = \sec x(2\sec^2 x - 1)$.

(b) Hence find the values of x for which the function $y = \sec x \tan x$ is decreasing in the interval $0 \le x \le 2\pi$.

9. Consider the function $f(x) = \tan x - \cot x - 4x$, defined for $0 < x < \pi$.

(a) Show that $f'(x) = (\tan x - \cot x)^2$.

(b) For what value of x in the domain $0 < x < \pi$ is $f(x)$ undefined?

(c) Find any stationary points and determine their nature.

(d) Sketch the graph of $f(x)$.

10. Consider the curve $y = 3\sqrt{3}\sec x - \operatorname{cosec} x$ over the domain $0 < x < 2\pi$.

(a) For what values of x is y undefined? (b) Show that $y' = 0$ when $\tan x = -\frac{1}{\sqrt{3}}$.

(c) Find the stationary points and determine their nature.

(d) Use a calculator to examine the behaviour of y as $x \to 0^+$, as $x \to \frac{\pi}{2}^+$, as $x \to \pi^+$, and as $x \to \frac{3\pi}{2}^+$, and also as $x \to \frac{\pi}{2}^-$, as $x \to \pi^-$, as $x \to \frac{3\pi}{2}^-$, and as $x \to 2\pi^-$.

(e) Hence sketch the curve.

11. Use a similar approach to the previous question to sketch $y = \operatorname{cosec} x + \sec x$ for $0 < x < 2\pi$.

12. (a) Show that $\dfrac{d}{dx}\tan^{-1}(\cot x) = -1$.

(b) Show that $\dfrac{d}{dx}\cos^{-1}(\sin x) = -1$, provided that $\cos x > 0$.

(c) Hence explain why each piece of $y = \cos^{-1}(\sin x) - \tan^{-1}(\cot x)$ is horizontal for $\cos x > 0$, and find the value of the constant when:

(i) x is in the first quadrant, (ii) x is in the fourth quadrant.

13. Differentiate with respect to x:

(a) $\cot \frac{1}{x}$ (b) $\log \log \sec x$ (c) $\dfrac{1}{\tan 3x - \sec 3x}$

14. A curve is defined parametrically by the equations $x = 2\sec\theta$, $y = 3\tan\theta$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3\sec\theta}{2\tan\theta}$.

(b) Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$.

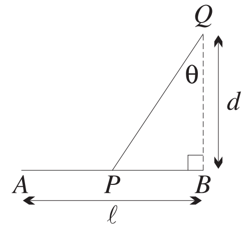
15. (a) Using the t-formulae, or otherwise, show that:

(i) $\dfrac{1 - \cos x}{\sin x} = \tan \frac{1}{2}x$ (ii) $\dfrac{1 + \sin x}{\cos x} = \tan(\frac{x}{2} + \frac{\pi}{4})$

(b) Hence show that:

(i) $\dfrac{d}{dx}\left(\ln \tan \frac{1}{2}x\right) = \operatorname{cosec} x$ (ii) $\dfrac{d}{dx}\log \tan(\frac{x}{2} + \frac{\pi}{4}) = \sec x$

16. In the diagram, AB is a major blood vessel and PQ is a minor blood vessel. Let $AB = \ell$ units, $BQ = d$ units and $\angle PQB = \theta$. It is known that the resistance to blood flow in a blood vessel is proportional to its length, and that the constant of proportionality varies from blood vessel to blood vessel. Let R be the sum of the resistances in AP and PQ.



(a) Show that $R = c_1(\ell - d\tan\theta) + c_2 d\sec\theta$, where c_1 and c_2 are constants of proportionality.

(b) If $c_2 = 2c_1$, find the value of θ that minimises R.

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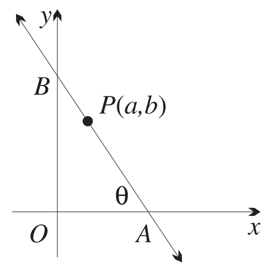
17. In the diagram, a line passes through the fixed point $P(a, b)$, where a and b are both positive, and meets the x-axis and y-axis at A and B respectively. Let $\angle OAB = \theta$.

(a) Show that $AB = a\sec\theta + b\,\mathrm{cosec}\,\theta$.

(b) Show that AB is minimum when $\tan\theta = \dfrac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$.

(c) Show that the minimum distance is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.



EXTENSION

18. Differentiate implicitly to find $\dfrac{dy}{dx}$, given:

(a) $\cot y = \mathrm{cosec}\,x$ (b) $xy = \sec(x + y)$

19. Sketch the graph of the function $y = \dfrac{4}{\mathrm{cosec}\,x - \sec x}$, for $0 < x < 2\pi$.

[HINT: First find any x-intercepts, vertical asymptotes and stationary points.]

20. Use the result in question 6(d) to sketch $y = \tan^{-1}(\mathrm{cosec}\,x + \cot x)$.

6 B Integration Using the Six Trigonometric Functions

Systematic integration of the trigonometric functions is not easy. The point of this section is learning the methods of integration — memorising results other than the six standard forms below is not required.

The Six Standard Forms: The first step is to reverse the six derivatives of the previous section to obtain the six standard forms for integration.

2 THE SIX STANDARD FORMS: Omitting constants of integration,

$$\int \cos x\,dx = \sin x \qquad \int \sec^2 x\,dx = \tan x \qquad \int \sec x\tan x\,dx = \sec x$$

$$\int \sin x\,dx = -\cos x \qquad \int \mathrm{cosec}^2\,x\,dx = -\cot x \qquad \int \mathrm{cosec}\,x\cot x\,dx = -\,\mathrm{cosec}\,x$$

Again, linear extensions follow easily. For example,

$$\int \sec(ax + b)\tan(ax + b)\,dx = \frac{1}{a}\sec(ax + b) + C.$$

The Primitives of the Squares of the Trigonometric Functions: We have already integrated the squares of the trigonometric functions.

First, the primitives of $\sec^2 x$ and $\mathrm{cosec}^2\,x$ are standard forms:

$$\int \sec^2 x\,dx = \tan x + C \qquad \text{and} \qquad \int \mathrm{cosec}^2\,x\,dx = -\cot x + C.$$

Secondly, $\tan^2 x$ and $\cot^2 x$ can be integrated by writing them in terms of $\sec^2 x$ and $\mathrm{cosec}^2\,x$ using the Pythagorean identities:

$$\tan^2 x = \sec^2 x - 1 \qquad \text{and} \qquad \cot^2 x = \mathrm{cosec}^2\,x - 1.$$

Thirdly, $\sin^2 x$ and $\cos^2 x$ can be integrated by writing them in terms of $\cos 2x$:

$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x \qquad \text{and} \qquad \sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x.$$

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The six results, and the methods of obtaining them, are listed below.

3 **PRIMITIVES OF THE SQUARES OF THE SIX TRIGONOMETRIC FUNCTIONS:**

$$\int \cos^2 x\,dx = \int (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C$$

$$\int \sin^2 x\,dx = \int (\tfrac{1}{2} - \tfrac{1}{2}\cos 2x)\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$$

$$\int \cot^2 x\,dx = \int (\operatorname{cosec}^2 x - 1)\,dx = -\cot x - x + C$$

The Primitives of the Six Trigonometric Functions: Surprisingly, it is harder to find primitives of the functions themselves than it is to find primitive of their squares. First, the primitives of $\sin x$ and $\cos x$ are standard forms:

$$\int \cos x\,dx = \sin x + C \qquad \text{and} \qquad \int \sin x\,dx = -\cos x + C.$$

Secondly, $\tan x$ and $\cot x$ can be integrated by the reverse chain rule:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\log(\cos x) + C$$

Let $u = \cos x$.
Then $u' = -\sin x$,
and $\displaystyle\int \frac{1}{u}\frac{du}{dx}\,dx = \log u$.

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \log(\sin x) + C$$

Let $u = \sin x$.
Then $u' = \cos x$,
and $\displaystyle\int \frac{1}{u}\frac{du}{dx}\,dx = \log u$.

Thirdly, the primitives of $\sec x$ and $\operatorname{cosec} x$ require some subtle tricks, whatever way they are found, and are beyond the 3 Unit course. One method is given here, but further details are left to the following exercise.

$$\begin{aligned}\int \sec x\,dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx \\ &= \log(\sec x + \tan x) + C\end{aligned}$$

Let $u = \sec x + \tan x$.
Then $u' = \sec x \tan x + \sec^2 x$,
and $\displaystyle\int \frac{1}{u}\frac{du}{dx}\,dx = \log u$.

$$\begin{aligned}\int \operatorname{cosec} x\,dx &= \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)}{\operatorname{cosec} x + \cot x}\,dx \\ &= \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x}\,dx \\ &= -\log(\operatorname{cosec} x + \cot x) + C\end{aligned}$$

Let $u = \operatorname{cosec} x + \cot x$.
Then $u' = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$,
and $\displaystyle\int \frac{1}{u}\frac{du}{dx}\,dx = \log u$.

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Here are the six results and the methods of obtaining them.

4

PRIMITIVES OF THE SIX TRIGONOMETRIC FUNCTIONS:

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\log(\cos x) + C$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \log(\sin x) + C$$

$$* \quad \int \sec x\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx = \log(\sec x + \tan x) + C$$

$$* \quad \int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x}\,dx = -\log(\operatorname{cosec} x + \cot x) + C$$

*These forms are not required in the 3 Unit course.

A Special Case of the Reverse Chain Rule: The two functions $y = \cos x \sin^n x$ and $y = \sin x \cos^n x$ can be integrated easily using the reverse chain rule.

WORKED EXERCISE: Find primitives of: (a) $y = \sin x \cos^4 x$ (b) $y = \cos x \sin^n x$

SOLUTION:

(a) $$\int \sin x \cos^4 x\,dx = -\int (-\sin x)\cos^4 x\,dx = -\tfrac{1}{5}\cos^5 x + C$$

Let $u = \cos x$.
Then $\dfrac{du}{dx} = -\sin x$,
and $\displaystyle\int u^4 \frac{du}{dx}\,dx = \tfrac{1}{5}u^5$.

(b) $$\int \cos x \sin^n x\,dx = \frac{\sin^{n+1} x}{n+1} + C$$

Let $u = \sin x$.
Then $\dfrac{du}{dx} = \cos x$,
and $\displaystyle\int u^n \frac{du}{dx}\,dx = \frac{u^n}{n+1}$.

WORKED EXERCISE: [A harder question]

(a) Find $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ by writing $\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x$.

(b) Find $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$ by writing $\cos^3 x = \cos x(1 - \sin^2 x)$.

(c) Find $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$ by writing $\cos^4 x = (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)^2$.

SOLUTION:

(a) $$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^2 x\,dx &= \int_0^{\frac{\pi}{2}} (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,dx \\ &= \Big[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\Big]_0^{\frac{\pi}{2}} \\ &= \tfrac{\pi}{4}\end{aligned}$$

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(b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

$\displaystyle = \int_0^{\frac{\pi}{2}} \cos x(1-\sin^2 x)\,dx$

$\displaystyle = \Big[\sin x - \tfrac{1}{3}\sin^3 x\Big]_0^{\frac{\pi}{2}}$

(using the previous worked exercise)

$= (1-\frac{1}{3}) - (0-0)$

$= \frac{2}{3}$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$

$\displaystyle = \int_0^{\frac{\pi}{2}} (\cos^2 x)^2\,dx$

$\displaystyle = \int_0^{\frac{\pi}{2}} (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)^2\,dx$

$\displaystyle = \int_0^{\frac{\pi}{2}} (\tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cos^2 2x)\,dx$

$\displaystyle = \int_0^{\frac{\pi}{2}} (\tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8} + \tfrac{1}{8}\cos 4x)\,dx$

$\displaystyle = \Big[\tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}x + \tfrac{1}{32}\sin 4x\Big]_0^{\frac{\pi}{2}}$

$= (\frac{\pi}{8} + 0 + \frac{\pi}{16} + 0) - (0+0+0+0)$

$= \frac{3\pi}{16}$

NOTE: Almost all the arguments above using primitives could have been replaced by arguments about symmetry. In particular, horizontal shifting and reflection in the x-axis will prove that

$$\int_0^{\frac{\pi}{2}} \cos 2x\,dx = \int_0^{\frac{\pi}{2}} \cos 4x\,dx = 0,$$

and arguments about reflection in $y = \frac{1}{2}$ will prove that

$$\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 2x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 4x\,dx = \frac{\pi}{4}.$$

Students taking the 4 Unit course may like to investigate the symmetries involved.

Exercise 6B

1. Find:

(a) $\displaystyle\int \cos 2x\,dx$

(b) $\displaystyle\int \sin 2x\,dx$

(c) $\displaystyle\int \sec^2 2x\,dx$

(d) $\displaystyle\int \operatorname{cosec}^2 2x\,dx$

(e) $\displaystyle\int \sec 2x \tan 2x\,dx$

(f) $\displaystyle\int \operatorname{cosec} 2x \cot 2x\,dx$

2. Find:

(a) $\displaystyle\int \cos \tfrac{1}{3}x\,dx$

(b) $\displaystyle\int \sin \tfrac{1}{2}(1-x)\,dx$

(c) $\displaystyle\int \sec^2(4-3x)\,dx$

(d) $\displaystyle\int \operatorname{cosec}^2 \tfrac{1}{5}(2x+3)\,dx$

(e) $\displaystyle\int \sec(ax+b)\tan(ax+b)\,dx$

(f) $\displaystyle\int \operatorname{cosec}(a-bx)\cot(a-bx)\,dx$

3. Calculate the exact area bounded by each curve, the x-axis and the two vertical lines.
NOTE: In each case, the region lies completely above the x-axis.

(a) $y = \sec x \tan x$, $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$,

(b) $y = \operatorname{cosec}^2 2x$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$,

(c) $y = \operatorname{cosec} \frac{1}{3}x \cot \frac{1}{3}x$, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{4}$,

(d) $y = \tan x$, $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$.

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4. (a) Show that $\dfrac{d}{dx}(\ln\sec x) = \tan x$, and hence find $\displaystyle\int_0^{\frac{\pi}{3}} \tan x\,dx$.

(b) Show that $\dfrac{d}{dx}(\ln\sin 3x) = 3\cot 3x$, and hence find $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \cot 3x\,dx$.

(c) Show that $\dfrac{d}{dx}\big(\ln(\sec x + \tan x)\big) = \sec x$, and hence find $\displaystyle\int_0^{\frac{\pi}{4}} \sec x\,dx$.

(d) Show that $\dfrac{d}{dx}\big(\tfrac{1}{2}\ln(\operatorname{cosec} 2x - \cot 2x)\big) = \operatorname{cosec} 2x$, and hence find $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} 2x\,dx$.

5. Express $\sin^2 x$ in terms of $\cos 2x$, and hence find:

(a) $\displaystyle\int \sin^2 x\,dx$ (b) $\displaystyle\int \sin^2 2x\,dx$ (c) $\displaystyle\int \sin^2 \tfrac{1}{4}x\,dx$ (d) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2 3x\,dx$

6. Express $\cos^2 x$ in terms of $\cos 2x$, and hence find:

(a) $\displaystyle\int \cos^2 x\,dx$ (b) $\displaystyle\int \cos^2 6x\,dx$ (c) $\displaystyle\int \cos^2 \tfrac{1}{2}x\,dx$ (d) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 2x\,dx$

7. (a) Find: (i) $\displaystyle\int \tan^2 2x\,dx$ (ii) $\displaystyle\int \cot^2 \tfrac{1}{2}x\,dx$

(b) Evaluate: (i) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} 3\tan^2 3x\,dx$ (ii) $\displaystyle\int_{\frac{\pi}{24}}^{\frac{\pi}{8}} \cot^2 4x\,dx$

8. (a) If $y' = \sin^2 x + \tan^2 x$ and $y = 1$ when $x = 0$, find y when $x = \frac{\pi}{4}$.

(b) Given that $f'(x) = -\operatorname{cosec} 2x(\cot 2x + \operatorname{cosec} 2x)$ and $f(\frac{\pi}{4}) = 1$, find $f(\frac{\pi}{12})$.

9. Find the volume of the solid generated when the given curve is rotated about the x-axis. [HINT: In part (f), use the reverse chain rule.]

(a) $y = \sec 2x$ between $x = \frac{\pi}{8}$ and $x = \frac{\pi}{6}$,

(b) $y = \tan \frac{1}{2}x$ between $x = 0$ and $x = \frac{\pi}{2}$,

(c) $y = \cos \pi x$ between $x = 0$ and $x = \frac{1}{2}$,

(d) $y = \sqrt{\cot x}$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$,

(e) $y = \cot \frac{\pi}{2}x$ between $x = \frac{1}{2}$ and $x = 1$,

(f) $y = \sec x \tan x$ between $x = 0$ and $x = \frac{\pi}{3}$.

DEVELOPMENT

10. Use the reverse chain rule to find:

(a) $\displaystyle\int \sin^3 x\cos x\,dx$ [Let $u = \sin x$.]

(b) $\displaystyle\int \cot^4 x\operatorname{cosec}^2 x\,dx$ [Let $u = \cot x$.]

(c) $\displaystyle\int \sec^7 x\tan x\,dx$ [Let $u = \sec x$, and write $\sec^7 x\tan x = \sec^6 x \times \sec x\tan x$.]

(d) $\displaystyle\int_0^{\pi} \cos^6 x\sin x\,dx$ (e) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan^3 x}\,dx$ (f) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^3 x\cot x\,dx$

11. Find:

(a) $\displaystyle\int 2x\sec x^2\tan x^2\,dx$

(b) $\displaystyle\int \frac{\operatorname{cosec}^2 x}{1+\cot x}\,dx$

(c) $\displaystyle\int e^x\cot e^x\,dx$

(d) $\displaystyle\int \sec 2x\tan 2x\,e^{\sec 2x}\,dx$

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12. Evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\sec 2x}\,dx$ (b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\,dx$ (c) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1+\sin^3 x}{\sin^2 x}\,dx$ (d) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\operatorname{cosec} x\cot x}{1+\operatorname{cosec} x}\,dx$

13. In each part, sketch the region defined by the given boundaries. Then find the area of the region, and the volume generated when the region is rotated about the x-axis.

(a) $y = 1+\sin x$, $x=0$, $x=\pi$, $y=0$ (b) $y=\sin x+\cos x$, $x=0$, $x=\frac{\pi}{3}$, $y=0$ (c) $y=\sin x\cos x$, $x=\frac{\pi}{8}$, $x=\frac{3\pi}{8}$, $y=0$ (d) $y=\tan x+\cot x$, $x=\frac{\pi}{6}$, $x=\frac{\pi}{4}$, $y=0$

(e) $y=1+\operatorname{cosec} x$, $x=\frac{\pi}{6}$, $x=\frac{5\pi}{6}$, $y=0$ [HINT: $\displaystyle\int \operatorname{cosec} x\,dx = -\ln(\operatorname{cosec} x+\cot x)+C$]

14. (a) Show that $\dfrac{d}{dx}(\ln\sin x - x\cot x) = x\operatorname{cosec}^2 x$, and hence find $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\operatorname{cosec}^2 x\,dx$.

(b) Show that $\dfrac{d}{dx}(\operatorname{cosec} x - \cot x) = \dfrac{1}{1+\cos x}$, and hence find $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{1+\cos x}\,dx$.

(c) Show that $\dfrac{d}{dx}(\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x) = \sec^3 x\tan^3 x$, and hence find $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x\tan^3 x\,dx$.

(d) Show that $\dfrac{d}{dx}\big(\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln(\sec x+\tan x)\big) = \sec^3 x$, hence find $\displaystyle\int_0^{\frac{\pi}{4}} \sec^3 x\,dx$.

(e) Show that $\dfrac{d}{dx}(\cot^3 x) = 3\operatorname{cosec}^2 x - 3\operatorname{cosec}^4 x$, and hence find $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^4 x\,dx$.

(f) Show that $\dfrac{d}{dx}(\cos^3 x\sin x) = 4\cos^4 x - 3\cos^2 x$, and hence find $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$.

15. Find the value of $\displaystyle\lim_{R\to\infty}\left(\frac{1}{R}\int_0^R \sin^2 t\,dt\right)$, explaining your reasoning carefully.

16. Starting with $\displaystyle\int \operatorname{cosec} x\,dx = \ln(\operatorname{cosec} x - \cot x)+C$, show that

$$\int \operatorname{cosec} x\,dx = \ln\left(\frac{1-\cos x}{\sin x}\right)+C = \ln\left(\frac{\sin x}{1+\cos x}\right)+C = \ln t + C,\quad \text{where } t=\tan\tfrac{1}{2}x.$$

EXTENSION

17. (a) Show that $\dfrac{d}{dx}\left(\dfrac{1}{n-1}\left(\tan x\sec^{n-2} x + (n-2)\displaystyle\int \sec^{n-2} x\,dx\right)\right) = \sec^n x$, for $n\ge 2$.

(b) Hence find the value of $\displaystyle\int_0^{\frac{\pi}{4}} \sec^7 x\,dx$.

6 C Integration by Substitution

The reverse chain rule as we have been using it so far does not cover all the situations where the chain rule can be used in integration. This section and the next develop a more general method called *integration by substitution*. The first stage, covered in this section, begins by translating the reverse chain rule into a slightly more flexible notation. It involves substitutions of the form

'Let $u=$ some function of x.'

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The Reverse Chain Rule — An Example: Here is an example of the reverse chain rule as we have been using it. The working is set out in full on the right.

WORKED EXERCISE: Find $\int x(1-x^2)^4\,dx$.

SOLUTION:
$$\begin{aligned}\int x(1-x^2)^4\,dx &= -\tfrac{1}{2}\int(-2x)(1-x^2)^4\,dx \\ &= -\tfrac{1}{2}\times\tfrac{1}{5}(1-x^2)^5+C \\ &= -\tfrac{1}{10}(1-x^2)^5+C\end{aligned}$$

Let $u=1-x^2$.
Then $\dfrac{du}{dx}=-2x$,
and $\displaystyle\int u^4\,\frac{du}{dx}\,dx=\tfrac{1}{5}u^5$.

Rewriting this Example as Integration by Substitution: We shall now rewrite this using a new notation. The key to this new notation is that the derivative $\dfrac{du}{dx}$ is treated as a fraction — the du and the dx are split apart, so that the statement

$$\frac{du}{dx}=-2x \qquad \text{is written instead as} \qquad du=-2x\,dx.$$

The new variable u no longer remains in the working column on the right, but is brought over into the main sequence of the solution on the left.

WORKED EXERCISE: Find $\int x(1-x^2)^4\,dx$, using the substitution $u=1-x^2$.

SOLUTION:
$$\begin{aligned}\int x(1-x^2)^4\,dx &= \int u^4\left(-\tfrac{1}{2}\right)du \\ &= -\tfrac{1}{2}\times\tfrac{1}{5}u^5+C \\ &= -\tfrac{1}{10}(1-x^2)^5+C\end{aligned}$$

Let $u=1-x^2$.
Then $du=-2x\,dx$,
and $x\,dx=-\tfrac{1}{2}\,du$.

WORKED EXERCISE: Find $\int \sin x\sqrt{1-\cos x}\,dx$, using the substitution $u=1-\cos x$.

SOLUTION:
$$\begin{aligned}\int \sin x\sqrt{1-\cos x}\,dx &= \int u^{\frac{1}{2}}\,du \\ &= \tfrac{2}{3}u^{\frac{3}{2}}+C \\ &= \tfrac{2}{3}(1-\cos x)^{\frac{3}{2}}+C\end{aligned}$$

Let $u=1-\cos x$.
Then $du=\sin x\,dx$.

An Advance on the Reverse Chain Rule: Some integrals which can be done in this way could only be done by the reverse chain rule in a rather clumsy manner.

WORKED EXERCISE: Find $\int x\sqrt{1-x}\,dx$, using the substitution $u=1-x$.

SOLUTION:
$$\begin{aligned}\int x\sqrt{1-x}\,dx &= \int(1-u)\sqrt{u}\,du \\ &= \int\left(u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)du \\ &= \tfrac{2}{3}u^{\frac{3}{2}}-\tfrac{2}{5}u^{\frac{5}{2}}+C \\ &= \tfrac{2}{3}(1-x)^{\frac{3}{2}}-\tfrac{2}{5}(1-x)^{\frac{5}{2}}+C\end{aligned}$$

Let $u=1-x$.
Then $du=-\,dx$,
and $x=1-u$.

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Substituting the Limits of Integration in a Definite Integral: A great advantage of this method is that the limits of integration can be changed from values of x to values of u. There is then no need ever to go back to x. The first worked exercise below repeats the previous integrand, but this time within a definite integral.

WORKED EXERCISE: Find $\int_0^1 x\sqrt{1-x}\,dx$, using the substitution $u = 1 - x$.

SOLUTION:

$$\begin{aligned}\int_0^1 x\sqrt{1-x}\,dx &= -\int_1^0 (1-u)\sqrt{u}\,du \\ &= -\int_1^0 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du \\ &= -\left[\tfrac{2}{3}u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right]_1^0 \\ &= -0 + \left(\tfrac{2}{3} - \tfrac{2}{5}\right) \\ &= \tfrac{4}{15}\end{aligned}$$

Let $u = 1 - x$.
Then $du = -\,dx$,
and $x = 1 - u$.
When $x = 0$, $u = 1$,
when $x = 1$, $u = 0$.

WORKED EXERCISE: Find $\int_0^\pi \sin x \cos^6 x\,dx$, using the substitution $u = \cos x$.

SOLUTION:

$$\begin{aligned}\int_0^\pi \sin x\cos^6 x\,dx &= -\int_1^{-1} u^6\,du \\ &= -\tfrac{1}{7}\Big[u^7\Big]_1^{-1} \\ &= -\tfrac{1}{7}\times(-1) + \tfrac{1}{7}\times 1 \\ &= \tfrac{2}{7}\end{aligned}$$

Let $u = \cos x$.
Then $du = -\sin x\,dx$.
When $x = 0$, $u = 1$,
when $x = \pi$, $u = -1$.

Exercise 6C

1. Consider the integral $\int 2x(1+x^2)^3\,dx$, and the substitution $u = 1 + x^2$.

(a) Show that $du = 2x\,dx$. (b) Show that the integral can be written as $\int u^3\,du$.

(c) Hence find the primitive of $2x(1+x^2)^3$. (d) Check your answer by differentiating it.

2. Repeat the previous question for each of the following indefinite integrals and substitutions.

(a) $\int 2(2x+3)^3\,dx$ [Let $u = 2x + 3$.]

(b) $\int 3x^2(1+x^3)^4\,dx$ [Let $u = 1 + x^3$.]

(c) $\int \dfrac{2x}{(1+x^2)^2}\,dx$ [Let $u = 1 + x^2$.]

(d) $\int \dfrac{3}{\sqrt{3x-5}}\,dx$ [Let $u = 3x - 5$.]

(e) $\int \sin^3 x\cos x\,dx$ [Let $u = \sin x$.]

(f) $\int \dfrac{4x^3}{1+x^4}\,dx$ [Let $u = 1 + x^4$.]

3. Consider the integral $\int \dfrac{x}{\sqrt{1-x^2}}\,dx$, and the substitution $u = 1 - x^2$.

(a) Show that $x\,dx = -\frac{1}{2}\,du$. (b) Show that the integral can be written as $-\frac{1}{2}\int u^{-\frac{1}{2}}\,du$.

(c) Hence find the primitive of $\dfrac{x}{\sqrt{1-x^2}}$.

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4. Repeat the previous question for each of the following indefinite integrals and substitutions.

(a) $\int x^3(x^4+1)^5\,dx$ [Let $u = x^4+1$.]

(b) $\int x^2\sqrt{x^3-1}\,dx$ [Let $u = x^3-1$.]

(c) $\int x^2 e^{x^3}\,dx$ [Let $u = x^3$.]

(d) $\int \dfrac{1}{\sqrt{x}(1+\sqrt{x}\,)^3}\,dx$ [Let $u = 1+\sqrt{x}$.]

(e) $\int \tan^2 2x\sec^2 2x\,dx$ [Let $u = \tan 2x$.]

(f) $\int \dfrac{e^{\frac{1}{x}}}{x^2}\,dx$ [Let $u = \dfrac{1}{x}$.]

5. Find the exact value of each definite integral, using the given substitution.

(a) $\int_0^1 x^2(2+x^3)^3\,dx$ [Let $u = 2+x^3$.]

(b) $\int_0^1 \dfrac{2x^3}{\sqrt{1+x^4}}\,dx$ [Let $u = 1+x^4$.]

(c) $\int_0^{\frac{\pi}{2}} \cos^2 x\sin x\,dx$ [Let $u = \cos x$.]

(d) $\int_{\frac{1}{2}\sqrt{3}}^1 x\sqrt{1-x^2}\,dx$ [Let $u = 1-x^2$.]

(e) $\int_1^{e^2} \dfrac{\ln x}{x}\,dx$ [Let $u = \ln x$.]

(f) $\int_0^4 \dfrac{e^{\sqrt{x}}}{4\sqrt{x}}\,dx$ [Let $u = \sqrt{x}\,$.]

(g) $\int_0^{\frac{\pi}{4}} \sin^4 2x\cos 2x\,dx$ [Let $u = \sin 2x$.]

(h) $\int_0^1 \dfrac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\,dx$ [Let $u = \sin^{-1}x$.]

(i) $\int_0^2 \dfrac{x+1}{\sqrt[3]{x^2+2x}}\,dx$ [Let $u = x^2+2x$.]

(j) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{\sec^2 x}{\tan x}\,dx$ [Let $u = \tan x$.]

DEVELOPMENT

6. Use the substitution $u = x^3$ to find:

(a) the exact area bounded by the curve $y = \dfrac{x^2}{1+x^6}$, the x-axis and the line $x = 1$,

(b) the exact volume generated when the region bounded by the curve $y = \dfrac{x}{(1-x^6)^{\frac{1}{4}}}$, the x-axis and the line $x = 1$ is rotated about the x-axis.

7. Evaluate each of the following, using the substitution $u = \sin x$.

(a) $\int_0^{\frac{\pi}{6}} \dfrac{\cos x}{1+\sin x}\,dx$

(b) $\int_0^{\frac{\pi}{2}} \dfrac{\cos x}{1+\sin^2 x}\,dx$

(c) $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

(d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \dfrac{\cos^3 x}{\sin^4 x}\,dx$

8. Find each indefinite integral, using the given substitution.

(a) $\int \dfrac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx$ [Let $u = e^{2x}$.]

(b) $\int \dfrac{1}{x\ln x}\,dx$ [Let $u = \ln x$.]

(c) $\int \dfrac{\tan x}{\ln\cos x}\,dx$ [Let $u = \ln\cos x$.]

(d) $\int \tan^3 x\sec^4 x\,dx$ [Let $u = \tan x$.]

9. (a) A curve has gradient function $\dfrac{e^{2x}}{1+e^{4x}}$ and passes through the point $(0, \frac{\pi}{8})$. Use the substitution $u = e^{2x}$ to find its equation.

(b) If $y'' = \dfrac{x}{(4-x^2)^{\frac{3}{2}}}$, and when $x = 0$, $y' = 1$ and $y = \frac{1}{2}$, use the substitution $u = 4-x^2$ to find y' and then find y as a function of x.

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10. (a) Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

(b) Use the substitution $u = \sec x$ to find:

(i) $\displaystyle\int_0^{\frac{\pi}{3}} 2^{\sec x} \sec x \tan x \, dx$ [HINT: $\displaystyle\int a^x \, dx = \frac{a^x}{\ln a}$] (ii) $\displaystyle\int_0^{\frac{\pi}{4}} \sec^5 x \tan x \, dx$

11. Evaluate each integral, using the given substitution.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\sin^2 x}\, dx$ [Let $u = \sin^2 x$.] (b) $\displaystyle\int_1^{e} \frac{\ln x + 1}{(x \ln x + 1)^2}\, dx$ [Let $u = x \ln x$.]

12. Use the substitution $u = \sqrt{x-1}$ to find $\displaystyle\int \frac{1}{2x\sqrt{x-1}}\, dx$.

13. The region R is bounded by the curve $y = \dfrac{x}{x+1}$, the x-axis and the vertical line $x = 3$.
Use the substitution $u = x + 1$ to find:

(a) the exact area of R,

(b) the exact volume generated when R is rotated about the x-axis.

14. (a) Use the substitution $u = \sqrt{x}$ to find $\displaystyle\int \frac{1}{\sqrt{x(1-x)}}\, dx$.

(b) Evaluate the integral in part (a) again, using the substitution $u = x - \frac{1}{2}$.

(c) Hence show that $\sin^{-1}(2x-1) = 2\sin^{-1}\sqrt{x} - \frac{\pi}{2}$, for $0 < x < 1$.

EXTENSION

15. Use the substitution $u = x - \dfrac{1}{x}$ to show that $\displaystyle\int_1^{\frac{\sqrt{6}+\sqrt{2}}{2}} \frac{1+x^2}{1+x^4}\, dx = \frac{\pi}{4\sqrt{2}}$.

6 D Further Integration by Substitution

The second stage of integration by substitution reverses the previous procedure and replaces x by a function of u. The substitutions are therefore of the form

'Let $x =$ some function of u.'

Substituting x by a Function of u: As a first example, here is a quite different substitution which solves the integral given in a worked example of the last section.

WORKED EXERCISE: Find $\displaystyle\int_0^1 x\sqrt{1-x}\, dx$, using the substitution $x = 1 - u^2$.

SOLUTION:

$$\begin{aligned}\int_0^1 x\sqrt{1-x}\, dx &= \int_1^0 (1-u^2)u(-2u)\, du \\ &= -2\int_1^0 (u^2 - u^4)\, du \\ &= -2\left[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right]_1^0 \\ &= -0 + 2(\tfrac{1}{3} - \tfrac{1}{5}) \\ &= \tfrac{4}{15}\end{aligned}$$

Let $x = 1 - u^2$.
Then $dx = -2u\, du$,
and $\sqrt{1-x} = u$.
When $x = 0$, $u = 1$,
when $x = 1$, $u = 0$.

This question is a good example of the fact that an integral may be evaluated in a variety of ways. The following integral uses a trigonometric substitution, but can also be done through areas of segments.

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WORKED EXERCISE: Find $\displaystyle\int_{3\sqrt{2}}^{6} \sqrt{36 - x^2}\, dx$:

(a) using the substitution $x = 6\sin u$,

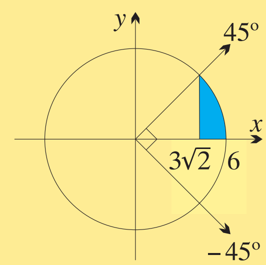
(b) using the formula for the area of a segment.

SOLUTION:

(a)
$$\begin{aligned}\int_{3\sqrt{2}}^{6} \sqrt{36 - x^2}\, dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 6\cos u \times 6\cos u\, du \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 36\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2u\right) du \\ &= \Big[18u + 9\sin 2u\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= (9\pi + 0) - \left(\tfrac{9\pi}{2} + 9\right) \\ &= \tfrac{9}{2}(\pi - 2)\end{aligned}$$

Let $x = 6\sin u$.
Then $dx = 6\cos u\, du$,
and $\sqrt{36 - x^2} = 6\cos u$.
When $x = 3\sqrt{2}$, $u = \frac{\pi}{4}$,
when $x = 6$, $u = \frac{\pi}{2}$.

(b) The integral is sketched opposite. The shaded area is half the segment subtending an angle of 90°.



Hence $\displaystyle\int_{3\sqrt{2}}^{6} \sqrt{36 - x^2}\, dx = \tfrac{1}{2} \times \tfrac{1}{2} \times 6^2\left(\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\right)$
$= 9\left(\tfrac{\pi}{2} - 1\right)$.

NOTE: Careful readers may notice a problem here, in that given the value $x = 3\sqrt{2}$, u is determined by $\sin u = \frac{1}{2}\sqrt{2}$, so there are infinitely many possible values of u. A similar problem occurred in the previous worked exercise, where $0 = 1 - u^2$ had two solutions. These problems arise because the functions involved in the substitutions were $x = 1 - u^2$ and $x = 6\sin u$, whose inverses were not functions. A full account of all this would require substitutions by restrictions of the functions given above so that they had inverse functions. In practice, however, this is rarely necessary, and it is certainly not a concern of this course.

As a rule of thumb, work with positive square roots, and with trigonometric functions, work in the same quadrants as were involved in the definitions of the inverse trigonometric functions in Chapter One.

Exercise 6D

1. Consider the integral $I = \displaystyle\int x(x-1)^5\, dx$, and let $x = u + 1$.

(a) Show that $dx = du$.

(b) Show that $I = \displaystyle\int u^5(u+1)\, du$.

(c) Hence find I.

(d) Check your answer by differentiating it.

2. Using the same substitution as in the previous question, find:

(a) $\displaystyle\int \frac{x}{\sqrt{x-1}}\, dx$

(b) $\displaystyle\int \frac{x}{(x-1)^2}\, dx$

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3. Consider the integral $J = \displaystyle\int x\sqrt{x+1}\,dx$, and let $x = u^2 - 1$.

(a) Show that $dx = 2u\,du$.

(b) Show that $J = 2\displaystyle\int (u^4 - u^2)\,du$.

(c) Hence find J.

(d) Check your answer by differentiating it.

4. Using the same substitution as in the previous question, find:

(a) $\displaystyle\int x^2\sqrt{x+1}\,dx$

(b) $\displaystyle\int \frac{2x+3}{\sqrt{x+1}}\,dx$

5. Find each of the following indefinite integrals using the given substitution.

(a) $\displaystyle\int \frac{x-2}{x+2}\,dx$ [Let $x = u - 2$.]

(b) $\displaystyle\int \frac{2x+1}{\sqrt{2x-1}}\,dx$ [Let $x = \frac{1}{2}(u+1)$.]

(c) $\displaystyle\int 3x\sqrt{4x-5}\,dx$ [Let $x = \frac{1}{4}(u^2+5)$.]

(d) $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$ [Let $x = (u-1)^2$.]

6. Evaluate, using the given substitution:

(a) $\displaystyle\int_0^1 x(x+1)^3\,dx$ [Let $x = u - 1$.]

(b) $\displaystyle\int_0^{\frac{1}{2}} \frac{1+x}{1-x}\,dx$ [Let $x = 1 - u$.]

(c) $\displaystyle\int_0^1 \frac{3x}{\sqrt{3x+1}}\,dx$ [Let $x = \frac{1}{3}(u-1)$.]

(d) $\displaystyle\int_0^1 \frac{2-x}{(2+x)^3}\,dx$ [Let $x = u - 2$.]

(e) $\displaystyle\int_0^4 x\sqrt{4-x}\,dx$ [Let $x = 4 - u^2$.]

(f) $\displaystyle\int_1^5 \frac{x}{(2x-1)^{\frac{3}{2}}}\,dx$ [Let $x = \frac{1}{2}(u^2+1)$.]

(g) $\displaystyle\int_0^4 \frac{1}{3+\sqrt{x}}\,dx$ [Let $x = (u-3)^2$.]

(h) $\displaystyle\int_0^7 \frac{x^2}{\sqrt[3]{x+1}}\,dx$ [Let $x = u^3 - 1$.]

DEVELOPMENT

7. (a) Consider the integral $I = \displaystyle\int \frac{1}{\sqrt{5-4x-x^2}}\,dx$, and let $x = u - 2$.

Show that $I = \displaystyle\int \frac{1}{\sqrt{9-u^2}}\,du$, and hence find I.

(b) Use a similar approach to find:

(i) $\displaystyle\int \frac{1}{x^2+2x+4}\,dx$ [Let $x = u - 1$.]

(ii) $\displaystyle\int \frac{1}{\sqrt{4-2x-x^2}}\,dx$ [Let $x = u - 1$.]

(iii) $\displaystyle\int_1^2 \frac{1}{\sqrt{3+2x-x^2}}\,dx$ [Let $x = u + 1$.]

(iv) $\displaystyle\int_3^7 \frac{1}{x^2-6x+25}\,dx$ [Let $x = u + 3$.]

8. (a) Consider the integral $J = \displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$, and let $x = 2\sin\theta$.

Show that $J = \displaystyle\int 1\,d\theta$, and hence show that $J = \sin^{-1}\frac{x}{2} + C$.

(b) Using a similar approach, find:

(i) $\displaystyle\int \frac{1}{9+x^2}\,dx$ [Let $x = 3\tan\theta$.]

(ii) $\displaystyle\int \frac{-1}{\sqrt{3-x^2}}\,dx$ [Let $x = \sqrt{3}\cos\theta$.]

(iii) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$ [Let $x = \frac{1}{2}\sin\theta$.]

(iv) $\displaystyle\int \frac{1}{1+16x^2}\,dx$ [Let $x = \frac{1}{4}\tan\theta$.]

(v) $\displaystyle\int_0^3 \frac{1}{\sqrt{36-x^2}}\,dx$ [Let $x = 6\sin\theta$.]

(vi) $\displaystyle\int_0^{\frac{2}{3}} \frac{1}{4+9x^2}\,dx$ [Let $x = \frac{2}{3}\tan\theta$.]

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9. (a) Consider the integral $I = \displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$, and let $x = \sin\theta$.

Show that $I = \displaystyle\int \sec^2\theta\,d\theta$, and hence show that $I = \frac{x}{\sqrt{1-x^2}} + C$.

(b) Similarly, use the given substitution to find:

(i) $\displaystyle\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\,dx$ [Let $x = 2\tan\theta$.]

(ii) $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$ [Let $x = \sin\theta$.]

(iii) $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$ [Let $x = 2\sin\theta$.]

(iv) $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ [Let $x = 5\cos\theta$.]

(v) $\displaystyle\int \frac{1}{x^2\sqrt{9+x^2}}\,dx$ [Let $x = 3\tan\theta$.]

(vi) $\displaystyle\int_2^4 \frac{1}{x^2\sqrt{x^2-4}}\,dx$ [Let $x = 2\sec\theta$.]

10. (a) Sketch the region R bounded by $y = \dfrac{1}{x^2+1}$, the x- and y-axes, and the line $x = 1$.

(b) Find the volume generated when R is rotated about the x-axis.
[HINT: Use the substitution $x = \tan\theta$.]

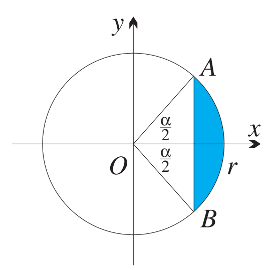
11. Find the equation of the curve $y = f(x)$ if $f'(x) = \dfrac{\sqrt{x^2-9}}{x}$ and $f(3) = 0$.
[HINT: Use the substitution $x = 3\sec\theta$.]

12. Find the exact area of the region bounded by $y = \dfrac{x^3}{\sqrt{3-x^2}}$, the x-axis and the line $x = 1$.
[HINT: Use the substitution $x = \sqrt{3}\,\sin\theta$, followed by the substitution $u = \cos\theta$.]

13. [These are confirmations rather than proofs, since the calculus of trigonometric functions was developed on the basis of the formulae in parts (a) and (b).]

(a) Use integration to confirm that the area of a circle is πr^2.
[HINT: Find the area bounded by the semicircle $y = \sqrt{r^2-x^2}$ and the x-axis and double it. Use the substitution $x = r\sin\theta$.]

(b) The shaded area in the diagram to the right is the segment of a circle of radius r cut off by the chord AB subtending an angle α at the centre O.



(i) Show that the area is $I = 2\displaystyle\int_{r\cos\frac{1}{2}\alpha}^{r} \sqrt{r^2-x^2}\,dx$.

(ii) Let $x = r\cos\theta$, and show that $I = -2r^2\displaystyle\int_{\frac{1}{2}\alpha}^{0} \sin^2\theta\,d\theta$.

(iii) Hence confirm that $I = \frac{1}{2}r^2(\alpha - \sin\alpha)$.

(c) Use a similar approach to confirm that the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab. Then justify the formula by regarding the ellipse as the unit circle stretched horizontally by a factor of a and vertically by a factor of b.

EXTENSION

14. (a) Multiply $\sec\theta$ by $\dfrac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$, and hence find $\displaystyle\int \sec\theta\,d\theta$.

(b) The region R is bounded by $y = \dfrac{x}{\sqrt{x^2+16}}$, the x-axis and line $x = 4$. Show that the volume generated by rotating R about the y-axis is $16\pi\left(\sqrt{2} - \ln(\sqrt{2}+1)\right)$ units^3.
[HINT: Use the substitution $y = \sin\theta$ and the result in part (a).]

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15. (a) Use the substitution $x = -u$ to show that $\displaystyle\int_{-2}^{2} \frac{x^2}{e^x+1}\,dx = \int_{-2}^{2} \frac{x^2 e^x}{e^x+1}\,dx$.

(b) Hence find $\displaystyle\int_{-2}^{2} \frac{x^2}{e^x+1}\,dx$.

6 E Approximate Solutions and Newton's Method

Most equations cannot be solved exactly. This section deals with two methods of finding approximate solutions, called *halving the interval* and *Newton's method*. Each method produces a sequence of approximate solutions with increasingly greater accuracy, with Newton's method converging to the solution very fast indeed.

Approaching an Unknown Equation: Given an unknown equation, there are three successive questions to ask:

5 THREE QUESTIONS TO ASK ABOUT AN UNKNOWN EQUATION:

1. Does the equation have a solution?
2. How many solutions are there, and roughly where are they?
3. How can approximations be found, correct to the required level of accuracy?

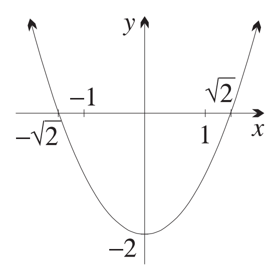
Any work on approximations should therefore be preceded by an exploratory table of values, and probably a graph, to give the rough locations of the solutions. These procedures were described in Section 3F of the Year 11 volume.

The easiest example of our methods is finding approximations to $\sqrt{2}$. This means finding the positive root of the equation $x^2 = 2$. We will write the equation as

$$x^2 - 2 = 0,$$

so that it has the form $f(x) = 0$, where $f(x) = x^2 - 2$. Then

x	-2	-1	0	1	2
$x^2 - 2$	2	-1	-2	-1	2



Hence there is solution between 1 and 2, and another between -2 and -1. We shall seek approximations to the solution $x = \sqrt{2}$ between 1 and 2.

Halving the Interval: This is simply a systematic approach to constructing a table of values near the solution. A function can only change sign at a zero or a discontinuity, hence we have trapped a solution between 1 and 2. If we keep halving the interval, the solution will be trapped successively within intervals of length $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ..., until the desired order of accuracy is obtained.

6 APPROXIMATING SOLUTIONS BY HALVING THE INTERVAL: Given the equation $f(x) = 0$:

1. Locate the solutions roughly by means of a table of values and/or a graph.
2. To obtain a sequence approximating a particular solution, trap the solution within an interval, then keep halving the interval where the solution is trapped.

Each successive application of the method will halve the uncertainty of the approximation. Since $2^{10} = 1024 \doteqdot 1000$, it will take roughly ten further steps to obtain three further decimal places.

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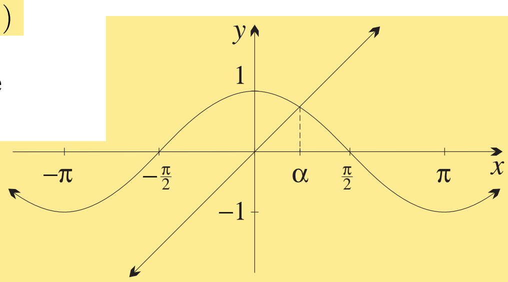
WORKED EXERCISE: Use the method of halving the interval to approximate $\sqrt{2}$ correct to three significant figures.

SOLUTION: We have already found that $\sqrt{2}$ lies between 1 and 2.
Let $f(x) = x^2 - 2$. Then by hand and by calculator,

x	1	2	$1\frac{1}{2}$	$1\frac{1}{4}$	$1\frac{3}{8}$	$1\frac{7}{16}$	$1\frac{13}{32}$	$1\frac{27}{64}$	$1\frac{53}{128}$	$1\frac{107}{256}$	$1\frac{213}{512}$	$1\frac{425}{1024}$	$1\frac{849}{2048}$	$1\frac{1697}{4096}$
$f(x)$	-1	2	$\frac{1}{4}$	$-\frac{7}{16}$	$-\frac{7}{64}$	$\frac{17}{256}$	$-$	$+$	$-$	$+$	$+$	$+$	$+$	$+$

Hence $\qquad 1\frac{53}{128} < \sqrt{2} < 1\frac{1697}{4096}$,
or in decimal form, $0\cdot4140 < \sqrt{2} < 1\cdot4144$, so that $\sqrt{2} \doteqdot 1\cdot414$.
(Strictly speaking, one should round down the left-hand bound, and round up the right-hand bound.)

WORKED EXERCISE: Solve $\cos x = x$ correct to three decimal places, by halving the interval.

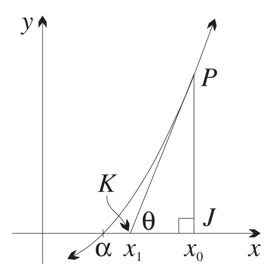


SOLUTION: The graph shows that there is exactly one solution, and that it lies between $x = 0$ and $x = 1$. Let the solution be $x = \alpha$, and consider the function $y = \cos x - x$.

x	0	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{23}{32}$	$\frac{47}{64}$	$\frac{95}{128}$	$\frac{189}{256}$	$\frac{379}{512}$	$\frac{757}{1024}$	$\frac{1513}{2048}$
y	1	$-0\cdot46$	$0\cdot38$	$-0\cdot02$	$0\cdot19$	$0\cdot09$	$0\cdot03$	$+$	$-$	$+$	$-$	$-$	$+$

Hence $\qquad \frac{1513}{2048} < \alpha < \frac{757}{1024}$,
or in decimal form, $0\cdot7387 < \alpha < 0\cdot7393$, so that $\alpha \doteqdot 0\cdot739$.

Newton's Method: The function graphed below has an unknown root at $x = \alpha$, and $x = x_0$ is a known approximation to that root. Let $J = (x_0, 0)$.
Draw a tangent at $P\big(x_0, f(x_0)\big)$,
and let it meet the x-axis at $K(x_1, 0)$ with angle of inclination θ.
Then x_1 will be a better approximation to α than x_0.
Now $\tan\theta$ is the gradient of $y = f(x)$ at $x = x_0$,



so $\qquad \tan\theta = f'(x_0)$.

In $\triangle JPK$, $\qquad JK = \dfrac{PJ}{\tan\theta}$,

that is, $\qquad x_0 - x_1 = \dfrac{f(x_0)}{f'(x_0)}$,

so $\qquad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

This formula is the basis of Newton's method.

7 NEWTON'S METHOD: Suppose that $x = x_0$ is an approximation to a root $x = \alpha$ of an equation $f(x) = 0$. Then, provided that the situation is favourable, a closer approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The formula can be applied successively to produce a sequence of successively closer approximations to the root.

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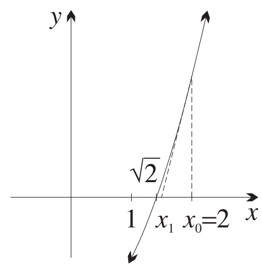
We will mention below some serious questions about what makes a 'favourable situation'. For now, notice from the accompanying diagram that the function was carefully chosen so that it was increasing and concave up, with $x_0 > \alpha$.

WORKED EXERCISE:



(a) Beginning with the approximation $x_0 = 2$ for $\sqrt{2}$, use one step of Newton's method to obtain a better approximation x_1.

(b) Show that in general, $x_n = \dfrac{{x_{n-1}}^2 + 2}{2x_{n-1}}$.

(c) Continue the process to obtain an approximation correct to eight decimal places.

SOLUTION:

(a) Here $f(x) = x^2 - 2$
so $f'(x) = 2x$.

Hence
$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 2 - \frac{f(2)}{f'(2)} \\ &= 2 - \tfrac{2}{4} \\ &= 1\tfrac{1}{2}. \end{aligned}$$

(b) In general,
$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{{x_{n-1}}^2 - 2}{2x_{n-1}} \\ &= \frac{2{x_{n-1}}^2 - {x_{n-1}}^2 + 2}{2x_{n-1}} \\ &= \frac{{x_{n-1}}^2 + 2}{2x_{n-1}}. \end{aligned}$$

(c) Continuing these calculations,
$$\begin{aligned} x_2 &= \frac{{x_1}^2 + 2}{2x_1} = 1{\cdot}416\,666\,666\ldots \\ x_3 &= \frac{{x_2}^2 + 2}{2x_2} = 1{\cdot}414\,215\,686\ldots \\ x_4 &= \frac{{x_3}^2 + 2}{2x_3} = 1{\cdot}414\,213\,562\ldots \\ x_5 &= \frac{{x_4}^2 + 2}{2x_4} = 1{\cdot}414\,213\,562\ldots. \end{aligned}$$

Hence $\sqrt{2} \doteqdot 1{\cdot}414\,213\,56$.

A NOTE ON CALCULATORS: On many new calculators, the formula only needs to be entered once, after which each successive approximation can be obtained simply by pressing $\boxed{=}$. Enter the initial value x_0 and press $\boxed{=}$, then enter the formula using the key labelled $\boxed{\text{Ans}}$ whenever x_0 occurs in the formula.

A NOTE ON THE SPEED OF CONVERGENCE: It should be obvious from the diagram above that Newton's method converges extremely rapidly once it gets going. As a rule of thumb, the number of correct decimal places doubles with each step. It would help intuition to continue these calculations using mathematical software capable of working with thirty or more decimal places.

Problem One — The Initial Approximation May Be on the Wrong Side: The original diagram above shows that Newton's method works when the curve bulges towards the x-axis in the region between $x = \alpha$ and $x = x_0$. In other situations, the method can easily run into problems. The first problem is hopefully only a nuisance — in the example below, x_0 is chosen on the wrong side of the root, but the next approximation x_1 is on the favourable side, and the sequence then converges rapidly as before.

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WORKED EXERCISE:

(a) Beginning with the approximate solution $x_0 = 0$ of $\cos x = x$, use one step of Newton's method to obtain x_1.

(b) Show that in general, $x_n = \dfrac{x_{n-1}\sin x_{n-1} + \cos x_{n-1}}{1 + \sin x_{n-1}}$.

(c) Find an approximation correct to eight decimal places.

SOLUTION: The graph below shows $f(x) = \cos x - x$ in the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. With $x_0 = 0$, the next approximation is $x = 1$, as shown in part (a). Were x_0 chosen further to the left, more serious problems could occur.

(a) Let $f(x) = \cos x - x$.
Then $f'(x) = -\sin x - 1$.

$$\text{Hence}\quad x_1 = x_0 - \frac{\cos x_0 - x_0}{-\sin x - 1}$$
$$= 0 + \frac{\cos 0 - 0}{\sin 0 + 1}$$
$$= 1.$$

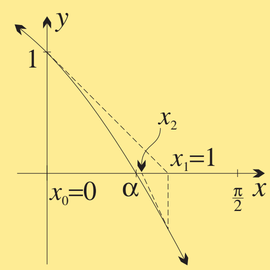
(b) Now that the approximation has crossed to the other side, convergence will be rapid.

$$\text{In general,}\quad x_n = x_{n-1} - \frac{\cos x_{n-1} - x^{n-1}}{-\sin x_{n-1} - 1}$$
$$= \frac{x_{n-1}(\sin x_{n-1} + 1) + (\cos x_{n-1} - x_{n-1})}{\sin x_{n-1} + 1}$$
$$= \frac{x_{n-1}\sin x_{n-1} + \cos x_{n-1}}{\sin x_{n-1} + 1}.$$

(c) Continuing the process,

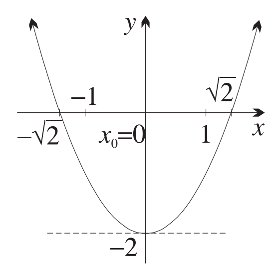
$$x_2 = \frac{x_1\sin x_1 + \cos x_1}{\sin x_1 + 1} = 0{\cdot}750\,363\,867\ \ldots$$
$$x_3 = \frac{x_2\sin x_2 + \cos x_2}{\sin x_2 + 1} = 0{\cdot}739\,112\,890\ \ldots$$
$$x_4 = \frac{x_3\sin x_3 + \cos x_3}{\sin x_3 + 1} = 0{\cdot}739\,085\,133\ \ldots$$
$$x_5 = \frac{x_4\sin x_4 + \cos x_4}{\sin x_4 + 1} = 0{\cdot}739\,085\,133\ \ldots.$$

Hence $\alpha \doteqdot 0{\cdot}739\,085\,13$.



Problem Two — The Tangent May Be Horizontal: If the tangent at $x = x_0$ is horizontal, it will never meet the x-axis, hence there will be no approximation x_1. The algebraic result is a zero denominator.

WORKED EXERCISE: Explain, algebraically and geometrically, why $x_0 = 0$ cannot be taken as a suitable first approximation when finding $\sqrt{2}$ by Newton's method.

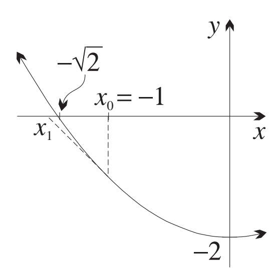


SOLUTION: Here $f(x) = x^2 - 2$ and $f'(x) = 2x$.

$$\text{Algebraically,}\quad x_1 = x_0 - \frac{{x_0}^2 - 2}{2x_0}$$
$$= 0 - \frac{0^2 - 2}{2 \times 0},\quad \text{which is undefined.}$$

Geometrically, the tangent at $P(0, -2)$ is horizontal, so it never meets the x-axis, and x_1 cannot be found.

Problem Three — The Sequence May Converge to the Wrong Root: In the previous example, if we were to choose $x_0 = -1$, beginning on the wrong side of the stationary point, then the sequence would converge to $-\sqrt{2}$ instead of to $\sqrt{2}$. The diagram shows this happening.



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Problem Four — The Sequence May Oscillate, or even Move Away from the Root: The diagram below shows the curve $y = x^3 - 5x$, which has an inflexion at the origin. If we try to approximate the root $x = 0$ using Newton's method, then neither side is favourable, and the sequence will keep crossing sides. Worse still, if $x_0 = 1$, the sequence will simply oscillate between 1 and -1, and if $x_0 > 1$, the sequence will move away from $x = 0$ instead of converging to it.

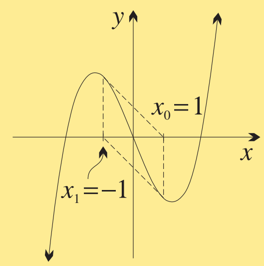
WORKED EXERCISE: Show that for $f(x) = x^3 - 5x$, one application of Newton's method will give $x_1 = \dfrac{2{x_0}^3}{3{x_0}^2 - 5}$.

(a) For $x_0 = 1$, show that the sequence of approximations oscillates.

(b) For $x_0 > 1$, show that the sequence will move away from $x = 0$.

SOLUTION: Since $f(x) = x^3 - 5x$, $f'(x) = 3x^2 - 5$.

Hence
$$\begin{aligned} x_1 &= x_0 - \frac{{x_0}^3 - 5x_0}{3{x_0}^2 - 5} \\ &= \frac{3{x_0}^3 - 5x_0 - {x_0}^3 + 5x_0}{3{x_0}^2 - 5} \\ &= \frac{2{x_0}^3}{3{x_0}^2 - 5}. \end{aligned}$$

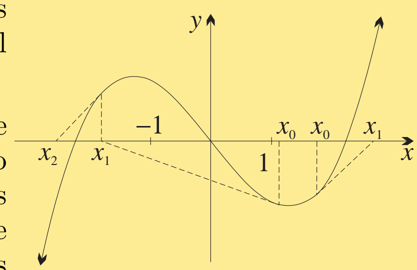


(a) Substituting $x_0 = 1$, $x_1 = \dfrac{2}{3-5}$
$= -1$.

Then because $f(x)$ has odd symmetry, the sequence oscillates:
$$x_2 = 1, \quad x_3 = -1, \quad x_4 = 1, \quad \ldots .$$

(b) When x_0 is to the right of the turning point, the tangent will slope upwards, and will meet the x-axis to the right of the positive zero — the sequence will then converge to that zero.

When x_0 is between $x = 1$ and the turning point, the tangent will be flatter than the tangent at $x = 1$, so x_1 will be to the left of -1. Once the sequence moves outside the two turning points, it will converge to one of the other two zeroes. But if any of $x_0, x_1, x_2, \ldots$ is ever at a turning point, the tangent will be horizontal and the method will terminate.



Problem Five — The Equation May Have No Solutions: The final Extension problem in the following exercise pursues the consequences when Newton's method is applied to the function $f(x) = 1 + x^2$, which has no zeroes at all. It is in such situations that Newton's method becomes a topic within modern chaos theory.

Exercise 6E

1. (a) If $P(x) = x^2 - 2x - 1$, show that $P(2) < 0$ and $P(3) > 0$, and therefore that there is a root of the equation $x^2 - 2x - 1 = 0$ between 2 and 3.

(b) Evaluate $P(\frac{5}{2})$ and hence show that the root to the equation $P(x) = 0$ lies in the interval $2 < x < 2\frac{1}{2}$.

(c) Which end of this interval is the root closer to? Justify your answer by using the halving the interval method a second time.

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2. (a) (i) Show that the equation $x^3 + x^2 + 2x - 3 = 0$ has a root between $x = 0$ and $x = 1$.
(ii) Use halving the interval twice to find an approximation to the root.
(b) (i) Show that the equation $x^4 + 2x^2 - 5 = 0$ has a root between 0·5 and 1·5.
(ii) Use halving the interval until you can approximate the root to one decimal place.

3. (a) (i) Show that the function $F(x) = x^3 - \log_e(x+1)$ has a zero between 0·8 and 0·9.
(ii) Use halving the interval once to approximate the root to one decimal place.
(b) (i) Show that the equation $\log_e x = \sin x$ has a root between 2 and 3.
(ii) Use halving the interval to approximate the root to one decimal place.
(c) (i) Show that the equation $e^x - \log_e x = 3$ has a root between 1 and 2.
(ii) Use halving the interval to approximate the root to one decimal place.

4. (a) Beginning with the approximate solution $x_0 = 2$ of $x^2 - 5 = 0$, use one step of Newton's method to obtain a better approximation x_1. Give your answer to one decimal place.
(b) Show that in general, $x_{n+1} = \dfrac{{x_n}^2 + 5}{2x_n}$.
(c) Use part (b) to find x_2, x_3, x_4 and x_5, which should confirm the accuracy of x_4 to at least eight decimal places.

NOTE: Your calculator may be able to obtain each successive approximation simply by pressing $\boxed{=}$. Try doing this — enter $x_0 = 2$ and press $\boxed{=}$, then enter the formula in part (b) using the key labelled $\boxed{\text{Ans}}$ whenever x_0 is needed, then press $\boxed{=}$ to get x_1. Now pressing $\boxed{=}$ successively should yield x_2, x_3, x_4 ....

5. Repeat the steps of the previous question in each of the following cases.
(a) $x^3 - 9x - 2 = 0$, $x_0 = 3$. Show that $x_{n+1} = \dfrac{2{x_n}^3 + 2}{3{x_n}^2 - 9}$.
(b) $e^x - 3x - 1 = 0$, $x_0 = 2$. Show that $x_{n+1} = \dfrac{e^{x_n}(x_n - 1) + 1}{e^{x_n} - 3}$.
(c) $2\sin x - x = 0$, $x_0 = 2$. Show that $x_{n+1} = \dfrac{2(\sin x_n - x_n \cos x_n)}{1 - 2\cos x_n}$.

6. Use Newton's method twice to find the indicated root of each equation, giving your answer correct to two decimal places. Then continue the process to obtain an approximation correct to eight decimal places.
(a) For $x^2 - 2x - 1 = 0$, approximate the root near $x = 2$.
(b) For $x^3 + x^2 + 2x - 3 = 0$, approximate the root near $x = 1$.
(c) For $x^4 + 2x^2 - 5 = 0$, approximate the root near $x = 1$.
(d) For $x^3 - \log_e(x+1) = 0$, approximate the root near $x = 0{\cdot}8$.
(e) For $\log_e x = \sin x$, approximate the root near $x = 2$.
(f) For $e^x - \log_e x = 3$, approximate the root near $x = 1$.

DEVELOPMENT

7. (a) Show that the equation $x^3 - 16 = 0$ has a root between 2 and 3.
(b) Use halving the interval three times to find a better approximation to the root.
(c) The actual answer to five decimal places is 2·51 984. Was the final number you substituted the best approximation to the root?

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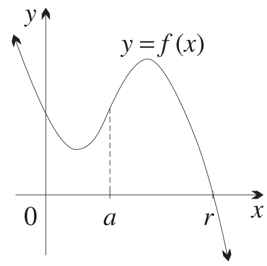
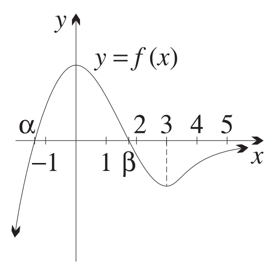
8. Use Newton's method to find approximations correct to two decimal places. Then continue the process to obtain an approximation correct to eight decimal places.
(a) $\sqrt{13}$ (b) $\sqrt[3]{35}$ (c) $\sqrt[5]{158}$

9. The closest integer to $\sqrt[4]{100}$ is 3. Use one application of Newton's method to show that $3\frac{19}{108}$ is a better approximation to $\sqrt[4]{100}$. Then obtain an approximation correct to eight decimal places.

10. Consider the polynomial $P(x) = 4x^3 + 2x^2 + 1$.
(a) Show that $P(x)$ has a real zero α in the interval $-1 < x < 0$.
(b) By sketching the graph of $P(x)$, show that α is the only real zero of $P(x)$.
(c) Use Newton's method with initial value $\alpha \doteqdot -\frac{1}{4}$ to obtain a second approximation.
(d) Explain from the graph of $P(x)$ why this second approximation is not a better approximation to α than $-\frac{1}{4}$ is.



11. Consider the graph of $y = f(x)$. The value a shown on the axis is taken as the first approximation to the solution r of $f(x) = 0$. Is the second approximation obtained by Newton's method a better approximation to r than a is? Give a reason for your answer.



12. The diagram shows the curve $y = f(x)$, which has turning points at $x = 0$ and $x = 3$ and a point of inflexion at $x = 4$. The equation $f(x) = 0$ has two real roots α and β. Determine which of the following cases applies when Newton's method is repeatedly applied with the given starting value x_0:
A. α is approximated. B. β is approximated.
C. The sequence $x_1, x_2, x_3, \ldots$ is moving away from both roots.
D. The method breaks down at the first application.

(a) $x_0 = -2$	(d) $x_0 = 0$	(g) $x_0 = 2$	(j) $x_0 = 3{\cdot}1$
(b) $x_0 = -1$	(e) $x_0 = 0{\cdot}1$	(h) $x_0 = 2{\cdot}9$	(k) $x_0 = 4$
(c) $x_0 = -0{\cdot}1$	(f) $x_0 = 1$	(i) $x_0 = 3$	(l) $x_0 = 5$

13. (a) On the same diagram, sketch the graphs of $y = e^{-\frac{1}{2}x}$ and $y = 5 - x^2$, showing all intercepts with the x and y axes.
(b) On your diagram, indicate the negative root α of the equation $x^2 + e^{-\frac{1}{2}x} = 5$.
(c) Show that $-2 < \alpha < -1$.
(d) Use one iteration of Newton's method, with starting value $x_1 = -2$, to show that α is approximately $\dfrac{-18}{e+8}$.

14. (a) Suppose that we apply Newton's method with starting value $x_0 = 0$ repeatedly to the function $y = e^{-kx}$, where k is a positive constant.
(i) Show that $x_{n+1} = x_n + \dfrac{1}{k}$.
(ii) Describe the resulting sequence $x_1, x_2, x_3, \ldots$.
(b) Repeat part (a) with the function $y = x^{-k}$ (where once again $k > 0$) and starting value $x_0 = 1$.
(c) What can we deduce from parts (a) and (b) about the rates at which e^{-kx} and x^{-k} approach zero as $x \to \infty$? Draw a diagram to illustrate this.

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EXTENSION

15. Suppose that $a \ge 2$ is an integer which is not a perfect square. Our aim is to approximate $\sqrt{a}$ by applying Newton's method to the equation $x^2 - a = 0$. Let x_0, x_1, x_2, ... be the approximations obtained by successive applications of Newton's method, where the initial value x_0 is the smallest integer greater than $\sqrt{a}$.

(a) Show that $x_{n+1} = \dfrac{{x_n}^2 + a}{2x_n}$, for $n \ge 0$.

(b) Prove by induction that for all integers $n \ge 0$,
$$x_n - \sqrt{a} \le 2\sqrt{a}\left(\frac{x_0 - \sqrt{a}}{2\sqrt{a}}\right)^{2^n}.$$
(Note that the index on the RHS is 2^n, not $2n$.)

(c) Show that when Newton's method is applied to finding $\sqrt{3}$, using the initial value $x_0 = 2$, the twentieth approximation x_{20} is correct to at least one million decimal places.

16. Let $f(x) = 1 + x^2$ and let x_1 be a real number. For $n = 1, 2, 3, \ldots$, define
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$
(You may assume that $f'(x_n) \ne 0$.)

(a) Show that $|x_{n+1} - x_n| \ge 1$, for $n = 1, 2, 3, \ldots$.

(b) Graph the function $y = \cot\theta$ for $0 < \theta < \pi$.

(c) Use the graph to show that there exists a real number θ_n such that $x_n = \cot\theta_n$ and $0 < \theta_n < \pi$.

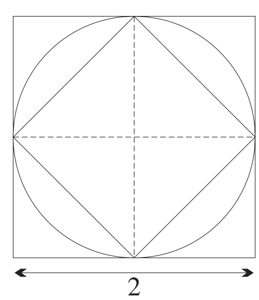
(d) By using the formula for $\tan 2A$, deduce that $\cot\theta_{n+1} = \cot 2\theta_n$, for $n = 1, 2, 3, \ldots$.

(e) Find all points x_1 such that $x_1 = x_{n+1}$, for some value of n.

6 F Inequalities and Limits Revisited

Arguments about inequalities and limits have occurred continually throughout our work. This demanding section is intended to revisit the subject and focus attention on some of the types of arguments being used. As mentioned in the Study Notes, it is intended for 4 Unit students — familiarity with arguments about inequalities and limits is required in that course — and for the more ambitious 3 Unit students, who may want to leave it until final revision.

A Geometrical Argument Proving an Inequality about π: The following worked exercise does nothing more than prove that π is between 2 and 4 — hardly a brilliant result — but it is a good illustration of the use of geometrical arguments.



WORKED EXERCISE: The outer square in the diagram to the right has side length 2. Find the areas of the circle and both squares, and hence prove that $2 < \pi < 4$.

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SOLUTION: The circle has radius 1,
so area of circle $= \pi \times 1^2$
$= \pi$.
The outer square has side length 2,
so area of outer square $= 2^2$
$= 4$.
The inner square has diagonals of length 2,
so area of inner square $= \frac{1}{2} \times 2 \times 2$
$= 2$.
But area of inner square $<$ area of circle $<$ area of outer square.
Hence $2 < \pi < 4$.

Arguments using Concavity and the Definite Integral: The following worked exercise applies two very commonly used principles to produce inequalities.

8 USING CONCAVITY AND THE DEFINITE INTEGRAL TO PRODUCE INEQUALITIES:
- If a curve is concave up in an interval, then the chord joining the endpoints of the curve lies above the curve.
- If $f(x) < g(x)$ in an interval $a < x < b$, then $\displaystyle\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$.

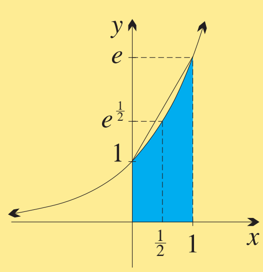
WORKED EXERCISE:
(a) Using the second derivative, prove that the chord joining the points $A(0,1)$ and $B(1,e)$ on the curve $y = e^x$ lies above the curve in the interval $0 < x < 1$.
(b) Find the equation of the chord, and hence prove that $\sqrt{e} < \frac{1}{2}(e+1)$.
(c) By integrating over the interval $0 \le x \le 1$, prove that $e < 3$.

SOLUTION:
(a) Since $y = e^x$, $y' = e^x$ and $y'' = e^x$.
Since y'' is positive for all x, the curve is concave up everywhere.
In particular, the chord joining A and B lies above the curve.

(b) The chord has gradient $= e - 1$ (the rise is $e - 1$, the run is 1),
so the chord is $y = (e-1)x + 1$ (using $y = mx + b$).
When $x = \frac{1}{2}$, the line is above the curve $y = e^x$,
so substituting $x = \frac{1}{2}$, $e^{\frac{1}{2}} < \frac{1}{2}(e-1) + 1$ (the chord is above the curve)
$\sqrt{e} < \frac{1}{2}(e+1)$, as required.

(c) Since $y = (e-1)x + 1$ is above $y = e^x$ in the interval $0 < x < 1$,

$$\int_0^1 e^x\,dx < \int_0^1 \big((e-1)x + 1\big)\,dx$$
$$\Big[e^x\Big]_0^1 < \Big[\tfrac{1}{2}(e-1)x^2 + x\Big]_0^1$$
$$e - 1 < \tfrac{1}{2}(e-1) + 1$$
$$2e - 2 < e - 1 + 2$$
$$e < 3, \text{ as required.}$$



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Extension — Algebraic Arguments about Inequalities: The result $\sqrt{e} < \frac{1}{2}(e+1)$ proven above is unremarkable, because it is true for any positive number x except 1. This is proven in the following worked exercise. The algebraic argument used there is normal in the 4 Unit course, but would seldom be required in the 3 Unit course.

WORKED EXERCISE: Show that $\sqrt{x} < \frac{1}{2}(x+1)$, for all $x \geq 0$ except $x = 1$.

SOLUTION: Suppose by way of contradiction that $\sqrt{x} \geq \frac{1}{2}(x+1)$.

Then $2\sqrt{x} \geq x + 1$.

Squaring, $4x \geq x^2 + 2x + 1$

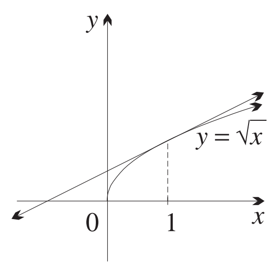
$0 \geq x^2 - 2x + 1$

$0 \geq (x-1)^2$.

This is impossible except when $x = 1$, because a square can never be negative.

NOTE: Question 1 in the following exercise proves this result using arguments involving tangents and concavity.

Exercise 6F

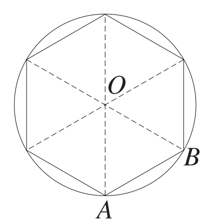


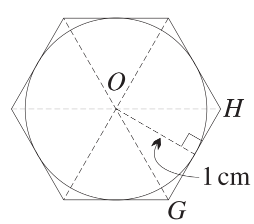
1. The diagram shows the curve $y = \sqrt{x}$ and the tangent at $x = 1$.

(a) Show that the tangent has equation $y = \frac{1}{2}(x+1)$.

(b) Find y'', and hence explain why the curve is concave down for $x > 0$.

(c) Hence prove graphically that $\sqrt{x} < \frac{1}{2}(x+1)$, for all $x \geq 0$ except $x = 1$. NOTE: This inequality was proven algebraically in the last worked exercise above.





2. (a) A regular hexagon is drawn inside a circle of radius 1 cm and centre O so that its vertices lie on the circumference, as shown in the first diagram.

(i) Show that $\triangle OAB$ is equilateral and hence find its area.

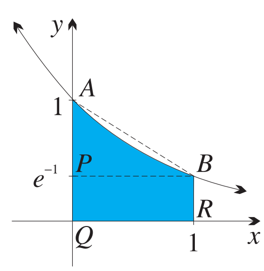
(ii) Hence find the exact area of this hexagon.

(b) Another regular hexagon is drawn outside the circle, as shown in the second diagram.

(i) Find the area of $\triangle OGH$.

(ii) Hence find the exact area of this outer hexagon.

(c) By considering the results in parts (a) and (b), show that $\dfrac{3\sqrt{3}}{2} < \pi < 2\sqrt{3}$.



3. The diagram shows the points $A(0, 1)$ and $B(1, e^{-1})$ on the curve $y = e^{-x}$.

(a) Show that the exact area of the region bounded by the curve, the x-axis and the vertical lines $x = 0$ and $x = 1$ is $(1 - e^{-1})$ square units.

(b) Find the area of:

(i) rectangle $PBRQ$, (ii) trapezium $ABRQ$.

(c) Use the areas found in the previous parts to show that $2 < e < 3$.

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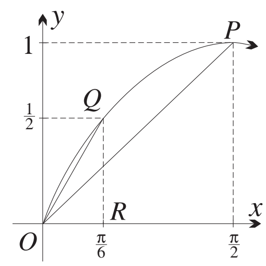
4. The diagram shows the curve $y = \sin x$ for $0 \le x \le \frac{\pi}{2}$. The points $P(\frac{\pi}{2}, 1)$ and $Q(\frac{\pi}{6}, \frac{1}{2})$ lie on the curve.



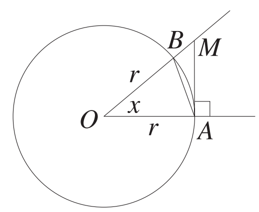
(a) Find the equation of the tangent at O.

(b) Find the equation of the chord OP, and hence show that $\frac{2x}{\pi} < \sin x < x$, for $0 < x < \frac{\pi}{2}$.

(c) Find the equation of the chord OQ, and hence show that $\frac{3x}{\pi} < \sin x < x$, for $0 < x < \frac{\pi}{6}$.

(d) By integrating $\sin x$ from 0 to $\frac{\pi}{6}$ and comparing this to the area of $\triangle ORQ$, show that $\pi < 12(2 - \sqrt{3}) \doteqdot 3{\cdot}2$.

5. The diagram shows a circle with centre O and radius r, and a sector OAB subtending an angle of x radians at O. The tangent at A meets the radius OB produced at M.



(a) Find, in terms of r and x, the areas of:

(i) $\triangle OAB$, (ii) sector OAB, (iii) $\triangle OAM$.

(b) Hence show that $\sin x < x < \tan x$, for $0 < x < \frac{\pi}{2}$.

6. (a) Prove, using mathematical induction, that for all positive integers n,

$$1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n+4) = \tfrac{1}{6}n(n+1)(2n+13).$$

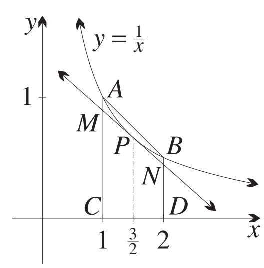
(b) Hence find $\displaystyle\lim_{n\to\infty} \frac{1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n+4)}{n^3}$.

7. Suppose that $f(x) = \ln(1+x) - \ln(1-x)$.

(a) Find the domain of $f(x)$.

(b) Find $f'(x)$, and hence explain why $f(x)$ is an increasing function.

8. The points A, P and B on the curve $y = \dfrac{1}{x}$ have x-coordinates 1, $1\frac{1}{2}$ and 2 respectively. The points C and D are the feet of the perpendiculars drawn from A and B to the x-axis. The tangent to the curve at P cuts AC and BD at M and N respectively.



(a) Show that the tangent at P has equation $4x + 9y = 12$.

(b) Find the coordinates of M and N.

(c) Find the areas of the trapezia $ABDC$ and $MNDC$.

(d) Hence show that $\frac{2}{3} < \ln 2 < \frac{3}{4}$.

DEVELOPMENT

9. Let $f(x) = \log_e x$.

(a) Show that $f'(1) = 1$.

(b) Use the definition of the derivative, that is, $f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, to show that

$$f'(1) = \lim_{h\to 0} \log_e (1+h)^{\frac{1}{h}}.$$

(c) Combine parts (a) and (b) and replace h with $\frac{1}{n}$ to show that $\displaystyle\lim_{n\to\infty} \log_e (1 + \tfrac{1}{n})^n = 1$.

(d) Hence show that $e = \displaystyle\lim_{n\to\infty} (1 + \tfrac{1}{n})^n$.

(e) To how many decimal places is the RHS of the equation in part (d) accurate when $n = 10,\ 10^2,\ 10^3,\ 10^4,\ 10^5,\ 10^6$?

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10. (a) Show, using calculus, that the graph of $y = \ln x$ is concave down throughout its domain.
(b) Sketch the graph of $y = \ln x$, and mark two points $A(a, \ln a)$ and $B(b, \ln b)$ on the curve, where $0 < a < b$.
(c) Find the coordinates of the point P that divides the interval AB in the ratio $2 : 1$.
(d) Using parts (b) and (c), deduce that $\frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln(\frac{1}{3}a + \frac{2}{3}b)$.

11. (a) Solve the equation $\sin 2x = 2\sin^2 x$, for $0 < x < \pi$.
(b) Show that if $0 < x < \frac{\pi}{4}$, then $\sin 2x > 2\sin^2 x$.

12. Evaluate $\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right)$. [HINT: Multiply by $\dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$.]

13. (a) Suppose that $f(x) = \sqrt{1+x}$. Find $f'(8)$.
(b) Sketch the curve $f(x) = \sqrt{1+x}$ and the tangent at $x = 8$. Hence show that $f'(x) < \frac{1}{6}$ for $x > 8$.
(c) Deduce that $\sqrt{1+x} \le 3 + \frac{1}{6}(x-8)$ when $x \ge 8$.

14. Let $f(x) = x^n\, e^{-x}$, where $n > 1$.
(a) Show that $f'(x) = x^{n-1}e^{-x}(n-x)$.
(b) Show that the graph of $f(x)$ has a maximum turning point at $(n, n^n e^{-n})$, and hence sketch the graph for $x \ge 0$. (Don't attempt to find points of inflexion.)
(c) Explain, by considering the graph of $f(x)$ for $x > n$, why $x^n e^{-x} < n^n e^{-n}$ for $x > n$.
(d) Deduce from part (c) that $(1 + \frac{1}{n})^n < e$. [HINT: Let $x = n + 1$.]

15. (a) Show that $\dfrac{1}{n-1} - \dfrac{1}{n+1} = \dfrac{2}{n^2-1}$.
(b) Hence find, as a fraction in lowest terms, the sum of the first 80 terms of the series $\frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \frac{2}{24} + \cdots$.
(c) Obtain an expression for $\displaystyle\sum_{r=2}^{n} \frac{1}{r^2-1}$, and hence find the limiting sum of the series.

16. A sequence is defined recursively by

$$t_1 = \tfrac{1}{3} \qquad \text{and} \qquad t_{n+1} = t_n + {t_n}^2, \ \text{ for } n \ge 1.$$

(a) Show that $\dfrac{1}{t_n} - \dfrac{1}{t_{n+1}} = \dfrac{1}{1+t_n}$.
(b) Hence find the limiting sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+t_n}$.

17. The function $f(x)$ is defined by $f(x) = x - \log_e(1+x^2)$.
(a) Show that $f'(x)$ is never negative.
(b) Explain why the graph of $y = f(x)$ lies completely above the x-axis for $x > 0$.
(c) Hence prove that $e^x > 1 + x^2$, for all positive values of x.

18. (a) Prove by induction that $2^n > n$, for all positive integers n.
(b) Hence show that $1 < \sqrt[n]{n} < 2$, if n is a positive integer greater than 1.
(c) Suppose that a and n are positive integers. It is known that if $\sqrt[n]{a}$ is a rational number, then it is an integer. What can we deduce about $\sqrt[n]{n}$, where n is a positive integer greater than 1?

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19. Consider the function $y = e^x \left(1 - \dfrac{x}{10}\right)^{10}$.

(a) Show that $y' = -\frac{1}{10}\, x\, e^x \left(1 - \dfrac{x}{10}\right)^9$.

(b) Find the two turning points of the graph of the function.

(c) Discuss the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.

(d) Sketch the graph of the function.

(e) From your graph, deduce that $e^x \le \left(1 - \dfrac{x}{10}\right)^{-10}$, for $x < 10$.

(f) Hence show that $\left(\dfrac{11}{10}\right)^{10} \le e \le \left(\dfrac{10}{9}\right)^{10}$.

20. (a) (i) Prove by induction that $(1+c)^n > 1 + cn$, for all integers $n \ge 2$, where c is a nonzero constant greater than -1.

(ii) Hence show that $(1 - \frac{1}{2n})^n > \frac{1}{2}$, for all integers $n \ge 2$.

(b) (i) Solve the inequation $x^2 > 2x + 1$.

(ii) Hence prove by induction that $2^n > n^2$, for all integers $n \ge 5$.

(c) Suppose that $a > 0$, $b > 0$, and n is a positive integer.

(i) Divide the expression $a^{n+1} - a^n b + b^{n+1} - b^n a$ by $a - b$, and hence show that $a^{n+1} + b^{n+1} \ge a^n b + b^n a$.

(ii) Hence prove by induction that $\left(\dfrac{a+b}{2}\right)^n \le \dfrac{a^n + b^n}{2}$.

21. Let $A(1,1)$ and $B(k, \frac{1}{k})$, where $k > 1$, be points on the hyperbola $y = \dfrac{1}{x}$.

(a) Show that the tangents to the hyperbola at A and B intersect at $T\left(\dfrac{2k}{k+1}, \dfrac{2}{k+1}\right)$.

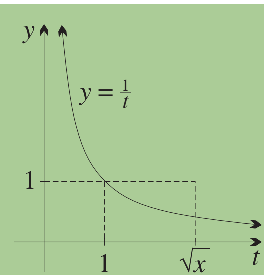
(b) Suppose that A', B' and T' are the feet of the perpendiculars drawn from A, B and T to the x-axis.

(i) Show that the sum of the areas of the two trapezia $AA'T'T$ and $TT'B'B$ is $\dfrac{2(k-1)}{k+1}$ square units.

(ii) Hence prove that $\dfrac{2u}{u+2} < \log(u+1) < u$, for all $u > 0$.

EXTENSION

22. The diagram shows the curve $y = \dfrac{1}{t}$, for $t > 0$.



(a) If $x > 1$, show that $\displaystyle\int_1^{\sqrt{x}} \frac{1}{t}\, dt = \tfrac{1}{2} \log x$.

(b) Explain why $0 < \frac{1}{2} \log x < \sqrt{x}$, for all $x > 1$.

(c) Hence show that $\displaystyle\lim_{x \to \infty} \left(\frac{\log x}{x}\right) = 0$.

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23. (a) Given that $\sin x > \dfrac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, show that:

(i) $e^{-\sin x} < e^{-\frac{2x}{\pi}}$ for $0 < x < \frac{\pi}{2}$,

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\,dx$.

(b) Use the substitution $u = \pi - x$ to show that $\displaystyle\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx = \int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx$.

(c) Hence show that $\displaystyle\int_0^{\pi} e^{-\sin x}\,dx < \frac{\pi}{e}(e-1)$.

24. (a) Show that $\dfrac{d}{dx}(x\ln x - x) = \ln x$.

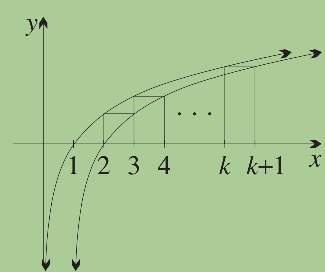
(b) Hence show that $\displaystyle\int_1^n \ln x\,dx = n\ln n - n + 1$.

(c) Use the trapezoidal rule on the intervals with endpoints 1, 2, 3, …, n to show that

$$\int_1^n \ln x\,dx \doteqdot \tfrac{1}{2}\ln n + \ln(n-1)!$$

(d) Hence show that $n! < e\,n^{n+\frac{1}{2}}\,e^{-n}$. NOTE: This is a preparatory lemma in the proof of Stirling's formula $n! \doteqdot \sqrt{2\pi}\,n^{n+\frac{1}{2}}\,e^{-n}$, which gives an approximation for $n!$ whose percentage error converges to 0 for large integers n.

25. The diagram shows the curves $y = \log x$ and $y = \log(x-1)$, and $k-1$ rectangles constructed between $x = 2$ and $x = k+1$, where $k \geq 2$.



(a) Using the result in part (a) of the previous question, show that:

(i) $\displaystyle\int_2^{k+1} \log x\,dx = (k+1)\log(k+1) - \log 4 - k + 1$

(ii) $\displaystyle\int_2^{k+1} \log(x-1)\,dx = k\log k - k + 1$

(b) Deduce that $k^k < k!\,e^{k-1} < \frac{1}{4}(k+1)^{k+1}$, for all $k \geq 2$.

26. (a) Show graphically that $\log_e x \leq x - 1$, for $x > 0$.

(b) Suppose that $p_1, p_2, p_3, \ldots, p_n$ are positive real numbers whose sum is 1. Show that

$$\sum_{r=1}^{n} \log_e(np_r) \leq 0.$$

(c) Let $x_1, x_2, x_3, \ldots, x_n$ be positive real numbers. Prove that

$$(x_1x_2x_3\cdots x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

When does equality apply in this relationship?
[HINT: Let $s = x_1 + x_2 + x_3 + \cdots + x_n$, and then use part (b) with $p_1 = \frac{x_1}{s}$, ….]

27. [The binomial theorem and differentiation by the product rule] Suppose that $y = uv$ is the product of two functions u and v of x.

(a) Show that $y'' = u''v + 2u'v' + uv''$, and develop formulae for y''', y'''' and y'''''.

(b) Find the fifth derivative of $y = (x^2 + x + 1)e^{-x}$.

(c) Use sigma notation to write down a formula for the nth derivative $y^{(n)}$.

GO Online Multiple Choice Quiz

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