CHAPTER **SEVEN**

The Derivative

Our study of functions has now prepared us for some quite new approaches known as *calculus*. Calculus begins with two processes called differentiation and integration, both based on limiting processes:

- *Differentiation* is the examination of the changing steepness of a curve as one moves along it.
- *Integration* is the examination of areas of regions bounded by curves.

These processes were used by the Greeks, for whom tangents and areas were routine parts of their geometry, but it was not until the late seventeenth century that Sir Isaac Newton in England and Gottfried Leibniz in Germany independently gave systematic accounts of them. These were based on the realisation that finding the gradients of tangents and finding areas are inverse processes — a surprising insight so central that it is called 'the fundamental theorem of calculus'. In this chapter we will be concerned with differentiation, introducing it in the context of functions, geometry and limiting processes.

Study Notes: The derivative is first defined geometrically using tangents in Section 7A, and is then characterised in Section 7B as a limiting process. Sections 7C–7G develop the standard algebraic techniques of differentiation, interlocked in the exercises with the geometry of tangents and normals, particularly to the circle, the parabola and the rectangular hyperbola. Rates of change are included in Section 7H of this introductory chapter, because this is one of the most illuminating interpretations of the derivative and so should occur at the outset. General remarks about limits, continuity and differentiability have been left until Section 7I and 7J, and the study of these two sections could well be delayed until later in the course. The final Section 7K on implicit differentiation is a 4 Unit topic, but the techniques are very useful in the 3 Unit course.

7 A The Derivative — Geometric Definition

Sketched below on graph paper is the graph of a function $y = f(x)$ — for reasons of convenience the cubic $y = \frac{1}{10}(x^3 - 12x)$ was chosen. Like any curve that is not a straight line, its steepness keeps changing as one moves along the curve. Tangents have been drawn at several points on the curve, because the steepness of the curve at any point is measured by drawing a tangent at the point and measuring the gradient of the tangent.

The gradient of each tangent can easily be found by measuring its rise and run against the grid lines behind it. Counting ten little divisions for the run and measuring the corresponding rise give a natural decimal value for the gradient. Here is the resulting table of values of the gradients:

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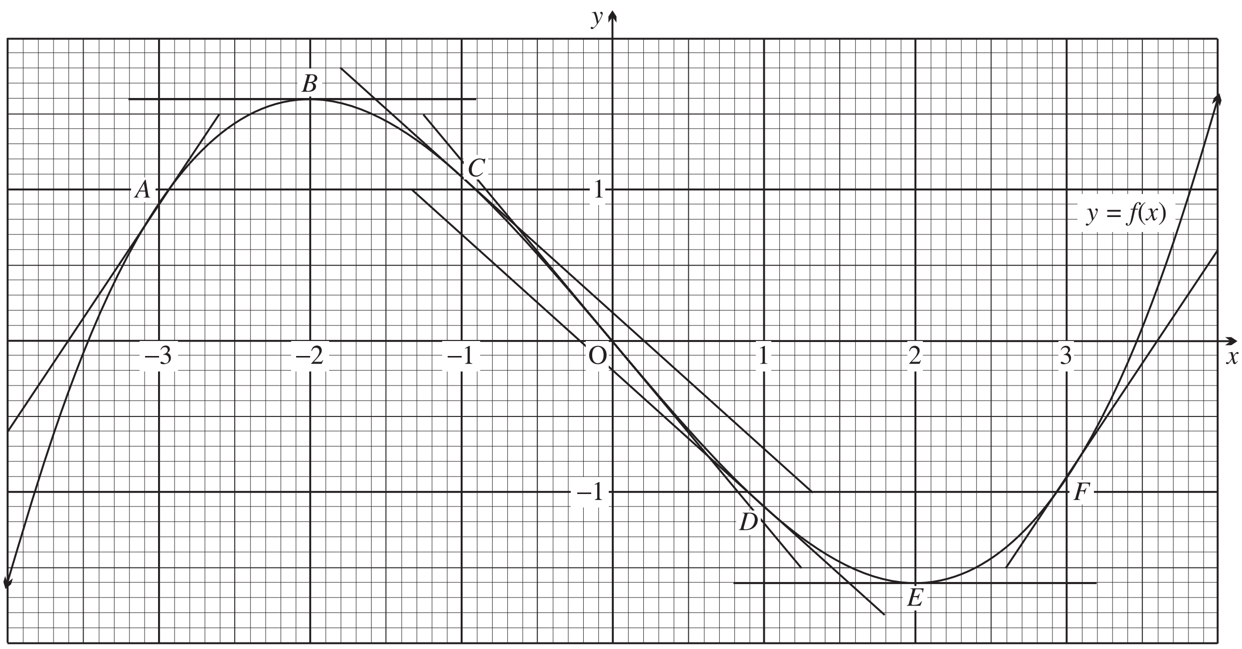
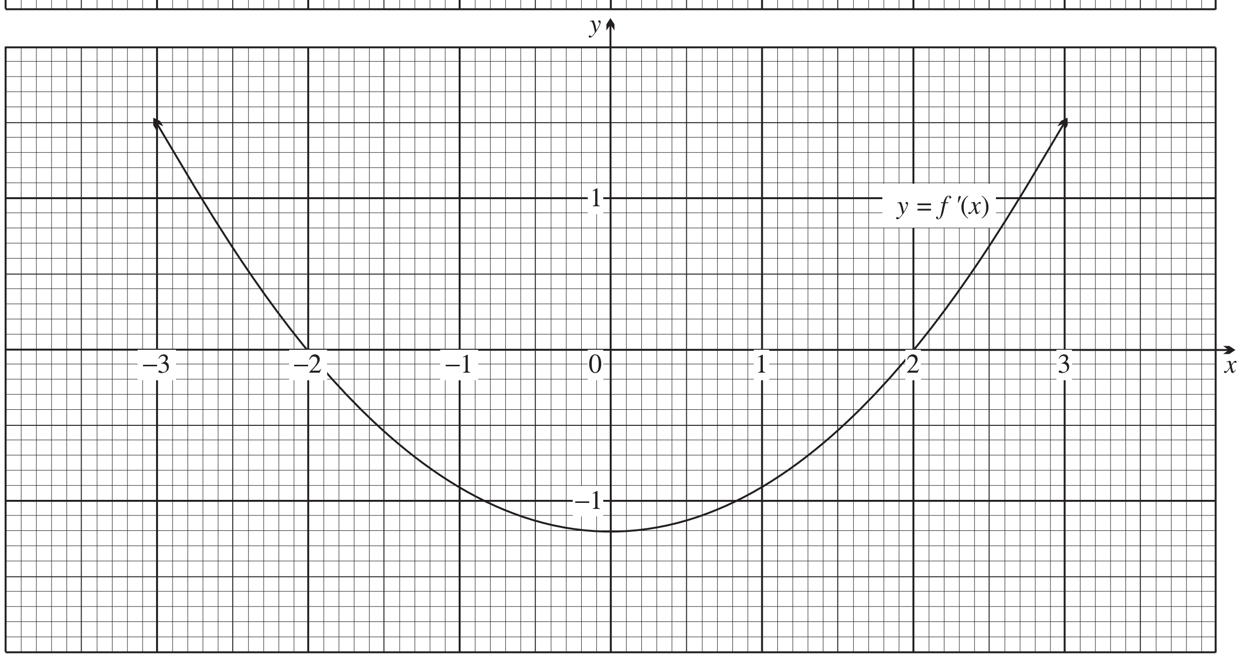
x	-3	-2	-1	0	1	2	3
gradient of tangent	$1{\cdot}5$	0	$-0{\cdot}9$	$-1{\cdot}2$	$-0{\cdot}9$	0	$1{\cdot}5$

Notice that the horizontal tangents at B and E have gradient zero. The tangents between B and E have negative gradients, because they slope downwards. Everywhere else the tangents slope upwards and their gradients are positive.

We can get a complete picture of all this by plotting these gradients on a second number plane and joining up the points. This gives the second graph below, which shows the gradient at each point on the curve $y = f(x)$. This second graph looks suspiciously like that of a quadratic function, and later we will be able to compute its equation exactly — it is $\frac{1}{10}(3x^2 - 12)$. But for now, it is enough to realise that the resulting function in this second sketch is a new function. This new function is called the *derivative of* $f(x)$, and is written as $f'(x)$.





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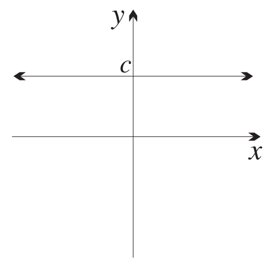
Geometric Definition of the Derivative: Here is the essential definition. Let $f(x)$ be a function. The *derivative* or *derived function* of $f(x)$, written as $f'(x)$, is defined by:

1 DEFINITION: $f'(x)$ is the gradient of the tangent to $y = f(x)$ at $P\big(x, f(x)\big)$.

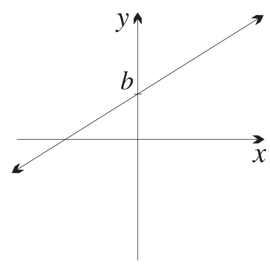
At present, circles are the only curves whose tangents we know much about, so the only functions we can apply our definition to are constant functions, linear functions and semicircular functions.

The Derivative of a Constant Function: Let $f(x) = c$ be a constant function. The tangent to the straight line $y = c$ at any point P on the line is of course just the line itself. So every tangent has gradient zero, and $f'(x)$ is the zero function.



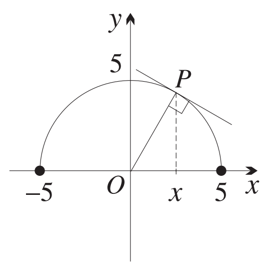
2 THEOREM: The derivative of a constant function $f(x) = c$ is the zero function $f'(x) = 0$.

The Derivative of a Linear Function: Let $f(x) = mx + b$ be a linear function. Again, the tangent to $y = mx + b$ at any point P on the line is just the line itself. So every tangent has gradient m, and $f'(x) = m$ is a constant function.



3 THEOREM: The derivative of a linear function $f(x) = mx + b$ is the constant function $f'(x) = m$.

The Derivative of a Semicircle Function: Let $f(x) = \sqrt{25 - x^2}$ be the upper semicircle with centre O and radius 5. We know from geometry that at any point $P(x, \sqrt{25 - x^2}\,)$ on the semicircle, the tangent at P is perpendicular to the radius OP.



Now gradient of radius $OP = \dfrac{\sqrt{25 - x^2}}{x}$,

so gradient of tangent at $P = -\dfrac{x}{\sqrt{25 - x^2}}$,

meaning that $f'(x) = -\dfrac{x}{\sqrt{25 - x^2}}$.

(This result is not to be memorised.)

The General Case: These examples of straight lines and circles are the only functions we can differentiate until we can use the methods developed in the next section. Question 1 in the following exercise continues the curve sketching methods used above, and asks for a reasonably precise construction of the derived function of $f(x) = x^2$, in preparation for Section 7B.

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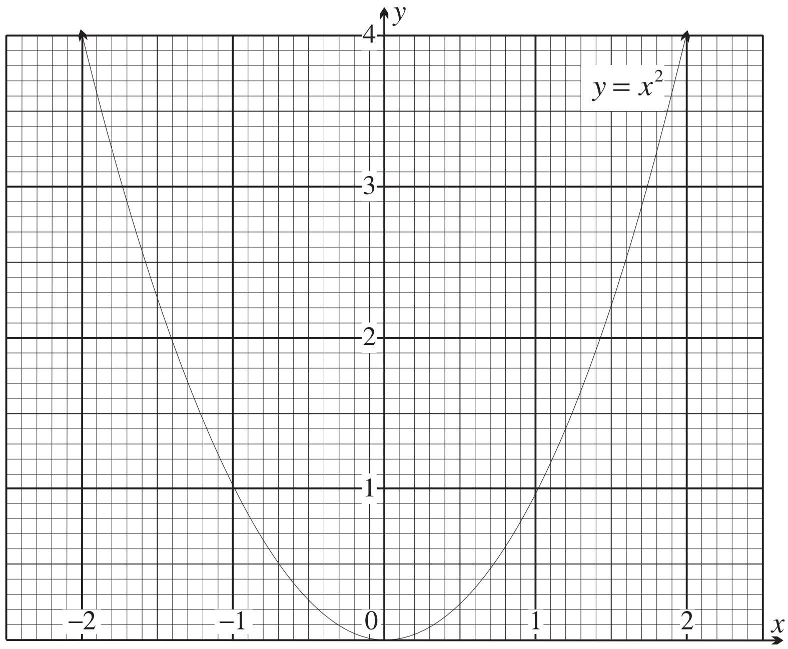
Exercise 7A

1.



(a) Photocopy the accompanying sketch of $f(x) = x^2$.

(b) At the point $P(1, 1)$, construct a tangent. Place the pencil point on P, bring the ruler to the pencil, then rotate the ruler about P until it seems reasonably like a tangent.

(c) Use the definition gradient $= \dfrac{\text{rise}}{\text{run}}$ to measure the gradient of the tangent to at most two decimal places. Choose the run to be 10 little divisions, and count how many vertical divisions the tangent rises as it runs across the 10 horizontal divisions.

(d) Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the nine points on the curve and measuring its gradient.

x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$f'(x)$									

(e) On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.

(f) Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

2. Write each function in the form $f(x) = mx + b$, and hence write down $f'(x)$:

(a) $f(x) = 2x + 3$

(b) $f(x) = 5 - 3x$

(c) $f(x) = \frac{1}{2}x - 7$

(d) $f(x) = -4$

(e) $f(x) = ax + b$

(f) $f(x) = \frac{2}{3}(x + 4)$

(g) $f(x) = \dfrac{3 - 5x}{4}$

(h) $f(x) = \frac{5}{2}(7 - \frac{4}{3}x)$

(i) $f(x) = \frac{1}{2} + \frac{1}{3}$

DEVELOPMENT

3. Write each function in the form $f(x) = mx+b$, and hence write down the derived function:

(a) $f(x) = \frac{1}{2}(3+5x) - \frac{1}{2}(5-2x)$

(b) $f(x) = (x+3)^2 - (x-3)^2$

(c) $f(x) = \dfrac{k - \ell x}{r} + \dfrac{k + \ell x}{r}$

4. Sketch the upper semicircle $f(x) = \sqrt{25 - x^2}$, mark the points $(4, 3)$, $(3, 4)$, $(0, 5)$, $(-3, 4)$ and $(-4, 3)$ on it, and sketch tangents and radii at these points. By using the fact that the tangent is perpendicular to the radius at the point of contact, find:

(a) $f'(4)$

(b) $f'(3)$

(c) $f'(0)$

(d) $f'(-4)$

(e) $f'(-3)$

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5. Use the fact that the tangent to a circle is perpendicular to the radius at the point of contact to find the derived functions of the following. Begin with a sketch.

(a) $f(x) = \sqrt{1 - x^2}$ (b) $f(x) = -\sqrt{1 - x^2}$ (c) $f(x) = \sqrt{4 - x^2}$

6. Sketch graphs of these functions, draw tangents at the points where $x = -2, -1, 0, 1, 2$, estimate their gradients, and hence draw a reasonable sketch of the derivative.

(a) $f(x) = 4 - x^2$ (b) $f(x) = \dfrac{1}{x}$ (c) $f(x) = 2^x$

EXTENSION

7. Use the radius and tangent theorem to find the derivatives of:

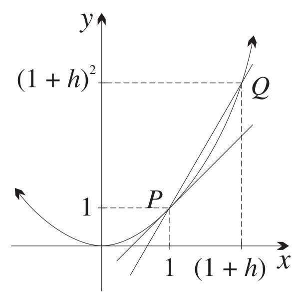
(a) $f(x) = \sqrt{9 - x^2} + 4$ (c) $f(x) = \sqrt{36 - (x - 7)^2}$

(b) $f(x) = 3 - \sqrt{16 - x^2}$ (d) $f(x) = 7 - \sqrt{2x - x^2}$

7 B The Derivative as a Limit

The gradient of a line is found by taking two distinct points on it and taking the ratio of rise over run. The difficulty with a tangent is that we only know one point on it — the point of contact — and unless the curve is a straight line, no other points on the curve near the point of contact actually lie on the tangent. The only general way to get at the tangent at some point P on the curve is to use a limiting process involving the family of lines through P.

The Tangent as the Limit of Secants: The diagram opposite shows the graph of $f(x) = x^2$ and the tangent at $P(1, 1)$ on the curve. Let $Q\big(1 + h, (1 + h)^2\big)$ be any other point on the curve. Then the straight line through P and Q is a secant whose gradient is the ratio of rise over run:



$$\begin{aligned}\text{gradient } PQ &= \frac{(1 + h)^2 - 1}{(1 + h) - 1}\\ &= \frac{2h + h^2}{h}\\ &= 2 + h, \quad \text{since } h \neq 0.\end{aligned}$$

As Q moves along the curve to the right of P (or to the left of P) the secant PQ rotates around P. But the closer Q is to P, the closer the secant PQ is to the tangent at P. In fact, we can make the gradient of the secant PQ 'as close as we like' to the gradient of the tangent by taking Q sufficiently close to P. That means we take the limit as $Q \to P$:

$$\begin{aligned}\text{gradient (tangent at } P) &= \lim_{Q \to P} (\text{gradient } PQ)\\ &= \lim_{h \to 0} (2 + h), \quad \text{because } h \to 0 \text{ as } Q \to P\\ &= 2, \quad \text{because } 2 + h \to 2 \text{ as } h \to 0.\end{aligned}$$

Thus the tangent at P has gradient 2, and so $f'(1) = 2$. Notice that Q cannot actually coincide with P, or both rise and run would be zero, and the calculation would be invalid.

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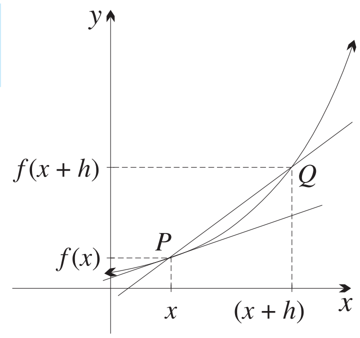
The Derivative as a Limit: This brings us to the formula for the derivative as a limit. The derivative of any function $f(x)$ is the new function $f'(x)$ defined at any value of x by:

4 DEFINITION: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



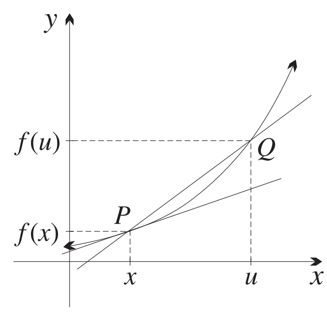
The accompanying diagram explains the definition. The secant PQ joins the point P with coordinates $(x, f(x))$ and the point Q with coordinates $\big(x+h, f(x+h)\big)$. Using the usual formula for gradient,

$$\text{gradient } PQ = \frac{f(x+h) - f(x)}{h} \quad \text{(rise over run)}.$$

Then the gradient of the tangent is the limit as $Q \to P$, that is as $h \to 0$.

NOTE: The diagram shows Q to the right of P. However, Q could as well be on the left, which corresponds algebraically to h being negative.

An Alternative Notation: There is an alternative notation which in some situations is more convenient to use. The diagram is the same, but we let Q have x-coordinate u and y-coordinate $f(u)$. In this case:



5 DEFINITION: $$f'(x) = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}$$

WORKED EXERCISE: As an example of this definition of the derivative as a limit, let us calculate the derivative of $f(x) = x^2$, using both notations above. (The graphical work in the first question of the previous exercise should already have obtained the answer $f'(x) = 2x$ for this derivative.) Calculating in this way is called differentiating 'from first principles' or 'from the definition of the derivative'.

SOLUTION:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h), \quad \text{since } h \neq 0, \\ &= 2x. \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{u \to x} \frac{f(u) - f(x)}{u - x} \\ &= \lim_{u \to x} \frac{u^2 - x^2}{u - x} \\ &= \lim_{u \to x} \frac{(u-x)(u+x)}{u - x} \\ &= \lim_{u \to x} (u + x), \quad \text{since } u \neq x, \\ &= 2x. \end{aligned}$$

What is a Tangent: The careful reader will realise that the word 'tangent' was introduced without definition in Section 7A. Whereas tangents to circles are well understood, tangents to more general curves are not so easily defined. It is possible to define a tangent geometrically, but it is far easier to take the formula for the derivative as part of its actual definition. So our strict definition of the tangent at a point $P\big(x, f(x)\big)$ is that it is the line through P with gradient $f'(x)$.

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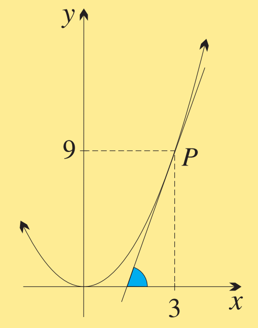
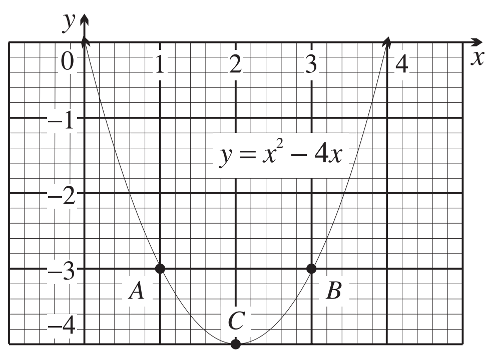
WORKED EXERCISE: Use the fact that the derivative of $f(x) = x^2$ is $f'(x) = 2x$ to find the gradient, the angle of inclination (to the nearest minute), and the equation of the tangent to the curve $y = x^2$ at the point $P(3, 9)$ on the curve.



SOLUTION: Substituting $x = 3$ into $f'(x) = 2x$ gives $f'(3) = 6$, so the tangent at P has gradient 6.
Hence the tangent is $y - 9 = 6(x - 3)$
$$y = 6x - 9.$$
Since the gradient is 6, the angle of inclination is about $80°32'$ (using the calculator to solve tan(angle of inclination) = 6).

Exercise 7B



1. Consider the function $f(x) = x^2 - 4x$.
(a) Simplify $\dfrac{f(x+h) - f(x)}{h}$.
(b) Show that $f'(x) = 2x - 4$, using the definition $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.
(c) Substitute $x = 1$ into $f'(x)$ to find the gradient of the tangent at $A(1, -3)$.
(d) Similarly find the gradients of the tangents at $B(3, -3)$ and $C(2, -4)$.
(e) The function $f(x) = x^2 - 4x$ is graphed above. Place your ruler on the curve at A, B and C to check the reasonableness of the results obtained above.

2. For each function below, simplify $\dfrac{f(x+h) - f(x)}{h}$, then take $\lim\limits_{h \to 0}$ to find the derivative. For part (i) you will need the result $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.
(a) $f(x) = 5x + 1$
(b) $f(x) = 4 - 3x$
(c) $f(x) = x^2 + 10$
(d) $f(x) = x^2 - 4x$
(e) $f(x) = x^2 + 3x + 2$
(f) $f(x) = 2x^2 + 3x$
(g) $f(x) = 9 - 4x^2$
(h) $f(x) = x^3$
(i) $f(x) = x^4$

3. For each function in question 2: (i) use the derivative to evaluate $f'(2)$, (ii) find the y-coordinate of the point P on $y = f(x)$ where $x = 2$, (iii) find the equation of the tangent at P, (iv) sketch the curve and the tangent.

4. For each of the functions in question 2, find any values of x for which the tangent is horizontal (that is, for which $f'(x) = 0$).

5. Find the derivatives of the functions in question 2 by first simplifying $\dfrac{f(u) - f(x)}{u - x}$, then taking $\lim\limits_{u \to x}$ to find the derivative. [HINT: Group the corresponding powers of u and x in the numerator, then factor each pair using difference of powers, then factor the whole numerator using grouping. For example in part (d):
$u^2 - 4u - x^2 + 4x = (u^2 - x^2) - 4(u - x) = (u + x)(u - x) - 4(u - x) = (u - x)(u + x - 4)$.]

DEVELOPMENT

6. (a) Sketch $f(x) = x^2 + 6x$, then use the $u \to x$ method to show that the derivative is $2x + 6$.

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(b) Hence find the gradient and angle of inclination of the tangent (nearest minute) at the point where: (i) $x = 0$ (ii) $x = -3$ (iii) $x = -2\frac{1}{2}$ (iv) $x = -3\frac{1}{2}$ (v) $x = -5$

7. (a) Use the $h \to 0$ method to show that the derivative of $f(x) = x^2 - 5x$ is $f'(x) = 2x - 5$.
(b) Hence find the points on $y = x^2 - 5x$ where the tangent has the following gradients. Then sketch the curve and the tangents. (i) 1 (ii) -1 (iii) 5 (iv) -5 (v) 0

8. (a) Use the $h \to 0$ method to show that the derivative of $f(x) = \frac{1}{4}x^2$ is $f'(x) = \frac{1}{2}x$.
(b) Hence find the x-coordinates of the points on $y = \frac{1}{4}x^2$ where the tangent has angle of inclination:
(i) $45°$ (ii) $135°$ (iii) $60°$ (iv) $120°$ (v) $30°$ (vi) $150°$ (vii) $37°$

9. (a) Use the $h \to 0$ method to show that the derivative of $f(x) = mx + b$ is $f'(x) = m$.
(b) Similarly show that the derivative of $f(x) = ax^2 + bx + c$ is $f'(x) = 2ax + b$.

10. (a) Sketch $f(x) = x^2 - 5x + 6$, then use the $u \to x$ method to show that the derivative is $2x - 5$. (b) Find the gradient at the y-intercept, the equation of the tangent there, and the x-intercept of that tangent.
(c) Find the gradients at the two x-intercepts $(2, 0)$ and $(3, 0)$ and show that they are opposites. Find the angles of inclination there and show that they are supplementary.
(d) Find, to the nearest minute, the angles of inclination when $x = 4$ and when $x = 1$.

11. Let $P(1, -3)$ and $Q\big(1 + h,\ (1 + h)^2 - 4\big)$ be two points on the graph of $f(x) = x^2 - 4$.
(a) Show that the gradient of the chord PQ is $2 + h$ and deduce that $f'(1) = 2$.
(b) Find the gradient of PQ when:
(i) $h = 2$ (ii) $h = -3$ (iii) $h = -2$ (iv) $h = 0{\cdot}01$
(c) Sketch the curve for $-2 \le x \le 3$, using a table of values, and add the chords PQ.

12. Use both the $u \to x$ method and the $h \to 0$ method to prove that:
(a) the derivative of $\dfrac{1}{x}$ is $-\dfrac{1}{x^2}$, (b) the derivative of $\dfrac{1}{x^2}$ is $-\dfrac{2}{x^3}$.

13. (a) Prove the identity $u - x = (\sqrt{u} + \sqrt{x})(\sqrt{u} - \sqrt{x})$, for positive values of u and x.
(b) Hence prove, using the $u \to x$ method, that the derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$.

14. Use the $u \to x$ method to show that the derivative of $f(x) = x^2 - ax$ is $f'(x) = 2x - a$, and find the value of a in the following cases: (a) the tangent at the origin has gradient 7, (b) $y = f(x)$ has a horizontal tangent at $x = 3$, (c) the tangent at the point where $x = 1$ has an angle of inclination of $45°$, (d) the tangent at the nonzero x-intercept has gradient 5, (e) the tangent at the vertex has y-intercept -9.

EXTENSION

15. [Algebraic differentiation of x^2] Let $P(a, a^2)$ be any point on the curve $y = x^2$, then the line ℓ through P with gradient m has equation $y - a^2 = m(x - a)$. Show that the x-coordinates of the points where ℓ meets the curve are $x = a$ and $x = m - a$. Find the value of the gradient m for which these two points coincide, and explain why it follows that the derivative of x^2 is $2x$.

16. [An alternative algebraic approach] Find the x-coordinates of the points where the line ℓ: $y = mx + b$ meets the curve $y = x^2$, and hence deduce that the derivative of x^2 is $2x$.

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7 C A Rule for Differentiating Powers of x

It was surely very obvious that the long calculations of the previous exercise had quite simple answers. Fortunately, there is a straightforward rule which allows the derivative of any power of x to be written down in one step.

6

> THEOREM: Let $f(x) = x^n$, where n is any real number.
> Then the derivative is $f'(x) = nx^{n-1}$.
>
> OR (expressing it as a process)
>
> Take the index as a factor, and reduce the index by 1.

The result will be proven in this section where n is a cardinal number or -1 or $\frac{1}{2}$. The proof will be extended in Section 7E to rational numbers — for the sake of convenience, however, the exercise of this section will use the general result for all real numbers. First, here are four examples of the theorem.

WORKED EXERCISE: Differentiate: (a) x^8 (b) x^{100} (c) x^{-4} (d) $x^{\frac{2}{3}}$

SOLUTION:

(a) $f(x) = x^8$
$f'(x) = 8x^7$

(b) $f(x) = x^{100}$
$f'(x) = 100x^{99}$

(c) $f(x) = x^{-4}$
$f'(x) = -4x^{-5}$

(d) $f(x) = x^{\frac{2}{3}}$
$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

Proof when n is a Cardinal Number: The result was proven in the last section for the cases where n was zero or 1. Suppose then that n is an integer with $n \ge 2$. The proof depends on the factorisation of the difference of nth powers, which was developed from partial sums of GPs in Section 6M of the previous chapter.

Using the definition,
$$\begin{aligned} f'(x) &= \lim_{u\to x} \frac{u^n - x^n}{u - x} \\ &= \lim_{u\to x} \frac{(u-x)(u^{n-1} + u^{n-2}x + \cdots + x^{n-1})}{u - x} \\ &= \lim_{u\to x} (u^{n-1} + u^{n-2}x + \cdots + x^{n-1}), \text{ since } u \ne x, \\ &= x^{n-1} + x^{n-1} + \cdots + x^{n-1} \quad (n \text{ terms}) \\ &= nx^{n-1}. \end{aligned}$$

The Derivatives of $1/x$ and $\sqrt{x}$: The derivatives of $1/x$ and $\sqrt{x}$ occur so often that they deserve special attention. Differentiating them from first principles:

A. Let $f(x) = \dfrac{1}{x}$.
$$\begin{aligned} f'(x) &= \lim_{u\to x} \frac{1/u - 1/x}{u - x} \times \frac{ux}{ux} \\ &= \lim_{u\to x} \frac{x - u}{ux(u - x)} \\ &= \lim_{u\to x} \frac{-1}{ux}, \text{ since } u \ne x, \\ &= -\frac{1}{x^2}, \text{ which is } -x^{-2}. \end{aligned}$$

B. Let $f(x) = \sqrt{x}$.
$$\begin{aligned} f'(x) &= \lim_{u\to x} \frac{\sqrt{u} - \sqrt{x}}{u - x} \\ &= \lim_{u\to x} \frac{\sqrt{u} - \sqrt{x}}{(\sqrt{u} - \sqrt{x})(\sqrt{u} + \sqrt{x})} \\ &= \lim_{u\to x} \frac{1}{\sqrt{u} + \sqrt{x}}, \text{ since } u \ne x, \\ &= \frac{1}{2\sqrt{x}}, \text{ which is } \tfrac{1}{2}x^{-\frac{1}{2}}. \end{aligned}$$

These are the same results as are obtained by applying the formula above for differentiating powers of x:

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A. $f(x) = \dfrac{1}{x}$
$= x^{-1}$
$f'(x) = -x^{-2}$

B. $f(x) = \sqrt{x}$
$= x^{\frac{1}{2}}$
$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

7 TWO SPECIAL FORMS: The derivative of $\dfrac{1}{x}$ is $-\dfrac{1}{x^2}$.

The derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$.

Linear Combinations of Functions: Compound functions formed by taking sums and multiples of simpler functions are quite straightforward to handle.

8 DERIVATIVE OF A SUM: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

DERIVATIVE OF A MULTIPLE: If $f(x) = kg(x)$, then $f'(x) = kg'(x)$.

PROOF:

For the first:

$$\begin{aligned} f'(x) &= \lim_{u\to x} \frac{f(u)-f(x)}{u-x} \\ &= \lim_{u\to x} \frac{g(u)+h(u)-g(x)-h(x)}{u-x} \\ &= \lim_{u\to x} \frac{g(u)-g(x)}{u-x} + \lim_{u\to x} \frac{h(u)-h(x)}{u-x} \\ &= g'(x) + h'(x). \end{aligned}$$

For the second:

$$\begin{aligned} f'(x) &= \lim_{u\to x} \frac{f(u)-f(x)}{u-x} \\ &= \lim_{u\to x} \frac{kg(u)-kg(x)}{u-x} \\ &= k \lim_{u\to x} \frac{g(u)-g(x)}{u-x} \\ &= kg'(x). \end{aligned}$$

WORKED EXERCISE: Differentiate: (a) $4x^2 - 3x + 2$ (b) $\frac{1}{2}x^6 - \frac{1}{6}x^3$ (c) $(2x-3)(3x-2)$

SOLUTION:

(a) $f(x) = 4x^2 - 3x + 2$
$f'(x) = 8x - 3$

(b) $f(x) = \frac{1}{2}x^6 - \frac{1}{6}x^3$
$f'(x) = 3x^5 - \frac{1}{2}x^2$

(c) $f(x) = (2x-3)(3x-2)$
$= 6x^2 - 13x + 6$
$f'(x) = 12x - 13$

WORKED EXERCISE: Differentiate: (a) $\dfrac{16}{x^3} - \dfrac{16}{x^2}$ (b) $\sqrt{5x}$ (c) $\dfrac{5}{12x}$

SOLUTION:

(a) $f(x) = \dfrac{16}{x^3} - \dfrac{16}{x^2}$
$= 16x^{-3} - 16x^{-2}$
$f'(x) = -48x^{-4} + 32x^{-3}$
$= -\dfrac{48}{x^4} + \dfrac{32}{x^3}$

(b) $f(x) = \sqrt{5x}$
$= \sqrt{5} \times \sqrt{x}$
$f'(x) = \dfrac{\sqrt{5}}{2\sqrt{x}}$

(c) $f(x) = \dfrac{5}{12x}$
$= \dfrac{5}{12} \times \dfrac{1}{x}$
$f'(x) = -\dfrac{5}{12x^2}$

Tangents and Normals to a Curve: Let P be a point on a curve $y = f(x)$. The tangent at P is, as we have said, the line through P with gradient equal to the derivative at P. The *normal* at P is defined to be the line through P perpendicular to the tangent at P. Equations of tangents and normals are easily calculated using the derivative.

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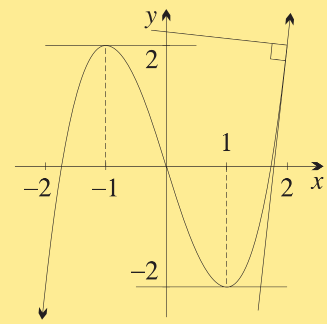
WORKED EXERCISE: Given that $f(x) = x^3 - 3x$, find the equations of the tangent and normal to the curve $y = f(x)$ at the point $P(2, 2)$ on the curve. Find also the points on the curve where the tangent is horizontal.

SOLUTION: Here $f'(x) = 3x^2 - 3$,
so at $P(2, 2)$, $f'(2) = 9$,
so the tangent has gradient 9 and the normal has gradient $-\frac{1}{9}$.
Hence the tangent is $y - 2 = 9(x - 2)$
$y = 9x - 16$,
and the normal is $y - 2 = -\frac{1}{9}(x - 2)$
$y = -\frac{1}{9}x + 2\frac{2}{9}$.
Also, the tangent has gradient zero when $3x^2 - 3 = 0$
$x = 1$ or -1,
so the tangent is horizontal at $(1, -2)$ and at $(-1, 2)$.



WORKED EXERCISE: Find the points on the graph of $f(x) = x + \dfrac{1}{x}$ where:

(a) the tangent is horizontal, (b) the normal has gradient -2,
(c) the tangent has angle of inclination $45°$.

SOLUTION: Since $f(x) = x + \dfrac{1}{x}$, $f'(x) = 1 - \dfrac{1}{x^2}$.

(a) Put $f'(x) = 0$, then $\dfrac{1}{x^2} = 1$
$x = 1$ or -1,
so the tangent is horizontal at $(1, 2)$ and $(-1, -2)$.

(b) When the normal has gradient -2, the tangent has gradient $\frac{1}{2}$,
so put $f'(x) = \frac{1}{2}$, then $\dfrac{1}{x^2} = \dfrac{1}{2}$
$x = \sqrt{2}$ or $-\sqrt{2}$,
so the normal has gradient -2 at $\left(\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$ and at $\left(-\sqrt{2}, -\frac{3}{2}\sqrt{2}\right)$.

(c) When the angle of inclination is $45°$, the tangent has gradient 1,
so put $f'(x) = 1$, then $\dfrac{1}{x^2} = 0$
which is impossible, so there is no such point.

Exercise 7C

1. Use the rule for differentiating x^n to differentiate (where a, b, c and ℓ are constants):

(a) $f(x) = x^7$
(b) $f(x) = 9x^5$
(c) $f(x) = \frac{1}{3}x^6$
(d) $f(x) = 3x^2 - 5x$
(e) $f(x) = x^4 + x^3 + x^2 + x + 1$
(f) $f(x) = 2 - 3x - 5x^3$
(g) $f(x) = \frac{1}{3}x^6 - \frac{1}{2}x^4 + x^2 - 2$
(h) $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 1$
(i) $f(x) = ax^4 - bx^2 + c$
(j) $f(x) = x^\ell$
(k) $f(x) = bx^{3b}$
(l) $f(x) = x^{5a+1}$

2. Find $f'(0)$ and $f'(1)$ for each function in the previous question.

3. Differentiate these functions by first expanding the products:

(a) $x(x^2 + 1)$
(b) $x^2(3 - 2x - 4x^2)$
(c) $(x + 4)(x - 2)$
(d) $(2x + 1)(2x - 1)$
(e) $(x^2 + 3)^2$
(f) $x(7 - x)^2$
(g) $(x^2 + 3)(x - 5)$
(h) $(ax - 5)^2$

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4. Use the rule for differentiating x^n to differentiate:
(a) $f(x) = 3x^{-1}$ (b) $f(x) = 5x^{-2}$ (c) $f(x) = -\frac{4}{3}x^{-3}$ (d) $f(x) = 2x^{-2} + \frac{1}{2}x^{-8}$

5. Write these functions using negative powers of x, then differentiate. Give the final answers in fractional form without negative indices.
(a) $f(x) = \dfrac{1}{x^2}$ (c) $f(x) = \dfrac{1}{2x^4}$ (e) $f(x) = \dfrac{c}{ax}$ (g) $f(x) = \dfrac{a}{x} - \dfrac{b}{x^2}$
(b) $f(x) = \dfrac{5}{x^3}$ (d) $f(x) = -\dfrac{3}{5x^5}$ (f) $f(x) = \dfrac{1}{x^6} - \dfrac{1}{x^8}$ (h) $f(x) = \dfrac{7}{2x^n}$

6. Use the fact that the derivative of $\dfrac{1}{x}$ is $-\dfrac{1}{x^2}$ to differentiate:
(a) $f(x) = \dfrac{3}{x}$ (b) $f(x) = \dfrac{1}{3x}$ (c) $f(x) = -\dfrac{7}{3x}$ (d) $f(x) = \dfrac{a}{x}$

7. Use the fact that the derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$ to differentiate:
(a) $f(x) = 3\sqrt{x}$ (b) $f(x) = 10\sqrt{x}$ (c) $f(x) = \sqrt{49x}$ (d) $f(x) = \sqrt{7x}$

8. Find the gradients of the tangent and normal at the point on $y = f(x)$ where $x = 3$:
(a) $f(x) = x^2 - 5x + 2$ (b) $f(x) = x^3 - 3x^2 - 10x$ (c) $f(x) = 2x^2 - 18x$ (d) $f(x) = 2\sqrt{x}$

9. Find the angles of inclination of the tangents and normals in the previous question.

10. Find the equations of the tangent and normal to the graph of $f(x) = x^2 - 8x + 15$ at:
(a) $A(1, 8)$ (b) $B(6, 3)$ (c) the y-intercept (d) $C(4, -1)$

11. Differentiate $f(x) = x^3$. Hence show that the tangents to $y = x^3$ have positive gradient everywhere except at the origin, and show that the tangent there is horizontal. Explain the situation using a sketch.

12. Find the equation of the tangent to $f(x) = 10x - x^3$ at the point $P(2, 12)$. Then find the points A and B where the tangent meets the x-axis and y-axis respectively, and find the length of AB and the area of $\triangle OAB$.

13. Find any points on the graph of each function where the tangent is parallel to the x-axis:
(a) $f(x) = 4 + 4x - x^2$ (c) $f(x) = 4ax - x^2$
(b) $f(x) = x^3 - 12x + 24$ (d) $f(x) = x^4 - 2x^2$

DEVELOPMENT

14. Find the tangent and normal to $f(x) = 12/x$ at: (a) $M(2, 6)$ (b) $N(6, 2)$

15. Show that the line $y = 3$ meets the parabola $y = 4 - x^2$ at $D(1, 3)$ and $E(-1, 3)$. Find the equations of the tangents to $y = 4 - x^2$ at D and E, and find the point where these tangents intersect. Sketch the situation.

16. The tangent and normal to $f(x) = 9 - x^2$ at the point $K(1, 8)$ meet the x-axis at A and B respectively. Sketch the situation, find the equations of the tangent and normal, find the coordinates of A and B, and hence find the length AB and the area of $\triangle AKB$.

17. The tangent and normal to the cubic $f(x) = x^3$ at the point $U(1, 1)$ meet the y-axis at P and Q respectively. Sketch the situation and find the equations of the tangent and normal. Find the coordinates of P and Q, and the area of $\triangle QUP$.

18. Find the derivative of the general quadratic $f(x) = ax^2 + bx + c$, and hence find the coordinates of the point on its graph where the tangent is horizontal.

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19. Show that the tangents at the x-intercepts of $f(x) = x^2 - 4x - 45$ have opposite gradients.

20. Find the derivative of the cubic $f(x) = x^3 + ax + b$, and hence find the x-coordinates of the points where the tangent is horizontal. For what values of a and b do such points exist?

21. [Change of pronumeral] (a) Find $G'(3)$, if $G(t) = t^3 - 4t^2 + 6t - 27$.
(b) Given that $\ell(H) = \frac{1}{H}$, find $\ell'(2)$.
(c) If $Q(k) = ak^2 - a^2k$, where a is constant, find $Q'(a)$, $Q'(0)$ and $|Q'(0) - Q'(a)|$.

22. Sketch the graph of $f(x) = x^2 - 6x$ and find the gradient of the tangent and normal at the point $A(a, a^2 - 6a)$ on the curve. Hence find the value of a if:
(a) the tangent has gradient (i) 0, (ii) 2, (iii) $\frac{1}{2}$,
(b) the normal has gradient (i) 4, (ii) $\frac{1}{4}$, (iii) 0,
(c) the tangent has angle of inclination 135°,
(d) the normal has angle of inclination 30°,
(e) the tangent is (i) parallel, (ii) perpendicular, to $2x - 3y + 4 = 0$.

23. (a) The tangent at $T(a, a^2)$ on the graph of $f(x) = x^2$ meets the x-axis at U and the y-axis at V. Find the equation of this tangent, and show that $\triangle OUV$ has area $|\frac{1}{4}a^3|$ square units. (b) Hence find the coordinates of T for which this area will be $31\frac{1}{4}$.

24. (a) Find the equation of the tangent to $y = x^2 + 9$ at the point P with x-coordinate x_0, and hence show that its x-intercept is $\dfrac{{x_0}^2 - 9}{2x_0}$. (b) Hence find the point(s) on the curve whose tangents pass through the origin. Draw a sketch of the situation.

25. Show that the equation of the tangent to $y = 1/x$ at the point $A(a, 1/a)$ is $x + a^2y = 2a$. Hence, with an explanatory sketch, find the point(s) where the tangent:
(a) has x-intercept 1,
(b) has y-intercept -1,
(c) passes through $(\frac{3}{2}, \frac{1}{2})$,
(d) passes through the origin.

26. (a) Find the equation of the tangent to $y = \sqrt{x} - 1$ at the point where $x = t$.
(b) Hence find t and the equation of the tangent if the tangent passes through the origin.
(c) Draw a sketch.

27. Using similar methods, find the points on $y = x^2 + 5$ where a line drawn from the origin can touch the curve (and draw a sketch of the situation).

28. Use the $u \to x$ method to differentiate $f(x) = x^7$ by first principles.

29. Differentiate $\sqrt{x}$ by the $h \to 0$ method, using the method of 'rationalising the numerator':

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}.$$

EXTENSION

30. Yet another formula for the derivative is $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h}$. Draw a diagram to justify this formula, then use it to find the derivatives of x^2, x^3, $1/x$ and $\sqrt{x}$.

31. The tangents to $y = x^2$ at two points $A(a, a^2)$ and $B(b, b^2)$ on the curve meet at K. Prove that the x-coordinate of K is the arithmetic mean of the x-coordinates of A and B, and the y-coordinate of K is the geometric mean of the y-coordinates of A and B.

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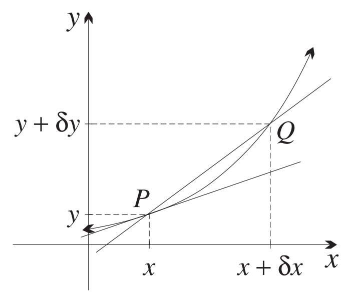
32. (a) Write down the equation of the tangent to the parabola $y = ax^2 + bx + c$ (where $a \neq 0$) at the point P where $x = x_0$, and show that the condition for the tangent at P to pass through the origin is $a{x_0}^2 - c = 0$. Hence find the condition on a, b and c for such tangents to exist, and the equations of these tangents. (b) Find the points A and B where the tangents from the origin touch the curve, and show that the y-intercept $C(0, c)$ is the midpoint of the interval joining the origin and the midpoint of the chord AB. Show also that the tangent at the y-intercept C is parallel to the chord AB. (c) Hence show that $\triangle OAB$ has four times the area of $\triangle OCA$, and find the area of $\triangle OAB$.

7 D The Notation $\dfrac{dy}{dx}$ for the Derivative

The purpose of this section is to introduce Leibniz's original notation for the derivative, which remains the most widely used and best known notation — it is even said that Dee Why Beach was named after the derivative dy/dx. The notation is extremely flexible, as will soon become evident, and clearly expresses the fact that the derivative is very like a fraction.

Small Changes in x and in y: Let $P(x, y)$ be any point on the graph of a function. Suppose that x changes by a small amount δx to $x + \delta x$, and let y change by a corresponding amount δy to $y + \delta y$. Let the new point be $Q(x + \delta x, y + \delta y)$. Then



$$\text{gradient } PQ = \frac{\delta y}{\delta x} \quad \text{(rise over run).}$$

When δx is small, the secant PQ is almost the same as the tangent at P, and, as before, the derivative is the limit of $\delta y/\delta x$ as $\delta x \to 0$. This is the basis for Leibniz's notation.

9 DEFINITION: Let δy be the small change in y resulting from a small change δx in x. Then the derivative dy/dx is

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

The object dx is intuitively understood as an 'infinitesimal change' in x, dy as the corresponding 'infinitesimal change' in y, and the derivative dy/dx as the ratio of these infinitesimal changes. Infinitesimal changes, however, are for the intuition only — the logic of the situation is that:

10 The derivative $\dfrac{dy}{dx}$ is not a fraction, but is the limit of the fraction $\dfrac{\delta y}{\delta x}$.

The genius of the notation is that the derivative is a gradient, and the gradient is a fraction, and the notation dy/dx preserves the intuition of fractions. The small differences δx and δy, and the infinitesimal differences dx and dy, are the origins of the word 'differentiation'.

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Operator Notation: The derivative $\dfrac{dy}{dx}$ can also be regarded as the *operator* $\dfrac{d}{dx}$ operating on the function y. This operator is also written as D_x, giving two further alternative notations for the derivative:

$$\frac{d}{dx}(x^2+x-1)=2x+1 \qquad \text{and} \qquad D_x(x^2+x-1)=2x+1.$$

WORKED EXERCISE: [These are examples of two further techniques used in differentiation — dividing through by the denominator, and using fractional indices.] Differentiate the following functions:

(a) $y=\dfrac{x^3+x^2+x+1}{x}$ (b) $y=6x\sqrt{x}$ (c) $y=\sqrt[3]{x^2}$ (d) $y=\dfrac{10x-2}{\sqrt{x}}$

SOLUTION:

(a) $$\begin{aligned} y&=\frac{x^3+x^2+x+1}{x}\\ &=x^2+x+1+\frac{1}{x}\\ \frac{dy}{dx}&=2x+1-\frac{1}{x^2}\end{aligned}$$

(b) $$\begin{aligned} y&=6x\sqrt{x}\\ &=6x^{1\frac{1}{2}}\\ \frac{dy}{dx}&=6\times\tfrac{3}{2}x^{\frac{1}{2}}\\ &=9\sqrt{x}\end{aligned}$$

(c) $$\begin{aligned} y&=\sqrt[3]{x^2}\\ &=x^{\frac{2}{3}}\\ \frac{dy}{dx}&=\tfrac{2}{3}x^{-\frac{1}{3}}\end{aligned}$$

(d) $$\begin{aligned} y&=\frac{10x-2}{\sqrt{x}}\\ &=10x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\\ \frac{dy}{dx}&=5x^{-\frac{1}{2}}+x^{-\frac{3}{2}}\\ &=\frac{5x+1}{x\sqrt{x}}\end{aligned}$$

WORKED EXERCISE: [These two worked exercises show how dy/dx notation is used to perform calculations on the geometry of a curve.] Find the equations of the tangent and normal to the curve $y=4-x^2$ at the point $P(1,3)$ on the curve.

SOLUTION: Here $\dfrac{dy}{dx}=-2x$, so at $P(1,3)$, $\dfrac{dy}{dx}=-2$.

Hence the tangent at P has gradient -2 and the normal has gradient $\frac{1}{2}$,

so the tangent is $y-3=-2(x-1)$

$y=-2x+5$,

and the normal is $y-3=\frac{1}{2}(x-1)$

$y=\frac{1}{2}x+2\frac{1}{2}$.

WORKED EXERCISE:

(a) Find the equation of the tangent to $y=x^2+x+1$ at $P(a,a^2+a+1)$.

(b) Hence find the equations of the tangents passing through the origin.

SOLUTION:

(a) Differentiating, $\dfrac{dy}{dx}=2x+1$,

so at P, $\dfrac{dy}{dx}=2a+1$,

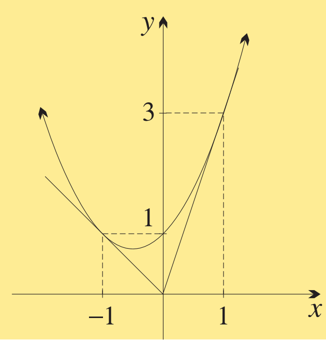
and the tangent is $y-(a^2+a+1)=(2a+1)(x-a)$

$y=(2a+1)x-a^2+1$.

(b) Substituting (0,0), $0=-a^2+1$

$a=1$ or -1,

and the tangents are $y=3x$ and $y=-x$.



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Exercise 7D

1. Find the derivative $\dfrac{dy}{dx}$ of each function, and the value of $\dfrac{dy}{dx}$ when $x = -1$:

(a) $y = x^4 - x^2 + 1$ (c) $y = (2x - 1)(x - 2)$ (e) $y = \dfrac{9}{x^3}$ (g) $y = \dfrac{a}{x} + \dfrac{a}{x^2}$

(b) $y = ax^2 + bx + c$ (d) $y = x^2(ax - c)$ (f) $y = 12\sqrt{x}$ (h) $y = \sqrt{121x}$

2. Differentiate each function by first dividing through by the denominator:

(a) $\dfrac{3x^4 - 5x^2}{x}$ (b) $\dfrac{5x^6 + 4x^5}{3x^3}$ (c) $\dfrac{4x^3 - 6}{2x^2}$ (d) $\dfrac{ax^3 - bx^2 + cx - d}{x^2}$

3. Find, in index form, the derivative $\dfrac{dy}{dx}$ of:

(a) $y = x^{2\frac{1}{2}}$ (b) $y = x^{-\frac{1}{2}}$ (c) $y = 4x^{\frac{3}{4}}$ (d) $y = 5x^{-\frac{2}{3}}$ (e) $y = -10x^{-0{\cdot}6}$

4. Differentiate each function by rewriting it using index form:

(a) $y = 12x\sqrt{x}$ (b) $y = 4x^2\sqrt{x}$ (c) $y = \dfrac{6}{\sqrt{x}}$ (d) $y = \dfrac{5}{x\sqrt{x}}$ (e) $y = 15\sqrt[5]{x}$

5. Using $\dfrac{dy}{dx}$ notation, find the tangent and normal to each curve at the point indicated:

(a) $y = x^2 - 6x$ at $O(0, 0)$,
(b) $y = \sqrt{x}$ at $K(4, 2)$,
(c) $y = x^2 - x^4$ at $J(-1, 0)$,
(d) $y = x^3 - 3x + 2$ at $P(1, 0)$.

6. Find any points on each graph where the tangent has gradient -1:

(a) $y = 1/x$ (b) $y = \frac{1}{2}x^{-2}$ (c) $y = 3 - \frac{1}{3}x^3$ (d) $y = x^3 + 1$ (e) $y = -\sqrt{x}$

7. Find any points on each curve where the tangent has the given angle of inclination:

(a) $y = \frac{1}{3}x^3 - 7$, $45°$ (b) $y = x^2 + \frac{1}{3}x^3$, $135°$ (c) $y = x^2 + 1$, $120°$ (d) $y = 2\sqrt{x}$, $30°$

8. Find the x-coordinates of any points on each curve where the normal is vertical:

(a) $y = 3 - 2x + x^2$ (b) $y = x^4 - 18x^2$ (c) $y = x + \dfrac{1}{x}$

DEVELOPMENT

9. (a) Find where $y = -2x$ meets $y = (x + 2)(x - 3)$.
(b) Find, to the nearest minute, the angles that the tangents to $y = (x + 2)(x - 3)$ at these points make with the x-axis. Sketch the situation.

10. Find, to four significant figures, the x-coordinate of the point where the tangent has the given angle of inclination:

(a) $y = x^2 + 3x$, $22°$ (b) $y = x^4$, $142°17'$ (c) $y = x^{-1}$, $70°$

11. For each curve below: (i) find the equation of the tangent at the point P where $x = a$, (ii) hence find the equations of any tangents passing through the origin.

(a) $y = x^2 - 10x + 9$ (b) $y = x^2 + 15x + 36$ (c) $y = 2x^2 - 7x + 6$

12. Differentiate $y = x^2 + bx + c$, and hence find b and c if:

(a) the parabola passes through the origin, and the tangent there has gradient 7,
(b) the parabola has y-intercept -3 and has gradient -2 there,
(c) the parabola is tangent to the x-axis at the point $(5, 0)$,
(d) when $x = 3$ the gradient is 5, and $x = 2$ is a zero,
(e) the parabola is tangent to $3x + y - 5 = 0$ at the point $T(3, -1)$,
(f) the line $3x + y - 5 = 0$ is a normal at the point $T(3, -1)$.

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13. Find the derivative $\frac{dy}{dx}$ of each of the following functions:

(a) $y = 3x^2\sqrt{x} - 2x\sqrt{x}$

(b) $y = 3x\sqrt{x} \times 4x^2\sqrt{x}$

(c) $y = 3\sqrt{2x} \times 2\sqrt{18x}$

(d) $y = \frac{1}{\pi}x^{\pi} + \pi x^{\frac{1}{\pi}}$

(e) $y = \dfrac{3x^2 - 2x + 4}{\sqrt{x}}$

(f) $y = \dfrac{x - 2\sqrt{x} + 1}{\sqrt{x}}$

(g) $y = \left(x + \dfrac{1}{x}\right)^2$

(h) $y = \sqrt{x^3}$

(i) $y = 4\left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2$

(j) $y = a\left(x^2 - \dfrac{1}{x^2}\right)^2$

14. For each of the following functions, find the value of $\frac{dy}{dx}$ when $x = 1$:

(a) $y = a^2x - ax^2$

(b) $y = \dfrac{a}{x} - \dfrac{x}{a}$

(c) $y = \left(\sqrt{x} - 3\right)\left(\sqrt{x} - 4\right)$

(d) $y = 3\left(4x^{-1} - 2x^{-2}\right)$

(e) $y = n^3x^2\left(nx + \dfrac{n}{x}\right)$

(f) $y = x^6 + x^5 + \cdots + 1$

(g) $y = 1 + x^{-1} + \cdots + x^{-6}$

(h) $y = \dfrac{1}{x^3\sqrt{x}}$

(i) $y = (2x)^n$

15. If $P = tx^2 + 3tu^2 + 3xu + t$, find $\frac{dP}{dx}$, $\frac{dP}{du}$ and $\frac{dP}{dt}$ (assuming that when differentiating with respect to one variable, the other pronumerals are constant).

16. The equation of the path of a ball thrown from the origin is $y = x(12 - x)$, with units in metres (the origin is at ground level). Sketch the curve and find its derivative, keeping in mind that the direction of motion at any point is the direction of the tangent at that point.

(a) How far from the origin does the ball land if the ground is level?

(b) Find the x-coordinate of the point H where the direction of motion is horizontal.

(c) Hence find the maximum height of the ball above the ground.

(d) Find at what angle the ball was initially thrown (find the gradient at O).

(e) Show that on level ground, it lands at the same acute angle to the ground.

(f) At what angle to the ground is the ball moving when it is at the point $P(2, 20)$?

(g) Show that the gradient of the flight path when $x = a$ is the opposite of the gradient of the flight path when $x = 12 - a$. What does this tell you about the two directions of flight?

(h) Let ℓ be the line of flight if there were no gravity to deflect the ball. Let A be the point on ℓ directly to the left of the point H, and B be the point on ℓ directly above H. Find the equation of ℓ and the distances HA and HB.

17. Show that the line $x + y + 2 = 0$ is a tangent to $y = x^3 - 4x$, and find the point of contact. [HINT: Find the equations of the tangents parallel to $x + y + 2 = 0$, and show that one of them is this very line.]

18. Find the tangent to the curve $y = x^4 - 4x^3 + 4x^2 + x$ at the origin, and show that this line is also the tangent to the curve at the point $(2, 2)$.

19. Find the points where the line $x + 2y = 4$ cuts the parabola $y = (x - 1)^2$, and show that the line is the normal to the curve at one of these points.

20. Find the equation of the tangent to $y = x^2 + 2x - 8$ at the point K on the curve with x-coordinate a. Hence find the points on the curve where the tangents from $H(2, -1)$ touch the curve.

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21. (a) If $y = Ax^n$, show that $x\dfrac{dy}{dx} = ny$. (b) If $y = \dfrac{C}{x^n}$, show that $x\dfrac{dy}{dx} = -ny$.

(c) (i) If $y = a\sqrt{x}$, show that $y\dfrac{dy}{dx}$ is a constant. (ii) Conversely, if $y = ax^n$ and $a \neq 0$, find $y\dfrac{dy}{dx}$ and show that it is constant if and only if $n = \frac{1}{2}$ or 0.

22. Find the equation of the tangent to the parabola $y = (x-3)^2$ at the point T where $x = \alpha$, find the coordinates of the x-intercept A and y-intercept B of the tangent, and find the midpoint M of AB. For what value of α does M coincide with T?

23. (a) Find the equation of the tangent to the hyperbola $xy = c$ at the point $T(t, c/t)$, find the points A and B where the tangent meets the x-axis and y-axis respectively, and show that A is independent of c.

(b) Find the area of $\triangle OAB$ and show that it is constant as T varies.

(c) Show that T bisects AB and that $OT = AT = BT$.

(d) Hence explain why the rectangle with diagonal OT has a constant area that is half the area of $\triangle OAB$.

(e) Find the perpendicular distance from the tangent to O, and the length AB, and hence calculate the area of $\triangle OAB$ by this alternative method. Draw sketches for c and t positive and negative.

24. [For discussion] Sketch the graph of $y = x^3$. Then choose any point in the plane and check by examining the graph that at least one tangent to the curve passes through every point in the plane. What points in the plane have three tangents to the curve passing through them? This problem can also be solved algebraically, but that is considerably harder.

EXTENSION

25. (a) Show that the tangent to $\mathcal{P}$: $y = ax^2 + bx + c$ with gradient m has y-intercept $c - \dfrac{(m-b)^2}{4a}$. (b) Hence find the equations of any quadratics that pass through the origin and are tangent to both $y = -2x - 4$ and to $y = 8x - 49$. (c) Find also any quadratics that are tangent to $y = -5x - 10$, to $y = -3x - 7$ and to $y = x - 7$.

26. Let $y = ax^3 + bx^2 + cx + d$ be a cubic (so that $a \neq 0$). Show that every point in the plane lies on at least one tangent to this cubic.

7 E The Chain Rule

Sections 7E, 7F and 7G will develop three methods that extend the rules for differentiation to cover compound functions of various types.

A Chain of Functions: The semicircle function $y = \sqrt{25 - x^2}$ is the composition of two functions — 'square and subtract from 25', followed by 'take the positive square root'. We can represent the situation by a chain of functions:

x				u				y
0	$\longrightarrow$	Square and subtract from 25	$\longrightarrow$	25	$\longrightarrow$	Take the positive square root	$\longrightarrow$	5
3	$\longrightarrow$		$\longrightarrow$	16	$\longrightarrow$		$\longrightarrow$	4
-4	$\longrightarrow$		$\longrightarrow$	9	$\longrightarrow$		$\longrightarrow$	3
x	$\longrightarrow$		$\longrightarrow$	$25 - x^2$	$\longrightarrow$		$\longrightarrow$	$\sqrt{25 - x^2}$

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The middle column is the output of the first function 'subtract the square from 25', and is then the input of the second function 'take the positive square root'. This decomposition of the original function $y = \sqrt{25 - x^2}$ into the chain of functions may be expressed as follows:

'Let $u = 25 - x^2$, then $y = \sqrt{u}$.'

The Chain Rule: Suppose then that y is a function of u, where u is a function of x. Using the dy/dx notation for the derivative:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x}\\ &= \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}\right) \quad \text{(multiplying top and bottom by } \delta u\text{)}\\ &= \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \quad \text{(because } \delta u \to 0 \text{ as } \delta x \to 0\text{)}\\ &= \frac{dy}{du} \times \frac{du}{dx}.\end{aligned}$$

In practice, although the proof uses limits, the usual attitude to this rule is that 'the du's cancel out'. The chain rule should thus be remembered in the form:

11 THE CHAIN RULE: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

WORKED EXERCISE: Use the chain rule to differentiate the functions:

(a) $(x^2 + 1)^6$ (b) $7(3x + 4)^5$ (c) $(ax + b)^n$ (d) $\sqrt{25 - x^2}$

NOTE: The working in the right-hand column is a recommended way to set out the calculation. The calculation should begin with that working, because the first step is the decomposition of the function into a chain of two functions.

SOLUTION:

(a) Let $y = (x^2 + 1)^6$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 6(x^2 + 1)^5 \times 2x$

$= 12x(x^2 + 1)^5$.

Let $u = x^2 + 1$,
then $y = u^6$.
So $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = 6u^5$.

(b) Let $y = 7(3x + 4)^5$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 35(3x + 4)^4 \times 3$

$= 105(3x + 4)^4$.

Let $u = 3x + 4$,
then $y = 7u^5$.
So $\dfrac{du}{dx} = 3$
and $\dfrac{dy}{du} = 35u^4$.

(c) Let $y = (ax + b)^n$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= n(ax + b)^{n-1} \times a$

$= an(ax + b)^{n-1}$.

Let $u = ax + b$,
then $y = u^n$.
So $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = nu^{n-1}$.

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(d) Let $y = \sqrt{25 - x^2}$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= \dfrac{1}{2\sqrt{25 - x^2}} \times (-2x)$

$= -\dfrac{x}{\sqrt{25 - x^2}}$,

Let $u = 25 - x^2$,
then $y = \sqrt{u}$.
So $\dfrac{du}{dx} = -2x$
and $\dfrac{dy}{du} = \dfrac{1}{2\sqrt{u}}$.

which agrees with the calculation by geometric methods in Section 7A.

Powers of a Linear Function: Part (c) of the previous exercise should be remembered as a formula for differentiating any linear function of x raised to a power.

12 POWERS OF A LINEAR FUNCTION: $\dfrac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$

WORKED EXERCISE: Use the standard form above to differentiate:

(a) $(4x - 1)^7$ (b) $\sqrt{5 - 3x}$ (c) $\dfrac{1}{7 - x}$

SOLUTION:

(a) $\dfrac{d}{dx}(4x - 1)^7 = 28(4x - 1)^6$ (with $a = 4$, $b = -1$ and $n = 7$).

(b) $\dfrac{d}{dx}\sqrt{5 - 3x} = \dfrac{-3}{2\sqrt{5 - 3x}}$ (with $a = -3$, $b = 5$ and $n = \frac{1}{2}$).

(c) $\dfrac{d}{dx}(7 - x)^{-1} = (7 - x)^{-2}$ (with $a = -1$, $b = 7$ and $n = -1$).

Parametric Differentiation: In many later situations, a curve will be specified by two equations giving x and y in terms of some third variable t, called a *parameter*. For example,

$$x = 2t, \qquad y = t^2$$

specifies the parabola $y = \frac{1}{4}x^2$, as can be seen by eliminating t from the two equations. In this situation it is very simple to calculate dy/dx directly using *parametric differentiation*. The formula below is another version of the chain rule, because 'the dt's just cancel out'.

13 PARAMETRIC FUNCTIONS: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

WORKED EXERCISE: In the example above, $\dfrac{dy}{dx} = \dfrac{2t}{2}$

$= t$.

Differentiating Inverse Functions: Suppose that y is a function of x, and that the inverse is also a function, so that x is a function of y. Then by the chain rule, $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$, and so:

14 INVERSE FUNCTIONS: $\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$ (provided neither is zero).

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WORKED EXERCISE: Differentiate $y = x^{\frac{1}{3}}$: (a) directly, (b) by first forming the inverse function and then differentiating.

SOLUTION:

(a) Using the usual rule for differentiating powers of x,

$$\frac{dy}{dx} = \tfrac{1}{3}x^{-\frac{2}{3}}.$$

(b) Solving for x, $\quad x = y^3$.

Then $\quad \dfrac{dx}{dy} = 3y^2$,

and taking reciprocals, $\quad \dfrac{dy}{dx} = \dfrac{1}{3y^2} = \tfrac{1}{3}x^{-\frac{2}{3}}$.

Completion of Proof that x^n has Derivative nx^{n-1}: The chain rule allows us to complete the proof of the derivative of x^n, at least for rational values of the index n.

THEOREM: $\dfrac{d}{dx}x^n = nx^{n-1}$, for all rational values of n.

PROOF: The result is already proven when n is a cardinal number, when $n = \frac{1}{2}$, and when $n = -1$.

A. Suppose that $y = x^{-m}$, where $m \geq 2$ is an integer.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \left(-\frac{1}{x^{2m}}\right) \times mx^{m-1} \\ &= -mx^{-m-1}, \quad \text{as required.} \end{aligned}$$

Let $u = x^m$,

then $y = \dfrac{1}{u}$.

So $\dfrac{du}{dx} = mx^{m-1}$,

and $\dfrac{dy}{du} = -\dfrac{1}{u^2}$ (proven earlier).

B. Suppose that $y = x^{\frac{1}{k}}$, where $k \geq 2$ is an integer.

Then $\quad x = y^k$,

so $\quad \dfrac{dx}{dy} = ky^{k-1}$, since k is a positive integer,

and $\quad \dfrac{dy}{dx} = \dfrac{1}{ky^{k-1}}$, since $\dfrac{dy}{dx}$ is the reciprocal of $\dfrac{dx}{dy}$,

$$= \frac{1}{kx^{\frac{k-1}{k}}}$$

$$= \tfrac{1}{k}x^{\frac{1}{k}-1}, \quad \text{as required.}$$

C. Suppose that $y = x^{\frac{m}{k}}$, where m and k are integers and $k \geq 2$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= mx^{\frac{m-1}{k}} \times \frac{1}{k}x^{\frac{1}{k}-1} \\ &= \frac{m}{k}x^{\frac{m}{k}-1}, \quad \text{as required.} \end{aligned}$$

Let $u = x^{\frac{1}{k}}$,

then $y = u^m$.

So $\dfrac{du}{dx} = \dfrac{1}{k}x^{\frac{1}{k}-1}$, (by B),

and $\dfrac{dy}{du} = mu^{m-1}$.

NOTE ON IRRATIONAL INDICES: We do not have a precise definition of powers like x^π or $x^{\sqrt{2}}$ with irrational indices, so we can hardly give a rigorous proof that the derivative of x^n is indeed nx^{n-1} for irrational values of n. Nevertheless, since every irrational is 'as close as we like' to a rational number for which the theorem is certainly true, the result is intuitively clear.

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Exercise 7E

1. Use the chain rule to differentiate each function. Be careful in each example to identify u as a function of x, and y as a function of u.

(a) $y = (3x + 7)^4$
(b) $y = (5 - 4x)^7$
(c) $y = (px + q)^8$
(d) $y = (x^2 + 1)^{12}$
(e) $y = 8(7 - x^2)^4$
(f) $y = (x^2 + 3x + 1)^9$
(g) $y = -3(x^3 + x + 1)^6$
(h) $y = \sqrt{5x + 4}$
(i) $y = \sqrt{3 - 2x}$
(j) $y = 7\sqrt{x^2 + 1}$
(k) $y = \sqrt{9 - x^2}$
(l) $y = -\sqrt{a^2 - b^2x^2}$

2. Use the standard form $\dfrac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate:

(a) $y = (5x - 7)^5$
(b) $y = (4 - 3x)^7$
(c) $y = (2 - 3x)^{-5}$
(d) $y = p(q - x)^{-4}$
(e) $y = \dfrac{1}{2 - x}$
(f) $y = \dfrac{1}{3 + 5x}$
(g) $y = -\dfrac{5}{(x + 1)^3}$
(h) $y = \sqrt{x + 4}$
(i) $y = \sqrt{4 - 3x}$
(j) $y = \sqrt{mx - b}$
(k) $y = (5 - x)^{-\frac{1}{2}}$

3. Use parametric differentiation to find dy/dx, then evaluate dy/dx when $t = -1$:

(a) $x = 5t$, $y = 10t^2$
(b) $x = ct$, $y = c/t$
(c) $x = at + b$, $y = bt + a$
(d) $x = 2t^2$, $y = 3t^3$

4. Differentiate each function, and hence find the coordinates of any points where the tangent is horizontal:

(a) $y = (x^2 - 1)^3$
(b) $y = (x^2 - 4x)^4$
(c) $y = (2x + x^2)^5$
(d) $y = \dfrac{1}{5x + 2}$
(e) $y = 24 - 7(x - 5)^2$
(f) $y = 4 + (x - 5)^6$
(g) $y = a(x - h)^2 + k$
(h) $y = \sqrt{3 - 2x}$
(i) $y = \dfrac{1}{1 + x^2}$
(j) $y = \sqrt{x^2 - 2x + 5}$
(k) $y = \sqrt{x^2 - 2x}$

5. Find the equations of the tangent and normal at the point where $x = 1$ to:

(a) $y = (5x - 4)^4$
(b) $y = (x^2 + 1)^3$
(c) $y = (x^2 + 1)^{-1}$
(d) $y = \sqrt{x - 2}$

6. Find the x-coordinates of any points on $y = (4x - 7)^3$ where the tangent is:

(a) parallel to $y = 108x + 7$
(b) perpendicular to $x + 12y + 6 = 0$

DEVELOPMENT

7. Find the tangent to each curve at the point where $t = 3$:

(a) $x = 5t^2$, $y = 10t$
(b) $x = (t - 1)^2$, $y = (t - 1)^3$

8. Find the value of a if:

(a) $y = \dfrac{1}{x + a}$ has gradient -1 when $x = 6$,
(b) $y = (x - a)^3$ has gradient 12 when $x = 6$.

9. (a) Find the equation of the tangent to $y = \dfrac{1}{x - 4}$ at the point L where $x = b$.

(b) Hence find the equations of the tangents passing through:

(i) the origin,
(ii) $W(6, 0)$.

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10. Differentiate:

(a) $y = \left(\sqrt{x} - 3\right)^{11}$

(b) $y = 3\sqrt{4 - \frac{1}{2}x}$

(c) $y = \dfrac{3}{1 - x\sqrt{2}}$

(d) $y = (5 - x)^{-\frac{1}{2}}$

(e) $y = \dfrac{-a}{\sqrt{1 + ax}}$

(f) $y = \dfrac{b}{\sqrt{c - \frac{1}{2}x}}$

(g) $y = -4\left(x + \dfrac{1}{x}\right)^4$

(h) $y = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^6$

11. Find the values of a and b if the parabola $y = a(x + b)^2 - 8$:

(a) has tangent $y = 2x$ at the point $P(4, 8)$,

(b) has a common tangent with $y = 2 - x^2$ at the point $A(1, 1)$.

12. Use the chain rule to show that: (a) $\dfrac{d}{dx}(x^2)^3 = 6x^5$ (b) $\dfrac{d}{dx}(x^k)^\ell = k\ell x^{k\ell - 1}$

13. (a) Differentiate the semicircle $y = \sqrt{169 - x^2}$, find the equation of the tangent at $P(12, 5)$, and find the x-intercept and y-intercept of the tangent.

(b) Show that the perpendicular distance from the tangent to the centre equals the radius.

(c) Find the area of the triangle enclosed by the tangent and the two axes.

(d) Find the perimeter of this triangle.

14. (a) Let the point $P(4, 3)$ lie on the semicircle $y = \sqrt{25 - x^2}$, and let $Q(4, \frac{9}{5})$ lie on the curve $y = \frac{3}{5}\sqrt{25 - x^2}$ (which is half an ellipse). Find the equations of the tangents at P and at Q, and show that they intersect on the x-axis.

(b) Find the equation of the tangent at the point P with x-coordinate $x_0 > 0$ on the curve $y = \lambda\sqrt{25 - x^2}$ (again, half an ellipse). Let the tangent meet the x-axis at T, let the ellipse meet the x-axis at $A(5, 0)$, and let the vertical line through P meet the x-axis at M. Show that the point T is independent of λ, and show that OA is the geometric mean of OM and OT.

15. (a) Find the x-coordinates of the points P and Q on $y = (x - 7)^2 + 3$ such that the tangents at P and Q have gradients 1 and -1 respectively.

(b) Show that the square formed by the tangents and normals at P and Q has area $\frac{1}{2}$.

EXTENSION

16. (a) Find the x-coordinates of the points P and Q on $y = (x - h)^2 + k$ such that the tangents at P and Q have gradients m and $-m$ respectively.

(b) Find the area of the quadrilateral formed by the tangents and normals at P and Q.

17. (a) Show that the tangent to $\mathcal{P}$: $y = a(x - h)^2 + k$ at the point T where $x = \alpha$ is $y = 2a(\alpha - h)x + k - a(\alpha^2 - h^2)$.

(b) Hence show that the vertical distance between the vertex $V(h, k)$ and the tangent at T is proportional to the square of the distance between α and the axis of symmetry.

(c) Find the equations of the tangents to $\mathcal{P}$ through the origin, and the x-coordinates of the points of contact.

18. (a) Develop a three-step chain rule for the derivative dy/dx, where y is a function of u, u is a function of v, and v is a function of x. Hence differentiate $y = \dfrac{1}{1 + \sqrt{1 - x^2}}$.

(b) Generalise the chain rule to n steps.

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7 F The Product Rule

The product rule extends the methods for differentiation to cover functions that are products of two simpler functions. Suppose then that $y = uv$ is the product of two functions u and v, each of which is a function of x. Then, as we shall prove after the worked exercise:

15 DERIVATIVE OF A PRODUCT: $\dfrac{dy}{dx} = v\,\dfrac{du}{dx} + u\,\dfrac{dv}{dx}$ or $y' = vu' + uv'$

The second form uses the convention of the dash $'$ to represent differentiation with respect to x, so $y' = dy/dx$ and $u' = du/dx$ and $v' = dv/dx$.

NOTE: The product rule can seem difficult to use with the algebraic functions under consideration at present, because the calculations can easily become quite involved. The rule will seem more straightforward later, in the context of exponential and trigonometric functions.

WORKED EXERCISE: Differentiate each function, expressing the result in fully factored form. Then state for what value(s) of x the derivative is zero.

(a) $x(x-10)^4$ (b) $x^2(3x+2)^3$ (c) $x\sqrt{x+3}$

SOLUTION:

(a) Let $y = x(x-10)^4$.

Then $\dfrac{dy}{dx} = v\,\dfrac{du}{dx} + u\,\dfrac{dv}{dx}$

$= (x-10)^4 \times 1 + x \times 4(x-10)^3$

$= (x-10)^3(x-10+4x)$

$= 5(x-10)^3(x-2)$.

So the derivative is zero for $x = 10$ and for $x = 2$.

Let $u = x$

and $v = (x-10)^4$.

Then $\dfrac{du}{dx} = 1$

and $\dfrac{dv}{dx} = 4(x-10)^3$.

(b) Let $y = x^2(3x+2)^3$.

Then $y' = vu' + uv'$

$= 2x(3x+2)^3 + 9x^2(3x+2)^2$

$= x(3x+2)^2(6x+4+9x)$

$= x(3x+2)^2(15x+4)$.

So the derivative is zero for $x = 0$, $x = -\frac{2}{3}$ and for $x = -\frac{4}{15}$.

Let $u = x^2$

and $v = (3x+2)^3$.

Then $u' = 2x$

and $v' = 9(3x+2)^2$.

(c) Let $y = x\sqrt{x+3}$.

Then $\dfrac{dy}{dx} = v\,\dfrac{du}{dx} + u\,\dfrac{dv}{dx}$

$= \sqrt{x+3} + \dfrac{x}{2\sqrt{x+3}}$

$= \dfrac{2(x+3)+x}{2\sqrt{x+3}}$ (common denominator)

$= \dfrac{3(x+2)}{2\sqrt{x+3}}$, which is zero for $x = -2$.

Let $u = x$

and $v = \sqrt{x+3}$.

Then $\dfrac{du}{dx} = 1$

and $\dfrac{dv}{dx} = \dfrac{1}{2\sqrt{x+3}}$.

Proof of the Product Rule: Suppose that x changes to $x + \delta x$, and that as a result, u changes to $u + \delta u$, v changes to $v + \delta v$, and y changes to $y + \delta y$.

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Here $\qquad y = uv,$

and $\qquad y + \delta y = (u + \delta u)(v + \delta v)$

$\qquad\qquad = uv + v\,\delta u + u\,\delta v + \delta u\,\delta v,$

so $\qquad \delta y = v\,\delta u + u\,\delta v + \delta u\,\delta v.$

Hence, dividing by δx, $\qquad \dfrac{\delta y}{\delta x} = v\,\dfrac{\delta u}{\delta x} + u\,\dfrac{\delta v}{\delta x} + \dfrac{\delta u}{\delta x} \times \dfrac{\delta v}{\delta x} \times \delta x,$

and taking limits as $\delta x \to 0$, $\qquad \dfrac{dy}{dx} = v\,\dfrac{du}{dx} + u\,\dfrac{dv}{dx} + 0,$ as required.

Exercise 7F

1. Differentiate each function: (i) by expanding the product and differentiating each term, (ii) using the product rule.

(a) $y = x^3(x - 2)$ (b) $y = (2x + 1)(x - 5)$ (c) $y = (x^2 - 3)(x^2 + 3)$

2. Differentiate these functions using the product rule, identifying the factors u and v in each example. Express your answers in fully factored form, and state the values of x for which the derivative is zero.

(a) $y = x(3 - 2x)^5$ (c) $y = x^5(1 - x)^7$ (e) $y = 2(x + 1)^3(x + 2)^4$

(b) $y = x^3(x + 1)^4$ (d) $y = (x - 1)(x - 2)^3$ (f) $y = (2x - 3)^4(2x + 3)^5$

3. Find the tangents and normals to these curves at the indicated points:

(a) $y = x(1 - x)^6$ at the origin (b) $y = (2x - 1)^3(x - 2)^4$ at $A(1, 1)$

DEVELOPMENT

4. Differentiate each function using the product rule, giving your answer in fully factored form. At least one of the factors will require the chain rule to differentiate it.

(a) $y = x(x^2 + 1)^5$ (c) $y = -2(x^2 + x + 1)^3 x$

(b) $y = 2\pi x^3(1 - x^2)^4$ (d) $y = (2 - 3x^2)^4(2 + 3x^2)^5$

5. Differentiate $y = (x^2 - 10)^3 x^4$, using the chain rule to differentiate the first factor. Hence find the points on the curve where the tangent is horizontal.

6. Differentiate each function using the product rule, combining terms using a common denominator and factoring the numerator completely. State the values of x for which the derivative is zero.

(a) $y = 6x\sqrt{x + 1}$ (b) $y = -4x\sqrt{1 - 2x}$ (c) $y = 10x^2\sqrt{2x - 1}$

7. (a) What is the domain of $y = x\sqrt{1 - x^2}$?

(b) Differentiate $y = x\sqrt{1 - x^2}$, using the chain rule to differentiate the second factor, then combine the terms using a common denominator.

(c) Find the points on the curve where the tangent is horizontal.

(d) Find the tangent and the normal at the origin.

8. (a) Differentiate $y = a(x - \alpha)(x - \beta)$ using the product rule.

(b) Show that the tangents at the x-intercepts have opposite gradients and meet at a point M whose x-coordinate is the average of the x-intercepts.

(c) Find the point V where the tangent is horizontal. Show that M is vertically above or below V and twice as far from the x-axis. Sketch the situation.

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9. Show that if a polynomial $f(x)$ can be written as a product $f(x) = (x-a)^n q(x)$ of the polynomials $(x-a)^n$ and $q(x)$, where $n \ge 2$, then $f'(x)$ can be written as a multiple of $(x-a)^{n-1}$. What does this say about the shape of the curve near $x = a$?

10. Show that the function $y = x^3(1-x)^5$ has a horizontal tangent at a point P with x-coordinate $\frac{3}{8}$. Show that the y-coordinate of P is $3^3 \times 5^5/8^8$.

11. Prove by mathematical induction that for all positive integers n, $\dfrac{d}{dx}x^n = nx^{n-1}$. Use only the product rule, and the fact that the derivative of the identity function $f(x) = x$ is $f'(x) = 1$.

EXTENSION

12. (a) Show that the function $y = x^r(1-x)^s$, where $r, s > 1$, has a horizontal tangent at a point P whose x-coordinate p lies between 0 and 1.
(b) Show that P divides the interval joining $O(0,0)$ and $A(1,0)$ in the ratio $r : s$, and find the y-coordinate of P. What are the coordinates of P if $s = r$?

13. Establish the rule for differentiating a product $y = uvw$, where u, v and w are functions of x. Hence find the derivative of these functions, and the values of x where the tangent is horizontal: (a) $x^5(x-1)^4(x-2)^3$ (b) $x(x-2)^4\sqrt{2x+1}$

14. Establish the rule for differentiating a product $y = u_1 u_2 \ldots u_n$ of n functions of x.

7 G The Quotient Rule

The last of these three methods extends the formulae for differentiation to cover functions that are quotients of two simpler functions. Suppose that $y = u/v$ is the quotient of two functions u and v, each of which is a function of x. Then we shall prove, again after the worked exercise:

16 DERIVATIVE OF A QUOTIENT: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ or $y' = \dfrac{vu' - uv'}{v^2}$

WORKED EXERCISE: Differentiate, stating when the derivative is zero:

(a) $\dfrac{2x+1}{2x-1}$ (b) $\dfrac{\sqrt{x+1}}{x}$

NOTE: Although these functions could be differentiated using the product rule by expressing them as $(2x+1)(2x-1)^{-1}$ and $x^{-1}\sqrt{x+1}$, the quotient rule makes the work much easier.

SOLUTION:

(a) Let $y = \dfrac{2x+1}{2x-1}$.

Then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$= \dfrac{2(2x-1) - 2(2x+1)}{(2x-1)^2}$

$= \dfrac{-4}{(2x-1)^2}$, which is never zero.

Let $u = 2x+1$
and $v = 2x-1$.
Then $\dfrac{du}{dx} = 2$
and $\dfrac{dv}{dx} = 2$.

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(b) Let $y = \dfrac{\sqrt{x+1}}{x}$.

Let $u = \sqrt{x+1}$
and $v = x$.
Then $u' = \dfrac{1}{2\sqrt{x+1}}$
and $v' = 1$.

$$\begin{aligned}
\text{Then } y' &= \frac{vu' - uv'}{v^2} \\
&= \frac{\dfrac{x}{2\sqrt{x+1}} - \sqrt{x+1}}{x^2} \times \frac{2\sqrt{x+1}}{2\sqrt{x+1}} \\
&= \frac{x - 2(x+1)}{2x^2\sqrt{x+1}} \\
&= \frac{-x-2}{2x^2\sqrt{x+1}},
\end{aligned}$$

which has no zeroes ($x = -2$ is outside the domain).

Proof of Quotient Rule: We differentiate uv^{-1} using the product rule.

Let $y = uv^{-1}$.

$$\begin{aligned}
\text{Then } \frac{dy}{dx} &= V\frac{dU}{dx} + U\frac{dV}{dx} \\
&= v^{-1}\frac{du}{dx} - uv^{-2}\frac{dv}{dx} \\
&= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}
\end{aligned}$$

(after multiplying by $\dfrac{v^2}{v^2}$).

Let $U = u$
and $V = v^{-1}$.
Then $\dfrac{dU}{dx} = \dfrac{du}{dx}$
and by the chain rule,

$$\begin{aligned}
\frac{dV}{dx} &= \frac{dV}{dv} \times \frac{dv}{dx} \\
&= -v^{-2}\frac{dv}{dx}.
\end{aligned}$$

Exercise 7G

1. Differentiate each function using the quotient rule, taking care to identify u and v first. Express your answer in fully factored form, and state any values of x for which the tangent is horizontal.

(a) $y = \dfrac{x+1}{x-1}$ (b) $y = \dfrac{2x}{x+2}$ (c) $y = \dfrac{3-2x}{x+5}$ (d) $y = \dfrac{x^2}{1-x}$ (e) $y = \dfrac{x^2-1}{x^2+1}$ (f) $y = \dfrac{mx+b}{bx+m}$ (g) $y = \dfrac{x^2-a}{x^2-b}$ (h) $y = \dfrac{x^n-3}{x^n+3}$

2. Differentiate $y = \dfrac{1}{3x-2}$:

(a) by using the chain rule with $u = 3x-2$ and $y = \dfrac{1}{u}$ (the better method),

(b) by using the quotient rule with $u = 1$ and $v = 3x-2$.

3. Differentiate $y = \dfrac{5+2x}{5-2x}$:

(a) by using the quotient rule (the better method),

(b) by using the product rule, with the function in the form $y = (5+2x)(5-2x)^{-1}$.

4. For each curve below, find the equations of the tangent and normal and their angles of inclination at the given point:

(a) $y = \dfrac{x}{5-3x}$ at $K(2,-2)$ (b) $y = \dfrac{x^2-4}{x-1}$ at $L(4,4)$

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DEVELOPMENT

5. Differentiate, stating any zeroes of the derivative: (a) $\dfrac{\sqrt{x}+1}{\sqrt{x}+2}$ (b) $\dfrac{x-3}{\sqrt{x+1}}$

6. (a) Find the value of c if $f'(c) = -3$, where $f(x) = \dfrac{x^2}{x+1}$.

(b) Find the value of k if $f'(-3) = 1$, where $f(x) = \dfrac{x^2+k}{x^2-k}$.

7. (a) Differentiate $y = \dfrac{x-\alpha}{x-\beta}$.

(b) Show that for $\alpha > \beta$, all tangents have positive gradient, and for $\alpha < \beta$, all tangents have negative gradient.

(c) What happens when $\alpha = \beta$?

8. (a) Find the normal to the curve $x = \dfrac{t}{t+1}$ and $y = \dfrac{t}{t-1}$ at the point T where $t = 2$.

(b) Eliminate t from the two equations (by solving the first for t and substituting into the second). Then differentiate this equation to find the gradient of the normal at T.

9. (a) Evaluate $f'(8)$ if $f(x) = \dfrac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}}$. (b) Evaluate $g'(5)$ if $g(r) = \dfrac{(r-6)^3-1}{(r-4)^3+1}$.

10. (a) Sketch the hyperbola $y = \dfrac{x}{x+1}$, showing the horizontal and vertical asymptotes, and state its domain and range.

(b) Show that the tangent at the point P where $x = a$ is $x - (a+1)^2y + a^2 = 0$.

(c) Let the tangent at $A(1, \frac{1}{2})$ meet the x-axis at I, and let G be the point on the x-axis below A. Show that the origin bisects GI.

(d) Let $T(c, 0)$ be any point on the x-axis. (i) Show that for $c > 0$, no tangents pass through T. (ii) Show that for $c < 0$ and $c \neq -1$, there are two tangents through T whose x-coordinates of their points of contact are opposites of each other. For what values of c are these two points of contact on the same and on different branches of the hyperbola?

11. (a) Suppose that $y = \dfrac{u}{x}$, where u is a function of x. Show that $y + x\dfrac{dy}{dx} = \dfrac{du}{dx}$.

(b) Suppose that $y = \dfrac{x}{u}$, where u is a function of x. Show that $y\dfrac{du}{dx} + u\dfrac{dy}{dx} = 1$.

EXTENSION

12. Sketch a point P on a curve $y = f(x)$ where x, $f(x)$ and $f'(x)$ are all positive. Let the tangent, normal and vertical at P meet the x-axis at T, N and M respectively. Let the (acute) angle of inclination of the tangent be $\theta = \angle PTN$, so that $y' = \tan\theta$.

(a) Using trigonometry, show that:

(i) $MN = yy'$
(ii) $TM = y/y'$
(iii) $\sec\theta = \sqrt{1+y'^2}$
(iv) $\operatorname{cosec}\theta = \sqrt{1+y'^2}\Big/y'$
(v) $PN = y\sqrt{1+y'^2}$
(vi) $PT = y\sqrt{1+y'^2}\Big/y'$

(b) Hence find the four lengths when $x = 3$ and: (i) $y = x^2$ (ii) $y = \dfrac{3x-1}{x+1}$

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7 H Rates of Change

The derivative has been defined geometrically in this chapter using tangents to the curve, but ever since its introduction the derivative has always been understood also as a rate of change. Let the variable on the horizontal axis be time t, then dy/dt is the ratio of the change in y corresponding to a change in t, when both changes are infinitesimally small. The fractional notation for the derivative as a ratio carries this interpretation of the derivative as a rate:

$$\frac{dy}{dt} = \lim_{\delta t \to 0} \frac{\delta y}{\delta t}\,.$$

This section deals with rates of change which are are not necessarily constant over time.

Using the Chain Rule to Compare Rates: The method is simply to use the chain rule to differentiate with respect to time. This will establish a relation between two rates.

17 RATES: Express one quantity as a function of the other quantity, then differentiate with respect to time using the chain rule.

WORKED EXERCISE: Suppose that water is flowing into a large spherical balloon at a constant rate of $50\,\text{cm}^3/\text{s}$.

(a) At what rate is the radius r increasing when the radius is $7\,\text{cm}$?

(b) At what rate is the radius increasing when the volume V is $4500\pi\,\text{cm}^3$?

(c) What should the flow rate be changed to so that when the radius is $7\,\text{cm}$, it is increasing at $1\,\text{cm/s}$?

There are two quantities varying with time here, the volume and the radius. The volume is increasing at a constant rate, but the radius is increasing at a rate that decreases as the balloon expands. The chain rule will allow the two rates of change to be related to each other.

SOLUTION: The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

Differentiating with respect to time t, $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$ (chain rule)

$$\frac{dV}{dt} = 4\pi r^2\,\frac{dr}{dt}\,.$$

(a) Substituting the known rate $dV/dt = 50$ and the radius $r = 7$,

$$50 = 4\pi \times 49 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{25}{98\pi}\ \text{cm/s} \quad (\doteqdot 0{\cdot}81\,\text{mm/s}),$$ the rate of increase of the radius.

(b) When $V = 4500\pi$,

$$\tfrac{4}{3}\pi r^3 = 4500\pi$$
$$r^3 = 3375$$
$$r = 15,$$

so substituting again,

$$50 = 4\pi \times 225 \times \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{18\pi} \quad (\doteqdot 0{\cdot}177\ \text{mm/s}).$$

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(c) Substituting $r = 7$ and $dr/dt = 1$ gives the rate of change of volume:

$$\frac{dV}{dt} = 4\pi \times 49 \times 1$$
$$= 196\pi \text{ cm}^3/\text{s} \quad (\doteqdot 616\,\text{cm}^3/\text{s}).$$

Some Formulae for Solids: These formulae were developed in earlier years.

18 VOLUME AND SURFACE AREA OF SOLIDS:

For a sphere:	For a cylinder:	For a cone:	For a pyramid:
$V = \frac{4}{3}\pi r^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$	$V = \frac{1}{3} \times \text{base} \times \text{height}$
$A = 4\pi r^2$	$A = 2\pi r^2 + 2\pi r h$	$A = \pi r^2 + \pi r \ell$	$A = \text{sum of faces}$
		($\ell = \sqrt{r^2 + h^2}$ is slant height of cone.)	

Exercise 7H

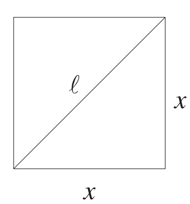
1. Given that $y = x^3 + x$, differentiate with respect to time, using the chain rule.
(a) If $dx/dt = 5$, find dy/dt when $x = 2$.
(b) If $dy/dt = -6$, find dx/dt when $x = -3$.

2. A circular oil stain of radius r and area A is spreading on water. Differentiate the area formula with respect to time to show that $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$. Hence find:
(a) the rate of increase of area when $r = 40$ cm if the radius is increasing at 3 cm/s,
(b) the rate of increase of the radius when $r = 60$ cm if the area is increasing at $10\,\text{cm}^2/\text{s}$.

3. A spherical bubble of radius r is shrinking so that its volume V is decreasing at a constant rate of $200\,\text{cm}^3/\text{s}$. Show that $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.
(a) At what rate is its radius decreasing when the radius is 5 cm?
(b) What is the radius when the radius is decreasing at 2π cm/s?
(c) At what rate is the radius decreasing when the volume is $36\pi\,\text{cm}^3$?
[HINT: First find the radius when the volume is $36\pi\,\text{cm}^3$.]

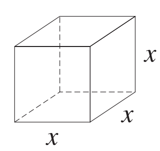
4. The side length x of a square shadow is increasing at 6 cm/s. If the area is A and the length of the diagonal is ℓ, show that

$$\frac{dA}{dt} = 2x\frac{dx}{dt} \qquad \text{and} \qquad \frac{d\ell}{dt} = \sqrt{2}\,\frac{dx}{dt}.$$



Hence find the rates of increase of the area and the diagonal when: (a) the side length is 70 cm, (b) the area is $1\,\text{m}^2$.

5. The side length x of a shrinking cube is decreasing at a constant rate of 5 mm per minute. Show that the rates of change of volume V, surface area A, and the total edge length ℓ are



$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \qquad \text{and} \qquad \frac{dA}{dt} = 12x\frac{dx}{dt} \qquad \text{and} \qquad \frac{d\ell}{dt} = 12\frac{dx}{dt}.$$

(a) Find the rate at which volume, surface area and edge length are decreasing when:
(i) the side length is 30 cm, (ii) the volume is $8000\,\text{cm}^3$.
(b) Find the side length when the volume is decreasing at $300\,\text{cm}^3/\text{min}$.

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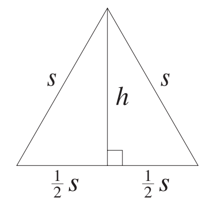
6. Show that in an equilateral triangle of side length s, the area is given by $A = \frac{1}{4}s^2\sqrt{3}$ and the height by $h = \frac{1}{2}s\sqrt{3}$.

(a) Find formulae for the rates of change of area and height.

(b) Hence find the rate at which the area and the height are increasing when the side length is 12 cm and is increasing at 3 mm/s.



DEVELOPMENT

7. A spherical balloon is to be filled with water so that its surface area increases at a constant rate of $1\,\text{cm}^2/\text{s}$.

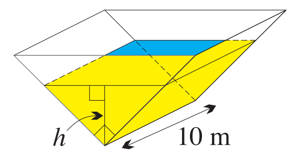
(a) Find, when the radius is 3 cm: (i) the required rate of increase of the radius, (ii) the rate at which the water must be flowing in at that time.

(b) Find the volume when the volume is increasing at $10\,\text{cm}^3/\text{s}$.

8. A water trough is 10 metres long, with cross section a right isosceles triangle. Show that when the water has depth h cm, its volume is $V = 1000h^2$ and its surface area is $A = 2000h$.

(a) Find the rates at which depth and surface area are increasing when the depth is 60 cm if the trough is filling at 5 litres per minute (remember that 1 litre is $1000\,\text{cm}^3$).

(b) Find the rates at which the volume and the surface area must increase when the depth is 40 cm, if the depth is required to increase at a constant rate of 0·1 cm/min.



9. The equation of the path of a bullet fired into the air is $y = -20x(x - 20)$, where x and y are displacements in metres horizontally and vertically from the origin. The bullet is moving horizontally at a constant rate of $\frac{1}{2}$ m/s.

(a) Find the rate at which the bullet is rising:
(i) when $x = 8$, (ii) when $x = 18$, (iii) when $x = 10$, (iv) when $y = 1500$.

(b) Find the height when the bullet is: (i) rising at 30 m/s, (ii) falling at 70 m/s.

(c) Use the gradient function dy/dx to find the angle of flight when the bullet is rising at 10 m/s.

(d) How high does the bullet go, and how far away does it land?

10. Sand being poured from a conveyor belt forms a cone with height h and semivertical angle 60°. Show that the volume of the pile is $V = \pi h^3$, and differentiate with respect to t.

(a) Suppose that the sand is being poured at a constant rate of $0{\cdot}3\,\text{m}^3/\text{min}$, and let A be the area of the base. Find the rate at which the height is increasing:
(i) when the height is 4 metres, (ii) when the radius is 4 metres.

(b) Show that $\dfrac{dA}{dt} = 6\pi h\,\dfrac{dh}{dt}$, and find the rate of increase of the base area at these times.

(c) At what rate must the sand be poured if it is required that the height increase at 8 cm/min, when the height is 4 metres?

11. An upturned cone of semivertical angle 45° is being filled with water at a constant rate of $20\,\text{cm}^3/\text{s}$. Find the rate at which the height, the area of the water surface, and the area of the cone wetted by the water, are increasing when the height is 50 cm.

12. A square pyramid has height twice its side length s.

(a) Show that the volume V and the surface area A are $V = \frac{2}{3}s^3$ and $A = (\sqrt{17} + 1)s^2$.

(b) Hence find the rate at which V and A are decreasing when the side length is 4 metres if the side length is shrinking at 3 mm/s.

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13. A ladder 13 metres long rests against a wall, with its base x metres from the wall and its top y metres high. Explain why $x^2 + y^2 = 169$, solve for y, and differentiate with respect to t. Hence find, when the base is 5 metres from the wall:

(a) the rate at which the top is slipping down when the base is slipping out at 1 cm/s,

(b) the rate at which the base is slipping out when the top is slipping down at 5 mm/s.

14. Water is flowing into a hemispherical container of radius 10 cm at a constant rate of 6 cm^3 per second. It is known that the formula for the volume of a solid segment cut off a sphere is $V = \frac{\pi}{3}h^2(3r - h)$, where r is the radius of the sphere and h is the height of the segment.

(a) Find the rate at which the height of the water is rising when the water height is 2 cm.

(b) Use Pythagoras' theorem to find the radius of the circular water surface when the height is h, and hence find the rate of increase of the surface area when the water height is 2 cm.

EXTENSION

15. Show that the volume V of a regular tetrahedron, all of whose side lengths are s, is $V = \frac{1}{12}s^3\sqrt{2}$ (all four faces of a regular tetrahedron are equilateral triangles). Hence find the rate of increase of the surface area when the volume is $144\sqrt{2}$ cm^3 and is increasing at a rate of 12 cm^3/s.

16. A large vase has a square base of side length 6 cm, and flat sides sloping outwards at an angle of 120° with the base. Water is flowing in at 12 cm^3/s. Find, to three significant figures, the rate at which the height of water is rising when the water has been flowing in for 3 seconds.

7 I Limits and Continuity

If a tangent can be drawn at a point on a curve, the curve must be smooth at that point without any sharp corner — the technical word is *differentiable*. The curve must also be *continuous* at the point, without any break. The purpose of Sections 7I and 7J is to make a little more precise what is meant by saying that a curve is continuous at a point and what is meant by saying that it is differentiable there. Both these definitions rest firmly on the idea of a limit.

Some Rules for Limits: It is not the intention of this course to provide anything more than an intuitive introduction to limits, and some fairly obvious rules about how to handle them. Here is the informal definition of a limit that we have been using.

19 DEFINITION: $\lim_{x\to a} f(x) = \ell$ means $f(x)$ is 'as close as we like' to ℓ when x is near a.

Here are some of the assumptions we have been making about the behaviour of limits.

20

LIMIT OF A SUM: $\lim_{x\to a}\big(f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$,

LIMIT OF A MULTIPLE: $\lim_{x\to a} kf(x) = k \times \lim_{x\to a} f(x)$,

LIMIT OF A PRODUCT: $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$,

LIMIT OF A QUOTIENT: $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.

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WORKED EXERCISE: Find: (a) $\displaystyle\lim_{x\to1}\frac{x^2-1}{x-1}$ (b) $\displaystyle\lim_{x\to3}\frac{x^2-7x+12}{x^2+x-12}$

SOLUTION:

(a) $$\begin{aligned}&\lim_{x\to1}\frac{x^2-1}{x-1}\\&=\lim_{x\to1}\frac{(x-1)(x+1)}{x-1}\\&=\lim_{x\to1}(x+1)\,,\quad\text{since } x\neq1,\\&=2\end{aligned}$$

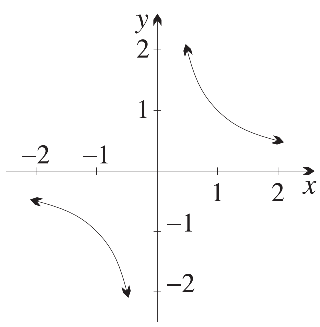
(The value at $x=1$ is irrelevant.)

(b) $$\begin{aligned}&\lim_{x\to3}\frac{x^2-7x+12}{x^2+x-12}\\&=\lim_{x\to3}\frac{(x-3)(x-4)}{(x-3)(x+4)}\\&=\lim_{x\to3}\frac{x-4}{x+4}\,,\quad\text{since } x\neq3,\\&=-\tfrac{1}{7}\end{aligned}$$

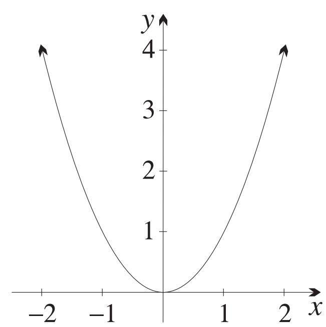
(The value at $x=3$ is irrelevant.)

Continuity at a Point — Informal Definition: As discussed already in Chapter 3, continuity at a point means that there is no break in the curve around that point.

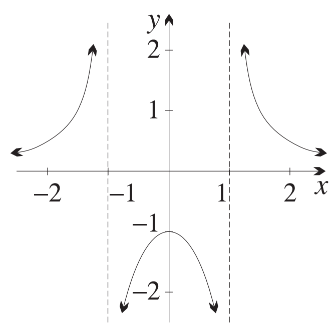
21 DEFINITION: A function $f(x)$ is called *continuous at $x=a$* if the graph of $y=f(x)$ can be drawn through the point where $x=a$ without any break. Otherwise we say that there is a *discontinuity* at $x=a$.



EXAMPLE: $y=1/x$ has a discontinuity at $x=0$, and is continuous everywhere else.



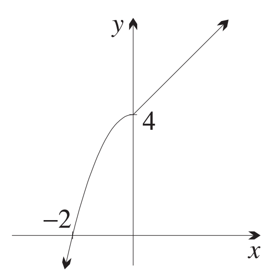
EXAMPLE: $y=x^2$ is continuous for all values of x.



EXAMPLE: $y=\dfrac{1}{x^2-1}$ has discontinuities at $x=1$ and at $x=-1$, and is continuous everywhere else.

Piecewise Defined Functions: A function can be *piecewise defined* by giving different definitions in different parts of its domain. For example,

$$f(x)=\begin{cases}4-x^2, & \text{for } x\le0,\\ 4+x, & \text{for } x>0.\end{cases}$$



Clearly the two pieces of this graph join up at the point $(0,4)$, making the function continuous at $x=0$.

The more formal way of talking about this involves analysing the behaviour of $f(x)$ on each side of $x=0$, and taking two limits, first when x is near zero and greater than zero, secondly when x is near zero and less than zero:

$\displaystyle\lim_{x\to0^+}f(x)$, meaning 'the limit as x approaches 0 from the positive side',

$\displaystyle\lim_{x\to0^-}f(x)$, meaning 'the limit as x approaches 0 from the negative side'.

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We now look at these two limits, as well as the value of $f(x)$ at $x = 0$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (4 - x^2) \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (4 + x) \qquad f(0) = 4 - 0^2$$
$$= 4 \qquad\qquad = 4 \qquad\qquad = 4$$

and the reason why $f(x)$ is continuous at $x = 0$ is that these three values all exist and are all equal.

Continuity at a Point — Formal Definition: Here then is a somewhat stricter definition of continuity at a point, using the machinery of limits.

22 DEFINITION: A function $f(x)$ is called *continuous* at $x = a$ if

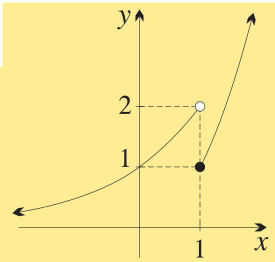
$$\lim_{x\to a^-} f(x) \text{ and } \lim_{x\to a^+} f(x) \text{ and } f(a)$$

all exist and are all equal.

WORKED EXERCISE: Examine for continuity at $x = 1$, then sketch, the function

$$f(x) = \begin{cases} 2^x, & \text{for } x < 1, \\ x^2, & \text{for } x \ge 1. \end{cases}$$

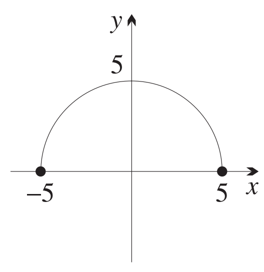
SOLUTION:
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2^x = 2,$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1,$$
$$f(1) = 1,$$
so the curve is not continuous at $x = 1$.



An Assumption of Continuity: It is intuitively obvious that a function like $y = x^2$ is continuous for every value of x. However, it is not possible in this course to give the sort of rigorous proof that mathematicians so enjoy, because the required rigorous treatment of limits is missing. It is therefore necessary to make a general assumption of continuity, loosely stated as follows.

23 ASSUMPTION: The functions in this course are continuous for every value of x in their domain, except where there is an obvious problem.

Continuity in a Closed Interval: The semicircular function with equation $f(x) = \sqrt{25 - x^2}$ presents an interesting test for the definition of continuity. At the right-hand endpoint $x = 5$ the curve is not continuous, because the right-hand limit $\lim_{x\to 5^+} f(x)$ does not exist. Neither is the curve continuous at the left-hand endpoint $x = -5$, because the left-hand limit $\lim_{x\to(-5)^-} f(x)$ does not exist.



In fact, no curve is continuous at an endpoint of its domain.

Nevertheless, it will be important later to say that $f(x)$ is continuous in the closed interval $-5 \le x \le 5$, and to justify this by the fact that the situation at the left and right-hand sides is

$$\lim_{x\to 5^-} f(x) = f(5) = 0 \quad \text{and} \quad \lim_{x\to(-5)^+} f(x) = f(-5) = 0.$$

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This leads to the following definition of continuity in a closed interval.

24 DEFINITION: $f(x)$ is called *continuous in the closed interval* $a \le x \le b$ if:

1. $f(x)$ is continuous for every value of x in the open interval $a < x < b$, and
2. $\lim_{x \to a^+} f(x)$ and $f(a)$ both exist and are equal, and
3. $\lim_{x \to b^-} f(x)$ and $f(b)$ both exist and are equal.

Then by this definition, $y = \sqrt{25 - x^2}$ is indeed continuous in the closed interval $-5 \le x \le 5$.

Continuous Functions: A function $f(x)$ is called *continuous* if it is continuous at every point in its domain. This turns out, however, to be a rather unsatisfactory definition for our purposes, because, for example, the function $y = 1/x$ is continuous at every value of x except $x = 0$, which lies outside its domain, and so we have to conclude that $y = 1/x$ is a continuous function with a discontinuity at $x = 0$. Consequently this course will rarely speak of continuous functions, and the emphasis will be on continuity at a point, and less often on continuity in a closed interval.

Exercise 7I

NOTE: In every graph, every curve must end with a closed circle if the endpoint is included, an open circle if it is not, or an arrow if it continues forever. After working on these limit questions, one should revise differentiation from first principles in Exercise 7B.

1. (a) Use the rule from Chapter Three, 'Divide top and bottom by the highest power of x in the denominator' to find the behaviour of these functions as $x \to \infty$:

(i) $y = \dfrac{x^2 - 4x + 3}{2x^2 - 7x + 6}$ (ii) $y = \dfrac{2 - 5x}{15x + 11}$ (iii) $y = \dfrac{x^2 + x + 1}{x^3 + x^2 + x + 1}$ (iv) $y = \dfrac{x^2 - 5}{1 + \sqrt{x}}$

(b) Use the same rule to find the behaviour of those functions as $x \to -\infty$.

2. First factor top and bottom and cancel any common factors, then find:

(a) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ (c) $\lim_{h \to 0} \dfrac{h^3 - 9h^2 + h}{h}$ (e) $\lim_{x \to 0} \dfrac{x^3 + 6x}{x^2 - 3x}$

(b) $\lim_{u \to 3} \dfrac{u^3 - 27}{u - 3}$ (d) $\lim_{h \to -3} \dfrac{h^2 - 9}{h^2 + 7h + 12}$ (f) $\lim_{x \to 0} \dfrac{x^4 - 4x^2}{x^2 - 2x}$

3. Discuss the behaviour of $y = \dfrac{x^2 - x - 12}{2x^2 + 7x + 3} = \dfrac{(x-4)(x+3)}{(2x+1)(x+3)}$:

(a) as $x \to \infty$ (c) as $x \to 0$ (e) as $x \to 4$ (g) as $x \to -\infty$

(b) as $x \to -3$ (d) as $x \to -\frac{1}{2}$ (f) as $x \to 1$ (h) as $x \to -1$

4. For each function below: (i) sketch the curve, (ii) find $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2^+} f(x)$ and $f(2)$, (iii) draw a conclusion about continuity at $x = 2$, (iv) give the domain and range.

(a) $f(x) = \begin{cases} x^3, & \text{for } x \le 2, \\ 10 - x, & \text{for } x > 2. \end{cases}$

(b) $f(x) = \begin{cases} 3^x, & \text{for } x < 2, \\ 13 - x^2, & \text{for } x > 2, \\ 4, & \text{for } x = 2. \end{cases}$

(c) $f(x) = \begin{cases} 1/x, & \text{for } 0 < x < 2, \\ 1 - \frac{1}{4}x, & \text{for } x > 2, \\ \frac{1}{2}, & \text{for } x = 2. \end{cases}$

(d) $f(x) = \begin{cases} x, & \text{for } x < 2, \\ 2 - x, & \text{for } x > 2, \\ 2, & \text{for } x = 2. \end{cases}$

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5. Cancel the algebraic fraction in each function, noting first the value of x for which the function is undefined. Then sketch the curve and state its domain and range:

(a) $y = \dfrac{x^2+2x+1}{x+1}$ (b) $y = \dfrac{x^4-x^2}{x^2-1}$ (c) $y = \dfrac{x-3}{x^2-4x+3}$ (d) $y = \dfrac{3x+3}{x+1}$

DEVELOPMENT

6. (a) Find the gradient of the secant joining the points $P(x, f(x))$ and $Q\big(x+h, f(x+h)\big)$ on the curve $y = x^2 - x + 1$, then take $\lim\limits_{h\to 0}$(gradient of PQ).

(b) Find the gradient of the secant joining the points $P(x, g(x))$ and $Q(u, g(u))$ on the curve $y = x^4 - 3x$, then take $\lim\limits_{u\to x}$(gradient of PQ).

7. (a) Find: (i) $\lim\limits_{x\to c} \dfrac{x^4-c^4}{x^2-c^2}$ (ii) $\lim\limits_{x\to c} \dfrac{x^4-c^4}{x^3-c^3}$ (iii) $\lim\limits_{x\to -c} \dfrac{x^5+c^5}{x^3+c^3}$

(b) Factor the difference of powers $x^n - a^n$ and hence find $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a}$.

(c) Factor the sum of odd powers $u^{2n+1} + 2^{2n+1}$ and hence find $\lim\limits_{u\to -2} \dfrac{u^{2n+1}+2^{2n+1}}{u+2}$.

8. For what values of a are these functions continuous:

(a) $f(x) = \begin{cases} ax^2, & \text{for } x \le 1, \\ 6-x, & \text{for } x > 1. \end{cases}$ (b) $g(x) = \begin{cases} \dfrac{a(x^2-9)}{x+3}, & \text{for } x \ne -3, \\ 12, & \text{for } x = -3. \end{cases}$

9. Find all zeroes and discontinuities of these functions:

(a) $y = \dfrac{x}{x-3}$ (c) $y = \operatorname{cosec} x^\circ$ (e) $y = \tan x^\circ$

(b) $y = \dfrac{x}{x^2-6x-7}$ (d) $y = \dfrac{1}{\cos x^\circ - 1}$ (f) $y = \dfrac{x^3-x}{x^3-9x}$

10. (a) Simplify $y = \dfrac{|x|}{x}$, find $\lim\limits_{x\to 0^+} y$, $\lim\limits_{x\to 0^-} y$ and $y(0)$, discuss the continuity at $x = 0$, and sketch it.

(b) Repeat the steps in part (a) for:

(i) $y = \dfrac{x^2}{x^2}$ (ii) $y = \dfrac{x^2}{\sqrt{x^2}}$ (iii) $y = \dfrac{|x|}{\sqrt{x}}$ (iv) $y = \dfrac{|x^2-2x|}{x}$

11. (a) Show that the GP $u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1}$ has common ratio $\dfrac{x}{u}$.

(b) Use the formula for the partial sum of a GP to show that its sum is $\dfrac{u^n-x^n}{u-x}$.

(c) Use this identity to find the derivative of $f(x) = x^n$ from first principles.

12. (a) Use the method of 'rationalising the numerator' to find these limits:

(i) $\lim\limits_{u\to x} \dfrac{\sqrt{u}-\sqrt{x}}{u-x}$ (ii) $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ (iii) $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x-h}}{2h}$

(b) Explain how each limit can be used to show that the derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$.

EXTENSION

13. Which of these functions are continuous in the closed interval $-1 \le x \le 1$:

(a) $\sqrt{x+1}$ (b) $\dfrac{1}{x^2-1}$ (c) $7\sqrt{1-x^2}$ (d) $\dfrac{x^2-1}{x^2-2}$

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14. Find zeroes and discontinuities of: (a) $y = \dfrac{\cos x^\circ + \sin x^\circ}{\cos x^\circ - \sin x^\circ}$ (b) $y = \dfrac{\cos x^\circ - \sin x^\circ}{\cos x^\circ + \sin x^\circ}$

15. Find these limits by rationalising the numerator or otherwise:

(a) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}$ (b) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x^2-1}$ (c) $\displaystyle\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$ (d) $\displaystyle\lim_{x\to 25} \frac{\frac{1}{\sqrt{x}}-\frac{1}{5}}{x-25}$

16. Sketch these functions over the whole real line:

(a) $y = \dfrac{|\sin 180x^\circ|}{\sin 180x^\circ}$

(b) $y = \dfrac{|\cos 180x^\circ|}{\cos 180x^\circ}$

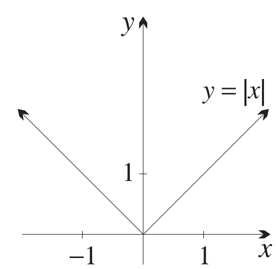
(c) $y = \dfrac{|\tan 180x^\circ|}{\tan 180x^\circ}$

(d) $y = \dfrac{|x(x^2-1)(x^2-4)|}{x(x^2-1)(x^2-4)}$

(e) $y = \displaystyle\lim_{n\to\infty} \frac{\left|x\prod_{k=1}^{2n}(x^2-k^2)\right|}{x\prod_{k=1}^{2n}(x^2-k^2)}$

7 J Differentiability

A tangent can only be drawn at a point P on a curve if the curve is smooth at that point, meaning that the curve can be drawn through P without any sharp change of direction. For example, the curve $y = |x|$ sketched opposite has a sharp point at the origin, where it changes gradient abruptly from -1 to 1. A tangent cannot be drawn there, and the function has no derivative at $x = 0$. This suggests the following definition.



25 DEFINITION: A function $f(x)$ is called *differentiable* (or *smooth*) at $x = a$ if the derivative $f'(a)$ exists there.

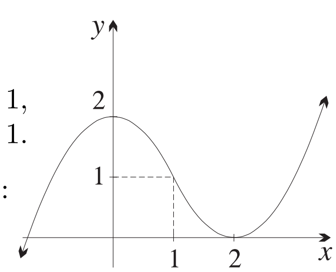
So $y = |x|$ is continuous at $x = 0$, but is not differentiable there.

Clearly a function that is not even continuous at some value $x = a$ cannot be differentiable there, because there is no way of drawing a tangent at a place where there is a break in the curve.

26 If $f(x)$ is not continuous at $x = a$, then it is certainly not differentiable there.

Piecewise Defined Functions: The sketch opposite shows

$$f(x) = \begin{cases} 2-x^2, & \text{for } x<1, \\ (x-2)^2 & \text{for } x \ge 1, \end{cases} \quad \text{so } f'(x) = \begin{cases} -2x, & \text{for } x<1, \\ 2(x-2) & \text{for } x>1. \end{cases}$$



The graph is continuous, because the two pieces join at $P(1,1)$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 1.$$

But in this case, when the two pieces join, they do so with the same gradient, so that the combined curve is smooth at the point $P(1,1)$. The reason for this is that the gradients on the left and right of $x = 1$ also converge to the same limit of -2:

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$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} (-2x) \qquad \text{and} \qquad \lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} 2(x-2)$$
$$= -2, \qquad\qquad\qquad\qquad = -2.$$

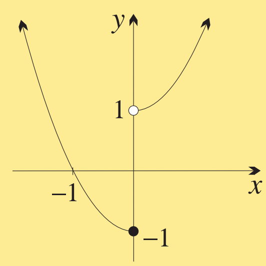
So the function does have a well-defined derivative of -2 when $x = 1$, and the curve is indeed differentiable there, with a well-defined tangent at the point $P(1, 1)$.

27 DIFFERENTIABILITY: To test a piecewise defined function for differentiability at a join $x = a$ between pieces:
1. Test whether the function is continuous at $x = a$.
2. Test whether $\lim\limits_{x \to a^-} f'(x)$ and $\lim\limits_{x \to a^+} f'(x)$ exist and are equal.

WORKED EXERCISE: Test the following functions for continuity and for differentiability at $x = 2$, and if they are differentiable, state the value of the derivative there. Then sketch the curves:

(a) $f(x) = \begin{cases} x^2 - 1, & \text{for } x \le 0, \\ x^2 + 1, & \text{for } x > 0. \end{cases}$ (b) $f(x) = \begin{cases} x, & \text{for } x \le 0, \\ x - x^2, & \text{for } x > 0. \end{cases}$

SOLUTION:



(a) First, $\lim\limits_{x \to 0^-} f(x) = -1$,

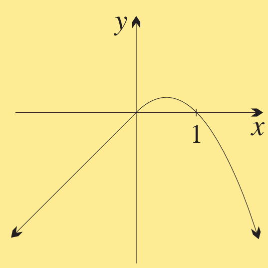
and $\lim\limits_{x \to 0^+} f(x) = 1$,

so the function is not even continuous at $x = 0$.

[Notice, however, that for $x \ne 0$, $f'(x) = 2x$,

so $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^-} f'(x) = 0$,

but this is irrelevant since the curve is not continuous at $x = 0$.]



(b) First, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0) = 0$,

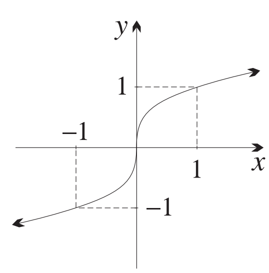
so the function is continuous at $x = 0$.

Secondly, $f'(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - 2x, & \text{for } x > 0, \end{cases}$

so $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^-} f'(x) = 1$,

and the function is differentiable at $x = 0$, with $f'(0) = 1$.

Tangents and Differentiability: It would be natural to think that being differentiable at $x = a$ and having a tangent at $x = a$ were equivalent. Some curves, however, have a point where there is a vertical tangent. But vertical tangents don't have a gradient, so at such a point the derivative is undefined, meaning that the curve is not differentiable there. For example, the curve on the right is



$$f(x) = x^{\frac{1}{3}}, \qquad \text{whose derivative is} \quad f'(x) = \tfrac{1}{3}x^{-\frac{2}{3}}.$$

This function is simply the inverse function of $y = x^3$, so its graph is the graph of $y = x^3$ reflected in the diagonal line $y = x$. There is no problem about the continuity of $f(x)$ at $x = 0$, where the graph passes through the origin. But $f'(0)$ is undefined, and

$$f'(x) \to \infty \text{ as } x \to 0^+ \qquad \text{and} \qquad f'(x) \to \infty \text{ as } x \to 0^-.$$

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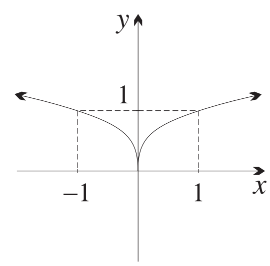
So the gradient of the curve becomes infinitely steep on both sides of the origin, and although the curve is not differentiable there, the y-axis is a vertical tangent. The complete story is:

28 TANGENTS AND DIFFERENTIABILITY: $f(x)$ is differentiable at $x = a$ if and only if there is a tangent there, and the tangent is non-vertical.

Cusps and Differentiability: A stranger picture is provided by the closely related function

$$f(x) = x^{\frac{2}{3}}, \qquad \text{whose derivative is} \quad f'(x) = \tfrac{2}{3}x^{-\frac{1}{3}}.$$



Again the function is continuous at $x = 0$ where the graph passes through the origin, and again $f'(0)$ is undefined. But this time the function can never be negative, since it is a square, and

$$f'(x) \to \infty \text{ as } x \to 0^+ \qquad \text{and} \qquad f'(x) \to -\infty \text{ as } x \to 0^-.$$

So although the gradient of the curve still becomes infinitely steep on both sides of the origin, the two sides are sloping one backwards and one forwards, and this time there is no tangent at $x = 0$. The point $(0, 0)$ is called a *cusp* of the curve.

Exercise 7J

1. Test these functions for continuity at $x = 1$. If the function is continuous there, find $\lim\limits_{x\to1^+} f'(x)$ and $\lim\limits_{x\to1^-} f'(x)$ to check for differentiability at $x = 1$. Then sketch the graph.

(a) $f(x) = \begin{cases} x^2, & \text{for } x \le 1, \\ 2x-1, & \text{for } x > 1. \end{cases}$

(b) $f(x) = \begin{cases} 3-2x, & \text{for } x < 1, \\ 1/x, & \text{for } x \ge 1. \end{cases}$

(c) $f(x) = \begin{cases} (x+1)^2, & \text{for } x \le 1, \\ 4x-2, & \text{for } x > 1. \end{cases}$

(d) $f(x) = \begin{cases} (x-1)^3, & \text{for } x \le 1, \\ (x-1)^2, & \text{for } x > 1. \end{cases}$

2. Sketch the graph of the function $y = \begin{cases} x^3 - x, & \text{for } -1 \le x \le 1, \\ x^2 - 1, & \text{for } x > 1 \text{ or } x < -1, \end{cases}$ after first checking for any values of x where the curve is not continuous or not differentiable.

DEVELOPMENT

3. (a) Sketch $f(x) = \begin{cases} \sqrt{1-(x+1)^2}\,, & \text{for } -2 \le x \le 0, \\ \sqrt{1-(x-1)^2}\,, & \text{for } 0 < x \le 2. \end{cases}$ Describe the situation at $x = 0$.

(b) Repeat part (a) for $f(x) = \begin{cases} \sqrt{1-(x+1)^2}\,, & \text{for } -2 \le x \le 0, \\ -\sqrt{1-(x-1)^2}\,, & \text{for } 0 < x \le 2. \end{cases}$

4. Sketch each function, giving any values of x where it is not continuous or not differentiable:

(a) $y = |x+2|$

(b) $y = \dfrac{1}{|x+2|}$

(c) $y = |x^2 - 4x + 3|$

(d) $y = \dfrac{1}{|x^2 - 4x + 3|}$

(e) $y = |x^2 + 2x + 2|$

(f) $y = \dfrac{1}{|x^2 + 2x + 2|}$

(g) $y = |x^3|$

(h) $y = |x^2(x-2)|$

(i) $y = |-(3-x)^2|$

(j) $y = \sqrt{(x-2)^2}$

5. (a) Differentiate $f(x) = x^{\frac{1}{5}}$, and find $\lim\limits_{x\to0^+} f'(x)$ and $\lim\limits_{x\to0^-} f'(x)$. Sketch the curve and state whether it has a vertical tangent or a cusp at $x = 0$. (b) Repeat for $f(x) = x^{\frac{2}{5}}$.

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6. Each example following gives a curve and two points P and Q on the curve. Find the gradient of the chord PQ, and find the x-coordinates of any points M on the curve between P and Q such that the tangent at M is parallel to PQ.

(a) $y = x^2 - 6x$, $P = (1, -5)$, $Q = (8, 16)$
(b) $y = x^3 - 9$, $P = (-1, -10)$, $Q = (2, -1)$
(c) $y = x^3$, $P = (-1, -1)$, $Q = (1, 1)$
(d) $y = \sqrt{x}$, $P = (1, 1)$, $Q = (4, 2)$
(e) $y = 1/x$, $P = (1, 1)$, $Q = \left(2, \frac{1}{2}\right)$
(f) $y = 1/x$, $P = (-1, -1)$, $Q = (1, 1)$
(g) $y = |x|$, $P = (-1, 1)$, $Q = (1, 1)$
(h) $y = x^2$, $P = (\alpha, \alpha^2)$, $Q = (-\alpha, \alpha^2)$
(i) $y = x^2$, $P = (\alpha, \alpha^2)$, $Q = (\beta, \beta^2)$
(j) $y = 1/x$, $P = (\alpha, 1/\alpha)$, $Q = (\beta, 1/\beta)$

NOTE: The existence of at least one such point is guaranteed by a theorem called the *mean value theorem*, provided that the curve is differentiable everywhere between the two points.

EXTENSION

7. Find the equation of the tangent at the point $P(a, ka^{\frac{4}{3}})$ on $y = kx^{\frac{4}{3}}$, where $k > 0$, and the coordinates of the points A and B where it meets the x-axis and y-axis respectively. Sketch the situation, and let the vertical and horizontal lines through P meet the x-axis and y-axis at G and H respectively. Show that A divides OG in the ratio $1 : 3$, and O divides BH in the ratio $1 : 3$. Find the ratio of the areas of the rectangle $OGPH$ and $\triangle OAB$.

8. Consider now the point $P(a, ka^n)$ on the general curve $y = kx^n$, where n is any nonzero real number and $k > 0$. Find the coordinates of the points A, B, G and H defined in the previous question, and show that A divides OG in the ratio $n - 1 : 1$ and that O divides BH in the ratio $n - 1 : 1$. Find the ratio of the areas of the rectangle $OGPH$ and $\triangle OAB$, and find when the rectangle is bigger. For what values of n is the point B above the origin? Can a be negative?

9. Suppose that p and q are integers with no common factors and with $q > 0$. Write down the derivative of $f(x) = x^{\frac{p}{q}}$, and hence find the conditions on p and q for which:

(a) $f(x)$ is defined for $x < 0$,
(b) $f(x)$ is defined at $x = 0$,
(c) $f(x)$ is defined for $x > 0$,
(d) $f(x)$ is continuous at $x = 0$,
(e) $f(x)$ is continuous for $x \ge 0$,
(f) $f(x)$ is differentiable at $x = 0$,
(g) there is a vertical tangent at the origin,
(h) there is a cusp at the origin.

7 K Extension — Implicit Differentiation

So far we have only been differentiating curves whose equation has the form $y = f(x)$, where $f(x)$ is a function. But solving an equation for y can sometimes be difficult or impossible, and sometimes the curve may not even be a function. The purpose of this rather more difficult section is to extend differentiation to curves like the circle $x^2 + y^2 = 25$, which may not be functions, yet are still defined by an algebraic equation in x and y. This is a 4 Unit topic which is useful, but not necessary, for the 3 Unit course.

Differentiating Expressions in x and y: The first step is using the chain, product and quotient rules to differentiate expressions in x and y where x and y are related. In this situation, neither x nor y is constant, and in particular y must be regarded as a function of x.

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WORKED EXERCISE: Differentiate the following expressions with respect to x:

(a) y^2 (b) x^2y (c) $\dfrac{x^2}{y^2}$ (d) $(x^2+y^2)^2$

SOLUTION:

(a) Using the chain rule:

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx}$$
$$= 2y\,\frac{dy}{dx}\,.$$

(b) Using the product rule:

$$\frac{d}{dx}(x^2y) = y\,\frac{d}{dx}(x^2) + x^2\,\frac{d}{dx}(y)$$
$$= 2xy + x^2\,\frac{dy}{dx}\,.$$

(c) Using the quotient rule:

$$\frac{d}{dx}\left(\frac{x^2}{y^2}\right) = \frac{y^2\,\dfrac{d}{dx}(x^2) - x^2\,\dfrac{d}{dx}(y^2)}{y^4}$$
$$= \frac{1}{y^4}\left(2xy^2 - 2x^2y\,\frac{dy}{dx}\right)$$
$$= \frac{2x}{y^3}\left(y - x\,\frac{dy}{dx}\right).$$

(d) Using the chain rule with $u = x^2 + y^2$:

$$\frac{d}{dx}(x^2+y^2)^2 = 2(x^2+y^2)\left(2x + 2y\,\frac{dy}{dx}\right)$$
$$= 4(x^2+y^2)\left(x + y\,\frac{dy}{dx}\right).$$

Finding Tangents to Implicitly Defined Curves: When a curve is defined by an algebraic equation in x and y, implicit differentiation will find the derivative as a function of x and y without solving the equation for y. Hence we can find the tangent at any given point on the curve.

WORKED EXERCISE: Use implicit differentiation to find the gradient of the tangent to $x^2 + y^2 = 25$ at the point $P(3,4)$ on the curve.

SOLUTION: Given that $x^2 + y^2 = 25,$

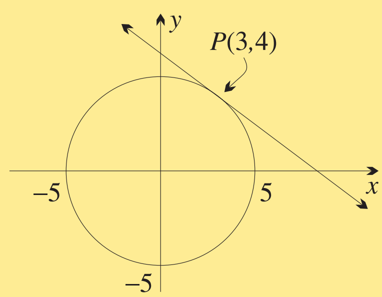
differentiating implicitly, $2x + 2y\,\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\,\frac{x}{y}\,.$$

Hence at $P(3,4)$, $\dfrac{dy}{dx} = -\frac{3}{4},$

and the tangent is $y - 4 = -\frac{3}{4}(x-3)$

$$3x + 4y = 25.$$



NOTE: In this particular case, the geometry of the circle is known independently of differentiation. The tangent is perpendicular to the radius joining the origin and $P(3,4)$, and since the radius has gradient $\frac{4}{3}$, the tangent must have gradient $-\frac{3}{4}$ (this geometric approach to differentiating the circle was used at the beginning of the chapter). This question could also be answered by differentiating the semicircular function, but implicit differentiation is much easier.

Exercise 7K

1. Differentiate the following expressions with respect to x (where x and y are related):

(a) y^4 (c) $1 - x + y - xy$ (e) $x^3y + y^3x$ (g) $(x+y)^3$ (i) $\sqrt{x+y}$

(b) xy (d) $3x^2 + 4y^2$ (f) $\dfrac{x}{y}$ (h) $\dfrac{x+y}{x-y}$ (j) $\sqrt{x^2+y^2}$

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2. Find dy/dx for the curves defined by these equations:

(a) $x^2 + y^2 = 49$ (d) $x^2 + 3xy + 2y^2 = 5$ (g) $ax^r + by^s = c$

(b) $3x^2 + 2y^2 = 25$ (e) $x^3 + xy^2 = x^2y + y^3$ (h) $\sqrt{x} + \sqrt{y} = 4$

(c) $x^2 - y^2 = 1$ (f) $x^2y^3 = 32$ (i) $\dfrac{x}{y} + \dfrac{y}{x} = 1$

DEVELOPMENT

3. (a) Differentiate the circle $x^2+y^2 = 169$ implicitly, and hence find the tangent and normal at the point $P(-5, 12)$. (Why does the normal pass through the origin?)

(b) Find the points A and B where the tangent meets the x-axis and the y-axis.

(c) Find the area of $\triangle AOB$: (i) using OA as the base, (ii) using AB as the base.

4. (a) Differentiate the rectangular hyperbola $xy = 6$ implicitly, and hence find the equations of the tangent and normal at the point $P(2, 3)$.

(b) Show that P is the midpoint of the interval cut off the tangent by the x-intercept and the y-intercept.

5. (a) Differentiate the curve $y^2 = x$ (which is a parabola whose axis of symmetry is the x-axis), and hence find the equation of the tangent and normal at the point $P(9, 3)$.

(b) Show that the y-intercept of the tangent at P bisects the interval joining P and its x-intercept.

6. (a) Use parametric differentiation to differentiate the function defined by $x = t + 1/t$ and $y = t - 1/t$, and find the tangent and normal at the point T where $t = 2$.

(b) Eliminate t from these equations, and use implicit differentiation to find the gradient of the curve at the same point T. [HINT: Square x and y and subtract.]

7. A ladder 8 metres long rests against a wall, with its base x metres from the wall and its top y metres high. Explain why $x^2 + y^2 = 64$, and differentiate the equation implicitly with respect to time. Find, to three significant figures:

(a) the rate at which the top is slipping down when the base is slipping out at a constant rate of 2 cm/s and is 2 metres from the wall,

(b) the rate at which the base is slipping out when the top is slipping down at a constant rate of 2 mm/s and is 7 metres high.

8. (a) Show that the volume and surface area of a sphere are related by $S^3 = 36\pi V^2$, and differentiate this equation with respect to time.

(b) A balloon is to be filled with water so that the rubber expands at a constant rate of 4 cm^2 per second. Use part (a) to find at what rate the water should be flowing in when the radius is 5 cm.

9. (a) Explain why the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ has line symmetry in the x-axis, the y-axis and the lines $y = x$ and $y = -x$.

(b) Explain why its domain is $-8 \le x \le 8$ and its range is $-8 \le y \le 8$.

(c) Show that $dy/dx = -x^{-\frac{1}{3}}y^{\frac{1}{3}}$, and examine the behaviour of the curve as $x \to 8^-$.

(d) Hence sketch the curve.

10. Using methods similar to those in the previous question, or otherwise, sketch:

(a) $x^{\frac{4}{3}} + y^{\frac{4}{3}} = 16$ (b) $x^4 + y^4 = 16$ (c) $|x| + |y| = 2$

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EXTENSION

11. (a) Differentiate $x^3 + y^3 = 8$, and hence find the equations of the tangents at the x- and y-intercepts and at the point where the curve meets $y = x$.

(b) Rewrite the equation as $1 + (y/x)^3 = 8/x^3$ and show that $y/x \to -1$ as $|x| \to \infty$. Show that $x + y = \dfrac{8}{x^2 - xy + y^2}$, and hence that $y = -x$ is an oblique asymptote.

(c) Sketch the curve.

12. (a) Differentiate $x^3 + y^3 = 3xy$, called the *Folium of Descartes*. Hence find the equation of the tangent at the point where the curve meets $y = x$, and find the points on the curve where the tangents are horizontal and vertical (leave the origin out of consideration at this stage). Sketch the curve after carrying out these steps:

(b) Show that $\dfrac{1 + (y/x)^3}{y/x} = \dfrac{3}{x}$, and hence that $y/x \to -1$ as $x \to \infty$.

(c) Show that $x + y = \dfrac{3}{x/y - 1 + y/x}$, and hence that $x + y = -1$ is an oblique asymptote.

13. Differentiate $(x^2 + y^2)^2 = 2(x^2 - y^2)$, called the *Lemniscate of Bernoulli*, and find the points where the tangents are horizontal or vertical (ignore the origin). Sketch the curve.

14. Assume that $\dfrac{d}{dx}x^n = nx^{n-1}$, for $n \in \mathbf{N}$. Implicit differentiation allows a slightly more elegant proof of the successive extensions of this rule to $n \in \mathbf{Z}$ and then to $n \in \mathbf{Q}$.

(a) Let $y = x^{-n}$, where $n \in \mathbf{N}$. Begin with $yx^n = 1$, and prove that $\dfrac{dy}{dx} = -nx^{-n-1}$.

(b) Let $y = x^{\frac{m}{k}}$, where m and k are integers with $k \neq 0$. Begin with $y^k = x^m$, and prove that $\dfrac{dy}{dx} = \frac{m}{k}x^{\frac{m}{k}-1}$.

GO Online Multiple Choice Quiz

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