CHAPTER **NINE**

The Geometry of the Parabola

In the previous chapter, parabolic graphs of quadratic functions were used to help solve problems that were mostly algebraic. All that is reversed in this chapter, where algebraic methods will merely be the tool used to establish the basic geometry of the parabola. In keeping with this geometric approach, the parabola will now be defined not as the graph of a quadratic, but will have a purely geometric definition in which the curve is completely determined by a point called the *focus* and a line called the *directrix*.

Although the motivation here is geometric, the methods used are mostly algebraic, and they make an interesting contrast with the Euclidean methods used to study the geometry of triangles, quadrilaterals and circles. The various configurations are placed almost at once on the coordinate plane so that the methods of functions, algebra and calculus can be applied to them. There will be particular emphasis on a new method of describing curves called *parametrisation*.

Study Notes: Sections 9A–9C complete the 2 Unit material on loci and on the parabola in particular — they form a convenient unit in themselves, the work on locus possibly having been covered previously. The remainder of the chapter presents the 3 Unit theory of the parabola and could be studied later, since it requires maturity in the use of algebra and coordinate geometry. Section 9D introduces parameters, then in Sections 9E–9H the theory of chords, tangents and normals is developed using parametric and non-parametric methods. Section 9I is a collection of general theorems about the parabola, and Section 9J deals with the solution of various locus problems generated by parabolas.

9 A A Locus and its Equation

A *locus* is a set of points. The word usually implies that the set has been described in terms of some geometric constraint. Usually a locus is some sort of curve, and can be thought of as the path traced out by a moving point. Our tasks in this section are to find the algebraic equation of a locus that has been geometrically specified, and to supply a geometric description of a locus specified algebraically.

Simple Loci — Sketch and Write Down the Equation: Some simple loci require no more than a sketch, after which the equation can be easily written down.

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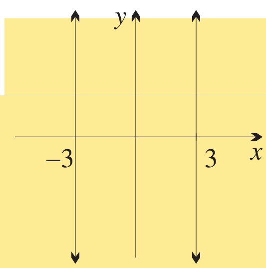
1 SIMPLE LOCI: Sketch the locus, then write down its equation.

WORKED EXERCISE: Sketch the locus of a point whose distance from the y-axis is 3 units, then write down its equation.



SOLUTION: From the sketch, the equation of the locus is

$$x = 3 \text{ or } x = -3.$$

Notice that this locus is not a function. It can also be written down algebraically as a single equation, $x^2 = 9$, or as $|x| = 3$.

Finding the Locus Algebraically: Generally, however, questions will require the use of algebraic methods to find the equation of a locus. Some sort of sketch should be made, with a general point $P(x, y)$ placed on the coordinate plane, then the formal algebraic work should begin as follows:

2 HARDER LOCI: 'Let $P(x, y)$ be any point in the plane.
The condition that P lie on the locus is . . . '.

We start with any point in the plane, because we seek an algebraic equation such that the point lies on the locus if and only if its coordinates satisfy the equation.

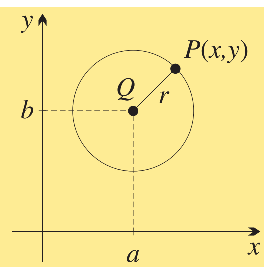
NOTE: When the distance formula is used, it is best to square the geometric condition first.

Example — Finding the Equation of a Circle: A *circle* is defined geometrically as a locus in terms of its centre and radius.

3 DEFINITION: A *circle* is the locus of all points in the plane that are a fixed distance (called the *radius*) from a given point (called the *centre*).

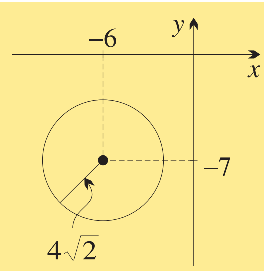
The well-known equation for a circle can easily be established from this definition.

WORKED EXERCISE: Use the definition of a circle to find the equation of the circle with centre $Q(a, b)$ and radius r.



SOLUTION: Let $P(x, y)$ be any point in the plane.
The condition that P lie on the locus is $PQ = r$.
Squaring both sides, $PQ^2 = r^2$.
Using the distance formula, $(x - a)^2 + (y - b)^2 = r^2$.

WORKED EXERCISE: Find the equation of the locus of a point which moves so that its distance from the point $A(2, 1)$ is twice its distance from the point $B(-4, -5)$. Describe the locus geometrically.



SOLUTION: Let $P(x, y)$ be any point in the plane.
The condition that P lie on the locus is

$$PA = 2 \times PB$$

$$\boxed{\text{square}} \qquad PA^2 = 4 \times PB^2$$

$$(x - 2)^2 + (y - 1)^2 = 4(x + 4)^2 + 4(y + 5)^2$$

$$x^2 - 4x + 4 + y^2 - 2y + 1 = 4x^2 + 32x + 64 + 4y^2 + 40y + 100$$

$$3x^2 + 36x + 3y^2 + 42y = -159.$$

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Then dividing by 3 and completing both squares,

$$x^2 + 12x + 36 + y^2 + 14y + 49 = -53 + 36 + 49$$
$$(x+6)^2 + (y+7)^2 = 32,$$

so the locus is a circle with centre $(-6,-7)$ and radius $4\sqrt{2}$.

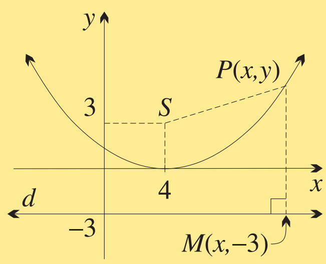
Example — Finding the Equation of a Parabola: The locus in the worked example below is a parabola, whose geometric definition is the subject of the next section.

WORKED EXERCISE: Find the equation of the locus of all points equidistant from the point $S(4,3)$ and the line $d : y = -3$.

SOLUTION: Let $P(x,y)$ be any point in the plane, and let $M(x,-3)$ be the foot of the perpendicular from P to d. The condition that P lie on the locus is

$$PS = PM$$

$\boxed{\text{square}}$ $$PS^2 = PM^2$$
$$(x-4)^2 + (y-3)^2 = (y+3)^2$$
$$(x-4)^2 = 12y.$$



Exercise 9A

NOTE: Solve question 1 by sketching the points $P(x,y)$ on the number plane. All other questions should be solved by starting with the phrase: 'The condition that $P(x,y)$ lie on the locus is …'.

1. Sketch each locus of the point $P(x,y)$, and hence write down its equation.
 (a) P is two units below the x-axis.
 (b) P is one unit to the left of the y-axis.
 (c) P is equidistant from the lines $y = -1$ and $y = 5$.
 (d) The distance of P from the x-axis is three times its distance from the y-axis.
 (e) P is three units from the origin.
 (f) P is equidistant from the lines with equations $y = x + 3$ and $y = x + 7$.
 (g) P is 3 units from the point $A(-3,1)$.

2. Derive the equation of the locus of the point $P(x,y)$ which moves so that it is always a distance of 4 units from the point $A(3,1)$.

3. (a) Use the distance formula to find the equation of the locus of the point $P(x,y)$ which moves so that it is equidistant from the points $R(-2,4)$ and $S(1,2)$.
 (b) Find the equation of the line through the midpoint of RS and perpendicular to RS. Comment on the result.

4. (a) Given the points $A(4,0)$, $B(-2,0)$ and $P(x,y)$, find the gradients of AP and BP.
 (b) Hence show that the equation of the locus of the point $P(x,y)$ which moves so that $\angle APB$ is a right angle is $x^2 + y^2 - 2x - 8 = 0$.
 (c) Complete the square in x, and hence describe the locus geometrically.

5. Given the points $A(1,4)$ and $B(-3,2)$, find the equation of each locus of the point $P(x,y)$, and describe each locus geometrically.
 (a) P is equidistant from A and B.
 (b) $\angle APB$ is a right angle.
 (c) P is equidistant from A and the x-axis.

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6. A point $P(x, y)$ moves so that it is equidistant from the point $K(-1, 3)$ and the line $y = 0$. Show that the locus is a parabola and find its vertex.

DEVELOPMENT

7. (a) Use the distance formula to find the square of the distance between the point $P(x, y)$ and each of the points $A(0, 4)$, $B(0, -4)$ and $C(6, 3)$.

(b) A point $P(x, y)$ moves so that the sum of the squares of its distances from the points A, B and C is 77. Show that the locus is a circle and find its centre and radius.

8. Find, and describe geometrically, the equation of the locus of the point $P(x, y)$ which moves so that the sum of squares of its distances from the points $A(1, 1)$, $B(-1, 1)$, $C(1, -1)$ and $D(-1, -1)$ is 12.

9. (a) Find the locus of the point $P(x, y)$ which moves so that its distance from the point $A(4, 0)$ is always twice its distance from the point $B(1, 0)$.

(b) Find the locus of the point $P(x, y)$ which moves so that its distance from the point $A(2, 5)$ is always twice its distance from the point $B(4, -1)$.

10. A point $P(x, y)$ moves so that its distance from the point $K(2, 5)$ is twice its distance from the line $x = -1$. Draw a diagram, and find the equation of the locus of P.

11. (a) By using the perpendicular distance formula, find the locus of a point $P(x, y)$ which is equidistant from the lines $3x + 4y = 36$ and $4x + 3y = 24$.

(b) Show that the locus consists of two perpendicular lines, and sketch all four lines on the same number plane.

12. Find the locus of a point $P(x, y)$ which lies above the x-axis so that the sum of its distances to the origin and the x-axis is 2.

13. If $P(x, y)$ is any point on the line $y = 4x + 3$, show that the midpoint M of OP has coordinates $\left(\frac{1}{2}x, \frac{1}{2}(4x + 3)\right)$. Hence find the locus of M.

14. $P(x, y)$ is a variable point on the line $2x - 3y + 6 = 0$, and the point Q divides OP in the ratio $3 : 2$. Show that Q has coordinates $\left(\frac{3}{5}x, \frac{2}{5}(x + 3)\right)$. Hence find the locus of Q.

15. (a) Show that the points $A(0, 2)$, $B(-\sqrt{3}, -1)$ and $C(\sqrt{3}, -1)$ form an equilateral triangle.

(b) Find the equations of the lines that form the sides AB, BC and CA.

(c) Find the locus of the point $P(x, y)$ which moves so that the sum of the squares of the distances from the sides of the triangle is 9.

(d) Describe the locus geometrically and state its relation to the triangle.

EXTENSION

16. (a) Show that if $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in three-dimensional space, then $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$.

(b) $P(x, y, z)$ is a variable point. The sum of the squares of the distances from P to the points $A(a, 0, 0)$ and $B(-a, 0, 0)$ is $4a^2$, where a is a constant. Find the equation of the locus of P and describe it geometrically.

(c) Consider the eight corners of a cube $A(1, 1, 1)$, $B(-1, 1, 1)$, $C(1, -1, 1)$, $D(-1, -1, 1)$, $E(1, 1, -1)$, $F(-1, 1, -1)$, $G(1, -1, -1)$ and $H(-1, -1, -1)$. A point $P(x, y, z)$ moves so that $PA^2 + PB^2 + PC^2 + PD^2 + PE^2 + PF^2 + PG^2 + PH^2 = 30$. Show that the locus of P is a sphere and find its centre and radius.

17. Let $P(x, y)$ be a point and L_1 and L_2 be two lines in the number plane. Let $\mathcal{C}$ be the set of all points P such that the sum of the squares of the distances of P from L_1 and L_2 is r^2. Prove that $\mathcal{C}$ is a circle if and only if L_1 and L_2 are perpendicular.

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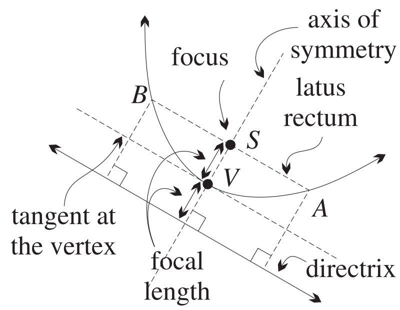
9 B The Geometric Definition of the Parabola

Until now, we have used the word 'parabola' to refer to any curve whose equation is a quadratic. The parabola, however, is a geometric object, and to be understood in its true form, needs to be defined geometrically. This section will define the parabola as a locus, similar to the locus definition of a circle, and the general algebraic equation will then be derived from this geometric definition. The definition is surprising in that the whole curve is entirely determined simply by taking any point in the plane and any line not through the point.

4 DEFINITION: A *parabola* is the locus of all points that are equidistant from a given point (called the *focus*) and given line (called the *directrix*) not passing through the focus.

Vertex, Axis of Symmetry, Latus Rectum and Focal Length: The diagram below begins to unpackage the definition and to define some special points and lines associated with a parabola. Suppose that the focus S and the directrix d have been given.

1. The *vertex* V is the point midway between the focus and the directrix.
2. The line through the vertex V parallel to the directrix is the tangent to the parabola at the vertex, because every point on it other than V is closer to the directrix than to S.
3. The line SV through the focus and vertex, perpendicular to the directrix, is called the *axis of symmetry* (or just *axis*), because the whole diagram is symmetric in that line.



4. Let A and B be the two points where the parabola meets the line through S parallel to the directrix. Being on the parabola, their distances from the focus equal the distance from the focus to the directrix.
5. The interval joining the points A and B passes through the focus and is parallel to the directrix. It is called the *latus rectum*.

Focal Length, Chords and Focal Chords: The *focal length* is the distance between the vertex and the focus (or between the vertex and the directrix) and is usually assigned the pronumeral a. Since it is a distance, $a > 0$. Then other important distances can be expressed in terms of the focal length, using the two squares formed by the axis, the directrix and the latus rectum.

5 THE FOCAL LENGTH:

Let	distance from focus to vertex	$= a$	(the focal length).
Then	distance from focus to directrix	$= 2a$	(twice focal length),
and	length of latus rectum	$= 4a$	(four times focal length).

By analogy with the circle, any interval joining two points on the parabola is called a *chord*, and the line through the two points is called a *secant*. A chord passing through the focus is called a *focal chord*. The latus rectum is then distinguished from all other chords because it is the focal chord parallel to the directrix.

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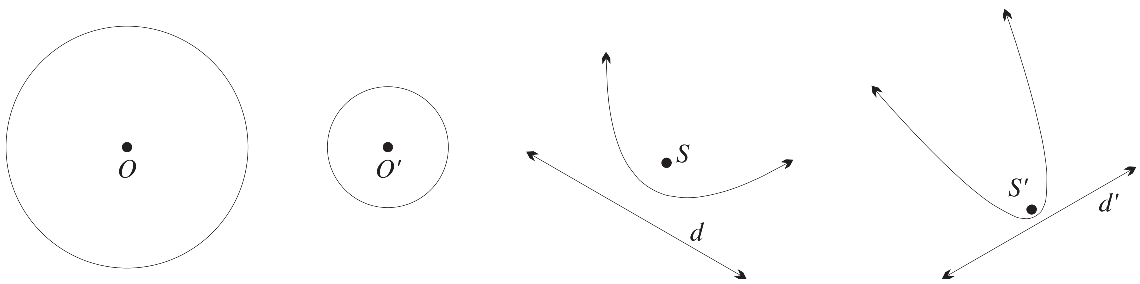
The focus of a parabola has some analogy with the centre of a circle, and focal chords are distinguished from other chords in a manner similar to the way that a diameter is a chord passing through a circle's centre. Either the length of the latus rectum, or the focal length, gives a measure of how opened out the arms of the parabola are, just as the diameter of a circle (or the radius) is the measure of a circle's size.



It is obvious from the circle's definition that any two circles of the same radius are congruent, and that any two circles are similar. In the same way, any two parabolas with the same focal length can be mapped to each other by congruence transformations — translate the second focus onto the first, then rotate the second directrix until it coincides with the first. Any two parabolas must then be similar, because an enlargement can be used to change the focal length.

6 SIMILARITY AND CONGRUENCE OF CIRCLES AND PARABOLAS:

Any two circles with the same radius are congruent.
Any two parabolas with the same focal length are congruent.
Any two circles are similar.
Any two parabolas are similar.

Using the Definition of a Parabola to Find its Equation: You must be able, by locus methods, to use the definition of a parabola to find its equation.

WORKED EXERCISE: [The locus method] Use the definition of the parabola to find the equation of the parabola with focus $S(0, 2)$ and directrix $d : y = -2$. What are the vertex, focal length, and length of the latus rectum?

SOLUTION: Let $P(x, y)$ be any point in the plane, and let $M(x, -2)$ be the foot of the perpendicular from P to d. The condition that P lie on the parabola is

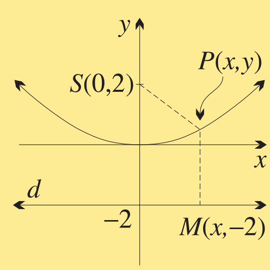
$$PS = PM$$

$\boxed{\text{square}}$ $$PS^2 = PM^2$$

$$x^2 + (y-2)^2 = (y+2)^2$$

$$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$

$$x^2 = 8y.$$



The diagram makes it clear that the vertex is $(0, 0)$ and the focal length is 2. Hence the length of the latus rectum is 8.

The Four Standard Positions of the Parabola: The intention of the rest of this chapter is to study the parabola using the methods of coordinate geometry. Although the parabola can be placed anywhere on the plane, in any orientation, the equation

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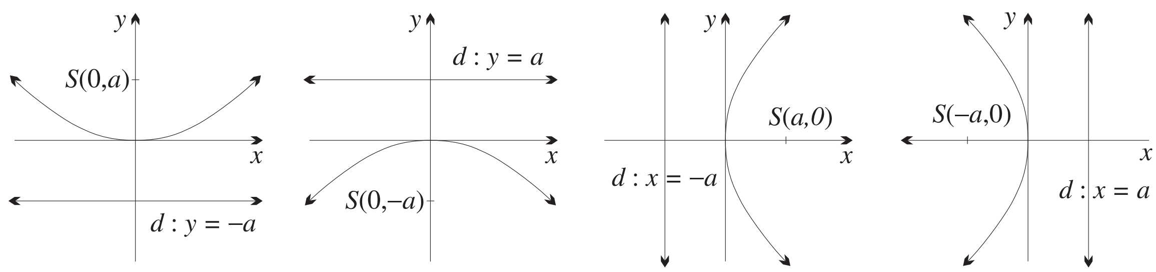
will obviously be simpler if the directrix is parallel to one of the axes, and even simpler if the vertex is at the origin. This gives four standard positions for a parabola with focal length a — facing up, facing down, facing right, and facing left. The four diagrams below show these four positions.

7 THE FOUR STANDARD POSITIONS OF THE PARABOLA: The equation of every parabola whose vertex is at the origin and whose axis is vertical or horizontal can be put into exactly one of the four forms

$$x^2 = 4ay \qquad x^2 = -4ay \qquad y^2 = 4ax \qquad y^2 = -4ax.$$

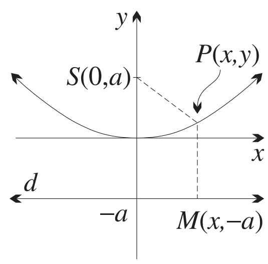


$x^2 = 4ay$ $\qquad$ $x^2 = -4ay$ $\qquad$ $y^2 = 4ax$ $\qquad$ $y^2 = -4ax$

The first position — vertex at the origin and facing upwards — is the most usual. We will prove that its equation is $x^2 = 4ay$; the other three equations then follow using reflections in the x-axis and in the line $y = x$.

PROOF: The parabola with vertex at the origin, with focal length a and facing upwards will have focus $S(0, a)$ and directrix $d : y = -a$.
Let $P(x, y)$ be any point in the plane,
and let $M(x, -a)$ be the foot of the perpendicular from P to d.
The condition that P lie on the parabola is

$$\begin{aligned} PS &= PM \\ \boxed{\text{square}} \qquad PS^2 &= PM^2 \\ x^2 + (y-a)^2 &= (y+a)^2 \\ x^2 &= (y+a)^2 - (y-a)^2 \\ x^2 &= 4ay. \end{aligned}$$

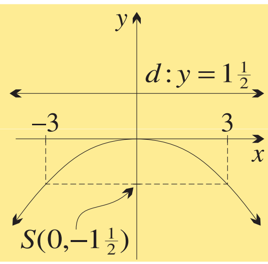


Using the Four Standard Equations of a Parabola:

Most of the time, there is no need to go back to the definition of a parabola. We can simply use the standard equations of the parabola established above. You need to be able to describe the parabola geometrically given its equation, and you need to be able to write down the equation of a parabola described geometrically.

8 FIND a FIRST: Establish the orientation, then find the values of $4a$ and of a.

WORKED EXERCISE: [Geometric description of an equation]
Sketch the parabola $x^2 = -6y$, showing focus, directrix, and the endpoints of the latus rectum.



SOLUTION: The parabola faces down, with $4a = 6$ and $a = 1\frac{1}{2}$.
So the focus is $S(0, -1\frac{1}{2})$, the directrix is $y = 1\frac{1}{2}$,
and the latus rectum has endpoints $(3, -1\frac{1}{2})$ and $(-3, -1\frac{1}{2})$.

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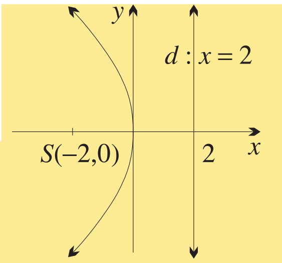

WORKED EXERCISE: [Writing down the equation] Write down the equation of the parabola with vertex at the origin and directrix $x = 2$.

SOLUTION: The parabola is facing left, with $a = 2$ and $4a = 8$.
So its equation is $y^2 = -8x$.

Exercise 9B

NOTE: When a question asks 'Use the definition of a parabola to find its equation', the solution should begin 'Let $P(x, y)$ be any point in the plane. The condition that P lie on the parabola is . . . .' Otherwise the four standard forms may be used.

1. [Construction using ruler and compasses] On a fresh piece of lined paper, rule a directrix d along a line about six lines from the bottom of the page, then mark a focus S two lines above d and horizontally centred on the page. With compasses, construct a set of concentric circles with centre S and radii 1, 2, 3, . . . times the distance between the lines on the page. Now use the definition of the parabola to mark points equidistant from the focus and directrix, then join them up by hand to form a parabola.

2. [An approach through the tangents to the parabola] On a blank piece of paper, mark a directrix d and a focus S about 4 cm apart. Fold the paper so that the focus S is positioned exactly on the directrix d. Make about twenty such folds, positioning S at different places along d. The set of folds will form an envelope of tangents to the parabola.

3. The variable point $P(x, y)$ moves so that it is equidistant from the point $S(0, 3)$ and the line $y + 3 = 0$. Draw a diagram, and let L be the point $(x, -3)$.
(a) Show that $PS^2 = x^2 + (y - 3)^2$ and $PL^2 = (y + 3)^2$.
(b) By setting $PS^2 = PL^2$, derive the equation of the locus of P.

4. Applying the method outlined in the previous question, use the definition of a parabola to derive the equations of:
(a) the parabola with focus $(0, 5)$ and directrix $y + 5 = 0$,
(b) the parabola with focus $(0, -1)$ and directrix $y - 1 = 0$,
(c) the parabola with focus $(2, 0)$ and directrix $x + 2 = 0$,
(d) the parabola with focus $(-\frac{3}{2}, 0)$ and directrix $x - \frac{3}{2} = 0$.

5. Derive the equation of the locus of the point $P(x, y)$ which moves so that:
(a) it is equidistant from the point $S(0, -a)$ and the line $y - a = 0$,
(b) it is equidistant from the point $S(a, 0)$ and the line $x + a = 0$.

6. For each of the following parabolas, find: (i) the focal length a, (ii) the coordinates of the vertex, (iii) the coordinates of the focus, (iv) the equation of the axis, (v) the equation of the directrix, (vi) the length of the latus rectum. Then sketch a graph of each parabola showing these features.

(a) $x^2 = 4y$	(e) $x^2 = -8y$	(i) $y^2 = 4x$	(m) $y^2 = -8x$
(b) $x^2 = 8y$	(f) $x^2 = -12y$	(j) $y^2 = x$	(n) $y^2 = -12x$
(c) $x^2 = y$	(g) $x^2 = -2y$	(k) $y^2 = 6x$	(o) $y^2 = -x$
(d) $x^2 = \frac{4}{3}y$	(h) $x^2 = -0{\cdot}4y$	(l) $y^2 = \frac{1}{2}x$	(p) $y^2 = -1{\cdot}2x$

7. Rearrange each equation into the form $x^2 = 4ay$, $x^2 = -4ay$, $y^2 = 4ax$ or $y^2 = -4ax$. Hence sketch a graph of each parabola, indicating the vertex, focus and directrix.
(a) $2x^2 = y$ (b) $4y + x^2 = 0$ (c) $9y^2 = 4x$ (d) $y^2 + 10x = 0$

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8. Use the four standard forms to find the equation of the parabola with vertex at the origin, axis vertical and:
(a) focus at $(0, 5)$, (c) directrix $y = -2$, (e) equation of latus rectum $y = 1$,
(b) focus at $(0, -3)$, (d) directrix $y = \frac{1}{2}$, (f) equation of latus rectum $y = -\frac{1}{8}$.

9. Use the four standard forms to find the equation of the parabola with vertex at the origin, axis horizontal and:
(a) focus at $(\frac{1}{2}, 0)$, (c) directrix $x = -4$, (e) equation of latus rectum $x = 3$,
(b) focus at $(-1, 0)$, (d) directrix $x = 2$, (f) equation of latus rectum $x = -\frac{3}{2}$.

10. Use the four standard forms to find the equation of the parabola with vertex at the origin and with the following properties:
(a) axis vertical, passing through $(4, 1)$,
(b) axis vertical, passing through $(-2, 8)$,
(c) axis horizontal, passing through $(2, -2)$,
(d) axis horizontal, passing through $(-1, 1)$.

DEVELOPMENT

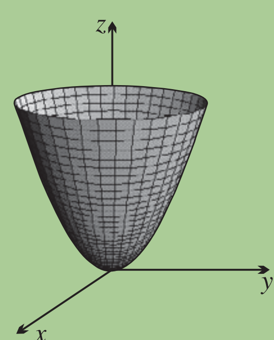
11. Find the equation of the parabola with vertex at the origin and:
(a) axis vertical, latus rectum 8 units in length (2 parabolas),
(b) focal length 3 units, and axis horizontal or vertical (4 parabolas),
(c) passing through $(1, 1)$, and axis horizontal or vertical (2 parabolas),
(d) axis horizontal, focal length $\frac{1}{2}$ (2 parabolas).

12. (a) The equation of a parabola is of the form $y = kx^2$. If the line $8x - y - 4 = 0$ is a tangent to the parabola, find the value of k.
(b) A parabola with vertical axis has its vertex at the origin. Find the equation of the parabola if the line $12x - 4y + 3 = 0$ is a tangent.

13. Use the definition of a parabola and perpendicular distance formula to find:
(a) the equation of the parabola with focus $S(-1, 1)$ and directrix $y = x - 2$,
(b) the equation of the parabola with focus $S(3, -3)$ and directrix $x - y + 6 = 0$.
Explain, without referring to its equation, why each parabola passes through the origin.

EXTENSION

14. The variable point $P(x, y, z)$ moves so that it is equidistant from the point $S(0, 0, a)$ and the plane $z = -a$ (such a surface is called a *paraboloid*). Show that the equation of the locus of P is $x^2 + y^2 = 4az$.



15. Find the equations of the following paraboloids:
(a) focus $(0, 3, 0)$, directrix $y = -3$,
(b) focus $(1, 0, 0)$, directrix $x = -1$,
(c) focus $(0, 0, -2)$, directrix $z = 2$,
(d) focus $(0, -\frac{3}{2}, 0)$, directrix $y - \frac{3}{2} = 0$.

16. Find the vertex and directrix of each of the following paraboloids:
(a) $x^2 + y^2 + 8z = 0$ (b) $y^2 + z^2 - 2x = 0$ (c) $x^2 + y + z^2 = 0$

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9 C Translations of the Parabola

When the vertex of a parabola is not at the origin, the normal rules for shifting curves around the plane apply — to move the vertex from $(0,0)$ to (h,k), replace x by $x-h$ and y by $y-k$. As with a parabola whose vertex is at the origin, there are two tasks to learn. First, one must be able to write down the equation of a parabola given its geometric description, and conversely, one must be able to describe a parabola geometrically given its equation.

9 THE FOUR SHIFTED STANDARD FORMS OF THE PARABOLA: Every parabola whose axis is vertical or horizontal has an equation that can be put into exactly one of the four forms

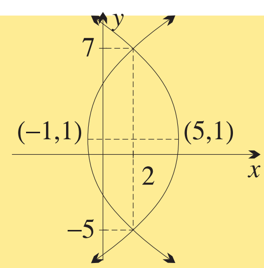
$$(x-h)^2 = 4a(y-k) \qquad (x-h)^2 = -4a(y-k)$$
$$(y-k)^2 = 4a(x-h) \qquad (y-k)^2 = -4a(x-h)$$

where $a > 0$ is the focal length, and (h,k) is the vertex.

Writing Down the Equation of a Given Parabola: A sketch is essential before anything else. Writing down the equation requires the focal length a, the vertex (h,k) and the orientation of the parabola.

WORKED EXERCISE: Write down the equations of the parabolas with focal length 3, focus $(2,1)$ and axis parallel to the x-axis. Sketch them, and find and describe their points of intersection.

SOLUTION: The parabola facing right has vertex $(-1,1)$, so its equation is $(y-1)^2 = 12(x+1)$.
The parabola facing left has vertex $(5,1)$, so its equation is $(y-1)^2 = -12(x-5)$.
The two parabolas meet at $(2,7)$ and $(2,-5)$, which are the endpoints of their common latus rectum.



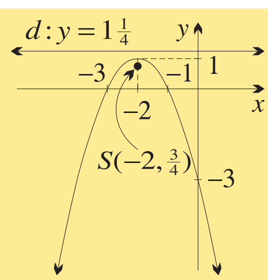
Describing a Parabola Given its Equation: If the equation of a parabola is given, the parabola should be forced into the appropriate standard form by completing the square. As always, find the focal length a. Then a sketch is essential.

WORKED EXERCISE: Find the focus, directrix, focal length and endpoints of the latus rectum of the parabola $y = -3 - 4x - x^2$.

SOLUTION: Completing the square,

$$x^2 + 4x = -y - 3$$
$$x^2 + 4x + 4 = -y - 3 + 4$$
$$(x+2)^2 = -(y-1).$$

So $4a = 1$ and $a = \frac{1}{4}$, the vertex is $(-2,1)$, and the parabola is concave down.
Thus the focus is $(-2, \frac{3}{4})$ and the directrix is $y = 1\frac{1}{4}$, and the endpoints of the latus rectum are $(-2\frac{1}{2}, \frac{3}{4})$ and $(-1\frac{1}{2}, \frac{3}{4})$.



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Exercise 9C

NOTE: When a question asks 'Use the definition of a parabola to find its equation', the solution should begin 'Let $P(x, y)$ be any point in the plane. The condition that P lie on the parabola is ....' Otherwise the four standard forms may be used.

1. The variable point $P(x, y)$ moves so that it is equidistant from the point $S(3, 3)$ and the line $y + 1 = 0$. Let L be the point $(x, -1)$.
(a) Show that $PS^2 = (x - 3)^2 + (y - 3)^2$ and $PL^2 = (y + 1)^2$.
(b) By setting $PS^2 = PL^2$, derive the equation of the locus of P.

2. Applying the method outlined in the previous question, use the definition of a parabola to derive the equations of the following parabolas:
(a) focus $(-7, -2)$, directrix $y + 8 = 0$
(b) focus $(0, 2)$, directrix $x + 2 = 0$

3. Sketch each of the following parabolas, clearly indicating the coordinates of the vertex and focus, and the equations of the axis and directrix:
(a) $x^2 = 4(y + 1)$
(b) $(x + 2)^2 = 4y$
(c) $(x - 3)^2 = 8(y + 5)$
(d) $(x - 4)^2 = -8y$
(e) $x^2 = -2(y + 3)$
(f) $(x + 5)^2 = -4(y - 3)$
(g) $y^2 = 6(x + 2)$
(h) $(y - 1)^2 = 16x$
(i) $(y + 7)^2 = 12(x - 5)$
(j) $(y + 8)^2 = -4(x - 3)$
(k) $y^2 = -10(x + 6)$
(l) $(y - 3)^2 = -2x$

4. Using the four standard forms, find the equation of the parabola with focus and vertex:
(a) $(-2, 6)$, $(-2, 4)$
(b) $(5, 1)$, $(1, 1)$
(c) $(2, -1)$, $(2, 2)$
(d) $(0, 0)$, $(1, 0)$
(e) $(-5, 4)$, $(-5, 2)$
(f) $(-3, -2)$, $(-7, -2)$
(g) $(8, -10)$, $(8, -7)$
(h) $(-3, -3)$, $(-1, -3)$
(i) $(6, 0)$, $(6, -3)$

5. Use the standard forms to find the equation of the parabola with vertex and directrix:
(a) $(2, -1)$, $y = -3$
(b) $(1, 0)$, $x = 0$
(c) $(-3, 4)$, $y - 6 = 0$
(d) $(2, 5)$, $x = 5$
(e) $(3, 1)$, $y = -1$
(f) $(-4, 2)$, $x = -7$
(g) $(0, -\frac{3}{2})$, $y = \frac{1}{2}$
(h) $(-1, -4)$, $x = 2$
(i) $(-7, -5)$, $y = -5\frac{1}{2}$

6. Use the standard forms to find the equation of the parabola with focus and directrix:
(a) $(0, 4)$, $y = 0$
(b) $(6, 0)$, $x = 0$
(c) $(0, -2)$, $y = 0$
(d) $(-4, 0)$, $x = 0$
(e) $(1, 7)$, $y = 3$
(f) $(3, -2)$, $x = 1$
(g) $(-1, 4)$, $y = 5$
(h) $(3, \frac{1}{2})$, $x = 5$
(i) $(5, -4)$, $y = -9$

DEVELOPMENT

7. Express the equation of each of the following parabolas in the form $(x - h)^2 = 4a(y - k)$ or $(x - h)^2 = -4a(y - k)$. Sketch a graph, clearly indicating the focus, vertex and directrix.
(a) $y = x^2 + 6x + 5$
(b) $x^2 = 1 - y$
(c) $6y = x^2 - 12x$
(d) $x^2 = 2(1 + 2y)$
(e) $y = (x + 8)(x - 2)$
(f) $(x + 3)(x + 5) = 8y - 25$
(g) $x^2 - 6x + 2y + 12 = 0$
(h) $x^2 - 8x + 12y + 4 = 0$

8. Express the equation of each of the following parabolas in the form $(y - k)^2 = 4a(x - h)$ or $(y - k)^2 = -4a(x - h)$. Sketch a graph, clearly indicating the focus, vertex and directrix.
(a) $y^2 - 4x = 0$
(b) $y^2 = 6 - 2x$
(c) $6x = y^2 + 18$
(d) $y^2 - 2y = 4x - 5$
(e) $y(y - 4) = 8x$
(f) $y^2 - 6y - 2x + 7 = 0$
(g) $y^2 + 4y + 6x - 26 = 0$
(h) $(y - 4)(y - 6) = 12x + 11$

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9. By using the general form $y = Ax^2 + Bx + C$ or $x = Ay^2 + By + C$, find the equation of the parabola with:
(a) axis parallel to the y-axis, passing through $(1, 0)$, $(-1, -6)$ and $(2, 9)$;
(b) axis parallel to the y-axis, passing through $(1, -5)$, $(-1, 5)$ and $(0, 1)$;
(c) axis parallel to the x-axis, passing through $(0, 1)$, $(8, -1)$ and $(-1, 2)$;
(d) axis parallel to the x-axis, passing through $(-4, 1)$, $(-6, -1)$ and $(-3, 0)$.

10. Find the equation of each of the following parabolas:
(a) vertex at $(1, 4)$, axis parallel to the y-axis, passing through $(3, 5)$;
(b) vertex at $(-2, 3)$, axis parallel to the y-axis, y-intercept at -1;
(c) vertex at $(-3, -2)$, axis parallel to the x-axis, passing through $(-1, 0)$;
(d) vertex at $(2, 5)$, axis parallel to the x-axis, passing through $(0, 4)$.

11. Find all possible equations of the parabolas with the following constraints, assuming that the axis is parallel to one of the coordinate axes:
(a) vertex at $(3, -1)$, focal length 2 units (4 parabolas);
(b) latus rectum has endpoints $(1, 3)$ and $(1, -5)$ (2 parabolas);
(c) focus at $(-2, 4)$, endpoint of latus rectum at $(0, 4)$ (2 parabolas);
(d) axis $y - 2 = 0$, vertex at $(3, 2)$, latus rectum has length 6 units (2 parabolas);
(e) focus $(6, -3)$, vertex on the line $y = x - 4$ (2 parabolas).

12. Find the equation of each of the following parabolas:
(a) vertex at $(3, -1)$, axis parallel to the y-axis and the line $4x + y - 7 = 0$ is a tangent;
(b) vertex at $(-4, 2)$, axis parallel to the x-axis and the line $x = 6 - 4y$ is a tangent.

13. Use the perpendicular distance formula to find the equation of the parabola with:
(a) focus $(-1, 4)$ and directrix $x - y - 1 = 0$;
(b) focus $(1, 2)$ and directrix $4x + 3y - 2 = 0$.

EXTENSION

14. Derive the equations of the following paraboloids:
(a) focus $(1, 2, 3)$, directrix $z + 1 = 0$,
(b) focus $(-1, 2, -1)$, directrix $x = 1$,
(c) focus $(0, -5, 3)$, directrix $y = 2$,
(d) focus $(4, -3, 7)$, directrix $z - 4 = 0$.

15. Find the focus, vertex and directrix of the following paraboloids:
(a) $x^2 + y^2 - 2x - 4y - 4z + 1 = 0$
(b) $y^2 + z^2 + 6y + 8x + 1 = 0$

9 D Parametric Equations of Curves

This section introduces an ingenious way of handling curves by making each coordinate a function of a single variable, called a *parameter*. In this way, each point on the curve is specified by a single number, rather than by a pair of coordinates. The main purpose here is to investigate further the geometry of the parabola, but the method is general, and some other curves will be considered, particularly circles and rectangular hyperbolas.

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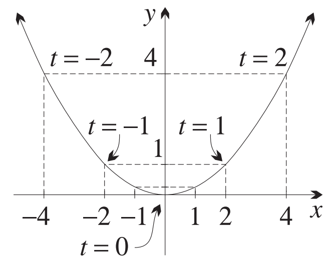
An Example of Parametrisation: The parabola $x^2 = 4y$ with focal length 1 can be *parametrised* by the pair of equations

$$x = 2t \qquad \text{and} \qquad y = t^2$$



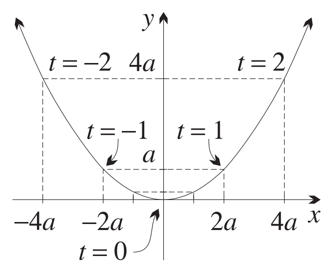
because by simple algebra, elimination of t gives $x^2 = 4y$. The variable point $(2t, t^2)$ now runs along the whole curve as the parameter t takes different values:

t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	-4	-2	-1	0	1	2	4
y	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4

The sketch shows the curve with the seven plotted points labelled by their parameter. In effect, the curve becomes a 'bent and stretched' number line. The original equation in x and y is called the *Cartesian equation* of the curve to distinguish it from the *parametric equations* of the curve.

The Standard Parametrisation of the Parabola: The parabola $x^2 = 4ay$ has a parametrisation that is so convenient that it is taken as the *standard parametrisation*:

$$x = 2at \qquad \text{and} \qquad y = at^2.$$



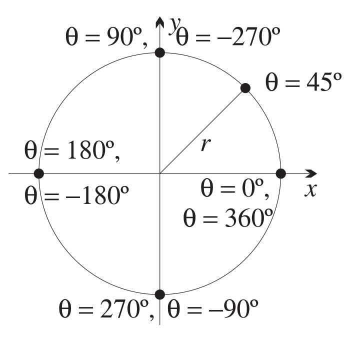
The parametrisation should be checked by elimination of t — notice that the previous example was a special case with $a = 1$. A short table of values shows how the parabola is divided neatly into four parts by this parametrisation:

t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	$-4a$	$-2a$	$-a$	0	a	$2a$	$4a$
y	$4a$	a	$\frac{1}{4}a$	0	$\frac{1}{4}a$	a	$4a$

The vertex has parameter $t = 0$, points to the right have positive parameter, and points to the left have negative parameter. The endpoints of the latus rectum have parameters $t = -1$ and $t = 1$, points on the curve between these endpoints have parameter with $|t| < 1$, and points above the latus rectum have parameters with $|t| > 1$. So the three chief points on the parabola are paired with the three most important numbers, 0, 1 and -1.

A Parametrisation of the Circle: The circle $x^2 + y^2 = r^2$ can be parametrised using trigonometric functions by

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$



As we saw in Chapter Four, this parametrisation works because of the Pythagorean identity $r^2\cos^2\theta + r^2\sin^2\theta = r^2$. Notice from the table of values below how in this case, each parameter corresponds to just one point, but each point corresponds to infinitely many different values of the parameter, all differing by multiples of 360°:

θ	$-360°$	$-270°$	$-180°$	$-90°$	0	$45°$	$90°$	$180°$	$270°$	$360°$
x	r	0	$-r$	0	r	$\frac{1}{2}r\sqrt{2}$	0	$-r$	0	r
y	0	r	0	$-r$	0	$\frac{1}{2}r\sqrt{2}$	r	0	$-r$	0

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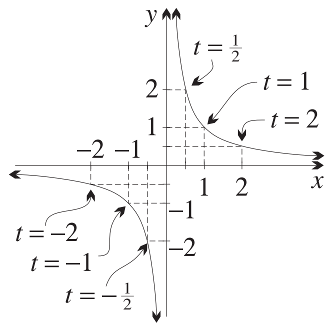
A Parametrisation of the Rectangular Hyperbola: The rectangular hyperbola $xy = 1$ can be parametrised algebraically by

$$x = t \qquad \text{and} \qquad y = \frac{1}{t}\,.$$

This time there is a one-to-one correspondence between the points on the curve and the real numbers, with the one exception that $t = 0$ does not correspond to any point:

t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	$-\frac{1}{2}$	-1	-2	$*$	2	1	$\frac{1}{2}$



WORKED EXERCISE: Find the Cartesian equations of the curves defined by the parametric equations: (a) $x = 4t,\ y = t^2 + 1$ (b) $x = \sec\theta,\ y = \sin\theta$
Describe part (a) geometrically.

SOLUTION:

(a) From the first, $t = \frac{1}{4}x$,
and substituting into the second,
$$y = \tfrac{1}{16}x^2 + 1$$
$$x^2 = 16(y - 1),$$
which is a parabola with vertex $(0, 1)$, concave up, with focal length 4.

(b) Squaring, $x^2 = \sec^2\theta$,
and $y^2 = \sin^2\theta$
$= 1 - \cos^2\theta$,
so $y^2 = 1 - \dfrac{1}{x^2}$
$x^2(1 - y^2) = 1.$

Exercise 9D

1. (a) Complete the table below for the curve $x = 2t$, $y = t^2$ and sketch its graph:

t	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
x									
y									

(b) Eliminate the parameter to find the Cartesian equation of the curve.
(c) State the coordinates of the vertex and focus of the parabola.
(d) What value of t gives the coordinates of the vertex?
(e) What are the coordinates of the endpoints of the latus rectum and what values of t give these coordinates?

2. Repeat the previous question for the curves: (a) $x = 4t,\ y = 2t^2$ (b) $x = t,\ y = \frac{1}{2}t^2$

3. (a) Show that the point $\left(ct, \dfrac{c}{t}\right)$ lies on the curve $xy = c^2$.

(b) Complete the table of values below for the curve $x = 2t$, $y = \dfrac{2}{t}$ and sketch its graph.

t	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
x										
y										

(c) Explain what happens as $t \to \infty$, $t \to -\infty$, $t \to 0^+$ and $t \to 0^-$.

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4. (a) Show that the point $(a\cos\theta, b\sin\theta)$ lies on the curve $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

(b) Complete a table of values for the curve $x=4\cos\theta$, $y=3\sin\theta$, where $0^\circ \le \theta \le 360^\circ$.

(c) Sketch the curve and state its Cartesian equation.

DEVELOPMENT

5. Eliminate the parameter and hence find the Cartesian equation of the curve.

(a) $x=3-t$, $y=2t+1$

(b) $x=1+2\tan\theta$, $y=3\sec\theta-4$

(c) $x=t+\dfrac{1}{t}$, $y=t^2+\dfrac{1}{t^2}$

(d) $x=\cos\theta+\sin\theta$, $y=\cos\theta-\sin\theta$

6. (a) Show that the point $(a\sec\theta, b\tan\theta)$ lies on the curve $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

(b) Complete a table of values for the curve $x=4\sec\theta$, $y=3\tan\theta$, where $0^\circ \le \theta \le 360^\circ$. What happens when $\theta=90^\circ$ and $\theta=270^\circ$.

(c) Sketch the curve (it has two asymptotes) and state its Cartesian equation.

7. (a) Show that $x=a+r\cos\theta$, $y=b+r\sin\theta$ defines a circle with centre (a,b) and radius r.

(b) Hence sketch a graph of the curve $x=1+2\cos\theta$, $y=-3+2\sin\theta$.

8. Different parametric representations may result in the same Cartesian equation. The graphical representation, however, may be different.

(a) Find the Cartesian equation of the curve $x=2-t$, $y=t-1$ and sketch its graph.

(b) Find the Cartesian equation of the curve $(\sin^2 t, \cos^2 t)$. Explain why $0 \le x \le 1$ and $0 \le y \le 1$ and sketch a graph of the curve.

(c) Find the Cartesian equation of the curve $x=4-t^2$, $y=t^2-3$. Explain why $x \le 4$ and $y \ge -3$ and sketch a graph of the curve.

9. Find the Cartesian equation of the curve $x=3+r\cos\theta$, $y=-2+r\sin\theta$, and describe it geometrically if: (a) r is constant and θ is variable, (b) θ is constant and r is variable.

EXTENSION

10. P_1 and P_2 are the points (x_1,y_1) and (x_2,y_2) respectively.

(a) Let $P(x,y)$ divide the interval P_1P_2 in the ratio $\lambda : 1$. Show that $x=\dfrac{x_1+\lambda x_2}{1+\lambda}$, $y=\dfrac{y_1+\lambda y_2}{1+\lambda}$ is a parametrisation of the line P_1P_2.

(b) What parameters do the points P_1 and P_2 have, and what happens near $\lambda=-1$?

(c) The line joining $P_1(1,5)$ and $P_2(4,9)$ meets the line $3x+y-11=0$ at the point P. Find the ratio $\dfrac{P_1P}{PP_2}$, without finding the coordinates of P.

(d) (i) The line P_1P_2 intersects the parabola $x^2=4ay$. Obtain an equation whose roots are the values of λ corresponding to the points of intersection. (ii) Find a necessary and sufficient condition for the line P_1P_2 to be a tangent to the parabola.

9 E Chords of a Parabola

In the remaining sections, the geometry of chords, tangents and normals is developed using parametric as well as using Cartesian methods. Although the more important equations in these sections are boxed as usual, it is not intended that they be learnt and applied — examination questions will either ask for them to be derived, or give the formulae in the question.

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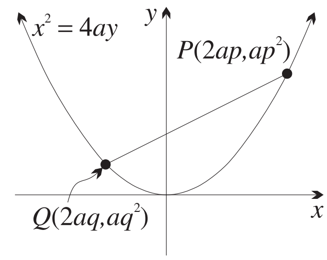
The Parametric Equation of the Chord: Suppose that $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two distinct points on the parabola $x^2 = 4ay$. We can find the equation of the chord PQ by finding the gradient of the chord and then using point–gradient form.

$$\begin{aligned}\text{Gradient of chord} &= \frac{ap^2 - aq^2}{2ap - 2aq}\\ &= \frac{a(p-q)(p+q)}{2a(p-q)}\\ &= \tfrac{1}{2}(p+q),\end{aligned}$$



so the chord is
$$\begin{aligned} y - ap^2 &= \tfrac{1}{2}(p+q)(x - 2ap)\\ y - ap^2 &= \tfrac{1}{2}(p+q)x - ap^2 - apq\\ y &= \tfrac{1}{2}(p+q)x - apq.\end{aligned}$$

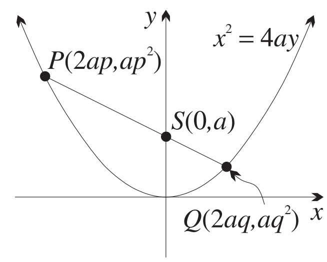
10 THE PARAMETRIC EQUATION OF THE CHORD: $y = \frac{1}{2}(p+q)x - apq$

NOTE: If the parameters p and q are exchanged, then the formulae for the gradient of the chord and the equation of the chord remain the same. Geometrically, this is because the chord PQ is the same line as the chord QP. Such expressions are called *symmetric in p and q*, and this can often be a good check that the calculations have been carried out accurately.

Parameters and Focal Chords: As defined in Section 9B, a chord that passes through the focus of a parabola is called a *focal chord*. Substituting the focus $(0, a)$ into the equation of the chord above gives

$$a = 0 - apq,$$

and dividing by a, $pq = -1$ (note that $a \neq 0$).
This is a condition for PQ to be a focal chord.



11 FOCAL CHORDS: PQ is a focal chord if and only if $pq = -1$.

WORKED EXERCISE: Two points $P(x_0, y_0)$ and $Q(x_1, y_1)$ lie on the parabola $x^2 = 4ay$.

(a) Show that $y_0 - y_1 = \dfrac{{x_0}^2 - {x_1}^2}{4a}$.

(b) Show that the chord PQ has equation $4ay = x(x_0 + x_1) - x_0 x_1$.

(c) Show that PQ is a focal chord if and only if $x_0 x_1 = -4a^2$.

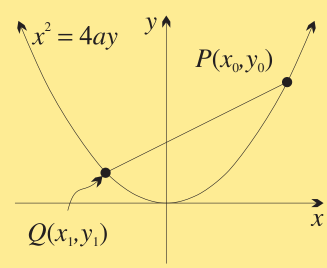
(d) Use part (a) to show that the chord joining the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on $x^2 = 4ay$ has equation $y = \frac{1}{2}(p+q)x - apq$.

SOLUTION:

(a) Since P and Q lie on $x^2 = 4ay$, ${x_0}^2 = 4ay_0$ and ${x_1}^2 = 4ay_1$,

hence $$y_0 - y_1 = \frac{{x_0}^2 - {x_1}^2}{4a}.$$

(b)
$$\begin{aligned}\text{Gradient } PQ &= \frac{y_0 - y_1}{x_0 - x_1}\\ &= \frac{{x_0}^2 - {x_1}^2}{4a(x_0 - x_1)}\\ &= \frac{x_0 + x_1}{4a},\end{aligned}$$



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so the chord is $\quad y - y_1 = \dfrac{x_0 + x_1}{4a}(x - x_1)$

$$4ay - 4ay_1 = x(x_0 + x_1) - x_0x_1 - {x_1}^2.$$

Since $4ay_1 = {x_1}^2$, $\quad 4ay = x(x_0 + x_1) - x_0x_1.$

(c) Substituting $(0, a)$ gives $4a^2 = 0 - x_0x_1$, as required.

(d) Substituting $x_0 = 2ap$ and $x_1 = 2aq$,

$$4ay = x(2ap + 2aq) - 4a^2pq$$

$\boxed{\div 4a}$ $\quad y = \frac{1}{2}x(p + q) - apq.$

NOTE: This worked exercise gives an alternative, but less elegant, derivation of the parametric equation of a chord.

Exercise 9E

1. Find the equation of the chord of the parabola joining the points with parameters:
(a) 1 and -3 on $x = 2t$, $y = t^2$
(b) $\frac{1}{2}$ and 2 on $x = 4t$, $y = 2t^2$
(c) -1 and -2 on $x = t$, $y = \frac{1}{2}t^2$
(d) -2 and 4 on $x = \frac{1}{2}t$, $y = \frac{1}{4}t^2$
Then use the formula $y = \frac{1}{2}(p + q)x - apq$ to obtain the chords in parts (a)–(d).

2. (a) Find the chord joining the points with parameters 2 and $-\frac{1}{2}$ on $x = 6t$, $y = 3t^2$.
(b) Find the Cartesian equation of the parabola and the coordinates of the focus.
(c) Show by substitution that the chord in part (a) is a focal chord.

DEVELOPMENT

3. The points P and Q on the curve $x = 2at$, $y = at^2$ have parameters p and q respectively.
(a) Show that the chord PQ has gradient $\frac{1}{2}(p + q)$.
(b) Hence show that the equation of the chord is $y - \frac{1}{2}(p + q)x + apq = 0$.
(c) Show that PQ intersects the directrix at $\left(\dfrac{2a(pq - 1)}{p + q}, -a\right)$.
(d) State the coordinates of the focus of the parabola.
(e) Show that if PQ is a focal chord, then $pq = -1$.
(f) Hence find the point of intersection of a focal chord and the directrix.

4. P and Q are the points with parameters p and q on the parabola $x = 2at$, $y = at^2$.
(a) State the coordinates of P, Q and the focus S.
(b) Use the distance formula to find an expression for the length of PS.
(c) Similarly find an expression for the length of QS.
(d) Hence show that $PS + QS = a(p^2 + q^2 + 2)$.
(e) If PQ is a focal chord (and hence $pq = -1$), show that $PQ = a(p + 1/p)^2$.

5. P and Q are the points with parameters p and q on the parabola $x = at^2$, $y = 2at$.
(a) Show that the chord PQ is $2x - (p + q)y + 2apq = 0$.
(b) If $OP \perp OQ$, show that the x-intercept of PQ is independent of p and q.

6. (a) The line $x + 2y - 8 = 0$ intersects the parabola $x = 4t$, $y = 2t^2$. By forming a quadratic in t, find the parameters at the points of intersection.
(b) Find the parameters of the points where $y = 3 - x$ intersects $x = 2t$, $y = t^2$.

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7. P and Q are the points with parameters p and q on the parabola $x = 2at$, $y = at^2$.

(a) Show that the chord PQ is $y - \frac{1}{2}(p+q)x + apq = 0$.

(b) If the chord when extended passes through the point $(0, -a)$, show that $pq = 1$.

(c) Hence, if S is the focus of the parabola, show that $\dfrac{1}{SP} + \dfrac{1}{SQ} = \dfrac{1}{a}$.

8. (a) The line $y = 2x + 10$ is a chord of the parabola $x = 4t$, $y = 2t^2$. By putting the line in the form $y = \frac{1}{2}(p+q)x - apq$, find $p+q$ and pq, and hence find the coordinates of the endpoints of the chord.

(b) Show, using a similar method, that $y = 2x - 8$ is a tangent to this parabola.

(c) Show, using a similar method, that $y = 2x - 10$ does not meet this parabola.

9. Using the equation $y = \frac{1}{2}(p+q)x - apq$, show that the midpoint of a chord of the parabola $x = 2at$, $y = at^2$ lies on the vertical line $x = k$ if and only if the chord has gradient $k/2a$.

10. The points P, Q, R and S lie on $x = 2at$, $y = at^2$ and have parameters p, q, r and s respectively. If the chords PQ and RS intersect on the axis, show that $p : r = s : q$.

EXTENSION

11. The parameters of the points P, Q and R on the parabola $x = 2at$, $y = at^2$ form a geometric sequence. Show that the y-intercepts of the chords PQ, PR and QR also form a geometric sequence.

12. A focal chord AB of a parabola meets the directrix at D. Prove that if the focus S divides AB internally in the ratio $k : 1$, then D divides AB externally in the ratio $k : 1$.

9 F Tangents and Normals: Parametric Approach

The easiest way to finding the equation of a tangent is to make an appeal to calculus. We can differentiate in order to find its gradient, and then use point–gradient form to find its equation. Alternatively, we can take the limit of the equation of a chord as its endpoints move together.

The Gradient of the Tangent: Suppose that $P(2ap, ap^2)$ is any point on the parabola with equation $x^2 = 4ay$.

Differentiating parametrically,
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dp}{dx/dp}\\ &= \frac{2ap}{2a}\\ &= p.\end{aligned}$$

12 THE PARAMETRIC GRADIENT OF THE TANGENT:

The gradient of the tangent at $P(2ap, ap^2)$ is p.

NOTE: The simplicity of this result is the essential reason why the standard parametrisation $x = 2at$ and $y = at^2$ is the most convenient parametrisation of the parabola $x^2 = 4ay$.

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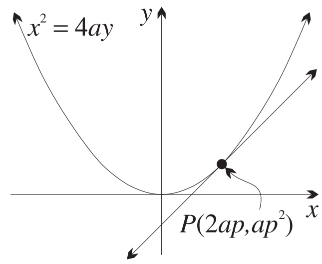
The Parametric Equation of the Tangent: Now the equation of the tangent can be found using the point–gradient form:

$$y - ap^2 = p(x - 2ap)$$
$$y - ap^2 = px - 2ap^2$$
$$y = px - ap^2.$$



13 PARAMETRIC EQUATION OF THE TANGENT:
The tangent at $P(2ap, ap^2)$ is $y = px - ap^2$.

The Tangent as the Limit of the Chord: The equation of the tangent can be developed in a completely different way by starting from the equation of the chord PQ. As the point Q moves closer to P, the line PQ becomes closer and closer to the tangent at P. In fact, the tangent is the limit of the chord PQ as $Q \to P$.

Algebraically, we can move Q towards P by taking the limit as $q \to p$.
The chord is $y = \frac{1}{2}(p+q)x - apq$,
and taking the limit as $q \to p$ gives $y = px - ap^2$, as before.

This process is identical to first principles differentiation, in that a tangent is characterised as the limit of the chord when the endpoints approach each other.

Tangents from an External Point: The parametric form of the tangent gives a straightforward way to find the equations of the two tangents from an external point.

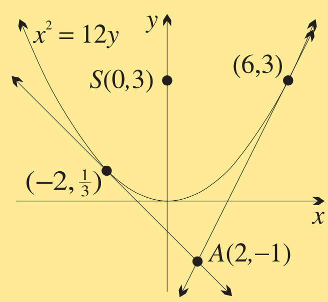
WORKED EXERCISE:

(a) Show that the tangent to $x^2 = 12y$ at the point $P(6p, 3p^2)$ has equation $y = px - 3p^2$.

(b) By substituting the point $A(2, -1)$ into this equation of the tangent, find the points of contact, and the equations, of the tangents to the parabola from A.

SOLUTION:

(a) Differentiating, $\dfrac{dy}{dx} = \dfrac{6p}{6}$
$= p$,
so the tangent is $y - 3p^2 = p(x - 6p)$
$y = px - 3p^2$
(this is the boxed equation above with $a = 3$).



(b) Substituting $A(2, -1)$ gives $-1 = 2p - 3p^2$
$3p^2 - 2p - 1 = 0$
$(3p+1)(p-1) = 0$
$p = 1$ or $-\frac{1}{3}$,
so the points of contact are $(6, 3)$ and $(-2, \frac{1}{3})$,
and the corresponding tangents are $y = x - 3$ and $y = -\frac{1}{3}(x+1)$.

The Intersection of Two Tangents: Simultaneous equations can give us the point T of intersection of the tangents at two distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on a parabola. Notice at the outset that the coordinates of T in the solution must be symmetric in p and q.

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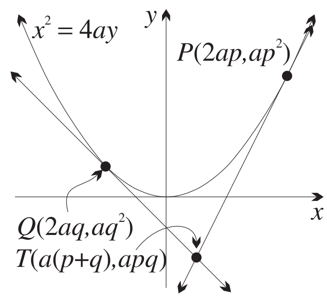


The tangents are $\quad y = px - ap^2 \quad (1)$

and $\quad y = qx - aq^2. \quad (2)$

Subtracting these, $\quad (p-q)x = a(p^2 - q^2)$

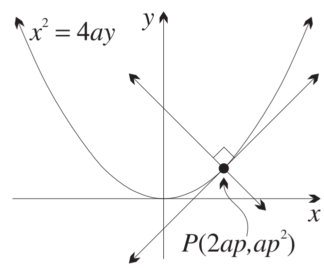
$\boxed{\div (p-q)}$ $\quad x = a(p+q), \;$ since $p \neq q$.

Substituting into (1), $\quad y = ap^2 + apq - ap^2$

$\quad y = apq.$

14 INTERSECTION OF TANGENTS:
The tangents at P and Q meet at $\big(a(p+q),\ apq\big)$.

The Parametric Equation of the Normal: Proceeding as usual,



$$\text{gradient of normal} = -\frac{1}{p},$$

so its equation is $\quad y - ap^2 = -\dfrac{1}{p}(x - 2ap)$

$\boxed{\times p}$ $\quad py - ap^3 = -x + 2ap$

$\quad x + py = 2ap + ap^3.$

15 PARAMETRIC EQUATION OF THE NORMAL:
The normal at $P(2ap, ap^2)$ has gradient $-\dfrac{1}{p}$ and equation $x + py = 2ap + ap^3$.

WORKED EXERCISE:

(a) Show that the normal to $x^2 = 20y$ at the point $P(10p, 5p^2)$ has equation $x + py = 10p + 5p^3$.

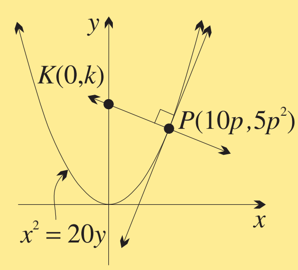
(b) Hence find the equations of the normals to $x^2 = 20y$ from $A(0, 30)$.

(c) Show that a normal passes through $K(0, k)$ if and only if $k > 10$ (apart from the normal at the vertex).

SOLUTION:

(a) Differentiating, $\dfrac{dy}{dx} = \dfrac{10p}{10}$

$\quad = p,$



so the normal has gradient $-\dfrac{1}{p}$ and its equation is

$$y - 5p^2 = -\frac{1}{p}(x - 10p)$$

$$x + py = 10p + 5p^3$$

(this is the boxed equation above with $a = 5$).

(b) Substituting $A(0, 30)$ gives $0 + 30p = 10p + 5p^3$

$\boxed{\div 5}$ $\quad 5p(p^2 - 4) = 0$

$\quad p = 0,\ 2 \;\text{ or }\; -2.$

So the normals from A are $x = 0$, $x + 2y = 60$ and $x - 2y = -60$.

(c) Substituting $K(0, k)$ gives $0 + pk = 10p + 5p^3$

$\quad p(5p^2 + 10 - k) = 0,$

which has nonzero solutions if and only if $k > 10$.

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Exercise 9F

1. Use the derivative to find the equations of the following tangents:
(a) at the point $t = 1$ on $x = 2t$, $y = t^2$
(b) at the point $t = \frac{1}{2}$ on $x = 4t$, $y = 2t^2$
(c) at the point $t = -3$ on $x = t$, $y = \frac{1}{2}t^2$
(d) at the point $t = q$ on $x = 6t$, $y = 3t^2$
Then use the formula $y = px - ap^2$ to obtain the tangents in parts (a)–(d).

2. Use the derivative to find the equations of the following normals:
(a) at the point $t = 2$ on $x = 2t$, $y = t^2$
(b) at $t = -\frac{1}{2}$ on $x = 4t$, $y = 2t^2$
(c) at the point $t = m$ on $x = 6t$, $y = 3t^2$
(d) at the point $t = q$ on $x = 2at$, $y = at^2$
Then use the formula $x + py = 2ap + ap^3$ to obtain the normals in parts (a)–(d).

3. (a) Find the equation of the tangent to the parabola $x = 2at$, $y = at^2$ at the point $t = p$.
(b) Find the coordinates of the points where the tangent intersects the coordinate axes.
(c) Find the area of the triangle formed by the two intercepts and the origin.

4. (a) Find the equation of the normal to $x = 2at$, $y = at^2$ at the point $t = p$.
(b) Find the points where the normal intersects the coordinate axes.
(c) Find the area of the triangle formed by the two intercepts and the origin.

5. (a) Show that the endpoints L and R of the latus rectum of the parabola $x = 2at$, $y = at^2$ have parameters 1 and -1 respectively. Then use these parameters to write down the equations of the tangents and normals at L and R.
(b) Show that these tangents and normals form a square, and find its vertices and its area.
(c) Sketch the parabola, showing the square, the focus and the directrix.

DEVELOPMENT

6. (a) Find the equation of the tangent to $x^2 = 4y$ at the point $(2t, t^2)$.
(b) If the tangent passes through the point $(2, -3)$, find the values of t.
(c) Hence state the equations of the tangents to $x^2 = 4y$ passing through $(2, -3)$.

7. (a) Using methods similar to the those in previous question, find the parameters of the points of contact of the tangents to $x = 10t$, $y = 5t^2$ from the point $P(24, -5)$.
(b) Hence find the gradients and points of contact of the tangents, and show that they are perpendicular. Show also that P lies on the directrix.

8. (a) Sketch the parabola $x = \frac{1}{2}t$, $y = \frac{1}{4}t^2$, and mark the points A and B with parameters $t = -2$ and $t = 4$ respectively.
(b) Find the tangents at A and B, and show that they intersect at $C(\frac{1}{2}, -2)$.
(c) Find the midpoint M of BC, then use the methods of the previous two questions to find the point D on the parabola between A and B such that the tangent at D passes through M. Show that the tangent at D is parallel to AB.

9. (a) Substitute the parabola $x = 6t$, $y = 3t^2$ into the line $x - y - 3 = 0$ to form a quadratic equation in t. Then use the discriminant to show that the line is a tangent, and find its point of contact.
(b) Similarly, show that $x - 2y - 1 = 0$ is a tangent to $x = 4t$, $y = 2t^2$, and find its perpendicular distance from the focus.
(c) By substituting the parabola $x = 2t$, $y = t^2$ into the line $x + y + a = 0$, show that the line is a tangent if and only if $a = 1$.
(d) Similarly, find the value of k if $y = kx - 12$ is a tangent to $x = 6t$, $y = 3t^2$.

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10. (a) Show that the endpoints of the latus rectum of the parabola $x = 2at$, $y = at^2$ have parameters $t = 1$ and $t = -1$.
(b) Hence find the normals to $x^2 = 4ay$ at the endpoints of the latus rectum.
(c) Show that the normals intersect the curve again when $x = 6a$ and $x = -6a$, and hence that the interval between these points of intersection has length $12a$.

11. (a) Show that the normal to $x^2 = 16y$ at the point $P(8p, 4p^2)$ on the parabola has equation $x + py = 8p + 4p^3$.
(b) By substituting $A(0, 44)$ into the normal, show that the normals at three points on the parabola pass through A, and find their coordinates.

12. P and Q are the points $t = p$ and $t = q$ on the parabola $x = 2at$, $y = at^2$.
(a) Find the equations of the normals to the curve at P and Q.
(b) Prove that $p^3 - q^3 = (p - q)(p^2 + pq + q^2)$.
(c) Show that the normals intersect at the point $(-apq(p + q), a(p^2 + q^2 + pq + 2))$.
(d) If $pq = 2$, show that the normals intersect on the parabola.

13. P and Q are the points $t = p$ and $t = q$ on the parabola $x = 2at$, $y = at^2$.
(a) Find the equations of the tangents to the curve at P and Q and show that they intersect at $R(a(p + q), apq)$.
(b) Find the equations of the normals at P and Q, and show that they intersect at $U(-apq(p + q), a(p^2 + q^2 + pq + 2)$.
(c) If PQ is a focal chord, show that the interval RU is parallel to the axis of the parabola.

14. (a) Find the equation of the tangent to $x^2 = 4ay$ at $P(2ap, ap^2)$, and find the point A where the tangent intersects the y-axis.
(b) Find the equation of the normal at P, and the point B where the normal intersects the y-axis.
(c) If S is the focus and C is the foot of the perpendicular from P to the axis of the parabola, show that: (i) $AS = SB$ (ii) $CB = 2a$ (iii) $AO = CO$

15. A line is drawn parallel to the axis of the parabola $x^2 = 4ay$, cutting the parabola at $P(2ap, ap^2)$ and the directrix at R. (a) State the coordinates of R. (b) Show that the normal at P is parallel to RS, where S is the focus.

16. The points P and Q on the parabola $x = 2at$, $y = at^2$ have parameters p and q respectively.
(a) Find the midpoint M of the chord PQ, the point T on the parabola where the tangent is parallel to PQ, and the point I where the tangents at P and Q intersect.
(b) Show that M, T and I lie in a vertical line, with T the midpoint of MI.
(c) Show that the tangent at T bisects the tangents PI and QI.
(d) What is the ratio of the areas of $\triangle PQI$ and $\triangle PQT$?

17. It was proven in the notes above that the tangents to the parabola $x^2 = 4ay$ at two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on the parabola intersect at the point $M\big(a(p + q), apq\big)$. Explain why this result can be restated as follows: 'The tangents at two points on the parabola $x^2 = 4ay$ meet at a point whose x-coordinate is the arithmetic mean of the x-coordinates of the points, and whose y-coordinate is one of the geometric means of the y-coordinates of the points.' Which geometric mean is it?

EXTENSION

18. Show that the common chord of any two circles having focal chords of $x^2 = 4ay$ as diameters passes through the vertex of the parabola. (You may assume that tangents at the extremities of a focal chord are perpendicular.)

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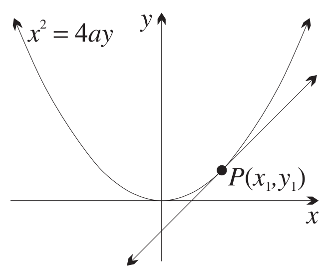
19. (a) The tangents to $x^2 = 4ay$ at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ meet at M. Show that the product of the distance PQ and the perpendicular distance from M to PQ is $a^2|p-q|^3$.
(b) Hence find the area of $\triangle MPQ$.
(c) Let T be the point on the parabola where the tangent is parallel to the chord PQ. Show that $\triangle TPQ$ has half the area of $\triangle MPQ$.

9 G Tangents and Normals: Cartesian Approach

The equations of tangents and normals can also be approached without reference to parameters, by simply using the standard theory, developed in Chapter Seven, of finding equations of tangents through differentiation of the equation.

The Cartesian Equation of the Tangent: Suppose then that $P(x_1, y_1)$ is any point on the parabola $x^2 = 4ay$.



Solving for y, $\quad y = \dfrac{x^2}{4a}$

and differentiating, $\quad \dfrac{dy}{dx} = \dfrac{x}{2a}$

so $\quad$ gradient at $P = \dfrac{x_1}{2a}$.

Hence the tangent is $\quad y - y_1 = \dfrac{x_1}{2a}(x - x_1)$

$$2ay - 2ay_1 = xx_1 - {x_1}^2.$$

Since P lies on the parabola, ${x_1}^2 = 4ay_1$ (this is a subtle point),

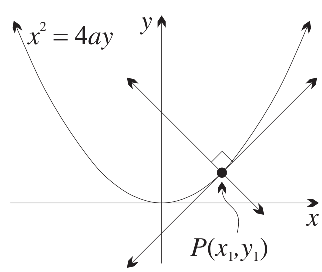
and so $\quad 2ay - 2ay_1 = xx_1 - 4ay_1$

$$2a(y + y_1) = xx_1.$$

16 CARTESIAN EQUATION OF THE TANGENT:
The tangent at $P(x_1, y_1)$ is $xx_1 = 2a(y + y_1)$.

NOTE: The relationship between this formula and the equation $x^2 = 4ay$ is quite striking. Notice how the degree 2 term x^2 has been split multiplicatively into $x \times x_1$, and the degree 1 term $4ay$ has been split additively into $2ay + 2ay_1$. It is a general result that the Cartesian equation of the tangent to any second-degree curve can be written down following this procedure. See the last question in Exercise 9G for a clear statement and proof.

The Cartesian Equation of the Normal: Suppose again that the point $P(x_1, y_1)$ lies on the parabola $x^2 = 4ay$. Using the formula for perpendicular gradients,



gradient of normal at $P = -\dfrac{2a}{x_1}$,

so the normal is $\quad y - y_1 = -\dfrac{2a}{x_1}(x - x_1)$

$$x_1 y - x_1 y_1 = -2ax + 2ax_1$$

$$x_1 y + 2ax = x_1 y_1 + 2ax_1.$$

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17 CARTESIAN EQUATION OF THE NORMAL:
The normal at $P(x_1, y_1)$ is $x_1 y + 2ax = x_1 y_1 + 2ax_1$.

Algebraic Approaches to the Tangents: Calculus is not necessary for parabolas, and the following worked example shows how to find tangents using the discrimimant.

WORKED EXERCISE: Write down the general form of a line with x-intercept 2, and hence use the discriminant to find the tangents to $x^2 = -6y$ with x-intercept 2.

SOLUTION: A line with gradient m and x-intercept 2 has equation $y = m(x-2)$.
Substituting into $x^2 = -6y$, $\quad x^2 = -6m(x-2)$

$$x^2 + 6mx - 12m = 0$$

$$\Delta = 36m^2 + 48m$$

$$= 12m(3m+4).$$

So $\Delta = 0$ when $m = 0$ or $m = -\frac{4}{3}$, and the required tangents are

$$y = 0 \text{ and } y = -\tfrac{4}{3}(x-2).$$

Exercise 9G

1. Use the derivative to find the equations of the following tangents:
(a) at the point $(2,1)$ on $x^2 = 4y$
(b) at the point $(3,3)$ on $x^2 = 3y$
(c) at the point $(-1,8)$ on $y = x^2 - 2x + 5$
(d) at the point $(2,1)$ on $y = 2x^2 - 4x + 1$
Then use the formula $xx_1 = 2a(y + y_1)$ to obtain the tangents in parts (a) and (b).

2. Use the derivative to find the equations of the following normals:
(a) at the point $(1,1)$ on $x^2 = y$
(b) at the point $(-6,9)$ on $x^2 = 4y$
(c) at the point $(-3,-7)$ on $y = x^2 + 3x - 7$
(d) at the point $(0,3)$ on $y = (2x+1)(x+3)$
Then use the formula $x_1 y + 2ax = x_1(2a + y_1)$ to obtain the normals in parts (a) and (b).

3. (a) Show that $y = 3x - 9$ is a tangent to the parabola $x^2 = 4y$ by solving the two equations simultaneously and showing that there is exactly one solution. What is the point of contact?
(b) Use a similar method to show that $8x + 4y - 27 = 0$ is a tangent to the parabola $y = x^2 - 3x + 7$, and find the point of contact.

4. (a) Find the equations of the normals to $x^2 = 4y$ at the points where $x = 2$ and $x = 4$.
(b) Find the point of intersection of the normals.

5. (a) Show that the endpoints of the latus rectum of the parabola $x^2 = 4ay$ are $A(-2a, a)$ and $B(2a, a)$.
(b) Find the equations of the tangents and normals to $x^2 = 4ay$ at these endpoints.
(c) Show that these tangents and normals form a square, and find its area.

DEVELOPMENT

6. (a) Find the equation of the parabola which is symmetrical about the y-axis and passes through the points $(1,1)$ and $(-2,2)$. [HINT: It will have the form $y = ax^2 + c$.]
(b) Find the tangent and the normal at the point $(1,1)$.

7. (a) Find where the line $y = 3x + 4$ intersects the parabola $2y = 5x^2$.
(b) Find the equations of the tangents to the parabola at the points of intersection.
(c) Find the point of intersection of the tangents.

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8. (a) Using the formula $xx_0 = 2a(y + y_0)$ for the equation of the tangent to $x^2 = 4ay$ at the point (x_0, y_0) on the curve, show that the tangents at $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the curve intersect at $M\left(\dfrac{2a(y_2 - y_1)}{x_2 - x_1}, \dfrac{x_1y_2 - x_2y_1}{x_2 - x_1}\right)$.

(b) Use the identities ${x_1}^2 = 4ay_1$ and ${x_2}^2 = 4ay_2$ to show that M is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{x_1x_2}{4a}\right)$.

(c) Check that the answer to the previous question agrees with this result.

(d) By substituting $x_1 = 2ap$ and $x_2 = 2aq$, deduce that the tangents to $x^2 = 4ay$ at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ meet at $\big(a(p + q), apq\big)$.

NOTE: Although approaches using calculus are usually more straightforward, tangents to parabolas can be found using purely algebraic methods based on the discriminant. The remaining questions in the exercise use these methods.

9. (a) Substitute the line $y = mx - 2$ into the parabola $x^2 = 2y$, and show that the resulting quadratic in x has discriminant $\Delta = 4m^2 - 16$.

(b) Hence find the two tangents to $x^2 = 2y$ with y-intercept -2.

(c) Use a similar method to find the two tangents to $x^2 = 8y$ with y-intercept -2, and show that they are perpendicular.

10. (a) Find the value of b for which $y = -2x + b$ will be a tangent to $x^2 = 6y$.

(b) Hence write down the tangent to $x^2 = 6y$ with gradient -2.

(c) Using a similar method, find the equations of the tangents:

(i) to $x^2 = 4y$ parallel to $y = 1 - 2x$, (ii) to $x^2 = 9y$ perpendicular to $y = 1 - 2x$.

(d) Repeat part (c) using the derivative to find the x-coordinate of the point of contact.

11. If $y = 1 - 2x$ is a tangent to $x^2 = 4ay$, find a and the point of contact.

12. (a) Use the discriminant to show that $y = mx + b$ is a tangent to the parabola $\mathcal{P}$: $x^2 = 4ay$ when $16a(am^2 + b) = 0$. Hence show that $y = mx - am^2$ is tangent to $\mathcal{P}$ for all m.

(b) Hence write down the tangent to $x^2 = 12y$ parallel to $y = 7x$.

13. (a) Use the discriminant to show that $mx - y + m^2 = 0$ touches the parabola $x^2 = -4y$, for all values of m.

(b) Hence find the equations of the tangents to $x^2 = -4y$ through the point $A(1, 2)$

14. Let ℓ: $y + 2 = m(x - 6)$ be a line with gradient m through $A(6, -2)$.

(a) Show that ℓ is a tangent to the parabola $\mathcal{P}$: $x^2 = 8y$ when $m^2 - 3m - 1 = 0$.

(b) Without solving this quadratic in m, show that there are two tangents from A to the parabola, and that they are perpendicular.

15. (a) Show that ℓ: $ax + by = 1$ is a tangent to $\mathcal{P}$: $x^2 = 12y$ when $3a^2 + b = 0$.

(b) Hence find the tangents to $\mathcal{P}$ with y-intercept -27.

(c) Show that if ℓ passes through $U(4, 1)$, then $4a + b = 1$. Hence find the tangents to $\mathcal{P}$ through U.

16. [Using the discriminant to derive the general equation of the tangent] Suppose that $P(2ap, ap^2)$ is any point on the parabola $\mathcal{P}$: $x^2 = 4ay$. Let ℓ: $y - ap^2 = m(x - 2ap)$ be a line with any gradient m through P.

(a) Show that solving the line ℓ and the parabola $\mathcal{P}$: $x^2 = 4ay$ simultaneously yields the quadratic equation $x^2 - 4amx + (8a^2mp - 4a^2p^2) = 0$.

(b) Show that the discriminant of this quadratic is $\Delta = 16a^2(m - p)^2$.

(c) Hence show that the line and the parabola touch when $m = p$, and that the equation of the tangent at P is $y = px - ap^2$.

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17. [An alternative algebraic approach] Using the pronumerals of the previous question:

(a) Show that substituting the parabola $x = 2at$, $y = at^2$ into the line ℓ yields the quadratic equation $t^2 - 2mt + (2mp - p^2) = 0$ in t.

(b) Show that the discriminant of this quadratic is $\Delta = 4(m-p)^2$, and hence that the line and the parabola touch when $m = p$.

18. [An algebraic approach without parameters] Suppose that $P(x_0, y_0)$ is any point on the parabola $\mathcal{P}$: $x^2 = 4ay$. Let ℓ: $y - y_0 = m(x - x_0)$ be a line with gradient m through P.

(a) Show that solving the line ℓ and the parabola $\mathcal{P}$: $x^2 = 4ay$ simultaneously, and substituting $y_0 = \dfrac{{x_0}^2}{4a}$, yields the equation $x^2 - 4amx + (4amx_0 - {x_0}^2) = 0$.

(b) Show that the discriminant of this quadratic is $\Delta = 4(x_0 - 2am)^2$, and hence show that the equation of the tangent to the parabola at P is $xx_0 = 2a(y + y_0)$.

EXTENSION

19. Show that the tangent at the point $P(x_0, y_0)$ on the general degree 2 curve

$$ax^2 + by^2 + 2cxy + 2dx + 2ey + f = 0$$

is $axx_0 + byy_0 + c(x_0y + xy_0) + d(x + x_0) + e(y + y_0) + f = 0$.

9 H The Chord of Contact

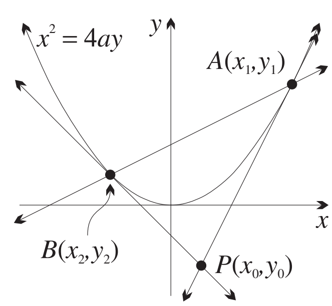
Establishing the equation of the chord of contact is the principal reason why the Cartesian equation of the tangent was introduced in the last section. The resulting relationships between tangents and chords go to the heart of the study of second-degree curves.

The Chord of Contact: Suppose that $P(x_0, y_0)$ is a point that lies outside the parabola $x^2 = 4ay$. A glance at the graph below will make it clear that there are two tangents to the parabola from P. These two tangents touch the curve at two points of contact, which we call here A and B. The chord AB joining these two points of contact is called the *chord of contact from P*.

It is a remarkable fact that this chord of contact has equation $xx_0 = 2a(y + y_0)$, exactly the same equation as the tangent, except that here the point P does not lie on the curve.

18 THE CHORD OF CONTACT: The chord of contact from $P(x_0, y_0)$ is
$$xx_0 = 2a(y + y_0).$$

PROOF: The proof is very elegant indeed, and involves no calculation whatsoever.
Let the points of contact be $A(x_1, y_1)$ and $B(x_2, y_2)$.
Then the tangent at A is $xx_1 = 2a(y + y_1)$,
and the tangent at B is $xx_2 = 2a(y + y_2)$.
Since $P(x_0, y_0)$ lies on the tangent at A,
$$x_0x_1 = 2a(y_0 + y_1),$$
and since P lies on the tangent at B,
$$x_0x_2 = 2a(y_0 + y_2).$$



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But the first identity shows that $A(x_1, y_1)$ lies on $xx_0 = 2a(y + y_0)$,
and the second identity shows that $B(x_2, y_2)$ lies on $xx_0 = 2a(y + y_0)$.
Since both A and B lie on $xx_0 = 2a(y + y_0)$, this equation must be the line AB.

Using the Chord of Contact to Find the Points of Contact: Given a point $P(x_1, y_1)$ outside a parabola, the two points of contact of the tangents from P can be found by finding the chord of contact from P, and then solving simultaneously the parabola and the chord of contact from P.

WORKED EXERCISE:

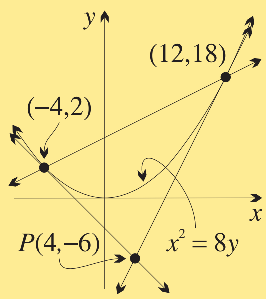
(a) Given the parabola $x^2 = 8y$, find the chord of contact from $P(4, -6)$.

(b) Hence find the two points of contact of the tangents from $P(4, -6)$.

SOLUTION:

(a) Here $4a = 8$, so $a = 2$,
so the chord of contact is $xx_0 = 4(y + y_0)$

$$4x = 4(y - 6)$$
$$y = x + 6.$$



(b) Solving simultaneously with the parabola $x^2 = 8y$,

$$x^2 = 8(x + 6)$$
$$x^2 - 8x - 48 = 0$$
$$(x - 12)(x + 4) = 0$$
$$x = 12 \text{ or } x = -4.$$

So the points of contact are $(12, 18)$ and $(-4, 2)$.

Exercise 9H

1. Find the equation of the chord of contact of $x^2 = 4y$ from each point:
(a) $(0, -2)$ (b) $(3, 0)$ (c) $(-2, -1)$ (d) $(4, -6)$

2. Find the equation of the chord of contact of $x = 6t$, $y = 3t^2$ from each point:
(a) $(0, -3)$ (b) $(-2, 0)$ (c) $(6, 1)$ (d) $(-5, -4)$

3. The point $P(2, 0)$ lies outside the parabola $x^2 = 8y$.
(a) Find the equation of the chord of contact from P.
(b) Find the points of intersection of the chord of contact and the parabola.
(c) Find the equations of the two tangents.

4. Each point $A(1, -2)$, $B(3, -2)$ and $C(-4, -2)$ lies on the directrix of the parabola $x^2 = 8y$.
(a) Write down the coordinates of the focus of the parabola.
(b) Find the equations of the chords of contact from A, B and C, and show that each chord is a focal chord.

5. (a) Write down the equation of the chord of contact of the parabola $y^2 = 4ax$ from the external point (x_0, y_0).
(b) Show by substitution that the chord of contact from the point $(-5, 2)$ to the parabola $y^2 = 20x$ is a focal chord. Why is it so?

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6. Tangents are drawn from the point $T(2,-1)$ to the parabola $x^2 = 4y$. P and Q are the points of contact of the tangents.
(a) Find the equation of the chord PQ.
(b) Show that the x-coordinates of P and Q are the roots of the quadratic $x^2 - 4x - 4 = 0$.
(c) Find the sum of the roots of the equation in part (b).
(d) Hence find the midpoint M of the chord PQ, and show that TM is parallel to the axis of the parabola.

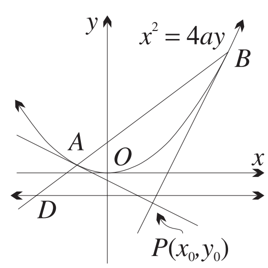
DEVELOPMENT

7. $P(x_1, -a)$ is any point on the directrix of the parabola $x^2 = 4ay$.
(a) Show that the chord of contact from P has equation $x_1 x = 2a(y - a)$.
(b) Hence show that the chord of contact passes through the focus of the parabola.

8. (a) Find the equation of the chord of contact of the parabola $x^2 = 8y$ from:
(i) $P(2,0)$ (ii) $Q(1,-1)$
(b) If the chords in part (a) intersect at R, show that the line PQ is a tangent to the parabola and that its point of contact is R.

9. (a) Write down the equation of the chord of contact of the parabola $x^2 = 4ay$ from the point $P(x_0, y_0)$.
(b) Suppose that the points of contact of the tangents are A and B. Find a quadratic equation whose roots are the x-coordinates of A and B.
(c) Hence find the coordinates of M, the midpoint of the chord AB.
(d) Show that PM is parallel to the axis of the parabola.
(e) Show that the midpoint N of PM lies on the parabola.

10. AB is the chord of contact of the parabola $x^2 = 4ay$ from the point $P(x_0, y_0)$. The line AB meets the directrix of the parabola at D.
(a) Write down the equation of AB.
(b) Show that D has coordinates $\left(\dfrac{2a(y_0 - a)}{x_0}, -a\right)$.
(c) Prove that PD subtends a right angle at the focus.



11. (a) Write down the equation of the chord of contact of the parabola $x^2 = 4ay$ from the point $P(x_0, y_0)$, then write it in gradient–intercept form $y = mx + b$.
(b) Let this chord meet the axis of the parabola at T, and let the line through P parallel to the axis meet the parabola at N. Use part (a) to show that:
(i) the points P and T are equidistant from the tangent at the origin,
(ii) the chord is parallel to the tangent to the parabola at N.

12. Tangents are drawn to the parabola $y = x^2$ from the point $T(1,-1)$. These tangents touch the parabola at P and Q.
(a) Obtain a quadratic equation whose roots are the x-coordinates of P and Q, and write down the sum and the product of these roots.
(b) Find a quadratic equation whose roots are the y-coordinates of P and Q, and write down the sum and the product of these roots.
(c) Prove the identity $(p - q)^2 = (p + q)^2 - 4pq$.
(d) Use the distance formula and this identity to find the length of the chord PQ.

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13. Repeat the previous question for the parabola $x^2 = 2y$ and $T(2, -3)$.

14. [An alternative derivation of the equation of the chord of contact] Let $T(2at, at^2)$ be any point on the parabola $x^2 = 4ay$.

(a) Show that the tangent at T has equation $y = tx - at^2$.

(b) If this tangent passes through the point $P(x_0, y_0)$, show that $at^2 - x_0 t + y_0 = 0$.

(c) What is the condition for this quadratic equation in t to have two real roots? Interpret this result geometrically.

(d) Suppose that t_1 and t_2 are the roots of the quadratic equation, and let T_1 and T_2 be the points on the parabola corresponding to $t = t_1$ and $t = t_2$ respectively.

(i) Show that the chord T_1T_2 has equation $(t_1 + t_2)x = 2y + 2at_1t_2$.

(ii) Show that $t_1 + t_2 = \dfrac{x_0}{a}$ and $t_1t_2 = \dfrac{y_0}{a}$.

(iii) Hence show that the chord T_1T_2 has equation $x_0x = 2a(y + y_0)$.

15. $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie outside the parabola $x^2 = 4ay$.

(a) Write down the equation of the chord of contact from P_1.

(b) If the line containing the chord of contact from P_1 passes through P_2, show that the line containing the chord of contact from P_2 passes through P_1.

16. Find the condition that must be satisfied if the chord of contact of the parabola $x^2 = 4ay$ from the point (x_0, y_0) is parallel to the line $y = mx + b$.

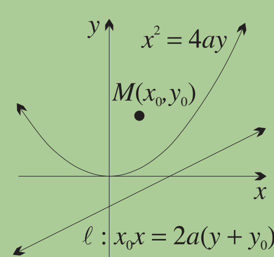
EXTENSION

17. The parabola $\mathcal{P}$ has equation $x^2 = 4ay$. The point $M(x_0, y_0)$ lies inside $\mathcal{P}$, so that ${x_0}^2 < 4ay_0$. The line ℓ has equation $x_0x = 2a(y + y_0)$.

(a) Show that $x_0x_1 \le \dfrac{{x_0}^2 + {x_1}^2}{2}$, for all real x_0.

(b) Prove that the line ℓ lies entirely outside $\mathcal{P}$. That is, show that if $P(x_1, y_1)$ is any point on ℓ, then ${x_1}^2 > 4ay_1$. (Use the result in part (a).)

(c) The chord of contact from any point $Q(x_2, y_2)$ outside $\mathcal{P}$ has equation $x_2x = 2a(y + y_2)$. Prove that M lies on the chord of contact from any point on ℓ.



18. (a) Use implicit differentiation to show that the equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $P(x_1, y_1)$ on the circle is $xx_1 + yy_1 = a^2$.

(b) Use the methods of this section to prove that if $P(x_0, y_0)$ is a point outside the circle, then $xx_0 + yy_0 = a^2$ is the equation of the chord of contact from P.

(c) Hence prove that the product of the distances from the centre O to the point P and to the chord of the contact is the square of the radius.

(d) Find the equation of the chord of contact of the circle $x^2 + y^2 = 25$ from the external point $P(4, 5)$. Then solve the circle and the chord simultaneously to find the points of contact of the tangents from P.

19. (a) Using similar methods, find the equation of the chord of contact to the hyperbola $xy = c^2$ from a point $P(x_0, y_0)$.

(b) Show that the product of the distances from O to the point P and to the chord of contact is the constant $2c^2$.

(c) Find the equation of the chord of contact of $xy = 25$ from the point $P(2, 8)$. Then solve the curve and the chord simultaneously to find the points of contact of the tangents from P.

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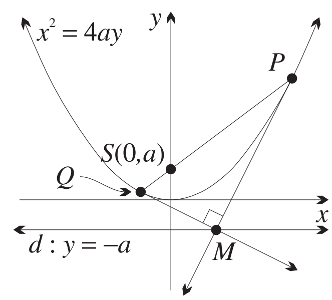
9 I Geometrical Theorems about the Parabola

This section is concerned with establishing purely geometric properties of the parabola, that is, properties of the parabola that do not depend on the particular way in which the parabola has been tied to the coordinate system of the plane. The machinery of coordinate geometry becomes here only a convenient set of tools for proving the theorems, but the whole process illustrates well how coordinate geometry has been able to create a unity between geometry and algebra.

Two Geometric Characterisations of Focal Chords: We have already shown that a chord PQ is a focal chord if and only if $pq = -1$. But p and q are the gradients of the tangents at P and Q, so the tangents at P and Q are perpendicular if and only if $pq = -1$, hence:

19 FOCAL CHORDS AND PERPENDICULAR TANGENTS: A chord that joins two points on a parabola is a focal chord if and only if the tangents at the endpoints of the chord are perpendicular.

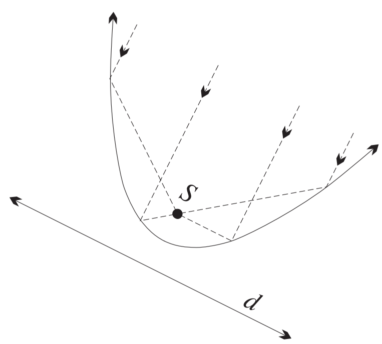
Secondly, the directrix is the line $y = -a$. Now we have seen that the tangents at P and Q meet at the point $M\big(a(p+q),\ apq\big)$, so the condition $pq = -1$ is the same as saying that the intersection M lies on the directrix. Putting all this together:



20 FOCAL CHORDS AND INTERSECTION OF TANGENTS: A chord joining two points on a parabola is a focal chord if and only if the tangents at the endpoints of the chord meet on the directrix.

NOTE: It should be stressed here that these two theorems are purely geometric. Although five pronumerals p, q, a, x and y were used in the proof, they have no place in the final statement of either theorem.

The Reflection Property of the Parabola: Parabolic bowls that are silvered on the inside have a most useful function in focusing light. When light travelling parallel to the axis falls on the bowl, the mirrored surface focuses it at the focus — hence the name focus for that point. Conversely, if a source of light is placed at the focus of the parabola, then it will be reflected from the bowl in a direction parallel to the axis.



Proving this requires the fairly obvious fact from physics that light is reflected from a surface so that the angle between the incident ray and the tangent at the point equals the angle between the reflected ray and the tangent (in physics one usually measures the angles with the normal, but the angle with the tangent is easier to handle in our case). Writing all this geometrically, the necessary theorem is as follows:

21 THE REFLECTION PROPERTY: The interval joining a point on a parabola to the focus, and the line through the point parallel to the axis, are equally inclined to the tangent at the point.

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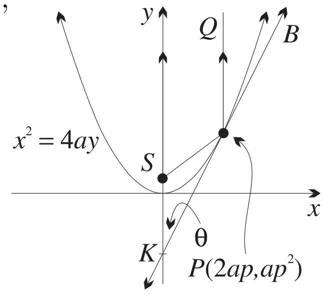
PROOF:

A. Label the diagram as shown, with the tangent at P meeting axis at K,
and let $\theta = \angle SKP$.
Then $\angle QPB = \theta$ (corresponding angles, $KS \parallel PQ$).
It will suffice to prove that the intervals SK and SP are equal,
because then $\angle SPK = \theta$ (base angles of isosceles $\triangle SPK$),
and so $\angle SPK = \angle SKP$, as required.



B. By the distance formula, $SP^2 = (2ap - 0)^2 + (ap^2 - a)^2$
$= a^2\left(4p^2 + (p^2 - 1)^2\right)$
$= a^2(p^2 + 1)^2$.
Also, the tangent $y = px - ap^2$ has y-intercept $-ap^2$,
so $K = (0, -ap^2)$ and $SK^2 = (a + ap^2)^2$
$= a^2(1 + p^2)^2$.
Hence $SP = SK$, and the result is proven.

Exercise 9I

NOTE: This exercise and the next are the culmination of the work on the parabola and its properties. The large number of questions is intended to be sufficient for later revision.

1. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two variable points on the parabola $x^2 = 4ay$. M is the midpoint of the chord PQ, and T is the point of intersection of the tangents at P and Q.
(a) Show that the tangent at P has equation $y = px - ap^2$ and write down the equation of the tangent at Q.
(b) Show that M has coordinates $\left(a(p+q), \frac{1}{2}a(p^2+q^2)\right)$.
(c) Show that T has coordinates $(a(p+q), apq)$.
(d) Show that MT is parallel to the axis of the parabola.
(e) Find the coordinates of the midpoint N of MT, and show that it lies on the parabola.

2. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are variable points on the parabola $x^2 = 4ay$, and PQ is a focal chord. The tangents at P and Q meet at T.
(a) Show that the chord PQ has equation $y = \frac{1}{2}(p+q)x - apq$. (b) Show that $pq = -1$.
(c) Show that the tangent at P has gradient p, and state the gradient of the tangent at Q.
(d) Show that T is the point $(a(p+q), apq)$. (e) Show that the tangents at P and Q are perpendicular and that they meet on the directrix.

3. $P(2at, at^2)$, where $t \neq 0$, is a variable point on the parabola $x^2 = 4ay$. The normal at P meets the axis of the parabola at N, and PB is the perpendicular from P to the axis of the parabola. The interval BN is called the *subnormal* corresponding to P.
(a) Show that the normal at P has equation $x + ty = 2at + at^3$.
(b) Write down the coordinates of B and N. (c) Hence prove that the length of the subnormal is constant, that is, independent of where P is on the parabola.

4. $P(at^2, 2at)$ is an arbitrary point on the parabola $y^2 = 4ax$ with focus $S(a, 0)$.
(a) Show that the tangent at P has equation $x = ty - at^2$.
(b) Show that the tangent at P meets the directrix at the point $Q\left(-a, a(t - \frac{1}{t})\right)$.
(c) Hence prove that $\angle PSQ = 90^\circ$.

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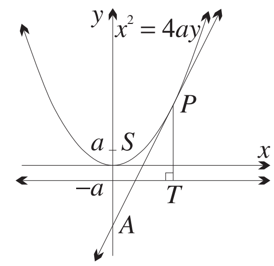
5. $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$. S is the focus, T is the foot of the perpendicular from P to the directrix, and A is the point where the tangent at P meets the y-axis.



(a) Write down the coordinates of T.
(b) Show that A is the point $(0, -at^2)$.
(c) Show that PA and ST bisect each other, by finding their midpoints.
(d) Show that PA and ST are perpendicular to each other.
(e) What type of quadrilateral is $SPTA$?

6. $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$. The normal at P meets the x-axis at A and the y-axis at B.
(a) Find the coordinates of A and B.
(b) If $C(c, d)$ is the fourth vertex of the rectangle $BOAC$, where O is the origin, show that $c = td$.

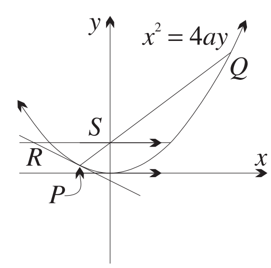
DEVELOPMENT

7. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ vary on the parabola $x^2 = 4ay$. The tangents at P and Q meet at right angles at T.
(a) Show that $pq = -1$. What does this result tell us about the chord PQ?
(b) Show that the tangent at P has equation $y = px - ap^2$, and write down the equation of the tangent at Q.
(c) Show that T has coordinates $(a(p + q), apq)$.
(d) Find the gradient of the chord PQ, and hence show that the line through T perpendicular to the chord PQ has equation $y = -\dfrac{2}{p+q}\,x + a$.
(e) Show that the line in part (d) meets the chord PQ at the focus of the parabola.

8. $P(2ap, ap^2)$, $Q(2aq, aq^2)$ and $T(2at, at^2)$ are variable points on the parabola $x^2 = 4ay$.
(a) Show that the chord PQ has equation $y - \frac{1}{2}(p + q)x + apq = 0$.
(b) Find the equation of the tangent at T.
(c) The tangent at T cuts the axis of the parabola at R. Find the coordinates of R. If the chord PQ, when extended, passes through R, show that p, t and q form a geometric progression.

9. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $\mathcal{P}$ whose equation is $x^2 = 4ay$.
(a) Find the point of intersection A of the tangents to $\mathcal{P}$ at P and Q. (You may use the fact that the tangent to $\mathcal{P}$ at any point $T(2at, at^2)$ on $\mathcal{P}$ has equation $y = tx - at^2$.)
(b) Suppose further that A lies on the line containing the latus rectum of $\mathcal{P}$.
  (i) Show that $pq = 1$.
  (ii) Show that the chord PQ intersects the axis of symmetry of $\mathcal{P}$ on the directrix.

10. P and Q are the points $t = p$ and $t = q$ respectively on the parabola $x = 2at$, $y = at^2$ with focus S. PQ is a focal chord and the tangent at P meets the latus rectum produced at R.



(a) Show that $pq = -1$. (b) Show that $SP = a(p^2 + 1)$.
(c) Show that R has coordinates $\left(\dfrac{a}{p}(p^2 + 1), a\right)$.
(d) Hence show that $SR^2 = SP \times SQ$.

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11. [In this question we prove the reflection property of the parabola.] $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$, and $t \neq 0$. S is the focus and T is the point where the tangent to the parabola at P meets the axis of the parabola.

(a) Show that the tangent at P has equation $y = tx - at^2$.

(b) Show that $SP = ST$.

(c) Hence show that $\angle SPT$ is equal to the acute angle between the tangent and the line through P parallel to the axis of the parabola.

12. [An alternative proof of the reflection property] P is the variable point (x_1, y_1) on the parabola $x^2 = 4ay$, and S is the focus. The tangent at P meets the tangent at the vertex of the parabola at Q and it meets the axis of the parabola at R.

(a) Explain why ${x_1}^2 = 4ay_1$. (b) Show that the tangent at P is $x_1x = 2a(y + y_1)$.

(c) Find the coordinates of Q and R.

(d) Show that $SQ \perp PQ$, and that the tangent at the vertex bisects PR.

(e) Hence, using congruent triangles, show that the tangent at P is equally inclined to the axis of the parabola and the focal chord through P.

13. P is a point on a parabola, and ℓ is the axis of symmetry of the parabola. The tangent and normal to the parabola at P meet ℓ at T and N respectively. Prove that P, T and N all lie on a circle whose centre is at the focus of the parabola.

14. The normal at the point $P(2at, at^2)$ on the parabola $x^2 = 4ay$ intersects the y-axis at Q. A is the point $(0, -a)$ and S is the focus. The midpoint of QS is R.

(a) Show that R has coordinates $\left(0, \frac{1}{2}a(t^2 + 3)\right)$. (b) Show that $AR^2 - RP^2 = 4a^2$.

15. $P(2ap, ap^2)$ is any point on the parabola $x^2 = 4ay$ other than its vertex. The normal at P meets the parabola again at Q.

(a) Show that the normal at P cannot pass through the focus of the parabola.

(b) Show that the x-coordinate of Q is one of the roots of the quadratic equation $px^2 + 4ax - 4a^2p(2 + p^2) = 0$. Then find the coordinates of Q.

16. $P(2p, p^2)$ is a variable point on the parabola $x^2 = 4y$, whose focus is S. The normal at P meets the y-axis at N and M is the midpoint of PN.

(a) Find the coordinates of M, and show that SM is parallel to the tangent at P.

(b) Suppose that $\triangle SNP$ is equilateral. Find the coordinates of P.

17. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are variable points on the parabola $x^2 = 4ay$. R is the intersection of the tangent at P and the line through Q parallel to the axis of the parabola, while U is the intersection of the tangent at Q and the line through P parallel to the axis.

(a) Show that $PQRU$ is a parallelogram.

(b) If $p > q$, show that parallelogram $PQRU$ has area $2a^2(p - q)^3$ units^2.

18. The points $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x = 2t$, $y = t^2$. S is the focus.

(a) Show that $PS = p^2 + 1$, by using the fact that any point on a parabola is equidistant from the focus and the directrix.

(b) Hence find an expression for $PS + QS$ in terms of p and q.

(c) If PQ is a focal chord, show that $PQ = \left(p + \dfrac{1}{p}\right)^2$.

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19. P is a variable point on the parabola $x^2 = 4y$. The normal at P meets the parabola again at Q. The tangents at P and Q meet at T. S is the focus and $QS = 2PS$.

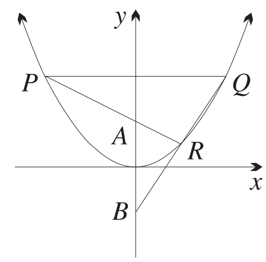
(a) Prove that $\angle PSQ = 90°$.

(b) Prove that $PQ = PT$.

20. The points $P(p, \frac{1}{2}p^2)$ and $Q(q, \frac{1}{2}q^2)$ vary on the parabola $2y = x^2$ with focus S. The line PQ passes through the point where the directrix meets the axis of symmetry.

(a) Show that $pq = 1$.

(b) Show that $PS \times QS = \frac{1}{2}(PS + QS)$.



21. The diagram shows a parabola. PQ is any chord parallel to the directrix. R is a third point on the parabola, and the lines RP and RQ cut the axis of the parabola at A and B respectively. Show that the interval AB is bisected by the vertex of the parabola.

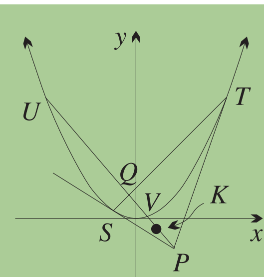
22. $P(2p, p^2)$ and $Q(2q, q^2)$, where $p \neq q$, are variable points on the parabola $x^2 = 4y$. You may assume that the chord PQ has equation $(p+q)x - 2y - 2pq = 0$, and that the tangents at P and Q meet at the point $T(p + q, pq)$.

(a) Show that, for each non-zero value of p, there are two values of q for which T lies on the parabola $x^2 = -4y$, and find these values in terms of p.

(b) For each value of q, show that PQ produced is a tangent to the parabola $x^2 = -4y$.

EXTENSION

23. In the diagram the point $P(x_0, y_0)$ lies outside the parabola $x^2 = 4ay$ (which means that ${x_0}^2 > 4ay_0$.) The two tangents to the parabola from P touch the parabola at S and T.

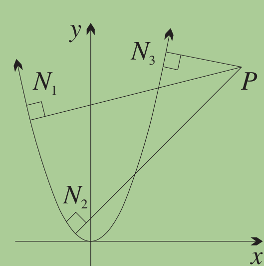


(a) Suppose that $Q(x_1, y_1)$ is any point between S and T on the chord of contact ST. Suppose that k is any real number, and let K be the point on PQ dividing the interval PQ in the ratio $k : 1$. Write down the coordinates of K in terms of x_0, x_1, y_0, y_1 and k.

(b) Find the condition that K should lie on the parabola, and rearrange this condition as a quadratic equation in k.

(c) Hence show that the two points U and V where the line PQ meets the parabola divide the interval PQ internally and externally in the same ratio.

24. The diagram shows the parabola $y = x^2$. P is a point that lies on three distinct normals (PN_1, PN_2 and PN_3) to the parabola.



(a) Show that the equation of the normal to $y = x^2$ at the variable point (t, t^2) is $t^3 + \left(\dfrac{1 - 2y}{2}\right) t - \dfrac{x}{2} = 0$.

(b) If the function $f(t) = t^3 + ct + d$ has three distinct real zeroes, prove that $27d^2 + 4c^3 < 0$. [HINT: The graph of $f(t)$ must have two points where the tangent is horizontal, one above the t-axis and one below.]

(c) Suppose that the normals at three distinct points N_1, N_2 and N_3 on the parabola $y = x^2$ all pass through $P(x_0, y_0)$. Use part (b) to show that $y_0 > 3\left(\dfrac{x_0}{4}\right)^{\frac{2}{3}} + \dfrac{1}{2}$.

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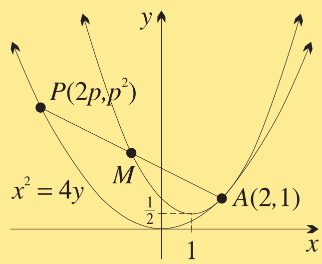
9 J Locus Problems

In many situations a variable point P on a parabola will determine another point M, so that M moves as the point moves. The problem then is to find the equation of the path or locus of M, and if possible to describe that locus in geometrical terms.

One-parameter Locus Problems: The usual method is to give the point P its parametric coordinates, and then find the coordinates of M in terms of the parameter p. The formulae for the coordinates of M then form two simultaneous equations, and the parameter p can be eliminated from them.

22 LOCUS PROBLEMS: Write the coordinates of the moving point as two simultaneous equations, then eliminate the parameter.

WORKED EXERCISE: Let A be the endpoint $(2, 1)$ of the latus rectum of the parabola $x^2 = 4y$, and let $P(2p, p^2)$ be any point on the parabola. Find and describe the locus of the midpoint M of PA.



SOLUTION: The coordinates of M are

$$x = p + 1 \qquad (1)$$
$$y = \tfrac{1}{2}(p^2 + 1). \qquad (2)$$

From (1), $p = x - 1$,

and substituting into (2), $y = \tfrac{1}{2}(x^2 - 2x + 1 + 1)$

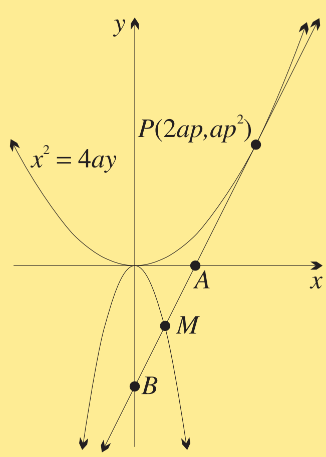
$$2y = x^2 - 2x + 2.$$

Completing the square, $2y - 1 = x^2 - 2x + 1$

$$(x - 1)^2 = 2(y - \tfrac{1}{2}),$$

so the locus is a parabola with vertex $(1, \tfrac{1}{2})$ and focal length $\tfrac{1}{2}$.

WORKED EXERCISE: The tangent at a point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ meets the x-axis at A and the y-axis at B. Find and describe the locus of the midpoint M of AB.



SOLUTION: We assume the tangent is $y = px - ap^2$,
so putting $x = 0$, the point B is $B(0, -ap^2)$,
and putting $y = 0$, the point A is $A(ap, 0)$,
so the coordinates of the midpoint M are

$$x = \tfrac{1}{2}ap \qquad (1)$$
$$y = -\tfrac{1}{2}ap^2. \qquad (2)$$

Squaring (1), $x^2 = \tfrac{1}{4}a^2p^2$

$$x^2 = -\tfrac{1}{2}ap^2 \times (-\tfrac{1}{2}a),$$

and using (2), $x^2 = -\tfrac{1}{2}ay$.

This is a parabola facing downwards
with vertex $(0, 0)$ and focal length $\tfrac{1}{8}a$,
so the focus is $(0, -\tfrac{1}{8}a)$ and the directrix is $y = \tfrac{1}{8}a$.

WORKED EXERCISE: Find the locus of the midpoint of the x-intercept and y-intercept of the normal at a variable point on the parabola $y^2 = 4ax$.

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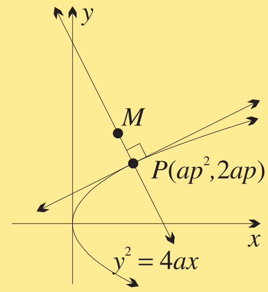
SOLUTION: We assume that the normal at $P(ap^2, 2ap)$ is $y + px = 2ap + ap^3$,
so putting $x = 0$, the y-intercept is $(0, 2ap + ap^3)$,
and putting $y = 0$, the x-intercept is $(2a + ap^2, 0)$,
so the coordinates of the midpoint M are



$$x = a + \tfrac{1}{2}ap^2 \qquad (1)$$
$$y = ap + \tfrac{1}{2}ap^3. \qquad (2)$$

From (1) and (2), $y = px$. (3)

From (1), $\tfrac{1}{2}ap^2 = x - a$

$$p^2 = \frac{2(x-a)}{a},$$

and substituting this into the square of (3),

$$y^2 = \frac{2x^2(x-a)}{a}$$
$$ay^2 - 2x^3 + 2x^2a = 0.$$

NOTE: This last locus is not a parabola because it involves a term in the cube of x. This is common with locus problems involving the normal, where the algebra of the elimination of the parameters is often more complicated.

Two-parameter Locus Problems: In a more difficult type of problem, the variable point depends on two points with parameters say p and q, but there is a relation between the two parameters. This case produces three simultaneous equations — expressing the coordinates of the variable point in terms of p and q gives two equations, and the relation between p and q is a third equation. From these three equations, both parameters p and q must be eliminated.

23 TWO-PARAMETER LOCUS PROBLEMS: Write the coordinates of the moving point and the relation between the two parameters as three simultaneous equations, then eliminate both parameters.

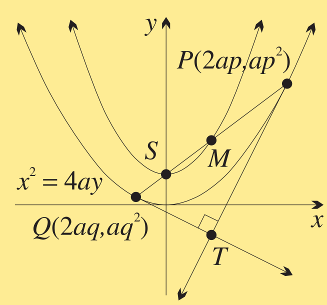
WORKED EXERCISE: Suppose that PQ is a focal chord of the parabola $x^2 = 4ay$.

(a) Find and describe the locus of the midpoint M of PQ.

(b) Find and describe also the locus of the intersection T of the tangents at P and Q, and show that MT is always parallel to the axis.

SOLUTION:

(a) Let the endpoints of the chord be $P(2ap, ap^2)$ and $Q(2aq, aq^2)$.
Then the coordinates of the midpoint M are



$$x = \tfrac{1}{2}(2ap + 2aq)$$
$$y = \tfrac{1}{2}(ap^2 + aq^2),$$

that is, $x = a(p+q)$ (1)

$$y = \tfrac{1}{2}a(p^2 + q^2). \qquad (2)$$

But the parameters p and q are related by

$$pq = -1. \qquad (3)$$

Squaring (1), $x^2 = a^2(p^2 + q^2 + 2pq)$,
and using (2) and (3), $x^2 = a(2y - 2a)$

$$x^2 = 2a(y - a),$$

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so the locus is a parabola with vertex $(0, a)$ and focal length $\frac{1}{2}a$.
Hence the focus is $(0, \frac{3}{2}a)$ and the directrix is $y = \frac{1}{2}a$.

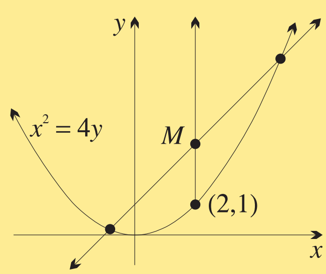
(b) We assume that $\big(a(p+q), apq\big)$ is the intersection of the tangents,
so T has coordinates $x = a(p+q)$ (4)
$y = apq$, (5)
and again, $pq = -1$. (6)
Substituting (6) into (5), $y = -a$ (notice that (4) was irrelevant),
so the locus of T is the line $y = -a$, which is the directrix of the parabola.
Since the x-coordinates of M and T are equal, MT is always vertical.

NOTE: We proved in Section 9F that the tangents at the endpoints of a focal chord intersect on the directrix. Part (b) of this locus question has simply proven the same result a different way.

Using Sum and Product of Roots: The previous work in Chapter Eight on sum and product of roots can be very useful in locus questions.

WORKED EXERCISE: Find and describe the locus of the midpoints of the chords cut off a parabola $x^2 = 4y$ by lines parallel to $y = x$. Make clear any restrictions on the locus.

SOLUTION: The family of chords parallel to $y = x$
has equation $y = x + b$, where b can vary.
Substituting the line $y = x + b$ into the parabola $x^2 = 4y$,
$$x^2 = 4x + 4b$$
$$x^2 - 4x - 4b = 0,$$
so average of roots $= 2$.
Hence the locus of the midpoints is the vertical line $x = 2$.
The tangent at the endpoint $(2, 1)$ of the latus rectum is the farthest right a line $y = x + b$ can be, yet still touch the curve,
so more carefully stated, the locus is $x = 2$, where $y \geq 1$.



Exercise 9J

1. Let $P(2at, at^2)$ be a variable point on the parabola $x^2 = 4ay$. Suppose that M is the midpoint of the interval OP, where O is the origin.
(a) Show that M has coordinates $(at, \frac{1}{2}at^2)$.
(b) Write down a pair of parametric equations representing the locus of M.
(c) Hence show that the locus of M has Cartesian equation $x^2 = 2ay$.
(d) Give a geometrical description of this locus.

2. The tangent at $P(2at, at^2)$ on the parabola $x^2 = 4ay$ meets the x-axis at T.
(a) Show that the tangent has equation $y = tx - at^2$.
(b) Find the coordinates of T.
(c) Find the coordinates of the midpoint M of PT.
(d) Show that, as t varies, the locus of M is the parabola $2x^2 = 9ay$.

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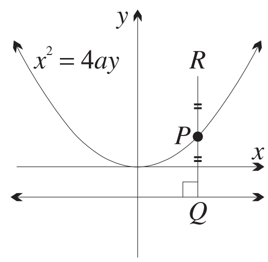
3. P is the variable point $(4t, 2t^2)$ on the parabola $x^2 = 8y$. The normal at P cuts the y-axis at A and R is the midpoint of AP.
(a) Show that the normal at P has equation $x + ty = 4t + 2t^3$.
(b) Show that R has coordinates $(2t, 2t^2 + 2)$.
(c) Show that the locus of R is a parabola, and show that the vertex of this parabola is the focus of the original parabola.

4. $P(2at, at^2)$ moves on the curve $x^2 = 4ay$. The tangent at P meets the x-axis at T and the normal at P meets the y-axis at N.
(a) Show that T and N are the points $(at, 0)$ and $(0, 2a + at^2)$ respectively.
(b) Find the coordinates of M, the midpoint of TN.
(c) Show that the locus of M is the parabola $x^2 = \frac{1}{2}a(y - a)$.

5. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are variable points on $x^2 = 4ay$. $S(0, a)$ is the focus, and M is the midpoint of the chord PQ. The chord PQ passes through S as p and q vary.
(a) Show that the chord PQ has equation $y = \frac{1}{2}(p + q)x - apq$.
(b) Use the fact that the chord PQ passes through S to show that $pq = -1$.
(c) Show that $x = a(p + q)$, $y = \frac{1}{2}a(p^2 + q^2)$ are parametric equations of the locus of M.
(d) Use the identity $(p + q)^2 = p^2 + q^2 + 2pq$ to show that the Cartesian equation of the locus of M is $x^2 = 2a(y - a)$.

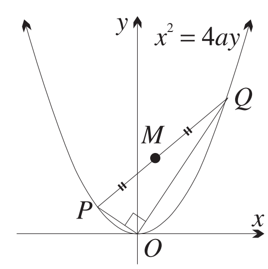
6. $P(2at, at^2)$ lies on the parabola $x^2 = 4ay$. Q is the foot of the perpendicular from P to the directrix of the parabola and the interval QP is extended to R so that $RP = PQ$.
(a) Write down the coordinates of Q.
(b) Show that R has coordinates $\big(2at, a(2t^2 + 1)\big)$.
(c) Hence find the Cartesian equation of the locus of R as t varies, then describe this locus geometrically.



7. $P(2ap, ap^2)$ is any point on the parabola $x^2 = 4ay$.
(a) Find the equation of the line ℓ through the focus parallel to the tangent at P.
(b) Show that the point T where ℓ meets the x-axis has coordinates $\left(-\dfrac{a}{p}, 0\right)$.
(c) Find the Cartesian equation of the locus of M, the midpoint of ST.
(d) Show that ℓ meets the normal at P to the parabola at the point $N\big(ap, a(p^2 + 1)\big)$.
(e) Find the Cartesian equation of the locus of N.

8. P is the variable point $(at^2, 2at)$ on the parabola $y^2 = 4ax$. The perpendiculars from P to the y- and x-axis meet them at A and B respectively. M is the midpoint of PB.
(a) Find the coordinates the midpoint N of MA.
(b) Show that as P varies on the parabola $y^2 = 4ax$, N moves on the parabola $2y^2 = 9ax$.

9. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on $x^2 = 4ay$, and the chord PQ subtends a right angle at the vertex O.
(a) Show that $pq = -4$.
(b) Show that the coordinates of the midpoint M of PQ are $\big(a(p + q), \frac{1}{2}a(p^2 + q^2)\big)$.
(c) Use the identity $p^2 + q^2 = (p + q)^2 - 2pq$ to show that the Cartesian equation of the locus of M is $x^2 = 2a(y - 4a)$.
(d) Give a geometrical description of this locus.



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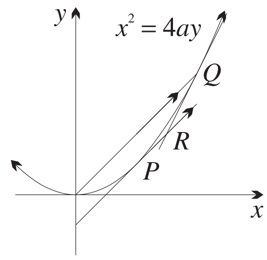
10. $P(2at, at^2)$ varies on the parabola $x^2 = 4ay$. $S(0, a)$ is the focus.
(a) Show that the tangent at P has equation $y = tx - at^2$.
(b) A line drawn from the focus perpendicular to the tangent at P meets it at T. Find the coordinates of T.
(c) What is the locus of T as P varies on the parabola?

DEVELOPMENT

11. P and Q are the points where $t = t_1$ and $t = t_2$ respectively on the parabola $x = 2at$, $y = at^2$. The chord PQ cuts the axis of the parabola at $(0, 3a)$.
(a) Show that $t_1 t_2 = -3$.
(b) Show that as t_1 and t_2 vary, the midpoint of the chord PQ moves on the parabola $x^2 = 2a(y - 3a)$.

12. The variable point $P(2ap, ap^2)$ lies on $x^2 = 4ay$, and the chord OQ is drawn parallel to the tangent at P. The tangents at P and Q meet at R.
(a) Derive the equation of the tangent at P.
(b) Write down the equation of the chord OQ.
(c) Show that the coordinates of Q are $(4ap, 4ap^2)$.
(d) Find the equation of the tangent at Q.
(e) Show that R has coordinates $(3ap, 2ap^2)$.
(f) Find the Cartesian equation of the locus of R as P varies on the parabola.



13. Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$, where $p > q$, move along the parabola $x^2 = 4ay$. At all times the x-coordinates of P and Q differ by $2a$.
(a) Find the midpoint M of the chord PQ, and the Cartesian equation of its locus.
(b) Give a geometrical description of this locus.

14. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ vary on the parabola $x^2 = 4ay$. The chord PQ subtends a right angle at the vertex. The tangents at P and Q meet at T, while the normals at P and Q meet at N.
(a) Show that $pq = -4$.
(b) Show that T has coordinates $(a(p + q), apq)$.
(c) Find the Cartesian equation of the locus of T.
(d) Show that N has coordinates $\big(-apq(p + q), a(p^2 + pq + q^2 + 2)\big)$.
(e) Find the Cartesian equation of the locus of N.

15. A parabola is defined by $x = 2at$, $y = at^2$. The points P and Q lie on the parabola and have parameters $t = p$ and $t = \dfrac{1}{p}$ respectively. The tangents at P and Q intersect at T.
(a) Find the equations of the tangents at P and Q.
(b) Prove that the locus of T is part of a line parallel to the directrix of the parabola.

16. $P(8p, 4p^2)$ and $Q(8q, 4q^2)$ are variable points on the parabola $x^2 = 16y$. The chord PQ produced passes through the fixed point $(4, 0)$. The tangents at P and Q meet at R.
(a) Show that the chord PQ has equation $y = \frac{1}{2}(p + q)x - 4pq$.
(b) Show that $p + q = 2pq$.
(c) Find the coordinates of R.
(d) Show that R moves on the line $x = 2y$.

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17. $A(2at_1, at_1{}^2)$ and $B(2at_2, at_2{}^2)$ are two variable points on the parabola $x^2 = 4ay$. The normals to the parabola at A and B meet at right angles at N.

(a) Show that $t_1 t_2 = -1$.

(b) Show that N has coordinates $\big(a(t_1 + t_2), a(t_1{}^2 + t_2{}^2 + 1)\big)$.

(c) Hence find the Cartesian equation of the locus of N.

18. $P(2p, p^2)$ and $Q(2q, q^2)$ are variable points on the parabola $x^2 = 4y$. PQ is a focal chord of gradient m, and the normals at P and Q intersect at N.

(a) Derive the equation of the normal at P.

(b) Show that N is the point $\big(-pq(p + q), p^2 + pq + q^2 + 2\big)$.

(c) Show that $p + q = 2m$ and that $pq = -1$.

(d) Write the coordinates of N in terms of m.

(e) Hence find the Cartesian equation of the locus of N as m varies.

19. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. The tangents at P and Q meet at R, and R lies on the parabola $x^2 = -4ay$.

(a) Show that R has coordinates $(a(p + q), apq)$.

(b) Show that $p^2 + q^2 + 6pq = 0$.

(c) As P and Q vary, show that the locus of the midpoint of the chord PQ is the parabola $3x^2 = 4ay$.

20. P is the point with parameter $t = p$ on the parabola $x = 6t$, $y = 3t^2$.

(a) Show that the tangent to the parabola at P has equation $y = px - 3p^2$.

(b) If Q is the point on the parabola where $t = 1 - p$, and P and Q are distinct, show that the tangents at P and Q meet at the point $T(3, 3p - 3p^2)$.

(c) Specify algebraically the locus of T.

(d) Comment on the points P, Q and T in the case where $p = \frac{1}{2}$.

21. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are variable points on the parabola $x^2 = 4ay$. It is given that the tangents at P and Q meet at $T\,(a(p + q), apq)$, and that the line PQ is a tangent to the parabola $x^2 = 2ay$.

(a) Show that the line PQ has equation $y = \frac{1}{2}(p + q)x - apq$.

(b) Show that $(p + q)^2 = 8pq$.

(c) Find the Cartesian equation of the locus of T.

22. $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$. The normal at P cuts the y-axis at Q, and R divides the interval PQ externally in the ratio $2 : 1$.

(a) Show that R has coordinates $(-2at, 4a + at^2)$.

(b) Find the Cartesian equation of the locus of R.

(c) Show that if the normal at P passes through the fixed point (h, k), then the parameter t satisfies $at^3 + (2a - k)t - h = 0$.

(d) What is the greatest number of normals to the parabola $x^2 = 4ay$ that can be drawn from any point in the number plane? Give a reason for your answer.

23. Tangents from the point $P(x_1, y_1)$ touch the parabola $x^2 = 8y$ at the points A and B.

(a) Show that the x-coordinates of A and B are the roots of the quadratic equation $x^2 - 2x_1 x + 8y_1 = 0$.

(b) Hence show that the midpoint M of the chord AB has coordinates $(x_1, \frac{1}{4}x_1{}^2 - y_1)$.

(c) Suppose that P varies on the line $x - y = 2$. Find the Cartesian equation of the locus of M.

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24. The parabola $x^2 = 4ay$ and the line $y = mx + k$ intersect at distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$.

(a) Show that the normal at P has equation $x + py = 2ap + ap^3$, and that the normals at P and Q intersect at the point $N\left(-apq(p+q), a(p^2+q^2+pq+2)\right)$.

(b) Show that: (i) $p + q = 2m$, (ii) $pq = -\frac{k}{a}$.

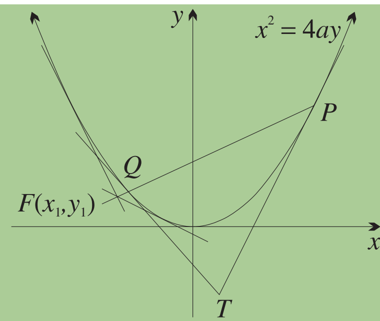
(c) Hence show that $p^2 + q^2 = \frac{2k}{a} + 4m^2$.

(d) Express the coordinates of N in terms of a, m and k.

(e) If the chord PQ has constant gradient m, show that the locus of N is a straight line, and show that this line is a normal to the parabola.

EXTENSION

25. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ vary on the parabola $x^2 = 4ay$ in such a way that the line PQ always passes through the fixed point $F(x_1, y_1)$, which lies outside the parabola. The tangents at P and Q meet at T.



(a) Show that $(p + q)x_1 = 2apq + 2y_1$.

(b) Hence show that the locus of T is part of a straight line. Show also that the other part of this straight line is the chord of contact of the tangents to the parabola from F.

26. $P_1(2at_1, at_1{}^2)$, $P_2(2at_2, at_2{}^2)$ and $P_3(2at_3, at_3{}^2)$ are variable points on $x^2 = 4ay$. Suppose that T is the point of intersection of the tangents to the parabola at P_2 and P_3.

(a) Show that T has coordinates $(a(t_2 + t_3), at_2t_3)$.

(b) Show that the line through T perpendicular to the tangent at P_1 meets the directrix at the point $D\,(a(t_1 + t_2 + t_3 + t_1t_2t_3), -a)$.

(c) Hence find the locus of the orthocentre of the triangle formed by the three tangents to the parabola drawn at P_1, P_2 and P_3. (The *orthocentre* of a triangle is the point of intersection of its three altitudes.)

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