THE UNIVERSITY OF SYDNEY

MATH1902 LINEAR ALGEBRA (ADVANCED)

Semester 1 | **Exercises for Week 9** | 2017

Preparatory exercises should be attempted before coming to the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) The *inverse* of a matrix A is a matrix A^{-1} such that, for some positive integer n,

$$AA^{-1} = A^{-1}A = I_n .$$

Only square matrices have inverses. When it exists, the inverse A^{-1} is unique.

(ii) Only half of the definition needs to be checked, in the following sense: if A is a square matrix and $AB = I$ or $BA = I$ then

$$AB = BA = I$$

so that the inverse A^{-1} exists and equals B.

(iii) A matrix is *invertible* if its inverse exists. If A and B are invertible matrices of the same size then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(iv) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define the *determinant* of A to be $\det A = ad - bc$. Then A is invertible if and only if $\det A \neq 0$, in which case $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(v) Let A be an invertible matrix. If n is an integer define

$$A^n = \begin{cases} I & \text{if } n = 0 \\ \underbrace{AA\dots A}_{n \text{ times}} & \text{if } n \text{ is positive} \\ \underbrace{A^{-1}A^{-1}\dots A^{-1}}_{-n \text{ times}} & \text{if } n \text{ is negative} \end{cases}$$

Then, for all integers m, n and all nonzero scalars λ,

$$A^mA^n = A^{m+n}, \qquad (A^{-1})^{-1} = A, \qquad (A^m)^n = A^{mn}, \qquad (\lambda A)^{-1} = \frac{1}{\lambda}A^{-1} .$$

(vi) A square matrix A is invertible if and only if the augmented matrix $\left[\, A \,\middle|\, I \,\right]$ can be row reduced to $\left[\, I \,\middle|\, B \,\right]$, in which case $A^{-1} = B$.

(vii) If a system of equations can be expressed in the form $A\mathbf{x} = \mathbf{b}$ where A is invertible, then $\mathbf{x} = A^{-1}\mathbf{b}$.

(viii) An $n \times n$ matrix is called *elementary* if it is the result of applying a single elementary row [column] operation to the identity matrix I_n.

(ix) If E is the elementary matrix obtained by applying the elementary row [column] operation ρ to I_n, and A is any matrix with n rows, then the matrix product EA $[AE]$ is the matrix obtained by applying ρ to A.

(x) The inverse of an elementary matrix is elementary.

(xi) Every invertible matrix is the product of elementary matrices.

Preparatory Exercises:

1. Use the formula for the inverse of a 2×2 matrix to invert the following matrices:

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 4 \\ 3 & 3 \end{bmatrix}.$$

2. Apply row reduction to the following augmented matrices to invert A and D of the previous exercise:

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right], \qquad \left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{array}\right].$$

3. Row reduce the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

to invert $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{bmatrix}$.

4. Solve each of the following systems by writing down an equivalent matrix equation and then multiplying through by A^{-1} or D^{-1} from the first exercise:

(i) $\begin{array}{rcrcr} x & - & 2y & = & 6 \\ -x & + & 3y & = & 5 \end{array}$ (ii) $\begin{array}{rcrcr} x & - & 2y & = & -2 \\ -x & + & 3y & = & 1 \end{array}$ (iii) $\begin{array}{rcrcr} 5x & + & 4y & = & -2 \\ 3x & + & 3y & = & -3 \end{array}$

5. Find the inverse of $\begin{bmatrix} 5 & -3 \\ 7 & -4 \end{bmatrix}$ and use it to solve for x, y, z, w where

$$\begin{bmatrix} 5 & -3 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 15 & 5 \end{bmatrix}.$$

Exercises:

16. Suppose that M is an invertible matrix such that the inverse of $5M$ is $\begin{bmatrix} 5 & 6 \\ 5 & 5 \end{bmatrix}$. Find M.

17. Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ and solve for x, y, z in terms of a, b, c where

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & a \\ 2x & + & 3y & + & z & = & b \\ 3x & + & y & + & 2z & = & c \end{array}$$

18. When is a diagonal matrix $\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$ invertible, and what is its inverse?

19.* Find elementary matrices E_1, E_2, E_3, E_4 and a reduced row echelon matrix B such that $A = E_1E_2E_3E_4B$, where

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ -2 & -3 & -4 & -5 \end{bmatrix}.$$

20.* Let n be a positive integer and J the $n \times n$ matrix each of whose entries is 1. Verify that $I - J$ is invertible if and only if $n \geq 2$, in which case

$$(I - J)^{-1} = I - \frac{1}{n-1} J .$$

21.* Use row reduction to determine the values of λ for which the matrix $A - \lambda I$ is *not* invertible in each case:

(i) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ (iii) $A = \begin{bmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{bmatrix}$

22.* A matrix A is called *symmetric* if $A = A^T$ and *skew-symmetric* if $A = -A^T$. Prove that inverses of symmetric and skew-symetric matrices, when they exist, are symmetric and skew-symmetric, respectively.

23.* Prove that every square matrix is the sum of a symmetric matrix and a skew-symmetric matrix.

24.* Can you find 2×2 matrices A and B such that

$$A \begin{bmatrix} x & y \\ z & w \end{bmatrix} B = \begin{bmatrix} y & w \\ x & z \end{bmatrix}$$

for all real numbers x, y, z, w?

25.** Suppose A and B are arrays of numbers where the rows and columns are indexed by positive integers, and each row and column contains finitely many nonzero entries. Then it is sensible to take the dot product of each row of A with each column of B (by ignoring summands involving infinitely many zeros) to form the product AB. Let I denote the infinite array with 1's down the diagonal and 0's elsewhere. Is it possible to find A and B such that $AB = I$ but $BA \neq I$?

Short Answers to Selected Exercises:

1. $A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad D^{-1} = \begin{bmatrix} 1 & -4/3 \\ -1 & 5/3 \end{bmatrix}$

2. $\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right],$

$\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4/5 & 1/5 & 0 \\ 0 & 3/5 & -3/5 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4/3 \\ 0 & 1 & -1 & 5/3 \end{array}\right]$

3. $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right]$

4. (i) $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 28 \\ 11 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -4/3 \\ -1 & 5/3 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

5. $\begin{bmatrix} -4 & 3 \\ -7 & 5 \end{bmatrix}, \; x = 1, \; y = -1, \; z = -2, \; w = -3$

7. (i) $\frac{1}{16} \begin{bmatrix} 2 & 2 \\ 3 & -5 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$ (vi) $\frac{1}{24} \begin{bmatrix} -3 & 0 & 3 \\ 3 & 6 & -9 \\ 1 & -6 & 13 \end{bmatrix}$

8. $B = (A^{-1}A)B(DD^{-1}) = A^{-1}(ABD)D^{-1} = A^{-1}(ACD)D^{-1} = (A^{-1}A)C(DD^{-1}) = C$

9. $\begin{bmatrix} 6 & -1 & -1 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}, \; x = 11, \; y = -5, \; z = -4$

12. (ii), (iii), (iv), (v), (vii), (viii), (ix)

13. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$

$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

14. $\lambda = 13$

16. $\begin{bmatrix} -1/5 & 6/25 \\ 1/5 & -1/5 \end{bmatrix}$

17. $\frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$, $x = \frac{1}{18}(-5a+b+7c)$, $y = \frac{1}{18}(a+7b-5c)$, $z = \frac{1}{18}(7a-5b+c)$

18. A diagonal matrix is invertible if and only if the diagonal entries are nonzero, in which case the inverse matrix is formed by inverting the diagonal entries.

19. $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

20. $(I - J)(I - \frac{1}{n-1}J) = I - \frac{n}{n-1}J + \frac{1}{n-1}J^2 = I - \frac{n}{n-1}J + \frac{n}{n-1}J = I$

21. (i) $2, -3$ (ii) $2, 3$ (iii) $1, -1, 3$