8027A Semester 2 2012

The University of Sydney
School of Mathematics and Statistics

MATH1903/1907
Integral Calculus and Modelling (Advanced)

November 2012

Lecturers: D Daners, J Parkinson

Time Allowed: **One and a half hours**

Family Name: ....

Other Names: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination;
there are 20 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination;
there are 4 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Approved non-programmable calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Marker's use only	

Extended Answer Section

*There are **four** questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

MARKS

1. (a) Let

$$G(x) = \int_0^x \frac{1}{1+t^3}\,dt.$$

(i) Find $\frac{d}{dx}G(x^2)$. **2**

(ii) Calculate the integral **3**

$$\int_0^1 xG(x)\,dx.$$

in terms of $G(1)$.

Question 1 continues on the next page

MARKS

(b) Let D be the region of the plane with $0 \leq x \leq 1$ and $0 \leq y \leq e^x$. Calculate the volume of the solid obtained by revolving D around the y-axis. **2**

QUESTION 1 CONTINUES ON THE NEXT PAGE

MARKS

(c) Calculate the value of the improper integral **3**

$$\int_1^{\infty} \left(\frac{1}{x+2} - \frac{5}{5x+1} \right) dx.$$

Question 2 begins on the next page

MARKS

2. (a) Calculate the length of the graph $y = \cosh x$ between $x = 0$ and $x = 1$. **2**

(b) Use a suitable comparison test to prove either convergence, or divergence, of the improper integral **2**

$$\int_1^\infty \frac{\cos x}{x^2+1}\,dx$$

QUESTION 2 CONTINUES ON THE NEXT PAGE

MARKS

(c) Let $f(x) = \sqrt{1+x}$.

(*i*) Calculate the second order Taylor polynomial $T_2(x)$ of $f(x)$ centred at 0. **2**

(*ii*) Use Taylor's Theorem to write down a formula for the second order remainder term $R_2(x) = f(x) - T_2(x)$. Hence show that **2**

$$0 \le f(x) - T_2(x) \le \frac{x^3}{16} \qquad \text{for all } x \ge 0.$$

(*iii*) Hence approximate the integral **2**

$$\int_0^1 \sqrt{1+x^3}\, dx$$

correct to 1 decimal place. (Note the x^3 in the integrand).

Question 3 begins on the next page

MARKS

3. (a) Consider the differential equation

$$u'' + 6u' + 13u = 0$$

(*i*) Find the general solution of the differential equation **2**

QUESTION 3 CONTINUES ON THE NEXT PAGE

MARKS

(*ii*) Find the particular solution of the differential equation satisfying the conditions $u(0) = 0$ and $u'(0) = 1$. **2**

(*iii*) Let $x(t) = u(t)$ and $y(t) = u'(t)$. Derive a first order system of differential equations for $x(t)$ and $y(t)$ which is equivalent to the given second order differential equation. **2**

QUESTION 3 CONTINUES ON THE NEXT PAGE

MARKS

(b) Find the solution of the differential equation 4

$$t^2 y'(t) = \frac{4+t}{y(t)}$$

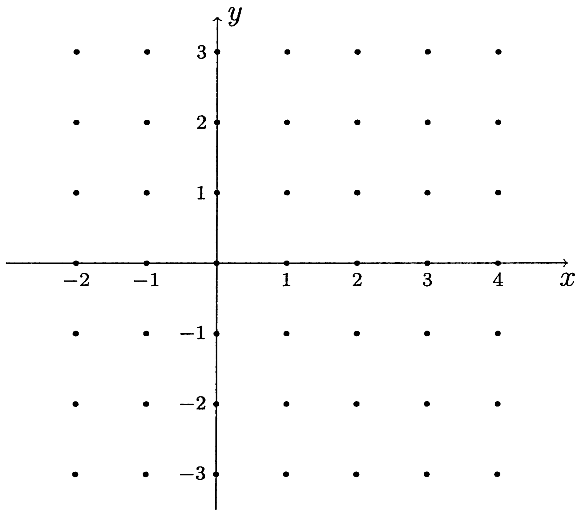
satisfying the initial condition $y(1) = -2$.

QUESTION 4 BEGINS ON THE NEXT PAGE

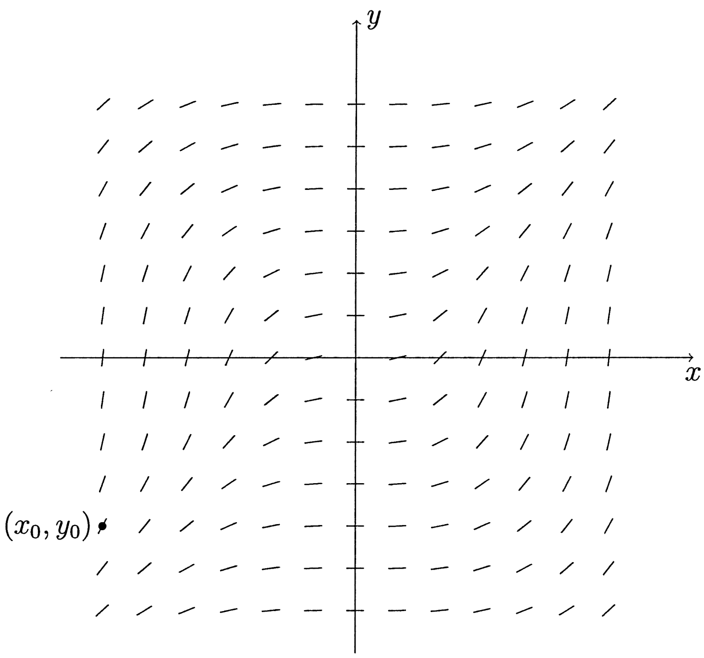
MARKS

4. (a) On the graph below, sketch the direction field of the differential equation **2**

$$y' = y^2 - 4.$$



(b) The following graph shows the direction field of a differential equation. Sketch the solution starting at the given value (x_0, y_0). **1**



Question 4 continues on the next page

MARKS

(c) Find the general solution of the linear inhomogeneous differential equation 3

$$t^2y' + y = t^3e^{1/t}.$$

Question 4 continues on the next page

MARKS

(d) Consider the system of differential equations **4**

$$\dot{x} = x + 3y$$
$$\dot{y} = 4x + 2y$$

Find the solution of the system with $x(0) = 5$ and $y(0) = 2$.

Table of Standard Integrals

1. $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

3. $\displaystyle\int e^x \, dx = e^x + C$

4. $\displaystyle\int \sin x \, dx = -\cos x + C$

5. $\displaystyle\int \cos x \, dx = \sin x + C$

6. $\displaystyle\int \tan x \, dx = -\ln\left|\cos x\right| + C$

7. $\displaystyle\int \cot x \, dx = \ln\left|\sin x\right| + C$

8. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

9. $\displaystyle\int \sec^2 x \, dx = \tan x + C$

10. $\displaystyle\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

11. $\displaystyle\int \sec x \, dx = \ln\left|\sec x + \tan x\right| + C$

12. $\displaystyle\int \operatorname{cosec} x \, dx = \ln\left|\operatorname{cosec} x - \cot x\right| + C$

13. $\displaystyle\int \sinh x \, dx = \cosh x + C$

14. $\displaystyle\int \cosh x \, dx = \sinh x + C$

15. $\displaystyle\int \tanh x \, dx = \ln\cosh x + C$

16. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (|x| < a)$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2 + a^2}\right) + C'$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2 - a^2}\right) + C' \quad (x > a)$

Linearity: $\displaystyle\int (\lambda f(x) + \mu g(x)) \, dx = \lambda \int f(x) \, dx + \mu \int g(x) \, dx$

Integration by substitution: $\displaystyle\int f(u(x)) \frac{du}{dx} \, dx = \int f(u) \, du$

Integration by parts: $\displaystyle\int f(x) g'(x) \, dx = f(x) g(x) - \int f'(x) g(x) \, dx$

End of Extended Answer Section

This is the last page of the question paper.