CHAPTER **TEN**

Probability and Counting

Probability arises when one performs an experiment that has various possible outcomes, but for which there is insufficient information to predict precisely which of these outcomes will occur. The classic examples of this are tossing a coin, throwing a die, or drawing a card from a pack. Probability, however, is involved in almost every experiment done in science, and is fundamental to understanding statistics. This chapter will first review some of the basic ideas of probability, using the language of sets, and then establish various systematic counting procedures that will allow more complicated probability questions to be solved. These counting procedures open up the link between probability theory and the binomial expansion, leading to the development of binomial probability.

Study Notes: Sections 10A–10C review in a more systematic manner the basic ideas of probability from earlier years. The language of sets is used here, particularly in Section 10B to deal with 'and', 'or' and 'not' — it may help to review Section 1J in the Year 11 volume, which presented a straightforward account of the elementary ideas about sets. Section 10D deals with probability tree diagrams, which will be new to most students. Further work in probability requires some systematic counting procedures, which are developed in Sections 10E–10G, and applied to probability in Sections 10H and 10I. Section 10J on binomial probability combines the binomial theorem from Chapter Five with the counting procedures.

Throughout the chapter, attention should be given to the fallacies and confusions that inevitably arise in any discussion of probability.

10 A Probability and Sample Spaces

Our first task is to develop a workable formula for probability that can serve as the foundation for the topic. That formula will rest on the idea of dividing the results of an experiment into equally likely possible outcomes. We will also need to make reference to probabilities that are experimentally determined.

Equally Likely Possible Outcomes: The idea of equally likely possible outcomes is well illustrated by the experiment of throwing a die. (A *die*, plural *dice*, is a cube with its corners and edges rounded so that it rolls easily, and with the numbers 1–6 printed on its six sides.) The outcome of this experiment is the number on the top face when the die stops rolling, giving six *possible outcomes* — 1, 2, 3, 4, 5, 6. This is a complete list of the possible outcomes, because each time the die is rolled, one and only one of these outcomes can occur.

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Provided that the die is completely symmetric, that is, it is not *biased* in any way, we have no reason to expect that any one outcome is more likely to occur than any of the other five, and we call these six possible outcomes *equally likely possible outcomes*. With the results of the experiment thus divided into six equally likely possible outcomes, we now assign the probability $\frac{1}{6}$ to each of these six outcomes. Notice that the six probabilities are equal and they all add up to 1. The general case is as follows.

1 EQUALLY LIKELY POSSIBLE OUTCOMES: Suppose that the possible results of an experiment can be divided into n *equally likely possible outcomes* — meaning that one and only one of these n outcomes will occur, and there is no reason to expect one outcome to be more likely than another.

Then the probability $\dfrac{1}{n}$ is assigned to each of these equally likely possible outcomes.

Randomness: Notice that it has been assumed that the terms 'more likely' and 'equally likely' already have a meaning in the mind of the reader. There are many ways of interpreting these words. In the case of a thrown die, one could interpret the phrase 'equally likely' as meaning that the die is perfectly symmetric. Alternatively, one could interpret it as saying that we lack entirely the knowledge to make any statement of preference for one outcome over another.

The word *random* can be used here. In the context of equally likely possible outcomes, saying that a die is thrown 'randomly' means that we are justified in assigning the same probability to each of the six possible outcomes. In a similar way, we will speak about drawing a card 'at random' from a pack, or forming a queue of people in a 'random order'.

The Fundamental Formula for Probability: Suppose that we need a throw of at least 3 on a die to win a game. Then getting at least 3 is called the particular *event* under discussion, the outcomes 3, 4, 5 and 6 are called *favourable* for this event, and the other two possible outcomes 1 and 2 are called *unfavourable*. The probability assigned to getting a score of at least 3 is then

$$\begin{aligned}\mathcal{P}(\text{scoring at least 3}) &= \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}\\ &= \frac{4}{6}\\ &= \frac{2}{3}\,.\end{aligned}$$

In general, if the results of an experiment can be divided into a number of equally likely possible outcomes, some of which are favourable for a particular event and the others unfavourable, then:

2 THE FUNDAMENTAL FORMULA FOR PROBABILITY:

$$\mathcal{P}(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

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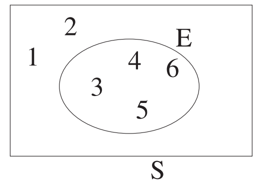
The Sample Space and the Event Space: The language of sets makes some of the theory of probability easier to explain — some review of Section 1J of the Year 11 volume may be helpful at this stage. The Venn diagram on the right shows the six possible outcomes when a die is thrown. The set of all these outcomes is



$$S = \{1, 2, 3, 4, 5, 6\}.$$

This set is called the *sample space* and is represented by the outer rectangular box. The event 'scoring at least 3' is the set

$$E = \{3, 4, 5, 6\},$$

which is called the *event space* and is represented by the ellipse. In general, the set of all equally likely possible outcomes is called the *sample space*, and the set of all favourable outcomes is called the *event space*. The basic probability formula can then be restated in set language.

3 THE SAMPLE SPACE AND THE EVENT SPACE: Suppose that an event E has sample space S. Then, using the symbol $|A|$ for the number of members of a set A,

$$\mathcal{P}(E) = \frac{|E|}{|S|}.$$

Probabilities Involving Playing Cards: So many questions in probability involve a pack of playing cards that any student of probability needs to be familiar with them — the reader is encouraged to acquire some cards and play some simple games with them. A pack of cards consists of 52 cards organised into four *suits*, each containing 13 cards. The four suits are

two black suits: $\clubsuit$ clubs, $\spadesuit$ spades,
two red suits: $\diamondsuit$ diamonds, $\heartsuit$ hearts.

Each of the four suits contains 13 cards:

A (Ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (Jack), Q (Queen), K (King).

An ace can also be regarded as a 1. It is assumed that when a pack of cards is shuffled, the order is totally *random*, meaning that there is no reason to expect any one ordering of the cards to be more likely to occur than any other.

WORKED EXERCISE: A card is drawn at random from a pack of playing cards. Find the probability that the card is:

(a) the seven of hearts, (b) a heart, (c) a seven, (d) a red card,
(e) a picture card (Jack, Queen, King), (f) a green card, (g) red or black.

SOLUTION: In each case, there are 52 equally likely possible outcomes.

(a) Since there is 1 seven of hearts, $\mathcal{P}(7\heartsuit) = \frac{1}{52}$.
(b) Since there are 13 hearts, $\mathcal{P}(\text{heart}) = \frac{13}{52} = \frac{1}{4}$.
(c) Since there are 4 sevens, $\mathcal{P}(\text{seven}) = \frac{4}{52} = \frac{1}{13}$.
(d) Since there are 26 red cards, $\mathcal{P}(\text{red card}) = \frac{26}{52} = \frac{1}{2}$.
(e) Since there are 12 picture cards, $\mathcal{P}(\text{picture card}) = \frac{12}{52} = \frac{3}{13}$.
(f) Since no card is green, $\mathcal{P}(\text{green card}) = \frac{0}{52} = 0$.
(g) Since all 52 cards are red or black, $\mathcal{P}(\text{red or black card}) = \frac{52}{52} = 1$.

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Impossible and Certain Events: Parts (f) and (g) of the previous worked exercise were intended to illustrate the probabilities of events that are impossible or certain. Since getting a green card is impossible, there are no favourable outcomes, so the probability is 0. Since all the cards are either red or black, and getting a red or black card is certain to happen, all possible outcomes are favourable outcomes, so the probability is 1. Notice that for the other five events, the probability lies between 0 and 1.

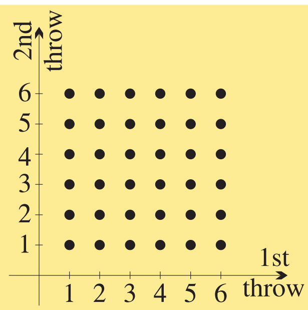
4

IMPOSSIBLE AND CERTAIN EVENTS:
- An event has probability 0 if and only if it cannot happen.
- An event has probability 1 if and only if it is certain to happen.
- For any other event, $0 < \mathcal{P}(\text{event}) < 1$.

Graphing the Sample Space: Many experiments consist of several *stages*. For example, when a die is thrown twice, the two throws can be regarded as two separate stages of the one experiment. The reason for using the word 'sample space' rather than 'sample set' is that the sample space of a multi-stage experiment takes on some of the characteristics of a space. In particular, the sample space of a two-stage experiment can be displayed on a two-dimensional graph, and the sample space of a three-stage experiment can be displayed in a three-dimensional graph. The following worked example shows how a two-dimensional dot diagram can be used for calculations with the sample space of a die thrown twice.

WORKED EXERCISE: A die is thrown twice. Find the probability that:

(a) the pair is a double,
(b) at least one number is four,
(c) both numbers are greater than four,
(d) both numbers are even,
(e) the sum of the two numbers is six,
(f) the sum is at most four.



SOLUTION: The horizontal axis in the diagram to the right represents the six possible outcomes of the first throw, and the vertical axis represents the six possible outcomes of the second throw. The 36 dots therefore represent the 36 different possible outcomes of the two-stage experiment, all equally likely, which is the full sample space. The various parts can now be answered by counting the dots representing the various event spaces.

(a) Since there are 6 doubles, $\mathcal{P}(\text{double}) = \frac{6}{36} = \frac{1}{6}$.
(b) Since 11 pairs contain a 4, $\mathcal{P}(\text{at least one is a 4}) = \frac{11}{36}$.
(c) Since 4 pairs consist only of 5 or 6, $\mathcal{P}(\text{both greater than 4}) = \frac{4}{36} = \frac{1}{9}$.
(d) Since 9 pairs have two even members, $\mathcal{P}(\text{both even}) = \frac{9}{36} = \frac{1}{4}$.
(e) Since 5 pairs have sum 6, $\mathcal{P}(\text{sum is 6}) = \frac{5}{36}$.
(f) Since 6 pairs have sum 2, 3 or 4, $\mathcal{P}(\text{sum at most 4}) = \frac{6}{36} = \frac{1}{6}$.

Tree Diagrams: Listing the sample space of a multi-stage experiment can be difficult, and the dot diagrams of the previous paragraph are hard to draw in more than two dimensions. Tree diagrams provide a very useful alternative way to display the sample space. Such diagrams have a column for each stage, plus an initial column labelled 'Start' and a final column listing the possible outcomes.

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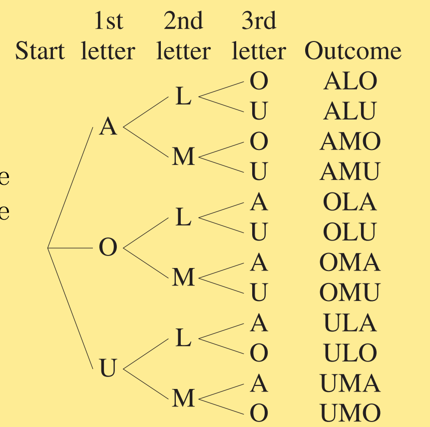
WORKED EXERCISE: A three-letter word is chosen in the following way. The first and last letters are chosen from the three vowels 'A', 'O' and 'U', with repetition not allowed, and the middle letter is chosen from 'L' and 'M'. List the sample space, then find the probability that:

(a) the word is 'ALO',
(b) the letter 'O' does not occur,
(c) 'M' and 'U' do not both occur,
(d) the letters are in alphabetical order.



SOLUTION: The tree diagram to the right lists all twelve equally likely possible outcomes. The two vowels must be different, because repetition was not allowed.

(a) $\mathcal{P}(\text{'ALO'}) = \frac{1}{12}$.
(b) $\mathcal{P}(\text{no 'O'}) = \frac{4}{12} = \frac{1}{3}$.
(c) $\mathcal{P}(\text{not both 'M' and 'U'}) = \frac{8}{12} = \frac{2}{3}$.
(d) $\mathcal{P}(\text{alphabetical order}) = \frac{4}{12} = \frac{1}{3}$.

The Meaning of 'Word': In the worked exercise above, and throughout this chapter, 'word' simply means an arrangement of letters — the arrangement doesn't have to have any meaning or be a word in the dictionary. Thus 'word' simply becomes a convenient device for discussing arrangements of things in particular orders.

Invalid Arguments: Arguments offered in probability theory can be invalid for all sorts of subtle reasons, and it is common for a question to ask for comment on a given argument. It is most important in such a situation that any fallacy in the given argument be explained — it is not sufficient only to offer an alternative argument with a different conclusion.

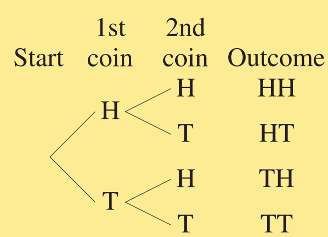
WORKED EXERCISE: Comment on the validity of these arguments.

(a) 'When two coins are tossed together, there are three outcomes: two heads, two tails, and one of each. Hence the probability of getting one of each is $\frac{1}{3}$.'
(b) 'Brisbane is one of fourteen teams in the Rugby League, so the probability that Brisbane wins the premiership is $\frac{1}{14}$.'

SOLUTION:

(a) [Identifying the fallacy] The division of the results into the three given outcomes is correct, but no reason is offered as to why these outcomes are equally likely.

[Supplying the correct argument] The diagram on the right divides the results of the experiment into four *equally likely* possible outcomes; since two of these outcomes, HT and TH, are favourable to the event 'one of each', it follows that $\mathcal{P}(\text{one of each}) = \frac{2}{4} = \frac{1}{2}$.



(b) [Identifying the fallacy] The division into fourteen possible outcomes is correct provided that one assumes that a tie for first place is impossible, but no reason has been offered as to why each team is equally likely to win, so the argument is invalid.

[Offering a replacement question] What can be said with confidence is that if a team is selected at random from the fourteen teams, then the probability that it is the premiership-winning team is $\frac{1}{14}$.

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NOTE: It is difficult to give a complete account of part (b). It is not clear that an exact probability can be assigned to the event 'Brisbane wins', although we can safely assume that those with knowledge of the game would have some idea of ranking the fourteen teams in order from most likely to win to least likely to win. If there is an organised system of betting, one might, or might not, agree to take this as an indication of the community's collective wisdom on the fourteen probabilities.

Experimental Probability: When a drawing pin is thrown, there are two possible outcomes, point-up and point-down. But these two outcomes are not equally likely, and there seems to be no way to analyse the results of the experiment into equally likely possible outcomes. In the absence of any fancy arguments from physics about rotating pins falling on a smooth surface, however, we can gain some estimate of the two probabilities by performing the experiment a number of times.

The questions in the following worked example could raise difficult issues beyond the scope of this course, but the intention here is only that they be answered briefly in a common-sense manner.

WORKED EXERCISE: A drawing pin is thrown 400 times, and falls point-up 362 times.

(a) What probability does this experiment suggest for the result 'point-up'?

(b) Discuss whether the results are inconsistent with a probability of:

(i) $\frac{9}{10}$ (ii) $\frac{11}{12}$ (iii) $\frac{1}{2}$ (iv) $\frac{4}{5}$

(c) A machine repeats the experiment 1 000 000 times and the pin falls point-up 916 203 times. What answers would you now give for part (b)?

SOLUTION:

(a) These results suggest $\mathcal{P}(\text{point-up}) \doteqdot 0{\cdot}905$, but with only 400 trials, there would be little confidence in this result past the second, or even the first, decimal place, since we would expect different runs of the same experiment to differ by small numbers. So we conclude that $\mathcal{P}(\text{point-up}) \doteqdot 0{\cdot}9$.

(b) Since a probability of $\frac{9}{10}$ would predict about 360 point-up results and $\frac{11}{12}$ would predict about 367 point-up results, both these fractions seem consistent with the experiment. If the probability were $\frac{1}{2}$, however, we would have expected about 200 point-up results, so this can be excluded. Similarly, a probability of $\frac{4}{5}$ would predict about 320 point-up results, and can reasonably be excluded.

(c) A probability of $\frac{9}{10}$ would predict about 900 000 point-up results, and $\frac{11}{12}$ would predict about 916 667 point-up results. Hence the estimate of $\frac{11}{12}$ seems reasonable, but the estimate of $\frac{9}{10}$ can now reasonably be rejected.

Interpreting Probability as an Experimental Limit: The experimental meaning of probability is a little elusive. The probability of getting 5 or 6 on a die is $\frac{1}{3}$, but what does this mean when we repeatedly throw a die? If we throw the die 60 times, we would expect to get 5 or 6 about $\frac{1}{3} \times 60 = 20$ times, although if it happened 17–23 times, it would probably not disturb our confidence. If we threw the die 6000 times, we would expect to get 5 or 6 about $\frac{1}{3} \times 6000 = 2000$ times (but interestingly enough we would be quite surprised if we got 5 or 6 exactly 2000 times). Thus one interpretation of the statement that getting 5 or 6 has probability $\frac{1}{3}$ is:

$$\text{As (number of trials)} \to \infty, \quad \frac{\text{number of 5s and 6s}}{\text{number of trials}} \to \frac{1}{3}\,.$$

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Some experiments, however, are inherently unrepeatable, and yet we seem to understand what it means to assign probabilities to them. For example, many people bet money on Australia beating England in a particular test series, and their bets seem to rely on some estimate of the probability that Australia will win that particular series. (Most would agree that this probability is much greater than $\frac{1}{2}$!)

Exercise **10A**

1. A coin is tossed. Find the probability that it shows:
(a) a head, (b) a tail, (c) either a head or a tail, (d) neither a head nor a tail.

2. If a die is rolled, find the probability that the uppermost face is:
(a) three, (b) an even number, (c) a number greater than four, (d) a multiple of three.

3. A bag contains eight red balls, seven yellow balls and three green balls. A ball is selected at random. Find the probability it is: (a) red, (b) yellow or green, (c) not yellow.

4. In a bag there are four red capsicums, three green capsicums, six red apples and five green apples. One item is chosen at random. Find the probability that it is:
(a) green,
(b) red,
(c) an apple,
(d) a capsicum,
(e) a red apple,
(f) a green capsicum.

5. A letter is randomly selected from the 26 letters in the English alphabet. Find the probability that the letter is:
(a) the letter S,
(b) a vowel,
(c) a consonant,
(d) the letter γ,
(e) either C, D or E,
(f) one of the letters of the word MATHS.
[NOTE: The letter Y is normally classified as a consonant.]

6. A number is selected at random from the integers 1, 2, 3, . . . , 19, 20. Find the probability of choosing:
(a) the number 4,
(b) a number greater than 15,
(c) an even number,
(d) an odd number,
(e) a prime number,
(f) a square number,
(g) a multiple of 4,
(h) the number e,
(i) a rational number.

7. From a regular pack of 52 cards, one card is drawn at random. Find the probability that:
(a) it is black,
(b) it is red,
(c) it is a king,
(d) it is the jack of hearts,
(e) it is a club,
(f) it is a picture card,
(g) it is a heart or a spade,
(h) it is a red five or a black seven,
(i) it is less than a four.

8. A book has 150 pages. The book is randomly opened at a page. Find the probability that the page number is:
(a) greater than 140,
(b) a multiple of 20,
(c) an odd number,
(d) a number less than 25,
(e) either 72 or 111,
(f) a three-digit number.

9. An integer x, where $1 \le x \le 200$, is chosen at random. Determine the probability that it:
(a) is divisible by 5,
(b) is a multiple of 13,
(c) has two digits,
(d) is a square number,
(e) is greater than 172,
(f) has three equal digits.

10. A bag contains three times as many yellow marbles as blue marbles. If a marble is chosen at random, find the probability that it is: (a) yellow, (b) blue.

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11. From a group of four students, Anna, Bill, Charlie and David, two are chosen at random to be on the Student Representative Council. List the sample space, and hence find the probability that:
(a) Bill and David are chosen,
(b) Anna is chosen,
(c) Charlie is chosen but Bill is not,
(d) neither Anna nor David is selected.

12. A fair coin is tossed twice. Use a tree diagram to list the possible outcomes. Hence find the probability that the two tosses result in:
(a) two heads, (b) a head and a tail, (c) a head on the first toss and a tail on the second.

13. A die is rolled and a coin is tossed. Use a tree diagram to list all the possible outcomes. Hence find the probability of obtaining:
(a) a head and an even number,
(b) a tail and a number greater than four,
(c) a tail and a number less than four,
(d) a head and a prime number.

14. From the integers 2, 3, 8 and 9, two-digit numbers are formed in which no digit can be repeated in the same number.
(a) Draw a tree diagram to illustrate the possible outcomes.
(b) If one of the two-digit numbers is chosen at random, find the probability that it is:
(i) the number 82,
(ii) a number greater than 39,
(iii) an even number,
(iv) a multiple of 3,
(v) a number ending in 2,
(vi) a perfect square.

15. A captain and vice-captain of a cricket team are to be chosen from Amanda, Belinda, Carol, Dianne and Emma.
(a) Use a tree diagram to list the possible pairings, noting that order is important.
(b) Find the probability that: (i) Carol is captain and Emma is vice-captain,
(ii) Belinda is either captain or vice-captain,
(iii) Amanda is not selected for either position,
(iv) Emma is vice-captain.

16. A hand of five cards contains a ten, jack, queen, king and ace. From the hand, two cards are drawn in succession, the first card not being replaced before the second card is drawn.
(a) Find the probability that: (i) the ace is selected, (ii) the king is not selected, (iii) the queen is the second card chosen.
(b) Repeat part (a) if the first card is replaced before the second card is drawn.

17. Two dice are thrown simultaneously. List the set of 36 possible outcomes on a two-dimensional graph, and hence find the probability of:
(a) obtaining a three on the first throw,
(b) obtaining a four on the second throw,
(c) a double five,
(d) a total score of seven,
(e) a total score greater than nine,
(f) an even number on both dice,
(g) at least one two,
(h) neither a one nor a four appearing,
(i) a five and a number greater than three,
(j) the same number on both dice.

18. Fifty tagged fish were released into a dam known to contain fish. Later a sample of thirty fish was netted from this dam, of which eight were found to be tagged. Estimate the total number of fish in the dam just prior to the sample of thirty being removed.

19. A biased coin is tossed 300 times, and lands on heads 227 times.
(a) What probability does this experiment suggest for the result 'heads'?
(b) Discuss whether the results are inconsistent with a probability of:
(i) $\frac{1}{2}$ (ii) $\frac{3}{4}$ (iii) $\frac{7}{9}$ (iv) $\frac{5}{8}$

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DEVELOPMENT

20. A coin is tossed three times. Draw a tree diagram to illustrate the possible outcomes. Then find the probability of obtaining:

(a) three heads,
(b) a head and two tails,
(c) at least two tails,
(d) at most one head,
(e) more heads than tails,
(f) a head on the second toss.

21. If the births of boys and girls are equally likely, determine the probability that:

(a) in a family of two children there are:
(i) two girls, (ii) no girls, (iii) one boy and one girl.

(b) in a family of three children there are:
(i) three boys, (ii) two girls and one boy, (iii) more boys than girls.

22. An unbiased coin is tossed four times. Find the probability of obtaining:

(a) four heads,
(b) exactly three tails,
(c) at least two heads,
(d) at most one head,
(e) two heads and two tails,
(f) more tails than heads.

23. A rectangular field is 60 metres long and 30 metres wide. A cow wanders randomly around the field. Find the probability that the cow is:

(a) more than 10 metres from the edge of the field,
(b) not more than 10 metres from a corner of the field.

24. Comment on the following arguments. Identify precisely any fallacies in the arguments, and if possible, give some indication of how to correct them.

(a) 'On every day of the year it either rains or it doesn't. Therefore the chance that it will rain tomorrow is $\frac{1}{2}$.'
(b) 'When the Sydney Swans play Hawthorn, either Hawthorn wins, the Swans win or the game is a draw. Therefore the probability that the next game between these two teams results in a draw is $\frac{1}{3}$.'
(c) 'When answering a multiple-choice test in which there are four possible answers given to each question, the chance that Peter answers a question correctly is $\frac{1}{4}$.'
(d) 'A bag contains a number of red, white and black beads. If you choose one bead at random from the bag, the probability that it is black is $\frac{1}{3}$.'
(e) 'Four players play in a knockout tennis tournament resulting in a single winner. A man with no knowledge of the game or the players declares that one particular player will win his semi-final, but lose the final. The probability that he is correct is $\frac{1}{4}$.'

25. If a coin is tossed n times, where $n > 1$, find the probability of obtaining:

(a) n heads, (b) at least one head and at least one tail.

EXTENSION

26. [Buffon's needle problem — an example of continuous probability]

(a) A needle, whose length is equal to the width of the floorboards, is thrown at random onto the floor. Show that if its inclination to the cracks is θ, then the probability that it lies across a crack is $\sin\theta$. Then, by integrating across all possible angles, show that the probability that it lies across a crack is $\frac{2}{\pi}$.
(b) Hence devise a probabilistic experiment to compute π.
(c) Find the probability if the needle has length half the width of the floorboards.
(d) Find the probability if the needle has length twice the width of the floorboards, but take care — some angles are a problem because probabilities cannot ever exceed 1.

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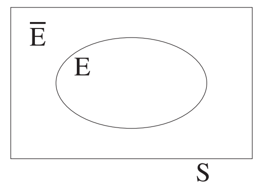
10 B Probability and Venn Diagrams

The language of sets was introduced in the previous section when speaking about the sample space and the event space. One outcome of this is that we can use Venn diagrams to visualise the possible outcomes, and calculations become easier.

Complementary Events and the Word 'Not': It is often easier to find the probability that an event does not occur than the probability that it does occur. The *complementary event* of an event E is the event 'E does not occur', and is written as $\overline{E}$. Using a Venn diagram, the complementary event $\overline{E}$ is represented by the region outside the circle in the diagram to the right.



Since $|\overline{E}| = |S| - |E|$, it follows that $\mathcal{P}(\overline{E}) = 1 - \mathcal{P}(E)$.

5 COMPLEMENTARY EVENTS: Suppose that E is an event with sample space S. Define the *complementary event* $\overline{E}$ to be the event 'E does not occur'. Then

$$\mathcal{P}(\overline{E}) = 1 - \mathcal{P}(E)\,.$$

In Section 1J of the Year 11 volume, we defined the *complement* $\overline{E}$ of a set E to be the set of things in S but not in E, and commented that the complement of a set is closely linked to the word 'not'. The new notation $\overline{E}$ for complementary event is quite deliberately the same notation as that for the complement of a set.

WORKED EXERCISE: What is the probability of failing to throw a double six when throwing a pair of dice?

SOLUTION: As discussed in the previous section, the double six is but one outcome amongst 36 possible outcomes, and so has probability $\frac{1}{36}$.

Hence $\mathcal{P}(\text{not throwing a double six}) = 1 - \mathcal{P}(\text{double six})$
$= \frac{35}{36}$.

WORKED EXERCISE: A card is drawn at random from a pack. Find the probability that it is an even number, or a picture card, or red.

NOTE: Remember that the word 'or' always means 'and/or' in logic and mathematics. Thus in this worked example, the words 'or any two of these, or all three of these' are understood, and need not be added.

SOLUTION: The complementary event $\overline{E}$ is drawing a card that is a black odd number less than ten. This complementary event has ten members:

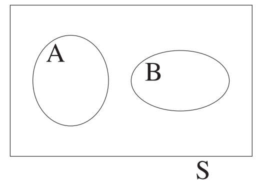
$$\overline{E} = \{\, A\clubsuit, 3\clubsuit,\ 5\clubsuit,\ 7\clubsuit,\ 9\clubsuit,\ A\spadesuit,\ 3\spadesuit,\ 5\spadesuit,\ 7\spadesuit,\ 9\spadesuit \,\}.$$

There are 52 possible cards to choose, so using the complementary event formula above,

$$\mathcal{P}(E) = 1 - \tfrac{10}{52} = \tfrac{42}{52} = \tfrac{21}{26}.$$

Mutually Exclusive Events and Disjoint Sets: Two events A and B with the same sample space S are called *mutually exclusive* if they cannot both occur. In the Venn diagram of such a situation, the two events A and B are represented as disjoint sets (*disjoint* means that their intersection is empty).

In this situation, the event 'A and B' is impossible, and has probability zero. On the other hand, the event 'A or B' is represented on the Venn diagram by the union $A \cup B$ of the two sets.



Since $|A \cup B| = |A| + |B|$ for disjoint sets, it follows that

$$\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B)\,.$$

6 MUTUALLY EXCLUSIVE EVENTS: Suppose that A and B are mutually exclusive events with sample space S. Then the event 'A or B' is represented by $A \cup B$, and

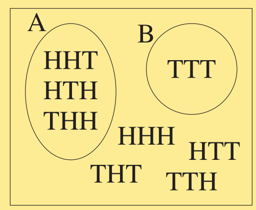
$$\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B)\,.$$

The event 'A and B' cannot occur, and has probability zero.

WORKED EXERCISE: If three coins are tossed, find the probability of tossing an odd number of tails.

SOLUTION: Let A be the event 'one tail' and B the event 'three tails'. Then A and B are mutually exclusive, and

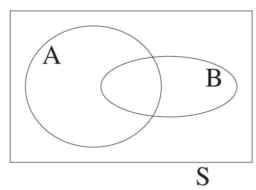
$$A = \{\,\text{HHT, HTH, THH}\,\} \qquad \text{and} \qquad B = \{\,\text{TTT}\,\}.$$



The full sample space has eight members altogether (question 20 in the previous exercise lists them all), so

$$\mathcal{P}(A \text{ or } B) = \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{1}{2}.$$

The Events 'A and B' and 'A or B' — The Addition Rule: More generally, suppose that A and B are any two events with the same sample space S, not necessarily mutually exclusive. The Venn diagram of the situation will now represent the two events A and B as overlapping sets within the same universal set S. The event 'A and B' will then be represented by the intersection $A \cap B$ of the two sets, and the event 'A or B' will be represented by the union $A \cup B$.



The general counting rule for sets is $|A \cup B| = |A| + |B| - |A \cap B|$, because the members of the intersection $A \cap B$ are counted in A and again in B, and so have to be subtracted. It follows then that $\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \text{ and } B)$ — this rule is often called the *addition rule* of probability.

7 THE EVENTS 'A OR B' AND 'A AND B': Suppose that A and B are two events with sample space S. Then the event 'A and B' is represented by the intersection $A \cap B$ and the event 'A or B' is represented by the union $A \cup B$, and

$$\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \text{ and } B).$$

It was explained in Section 1J of the Year 11 volume that the word 'or' is closely linked with the union of sets, and the word 'and' is closely linked with the intersection of sets. For this reason, the event 'A or B' is often written as '$A \cup B$', and the event 'A and B' is often written as '$A \cap B$' or just 'AB'.

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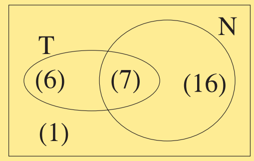
WORKED EXERCISE: In a class of 30 girls, 13 play tennis and 23 play netball. If 7 girls play both sports, what is the probability that a girl chosen at random from the class plays neither sport?

SOLUTION: Let T be the event 'she plays tennis', and let N be the event 'she plays netball'.



Then $\mathcal{P}(T) = \frac{13}{30}$

$\mathcal{P}(N) = \frac{23}{30}$

and $\mathcal{P}(N \text{ and } T) = \frac{7}{30}$.

Hence $\mathcal{P}(N \text{ or } T) = \frac{13}{30} + \frac{23}{30} - \frac{7}{30}$

$= \frac{29}{30}$,

and $\mathcal{P}(\text{neither sport}) = 1 - \mathcal{P}(N \text{ or } T)$

$= \frac{1}{30}$.

NOTE: An alternative approach is shown in the diagram. Starting with the 7 girls in the intersection, the numbers 6 and 16 can then be written into the respective regions 'tennis but not netball' and 'netball but not tennis'. Since these numbers add to 29, this leaves only one girl playing neither tennis nor netball.

Exercise 10B

1. [A brief review of set notation. Students should refer to Exercise 1J of the Year 11 volume for a more substantial list of questions.]

(a) Find $A \cup B$ and $A \cap B$ for each pair of sets:

(i) $A = \{1, 3, 5\}$, $B = \{3, 5, 7\}$ (ii) $A = \{1, 3, 4, 8, 9\}$, $B = \{2, 4, 5, 6, 9, 10\}$

(iii) $A = \{\text{h, o, b, a, r, t}\}$, $B = \{\text{b, i, c, h, e, n, o}\}$

(iv) $A = \{\text{prime numbers less than } 10\}$, $B = \{\text{odd numbers less than } 10\}$

(b) Let $A = \{1, 3, 7, 10\}$ and $B = \{4, 6, 7, 9\}$, and take the universal set to be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. List the members of:

(i) $\overline{A}$ (ii) $\overline{B}$ (iii) $A \cap B$ (iv) $\overline{A \cap B}$ (v) $A \cup B$ (vi) $\overline{A \cup B}$

2. A student has a 22% chance of being chosen as a prefect. What is the chance that he will not be chosen as a prefect?

3. When breeding labradors, the probability of breeding a black dog is $\frac{3}{7}$.

(a) What is the probability of breeding a dog that is not black?

(b) If you breed 56 dogs, how many would you expect to be not black?

4. The chance of a new light globe being defective is $\frac{1}{15}$.

(a) What is the probability that a new light globe will not be defective?

(b) If 120 new light globes were checked, how many would you expect to be defective?

5. A die is rolled. If n denotes the number on the uppermost face, find:

(a) $\mathcal{P}(n = 5)$

(b) $\mathcal{P}(n \neq 5)$

(c) $\mathcal{P}(n = 4 \text{ or } n = 5)$

(d) $\mathcal{P}(n = 4 \text{ and } n = 5)$

(e) $\mathcal{P}(n \text{ is even or odd})$

(f) $\mathcal{P}(n \text{ is neither even nor odd})$

(g) $\mathcal{P}(n \text{ is even and divisible by three})$

(h) $\mathcal{P}(n \text{ is even or divisible by three})$

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6. A card is selected from a regular pack of 52 cards. Find the probability that the card:

(a) is a jack,
(b) is a ten,
(c) is a jack or a ten,
(d) is a jack and a ten,
(e) is neither a jack nor a ten,
(f) is black,
(g) is a picture card,
(h) is a black picture card,
(i) is black or a picture card,
(j) is neither black nor a picture card.

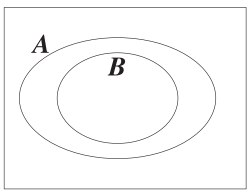
7. A die is thrown. Let A be the event that an even number appears. Let B be the event that a number greater than two appears.

(a) Are A and B mutually exclusive?
(b) Find: (i) $\mathcal{P}(A)$ (ii) $\mathcal{P}(B)$ (iii) $\mathcal{P}(A \text{ and } B)$ (iv) $\mathcal{P}(A \text{ or } B)$

8. Two dice are thrown. Let a and b denote the numbers rolled. Find:

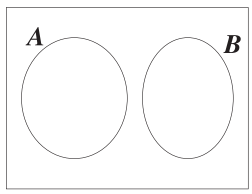
(a) $\mathcal{P}(a \text{ is odd})$
(b) $\mathcal{P}(b \text{ is odd})$
(c) $\mathcal{P}(a \text{ and } b \text{ are odd})$
(d) $\mathcal{P}(a \text{ or } b \text{ is odd})$
(e) $\mathcal{P}(\text{neither } a \text{ nor } b \text{ is odd})$
(f) $\mathcal{P}(a = 1)$
(g) $\mathcal{P}(b = a)$
(h) $\mathcal{P}(a = 1 \text{ and } b = a)$
(i) $\mathcal{P}(a = 1 \text{ or } b = a)$
(j) $\mathcal{P}(a \neq 1 \text{ and } a \neq b)$

9. (a)



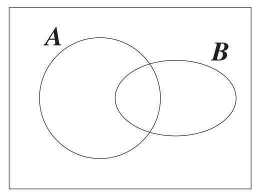
If $\mathcal{P}(A) = \frac{1}{2}$ and $\mathcal{P}(B) = \frac{1}{3}$, find:
(i) $\mathcal{P}(\overline{A})$ (ii) $\mathcal{P}(\overline{B})$
(iii) $\mathcal{P}(A \text{ and } B)$
(iv) $\mathcal{P}(A \text{ or } B)$
(v) $\mathcal{P}(\text{neither } A \text{ nor } B)$

(b)



If $\mathcal{P}(A) = \frac{2}{5}$ and $\mathcal{P}(B) = \frac{1}{5}$, find:
(i) $\mathcal{P}(\overline{A})$ (ii) $\mathcal{P}(\overline{B})$
(iii) $\mathcal{P}(A \text{ or } B)$
(iv) $\mathcal{P}(A \text{ and } B)$
(v) $\mathcal{P}(\text{not both } A \text{ and } B)$

(c)



If $\mathcal{P}(A) = \frac{1}{2}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(A \text{ and } B) = \frac{1}{6}$, find:
(i) $\mathcal{P}(\overline{A})$ (ii) $\mathcal{P}(\overline{B})$
(iii) $\mathcal{P}(A \text{ or } B)$
(iv) $\mathcal{P}(\text{neither } A \text{ nor } B)$
(v) $\mathcal{P}(\text{not both } A \text{ and } B)$

10. Use the addition rule $\mathcal{P}(A \text{ or } B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \text{ and } B)$ to answer the following questions:

(a) If $\mathcal{P}(A) = \frac{1}{5}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(A \text{ and } B) = \frac{1}{15}$, find $\mathcal{P}(A \text{ or } B)$.
(b) If $\mathcal{P}(A) = \frac{1}{2}$, $\mathcal{P}(B) = \frac{1}{3}$ and $\mathcal{P}(A \text{ or } B) = \frac{5}{6}$, find $\mathcal{P}(A \text{ and } B)$.
(c) If $\mathcal{P}(A \text{ or } B) = \frac{9}{10}$, $\mathcal{P}(A \text{ and } B) = \frac{1}{5}$ and $\mathcal{P}(A) = \frac{1}{2}$, find $\mathcal{P}(B)$.
(d) If A and B are mutually exclusive and $\mathcal{P}(A) = \frac{1}{7}$ and $\mathcal{P}(B) = \frac{4}{7}$, find $\mathcal{P}(A \text{ or } B)$.

DEVELOPMENT

11. An integer n is picked at random, where $1 \le n \le 20$. The events A, B, C and D are:

A: an even number is chosen,
B: a number greater than 15 is chosen,
C: a multiple of 3 is chosen,
D: a one-digit number is chosen.

(a) (i) Are the events A and B mutually exclusive?
(ii) Find $\mathcal{P}(A)$, $\mathcal{P}(B)$, $\mathcal{P}(A \text{ and } B)$ and hence evaluate $\mathcal{P}(A \text{ or } B)$.
(b) (i) Are the events A and C mutually exclusive?
(ii) Find $\mathcal{P}(A)$, $\mathcal{P}(C)$, $\mathcal{P}(A \text{ and } C)$ and hence evaluate $\mathcal{P}(A \text{ or } C)$.
(c) (i) Are the events B and D mutually exclusive?
(ii) Find $\mathcal{P}(B)$, $\mathcal{P}(D)$, $\mathcal{P}(B \text{ and } D)$ and hence evaluate $\mathcal{P}(B \text{ or } D)$.

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12. List the twenty-five primes less than 100. A number is drawn at random from the integers from 1 to 100. Find the probability that:

(a) it is prime, (b) it has remainder 1 after division by 4,

(c) it is prime and it has remainder 1 after division by 4,

(d) it is either prime or it has remainder 1 after division by 4.

13. In a group of 50 students, there are 26 who study Latin and 15 who study Greek and 8 who take both languages. Draw a Venn diagram and find the probability that a student chosen at random:

(a) studies only Latin, (b) studies only Greek, (c) does not study either language.

14. During a game, all 21 members of an Australian Rules football team consume liquid. Some players drink only water, some players drink only Gatorade™ and some players drink both. If there are 14 players who drink water and 17 players who drink Gatorade™:

(a) How many drink both water and Gatorade™?

(b) If one team member is selected at random, find the probability that:

(i) he drinks water but not Gatorade™,

(ii) he drinks Gatorade™ but not water.

15. Each student in a music class of 28 studies either the piano or the violin or both. It is known that 20 study the piano and 15 study the violin. Find the probability that a student selected at random studies both instruments.

16. A group of 60 students was invited to try out for three sports: rugby, soccer and cross country — 32 tried out for rugby, 29 tried out for soccer, 15 tried out for cross country, 11 tried out for rugby and soccer, 9 tried out for soccer and cross country, 8 tried out for rugby and cross country, and 5 tried out for all three sports. Draw a Venn diagram and find the probability that a student chosen at random:

(a) tried out for only one sport,

(b) tried out for exactly two sports,

(c) tried out for at least two sports,

(d) did not try out for a sport.

17. 43 people were surveyed and asked whether they drank Coke™, Sprite™ or Fanta™. Three people drank all of these, while four people did not drink any of them. 19 drank Coke™, 21 drank Sprite™ and 17 drank Fanta™. One person drank Coke™ and Fanta™ but not Sprite™. Find the probability that a person selected at random from the group drank Sprite™ only.

EXTENSION

18. [The inclusion–exclusion principle] This rule allows you to count the number of elements contained in the union of the sets without counting any element more than once.

(a) Given three finite sets A, B and C, find a rule for calculating $n(A \cup B \cup C)$. [HINT: Use a Venn diagram and pay careful to attention to the elements in $A \cap B \cap C$.]

(b) A new car dealer offers three options to his customers: power steering, air conditioning and a CD player. He sold 72 cars without any options, 12 with all three options, 38 included power steering and air conditioning, 25 included power steering and a CD player, 22 included air conditioning and a CD player, 83 included power steering, 55 included air conditioning and 70 included a CD player. Using the formula established in (a), how many cars did he sell?

(c) Extend the formula to the unions of four sets, each with finitely many elements. Is it possible to draw a sensible Venn diagram of four sets?

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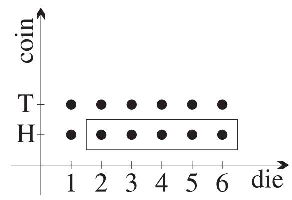
10 C Multi-Stage Experiments

This section deals with experiments that have a number of stages. The full sample space of such an experiment can quickly become too large to be conveniently listed, and instead we shall develop a rule for multiplying together the probabilities associated with each stage.

Two-Stage Experiments — The Product Rule: Here is a simple question about a two-stage experiment:

> 'Throw a die, then toss a coin. What is the probability of obtaining at least two on the die followed by a head?'



Graphed on the right are the twelve possible outcomes of the experiment, all equally likely, with a box drawn around the five favourable outcomes. Thus

$$\mathcal{P}(\text{at least two and a head}) = \tfrac{5}{12}.$$

Now let us consider the two stages separately. The first stage is throwing a die, and we want the outcome A = 'getting at least two' — here there are six possible outcomes and five favourable outcomes, giving probability $\frac{5}{6}$. The second stage is tossing a coin, and we want the outcome B = 'tossing a head' — here there are two possible outcomes and one favourable outcome, giving probability $\frac{1}{2}$.

The full experiment then has $6 \times 2 = 12$ possible outcomes, and there are $5 \times 1 = 5$ favourable outcomes. Hence

$$\mathcal{P}(AB) = \frac{5 \times 1}{6 \times 2} = \frac{5}{6} \times \frac{1}{2} = \mathcal{P}(A) \times \mathcal{P}(B)\,.$$

Thus the probability of the compound event 'getting at least two and a head' can be found by multiplying together the probabilities of the two stages. The argument here can easily be generalised to any two-stage experiment.

8 TWO-STAGE EXPERIMENTS: If A and B are independent events in successive stages of a two-stage experiment, then

$$\mathcal{P}(AB) = \mathcal{P}(A) \times \mathcal{P}(B)\,,$$

where the word 'independent' means that the outcome of one stage does not affect the outcome of the other stage.

Independent Events: The word 'independent' needs further discussion. In our example above, the throwing of the die clearly does not affect the tossing of the coin, so the two events are independent.

Here is a very common and important type of experiment where the two stages are not independent:

> 'Choose an elector at random from the NSW population. First note the elector's gender. Then ask whether the elector voted Labor or non-Labor in the last State election.'

In this example, we would suspect that the gender and the political opinion of a person may not be independent, and that there is *correlation* between them. This is in fact the case, as almost every opinion poll has shown over the years. Correlation is beyond this course, but is one of the things statisticians most commonly study in their routine work.

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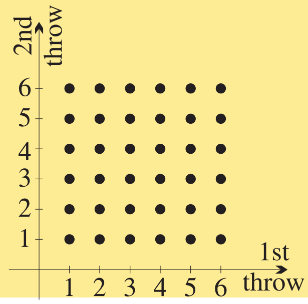
WORKED EXERCISE: A pair of dice are thrown twice. What is the probability that the first throw is a double and the second throw gives a sum of at least four?

SOLUTION: We saw in Section 10A that when two dice are thrown, there are 36 possible outcomes, graphed in the diagram to the right.



There are six doubles amongst the 36 possible outcomes,

so $\quad \mathcal{P}(\text{double}) = \frac{6}{36} = \frac{1}{6}$.

All but the pairs $(1,1)$, $(2,1)$ and $(1,2)$ give a sum at least four,

so $\quad \mathcal{P}(\text{sum is at least four}) = \frac{33}{36} = \frac{11}{12}$.

Since the two stages are independent,

$$\mathcal{P}(\text{double, sum at least four}) = \frac{1}{6} \times \frac{11}{12} = \frac{11}{72}.$$

Multi-Stage Experiments — The Product Rule: The same arguments clearly apply to an experiment with any number of stages.

9 MULTI-STAGE EXPERIMENTS: If $A_1, A_2, \ldots, A_n$ are independent events, then

$$\mathcal{P}(A_1 A_2 \ldots A_n) = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n).$$

WORKED EXERCISE: A coin is tossed four times. Find the probability that:

(a) every toss is a head,
(b) there is at least one head.

SOLUTION:

(a) $\mathcal{P}(\text{HHHH}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$.

(b) $\mathcal{P}(\text{at least one head}) = 1 - \mathcal{P}(\text{TTTT}) = 1 - \frac{1}{16} = \frac{15}{16}$.

Listing the Favourable Outcomes: The product rule is often combined with a listing of the favourable outcomes. A tree diagram may help in producing that listing, although this is hardly necessary in the straightforward worked exercise below, which continues the previous example.

WORKED EXERCISE: A coin is tossed four times. Find the probability that:

(a) the first three coins are heads,
(b) the middle two coins are tails,
(c) there are at least three heads,
(d) there are exactly two heads.

SOLUTION:

(a) $\mathcal{P}(\text{the first three coins are heads}) = \mathcal{P}(\text{HHHH}) + \mathcal{P}(\text{HHHT})$

(notice that the two events HHHH and HHHT are mutually exclusive)

$= \frac{1}{16} + \frac{1}{16}$

(since each of these two probabilities is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$)

$= \frac{1}{8}$.

(b) $\mathcal{P}(\text{middle two are tails}) = \mathcal{P}(\text{HTTH}) + \mathcal{P}(\text{HTTT}) + \mathcal{P}(\text{TTTH}) + \mathcal{P}(\text{TTTT})$
$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}$.

(c) $\mathcal{P}(\text{at least 3 heads}) = \mathcal{P}(\text{HHHH}) + \mathcal{P}(\text{HHHT}) + \mathcal{P}(\text{HHTH}) + \mathcal{P}(\text{HTHH}) + \mathcal{P}(\text{THHH})$
$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{5}{16}$.

(d) $\mathcal{P}(\text{exactly 2 heads}) = \mathcal{P}(\text{HHTT}) + \mathcal{P}(\text{HTHT}) + \mathcal{P}(\text{THHT})$
$+ \mathcal{P}(\text{HTTH}) + \mathcal{P}(\text{THTH}) + \mathcal{P}(\text{TTHH})$

(since these are all the six possible orderings of H, H, T and T)

$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{8}$.

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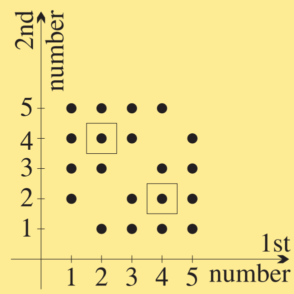
Sampling Without Replacement — An Extension of the Product Rule: The product rule can be extended to the following question, where the two stages of the experiment are not independent.

WORKED EXERCISE: A box contains five discs numbered 1, 2, 3, 4 and 5. Two numbers are drawn in succession, without replacement. What is the probability that both are even?

SOLUTION: The probability that the first number is even is $\frac{2}{5}$.
When this even number is removed, one even and three odd numbers remain, so the probability that the second number is also even is $\frac{1}{4}$.

Hence $\mathcal{P}(\text{both even}) = \frac{2}{5} \times \frac{1}{4}$
$= \frac{1}{10}$.



The graph on the right allows the calculation to be checked by examining its full sample space. Because doubles are not allowed, there are only 20 possible outcomes. The two boxed outcomes are the only outcomes that consist of two even numbers, giving the same probability of $\frac{2}{20} = \frac{1}{10}$.

Retelling the Experiment: Sometimes, the manner in which an experiment is told makes calculation difficult, but the experiment can be retold in a different manner so that the probabilities are the same, but the calculations are much simpler.

WORKED EXERCISE: Wes is sending Christmas cards to ten friends. He has two cards with Christmas trees, two with angels, two with snow, two with reindeer, and two with Santa Claus. What is the probability that Harry and Helmut get matching cards?

SOLUTION: Retell the process as follows. 'Wes decides to pair up the friends who will receive matching cards. First he writes down Harry's name, then he chooses a pair for Harry. He then proceeds similarly with the remaining eight names.'

All that now matters is the person he chooses to pair with Harry. Since there are nine names remaining, the probability that it is Helmut is $\frac{1}{9}$.

WORKED EXERCISE: From a room of ten people, five are chosen at random to be seated on the balcony for dinner. What is the probability that Sandra and Dinesh both sit on the balcony?

SOLUTION: Retell the method of choosing the random seating as
'Choose where Sandra sits, then choose where Dinesh sits'.
Sandra then has five chances out of ten of sitting on the balcony.
If she does, then Dinesh has four chances out of nine of sitting on the balcony.
Hence $\mathcal{P}(\text{both on balcony}) = \frac{5}{10} \times \frac{4}{9}$
$= \frac{2}{9}$.

Alternatively, retell the experiment as 'Choose, in order, the five people to sit inside, and find the probability that neither Sandra nor Dinesh sits inside'.
Then $\mathcal{P}(\text{neither inside}) = \frac{8}{10} \times \frac{7}{9} \times \frac{6}{8} \times \frac{5}{7} \times \frac{4}{6}$
$= \frac{2}{9}$.

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Exercise **10C**

1. Suppose that A, B, C and D are independent events, with $\mathcal{P}(A) = \frac{1}{8}$, $\mathcal{P}(B) = \frac{1}{3}$, $\mathcal{P}(C) = \frac{1}{4}$ and $\mathcal{P}(D) = \frac{2}{7}$. Use the product rule to find:
(a) $\mathcal{P}(AB)$ (b) $\mathcal{P}(AD)$ (c) $\mathcal{P}(BC)$ (d) $\mathcal{P}(ABC)$ (e) $\mathcal{P}(BCD)$ (f) $\mathcal{P}(ABCD)$

2. A coin and a die are tossed. Use the product rule to find the probability of obtaining:
(a) a three and a head,
(b) a six and a tail,
(c) an even number and a tail,
(d) a number less than five and a head.

3. One set of cards contains the numbers 1, 2, 3, 4 and 5, and another set contains the letters A, B, C, D and E. One card is drawn at random from each set. Use the product rule to find the probability of drawing:
(a) 4 and B,
(b) 2 or 5, then D,
(c) 1, then A or B or C,
(d) an odd number and C,
(e) an even number and a vowel,
(f) a number less than 3, and E,
(g) the number 4, followed by a letter from the word MATHS.

4. Two marbles are picked at random, one from a bag containing three red and four blue marbles, and the other from a bag containing five red and two blue marbles. Find the probability of drawing:
(a) two red marbles,
(b) two blue marbles,
(c) a red marble from the first bag and a blue marble from the second.

5. A box contains five light globes, two of which are faulty. Two globes are selected, one at a time without replacement. Find the probability that:
(a) both globes are faulty,
(b) neither globe is faulty,
(c) the first globe is faulty and the second one is not,
(d) the second globe is faulty and the first one is not.

6. A box contains twelve red and ten green discs. Three discs are selected, one at a time without replacement.
(a) What is the probability that the discs selected are red, green, red in that order?
(b) What is the probability of this event if the disc is replaced after each draw?

7. (a) From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:
(i) a spade then a heart,
(ii) two clubs,
(iii) a jack then a queen,
(iv) the king of diamonds then the ace of clubs.
(b) Repeat the question if the first card is replaced before the second card is drawn.

8. A coin is weighted so that it is twice as likely to fall heads as it is tails.
(a) Write down the probabilities that the coin falls: (i) heads, (ii) tails.
(b) If you toss the coin three times, find the probability of:
(i) three heads, (ii) three tails, (iii) head, tail, head in that order.

9. [Valid and invalid arguments] Identify any fallacies in the following arguments. If possible, give some indication of how to correct them.
(a) 'The probability that a Year 12 student chosen at random likes classical music is 50%, and the probability that a student plays a classical instrument is 20%. Therefore the probability that a student chosen at random likes classical music and plays a classical instrument is 10%.'

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(b) 'The probability of a die showing a prime is $\frac{1}{2}$, and the probability that it shows an odd number is $\frac{1}{2}$. Hence the probability that it shows an odd prime number is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.'
(c) 'I choose a team at random from an eight-team competition. The probability that it wins any game is $\frac{1}{2}$, so the probability that it defeats all of the other seven teams is $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$.'
(d) 'A normal coin is tossed and shows heads eight times. Nevertheless, the probability that it shows heads the next time is still $\frac{1}{2}$.'

DEVELOPMENT

10. An archer fires three shots at a bulls-eye. He has a 90% chance of hitting the bulls-eye. Using H for hit and M for miss, list all eight possible outcomes. Then, assuming that successive shots are independent, use the product rule to find the probability that he will:
(a) hit the bulls-eye three times,
(b) miss the bulls-eye three times,
(c) hit the bulls-eye on the first shot only,
(d) hit the bulls-eye exactly once,
(e) miss the bulls-eye on the first shot only,
(f) miss the bulls-eye exactly once.
[HINT: Part (d) requires adding the probabilities of HMM, MHM and MMH, and part (f) requires a similar calculation.]

11. A die is rolled twice. Using the product rule, find the probability of rolling:
(a) a double two,
(b) any double,
(c) a number greater than three, then an odd number,
(d) a one and then a four,
(e) a four and then a one,
(f) a one and a four in any order,
(g) an even number, then a five,
(h) a five and then an even number,
(i) an even number and a five in any order.

12. There is a one-in-five chance that you will guess the correct answer to a multiple-choice question. The test contains five such questions — label the various possible results of the test as CCCCC, CCCCI, CCCIC, ... What is the chance that you will answer:
(a) all five correctly,
(b) all five incorrectly,
(c) the first, third and fifth correctly, and the second and fourth incorrectly,
(d) the first correctly and the remainder incorrectly,
(e) exactly one correctly, [HINT: Add the probabilities of CIIII, ICIII, IICII, IIICI, IIIIC.]
(f) exactly four correctly. [HINT: List the possible outcomes first.]

13. A die is thrown six times.
(a) What is the probability that the nth throw is n on each occasion?
(b) What is the probability that the nth throw is n on exactly five occasions?

14. From a bag containing two red and two green marbles, marbles are drawn one at a time without replacement until two green marbles have been drawn. Find the probability that:
(a) exactly two draws are required,
(b) at least three draws are required,
(c) exactly four draws are required,
(d) exactly three draws are required.
[HINT: In part (c), consider the colour of the fourth marble drawn.]

15. Sophia, Gabriel and Elizabeth take their driving test. The chances that they pass are $\frac{1}{2}$, $\frac{5}{8}$ and $\frac{3}{4}$ respectively.
(a) Find the probability that Sophia passes and the other two fail.
(b) By listing the possible outcomes for one of the girls passing and the other two failing, find the probability that exactly one of the three passes.
(c) If only one of them passes, find the probability that it is Gabriel.

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16. (a) If a coin is tossed repeatedly, find the probability of obtaining at least one head in:
(i) two tosses, (ii) five tosses, (iii) ten tosses.
(b) Write down the probability of obtaining at least one head in n tosses.
(c) How many times would you need to toss a coin so that the probability of tossing at least one head is greater than $0{\cdot}9999$?

17. (a) When rolling a die n times, what is the probability of not rolling a six?
(b) Show that the probabilities of not rolling a six on 1, 2, 3, ... tosses of the coin form a GP, and write down the first term and common ratio.
(c) How many times would you need to roll a die so that the probability of rolling at least one six was greater than $\frac{9}{10}$?

18. One layer of tinting material on a window cuts out $\frac{1}{5}$ of the sun's UV rays.
(a) What fraction would be cut out by using two layers?
(b) How many layers would be required to cut out at least $\frac{9}{10}$ of the sun's UV rays?

19. In a lottery, the probability of the jackpot being won in any draw is $\frac{1}{60}$.
(a) What is the probability that the jackpot prize will be won in each of four consecutive draws?
(b) How many consecutive draws need to be made for there to be a greater than 98% chance that at least one jackpot prize will have been won?

20. [This question and the next are best done by retelling the story of the experiment, as explained in the notes above.] Nick has five different pairs of socks to last the week, and they are scattered loose in his drawer. Each morning, he gets up before light and chooses two socks at random. Find the probability that he wears a matching pair:
(a) on the first morning,
(b) on the last morning,
(c) on the third morning,
(d) the first two mornings,
(e) every morning,
(f) every morning but one.

21. Kia and Abhishek are two of twelve guests at a tennis party, where people are playing doubles on three courts. The twelve have been divided randomly into three groups of four. Find the probability that:
(a) Kia and Abhishek play on the same court,
(b) Kia and Abhishek both play on River Court,
(c) Kia is on River Court and Abhishek is on Rose Court.

EXTENSION

22. [A notoriously confusing problem] In a television game show, the host shows the contestant three doors, only one of which conceals the prize, and the game proceeds as follows. First, the contestant chooses a door. Secondly, the host opens one of the other two doors, showing the contestant that it is not the prize door. Thirdly, the host invites the contestant to change her choice, if she wishes. Analyse the game, and advise the contestant what to do.

23. In a game, a player draws a card from a pack of 52. If he draws a two he wins. If he draws a three, four or five he loses. If he draws a heart that is not a two, three, four or five then he must roll a die. He wins only if he rolls a one. If he draws one of the other three suits and the card is not a two, three, four or five, then he must toss a coin. He wins only if he tosses a tail. Given that the player wins the game, what is the probability that he drew a black card?

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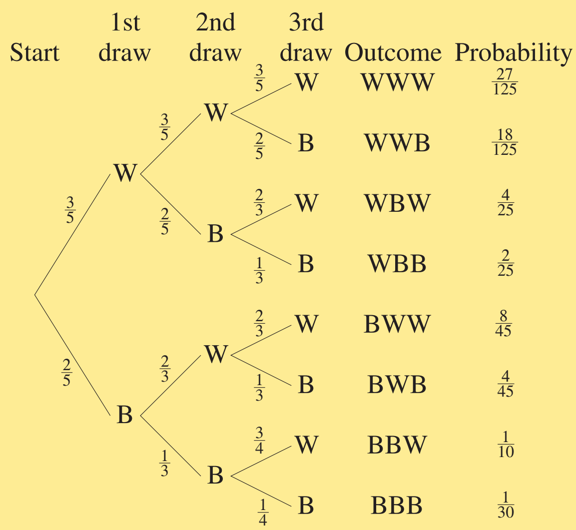
10 D Probability Tree Diagrams

In more complicated problems, and particularly in unsymmetric situations, a *probability tree diagram* can be very useful in organising the various cases, in preparation for the application of the product rule.

Constructing a Probability Tree Diagram: A probability tree diagram differs from the tree diagrams used in Section 10A for counting possible outcomes, in that the relevant probabilities are written on the branches and then multiplied together in accordance with the product rule. An example should demonstrate the method. Notice that, as before, these diagrams have one column labelled 'Start', a column for each stage, and a column listing the outcomes, but there is now an extra column labelled 'Probability' at the end.

WORKED EXERCISE: A bag contains six white marbles and four blue marbles. Three marbles are drawn in succession. At each draw, if the marble is white, it is replaced, and if it is blue, it is not replaced. Find the probabilities of drawing zero, one, two and three blue marbles.

SOLUTION: With the ten marbles in the bag, the probabilities are $\frac{3}{5}$ and $\frac{2}{5}$.
If one blue marble is removed, the probabilities become $\frac{2}{3}$ and $\frac{1}{3}$.
If two blue marbles are removed, the probabilities become $\frac{3}{4}$ and $\frac{1}{4}$.



Multiplying probabilities along each arm, and then adding the cases,

$$\mathcal{P}(\text{no blue marbles}) = \tfrac{27}{125},$$
$$\mathcal{P}(\text{one blue marble}) = \tfrac{18}{125} + \tfrac{4}{25} + \tfrac{8}{45} = \tfrac{542}{1125},$$
$$\mathcal{P}(\text{two blue marbles}) = \tfrac{2}{25} + \tfrac{4}{45} + \tfrac{1}{10} = \tfrac{121}{450},$$
$$\mathcal{P}(\text{three blue marbles}) = \tfrac{1}{30}.$$

NOTE: Your calculator will show that the eight probabilities in the last column of the diagram add exactly to 1, and that the four answers above also add to 1. These are useful checks on the working.

An Infinite Probability Tree Diagram: Some situations generate an infinite probability tree diagram. In the following, more difficult worked example, the limiting sum of a GP is used to evaluate the resulting sum.

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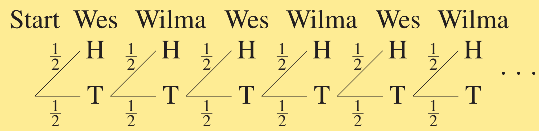
WORKED EXERCISE: Wes and Wilma toss a coin alternately, beginning with Wes. The first to toss heads wins. What probability of winning does Wes have?

SOLUTION: The branches on the tree diagram below terminate when a head is tossed, and the person who tosses that head wins the game. Space precludes writing in either the final outcome or the product of the probabilities!

From the diagram, $\mathcal{P}(\text{Wes wins}) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$.

This is the limiting sum of the GP with $a = \frac{1}{2}$ and $r = \frac{1}{4}$,

so
$$\begin{aligned}\mathcal{P}(\text{Wes wins}) &= \frac{a}{1-r}\\ &= \tfrac{1}{2} \div \tfrac{3}{4}\\ &= \tfrac{2}{3}.\end{aligned}$$



Exercise 10D

1. A bag contains three black and four white discs. A disc is selected from the bag, its colour is noted, and it is then returned to the bag before a second disc is drawn.

(a) Draw a probability tree diagram and hence find the probability that:

(i) both discs drawn are white, (ii) the discs have different colours.

(b) Repeat the question if the first ball is not replaced before the second one is drawn.

2. Two light bulbs are selected at random from a large batch of bulbs in which 5% are defective. Draw a probability tree diagram and find the probability that:

(a) both bulbs are defective, (b) at least one bulb works.

3. One bag contains three red and two blue balls and another bag contains two red and three blue balls. A ball is drawn at random from each bag. Draw a probability tree diagram and hence find the probability that:

(a) the balls have the same colour, (b) the balls have different colours.

4. In group A there are three girls and seven boys, and in group B there are six girls and four boys. One person is chosen at random from each group. Draw a probability tree diagram and hence find the probability that:

(a) both people chosen are of the same sex, (b) one boy and one girl are chosen.

5. There is an 80% chance that Gary will pass his driving test and a 90% chance that Emma will pass hers. Draw a probability tree diagram, and find the chance that:

(a) Gary passes and Emma fails,
(b) Gary fails and Emma passes,
(c) only one of Gary and Emma passes,
(d) at least one fails.

6. In an aviary there are four canaries, five cockatoos and three budgerigars. If two birds are selected at random, find the probability that:

(a) both are canaries,
(b) neither is a canary,
(c) one is a canary and one is a cockatoo,
(d) at least one is a canary.

7. In a large co-educational school, the population is 47% female and 53% male. Two students are selected at random. Find, correct to two decimal places, the probability that:

(a) both are male, (b) they are of different sexes.

8. The probability of a woman being alive at 80 years of age is 0·2, and the probability of her husband being alive at 80 years of age is 0·05. What is the probability that:

(a) they will both live to be 80 years of age, (b) only one of them will live to be 80?

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9. The numbers 1, 2, 3, 4 and 5 are each written on a card. The cards are shuffled and one card is drawn at random. The number is noted and the card is then returned to the pack. A second card is selected, and in this way a two-digit number is recorded. For example, a 2 on the first draw and a 3 on the second results in the number 23.

(a) What is the probability of:

(i) the number 35 being recorded, (ii) an odd number being recorded?

(b) Repeat the question if the first card is not returned to the pack before the second one is drawn.

10. A factory assembles calculators. Each calculator requires a chip and a battery. It is known that 1% of chips and 4% of batteries are defective. Find the probability that a calculator selected at random will have at least one defective component.

11. Alex and Julia are playing in a tennis tournament. They will play each other twice and each has an equal chance of winning the first game. If Alex wins the first game his confidence increases, and his probability of winning the second game is increased to 0·55. If he loses the first game he loses heart so that his probability of winning the second game is reduced to 0·25. Find the probability that Alex wins exactly one game.

12. In a raffle in which there are 200 tickets, the first prize is drawn and then the second prize is drawn without replacing the winning ticket. If you buy 15 tickets, find the probability that you win:

(a) both prizes, (b) at least one prize.

13. The probability that a set of traffic lights will be green when you arrive at them is $\frac{3}{5}$. A motorist drives through two sets of lights. Assuming that the two sets of traffic lights are not synchronised, find the probability that:

(a) both sets of lights will be green, (b) at least one set of lights will be green.

DEVELOPMENT

14. A box contains ten chocolates, all of identical appearance. Three of the chocolates have caramel centres and the other seven have mint centres. Hugo randomly selects and eats three chocolates from the box. Find the probability that:

(a) the first chocolate Hugo eats is caramel, (b) Hugo eats three mint chocolates,

(c) Hugo eats exactly one caramel chocolate,

(d) Hugo eats at least two caramel chocolates.

15. In a bag there are four green, three blue and five red discs.

(a) Two discs are drawn at random, and the first disc is not replaced before the second disc is drawn. Find the probability of drawing:

(i) two red discs, (iv) a blue disc on the first draw,

(ii) one red and one blue disc, (v) two discs of the same colour,

(iii) at least one green disc, (vi) two differently coloured discs.

(b) Repeat the above questions if the first disc is replaced before the second disc is drawn.

16. Max and Jack each throw a die. Find the probability that:

(a) they do not throw the same number,

(b) the number thrown by Max is greater than the number thrown by Jack,

(c) the numbers they throw differ by three,

(d) the product of the numbers is even.

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17. In basketball, the chance of a girl making a basket from the free-throw line is 0·7 and the chance of a boy making the basket is 0·65. Therefore if a boy and a girl are selected at random, the chance that at least one of them will shoot a basket is 1·35. Explain the problem with this argument.

18. A game is played in which two coloured dice are rolled once. The six faces of the black die are numbered 5, 7, 8, 10, 11, 14. The six faces of the white die are numbered 3, 6, 9, 12, 13, 15. The player wins if the number on the black die is bigger than the number on the white die.

(a) Calculate the probability of a player winning the game.

(b) Calculate the probability that a player will lose at least once in two consecutive games.

(c) How many games must be played before you have a 90% chance of winning at least one game?

19. Two dice are rolled. A three appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than seven.

20. In a game, two dice are rolled and the score given is the maximum of the two numbers on the uppermost faces. For example, if the dice show a three and a five, the score is a five.

(a) Find the probability that you score a one in a single throw of the two dice.

(b) What is the probability of scoring three consecutive ones in three rolls of the dice?

(c) Find the probability of a six in a single roll of the dice.

(d) Given that one of the dice shows a three, what is the probability of getting a score greater than five?

21. A set of four cards contains two jacks, a queen and a king. Bob selects one card and then, without replacing it, selects another. Find the probability that:

(a) both Bob's cards are jacks,

(b) at least one of Bob's cards is a jack,

(c) given that one of Bob's cards is a jack, the other one is also.

22. A twenty-sided die has the numbers from 1 to 20 on its faces.

(a) If it is rolled twice, what is the probability that the same number appears on the uppermost face each time?

(b) If it is rolled three times, what is the probability that the number 15 appears on the uppermost face (i) exactly twice, (ii) at most once?

23. In each game of Sic Bo, three regular six-sided dice are thrown once.

(a) In a single game, what is the probability that all three dice show six?

(b) What is the probability that exactly two of the dice show six?

(c) What is the probability that exactly two of the dice show the same number?

(d) What is the probability of rolling three different numbers on the dice?

24. Shadia has invented a game for one person. She throws two dice repeatedly until the sum of the two numbers shown is either six or eight. If the sum is six, she wins. If the sum is eight, she loses. If the sum is any other number, she continues to throw until the sum is six or eight.

(a) What is the probability that she wins on the first throw?

(b) What is the probability that a second throw is needed?

(c) Find an expression for the probability that Shadia wins on her first, second or third throw.

(d) Calculate the probability that Shadia wins the game.

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25. In a game, Anna and Ingrid take turns at drawing, and immediately replacing, a ball from an urn containing three blue and four green balls. The game is won when Anna draws a blue ball, or when Ingrid draws a green ball. Anna goes first. Find the probability that:

(a) Anna wins on her first draw,

(b) Ingrid wins on her first draw,

(c) Anna wins in less than four of her turns,

(d) Anna wins the game.

26. A bag contains two green and two blue marbles. Marbles are drawn at random, one by one, without replacement, until two green marbles have been drawn.

(a) What is the probability that exactly three draws will be required?

(b) If the marbles are replaced after each draw, and the first one drawn is green, find the probability that three draws will be required.

27. Prasad and Wilson are going to enlist in the Australian Army. The recruiting officer will be in town for seven consecutive days, starting on Monday and finishing the following Sunday. The boys must nominate three consecutive days on which to attend the recruitment office. They do this randomly and independently of one another.

(a) By listing all the ways in which to choose the three consecutive days, find the probability that they both go on Monday.

(b) What is the probability that they meet at the recruitment office on Tuesday for the first time?

(c) Find the probability that Prasad and Wilson will not meet at the recruitment office.

(d) Hence find the probability that they will meet on at least one day at the recruitment office.

28. There are two white and three black discs in a bag. Two players A and B are playing a game in which they draw a disc alternately from the bag and then replace it. Player A must draw a white disc to win and player B must draw a black disc. Player A goes first. Find the probability that:

(a) player A wins on the first draw,

(b) player B wins on her first draw,

(c) player A wins in less than four of her draws,

(d) player A wins the game.

29. A coin is tossed continually until, for the first time, the same result appears twice in succession. That is, you continue tossing until you toss two heads or two tails in a row.

(a) Find the probability that the game ends before the sixth toss of the coin.

(b) Find the probability that an even number of tosses will be required.

EXTENSION

30. A bag contains g green and b blue marbles. Three marbles are randomly selected from the bag.

(a) Write down an expression for the probability that the three marbles chosen were green.

(b) If the bag had initially contained one additional green marble, the probability that the three marbles chosen were green would have been double that found in part (a).

(i) Show that $b = \dfrac{g^2 - g - 2}{5 - g}$.

(ii) Show that $b = -g - 4 + \dfrac{18}{5 - g}$.

(iii) Sketch a graph of b against g, indicating the g-coordinates of any stationary points.

(iv) Hence determine all possible numbers of green and blue marbles.

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10 E Counting Ordered Selections

It should be clear by now that probability problems would become easier if we could develop greater sophistication in counting methods. This section and the next two take a break from probability questions to develop a more systematic approach to counting. Two questions dominate these sections:

1. Are the selections we are counting ordered or unordered?
2. If they are ordered, is repetition allowed or not?

Sections 10E and 10F develop the theory of counting ordered selections, with and without repetition, then Section 10G will deal with unordered selections.

The Multiplication Principle for Ordered Selections: An ordered selection can usually be regarded as a sequence of choices made one after the other. An efficient setting-out here is to use a box diagram to keep track of these successive choices.

WORKED EXERCISE: How many five-letter words can be formed in which the second and fourth letters are vowels and the other three letters are consonants?

NOTE: Unless otherwise indicated, always take 'y' as a consonant.

SOLUTION: We can select each letter in order:

1st letter	2nd letter	3rd letter	4th letter	5th letter
21	5	21	5	21

$$\text{Number of words} = 21 \times 5 \times 21 \times 5 \times 21$$
$$= 231\,525.$$

10 MULTIPLICATION PRINCIPLE: Suppose that a selection is to be made in k stages. Suppose that the first stage can be chosen in n_1 ways, the second in n_2 ways, the third in n_3 ways, ..., the kth in n_k ways. Then the number of ways of choosing the complete selection is $n_1 \times n_2 \times \cdots \times n_k$.

Ordered Selections With Repetition: A general formula can now be found for the number of ordered selections with repetition. Suppose that k-letter words are to be formed from n distinct letters, where any letter can be used any number of times. Then each successive letter in the word can be chosen in n ways:

1st letter	2nd letter	3rd letter	...	kth letter
n	n	n	...	n

giving n^k distinct words altogether.

11 ORDERED SELECTIONS WITH REPETITION: The number of k-letter words formed from n distinct letters, allowing repetition, is n^k.

WORKED EXERCISE:

(a) How many six-digit numbers can be formed entirely from odd digits?

(b) How many of these numbers contain at least one seven?

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SOLUTION:

(a) There are five odd digits, so the number of such numbers is 5^6.

(b) We first count the number of these six-digit numbers not containing 7.
Such numbers are formed from the digits 1, 3, 5 and 9, so there are 4^6 of them.
Subtracting this from the answer to part (a),
number of numbers $= 5^6 - 4^6 = 11\,529$.

Ordered Selections Without Repetition: Counting ordered selections without repetition typically involves factorials, because as each stage is completed, the number of objects to choose from diminishes by 1.

WORKED EXERCISE: In how many ways can a class of 16 select a committee consisting of a president, a vice-president, a treasurer and a secretary?

SOLUTION: Select in order the president, the vice-president, the treasurer and the secretary (we assume that the same person cannot fill two roles).

president	vice-president	treasurer	secretary
16	15	14	13

Hence there are $16 \times 15 \times 14 \times 13 = \dfrac{16!}{12!}$ possible committees.

The General Case — Permutations: A *permutation* or *ordered set* is an arrangement of objects chosen from a certain set. For example, the words ABC, CED, EAB and DBC are some of the permutations of three letters taken from the 5-member set $\{\,A, B, C, D, E\,\}$.

The symbol ${}^n\mathrm{P}_k$ is used to denote the number of permutations of k letters chosen without repetition from a set of n distinct letters. The previous worked exercise is easily generalised to show that there are $\dfrac{n!}{(n-k)!}$ such permutations, so this becomes the formula for ${}^n\mathrm{P}_k$:

1st letter	2nd letter	3rd letter	4th letter	...	kth letter
n	$n-1$	$n-2$	$n-3$	...	$n-k+1$

$$\begin{aligned}\text{Hence } {}^n\mathrm{P}_k &= n(n-1)(n-2)(n-3)\cdots(n-k+1)\\ &= \frac{n(n-1)(n-2)(n-3)\times\cdots\times 2\times 1}{(n-k)(n-k-1)\times\cdots\times 2\times 1}\\ &= \frac{n!}{(n-k)!}\,.\end{aligned}$$

12 ORDERED SELECTIONS WITHOUT REPETITION (PERMUTATIONS): The number of k-letter words that can be formed without repetition from a set of n distinct letters is

$${}^n\mathrm{P}_k = \frac{n!}{(n-k)!}\,.$$

For example, the number of permutations of three distinct letters taken from the 5-member set $\{\,A, B, C, D, E\,\}$ is ${}^5\mathrm{P}_3 = \dfrac{5!}{2!} = 60$.

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WORKED EXERCISE: How many eight-digit numbers, and how many nine-digit numbers, can be formed from the nine nonzero digits if no repetition is allowed?

SOLUTION:

Number of 8-digit numbers $= {}^9\mathrm{P}_8 = \dfrac{9!}{1!} = 9!$

Number of 9-digit numbers $= {}^9\mathrm{P}_9 = \dfrac{9!}{0!} = 9!$

NOTE: It may seem surprising that these two results are equal. Notice, however, that every eight-digit number with distinct nonzero digits can be extended to a nine-digit number of this type simply by tacking the missing digit onto the right-hand end. This establishes a one-to-one correspondence between the two sets of numbers, for example,

$$96281745 \longleftrightarrow 962817453 \quad \text{(tacking the missing 3 onto the right),}$$

which explains why the two answers are the same. In general, ${}^n\mathrm{P}_{n-1}$ and ${}^n\mathrm{P}_n$ are both equal to $n!$

The Permutations of a Set: A *permutation of a set* is an arrangement of all the members of the set in a particular order. The number of such permutations is a special case of the previous paragraph — if the set has n members, then the number of permutations is ${}^n\mathrm{P}_n = \dfrac{n!}{0!} = n!$

13 PERMUTATIONS OF A SET: The number of permutations of an n-member set, that is, the number of distinct orderings of the set, is ${}^n\mathrm{P}_n = n!$

This result is so important that it should also be proven directly using a box diagram. In the boxes below, members are selected in turn to go into the first position, the second position, and so on.

1st position	2nd position	3rd position	...	nth position
n	$n-1$	$n-2$	...	1

Hence the number of orderings is $n(n-1)(n-2)\dots 1 = n!$ as expected.

WORKED EXERCISE: In how many ways can 20 people form a queue? Will the number of ways double with 40 people?

SOLUTION: With 20 people, number of ways $= 20!$ $[\doteqdot 2{\cdot}4 \times 10^{18}]$.
With 40 people, number of ways $= 40!$ $[\doteqdot 8{\cdot}2 \times 10^{47}]$.
Hence 40 people can form a queue in about 3×10^{29} times more ways than 20 people.

A General Counting Principle — Deal with the Restrictions First: Many problems have some restrictions in the way things can be arranged. These restrictions should be dealt with first. It is also important to keep in mind that the order of the boxes represents the order in which the choices are made, not the final ordering of the objects, and that they can be used in surprisingly flexible ways.

14 DEAL WITH THE RESTRICTIONS FIRST: When using boxes for counting ordered selections, deal with any restrictions first.
Remember that the boxes are in the order in which the selections are made.

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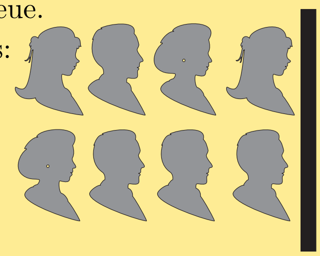
WORKED EXERCISE: Eight people form two queues, each with four people. Albert will only stand in the left-hand queue, Beth will only stand in the right-hand queue, and Charles and Diana insist on standing in the same queue. In how many ways can the two queues be formed?

SOLUTION: Place Albert in any of the 4 possible positions in the left-hand queue,
Then place Beth in any of the 4 positions in the right-hand queue.
Place Charles in any of the remaining 6 positions.
Place Diana in one of the 2 remaining positions in the same queue.
There remain 4 unfilled positions, which can be filled in 4! ways:

Albert	Beth	Charles	Diana	last four positions
4	4	6	2	4!



Hence number of ways $= 4 \times 4 \times 6 \times 2 \times 4!$
$= 4608.$

A General Counting Principle — Compound Orderings: In some ordering problems, particular members must be grouped together. This produces a *compound ordering*, in which the various groups must first be ordered, and then the individuals ordered within each group.

15 COMPOUND ORDERINGS: First order the groups, then order the individuals within each group. (In this context, a group may sometimes consist of a single individual.)

WORKED EXERCISE: Four boys and four girls form a queue at the bus stop. There is one couple who want to stand together, the other three girls want to stand together, but the other three boys don't care where they stand. How many acceptable ways are there of forming the queue?

SOLUTION: There are five groups — the couple, the group of three girls, and the three groups each consisting of one individual boy. These five groups can be ordered in 5! ways. Then the individuals within each group must be ordered.

order the 5 groups	order the couple	order the 3 girls
5!	2!	3!

Hence number of ways $= 5! \times 2! \times 3! = 1440.$

Exercise 10E

1. Evaluate, using the formula ${}^n\mathrm{P}_r = \dfrac{n!}{(n-r)!}$:

(a) ${}^6\mathrm{P}_2$ (c) ${}^3\mathrm{P}_3$ (e) ${}^5\mathrm{P}_5$ (g) ${}^9\mathrm{P}_5$
(b) ${}^{10}\mathrm{P}_2$ (d) ${}^3\mathrm{P}_2$ (f) ${}^4\mathrm{P}_3$ (h) ${}^{10}\mathrm{P}_8$

2. List all the permutations of the letters of the word DOG. How many are there?

3. List all the permutations of the letters EFGHI, beginning with F, taken three at a time.

4. Find how many arrangements of the letters of the word FRIEND are possible if the letters are taken: (a) four at a time, (b) six at a time.

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5. Find how many four-digit numbers can be formed using the digits 5, 6, 7, 8 and 9 if:
(a) no digit is to be repeated, (b) any of the digits can occur more than once.

6. How many three-digit numbers can be formed using the digits 2, 3, 4, 5 and 6 if no digit can be repeated? How many of these are greater than 400?

7. In how many ways can seven people be seated in a row of seven different chairs?

8. Eight runners are participating in a 400-metre race.
(a) In how many ways can they finish?
(b) In how many ways can the gold, silver and bronze medals be awarded?

9. (a) If you toss a coin and roll a die, how many outcomes are possible?
(b) If you toss two coins and roll three dice, how many outcomes are possible?

10. A woman has four hats, three blouses, five skirts, two handbags and six pairs of shoes. In how many ways can she be attired, assuming that she wears one of each item?

11. Jack has six different football cards and Meg has another eight different football cards. In how many ways can one of Jack's cards be exchanged for one of Meg's cards?

DEVELOPMENT

12. In Sydney, phone numbers at present consist of eight digits, starting with the digit 9.
(a) How many phone numbers are possible?
(b) How many of these end in an odd number?
(c) How many consist of odd digits only?
(d) How many are there that do not contain a zero, and in which the consecutive digits alternate between odd and even?

13. Users of automatic teller machines are required to enter a four-digit pin number. Find how many pin numbers:
(a) are possible,
(b) consist of four distinct digits,
(c) consist of odd digits only,
(d) start and end with the same digit.

14. (a) If repetitions are not allowed, how many four-digit numbers can be formed from the digits 1, 2, . . . 8, 9?
(b) How many of these end in 1?
(c) How many of these are even?
(d) How many are divisible by 5?
(e) How many are greater than 7000?

15. Repeat the previous question if repetitions are allowed.

16. In Tasmania, a car licence plate consists of two letters followed by four digits. Find how many of these are possible:
(a) if there are no restrictions,
(b) if there is no repetition of letters or digits,
(c) if the second letter is X and the third digit is 3,
(d) if the letters are D and Q and the digits are 3, 6, 7 and 9.

17. (a) In how many ways can the letters of the word NUMBER be arranged?
(b) How many begin with N?
(c) How many begin with N and end with U?
(d) In how many is the N somewhere to the left of the U?

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18. Find how many arrangements of the letters of the word UNIFORM are possible:
(a) if the vowels must occupy the first, middle and last positions,
(b) if the word must start with U and end with M,
(c) if all the consonants must be in a group at the end of the word,
(d) if the M is somewhere to the right of the U.

19. Find how many arrangements of the letters of the word BEHAVING:
(a) end in NG, (b) begin with three vowels, (c) have three vowels occurring together.

20. In how many ways can three Maths books, six Science books and four English books be placed on a shelf, if the books relating to each subject are to be kept together?

21. A Maths test is to consist of six questions. In how many ways can it be arranged so that:
(a) the shortest question is first and the longest question is last,
(b) the shortest and longest questions are next to one another?

22. In how many ways can a boat crew of eight women be arranged if three of the women can only row on the bow side and two others can only row on the stroke side?

23. A motor bike can carry three people: the driver, one passenger behind the driver and one in the sidecar. If among five people, only two can drive, in how many ways can the bike be filled?

24. In Morse code, letters are formed by a sequence of dashes and dots. How many different letters is it possible to represent if a maximum of ten symbols are used?

25. Four boys and four girls are to sit in a row. Find how many ways this can be done if:
(a) the boys and girls alternate, (b) the boys and girls sit in distinct groups.

26. Five-letter words are formed without repetition from the letters of PHYSICAL.
(a) How many consist only of consonants?
(b) How many begin with P and end with S?
(c) How many begin with a vowel?
(d) How many contain the letter Y?
(e) How many have the two vowels occurring next to one another?
(f) How many have the letter A immediately following the letter L?

27. (a) How many seven-letter words can be formed without repetition from the letters of the word INCLUDE?
(b) How many of these do not begin with I? (c) How many end in L?
(d) How many have the vowels and consonants alternating?
(e) How many have the C immediately following the D?
(f) How many have the letters N and D separated by exactly two letters?
(g) How many have the letters N and D separated by more than two letters?

28. Repeat parts (a)–(d) of the previous question if repetition is allowed.

29. (a) In how many ways can ten people be arranged in a line:
(i) without restriction,
(ii) if one particular person must sit at either end,
(iii) if two particular people must sit next to one another,
(iv) if neither of two particular people can sit on either end of the row?
(b) In how many ways can n people be placed in a row of n chairs:
(i) if one particular person must be on either end of the row,
(ii) if two particular people must sit next to one another,
(iii) if two of them are not permitted to sit at either end?

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30. Five boys and four girls form a queue at the cinema. There are two brothers who want to stand together, the remaining three boys wish to stand together, and the four girls don't mind where they stand. In how many ways can the queue be formed?

31. Eight people are to form two queues of four. In how many ways can this be done if:
(a) there are no restrictions, (b) Jim will only stand in the left hand queue,
(c) Sean and Liam must stand in the same queue?

32. (a) How many five-digit numbers can be formed from the digits 2, 3, 4, 5 and 6?
(b) How many of these numbers are greater than 56 432?
(c) How many of these numbers are less than 56 432?

33. There are eight swimmers in a race. In how many ways can they finish if there are no dead heats and the swimmer in Lane 2 finishes:
(a) immediately after the swimmer in Lane 5, (b) after the swimmer in Lane 5?

34. (a) Integers are formed from the digits 2, 3, 4 and 5, with repetitions not allowed.
(i) How many such numbers are there? (ii) How many of them are even?
(b) Repeat the two parts to this question if repetitions are allowed.

35. (a) How many five-digit numbers can be formed from the digits 0, 1, 2, 3 and 4 if repetitions are not allowed?
(b) How many of these are odd? (c) How many are divisible by 5?

36. Five backpackers arrive in a city where there are five youth hostels.
(a) How many different accommodation arrangements are there if there are no restrictions on where the backpackers stay?
(b) How many different accommodation arrangements are there if each backpacker stays at a different youth hostel?
(c) Suppose that two of the backpackers are brother and sister and wish to stay in the same youth hostel. How many different accommodation arrangements are there if the other three can go to any of the other youth hostels?

37. Numbers less than 4000 are formed from the digits 1, 3, 5, 8 and 9, without repetition.
(a) How many such numbers are there? (c) How many of them are divisible by 5?
(b) How many of them are odd? (d) How many of them are divisible by 3?

38. (a) If ${}^8\mathrm{P}_r = 336$, find the value of r. (b) If $7\,{}^{2n}\mathrm{P}_n = 4\,{}^{2n+1}\mathrm{P}_n$, find n.
(c) Using the result ${}^n\mathrm{P}_r = \dfrac{n!}{(n-r)!}$, prove that:
(i) ${}^{n+1}\mathrm{P}_r = {}^n\mathrm{P}_r + r\,{}^n\mathrm{P}_{r-1}$ (ii) ${}^n\mathrm{P}_r = {}^{n-2}\mathrm{P}_r + 2r\,{}^{n-2}\mathrm{P}_{r-1} + r(r-1)\,{}^{n-2}\mathrm{P}_{r-2}$

EXTENSION

39. [Derangements] A *derangement* of n distinct letters is a permutation of them so that no letter appears in its original position. For example, DABC is a derangement of ABCD, but DACB is not. Denote the number of derangements of n letters by $D(n)$.
(a) By listing all the derangements of A, AB, ABC and ABCD, find the values of $D(1)$, $D(2)$, $D(3)$ and $D(4)$.
(b) Suppose that we have formed a derangement of the eight letters ABCDEFGH. Let the last letter in the derangement be X, and exchange X with H. Either X is now in its original position so that six letters are away from their original positions, or X is

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not in its original position so that seven letters are away from their original positions. Hence explain why $D(8) = 7 \times D(7) + 7 \times D(6)$.

(c) Use the same idea to write down a formula for $D(n)$ in terms of $D(n-1)$ and $D(n-2)$. Use it to calculate $D(5)$, $D(6)$, $D(7)$ and $D(8)$, and then find, for $n = 2, 3, \ldots, 8$, the ratio of the number of permutations to the number of derangements.

(d) Use the formula established in part (c), and the values for $D(1)$ and $D(2)$, to prove by mathematical induction that $D(n) = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$.

10 F Counting with Identical Elements, and Cases

This section covers two ideas — counting permutations in which there are identical elements, and counting in which the box methods of the previous section break down and cases have to be considered.

Counting with Identical Elements: Finding the number of different words formed using all the letters of the word 'PRESSES' is complicated by the fact that there are three Ss and two Es. If the seven letters were all different, we would conclude that

$$\text{number of ways} = 7!$$

But we have *overcounted by a factor of* $2! = 2$, because the Es can be interchanged without changing the word. We have also *overcounted by a factor of* $3! = 6$, because the three Ss can be permuted amongst themselves in $3!$ ways without changing the word. Taking account of both overcountings,

$$\text{number of ways} = \frac{7!}{2! \times 3!} = 420.$$

This method is easily generalised. In the language of 'words':

16 COUNTING WITH IDENTICAL ELEMENTS: Suppose that a word of n letters has ℓ_1 alike of one type, ℓ_2 alike of another type, $\ldots$, ℓ_k alike of a final type.
Then the number of distinct words that can be formed from the letters is

$$\text{number of words} = \frac{n!}{\ell_1! \times \ell_2! \times \cdots \times \ell_k!}.$$

WORKED EXERCISE: Four identical wine glasses and four identical tumblers are to be arranged in a line across the front of a cupboard.

(a) In how many ways can this be done?

(b) How does this change if one of the wine glasses now is chipped, and two tumblers have now been replaced by two identical tumblers slightly different from the other two?

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SOLUTION:

(a) Number of ways $= \dfrac{8!}{4! \times 4!}$
$= 70.$

(b) Number of ways $= \dfrac{8!}{3! \times 1! \times 2! \times 2!}$
$= 1680.$

Words Containing Only Two Different Letters: The theory about counting with identical letters assumes particular importance when there are only two types of letters, as in part (a) of the previous worked exercise. The answer to part (a) was $\dfrac{8!}{4! \times 4!}$, which is none other than ${}^8C_4$. This is true in general, and provides the essential link between the binomial theorem and probability.

17 PERMUTATIONS OF WORDS CONTAINING ONLY TWO DIFFERENT LETTERS: Suppose that a word with n letters has k alike of one type and the remaining $n-k$ alike of another type. Then the number of distinct words that can be formed from the letters is

$$\text{number of words} = \frac{n!}{k! \times (n-k)!} = {}^nC_k.$$

WORKED EXERCISE: In a ten-question test, one mark is awarded for each question. In how many ways can a pupil score 7/10? For what other mark is there the same number of ways of achieving it?

SOLUTION: The mark of 7/10 means there were 7 correct and 3 incorrect,

so number of ways $= \dfrac{10!}{7! \times 3!} = 120.$

The other mark that would give 120 ways is 3/10, because for this mark there would be 3 correct and 7 incorrect. Notice that ${}^{10}C_7 = {}^{10}C_3 = 120$.

The Connection with the Binomial Theorem: The fact that the number of words with k As and $n-k$ Bs is ${}^nC_k$ can be seen directly from the binomial expansion. Below is the expansion of $(A+B)^5$ into 32 terms, initially without any simplification or collection of terms. There are $2^5 = 32$ terms, because each term involves choosing either A or B from each of the five brackets in turn.

$$\begin{aligned}
(A+B)^5 &= (A+B)(A+B)(A+B)(A+B)(A+B) \\
&= AAAAA \\
&\quad + AAAAB + AAABA + AABAA + ABAAA + BAAAA \\
&\quad + AAABB + AABAB + ABAAB + BAAAB + AABBA \\
&\qquad + ABABA + BAABA + ABBAA + BABAA + BBAAA \\
&\quad + AABBB + ABABB + BAABB + ABBAB + BABAB \\
&\qquad + BBAAB + ABBBA + BABBA + BBABA + BBBAA \\
&\quad + ABBBB + BABBB + BBABB + BBBAB + BBBBA \\
&\quad + BBBBB \\
&= A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5
\end{aligned}$$

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This should make it clear that, for example, the coefficient ${}^5C_3 = 10$ of A^3B^2 is simply the number of all possible words formed from 3 As and 2 Bs. More generally, the coefficient ${}^nC_k$ of A^kB^{n-k} in the expansion of $(A+B)^n$ is the number of all possible words formed from k As and $n-k$ Bs.

Using Cases: Many counting problems are too complicated to be analysed completely by a single box diagram. In such situations, the use of cases is unavoidable. Attention should be given, however, to minimising the number of different cases that need to be considered.

WORKED EXERCISE: How many six-letter words can be formed by using the letters of the word 'PRESSES'?

SOLUTION: We omit in turn each of the four letters 'P', 'R', 'E' and 'S'.
This leaves six letters which we must then arrange in order.

1. If an S is omitted, there are then 2 Es and 2 Ss,

   so number of words $= \dfrac{6!}{2! \times 2!} = 180.$
2. If an E is omitted, there are then 3 Ss,

   so number of words $= \dfrac{6!}{3!} = 120.$
3. If P or R is omitted (2 cases), there are then 2 Es and 3 Ss,

   so number of words $= \dfrac{6!}{3! \times 2!} \times 2$ (doubling for the two cases)

   $= 120.$

Hence the total number of words is $180 + 120 + 120 = 420$.

Exercise **10F**

1. Find the number of permutations of the following words if all the letters are used:
   (a) BOB
   (b) ALAN
   (c) SNEEZE
   (d) TASMANIA
   (e) BEGINNER
   (f) FOOTBALLS
   (g) EQUILATERAL
   (h) COMMITTEE
   (i) WOOLLOOMOOLOO

2. The six digits 1, 1, 1, 2, 2, 3 are used to form a six-digit number. How many numbers can be formed?

3. Six coins are lined up on a table. Find how many patterns are possible if there are:
   (a) five tails and one head, (b) four heads and two tails, (c) three tails and three heads.

4. Eight balls, identical except for colour, are arranged in a line. Find how many different arrangements are possible if:
   (a) all balls are of a different colour, (b) there are seven red balls and one white ball,
   (c) there are six red balls, one white ball and one black ball,
   (d) there are three red balls, three white balls and two black balls.

5. Five identical green chairs and three identical red chairs are arranged in a row. Find how many arrangements are possible:
   (a) if there are no restrictions, (b) if there must be a green chair on either end.

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DEVELOPMENT

6. A motorist travels through eight sets of traffic lights, each of which is red or green. He is forced to stop at three sets of lights.
(a) In how many ways could this happen?
(b) What other number of red lights would give an identical answer to part (a)?

7. In how many ways can the letters of the word SOCKS be arranged in a line:
(a) without restriction,
(b) so that the two Ss are together,
(c) so that the two Ss are separated by at least one other letter,
(d) so that the K is somewhere to the left of the C?

8. (a) Find the number of arrangements of the letters in SLOOPS if:
(i) there are no restrictions,
(ii) the two Os are together,
(iii) the two Os are to be separated,
(iv) the Os are together and the Ss are together.
(b) In how many arrangements of the letters in TATTOO are the two Os separated?

9. In how many ways can the letters of the word DECISIONS be arranged:
(a) without restriction,
(b) so that the vowels and consonants alternate,
(c) so that the vowels come first followed by the consonants,
(d) so that the N is somewhere to the right of the D?

10. In how many ways can the letters of the word PROPORTIONALITY be arranged so that the vowels and consonants still occupy the same places?

11. A form has ten questions in order, each of which requires the answer 'Yes' or 'No'. Find the number of ways the form can be filled in:
(a) without restriction,
(b) if the first and last answers are 'Yes',
(c) if two are 'Yes' and eight are 'No',
(d) if five are 'Yes' and five are 'No',
(e) if more than seven answers are 'Yes',
(f) if an odd number of answers are 'Yes',
(g) if exactly three answers are 'Yes', and they occur together,
(h) if the first and last answers are 'Yes' and exactly four more are 'Yes'.

12. Containers are identified by a row of coloured dots on their lids. The colours used are yellow, green and purple. In any arrangement, there are to be no more than three yellow dots, no more than two green dots and no more than one purple dot.
(a) If six dots are used, what is the number of possible codes?
(b) What is the number of different codes possible if only five dots are used?

13. (a) How many five-letter words can be formed by using the letters of the word STRESS?
(b) How many five-letter words can be formed by using the letters of the word BANANA?

14. Six-letter words are formed using two As and four Bs. By referring to the relevant section of the notes, show how all the different possible words can be derived from the expansion of $(A+B)^6$.

15. Find how many arrangements of the letters of the word TRANSITION are possible if:
(a) there are no restrictions,
(b) the Is are together,
(c) the Is are together, and so are the Ns, and so are the Ts,
(d) the Ns occupy the end positions,
(e) an N occupies the first but not the last position,
(f) the letter N is not at either end,
(g) the vowels are together.

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16. Ten coloured marbles are placed in a row.

(a) If they are all of different colours, how many arrangements are possible?

(b) What is the minimum number of colours needed to guarantee at least 10 000 different patterns? [HINT: This will need a guess-and-check approach.]

EXTENSION

17. Find how many arrangements there are of the letters of the word GUMTREE if:

(a) there are no restrictions,

(b) the Es are together,

(c) the Es are separated by: (i) one, (ii) two, (iii) three, (iv) four, (v) five letters.

(d) the G is somewhere between the two Es,

(e) the M is somewhere to the left of both Es and the U is somewhere between them,

(f) the G is somewhere to the left of the U and the M is somewhere to the right of the U.

18. If the letters of the word GUMTREE and the letters of the word KOALA are combined and arranged into a single twelve-letter word, in how many of these arrangements do the letters of KOALA appear in their correct order, but not necessarily together?

19. Bob is about to hang his eight shirts in the wardrobe. He has four different styles of shirt, with two identical shirts in each style. How many different arrangements are possible if no two identical shirts are next to one another?

20. Binary numbers consist of a sequence of 0s and 1s, and we shall allow leading zeroes in this question. One such number contains exactly a 0s and at most b 1s.

(a) If the number contains $a+b$ digits, show that there are ${}^{a+b}\mathrm{C}_b$ possible sequences. You will need to recall that $\dfrac{n!}{r!\,(n-r)!} = {}^{n}\mathrm{C}_r$.

(b) Prove that ${}^{a}\mathrm{C}_0 + {}^{a+1}\mathrm{C}_1 + {}^{a+2}\mathrm{C}_2 + \cdots + {}^{a+b}\mathrm{C}_b = {}^{a+b+1}\mathrm{C}_b$.

(c) Hence show that the total number of possible sequences is ${}^{a+b+1}\mathrm{C}_b$.

10 G Counting Unordered Selections

This section now turns from the ordered selections of the previous two sections to the counting of unordered selections. An unordered selection of distinct objects chosen from a certain set can be regarded simply as a subset of the set. For example, the four-member set { A, B, C, D } has six two-member subsets:

{ A, B } { A, C } { A, D } { B, C } { B, D } { C, D }

The central result of the section is the following.

18 UNORDERED SELECTIONS: If a set S has n members, then the number of unordered selections of k members from the set is

$$\text{number of } k\text{-member subsets} = \frac{n!}{k! \times (n-k)!} = {}^{n}\mathrm{C}_k.$$

The total number of subsets of S is 2^n.

For example, the number of unordered selections, or subsets, of two distinct letters taken from the four-member set { A, B, C, D } is ${}^{4}\mathrm{C}_2 = \dfrac{4!}{2! \times 2!} = 6$. These are the six subsets listed above.

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This result brings the whole theory of the binomial expansion into probability and counting, and it will dominate the remainder of this chapter. Two different proofs of the result will be offered.

An Example of the Result: Before giving any proof, however, here is an example to illustrate the result. Let S be the five-member set

$$S = \{\,\mathrm{A, B, C, D, E}\,\}.$$

The total number of subsets of S is $2^5 = 32$, because choosing a subset or unordered selection from S requires looking at every member of S and deciding whether to include it in the subset or not — using the box notation:

Is A in or out?	Is B in or out?	Is C in or out?	Is D in?	Is E in?
2	2	2	2	2

Here is a list of all 32 subsets, arranged according to the number of members:

1 0-member subset : $\emptyset$
5 1-member subsets: $\{\mathrm{A}\}$, $\{\mathrm{B}\}$, $\{\mathrm{C}\}$, $\{\mathrm{D}\}$, $\{\mathrm{E}\}$
10 2-member subsets: $\{\mathrm{A,B}\}$, $\{\mathrm{A,C}\}$, $\{\mathrm{A,D}\}$, $\{\mathrm{A,E}\}$, $\{\mathrm{B,C}\}$, $\{\mathrm{B,D}\}$, $\{\mathrm{B,E}\}$, $\{\mathrm{C,D}\}$, $\{\mathrm{C,E}\}$, $\{\mathrm{D,E}\}$
10 3-member subsets: $\{\mathrm{A,B,C}\}$, $\{\mathrm{A,B,D}\}$, $\{\mathrm{A,B,E}\}$, $\{\mathrm{A,C,D}\}$, $\{\mathrm{A,C,E}\}$, $\{\mathrm{A,D,E}\}$, $\{\mathrm{B,C,D}\}$, $\{\mathrm{B,C,E}\}$, $\{\mathrm{B,D,E}\}$, $\{\mathrm{C,D,E}\}$
5 4-member subsets: $\{\mathrm{A,B,C,D}\}$, $\{\mathrm{A,B,C,E}\}$, $\{\mathrm{A,B,D,E}\}$, $\{\mathrm{A,C,D,E}\}$, $\{\mathrm{B,C,D,E}\}$
1 5-member subset : $\{\mathrm{A,B,C,D,E}\}$

The numbers 1, 5, 10, 10, 5, 1 form the row of the Pascal triangle indexed by $n = 5$ and add to $2^5 = 32$. Thus at least for $n = 5$, the number of k-member subsets of an n-member set is indeed ${}^n\mathrm{C}_k$.

A Proof Using Words Containing only Two Distinct Letters: Continuing with the example above, every subset of S can be described by visiting each member A, B, C, D and E of S in turn, answering 'yes' or 'no' as to whether that member is included in the subset. Thus every subset of S corresponds to a five-letter word consisting entirely of Ys and Ns. For example, each of the ten three-member subsets of S is paired below with the corresponding word consisting of three Ys and two Ns:

$$\begin{array}{ll}
\{\mathrm{A,B,C}\} \longleftrightarrow \mathrm{YYYNN} & \{\mathrm{A,D,E}\} \longleftrightarrow \mathrm{YNNYY} \\
\{\mathrm{A,B,D}\} \longleftrightarrow \mathrm{YYNYN} & \{\mathrm{B,C,D}\} \longleftrightarrow \mathrm{NYYYN} \\
\{\mathrm{A,B,E}\} \longleftrightarrow \mathrm{YYNNY} & \{\mathrm{B,C,E}\} \longleftrightarrow \mathrm{NYYNY} \\
\{\mathrm{A,C,D}\} \longleftrightarrow \mathrm{YNYYN} & \{\mathrm{B,D,E}\} \longleftrightarrow \mathrm{NYNYY} \\
\{\mathrm{A,C,E}\} \longleftrightarrow \mathrm{YNYNY} & \{\mathrm{C,D,E}\} \longleftrightarrow \mathrm{NNYYY}
\end{array}$$

From the previous section, we know that the number of five-letter words consisting of three Ys and two Ns is $\dfrac{5!}{3! \times 2!} = {}^5\mathrm{C}_3 = 10$, so this must be the number of three-member subsets. Moreover, the total number of five-letter words consisting entirely of Ys and Ns is $2^5 = 32$, because there are two choices for each letter in turn, so this must be the total number of subsets.

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In general, let the members of the n-member set S be listed in some order:

$$S = \{ S_1, S_2, S_3, S_4, S_5, \dots , S_n \}.$$

Again, choosing a subset T of S means visiting every member of S and saying 'yes' or 'no' as to whether that member is to be included in the subset T. Thus there is a one-to-one correspondence between the subsets of S and the set of 2^n words of n letters consisting entirely of Ys and Ns. In particular, there is a one-to-one correspondence between the k-member subsets of S and the ${}^n\mathrm{C}_k$ words consisting of k Ys and $n - k$ Ns.

A Proof Moving from Ordered Selections to Unordered Selections: We saw in Section 10E that the number of three-letter words formed without repetition from the five letters A, B, C, D and E is ${}^5\mathrm{P}_3 = 5 \times 4 \times 3 = 60$. When we turn to unordered selections, however, there are six distinct words which all correspond, for example, to the three-member subset $\{ \mathrm{B, C, E} \}$:

$$\mathrm{BCE,\ BEC,\ CBE,\ CEB,\ EBC,\ ECB} \longleftrightarrow \{ \mathrm{B, C, E} \}$$

The reason for this is that there are ${}^3\mathrm{P}_3 = 3 \times 2 \times 1 = 6$ ways of ordering the subset $\{ \mathrm{B, C, E} \}$. Thus the correspondence between three-letter words and three-member subsets is many-to-one, with a six-fold overcounting. Hence the number of three-member subsets is $60 \div 6 = 10$, as required.

In general, ${}^n\mathrm{P}_k = \dfrac{n!}{(n-k)!}$ words of k letters can be formed without repetition from the members of S. But every k-member subset can be ordered in ${}^k\mathrm{P}_k = k!$ ways, so the correspondence between the ordered selections and the unordered selections is many-to-one with overcounting by a factor of $k!$ Hence the number of (unordered) k-member subsets of S is

$${}^n\mathrm{P}_k \div {}^k\mathrm{P}_k = \frac{n!}{(n-k)!} \div k! = \frac{n!}{k! \times (n-k)!} = {}^n\mathrm{C}_k.$$

The Meaning of the Notation ${}^n\mathrm{C}_k$: The notation ${}^n\mathrm{C}_k$ does not originate with the binomial theorem, but with this piece of counting theory. Unordered selections are also called *combinations*, and the letter C in ${}^n\mathrm{C}_k$ stands for 'combination', just as the letter P in ${}^n\mathrm{P}_k$ stands for 'permutation'.

By a convenient, but false, etymology, the letter C also stands for 'choose'. This is the origin of the more recent convention of calling ${}^n\mathrm{C}_k$ 'n choose k'.

WORKED EXERCISE: Ten people meet to play doubles tennis.

(a) In how many ways can four people be selected from this group to play the first game? (Ignore the subsequent organisation into pairs.)

(b) How many of these ways will include Maria and exclude Alex?

(c) If there are four women and six men, in how many ways can two men and two women be chosen for this game?

(d) Again with four women and six men, in how many ways will women be in the majority?

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SOLUTION:

(a) Number of ways $= {}^{10}\mathrm{C}_4 = 210$.

(b) Since Maria is included, three further people must be chosen, and since Alex is excluded, there are now eight people to choose these three from.
Hence number of ways $= {}^{8}\mathrm{C}_3 = 56$.

(c) Number of ways of choosing the women $= {}^{4}\mathrm{C}_2 = 6$,
number of ways of choosing the men $= {}^{6}\mathrm{C}_2 = 15$,
so number of ways of choosing all four $= 15 \times 6 = 90$.

(d) Number of ways with one man and three women $= {}^{6}\mathrm{C}_1 \times {}^{4}\mathrm{C}_3 = 24$,
number of ways with four women $= 1$,
so number of ways with a majority of women $= 24 + 1 = 25$.

A Natural (or Canonical) Correspondence — ${}^n\mathrm{C}_k = {}^n\mathrm{C}_{n-k}$: Suppose that two people are to be chosen from five to make afternoon tea. This task can be done equally well by choosing the three people who will *not* be making tea. Thus for every choice of two people out of five, the remaining three people is a corresponding choice of three people out of five. Not only does this confirm that ${}^5\mathrm{C}_2 = {}^5\mathrm{C}_3$, but it gives a one-to-one correspondence between the two-member and three-member subsets of a five-member set:

$$\begin{array}{lll}
\{\mathrm{A,B}\} \longleftrightarrow \{\mathrm{C,D,E}\} & \qquad & \{\mathrm{B,D}\} \longleftrightarrow \{\mathrm{A,C,E}\} \\
\{\mathrm{A,C}\} \longleftrightarrow \{\mathrm{B,D,E}\} & & \{\mathrm{B,E}\} \longleftrightarrow \{\mathrm{A,C,D}\} \\
\{\mathrm{A,D}\} \longleftrightarrow \{\mathrm{B,C,E}\} & & \{\mathrm{C,D}\} \longleftrightarrow \{\mathrm{A,B,E}\} \\
\{\mathrm{A,E}\} \longleftrightarrow \{\mathrm{B,C,D}\} & & \{\mathrm{C,E}\} \longleftrightarrow \{\mathrm{A,B,D}\} \\
\{\mathrm{B,C}\} \longleftrightarrow \{\mathrm{A,D,E}\} & & \{\mathrm{D,E}\} \longleftrightarrow \{\mathrm{A,B,C}\}
\end{array}$$

In this correspondence, every subset T corresponds to its complement $\overline{T}$:

$$T \longleftrightarrow \overline{T}$$

This correspondence is a natural or *canonical* correspondence, but the correspondence is by no means restricted to mathematics. In normal language, a situation can often be described just as well by saying what it is not as by saying what it is, and we are all familiar with 'invisible enemies' or 'unforgivable actions' or 'days when no rain fell'.

Words such as 'at least', 'at most', 'not' and 'excluding' should always be regarded as warnings that the problem may best be solved by considering the complementary situation.

WORKED EXERCISE: Let $S = \{2, 4, 6, 8, 10, 12\}$ be the set consisting of the first six positive even numbers.

(a) How many subsets of S contain at least two numbers?

(b) How many subsets with at least two numbers do not contain 8?

(c) How many subsets with at least two numbers do not contain 8 but do contain 10?

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SOLUTION:

(a) Number of 1-member and 0-member subsets $= {}^6\mathrm{C}_1 + {}^6\mathrm{C}_0 = 7$,
so number with at least 2 members $= 2^6 - 7 = 57$.

(b) We consider now the 5-member set $T = \{2, 4, 6, 10, 12\}$.
Number of 1-member and 0-member subsets $= {}^5\mathrm{C}_1 + {}^5\mathrm{C}_0 = 6$,
so number with at least 2 members $= 2^5 - 6 = 26$.

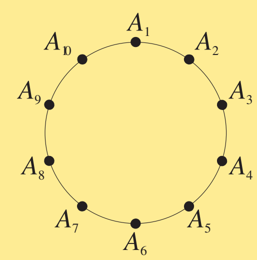
(c) Since 10 has already been chosen, we need to choose subsets with at least one member from the four-member set $U = \{2, 4, 6\ 12\}$.
Number of 0-member subsets $= 1$ (the empty set),
so number of such subsets $= 2^4 - 1 = 15$.

WORKED EXERCISE: [A harder question] Ten points $A_1, A_2, \ldots, A_{10}$ are arranged in order around a circle.

(a) How many triangles can be drawn with these points as vertices?

(b) How many pairs of such triangles can be drawn, if the vertices of the two triangles are distinct?

(c) In how many such pairs will the triangles:
(i) not overlap, (ii) overlap?



SOLUTION:

(a) To form a triangle, we must choose 3 points out of 10,
so number of triangles $= {}^{10}\mathrm{C}_3 = 120$.

(b) To form two triangles, first choose 6 points out of 10,
which can be done in ${}^{10}\mathrm{C}_6 = 210$ ways.
Take any one of those 6 points, and choose the other 2 points in its triangle;
this can be done in ${}^5\mathrm{C}_2 = 10$ ways.
Hence number of pairs of triangles $= 210 \times 10 = 2100$.

(c) To form two non-overlapping triangles, we first choose 6 points out of 10,
which again can be done in ${}^{10}\mathrm{C}_6 = 210$ ways.
These 6 points can be made into two non-overlapping triangles in 3 ways,
by arranging the 6 points in cyclic order, and choosing 3 adjacent points.
(i) Hence number of non-overlapping pairs $= 210 \times 3 = 630$.
(ii) By subtraction, number of overlapping pairs $= 2100 - 630 = 1470$.

Each Row of the Pascal Triangle Adds to 2^n: One identity about the Pascal triangle needs review here. The binomial expansion is

$$(x + y)^n = {}^n\mathrm{C}_0\, x^n\, y^0 + {}^n\mathrm{C}_1\, x^{n-1}\, y^1 + {}^n\mathrm{C}_2\, x^{n-2}\, y^2 + \cdots + {}^n\mathrm{C}_n\, x^0\, y^n .$$

Substituting $x = y = 1$ gives $2^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n$, which means that the row indexed by n in the Pascal triangle adds to 2^n.

We now have another interpretation of this identity. An n-member set has ${}^n\mathrm{C}_k$ k-member subsets, and the total number of subsets is

$${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n = 2^n .$$

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Exercise 10G

1. Two people are chosen from a group of five people called P, Q, R, S and T. List all possible combinations, and find how many there are.

2. Find in how many ways you can form a group of:

(a) two people from a choice of seven,
(b) three people from a choice of seven,
(c) two people from a choice of six,
(d) five people from a choice of nine.

3. (a) Find how many possible combinations there are if, from a group of ten people:
(i) two people are chosen,
(ii) eight people are chosen.
(b) Why are the answers identical?

4. From a party of twelve men and eight women, find how many groups there are of:

(a) five men and three women,
(b) four women and four men.

5. Four numbers are to be selected from the set of the first eight positive integers. Find how many possible combinations there are if:

(a) there are no restrictions,
(b) there are two odd numbers and two even numbers,
(c) there is exactly one odd number,
(d) all the numbers must be even,
(e) there is at least one odd number.

6. Four balls are simultaneously drawn from a bag containing three green and six blue balls. Find how many ways there are of drawing the four balls if:

(a) the balls may be of any colour,
(b) there are exactly two green balls,
(c) there are at least two green balls,
(d) there are more blue balls than green balls.

7. A committee of five is to be chosen from six men and eight women. Find how many committees are possible if:

(a) there are no restrictions,
(b) all members are to be female,
(c) all members are to be male,
(d) there are exactly two men,
(e) there are four women and one man,
(f) there is a majority of women,
(g) a particular man must be included,
(h) a particular man must not be included.

DEVELOPMENT

8. (a) What is the number of combinations of the letters of the word EQUATION taken four at a time (without repetition)?
(b) How many contain four vowels?
(c) How many contain the letter Q?

9. A team of seven netballers is to be chosen from a squad of twelve players A, B, C, D, E, F, G, H, I, J, K and L. In how many ways can they be chosen:

(a) with no restrictions,
(b) if the captain C is to be included,
(c) if J and K are both to be excluded,
(d) if A is included but H is not,
(e) if one of F and L is to be included and the other excluded?

10. (a) Consider the digits 9, 8, 7, 6, 5, 4, 3, 2, 1 and 0. Find how many five-digit numbers are possible if the digits are to be in: (i) descending order, (ii) ascending order.
(b) Why do these two questions involve unordered selections?

11. Twelve people arrive at a restaurant. There is one table for six, one table for four and one table for two. In how many ways can they be assigned to a table?

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12. Twenty students, ten male and ten female, are to travel from school to the sports ground. Eight of them go in a minibus, six of them in cars, four of them on bikes and two walk.

(a) In how many ways can they be distributed for the trip?

(b) In how many ways can they be distributed if none of the boys walk?

13. Ten points P_1, P_2, ..., P_{10} are chosen in a plane, no three of the points being collinear.

(a) How many lines can be drawn through pairs of the points?

(b) How many triangles can be drawn using the given points as vertices?

(c) How many of these triangles have P_1 as one of their vertices?

(d) How many of these triangles have P_1 and P_2 as vertices?

14. Ten points are chosen in a plane. Five of the points are collinear, but no other set of three of the points is collinear.

(a) How many sets of three points can be selected from the five that are collinear?

(b) How many triangles can be formed using the ten points as vertices?

15. From a standard deck of 52 playing cards, find how many five-card hands can be dealt:

(a) consisting of black cards only,

(b) consisting of diamonds only,

(c) containing all four kings,

(d) consisting of three diamonds and two clubs,

(e) consisting of three twos and another pair,

(f) consisting of one pair and three of a kind.

16. (a) In how many ways can a group of six people be divided into:

(i) two unequal groups (neither group being empty), (ii) two equal groups?

(b) Repeat part (a) for four people.

(c) Repeat part (a) for eight people.

17. Find how many diagonals there are in:

(a) a quadrilateral, (b) a pentagon, (c) a decagon, (d) a polygon with n sides.

18. Twelve points are arranged in order around a circle.

(a) How many triangles can be drawn with these points as vertices?

(b) In how many pairs of such triangles are the vertices of the two triangles distinct?

(c) In how many such pairs will the triangles: (i) not overlap, (ii) overlap?

19. Let $S = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ be the set of the first ten positive odd integers.

(a) How many subsets does S have?

(b) How many subsets of S contain at least three numbers?

(c) How many subsets with at least three numbers do not contain 7?

(d) How many subsets with at least three numbers do not contain 7 but do contain 19?

20. In how many ways can two numbers be selected from 1, 2, ..., 8, 9 so that their sum is:

(a) even, (b) odd, (c) divisible by 3, (d) divisible by 5, (e) divisible by 6?

21. There are ten basketballers in a team. Find in how many ways:

(a) the starting five can be chosen,

(b) they can be split into two teams of five.

22. Nine players are to be divided into two teams of four and one umpire.

(a) In how many ways can the teams be formed?

(b) If two particular people cannot be on the same team, how many different combinations are possible?

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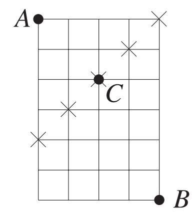
23. By considering their prime factorisations, find the number of positive divisors of:
(a) $2^3 \times 3^2$ (b) 1 000 000 (c) 315 000 (d) $2^a \times 5^b \times 13^c$

24. (a) The six faces of a number of identical cubes are painted in six distinct colours. How many different cubes can be formed?
(b) A die fits perfectly into a cubical box. How many ways are there of putting the die into the box?

25. The diagram shows a 6×4 grid. The aim is to walk from the point A in the top left-hand corner to the point B in the bottom right-hand corner by walking along the black lines either downwards or to the right. A single move is defined as walking along one side of a single small square, thus it takes you ten moves to get from A to B.



(a) Find how many different routes are possible:
(i) without restriction, (ii) if you must pass through C,
(iii) if you cannot move along the top line of the grid,
(iv) if you cannot move along the second row from the top of the grid.
(b) Notice that every route must pass through one and only one of the five crossed points. Hence prove that ${}^{10}C_4 = {}^4C_0 \times {}^6C_4 + {}^4C_1 \times {}^6C_3 + {}^4C_2 \times {}^6C_2 + {}^4C_3 \times {}^6C_1 + {}^4C_4 \times {}^6C_0$.
(c) Draw another suitable diagonal and, using a method similar to that in part (b), prove that ${}^{10}C_4 = {}^5C_0 \times {}^5C_4 + {}^5C_1 \times {}^5C_3 + {}^5C_2 \times {}^5C_2 + {}^5C_3 \times {}^5C_1 + {}^5C_4 \times {}^5C_0$.
(d) Draw up a similar 6×6 grid, then using the same idea as that used in parts (b) and (c), prove that ${}^{12}C_6 = \left({}^6C_0\right)^2 + \left({}^6C_1\right)^2 + \left({}^6C_2\right)^2 + \left({}^6C_3\right)^2 + \left({}^6C_4\right)^2 + \left({}^6C_5\right)^2 + \left({}^6C_6\right)^2$.

26. Use the fact that ${}^nC_r$ is the number of unordered selections of r objects from n objects to provide combinatoric proofs of each of the following:
(a) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$
(b) ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$
(c) ${}^{n+2}C_r = {}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2}$
(d) ${}^{n+3}C_r = {}^nC_r + 3\,{}^nC_{r-1} + 3\,{}^nC_{r-2} + {}^nC_{r-3}$
(e) ${}^{m+n}C_2 = {}^mC_2 + {}^mC_1\,{}^nC_1 + {}^nC_2$
(f) ${}^{m+n}C_3 = {}^mC_3 + {}^mC_2\,{}^nC_1 + {}^mC_1\,{}^nC_2 + {}^nC_3$

EXTENSION

27. A piece of art receives an integer mark from zero to 100 for each of the categories design, technique and originality. In how many ways is it possible to score a total mark of 200?

28. How many different combinations are there of three different integers between one and thirty inclusive such that their sum is divisible by three?

29. (a) How many doubles tennis games are possible, given a group of four players?
(b) In how many ways can two games of doubles tennis be arranged, given a group of eight players?
(c) Six married couples are to play in three games of doubles tennis. Find how many ways the pairings can be arranged if:
(i) there are no restrictions, (ii) each game is to be a game of mixed doubles.

10 H Using Counting in Probability

The purpose of this section is to apply the counting procedures of the last three sections to questions about probability. In these more complicated questions, counting procedures are required for counting both the sample space and the event space.

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Counting the Sample Space and the Event Space: It is usually easier to use unordered selections, when this is possible, but as always, the two questions that need to be asked are: 'Is the selection ordered?' and if so, 'Is repetition allowed?'

WORKED EXERCISE: Three cards are dealt from a pack of 52.
(a) Find the probability that one club and two hearts are dealt, in any order.
(b) Find the probability that one club and two hearts are dealt in that order.

SOLUTION:
(a) Let the sample space be the set of all unordered selections of 3 cards from 52,

so $\qquad$ number of unordered hands $= {}^{52}\mathrm{C}_3 = \dfrac{52 \times 51 \times 50}{3 \times 2 \times 1}$.

We can now choose the hand by choosing 1 club from 13 in ${}^{13}\mathrm{C}_1 = 13$ ways, and choosing the 2 hearts from 13 in ${}^{13}\mathrm{C}_2 = 78$ ways, so the hand can be chosen in 13×78 ways.

Hence $\qquad \mathcal{P}(\text{1 club and 2 hearts}) = 13 \times 78 \times \dfrac{3 \times 2 \times 1}{52 \times 51 \times 50} = \dfrac{39}{850}$.

(b) Let the sample space be the set of all ordered selections of 3 cards from 52,

so $\qquad$ number of ordered hands $= {}^{52}\mathrm{P}_3 = 52 \times 51 \times 50$,

and $\qquad$ number of such hands in the order $\clubsuit\heartsuit\heartsuit = 13 \times 13 \times 12$.

Hence $\qquad \mathcal{P}(\clubsuit\heartsuit\heartsuit) = \dfrac{13 \times 13 \times 12}{52 \times 51 \times 50} = \dfrac{13}{850}$.

NOTE: The answer to part (b) must be $\frac{1}{3}$ of the answer to part (a), because in a hand with one club and two hearts, the club can be any one of three positions. This indicates that it would be quite reasonable to do part (a) using ordered selections, and to do part (b) using unordered selections, although the methods chosen above are more natural to the way in which each question was worded.

Problems Requiring a Variety of Methods: The sample spaces in the two worked examples following are easily found, but a variety of methods is needed to establish the sizes of the various event spaces.

WORKED EXERCISE: A five-digit number is chosen at random. Find the probability:
(a) that it is at least 60 000,
(b) that it consists only of even digits,
(c) that the digits are distinct,
(d) that the digits are distinct and in increasing order.

SOLUTION: The first digit of a five-digit number cannot be zero, giving nine choices, but the other digits can be any one of the ten digits.
Hence the number of five-digit numbers $= 9 \times 10 \times 10 \times 10 \times 10 = 90\,000$.

(a) To be at least 60 000, the first digit can be 6, 7, 8 or 9, so the number of favourable numbers is $4 \times 10 \times 10 \times 10 \times 10 = 40\,000$.
Hence $\mathcal{P}(\text{at least } 60\,000) = \frac{4}{9}$.

(b) If all the digits are even, there are four choices for the first digit (it cannot be zero) and five choices for each of the other four.
Hence number of such numbers $= 4 \times 5 \times 5 \times 5 \times 5 = 2500$,

and $\qquad \mathcal{P}(\text{all digits are even}) = \dfrac{2500}{90\,000} = \dfrac{1}{36}$.

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(c) This is counting without replacement:

1st digit	2nd digit	3rd digit	4th digit	5th digit
9	9	8	7	6

so number of such numbers $= 9 \times 9 \times 8 \times 7 \times 6$

and $\mathcal{P}(\text{digits are distinct}) = \dfrac{9 \times 9 \times 8 \times 7 \times 6}{90\,000} = \dfrac{189}{625}$.

(d) Every *unordered* five-member subset of the set of nine nonzero digits can be arranged in exactly one way into a five-digit number with the digits in increasing order. (Note that the digit zero cannot be used, since a number can't begin with the digit zero.)
Hence number of such numbers
$= \text{number of unordered subsets of } \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$= {}^9C_5 = 126$,

so $\mathcal{P}(\text{digits are distinct and in increasing order}) = \dfrac{126}{90\,000} = \dfrac{7}{5000}$.

WORKED EXERCISE: [A harder example] Continuing with the previous worked exercise, find the probability that:
(a) the number contains at least one four,
(b) the number contains at least one four and at least one five,
(c) the number contains exactly three sevens,
(d) the number contains at least three sevens.

SOLUTION:
(a) This can be approached using the complementary event:
number of five-digit numbers without a $4 = 8 \times 9 \times 9 \times 9 \times 9 = 52\,488$,
so number of five-digit numbers with a $4 = 90\,000 - 52\,488 = 37\,512$.
Hence $\mathcal{P}(\text{at least one } 4) = \dfrac{37\,512}{90\,000} = \dfrac{521}{1250}$.

(b) This can be approached using the addition theorem:
number without a $4 = 52\,488$,
similarly number without a $5 = 52\,488$,
and number with no 5 and no $4 = 7 \times 8 \times 8 \times 8 \times 8 = 28\,672$.
Hence number with no 5 or no $4 = 52\,488 + 52\,488 - 28\,672 = 76\,304$,
and number with at least one 5 and at least one $4 = 90\,000 - 76\,304 = 13\,696$.
Hence $\mathcal{P}(\text{at least one 4 and at least one } 5) = \dfrac{13\,696}{90\,000} = \dfrac{856}{5625}$.

(c) Counting the number of five-digit numbers with exactly three 7s requires cases. First we count the five-digit strings with exactly three 7s, by first placing the three 7s and then choosing the first and second non-7 digits:

position of the three 7s	choose first non-7	choose second non-7
${}^5C_3 = 10$	9	9

giving 810 such strings. Secondly, we must subtract the number of five-digit strings with exactly three 7s and beginning with zero:

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position of the three 7s	choose the other non-7
${}^4C_3 = 4$	9

giving 36 such strings. Hence there are $810 - 36 = 774$ such numbers, and

$$\mathcal{P}(\text{number has exactly three 7s}) = \frac{774}{90\,000} = \frac{43}{5000}.$$

(d) The number 77 777 is the only five-digit number with five 7s.
Any five-digit number with exactly four 7s has one of the five forms

$$*7\,777, \qquad 7*777, \qquad 77*77, \qquad 777*7, \qquad 7777*,$$

where the $*$ in $*7\,777$ is a nonzero digit. There are eight numbers of the first form, and nine of the other four forms, giving 44 numbers altogether.
Hence the number with at least three 7s $= 774 + 44 + 1 = 819$,

and $$\mathcal{P}(\text{at least three 7s}) = \frac{819}{90\,000} = \frac{91}{10\,000}.$$

Exercise **10H**

1. A committee of three is to be selected from the nine members in a club.
(a) How many different committees can be formed?
(b) If there are five men in the club, what is the probability that the selected committee consists entirely of males?

2. The integers from 1 to 10 inclusive are written on ten separate pieces of paper. Four pieces of paper are drawn at random. Find the probability that:
(a) the four numbers drawn are 1, 2, 3 and 6,
(b) the number 9 is one of the numbers drawn,
(c) the number 8 is not drawn,
(d) the number 7 is drawn but the number 1 is not.

3. A bag contains three red, seven yellow and five blue balls. If three balls are drawn from the bag simultaneously, find the probability that:
(a) all three balls are yellow,
(b) all the balls are of the same colour,
(c) there are two red balls and one blue ball,
(d) all the balls are of different colours.

4. A sports committee of five members is to be chosen from eight AFL footballers and seven soccer players. Find the probability that the committee will contain:
(a) only AFL footballers,
(b) only soccer players,
(c) three soccer players and two AFL footballers,
(d) at least one soccer player,
(e) at most one soccer player,
(f) Ian, a particular soccer player.

5. From a standard pack of 52 cards, three are selected at random. Find the probability that:
(a) they are the jack of spades, the two of clubs and the seven of diamonds,
(b) all three are aces,
(c) they are all diamonds,
(d) they are all of the same suit,
(e) they are all picture cards,
(f) two are red and one is black,
(g) one is a seven, one is an eight and one is a nine,
(h) two are 7s and one is a 6,
(i) exactly one is a diamond,
(j) at least two of them are diamonds.

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6. Repeat the previous question if the cards are selected from the pack one at a time, and each card is replaced before the next one is drawn.

7. Three boys and three girls are to sit in a row. Find the probability that:
(a) the boys and girls alternate,
(b) the boys and girls sit together,
(c) two specific girls sit next to one another.

8. A family of five are seated in a row at the cinema. Find the probability that:
(a) the parents sit on the end and the three children are in the middle,
(b) the parents sit next to one another.

9. Six people, of whom Patrick and Jessica are two, arrange themselves in a row. Find the probability that:
(a) Patrick and Jessica occupy the end positions,
(b) Patrick and Jessica are not next to each other.

DEVELOPMENT

10. The letters of PROMISE are arranged randomly in a row. Find the probability that:
(a) the word starts with R and ends with S,
(b) the letters P and R are next to one another,
(c) the letters P and R are separated by at least three letters,
(d) the vowels and the consonants alternate,
(e) the vowels are together.

11. The digits 3, 3, 4, 4, 4 and 5 are placed in a row to form a six-digit number. If one of these numbers is selected at random, find the probability that:
(a) it is even,
(b) it ends in 5,
(c) the 4s occur together,
(d) the number starts with 5 and then the 4s and 3s alternate,
(e) the 3s are separated by at least one other number.

12. The letters of the word PRINTER are arranged in a row. Find the probability that:
(a) the word starts with the letter E,
(b) the letters I and P are next to one another,
(c) there are three letters between N and T,
(d) there are at least three letters between N and T.

13. The letters of KETTLE are arranged randomly in a row. Find the probability that:
(a) the two letters E are together,
(b) the two letters E are not together,
(c) the two letters E are together and the two letters T are together,
(d) the Es and Ts are together in one group.

14. The letters of ENTERTAINMENT are arranged in a row. Find the probability that:
(a) the letters E are together,
(b) two Es are together and one is apart,
(c) all the letters E are apart,
(d) the word starts and ends with E.

15. A tank contains 20 tagged fish and 80 untagged fish. On each day, four fish are selected at random, and after noting whether they are tagged or untagged, they are returned to the tank. Answer the following questions, correct to three significant figures.
(a) What is the probability of selecting no tagged fish on a given day?
(b) What is the probability of selecting at least one tagged fish on a given day?
(c) Calculate the probability of selecting no tagged fish on every day for a week.
(d) What is the probability of selecting no tagged fish on exactly three of the seven days during the week?

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16. A bag contains seven white and five black discs. Three discs are chosen from the bag. Find the probability that all three discs are black, if the discs are chosen:

(a) without replacement, (b) with replacement,

(c) so that after each draw the disc is replaced with one of the opposite colour.

17. Six people are to be divided into two groups, each with at least one person. Find the probability that:

(a) there will be three in each group,

(b) there will be two in one group and four in the other,

(c) there will be one group of five and an individual.

18. A three-digit number is formed from the digits 3, 4, 5, 6 and 7 (no repetitions allowed). Find the probability that:

(a) the number is 473,

(b) the number is odd,

(c) the number is divisible by 5,

(d) the number is divisible by 3,

(e) the number starts with 4 and ends with 7,

(f) the number contains the digit 3,

(g) the number contains the digits 3 and 5,

(h) the number contains the digits 3 or 5,

(i) all digits in the number are odd,

(j) the number is greater than 500.

19. The digits 1, 2, 3 and 4 are used to form numbers that may have 1, 2, 3 or 4 digits in them. If one of the numbers is selected at random, find the probability that:

(a) it has three digits, (b) it is even, (c) it is greater than 200,

(d) it is odd and greater than 3000, (e) it is divisible by 3.

20. (a) A senate committee of five members is to be selected from six Labor and five Liberal senators. What is the probability that Labor will have a majority on the committee?

(b) The senate committee is to be selected from N Labor and five Liberal senators. Use trial and error to find the minimum value of N, given that the probability of Labor having a majority on the committee is greater than $\frac{3}{4}$.

21. Four basketball teams A, B, C and D each consist of ten players, and in each team, the players are numbered 1, 2, ... 9, 10. Five players are to be selected at random from the four teams. Find the probability that of the five players selected:

(a) three are numbered 4 and two are numbered 9,

(b) at least four are from the same team.

22. A poker hand of five cards is dealt from a standard pack of 52. Find the probability of obtaining:

(a) one pair, (b) two pairs, (c) three of a kind, (d) four of a kind,

(e) a full house (one pair and three of a kind),

(f) a straight (five cards in sequence regardless of suit, ace high or low),

(g) a flush, (five cards of the same suit),

(h) a royal flush (ten, jack, queen, king and ace in a single suit).

23. Four adults are standing in a room that has five exits. Each adult is equally likely to leave the room through any one of the five exits.

(a) What is the probability that all four adults leave the room via the same exit?

(b) What is the probability that three particular adults use the same exit and the fourth adult uses a different exit?

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(c) What is the probability that any three of the four adults come out the same exit, and the remaining adult comes out a different exit?

(d) What is the probability that no more than two adults come out any one exit?

24. (a) Five diners in a restaurant choose randomly from a menu featuring five main courses. Find the probability that exactly one of the main courses is not chosen by any of the diners.

(b) Repeat the question if there are n diners and a choice of n main courses.

25. [The birthday problem]

(a) Assuming a 365-day year, find the probability that in a group of three people there will be at least one birthday in common. Answer correct to two significant figures.

(b) If there are n people in the group, find an expression for the probability of at least one common birthday.

(c) By choosing a number of values of n, plot a graph of the probability of at least one common birthday against n for $n \le 50$.

(d) How many people need to be in the group before the probability exceeds 0·5?

(e) How many people need to be in the group before the probability exceeds 90%?

EXTENSION

26. During the seven games of the football season, Max and Bert must each miss three consecutive games. The games to be missed by each player are randomly and independently selected.

(a) What is the probability that they both have the first game off together?

(b) What is the probability that the second game is the first one missed by both players?

(c) What is the probability that Max and Bert miss at least one of the same games?

27. Eight players make the quarter-finals at Wimbledon. The winner of each of the quarter-finals plays a semi-final to see who enters the final.

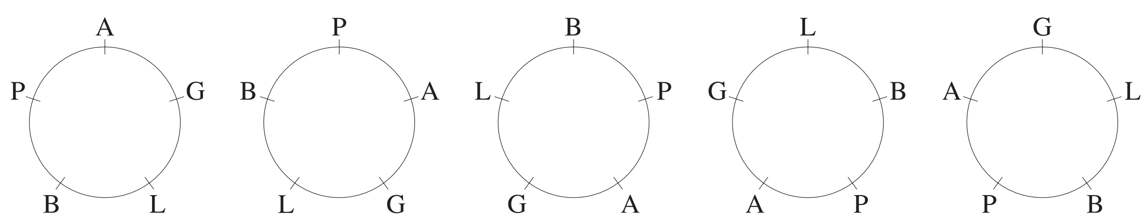
(a) Assuming that all eight players are equally likely to win a match, show that the probability that any two particular players will play each other is $\frac{1}{4}$.

(b) What is the probability that two particular people will play each other if the tournament starts with 16 players?

(c) What is the probability that two particular players will meet in a similar knockout tournament if 2^n players enter?

10 I Arrangements in a Circle

Arrangements in a circle or around a round table are complicated by the fact that two arrangements are regarded as equivalent if one can be rotated to produce the other. For example, all the five round-table seatings below of King Arthur, Queen Guinevere, Sir Lancelot, Sir Bors and Sir Percival are to be regarded as the same:



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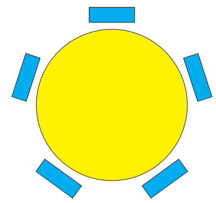
The Basic Algorithm: The most straightforward way of counting arrangements in a circle is to seat the people in order, dealing with the restrictions first as always, but reckoning that there is essentially only one way to seat the first person who sits down, because until that time, all the seats are identical.

19 COUNTING ARRANGEMENTS IN A CIRCLE: There is essentially only one way to seat the first person, because until then, all the seats are equivalent.

WORKED EXERCISE: Arthur, Guinevere, Lancelot, Bors and Percival sit around a round table. Find in how many ways this can be done:



(a) without restriction,
(b) if Guinevere sits at Arthur's right hand,
(c) if Guinevere sits between Lancelot and Bors.
(d) if Arthur and Lancelot do not sit together.

SOLUTION:

(a)

Seat Arthur	Seat Guinevere	Seat Lancelot	Seat Bors	Seat Percival
1	4	3	2	1

Number of ways = 24.

(b)

Seat Arthur	Seat Guinevere	Seat Lancelot	Seat Bors	Seat Percival
1	1	3	2	1

Number of ways = 6.

(c)

Seat Guinevere	Seat Lancelot	Seat Bors	Seat Arthur	Seat Percival
1	2	1	2	1

Number of ways = 4.

(d)

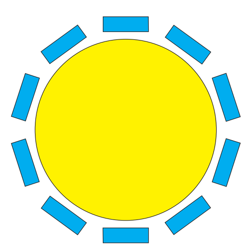
Seat Arthur	Seat Lancelot	Seat Guinevere	Seat Bors	Seat Percival
1	2	3	2	1

Number of ways = 12.

Arranging Groups Around a Circle: When arranging groups around a circle, the principle is the same as the principle for compound orderings established in Section 10E.

20 ARRANGING GROUPS AROUND A CIRCLE: First choose an order for each group. Then arrange the groups around the circle, reckoning that there is essentially only one way to place the first group.

WORKED EXERCISE: Five boys and five girls are to sit around a table. Find in how many ways this can be done:



(a) without restriction,
(b) if the boys and girls alternate,
(c) if there are five couples, all of whom sit together,
(d) if the boys sit together and the girls sit together,
(e) if four couples sit together, but Walter and Maude do not.

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SOLUTION:

(a)

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1	9	8	7	6	5	4	3	2	1

Number of ways $= 9! = 362\,880$.

(b)

1st boy	2nd boy	3rd boy	4th boy	5th boy	1st girl	2nd girl	3rd girl	4th girl	5th girl
1	4	3	2	1	5	4	3	2	1

Number of ways $= 5! \times 4! = 2880$.

(c) Each couple can be ordered in 2 ways, giving 2^5 orderings of the five couples. Then seat the five couples around the table:

1st couple	2nd couple	3rd couple	4th couple	5th couple
1	4	3	2	1

Number of ways $= 2^5 \times 4! = 768$.

(d) The boys can be ordered in 5! ways, and the girls in 5! ways also. Then seat the two groups around the table:

group of boys	group of girls
1	1

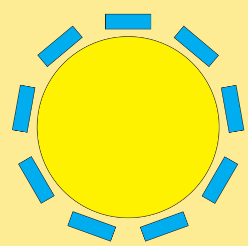
Number of ways $= 5! \times 5! \times 1 = 14\,400$.

(e) Order each of the four couples in 2 ways, giving 16 orderings of the couples. There are now four couples and two individuals to seat around the table, with the restriction that Maude does not sit next to Walter:

Walter	Maude	1st couple	2nd couple	3rd couple	4th couple
1	3	4	3	2	1

Number of ways $= 2^4 \times 3 \times 4! = 1152$.

Probability in Arrangements Around a Circle: As always, counting allows probability problems to be solved by counting the sample space and the event space.



WORKED EXERCISE: Three Tasmanians, three New Zealanders and three people from NSW are seated at random around a round table. What is the probability that the three groups are seated together?

SOLUTION: Using the same boxes as before, there are $1 \times 8!$ possible orderings.
To find the number of favourable orderings,
first order each group in $3! = 6$ ways,
then order the three groups around the table in $1 \times 2 \times 1 = 2$ ways,
so the total number of favourable orderings is $6 \times 6 \times 6 \times 2$.

$$\text{Hence } \mathcal{P}(\text{groups are together}) = \frac{6 \times 6 \times 6 \times 2}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{3}{280}.$$

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Exercise 10I

1. (a) In how many ways can five people be arranged: (i) in a line, (ii) in a circle?
(b) In how many ways can ten people be arranged: (i) in a line, (ii) in a circle?

2. Eight people are arranged in: (a) a straight line, (b) a circle.
In how many ways can they be arranged so that two particular people sit together?

3. Bob, Betty, Ben, Brad and Belinda are to be seated at a round table. In how many ways can this be done:
(a) if there are no restrictions,
(b) if Betty sits on Bob's right-hand side,
(c) if Brad is to sit between Bob and Ben,
(d) if Belinda and Betty sit apart,
(e) if Ben and Belinda sit apart, but Betty sits next to Bob?

4. Four boys and four girls are arranged in a circle. In how many ways can this be done:
(a) if there are no restrictions,
(b) if the boys and the girls alternate,
(c) if the boys and girls are in distinct groups,
(d) if a particular boy and girl wish to sit next to one another,
(e) if two particular boys do not wish to sit next to one another,
(f) if one particular boy wants to sit between two particular girls?

DEVELOPMENT

5. The letters A, E, I, P, Q and R are arranged in a circle. Find the probability that:
(a) the vowels are together,
(b) A is opposite R,
(c) the vowels and consonants alternate,
(d) at least two vowels are next to one another.

6. In how many ways can the integers 1, 2, 3, 4, 5, 6, 7, 8 be placed in a circle if:
(a) there are no restrictions,
(b) all the even numbers are together,
(c) the odd and even numbers alternate,
(d) at least three odd numbers are together,
(e) the numbers 1 and 7 are adjacent,
(f) the numbers 3 and 4 are separated?

7. A committee of three women and seven men is to be seated randomly at a round table.
(a) What is the probability that the three females will sit together?
(b) The committee elects a president and a vice-president. What is the probability that they are sitting opposite one another?

8. Find how many arrangements of n people around a circle are possible if:
(a) there are no restrictions,
(b) two particular people must sit together,
(c) two particular people sit apart,
(d) three particular people sit together.

9. Twelve marbles are to be placed in a circle. In how many ways can this be done if:
(a) all the marbles are of different colours,
(b) there are eight red, three blue and one green marble?

10. There are two round tables, one oak and one mahogany, each with five seats. In how many ways may a group of ten people be seated?

11. A sports committee consisting of four rowers, three basketballers and two cricketers sits at a circular table.
(a) How many different arrangements of the committee are possible if the rowers and basketballers both sit together in groups, but no rower sits next to a basketballer?
(b) One rower and one cricketer are related. If the conditions in (a) apply, what is the probability that these two members of the committee will sit next to one another?

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EXTENSION

12. A group of n men and $n+1$ women sit around a circular table. What is the probability that no two men sit next to one another?

13. (a) Consider a necklace of six differently coloured beads. Because the necklace can be turned over, clockwise and anticlockwise arrangements of the beads do not yield different orders. Hence find how many different arrangements there are of the six beads on the necklace.

(b) In how many ways can ten different keys be placed on a key ring?

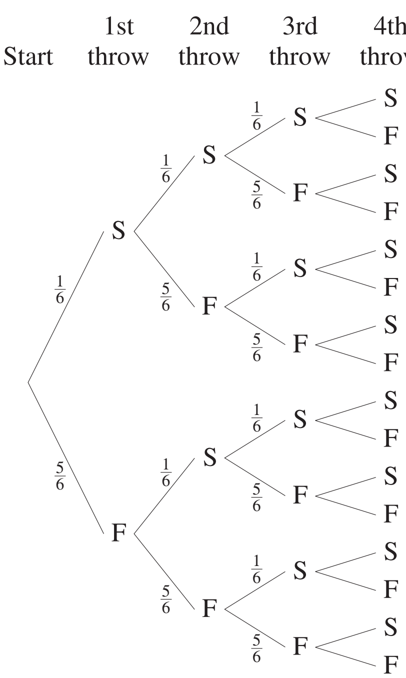
(c) In how many ways can one yellow, two red and four green beads be placed on a bracelet if the beads are identical apart from colour? [HINT: This will require a listing of patterns to see if they are identical when turned over.]

10 J Binomial Probability

This final section combines probability with the theory of the expansion of the binomial $(x+y)^n$, and provides a beautiful example of the relationship between probability and the theory of polynomials. We shall be concerned with multistage experiments in which:

- all the stages are identical, and
- each stage has only two possible outcomes, conventionally called 'success' and 'failure', and not necessarily equally likely.

Example — Repeatedly Attempting to Throw a Six on a Die: If a die is thrown four times, what is the probability of getting exactly two sixes? Let S (success) be the outcome 'throwing a six' and F (failure) be the outcome 'not throwing a six'. Here is the probability tree diagram showing the sixteen possible outcomes, and their respective probabilities, when a die is thrown four times.



Outcome	Probability
SSSS	$\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{1}{1296}$
SSSF	$\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{5}{1296}$
SSFS	$\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{5}{1296}$
SSFF	$\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{25}{1296}$
SFSS	$\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{5}{1296}$
SFSF	$\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{25}{1296}$
SFFS	$\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{25}{1296}$
SFFF	$\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{125}{1296}$
FSSS	$\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{5}{1296}$
FSSF	$\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{25}{1296}$
FSFS	$\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{25}{1296}$
FSFF	$\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{125}{1296}$
FFSS	$\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{25}{1296}$
FFSF	$\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{125}{1296}$
FFFS	$\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{125}{1296}$
FFFF	$\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{625}{1296}$

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The outcome 'two sixes' can be obtained in ${}^4C_2 = 6$ different ways:

SSFF, SFSF, SFFS, FSSF, FSFS, FFSS,

because there are $\dfrac{4!}{2! \times 2!} = {}^4C_2$ ways of arranging two Ss and two Fs.

Each of these six outcomes has the same probability $(\frac{1}{6})^2 \times (\frac{5}{6})^2$, so

$$\mathcal{P}(\text{two sixes}) = {}^4C_2 \times (\tfrac{1}{6})^2 \times (\tfrac{5}{6})^2.$$

Similar arguments apply to the probabilities of getting 0, 1, 2, 3 and 4 sixes:

Result	Probability	Approximation
4 sixes	${}^4C_4 \times (\frac{1}{6})^4 \times (\frac{5}{6})^0$	0·000 77
3 sixes	${}^4C_3 \times (\frac{1}{6})^3 \times (\frac{5}{6})^1$	0·015 43
2 sixes	${}^4C_2 \times (\frac{1}{6})^2 \times (\frac{5}{6})^2$	0·115 74
1 six	${}^4C_1 \times (\frac{1}{6})^1 \times (\frac{5}{6})^3$	0·385 80
0 sixes	${}^4C_0 \times (\frac{1}{6})^0 \times (\frac{5}{6})^4$	0·482 25

The five probabilities of course add up to 1. Moreover, the five probabilities are the successive terms in the binomial expansion of

$$(\tfrac{1}{6} + \tfrac{5}{6})^4 = {}^4C_0 \times (\tfrac{1}{6})^4 \times (\tfrac{5}{6})^0 \;+\; {}^4C_1 \times (\tfrac{1}{6})^3 \times (\tfrac{5}{6})^1 \;+\; {}^4C_2 \times (\tfrac{1}{6})^2 \times (\tfrac{5}{6})^2$$
$$+\, {}^4C_3 \times (\tfrac{1}{6})^1 \times (\tfrac{5}{6})^3 \;+\; {}^4C_4 \times (\tfrac{1}{6})^0 \times (\tfrac{5}{6})^4.$$

Binomial Probability — The General Case: Suppose that a multi-stage experiment consists of n identical stages, and at each stage the probability of 'success' is p and of 'failure' is q, where $p + q = 1$. Then

$$\mathcal{P}(k \text{ successes and } n-k \text{ failures in that order}) = p^k\, q^{n-k}.$$

But there are ${}^nC_k$ ways of ordering k successes and $(n-k)$ failures, so

$$\mathcal{P}(k \text{ successes and } n-k \text{ failures in any order}) = {}^nC_k\, p^k\, q^{n-k},$$

which is the term in $p^k q^{n-k}$ in the expansion of the binomial $(p+q)^n$.

21 BINOMIAL PROBABILITY: Suppose that the probabilities of 'success' and 'failure' in any stage of an n-stage experiment are p and q respectively. Then

$$\mathcal{P}(k \text{ successes and } n-k \text{ failures in any order}) = {}^nC_k\, p^k\, q^{n-k}.$$

This is the term in $p^k q^{n-k}$ in the expansion of $(p+q)^n$.

WORKED EXERCISE: [This example shows also how to use complementary events and cases to answer questions.] Six cards are drawn at random from a pack of 52 playing cards, each being replaced before the next is drawn. Find, as fractions with denominator 4^6, the probability that:

(a) two are clubs,

(b) one is a club,

(c) at least one is a club,

(d) at least four are clubs.

SOLUTION: There are 13 clubs in the pack, so at each stage the probability of drawing a club is $\frac{1}{4}$. Applying the formula with $p = \frac{1}{4}$ and $q = \frac{3}{4}$:

(a) $\mathcal{P}(\text{two are clubs}) = {}^6C_2 \times (\frac{1}{4})^2 \times (\frac{3}{4})^4 = \dfrac{15 \times 3^4}{4^6} = \dfrac{1215}{4^6}$,

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(b) $\mathcal{P}(\text{one is a club}) = {}^6C_1 \times (\frac{1}{4})^1 \times (\frac{3}{4})^5 = \dfrac{6 \times 3^5}{4^6} = \dfrac{1458}{4^6}$,

(c) $\mathcal{P}(\text{at least one is a club}) = 1 - \mathcal{P}(\text{all are non-clubs})$

$$= 1 - (\tfrac{3}{4})^6 \quad (\text{or } 1 - {}^6C_0 \times (\tfrac{1}{4})^0 \times (\tfrac{3}{4})^6)$$

$$= \frac{3367}{4^6},$$

(d) $\mathcal{P}(\text{at least four are clubs})$

$$= \mathcal{P}(\text{four are clubs}) + \mathcal{P}(\text{five are clubs}) + \mathcal{P}(\text{six are clubs})$$

$$= {}^6C_4 \times (\tfrac{1}{4})^4 \times (\tfrac{3}{4})^2 + {}^6C_5 \times (\tfrac{1}{4})^5 \times (\tfrac{3}{4})^1 + {}^6C_6 \times (\tfrac{1}{4})^6 \times (\tfrac{3}{4})^0$$

$$= \frac{15 \times 3^2 + 6 \times 3 + 1}{4^6}$$

$$= \frac{154}{4^6}.$$

An Example where $p = q = \frac{1}{2}$: A particular case of binomial probability is when the probabilities p and q of 'success' and 'failure' are both $\frac{1}{2}$.

WORKED EXERCISE: If a coin is tossed 100 times, what is the probability that it comes up heads exactly 50 times (correct to four significant figures)?

SOLUTION: Taking $p = q = \frac{1}{2}$,

$\mathcal{P}(50 \text{ heads}) = {}^{100}C_{50} \times (\frac{1}{2})^{50} \times (\frac{1}{2})^{50} \doteqdot 0{\cdot}0796$.

NOTE: This is a fairly low probability. We might, without careful thought, have expected a higher probability than this, but of course any result from about 45 to 55 heads would be unlikely to surprise us. It should now be clear that in general

$$\mathcal{P}(k \text{ heads in } n \text{ tosses of a coin}) = {}^nC_k \times (\tfrac{1}{2})^n.$$

Experimental Probabilities and Binomial Probability: Some of the most straightforward and important applications of binomial theory arise in situations where the probabilities of 'success' and 'failure' are determined experimentally.

WORKED EXERCISE: A light bulb is classed as 'defective' if it burns out in under 1000 hours. A company making light bulbs finds, after careful testing, that 1% of its bulbs are defective. If it packs its bulbs in boxes of 50, find, correct to three significant figures:

(a) the probability that a box will contain no defective bulbs,

(b) the probability that at least two bulbs in a box are defective.

SOLUTION: In this case, $p = 0{\cdot}01$ and $q = 0{\cdot}99$. Let X be the number of defective bulbs in the box. Then:

(a) $\mathcal{P}(X = 0) = 0{\cdot}99^{50} \doteqdot 0{\cdot}605$,

(b) $\mathcal{P}(X \ge 2) = 1 - \Big(\mathcal{P}(X = 0) + \mathcal{P}(X = 1)\Big)$

$$= 1 - \Big(0{\cdot}99^{50} + {}^{50}C_1 \times 0{\cdot}01^1 \times 0{\cdot}99^{49}\Big)$$

$$\doteqdot 0{\cdot}089.$$

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Binomial Probability and the Maximum Term in a Binomial Expansion: The earlier theory in Chapter Seven on the maximum term in a binomial expansion has an important application in binomial probability, because it allows us to find the event with the greatest probability.

WORKED EXERCISE: A die is thrown 100 times.

(a) What is the most likely number of sixes that will be thrown?

(b) What is the probability that that particular number of sixes is thrown (answer correct to four significant figures)?

SOLUTION: Here $p = \frac{1}{6}$ and $q = \frac{5}{6}$.

(a) Let $P_k = \mathcal{P}(k \text{ sixes})$.

Then $P_k = {}^{100}\mathrm{C}_k\, p^k\, q^{100-k}$

and $P_{k+1} = {}^{100}\mathrm{C}_{k+1}\, p^{k+1}\, q^{99-k}$,

so
$$\frac{P_{k+1}}{P_k} = \frac{100! \times p^{k+1}\, q^{99-k}}{(k+1)! \times (99-k)!} \times \frac{k! \times (100-k)!}{100! \times p^k\, q^{100-k}}$$
$$= \frac{(100-k)p}{(k+1)q}$$
$$= \frac{100-k}{5k+5}\,, \quad \text{substituting } p = \tfrac{1}{6} \text{ and } q = \tfrac{5}{6}.$$

To find the greatest value of P_k, we proceed as in Section 5E,

solving $P_{k+1} > P_k$, that is,
$$\frac{P_{k+1}}{P_k} > 1 \qquad (1)$$
$$\frac{100-k}{5k+5} > 1$$
$$5k + 5 < 100 - k$$
$$6k < 95$$
$$k < 15\tfrac{5}{6},$$

so the inequation (1) is true for $k = 0, 1, 2, \ldots, 15$ and false otherwise.

Hence $P_0 < P_1 < \cdots < P_{15} < P_{16} > P_{17} \cdots > P_{100}$.

Thus the most likely number of sixes is 16.

(b) Also,
$$P_{16} = {}^{100}\mathrm{C}_{16} \times (\tfrac{1}{6})^{16} \times (\tfrac{5}{6})^{84}$$
$$\doteqdot 0{\cdot}1065.$$

An Example where Each Stage is a Compound Event: Sometimes, when each stage of the experiment is itself a compound event, it may take some work to find the probability of success at each stage.

WORKED EXERCISE: Joe King and his sister Fay make shirts for a living. Joe works more slowly but more accurately, making 20 shirts a day, of which 2% are defective. Fay works faster, making 30 shirts a day, of which 4% are defective. If they send out their shirts in randomly mixed parcels of 30 shirts, what is the probability (to three significant figures) that no more than two shirts in a box are defective?

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SOLUTION: If a shirt is chosen at random from one parcel, then using the product rule and the addition rule, the probability p that the shirt is defective is

$$\begin{aligned} p &= \mathcal{P}(\text{Joe made it, and it is defective}) \\ &\quad + \mathcal{P}(\text{Fay made it, and it is defective}) \\ &= \frac{20}{50}\times\frac{2}{100} + \frac{30}{50}\times\frac{4}{100} \\ &= \frac{4}{125}, \end{aligned}$$

so $p = \frac{4}{125}$ and $q = \frac{121}{125}$.

Let X be the number of defective shirts.

$$\begin{aligned} \text{Then } \mathcal{P}(X \le 2) &= \mathcal{P}(X=0) + \mathcal{P}(X=1) + \mathcal{P}(X=2) \\ &= (\tfrac{121}{125})^{30} + {}^{30}\mathrm{C}_1 \times (\tfrac{121}{125})^{29} \times \tfrac{4}{125} + {}^{30}\mathrm{C}_2 \times (\tfrac{121}{125})^{28} \times (\tfrac{4}{125})^2 \\ &\doteqdot 0{\cdot}930. \end{aligned}$$

Exercise 10J

NOTE: Unless otherwise specified, leave your answers in unsimplified form.

1. Assume that the probability that a child is female is $\frac{1}{2}$, and that sex is independent from child to child. Giving your answers as fractions in simplest form, find the probability that in a family of five children:

(a) all are boys,

(b) there are two girls and three boys,

(c) there are four boys and one girl,

(d) at least one will be a girl.

2. In a one-day cricket game, a batsman has a chance of $\frac{1}{5}$ of hitting a boundary every time he faces a ball. If he faces all six balls in an over, what is the probability that he will hit exactly two boundaries, assuming that successive strikes are independent?

3. During the wet season, it rains on average every two days out of three. Let W denote the number of wet days in a week. Assuming that the types of weather on successive days are independent events, find:

(a) $\mathcal{P}(W = 3)$ (b) $\mathcal{P}(\overline{W} = 2)$ (c) $\mathcal{P}(W = 0)$ (d) $\mathcal{P}(W \ge 1)$

4. A marksman finds that on average he hits the target five times out of six. Assuming that successive shots are independent events, find the probability that in four shots:

(a) he has exactly three hits,

(b) he has exactly two misses.

5. A jury roll contains 200 names, 70 of females and 130 of males. If twelve jurors are randomly selected, what is the probability of ending up with an all-male jury?

6. A die is rolled twelve times. Find the probability that 5 appears on the uppermost face:

(a) exactly three times,

(b) exactly eight times,

(c) ten or more times (that is ten, eleven or twelve times).

7. A die is rolled six times. Let N denote the number of times that the number 3 is shown on the uppermost face. Find, correct to four decimal places:

(a) $\mathcal{P}(N = 2)$ (b) $\mathcal{P}(N < 2)$ (c) $\mathcal{P}(N \ge 2)$

8. Five out of six people surveyed think that Tasmania is the most beautiful state in Australia. What is the probability that in a group of 15 randomly selected people, at least 13 of them think that Tasmania is the most beautiful state in Australia?

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9. An archer finds that on average he hits the bulls-eye nine times out of ten. Assuming that successive attempts are independent, find the probability that in twenty attempts:
(a) he scores at least eighteen hits, (b) he misses at least once.

10. A torch manufacturer finds that on average 9% of the bulbs are defective. What is the probability that in a randomly selected batch of ten one-bulb torches:
(a) there will be no more than two with defective bulbs,
(b) there will be at least two with defective bulbs.

DEVELOPMENT

11. A poll indicates that 55% of people support Labor party policy. If five people are selected at random, what is the probability that a majority of them will support Labor party policy? Give your answer correct to three decimal places.

12. In a multiple-choice test, there are ten questions, and each question has five possible answers, only one of which is correct. What is the probability of answering exactly seven questions correctly by chance alone? Give your answer correct to six significant figures.

13. The probability that a small earthquake occurs somewhere in the world on any one day is $0{\cdot}95$. Assuming that earthquake frequencies on successive days are independent, what is the probability that a small earthquake occurs somewhere in the world on exactly 28 of January's 31 days? Leave your answer in index form.

14. The probability that a jackpot prize will be won in a given lottery is $0{\cdot}012$.
(a) Find, correct to five decimal places, the probability that the jackpot prize will be won:
(i) exactly once in ten independent lottery draws,
(ii) at least once in ten independent lottery draws.
(b) The jackpot prize is initially \$10 000 and increases by \$10 000 each time the prize is not won. Find, correct to five decimal places, the probability that the jackpot prize will exceed \$200 000 when it is finally won.

15. (a) How many times must a die be rolled so that the probability of rolling at least one six is greater than 95%?
(b) How many times must a coin be tossed so that the probability of tossing at least one tail is greater than 99%?

16. Five families have three children each.
(a) Find, correct to three decimal places, the probability that:
(i) at least one of these families has three boys,
(ii) each family has more boys than girls.
(b) What assumptions have been made in arriving at your answer?

17. Comment on the validity of the following arguments:
(a) 'In the McLaughlin Library, 10% of the books are mathematics books. Hence if I go to a shelf and choose five books from that shelf, then the probability that all five books are mathematics books is 10^{-5}.'
(b) 'During an election, 45% of voters voted for party A. Hence if I select a street at random, and then select a voter from each of four houses in the street, the probability that exactly two of those voters voted for party A is ${}^4C_2 \times (0{\cdot}45)^2 \times (0{\cdot}55)^2$.'

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18. During winter it rains on average 18 out of 30 days. Five winter days are selected at random. Find, correct to four decimal places, the probability that:
(a) the first two days chosen will be fine and the remainder wet,
(b) more rainy days than fine days have been chosen.

19. A tennis player finds that on average he gets his serve in eight out of every ten attempts and serves an ace once every fifteen serves. He serves four times. Assuming that successive serves are independent events, find, correct to six decimal places, the probability that:
(a) all four serves are in,
(b) he hits at least three aces,
(c) he hits exactly three aces and the other serve lands in.

20. A man is restoring ten old cars, six of them manufactured in 1955 and four of them manufactured in 1962. When he tries to start them, on average the 1955 models will start 65% of the time and the 1962 models will start 80% of the time. Find, correct to four decimal places, the probability that at any time:
(a) exactly three of the 1955 models and one of the 1962 models will start,
(b) exactly four of the cars will start. [HINT: You will need to consider five cases.]

21. An apple exporter deals in two types of apples, Red Delicious and Golden Delicious. The ratio of Red Delicious to Golden Delicious is 4 : 1. The apples are randomly mixed together before they are boxed. One in every fifty Golden Delicious and one in every one hundred Red Delicious apples will need to be discarded because they are undersized.
(a) What is the probability that an apple selected from a box will need to discarded?
(b) If ten apples are randomly selected from a box, find the probability that:
  (i) all the apples will have to be discarded,
  (ii) half of the apples will have to be discarded,
  (iii) less than two apples will be discarded.

22. One bag contains three red and five white balls, and another bag contains four red and four white balls. One bag is chosen at random, a ball is selected from that bag, its colour is noted and then it is replaced.
(a) Find the probability that the ball chosen is red.
(b) If the operation is carried out eight times, find the probability that:
  (i) exactly three red balls are drawn,
  (ii) at least three red balls are drawn.

23. (a) If six dice are rolled one hundred times, how many times would you expect the number of even numbers showing to exceed the number of odd numbers showing?
(b) If eight coins are tossed sixty times, how many times would you expect the number of heads to exceed the number of tails?

24. [Probability and the greatest term in a binomial series] In each part, you will need to find the greatest term in the expansion of $(p + q)^n$, where p and q are the respective probabilities of 'success' and 'failure'. Give each probability unsimplified, and then correct to four significant figures.
(a) A die is rolled 200 times. What is the most likely number of twos that will be thrown and what is the probability that that particular number of twos is thrown?
(b) A coin is tossed 41 times. What is the most likely number of heads that will be tossed, and what is the probability that that particular number of heads is tossed?

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(c) From a 52-card pack, a card is drawn 35 times, and is replaced after each draw. What is the most likely number of aces that will be drawn, and what is the probability that that particular number of aces is drawn?

(d) Repeat part (iii) for the number of spades that will be drawn.

25. [The special case of a coin tossed an even number of times] If a fair coin is tossed $2n$ times, the probability of observing k heads and $2n-k$ tails is given by $P_k = {}^{2n}\mathrm{C}_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{2n-k}$.

(a) Show that the most likely outcome is $k = n$.

(b) Show that $P_n = \dfrac{(2n)!}{2^{2n} \times (n!)^2}$.

EXTENSION

26. A game is played using a barrel containing twenty similar balls numbered 1 to 20. The game consists of drawing four balls, without replacement, from the twenty balls in the barrel. Thus the probability that any particular number is drawn in any game is 0·2.

(a) Find the probability that the number 19 is drawn in exactly two of the next five games played.

(b) Find the probability that the number 19 is drawn in at least two of the next five games played.

(c) Let n be an integer, where $4 \le n \le 20$.

(i) What is the probability that, in any one game, all four selected numbers are less than or equal to n?

(ii) Show that the probability that, in any one game, n is the largest of the four numbers drawn is $\dfrac{{}^{n-1}\mathrm{C}_3}{{}^{20}\mathrm{C}_4}$.

27. (a) Expand $(a+b+c)^3$.

(b) In a survey of football supporters, 65% supported Hawthorn, 24% followed Collingwood and 11% followed Sydney. Use the expansion in part (a) to find, correct to five decimal places, the probability that if three people are randomly selected:

(i) one supports Hawthorn, one supports Collingwood and one supports Sydney,

(ii) exactly two of them support Collingwood,

(iii) at least two of them support the same team.

28. [Wallis' product and probability] An Extension question in Exercise 4E mentioned that the famous limit called *Wallis' product* was accessible by methods of the 4 Unit course:

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2^2}{1\times 3} \times \frac{4^2}{3\times 5} \times \frac{6^2}{5\times 7} \times \cdots \times \frac{(2n)^2}{(2n-1)(2n+1)} \right).$$

(a) Show that the expression in brackets can be rewritten as $\left(\dfrac{2^{2n} \times (n!)^2}{(2n)!} \right)^2 \times \dfrac{1}{2n+1}$.

(b) Hence prove that $\displaystyle\lim_{n\to\infty} \frac{(2n)! \times \sqrt{n\pi}}{2^{2n} \times (n!)^2} = 1$.

(c) Look again at question 25 above, and explain the relevance of the limit in part (b) of this question to the tossing of a coin.

GO Online Multiple Choice Quiz

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