Extended Answer Section

There are **three** *questions in this section, each with a number of parts. Write your answers in the space provided below each part. If you need more space there are extra pages at the end of the examination paper.*

1. The parametric vector form of the line $\mathcal{L}_1$ is given as $\mathbf{r}_1 = \mathbf{u}_1 + r\mathbf{v}_1$ ($r \in \mathbb{R}$) where $\mathbf{u}_1$ is the position vector of $P_1 = (1, 1, -3)$ and $\mathbf{v}_1 = \overrightarrow{P_1Q_1}$ where $Q_1 = (3, 3, -2)$. The parametric vector form of the line $\mathcal{L}_2$ is given as $\mathbf{r}_2 = \mathbf{u}_2 + s\mathbf{v}_2$ ($s \in \mathbb{R}$) where $\mathbf{u}_2$ is the position vector of $P_2 = (-2, 0, 2)$ and $\mathbf{v}_2 = -\mathbf{j} - \mathbf{k}$.

(a) **[2 marks]** Give the parametric scalar equations of $\mathcal{L}_1$.

$u_1 = \overrightarrow{OP_1} = i + j - 3k$ and $v_1 = \overrightarrow{P_1Q_1} = \overrightarrow{OQ_1} - \overrightarrow{OP_1}$

$= (3-1)i + (3-1)j + (-2-(-3))k = 2i + 2j + k$.

So parametric vector form of equation of $\mathcal{L}_1$ becomes

$r_1 = (i + j - 3k) + r(2i + 2j + k) = (1+2r)i + (1+2r)j + (-3+r)k$

If (x, y, z) is a point on $\mathcal{L}_1$ then $r_1 = xi + yj + zk$

ie $\left.\begin{aligned} x &= 1 + 2r \\ y &= 1 + 2r \\ z &= -3 + r \end{aligned}\right\}$ where $r \in \mathbb{R}$. (These are the parametric scalar equations of $\mathcal{L}_1$)

(b) **[2 marks]** Find a *unit* vector $\hat{\mathbf{n}}$ that is perpendicular to both $\mathcal{L}_1$ and $\mathcal{L}_2$.

Question 1 continues on the next page

v_1 & v_2 give the directions of $\mathcal{L}_1$ & $\mathcal{L}_2$, & so $v_1 \times v_2$ is perpendicular to both $\mathcal{L}_1$ and $\mathcal{L}_2$.

$v_1 \times v_2 : \begin{vmatrix} i & j & k \\ 2 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} = i(-2+1) - j(-2-0) + k(-2-0)$

$v_1 \times v_2 = -i + 2j - 2k$

To obtain a unit vector in the direction of $v_1 \times v_2$, divide $v_1 \times v_2$ by its length, $|v_1 \times v_2| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$.

So $\hat{n} = -\frac{1}{3}i + \frac{2}{3}j - \frac{2}{3}k$

[Note: $\frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k$ would be equally correct.

You can also calculate $v_1 \times v_2$ as follows:

$(2i + 2j + k) \times (-j - k) = (2i \times (-j)) + (2i \times -k) + (2j \times (-j)) + (2j \times (-k))$
$+ (k \times (-j)) + (k \times (-k))$

$= -2k + 2j - 2i + i$ (since $i \times j = k$, $i \times k = -j$, $j \times k = i$,

$= -i + 2j - 2k$ $k \times j = -i$ and $j \times j = k \times k = 0$)]

(c) [**4 marks**] The shortest distance between two lines is the length of a vector that connects the two lines and is perpendicular to both lines. For $\mathcal{L}_1$ and $\mathcal{L}_2$ this is expressed in the vector equation $\mathbf{r}_1 + t\hat{\mathbf{n}} = \mathbf{r}_2$ where $t \in \mathbb{R}$ is a parameter. The corresponding system of linear equations is

$$\begin{aligned} 2r - \frac{1}{3}t &= -3 \\ 2r + s + \frac{2}{3}t &= -1 \\ r + s - \frac{2}{3}t &= 5. \end{aligned}$$

These equations come from
(pt on $\mathcal{L}_1$) + (scalar mult. of $\hat{\underset{\sim}{n}}$) = (pt on $\mathcal{L}_2$)
$(\underset{\sim}{u}_1 + r\underset{\sim}{v}_1) + t\hat{\underset{\sim}{n}} = \underset{\sim}{u}_2 + s\underset{\sim}{v}_2$
ie $r\underset{\sim}{v}_1 - s\underset{\sim}{v}_2 + t\hat{\underset{\sim}{n}} = \underset{\sim}{u}_2 - \underset{\sim}{u}_1$

Solve this system of linear equations.

Question 1 continues on the next page

We must solve $\begin{bmatrix} 2 & 0 & -\frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 1 & 1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 0 & -\frac{1}{3} & -3 \\ 2 & 1 & \frac{2}{3} & -1 \\ 1 & 1 & -\frac{2}{3} & 5 \end{array}\right]$$

$$\xrightarrow{\substack{(R_2 \to R_2 - R_1, \\ R_3 \to R_3 - \frac{1}{2}R_1)}} \left[\begin{array}{ccc|c} 2 & 0 & -\frac{1}{3} & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & \frac{-4+1}{6} & \frac{10+3}{2} \end{array}\right]$$

$$\xrightarrow{(R_3 \to R_3 - R_2)} \left[\begin{array}{ccc|c} 2 & 0 & -\frac{1}{3} & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{3}{2} & 9/2 \end{array}\right]$$

$$\xrightarrow{\substack{(R_1 \to \frac{1}{2}R_1 \\ R_3 \to -\frac{2}{3}R_3)}} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{6} & -3/2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Last equation says $t = -3$, & by back substitution

$$s = 2 - t = 2 - (-3) = 5$$

$$r = -3/2 + \frac{1}{6}t = -\frac{3}{2} - \frac{1}{2} = -2.$$

Solution: $\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -3 \end{bmatrix}$.

(d) [**2 marks**] Hence find the shortest distance between $\mathcal{L}_1$ and $\mathcal{L}_2$ and find the point Q on $\mathcal{L}_1$ that is closest to $\mathcal{L}_2$.

The point on $\mathcal{L}_1$ corresponding to $r = -2$ has coordinates

$$\left.\begin{aligned} x &= 1 + 2r = -3 \\ y &= 1 + 2r = -3 \\ z &= -3 + r = -5 \end{aligned}\right\} \text{ (by Part (a))}$$

So $Q = (-3, -3, -5)$ is the point on $\mathcal{L}_1$ closest to $\mathcal{L}_2$.

[The question does not ask you to find the point M on $\mathcal{L}_2$ that is closest to $\mathcal{L}_1$, but it is given by the parameter $s = 5$, & so the parametric vector form of the equation of $\mathcal{L}_2$ gives

$$\overrightarrow{OM} = \underset{\sim}{u}_2 + 5\underset{\sim}{v}_2 = (-2\underset{\sim}{i} + 0\underset{\sim}{j} + 2\underset{\sim}{k}) + 5(-\underset{\sim}{j} - \underset{\sim}{k})$$

ie M is $(-2, -5, -3)$].

If M is the point on $\mathcal{L}_2$ closest to $\mathcal{L}_1$ then

$$\overrightarrow{QM} = t\hat{\underset{\sim}{n}}$$

where t is given by the solution of the system in Part (c), ie $t = -3$.

So shortest distance between $\mathcal{L}_1$ & $\mathcal{L}_2$ is

$$|\overrightarrow{QM}| = |t\hat{\underset{\sim}{n}}| = |t||\hat{\underset{\sim}{n}}|$$

$$= |-3| \quad \text{since } |\hat{n}| = 1$$

$$= \underline{\underline{3}}$$

[You can check this using the coordinates of Q & M:

$$|\overrightarrow{QM}| = \sqrt{(-3+2)^2 + (-3+5)^2 + (-5+3)^2} = \sqrt{1+4+4} = 3$$]

2. (a) Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(i) [**3 marks**] Find the inverse A^{-1}.

Question 2 continues on the next page

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + \frac{1}{3}R_1]{R_2 \to R_2 - \frac{1}{3}R_1} \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 8/3 & 2 & -1/3 & 1 & 0 \\ 0 & 4/3 & 2 & \frac{1}{3} & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - \frac{1}{2}R_2]{} \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 8/3 & 2 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 2R_3]{} \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 8/3 & 0 & -4/3 & 2 & -2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to \frac{3}{8}R_2]{} \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 3/4 & -3/4 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - R_2]{} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 3/2 & -3/4 & 3/4 \\ 0 & 1 & 0 & -1/2 & 3/4 & -3/4 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \to \frac{1}{3}R_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/4 & 1/4 \\ 0 & 1 & 0 & -1/2 & 3/4 & -3/4 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 1/2 & -1/4 & 1/4 \\ -1/2 & 3/4 & -3/4 \\ 1/2 & -1/2 & 1 \end{bmatrix}$

(*ii*) [**2 marks**] Hence find the 3×3 lower triangular matrix X such that $AX = B$.

$AX = B$ gives $A^{-1}AX = A^{-1}B$ & so $X = A^{-1}B$.

$$\text{ie } X = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} & -\frac{3}{4} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(*iii*) [**1 mark**] Find $\det(A)$. (*Hint: This may be done directly, but also follows easily from a correct answer to part (ii).*)

Question 2 continues on the next page

With X as in (ii), $\det X = \frac{1}{2}$ (product of diagonal entries – since X is lower triangular).
Also $\det B = 4$ (product of diagonal entries – since B is upper triangular)

$AX = B$ gives $(\det A)(\det X) = (\det B)$

ie $\frac{1}{2}(\det A) = 4$

$$\underline{\underline{\det A = 8}}$$

[In truth, it is easier to use 1st row expansion:

$$\det \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} = 3(6-2) - (2+2) = 8 \text{]}$$

(b) Consider the matrix $C = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 1 & -4 \\ 1 & 0 & -1 \end{bmatrix}$.

(*i*) **[2 marks]** Find all eigenvalues of C.

Question 2 continues on the next page

$$C - \lambda I = \begin{bmatrix} -\lambda & 0 & 2 \\ 2 & 1-\lambda & -4 \\ 1 & 0 & -1-\lambda \end{bmatrix}$$

It is easiest to evaluate $\det(C-\lambda I)$ by the 2^{nd} column expansion:

$$\det(C-\lambda I) = -0 + (1-\lambda)\det\begin{bmatrix} -\lambda & 2 \\ 1 & -1-\lambda \end{bmatrix} - 0$$

$$= (1-\lambda)(\lambda(\lambda+1) - 2)$$

$$= (1-\lambda)(\lambda^2+\lambda-2)$$

$$= (1-\lambda)(\lambda+2)(\lambda-1)$$

So the eigenvalues are 1, 1 and 2 (ie 1 is repeated)

OR by 1^{st} row expansion

$$\det(C-\lambda I) = (-\lambda)\det\begin{bmatrix} 1-\lambda & -4 \\ 0 & -1-\lambda \end{bmatrix} - 0 + 2\det\begin{bmatrix} 2 & 1-\lambda \\ 1 & 0 \end{bmatrix}$$

$$= -\lambda\left[(1-\lambda)(-1-\lambda) - 0\right] + 2\left[0 - (1-\lambda)\right]$$

It helps at this point to notice that $1-\lambda$ is a factor of both nonzero terms. So

$$\det(C-\lambda I) = (1-\lambda)\left[-\lambda(-1-\lambda) - 2\right]$$

$$= (1-\lambda)(\lambda^2+\lambda-2) \text{ as above.}$$

If you don't notice the common factor, you can still do it:

$$\det(C-\lambda I) = -\lambda\left[\lambda^2-1\right] + 2\lambda - 2$$

$$= -\lambda^3 + 3\lambda - 2$$

Here you need to notice that $\lambda = 1$ is obviously a root, & so $\lambda - 1$ is a factor (Anyway, Part(ii) tells you that 1 is an eigenvalue)

$$\det(C-\lambda I) = (\lambda-1)(-\lambda^2-\lambda+2), \text{ etc.}$$

(*ii*) **[2 marks]** Find the eigenspace of C corresponding to eigenvalue 1.

We must solve $(C - I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

ie $\begin{bmatrix} -1 & 0 & 2 \\ 2 & 0 & -4 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

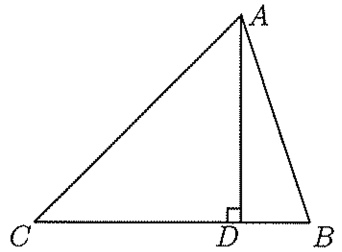
Applying ERO's $R_2 \to R_2 + 2R_1$ & $R_3 \to R_3 + R_1$ etc. shows that this system is equivalent to

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

y & z are free variables & the equation gives $x = 2z$. So if we let $y = s$ & $z = t$ (arbitrary parameters) we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ s \\ t \end{bmatrix}$.

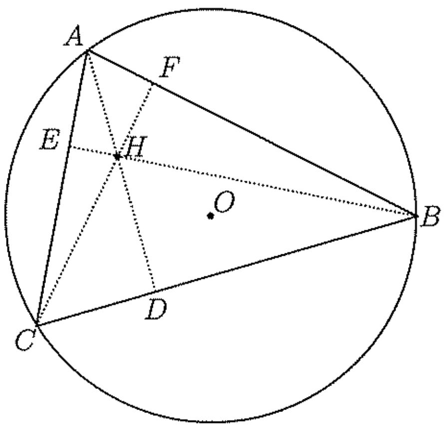
ie the 1-eigenspace is $\left\{ \begin{bmatrix} 2t \\ s \\ t \end{bmatrix} \middle| \, s, t \in \mathbb{R} \right\}$.

3. Recall the definition of an *altitude* of a triangle ABC:



It is the line joining a vertex A to the opposite side at D so that $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are orthogonal.

Let O be the centre of a circumscribing circle around a triangle ABC, and let H be the point of intersection of the altitudes AD, BE and CF:



Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, and $\overrightarrow{OC} = \mathbf{c}$.

(a) [**2 marks**] Write the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Question 3 continues on the next page

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \mathbf{c} - \mathbf{b}$.

(b) [**2 marks**] Show that $\mathbf{b} + \mathbf{c}$ and $\mathbf{b} - \mathbf{c}$ are orthogonal.

$(\underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{b} - \underset{\sim}{c}) = \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{c} \cdot \underset{\sim}{c}$

$= |\underset{\sim}{b}|^2 - |\underset{\sim}{c}|^2$ since $\underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{c} \cdot \underset{\sim}{b}$.

Since O is the centre of the circle, the line segments OB and OC have the same length. So $|\underset{\sim}{b}| = |\underset{\sim}{c}|$, & hence $(\underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{b} - \underset{\sim}{c}) = 0$.

By definition, this says that $\underset{\sim}{b} + \underset{\sim}{c}$ & $\underset{\sim}{b} - \underset{\sim}{c}$ are orthogonal.

[N.B. $\underset{\sim}{b} - \underset{\sim}{c} \neq \underset{\sim}{0}$, since B & C are distinct points, but $\underset{\sim}{b} + \underset{\sim}{c}$ could be $\underset{\sim}{0}$ — if the centre O lies on BC.]

(c) [**2 marks**] Hence, or otherwise, justify the statement:

A parametric vector equation for the line through A and D is

$$\mathcal{L}_1: \quad \mathbf{r}_1 = \mathbf{a} + t(\mathbf{b} + \mathbf{c}), \quad t \in \mathbb{R}.$$

Question 3 continues on the next page

The statement is not actually true if $\underset{\sim}{b} + \underset{\sim}{c} = 0$ — error in the question! But it is OK if $\underset{\sim}{b} + \underset{\sim}{c} \neq 0$. So let us assume that $\underset{\sim}{b} + \underset{\sim}{c} \neq \underset{\sim}{0}$.

Since AD is perpendicular to BC the vector $\overrightarrow{AD}$ must be parallel to any nonzero vector orthogonal to $\overrightarrow{CB}$. Since $\overrightarrow{CB} = \underset{\sim}{b} - \underset{\sim}{c}$ it follows that $\overrightarrow{AD}$ is parallel to $\underset{\sim}{b} + \underset{\sim}{c}$, ie $\overrightarrow{AD} = \alpha(\underset{\sim}{b} + \underset{\sim}{c})$ for some scalar α.

~~The parametric vector form of the equation of~~ the line through A in the direction of $\underset{\sim}{b} + \underset{\sim}{c}$ is $\underset{\sim}{r} = \overrightarrow{OA} + t(\underset{\sim}{b} + \underset{\sim}{c})$ $(t \in R)$

ie $\underset{\sim}{r} = \underset{\sim}{a} + t(\underset{\sim}{b} + \underset{\sim}{c})$, as required.

(d) [**4 marks**] Express the vector $\overrightarrow{OH}$ in terms of **a**, **b** and **c**.
(*Hint: use a parametric vector equation for the line through C and F.*)

There are no more questions.
Extra blank pages are provided in case you need more space for your answers.

Assuming that $\underset{\sim}{a} + \underset{\sim}{b} \neq \underset{\sim}{0}$ a parametric form for the equation of the line through C & F is $\underset{\sim}{r} = \underset{\sim}{c} + s(\underset{\sim}{a} + \underset{\sim}{b})$ $(s \in \mathbb{R})$ by the same reasoning as used for AD.

Since H lies on both AD & CF there must exist scalars t & s such that

$$\overrightarrow{OH} = \underset{\sim}{a} + t(\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{c} + s(\underset{\sim}{a} + \underset{\sim}{b}) \quad (*)$$

Since the lines AD and CF have a unique point of intersection, there can only be one value of t and one value of s to satisfy the equation $(*)$. But $t = 1$ & $s = 1$ clearly works, making both sides equal to $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$.

So $\overrightarrow{OH} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$.

[Note: this answer is valid in all cases, despite our assumptions that $\underset{\sim}{a} + \underset{\sim}{b} \neq \underset{\sim}{0}$ and $\underset{\sim}{b} + \underset{\sim}{c} \neq \underset{\sim}{0}$. The point H defined by $\overrightarrow{OH} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ always lies on all three altitudes, eg. $\overrightarrow{AH} = \overrightarrow{AO} + \overrightarrow{OH} = -\underset{\sim}{a} + (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{b} + \underset{\sim}{c}$, which is orthogonal to $\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b}$, etc.]