Semester 1 **Board Tutorial for Week 4** 2017

8. Given that $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $\mathbf{v}\times\mathbf{w} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ find

(i) $\mathbf{w}\times\mathbf{v}$ (ii) $\big(\mathbf{v}+3\mathbf{w}\big)\times\big(2\mathbf{w}-\mathbf{v}\big)$

9. Calculate $|\mathbf{a}\times\mathbf{b}|$ given that $|\mathbf{a}| = 7$, $|\mathbf{b}| = 4$ and $\mathbf{a}\cdot\mathbf{b} = -21$.

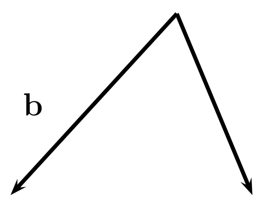
10. Use the algebraic definition of the cross product to verify the following properties for any vectors $\mathbf{v}$ and $\mathbf{w}$:

(i) $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{v} = 0$ (ii) $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{w} = 0$

(iii) $\mathbf{v}\times\mathbf{w} = -(\mathbf{w}\times\mathbf{v})$ (iv) $\mathbf{v}\times\mathbf{v} = \mathbf{0}$

11. Use the parallelogram property of the cross product to deduce quickly that vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.

12. Let $\mathbf{a}$ and $\mathbf{b}$ be the following vectors in the page:



True or false:

(i) $\mathbf{a}\times\mathbf{b}$ points upwards, away from the page, towards the ceiling

(ii) $\mathbf{b}\times\big(\mathbf{a}-\mathbf{b}\big)$ points downwards, away from the page, towards the floor

(iii) $\mathbf{a}\times\big(\mathbf{b}\times\mathbf{a}\big)$ is perpendicular to $\mathbf{a}$ but not to $\mathbf{b}$

(iv) $\mathbf{b}\times\big(\mathbf{b}\times\mathbf{a}\big)$ is the zero vector

13. Does the expression $\mathbf{u}\times\mathbf{v}\times\mathbf{w}$ make sense? Does the equation $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w}$ imply $\mathbf{v} = \mathbf{w}$ whenever $\mathbf{u} \neq \mathbf{0}$?

14.* A tetrahedron has four faces. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ be vectors perpendicular to the faces, pointing outwards, of length equal to the respective areas of the faces. Verify that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}\,.$$

15.* Verify that, for any geometric vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,

$$\mathbf{a}\cdot\big(\mathbf{b}\times\mathbf{c}\big) = \big(\mathbf{a}\times\mathbf{b}\big)\cdot\mathbf{c}\,.$$

Give both algebraic and geometric verifications. Use anticommutativity of the cross-product to deduce that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -(\mathbf{b}\times\mathbf{a})\cdot\mathbf{c}\,.$$

Important Ideas and Useful Facts:

(i) Algebraic definition of cross product: If $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$ then

$$\mathbf{v} \times \mathbf{w} \;=\; (v_2 w_3 - v_3 w_2)\,\mathbf{i} + (v_3 w_1 - v_1 w_3)\,\mathbf{j} + (v_1 w_2 - v_2 w_1)\,\mathbf{k}\;.$$

which can be evaluated by

(a) using the "up-and-down-diagonal" method;

(b) using the "expanding brackets" method and the facts that

$$\mathbf{i} \times \mathbf{j} \;=\; \mathbf{k} \;=\; -\big(\mathbf{j} \times \mathbf{i}\big)\;, \quad \mathbf{j} \times \mathbf{k} \;=\; \mathbf{i} \;=\; -\big(\mathbf{k} \times \mathbf{j}\big)\;, \quad \mathbf{k} \times \mathbf{i} \;=\; \mathbf{j} \;=\; -\big(\mathbf{i} \times \mathbf{k}\big)\;,$$

$$\mathbf{i} \times \mathbf{i} \;=\; \mathbf{j} \times \mathbf{j} \;=\; \mathbf{k} \times \mathbf{k} \;=\; 0\;;$$

(c) evaluating a 3×3 determinant (explained later): $\mathbf{v} \times \mathbf{w} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\;.$

(ii) The cross product $\mathbf{v} \times \mathbf{w}$ is always perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ so that

$$\big(\mathbf{v} \times \mathbf{w}\big) \cdot \mathbf{v} \;=\; \big(\mathbf{v} \times \mathbf{w}\big) \cdot \mathbf{w} \;=\; 0\;.$$

(iii) Anti-commutativity of cross product: $\mathbf{v} \times \mathbf{w} \;=\; -\big(\mathbf{w} \times \mathbf{v}\big)\;.$

(iv) Distributivity of cross over plus: $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} \;=\; \mathbf{u} \times \mathbf{w} \;+\; \mathbf{v} \times \mathbf{w}\;.$

(v) If $\mathbf{v}$ and $\mathbf{w}$ are vectors and λ is a scalar then

$$(\lambda \mathbf{v}) \times \mathbf{w} = \lambda(\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (\lambda \mathbf{w}) \qquad \text{and} \qquad \mathbf{v} \times \mathbf{v} = \mathbf{0}\;.$$

(vi) The area of the parallelogram inscribed by $\mathbf{v}$ and $\mathbf{w}$ is $|\mathbf{v} \times \mathbf{w}|\,.$

(vii) The area of the triangle inscribed by $\mathbf{v}$ and $\mathbf{w}$ is $\dfrac{|\mathbf{v} \times \mathbf{w}|}{2}\,.$

(viii) Geometric formula for cross product: if θ is the angle between vectors $\mathbf{v}$ and $\mathbf{w}$ chosen so that $0 \le \theta \le \pi$ then

$$\mathbf{v} \times \mathbf{w} \;=\; |\mathbf{v}||\mathbf{w}|\sin\theta\;\; \mathbf{u}\;,$$

where $\mathbf{u}$ is the unit vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ such that the triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is right-handed. In particular

$$|\mathbf{v} \times \mathbf{w}| \;=\; |\mathbf{v}||\mathbf{w}|\sin\theta\;.$$

(ix) Triple product: If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \;=\; (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

and its magnitude is the volume of the parallelopiped spanned by the three vectors, when placed tail-to-tail in space. If nonzero, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is positive if and only if the triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is right-handed.