CHAPTER **SEVEN**

Rates and Finance

The various topics of this chapter are linked in three ways. First, exponential functions, to various bases, underlie the mathematics of natural growth, compound interest, geometric sequences and housing loans. Secondly, the rate of change in a quantity over time can be studied using the continuous functions presented towards the end of the chapter, or by means of the sequences that describe the changing values of salaries, loans and capital values. Thirdly, many of the applications in the chapter are financial. It is intended that by juxtaposing these topics, the close relationships amongst them in terms of content and method will be made clearer.

Study Notes: Sections 7A and 7B review the earlier formulae of APs and GPs in the context of various practical applications, including salaries, simple interest and compound interest. Sections 7C and 7D concern the specific application of the sums of GPs to financial calculations that involve the payment of regular instalments while compound interest is being charged — superannuation and housing loans are typical examples. Sections 7E and 7F deal with the application of the derivative and the integral to general rates of change, Section 7E being a review of work on related rates of change in Chapter Seven of the Year 11 volume. Section 7G reviews natural growth and decay, in preparation for the treatment in Section 7H of modified equations of growth and decay.

For those who prefer to study the continuous rates of change first, it is quite possible to study Sections 7E–7G first and then return to the applications of APs and GPs in Sections 7A–7D. A handful of questions in Section 7G are designed to draw the essential links between exponential functions, GPs and compound interest, and these can easily be left until Sections 7A–7D have been completed.

Prepared spreadsheets may be useful here in providing experience of how superannuation funds and housing loans behave over time, and computer programs may be helpful in modelling rates of change of some quantities. The intention of the course, however, is to establish the relationships between these phenomena and the known theories of sequences, exponential functions and calculus.

7 A Applications of APs and GPs

Arithmetic and geometric sequences were studied in Chapter Six of the Year 11 volume — this section will review the main results about APs and GPs and apply them to problems. Many of the applications will be financial, in preparation for the next three sections.

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Formulae for Arithmetic Sequences: At this stage, it should be sufficient simply to list the essential definitions and formulae concerning arithmetic sequences.

ARITHMETIC SEQUENCES:

- A sequence T_n is called an *arithmetic sequence* if

$$T_n - T_{n-1} = d, \quad \text{for } n \ge 2,$$

where d is a constant, called the *common difference*.

- The nth term of an AP is given by

$$T_n = a + (n-1)d,$$

1 where a is the first term T_1.

- Three terms T_1, T_2 and T_3 are in AP if $T_3 - T_2 = T_2 - T_1$.
- The *arithmetic mean* of a and b is $\frac{1}{2}(a+b)$.
- The sum S_n of the first n terms of an AP is

$$S_n = \tfrac{1}{2}n(a+\ell) \qquad \text{(use when } \ell = T_n \text{ is known)},$$

$$\text{or } S_n = \tfrac{1}{2}n\big(2a + (n-1)d\big) \qquad \text{(use when } d \text{ is known)}.$$

WORKED EXERCISE: [A simple AP] Gulgarindi Council is sheltering 100 couples taking refuge in the Town Hall from a flood. They are providing one chocolate per day per person. Every day after the first day, one couple is able to return home. How many chocolates will remain from an initial store of 12 000 when everyone has left?

SOLUTION: The chocolates eaten daily form a series $200 + 198 + \cdots + 2$, which is an AP with $a = 200$, $\ell = 2$ and $n = 100$,

$$\begin{aligned} \text{so number of chocolates eaten} &= \tfrac{1}{2}n(a+\ell) \\ &= \tfrac{1}{2} \times 100 \times (200+2) \\ &= 10\,100. \end{aligned}$$

Hence 1900 chocolates will remain.

WORKED EXERCISE: [Salaries and APs] Georgia earns \$25 000 in her first year, then her salary increases every year by a fixed amount $\$D$. If the total amount earned at the end of twelve years is \$600 000, find, correct to the nearest dollar:

(a) the value of D, (b) her final salary.

SOLUTION: Her annual salaries form an AP with $a = 25\,000$ and $d = D$.

(a) Put

$$\begin{aligned} S_{12} &= 600\,000. \\ \tfrac{1}{2}n\big(2a + (n-1)d\big) &= 600\,000 \\ 6(2a + 11d) &= 600\,000 \\ 6(50\,000 + 11D) &= 600\,000 \\ 50\,000 + 11D &= 100\,000 \\ D &= 4545\tfrac{5}{11} \end{aligned}$$

Hence the annual increment is about \$4545.

(b)

$$\begin{aligned} \text{Final salary } &= T_{12} \\ &= a + 11d \\ &= 25\,000 + 11 \times 4545\tfrac{5}{11} \\ &= \$75\,000. \end{aligned}$$

OR

$$\begin{aligned} S_n &= \tfrac{1}{2}n(a+\ell) \\ 600\,000 &= \tfrac{1}{2} \times 12 \times (25\,000 + \ell) \\ 100\,000 &= 25\,000 + \ell \\ \ell &= 75\,000, \end{aligned}$$

so her final salary is \$75 000.

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Formulae for Geometric Sequences: Geometric sequences involve the one further idea of the limiting sum.

GEOMETRIC SEQUENCES:

- A sequence T_n is called a *geometric sequence* if
$$\frac{T_n}{T_{n-1}} = r, \quad \text{for } n \ge 2,$$
where r is a constant, called the *common ratio*.
- The nth term of a GP is given by
$$T_n = ar^{n-1}.$$
- Three terms T_1, T_2 and T_3 are in GP if $\dfrac{T_3}{T_2} = \dfrac{T_2}{T_1}$.

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- The *geometric mean* of a and b is $\sqrt{ab}$ or $-\sqrt{ab}$.
- The sum S_n of the first n terms of a GP is
$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{(easier when } r > 1\text{)},$$
$$\text{or } S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{(easier when } r < 1\text{)}.$$
- The limiting sum S_∞ exists if and only if $-1 < r < 1$, and then
$$S_\infty = \frac{a}{1 - r}.$$

The following worked example is a typical problem on GPs, involving both the nth term T_n and the nth partial sum S_n. Notice the use of the change-of-base formula to solve exponential equations by logarithms. For example,
$$\log_{1{\cdot}05} 1{\cdot}5 = \frac{\log_e 1{\cdot}5}{\log_e 1{\cdot}05}.$$

WORKED EXERCISE: [Inflation and GPs] The General Widget Company sells 2000 widgets per year, beginning in 1991, when the price was \$300 per widget. Each year, the price rises 5% due to cost increases.

(a) Find the total sales in 1996.

(b) Find the first year in which total sales will exceed \$900 000.

(c) Find the total sales from the foundation of the company to the end of 2010.

(d) During which year will the total sales of the company since its foundation first exceed \$20 000 000?

SOLUTION: The annual sales form a GP with $a = 600\,000$ and $r = 1{\cdot}05$.

(a) The sales in any one year constitute the nth term T_n of the series,

and
$$\begin{aligned} T_n &= ar^{n-1} \\ &= 600\,000 \times 1{\cdot}05^{n-1}. \end{aligned}$$

Hence
$$\begin{aligned} \text{sales in 1996} &= T_6 \\ &= 600\,000 \times 1{\cdot}05^5 \\ &\doteqdot \$765\,769. \end{aligned}$$

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(b) Put $T_n > 900\,000.$

Then $600\,000 \times 1{\cdot}05^{n-1} > 900\,000$

$1{\cdot}05^{n-1} > 1{\cdot}5$

$n - 1 > \log_{1\cdot05} 1{\cdot}5,$

and using the change-of-base formula, $n - 1 > \dfrac{\log_e 1{\cdot}5}{\log_e 1{\cdot}05} \doteqdot 8{\cdot}31$

$n > 9{\cdot}31.$

Hence $n = 10$, and sales first exceed \$900 000 in 2000.

(c) The total sales since foundation constitute the nth partial sum S_n of the series,

and
$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{600\,000 \times (1{\cdot}05^n - 1)}{0{\cdot}05} \\ &= 12\,000\,000 \times (1{\cdot}05^n - 1). \end{aligned}$$

Hence total sales to 2010 $= S_{20}$

$= 12\,000\,000(1{\cdot}05^{20} - 1)$

$\doteqdot \$19\,839\,572.$

(d) Put $S_n > 20\,000\,000.$

Then $12\,000\,000 \times (1{\cdot}05^n - 1) > 20\,000\,000$

$1{\cdot}05^n > 2\tfrac{2}{3}$

$n > \log_{1\cdot05} 2\tfrac{2}{3},$

and using the change-of-base formula, $n > \dfrac{\log 2\frac{2}{3}}{\log 1{\cdot}05} \doteqdot 20{\cdot}1.$

Hence $n = 21$, and cumulative sales will first exceed \$20 000 000 in 2011.

Taking Logarithms when the Base is Less than 1, and Limiting Sums: When the base is less than 1, passing from an index inequation to a log inequation reverses the inequality sign. For example,

$$(\tfrac{1}{2})^n < \tfrac{1}{8} \qquad \text{means} \qquad n > 3.$$

The following worked exercise demonstrates this. Moreover, the GP in the exercise has a limiting sum because the ratio is positive and less than 1. This limiting sum is used to interpret the word 'eventually'.

WORKED EXERCISE: Sales from the Gumnut Softdrinks Factory in Wadelbri were 50 000 bottles in 2001, but are declining by 6% every year. Nevertheless, the company will always continue to trade.

(a) In what year will sales first fall below 20 000?

(b) What will the total sales from 2001 onwards be eventually?

(c) What proportion of those sales will occur by the end of 2020?

SOLUTION: The sales form a GP with $a = 50\,000$ and $r = 0{\cdot}94$.

(a) Put $T_n < 20\,000.$

Then $ar^{n-1} < 20\,000$

$50\,000 \times 0{\cdot}94^{n-1} < 20\,000$

$0{\cdot}94^{n-1} < 0{\cdot}4$

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$$n - 1 > \log_{0{\cdot}94} 0{\cdot}4 \quad \text{(the inequality reverses)}$$
$$n - 1 > \frac{\log_e 0{\cdot}4}{\log_e 0{\cdot}94} \doteqdot 14{\cdot}8$$
$$n \geq 15{\cdot}8.$$

Hence $n = 16$, and sales will first fall below 20 000 in 2016.

Since $-1 < r < 1$, the series has a limiting sum.

(b) Eventual sales $= S_\infty$

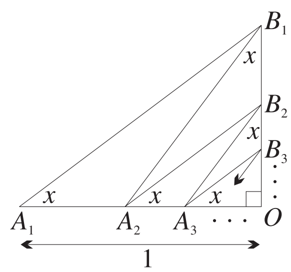
$$= \frac{a}{1-r} = \frac{50\,000}{0{\cdot}06} \doteqdot 833\,333.$$

(c) $$\frac{\text{Sales to 2020}}{\text{eventual sales}} = \frac{a(1-r^{20})}{1-r} \div \frac{a}{1-r} = 1 - r^{20} = 1 - 0{\cdot}94^{20} \doteqdot 71\%.$$

WORKED EXERCISE: [A harder trigonometric application]

(a) Consider the series $1 - \tan^2 x + \tan^4 x - \cdots$, where $-90° < x < 90°$.
  (i) For what values of x does the series converge?
  (ii) What is the limit when it does converge?

(b) In the diagram, $\triangle OA_1B_1$ is right-angled at O, OA_1 has length 1, and $\angle OA_1B_1 = x$, where $x < 45°$. Construct $\angle OB_1A_2 = x$, and construct $A_2B_2 \parallel A_1B_1$. Continue the construction of A_3, B_3, A_4, ...
  (i) Show that $A_1A_2 = 1 - \tan^2 x$ and $A_3A_4 = \tan^4 - \tan^6 x$.
  (ii) Find the limiting sum of $A_1A_2 + A_3A_4 + A_5A_6 + \cdots$.
  (iii) Find the limiting sum of $A_2A_3 + A_4A_5 + A_6A_7 + \cdots$.



SOLUTION:

(a) The series is a GP with $a = 1$ and $r = -\tan^2 x$.

  (i) Hence the series converges when $\tan^2 x < 1$,
  that is, when $-1 < \tan x < 1$,
  so from the graph of $\tan x$, $-45° < x < 45°$.

  (ii) When the series converges, $$S_\infty = \frac{a}{1-r} = \frac{1}{1+\tan^2 x} = \cos^2 x, \text{ since } 1 + \tan^2 x = \sec^2 x.$$

(b) (i) In $\triangle OA_1B_1$, $OB_1 = \tan x$.
  In $\triangle OB_1A_2$, $\dfrac{OA_2}{OB_1} = \tan x$,
  so $OA_2 = \tan^2 x$,
  hence $A_1A_2 = 1 - \tan^2 x$.

  (ii) Continuing the process, $OA_3 = OA_2 \times \tan^2 x = \tan^4 x$,
  and $OA_4 = OA_3 \times \tan^2 x = \tan^6 x$,
  so $A_3A_4 = \tan^4 x - \tan^6 x$.
  Hence $$A_1A_2 + A_3A_4 + \cdots = 1 - \tan^2 x + \tan^4 x - \tan^6 x + \cdots = \cos^2 x, \text{ by part (a).}$$

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(iii) Every piece of OA_1 is on $A_1A_2 + A_3A_4 + \cdots$ or on $A_2A_3 + A_4A_5 + \cdots$,

$$\begin{aligned} \text{so } A_2A_3 + A_4A_5 + \cdots &= OA_1 - (A_1A_2 + A_3A_4 + \cdots) \\ &= 1 - \cos^2 x \\ &= \sin^2 x. \end{aligned}$$

Exercise 7A

NOTE: The theory for this exercise was covered in Chapter Six of the Year 11 volume. This exercise is therefore a medley of problems on APs and GPs, with two introductory questions to revise the formulae for APs and GPs.

1. (a) Five hundred terms of the series $102 + 104 + 106 + \cdots$ are added. What is the total?
(b) In a particular arithmetic series, there are 48 terms between the first term 15 and the last term -10. What is the sum of all the terms in the series?
(c) (i) Show that the series $100+97+94+\cdots$ is an AP, and find the common difference.
(ii) Show that the nth term is $T_n = 103 - 3n$, and find the first negative term.
(iii) Find an expression for the sum S_n of the first n terms, and show that 68 is the minimum number of terms for which S_n is negative.

2. (a) The first few terms of a particular series are $2000 + 3000 + 4500 + \cdots$.
(i) Show that it is a geometric series, and find the common ratio.
(ii) What is the sum of the first five terms?
(iii) Explain why the series does not converge.
(b) Consider the series $18 + 6 + 2 + \cdots$.
(i) Show that it is a geometric series, and find the common ratio.
(ii) Explain why this geometric series has a limiting sum, and find its value.
(iii) Show that the limiting sum and the sum of the first ten terms are equal, correct to the first three decimal places.

3. A secretary starts on an annual salary of \$30 000, with annual increments of \$2000.
(a) Find his annual salary, and his total earnings, at the end of ten years.
(b) In which year will his salary be \$42 000?

4. An accountant receives an annual salary of \$40 000, with 5% increments each year.
(a) Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.
(b) In which year will her salary first exceed \$70 000?

5. Lawrence and Julian start their first jobs on low wages. Lawrence starts at \$25 000 per annum, with annual increases of \$2500. Julian starts at the lower wage of \$20 000 per annum, with annual increases of 15%.
(a) Find Lawrence's annual wages in each of the first three years, and explain why they form an arithmetic sequence.
(b) Find Julian's annual wages in each of the first three years, and explain why they form a geometric sequence.
(c) Show that the first year in which Julian's annual wage is the greater of the two will be the sixth year, and find the difference, correct to the nearest dollar.

6. (a) An initial salary of \$50 000 increases each year by \$3000. In which year will the salary first be at least twice the original salary?
(b) An initial salary of \$50 000 increases by 4% each year. In which year will the salary first be at least twice the original salary?

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7. A certain company manufactures three types of shade cloth. The product with code SC50 cuts out 50% of harmful UV rays, SC75 cuts out 75% and SC90 cuts out 90% of UV rays. In the following questions, you will need to consider the amount of UV light let through.
(a) What percentage of UV light does each cloth let through?
(b) Show that two layers of SC50 would be equivalent to one layer of SC75 shade cloth.
(c) Use trial and error to find the minimum number of layers of SC50 that would be required to cut out at least as much UV light as one layer of SC90.
(d) Similarly, find how many layers of SC50 would be required to cut out 99% of UV rays.

8. Olim, Pixi, Thi (pronounced 'tea'), Sid and Nee work in the sales division of a calculator company. Together they find that sales of scientific calculators are dropping by 150 per month, while sales of graphics calculators are increasing by 150 per month.
(a) Current sales of all calculators total 20 000 per month, and graphics calculators account for 10% of sales. How many graphics calculators are sold per month?
(b) How many more graphics calculators will be sold per month by the sales team six months from now?
(c) Assuming that current trends continue, how long will it be before all calculators sold by the company are graphics calculators?

DEVELOPMENT

9. One Sunday, 120 days before Christmas, Franksworth store publishes an advertisement saying '120 shopping days until Christmas'. Franksworth subsequently publishes similar advertisements every Sunday until Christmas.
(a) How many times does Franksworth advertise?
(b) Find the sum of the numbers of days published in all the advertisements.
(c) On which day of the week is Christmas?

10. A farmhand is filling a row of feed troughs with grain. The distance between adjacent troughs is 5 metres, and he has parked the truck with the grain 1 metre from the closest trough. He decides that he will fill the closest trough first and work his way to the far end. Each trough requires three bucketloads to fill it completely .
(a) How far will the farmhand walk to fill the 1st trough and return to the truck? How far for the 2nd trough? How far for the 3rd trough?
(b) How far will the farmhand walk to fill the nth trough and return to the truck?
(c) If he walks a total of 156 metres to fill the furthest trough, how many feed troughs are there?
(d) What is the total distance he will walk to fill all the troughs?

11. Yesterday, a tennis ball used in a game of cricket in the playground was hit onto the science block roof. Luckily it rolled off the roof. After bouncing on the playground it reached a height of 3 metres. After the next bounce it reached 2 metres, then $1\frac{1}{3}$ metres and so on.
(a) What was the height reached after the nth bounce?
(b) What was the height of the roof the ball fell from?
(c) The last time the ball bounced, its height was below 1 cm for the first time. After that it rolled away across the playground.
(i) Show that $(\frac{3}{2})^{n-1} > 300$. (ii) How many times did the ball bounce?

12. A certain algebraic equation is being solved by the method of halving the interval, with the two starting values 4 units apart. The pen of a plotter begins at the left-hand value, and then moves left or right to the location of each successive midpoint. What total distance will the pen have travelled eventually?

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13. Theodor earns \$30 000 in his first year, and his salary increases each year by a fixed amount \$$D$.

(a) Find D if his salary in his tenth year is \$58 800.

(b) Find D if his total earnings in the first ten years are \$471 000.

(c) If $D = 2200$, in which year will his salary first exceed \$60 000?

(d) If $D = 2000$, show that his total earnings first exceed \$600 000 during his 14th year.

14. Madeline opens a business selling computer stationery. In its first year, the business has sales of \$200 000, and each year sales are 20% more than the previous year's sales.

(a) In which year do annual sales first exceed \$1 000 000?

(b) In which year do total sales since foundation first exceed \$2 000 000?

15. Madeline's sister opens a hardware store. Sales in successive years form a GP, and sales in the fifth year are half the sales in the first year. Let sales in the first year be \$$F$.

(a) Find, in exact form, the ratio of the GP.

(b) Find the total sales of the company as time goes on, as a multiple of the first year's sales, correct to two decimal places.

16. [Limiting sums of trigonometric series]

(a) Find when each series has a limiting sum, and find that limiting sum:

(i) $1 + \cos^2 x + \cos^4 x + \cdots$ (ii) $1 + \sin^2 x + \sin^4 x + \cdots$

(b) Find, in terms of $t = \tan \frac{1}{2}x$, the limiting sums of these series when they converge:

(i) $1 - \sin x + \sin^2 x - \cdots$ (ii) $1 + \sin x + \sin^2 x + \cdots$

(c) Show that when these series converge:

(i) $1 - \cos x + \cos^2 x - \cdots = \frac{1}{2}\sec^2 \frac{1}{2}x$ (ii) $1 + \cos x + \cos^2 x + \cdots = \frac{1}{2}\operatorname{cosec}^2 \frac{1}{2}x$

17.

Two bulldozers are sitting in a construction site facing each other. Bulldozer A is at $x = 0$, and bulldozer B is 36 metres away at $x = 36$. A bee is sitting on the scoop at the very front of bulldozer A. At 7:00 am the workers start up both bulldozers and start them moving towards each other at the same speed V m/s. The bee is disturbed by the commotion and flies at twice the speed of the bulldozers to land on the scoop of bulldozer B.

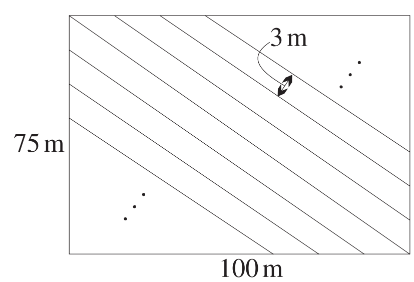
(a) Show that the bee reaches bulldozer B when it is at $x = 24$.

(b) Immediately the bee lands, it takes off again and flies back to bulldozer A. Where is bulldozer A when the two meet?

(c) Assume that the bulldozers keep moving towards each other and the bee keeps flying between the two, so that the bee will eventually be squashed.

(i) Where will this happen? (ii) How far will the bee have flown?

18. The area available for planting in a particular paddock of a vineyard measures 100 metres by 75 metres. In order to make best use of the sun, the grape vines are planted in rows diagonally across the paddock, as shown in the diagram, with a 3-metre gap between adjacent rows.



(a) What is the length of the diagonal of the field?

(b) What is the length of each row on either side of the diagonal?

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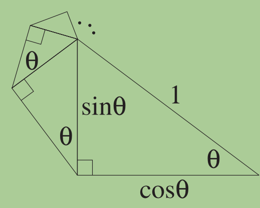
(c) Confirm that each row two away from the diagonal is 112·5 metres long.
(d) Show that the lengths of these rows form an arithmetic sequence.
(e) Hence find the total length of all the rows of vines in the paddock.

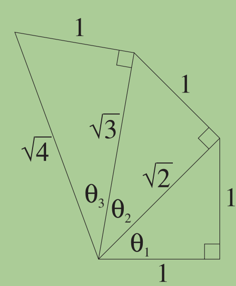
EXTENSION

19. The diagram shows the first few triangles in a spiral of similar right-angled triangles, each successive one built with its hypotenuse on a side of the previous one.



(a) What is the area of the largest triangle?
(b) Use the result for the ratio of areas of similar figures to show that the areas of successive triangles form a geometric sequence. What is the common ratio?
(c) Hence show that the limiting sum of the areas of the triangles is $\frac{1}{2}\tan\theta$.

20. The diagram shows the beginning of a spiral created when each successive right-angled triangle is constructed on the hypotenuse of the previous triangle. The altitude of each triangle is 1, and it is easy to show by Pythagoras' theorem that the sequence of hypotenuse lengths is $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \cdots$. Let the base angle of the nth triangle be θ_n. Clearly θ_n gets smaller, but does this mean that the spiral eventually stops turning? Answer the following questions to find out.



(a) Write down the value of $\tan\theta_n$.
(b) Show that $\displaystyle\sum_{n=1}^{k}\theta_n \ge \frac{1}{2}\sum_{n=1}^{k}\frac{1}{n}$. [HINT: $\theta \ge \frac{1}{2}\tan\theta$, for $0 \le \theta \le \frac{\pi}{4}$.]
(c) By sketching $y = \dfrac{1}{x}$ and constructing the upper rectangle on each of the intervals $1 \le x \le 2,\ 2 \le x \le 3,\ 3 \le x \le 4,\ \ldots$, show that $\displaystyle\sum_{n=1}^{k}\frac{1}{n} \ge \int_1^k \frac{1}{n}\,dn$.
(d) Does the total angle through which the spiral turns approach a limit?

7 B Simple and Compound Interest

This section will review the formulae for simple and compound interest, but with greater attention to the language of functions and of sequences. Simple interest can be understood mathematically both as an arithmetic sequence and as a linear function. Compound interest or depreciation can be understood both as a geometric sequence and as an exponential function.

Simple Interest, Arithmetic Sequences and Linear Functions: The well-known formula for simple interest is $I = PRn$. But if we want the total amount A_n at the end of n units of time, we need to add the principal P — this gives $A_n = P + PRn$, which is a linear function of n. Substituting into this function the positive integers $n = 1, 2, 3, \ldots$ gives the sequence

$$P + PR,\quad P + 2PR,\quad P + 3PR,\ \ldots$$

which is an AP with first term $P + PR$ and common difference PR.

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3 SIMPLE INTEREST: Suppose that a principal $\$P$ earns simple interest at a rate R per unit time for n units of time. Then the simple interest $\$I$ earned is

$$I = PRn.$$

The total amount $\$A_n$ after n units of time is a linear function of n,

$$A_n = P + PRn.$$

This forms an AP with first term $P + PR$ and common difference PR.

Be careful that the interest rate here is a number, not a percentage. For example, if the interest rate is 7% pa, then $R = 0{\cdot}07$. (The initials 'pa' stand for 'per annum', which is Latin for 'per year'.)

WORKED EXERCISE: Find the principal $\$P$, if investing $\$P$ at 6% pa simple interest yields a total of \$6500 at the end of five years.

SOLUTION: Put $P + PRn = 6500.$
Since $R = 0{\cdot}06$ and $n = 5$, $P(1 + 0{\cdot}30) = 6500$
$\boxed{\div 1{\cdot}3}$ $P = \$5000.$

Compound Interest, Geometric Sequences and Exponential Functions: The well-known formula for compound interest is $A_n = P(1 + R)^n$. First, this is an exponential function of n, with base $1 + R$. Secondly, substituting $n = 1, 2, 3, \ldots$ into this function gives the sequence

$$P(1 + R),\ \ P(1 + R)^2,\ \ P(1 + R)^3,\ \ \ldots$$

which is a GP with first term $P(1 + R)$ and common ratio $1 + R$.

4 COMPOUND INTEREST: Suppose that a principal $\$P$ earns compound interest at a rate R per unit time for n units of time, compounded every unit of time. Then the total amount after n units of time is an exponential function of n,

$$A_n = P(1 + R)^n.$$

This forms a GP with first term $P(1 + R)$ and common ratio $1 + R$.

Note that the formula only works when compounding occurs after every unit of time. For example, if the interest rate is 18% per year with interest compounded monthly, then the units of time must be months, and the interest rate per month is $R = 0{\cdot}18 \div 12 = 0{\cdot}015$. Unless otherwise stated, compounding occurs over the unit of time mentioned when the interest rate is given.

PROOF: Although the formula was developed in earlier years, it is vital to understand how it arises, and how the process of compounding generates a GP.
The initial principal is P, and the interest is R per unit time.
Hence the amount A_1 at the end of one unit of time is

$$A_1 = \text{principal} + \text{interest} = P + PR = P(1 + R).$$

This means that adding the interest is effected by multiplying by $1 + R$.
Similarly, the amount A_2 is obtained by multiplying A_1 by $1 + R$:

$$A_2 = A_1(1 + R) = P(1 + R)^2.$$

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Then, continuing the process,

$$A_3 = A_2(1+R) = P(1+R)^3,$$
$$A_4 = A_3(1+R) = P(1+R)^4,$$

so that when the money has been invested for n units of time,

$$A_n = A_{n-1}(1+R) = P(1+R)^n.$$

WORKED EXERCISE: Amelda takes out a loan of \$5000 at a rate of 12% pa, compounded monthly. She makes no repayments.

(a) Find the total amount owing at the end of five years.

(b) Find when, correct to the nearest month, the amount owing doubles.

SOLUTION: Because the interest is compounded every month, the units of time must be months. The interest rate is therefore 1% per month, and $R = 0{\cdot}01$.

(a) $A_{60} = P \times 1{\cdot}01^{60}$ (5 years is 60 months),
$\doteqdot \$9083.$

(b) Put $A_n = 10\,000.$

$$\begin{aligned} \text{Then } 5000 \times 1{\cdot}01^n &= 10\,000 \\ 1{\cdot}01^n &= 2 \\ n &= \log_{1{\cdot}01} 2 \\ &= \frac{\log 2}{\log 1{\cdot}01}, \text{ using the change-of-base formula,} \\ &\doteqdot 70 \text{ months.} \end{aligned}$$

Depreciation: Depreciation is usually expressed as the loss per unit time of a percentage of the current price of an item. The formula for depreciation is therefore the same as the formula for compound interest, except that the rate is negative.

5 DEPRECIATION: Suppose that goods originally costing $\$P$ depreciate at a rate R per unit time for n units of time. Then the total amount after n units of time is

$$A_n = P(1-R)^n.$$

WORKED EXERCISE: An espresso machine bought on 1st January 2001 depreciates at $12\frac{1}{2}\%$ pa. In which year will the value drop below 10% of the original cost, and what will be the loss of value during that year, as a percentage of the original cost?

SOLUTION: In this case, $R = -0{\cdot}125$ is negative, because the value is decreasing.

Let the initial value be P. Then $A_n = P \times 0{\cdot}875^n$.

Put $A_n = 0{\cdot}1 \times P$, to find when the value has dropped to 10%.

$$\begin{aligned} \text{Then } P \times 0{\cdot}875^n &= 0{\cdot}1 \times P \\ n &= \frac{\log 0{\cdot}1}{\log 0{\cdot}875} \\ &\doteqdot 17{\cdot}24. \end{aligned}$$

Hence the depreciated value will drop below 10% during 2018.

$$\begin{aligned} \text{Loss during that year } &= A_{17} - A_{18} \\ &= (0{\cdot}875^{17} - 0{\cdot}875^{18})P, \\ \text{so} \quad \text{percentage loss} &= (0{\cdot}875^{17} - 0{\cdot}875^{18}) \times 100\% \\ &\doteqdot 1{\cdot}29\%. \end{aligned}$$

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Exercise 7B

NOTE: This exercise combines the work on series from Chapter Six of the Year 11 volume, and simple and compound interest from Years 9 and 10.

1. (a) Find the total value of an investment of \$5000 that earns 7% per annum simple interest for three years.
(b) A woman invested an amount for nine years at a rate of 6% per annum. She earned a total of \$13 824 in simple interest. What was the initial amount she invested?
(c) A man invested \$23 000 at 3·25% per annum simple interest, and at the end of the investment period he withdrew all the funds from the bank, a total of \$31 222.50. How many years did the investment last?
(d) The total value of an investment earning simple interest after six years is \$22 610. If the original investment was \$17 000, what was the interest rate?

2. At the end of each year, a man wrote down the value of his investment of \$10 000, invested at 6·5% per annum simple interest for five years. He then added up these five values and thought that he was very rich.
(a) What was the total he arrived at?
(b) What was the actual value of his investment at the end of five years?

3. Howard is arguing with Juno over who has the better investment. Each invested \$20 000 for one year. Howard has his invested at 6·75% per annum simple interest, while Juno has hers invested at 6·6% per annum compound interest.
(a) On the basis of this information, who has the better investment, and what are the final values of the two investments?
(b) Juno then points out that her interest is compounded monthly, not yearly. Now who has the better investment?

4. (a) Calculate the value to which an investment of \$12 000 will grow if it earns compound interest at a rate of 7% per annum for five years.
(b) The final value of an investment, after ten years earning 15% per annum, compounded yearly, was \$32 364. Find the amount invested, correct to the nearest dollar.
(c) A bank customer earned \$7824.73 in interest on a \$40 000 investment at 6% per annum, compounded quarterly.
(i) Show that $1{\cdot}015^n \doteqdot 1{\cdot}1956$, where n is the number of quarters.
(ii) Hence find the period of the investment, correct to the nearest quarter.
(d) After six years of compound interest, the final value of a \$30 000 investment was \$45 108.91. What was the rate of interest, correct to two significant figures, if it was compounded annually?

5. What does \$1000 grow to if invested for a year at 12% pa compound interest, compounded:
(a) annually, (c) quarterly, (e) weekly (for 52 weeks),
(b) six-monthly, (d) monthly, (f) daily (for 365 days)?
Compare these values with $1000 \times e^{0{\cdot}12}$. What do you notice?

6. A company has bought several cars for a total of \$229 000. The depreciation rate on these cars is 15% per annum. What will be the net worth of the fleet of cars five years from now?

DEVELOPMENT

7. Find the total value A_n when a principal P is invested at 12% pa simple interest for n years. Hence find the smallest number of years required for the investment:
(a) to double, (b) to treble, (c) to quadruple, (d) to increase by a factor of 10.

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8. Find the total value A_n when a principal P is invested at 12% pa compound interest for n years. Hence find the smallest number of years for the investment:
(a) to double, (b) to treble, (c) to quadruple, (d) to increase by a factor of 10.

9. A student was asked to find the original value, correct to the nearest dollar, of an investment earning 9% per annum, compounded annually for three years, given its current value of \$54 391.22.
(a) She incorrectly thought that since she was working in reverse, she should use the depreciation formula. What value did she get?
(b) What is the correct answer?

10. An amount of \$10 000 is invested for five years at 4% pa interest, compounded monthly.
(a) Find the final value of the investment.
(b) What rate of simple interest, correct to two significant figures, would be needed to yield the same final balance?

11. Xiao and Mai win a prize in the lottery and decide to put \$100 000 into a retirement fund offering 8·25% per annum interest, compounded monthly. How long will it be before their money has doubled? Give your answer correct to the nearest month.

12. The present value of a company asset is \$350 000. If it has been depreciating at $17\frac{1}{2}$% per annum for the last six years, what was the original value of the asset, correct to the nearest \$1000?

13. Thirwin, Neri, Sid and Nee each inherit \$10 000. Each invests the money for one year. Thirwin invests his money at 7·2% per annum simple interest. Neri invests hers at 7·2% per annum, compounded annually. Sid invests his at 7% per annum, compounded monthly. Nee invests in certain shares with a return of 8·1% per annum, but must pay stockbrokers' fees of \$50 to buy the shares initially and again to sell them at the end of the year. Who is furthest ahead at the end of the year?

14. (a) A principal P is invested at a compound interest rate of r per period.
(i) Write down A_n, the total value after n periods.
(ii) Hence find the number of periods required for the total value to double.
(b) Suppose that a simple interest rate of R per period applied instead.
(i) Write down B_n, the total value after n periods.
(ii) Further suppose that for a particular value of n, $A_n = B_n$. Derive a formula for R in terms of r and n.

EXTENSION

15. [Compound interest and e^x] Referring to question 5, explain the significance for compound interest of $\lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = e^x$, proven in Exercise 12B of the Year 11 volume.

16. (a) Find the total value A_n if P is invested at a simple interest rate R for n periods.
(b) Show, by means of the binomial theorem, that the total value of the investment when compound interest is applied may be written as $A_n = P + PRn + P\displaystyle\sum_{k=2}^{n} {}^n\mathrm{C}_k\, R^k$.
(c) Explain what each of the three terms of the formula in part (b) represents.

17. (a) Write out the terms of $P(1+R)^n$ as a binomial expansion.
(b) Show that the term $P\,{}^n\mathrm{C}_k\,R^k$ is the sum of interest earned for any, not necessarily consecutive, k years over the life of the investment.
(c) What is the significance of the greatest term in the binomial expansion, in this context?

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7 C Investing Money by Regular Instalments

Many investment schemes, typically superannuation schemes, require money to be invested at regular intervals such as every month or every year. This makes things difficult, because each individual instalment earns compound interest for a different length of time. Hence calculating the value of these investments at some future time requires the theory of GPs.

This topic is intended to be an application of GPs, and learning formulae is not recommended.

Developing the GP and Summing It: The most straightforward way to solve these problems is to find what each instalment grows to as it accrues compound interest. These final amounts form a GP, which can then be summed.

WORKED EXERCISE: Robin and Robyn are investing \$10 000 in a superannuation scheme on 1st July each year, beginning in 2000. The money earns compound interest at 8% pa, compounded annually.

(a) How much will the fund amount to by 30th June 2020?

(b) Find the year in which the fund first exceeds \$700 000 on 30th June.

(c) What annual instalment would have produced \$1 000 000 by 2020?

SOLUTION: Because of the large numbers involved, it is usually easier to work with pronumerals, apart perhaps from the (fixed) interest rate.

Let M be the annual instalment, so $M = 10\,000$ in parts (a) and (b),
and let A_n be the value of the fund at the end of n years.
After the first instalment is invested for n years, it amounts to $M \times 1{\cdot}08^n$,
after the second instalment is invested for $n-1$ years, it amounts to $M \times 1{\cdot}08^{n-1}$,
after the nth instalment is invested for just 1 year, it amounts to $M \times 1{\cdot}08$,
so $\quad A_n = 1{\cdot}08M + 1{\cdot}08^2 M + \cdots + 1{\cdot}08^n M$.
This is a GP with first term $a = 1{\cdot}08M$, ratio $r = 1{\cdot}08$, and n terms.

$$\begin{aligned} \text{Hence } A_n &= \frac{a(r^n-1)}{r-1} \\ &= \frac{1{\cdot}08M \times (1{\cdot}08^n - 1)}{0{\cdot}08} \\ A_n &= 13{\cdot}5M \times (1{\cdot}08^n - 1). \end{aligned}$$

(a) Substituting $n = 20$ and $M = 10\,000$,

$$\begin{aligned} A_n &= 13{\cdot}5 \times 10\,000 \times (1{\cdot}08^{20} - 1) \\ &\doteqdot \$494\,229. \end{aligned}$$

(b) Substituting $M = 10\,000$ and $A_n = 700\,000$,

$$\begin{aligned} 700\,000 &= 13{\cdot}5 \times 10\,000 \times (1{\cdot}08^n - 1) \\ 1{\cdot}08^n - 1 &= \tfrac{70}{13\cdot5} \\ n &= \frac{\log(\frac{70}{13\cdot5}+1)}{\log 1{\cdot}08} \\ &\doteqdot 23{\cdot}68. \end{aligned}$$

Hence the fund first exceeds \$700 000 on 30th June 2024.

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(c) Substituting $A_n = 1\,000\,000$ and $n = 20$,

$$1\,000\,000 = 13{\cdot}5 \times M \times (1{\cdot}08^{20} - 1)$$

$$M = \frac{1\,000\,000}{13{\cdot}5 \times (1{\cdot}08^{20} - 1)}$$

$$\fallingdotseq \$20\,234.$$

WORKED EXERCISE: Charmaine is offered the choice of two superannuation schemes, both of which will yield the same amount at the end of ten years.

- Pay \$600 per month, with interest of 7·8% pa, compounded monthly.
- Pay weekly, with interest of 7·8% pa, compounded weekly.

(a) What is the final value of the first scheme?
(b) What are the second scheme's weekly instalments?
(c) Which scheme would cost her more per year?

SOLUTION: The following solution begins by generating the general formula for the amount A_n after n units of time, in terms of the instalment M and the rate R, and this formula is then applied in parts (a) and (b). An alternative approach would be to generate separately each of the formulae required in parts (a) and (b). Whichever approach is adopted, the formulae must be derived rather than just quoted from memory.

Let M be the instalment and R the rate per unit time,
and let A_n be the value of the fund at the end of n units of time.
The first instalment is invested for n months, and so amounts to $M(1+R)^n$,
the second instalment is invested for $n-1$ months, and so amounts to $M(1+R)^{n-1}$,
and the last instalment is invested for 1 month, and so amounts to $M(1+R)$,
so $\quad A_n = M(1+R) + M(1+R)^2 + \cdots + M(1+R)^n$.
This is a GP with first term $a = M(1+R)$, ratio $r = (1+R)$, and n terms.

Hence $\quad A_n = \dfrac{a(r^n - 1)}{r - 1}$

$$A_n = \frac{M(1+R) \times \big((1+R)^n - 1\big)}{R}.$$

(a) For the first scheme, the interest rate is $\frac{7{\cdot}8}{12}\% = 0{\cdot}65\%$ per month,
so substitute $n = 120$, $M = 600$ and $R = 0{\cdot}0065$.

$$A_n = \frac{600 \times 1{\cdot}0065 \times (1{\cdot}0065^{120} - 1)}{0{\cdot}0065}$$

$$\fallingdotseq \$109\,257 \;\big(\text{retain in the memory for part(b)}\big).$$

(b) For the second scheme, the interest rate is $\frac{7{\cdot}8}{52}\% = 0{\cdot}15\%$ per week,
so substituting $R = 0{\cdot}0015$,

$$A_n = \frac{M \times 1{\cdot}0015 \times (1{\cdot}0015^n - 1)}{0{\cdot}0015}.$$

Writing this formula with M as the subject,

$$M = \frac{A_n \times 0{\cdot}0015}{1{\cdot}0015 \times (1{\cdot}0015^n - 1)},$$

and substituting $n = 520$ and $A_n = 109\,257$ (from memory),

$$M \fallingdotseq \$138{\cdot}65 \;\big(\text{retain in the memory for part(c)}\big).$$

(c) This is about \$7210·04 per year, compared with \$7200 per year for the first.

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An Alternative Approach Using Recursion: There is an alternative approach, using recursion, to developing the GPs involved in these calculations. Because the working is slightly longer, we have chosen not to display this method in the notes. It has, however, the advantage that its steps follow the progress of a banking statement. For those who are interested in the recursive method, it is developed in two structured questions at the end of the Development section in the following exercise.

Exercise 7C

1. A company makes contributions of \$3000 on 1st July each year to the superannuation fund of one of its employees. The money earns compound interest at 6·5% per annum. In the following parts, round all currency amounts correct to the nearest dollar.

(a) Let M be the annual contribution, and let A_n be the value of the fund at the end of n years.

(i) How much does the first instalment amount to at the end of n years?

(ii) How much does the second instalment amount to at the end of $n-1$ years?

(iii) What is the worth of the last contribution, invested for just one year?

(iv) Hence write down a series for A_n.

(b) Hence show that $A_n = \dfrac{1{\cdot}065\,M(1{\cdot}065^n - 1)}{0{\cdot}065}$.

(c) What will be the value of the fund after 25 years, and what will be the total amount of the contributions?

(d) Suppose that the employee wanted to achieve a total investment of \$300 000 after 25 years, by topping up the contributions.

(i) What annual contribution would have produced this amount?

(ii) By how much would the employee have to top up the contributions?

2. A company increases the annual wage of an employee by 4% on 1st January each year.

(a) Let M be the annual wage in the first year of employment, and let W_n be the wage in the nth year. Write down W_1, W_2 and W_n in terms of M.

(b) Hence show that the total amount paid to the employee is $A_n = \dfrac{M(1{\cdot}04^n - 1)}{0{\cdot}04}$.

(c) If the employee starts on \$30 000 and stays with the company for 20 years, how much will the company have paid over that time? Give your answer correct to the nearest dollar.

3. A person invests \$10 000 each year in a superannuation fund. Compound interest is paid at 10% per annum on the investment. The first payment is on 1st January 2001 and the last payment is on 1st January 2020.

(a) How much did the person invest over the life of the fund?

(b) Calculate, correct to the nearest dollar, the amount to which the 2001 payment has grown by the beginning of 2021.

(c) Find the total value of the fund when it is paid out on 1st January 2021.

DEVELOPMENT

4. Each year on her birthday, Jane's parents put \$20 into an investment account earning $9\frac{1}{2}\%$ per annum compound interest. The first deposit took place on the day of her birth. On her 18th birthday, Jane's parents gave her the account and \$20 cash in hand.

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(a) How much money had Jane's parents deposited in the account?
(b) How much money did she receive from her parents on her 18th birthday?

5. A man about to turn 25 is getting married. He has decided to pay \$5000 each year on his birthday into a combination life insurance and superannuation scheme that pays 8% compound interest per annum. If he dies before age 65, his wife will inherit the value of the insurance to that point. If he lives to age 65, the insurance company will pay out the value of the policy in full. Answer the following correct to the nearest dollar.
(a) The man is in a dangerous job. What will be the payout if he dies just before he turns 30?
(b) The man's father died of a heart attack just before age 50. Suppose that the man also dies of a heart attack just before age 50. How much will his wife inherit?
(c) What will the insurance company pay the man if he survives to his 65th birthday?

6. In 2001, the school fees at a private girls' school are \$10 000 per year. Each year the fees rise by $4\frac{1}{2}\%$ due to inflation.
(a) Susan is sent to the school, starting in Year 7 in 2001. If she continues through to her HSC year, how much will her parents have paid the school over the six years?
(b) Susan's younger sister is starting in Year 1 in 2001. How much will they spend on her school fees over the next twelve years if she goes through to her HSC?

7. A woman has just retired with a payment of \$500 000, having contributed for 25 years to a superannuation fund that pays compound interest at the rate of $12\frac{1}{2}\%$ per annum. What was the size of her annual premium, correct to the nearest dollar?

8. John is given a \$10 000 bonus by his boss. He decides to start an investment account with a bank that pays $6\frac{1}{2}\%$ per annum compound interest.
(a) If he makes no further deposits, what will be the balance of his account, correct to the nearest cent, 15 years from now?
(b) If instead he also makes an annual deposit of \$1000 at the beginning of each year, what will be the balance at the end of 15 years?

9. At age 20, a woman takes out a life insurance policy in which she agrees to pay premiums of \$500 per year until she turns 65, when she is to be paid a lump sum. The insurance company invests the money and gives a return of 9% per annum, compounded annually. If she dies before age 65, the company pays out the current value of the fund plus 25% of the difference had she lived until 65.
(a) What is the value of the payout, correct to the nearest dollar, at age 65?
(b) Unfortunately she dies at age 53, just before her 35th premium is due.
(i) What is the current value of the life insurance?
(ii) How much does the life insurance company pay her family?

10. A finance company has agreed to pay a retired couple a pension of \$15 000 per year for the next twenty years, indexed to inflation which is $3\frac{1}{2}\%$ per annum.
(a) How much will the company have paid the couple at the end of twenty years?
(b) Immediately after the tenth annual pension payment is made, the finance company increases the indexed rate to 4% per annum to match the increased inflation rate. Given these new conditions, how much will the company have paid the couple at the end of twenty years?

11. A person pays \$2000 into an investment fund every six months, and it earns interest at a rate of 6% pa, compounded monthly. How much is the fund worth at the end of ten years?

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NOTE: The following two questions illustrate an alternative approach to superannuation questions, using a recursive method to generate the appropriate GP. As mentioned in the notes above, the method has the disadvantage of requiring more steps in the working, but has the advantage that its steps follow the progress of a banking statement.

12. Cecilia deposits $\$M$ at the start of each month into a savings scheme that pays interest of 1% per month, compounded monthly. Let A_n be the amount in her account at the end of the nth month.

(a) Explain why $A_1 = 1{\cdot}01\,M$.

(b) Explain why $A_2 = 1{\cdot}01(M + A_1)$, and why $A_{n+1} = 1{\cdot}01(M + A_n)$, for $n \geq 2$.

(c) Use the recursive formulae in part (b), together with the value of A_1 in part (a), to obtain expressions for $A_2, A_3, \ldots, A_n$.

(d) Use the formula for the nth partial sum of a GP to show that $A_n = 101M(1{\cdot}01^n - 1)$.

(e) If each deposit is \$100, how much will be in the fund after three years?

(f) Hence find, correct to the nearest cent, how much each deposit M must be if Cecilia wants the fund to amount to \$30 000 at the end of five years.

13. A couple saves \$100 at the start of each week in an account paying 10·4% pa interest, compounded weekly. Let A_n be the amount in the account at the end of the nth week.

(a) Explain why $A_1 = 1{\cdot}002 \times 100$, and why $A_{n+1} = 1{\cdot}002(100 + A_n)$, for $n \geq 2$.

(b) Use these recursive formulae to obtain expressions for $A_2, A_3, \ldots, A_n$.

(c) Using GP formulae, show that $A_n = 50\,100(1{\cdot}01^n - 1)$.

(d) Hence find how many weeks it will be before the couple has \$100 000.

EXTENSION

14. Let V be the value of an investment of \$1000 earning compound interest at the rate of 10% per annum for n years.

(a) Draw up a table of values for V of values of n between 0 and 7.

(b) Plot these points and join them with a smooth curve. What type of curve is this?

(c) On the same graph add upper rectangles of width 1, add the areas of these rectangles, and give your answer correct to the nearest dollar.

(d) Compare your answer with the value of superannuation after seven years if \$1000 is deposited each year at the same rate of interest.

(i) What do you notice? (ii) What do you conclude?

15. (a) If you have access to a program like Excel™ for Windows 98™, try checking your answers to questions 1 to 10 using the built-in financial functions. In particular, the built-in Excel™ function `FV(rate, nper, pmt, pv, type)` seems to produce an answer different from what might be expected. Investigate this and explain the difference.

(b) If you have access to a program like Mathematica™, try checking your answers to questions 1 to 10, using the following function definitions.

(i) Calculate the final value of a superannuation fund, invested for n years at a rate of r per annum with annual premiums of $\$m$, using

```
Super[n_, r_, m_]:= m * (1 + r) * ((1 + r) ^ n - 1) / r.
```

(ii) Calculate the premiums if the final value of the fund is p, using

```
SupContrib[p_, n_, r_]:= p * r / ((1 + r) * ((1 + r) ^ n - 1)).
```

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7 D Paying Off a Loan

Long-term loans such as housing loans are usually paid off by regular instalments, with compound interest charged on the balance owing at any time. The calculations associated with paying off a loan are therefore similar to the investment calculations of the previous section. The extra complication is that an investment fund is always in credit, whereas a loan account is always in debit because of the large initial loan that must be repaid.

Developing the GP and Summing It: As with superannuation, the most straightforward method is to calculate the final value of each payment as it accrues compound interest, and then add these final values up using the theory of GPs. We must also deal with the final value of the initial loan.

WORKED EXERCISE: Natasha and Richard take out a loan of \$200 000 on 1st January 2002 to buy a house. Interest is charged at 12% pa, compounded monthly, and they will repay the loan in monthly instalments of \$2200.

(a) Find the amount owing at the end of n months.

(b) Find how long it takes to repay: (i) the full loan, (ii) half the loan.

(c) How long would repayment take if they were able to pay \$2500 per month?

(d) Why would instalments of \$1900 per month never repay the loan?

NOTE: The first repayment is normally made at the end of the first repayment period. In this example, that means on the last day of each month.

SOLUTION: Let $P = 200\,000$ be the principal, let M be the instalment, and let A_n be the amount still owing at the end of n months.

To find a formula for A_n, we need to calculate the value of each instalment under the effect of compound interest of 1% per month, from the time that it is paid.

The first instalment is invested for $n-1$ months, and so amounts to $M \times 1{\cdot}01^{n-1}$,

the second instalment is invested for $n-2$ months, and so amounts to $M \times 1{\cdot}01^{n-2}$,

the nth instalment is invested for no time at all, and so amounts to M.

The initial loan, after n months, amounts to $P \times 1{\cdot}01^n$.

Hence $\qquad A_n = P \times 1{\cdot}01^n \;-\; (M + 1{\cdot}01M + \cdots + 1{\cdot}01^{n-1}M)$.

The bit in brackets is a GP with first term $a = M$, ratio $r = 1{\cdot}01$, and n terms.

Hence
$$\begin{aligned} A_n &= P \times 1{\cdot}01^n \;-\; \frac{a(r^n - 1)}{r - 1} \\ &= P \times 1{\cdot}01^n \;-\; \frac{M(1{\cdot}01^n - 1)}{0{\cdot}01} \\ &= P \times 1{\cdot}01^n \;-\; 100M(1{\cdot}01^n - 1) \end{aligned}$$
or, reorganising, $A_n = 100M \;-\; 1{\cdot}01^n(100M - P)$.

(a) Substituting $P = 200\,000$ and $M = 2200$ gives
$$\begin{aligned} A_n &= 100 \times 2200 \;-\; 1{\cdot}01^n \times 20\,000 \\ &= 220\,000 \;-\; 1{\cdot}01^n \times 20\,000. \end{aligned}$$

(b) (i) To find when the loan is repaid, put $A_n = 0$:
$$\begin{aligned} 1{\cdot}01^n \times 20\,000 &= 220\,000 \\ n &= \frac{\log 11}{\log 1{\cdot}01} \\ &\doteqdot 20 \text{ years and 1 month.} \end{aligned}$$

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(ii) To find when the loan is half repaid, put $A_n = 100\,000$:

$$1{\cdot}01^n \times 20\,000 = 120\,000$$
$$n = \frac{\log 6}{\log 1{\cdot}01}$$
$$\doteqdot 15 \text{ years.}$$

(c) Substituting instead $M = 2500$ gives 100M =250 000,

so $A_n = 250\,000 - 1{\cdot}01^n \times 50\,000$.

Put $A_n = 0$, for the loan to be repaid.

Then $1{\cdot}01^n \times 50\,000 = 250\,000$

$$n = \frac{\log 5}{\log 1{\cdot}01}$$
$$\doteqdot 13 \text{ years and 6 months.}$$

(d) Substituting $M = 1900$ gives 100M =190 000,

so $A_n = 190\,000 - 1{\cdot}01^n \times (-10\,000)$, which is always positive.

This means that the debt would be increasing rather than decreasing.

Another way to understand this is to calculate

initial interest per month $= 200\,000 \times 0{\cdot}01$
$= 2000$,

so initially, \$2000 of the instalment is required just to pay the interest.

The Alternative Approach Using Recursion: As with superannuation, the GP involved in loan-repayment calculations can be developed using an alternative recursive method, whose steps follow the progress of a banking statement. Again, this method is developed in two structured questions at the end of the Development section in the following exercise.

Exercise **7D**

1. I took out a personal loan of \$10 000 with a bank for five years at an interest rate of 18% per annum, compounded monthly.

(a) Let P be the principal, let M be the size of each repayment to the bank, and let A_n be the amount owing on the loan after n months.

(i) To what does the initial loan amount after n months?

(ii) Write down the amount to which the first instalment grows by the end of the nth month.

(iii) Do likewise for the second instalment and for the nth instalment.

(iv) Hence write down a series for A_n.

(b) Hence show that $A_n = P \times 1{\cdot}015^n - \dfrac{M(1{\cdot}015^n - 1)}{0{\cdot}015}$.

(c) When the loan is paid off, what is the value of A_n?

(d) Hence find an expression for M in terms of P and n.

(e) Given the values of P and n above, find M, correct to the nearest dollar.

2. A couple takes out a \$250 000 mortgage on a house, and they agree to pay the bank \$2000 per month. The interest rate on the loan is 7·2% per annum, compounded monthly, and the contract requires that the loan be paid off within twenty years.

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(a) Again let A_n be the balance on the loan after n months, let P be the amount borrowed, and let M be the amount of each instalment. Find a series expression for A_n.

(b) Hence show that $A_n = P \times 1{\cdot}006^n - \dfrac{M(1{\cdot}006^n - 1)}{0{\cdot}006}$.

(c) Find the amount owing on the loan at the end of the tenth year, and state whether this is more or less than half the amount borrowed.

(d) Find A_{240}, and hence show that the loan is actually paid out in less than twenty years.

(e) If it is paid out after n months, show that $1{\cdot}006^n = 4$, and hence that $n = \dfrac{\log 4}{\log 1{\cdot}006}$.

(f) Find how many months early the loan is paid off.

3. As can be seen from the last two questions, the calculations involved with reducible loans are reasonably complex. For that reason, it is sometimes convenient to convert the reducible interest rate into a simple interest rate. Suppose that a mortgage is taken out on a \$180 000 house at 6·6% reducible interest per annum for a period of 25 years, with payments made monthly.

(a) Using the usual pronumerals, explain why $A_{300} = 0$.

(b) Find the size of each repayment to the bank.

(c) Hence find the total paid to the bank, correct to the nearest dollar, over the life of the loan.

(d) What amount is therefore paid in interest? Use this amount and the simple interest formula to calculate the simple interest rate per annum over the life of the loan, correct to two significant figures.

DEVELOPMENT

4. What is the monthly instalment necessary to pay back a personal loan of \$15 000 at a rate of $13\frac{1}{2}$% per annum over five years? Give your answer correct to the nearest dollar.

5. Most questions so far have asked you to round monetary amounts correct to the nearest dollar. This is not always wise, as this question demonstrates. A personal loan for \$30 000 is approved with the following conditions. The reducible interest rate is 13·3% per annum, with payments to be made at six-monthly intervals over five years.

(a) Find the size of each instalment, correct to the nearest dollar.

(b) Using this amount, show that $A_{10} \neq 0$, that is, the loan is not paid off in five years.

(c) Explain why this has happened.

6. A couple have worked out that they can afford to pay \$19 200 each year in mortgage payments. If the current home loan rate is 7·5% per annum, with payments made monthly over a period of 25 years, what is the maximum amount that the couple can borrow and still pay off the loan?

7. A company borrows \$500 000 from the bank at an interest rate of 5% per annum, to be paid in monthly instalments. If the company repays the loan at the rate of \$10 000 per month, how long will it take? Give your answer in whole months with an appropriate qualification.

8. Some banks offer a 'honeymoon' period on their loans. This usually takes the form of a lower interest rate for the first year. Suppose that a couple borrowed \$170 000 for their first house, to be paid back monthly over 15 years. They work out that they can afford to pay \$1650 per month to the bank. The standard rate of interest is $8\frac{1}{2}$% pa, but the bank also offers a special rate of 6% pa for one year to people buying their first home.

(a) Calculate the amount the couple would owe at the end of the first year, using the special rate of interest.

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(b) Use this value as the principal of the loan at the standard rate for the next 14 years. Calculate the value of the monthly payment that is needed to pay the loan off. Can the couple afford to agree to the loan contract?

9. A company buys machinery for \$500 000 and pays it off by 20 equal six-monthly instalments, the first payment being made six months after the loan is taken out. If the interest rate is 12% pa, compounded monthly, how much will each instalment be?

10. The current rate of interest on Bankerscard is 23% per annum, compounded monthly.

(a) If a cardholder can afford to repay \$1500 per month on the card, what is the maximum value of purchases that can be made in one day if the debt is to be paid off in two months?

(b) How much would be saved in interest payments if the cardholder instead saved up the money for two months before making the purchase?

11. Over the course of years, a couple have saved up \$300 000 in a superannuation fund. Now that they have retired, they are going to draw on that fund in equal monthly pension payments for the next twenty years. The first payment is at the beginning of the first month. At the same time, any balance will be earning interest at $5\frac{1}{2}\%$ per annum, compounded monthly. Let B_n be the balance left immediately after the nth payment, and let M be the amount of the pension instalment. Also, let $P = 300\,000$ and R be the monthly interest rate.

(a) Show that $B_n = P \times (1+R)^{n-1} - \dfrac{M\big((1+R)^n - 1\big)}{R}$.

(b) Why is $B_{240} = 0$? (c) What is the value of M?

NOTE: The following two questions illustrate the alternative approach to loan repayment questions, using a recursive method to generate the appropriate GP.

12. A couple buying a house borrow $\$P = \$150\,000$ at an interest rate of 6% pa, compounded monthly. They borrow the money at the beginning of January, and at the end of every month, they pay an instalment of $\$M$. Let A_n be the amount owing at the end of n months.

(a) Explain why $A_1 = 1{\cdot}005\,P - M$.

(b) Explain why $A_2 = 1{\cdot}005\,A_1 - M$, and why $A_{n+1} = 1{\cdot}005\,A_n - M$, for $n \ge 2$.

(c) Use the recursive formulae in part (b), together with the value of A_1 in part (a), to obtain expressions for A_2, A_3, …, A_n.

(d) Using GP formulae, show that $A_n = 1{\cdot}005^n\,P - 200M(1{\cdot}005^n - 1)$.

(e) Hence find, correct to the nearest cent, what each instalment should be if the loan is to be paid off in twenty years?

(f) If each instalment is \$1000, how much is still owing after twenty years?

13. Eric and Enid borrow $\$P$ to buy a house at an interest rate of 9·6% pa, compounded monthly. They borrow the money on 15th September, and on the 14th day of every subsequent month, they pay an instalment of $\$M$. Let A_n be the amount owing after n months have passed.

(a) Explain why $A_1 = 1{\cdot}008\,P - M$, and why $A_{n+1} = 1{\cdot}008\,A_n - M$, for $n \ge 2$.

(b) Use these recursive formulae to obtain expressions for A_2, A_3, …, A_n.

(c) Using GP formulae, show that $A_n = 1{\cdot}008^n\,P - 125M(1{\cdot}008^n - 1)$.

(d) If the maximum instalment they can afford is \$1200, what is the maximum they can borrow, if the loan is to be paid off in 25 years? (Answer correct to the nearest dollar.)

(e) Put $A_n = 0$ in part (c), and solve for n. Hence find how long will it take to pay off the loan of \$100 000 if each instalment is \$1000. (Round up to the next month.)

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EXTENSION

14. A finance company has agreed to pay a retired couple a pension of \$19 200 per year for the next twenty years, indexed to inflation that is $3\frac{1}{2}\%$ per annum.

(a) How much will the company have paid the couple at the end of twenty years?

(b) In return, the couple pay an up-front fee which the company invests at a compound interest rate of 7% per annum. The total value of the fee plus interest covers the pension payouts over the twenty-year period. How much did the couple pay the firm up front, correct to the nearest dollar?

15. [This question will be much simpler to solve using a computer for the calculations.] Suppose, using the usual notation, that a loan of $\$P$ at an interest rate of R per month is repaid over n monthly instalments of $\$M$.

(a) Show that $M - (M + P)K^n + PK^{1+n} = 0$, where $K = 1 + R$.

(b) Suppose that I can afford to repay \$650 per month on a \$20 000 loan to be paid back over three years. Use these figures in the equation above and apply Newton's method in order to find the highest rate of interest I can afford to meet. Give your answer correct to three significant figures.

(c) Repeat the same problem using the bisection method, in order to check your answer.

16. A man aged 25 is getting married, and has decided to pay \$3000 each year into a combination life insurance and superannuation scheme that pays 8% compound interest per annum. Once he reaches 65, the insurance company will pay out the value of the policy as a pension in equal monthly instalments over the next 25 years. During those 25 years, the balance will continue to earn interest at the same rate, but compounded monthly.

(a) What is the value of the policy when he reaches 65, correct to the nearest dollar?

(b) What will be the size of pension payments, correct to the nearest dollar?

7 E Rates of Change — Differentiating

A *rate of change* is the rate at which some quantity Q is changing. It is therefore the derivative $\dfrac{dQ}{dt}$ of Q with respect to time t, and is the gradient of the tangent to the graph of Q against time. A rate of change is always instantaneous unless otherwise stated, and should not be confused with an *average rate of change*, which is the gradient of a chord. This section will review the work on rates of change in Section 7H of the Year 11 volume, where the emphasis is on using the chain rule to calculate the rate of change of a given function. The next section will deal with the integration of rates.

Calculating Related Rates: As explained previously, the calculation of the relationship between two rates is simply an exercise in applying the chain rule.

6 RELATED RATES: Find a relation between the two quantities, then differentiate with respect to time, using the chain rule.

WORKED EXERCISE: Sand is being poured onto the top of a pile at the rate of $3\,\text{m}^3/\text{min}$. The pile always remains in the shape of a cone with semi-vertical angle 45°. Find the rate at which:

(a) the height, (b) the base area,

is changing when the height is 2 metres.

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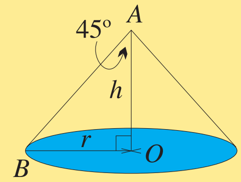
SOLUTION: Let the cone have volume V, height h and base radius r.

Since the semi-vertical angle is 45°, $r = h$ (isosceles $\triangle AOB$).

The rate of change of volume is known to be $\dfrac{dV}{dt} = 3\,\text{m}^3/\text{min}$.



(a) We know that $V = \frac{1}{3}\pi r^2 h$,

and since $r = h$, $V = \frac{1}{3}\pi h^3$.

Differentiating with respect to time (using the chain rule with the RHS),

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$
$$= \pi h^2 \, \frac{dh}{dt}\,.$$

Substituting, $$3 = \pi \times 2^2 \times \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{3}{4\pi}\,\text{m/min}.$$

(b) The base area is $A = \pi h^2$ (since $r = h$).

Differentiating, $$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$$
$$= 2\pi h \, \frac{dh}{dt}\,.$$

Substituting, $$\frac{dA}{dt} = 2 \times \pi \times 2 \times \frac{3}{4\pi}$$
$$= 3\,\text{m}^2/\text{min}.$$

WORKED EXERCISE: A 10 metre ladder is leaning against a wall, and the base is sliding away from the wall at 1 cm/s. Find the rate at which:

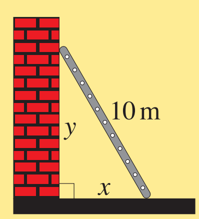
(a) the height, (b) the angle of inclination,

is changing when the foot is already 6 metres from the wall.

SOLUTION: Let the height be y and the distance from the wall be x, and let the angle of inclination be θ. We know that $\dfrac{dx}{dt} = 0{\cdot}01\,\text{m/s}$.

(a) By Pythagoras' theorem, $x^2 + y^2 = 10^2$,

hence $$y = \sqrt{100 - x^2}\,.$$



Differentiating, $$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$= \frac{-2x}{2\sqrt{100 - x^2}} \times \frac{dx}{dt}$$
$$= -\,\frac{x}{\sqrt{100 - x^2}} \times \frac{dx}{dt}\,.$$

Substituting $x = 6$ and $\dfrac{dx}{dt} = 0{\cdot}01$,

$$\frac{dy}{dt} = -\,\frac{6}{\sqrt{100 - 36}} \times 0{\cdot}01$$
$$= -0{\cdot}0075.$$

Hence the height is decreasing at $\frac{3}{4}$ cm/s.

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[Alternatively we can differentiate $x^2 + y^2 = 10^2$ implicitly.

This gives $\quad 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} = 0.$

When $x = 6$, $y = 8$ by Pythagoras' theorem, so substituting,

$$12 \times 0{\cdot}01 \;+\; 16\,\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -0{\cdot}0075.$$

Hence the height is decreasing at $\frac{3}{4}$ cm/s.]

(b) By trigonometry, $\quad x = 10\cos\theta\,.$

Differentiating, $$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$
$$= -10\sin\theta \times \frac{d\theta}{dt}\,.$$

When $x = 6$, $\sin\theta = \dfrac{y}{10} = \dfrac{8}{10}$, so substituting,

$$0{\cdot}01 = -10 \times \frac{8}{10} \times \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -0{\cdot}01 \times \frac{1}{8}$$

$$= -\frac{1}{800}\,.$$

Hence the angle of inclination is decreasing by $\frac{1}{800}$ radians per second, or, multiplying by $\frac{180}{\pi}$, by about $0{\cdot}072°$ per second.

Exercise 7E

NOTE: This exercise reviews material already covered in Exercise 7H of the Year 11 volume.

1. The sides of a square of side length x metres are increasing at a rate of $0{\cdot}1$ m/s.

(a) Show that the rate of increase of the area is given by $\dfrac{dA}{dt} = 0{\cdot}2\,x\,\text{m}^2/\text{s}$.

(b) At what rate is the area of the square increasing when its sides are 5 metres long?

(c) What is the side length when the area is increasing at $1{\cdot}4\,\text{m}^2/\text{s}$?

(d) What is the area when the area is increasing at $0{\cdot}6\,\text{m}^2/\text{s}$?

2. The diagonal of a square is decreasing at a rate of $\frac{1}{2}$ m/s.

(a) Find the area A of a square with a diagonal of length ℓ.

(b) Hence show that the rate of change of area is $\dfrac{dA}{dt} = -\frac{1}{2}\ell\,\text{m}^2/\text{s}$.

(c) Find the rate at which the area is decreasing when:

(i) the diagonal is 10 metres, (ii) the area is $18\,\text{m}^2$.

(d) What is the length of the diagonal when the area is decreasing at $17\,\text{m}^2/\text{s}$?

3. The radius r of a sphere is increasing at a rate of $0{\cdot}3$ m/s. In both parts, approximate π using a calculator and give your answer correct to three significant figures.

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(a) Show that the sphere's rate of change of volume is $\dfrac{dV}{dt} = 1{\cdot}2\pi r^2$, and find the rate of increase of its volume when the radius is 2 metres.

(b) Show that the sphere's rate of change of surface area is $\dfrac{dS}{dt} = 2{\cdot}4\pi r$, and find the rate of increase of its surface area when the radius is 4 metres.

4. Jules is blowing up a spherical balloon at a constant rate of $200\,\text{cm}^3/\text{s}$.

(a) Show that $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.

(b) Hence find the rate at which the radius is growing when the radius is 15 cm.

(c) Find the radius and volume when the radius is growing at $0{\cdot}5$ cm/s.

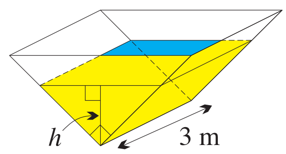
5. A lathe is used to shave down the radius of a cylindrical piece of wood 500 mm long. The radius is decreasing at a rate of 3 mm/min.

(a) Show that the rate of change of volume is $\dfrac{dV}{dt} = -3000\pi r$, and find how fast the volume is decreasing when the radius is 30 mm.

(b) How fast is the circumference decreasing when the radius is: (i) 20 mm, (ii) 37 mm?

DEVELOPMENT

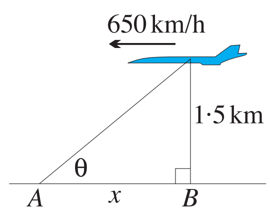
6. The water trough in the diagram is in the shape of an isosceles right triangular prism, 3 metres long. A jackaroo is filling the trough with a hose at the rate of 2 litres per second.



(a) Show that the volume of water in the trough when the depth is h cm is $V = 300h^2\ \text{cm}^3$.

(b) Given that 1 litre is $1000\,\text{cm}^3$, find the rate at which the depth of the water is changing when $h = 20$.

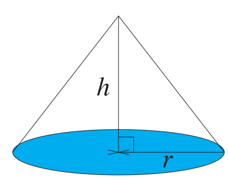
7. An observer at A in the diagram is watching a plane at P fly overhead, and he tilts his head so that he is always looking directly at the plane. The aircraft is flying at 650 km/h at an altitude of $1{\cdot}5$ km. Let θ be the angle of elevation of the plane from the observer, and suppose that the distance from A to B, directly below the aircraft, is x km.



(a) By writing $x = \dfrac{3}{2\tan\theta}$, show that $\dfrac{dx}{d\theta} = -\dfrac{3}{2\sin^2\theta}$.

(b) Hence find the rate at which the observer's head is tilting when the angle of inclination to the plane is $\frac{\pi}{3}$. Convert your answer from radians per hour to degrees per second, correct to the nearest degree.

8. Sand is poured at a rate of $0{\cdot}5\,\text{m}^3/\text{s}$ onto the top of a pile in the shape of a cone, as shown in the diagram. Let the base have radius r, and let the height of the cone be h. The pile always remains in the same shape, with $r = 2h$.



(a) Find the cone's volume, and show that it is the same as that of a sphere with radius equal to the cone's height.

(b) Find the rate at which the height is increasing when the radius of the base is 4 metres.

9. A boat is observed from the top of a 100-metre-high cliff. The boat is travelling towards the cliff at a speed of 50 m/min. How fast is the angle of depression changing when the angle of depression is 15°? Convert your answer from radians per minute to degrees per minute, correct to the nearest degree.

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10. The volume of a sphere is increasing at a rate numerically equal to its surface area at that instant. Show that $\dfrac{dr}{dt} = 1$.

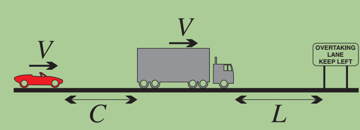
11. A point moves anticlockwise around the circle $x^2 + y^2 = 1$ at a uniform speed of $2\,\text{m/s}$.

(a) Find an expression for the rate of change of its x-coordinate in terms of x, when the point is above the x-axis. (The units on the axes are metres.)

(b) Use your answer to part (a) to find the rate of change of the x-coordinate as it crosses the y-axis at $P(0, 1)$. Why should this answer have been obvious without this formula?

EXTENSION

12. A car is travelling C metres behind a truck, both travelling at a constant speed of V m/s. The road widens L metres ahead of the truck and there is an overtaking lane. The car accelerates at a uniform rate so that it is exactly alongside the truck at the beginning of the overtaking lane.



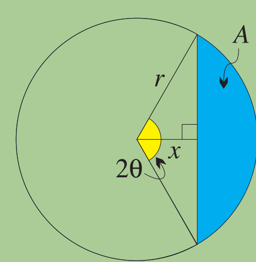
(a) What is the acceleration of the car?

(b) Show that the speed of the car as it passes the truck is $V\left(1 + \dfrac{2C}{L}\right)$.

(c) The objective of the driver of the car is to spend as little time alongside the truck as possible. What strategies could the driver employ?

(d) The speed limit is $100\,\text{km/h}$ and the truck is travelling at $90\,\text{km/h}$, and is 50 metres ahead of the car. How far before the overtaking lane should the car begin to accelerate if applying the objective in part (c)?

13. The diagram shows a chord distant x from the centre of a circle. The radius of the circle is r, and the chord subtends an angle 2θ at the centre.



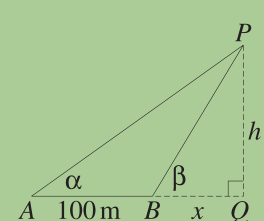
(a) Show that the area of the segment cut off by this chord is $A = r^2(\theta - \sin\theta\cos\theta)$.

(b) Explain why $\dfrac{dA}{dt} = \dfrac{dA}{d\theta} \times \dfrac{d\theta}{dx} \times \dfrac{dx}{dt}$.

(c) Show that $\dfrac{d\theta}{dx} = -\dfrac{1}{\sqrt{r^2 - x^2}}$.

(d) Given that $r = 2$, find the rate of increase in the area if $\dfrac{dx}{dt} = -\sqrt{3}$ when $x = 1$.

14. The diagram shows two radars at A and B 100 metres apart. An aircraft at P is approaching and the radars are tracking it, hence the angles α and β are changing with time.



(a) Show that $x\tan\beta = (x + 100)\tan\alpha$.

(b) Keeping in mind that x, α and β are all functions of time, use implicit differentiation to show that

$$\frac{dx}{dt} = \frac{\dot\alpha(x + 100)\sec^2\alpha - \dot\beta x\sec^2\beta}{\tan\beta - \tan\alpha}.$$

(c) Use part (a) to find the value of x and the height of the plane when $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{4}$.

(d) At the angles given in part (c), it is found that $\dfrac{d\alpha}{dt} = \frac{5}{36}(\sqrt{3} - 1)$ radians per second and $\dfrac{d\beta}{dt} = \frac{5}{18}(\sqrt{3} - 1)$ radians per second. Find the speed of the plane.

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7 F Rates of Change — Integrating

In some situations, only the rate of change of a quantity as a function of time is known. The original function can then be obtained by integration, provided that the value of the function is known initially or at some other time.

WORKED EXERCISE: During a drought, the flow $\dfrac{dV}{dt}$ of water from Welcome Well gradually diminishes according to the formula $\dfrac{dV}{dt} = 3e^{-0\cdot02t}$, where t is time in days after time zero, and V is the volume in megalitres of water that has flowed out.

(a) Show that $\dfrac{dV}{dt}$ is always positive, and explain this physically.

(b) Find an expression for the volume of water obtained after time zero.

(c) How much will flow from the well during the first 100 days?

(d) Describe the behaviour of V as $t \to \infty$, and find what percentage of the total flow comes in the first 100 days. Sketch the function.

SOLUTION:

(a) Since $e^x > 0$ for all x, $\dfrac{dV}{dt} = 3e^{-0\cdot02t}$ is always positive.

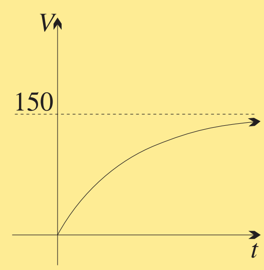
V is always increasing, because V is the amount that has flowed out.

(b) We are given that $\dfrac{dV}{dt} = 3e^{-0\cdot02t}$.

Integrating, $V = -150e^{-0\cdot02t} + C$.

When $t = 0$, $V = 0$, so $0 = -150 + C$,

so $C = 150$, and $V = 150(1 - e^{-0\cdot02t})$.



(c) When $t = 100$, $V = 150(1 - e^{-2})$

$\doteqdot 129\cdot7$ megalitres.

(d) As $t \to \infty$, $V \to 150$, since $e^{-0\cdot02t} \to 0$.

Hence proportion of flow in first 100 days $= \dfrac{150(1 - e^{-2})}{150}$

$= 1 - e^{-2}$

$\doteqdot 86\cdot5\%$.

WORKED EXERCISE: The rate at which ice on the side of Black Mountain is melting during spring changes with the time of day according to $\dfrac{dI}{dt} = -5 + 5\cos\frac{\pi}{12}t$, where I is the mass in tonnes of ice remaining on the mountain, and t is the time in hours after midnight on the day measuring began.

(a) Initially, there were 2400 tonnes of ice. Find I as a function of t.

(b) Show that for all t, I is decreasing or stationary, and find when I is stationary.

(c) Show that the ice disappears at the end of the 20th day.

SOLUTION:

(a) We are given that $\dfrac{dI}{dT} = -5 + 5\cos\frac{\pi}{12}t$.

Integrating, $I = -5t + \frac{60}{\pi}\sin\frac{\pi}{12}t + C$, for some constant C.

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When $t = 0$, $I = 2400$, so $2400 = -0 - 0 + C$,
so $C = 2400$, and $\quad I = -5t + \frac{60}{\pi}\sin\frac{\pi}{12}t + 2400.$

(b) Since $-5 \le 5\cos\frac{\pi}{12}t \le 5$, $\dfrac{dI}{dt}$ can never be positive.

I is stationary when $\dfrac{dI}{dt} = 0$, that is, when $\cos\frac{\pi}{12}t = 1$.

The general solution for $t \ge 0$ is $\quad \frac{\pi}{12}t = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$
$t = 0,\ 24,\ 48,\ 72,\ \ldots.$

That is, melting ceases at midnight on each successive day.

(c) When $t = 480$, $I = -2400 + 0 + 2400 = 0$,
so the ice disappears at the end of the 20th day.
(Notice that I is never increasing, so there can only be one solution for t.)

Exercise 7F

1. Water is flowing out of a tank at the rate of $\dfrac{dV}{dt} = 5(2t - 50)$, where V is the volume in litres remaining in the tank at time t minutes after time zero.
 (a) When does the water stop flowing?
 (b) Given that the tank still has 20 litres left in it when the water flow stops, find V as a function of t.
 (c) How much water was initially in the tank?

2. The rate at which a perfume ball loses its scent over time is $\dfrac{dP}{dt} = -\dfrac{2}{t+1}$, where t is measured in days.
 (a) Find P as a function of t if the initial perfume content is 6·8.
 (b) How long will it be before the perfume in the ball has run out and it needs to be replaced? (Answer correct to the nearest day.)

3. A tap on a large tank is gradually turned off so as not to create any hydraulic shock. As a consequence, the flow rate while the tap is being turned off is given by $\dfrac{dV}{dt} = -2 + \frac{1}{10}t\ \text{m}^3/\text{s}$.
 (a) What is the initial flow rate, when the tap is fully on?
 (b) How long does it take to turn the tap off?
 (c) Given that when the tap has been turned off there are still $500\,\text{m}^3$ of water left in the tank, find V as a function of t.
 (d) Hence find how much water is released during the time it takes to turn the tap off.
 (e) Suppose that it is necessary to let out a total of $300\,\text{m}^3$ from the tank. How long should the tap be left fully on before gradually turning it off?

4. The velocity of a particle is given by $\dfrac{dx}{dt} = e^{-0\cdot 4t}$.
 (a) Does the particle ever stop moving?
 (b) If the particle starts at the origin, find its displacement x as a function of time.
 (c) When does the particle reach $x = 1$? (Answer correct to two decimal places.)
 (d) Where does the particle move to eventually? (That is, find its limiting position.)

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DEVELOPMENT

5. A ball is falling through the air and experiences air resistance. Its velocity, in metres per second at time t, is given by $\dfrac{dx}{dt} = 250(e^{-0\cdot 2t} - 1)$, where x is the height above the ground.

(a) What is its initial speed? (b) What is its eventual speed?
(c) Find x as a function of t, if it is initially 200 metres above the ground.

6. Over spring and summer, the snow and ice on White Mountain is melting with the time of day according to $\dfrac{dI}{dt} = -5 + 4\cos\frac{\pi}{12}t$, where I is the tonnage of ice on the mountain at time t in hours since 2:00 am on 20th October.

(a) It was estimated at that time that there was still 18 000 tonnes of snow and ice on the mountain. Find I as a function of t.
(b) Explain, from the given rate, why the ice is always melting.
(c) The beginning of the next snow season is expected to be four months away (120 days). Show that there will still be snow left on the mountain then.

7. As a particle moves around a circle, its angular velocity is given by $\dfrac{d\theta}{dt} = \dfrac{1}{1+t^2}$.

(a) Given that the particle starts at $\theta = \frac{\pi}{4}$, find θ as a function of t.
(b) Hence find t as a function of θ.
(c) Using the result of part (a), show that $\frac{\pi}{4} \le \theta < \frac{3\pi}{4}$, and hence explain why the particle never moves through an angle of more than $\frac{\pi}{2}$.

8. The flow of water into a small dam over the course of a year varies with time and is approximated by $\dfrac{dW}{dt} = 1\cdot 2 - \cos^2\frac{\pi}{12}t$, where W is the volume of water in the dam, measured in thousands of cubic metres, and t is the time measured in months from the beginning of January.

(a) What is the maximum flow rate into the dam and when does this happen?
(b) Given that the dam is initially empty, find W.
(c) The capacity of the dam is 25 200 m^3. Show that it will be full in three years.

9. A certain brand of medicine tablet is in the shape of a sphere with diameter $\frac{1}{2}$ cm. The rate at which the pill dissolves is proportional to its surface area at that instant, that is, $\dfrac{dV}{dt} = kS$ for some constant k, and the pill lasts 12 hours before dissolving completely.

(a) Show that $\dfrac{dr}{dt} = k$, where r is the radius of the sphere at time t hours.
(b) Hence find r as a function of t. (c) Thus find k.

10. Sand is poured onto the top of a pile in the shape of a cone at a rate of $0\cdot 5\,\text{m}^3/\text{s}$. The apex angle of the cone remains constant at 90°. Let the base have radius r and let the height of the cone be h.

(a) Find the volume of the cone, and show that it is one quarter of the volume of a sphere with the same radius.
(b) Find the rate of change of the radius of the cone as a function of r.
(c) By taking reciprocals and integrating, find t as a function of r, given that the initial radius of the pile was 10 metres.
(d) Hence find how long it takes, correct to the nearest second, for the pile to grow another 2 metres in height.

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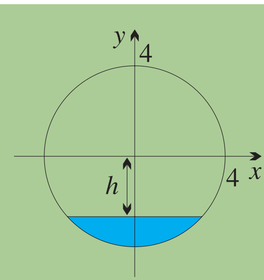
EXTENSION

11. (a) The diagram shows the spherical cap formed when the region between the lower half of the circle $x^2 + y^2 = 16$ and the horizontal line $y = -h$ is rotated about the y-axis. Find the volume V so formed.



(b) The cap represents a shallow puddle of water left after some rain. When the sun comes out, the water evaporates at a rate proportional to its surface area (which is the circular area at the top of the cap).

(i) Find this surface area A.

(ii) We are told that $\dfrac{dV}{dt} = -kA$. Show that the rate at which the depth of the water changes is $-k$.

(iii) The puddle is initially 2 cm deep and the evaporation constant is known to be $k = 0{\cdot}025$ cm/min. Find how long it takes for the puddle to evaporate.

7 G Natural Growth and Decay

This section will review the approaches to natural growth and decay developed in Section 13F of the Year 11 volume. The key idea here is that the exponential function $y = e^t$ is its own derivative, that is,

$$\text{if } y = e^t, \text{ then } \frac{dy}{dt} = e^t = y.$$

This means that at each point on the curve, the gradient is equal to the height. More generally,

$$\text{if } y = y_0\, e^{kt}, \text{ then } \frac{dy}{dt} = ky_0\, e^{kt} = ky.$$

This means that the rate of change of $y = Ae^{kt}$ is proportional to y.

The natural growth theorem says that, conversely, the only functions where the rate of growth is proportional to the value are functions of the form $y = Ae^{kt}$.

7 NATURAL GROWTH: Suppose that the rate of change of y is proportional to y:

$$\frac{dy}{dt} = ky, \quad \text{where } k \text{ is a constant of proportionality.}$$

Then $y = y_0\, e^{kt}$, where y_0 is the value of y at time $t = 0$.

WORKED EXERCISE: The value V of some machinery is depreciating according to the law of natural decay $\dfrac{dV}{dt} = -kV$, for some positive constant k. Each year its value drops by 15%.

(a) Show that $V = V_0\, e^{-kt}$ satisfies this differential equation, where V_0 is the initial cost of the machinery.

(b) Find the value of k, in exact form, and correct to four significant figures.

(c) Find, correct to four significant figures, the percentage drop in value over five years.

(d) Find, correct to the nearest 0·1 years, when the value has dropped by 90%.

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SOLUTION:

(a) Substituting $V = V_0\, e^{-kt}$ into $\dfrac{dV}{dt} = -kV$,

$$\text{LHS} = \frac{d}{dt}\left(V_0\, e^{-kt}\right) = -kV_0\, e^{-kt}, \qquad \text{RHS} = -k \times V_0\, e^{-kt} = \text{LHS}.$$

Also, substituting $t = 0$ gives $V = V_0\, e^0 = V_0$, as required.

(b) When $t = 1$, $V = 0{\cdot}85\, V_0$, so

$$\begin{aligned} 0{\cdot}85\, V_0 &= V_0\, e^{-k} \\ e^{-k} &= 0{\cdot}85 \\ k &= -\log_e 0{\cdot}85 \\ &\doteqdot 0{\cdot}1625. \end{aligned}$$

(c) When $t = 5$,

$$\begin{aligned} V &= V_0\, e^{-5k} \\ &\doteqdot 0{\cdot}4437\, V_0, \end{aligned}$$

so the value has dropped by about $55{\cdot}63\%$ over the 5 years.

(d) Put $V = 0{\cdot}1\, V_0$.

$$\begin{aligned} \text{Then } V_0\, e^{-kt} &= 0{\cdot}1\, V_0 \\ -kt &= \log_e 0{\cdot}1 \\ t &\doteqdot 14{\cdot}2 \text{ years.} \end{aligned}$$

Natural Growth and GPs: There are very close relationships between GPs and natural growth, as the following worked exercise shows.

WORKED EXERCISE: Continuing with the previous worked exercise:

(a) show that the values of the machinery after 0, 1, 2, ... years forms a GP, and find the ratio of the GP,

(b) find the loss of value during the 1st, 2nd, 3rd, ... years. Show that these losses form a GP, and find the ratio of the GP.

SOLUTION:

(a) The values after 0, 1, 2, ... years are V_0, $V_0\, e^{-k}$, $V_0\, e^{-2k}$, ...
This sequence forms a GP with first term V_0 and ratio $e^{-k} = 0{\cdot}85$.

(b)

$$\begin{aligned} \text{Loss of value during the first year} &= V_0 - V_0\, e^{-k} \\ &= V_0(1 - e^{-k}), \\ \text{loss of value during the second year} &= V_0\, e^{-k} - V_0\, e^{-2k} \\ &= V_0\, e^{-k}(1 - e^{-k}), \\ \text{loss of value during the third year} &= V_0\, e^{-2k} - V_0\, e^{-3k} \\ &= V_0\, e^{-2k}(1 - e^{-k}). \end{aligned}$$

These losses form a GP with first term $V_0(1 - e^{-k})$ and ratio $e^{-k} = 0{\cdot}85$.

A Confusing Term — The 'Growth Rate': Suppose that a population P is growing according to the equation $P = P_0\, e^{0{\cdot}08t}$. The constant $k = 0{\cdot}08$ is sometimes called the '*growth rate*', but this is a confusing term, because 'growth rate' normally refers to the instantaneous increase $\dfrac{dP}{dt}$ of the number of individuals per unit time. The constant k is better described as the *instantaneous proportional growth rate*, because the differential equation $\dfrac{dP}{dt} = kP$ shows that k is the proportionality constant relating the instantaneous rate of growth and the population.

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It is important in this context not to confuse average rates of growth, represented by chords on the exponential graph, with instantaneous rates of growth, represented by tangents on the exponential graph. There are in fact four different rates — two instantaneous rates, one absolute and one proportional, and two average rates, one absolute and one proportional. The following worked exercise on inflation asks for all four of these rates.

WORKED EXERCISE: [Four different rates associated with natural growth]
The cost C of building an average house is rising according to the natural growth equation $C = 150\,000\, e^{0{\cdot}08t}$, where t is time in years since 1st January 2000.

(a) Show that $\dfrac{dC}{dt}$ is proportional to C, and find the constant of proportionality (this is the so-called 'growth rate', or, more correctly, the 'instantaneous proportional growth rate').

(b) Find the instantaneous rates at which the cost is increasing on 1st January 2000, 2001, 2002 and 2003, correct to the nearest dollar per year, and show that they form a GP.

(c) Find the value of C when $t = 1$, $t = 2$ and $t = 3$, and the average increases in cost over the first year, the second year and the third year, correct to the nearest dollar per year, and show that they form a GP.

(d) Show that the average increase in cost over the first year, the second year and the third year, expressed as a proportion of the cost at the start of that year, is constant.

SOLUTION:

(a) Differentiating, $\dfrac{dC}{dt} = 0{\cdot}08 \times 150\,000 \times e^{0{\cdot}08t} = 0{\cdot}08\, C$,

so $\dfrac{dC}{dt}$ is proportional to C, with constant of proportionality $0{\cdot}08$.

(b) Substituting into $\dfrac{dC}{dt} = 12\,000\, e^{0{\cdot}08t}$,

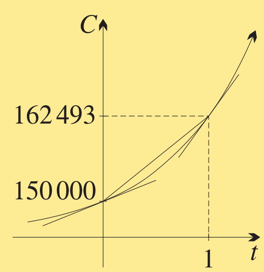
on 1st January 2000, $\dfrac{dC}{dt} = 12\,000\, e^{0} = \$12\,000$ per year,

on 1st January 2001, $\dfrac{dC}{dt} = 12\,000\, e^{0{\cdot}08} \doteqdot \$12\,999$ per year,

on 1st January 2002, $\dfrac{dC}{dt} = 12\,000\, e^{0{\cdot}16} \doteqdot \$14\,082$ per year,

on 1st January 2003, $\dfrac{dC}{dt} = 12\,000\, e^{0{\cdot}24} \doteqdot \$15\,255$ per year.

These form a GP with ratio $r = e^{0{\cdot}08} \doteqdot 1{\cdot}0833$.



(c) The values of C when $t = 0$, $t = 1$, $t = 2$ and $t = 3$ are respectively

$\$150\,000$, $150\,000\, e^{0{\cdot}08}$, $150\,000\, e^{0{\cdot}16}$ and $150\,000\, e^{0{\cdot}16}$,

$$\begin{aligned}
\text{so over the first year, increase } &= 150\,000(e^{0{\cdot}08} - 1) \doteqdot \$12\,493,\\
\text{over the second year, increase } &= 150\,000(e^{0{\cdot}16} - e^{0{\cdot}08})\\
&= 150\,000 \times e^{0{\cdot}08}(e^{0{\cdot}08} - 1) \doteqdot \$13\,534,\\
\text{over the third year, increase } &= 150\,000(e^{0{\cdot}24} - e^{0{\cdot}16})\\
&= 150\,000 \times e^{0{\cdot}16}(e^{0{\cdot}08} - 1) \doteqdot \$14\,661.
\end{aligned}$$

These increases form a GP with ratio $e^{0{\cdot}08} \doteqdot 1{\cdot}0833$.

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(d) The three proportional increases are

over the first year, $\dfrac{150\,000(e^{0\cdot08}-1)}{150\,000} = e^{0\cdot08}-1$,

over the second year, $\dfrac{150\,000 \times e^{0\cdot08} \times (e^{0\cdot08}-1)}{150\,000 \times e^{0\cdot08}} = e^{0\cdot08}-1$,

over the third year, $\dfrac{150\,000 \times e^{0\cdot16} \times (e^{0\cdot08}-1)}{150\,000 \times e^{0\cdot16}} = e^{0\cdot08}-1$,

so the proportional increases are all equal to $e^{0\cdot08}-1 \doteqdot 8\cdot33\%$.

Exercise 7G

NOTE: This exercise is a review of the material covered in Section 13E of the Year 11 volume, with a little more stress laid on the rates.

1. It is found that under certain conditions, the number of bacteria in a sample grows exponentially with time according to the equation $B = B_0\,e^{0\cdot1t}$, where t is measured in hours.
   (a) Show that B satisfies the differential equation $\dfrac{dB}{dt} = \frac{1}{10}B$.
   (b) Initially, the number of bacteria is estimated to be 1000. Find how many bacteria there are after three hours. Answer correct to the nearest bacterium.
   (c) Use parts (a) and (b) to find how fast the number of bacteria is growing after three hours.
   (d) By solving $1000\,e^{0\cdot1t} = 10\,000$, find, correct to the nearest hour, when there will be 10 000 bacteria.

2. Twenty grams of salt is gradually dissolved in hot water. Assume that the amount S left undissolved after t minutes satisfies the law of natural decay, that is, $\dfrac{dS}{dt} = -kS$, for some positive constant k.
   (a) Show that $S = 20e^{-kt}$ satisfies the differential equation.
   (b) Given that only half the salt is left after three minutes, show that $k = \frac{1}{3}\log 2$.
   (c) Find how much salt is left after five minutes, and how fast the salt is dissolving then. (Answer correct to two decimal places.)
   (d) After how long, correct to the nearest second, will there be 4 grams of salt left undissolved?
   (e) Find the amounts of undissolved salt when $t = 0$, 1, 2 and 3, correct to the nearest 0·01 g, show that these values form a GP, and find the common ratio.

3. The population P of a rural town has been declining over the last few years. Five years ago the population was estimated at 30 000 and today it is estimated at 21 000.
   (a) Assume that the population obeys the law of natural decay $\dfrac{dP}{dt} = -kP$, for some positive constant k, where t is time in years from the first estimate, and show that $P = 30\,000e^{-kt}$ satisfies this differential equation.
   (b) Find the value of the positive constant k.
   (c) Estimate the population ten years from now.
   (d) The local bank has estimated that it will not be profitable to stay open once the population falls below 16 000. When will the bank close?

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4. A chamber is divided into two identical parts by a porous membrane. The left part of the chamber is initially more full of a liquid than the right. The liquid is let through at a rate proportional to the difference in the levels x, measured in centimetres. Thus $\dfrac{dx}{dt} = -kx$.

(a) Show that $x = Ae^{-kt}$ is a solution of this equation.

(b) Given that the initial difference in heights is 30 cm, find the value of A.

(c) The level in the right compartment has risen 2 cm in five minutes, and the level in the left has fallen correspondingly by 2 cm.

(i) What is the value of x at this time?

(ii) Hence find the value of k.

5. A radioactive substance decays with a half-life of 1 hour. The initial mass is 80 g.

(a) Write down the mass when $t = 0$, 1, 2 and 3 hours (no need for calculus here).

(b) Write down the average loss of mass during the 1st, 2nd and 3rd hour, then show that the percentage loss of mass per hour during each of these hours is the same.

(c) The mass M at any time satisfies the usual equation of natural decay $M = M_0\, e^{-kt}$, where k is a constant. Find the values of M_0 and k.

(d) Show that $\dfrac{dM}{dt} = -kM$, and find the instantaneous rate of mass loss when $t = 0$, $t = 1$, $t = 2$ and $t = 3$.

(e) Sketch the M–t graph, for $0 \le t \le 1$, and add the relevant chords and tangents.

DEVELOPMENT

6. [The formulae for compound interest and for natural growth are essentially the same.] The cost C of an article is rising with inflation in such a way that at the start of every month, the cost is 1% more than it was a month before. Let C_0 be the cost at time zero.

(a) Use the compound interest formula of Section 7B to construct a formula for the cost C after t months. Hence find, in exact form and then correct to four significant figures:

(i) the percentage increase in the cost over twelve months,

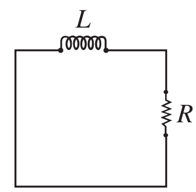
(ii) the time required for the cost to double.

(b) The natural growth formula $C = C_0\, e^{kt}$ also models the cost after t months. Use the fact that when $t = 1$, $C = 1{\cdot}01\, C_0$ to find the value of k. Hence find, in exact form and then correct to four significant figures:

(i) the percentage increase in the cost over twelve months,

(ii) the time required for the cost to double.

7. A current i_0 is established in the circuit shown on the right. When the source of the current is removed, the current in the circuit decays according to the equation $L\,\dfrac{di}{dt} = -iR$.



(a) Show that $i = i_0\, e^{-\frac{R}{L}t}$ is a solution of this equation.

(b) Given that the resistance is $R = 2$ and that the current in the circuit decays to 37% of the initial current in a quarter of a second, find L. (NOTE: 37% $\doteqdot \frac{1}{e}$)

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8. A tank in the shape of a vertical hexagonal prism with base area A is filled to a depth of 25 metres. The liquid inside is leaking through a small hole in the bottom of the tank, and it is found that the change in volume at any instant t hours after the tank starts leaking is proportional to the depth h metres, that is, $\dfrac{dV}{dt} = -kh$.

(a) Show that $\dfrac{dh}{dt} = -\dfrac{kh}{A}$.

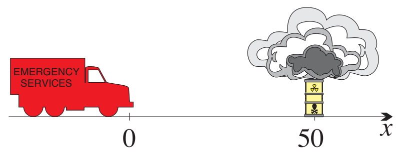
(b) Show that $h = h_0\, e^{-\frac{k}{A}t}$ is a solution of this equation.

(c) What is the value of h_0?

(d) Given that the depth in the tank is 15 metres after 2 hours, find $\frac{k}{A}$.

(e) How long will it take to empty to a depth of just 5 metres? Answer correct to the nearest minute.

9. The emergency services are dealing with a toxic gas cloud around a leaking gas cylinder 50 metres away. The prevailing conditions mean that the concentration C in parts per million (ppm) of the gas increases proportionally to the concentration as one moves towards the cylinder. That is, $\dfrac{dC}{dx} = kC$, where x is the distance in metres towards the cylinder from their current position.



(a) Show that $C = C_0\, e^{kx}$ is a solution of the above equation.

(b) At the truck, where $x = 0$, the concentration is $C = 20\,000$ ppm. Five metres closer, the concentration is $C = 22\,500$ ppm. Use this information to find the values of the constants C_0 and k. (Give k exactly, then correct to three decimal places.)

(c) Find the gas concentration at the cylinder, correct to the nearest part per million.

(d) The accepted safe level for this gas is 30 parts per million. The emergency services calculate how far back from the cylinder they should keep the public, rounding their answer up to the nearest 10 metres.

(i) How far do they keep the public back?

(ii) Why do they round their answer up and not round it in the normal way?

10. Given that $y = A_0\, e^{kt}$, it is found that at $t = 1$, $y = \frac{3}{4}A_0$.

(a) Show that it is not necessary to evaluate k in order to find y when $t = 3$.

(b) Find $y(3)$ in terms of A_0.

11. (a) The price of shares in Bravo Company rose in one year from \$5.25 to \$6.10.

(i) Assuming the law of natural growth, show that the share price in cents is given by $B = 525e^{kt}$, where t is measured in months.

(ii) Find the value of k.

(b) A new information technology company, ComIT, enters the stock market at the same time with shares at \$1, and by the end of the year these are worth \$2.17.

(i) Again assuming natural growth, show that the share price in cents is given by $C = 100\, e^{\ell t}$.

(ii) Find the value of ℓ.

(c) During which month will the share prices in both companies be equal?

(d) What will be the (instantaneous) rate of increase in ComIT shares at the end of that month, correct to the nearest cent per month?

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NOTE: The following two questions deal with finance, where rates are usually expressed not as instantaneous rates, but as average rates. It will usually take some work to relate the value k of the instantaneous rate to the average rate.

12. At any time t, the value V of a certain item is depreciating at an instantaneous rate of 15% of V per annum.

(a) Express $\dfrac{dV}{dt}$ in terms of V.

(b) The cost of purchasing the item was \$12 000. Write V as a function of time t years since it was purchased, and show that it is a solution of the equation in part (a).

(c) Find V after one year, and find the decrease as a percentage of the initial value.

(d) Find the instantaneous rate of decrease when $t = 1$.

(e) How long, correct to the nearest 0·1 years, does it take for the value to decrease to 10% of its cost?

13. An investment of \$5000 is earning interest at the advertised rate of 7% per annum, compounded annually. (This is the average rate, not the instantaneous rate.)

(a) Use the compound interest formula to write down the value A of the investment after t years.

(b) Use the result $\dfrac{d}{dt}(a^t) = a^t \log a$ to show that $\dfrac{dA}{dt} = A \log 1{\cdot}07$.

(c) Use the result $a = e^{\log a}$ to re-express the exponential term in A with base e.

(d) Hence confirm that $\dfrac{dA}{dt} = A \log 1{\cdot}07$.

(e) Use your answer to either part (a) or part (c) to find the value of the investment after six years, correct to the nearest cent.

(f) Hence find the instantaneous rate of growth after six years, again to the nearest cent.

14. (a) The population P_1 of one town is growing exponentially, with $P_1 = Ae^t$, and the population P_2 of another town is growing at a constant rate, with $P_2 = Bt + C$, where A, B and C are constants. When the first population reaches $P_1 = Ae$, it is found that $P_1 = P_2$, and also that both populations are increasing at the same rate.

(i) Show that the second population was initially zero (that is, that $C = 0$).

(ii) Draw a graph showing this information.

(iii) Show that the result in part (i) does not change if $P_1 = Aa^t$, for some $a > 1$. [HINT: You may want to use the identity $a^t = e^{t \log a}$.]

(b) Two graphs are drawn on the same axes, one being $y = \log x$ and the other $y = mx + b$. It is found that the straight line is tangent to the logarithmic graph at $x = e$.

(i) Show that $b = 0$, and draw a graph showing this information.

(ii) Show that the result in part (i) does not change if $y = \log_a x$, for some $a > 1$.

(c) Explain the effect of the change of base in parts (a) and (b) in terms of stretching.

(d) Explain in terms of a reflection why the questions in parts (a) and (b) are equivalent.

EXTENSION

15. The growing population of rabbits on Brair Island can initially be modelled by the law of natural growth, with $N = N_0\, e^{\frac{1}{2}t}$. When the population reaches a critical value, $N = N_c$, the model changes to $N = \dfrac{B}{C + e^{-t}}$, with the constants B and C chosen so that both models predict the same rate of growth at that time .

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(a) Find the values of B and C in terms of N_c and N_0.
(b) Show that the population reaches a limit, and find that limit in terms of N_c.

7 H Modified Natural Growth and Decay

In many situations, the rate of change of a quantity P is proportional not to P itself, but to the amount $P - B$ by which P exceeds some fixed value B. Mathematically, this means shifting the graph upwards by B, which is easily done using theory previously established.

The General Case: Here is the general statement of the situation.

8 MODIFIED NATURAL GROWTH: Suppose that the rate of change of a quantity P is proportional to the difference $P - B$, where B is some fixed value of P:

$$\frac{dP}{dt} = k(P - B), \text{ where } k \text{ is a constant of proportionality.}$$

Then $P = B + Ae^{kt}$, where A is the value of $P - B$ at time zero.

NOTE: Despite the following proof, memorisation of this general solution is not required. Questions will always give a solution in some form, and may then ask to verify by substitution that it is a solution of the differential equation.

PROOF: Let $y = P - B$ be the difference between P and B.

Then $\dfrac{dy}{dt} = \dfrac{dP}{dt} - 0$, since B is a constant,

$= k(P - B)$, since we are given that $\dfrac{dP}{dt} = k(P - B)$,

so $\dfrac{dy}{dt} = ky$, since we defined y by $y = P - B$.

Hence, using the previous theory of natural growth,

$$y = y_0 e^{kt}, \text{ where } y_0 \text{ is the initial value of } y,$$

and substituting $y = P - B$,

$$P = B + Ae^{kt}, \text{ where } A \text{ is the initial value of } P - B.$$

WORKED EXERCISE: The large French tapestries that are hung in the permanently air-conditioned La Châtille Hall have a normal water content W of 8 kg. When the tapestries were removed for repair, they dried out in the workroom atmosphere. When they were returned, the rate of increase of the water content was proportional to the difference from the normal 8 kg, that is,

$$\frac{dW}{dt} = k(8 - W), \text{ for some positive constant } k \text{ of proportionality.}$$

(a) Prove that for any constant A, $W = 8 - Ae^{-kt}$ is a solution of the differential equation.
(b) Weighing established that $W = 4$ initially, and $W = 6{\cdot}4$ after 3 days.
  (i) Find the values of A and k.
  (ii) Find when the water content has risen to $7{\cdot}9$ kg.
  (iii) Find the rate of absorption of the water after 3 days.
  (iv) Sketch the graph of water content against time.

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SOLUTION:

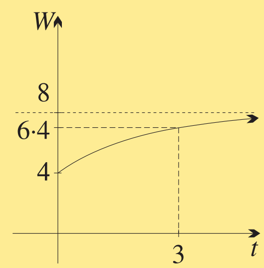
(a) Substituting $W = 8 - Ae^{-kt}$ into $\dfrac{dW}{dt} = k(8 - W)$,

$$\text{LHS} = \frac{dW}{dt} = kA\,e^{-kt}, \qquad \text{RHS} = k(8 - 8 + Ae^{-kt}) = \text{LHS}, \text{ as required.}$$

(b) (i) When $t = 0$, $W = 4$, so $4 = 8 - A$
$A = 4$.

When $t = 3$, $W = 6{\cdot}4$, so $6{\cdot}4 = 8 - 4e^{-3k}$
$e^{-3k} = 0{\cdot}4$
$k = -\frac{1}{3}\log 0{\cdot}4$ (calculate and leave in memory).

(ii) Put $W = 7{\cdot}9$, then $7{\cdot}9 = 8 - 4e^{-kt}$
$e^{-kt} = 0{\cdot}025$
$t = -\dfrac{1}{k}\log 0{\cdot}025$
$\doteqdot 12$ days.



(iii) We know that $\dfrac{dW}{dt} = k(8 - W)$.

When $t = 3$, $W = 6{\cdot}4$, so $\dfrac{dW}{dt} = k \times 1{\cdot}6$
$\doteqdot 0{\cdot}49$ kg/day.

Newton's Law of Cooling: Newton's law of cooling is a well-known example of natural decay. When a hot object is placed in a cool environment, the rate at which the temperature decreases is proportional to the difference between the temperature T of the object and the temperature E of the environment:

$$\frac{dT}{dt} = -k(T - E), \quad \text{where } k \text{ is a constant of proportionality.}$$

The same law applies to a cold body placed in a warmer environment.

WORKED EXERCISE: In a kitchen where the temperature is 20°C, Stanley takes a kettle of boiling water off the stove at time zero. Five minutes later, the temperature of the water is 70°C.

(a) Show that $T = 20 + 80e^{-kt}$ satisfies the cooling equation $\dfrac{dT}{dt} = -k(T - 20)$, and gives the correct value of 100°C at $t = 0$. Then find k.

(b) How long will it take for the water temperature to drop to 25°C?

(c) Graph the temperature–time function.

SOLUTION:

(a) Substituting $T = 20 + 80e^{-kt}$ into $\dfrac{dT}{dt} = -k(T - 20)$,

$$\text{LHS} = \frac{dT}{dt} = -80ke^{-kt}, \qquad \text{RHS} = -k(20 + 80e^{kt} - 20) = \text{LHS}, \text{ as required.}$$

Substituting $t = 0$, $T = 20 + 80 \times 1 = 100$, as required.

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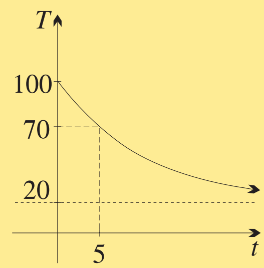
When $t = 5$, $T = 70$, so
$$70 = 20 + 80e^{-5k}$$
$$e^{-5k} = \tfrac{5}{8}$$
$$k = -\tfrac{1}{5}\log\tfrac{5}{8}.$$

(b) Substituting $T = 25$,
$$25 = 20 + 80e^{-kt}$$
$$e^{-kt} = \tfrac{1}{16}$$
$$t = -\frac{1}{k}\log\tfrac{1}{16}$$
$$\doteqdot 29\tfrac{1}{2} \text{ minutes.}$$



Exercise 7H

1. (a) Suppose that $P = 10\,000 + 2\,000\,e^{0{\cdot}1t}$. (i) Show that $\dfrac{dP}{dt} = \tfrac{1}{10}(P - 10\,000)$.
(ii) Find the value of P when $t = 0$, and state what happens as $t \to \infty$.

(b) Suppose that $P = 10\,000 + 2\,000\,e^{-0{\cdot}1t}$. (i) Show that $\dfrac{dP}{dt} = -\tfrac{1}{10}(P - 10\,000)$.
(ii) Find the value of P when $t = 0$, and state what happens as $t \to \infty$.

(c) Suppose that $P = 10\,000 - 2\,000\,e^{-0{\cdot}1t}$. (i) Show that $\dfrac{dP}{dt} = -\tfrac{1}{10}(P - 10\,000)$.
(ii) Find the value of P when $t = 0$, and state what happens as $t \to \infty$.

2. The rate of increase of a population P of green and purple flying bugs is proportional to the excess of the population over 2000, that is, $\dfrac{dP}{dt} = k(P - 2000)$, for some constant k. Initially, the population is 3000, and three weeks later the population is 8000.
(a) Show that $P = 2000 + Ae^{kt}$ satisfies the differential equation, where A is constant.
(b) By substituting $t = 0$ and $t = 3$, find the values of A and k.
(c) Find the population after seven weeks, correct to the nearest ten bugs.
(d) Find when the population reaches 500 000, correct to the nearest 0·1 weeks.

3. During the autumn, the rate of decrease of the fly population F in Wanzenthal Valley is proportional to the excess over 30 000, that is, $\dfrac{dF}{dt} = -k(F - 30\,000)$, for some positive constant k. Initially, there are 1 000 000 flies in the valley, and ten days later the number has halved.
(a) Show that $F = 30\,000 + Be^{-kt}$ satisfies the differential equation, where B is constant.
(b) Find the values of B and k.
(c) Find the population after four weeks, correct to the nearest 1000 flies.
(d) Find when the population reaches 35 000, correct to the nearest day.

4. A hot cup of coffee loses heat in a colder environment according to Newton's law of cooling, $\dfrac{dT}{dt} = -k(T - T_e)$, where T is the temperature of the coffee in degrees Celsius at time t minutes, T_e is the temperature of the environment and k is a positive constant.
(a) Show that $T = T_e + Ae^{-kt}$ is a solution of this equation, for any constant A.
(b) I make myself a cup of coffee and find that it has already cooled from boiling to 90°C. The temperature of the air in the office is 20°C. What are the values of T_e and A?
(c) The coffee cools from 90°C to 50°C after six minutes. Find k.
(d) Find how long, correct to the nearest second, it will take for the coffee to reach 30°C.

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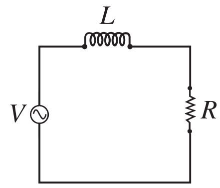
5. A tray of meat is taken out of the freezer at $-9°$C and allowed to thaw in the air at $25°$C. The rate at which the meat warms follows Newton's law of cooling and so $\dfrac{dT}{dt} = -k(T-25)$, with time t measured in minutes.
(a) Show that $T = 25 - Ae^{-kt}$ is a solution of this equation, and find the value of A.
(b) The meat reaches $8°$C in 45 minutes. Find the value of k.
(c) Find the temperature it reaches after another 45 minutes.

6. A 1 kilogram weight falls from rest through the air. When both gravity and air resistance are taken into account, it is found that its velocity is given by $v = 160(1 - e^{-\frac{1}{16}t})$. The velocity v is measured in metres per second, and downwards has been taken as positive.
(a) Confirm that the initial velocity is zero. Show that the velocity is always positive for $t > 0$, and explain this physically.
(b) Show that $\dfrac{dv}{dt} = \frac{1}{16}(160 - v)$, and explain what this represents.
(c) What velocity does the body approach?
(d) How long does it take to reach one eighth of this speed?

DEVELOPMENT

7. A chamber is divided into two identical parts by a porous membrane. The left compartment is initially full and the right is empty. The liquid is let through at a rate proportional to the difference between the level x cm in the left compartment and the average level. Thus $\dfrac{dx}{dt} = k(15 - x)$.
(a) Show that $x = 15 + Ae^{-kt}$ is a solution of this equation.
(b) (i) What value does the level in the left compartment approach?
(ii) Hence explain why the initial height is 30 cm.
(iii) Thus find the value of A.
(c) The level in the right compartment has risen 6 cm in 5 minutes. Find the value of k.

8. The diagram shows a simple circuit containing an inductor L and a resistor R with an applied voltage V. Circuit theory tells us that $V = RI + L\,\dfrac{dI}{dt}$, where I is the current at time t seconds.



(a) Prove that $I = \dfrac{V}{R} + Ae^{-\frac{R}{L}t}$ is a solution of the differential equation, for any constant A.
(b) Given that initially the current is zero, find A in terms of V and R.
(c) Find the limiting value of the current in the circuit.
(d) Given that $R = 12$ and $L = 8 \times 10^{-3}$, find how long it takes for the current to reach half its limiting value. Give your answer correct to three significant figures.

9. When a person takes a pill, the medicine is absorbed into the bloodstream at a rate given by $\dfrac{dM}{dt} = -k(M - a)$, where M is the concentration of the medicine in the blood t minutes after taking the pill, and a and k are constants.
(a) Show that $M = a(1 - e^{-kt})$ satisfies the given equation, and gives an initial concentration of zero.
(b) What is the limiting value of the concentration?
(c) Find k, if the concentration reaches 99% of the limiting value after 2 hours.

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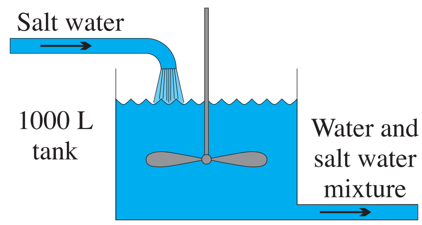
(d) The patient starts to notice relief when the concentration reaches 10% of the limiting value. When will this occur, correct to the nearest second?

10. In the diagram, a tank initially contains 1000 litres of pure water. Salt water begins pouring into the tank from a pipe and a stirring blade ensures that it is completely mixed with the pure water. A second pipe draws the water and salt water mixture off at the same rate, so that there is always a total of 1000 litres in the tank.



(a) If the salt water entering the tank contains 2 grams of salt per litre, and is flowing in at the constant rate of w litres/min, how much salt is entering the tank per minute?

(b) If there are Q grams of salt in the tank at time t, how much salt is in 1 litre at time t?

(c) Hence write down the amount of salt leaving the tank per minute.

(d) Use the previous parts to show that $\dfrac{dQ}{dt} = -\dfrac{w}{1000}(Q - 2000)$.

(e) Show that $Q = 2000 + Ae^{-\frac{wt}{1000}}$ is a solution of this differential equation.

(f) Determine the value of A.

(g) What happens to Q as $t \to \infty$?

(h) If there is 1 kg of salt in the tank after $5\frac{3}{4}$ hours, find w.

EXTENSION

11. [Alternative proof of the modified natural growth theorem] Suppose that a quantity P changes at a rate proportional to the difference between P and some fixed value B, that is, $\dfrac{dP}{dt} = k(P - B)$.

(a) Take reciprocals, integrate, and hence show that $\log(P - B) = kt + C$.

(b) Take exponentials and finally show that $P - B = Ae^{kt}$.

12. It is assumed that the population of a newly introduced species on an island will usually grow or decay in proportion to the difference between the current population P and the ideal population I, that is, $\dfrac{dP}{dt} = k(P - I)$, where k may be positive or negative.

(a) Prove that $P = I + Ae^{kt}$ is a solution of this equation.

(b) Initially 10 000 animals are released. A census is taken 7 weeks later and again at 14 weeks, and the population grows to 12 000 and then 18 000. Use these data to find the values of I, A and k.

(c) Find the population after 21 weeks.

13. [The coffee drinkers' problem] Two coffee drinkers pour themselves a cup of coffee each just after the kettle has boiled. The woman adds milk from the fridge, stirs it in and then waits for it to cool. The man waits for the coffee to cool first, then just before drinking adds the milk and stirs. If they both begin drinking at the same time, whose coffee is cooler? Justify your answer mathematically. Assume that the air temperature is colder than the coffee and that the milk is colder still. Also assume that after the milk is added and stirred, the temperature drops by a fixed percentage.

GO Online Multiple Choice Quiz

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