CHAPTER **SIX**

Sequences and Series

Many situations give rise to a sequence of numbers with a simple pattern. For example, the weight of a tray carrying a stack of plates increases steadily as each new plate is added. When cells continually divide into two, then the numbers in successive generations descending from a single cell form the sequence 1, 2, 4, 8, … of powers of 2. Someone thinking about the half-life of a radioactive substance will need to ask what happens when we add up more and more terms of the series

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \cdots.$$

The highly structured world of mathematics is full of sequences, and, in particular, knowledge of them will be needed to establish important results in calculus in the next chapter.

STUDY NOTES: Sections 6A and 6B are a review of the algebraic work of indices and logarithms at a more demanding level, and could be studied independently. In Sections 6C–6L on sequences and series, computers and graphics calculators could perhaps help to represent examples in alternative graphical forms, to emphasise the linear and exponential functions that lie behind arithmetic and geometric sequences, and to give some interactive experience of the limits of sequences. Section 6M generalises the difference-of-squares and difference-of-cubes identities in preparation for their use in calculus. The final Section 6N on mathematical induction could also be studied at some other time — it is included in this chapter because it involves recursion like APs and GPs. Applications of series, particularly to financial situations, will be covered in the Year 12 text.

6 A Indices

We begin with a review of the index laws, which will be needed in calculations later in the chapter. The accompanying exercise is intended to cover a wide variety of arithmetic and algebraic manipulations.

Definition of Indices: An expression of the form a^x is called a *power*. The number a is called the *base* of the power, and the number x is called the *index* (plural *indices*) or *exponent*. The power a^x is defined in different ways for various types of indices.

First, we define $a^0 = 1$. Then for integers $n \geq 1$, we define $a^n = a \times a^{n-1}$, so that $a^1 = a \times a^0$, $a^2 = a \times a^1$, $a^3 = a \times a^2$, …. This is called a *recursive* definition.

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1
$$a^0 = 1,$$
$$a^n = a \times a^{n-1}, \quad \text{for integers } n \geq 1.$$

For positive rational indices m/n, where m and n are positive integers, we define:

2
$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m, \quad \text{where if } n \text{ is even, } \sqrt[n]{a} \text{ means the positive } n\text{th root.}$$

For negative rational indices $-q$, we define:

3
$$a^{-q} = \frac{1}{a^q}.$$

NOTE: Defining powers like 2^π and $3^{\sqrt{2}}$ with irrational indices is quite beyond this course. We will make the quite reasonable assumption that powers with irrational indices can be formed, and that they work as expected. One approach to a power like 2^π is to consider powers with rational indices increasingly close to π, such as 2^3, $2^{3\frac{1}{7}}$, $2^{3\frac{16}{113}}$ and so on. This point will be taken up later in the chapters on the logarithmic and exponential functions.

WORKED EXERCISE: Simplify the following powers:
(a) 7^{-1} (b) $(\frac{3}{5})^{-1}$ (c) $(1\frac{1}{2})^{-2}$ (d) $121^{\frac{1}{2}}$ (e) $49^{-\frac{1}{2}}$

SOLUTION:

(a) $7^{-1} = \frac{1}{7}$

(b) $(\frac{3}{5})^{-1} = \frac{5}{3}$

(c) $(1\frac{1}{2})^{-2} = (\frac{3}{2})^{-2} = (\frac{2}{3})^2 = \frac{4}{9}$

(d) $(121)^{\frac{1}{2}} = 11$

(e) $(49)^{-\frac{1}{2}} = (\frac{1}{49})^{\frac{1}{2}} = \frac{1}{7}$

WORKED EXERCISE: Simplify: (a) $(\frac{4}{9})^{\frac{1}{2}}$ (b) $9^{\frac{3}{2}}$ (c) $125^{-\frac{2}{3}}$ (d) $(2\frac{1}{4})^{-\frac{5}{2}}$

SOLUTION:

(a) $(\frac{4}{9})^{\frac{1}{2}} = \frac{2}{3}$

(b) $9^{\frac{3}{2}} = 3^3 = 27$

(c) $125^{-\frac{2}{3}} = (\frac{1}{125})^{\frac{2}{3}} = (\frac{1}{5})^2 = \frac{1}{25}$

(d) $(2\frac{1}{4})^{-\frac{5}{2}} = (\frac{4}{9})^{\frac{5}{2}} = (\frac{2}{3})^5 = \frac{32}{243}$

Laws for Indices: These rules should be well known from earlier years. Group A restates some of the definitions, Group B involves *compound indices*, and Group C involves *compound bases*.

4
INDEX LAWS:

A. $a^{\frac{1}{2}} = \sqrt{a}$, $\quad a^{-1} = \dfrac{1}{a}$, $\quad a^{-\frac{1}{2}} = \dfrac{1}{\sqrt{a}}$

B. $a^{x+y} = a^x\, a^y$, $\quad a^{x-y} = \dfrac{a^x}{a^y}$, $\quad (a^x)^n = a^{xn}$

C. $(ab)^x = a^x\, b^x$, $\quad \left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

The Use of Primes: Bases that are composite numbers are often best factored into primes when calculations are required.

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WORKED EXERCISE: Simplify: (a) $12^n \times 18^{-2n}$ (b) $9^{-\frac{4}{3}} \times 81^{\frac{2}{3}}$

SOLUTION:

(a) $12^n \times 18^{-2n}$
$= (2^2 \times 3)^n \times (2 \times 3^2)^{-2n}$
$= 2^{2n} \times 3^n \times 2^{-2n} \times 3^{-4n}$
$= 3^{-3n}$

(b) $9^{-\frac{4}{3}} \times 81^{\frac{2}{3}} = (3^2)^{-\frac{4}{3}} \times (3^4)^{\frac{2}{3}}$
$= 3^{-\frac{8}{3}} \times 3^{\frac{8}{3}}$
$= 3^0$
$= 1$

WORKED EXERCISE: Solve: (a) $16^x = \sqrt{8}$ (b) $27^x = 9^{1-x}$

SOLUTION:

(a) $16^x = \sqrt{8}$
$(2^4)^x = (2^3)^{\frac{1}{2}}$
$2^{4x} = 2^{\frac{3}{2}}$
$4x = \frac{3}{2}$
$x = \frac{3}{8}$

(b) $27^x = 9^{1-x}$
$3^{3x} = 3^{2-2x}$
$3x = 2 - 2x$
$x = \frac{2}{5}$

Negative and Zero Bases: Negative numbers do not have square roots, and so negative bases are impossible if a square root or fourth root or any even root is involved. For example, $(-64)^{\frac{1}{2}}$ is undefined, but $(-64)^{\frac{1}{3}} = -4$.

Powers of zero are only defined for positive indices x, in which case $0^x = 0$ — for example, $0^3 = 0$. If the index x is negative, then 0^x is undefined, being the reciprocal of zero — for example, 0^{-3} is undefined. If the index is zero, then 0^0 is also undefined.

Exercise 6A

NOTE: Do not use the calculator in this exercise at all.

1. Simplify these numerical expressions:

(a) 3^0 (b) 5^{-1} (c) $(\frac{2}{7})^{-1}$ (d) $(5\frac{1}{4})^{-1}$ (e) 7^{-2} (f) $(\frac{1}{4})^{-3}$ (g) $(\frac{2}{3})^{-4}$ (h) $(2\frac{1}{2})^{-2}$ (i) $(3\frac{1}{3})^{-3}$ (j) $(45\frac{3}{7})^0$

2. Simplify:

(a) $25^{\frac{1}{2}}$ (b) $27^{\frac{1}{3}}$ (c) $27^{\frac{2}{3}}$ (d) $8^{\frac{2}{3}}$ (e) $16^{\frac{3}{4}}$ (f) $9^{\frac{3}{2}}$ (g) $27^{\frac{4}{3}}$ (h) $(\frac{25}{4})^{\frac{3}{2}}$ (i) $(\frac{4}{9})^{1\frac{1}{2}}$ (j) $(5\frac{1}{16})^{0\cdot 25}$

3. Simplify:

(a) $\dfrac{1}{13^{-2}}$ (b) $\dfrac{2^{-3}}{3^{-2}}$ (c) $\dfrac{4^{-2}}{2^{-3}}$ (d) $\dfrac{5^3}{2^{-5}}$ (e) $\dfrac{2^{-2}}{15^2}$

4. Write as single fractions without negative indices:

(a) $7x^{-1}$ (b) $3x^{-3}$ (c) $x^2(y+1)^{-2}$ (d) $\dfrac{s^{-2}}{(t+3)^{-1}}$ (e) $\dfrac{2^{-3}x^2}{3^{-2}y^{-2}}$

5. Simplify, giving the answers without negative indices:

(a) $x^{-5}y^3 \times x^3y^{-2}$ (b) $3x^{-2} \times 7x$ (c) $(s^2y^{-3})^3$ (d) $(5c^{-2}d^3)^{-1}$ (e) $7m^{\frac{4}{3}} \times 3m^{\frac{2}{3}}$ (f) $(a^{-2}b^4)^{\frac{1}{2}}$ (g) $(8x^3y^{-6})^{\frac{1}{3}}$ (h) $(p^{\frac{1}{5}}q^{-\frac{3}{5}})^{10}$

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6. Write down the solutions of these index equations:

(a) $17^x = \frac{1}{17}$ (b) $9^x = \frac{1}{81}$ (c) $64^x = 4$ (d) $4^x = 8$ (e) $(\frac{1}{25})^x = \frac{1}{5}$ (f) $49^x = \frac{1}{7}$

DEVELOPMENT

7. Simplify:

(a) $16^{-\frac{1}{4}}$ (b) $32^{-1\cdot4}$ (c) $(\frac{8}{125})^{-\frac{2}{3}}$ (d) $(\frac{16}{49})^{-\frac{3}{2}}$ (e) $(3\frac{3}{8})^{-\frac{4}{3}}$ (f) $(0\cdot4)^{-3}$ (g) $(1\cdot2)^{-2}$ (h) $(0\cdot25)^{1\cdot5}$ (i) $(2\cdot25)^{-0\cdot5}$ (j) $(0\cdot36)^{-1\cdot5}$

8. Write these expressions using fractional and negative indices:

(a) $\sqrt[3]{7}$ (b) $\sqrt[4]{5^3}$ (c) $\dfrac{\sqrt[3]{x^4}}{5}$ (d) $\dfrac{\sqrt{a}}{\sqrt[5]{11}}$ (e) $\dfrac{7}{\sqrt[5]{y^4}}$ (f) $x\sqrt{x}$ (g) $x^2\sqrt{x}$ (h) $\dfrac{2\sqrt{x}}{x}$ (i) $\dfrac{1}{8x}$ (j) $\dfrac{7}{3x+2}$

9. Given that $x = 16$ and $y = 25$, evaluate:

(a) $x^{\frac{1}{2}} + y^{\frac{1}{2}}$ (b) $x^{\frac{3}{4}} - y^{\frac{1}{2}}$ (c) $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$ (d) $(y-x)^{\frac{1}{2}} \times (4y)^{-\frac{1}{2}}$

10. Expand and simplify, answering without using negative indices:

(a) $(x + 5x^{-1})^2$ (b) $(x^2 - 7x^{-2})^2$ (c) $(3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}})^2$

11. Explain why $8^n = (2^3)^n = 2^{3n}$. Using similar methods, write these expressions with prime bases and simplify:

(a) $2^n \times 8^n$ (b) $\dfrac{25^n}{5^{n+1}}$ (c) $3 \times 9^x \times 81^x$ (d) $\dfrac{121^{-n} \times 11}{11^{3n}}$ (e) $\frac{1}{7} \times 49^n \times \sqrt{7}$ (f) $\dfrac{32^x \times \frac{1}{2}}{4^x \times 16}$

12. Explain why if $3^{3x-1} = 9$, then $3x - 1 = 2$, and so $x = 1$. Similarly, by reducing both sides to powers of the same base, solve:

(a) $125^x = \frac{1}{5}$ (b) $25^x = \sqrt{5}$ (c) $8^x = \frac{1}{4}$ (d) $64^x = \sqrt{32}$ (e) $8^{x+1} = 2 \times 4^{x-1}$ (f) $(\frac{1}{9})^x = 3^4$

13. By taking appropriate powers of both sides, solve:

(a) $b^{\frac{1}{3}} = \frac{1}{7}$ (b) $n^{-2} = 121$ (c) $x^{-\frac{3}{4}} = 27$

14. Solve simultaneously:

(a) $7^{2x-y} = 49$, $2^{x+y} = 128$ (b) $8^x \div 4^y = 4$, $11^{y-x} = \frac{1}{11}$ (c) $13^{x+4y} = 1$, $25^{x+5y} = 5$

15. Write as a single fraction, without negative indices, and simplify:

(a) $a^{-1} - b^{-1}$ (b) $\dfrac{1-y^{-1}}{1-y^{-2}}$ (c) $(x^{-2} - y^{-2})^{-1}$ (d) $\dfrac{a^{-1}+b^{-1}}{a^{-2}-b^{-2}}$ (e) $x^{-2}y^{-2}(x^2y^{-1} - y^2x^{-1})$ (f) $\dfrac{(a^2-1)^{-1}}{(a-1)^{-1}}$

16. Explain why $12^n = (2^2 \times 3)^n = 2^{2n} \times 3^n$. Using similar methods, write these expressions with prime bases and simplify:

(a) $2^n \times 4^n \times 8^n$ (b) $\dfrac{9^{n+2} \times 3^{n+1}}{3 \times 27^n}$ (c) $6^x \times 4^x \div 3^x$ (d) $\dfrac{12^x \times 18^x}{3^x \times 2^x}$ (e) $100^{2n-1} \times 25^{-1} \times 8^{-1}$ (f) $\dfrac{24^{x+1} \times 8^{-1}}{6^{2x}}$

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17. Explain why $3^n + 3^{n+1} = 3^n(1+3) = 4 \times 3^n$. Then use similar methods to simplify:

(a) $7^{n+2} + 7^n$
(b) $\dfrac{3^{n+3} - 3^n}{3^n}$
(c) $5^n - 5^{n-3}$
(d) $\dfrac{7^n + 7^{n+2}}{7^{n-1} + 7^{n+1}}$
(e) $2^{2n+2} - 2^{2n-1}$
(f) $\dfrac{2^{2n} - 2^{n-1}}{2^n - 2^{-1}}$

18. Use the methods of the last two questions to simplify:

(a) $\dfrac{6^n + 3^n}{2^{n+1} + 2}$
(b) $\dfrac{12^x + 1}{6^{2x} + 3^x}$
(c) $\dfrac{12^n - 18^n}{3^n - 2^n}$

19. By taking 6th powers of both sides, show that $11^{\frac{1}{3}} < 5^{\frac{1}{2}}$. Using similar methods (followed perhaps by a check on the calculator), compare:

(a) $3^{\frac{1}{3}}$ and $2^{\frac{1}{2}}$
(b) $2^{\frac{1}{2}}$ and $5^{\frac{1}{5}}$
(c) $7^{\frac{3}{2}}$ and 20
(d) $5^{\frac{1}{5}}$ and $3^{\frac{1}{3}}$
(e) $2^{\frac{1}{2}} \times 5^{\frac{1}{3}}$ and $2 \times 3^{\frac{1}{6}}$
(f) $5^{\frac{1}{3}}$ and $2^{\frac{1}{4}} \times 3^{\frac{1}{12}} \times 5^{\frac{1}{6}}$

20. If $a = 2^{\frac{1}{2}} + 2^{-\frac{1}{2}}$ and $b = 2^{\frac{1}{2}} - 2^{-\frac{1}{2}}$, find:

(a) ab
(b) a^2
(c) $a^2 + b^2$
(d) $a^2 - b^2$
(e) $(a+b)a$
(f) $a^3 + b^3$

21. (a) If $x = 2^{\frac{1}{3}} + 4^{\frac{1}{3}}$, show that $x^3 = 6(1+x)$.

(b) If $x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, show that $\dfrac{x^2 + x^{-2}}{x - x^{-1}} = 3$.

(c) Show that $\dfrac{pq^{-1} - p^{-1}q}{p^2q^{-2} - p^{-2}q^2} = \dfrac{pq}{p^2 + q^2}$.

EXTENSION

22. Find the smallest positive integers m and n for which:

(a) $12 < 2^{m/n} < 13$
(b) $13 < 2^{m/n} < 14$

23. Find $\lim\limits_{x \to 0^+} 0^x$ and $\lim\limits_{x \to 0} x^0$. Explain what these two limits have to do with the remark made in the notes that 0^0 is undefined.

6 B Logarithms

The most important thing to learn about logarithms is that the logarithmic function $y = \log_a x$ is simply the inverse function of the exponential function $y = a^x$. The ability to convert between statements about indices and statements about logarithms is a fundamental skill that must be developed.

Definition of Logarithms: Suppose that a and x are positive numbers, with $a \neq 1$. Then $\log_a x$ is the index, when x is written as a power of a.

5 DEFINITION: $\log_a x$ is the index, when x is written as a power of a.
In symbols, $y = \log_a x$ means $x = a^y$.

We read $\log_a x$ as 'the logarithm of x base a', or 'log x base a'.

WORKED EXERCISE:

$\log_2 8 = 3$, because $2^3 = 8$.
$\log_5 25 = 2$, because $5^2 = 25$.
$\log_7 7 = 1$, because $7^1 = 7$.
$\log_3 1 = 0$, because $3^0 = 1$.
$\log_{10} \frac{1}{10} = -1$, because $10^{-1} = \frac{1}{10}$.
$\log_6 \sqrt{6} = \frac{1}{2}$, because $6^{\frac{1}{2}} = \sqrt{6}$.

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WORKED EXERCISE: Rewrite these equations in index form, then solve:

(a) $x = \log_3 \sqrt{3}$ (b) $\log_2 x = 5$ (c) $\log_x 10\,000 = 4$

SOLUTION: When rewriting the equation in index form, remember that 'the base of the log is the base of the power', and 'the log is the index'.

(a) $x = \log_3 \sqrt{3}$
$3^x = \sqrt{3}$
$x = \frac{1}{2}$

(b) $\log_2 x = 5$
$x = 2^5$
$x = 32$

(c) $\log_x 10\,000 = 4$
$x^4 = 10\,000$
$x = 10$

Laws for Logarithms: The laws in Group A are four important special cases. Those in Group B are the Group B index laws written in logarithmic form.

6

A.
$\log_a 1 = 0$ (because $1 = a^0$)
$\log_a a = 1$ (because $a = a^1$)
$\log_a \frac{1}{a} = -1$ (because $\frac{1}{a} = a^{-1}$)
$\log_a \sqrt{a} = \frac{1}{2}$ (because $\sqrt{a} = a^{\frac{1}{2}}$)

B.
$\log_a xy = \log_a x + \log_a y$
$\log_a \dfrac{x}{y} = \log_a x - \log_a y$
$\log_a x^n = n \log_a x$

WORKED EXERCISE: Write each of the following in terms of $\log_2 3$:

(a) $\log_2 81$ (b) $\log_2 2\sqrt{3}$ (c) $\log_2 \frac{8}{9}$

SOLUTION: Each number must be written in terms of powers of 2 and 3.

(a) $\log_2 81$
$= \log_2 3^4$
$= 4 \log_2 3$

(b) $\log_2 2\sqrt{3}$
$= \log_2 2 + \log_2 3^{\frac{1}{2}}$
$= \log_2 2 + \frac{1}{2}\log_2 3$
$= 1 + \frac{1}{2}\log_2 3$,
since $\log_2 2 = 1$.

(c) $\log_2 \frac{8}{9}$
$= \log_2 2^3 - \log_2 3^2$
$= 3\log_2 2 - 2\log_2 3$
$= 3 - 2\log_2 3$,
since $\log_2 2 = 1$.

The Change of Base Law: Conversion of logarithms from one base to another is often needed. For example, the calculator only gives approximations to logarithms to the two bases 10 and e, so the ability to change the base is necessary to approximate logarithms to other bases.

7

CHANGE OF BASE: $\log_b x = \dfrac{\log_a x}{\log_a b}$

'Take the log of the number over the log of the base.'

PROOF: To prove this formula, let $y = \log_b x$.
Then by the definition of logs, $x = b^y$
and taking logs base a of both sides, $\log_a x = \log_a b^y$.
Now by the third law in Group B above, $\log_a x = y \log_a b$
and rearranging, $y = \dfrac{\log_a x}{\log_a b}$, as required.

WORKED EXERCISE: $\log_2 7 = \dfrac{\log_{10} 7}{\log_{10} 2}$
$\doteqdot 2{\cdot}807$ (using the calculator)

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WORKED EXERCISE: Solve $2^x = 7$, correct to four significant figures.

SOLUTION:

$$\begin{aligned} 2^x &= 7 \\ x &= \log_2 7 \\ &= \frac{\log_{10} 7}{\log_{10} 2} \\ &\doteqdot 2{\cdot}807 \end{aligned}$$

Alternatively, taking logs of both sides,

$$\begin{aligned} \log_{10} 2^x &= \log_{10} 7 \\ x \log_{10} 2 &= \log_{10} 7 \\ x &= \frac{\log_{10} 7}{\log_{10} 2} \\ &\doteqdot 2{\cdot}807. \end{aligned}$$

WORKED EXERCISE: How many positive integer powers of 7 are less than 10^9?

SOLUTION: Put $7^n < 1\,000\,000\,000$.

Then $\quad n < \log_7 1\,000\,000\,000$

so $\quad n < \dfrac{\log_{10} 1\,000\,000\,000}{\log_{10} 7}$

$$n < \frac{9}{\log_{10} 7} \doteqdot 10{\cdot}649\ldots$$

$$n = 1,\ 2,\ \ldots,\ 10,$$

so 10 powers of 7 are less than $1\,000\,000\,000$.

NOTE: The calculator key marked $\boxed{\log}$ gives logarithms base 10, and should be the only log key used for the moment. The key $\boxed{\ln}$ also gives logarithms, but to a different base $e \doteqdot 2{\cdot}7183$ — it will be needed in Chapter 12.

Exponential and Logarithmic Functions: The definition of logarithms means that for any base a, the *logarithmic function* $y = \log_a x$ and the *exponential function* $y = a^x$ are inverse functions, as discussed in Section 2H. This means that when the functions are applied in succession they cancel each other out.

8 POWERS AND LOGARITHMS: $y = \log_a x$ and $y = a^x$ are inverse functions.
Applying them in succession, $\log_a a^x = x$ and $a^{\log_a x} = x$.

For example, $\log_2 2^3 = \log_2 8 = 3$, and $2^{\log_2 8} = 2^3 = 8$.

NOTE: The calculator reflects this structure. On most calculators, the function $\boxed{10^x}$ is reached by pressing $\boxed{\text{inv}}$ or $\boxed{\text{shift}}$ followed by $\boxed{\log}$. In Chapter 13, we will need to reach the function $\boxed{e^x}$ by pressing $\boxed{\text{inv}}$ followed by $\boxed{\ln}$.

WORKED EXERCISE:

(a) Simplify $\log_3 3^{17}$ and $2^{\log_2 7}$. (b) Express 5 and x as powers of 2.

SOLUTION:

(a) $\log_3 3^{17} = 17$, since $y = \log_3 x$ and $y = 3^x$ are inverse functions.
$2^{\log_2 7} = 7$, since $y = \log_2 x$ and $y = 2^x$ are inverse functions.

(b) Similarly, $5 = 2^{\log_2 5}$, and $x = 2^{\log_2 x}$.

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Exercise 6B

1. Rewrite each equation in index form, then solve:

(a) $x = \log_3 9$ (b) $x = \log_2 16$ (c) $x = \log_5 125$ (d) $x = \log_{10} \frac{1}{10}$ (e) $x = \log_7 \frac{1}{49}$ (f) $x = \log_{\frac{1}{3}} \frac{1}{81}$ (g) $x = \log_5 \sqrt{5}$ (h) $x = \log_{11} \dfrac{1}{\sqrt{11}}$

2. Copy and complete the tables of values below, and verify that the functions $y = 2^x$ and $y = \log_2 x$ are inverse functions. Then sketch both curves on the one set of axes, and verify that they are reflections of each other in $y = x$.

x	-3	-2	-1	0	1	2	3
2^x							

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2 x$							

3. Rewrite each equation in index form, then solve:

(a) $\log_4 x = 3$ (b) $\log_{13} x = -1$ (c) $\log_9 x = \frac{1}{2}$ (d) $\log_{10} x = -2$ (e) $\log_{\frac{1}{16}} x = -\frac{1}{4}$ (f) $\log_7 x = -\frac{1}{2}$ (g) $\log_{36} x = 1{\cdot}5$ (h) $\log_8 x = -\frac{2}{3}$

4. Rewrite each equation in index form, then solve:

(a) $\log_x 27 = 3$ (b) $\log_x \frac{1}{7} = -1$ (c) $\log_x 1000 = 3$ (d) $\log_x 3 = \frac{1}{2}$ (e) $\log_x 25 = -2$ (f) $\log_x \frac{4}{9} = 2$ (g) $\log_x 16 = \frac{4}{3}$ (h) $\log_x 9 = -\frac{1}{2}$

5. Given that a is a positive real number not equal to 1, evaluate:

(a) $\log_a a$ (b) $\log_a \dfrac{1}{a}$ (c) $\log_a a^3$ (d) $\log_a \dfrac{1}{a^2}$ (e) $\log_a \sqrt{a}$ (f) $\log_a \dfrac{1}{\sqrt{a}}$ (g) $\log_a 1$ (h) $\log_a \dfrac{1}{a\sqrt{a}}$

6. Find which two integers these expressions lie between. Then use the change of base formula and the calculator to find, correct to three significant figures:

(a) $\log_2 11$ (b) $\log_5 127$ (c) $\log_{11} 200$ (d) $\log_{\sqrt{2}} 20$

7. Express in terms of $\log_2 3$ and $\log_2 5$ (remember that $\log_2 2 = 1$):

(a) $\log_2 9$ (b) $\log_2 18$ (c) $\log_2 \frac{1}{6}$ (d) $\log_2 2\frac{1}{2}$

DEVELOPMENT

8. Rewrite in logarithmic form, then solve using the change of base formula. Give your answers in exact form, then to four significant figures:

(a) $2^x = 13$ (b) $3^{x-2} = 20$ (c) $7^x > 1000$ (d) $2^{x+1} < 10$ (e) $5^x < 0{\cdot}04$ (f) $(\frac{1}{2})^{x+1} = 10$ (g) $(\frac{1}{3})^x > 100$ (h) $(0{\cdot}06)^x < 0{\cdot}001$

9. (a) How many positive integer powers of 2 are less than 10^{10}?

(b) How many positive integer powers of $\frac{1}{5}$ are greater than 10^{-10}?

10. If $x = \log_a 2$, $y = \log_a 3$ and $z = \log_a 5$, simplify:

(a) $\log_a 64$ (b) $\log_a \dfrac{1}{30}$ (c) $\log_a 27a^5$ (d) $\log_a \dfrac{100}{a}$ (e) $\log_a 1{\cdot}5$ (f) $\log_a \dfrac{18}{25a}$ (g) $\log_a 0{\cdot}04$ (h) $\log_a \dfrac{8}{15a^2}$

11. Express in terms of $\log_2 3$ and $\log_2 5$:

(a) $\log_2 \frac{5}{8}$ (b) $\log_2 15\sqrt{3}$ (c) $\log_2 \frac{1}{6}\sqrt{2}$ (d) $\log_2 \frac{3}{25}\sqrt{30}$

12. Use the identities $\log_a a^x = x$ and $a^{\log_a x} = x$ to simplify:

(a) $\log_7 7^5$ (b) $3^{\log_3 7}$ (c) $\log_{12} 12^n$ (d) $6^{\log_6 y}$

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13. Using the identity $x = a^{\log_a x}$:

(a) express 3 as a power of 2,
(b) express u as a power of 3,
(c) express 7 as a power of a,
(d) express u as a power of v.

14. Simplify these expressions:

(a) $5^{-\log_5 2}$
(b) $12^{2\log_{12} 7}$
(c) $2^{\log_2 3 + \log_2 5}$
(d) $a^{n \log_a x}$
(e) $7^{-\log_7 x}$
(f) $5^{x + \log_5 x}$
(g) $2^{x \log_2 x}$
(h) $3^{\frac{\log_3 x}{x}}$

15. Rewrite these relations in index form (that is, without using logarithms):

(a) $\log_a(x+y) = \log_a x + \log_a y$
(b) $\log_{10} x = 3 + \log_{10} y$
(c) $\log_3 x = 4\log_3 y$
(d) $2\log_2 x + 3\log_2 y - 4\log_2 z = 0$
(e) $x\log_a 2 = \log_a y$
(f) $\log_a x - \log_a y = n\log_a z$
(g) $\frac{1}{2}\log_2 x = \frac{1}{3}\log_2 y - 1$
(h) $2\log_3(2x+1) = 3\log_3(2x-1)$

16. (a) Prove the identities: (i) $\log_a x = -\log_a \frac{1}{x}$ (ii) $\log_a x = -\log_{\frac{1}{a}} x$

(b) Check these identities by evaluating $\log_5 25$, $\log_5 \frac{1}{25}$ and $\log_{\frac{1}{5}} 25$.

17. Prove by contradiction that $\log_2 7$ and $\log_7 3$ are irrational (see the relevant worked exercise in Section 2B).

EXTENSION

18. Let $S = \frac{1}{2}(2^x + 2^{-x})$ and $D = \frac{1}{2}(2^x - 2^{-x})$.

(a) Simplify SD, $S + D$, $S - D$ and $S^2 - D^2$.

(b) Rewrite the formulae for S and D as quadratic equations in 2^x. Hence express x in terms of S, and in terms of D, in the case where $x > 1$.

(c) Show that $x = \frac{1}{2}\log_2 \dfrac{1+y}{1-y}$, where $y = DS^{-1}$.

6 C Sequences and How to Specify Them

A typical *infinite sequence* is formed by the positive odd integers, arranged in increasing order:

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots .$$

The three dots … indicate that the sequence goes on forever, with no last term. Notice, however, that the sequence starts abruptly with first term 1, then has second term 3, third term 5, and so on. Using the symbol T_n to stand for the nth term:

$$T_1 = 1,\quad T_2 = 3,\quad T_3 = 5,\quad T_4 = 7,\quad T_5 = 9,\quad \ldots .$$

The two-digit odd numbers arranged in increasing order form a *finite sequence*:

$$11, 13, 15, \ldots, 99,$$

where the dots … here stand for the 41 terms that have been omitted.

There are three different ways to specify a sequence, and it is important to be able to display a given sequence in these different ways.

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Write Out the First Few Terms: The easiest way is to write out the first few terms until the pattern is clear. Our example of the positive odd integers could be written

$$1, 3, 5, 7, 9, \ldots.$$

This sequence clearly continues $\ldots, 11, 13, 15, 17, 19, \ldots$, and with a few more calculations, it is clear that $T_{11} = 21$, $T_{14} = 27$, and $T_{16} = 31$.

Give a Formula for the nth Term: The formula for the nth term of this sequence is

$$T_n = 2n - 1,$$

because the nth term is always one less than $2n$. Notice that n must be a positive integer. Giving the formula does not rely on the reader recognising a pattern, and any particular term of the sequence can now be calculated quickly:

$$\begin{aligned} T_{30} &= 60 - 1 \\ &= 59 \end{aligned} \qquad \begin{aligned} T_{100} &= 200 - 1 \\ &= 199 \end{aligned} \qquad \begin{aligned} T_{244} &= 488 - 1 \\ &= 487 \end{aligned}$$

Say Where to Start and How to Proceed (Recursive Formula): This sequence of odd cardinals starts with 1, then each term is 2 more than the previous one. Thus the sequence is completely specified by writing down these two statements:

$$\begin{aligned} T_1 &= 1, \\ T_n &= T_{n-1} + 2, \ \text{ for } n \ge 2. \end{aligned}$$

Such a specification is called a *recursive* formula of a sequence, and some important definitions later in this chapter are based on this idea.

WORKED EXERCISE: Give all three specifications of the sequence of positive multiples of seven, arranged in increasing order.

SOLUTION: The sequence is $7, 14, 21, 28, \ldots$.
The formula for the nth term is $T_n = 7n$.
The recursive formula is $T_1 = 7$, and $T_n = T_{n-1} + 7$ for $n \ge 2$.

WORKED EXERCISE: Find the first five terms, and the formula for the nth term, of the sequence given by

$$T_1 = 1 \quad \text{and} \quad T_n = \frac{n-1}{n}\, T_{n-1}\,, \ \text{ for } n \ge 2.$$

SOLUTION: Using the formula, the first five terms are $T_1 = 1$,

$$\begin{aligned} T_2 &= \tfrac{1}{2} \times T_1 \\ &= \tfrac{1}{2}, \end{aligned} \qquad \begin{aligned} T_3 &= \tfrac{2}{3} \times T_2 \\ &= \tfrac{1}{3}, \end{aligned} \qquad \begin{aligned} T_4 &= \tfrac{3}{4} \times T_3 \\ &= \tfrac{1}{4}, \end{aligned} \qquad \begin{aligned} T_5 &= \tfrac{4}{5} \times T_4 \\ &= \tfrac{1}{5}. \end{aligned}$$

From this pattern it is clear that the formula for the nth term is $T_n = \dfrac{1}{n}$.

WORKED EXERCISE: Find whether 411 and 500 are members of the sequence whose nth term is $T_n = n^2 - 30$.

SOLUTION:

Put $T_n = 411$.
Then $n^2 - 30 = 411$
$n^2 = 441$
$n = 21$ or -21.
But n cannot be negative,
so 411 is the 21st term.

Put $T_n = 500$.
Then $n^2 - 30 = 500$
$n^2 = 530$.
But $\sqrt{530}$ is not a positive integer,
so 500 is not a term of the sequence.

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WORKED EXERCISE:

(a) Find how many negative terms there are in the sequence $T_n = 12n - 100$, and find the first positive term (its number and its value).

(b) How many positive terms are less than 200?

SOLUTION:

(a) Put $T_n < 0$.
Then $12n - 100 < 0$
$n < 8\frac{1}{3}$,
so there are eight negative terms, and the first positive term is $T_9 = 8$.

(b) Put $0 < T_n < 200$.
Then $0 < 12n - 100 < 200$
$8\frac{1}{3} < n < 25$,
so the 16 terms from T_9 to T_{24} inclusive are positive and less than 200.

Exercise 6C

1. Write down the next four terms of each sequence:

(a) 9, 13, 17, ...
(b) 3, 6, 12, ...
(c) 26, 17, 8, ...
(d) 81, 27, 9, ...
(e) $-1, 1, -1, \ldots$
(f) 25, 36, 49, ...
(g) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$
(h) $16, -8, 4, \ldots$

2. Write down the first four terms of the sequence whose nth term is:

(a) $T_n = 5n - 2$
(b) $T_n = 5^n$
(c) $T_n = n^3$
(d) $T_n = 12 - 7n$
(e) $T_n = 4 \times 3^n$
(f) $T_n = 2n(n+1)$
(g) $T_n = (-1)^n \times n$
(h) $T_n = (-3)^n$

3. Write down the first four terms of the sequence defined recursively by:

(a) $T_1 = 5$, $T_n = T_{n-1} + 12$
(b) $T_1 = \frac{3}{4}$, $T_n = 2T_{n-1}$
(c) $T_1 = 1$, $T_n = nT_{n-1}$
(d) $T_1 = 28$, $T_n = -\frac{1}{2}T_{n-1}$
(e) $T_1 = 37$, $T_n = T_{n-1} - 24$
(f) $T_1 = 2\sqrt{2}$, $T_n = T_{n-1}\sqrt{2}$
(g) $T_1 = 5$, $T_n = T_{n-1} + 5n$
(h) $T_1 = \frac{1}{2}$, $T_n = T_{n-1} + (\frac{1}{2})^n$

4. The nth term of a sequence is given by $T_n = 6n + 17$.

(a) By forming equations and solving them, find whether each of the numbers 77, 349 and 1577 is a member of the sequence, and if so, which term it is.

(b) By forming an inequation and solving it, find how many terms of the sequence are less than 400, and find the value of the first term greater than 400.

5. The nth term of a sequence is given by $T_n = 5n^2$.

(a) By forming an equation and solving it, find whether each of the numbers 60, 80 and 605 is a member of the sequence, and if so, which term it is.

(b) By forming an inequation and solving it, find how many terms of the sequence are less than 1000, and find the value of the first term greater than 1000.

DEVELOPMENT

6. Write down the first four terms of these sequences (where a and x are constants):

(a) $T_n = 1 + (-1)^n$
(b) $T_n = 25 \times (-2)^n$
(c) $T_n = -36x \times (-\frac{1}{2})^{n-1}$
(d) $T_n = 7a - 2an$
(e) $T_n = 4a \times 2^{n-1}$
(f) $T_n = 3^n - 2^n$
(g) $T_n = (-1)^n(4n - 7)$
(h) $T_n = (2\sqrt{2})^{n-1}$
(i) $T_n = \frac{3}{4}n^2x$

7. Give a recursive formula for the nth term T_n of each sequence in terms of the $(n-1)$th term T_{n-1}:

(a) 16, 21, 26, ...
(b) 7, 14, 28, ...
(c) $9, 2, -5, \ldots$
(d) $4, -4, 4, \ldots$

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8. (a) Find whether -10 and -15 are members of the sequence $T_n = 48 - 7n$, and if so, what terms they are. (b) How many terms in this sequence are greater than -700?

9. (a) Find whether 28 and 70 are members of the sequence $T_n = n^2 - 3n$, and if so, what terms they are. (b) How many terms of this sequence are less than 18?

10. (a) Find whether $1\frac{1}{2}$ and 96 are members of the sequence $T_n = \frac{3}{32} \times 2^n$, and if so, what terms they are. (b) Find the first term in this sequence which is greater than 10.

11. The rigorous definition of a sequence is: 'A *sequence* is a function whose domain is the set of positive integers'. Graph the sequences in question 2, with n on the horizontal axis and T_n on the vertical axis. If there is a simple curve joining the points, draw it and give its equation.

12. Write down the first four terms, then state which terms are zero:

(a) $T_n = \sin 90n^\circ$ (c) $T_n = \cos 180n^\circ$ (e) $T_1 = -1$ and $T_n = T_{n-1} + \cos 180n^\circ$
(b) $T_n = \cos 90n^\circ$ (d) $T_n = \sin 180n^\circ$ (f) $T_1 = 1$ and $T_n = T_{n-1} + \sin 90n^\circ$

13. (a) A sequence satisfies $T_n = \frac{1}{2}(T_{n-1} + T_{n+1})$, with $T_1 = 3$ and $T_2 = 7$. Find T_3 and T_4.

(b) A sequence satisfies $T_n = \sqrt{T_{n-1} \times T_{n+1}}$, with $T_1 = 1$ and $T_2 = 2$. Find T_3 and T_4.

14. A sequence is defined by $T_n = \dfrac{1}{n} - \dfrac{1}{n+1}$. (a) Find $T_1 + T_2 + T_3 + T_4$.

(b) Give a formula for $T_1 + T_2 + \cdots + T_n$. (c) Show that $T_n = \dfrac{1}{n(n+1)}$.

(d) Which term of the sequence is $\frac{1}{30}$?

15. (a) Which terms of the sequence $T_n = \dfrac{n-1}{n}$ are 0·9 and 0·99? (b) Find $T_{n+1} : T_n$.

(c) Prove that $\dfrac{T_n}{T_{n+1}} + \dfrac{1}{n^2} = 1$. (d) Find $T_2 \times T_3 \times \cdots \times T_n$.

(e) Prove that $T_{n+1} - T_{n-1} = \dfrac{2}{n^2 - 1}$.

EXTENSION

16. [The Fibonacci and Lucas sequences] These sequences are defined recursively by

$$\begin{aligned} &F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \ge 3, \\ &L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2}, \quad \text{for } n \ge 3. \end{aligned}$$

(a) Write out the first 12 terms of each sequence. Explain why every third term of each sequence is even and the rest are odd.

(b) Write out the sequences $L_1 + F_1$, $L_2 + F_2$, $L_3 + F_3$, ... and $L_1 - F_1$, $L_2 - F_2$, $L_3 - F_3$, ... How are these two new sequences related to the Fibonacci sequence, and why?

(c) Expand and simplify the first four terms of the sequence $T_n = \left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^n$. Let the two sequences A_n and B_n of rational numbers be defined by $T_n = \frac{1}{2}A_n + \frac{1}{2}B_n\sqrt{5}$. Show that

$$A_{n+2} = A_{n+1} + A_n \qquad \text{and} \qquad B_{n+2} = B_{n+1} + B_n,$$

and hence that A_n is the Lucas sequence and B_n is the Fibonacci sequence.

(d) Examine similarly the sequence $U_n = \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n$.

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6 D Arithmetic Sequences

A very simple type of sequence is an *arithmetic sequence*. This is a sequence like

$$3, 7, 11, 15, 19, 23, 27, 31, 35, 39, \ldots,$$

in which the difference between successive terms is constant — in this example each term is 4 more than the previous term. Because the difference is constant, all the terms can be generated from the first term 3 by repeated addition of this common difference 4. Arithmetic sequences are called APs for short, standing for 'arithmetic progression', an old name for the same thing.

Definition of an Arithmetic Sequence: An arithmetic sequence is a sequence in which the difference between successive terms is constant.

9 DEFINITION: A sequence T_n is called an *arithmetic sequence* if

$$T_n - T_{n-1} = d, \quad \text{for } n \geq 2,$$

where d is a constant, called the *common difference*.

This definition is essentially a recursive definition, because if a is the first term, then the terms of the sequence are defined by

$$T_1 = a \quad \text{and} \quad T_n = T_{n-1} + d, \quad \text{for } n \geq 2.$$

The first few terms of the sequence are

$$T_1 = a, \quad T_2 = a + d, \quad T_3 = a + 2d, \quad T_4 = a + 3d, \quad \ldots$$

and from this pattern it is clear that the general formula for the nth term of an AP is:

10 THE nTH TERM OF AN AP: $T_n = a + (n-1)d$

WORKED EXERCISE: Write out the first five terms, and calculate the 20th term, of the AP with: (a) $a = 2$ and $d = 5$, (b) $a = 20$ and $d = -3$.

SOLUTION:

(a) 2, 7, 12, 17, 22, ...

$$\begin{aligned} T_{20} &= a + 19d \\ &= 2 + 5 \times 19 \\ &= 97 \end{aligned}$$

(b) 20, 17, 14, 11, 8, ...

$$\begin{aligned} T_{20} &= a + 19d \\ &= 20 - 3 \times 19 \\ &= -37 \end{aligned}$$

WORKED EXERCISE: Show that the sequence 200, 193, 186, ... is an AP. Then find a formula for the nth term, and find the first negative term.

SOLUTION:

Since $T_2 - T_1 = -7$
and $T_3 - T_2 = -7$,
it is an AP with $a = 200$ and $d = -7$,
so $T_n = 200 - 7(n-1)$
$= 207 - 7n$.

Put $T_n < 0$.
Then $207 - 7n < 0$
$7n > 207$
$n > 29\frac{4}{7}$,
so the first negative term is $T_{30} = -3$.

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WORKED EXERCISE: Test whether these sequences are APs:

(a) 3, 9, 27, ...

(b) $\log_5 6,\ \log_5 12,\ \log_5 24,\ \ldots$

SOLUTION:

(a) $T_2 - T_1 = 6$
and $T_3 - T_2 = 18$,
so it is not an AP.

(b) $T_2 - T_1 = \log_5 2$
and $T_3 - T_2 = \log_5 2$,
so it is an AP, and $d = \log_5 2$.

Further Problems: The first example below uses simultaneous equations. The second uses a double inequality to find the number of terms between two given numbers.

WORKED EXERCISE: The third term of an AP is 16, and the 12th term is 79. Find the 41st term.

SOLUTION: Let the first term be a and the common difference be d.

Since $T_3 = 16$, $\quad a + 2d = 16, \quad (1)$

and since $T_{12} = 79$, $\quad a + 11d = 79. \quad (2)$

Subtracting (1) from (2), $\quad 9d = 63$

$d = 7.$

Substituting into (1) gives $a = 2$, and so $\quad T_{41} = a + 40d$

$= 282.$

WORKED EXERCISE: Use the fact that the positive multiples of 7 form an AP to find how many multiples of 7 lie between 1000 and 10 000.

SOLUTION: The positive multiples of 7 form an AP $7, 14, 21, \ldots$
in which $a = 7$ and $d = 7$.

The nth term of the AP is $\quad T_n = 7 + 7(n - 1)$

$= 7n$

(or one can simply claim that it's obvious that $T_n = 7n$).

To find the multiples of 7 between 1000 and 10 000, put

$$1000 < T_n < 10\,000$$

$$1000 < 7n < 10\,000$$

$\boxed{\div 7}$ $\quad 142\frac{6}{7} < n < 1428\frac{4}{7},$

so there are 1428 multiples of 7 less than 10 000, and 142 less than 1000,
leaving $1428 - 142 = 1286$ multiples of 7 between 1000 and 10 000.

Exercise 6D

1. Find $T_3 - T_2$ and $T_2 - T_1$ to test whether each sequence is an AP. If it is, write down the common difference d, find T_{10}, then find a formula for the nth term T_n:

(a) 8, 11, 14, ...
(b) 21, 15, 9, ...
(c) 8, 4, 2, ...
(d) −3, 1, 5, ...
(e) $1\frac{3}{4},\ 3,\ 4\frac{1}{4},\ \ldots$
(f) 12, −5, −22, ...
(g) $5 + \sqrt{2},\ 5,\ 5 - \sqrt{2},\ \ldots$
(h) 1, 4, 9, 16, ...
(i) $-2\frac{1}{2},\ 1,\ 4\frac{1}{2},\ \ldots$

2. Find T_n for each AP, then find T_{25} and the first negative term:

(a) 82, 79, 76, ...
(b) 345, 337, 329, ...
(c) $24\frac{1}{2},\ 23\frac{1}{4},\ 22,\ \ldots$

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3. Find x and the common difference if the following numbers form an arithmetic sequence. [HINT: Form an equation using the identity $T_2 - T_1 = T_3 - T_2$, then solve it to find x.]
(a) 14, x, 32 (b) x, 14, 32 (c) $x-1$, 17, $x+15$ (d) $2x+2$, $x-4$, $5x$

DEVELOPMENT

4. The price of windows in a house is \$500 for the first window, then \$300 for each additional window. (a) Find a formula for the cost of n windows. (b) How much will fifteen windows cost? (c) What is the maximum number of windows whose total cost is less than \$10 000?

5. [Simple interest and APs] A principal of \$2000 is invested at 6% per annum simple interest. Let $\$A_n$ be the total amount (principal plus interest) at the end of n years. (a) Write out the values of A_1, A_2, A_3 and A_4. (b) Find a formula for A_n, and evaluate A_{12}. (c) How many years will it take before the total amount exceeds \$6000?

6. Use the formula $T_n = a + (n-1)d$ to find how many terms there are in each sequence:
(a) 2, 5, 8, ..., 2000 (b) 100, 92, 84, ..., -244 (c) -12, $-10\frac{1}{2}$, -9, ..., 108

7. The nth term of an arithmetic sequence is $T_n = 7 + 4n$.
(a) Write out the first four terms, and hence find the values of a and d.
(b) Find the sum and the difference of the 50th and the 25th terms.
(c) Prove that $5T_1 + 4T_2 = T_{27}$. (d) Which term of the sequence is 815?
(e) Find the last term less than 1000 and the first term greater than 1000.
(f) Find which terms are between 200 and 300, and how many of them there are.

8. (a) Let T_n be the sequence of positive multiples of 8. (i) Find the first term of the sequence greater than 500. (ii) Find the last term of the sequence less than 850. (iii) Hence find the number of positive multiples of 8 between 500 and 850.
(b) Use similar methods to find: (i) the number of multiples of 11 between 1000 and 2000, (ii) the number of multiples of 7 between 800 and 2000.

9. Find the first term and the common difference of the AP $T_n = a + (n-1)d$ with:
(a) $T_2 = 3$ and $T_{10} = 35$
(b) $T_5 = 24$ and $T_9 = -12$
(c) $T_4 = 6$ and $T_{12} = 34$
(d) $T_7 = \sqrt{5} - 4$ and $T_{13} = 8 - 5\sqrt{5}$

10. (a) The third term of an AP is 7, and the seventh term is 31. Find the eighth term.
(b) The fourth, sixth and eighth terms of an AP add to -6. Find the sixth term.

11. Find the common difference of each AP, then find x, given that $T_{11} = 36$:
(a) $5x-9$, $5x-5$, $5x-1$, ...
(b) 16, $16+6x$, $16+12x$, ...
(c) $2x+10$, $7-x$, $4-4x$, ...

12. Find the common difference and a formula for the nth term of each AP:
(a) $\log_3 2$, $\log_3 4$, $\log_3 8$, ...
(b) $\log_a 54$, $\log_a 18$, $\log_a 6$, ...
(c) $x-3y$, $2x+y$, $3x+5y$, ...
(d) $5-6\sqrt{5}$, $1+\sqrt{5}$, $-3+8\sqrt{5}$, ...
(e) 1·36, $-0{\cdot}52$, $-2{\cdot}4$, ...
(f) $\log_a 3x^2$, $\log_a 3x$, $\log_a 3$, ...

13. How many terms of the series 100, 97, 94, ... have squares less than 400?

14. [APs are essentially linear functions.] (a) Show that if $f(x) = mx + b$ is any linear function, then the sequence T_n defined by $T_n = mn + b$ is an AP, and find its first term and common difference. (b) Conversely, if T_n is an AP with first term a and difference d, find the linear function $f(x)$ such that $T_n = f(n)$. (c) Plot on the same axes the points of the AP $T_n = 8 - 3(n-1)$ and the graph of the continuous linear function $y = 8 - 3(x-1)$.

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EXTENSION

15. [The set of all APs forms a two-dimensional space.] Let $\mathcal{A}(a, d)$ represent the AP whose first term is a and difference is d.

(a) The *sum* of the two sequences T_n and U_n is defined to be the sequence whose nth term is $T_n + U_n$. Show that for all constants λ and μ, and for all values of a_1, a_2, d_1 and d_2, the sequence $\lambda\mathcal{A}(a_1, d_1) + \mu\mathcal{A}(a_2, d_2)$ is an AP, and find its first term and common difference.

(b) Write out the sequences $\mathcal{A}(1, 0)$ and $\mathcal{A}(0, 1)$. Show that any AP $\mathcal{A}(a, d)$ with first term a and difference d can be written in the form $\lambda\mathcal{A}(1, 0) + \mu\mathcal{A}(0, 1)$, and find λ and μ.

(c) Show more generally that, provided $a_1 : a_2 \neq d_1 : d_2$, any AP $\mathcal{A}(a, d)$ can be written in the form $\lambda\mathcal{A}(a_1, d_1) + \mu\mathcal{A}(a_2, d_2)$, and find expressions for λ and μ.

6 E Geometric Sequences

A *geometric sequence* is a sequence like this:

$$6,\ 18,\ 54,\ 162,\ 486,\ 1458,\ 4374,\ \dots,$$

in which the ratio of successive terms is constant — in this example, each term is 3 times the previous term. This is a very similar situation to the APs of the last section, where the difference of successive terms was constant. Because the ratio is constant, all the terms can be generated from the first term 2 by repeated multiplication by this common ratio 3. The old name was ‘geometric progession’ and so geometric sequences are called GPs for short.

Definition of a Geometric Sequence: A geometric sequence is a sequence in which the ratio of successive terms is constant.

11 DEFINITION: A sequence T_n is called a *geometric sequence* if

$$\frac{T_n}{T_{n-1}} = r, \quad \text{for } n \geq 2,$$

where r is a constant, called the *common ratio*.

This definition, like the definition of an AP, is a recursive definition. If a is the first term, then the terms of the sequence are

$$T_1 = a \quad \text{and} \quad T_n = rT_{n-1}, \quad \text{for } n \geq 2.$$

The first few terms of the sequence are

$$T_1 = a, \quad T_2 = ar, \quad T_3 = ar^2, \quad T_4 = ar^3, \quad \dots$$

and it follows from this pattern that the general formula for the nth term of a GP is:

12 THE nTH TERM OF A GP: $T_n = ar^{n-1}$, for $n \geq 1$.

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WORKED EXERCISE: The sequence 6, 18, 54, 162, 486, 1458, 4374, ... at the start of this section is a GP with first term $a = 6$ and common ratio $r = 3$,

so $$\begin{aligned} T_n &= ar^{n-1} \\ &= 6 \times 3^{n-1}. \end{aligned}$$

For example, $T_6 = 6 \times 3^5 = 1458$,

and $T_{15} = 6 \times 3^{14}$ (the large number is best left factored).

Negative Ratios and Alternating Signs: The sequence 6, −18, 54, −162, ... formed by alternating the signs of the previous sequence is also a GP — its first term is still $a = 6$ but its ratio is $r = -3$. The repeated multiplication by −3 makes the terms alternate in sign.

WORKED EXERCISE: Find T_n, T_6 and T_{15} for 6, −18, 54, −162, ...

SOLUTION:

The sequence is a GP with $a = 6$ and $r = -3$,

so $$\begin{aligned} T_n &= ar^{n-1} \\ &= 6 \times (-3)^{n-1}. \end{aligned}$$

Also, $$\begin{aligned} T_6 &= (-3)^5 \times 6 \\ &= -1458, \end{aligned}$$

and $$\begin{aligned} T_{15} &= (-3)^{14} \times 6 \\ &= 6 \times 3^{14}. \end{aligned}$$

A Condition for Three Numbers to be in AP or GP: Three numbers T_1, T_2, T_3 form an AP when the differences $T_3 - T_2$ and $T_2 - T_1$ are equal. Similarly, they form a GP when the ratios T_3/T_2 and T_2/T_1 are equal.

13

CONDITION FOR AN AP: $T_3 - T_2 = T_2 - T_1$

CONDITION FOR A GP: $\dfrac{T_3}{T_2} = \dfrac{T_2}{T_1}$

WORKED EXERCISE: Find the value of x such that 3, $x + 4$ and $x + 10$ form:

(a) an arithmetic sequence, (b) a geometric sequence.

SOLUTION:

(a) Put $$\begin{aligned} T_3 - T_2 &= T_2 - T_1 \\ (x+10) - (x+4) &= (x+4) - 3 \\ 6 &= x + 1, \end{aligned}$$

so $x = 5$,

and the numbers are 3, 9 and 15.

(b) Put $$\begin{aligned} \frac{x+10}{x+4} &= \frac{x+4}{3} \\ 3(x+10) &= (x+4)^2 \\ x^2 + 5x - 14 &= 0 \\ (x+7)(x-2) &= 0, \end{aligned}$$

so $x = 2$, giving 3, 6 and 12,

or $x = -7$, giving 3, −3 and 3.

Further Problems: The first example below uses elimination to solve simultaneous equations, but takes the ratio rather than the difference of the two equations. The second shows the solution of an inequality involving indices.

WORKED EXERCISE: Find the first term a and the common ratio r of a GP in which the fourth term is 30 and the sixth term is 480.

SOLUTION: Since $T_4 = 30$, $ar^3 = 30$ (1)

and since $T_6 = 480$, $ar^5 = 480$ (2)

Dividing (2) by (1), $r^2 = 16$,

so $r = 4$ and $a = \frac{15}{32}$, or $r = -4$ and $a = -\frac{15}{32}$.

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WORKED EXERCISE: [A harder example] (a) Show that the sequence whose terms are 1000, 400, 160, ... forms a GP, and then find the formula for the nth term.
(b) Find the first term less than $\frac{1}{1000}$.

SOLUTION:

(a) Since $\frac{400}{1000} = \frac{2}{5}$
and $\frac{160}{400} = \frac{2}{5}$,
it is a GP with $a = 1000$ and $r = \frac{2}{5}$,
so $T_n = ar^{n-1}$
$= 1000 \times (\frac{2}{5})^{n-1}$.

(b) Put $T_n < \frac{1}{1000}$.
Using the calculator,
$T_{16} = 0{\cdot}001\,07\ldots$
and $T_{17} = 0{\cdot}000\,4\ldots$,
so the first term less than $0{\cdot}001$ is
$T_{17} = 1000 \times (\frac{2}{5})^{16}$
$\doteqdot 0{\cdot}000\,429$.

Alternatively, the inequation can be solved analytically using logarithms:

$$\begin{aligned}
\text{Put}\quad T_n &< \tfrac{1}{1000}. \\
\text{Then } 1000 \times (\tfrac{2}{5})^{n-1} &< \tfrac{1}{1000} \\
(\tfrac{2}{5})^{n-1} &< \tfrac{1}{1\,000\,000} \\
(\tfrac{5}{2})^{n-1} &> 1\,000\,000 \\
n - 1 &> \log_{\frac{5}{2}} 1\,000\,000 \\
n - 1 &> \frac{\log_{10} 1\,000\,000}{\log_{10} 2\frac{1}{2}} \\
n - 1 &> 15{\cdot}07\ldots \\
n &> 16{\cdot}07\ldots .
\end{aligned}$$

Hence the first term less than $\frac{1}{1000}$ is $T_{17} = 1000 \times (\frac{2}{5})^{16} \doteqdot 0{\cdot}000\,429$.

Exercise 6E

1. Find the first four terms, and the formula for the nth term, of the GP with:

(a) $a = 1$ and $r = 3$
(b) $a = 5$ and $r = -2$
(c) $a = 18$ and $r = \frac{1}{3}$
(d) $a = 6$ and $r = -\frac{1}{2}$
(e) $a = 1$ and $r = \sqrt{2}$
(f) $a = -7$ and $r = -1$

2. Find $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ to test whether each sequence is a GP. If it is, write down the common ratio, find T_6, then find a formula for the nth term:

(a) 10, 20, 40, ...
(b) 180, 60, 20, ...
(c) 64, 81, 100, ...
(d) 35, 50, 65, ...
(e) $\frac{3}{4}$, 3, 12, ...
(f) -24, -6, $-1\frac{1}{2}$, ...

3. Find the common ratio, find a formula for T_n, and find T_6:

(a) 1, -1, 1, ...
(b) -2, 4, -8, ...
(c) -8, 24, -72, ...
(d) 60, -30, 15, ...
(e) -1024, 512, -256, ...
(f) $\frac{3}{8}$, $-\frac{9}{2}$, 54, ...

4. Use the formula $T_n = ar^{n-1}$ to find r for a GP where:

(a) $a = 3$ and $T_6 = 96$
(b) $a = 1$ and $T_5 = 81$
(c) $a = 486$ and $T_5 = \frac{2}{27}$
(d) $a = 32$ and $T_6 = -243$
(e) $a = 1000$ and $T_7 = 0{\cdot}001$
(f) $a = 5$ and $T_7 = 40$

DEVELOPMENT

5. Use the formula $T_n = ar^{n-1}$ to find a and r for a GP with:

(a) $T_3 = 1$ and $T_6 = 64$
(b) $T_2 = \frac{1}{3}$ and $T_6 = 27$
(c) $T_9 = 24$ and $T_5 = 6$
(d) $T_7 = 2\sqrt{2}$ and $T_{12} = \frac{1}{16}\sqrt{2}$

6. Find the nth term of each GP:

(a) $\sqrt{6}$, $2\sqrt{3}$, $2\sqrt{6}$, ...
(b) ax, a^2x^3, a^3x^5, ...
(c) $-x/y$, -1, $-y/x$, ...

7. The nth term of a geometric sequence is $T_n = 25 \times 2^n$.

(a) Write out the first six terms, and hence find the values of a and r.

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(b) Find in factored form $T_{50} \times T_{25}$ and $T_{50} \div T_{25}$.
(c) Prove that $T_9 \times T_{11} = 25T_{20}$. (d) Which term of the sequence is 6400?
(e) Verify that $T_{11} = 51\,200$ is the last term less than 100 000, and $T_{12} = 102\,400$ is the first term greater than 100 000.
(f) Find which terms are between 1000 and 100 000, and how many of them there are.

8. Find x and the common difference or ratio, if these form (i) an AP, (ii) a GP:
(a) x, 24, 96 (b) 24, x, 96 (c) $x-4$, $x+1$, $x+11$ (d) $x-2$, $x+2$, $5x-2$

9. Find T_n for each GP, then find how many terms there are:
(a) 7, 14, 28, ..., 224 (b) 2, 14, 98, ..., 4802 (c) $\frac{1}{25}$, $\frac{1}{5}$, 1, ..., 625

10. Use logs to find how many terms in each of the previous sequences are less than 1 000 000.

11. How many terms are between 1000 and 1 000 000 in the sequences in question 9?

12. [Compound interest and GPs] A principal $\$P$ is invested at 7% per annum compound interest. Let A_n be the total amount at the end of n years.
(a) Write down A_1, A_2 and A_3.
(b) Show that the total amount at the end of n years forms a GP with first term $1{\cdot}07 \times P$ and ratio 1·07, and find the nth term A_n.
(c) How many full years does it take for the amount to double, and how many years does it take for it to become ten times the original principal?

13. Find T_n for each GP, then use logs to find how many terms exceed 10^{-6}:
(a) 98, 14, 2, ... (b) 25, 5, 1, ... (c) 1, 0·9, 0·81, ...

14. [Depreciation and GPs] A car originally costs \$20 000, then at the end of every year, it is worth only 80% of what it was worth a year before. Let W_n be its worth at the end of n years.
(a) Write down expressions for W_1, W_2 and W_3, and find a formula for W_n.
(b) Find how many complete years it takes for the value to fall below \$2000.

15. When light passes through one sheet of very thin glass, its intensity is reduced by 3%. What is the minimum number of sheets that will reduce the intensity below 1%?

16. (a) Find a formula for T_n in $2x$, $2x^2$, $2x^3$, ..., then find x given that $T_6 = 2$.
(b) Find a formula for T_n in x^4, x^2, 1, ..., then find x given that $T_6 = 3^6$.
(c) Find a formula for T_n in $2^{-16}x$, $2^{-12}x$, $2^{-8}x$, ..., then find x given that $T_6 = 96$.

17. (a) Find a and b if a, b, 1 forms a GP, and b, a, 10 forms an AP.
(b) Find a and b if a, 1, $a+b$ forms a GP, and b, $\frac{1}{2}$, $a-b$ forms an AP.
(c) Find the first term of the AP with common difference -7 in which $T_{10} = 3$.
(d) Find the first term of the GP with common ratio 2 in which $T_6 = 6$.
(e) Find a and d of the AP in which $T_6 + T_8 = 44$ and $T_{10} + T_{13} = 35$.
(f) Find a and r of the GP in which $T_2 + T_3 = 4$ and $T_4 + T_5 = 36$.

18. (a) Show that if the first, second and fourth terms of an AP form a geometric sequence, then either the sequence is a constant sequence, or the terms are the positive integer multiples of the first term.
(b) Show that if the first, second and fifth terms of an AP form a geometric sequence, then either the sequence is a constant sequence, or the terms are the odd positive integer multiples of the first term.

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(c) Find the common ratio of the GP in which the first, third and fourth terms form an arithmetic sequence. [HINT: $r^3 - 2r^2 + 1 = (r-1)(r^2 - r - 1)$]

(d) Find the GP in which each term is one more than the sum of all the previous terms.

19. (a) Show that $2^5, 2^2, 2^{-1}, 2^{-4}, \ldots$ is a GP, and find its nth term.

(b) Show that $\log_2 96$, $\log_2 24$, $\log_2 6$, ... is an AP, and show that $T_n = 7 - 2n + \log_2 3$.

20. [The relationship between APs and GPs]

(a) Suppose that $T_n = a + (n-1)d$ is an AP with first term a and difference d. Show that the sequence $U_n = 2^{T_n}$ is a GP, and find its first term and ratio.

(b) Suppose that $T_n = ar^{n-1}$ is a GP with first term a and ratio r. Show that the sequence $U_n = \log_2 T_n$ is an AP, and find its first term and difference.

(c) Does the base have to be 2?

21. [GPs are essentially exponential functions.]

(a) Show that if $f(x) = kb^x$ is any exponential function, then the sequence $T_n = kb^n$ is a GP, and find its first term and common ratio.

(b) Conversely, if T_n is a GP with first term a and ratio r, find the exponential function $f(x)$ such that $T_n = f(n)$.

(c) Plot on the same axes the points of the GP $T_n = 2^{4-n}$ and the graph of the continuous function $y = 2^{4-x}$.

EXTENSION

22. [Products and sums of GPs] Suppose that $T_n = ar^{n-1}$ and $U_n = AR^{n-1}$ are two GPs.

(a) Show that the sequence $V_n = T_n U_n$ is a GP, and find its first term and common ratio.

(b) Show that the sequence $W_n = T_n + U_n$ is a GP if and only if $r = R$. [HINT: The condition for W_n to be a GP is $W_n W_{n+2} = {W_{n+1}}^2$ — substitute into this condition, and deduce that $(R-r)^2 = 0$.]

23. [The set of all GPs] Let $\mathcal{G}(a, r)$ represent the GP whose first term is a and ratio is r.

(a) The *product* of two sequences T_n and U_n is defined to be the sequence whose nth term is $T_n U_n$. Show that for all positive constants λ and μ, and for all non-zero a_1, a_2, r_1 and r_2, the sequence $\mathcal{G}(a_1, r_1)^\lambda \mathcal{G}(a_2, r_2)^\mu$ is a GP, and find its first term and common difference.

(b) Write out the sequences $\mathcal{G}(2, 1)$ and $\mathcal{G}(1, 2)$. Show that any GP $\mathcal{G}(a, r)$ with first term a and ratio r can be written in the form $\mathcal{G}(2, 1)^\lambda \mathcal{G}(1, 2)^\mu$, and find the values of λ and μ.

6 F Arithmetic and Geometric Means

What number x should be placed between 3 and 12 to make a satisfactory pattern 3, x, 12? There are three obvious answers to this question:

$$3,\ 7\tfrac{1}{2},\ 12 \qquad \text{and} \qquad 3,\ 6,\ 12 \qquad \text{and} \qquad 3,\ -6,\ 12.$$

The number $7\frac{1}{2}$ makes the sequence an AP and is called the *arithmetic mean* of 3 and 12. Notice that $7\frac{1}{2}$ is calculated by taking half the sum of 3 and 12.

The numbers 6 and -6 each make the sequence a GP, with ratio 2 and -2 respectively, and are both called *geometric means* of 3 and 12. Notice that the numbers 6 and -6 can easily be calculated, being the positive and negative square roots of the product 36 of 3 and 12.

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Definition — Arithmetic and Geometric Means: Let a and b be two numbers.

ARITHMETIC MEAN: The *arithmetic mean* (AM) of a and b is the number x such that a, x, b forms an AP. Then $b - x = x - a$, so $2x = a + b$, giving

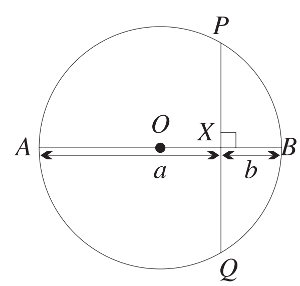
$$\text{AM} = \tfrac{1}{2}(a + b).$$

14 GEOMETRIC MEAN: A *geometric mean* (GM) of a and b is a number x such that a, x, b forms a GP. Then $\dfrac{b}{x} = \dfrac{x}{a}$, so $x^2 = ab$, giving

$$\text{GM} = \sqrt{ab} \text{ or } -\sqrt{ab}.$$

Geometric Interpretation of the Means: Arithmetic and geometric means occur often in geometry. The diagram on the right is particularly interesting in that it illustrates both means.

Let a and b be two given lengths. Construct the interval AXB with $AX = a$ and $XB = b$. Construct the midpoint O of AB, and construct the circle with diameter AB. Construct the chord through X perpendicular to AB, meeting the circle at P and Q.



First, the radius AO is the arithmetic mean of a and b, because $AO = \frac{1}{2}(AX + XB)$. Secondly, by circle geometry,

$$AX \times XB = PX \times XQ$$

because the chords AB and PQ intersect at X. So, since $PX = XQ$, it follows that $PX^2 = a \times b$, and hence PX is the geometric mean of a and b.

The semichord PX cannot exceed the radius AO. This gives a geometric proof of the following important inequality (not explicitly part of the course).

THEOREM: The GM of two positive numbers cannot exceed their AM.

Inserting More than One Mean: One can also insert several terms in arithmetic or geometric sequence between two given numbers. This process is called *inserting arithmetic or geometric means*. It should be done by forming an AP or GP with the given numbers as first and last terms.

WORKED EXERCISE: (a) Insert four numbers in arithmetic sequence (that is, insert four arithmetic means) between 10 and 30. (b) Insert three numbers in geometric sequence (geometric means) between 10 and 40.

SOLUTION:

(a) Form an AP with $a = 10$ and $T_6 = 30$.

Then $a + 5d = 30$

$5d = 20$

$d = 4$,

so the means are 14, 18, 22 and 26.

(b) Form a GP with $a = 10$ and $T_5 = 40$.

Then $ar^4 = 40$

$r^4 = 4$

$r = \sqrt{2}$ or $-\sqrt{2}$,

so the means are $10\sqrt{2}$, 20 and $20\sqrt{2}$, or $-10\sqrt{2}$, 20 and $-20\sqrt{2}$.

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Exercise **6F**

1. Find the arithmetic and geometric means of the following pairs of numbers:

(a) 4 and 16 (d) 10 and -40 (g) a and $-a$ (j) 2^4 and 2^7
(b) 16 and 25 (e) $1\frac{1}{2}$ and 6 (h) 1 and a (k) a^3 and a^5
(c) -5 and -20 (f) a^2 and $49a^2$ (i) 2^4 and 2^6 (l) x^{-3} and x^3

2. Find the value of x, then write out the three numbers, if:

(a) 5 is the AM of $x-3$ and $2x+7$,
(b) x is the AM of $3x-2$ and $x+10$,
(c) $x-1$ is the GM of $x-3$ and $x+4$,
(d) 2 is the GM of $2-x$ and $5-x$.

3. (a) Insert four numbers in arithmetic sequence between 7 and 42.
(b) Insert two numbers in geometric sequence between 27 and 8.
(c) Insert nine arithmetic means between 40 and 5.
(d) Insert five geometric means between 1 and 1000.

4. Find a, b and c such that 3, a, b, c, 48 is: (a) an AP, (b) a GP.

DEVELOPMENT

5. Find the arithmetic and geometric means of the following pairs of numbers:

(a) $\sqrt{5}+1$ and $\sqrt{5}-1$ (d) $(x-y)^2$ and $(x+y)^2$ (g) $\log_2 3$ and $\log_2 27$
(b) $\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{\sqrt{8}}$ (e) $\dfrac{1}{x-y}$ and $\dfrac{1}{x+y}$ (h) $\log_b 4$ and $\log_b 256$
(c) $x-y$ and $x+y$ (f) $\log_2 3$ and $\log_2 81$ (i) $\dfrac{1}{\sqrt{5}+1}$ and $\dfrac{1}{\sqrt{5}-1}$

6. (a) Find the arithmetic mean and geometric mean of 0·2 and 0·000 02.
(b) Insert three arithmetic means and three geometric means between 0·2 and 0·000 02.

7. Suppose that x and y are positive numbers. (a) Find the arithmetic mean and the positive geometric mean of $\dfrac{y}{x}$ and $\dfrac{x}{y}$. (b) Show that the difference between the two means is $\dfrac{(x-y)^2}{2xy}$. (c) What is the condition on x and y for the two means to be equal?

8. (a) Show that if a and b have opposite signs, then they do not have a geometric mean.
(b) If a and b have opposite signs, what determines the sign of the arithmetic mean?
(c) Three nonzero numbers form both an AP and a GP. Prove that they are all equal. [HINT: Let the numbers be $x-d$, x and $x+d$, and prove that $d=0$.]
(d) Show that the fourth term of an AP is the arithmetic mean of the first and seventh terms.
(e) Show that the fourth term of a GP is a geometric mean of the first and seventh terms.
(f) Show that if the fifth term of an AP is a geometric mean of the third and eighth terms, then the seventh term is a geometric mean of the third and fifteenth terms.

9. [The relationship between arithmetic means and geometric means] (a) Show that if m is the arithmetic mean of a and b, then 3^m is the geometric mean of 3^a and 3^b.
(b) Show that if m is the positive geometric mean of a and b, then $\log_3 m$ is the arithmetic mean of $\log_3 a$ and $\log_3 b$.

10. [An algebraic proof of the AM/GM inequality] Suppose that a and b are two positive numbers. (a) Expand $(a-b)^2$. (b) Use the fact that $(a-b)^2$ cannot be negative to prove that the arithmetic mean of a and b is never less than the geometric mean. (c) When are the two means equal?

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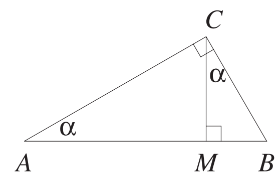
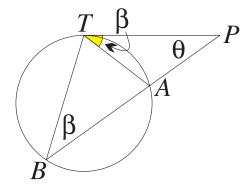
11. [The altitude to the hypotenuse of a right-angled triangle] Let $\triangle ABC$ be right-angled at C. Let CM be the altitude from C to the side AB.



(a) Show that $\triangle ABC \,|||\, \triangle ACM \,|||\, \triangle CBM$.
(b) Show that CM is the geometric mean of AM and BM.
(c) Show that BC is the geometric mean of AB and BM.

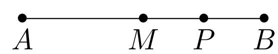
12. [The tangent as the GM of two secants] Let PT be a tangent from a point P outside a circle, touching the circle at T. Let PAB be a secant through P meeting the circle at A and B.



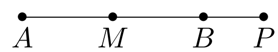
(a) Show that $\triangle PTA \,|||\, \triangle PBT$ (recall the alternate segment theorem that tells us that $\angle TBP = \angle PTA$).
(b) Use this similarity to show that PT is the geometric mean of PA and PB.

13. [Arithmetic means and midpoints] Let A and B be two distinct points, and let M be the midpoint of AB.

(a) Suppose that P is any point between A and B. Explain why AM is the arithmetic mean of AP and PB.



(b) Suppose now that P lies on AB, but beyond B. Where is the point X so that AX is the arithmetic mean of AP and PB?



14. Let $\triangle ABC$ be right-angled at C, so $a^2 + b^2 = c^2$. Find the ratio $c : a$ if:
(a) b is the AM of a and c, (b) [The golden mean] b is the GM of a and c.

15. [Geometric means in musical intruments] The pipe length in a modern rank of organ pipes decreases from left to right in such a way that the lengths form a GP, and the thirteenth pipe along is exactly half the length of the first pipe (making an interval called an *octave*).
(a) Show that the ratio of the GP is $r = (\frac{1}{2})^{\frac{1}{12}}$.
(b) Show that the 8th pipe along is just over two-thirds the length of the first pipe (this interval is called a *perfect fifth*).
(c) Show that the 5th pipe along is just under four-fifths the length of the first pipe (a *major third*).
(d) Find which pipes are about three-quarters (a *perfect fourth*) and five-sixths (a *minor third*) the length of the first pipe.
(e) What simple fractions are closest to the relative lengths of the third pipe (a *major second*) and the second pipe (a *minor second*)?

EXTENSION

16. [The golden mean] (a) The point M divides the interval AB in the ratio $1 : \lambda$ in such a way that AM is the geometric mean of BM and BA. Find λ, and draw a diagram.
(b) The point M divides the interval AB externally in the ratio $1 : \lambda$ in such a way that AB is the geometric mean of AM and BM. Find λ, and draw a diagram.

17. (a) Let $A(a, 2^a)$, $M(m, 2^m)$ and $B(b, 2^b)$ be three points on the curve $y = 2^x$. (i) Show that the x-coordinates form an AP if and only if the y-coordinates form a GP. (ii) Sketch $y = 2^x$, then use the fact that the chord AB lies above the curve $y = 2^x$

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to show that the geometric mean of two distinct positive numbers is less than their arithmetic mean.

(b) Let A, M and B be three points on the curve $y = \log_2 x$. Show that the x-coordinates form a GP if and only if the y-coordinates form an AP. Sketch $y = \log_2 x$, then use the fact that the chord AB lies below the curve $y = \log_2 x$ to show that the GM of two distinct positive numbers is less that their AM.

18. Let a and b be positive numbers, with $a < b$, and let m and g be the arithmetic and positive geometric means respectively of a and b.

(a) Show that g is the arithmetic mean of a and m if and only if $b = 9a$.

(b) Show that g is closer to a than to m if and only if $b > 9a$.

19. (a) Using the fact that the GM of two numbers cannot exceed their AM, prove that if a, b, c and d are any four positive numbers, then $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

(b) By letting $d = (abc)^{\frac{1}{3}}$ in part (a), prove that $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$.

6 G Sigma Notation

We turn now to the problem of adding up some of the terms of a sequence. For example, we may want to evaluate the sum

$$1 + 4 + 9 + \cdots + 100$$

of the first ten positive square numbers. The purpose of this section is to introduce a concise notation for such sums, called *sigma notation*.

Sigma Notation: The notation for the sum above is

$$\sum_{n=1}^{10} n^2 = 1 + 4 + 9 + \cdots + 81 + 100 = 385,$$

which says 'evaluate the function n^2 for all the integers from $n = 1$ to $n = 10$, then add up the resulting values', giving the final answer 385. More generally, if k and ℓ are integers and T_n is defined for all integers from $n = k$ to $n = \ell$, then:

15 DEFINITION: $$\sum_{n=k}^{\ell} T_n = T_k + T_{k+1} + T_{k+2} + \cdots + T_\ell$$

The symbol $\sum$ used here is a large version of the Greek capital letter called 'sigma', which is pronounced 's'. It stands for the word 'sum'.

WORKED EXERCISE: Evaluate: (a) $\displaystyle\sum^{7}(5n+1)$ (b) $\displaystyle\sum^{5} 3 \times (-2)^n$.

SOLUTION:

(a) $$\sum_{n=4}^{7}(5n+1) = 21 + 26 + 31 + 36 = 114$$

(b) $$\sum_{n=1}^{5} 3 \times (-2)^n = -6 + 12 - 24 + 48 - 96 = -66$$

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WORKED EXERCISE: Express the sum $\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}$ in sigma notation.

SOLUTION: $\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12} = \sum_{n=1}^{5} \frac{1}{2n+2}$, or $\sum_{n=2}^{6} \frac{1}{2n}$, or $\sum_{n=3}^{7} \frac{1}{2n-2}$.

There are many answers, depending on the initial value of n.

Exercise 6G

1. Rewrite each sum without sigma notation, and evaluate:

(a) $\sum_{n=1}^{6}(3n+2)$ (d) $\sum_{n=1}^{5}(n^2-n)$ (g) $\sum_{k=4}^{6} 5^{k-4}$ (j) $\sum_{\ell=1}^{31}(-1)^{\ell}$

(b) $\sum_{n=1}^{5} n^2$ (e) $\sum_{n=5}^{105} 4$ (h) $\sum_{n=0}^{4}(-1)^n n^2$ (k) $\sum_{\ell=1}^{31}(-1)^{\ell-1}$

(c) $\sum_{n=-2}^{2} n(n+1)$ (f) $\sum_{n=-4}^{4} n^3$ (i) $\sum_{n=0}^{4}(-1)^{n+1}n^2$ (l) $\sum_{a=1}^{4}(3^a-3^{a-1})$

DEVELOPMENT

2. Rewrite each sum in sigma notation, starting each sum at $n=1$ (do not evaluate):

(a) $1^3+2^3+3^3+\cdots+40^3$

(b) $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{40}$

(c) $3+4+5+\cdots+22$

(d) $2+2^2+2^3+\cdots+2^{12}$

(e) $1+2+2^2+\cdots+2^{12}$

(f) $a+ar+ar^2+\cdots+ar^{k-1}$

(g) $a+(a+d)+(a+2d)+\cdots+\big(a+(k-1)d\big)$

(h) $-1+2-3+\cdots+10$

(i) $1-2+3-\cdots-10$

(j) $1-x+x^2-x^3+\cdots+x^{2k}$

3. (a) By writing out the terms, show that $\sum_{r=1}^{6} r^3 = \sum_{t=5}^{10}(t-4)^3$.

(b) Show similarly by writing out the terms that $\sum_{k=1}^{5} \frac{3k-1}{k+2} = \sum_{k=4}^{8} \frac{3k-10}{k-1}$.

(c) Write $1+4+7+\cdots+19$ as: (i) $\sum_{n=0}^{6}\cdots$ (ii) $\sum_{n=2}^{8}\cdots$ (iii) $\sum_{n=7}^{\cdots}\cdots$

4. Write out the terms of $\sum_{r=1}^{10}\left(\frac{1}{r}-\frac{1}{r+1}\right)$, and hence show that the sum is $\frac{10}{11}$.

EXTENSION

5. (a) Show that $\dfrac{1}{(\sqrt{k}+\sqrt{k+1})(\sqrt[4]{k}+\sqrt[4]{k+1})} = \sqrt[4]{k+1}-\sqrt[4]{k}$.

(b) Hence evaluate $\sum_{k=1}^{255} \dfrac{1}{(\sqrt{k}+\sqrt{k+1})(\sqrt[4]{k}+\sqrt[4]{k+1})}$.

6. Evaluate: (a) $\frac{1}{2}\sum_{r=1}^{4}\left(\frac{1}{2}\sum_{s=1}^{4}\left(\frac{1}{2}\sum_{t=1}^{4} rst\right)\right)$ (b) $\sum_{r=1}^{4}\left(\sum_{s=1}^{4}\left(\sum_{t=1}^{4}(r-s)(s-t)(t-r)\right)\right)$

(c) $\sum_{n=1}^{6}\left(\prod_{k=1}^{n} k\right)$, where $\prod_{n=k}^{\ell} u_n = u_k \times u_{k+1} \times \cdots \times u_\ell$ (d) $\prod_{n=1}^{6}\left(\sum_{k=1}^{n} k\right)$

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6 H Partial Sums of a Sequence

The *nth partial sum* S_n of a sequence $T_1, T_2, T_3, T_4, \ldots$ is the sum of the first n terms.

16 THE nTH PARTIAL SUM: $S_n = T_1 + T_2 + T_3 + \cdots + T_n$

For example, the first ten partial sums of the sequence 1, 2, 4, 8, ... are:

T_n	1	2	4	8	16	32	64	128	256	512	$\cdots$
S_n	1	3	7	15	31	63	127	255	511	1023	$\cdots$

There may be a simple formula for the nth partial sum — in this example it should be reasonably clear that

$$S_n = 2^n - 1.$$

For most sequences, however, it will require somewhat more effort than this to arrive at the formula for the nth partial sum. Notice that the partial sums form a second sequence $S_1, S_2, S_3, \ldots$, although this will usually not concern us explicitly.

Recovering the Sequence from the Partial Sums: The partial sums S_n have a very simple recursive definition as follows:

$$S_1 = T_1 \qquad \text{and} \qquad S_n = S_{n-1} + T_n, \text{ for } n \ge 2,$$

because each partial sum is just the previous partial sum plus the next term. Rearranging these equations so that T_1 and T_n are the subjects gives a formula for T_n.

17 RECOVERING THE SEQUENCE: $T_1 = S_1$ and $T_n = S_n - S_{n-1}$, for $n \ge 2$.

The formula $T_n = S_n - S_{n-1}$ should also be understood as a subtraction:

$$T_n = (T_1 + T_2 + \cdots + T_n) - (T_1 + T_2 + \cdots + T_{n-1}).$$

These equations allow the original sequence to be recovered from the partial sums.

WORKED EXERCISE: Given that $S_n = n^2$, find a formula for the nth term.

SOLUTION: For $n \ge 2$,
$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= n^2 - (n-1)^2 \\ &= 2n - 1. \end{aligned}$$
Also $T_1 = S_1 = 1$, so T_1 satisfies this formula.
So $T_n = 2n - 1$, for all $n \ge 1$.

Since $T_n = 2n - 1$ is the formula for the nth odd cardinal, this particular example establishes the following well-known and important result (not an explicit part of our course).

THEOREM: The sum of the first n odd cardinals is n^2:

$$1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2, \text{ for } n \ge 1.$$

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Series: The word *series* is a rather imprecise term, but it always refers to the activity of adding up terms of a sequence. For example,

'the series $1 + 4 + 9 + \cdots + 81 + 100$'

means the expression giving the sum of the first 10 terms of the sequence of positive squares, and the value of this series is 385. One can also speak of

'the series $1 + 4 + 9 + \cdots$',

which means that one is considering the sequence of positive squares and their successive partial sums. In practice, the words 'series' and 'sequence' tend to be used interchangeably.

Exercise 6H

1. Copy and complete these tables of a sequence and its partial sums. Then describe each sequence:

(a)

T_n	2	5	8	11	14	17	20
S_n							

(b)

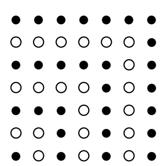
T_n							
S_n	2	6	14	30	62	126	254

2. The maximum numbers of electrons in the successive electron shells of an atom are 2, 8, 18, 32, ... By taking successive differences, make sense of these numbers as the partial sums of a simple series.

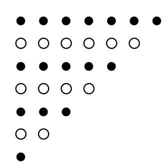
3. The nth partial sum of a series is $S_n = n^2 + 2n$. (a) Write out the first five partial sums.
(b) Take differences to write out the first five terms of the original sequence.
(c) Find S_{n-1}, then use the result $T_n = S_n - S_{n-1}$ to find a formula for T_n.

4. Repeat the steps of the previous question for the sequence whose nth partial sum is:
(a) $S_n = 4n - n^2$ (b) $S_n = 3n^2 - 5n$ (c) $S_n = 6n - 5n^2$

5. (a) Use the dot diagram on the right to explain why the sum of the first n odd positive integers is n^2.



(b) Use the dot diagram on the right to explain why the sum of the first n positive integers is $\frac{1}{2}n(n + 1)$.



DEVELOPMENT

6. The nth partial sum of a series is $S_n = 3^n - 1$. (a) Write out the first five partial sums.
(b) Take differences to find the first five terms of the original sequence.
(c) Find S_{n-1}, then use the result $T_n = S_n - S_{n-1}$ to find a formula for T_n.
[HINT: This will need the factorisation $3^n - 3^{n-1} = 3^{n-1}(3 - 1) = 2 \times 3^{n-1}$.]

7. Repeat the steps of the previous question for the sequence whose nth partial sum is:
(a) $S_n = 10(2^n - 1)$ (b) $S_n = 4(5^n - 1)$ (c) $S_n = \frac{1}{4}(4^n - 1)$
[HINT: You will need factorisations such as $2^n - 2^{n-1} = 2^{n-1}(2 - 1)$.]

8. Find the nth term and the first three terms of the sequence for which S_n is:
(a) $S_n = 3n(n + 1)$
(b) $S_n = \frac{1}{2}n^2 + \frac{3}{2}n$
(c) $S_n = 5n - n^2$
(d) $S_n = 4n$
(e) $S_n = n^3$
(f) $S_n = 1 - 3^{-n}$
(g) $S_n = (\frac{1}{7})^n - 1$
(h) $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$
(i) $S_n = \frac{1}{6}n(n + 1)(2n + 1)$
(j) $S_n = \frac{1}{4}n^2(n + 1)^2$
(k) $S_n = \dfrac{a(r^n - 1)}{r - 1}$

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EXTENSION

9. In these sequences, the first term will not necessarily obey the same rule as the succeeding terms, in which case the formula for the sequence will need to be given piecewise:

(a) $S_n = n^2 + 4n + 3$ (b) $S_n = 7(3^n - 4)$ (c) $S_n = \dfrac{1}{n}$ (d) $S_n = n^3 + n^2 + n$

Find T_1 and a formula for T_n for each sequence. How could you have predicted whether or not the general formula would hold for T_1?

10. [Fibonacci and Lucas sequences] Examine the sequence of differences between successive terms of the Fibonacci and Lucas sequences.

11. (a) Write down the fourth powers of the positive integers, form the new sequence of differences between successive terms, repeat the process with the new sequence, and continue the process until the resulting sequence is constant. Why is the result 24? What happens when this process is applied to the sequence of some other fixed powers of the integers?

(b) Apply this same repeated process to the sequence of positive integer powers of 2, or of 3, or of some other base. Examine the situation and justify what you observe.

6 I Summing an Arithmetic Series

There is a clever way to add up the terms of an arithmetic series. Here is an example of adding up the first ten terms of the AP with $a = 4$ and $d = 5$:

$$S_{10} = 4 + 9 + 14 + 19 + 24 + 29 + 34 + 39 + 44 + 49.$$

Reversing the sum, $S_{10} = 49 + 44 + 39 + 34 + 29 + 24 + 19 + 14 + 9 + 4,$

and adding the two, $2S_{10} = 53 + 53 + 53 + 53 + 53 + 53 + 53 + 53 + 53 + 53,$

$$2S_{10} = 10 \times 53 \quad (53 \text{ is the sum of } T_1 = 4 \text{ and } T_{10} = 49).$$

Hence $S_{10} = \frac{1}{2} \times 10 \times 53$
$= 265.$

This process can be done just as well with the general arithmetic series. Let the first term be a, the common difference be d, and the last term T_n be ℓ:

$$S_n = a + (a + d) + (a + 2d) + \cdots + (\ell - 2d) + (\ell - d) + \ell.$$

Reversing the sum, $S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + (a + 2d) + (a + d) + a,$

and adding, $2S_n = (a + \ell) + (a + \ell) + \cdots + (a + \ell) + (a + \ell) + (a + \ell)$

$$2S_n = n(a + \ell) \quad (\text{there are } n \text{ terms}),$$

hence $S_n = \frac{1}{2}n(a + \ell).$

Substituting $\ell = a + (n - 1)d$ gives a second equally useful form of this formula:

$$S_n = \tfrac{1}{2}n\Big(2a + (n - 1)d\Big).$$

Method for Summing an AP: The two formulae to remember are:

18 PARTIAL SUMS OF APS: $S_n = \frac{1}{2}n(a + \ell)$ (use when $\ell = T_n$ is known)
$S_n = \frac{1}{2}n\Big(2a + (n - 1)d\Big)$ (use when d is known)

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WORKED EXERCISE: Add up all the integers from 100 to 200 inclusive.

SOLUTION: The sum $100 + 101 + \cdots + 200$ is an AP with 101 terms, in which the first term is $a = 100$ and the last term is $\ell = 200$.

$$\begin{aligned} \text{So}\quad S_{101} &= \tfrac{1}{2}n(a+\ell) \\ &= \tfrac{1}{2} \times 101 \times 300 \\ &= 15\,150. \end{aligned}$$

WORKED EXERCISE: (a) Given the AP $40+37+34+\cdots$, find S_{10} and an expression for S_n. (b) What is the first negative partial sum?

SOLUTION:

(a) Here $a = 40$ and $d = -3$,

$$\begin{aligned} \text{so}\quad S_{10} &= 5(2a+9d) \\ &= 5 \times (80-27) \\ &= 265, \\ \text{and}\quad S_n &= \tfrac{1}{2}n\big(2a+(n-1)d\big) \\ &= \tfrac{1}{2}n\big(80-3(n-1)\big) \\ &= \tfrac{1}{2}n(83-3n). \end{aligned}$$

(b) Put $S_n < 0$.

Then $\tfrac{1}{2}n(83-3n) < 0$.

Since n must be positive,

$$\begin{aligned} 83-3n &< 0 \\ n &> 27\tfrac{2}{3}. \end{aligned}$$

S_{28} is the first negative partial sum, and by the formula, $S_{28} = -14$.

WORKED EXERCISE: The sum of the first ten terms of an AP is zero, and the sum of the first and second terms is 24. Find the first three terms.

SOLUTION:

$$\begin{aligned} \text{First,}\quad S_{10} &= 0 \\ 5(2a+9d) &= 0 \\ 2a+9d &= 0. \qquad (1) \\ \text{Secondly,}\quad a+(a+d) &= 24 \\ 2a+d &= 24. \qquad (2) \end{aligned}$$

$$\begin{aligned} (1)-(2)\qquad 8d &= -24 \\ d &= -3, \\ \text{so from (2),}\quad 2a-3 &= 24 \\ a &= 13\tfrac{1}{2}. \end{aligned}$$

Hence the AP is $13\frac{1}{2} + 10\frac{1}{2} + 7\frac{1}{2} + \cdots$.

Exercise 6I

1. Let $S_{10} = 5+8+11+14+17+20+23+26+29+32$. By reversing the sum and adding in columns, evaluate S_{10}.

2. Use the formula $S_n = \frac{1}{2}n\big(2a+(n-1)d\big)$ to find these sums:

(a) $2+5+8+\cdots$ (12 terms)
(b) $40+33+26+\cdots$ (21 terms)
(c) $-6-2+2+\cdots$ (200 terms)
(d) $33+30+27+\cdots$ (23 terms)
(e) $-10-7\frac{1}{2}-5+\cdots$ (13 terms)
(f) $10\frac{1}{2}+10+9\frac{1}{2}+\cdots$ (40 terms)

3. First use the formula $T_n = a+(n-1)d$ to find the number of terms in each sum. Then find the sum using the formula $S_n = \frac{1}{2}n(a+\ell)$, where ℓ is the last term T_n:

(a) $50+51+52+\cdots+150$
(b) $8+15+22+\cdots+92$
(c) $-10-3+4+\cdots+60$
(d) $4+7+10+\cdots+301$
(e) $6\frac{1}{2}+11+15\frac{1}{2}+\cdots+51\frac{1}{2}$
(f) $-1\frac{1}{3}+\frac{1}{3}+2+\cdots+13\frac{2}{3}$

4. Find these sums by any appropriate method:

(a) $2+4+6+\cdots+1000$
(b) $1000+1001+\cdots+3000$
(c) $1+5+9+\cdots$ (40 terms)
(d) $10+30+50+\cdots$ (12 terms)

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5. Find the formula for the nth partial sums of the series:

(a) $3 + 7 + 11 + \cdots$

(b) $-9 - 4 + 1 + \cdots$

(c) $5 + 4\frac{1}{2} + 4 + \cdots$

(d) $(1 - \sqrt{2}\,) + 1 + (1 + \sqrt{2}) + \cdots$

6. Find formulae for the sums of the first n:

(a) positive integers,

(b) odd positive integers,

(c) positive integers divisible by 3,

(d) odd positive multiples of 100.

7. (a) How many legs are there on 15 fish, 15 ducks, 15 dogs, 15 beetles, 15 spiders, and 15 ten-legged grubs? How many of these creatures have the mean number of legs?

(b) A school has 1560 pupils, with equal numbers of each age from 6 to 17 years inclusive. It also has 120 teachers and ancilliary staff all aged 32 years, and one Principal aged 55 years. What is the total of the ages of everyone in the school?

(c) A graduate earns \$28 000 per annum in her first year, then each successive year her salary rises by \$1600. What are her total earnings over ten years?

DEVELOPMENT

8. Find these sums:

(a) $x + 2x + 3x + \cdots + nx$

(b) $3 + (3 + d) + (3 + 2d) + \cdots$ (20 terms)

(c) $a + (a - 5) + (a - 10) + \cdots + (a - 100)$

(d) $3b + 5b + 7b + \cdots$ (200 terms)

(e) $(1 + \sqrt{2}\,) + (2 + 3\sqrt{2}\,) + (3 + 5\sqrt{2}\,) + \cdots$ (12 terms)

(f) $\sqrt{12} + \sqrt{27} + \sqrt{48} + \cdots + 21\sqrt{3}$

9. (a) Show that the nth partial sum of the series $60 + 52 + 44 + 36 + \cdots$ is $S_n = 4n(16 - n)$.

(b) Hence find how many terms must be taken to make the sum: (i) zero, (ii) negative.

(c) Find the two values of n for which the partial sum S_n is 220.

(d) Show that $S_n = -144$ has two integer solutions, but that only one has meaning.

(e) For what values of n does the partial sum S_n exceed 156?

(f) Prove that no partial sum can exceed 256.

(g) Write out the first 16 terms and partial sums, and check your results.

10. (a) Prove that the sum of the first n positive integers is $S_n = \frac{1}{2}n(n + 1)$.

(b) Find n if the sum is: (i) 6 (ii) 55 (iii) 820

(c) How many terms must be taken for the sum to exceed 210?

(d) Show that the sum can never be 50.

11. (a) Logs of wood are stacked with 10 on the top row, 11 on the next, and so on. If there are 390 logs, find the number of rows, and the number of logs on the bottom row.

(b) A stone dropped from the top of a 245 metre cliff falls 5 metres in the first second, 15 metres in the second second, and so on in arithmetic sequence. Find a formula for the distance after n seconds, and find how long the stone takes to fall to the ground.

(c) A truck spends the day depositing truckloads of gravel from a quarry at equally spaced intervals along a straight road. The first load is deposited 20 km from the quarry, the last is 10 km further along the road. If the truck travels 550 km during the day, how many trips does it make, and how far apart are the deposits?

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12. Find the sums of these APs whose terms are logarithms:

(a) $\log_a 2 + \log_a 4 + \log_a 8 + \cdots + \log_a 1024$

(b) $\log_5 243 + \log_5 81 + \log_5 27 + \cdots + \log_5 \frac{1}{243}$

(c) $\log_b 36 + \log_b 18 + \log_b 9 + \cdots + \log_b \frac{9}{8}$

(d) $\log_x \frac{27}{8} + \log_x \frac{9}{4} + \log_x \frac{3}{2} + \cdots$ (10 terms)

13. (a) Find the common difference, if a series with 8 terms and first term 5 has sum 348.

(b) Find the last term, if a series with 10 terms and first term -23 has sum -5.

(c) Find the first term, if a series with 40 terms and last term $8\frac{1}{2}$ has sum 28.

(d) Find the first term, if a series with 15 terms and difference $\frac{2}{7}$ has sum -15.

(e) The sum of the first and fourth terms of an AP is 16, and the sum of the third and eighth terms is 4. Find the sum of the first ten terms.

(f) The tenth partial sum of an AP is zero, and the tenth term is -9. Find the first and second terms.

(g) The sum to 16 terms of an AP is 96, and the sum of the second and fourth terms is 45. Find the fourth term, and show the sum to four terms is also 96.

14. (a) Prove that if the tenth and twentieth partial sums of an AP are equal, then the thirtieth partial sum must be zero.

(b) Prove that if the twelfth partial sum of an AP is twice the sixth partial sum, then the sequence is a constant sequence.

(c) Find the first term and common difference of an AP in which the sum to ten terms is three times the sum to four terms, and the 28th term is -81.

(d) Find n, if the sum of the first n terms of the series $48 + 44 + 40 + \cdots$ equals the sum of the first n terms of the series $-1 + 2 + 5 + \cdots$.

15. (a) Insert 9 arithmetic means between 29 and 109, then find their sum.

(b) Show that the sum of n arithmetic means inserted between a and b is $\frac{1}{2}n(a+b)$.

(c) Find n, if n arithmetic means inserted between 10 and 82 have sum 506.

(d) How many arithmetic means must be inserted between 1 and 2 if their sum exceeds 1 000 000?

16. (a) Use the formula $S_n = \frac{1}{2}n(a+\ell)$ to simplify $S_n = \sum_{k=1}^{n}(44-2k)$, and find n if $S_n = 0$.

(b) Solve similarly: (i) $\sum_{k=1}^{n}(63-3k) = 0$ (ii) $\sum_{k=1}^{n}(39+6k) = 153$ (iii) $\sum_{r=1}^{n}(2+\frac{1}{2}r) = 22\frac{1}{2}$

17. (a) (i) Find the sum of all positive multiples of 3 less than 300.

(ii) Find the sum of all the other positive integers less than 300.

(b) What is the sum of all numbers ending in 5 between 1000 and 2000?

(c) What is the sum of all numbers ending in 2 or 9 between 1000 and 2000?

(d) How many multiples of 7 lie between 250 and 2500, and what is their sum?

18. Find the first term and the number of terms if a series has:

(a) $d = 4$, $\ell = 32$ and $S_n = 0$ (b) $d = -3$, $\ell = -10$ and $S_n = 55$

19. (a) Find $1 + 2 + \cdots + 24$. (b) Show that $\dfrac{1}{n} + \dfrac{2}{n} + \cdots + \dfrac{n}{n} = \dfrac{n+1}{2}$.

(c) Hence find the sum of the first 300 terms of $\frac{1}{1} + \frac{1}{2} + \frac{2}{2} + \frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} + \cdots$.

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EXTENSION

20. (a) Show that $\dfrac{1}{r(r+1)} = \dfrac{1}{r} - \dfrac{1}{r+1}$, for all $r \geq 1$, and hence, by writing out the first few terms, evaluate $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)}$.

(b) Use similar methods to evaluate $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)}$ and $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$.

6 J Summing a Geometric Series

The method used to find a partial sum of an AP will not work for a GP. There is, however, another equally clever way available. Suppose that a GP has first term a and common ratio r.

Let $$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiplying both sides by the ratio r,

$$rS_n = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \qquad (2)$$

Taking (2) − (1), $\quad (r-1)S_n = ar^n - a$

and provided $r \neq 1$, $\quad S_n = \dfrac{a(r^n - 1)}{r - 1}$.

Taking opposites of numerator and denominator gives an alternative form:

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Method for Summing a GP: Both forms of the formula are useful, depending on whether the ratio is greater or less than 1.

19 PARTIAL SUMS OF GPS: $S_n = \dfrac{a(r^n - 1)}{r - 1}$ (easier when $r > 1$)

$S_n = \dfrac{a(1 - r^n)}{1 - r}$ (easier when $r < 1$)

WORKED EXERCISE:

(a) Find the sum of all the powers of 5 from 5^0 to 5^7.

(b) Find the sixth partial sum of the GP $2 - 6 + 18 - \cdots$.

SOLUTION:

(a) The sum $5^0 + 5^1 \cdots + 5^7$ is a GP with 8 terms, with $a = 1$ and $r = 5$.

So $S_8 = \dfrac{a(r^8 - 1)}{r - 1}$ (here $r > 1$)

$= \dfrac{1 \times (5^8 - 1)}{5 - 1}$

$= 97\,656$.

(b) The series is a GP in which $a = 2$ and $r = -3$.

So $S_6 = \dfrac{a(1 - r^6)}{1 - r}$ (here $r < 1$)

$= \dfrac{2 \times \left(1 - (-3)^6\right)}{1 + 3}$

$= -364$.

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WORKED EXERCISE: [A harder example] How many terms of the GP $2+6+18+\cdots$ must be taken for the partial sum to exceed one billion?

SOLUTION:

Here $a = 2$ and $r = 3$,
so the nth partial sum is

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{2(3^n - 1)}{3 - 1} = 3^n - 1.$$

Put $S_n > 1\,000\,000\,000$.

Then $3^n - 1 > 1\,000\,000\,000$

$$3^n > 1\,000\,000\,001$$

$$n > \frac{\log_{10} 1\,000\,000\,001}{\log_{10} 3}$$

$$n > 18{\cdot}86\ldots,$$

so S_{19} is the first sum over one billion (or use trial and error).

Two Exceptional Cases: There are two particular values of the common ratio that need special attention, namely 0 and 1. First, according to our definition of a GP, the ratio cannot ever be zero, for then the second and third terms would be zero and the quotient T_3/T_2 would be undefined.

Secondly, if the ratio is 1, then the formula above for S_n doesn't work, because the denominator $r - 1$ would be zero. All the terms, however, are equal to the first term a, and so the formula for the nth partial sum is just $S_n = an$.

Exercise 6J

1. Let $S_7 = 2+6+18+54+162+486+1458$. By taking $3S_7$ and subtracting S_7 in columns, evaluate S_7.

2. 'As I was going to St Ives, I met a man with seven wives. Each wife had seven sacks, each sack had seven cats, each cat had seven kits. Kits, cats, sacks and wives, how many were going to St Ives?' Only the speaker was going to St Ives, but how many were going the other way?

3. Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ (when $r > 1$) or $S_n = \dfrac{a(1 - r^n)}{1 - r}$ (when $r < 1$) to find these sums, then find a formula for the sum to n terms:

(a) $1 + 2 + 4 + 8 + \cdots$ (10 terms)
(b) $1 - 2 + 4 - 8 + \cdots$ (10 terms)
(c) $2 + 6 + 18 + \cdots$ (5 terms)
(d) $2 - 6 + 18 - \cdots$ (5 terms)
(e) $8 + 4 + 2 + \cdots$ (10 terms)
(f) $8 - 4 + 2 - \cdots$ (10 terms)
(g) $9 + 3 + 1 + \cdots$ (6 terms)
(h) $9 - 3 + 1 - \cdots$ (6 terms)
(i) $45 + 15 + 5 + \cdots$ (6 terms)
(j) $-1 - 10 - 100 - \cdots$ (5 terms)
(k) $-1 + 10 - 100 + \cdots$ (5 terms)
(l) $\frac{2}{3} + 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8}$

4. Find an expression for S_n. Hence approximate S_{10} to four significant figures:

(a) $1 + 1{\cdot}2 + (1{\cdot}2)^2 + \cdots$
(b) $1 + 0{\cdot}95 + (0{\cdot}95)^2 + \cdots$
(c) $1 + 1{\cdot}01 + (1{\cdot}01)^2 + \cdots$
(d) $1 + 0{\cdot}99 + (0{\cdot}99)^2 + \cdots$

5. The King takes a chessboard of 64 squares, and places 1 grain of wheat on the first square, 2 on the next, 4 on the next and so on.

(a) How many grains are on: (i) the last square (ii) the whole chessboard?
(b) Given that 1 litre of wheat contains about 30 000 grains, how many cubic kilometres of wheat are on the chessboard?

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DEVELOPMENT

6. Find the sum to n terms of each series, where c, x and y are constants:

(a) $cx + 3cx^2 + 9cx^3 + \cdots$

(b) $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \cdots$

(c) $cx - 3cx^2 + 9cx^3 - \cdots$

(d) $1 + \dfrac{x}{y} + \dfrac{x^2}{y^2} + \cdots$

7. Find S_n and S_{10}, rationalising denominators:

(a) $1 + \sqrt{2} + 2 + \cdots$

(b) $\dfrac{1}{5} - \dfrac{1}{\sqrt{5}} + 1 - \cdots$

8. (a) Find: (i) $\displaystyle\sum_{n=3}^{8} 3^{n-4}$ (ii) $\displaystyle\sum_{n=3}^{8} \log_a 3^{n-4}$ (iii) $\displaystyle\sum_{n=1}^{8} 3 \times 2^{3-n}$

(b) Insert 3 geometric means between $\frac{1}{8}$ and 162, then find their sum.

9. (a) Show that the nth partial sum of the series $7 + 14 + 28 + \cdots$ is $S_n = 7(2^n - 1)$.

(b) For what value of n is the partial sum equal to 1785?

(c) Show that $T_n = 7 \times 2^{n-1}$, and find how many terms are less than 70 000.

(d) Use trial and error to find the first partial sum greater than 70 000.

(e) Prove that the nth partial sum is always 7 less than the $(n + 1)$th term.

10. The powers of 3 greater than 1 form a GP 3, 9, 27, ...

(a) Find using logs how many powers of 3 there are between 2 and 10^{20}.

(b) Show that $S_n = \frac{3}{2}(3^n - 1)$, and find how many terms must be added for the sum to exceed 10^{20}.

11. (a) Each year when a paddock is weeded, only half the previous weight of weed is dug out. In the first year, 6 tonnes of weed is dug out. (i) How much is dug out in the tenth year? (ii) What is the total dug out over the ten years (to four significant figures)?

(b) Every two hours, half of a particular medical isotope decays. If there was originally 20 g, how much remains after a day (to two significant figures)?

(c) The price of shoes is increasing with inflation over a ten-year period by 10% per annum, so that the price in each of those ten years is P, $1{\cdot}1P$, $(1{\cdot}1)^2P$, ... I buy one pair of these shoes each year.

(i) Find an expression for the total amount I pay over the ten years.

(ii) Hence find the initial price P (to the nearest cent) if the total paid is \$900.

12. The number of people attending the yearly Abletown Show is rising by 5% per annum, and the number attending the yearly Bush Creek Show is falling by 5% per annum. In the first year under consideration, 5000 people attended both shows.

(a) Find the total number attending each show during the first six years.

(b) Show that the number attending the Abletown Show first exceeds ten times the number attending the Bush Creek Show in the 25th year.

(c) What is the ratio (to three significant figures) of the total number attending the Abletown Show over these 25 years to the total attending the Bush Creek Show?

13. Find the nth terms of the sequences:

(a) $\dfrac{2}{1}, \dfrac{2+4}{1+3}, \dfrac{2+4+6}{1+3+5}, \ldots$

(b) $\dfrac{1}{1}, \dfrac{1+2}{1+4}, \dfrac{1+2+4}{1+4+16}, \ldots$

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14. (a) Show that the nth partial sum of the series $4 - 12 + 36 - \cdots$ is $S_n = \big(1 - (-3)^n\big)$.

(b) For what value of n is the partial sum equal to -728?

(c) What is the last term with absolute value less than $1\,000\,000$?

(d) Find the first partial sum with absolute value greater than $1\,000\,000$.

15. Show that the formula for the nth partial sum of a GP can also be written independently of n, in terms only of a, r and the last term $\ell = T_n = ar^{n-1}$, as

$$S_n = \frac{r\ell - a}{r - 1} \qquad \text{or} \qquad S_n = \frac{a - r\ell}{1 - r}\,.$$

(a) Hence find: (i) $1 + 2 + 4 + \cdots + 1\,048\,576$ (ii) $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{2187}$

(b) Find n and r if $a = 1$, $\ell = 64$, $S_n = 85$.

(c) Find ℓ and n if $a = 5$, $r = -3$, $S_n = -910$.

16. (a) Show that in any GP, $S_{2n} : S_n = (r^n + 1) : 1$. Hence find the common ratio of the GP if $S_{12} : S_6 = 65 : 1$.

(b) Show that if S_n and Σ_n are the sums to n terms of GPs with ratios r and r^2 respectively, but the same first term, then $\Sigma_n : S_n = (r^n + 1) : (r + 1)$.

(c) In any GP, let $R_n = T_{n+1} + T_{n+2} + \cdots + T_{2n}$. Show that $R_n : S_n = r^n : 1$, and hence find r if $R_8 : S_8 = 1 : 81$.

17. (a) The sequence $T_n = 2 \times 3^n + 3 \times 2^n$ is the sum of two GPs. Find S_n.

(b) The sequence $T_n = 2n + 3 + 2^n$ is the sum of an AP and a GP. Use a combination of AP and GP formulae to find S_n.

(c) It is given that the sequence 10, 19, 34, 61, ... has the form $T_n = a + nd + b2^n$, for some values of a, d and b. Find these values, and hence find S_n.

EXTENSION

18. Given a GP in which $T_1 + T_2 + \cdots + T_{10} = 2$ and $T_{11} + T_{12} + \cdots + T_{30} = 12$, find $T_{31} + T_{32} + \cdots + T_{60}$.

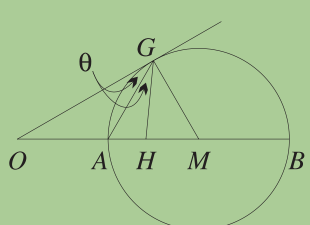
19. Show that if n geometric means are inserted between 1 and 2, then their sum is given by $S_n = \dfrac{1}{2^{\frac{1}{n+1}} - 1} - 1$. Show that $S_n \to \infty$ as $n \to \infty$, and find how many means must be inserted for the sum to be at least $1\,000$.

20. [The harmonic mean] The *harmonic mean* of two positive numbers a and b is the number h such that $1/h$ is the arithmetic mean of $1/a$ and $1/b$.

(a) Show that $h = \dfrac{2ab}{a+b}$ and $\dfrac{b-h}{h-a} = \dfrac{b}{a}$.

(b) Given a line $OAHB$, show that OH is the harmonic mean of OA and OB if and only if H divides AB internally in the same ratio as O divides AB externally.

(c) Given a line OAB, construct the circle with diameter AB, construct the centre M, and construct a tangent from O touching the circle at G. Construct H between A and B so that $\angle OGA = \angle HGA$. Show that OM, OG and OH are respectively the arithmetic, geometric and harmonic means of OA and OB. [HINT: Use the sine rule to show that $OG : GH = OA : AH = OB : BH$.]



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6 K The Limiting Sum of a Geometric Series

There is a sad story of a perishing frog, dying of thirst only 8 metres from the edge of a waterhole. He first jumps 4 metres towards it, his second jump is 2 metres, then each successive jump is half the previous jump. Does the frog perish?

The jumps form a GP, whose terms and partial sums are as follows:

T_n	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\ldots$
S_n	4	6	7	$7\frac{1}{2}$	$7\frac{3}{4}$	$7\frac{7}{8}$	$7\frac{15}{16}$	$\ldots$

The successive jumps have limit zero, meaning they get 'as close as we like' to zero. It seems too that the successive partial sums have limit 8, meaning that the frog's total distance gets 'as close as we like' to 8 metres. So provided the frog can stick his tongue out even the merest fraction of a millimetre, eventually he will get some water to drink and be saved.

The General Case: Suppose now that T_n is a GP with first term a and ratio r, so that

$$T_n = ar^{n-1} \qquad \text{and} \qquad S_n = \frac{a(1-r^n)}{1-r}.$$

A. When $r > 1$ or $r < -1$, then r^n increases in size without bound. This means that there is no limit for the nth term, and no limit for the nth partial sum. For example, if the ratio is 2 or -2, then the terms and partial sums are:

For $r = 2$:

T_n	a	$2a$	$4a$	$8a$	$16a$	$\ldots$
S_n	a	$3a$	$7a$	$15a$	$31a$	$\ldots$

For $r = -2$:

T_n	a	$-2a$	$4a$	$-8a$	$16a$	$\ldots$
S_n	a	$-a$	$3a$	$-5a$	$11a$	$\ldots$

B. When $r = 1$ the terms are all the same, and when $r = -1$ the terms have the same size but alternate in sign. Again the partial sums do not have a limit:

For $r = 1$:

T_n	a	a	a	a	a	$\ldots$
S_n	a	$2a$	$3a$	$4a$	$5a$	$\ldots$

For $r = -1$:

T_n	a	$-a$	a	$-a$	a	$\ldots$
S_n	a	0	a	0	a	$\ldots$

C. When $-1 < r < 1$, however, $\quad r^n \to 0$ as $n \to \infty$,
and so $\quad 1 - r^n \to 1$ as $n \to \infty$.
Hence as n tends to infinity, both the nth term T_n and the nth partial sum S_n tend to a limit, or as we also say, they *converge* to a limit:

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} ar^{n-1} = 0, \qquad \text{and} \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

The new notation $\lim\limits_{n\to\infty} T_n = 0$ means that $T_n \to 0$ as $n \to \infty$.

Similarly, $\lim\limits_{n\to\infty} S_n = \dfrac{a}{1-r}$ means that $S_n \to \dfrac{a}{1-r}$ as $n \to \infty$.

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For example, if $r = \frac{1}{2}$, then $\displaystyle\lim_{n\to\infty} S_n = \frac{a}{1-\frac{1}{2}} = 2a$,

and if $r = -\frac{1}{2}$, then $\displaystyle\lim_{n\to\infty} S_n = \frac{a}{1+\frac{1}{2}} = \frac{2}{3}a$:

For $r = \frac{1}{2}$:

T_n	a	$\frac{1}{2}a$	$\frac{1}{4}a$	$\frac{1}{8}a$	$\frac{1}{16}a$	$\to 0$
S_n	a	$\frac{3}{2}a$	$\frac{7}{4}a$	$\frac{15}{8}a$	$\frac{31}{16}a$	$\to 2a$

For $r = -\frac{1}{2}$:

T_n	a	$-\frac{1}{2}a$	$\frac{1}{4}a$	$-\frac{1}{8}a$	$\frac{1}{16}a$	$\to 0$
S_n	a	$\frac{1}{2}a$	$\frac{3}{4}a$	$\frac{5}{8}a$	$\frac{11}{16}a$	$\to \frac{2}{3}a$

To summarise all this in a single statement:

20 LIMITING SUMS OF GEOMETRIC SERIES:

The partial sums S_n converge to a limit if and only if $-1 < r < 1$.

The value of the limit is $S_\infty = \dfrac{a}{1-r}$.

WORKED EXERCISE: Explain why these series have limiting sums and find them:

(a) $18 - 6 + 2 - \cdots$

(b) $2 + \sqrt{2} + 1 + \cdots$

SOLUTION:

(a) Here $a = 18$ and $r = -\frac{1}{3}$.
Since $-1 < r < 1$,
we know that the series converges.

$$S_\infty = \frac{18}{1+\frac{1}{3}} = 18 \times \frac{3}{4} = 13\tfrac{1}{2}$$

(b) Here $a = 1$ and $r = \frac{1}{2}\sqrt{2}$.
Since $-1 < r < 1$,
we know that the series converges.

$$S_\infty = \frac{2}{1-\frac{1}{2}\sqrt{2}} \times \frac{1+\frac{1}{2}\sqrt{2}}{1+\frac{1}{2}\sqrt{2}} = \frac{2+\sqrt{2}}{1-\frac{1}{2}} = 4 + 2\sqrt{2}$$

WORKED EXERCISE: For what values of x does $1 + (x-2) + (x-2)^2 + \cdots$ converge, and what is the limiting sum?

SOLUTION:

The GP converges when

$$-1 < r < 1$$
$$-1 < x - 2 < 1$$
$$\boxed{+2} \quad 1 < x < 3.$$

The limiting sum is then

$$S_\infty = \frac{1}{1-(x-2)} = \frac{1}{3-x}.$$

The Notation for Infinite Sums: When $-1 < r < 1$ and the GP converges, the limiting sum S_∞ can also be written as an infinite sum, either using sigma notation or using dots, so that

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{or} \quad a + ar + ar^2 + \cdots = \frac{a}{1-r},$$

and we say that 'the series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ converges to $\dfrac{a}{1-r}$'.

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Exercise **6K**

1. Copy and complete the table of values for the GP with $a = 18$ and $r = \frac{1}{3}$. Then find the limiting sum S_∞, and the difference $S_\infty - S_6$.

T_n	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$
S_n						

2. Test whether these GPs have limiting sums, and find them if they do:

(a) $1 + \frac{1}{2} + \frac{1}{4} + \cdots$

(b) $1 - \frac{1}{2} + \frac{1}{4} - \cdots$

(c) $12 + 4 + \frac{4}{3} + \cdots$

(d) $1 - 1 + 1 - \cdots$

(e) $100 + 90 + 81 + \cdots$

(f) $-2 + \frac{2}{5} - \frac{2}{25} + \cdots$

(g) $-\frac{2}{3} - \frac{2}{15} - \frac{2}{75} - \cdots$

(h) $1 + (1{\cdot}01) + (1{\cdot}01)^2 + \cdots$

(i) $1 - 0{\cdot}99 + (0{\cdot}99)^2 - \cdots$

(j) $1 + (1{\cdot}01)^{-1} + (1{\cdot}01)^{-2} + \cdots$

(k) $0{\cdot}72 - 0{\cdot}12 + 0{\cdot}02 - \cdots$

(l) $16\sqrt{5} + 4\sqrt{5} + \sqrt{5} + \cdots$

3. Find the value of x, given the limiting sums of these GPs:

(a) $5 + 5x + 5x^2 + \cdots = 10$ (b) $5 + 5x + 5x^2 + \cdots = 3$ (c) $5 - 5x + 5x^2 - \cdots = 15$

4. Find the value of a, given the limiting sums of these GPs:

(a) $a + \dfrac{a}{3} + \dfrac{a}{9} + \cdots = 2$ (b) $a - \dfrac{a}{3} + \dfrac{a}{9} - \cdots = 2$ (c) $a + \frac{2}{3}a + \frac{4}{9}a + \cdots = 2$

5. Find the condition for each GP to have a limiting sum, then find that limiting sum:

(a) $1 + (x-1) + (x-1)^2 + \cdots$

(b) $1 + (1+x) + (1+x)^2 + \cdots$

(c) $1 + (3x-2) + (3x-2)^2 + \cdots$

(d) $1 - (3x+2) + (3x+2)^2 - \cdots$

DEVELOPMENT

6. Find the limiting sums if they exist, rationalising denominators:

(a) $7 + \sqrt{7} + 1 + \cdots$

(b) $4 - 2\sqrt{2} + 2 - \cdots$

(c) $5 - 2\sqrt{5} + 4 - \cdots$

(d) $9 + 3\sqrt{10} + 10 + \cdots$

(e) $1 + (1-\sqrt{3}\,) + (1-\sqrt{3}\,)^2 + \cdots$

(f) $1 + (2-\sqrt{3}\,) + (2-\sqrt{3}\,)^2 + \cdots$

(g) $(\sqrt{5}+1) + 2 + (\sqrt{5}-1) + \cdots$

(h) $(\sqrt{5}-1) + 2 + (\sqrt{5}+1) + \cdots$

7. When a council offers free reflective house numbers, 30% of residents install them in the first month, the numbers in the second month are only 30% of those in the first month, and so on. What proportion of residents eventually install them?

8. A bouncy ball drops from a height of 9 metres and bounces continually, each successive height being $\frac{2}{3}$ of the previous height. (a) Show that the first distance travelled down-and-up is 15 metres, and show that the successive down-and-up distances form a GP. (b) Through what distance does the ball eventually travel?

9. Verify the convergence of each of the following series, then find the limit:

(a) $\displaystyle\sum_{n=1}^{\infty} 7 \times \left(\tfrac{1}{2}\right)^{n-1}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \times \tfrac{25}{4} \times \left(\tfrac{4}{25}\right)^n$ (c) $\displaystyle\sum_{n=0}^{\infty} 4 \times 5^{-n} + 5 \times 4^{-n}$

10. For the GP $\left(\sqrt{5}+\sqrt{3}\right) + \left(\sqrt{5}-\sqrt{3}\right) + \cdots$, verify that $S_\infty = T_1 + \frac{1}{3}\sqrt{3}$.

11. Suppose that $T_n = ar^{n-1}$ is a GP with a limiting sum.

(a) Find the common ratio r if the limiting sum equals 5 times the first term.

(b) Find the first three terms if the second term is 6 and the limiting sum is 27.

(c) Find the ratio if the sum of all terms except the first equals 5 times the first term.

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(d) Show that the sum S of all terms from the third on is $\dfrac{ar^2}{1-r}$. Hence find r if S equals:
(i) the first term, (ii) the second term, (iii) the sum of the first and second terms.

(e) Find the ratio r if the sum of the first three terms equals half the limiting sum.

12. (a) Suppose that $a+ar+ar^2+\cdots$ is a GP with limiting sum. Show that the four sequences

$$a+ar+ar^2+\cdots\,, \quad a-ar+ar^2+\cdots\,, \quad a+ar^2+ar^4+\cdots\,, \quad ar+ar^3+ar^5+\cdots\,,$$

are all GPs, and that their limiting sums are in the ratio $1+r : 1-r : 1 : r$.

(b) Find the limiting sums of these four GPs, and verify the ratio proven above:
(i) $48+24+12+\cdots$ (ii) $48-24+12+\cdots$ (iii) $48+12+\cdots$ (iv) $24+6+\cdots$

13. Each spring, fresh flowers are gathered from a patch of bush. Each annual yield, however, is only 90% of the previous year's yield. (a) Find the ratio between the first year's yield and the total yield, if the gathering continues indefinitely into the future. (b) Find also in which year the annual yield will first drop to less than 1% of the first year's yield.

14. A clever new toy comes onto the market, and sells 20 000 units in the first month. Popularity wanes, and each month the sales are only 70% of the sales in the previous month.

(a) How many units are sold eventually?

(b) What proportion are sold in the first 6 months?

(c) In which month will monthly sales first drop below 500 per month?

(d) What proportion are sold before this month?

15. (a) Show that a GP has a limiting sum if $0 < 1-r < 2$.

(b) By calculating the common ratio, show that there is no GP with first term 8 and limiting sum 2.

(c) A GP has positive first term a, and has a limiting sum S_∞. Show that $S_\infty > \frac{1}{2}a$.

(d) Find the range of values of the limiting sum of a GP with:
(i) $a=6$ (ii) $a=-8$ (iii) $a>0$ (iv) $a<0$

16. Suppose that $T_n = ar^{n-1}$ is a GP with limiting sum S_∞. For any value of n, define the *defect* D_n to be the difference $D_n = S_\infty - S_n$ between the limiting sum and the nth partial sum.

(a) Show that $D_n = r^n S_\infty$, show that it is a GP, find its first term and common ratio, and prove that it converges to zero.

(b) Find the defect D_n for the GP $18+6+2+\cdots$, find the defect if 5 terms are taken, and find how many terms must be taken for the defect to be less than $\frac{1}{1\,000\,000}$.

(c) How many terms of the sequence $75, 15, 3, \cdots$ must be taken for the partial sum of those terms to differ from the limiting sum by less than $\frac{1}{100\,000}$?

17. Find the condition for each GP to have a limiting sum, then find that limiting sum:

(a) $1+(x^2-1)+(x^2-1)^2+\cdots$

(b) $1-(2-x^2)+(2-x^2)^2-\cdots$

(c) $1+\dfrac{1}{5x}+\dfrac{1}{(5x)^2}+\cdots$

(d) $1-\dfrac{2}{x}+\dfrac{4}{x^2}-\cdots$

(e) $1+\dfrac{1}{1+x^2}+\dfrac{1}{(1+x^2)^2}+\cdots$

(f) $1-\dfrac{1}{3-x}+\dfrac{1}{(3-x)^2}-\cdots$

(g) $1+\dfrac{2x}{1+x^2}+\dfrac{(2x)^2}{(1+x^2)^2}+\cdots$

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18. (a) By writing out the terms, show that $\displaystyle\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)=1-\frac{1}{n+1}$.

(b) Hence explain why $\displaystyle\sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{r+1}\right)=1$.

(c) Prove the identity $\dfrac{1}{r(r+1)}=\dfrac{1}{r}-\dfrac{1}{r+1}$, and hence show that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots=1.$$

EXTENSION

19. Consider the series $1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\frac{6}{32}+\cdots$.
(a) Write out the terms of S_n and $2S_n$.
(b) Subtract to get an expression for S_n.
(c) Find the limit as $n\to\infty$, and hence find S_∞.

20. Now consider the general series $a+\dfrac{a+d}{x}+\dfrac{a+2d}{x^2}+\dfrac{a+3d}{x^3}+\cdots$, where a, d and x are constants with $|x|>1$.
(a) Write out the terms of S_n and xS_n.
(b) Subtract to get an expression for $(x-1)S_n$ and hence for S_n.
(c) Find the limit as $n\to\infty$, and hence find S_∞.

21. The series $4+12+36+\cdots$ has no limiting sum because $r>1$. Nevertheless, substitution into the formula for the limiting sum gives

$$S_\infty=\frac{4}{1-3}=-2.$$

Can any meaning be given to this calculation and its result? [HINT: Look at the extension of the series to the *left* of the first term.]

6 L Recurring Decimals and Geometric Series

It is now possible to give a precise explanation of recurring decimals. They are infinite GPs, and their value is the limiting sum of that GP.

WORKED EXERCISE: Express the repeating decimals $0{\cdot}\dot{2}\dot{7}$ and $2{\cdot}6\dot{4}\dot{5}$ as infinite GPs, and use the formula for the limiting sum to find their values as fractions reduced to lowest terms.

SOLUTION:

$$\begin{aligned}0{\cdot}\dot{2}\dot{7}&=0{\cdot}272727\ldots\\&=0{\cdot}27+0{\cdot}0027+0{\cdot}000027+\cdots.\end{aligned}$$

This is an infinite GP
with $a=0{\cdot}27$ and $r=0{\cdot}01$,

$$\text{so}\quad\begin{aligned}0{\cdot}\dot{2}\dot{7}&=\frac{a}{1-r}\\&=\tfrac{0{\cdot}27}{0{\cdot}99}\\&=\tfrac{27}{99}\\&=\tfrac{3}{11}.\end{aligned}$$

$$\begin{aligned}2{\cdot}6\dot{4}\dot{5}&=2{\cdot}645454545\ldots\\&=2{\cdot}6+(0{\cdot}045+0{\cdot}00045+\cdots)\end{aligned}$$

This is 2·6 plus an infinite GP
with $a=0{\cdot}045$ and $r=0{\cdot}01$,

$$\text{so}\quad\begin{aligned}2{\cdot}6\dot{4}\dot{5}&=2{\cdot}6+\tfrac{0{\cdot}045}{0{\cdot}99}\\&=\tfrac{26}{10}+\tfrac{45}{990}\\&=\tfrac{286}{110}+\tfrac{5}{110}\\&=\tfrac{291}{110}.\end{aligned}$$

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Exercise 6L

NOTE: The following prime factorisations will be useful in this exercise:

$$9 = 3^2 \qquad 999 = 3^3 \times 37 \qquad 99\,999 = 3^2 \times 41 \times 271$$
$$99 = 3^2 \times 11 \qquad 9999 = 3^2 \times 11 \times 101 \qquad 999\,999 = 3^3 \times 7 \times 11 \times 13 \times 37$$

1. Write each of these recurring decimals as an infinite GP, and hence use the formula for the limiting sum of a GP to express it as a rational number in lowest terms:

(a) $0{\cdot}\dot{7}$ (b) $0{\cdot}\dot{6}$ (c) $0{\cdot}\dot{2}\dot{7}$ (d) $0{\cdot}\dot{7}\dot{8}$ (e) $0{\cdot}\dot{4}\dot{5}$ (f) $0{\cdot}\dot{0}2\dot{7}$ (g) $0{\cdot}\dot{1}3\dot{5}$ (h) $0{\cdot}\dot{1}8\dot{5}$

2. Write each recurring decimal as the sum of an integer or terminating decimal and an infinite GP, and hence express it as a fraction in lowest terms:

(a) $12{\cdot}\dot{4}$ (b) $7{\cdot}\dot{8}\dot{1}$ (c) $8{\cdot}4\dot{6}$ (d) $0{\cdot}2\dot{3}\dot{6}$

3. Apply the earlier method — multiplying by 10^n where n is the cycle length (see Section 2A), then subtracting — to every second recurring decimal in the previous two questions.

DEVELOPMENT

4. (a) Express the repeating decimal $0{\cdot}\dot{9}$ as an infinite GP, and hence show that it equals 1.

(b) If $0{\cdot}\dot{9}$ were not equal to 1, then the difference $1 - 0{\cdot}\dot{9}$ would be positive. Let $\varepsilon = 1 - 0{\cdot}\dot{9}$, explain why ε must be less than every positive number, and hence deduce that $\varepsilon = 0$.

(c) Express $12{\cdot}47\dot{9}$ as $12{\cdot}47$ plus an infinite GP, and hence show that it equals $12{\cdot}48$.

(d) Express 74 and $7{\cdot}282$ as recurring decimals ending in repeated 9s.

5. Use GPs to express these as fractions in lowest terms:

(a) $0{\cdot}\dot{0}95\dot{7}$ (b) $0{\cdot}\dot{2}47\dot{5}$ (c) $0{\cdot}\dot{2}30\,76\dot{9}$ (d) $0{\cdot}\dot{4}28\,57\dot{1}$ (e) $0{\cdot}25\dot{5}\dot{7}$ (f) $1{\cdot}1\dot{0}3\dot{7}$ (g) $0{\cdot}0\dot{0}0\,27\dot{1}$ (h) $7{\cdot}7\dot{7}1\,428\,\dot{5}$

6. The earlier method of handling recurring GPs is a special case of the method of deriving the formula for the nth partial sum of a GP. Compare the proof of that formula (Section 6J) with the earlier method of handling $0{\cdot}\dot{1}\dot{8}$ (Section 2A), and find the correspondence between them.

EXTENSION

7. (a) Write the base 2 'decimals' $0{\cdot}01$, $0{\cdot}1101$ and $0{\cdot}011\,011$ as normal fractions. (b) Express $\frac{1}{2}$, $\frac{3}{4}$, $\frac{3}{8}$ and $\frac{11}{16}$ as 'decimals' base 2. (c) By writing them as infinite GPs, express the base 2 'decimals' $0{\cdot}\dot{1}\dot{0}$, $0{\cdot}\dot{1}0\dot{1}$, $0{\cdot}\dot{0}01\dot{1}$ and $0{\cdot}\dot{1}$ as normal fractions. (d) Express $\frac{1}{3}$, $\frac{4}{5}$ and $\frac{1}{7}$ as recurring 'decimals' base 2. (e) Experiment with 'decimals' written to other bases.

8. (a) [The periods of recurring decimals] Let p be any prime other than 2 or 5. Explain why the cycle length of the recurring decimal equal to $1/p$ is n digits, where n is the least power of 10 that has remainder 1 when divided by p.

(b) Use the factorisations of $10^k - 1$ given at the start of this exercise to predict the periods of the decimal representations of $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{27}$, $\frac{1}{37}$, $\frac{1}{41}$, $\frac{1}{101}$ and $\frac{1}{271}$, then write each as a recurring decimal.

9. [Extension — for further reading] *Fermat's little theorem* says that if p is a prime, and a is not a multiple of p, then a^{p-1} has remainder 1 after division by p. Using this theorem with $a = 10$, deduce that for all primes p except 2 and 5, the period of the decimal representation of $1/p$ is a divisor of $p - 1$.

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6 M Factoring Sums and Differences of Powers

The well-known difference of squares identity can now be generalised to sums and differences of nth powers. This factorisation will be needed in Section 7C in the proof of one of the fundamental results of the next chapter.

Differences of nth Powers: The polynomial $1+x+x^2+\cdots+x^{n-1}$ is a GP with $a=1$ and $r=x$. So, using the formula for the sum of a GP,

$$1+x+x^2+\cdots+x^{n-1}=\frac{x^n-1}{x-1},$$

and rearranging, this becomes a factorisation of x^n-1:

$$x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1).$$

More generally, here is the factorisation of the difference of nth powers.

21 DIFFERENCE OF POWERS: $x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+y^{n-1})$

PROOF: The last identity is easily proven directly by multiplying out the RHS:

$$\begin{aligned}\text{RHS}&=x^n+x^{n-1}y+x^{n-2}y^2+\cdots+xy^{n-1}\\&\qquad-x^{n-1}y-x^{n-2}y^2-x^{n-3}y^3-\cdots-y^n\\&=x^n-y^n\end{aligned}$$

WORKED EXERCISE: Here are some examples, beginning with the difference of squares:

$$\begin{aligned}x^2-49&=(x-7)(x+7)\\x^3-1&=(x-1)(x^2+x+1)\\x^4-81y^4&=(x-3y)(x^3+3x^2y+9xy^2+27y^3)\end{aligned}$$

Sums of Odd Powers: The sum of two squares cannot be factored. The sum of two cubes, however, can easily be converted to the difference of powers, and can then be factored:

$$\begin{aligned}x^3+y^3&=x^3-(-y)^3\\&=\Big(x-(-y)\Big)\Big(x^2+x(-y)+(-y)^2\Big)\\&=(x+y)(x^2-xy+y^2).\end{aligned}$$

The same device works for all sums of odd powers, so if n is an odd positive integer:

22 SUMS OF ODD POWERS: $x^n+y^n=(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\cdots+y^{n-1})$

WORKED EXERCISE: Some further examples of factoring sums of odd powers:

$$\begin{aligned}x^3+125&=(x+5)(x^2-5x+25)\\x^5+32y^5&=(x+2y)(x^4-2x^3y+4x^2y^2-8xy^3+16y^4)\\1+a^7&=(1+a)(1-a+a^2-a^3+a^4-a^5+a^6)\end{aligned}$$

WORKED EXERCISE: Factor x^6-64 completely.

SOLUTION: $x^6-64=(x^3-8)(x^3+8)$ (using difference of squares)
$\qquad\qquad=(x-2)(x^2+2x+4)(x+2)(x^2-2x+4)$

(Neither quadratic can be factored, since $b^2-4ac=-12<0$.)

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Exercise 6M

1. Factor these expressions, using the difference or sum of nth powers:

(a) $x^2 - 1$ (b) $x^3 - 1$ (c) $x^5 - 1$ (d) $t^7 - 1$

(e) $t^3 + 1$ (f) $t^5 + 1$ (g) $x^7 + 1$ (h) $x^3 - 125$

(i) $x^3 + 8$ (j) $x^5 - 243$ (k) $x^3 + 125$ (l) $x^5 + y^5$

(m) $x^5 + 32$ (n) $32t^5 + 1$ (o) $1 - a^7x^7$ (p) $27t^3 + 8a^3$

2. Factor numerator and denominator, then simplify:

(a) $\dfrac{x^3 - y^3}{x - y}$ (b) $\dfrac{x^4 - y^4}{x - y}$ (c) $\dfrac{x^3 + 1}{x + 1}$ (d) $\dfrac{32x^5 + y^5}{2x + y}$ (e) $\dfrac{x^7 - y^7}{x^5 - y^5}$ (f) $\dfrac{x^7 + y^7}{x^5 + y^5}$

DEVELOPMENT

3. (a) Factor $x^4 - 1$ first as $\left(x^2\right)^2 - 1$, then go on.

(b) Factor $x^6 - 1$ first as $\left(x^3\right)^2 - 1$, then go on.

(c) Similarly factor: (i) $x^8 - a^8$ (ii) $x^{10} - 1$

4. By expressing x as $(\sqrt{x})^2$ and y as $(\sqrt{y})^2$, factor:

(a) $x - y$ (b) $x^{\frac{3}{2}} - y^{\frac{3}{2}}$ (c) $x^{\frac{3}{2}} + y^{\frac{3}{2}}$

5. By the same method, simplify these algebraic fractions:

(a) $\dfrac{\sqrt{x} - \sqrt{y}}{x - y}$ (b) $\dfrac{x - y}{\sqrt{x} + \sqrt{y}}$ (c) $\dfrac{x^{\frac{3}{2}} - y^{\frac{3}{2}}}{\sqrt{x} - \sqrt{y}}$ (d) $\dfrac{\sqrt{x} + \sqrt{y}}{x^{\frac{3}{2}} + y^{\frac{3}{2}}}$

6. Simplify, using sums and differences of nth powers or otherwise:

(a) $(n+1)^2 - n^2$ (b) $(n+1)^2 - (n-1)^2$ (c) $(n+1)^3 - n^3$ (d) $(n+1)^3 - (n-1)^3$ (e) $(n+a)^2 - (n-a)^2$ (f) $(6n+a)^2 - (n+6a)^2$

7. Simplify $\dfrac{f(u) - f(x)}{u - x}$ when:

(a) $f(x) = x^2$ (b) $f(x) = x^3$ (c) $f(x) = x^4$ (d) $f(x) = \dfrac{1}{x}$ (e) $f(x) = \sqrt{x}$ (f) $f(x) = \dfrac{1}{x^2}$

EXTENSION

8. (a) Show by rearranging each LHS as a difference of squares, or by expansion, that:

(i) $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$

(ii) $x^4 - x^2 + 1 = (x^2 + x\sqrt{3} + 1)(x^2 - x\sqrt{3} + 1)$

(iii) $x^4 + 1 = (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)$

(b) Hence factor completely:

(i) $x^6 + 1$ (ii) $x^8 - 1$ (iii) $x^{12} - 1$

9. (a) [Mersenne primes] Use the factorisation of differences of powers to show that $M_k = 2^k - 1$ can only be prime if k is a prime number p. Such primes M_p are called *Mersenne primes*. List the first few Mersenne primes, and find the first prime p such that M_p is not prime.

(b) [Fermat primes] Use the factorisation of sums of odd powers to show that $2^k + 1$ can only be prime if k is a power of 2. Such primes are called *Fermat primes*. List the first few Fermat primes, but accept the fact that $2^{32} + 1 = 641 \times 6\,700\,417$ is not prime.

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(c) [Mersenne primes and perfect numbers] Prove that if $M_p = 2^p - 1$ is a Mersenne prime, then $N = 2^{p-1}M_p = 2^{p-1}(2^p - 1)$ is a *perfect number*, meaning that the sum of all factors of N less than N is N itself. Hence list some perfect numbers.

(d) Let $F_n = 2^{2^n} + 1$, and prove that $F_{n+1} = F_0 F_1 F_2 \ldots F_n + 2$. Deduce that F_n and F_m are relatively prime when m and n are distinct. By considering the difference $M_p - M_q$, prove also that M_p and M_q are relatively prime when p and q are distinct primes. [NOTE: Two numbers are called *relatively prime* if their only common factor is 1.]

6 N Proof by Mathematical Induction

Mathematical induction is a method of proof quite different from other methods of proof seen so far. It is based on recursion, which is why it belongs with the work on sequences and series, and it is used for proving theorems which claim that a certain statement is true for integer values of some variable.

As far as this course is concerned, proof by mathematical induction can only be applied after a clear statement of the theorem to be proven has already been obtained. So let us examine a typical situation in which a clear pattern is easily generated, but no obvious explanation emerges for why that pattern occurs.

Example 1 — Proving a Formula for the Sum of a Series: Find a formula for the sum of the first n cubes, and prove it by mathematical induction.

SOME CALCULATIONS FOR LOW VALUES OF n: Here is a table of values of the first 10 cubes and their partial sums:

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
n^3	1	8	27	64	125	216	343	512	729	1000	$\cdots$
$1^3 + 2^3 + \cdots + n^3$	1	9	36	100	225	441	784	1296	2025	3025	$\cdots$
Form	1^2	3^2	6^2	10^2	15^2	21^2	28^2	36^2	45^2	55^2	$\cdots$

The surprising thing here is that the last row is the square of the *triangular numbers*, where the nth triangular number is the sum of all the positive integers up to n. Using the formula for the sum of an AP (the number of terms times the average of first and last term), the formula for the nth triangular number is $\frac{1}{2}n(n+1)$. So the sum of the first n cubes seems to be $\frac{1}{4}n^2(n+1)^2$.

Thus we have arrived at a *conjecture*, meaning that we appear to have a true theorem, but we have no clear idea why it is true. We cannot really be sure yet even whether it is true, because showing that a statement is true for the first 10 positive integers is most definitely not a proof that it is true for all integers. The following worked exercise gives a precise statement of the result we want to prove.

WORKED EXERCISE: Prove by mathematical induction that for all integers $n \geq 1$,

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

The proof below is a proof by mathematical induction. Read it carefully, then read the explanation of the proof in the notes below.

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PROOF (BY MATHEMATICAL INDUCTION):

A. When $n = 1$, $\text{RHS} = \frac{1}{4} \times 1 \times 2^2$
$= 1$
$= \text{LHS}.$
So the statement is true for $n = 1$.

B. Suppose that k is a positive integer for which the statement is true.
That is, suppose $1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2.$ $(**)$
We prove the statement for $n = k + 1$.
That is, we prove $1^3 + 2^3 + 3^3 + 4^3 + \cdots + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$.

$$\begin{aligned}
\text{LHS} &= 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 \\
&= \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3, \quad \text{by the induction hypothesis } (**), \\
&= \tfrac{1}{4}(k+1)^2\Big(k^2 + 4(k+1)\Big) \\
&= \tfrac{1}{4}(k+1)^2(k^2 + 4k + 4) \\
&= \tfrac{1}{4}(k+1)^2(k+2)^2 \\
&= \text{RHS}.
\end{aligned}$$

C. It follows from parts A and B by mathematical induction that the statement is true for all positive integers n.

Notes on the Proof: First, there are three clear parts. Part A proves the statement for the starting value 1. Part B is the most complicated, and proves that whenever the statement is true for some integer $k \ge 1$, then it is also true for the next integer $k + 1$. Part C simply appeals to the principle of mathematical induction to write a conclusion.

Secondly, any question on proof by mathematical induction is testing your ability to write a coherent account of the proof — you are advised not to deviate from the structure given here. The language of Part B is particularly important. It begins with four sentences, and these four sentences should be repeated strictly in all proofs. The first and second sentences of Part B set up what is assumed about k, writing down the specific statement for $n = k$, a statement later referred to as 'the induction hypothesis'. The third and fourth sentences set up specifically what it is that we intend to prove.

Statement of the Principle of Mathematical Induction: With this proof as an example, here is a formal statement of the principle of mathematical induction.

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MATHEMATICAL INDUCTION: Suppose that some statement is to be proven for all integers n greater than or equal to some starting value n_0. Suppose also that it has been proven that:
1. the statement is true for $n = n_0$,
2. whenever the statement is true for some positive integer $k \ge n_0$, then it is also true for the next integer $k + 1$.

Then the statement must be true for all positive integers $n \ge n_0$.

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Example 2 — Proving Divisibility: Find the largest integer that is a divisor of $3^{4n} - 1$, where $n \ge 0$ is any integer, and prove the result by mathematical induction.

SOME CALCULATIONS FOR LOW VALUES OF n: Again, here is a table of values for the first four values of n:

n	0	1	2	3	$\cdots$
$3^{4n} - 1$	0	80	6560	531440	$\cdots$

It seems likely from this that 80 is a divisor of all the numbers. Certainly no number bigger than 80 can be a divisor. So we write down the theorem, and try to provide a proof. Various proofs are available, but here is the proof by mathematical induction:

WORKED EXERCISE: Prove by mathematical induction that for all cardinals n,

$3^{4n} - 1$ is divisible by 80.

PROOF (BY MATHEMATICAL INDUCTION):

A. When $n = 0$, $3^{4n} - 1 = 0$, which is divisible by 80
(remember that every number is a divisor of zero).
So the statement is true for $n = 0$.

B. Suppose that k is a cardinal for which the statement is true.
That is, suppose $3^{4k} - 1 = 80m$, for some integer m. $(**)$
We prove the statement for $n = k + 1$.
That is, we prove $3^{4k+4} - 1$ is divisible by 80.

$$\begin{aligned} 3^{4k+4} - 1 &= 3^{4k} \times 3^4 - 1 \\ &= (80m + 1) \times 81 - 1, \quad \text{by the induction hypothesis } (**), \\ &= 80 \times 81m + 81 - 1 \\ &= 80m \times 81 + 80 \\ &= 80(81m + 1), \quad \text{which is divisible by 80, as required.} \end{aligned}$$

C. It follows from parts A and B by mathematical induction that the statement is true for all cardinals n.

NOTES ON THE PROOF: Notice that the induction hypothesis $(**)$ interprets divisibility by 80 as being $80m$ where m is an integer, whereas the fourth sentence of Part B stating what is to be proven does not interpret divisibility. Proofs of divisibility work more easily this way.

Example 3 — Proving an Inequality: For what integer values of n is 2^n greater than n^2?

SOME CALCULATIONS FOR LOW VALUES OF n: Here is a table of values:

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
n^2	0	1	4	9	16	25	36	49	64	81	100	$\cdots$
2^n	1	2	4	8	16	32	64	128	256	512	1024	$\cdots$

It seems obvious now that from $n = 5$ onwards, 2^n quickly becomes far larger than n^2. Reasons for this may seem clearer here, and other proofs are available, but here is the proof by mathematical induction.

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WORKED EXERCISE: Prove by mathematical induction that for all integers $n \geq 5$, n^2 is less than 2^n.

PROOF (BY MATHEMATICAL INDUCTION):

A. When $n = 5$, $n^2 = 25$ and $2^n = 32$. So the statement is true for $n = 5$.

B. Suppose that $k \geq 5$ is an integer for which the statement is true.
That is, suppose $k^2 < 2^k$. (∗∗)
We prove the statement for $n = k + 1$.
That is, we prove $(k + 1)^2 < 2^{k+1}$.
This is best done by proving that RHS − LHS > 0:

$$\begin{aligned}\text{RHS} - \text{LHS} &= 2 \times 2^k - (k+1)^2 \\ &> 2k^2 - (k^2 + 2k + 1), \text{ by the induction hypothesis } (**), \\ &= k^2 - 2k - 1 \\ &= (k-1)^2 - 2, \text{ completing the square,} \\ &> 0, \text{ since } k \geq 5 \text{ and so } (k-1)^2 \geq 16.\end{aligned}$$

C. It follows from parts A and B by mathematical induction that the statement is true for all integers $n \geq 5$.

NOTES ON THE PROOF: Proofs of inequalities can be difficult. The most systematic approach is probably the method shown here, 'Prove RHS − LHS > 0'.

Further Remarks on Mathematical Induction: Summing a series, divisibility, and inequalities are the major places where proof by mathematical induction is applied. The method is, however, quite general, and the following exercise gives further applications beyond those three situations. In particular, some geometrical situations require proof by mathematical induction. In all cases, the precise structure and words given in these three examples should be followed exactly.

Finally, all the proofs given earlier in the chapter of formulae associated with APs and GPs really require the axiom of mathematical induction for their validity. In fact, if one were to be very strict about logic, any situation where there are dots . . . only has meaning because of the axiom of mathematical induction.

Exercise 6N

1. Use mathematical induction to prove that for all positive integers n:

(a) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

(b) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

(c) $1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$

(d) $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n+1)$

(e) $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$

(f) $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

(g) $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

(h) $\dfrac{1}{1 \times 4} + \dfrac{1}{4 \times 7} + \dfrac{1}{7 \times 10} + \cdots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$

(i) $2 \times 2^0 + 3 \times 2^1 + 4 \times 2^2 + \cdots + (n+1)2^{n-1} = n \times 2^n$

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(j) $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

(k) $\dfrac{1}{1 \times 2 \times 3} + \dfrac{1}{2 \times 3 \times 4} + \dfrac{1}{3 \times 4 \times 5} + \cdots + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$

(l) $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$, provided $r \neq 1$

(m) $a + (a+d) + (a+2d) + \cdots + \big(a + (n-1)d\big) = \frac{1}{2}n\big(2a + (n-1)d\big)$

2. Hence find the limiting sums of the series in parts (g), (h) and (k) of the previous question.

DEVELOPMENT

3. Use mathematical induction to prove these divisibility results for all positive integers n:

(a) $5^n + 3$ is divisible by 4

(b) $9^n - 3$ is a multiple of 6

(c) $11^n - 1$ is divisible by 10

(d) $5^n + 2 \times 11^n$ is a multiple of 3

(e) $5^{2n} - 1$ is a multiple of 24

(f) $x^n - 1$ is divisible by $x - 1$

4. Prove these divisibility results, advancing in part B of the proof from k to $k+2$:

(a) For even n: (i) $n^3 + 2n$ is divisible by 12 (ii) $n^2 + 2n$ is a multiple of 8

(b) For odd n: (i) $3^n + 7^n$ is divisible by 10 (ii) $7^n + 6^n$ is divisible by 13

5. Examine the divisors of $n^3 - n$ for low odd values of n, make a judgement about the largest integer divisor, and prove your result by induction.

6. Prove these inequalities by mathematical induction:

(a) $n^2 > 10n + 7$, for $n \geq 11$

(b) $2^n > 3n^2$, for $n \geq 8$

(c) $3^n > n^2$, for $n \geq 2$ (and also for $n = 0$ and 1)

(d) $(1+\alpha)^n \geq 1 + n\alpha$, for $n \geq 1$, where $\alpha > -1$

7. Examine 2^n and $2n^3$ for low values of n, make a judgement about which is eventually bigger, and prove your result by induction.

8. Prove: (a) $\displaystyle\sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n}$, for $n \geq 1$ (b) $\dfrac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times 2n} \geq \dfrac{1}{2n}$, for $n \geq 1$

9. (a) Given that $T_n = 2T_{n-1} + 1$ and $T_1 = 5$, prove that $T_n = 6 \times 2^{n-1} - 1$.

(b) Given that $T_n = \dfrac{3T_{n-1} - 1}{4T_{n-1} - 1}$ and $T_1 = 1$, prove that $T_n = \dfrac{n}{2n-1}$.

10. Prove by induction that the sum of the angles of a polygon with n sides is $n - 2$ straight angles. [HINT: Dissect the $(k+1)$-gon into a k-gon and a triangle.]

11. Prove by induction that n lines in the plane, no two being parallel and no three concurrent, divide the plane into $\frac{1}{2}n(n+1) + 1$ regions. [HINT: The $(k+1)$th line will cross k lines in k distinct points, and so will add $k+1$ regions.]

12. Prove by mathematical induction that every set with n members has 2^n subsets. [HINT: When a new member is added to a k-member set, then every subset of the resulting $(k+1)$-member set either contains or does not contain the new member.]

13. Defining $n! = 1 \times 2 \times 3 \times \cdots \times n$ (this is called 'n factorial'), prove that:

(a) $\displaystyle\sum_{r=1}^{n} r \times r! = (n+1)! - 1$ (b) $\displaystyle\sum_{r=1}^{n} \frac{r-1}{r!} = 1 - \frac{1}{n!}$

14. Prove: (a) $\displaystyle\sum_{r=1}^{n} r^4 = \tfrac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$ (b) $\displaystyle\sum_{r=1}^{n} r^5 = \tfrac{1}{12}n^2(n+1)^2(2n^2+2n-1)$

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15. (a) By rationalising the numerator, prove that $\sqrt{n+1} - \sqrt{n} > \dfrac{1}{2\sqrt{n+1}}$.

(b) Hence prove by induction that $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \sqrt{n}$, for $n \geq 7$.

16. (a) Show that $f(n) = n^2 - n + 17$ is prime for $n = 0, 1, 2, \ldots, 16$. Show, however, that $f(17)$ is not prime. Which step of proof by induction does this counterexample show is necessary?

(b) Begin to show that $f(n) = n^2 + n + 41$ is prime for $n = 0, 1, 2, \ldots, 40$, but not for 41.

NOTE: There is no formula for generating prime numbers — these quadratics are interesting because of the long unbroken sequences of primes they produce.

EXTENSION

17. These proofs require a stronger form of mathematical induction in which 'the statement' is assumed true not only for $n = k$, but for any integer from the starting integer up to k.

(a) Given that $T_{n+2} = 3T_{n+1} - 2T_n$, where $T_1 = 5$ and $T_2 = 7$, prove that $T_n = 3 + 2^n$.

(b) The Fibonacci series F_n is defined by $F_{n+2} = F_{n+1} + F_n$, where $F_1 = F_2 = 1$. Prove:

(i) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ (ii) $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ (iii) $F_n \leq \left(\frac{5}{3}\right)^{n-1}$

(c) Prove that $F_n = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n$.

(d) The Lucas series L_n is defined by $L_{n+2} = L_{n+1} + L_n$, where $L_1 = 1$ and $L_2 = 3$. Use the observation that $L_n = F_n + 2F_{n-1}$ to generate a formula for L_n, then prove it by induction.

(e) Prove that every integer greater than or equal to 2 is the product of prime numbers. (Further reading: Find how to prove that this factorisation into primes is unique.)

18. [A rather difficult proof] (a) Prove by induction on n that the geometric mean of 2^n positive numbers never exceeds their arithmetic mean, that is, for all cardinals n,

$$\frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n} \geq \left(a_1 a_2 \ldots a_{2^n}\right)^{\frac{1}{2^n}}, \quad \text{for all positive numbers } a_1, a_2, \ldots, a_{2^n}.$$

(b) Induction can work backwards as well as forwards. Suppose that for some integer $k \geq 2$

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq \left(a_1 a_2 \ldots a_k\right)^{\frac{1}{k}}, \quad \text{for all positive numbers } a_1, a_2, \ldots, a_k.$$

By substituting $a_k = \left(a_1 a_2 \ldots a_{k-1}\right)^{\frac{1}{k-1}}$, show that it follows that

$$\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} \geq \left(a_1 a_2 \ldots a_{k-1}\right)^{\frac{1}{k-1}}, \quad \text{for all positive numbers } a_1, a_2, \ldots, a_{k-1}.$$

(c) Deduce from all this that the geometric mean of any set of positive numbers never exceeds their arithmetic mean.

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