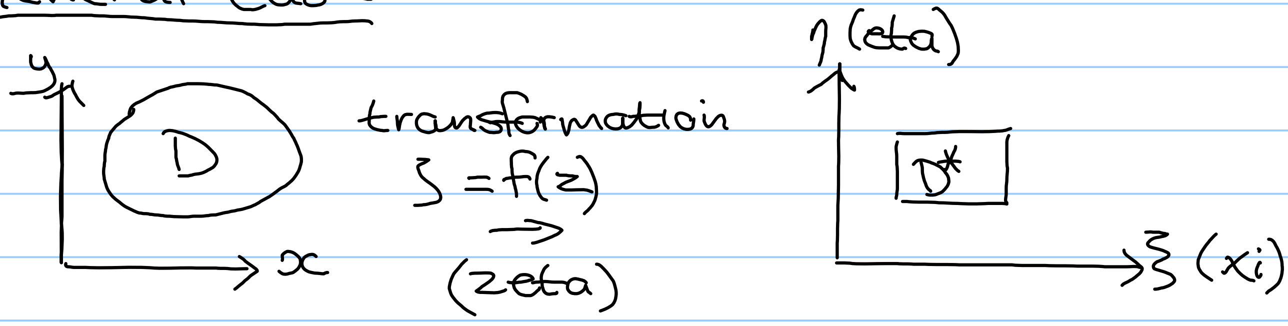
Conformal transformations in fluid dynamics

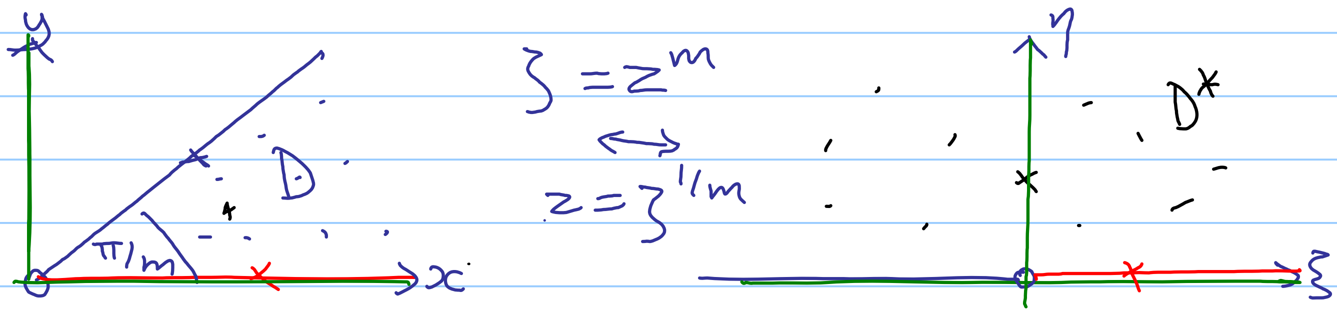
General case



z-plane $z = x + iy$ ζ-plane $\zeta = \xi + i\eta$

Useful conformal transformations

1. Let D be the wedge $0 \leq \arg z \leq \frac{\pi}{m}$, for $m \geq \frac{1}{2}$

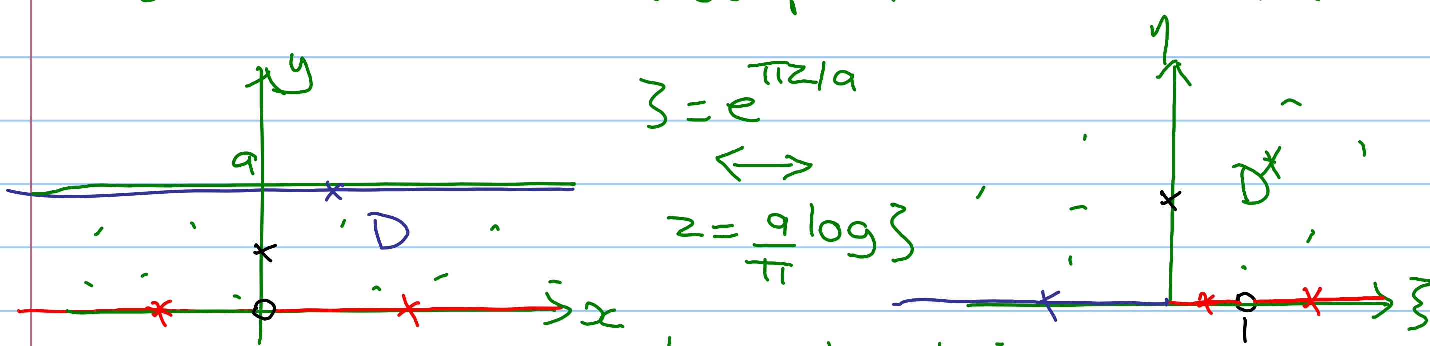


If $z = |z| e^{i\pi/m}$ so $z^m = |z|^m e^{i\pi} = -|z|^m$

If $z = |z| e^{i\pi/2m}$ so $z^m = |z|^m e^{i\pi/2}$

Thus, $\zeta = z^m$ maps the wedge to the upper half plane.

2. Let D be an infinite strip of width a, $0 \leq \mathrm{Im}(z) \leq a$



$\log w = \ln|w| + i \arg w$

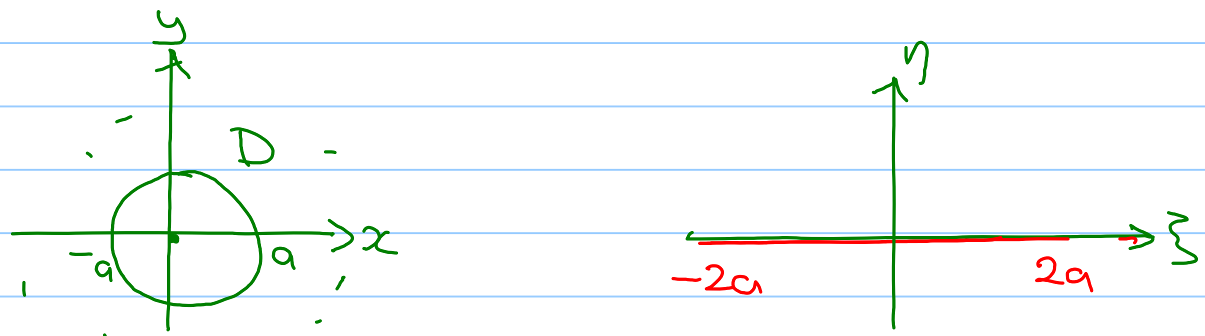
If $z = x + ia$ then $\zeta = e^{\pi(x+ia)/a} = e^{\pi x/a} e^{i\pi}$

If $z = \frac{ia}{2}$ then $\zeta = e^{i\pi/2}$.

Then $\zeta = e^{\pi z/a}$ maps D to the upper half plane

3. Joukowski transformation

$\zeta = z + \dfrac{a^2}{z}$. This maps the circle $|z| = a$ onto a flat plate of length $4a$.



Centre zero.

Let $z = re^{i\theta}$ $(z \neq 0)$ then $\zeta = re^{i\theta} + \dfrac{a^2}{r}e^{-i\theta}$

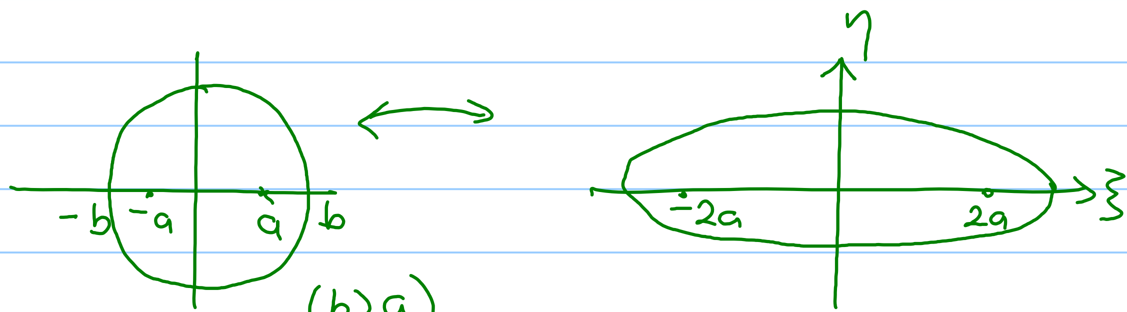
When $r = a$: $\zeta = ae^{i\theta} + ae^{-i\theta} = 2a\cos\theta$

$$\zeta = \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta$$

So $\xi = \left(r + \dfrac{a^2}{r}\right)\cos\theta$; $\eta = \left(r - \dfrac{a^2}{r}\right)\sin\theta$

$$\frac{\xi^2}{\left(r + \frac{a^2}{r}\right)^2} + \frac{\eta^2}{\left(r - \frac{a^2}{r}\right)^2} = \cos^2\theta + \sin^2\theta = 1.$$

ellipse.



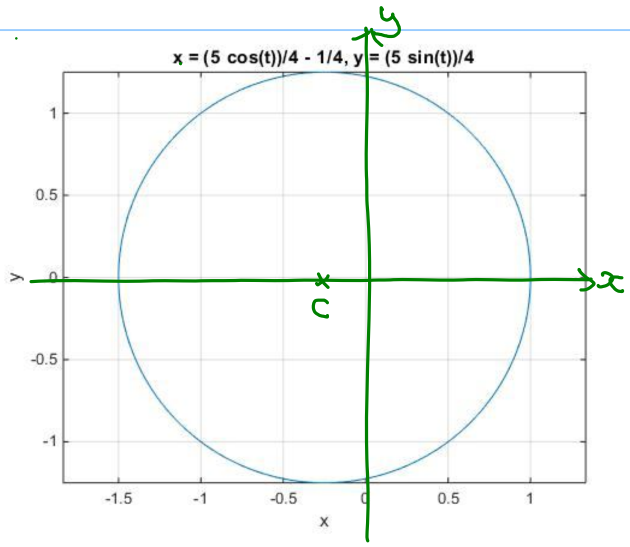
$(b > a)$

Practical applications are for $b > a$.

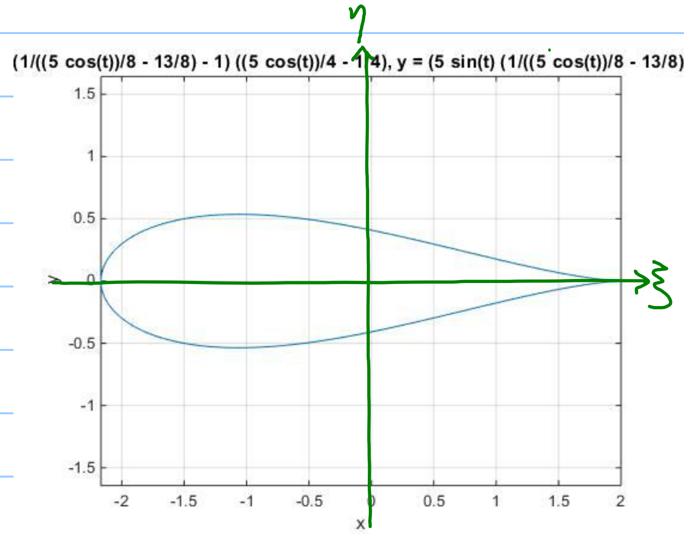
If we use the transformation on a circle in the z-plane which passes through $z = a$ but which encloses $z = -a$, we obtain a symmetric aerofoil with a rounded nose and a sharp trailing edge.

Example

$a = 1$ and the circle $|z + 1/4| = 1.25$



z-plane

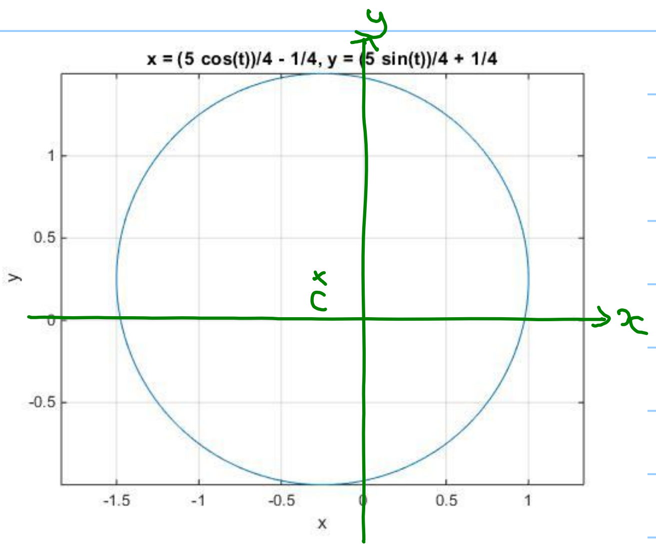


ζ-plane

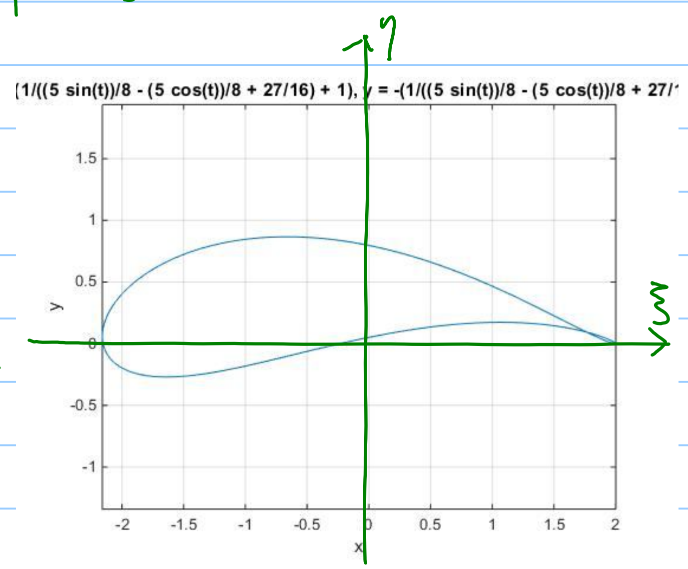
If the centre of the circle is off the x and y axes we get a cambered aerofoil.

Example

$a = 1$ and the circle $|z + \frac{1}{4} - \frac{i}{4}| = 1.25$



z-plane



ζ-plane