The University of Sydney
School of Mathematics and Statistics

Non-tutorial for Week 1

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

Lecturers: Daniel Daners and James Parkinson

Welcome to *Integral Calculus and Modelling (Advanced).* There is no tutorial in week 1. This sheet has some questions to get the ball rolling, and to remind you of some things from last semester.

The Mean Value Theorem

The *Mean Value Theorem* is one of the most important theorems in calculus, and we will be using it quite a few times in this course. It says that if f is continuous on $[a, b]$ and differentiable on (a, b) then there is at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

1. Draw a picture that makes the Mean Value Theorem obvious. Hence, or otherwise, remember the Mean Value Theorem forever.

2. Suppose that f is continuous on $[a, b]$. Use the Mean Value Theorem to show that if $f'(x) = 0$ for all $x \in (a, b)$ then f is a constant function on $[a, b]$.

3. Suppose that f is continuous on $[a, b]$. Use the Mean Value Theorem to show that if $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on $[a, b]$. Is the converse true?

4. (Challenging) *Rolle's Theorem* says that if g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = g(b)$, then there exists $c \in (a, b)$ such that $g'(c) = 0$.

(a) Use Rolle's Theorem to prove the Mean Value Theorem.
Hint: Choose α such that $g(x) = f(x) - \alpha x$ satisfies $g(a) = g(b)$.

(b) Now prove Rolle's Theorem. You may assume the *Extreme Value Theorem*: If f is continuous on $[a, b]$ then f attains a maximum and minimum on $[a, b]$.

5. (Challenging) Use the Mean Value Theorem (multiple times) to show that if $f''(x) \geq 0$ on an interval $[a, b]$ then

$$f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b) \qquad \text{for all } t \in [0, 1].$$

Geometrically this says that if $f''(x) \geq 0$ on the interval $[a, b]$ then f is concave up on the interval $[a, b]$ (draw a picture).

Hint: Let $p_t = ta + (1 - t)b$. Consider the intervals $[a, p_t]$ and $[p_t, b]$.

Numbers

Recall the following notations for the various number systems:

$\mathbb{N} = \{0, 1, 2, \ldots\}$	*natural numbers*
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	*integers*
$\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z},\ q \neq 0\}$	*rational numbers*
$\mathbb{R} = \{\text{real numbers}\}$	*real numbers*
$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$	*complex numbers.*

A real number which is not rational is called *irrational.* There is no standard notation for the set of irrational numbers, but using set notation it is $\mathbb{R} \backslash \mathbb{Q}$.

6. Show that $\sqrt{2}$ is irrational.

Hint: If $\sqrt{2}$ is rational then we can write $\sqrt{2} = \frac{p}{q}$ with $p, q > 0$ integers with no factors in common. Rearranging gives $p^2 = 2q^2$. Use this equation to show that both p and q are divisible by 2, giving a contradiction.

7. (a) Show the $\log_2 3$ is irrational. Show that $\log_3 6$ is irrational.

(b) Suppose that $a, b > 0$ are integers. When is $\log_a b$ rational/irrational?

8. Prove the following important property of the real numbers: Given any two rational numbers $a < b$ there is an irrational number c with $a < c < b$, and given any two irrational numbers $a < b$ there is a rational number c with $a < c < b$.

9. (Challenging) Suppose that x is a solution to the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$.

(a) Show that x is either an integer or an irrational number.

(b) Hence give another proof that $\sqrt{2}$ is irrational.

(c) Show that the golden ratio $\frac{1+\sqrt{5}}{2}$ is irrational.

10. (Challenging) Let ξ be the real number

$$\xi = 0.011010100010100010100001\ldots$$

where the ith decimal is 0 if i is composite and 1 if i is prime. Do you think ξ is rational or irrational? Try to prove your claim.

11. Show that there exist irrational numbers a and b such that a^b is rational.

Hint: Consider $\xi = \sqrt{2}^{\sqrt{2}}$, and consider the possibilities of ξ rational or irrational.

12. Show that 2^e is irrational ☺