University of New South Wales

MATH 2901

Higher Theory of Statistics

# Assignment 2

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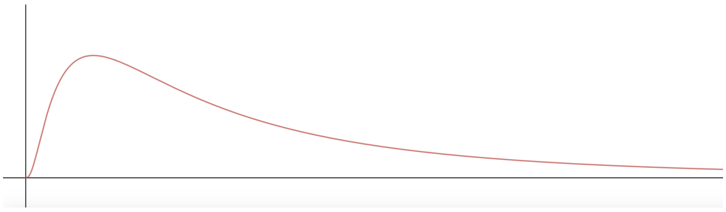
1. $X_1$ and $X_2$ have the following density functions:

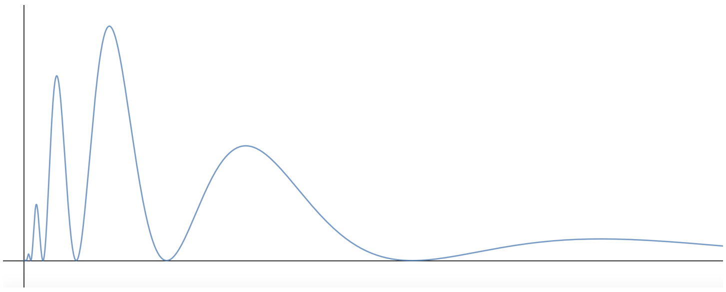
$$f_{X_1}(x) = \frac{1}{x\sqrt{2\pi}} e^{-(\ln x)^2/2} \qquad x > 0$$
$$f_{X_2}(x) = f_{X_1}(x)[1 + \sin(2\pi \ln x)] \qquad x > 0$$

(a) The graph for $f_{X_1}(x)$ is shown in the following figure.



As $x$ approaches $0$, $\ln x$ approaches $-\infty$. Thus, $e^{-(\ln x)^2/2}$ approaches $0$. As $e^x$ dominates $\frac{1}{x}$, $f_{X_1}(x)$ approaches $0$ as $x$ approaches $0$. As $x$ approaches $\infty$, $\ln x$ approaches $\infty$. Thus, $e^{-(\ln x)^2/2}$ approaches $0$. As $e^x$ dominates $\frac{1}{x}$, $f_{X_1}(x)$ approaches $0$ as $x$ approaches $\infty$.

The graph for $f_{X_2}(x)$ is shown in the following figure.



$f_{X_2}(x)$ behaves in the same manner as $f_{X_1}(x)$ when $x$ approaches $0$ or $\infty$, due to the same reasons, and $e^x$ dominating $\sin x$. Furthermore, the multiplication by the sine function accounts for the sinusoidal-esque shape.

(b)

$$\begin{aligned}
\mathbb{E}\big[X_1^r\big] &= \int_{-\infty}^{\infty} x^r I_{(0,\infty)}(x) f_{X_1}(x)dx \\
&= \int_{0}^{\infty} x^r f_{X_1}(x)dx \\
&= \int_{0}^{\infty} \frac{x^r}{x\sqrt{2\pi}} e^{-(\ln x)^2/2} dx \\
&= \int_{0}^{\infty} \frac{x^{r-1}}{\sqrt{2\pi}} e^{-(\ln x)^2/2} dx
\end{aligned}$$

Using the substitution $u = \ln x$, $x = e^u$, and $dx = e^u du$. The limits are now $-\infty$ and $\infty$.

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{\left(e^u\right)^{r-1}}{\sqrt{2\pi}} e^{-u^2/2} e^u du \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{u(r-1)} e^{(-u^2+2u)/2} du \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left[-u^2+2u+2u(r-1)\right]/2} du \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-u^2+2ur)/2} du \\
&= \int_{-\infty}^{\infty} e^{r^2/2} \frac{1}{\sqrt{2\pi}} e^{-(u^2-2ur+r^2)/2} du \\
&= \int_{-\infty}^{\infty} e^{r^2/2} \frac{1}{\sqrt{2\pi}} e^{-(u-r)^2/2} du
\end{aligned}$$

Using the substitution $y = u - r$, $u = y + r$, and $du = dy$. The limits remain the same.

$$\begin{aligned}
&= e^{r^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= e^{r^2/2}(1) \\
\therefore \mathbb{E}\big[X_1^r\big] &= e^{r^2/2}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbb{E}\big[X_2^r\big] &= \int_{-\infty}^{\infty} x^r I_{(0,\infty)}(x) f_{X_2}(x)dx \\
&= \int_{0}^{\infty} x^r f_{X_2}(x)dx \\
&= \int_{0}^{\infty} x^r f_{X_1}(x)[1 + \sin(2\pi \ln x)]dx \\
&= \int_{0}^{\infty} x^r f_{X_1}(x) + x^r f_{X_1}(x) \sin(2\pi \ln x)dx \\
&= \int_{0}^{\infty} x^r f_{X_1}(x)dx + \int_{0}^{\infty} x^r f_{X_1}(x) \sin(2\pi \ln x)dx \\
&= \mathbb{E}\big[X_1^r\big] + \int_{0}^{\infty} x^r f_{X_1}(x) \sin(2\pi \ln x)dx \\
\therefore \mathbb{E}\big[X_2^r\big] &= \mathbb{E}\big[X_1^r\big] + \int_{0}^{\infty} x^r f_{X_1}(x) \sin(2\pi \ln x)dx
\end{aligned}$$

(d)

$$I = \int_0^\infty x^r f_{X_1}(x) \sin(2\pi \ln x) dx$$

Using the substitution $u = \ln x$, $x = e^u$, and $dx = e^u du$. The limits are now $-\infty$ and $\infty$.

$$\begin{aligned}
\therefore I &= \int_{-\infty}^{\infty} \frac{(e^u)^{r-1}}{\sqrt{2\pi}} e^{-u^2/2} \sin(2\pi u) e^u du \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{u(r-1)} e^{(-u^2+2u)/2} \sin(2\pi u) du \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left[-u^2+2u+2u(r-1)\right]/2} \sin(2\pi u) du \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-u^2+2ur)/2} \sin(2\pi u) du \\
&= \int_{-\infty}^{\infty} e^{r^2/2} \frac{1}{\sqrt{2\pi}} e^{-(u^2-2ur+r^2)/2} \sin(2\pi u) du \\
&= \int_{-\infty}^{\infty} e^{r^2/2} \frac{1}{\sqrt{2\pi}} e^{-(u-r)^2/2} \sin(2\pi u) du
\end{aligned}$$

Now, using integration by parts, we set

$$\begin{aligned}
u' &= e^{r^2/2} \frac{1}{\sqrt{2\pi}} e^{-(u-r)^2/2} & u &= e^{r^2/2} \\
v' &= 2\pi \cos(2\pi u) & v &= \sin(2\pi u)
\end{aligned}$$

$$\begin{aligned}
\therefore I &= e^{r^2/2} \sin(2\pi u)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 2\pi e^{r^2/2} \cos(2\pi u) du \\
&= \lim_{a\to\infty} \left[ e^{r^2/2} \sin(2\pi u)\Big|_{-a}^{a} \right] - \lim_{a\to\infty} \left[ \int_{-a}^{a} 2\pi e^{r^2/2} \cos(2\pi u) du \right] \\
&= \lim_{a\to\infty} \left[ e^{r^2/2} \sin(2\pi u)\Big|_{-a}^{a} \right] - \lim_{a\to\infty} \left[ 2\pi e^{r^2/2} \frac{1}{2\pi} \sin(2\pi u)\Big|_{-a}^{a} \right] \\
&= e^{r^2/2} \lim_{a\to\infty} \left[ \sin(2\pi u)\Big|_{-a}^{a} - \sin(2\pi u)\Big|_{-a}^{a} \right] \\
&= e^{r^2/2} \lim_{a\to\infty} 0 \\
&= 0 \\
\therefore I &= 0
\end{aligned}$$

2. A random variable $X$ is said to follow a Pareto$(\alpha, k)$ distribution if the density function of $X$ is:

$$f_X(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}} \qquad \alpha, k > 0 \text{ and } x > k$$

Suppose, for $n > 2$, we have a sequence of iid Pareto$(\alpha, k)$ random variables $X_1, \ldots, X_n$

(a) In order to compute the MLE for $k$ and $\alpha$, we derive the likelihood function, and the log-likelihood function.

$$\begin{aligned}
L(\alpha, k) &= \prod_{i=1}^{n} \frac{\alpha k^\alpha}{x_i^{\alpha+1}} \\
\therefore l(\alpha, k) &= \ln\left(\prod_{i=1}^{n} \frac{\alpha k^\alpha}{x_i^{\alpha+1}}\right) \\
&= \sum_{i=1}^{n} \ln\left(\frac{\alpha k^\alpha}{x_i^{\alpha+1}}\right) \\
&= \sum_{i=1}^{n} \Big[\ln\left(\alpha k^\alpha\right) - \ln\left(x_i^{\alpha+1}\right)\Big] \\
&= \sum_{i=1}^{n} \ln\left(\alpha k^\alpha\right) - \sum_{i=1}^{n} \ln\left(x_i^{\alpha+1}\right) \\
&= n \ln\left(\alpha k^\alpha\right) - \sum_{i=1}^{n} \ln\left(x_i^{\alpha+1}\right) \\
&= n \ln(\alpha) + n\alpha \ln(k) - (\alpha + 1) \sum_{i=1}^{n} \ln(x_i) \ldots (1)
\end{aligned}$$

Considering (1) as an equation in $k$, we note that (1) is increasing over all values of $k$, as $n, \alpha > 0$. Thus, $l(\alpha, k)$ is maximised when $k$ takes its maximum value. As $k \leq x_i$, the maximum value that $k$ can take is:

$$\begin{aligned}
k &= \min(x_i) \\
\therefore \widehat{k} &= \min(X_i)
\end{aligned}$$

Thus, $\widehat{k}$ is the MLE for $k$.

Now, considering (1) as an equation in $\alpha$, we have:

$$
\begin{aligned}
\therefore \frac{\partial l(\alpha,k)}{\partial \alpha} &= \frac{n}{\alpha} + n\ln(k) - \sum_{i=1}^{n} \ln(x_i) \\
\frac{\partial l(\alpha,k)}{\partial \alpha} &= 0 \\
0 &= \frac{n}{\alpha} + n\ln(k) - \sum_{i=1}^{n} \ln(x_i) \\
\frac{n}{\alpha} &= \sum_{i=1}^{n} \ln(x_i) - n\ln(k) \\
\therefore \alpha &= \frac{n}{\sum_{i=1}^{n} \ln(x_i) - n\ln(k)} \\
\therefore \widehat{\alpha} &= \frac{n}{\sum_{i=1}^{n} \ln(X_i) - n\ln\left(\widehat{k}\right)} \\
\therefore \widehat{\alpha} &= \frac{n}{\sum_{i=1}^{n} \left[\ln(X_i) - \ln(\min(X_i))\right]} \\
\frac{\partial^2 l(\alpha,k)}{\partial \alpha^2} &= \frac{-n}{\alpha^2} \\
\therefore \frac{\partial^2 l(\alpha,k)}{\partial \alpha^2} &< 0 \quad \forall \alpha
\end{aligned}
$$

Thus, $\widehat{\alpha}$ maximises the log-likelihood function, and thus maximises the likelihood function, and therefore is the MLE for $\alpha$.

(b) The MLE of $k$ is $\widehat{k} = \min(X_i)$. In order to derive the distribution of $\widehat{k}$, we must consider the CDF of a minimum of a sequence of random variables. Let $Y = \min(X_i)$.

$$
\begin{aligned}
F_Y(y) &= \mathbb{P}(Y \leq y) \\
&= \mathbb{P}(\min(X_i) \leq y) \\
&= 1 - \mathbb{P}(\min(X_i) > y) \\
&= 1 - \left([1 - F_{X_1}(y)][1 - F_{X_2}(y)] \ldots [1 - F_{X_n}(y)]\right) \\
&= 1 - \left[1 - F_X(y)\right]^n \quad \text{as } X_i \ \forall i \text{ are iid} \\
\therefore F_{\widehat{k}}(x) &= 1 - \left[1 - F_X(x)\right]^n \\
&= 1 - \left[1 - \left[1 - \left(\frac{k}{x}\right)^{\alpha}\right]\right]^n \\
\therefore F_{\widehat{k}}(x) &= 1 - \left(\frac{k}{x}\right)^{n\alpha}
\end{aligned}
$$

Thus, $\widehat{k} \sim$ Pareto$(n\alpha, k)$

(c) The Bias of $\widehat{k}$ is given by:

$$\begin{aligned}
\text{Bias}\left(\widehat{k}\right) &= \mathbb{E}\left(\widehat{k}\right) - k \\
&= \frac{n\alpha k}{n\alpha - 1} - k \qquad \text{as } n\alpha > 1 \\
&= \frac{n\alpha k}{n\alpha - 1} - \frac{(n\alpha - 1)k}{n\alpha - 1} \\
\text{Bias}\left(\widehat{k}\right) &= \frac{k}{n\alpha - 1} \\
\therefore \text{Bias}\left(\widehat{k}\right) &> 0 \qquad \text{as } k > 0
\end{aligned}$$

Thus, the MLE $\widehat{k}$ is a biased estimator for $k$. An unbiased estimator for $k$ requires:

$$\begin{aligned}
\left[\frac{n\alpha k}{n\alpha - 1}\right]C - k &= 0 \qquad \text{for } C \text{ some constant} \\
\left[\frac{n\alpha k}{n\alpha - 1}\right]C &= k \\
\left[\frac{n\alpha}{n\alpha - 1}\right]C &= 1 \\
\therefore C &= \left[\frac{n\alpha - 1}{n\alpha}\right]
\end{aligned}$$

Thus, an MLE for $k$ that is unbiased is:

$$\widehat{k} = \left[\frac{n\alpha - 1}{n\alpha}\right]\min(X_i)$$

3. Let $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$ be two independent random variables, and define $Z = \min(X, Y)$.

$$\begin{aligned}
\mathbb{P}(Z \leq z) &= \mathbb{P}(\min(X,Y) \leq z) \\
&= 1 - \mathbb{P}(\min(X,Y) > z) \\
&= 1 - \mathbb{P}(X > z, Y > z) \\
&= 1 - \mathbb{P}(X > z)\mathbb{P}(Y > z)\ldots(A) \qquad \text{independence}
\end{aligned}$$

Now, considering $Z^2$

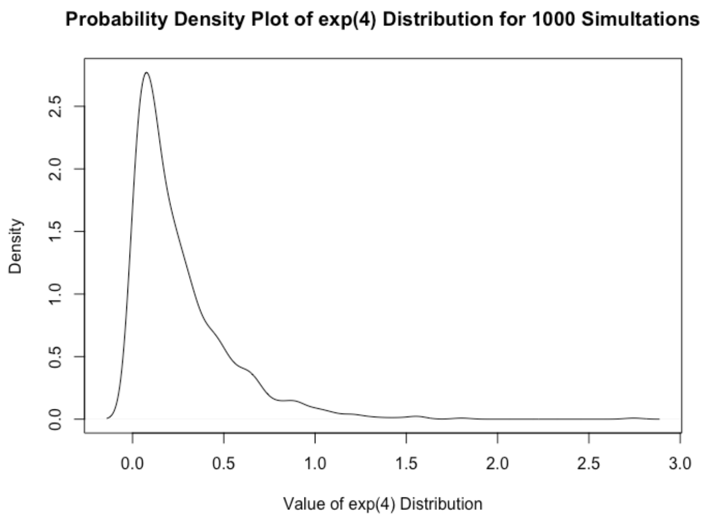
$$\begin{aligned}
\mathbb{P}(Z^2 \leq z) &= \mathbb{P}(-\sqrt{z} \leq Z \leq \sqrt{z}) \\
&= \mathbb{P}(Z \leq \sqrt{z}) - \mathbb{P}(Z \leq -\sqrt{z}) \\
&= 1 - \mathbb{P}(X > \sqrt{z})\mathbb{P}(Y > \sqrt{z}) - [1 - \mathbb{P}(X > -\sqrt{z})\mathbb{P}(Y > -\sqrt{z})] \qquad \text{from } (A) \\
&= \mathbb{P}(X > -\sqrt{z})\mathbb{P}(Y > -\sqrt{z}) - \mathbb{P}(X > \sqrt{z})\mathbb{P}(Y > \sqrt{z}) \\
&= \mathbb{P}(X > -\sqrt{z})[1 - \mathbb{P}(Y \leq -\sqrt{z})] - \mathbb{P}(X > \sqrt{z})[1 - \mathbb{P}(Y \leq \sqrt{z})] \\
&= \mathbb{P}(X > -\sqrt{z}) - \mathbb{P}(X > \sqrt{z}) - \mathbb{P}(X > -\sqrt{z})\mathbb{P}(Y \leq -\sqrt{z}) + \mathbb{P}(X > \sqrt{z})\mathbb{P}(Y \leq \sqrt{z}) \\
&= \mathbb{P}(X > -\sqrt{z}) - \mathbb{P}(X > \sqrt{z}) - \mathbb{P}(X \leq \sqrt{z})\mathbb{P}(Y > \sqrt{z}) + \mathbb{P}(X > \sqrt{z})\mathbb{P}(Y \leq \sqrt{z}) \\
&= \mathbb{P}(X > -\sqrt{z}) - \mathbb{P}(X > \sqrt{z}) - \mathbb{P}(Y \leq \sqrt{z})\mathbb{P}(X > \sqrt{z}) + \mathbb{P}(X > \sqrt{z})\mathbb{P}(Y \leq \sqrt{z}) \\
\therefore \mathbb{P}(Z^2 \leq z) &= \mathbb{P}(X > -\sqrt{z}) - \mathbb{P}(X > \sqrt{z}) \\
&= 1 - \mathbb{P}(X \leq -\sqrt{z}) - [1 - \mathbb{P}(X \leq \sqrt{z})] \\
&= \mathbb{P}(X \leq \sqrt{z}) - \mathbb{P}(X \leq -\sqrt{z}) \\
&= \mathbb{P}(-\sqrt{z} \leq X \leq \sqrt{z}) \\
\therefore \mathbb{P}(Z^2 \leq z) &= \mathbb{P}(X^2 \leq z)
\end{aligned}$$

Since $X \sim \mathcal{N}(0,1)$, $X^2 \sim \chi^2_1$ and thus $Z^2 \sim \chi^2_1$

4. (a) Using the following code,

```
sim1=rexp(1000,4)
densim1=(density(sim1))
plot(densim1, xlab = "Value of exp(4) Distribution",
   main = "Probability Density Plot of exp(4) Distribution for 1000 Simultations")
```

we get the following figure.

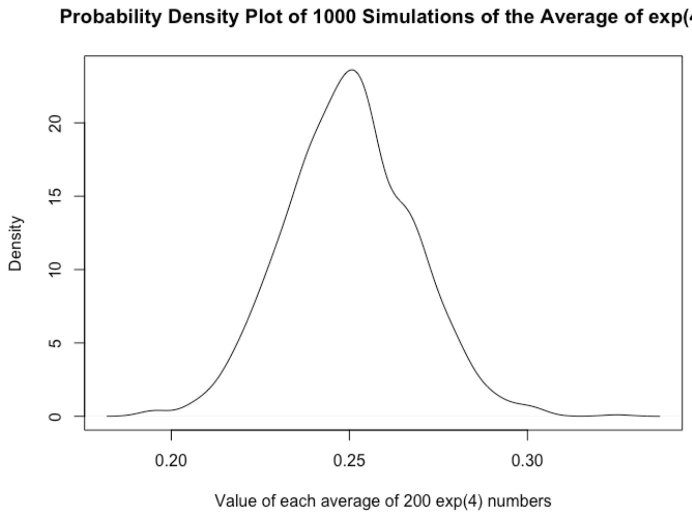



The above figure's right tail is longer than the left tail, and the mean sits further to the right than the median, thus the data is right skewed.

(b) Using the following code,

```
sim2=replicate(1000,((1/200)*(sum(rexp(200,4)))))
densim2=density(sim2)
plot(densim2, xlab = “Value of each average of 200 exp(4) numbers”,
   main = “Probability Density Plot of 1000 Simulations of the Average of exp(4)”)
```

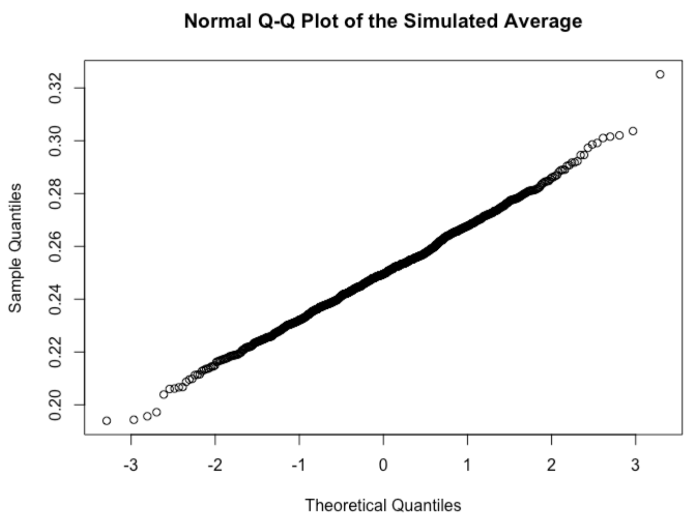
we get the following figure.




Using the following code,

```
qqnorm(sim2, main = ,“Normal Q-Q Plot of the Simulated Average”)
```

we get the following figure.

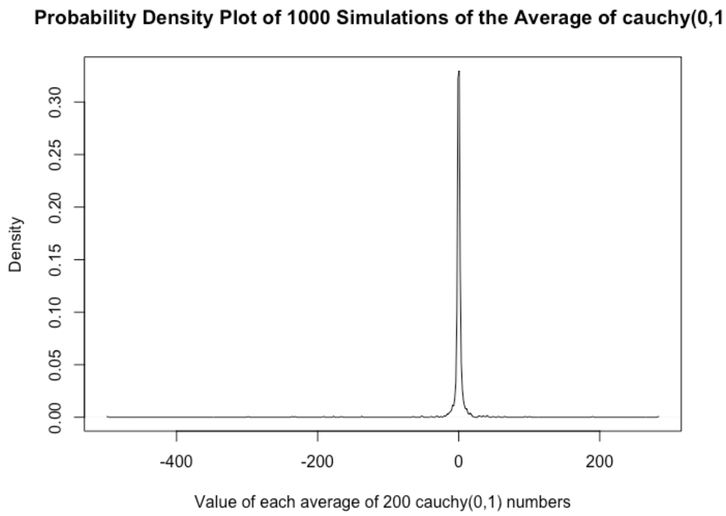



(c) The Probability Density Plot demonstrates that the distribution of averages of 200 simulations of the exp(4) distribution approximates a normal distribution. This indeed verifies the Central Limit Theorem.

(d) Using the following code,

```
sim3=replicate(1000,((1/200)*(sum(rcauchy(200,0,1)))))
densim3=density(sim3)
plot(densim3, xlab = “Value of each average of 200 cauchy(0,1) numbers”,
   main = “Probability Density Plot of 1000 Simulations of the Average of cauchy(0,1)”)
```

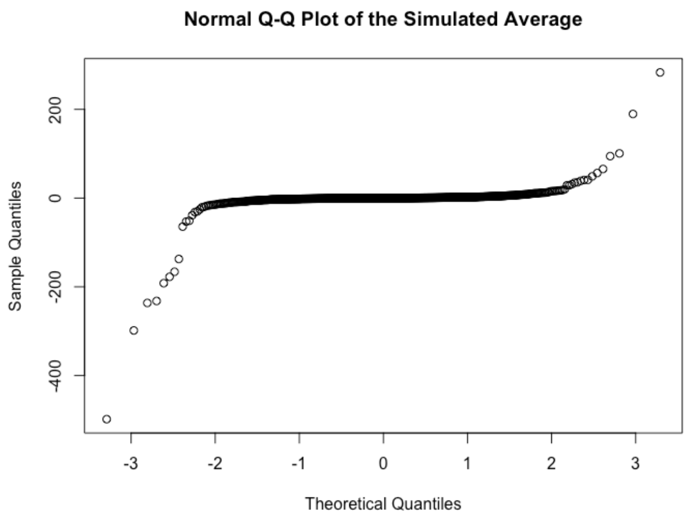
we get the following figure.




Using the following code,

```
qqnorm(sim3, main = ,“Normal Q-Q Plot of the Simulated Average”)
```

we get the following figure.




(e) The distribution of the mean of n independent identically distributed samples from a Cauchy Distribution has the same distribution as the original Cauchy Distribution, regardless of n. So the sample mean will never approximate a normal distribution, and in this way, the Central Limit Theorem is not verified.