(A)

MATH 1902 2004 exam solutions 10/6/2010

Q1/ (i) $\underline{u} = 2\underline{i} + \underline{j} - 2\underline{k}$, $\underline{v} = 3\underline{i} + 4\underline{k}$

(a) $|\underline{u}| = \sqrt{4+1+4} = \sqrt{9} = 3$,

$\underline{u}\cdot\underline{v} = 6 - 8 = -2$.

(b) $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|} = \dfrac{-2}{3\sqrt{9+16}} = \dfrac{-2}{15}$.

(c) projection of $\underline{v}$ in direction of $\underline{u}$ is

$$\frac{\underline{v}\cdot\underline{u}}{|\underline{u}|^2}\underline{u} = \frac{-2}{9}(2\underline{i}+\underline{j}-2\underline{k})$$

so take $\underline{a} = -\frac{2}{9}(2\underline{i}+\underline{j}-2\underline{k})$

and $\underline{b} = \underline{v} - \underline{a} = 3\underline{i} + 4\underline{k} + \frac{2}{9}(2\underline{i}+\underline{j}-2\underline{k})$

$$= \frac{1}{9}(31\underline{i} + 2\underline{j} + 32\underline{k})$$

(ii) l is line $\dfrac{x-1}{2} = \dfrac{y+2}{-3} = \dfrac{z-3}{4}$.

(a) $A = (1,-2,3)$ lies on l and $\underline{u} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

is in the direction of l.

(b) $\underline{v} = \overrightarrow{AB} = 2\underline{i} - 2\underline{k}$, and $\underline{u}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & 4 \\ 2 & 0 & -2 \end{vmatrix} = 6\underline{i} + 12\underline{j} + 6\underline{k}$

so $\underline{n} = \underline{i} + 2\underline{j} + \underline{k}$ is normal to P so has Cartesian equation

$$x + 2y + z = 3 - 4 + 1 = 0.$$

Q1/ (ii) (c) Want $ax+by+cz=1$ to contain $A(1,-2,3)$ and have normal $\underline{n} = a\underline{i}+b\underline{j}+c\underline{k}$ perpendicular to $\underline{u} = 2\underline{i}-3\underline{j}+4\underline{k}$, giving system

$$a - 2b + 3c = 1$$
$$2a - 3b + 4c = 0.$$

Solving: $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & -2 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & -2 & -2 \end{array}\right]$ $\quad \begin{aligned} a \quad -c &= -3 \\ b - 2c &= -2 \end{aligned}$

Put $c = t$, so $b = -2+2t$, $a = -3+t$.

Solution: $\boxed{(a,b,c) = (-3+t, -2+2t, t)}$ $(t \in \mathbb{R})$

Q2/ (i) (a) $|\underline{a}\times\underline{b}|^2 + (\underline{a}\cdot\underline{b})^2 = |\underline{a}|^2|\underline{b}|^2\sin^2\theta + |\underline{a}|^2|\underline{b}|^2\cos^2\theta$

$= |\underline{a}|^2|\underline{b}|^2(\sin^2\theta + \cos^2\theta)$

$= a^2b^2$, as required.

(where θ is the angle between $\underline{a}, \underline{b}$).

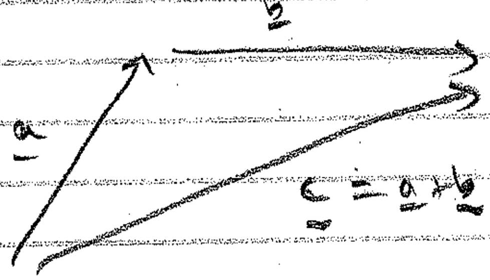
(b) $c^2 = (\underline{a}+\underline{b})\cdot(\underline{a}+\underline{b}) = \underline{a}\cdot\underline{a} + \underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{a} + \underline{b}\cdot\underline{b}$

$= |\underline{a}|^2 + 2\underline{a}\cdot\underline{b} + |\underline{b}|^2 = a^2+b^2+2\underline{a}\cdot\underline{b}$

So $2\underline{a}\cdot\underline{b} = c^2 - a^2 - b^2$, as required.

(c)

Q2/ (i) (c)



The area of the triangle is

$$\frac{1}{2}|\underline{a}\times\underline{b}| = \frac{1}{2}\sqrt{a^2b^2-(\underline{a}\cdot\underline{b})^2}$$

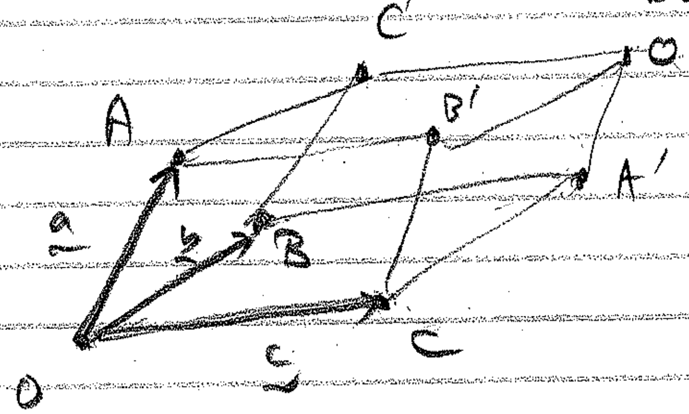
$$= \frac{1}{2}\sqrt{a^2b^2-\frac{(c^2-a^2-b^2)^2}{4}}$$

$$= \frac{1}{4}\sqrt{4a^2b^2-(c^4+a^4+b^4-2a^2c^2-2b^2c^2+2a^2b^2)}$$

$$= \frac{1}{4}\sqrt{2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4}$$

as required.

(ii)



(a) $\overrightarrow{OA'} = \underline{c}+\underline{b}$, $\overrightarrow{OB'} = \underline{c}+\underline{a}$, $\overrightarrow{OC'} = \underline{b}+\underline{a}$, $\overrightarrow{OO'} = \underline{a}+\underline{b}+\underline{c}$

so $\overrightarrow{AA'} = \overrightarrow{AO}+\overrightarrow{OA'} = \underline{c}+\underline{b}-\underline{a}$, $\overrightarrow{BB'} = \overrightarrow{BO}+\overrightarrow{OB'} = \underline{c}+\underline{a}-\underline{b}$,

$\overrightarrow{CC'} = \overrightarrow{CO}+\overrightarrow{OC'} = \underline{b}+\underline{a}-\underline{c}$, $\overrightarrow{OO'} = \underline{a}+\underline{b}+\underline{c}$.

Ⓓ

Q2/ (ii) (b) Let P be the midpoint of OO', so

$\vec{OP} = \frac{1}{2}(\underline{a}+\underline{b}+\underline{c})$. Then

$\vec{AP} = \vec{AO} + \vec{OP} = \frac{1}{2}(\underline{a}+\underline{b}+\underline{c}) - \underline{a} = \frac{1}{2}(-\underline{a}+\underline{b}+\underline{c})$

$= \frac{1}{2}\vec{AA'}$,

so A, P, A' are collinear, and P is the midpoint.

Similarly P is the midpoint of BB' and CC',

so all diagonals bisect each other.

Q3/ (i)
$$x + (p+2)y + pz = p+1$$
$$y + (2-p)z = 1$$
$$2x + 2py + (p^2+8p-23)z = 3p+3$$

(a)
$$\left[\begin{array}{ccc|c} 1 & p+2 & p & p+1 \\ 0 & 1 & 2-p & 1 \\ 2 & 2p & p^2+8p-23 & 3p+3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & p+2 & p & p+1 \\ 0 & 1 & 2-p & 1 \\ 0 & -4 & p^2+6p-23 & p+1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & p+2 & p & p+1 \\ 0 & 1 & 2-p & 1 \\ 0 & 0 & p^2+2p-15 & p+5 \end{array}\right]$$

$p^2+2p-15 = (p+5)(p-3)$, so $\boxed{p=3}$ gives no solution

(b) $\boxed{p \neq -5 \text{ and } p \neq 3}$ give a unique solution.

(E)

Q3/ (i) (c) For an infinite solution, require $p = -5$, so

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & -4 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 16 & -1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x \quad + 16z &= -1 \\ y + 7z &= 1 \end{aligned}$$

Putting $z = t$ gives $y = 1 - 7t$, $x = -1 - 16t$, so

$$\boxed{(x, y, z) = (-1-16t,\ 1-7t,\ t)} \qquad (t \in \mathbb{R})$$

(ii)
$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -2 & 0 & 1 \end{array}\right] R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right] R_3 - R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -1 \\ 0 & 2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ -R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right] R_2/2 \quad \text{so } A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 0 & \frac{1}{2} \\ 2 & 1 & -1 \end{bmatrix}$$

(iii) Working backwards from $E_1 E_2 E_3 E_4 E_5 E_6 A = I$

$$A = E_6^{-1} E_5^{-1} E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(F)

Q3/ (iv) (a) False

e.g. $0x + 0y = 1$ has no solutions.

(b) False

e.g. $\left.\begin{array}{l} x = 0 \\ 2x = 0 \end{array}\right\}$ is consistent $(x=0)$

(c) False

e.g. $\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{B}$

$A \neq I$, $B \neq 0$

Q4/ (i) $A = \begin{bmatrix} -3 & 4 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix}$

(a)

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 4 & 0 \\ 0 & 1-\lambda & 0 \\ -4 & 4 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -3-\lambda & 4 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-3-\lambda)(1-\lambda)$$

$$= (\lambda+3)(\lambda-1)(1-\lambda)$$

(b) eigenvalues are $\lambda = 1, -3$.

Q4 (c)

$$A - I = \begin{bmatrix} -4 & 4 & 0 \\ 0 & 0 & 0 \\ -4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so eigenspace $= \left\{ \begin{bmatrix} s \\ s \\ t \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$.

Q5 (i) $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X^2 = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

So $a^2+bc=1$, $b(a+d)=1$, $c(a+d)=0$, $bc+d^2=1$.

But $a+d \neq 0$, since $b(a+d) = 1 \neq 0$, so $c = 0$.

Hence $a^2 = d^2 = 1$, so $a = d = 1$ or $a = d = -1$

and $b = \pm\frac{1}{2}$, giving $X = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & -1 \end{bmatrix}$.

(ii)(a) $$\begin{bmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ 0 & x & -1 & \cdots & 0 & 0 \\ 0 & 0 & x & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & \cdots & & x & -1 \\ c_0 & c_1 & c_2 & \cdots & c_{n-2} & x+c_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \end{bmatrix} = \begin{bmatrix} x-x \\ x^2-x^2 \\ x^3-x^3 \\ \vdots \\ x^n-x^n \\ c_0+c_1x+\cdots+c_{n-1}x^{n-1}+x^n \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p(x) \end{bmatrix}$$

where $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$,

(H)

Q5/ (ii) (b) $\underline{v}(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix}$

If $p(\lambda) = 0$ then

$$M\underline{v}(\lambda) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & \ddots & & \\ \vdots & & & & & \\ 0 & & & & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \\ \vdots \\ \lambda^{n-2} \\ * \end{bmatrix}$$

$$-c_0 - c_1\lambda - c_2\lambda^2 + \cdots - c_{n-1}\lambda^{n-1}$$
$$= -(p(\lambda) - \lambda^n) = \lambda^n$$

So $M\underline{v}(\lambda) = \lambda\underline{v}(\lambda)$, so $\underline{v}(\lambda)$ is an eigenvector with eigenvalue λ.