The University of Sydney

MATH903 Integral Calculus and Modelling (Advanced)

Semester 2 **First Assignment Solutions** 2017

1. We have, for $x \in \Delta$, that

$$g(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = g(x) \ ,$$

and

$$h(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -h(x) \ ,$$

which demonstrates that g is even and h is odd.

2. Let $f \in \mathbb{R}^{\Delta}$. Put $f_{\text{even}} = g$ and $f_{\text{odd}} = h$, where g and h are defined in the previous question, so that f_{even} is even and f_{odd} is odd. But, for all $x \in \Delta$,

$$\begin{aligned}(f_{\text{even}} + f_{\text{odd}})(x) &= f_{\text{even}}(x) + f_{\text{odd}}(x) = g(x) + h(x) \\ &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x) \ ,\end{aligned}$$

so that $f = f_{\text{even}} + f_{\text{odd}}$, proving existence. To prove uniqueness, suppose that $f = f_1 + f_2$ where f_1 is even and f_2 is odd. Then, for all $x \in \Delta$,

$$f_1(x) + f_2(x) = f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x) \ ,$$

so that

$$(f_1 - f_{\text{even}})(x) = f_1(x) - f_{\text{even}}(x) = f_{\text{odd}}(x) - f_2(x) = (f_{\text{odd}} - f_2)(x) \ .$$

This shows that

$$f_1 - f_{\text{even}} = f_{\text{odd}} - f_2 \ .$$

But the left-hand side is a linear combination of even functions, so is even, and the right-hand side is a linear combination of odd functions, so is odd. Hence both sides must become the zero function, that is,

$$f_1 - f_{\text{even}} = \mathbf{0} = f_{\text{odd}} - f_2 \ ,$$

and it follows quickly that $f_1 = f_{\text{even}}$ and $f_2 = f_{\text{odd}}$, proving uniqueness.

3. (a) Here $f_{\text{even}}(x) = \frac{e^x + e^{-x}}{2} = \cosh x$ and $f_{\text{odd}}(x) = \frac{e^x - e^{-x}}{2} = \sinh x$, for all $x \in \mathbb{R}$.

(b) Here, for $x \neq \pm 1$, we have

$$f_{\text{even}}(x) = \frac{\frac{1}{1-x} + \frac{1}{1+x}}{2} = \frac{1}{1 - x^2} \ ,$$

and

$$f_{\text{odd}}(x) \;=\; \frac{\frac{1}{1-x} - \frac{1}{1+x}}{2} \;=\; \frac{x}{1-x^2} \,.$$

(c) Here, for $x \notin \mathbb{Z}$, we have $z < x < z+1$ for some $z \in \mathbb{Z}$, so that

$$-z - 1 < -x < -z \,,$$

giving

$$f_{\text{even}}(x) \;=\; \frac{\lfloor x \rfloor + \lfloor -x \rfloor}{2} \;=\; \frac{z + (-z-1)}{2} \;=\; -\frac{1}{2} \,,$$

and

$$f_{\text{odd}}(x) \;=\; \lfloor x \rfloor - f_{\text{even}}(x) \;=\; \lfloor x \rfloor + \frac{1}{2} \,.$$

4. (a) Let $a \in \mathbb{R}$. Put $u = -x$ so that $du = -dx$. If f is even then, using the fact that $f(-u) = f(u)$,

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx \;&=\; \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&=\; \int_{a}^{0} f(-u)(-1)\,du + \int_{0}^{a} f(x)\,dx \\
&=\; -\int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx \\
&=\; \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \;=\; 2\int_{0}^{a} f(x)\,dx \,,
\end{aligned}$$

verifying the first part of the formula. By contrast, if f is odd then, using the fact that $f(-u) = -f(u)$,

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx \;&=\; \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&=\; \int_{a}^{0} f(-u)(-1)\,du + \int_{0}^{a} f(x)\,dx \\
&=\; \int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx \\
&=\; -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \;=\; 0 \,,
\end{aligned}$$

verifying the second part of the formula.

(b) Suppose first that A is even, so that $A(-t) = A(t)$ for all $t \in \mathbb{R}$. Then, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
f(-t) \;&=\; A'(-t) \;=\; \lim_{h\to 0} \frac{A(-t+h) - A(-t)}{h} \;=\; \lim_{h\to 0} \frac{A(t-h) - A(t)}{h} \\
&=\; -\lim_{h\to 0} \frac{A(t+(-h)) - A(t)}{(-h)} \;=\; -\lim_{k\to 0} \frac{A(t+k) - A(t)}{k} \\
&=\; -A'(t) \;=\; -f(t) \,,
\end{aligned}$$

verifying that f is odd. Suppose conversely that f is odd. Then, for all $x \in \mathbb{R}$, we have, using the substitution $u = -t$ and the second part of the formula from part (a),

$$\begin{aligned}
A(-x) &= \int_a^{-x} f(t)\,dt \;=\; \int_{-a}^{x} f(-u)(-1)\,du \;=\; \int_{-a}^{x} f(u)\,du \\
&= \int_{-a}^{a} f(u)\,du + \int_a^x f(u)\,du \;=\; 0 + \int_a^x f(u)\,du \\
&= \int_a^x f(t)\,dt \;=\; A(x)\;,
\end{aligned}$$

verifying that A is even, completing the proof of part (i).

Suppose now that A is odd, so that $A(-t) = -A(t)$ for all $t \in \mathbb{R}$. In particular $A(0) = -A(0)$, so that $A(0) = 0$. Again, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
f(-t) &= A'(-t) = \lim_{h\to 0} \frac{A(-t+h) - A(-t)}{h} = \lim_{h\to 0} \frac{-A(t-h) + A(t)}{h} \\
&= \lim_{h\to 0} \frac{A(t+(-h)) - A(t)}{(-h)} \;=\; \lim_{k\to 0} \frac{A(t+k) - A(t)}{k} \\
&= A'(t) \;=\; f(t)\;,
\end{aligned}$$

verifying that f is even. Suppose conversely that f is even and $A(0) = 0$. Then, for all $x \in \mathbb{R}$, we have, using the substitution $u = -t$ and the first part of the formula from part (a),

$$\begin{aligned}
A(-x) &= \int_a^{-x} f(t)\,dt \;=\; \int_{-a}^{x} f(-u)(-1)\,du \;=\; -\int_{-a}^{x} f(u)\,du \\
&= -\int_{-a}^{a} f(u)\,du - \int_a^x f(u)\,du \;=\; -2\int_0^a f(u)\,du - \int_a^x f(u)\,du \\
&= 2\int_a^0 f(u)\,du - \int_a^x f(u)\,du \;=\; 2A(0) - \int_a^x f(t)\,dt \\
&= 0 - \int_a^x f(t)\,dt \;=\; -A(x)\;,
\end{aligned}$$

verifying that A is odd, completing the proof of part (ii).