THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 10

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Taylor's Theorem.
- ☐ Curves in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Outcomes

After completing this tutorial you should

- ☐ use Taylor's Theorem to approximate functions and integrals;
- ☐ use Taylor's Theorem to compute limits;
- ☐ sketch curves and find parametrisations in simple cases.

Summary of essential material

Taylor's Theorem: Let $f(x)$ be n times differentiable at $x = x_0$. Then $f(x) = T_n(x) + R_n(x)$ where

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is the n-th order Taylor polynomial of f centred at $x = x_0$. It is the *best polynomial approximation* of order n near x_0. If f has at least $n + 1$ derivatives in a neighbourhood of x_0, then the *remainder* can be written in the form

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \qquad \text{for some } c \text{ strictly between } x_0 \text{ and } x.$$

This is called the *Lagrange form* of the remainder. The remainder looks like the general term in $T_n(x)$, but $f^{(n+1)}$ is evaluated at an intermediate c value between x and x_0 and not at x_0. A special case is the Mean Value Theorem ($n = 0$).

To estimate $R_n(x)$ we maximise $|f^{(n+1)}(c)|$ over all c between x and x_0. A common method is to use the "worst case scenario" for an estimate:

- If $|f^{(n)}(c)|$ is monotone in c set $c = 0$ or $c = x_0$ depending on whether it is increasing or decreasing;
- If $|f^{(n)}(c)|$ is a fraction maximise the numinator and minimize the denominator, often by setting $c = 0$ or $c = x_0$.

Often this is achieved by setting $c = 0$ or $c = x_0$, or if it is a fraction to minimize the denominator and maximise the numerator. One can also use calculus to find the maximum, but that is not necessary most of the time.

Curves: A *curve* in $\mathbb{R}^2$ is a function $C\colon [a, b] \to \mathbb{R}^2$. We are often a little relaxed here, and either regard the curve as the the actual function, or as the image of $[a, b]$ under C in $\mathbb{R}^2$. A curve is given by its "component functions"

$$\big(x(t), y(t)\big) \quad \text{or} \quad x(t)\boldsymbol{i} + y(t)\boldsymbol{j} \quad \text{or} \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad t \in [a, b].$$

The points $\big(x(t), y(t)\big)$ trace out a curve as t increases from a to b. Similarly, a curve in $\mathbb{R}^3$ is a function $C\colon [a, b] \to \mathbb{R}^3$ with the same interpretation as above, but three components.

Standard Taylor Polynomials

Function	Taylor polynomial about $x = 0$
e^x	$T_n(x) = 1 + x + \frac{x}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
$\sin(x)$	$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\cos(x)$	$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$
$\frac{1}{1-x}$	$T_n(x) = 1 + x^2 + x^3 + \cdots + x^n$
$\ln(1+x)$	$T_n(x) = 1 - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^n}{n}$
$(1+x)^\alpha$	$T_n(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k$ where $\binom{\alpha}{0} := 1$ and $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-3)\cdots(\alpha-k+1)}{k!}$ if $k \geq 1$.

Questions to complete during the tutorial

Questions marked with * are more difficult.

1. Prove that for all $n \geq 0$, we have the estimate $\left| \sin(1) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \right| \leq \frac{1}{(2n+2)!}$.

Solution: Write

$$\sin x = T_{2n+1}(x) + R_{2n+1}(x),$$

where $T_{2n+1}(x)$ is the Taylor polynomial of order $2n+1$ of $f(x) = \sin x$ centred at $x = 0$. Thus

$$T_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

(this is a very common Taylor polynomial, and we should remember it!). By Taylor's Theorem the remainder can be written

$$R_{2n+1}(x) = \frac{f^{(2n+2)}(c)}{(2n+2)!} x^{2n+2} \quad \text{for some } c \text{ between } 0 \text{ and } x.$$

Note that $f^{(2n+2)}(x) = (-1)^{n+1} \sin x$. Thus

$$\sin(1) = T_{2n+1}(1) + R_{2n+1}(1) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} + (-1)^{n+1} \frac{\sin c}{(2n+2)!} \quad \text{for some } c \in (0, 1).$$

So

$$\left| \sin(1) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \right| = \frac{|\sin c|}{(2n+2)!} \leq \frac{1}{(2n+2)!}.$$

Remark: In fact we have the better inequality:

$$\left| \sin(1) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \right| \leq \frac{1}{(2n+3)!}.$$

To see this, repeat the above working using the Taylor polynomial $T_{2n+2}(x)$ of $\sin x$. Note that $T_{2n+2}(x) = T_{2n+1}(x)$, because the "even" terms in the Taylor polynomial are zero. The remainder $R_{2n+2}(x)$ can be written as

$$R_{2n+2}(x) = \frac{f^{(2n+3)}(c)}{(2n+3)!} x^{2n+3} = \frac{\pm \cos c}{(2n+3)!} x^{2n+3},$$

and the result follows. This is a common (and sometimes useful) trick to improve approximations when the even (or odd) terms of the Taylor polynomials are zero.

2. Let $p(x) := x^4 - 7x^3 + 13x^2 + 2x - 10$ be a polynomial.

(a) Determine the Taylor polynomials of order n about $x = 0$ for $n \geq 0$.

Solution: We clearly have $p(0) = -10$, $p'(0) = 2$, $p''(0) = 13 \cdot 2$, $p'''(x) = -7 \cdot 3!$, $p^{(4)}(0) = 4!$ and $p^{(n)} = 0$ for $n > 4$. Hence the n-th order Hence, $T_0(x) = -10$, $T_1(x) = -10 + 2x$, $T_2(x) = -10 + 2x + 13x^2$, $T_3(x) = -10 + 2x + 13x^2 - 7x^3$ and $T_n(x) = p(x)$ for $n \geq 4$.

(b) Express the polynomial p as a polynomial in terms of $(x - 3)$ by computing a relevant Taylor polynomial.

Solution: We compute the 4th order Taylor polynomial centred at 3. This polynomial has to coincide with $p(x)$. We have

$$\begin{aligned}
p(3) &= 5 \\
p'(3) &= 4x^3 - 21x^2 + 26x + 2\Big|_{x=3} = -1 \\
p''(3) &= 2!(6x^2 - 21x + 13)\Big|_{x=3} = 2! \times 4 \\
p'''(3) &= 3!(4x - 7)\Big|_{x=3} = 3! \times 5 \\
p^{(4)}(3) &= 4! \times 1
\end{aligned}$$

Hence the polynomial is $p(x) = T_4(x) = (x-3)^4 + 5(x-3)^3 + 4(x-3)^2 - (x-3) + 5$.

3. (a) Calculate the second order Taylor polynomial $T_2(x)$ for $f(x) = \sqrt{1+x}$ about $x = 0$, and write down a formula for the remainder term $R_2(x) = f(x) - T_2(x)$. Hence show that

$$1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 \leq \sqrt{1+x^4} \leq 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} \quad \text{for all } x \in \mathbb{R}$$

Solution: We have

$$\begin{aligned}
f(x) &= (1+x)^{1/2} & &\Rightarrow & f(0) &= 1 \\
f'(x) &= (1/2)(1+x)^{-1/2} & &\Rightarrow & f'(0) &= 1/2 \\
f^{(2)}(x) &= -(1/4)(1+x)^{-3/2} & &\Rightarrow & f^{(2)}(0) &= -1/4.
\end{aligned}$$

Thus

$$T_2(x) = 1 + \frac{x}{2} - \frac{x^2}{8}.$$

By Taylor's Theorem, the remainder may be expressed as

$$R_2(x) = \frac{f^{(3)}(c)}{3!}x^3 = \frac{3}{8 \cdot 3!}(1+c)^{-5/2}x^3 = \frac{x^3}{16(1+c)^{5/2}}$$

for some $c = c(x)$ between 0 and x.

We have $\sqrt{1+x} - T_2(x) = R_2(x)$, and replacing x by x^2 gives

$$\sqrt{1+x^4} - \left(1 + \frac{1}{2}x^4 - \frac{1}{8}x^8\right) = R_2(x^4) = \frac{x^{12}}{16(1+c)^{5/2}}$$

for some c between 0 and x^4. Thus $0 \leq c \leq x^4$, and so $0 \leq 1/(1+c)^{5/2} \leq 1$. Thus

$$0 \leq \sqrt{1+x^4} - \left(1 + \frac{1}{2}x^4 - \frac{1}{8}x^8\right) \leq \frac{1}{16}x^{12} \quad \text{for all } x \in \mathbb{R}, \tag{1}$$

hence the result.

(b) Hence, or otherwise, calculate $\lim_{x\to 0} \dfrac{2\sqrt{1+x^4}-2-x^4}{x^8}$.

Solution: Rearranging the above inequality gives

$$-\frac{1}{8}x^8 \le \sqrt{1+x^4} - 1 - \frac{1}{2}x^4 \le -\frac{1}{8}x^8 + \frac{1}{16}x^{12}$$

Multiplying through by 2, and dividing by x^8, gives

$$-\frac{1}{8} \le \frac{2\sqrt{1+x^4}-2-x^4}{x^8} \le -\frac{1}{8} + \frac{1}{16}x^4 \quad \text{for } x \neq 0.$$

Thus, by the squeeze law,

$$\lim_{x\to 0} \frac{2\sqrt{1+x^4}-2-x^4}{x^8} = -\frac{1}{8}.$$

4. Let $f(x) := (1+x)^\alpha$, where $\alpha \in \mathbb{R}$ is fixed. We define

$$\binom{\alpha}{0} := 1 \qquad \text{and} \qquad \binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-3)\cdots(\alpha-k+1)}{k!} \quad \text{for all integers } k \ge 1.$$

If $\alpha \ge 1$ is an integer these coincide with the binomial coefficients.

(a) Show that the n-th Taylor polynomial of f about $x = 0$ is given by $T_n(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k$.

Solution: Each time we differentiate $(1+x)^\alpha$ the exponent comes down as a factor and the new exponent is reduced by one. Hence the k-th derivative is

$$\frac{d^k}{dx^k}(1+x)^\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+x)^{\alpha-k}$$

Hence the coefficient of x^k in the Taylor polynomial is

$$\frac{f^{(k)}(0)}{k!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} = \binom{\alpha}{k}$$

for all $k \ge 1$. Clearly $f(0) = 1$ and hence the Taylor polynomial is as stated in the question.

(b) Hence, write down the 4th order Taylor polynomial of $\dfrac{1}{\sqrt{1+x}}$ about $x = 0$.

Solution: We apply the formula from the previous part to compute the coefficients:

$$\begin{aligned}
\binom{-\frac{1}{2}}{0} &= 1 \\
\binom{-\frac{1}{2}}{1} &= -\frac{1}{2}, \\
\binom{-\frac{1}{2}}{2} &= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2} = \frac{3}{8}, \\
\binom{-\frac{1}{2}}{3} &= \frac{-\frac{1}{2}\left(\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!} = -\frac{5}{16}, \\
\binom{-\frac{1}{2}}{3} &= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)}{4!} = \frac{35}{128}.
\end{aligned}$$

Hence the Taylor polynomial of order 4 is

$$T_4(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + \frac{35}{128}x^4.$$

(c) This part shows that the remainder $R_n(x)$ approaches zero as n gets large provided that $0 \le x < 1$.

*(i) Fix $x \in \mathbb{R}$ with $|x| < 1$. Using the Lagrange form of the remainder, show that

$$R_n(x) = (-1)^n \alpha \frac{(1+c)^\alpha}{n+1}\Big(1-\alpha\Big)\Big(1-\frac{\alpha}{2}\Big)\Big(1-\frac{\alpha}{3}\Big)\cdots\Big(1-\frac{\alpha}{n}\Big)\Big(\frac{x}{1+c}\Big)^{n+1}$$

for some c strictly between 0 and x. Derive that for $1 \le m < n$ and $x \in (0,1)$

$$|R_n(x)| \le \frac{2^\alpha}{n+1}\big(1+|\alpha|\big)^m\Big(1+\frac{|\alpha|}{m}\Big)^{n+1}|x|^{n+1}.$$

Solution: Using the Lagrange form of the remainder, there exists c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} = \binom{\alpha}{n+1}(1+c)^{\alpha-n-1}x^{n+1}$$

We rewrite the generalised binomial term. We pair the $n+1$ terms in the numerator and denominator and then simplify

$$\begin{aligned}\binom{\alpha}{n+1} &= \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-n)}{(n+1)!}\\ &= \frac{\alpha}{n+1}\cdot\frac{\alpha-1}{1}\cdot\frac{\alpha-2}{2}\cdot\frac{\alpha-3}{3}\cdots\frac{\alpha-n}{n}\\ &= \frac{\alpha}{n+1}\Big(\alpha-1\Big)\ \Big(\frac{\alpha}{2}-1\Big)\Big(\frac{\alpha}{3}-1\Big)\cdots\Big(\frac{\alpha}{n}-1\Big)\\ &= (-1)^n\frac{\alpha}{n+1}\Big(1-\alpha\Big)\ \Big(1-\frac{\alpha}{2}\Big)\Big(1-\frac{\alpha}{3}\Big)\cdots\Big(1-\frac{\alpha}{n}\Big).\end{aligned}$$

We can also write

$$(1+c)^{\alpha-n-1}x^{n+1} = (1+c)^\alpha\Big(\frac{x^{n+1}}{1+c}\Big).$$

Combining everything we obtain

$$R_n(x) = (-1)^n\alpha\frac{(1+c)^\alpha}{n+1}\Big(1-\alpha\Big)\Big(1-\frac{\alpha}{2}\Big)\Big(1-\frac{\alpha}{3}\Big)\cdots\Big(1-\frac{\alpha}{n}\Big)\Big(\frac{x}{1+c}\Big)^{n+1}.$$

By taking absolute values and using the triangle inequality on each term we deduce that

$$\begin{aligned}|R_n(x)| &= |\alpha|\frac{(1+c)^\alpha}{n+1}\Big|1-\alpha\Big|\Big|1-\frac{\alpha}{2}\Big|\Big|1-\frac{\alpha}{3}\Big|\cdots\Big|1-\frac{\alpha}{n}\Big|\Big(\frac{|x|}{1+c}\Big)^{n+1}\\ &\le |\alpha|\frac{(1+c)^\alpha}{n+1}\Big(1+|\alpha|\Big)\Big(1+\frac{|\alpha|}{2}\Big)\Big(1+\frac{|\alpha|}{3}\Big)\cdots\Big(1+\frac{|\alpha|}{n}\Big)\Big(\frac{|x|}{1+c}\Big)^{n+1}.\end{aligned}$$

Now assume that $0 < x < 1$. As $c \in (0,x)$ it follows that $0 < 1+c < 2$ and so we have $(1+c)^\alpha \le 2^\alpha$ and $(\frac{|x|}{1+c} \le |x|$. Therefore,

$$|R_n(x)| \le |\alpha|\frac{2^\alpha}{n+1}\Big(1+|\alpha|\Big)\Big(1+\frac{|\alpha|}{2}\Big)\Big(1+\frac{|\alpha|}{3}\Big)\cdots\Big(1+\frac{|\alpha|}{n}\Big)|x|^{n+1}.$$

Finally, if $1 < m < n$, then

$$\begin{aligned}&|\alpha|\Big(1+|\alpha|\Big)\Big(1+\frac{|\alpha|}{2}\Big)\Big(1+\frac{|\alpha|}{3}\Big)\cdots\Big(1+\frac{|\alpha|}{n}\Big)\\ &\quad= \Big[|\alpha|\Big(1+|\alpha|\Big)\Big(1+\frac{|\alpha|}{2}\Big)\cdots\Big(1+\frac{|\alpha|}{m-1}\Big)\Big]\ \Big[\Big(1+\frac{|\alpha|}{m}\Big)\cdots\Big(1+\frac{|\alpha|}{n}\Big)\Big]\\ &\quad\le \Big(1+|\alpha|\Big)^m\Big(1+\frac{|\alpha|}{m}\Big)^{n+1-m}\\ &\quad\le \Big(1+|\alpha|\Big)^m\Big(1+\frac{|\alpha|}{m}\Big)^{n+1}\end{aligned}$$

by replacing each factor in the square brackets by the largest term in the product in the second last step, then dropping all factors that are less than one in the last step. Combining everything we obtain

$$|R_n(x)| \le \frac{2^\alpha}{n+1}\big(1+|\alpha|\big)^m\Big(1+\frac{|\alpha|}{m}\Big)^{n+1}|x|^{n+1}$$

as required.

(ii) Hence show that $R_n(x) \to 0$ for all $x \in (0,1)$, that is, T_n approximates $(1+x)^\alpha$ well if $x \in (0,1)$. One can show that the same is true if $x \in (-1,0)$, but this is much harder.

Solution: As $x \in (0,1)$ we can choose $m < n$ large enough so that

$$A := \Big(1+\frac{|\alpha|}{m}\Big)|x| \le 1.$$

Hence, by the previous part

$$R_n(x)| \le \frac{2^\alpha}{n+1}\big(1+|\alpha|\big)^m A^{n+1} \le \frac{2^\alpha}{n+1}\big(1+|\alpha|\big)^m \to 0$$

as $n \to \infty$.

5. Find parametrisations of the following curves.

(a) The line segment in $\mathbb{R}^3$ between $A(1,3,2)$ and $B(5,2,4)$ going from B to A.

Solution: This is by linear interpolation

$$\begin{bmatrix} x(t)\\ y(t)\\ z(t)\end{bmatrix} = (1-t)\begin{bmatrix}1\\3\\2\end{bmatrix} + t\begin{bmatrix}5\\2\\4\end{bmatrix} = \begin{bmatrix}1\\3\\2\end{bmatrix} + t\begin{bmatrix}4\\-1\\2\end{bmatrix}, \qquad t \in [0,1].$$

(b) The circle in $\mathbb{R}^2$ centred at the origin with radius r, clockwise starting from the positive y-axis.

Solution: The angle t clockwise from the y-axis is the same as the angle $\pi/2-t$ from the positive x-axis. As t increases, the point with angle $\pi/2-t$ moves in the clockwise direction. Hence the required parametrisation is

$$x(t) = r\cos(\pi/2-t) = r\sin(t) \qquad \text{and} \qquad y(t) = r\sin(\pi/2-t) = r\cos(t).$$

6. (a) Find the points of intersection of the helix whose general point is given parametrically as $(\cos t, \sin t, t)$, $t \in \mathbb{R}$, with the sphere whose cartesian equation is $x^2+y^2+z^2 = 4$.

Solution: Put $x = \cos t$, $y = \sin t$ and $z = t$ in the equation, $x^2+y^2+z^2 = 4$, of the sphere. Then $\cos^2 t + \sin^2 t + t^2 = 4$ gives $t = \pm\sqrt{3}$. That is, the points of intersection are: $(\cos\sqrt{3}, \sin\sqrt{3}, \sqrt{3})$ and $(\cos\sqrt{3}, -\sin\sqrt{3}, -\sqrt{3})$.

The point $(\cos\sqrt{3}, \sin\sqrt{3}, \sqrt{3})$ lies on the circle centre $(0,0,\sqrt{3})$ and radius 1, in the plane $z = \sqrt{3}$. The point $(\cos\sqrt{3}, -\sin\sqrt{3}, -\sqrt{3})$ lies on the circle centre $(0,0,-\sqrt{3})$ and radius 1, in the plane $z = -\sqrt{3}$.

(b) Find all points common to the helices C_1 and C_2, where

$$C_1(t) = (\cos t, \sin t, t), \quad t \in \mathbb{R}, \qquad C_2(s) = (\cos s, s, \sin s), \quad s \in \mathbb{R}.$$

Solution: If there is a point on both C_1 and C_2, then there exist $t \in \mathbb{R}$ and $s \in \mathbb{R}$ such that

$$\cos t = \cos s \qquad (1)$$
$$\sin t = s \qquad (2)$$
$$t = \sin s \qquad (3)$$

From (1), $t = s + 2k\pi$ or $t = -s + 2k\pi$ (where $k \in \mathbb{Z}$). Substituting into (2) gives $\sin(s+2k\pi) = s$ or $\sin(-s+2k\pi) = s$, that is, $\sin s = s$ or $-\sin s = s$. But $\pm\sin s = s$ if and only if $s = 0$ (a quick sketch of the graphs $y = x$ and $y = \sin x$ illustrates this). Then from (3), we must have $t = s = 0$. The unique common point on the two helices is the point

$$(\cos 0, \sin 0, 0) = (\cos 0, 0, \sin 0) = (1,0,0).$$

Extra questions for further practice

7. Let $f(x) = \ln(1+x)$.

(a) Calculate the fourth order Taylor polynomial $T_4(x)$ for $f(x)$ centred at 0.

Solution: We did this last week,

$$T_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

(b) Use Taylor's Theorem to write down a formula for the fourth remainder term $R_4(x)$, and deduce that

$$\frac{x^5}{5(1+x)^5} \leq f(x) - T_4(x) \leq \frac{x^5}{5} \qquad \text{for all } x > 0.$$

Solution: We have, for some c between 0 and x,

$$R_4(x) = \frac{f^{(5)}(c)}{5!}x^5 = \frac{(-1)(-2)(-3)(-4)(1+c)^{-5}}{5!}x^5 = \frac{1}{5(1+c)^5}x^5.$$

Thus

$$f(x) - T_4(x) = \frac{1}{5(1+c)^5}x^5 \quad \text{for some } c \text{ between 0 and } x.$$

For $x > 0$ we have $0 < c < x$, and hence

$$\frac{x^5}{5(1+x)^5} \leq \frac{1}{5(1+c)^5}x^5 \leq \frac{x^5}{5},$$

hence the result.

(c) Use the previous part to compute the limit

$$\lim_{x\to 0^+} \frac{12\ln(1+x) - 12x + 6x^2 - 4x^3 + 3x^4}{\sin^5 x}$$

Solution: From the previous part, multiplying by 12,

$$\frac{12x^5}{5(1+x)^5} \leq 12\ln(1+x) - 12x + 6x^2 - 4x^3 + 3x^4 \leq \frac{12x^5}{5}.$$

Dividing by $\sin^5 x$ with $x > 0$ small (so that $\sin x > 0$) we have

$$\left(\frac{x}{\sin x}\right)^5 \frac{12}{5(1+x)^5} \leq \frac{12\ln(1+x) - 12x + 6x^2 - 4x^3 + 3x^4}{\sin^5 x} \leq \frac{12}{5}\left(\frac{x}{\sin x}\right)^5.$$

Taking limits as $x \to 0^+$, and using the squeeze law, we have

$$\lim_{x\to 0^+} \frac{12\ln(1+x) - 12x + 6x^2 - 4x^3 + 3x^4}{\sin^5 x} = \frac{12}{5}.$$

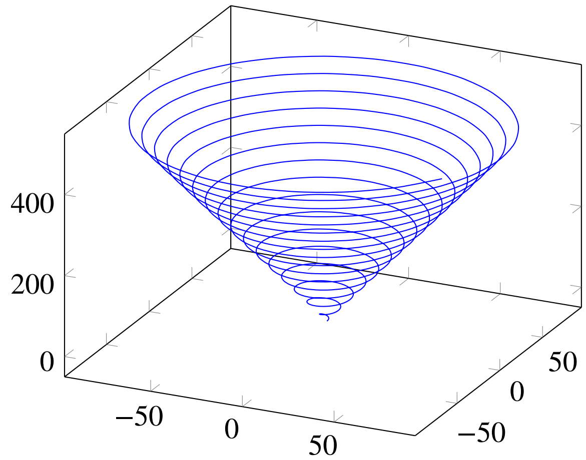
8. Show that the curve C with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$, $t \in \mathbb{R}$, passes through $(1, 4, 0)$ and $(9, -8, 28)$ but not $(4, 7, -6)$.

Solution: Let $C(t) = (t^2, 1 - 3t, 1 + t^3)$. Then $C(-1) = (1, 4, 0)$ and $C(3) = (9, -8, 28)$, so C passes through $(1, 4, 0)$ and $(9, -8, 28)$. If C passes through $(4, 7, -6)$, then for some t we must have $t^2 = 4$, $1 - 3t = 7$, and $1 + t^3 = -6$, which is impossible; no t can satisfy all equations simultaneously. (The second equation shows that $t = -2$, and while this satisfies the first equation, it doesn't satisfy the third.) Hence C does not pass through $(4, 7, -6)$.

9. Sketch the curves in $\mathbb{R}^3$ given by the following parametric equations.

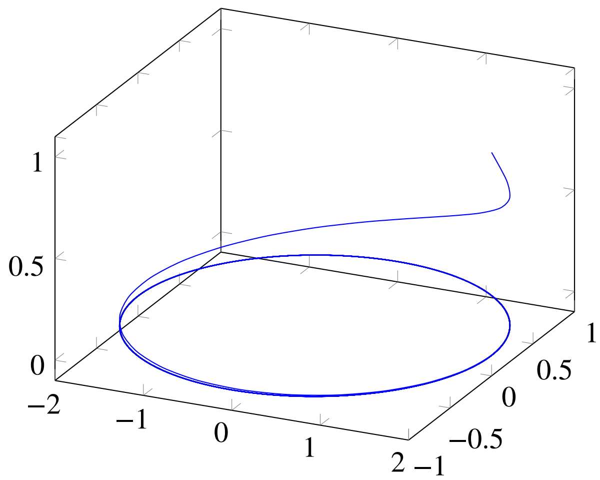
(a) $x = t\cos t,\ y = t\sin t,\ z = 5t, \quad t \in [0, 100]$.

Solution: As t increases, so does z. For any t, we have $x^2 + y^2 = t^2$. Hence the points lie on a circular spiral of increasing radius. The starting point is $(0, 0, 0)$ and the finishing point is $(100\cos 100, 100\sin 100, 500) \approx (86.2, -50.6, 500)$.



(b) $x = 2\cos t,\ y = \sin t,\ z = e^{-t}, \quad t \geq 0$.

Solution: The curve is confined to that part of space corresponding to values of z in the interval $(0, 1]$. It is a helix tracing out an elliptical orbit, with starting point $(2, 0, 1)$. As t takes values in any interval of the form $[a, a + 2\pi]$, the curve makes one revolution of the z axis with its height above the xy plane decreasing.



10. Suppose that f has at least $n + 1$ derivatives in a neighbourhood of x_0, and that $f^{(n+1)}$ is bounded near x_0. Let T_n be the n-th order Taylor theorem about x_0 and let R_n be the remainder term. We know that

$$\lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} = \lim_{x \to x_0} \frac{R_n(x)}{(x - x_0)^n} = 0.$$

Show that T_n is the only polynomial with the above property, that is, if $P(x)$ is a polynomial of at most degree n and

$$\lim_{x \to x_0} \frac{f(x) - P(x)}{(x - x_0)^n} = 0,$$

then $P = T_n$. Proceed as follows:

(a) Show that $\lim_{x \to x_0} \dfrac{T_n(x) - P(x)}{(x - x_0)^k} = 0$ for every $k = 0, 1, \ldots, n$.

Solution: By assumption

$$\frac{T_n(x) - P(x)}{(x - x_0)^n} = \frac{f(x) - P(x)}{(x - x_0)^n} - \frac{f(x) - T_n(x)}{(x - x_0)^n} \to 0 - 0 = 0$$

as $x \to x_0$. Hence, for $k = 0, \ldots, n-1$,

$$\frac{T_n(x) - P(x)}{(x - x_0)^k} = \frac{T_n(x) - P(x)}{(x - x_0)^n}(x - x_0)^{n-k} \to 0 \times 0 = 0$$

as $x \to x_0$.

(b) Setting $Q(x) := T_n(x) - P(x)$ and writing it as a polynomial in $(x - x_0)$, use induction by k to show that the coefficient a_k of x^k is zero for every $k = 0, \ldots, n$. Conclude that $T_n = P$.

Solution: Let $Q(x) = T_n(x) - P(x) = \sum_{k=0}^{n} a_k(x - x_0)^k$. From part (a) with $k = 0$ we have

$$0 = \lim_{x \to x_0} Q(x) = a_0$$

This provides the initial case. For the induction step assume that $a_j = 0$ for $j = 0, \ldots k < n$. Then from part (a) we have

$$0 = \lim_{x \to x_0} \frac{T_n(x) - P(x)}{(x - x_0)^k} = \lim_{x \to x_0} \frac{Q(x)}{(x - x_0)^k} = a_k$$

Hence $a_0 = a_1 = \cdots = a_n = 0$ and thus $T_n(x) = P(x)$ for all x.

Challenge questions (optional)

11. The *error function* from probability and statistics is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Show that for each $n \geq 1$ and $x \geq 0$ we have

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)k!} x^{2k+1} + E_n(x) \quad \text{with} \quad |E_n(x)| \leq \frac{2x^{2n+3}}{\sqrt{\pi}(2n+3)(n+1)!},$$

and hence find a numerical approximation α to $\operatorname{erf}(1)$ such that

$$|\operatorname{erf}(1) - \alpha| < \frac{1}{1000}.$$

Solution: By Taylor's Theorem, and familiar formulae for the Taylor polynomial of e^t, we have

$$e^t = T_n(t) + R_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!} + \frac{e^c}{(n+1)!} t^{n+1} \quad \text{for some } c \text{ between 0 and } t.$$

Replacing t by $-t^2$ we have

$$e^{-t^2} = \sum_{k=0}^{n} \frac{(-1)^k t^{2k}}{k!} + \frac{(-1)^{n+1} e^c}{(n+1)!} t^{2n+2} \quad \text{for some } c \text{ between 0 and } -t^2.$$

Thus

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \\
&= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \int_0^x t^{2k}\, dt + E_n(x) \\
&= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)k!} x^{2k+1} + E_n(x),
\end{aligned}$$

where

$$|E_n(x)| = \left|\frac{2}{\sqrt{\pi}}\int_0^x \frac{(-1)^{n+1}e^c}{(n+1)!}t^{2n+2}\,dt\right| \le \frac{2}{\sqrt{\pi}}\int_0^x \frac{t^{2n+2}}{(n+1)!}\,dt = \frac{2x^{2n+3}}{\sqrt{\pi}(2n+3)(n+1)!}.$$

In this inequality we have used the fact that $-t^2 \le c \le 0$ to give $e^c \le 1$.

Using the inequality for $|E_n(1)|$ with $n = 6$ gives $|E_6(1)| \le 0.0000149\ldots < 1/1000$. Thus we can take

$$\alpha = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{6}\frac{(-1)^k}{(2k+1)k!} = 0.84271422\ldots$$

Therefore,

$$|\mathrm{erf}(1) - 0.84271422\ldots| \le 1/1000$$

12. Use Question 1 to prove that $\sin(1)$ is irrational. Show also that $\cos(1)$ is irrational.

Solution: Recalling the solution for Question 1 we have

$$\left|\sin(1) - \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}\right| = \frac{|\sin c|}{(2n+2)!} \le \frac{1}{(2n+2)!}.$$

We also note that since $c \in (0,1)$ we have $|\sin c| \ne 0$, and therefore in fact

$$0 < \left|\sin(1) - \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}\right| \le \frac{1}{(2n+2)!} \quad \text{for all } n \ge 0.$$

Our plan now is to adapt the proof from lectures where we showed that e is irrational. So suppose that $\sin(1) = p/q$ is rational, with $p, q \in \mathbb{N}$ (note that $\sin(1) > 0$). Let q' be the smallest odd integer with $q' \ge q$. Thus $q' = q$ if q is odd, and if $q' = q + 1$ if q is even. Now let n be such that $q' = 2n + 1$. Then the above inequality gives (after multiplying by $(2n + 1)!$ and replacing $\sin(1)$ be p/q)

$$0 < (2n+1)!\left|\frac{p}{q} - \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}\right| \le \frac{1}{2n+2}.$$

Note that

$$(2n+1)! \times \frac{p}{q}$$

is an integer (because $q \le q' = 2n + 1$), and also that

$$(2n+1)! \times \frac{(-1)^k}{(2k+1)!}$$

is an integer for each $0 \le k \le n$. Therefore

$$(2n+1)!\left|\frac{p}{q} - \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}\right| = N$$

is an integer, and hence

$$0 < N < \frac{1}{2n+2} \le \frac{1}{2},$$

a contradiction. So $\sin(1)$ is not rational, and so it is irrational.

We omit the proof that $\cos(1)$ is irrational – it is rather similar, using the Taylor polynomial of $\cos x$.

13. There are limitations to Taylor polynomials. This question describes a function whose Taylor polynomials are all identically zero. That is, $T_n(x) = 0$ for all n, and so $R_n(x) = f(x) - T_n(x) = f(x)$. Thus this function is "all remainder"!

Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show, by induction, that for all $n \geq 0$ and for $x \neq 0$,

$$f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$$

where $P_n(t)$ is a polynomial with integer coefficients.

Solution: The result is true for $n = 0$, with $P_0(t) = 1$, starting the induction. Suppose that $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$ for $x \neq 0$. Then, for $x \neq 0$,

$$\begin{aligned} f^{(n+1)}(x) &= \frac{d}{dx} f^{(n)}(x) \\ &= \left(\frac{d}{dx} P_n(1/x)\right) e^{-1/x^2} - \frac{2}{x^3} P_n(x)e^{-1/x^2} \\ &= -\frac{1}{x^2} P_n(1/x)e^{-1/x^2} - \frac{2}{x^3} P_n(x)e^{-1/x^2} \\ &= \left(-\frac{1}{x^2} P_n(1/x) - \frac{2}{x^3} P_n(x)\right) e^{-1/x^2} \\ &= P_{n+1}(1/x)e^{-1/x^2}, \end{aligned}$$

where $P_{n+1}(t) = -t^2 P_n(t) - 2t^3 P_n(t)$ is a polynomial with integer coefficients. Hence the result is true by induction.

(b) Show that $\lim_{x \to 0} |x|^{-k} e^{-1/x^2} = 0$ for all integers $k \geq 0$.

Solution: It is easiest to do this limit by "changing variable" in the limit. Let $y = 1/x^2$, so that $|x| = 1/\sqrt{y}$. As $x \to 0$ we have $y \to \infty$, and so

$$\lim_{x \to 0} |x|^{-k} e^{-1/x^2} = \lim_{y \to \infty} \frac{y^{k/2}}{e^y}.$$

By repeated applications of L'Hôpital's Rule we arrive at

$$\lim_{x \to 0} |x|^{-k} e^{-1/x^2} = \lim_{y \to \infty} \frac{y^{k/2}}{e^y} = \begin{cases} \lim_{y \to \infty} \frac{1}{e^y} = 0 & \text{if } k \text{ is even} \\ \lim_{y \to \infty} \frac{1}{\sqrt{y}e^y} = 0 & \text{if } k \text{ is odd} \end{cases}$$

(c) Show, by induction, that $f(x)$ is differentiable as many times as we please at the point $x = 0$, and that $f^{(n)}(0) = 0$. Thus the nth order Taylor polynomial of $f(x)$ centred at $x = 0$ is identically zero.

Solution: For $n = 0$ we have $f^{(n)}(0) = f^{(0)}(0) = f(0) = 0$ by definition, starting the induction. Suppose that $f^{(n)}(0) = 0$. Then

$$\lim_{h \to 0} \frac{f^{(n)}(h) - f^{(n)}(0)}{h} = \lim_{h \to 0} \frac{P_n(1/h)e^{-1/h^2} - 0}{h} = \lim_{h \to 0} Q(1/h)e^{-1/h^2},$$

where we have used the first part, and the induction hypothesis, and set $Q(t) = tP_n(t)$ (a polynomial with integer coefficients). Writing

$$Q(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N$$

with $a_0, \dots, a_N \in \mathbb{Z}$ we have

$$0 \leq |Q(1/h)e^{-1/h^2}| \leq \sum_{k=0}^{N} |a_k||h|^{-k}e^{-1/h^2}.$$

By the previous part, and the limit laws, we have

$$\lim_{h\to 0} \sum_{k=0}^{N} |a_k||h|^{-k}e^{-1/h^2} = \sum_{k=0}^{N} |a_k| \lim_{h\to 0} |h|^{-k}e^{-1/h^2} = \sum_{k=0}^{N} |a_k| \times 0 = 0,$$

and so by the squeeze law we have

$$\lim_{h\to 0} Q(1/h)e^{-1/h^2} = 0.$$

Therefore

$$\lim_{h\to 0} \frac{f^{(n)}(h) - f^{(n)}(0)}{h} = 0,$$

and so $f^{(n+1)}(0) = 0$. So the result is true by induction.