The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Assignment 1 (due early in Week 9)** 2012

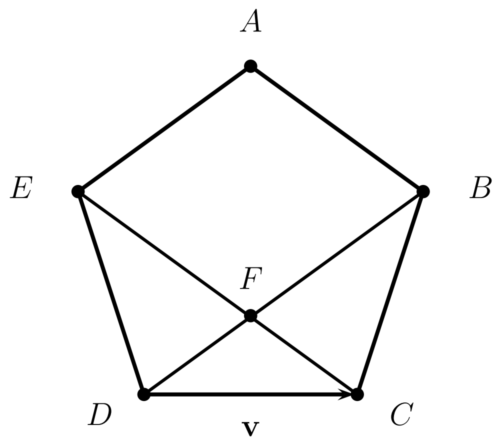
This assignment is due by **4:00 pm, Tuesday 8 May**. Your assignment should be posted in the locked collection boxes at the western end of the verandah (closest to Eastern Avenue) on Carslaw Level 3. Please do not post your assignment before the due date since the boxes are also used for the collection of assignments in other units. Your assignment should be stapled inside a manila folder, on the front of which you should write the initial of your family name as a LARGE letter, and a cover sheet signed and attached. The cover sheet may be downloaded from the MATH1902 website. Assignments without a cover sheet may not be marked. This assignment is worth 10 per cent of your assessment for MATH1902. There are three questions worth a total of 30 marks, and the breakdown of marks is indicated.

1. (i) Let P, Q, R and S be any points in space.

(a) Use vector arithmetic to verify that $\overrightarrow{PQ} + \overrightarrow{RS} = \overrightarrow{RQ} + \overrightarrow{PS}$.

(b) Suppose now that P, R and S are not collinear, that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{SR}$ and that $\overrightarrow{QR}$ is parallel to $\overrightarrow{PS}$. Prove that $\overrightarrow{PQ} = \overrightarrow{SR}$. (Thus $PQRS$ is a nondegenerate parallelogram.)

(ii) Let the figure $ABCDE$ be the regular pentagon depicted below (so that all the sides lengths are equal and all of the internal angles between adjacent sides are equal). Let F be the intersection point of the line segments BD and EC, and put $\mathbf{v} = \overrightarrow{DC}$.



We have $\mathbf{v} = \lambda\overrightarrow{EB}$ for some positive scalar λ (since DC is parallel to EB by reflectional symmetry of the regular pentagon, and you may take this as granted).

(a) Explain briefly why $ABFE$ is a rhombus. [Hint: use (i)(b).]

(b) Prove that $\lambda\mathbf{v} = \overrightarrow{CB} + \overrightarrow{ED}$. [Hint: $\mathbf{v} = \overrightarrow{DA} + \overrightarrow{AC}$.]

(c) Deduce that $\lambda\mathbf{v} = \overrightarrow{EB} + \overrightarrow{CD}$ and that $\lambda^2 = 1 - \lambda$. [Hint: use (i)(a).]

(d) Deduce that $\dfrac{|BF|}{|BD|} = \dfrac{\sqrt{5}-1}{2}$ (which is approximately 3/5).

(2+3+3+2+3+2=15 marks)

2. Denote the xy-plane by $\mathcal{P}$. Let $\mathcal{C}$ be some general curve in $\mathcal{P}$ defined by the equation

$$f(x,y) \;=\; 0\,,$$

where $f(x,y)$ is some algebraic expression involving x and y. Let x_0, y_0 and θ be real numbers and define bijections $T, R : \mathcal{P} \to \mathcal{P}$ by the rules

$$T(x,y) \;=\; (x-x_0, y-y_0) \quad \text{and} \quad R(x,y) \;=\; (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\,.$$

Thus T is the parallel translation of $\mathcal{P}$ that takes (x_0, y_0) to the origin, and R is the rotation θ radians anticlockwise about the origin. (You do not need to verify these facts.)

(i) Verify carefully that if $B : \mathcal{P} \to \mathcal{P}$ is any bijection then $B(\mathcal{C})$ is defined by the equation

$$f(B^{-1}(x,y)) \;=\; 0\,.$$

(ii) Deduce that $T(\mathcal{C})$ is the curve defined by the equation

$$f(x+x_0, y+y_0) \;=\; 0$$

and $R(\mathcal{C})$ by the equation

$$f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) \;=\; 0\,.$$

(iii) Find an equation that defines the curve $T^{-1}(R(T(\mathcal{C})))$.

(iv) Let a, b, c be constants. Use your answer to part (iii) to deduce that, if we rotate the line with equation $ax + by = c$ about the point (x_0, y_0) by θ radians anticlockwise, we obtain the line with equation

$$\begin{aligned}(a\cos\theta - b\sin\theta)x &+ (a\sin\theta + b\cos\theta)y \\ &= c + a\big(x_0(\cos\theta - 1) + y_0\sin\theta\big) - b\big(x_0\sin\theta + y_0(1-\cos\theta)\big)\,.\end{aligned}$$

$(4+4+4+3 = 15$ marks)

3. In this exercise we will prove the identity

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) \;=\; (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))\,\mathbf{a} \qquad (*)$$

for all geometric vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Note that this holds trivially if $\mathbf{a}$ is the zero vector, so throughout we will assume $\mathbf{a}$ is nonzero.

Now, you can find proofs of $(*)$ on the internet, which is reassuring, but please don't hand in something you read from the internet. To get credit for this question you have to follow the steps below, which proves $(*)$ using the Conjugation Principle. The proof that you construct in this way almost certainly does not appear on the internet.

(i) Verify $(*)$ in the easy case when $\mathbf{a} = \mathbf{i}$, the unit vector pointing in the x-direction. Do this by expressing $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ and calculating directly with components.

(ii) Now consider a general nonzero vector $\mathbf{a}$. Form the unit vector $\widehat{\mathbf{a}}$ and translate $\widehat{\mathbf{a}}$ and $\mathbf{i}$ so that their tails are joined at the origin. These vectors then determine a plane through the origin and there will be a natural rotation $\mathcal{R}$ of space about an axis perpendicular to this plane that takes $\widehat{\mathbf{a}}$ to $\mathbf{i}$. Then $\mathcal{R}$ is a function that uses geometric vectors as inputs and outputs and preserves lengths, angles and all geometric vector operations. In particular, for any vectors $\mathbf{v}$ and $\mathbf{w}$ and scalars λ, it follows quickly that

$$\mathcal{R}(\lambda\mathbf{v}) \;=\; \lambda\mathcal{R}(\mathbf{v})\,, \quad \mathbf{v}\cdot\mathbf{w} \;=\; \mathcal{R}(\mathbf{v})\cdot\mathcal{R}(\mathbf{w})\,, \quad \mathcal{R}(\mathbf{v}\times\mathbf{w}) \;=\; \mathcal{R}(\mathbf{v})\times\mathcal{R}(\mathbf{w})\,,$$

and you may take these as granted, without proof. Use the result from part (i), these facts and corresponding facts about the inverse rotation $\mathcal{S} = \mathcal{R}^{-1}$ to verify the identity

$$(\widehat{\mathbf{a}} \times \mathbf{b}) \times (\widehat{\mathbf{a}} \times \mathbf{c}) \;=\; (\widehat{\mathbf{a}} \cdot (\mathbf{b} \times \mathbf{c}))\,\widehat{\mathbf{a}} \qquad (**)$$

[Hint: $\mathcal{R}(\widehat{\mathbf{a}}) = \mathbf{i}$ and $\mathcal{S}(\mathbf{i}) = \widehat{\mathbf{a}}$.]

(iii) Deduce $(*)$ from $(**)$.

$(5 + 6 + 4 = 15$ marks$)$