THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Exercises for Week 2 (beginning 7 August)** 2017

It might be useful to attempt the Revision and Exploration Exercises before the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) Areas under curves and the definite integral:

Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is a function. Divide $[a, b]$ into n subintervals using

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\ .$$

Choose $x_i \in [t_{i-1}, t_i]$ for each i, and put $\Delta x_i = t_i - t_{i-1}$, the width of the ith subinterval. Then the (signed) area under the curve $y = f(x)$ over $[a, b]$ is approximated by the *Riemann sum*

$$\sum_{i=1}^{n} f(x_i)\Delta x_i \ = \ f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n\ .$$

Each summand $f(x_i)\Delta x_i$ is the area of an approximating rectangle of width Δx_i and height $f(x_i)$. The exact value of the (signed) area under the curve is the limit, when it exists, of the Riemann sums

$$\int_a^b f(x)dx \ = \ \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i\ ,$$

where we assume that widths of subintervals tend to zero as n increases. If this limit exists, independently of all possible choices of partitions and x_i, then f is called *Riemann integrable*, and $\int_a^b f(x)dx$ is called the *definite integral of f from a to b.*

(ii) Main Theorem: Continuous functions are Riemann integrable.

(iii) Approximations using upper and lower Riemann sums:

If we choose x_i within the ith subinterval so that $f(x_i)$ is a maximum value over the subinterval, then the Riemann sum is called an *upper sum*, denoted by U_n if there are n subintervals. If we choose x_i so that $f(x_i)$ is a minimum value over the subinterval, then the Riemann sum is called a *lower sum*, denoted by L_n. Then

$$L_n \ \le \ \int_a^b f(x)dx \ \le \ U_n\ ,$$

providing lower and upper bounds respectively for the area under the curve.

In the special case that the curve is increasing or decreasing over the entire interval $[a, b]$, then the various x_i used to form the Riemann sums will be either left-hand or right-hand endpoints of the subintervals (depending on whether the sum is upper or lower, and whether the function is increasing or decreasing).

(iv) Properties of the Definite Integral:

(a) $\int_a^b c\,dx = c(b-a)$ and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c

(b) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(c) $\int_a^a f(x)\,dx = 0$ and $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (defined also when $a \geq b$)

(d) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

(e) If $f(x) \leq g(x)$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

(f) If $m \leq f(x) \leq M$, for constants m and M, then

$$m(b-a) \;\leq\; \int_a^b f(x)\,dx \;\leq\; M(b-a)\,.$$

Revision and Exploration:

1. Write out the definitions of *even function* and *odd function*. Verify that if $f : \mathbb{R} \to \mathbb{R}$ is both even and odd then f is the zero function.

2. Differentiate $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \tan^{-1} x$, $y = \frac{1}{x^2+1}$ and $y = \frac{x}{x^2+1}$.

3. Which of the functions in the previous exercise and their derivatives are even or odd? Can you make and prove a conjecture about derivatives of odd and even functions?

4. Write out the Mean Value Theorem. Use it to prove that if $f : [0, \infty) \to \mathbb{R}$ is a function such that the derivative $f'(x)$ is positive for all $x > 0$, then f is strictly increasing.

5. Use the derivative to explain why the curve $y = \sqrt{1+x^3}$ is strictly increasing on $[0, \infty)$. Can you see this also without using calculus?

Tutorial Exercises:

6. Use upper and lower Riemann sums with 5 equal subintervals to estimate $\int_1^2 \sqrt{1+x^3}\,dx$.

7. A speedboat accelerates from rest (with increasing velocity), reaching a speed of 40 m/sec as it moves in a straight line over a period of 20 seconds. Velocities were measured every 4 seconds, and recorded in the following table:

time (sec)	0	4	8	12	16	20
vel (m/sec)	0	8	19	29	36	40

(i) Use lower and upper Riemann sums to find bounds for the distance travelled by the boat.

*(ii) How often would measurements need to be taken to guarantee that lower and upper Riemann sums differ from the actual distance travelled by less than 10m?

8. Suppose f is an odd function and g is an even function such that

$$\int_0^2 \frac{f(x)}{2}\,dx = 5\,, \qquad \int_0^3 f(x)\,dx = 7\,, \qquad \int_0^3 g(x)\,dx = -2\,.$$

Find

(i) $\displaystyle\int_2^3 f(x)\,dx$ (ii) $\displaystyle\int_0^3 \frac{f(x)-3g(x)}{2}\,dx$ (iii) $\displaystyle\int_{-3}^3 f(x)\,dx$

(iv) $\displaystyle\int_3^{-3} g(x)\,dx$ *(v) $\displaystyle\int_{-3}^3 (x - f(x)g(x))^{\frac{1}{3}}\,dx$

***9.** Recall from lectures that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}\,.$$

Now find

$$\sum_{i=1}^{n} i^3 \;=\; 1^3 + 2^3 + 3^3 + \ldots + n^3\,.$$

[Hint: pretend n^4 is a telescope.]

10. Use your answer to the previous exercise and upper Riemann sums for the partition

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < \frac{n}{n} = 1$$

to find $\displaystyle\int_0^1 x^3\,dx$.

11. (for general discussion) Is it obvious that π is sensibly defined? Given circles $\mathcal{C}_1$, $\mathcal{C}_2$ with radii r_1, r_2 and perimeters P_1, P_2 respectively, show that we get the common ratio (being the definition of π):

$$\frac{P_1}{2r_1} = \frac{P_2}{2r_2}$$

[Hint: consider a corresponding property for similar triangles and take limits.]

Further Exercises:

12. In a chemical reaction, the rate at which a precipitate is formed is a decreasing function of time. In an experiment the following rates were recorded:

time (sec)	0	1	2	3	4	5	6
rate (g/sec)	12	8.4	5.9	4.1	2.9	2.0	1.4

(i) Find lower and upper bounds for the total mass of precipitate formed in these 6 seconds.

(ii) How often would measurements need to be taken to guarantee that lower and upper bounds differ from the actual mass of precipitate by less than 1g?

***13.** Use the addition limit law to verify that the definite integral is additive.

***14.** Let $f(x) = 1/x^2$ for $0 < a \le x \le b$. Let

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

be any partition of $[a, b]$. For each i, choose $x_i = \sqrt{t_{i-1} t_i}$ (the geometric mean) and as usual put $\Delta x_i = t_i - t_{i-1}$. Verify that $x_i \in [t_{i-1}, t_i]$ and that

$$f(x_i)\Delta x_i = \frac{1}{t_{i-1}} - \frac{1}{t_i}.$$

Deduce quickly the value of $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x_i$ and observe that it is independent of the partition. Therefore write down $\displaystyle\int_a^b f(x)\,dx$.

***15.** Prove that between any two distinct real numbers lies both a rational and an irrational real number. (You may take it for granted that $\sqrt{2}$ is irrational.)

****16.** Let f be the function defined by the following rule:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

Is f continuous? Is f differentiable at any point? Is f Riemann integrable on $[0, 1]$?

****17.** Suppose that f is continuous and nonnegative on $[a, b]$. Show that if $\int_a^b f(x)dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$. What happens if we drop the assumption of continuity?

Short Answers to Selected Exercises:

6. lower bound 1.97, upper bound 2.29

7. (i) lower bound 368, upper bound 528 (ii) 4 times per second

8. (i) -3 (ii) $\frac{13}{2}$ (iii) 0 (iv) 4 (v) 0

9. $\left(\frac{n(n+1)}{2}\right)^2$

10. $\frac{1}{4}$

12. (i) lower bound 24.7, upper bound 35.3 (ii) about 11 times per second

14. $\frac{1}{a} - \frac{1}{b}$