School of Mathematics University of New South Wales

MATH2701: Abstract Algebra and Fundamental Analysis

Main Assignment

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1. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{R})$ and consider the function

$$f_A(x) = \frac{ax+b}{cx+d}, \quad \text{where } x \text{ is a real number.}$$

(a) Assume $c \neq 0$. Find the point $x_0 \in \mathbb{R}$ at which the function f_A is not defined, and find $y_0 \in \mathbb{R}$ such that $y_0 \notin f_A(\mathbb{R} - \{x_0\})$.

Consider the vertical asymptote $x = -\dfrac{d}{c}$. Let $x_0 = -\dfrac{d}{c}$. Thus, $f_A(x_0)$ is clearly undefined. Consider now the horizontal asymptote, found by taking the limit as x approaches ∞, $y = \dfrac{a}{c}$. Let $y_0 = \dfrac{a}{c}$. There does not exist an $x \in \mathbb{R} - \{x_0\}$ such that $f_A(x) = y_0$. Thus, $y_0 \notin f_A(\mathbb{R} - \{x_0\})$.

(b) Consider the projective line $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$ and assume $c \neq 0$. Define $f_A : \mathbb{R}P^1 \to \mathbb{R}P^1$ by

$$f_A(x) = \begin{cases} \frac{ax+b}{cx+d}, & \text{if } x \neq x_0, \infty; \\ \infty & \text{if } x = x_0; \\ y_0 & \text{if } x = \infty. \end{cases}$$

Show that f_A is bijective (i.e., a transformation on $\mathbb{R}P^1$).

To prove the above definition for f_A is a bijection, we must show that f_A is both injective and surjective. For injectivity, we must show that $f(x_1) = f(x_2) \implies x_1 = x_2$, so we consider the following three cases.

1) Consider $f(x_1) = f(x_2) = y_0$. Thus, $x_1 = \infty$ and $x_2 = \infty$, so $x_1 = x_2$.

2) Consider $f(x_1) = f(x_2) = \infty$. Thus, $x_1 = x_0$ and $x_2 = x_0$, so $x_1 = x_2$.

3) Consider $f(x_1) = f(x_2) \neq y_0, \infty$. Thus,

$$\begin{aligned} \frac{ax_1+b}{cx_1+d} &= \frac{ax_2+b}{cx_2+d} \\ (ax_1+b)(cx_2+d) &= (ax_2+b)(cx_1+d) \\ acx_1x_2 + adx_1 + bcx_2 + bd &= acx_1x_2 + adx_2 + bcx_1 + bd \\ adx_1 + bcx_2 &= adx_2 + bcx_1 \\ (ad-bc)x_1 &= (ad-bc)x_2 \\ \therefore x_1 &= x_2 \quad (ad-bc) \neq 0 \text{ as } A \in GL_2(\mathbb{R}). \end{aligned}$$

Clearly, f_A is injective, as it is injective for each of the cases, which correspond to the piecewise branches of the definition of f_A.

For surjectivity, we must show that $\forall y \in \text{im}(f_A), \exists\, x$ s.t. $f_A(x) = y$, so we again consider the same three cases.
1) Consider $f(x) = y_0$. Thus, $x = \infty$.
2) Consider $f(x) = \infty$. Thus, $x = x_0$.
3) Consider $f(x) = \dfrac{ax+b}{cx+d}$. Thus,

$$\begin{aligned}
f_A(x) &= \frac{ax+b}{cx+d} \\
(cx+d)f_A(x) &= ax+b \\
df_A(x) - b &= ax - cxf_A(x) \\
\therefore x &= \frac{df_A(x) - b}{-cf_A(x) + a}
\end{aligned}$$

Note that $f_A(x) \neq y_0$ in case **3)**, as it has been covered in case **1)**, so there always exists an x in the final line of case **3)**. Clearly, f_A is surjective, as it is surjective for each of the cases, which correspond to the piecewise branches of the definition of f_A. Thus, f_A is bijective.

(c) If $c = 0$, give a definition analogous to (b) for the bijection $f_A : \mathbb{R}P^1 \to \mathbb{R}P^1$. Verify that the function f_A you define is bijective.

If $c = 0$, then $f_A(x) = \dfrac{a}{d}x + \dfrac{b}{d}$. Clearly, f_A has no asymptotes, and only needs to be defined $x = \infty$. Thus, the analogous definition is

$$f_A(x) = \begin{cases} \frac{a}{d}x + \frac{b}{d}, & \text{if } x \neq \infty; \\ \infty, & \text{if } x = \infty. \end{cases}$$

To prove the above definition for f_A is a bijection, we must show that f_A is both injective and surjective. For injectivity, we must show that $f(x_1) = f(x_2) \implies x_1 = x_2$, so we consider the following two cases.
1) Consider $f(x_1) = f(x_2) = \infty$. Thus, $x_1 = \infty$ and $x_2 = \infty$, so $x_1 = x_2$.
2) Consider $f(x_1) = f(x_2) \neq \infty$. Thus,

$$\begin{aligned}
\frac{a}{d}x_1 + \frac{b}{d} &= \frac{a}{d}x_2 + \frac{b}{d} \\
\frac{a}{d}x_1 &= \frac{a}{d}x_2 \\
\therefore x_1 &= x_2 \qquad a, d \neq 0 \text{ as } c = 0 \text{ and } A \in GL_2(\mathbb{R}).
\end{aligned}$$

Clearly, f_A is injective, as it is injective for each of the cases, which correspond to the piecewise branches of the definition of f_A.

For surjectivity, we must show that $\forall y \in \text{im}(f_A), \exists\, x$ s.t. $f_A(x) = y$, so we again consider the same two cases.

1) Consider $f(x) = \infty$. Thus, $x = \infty$.

2) Consider $f(x) = \dfrac{a}{d}x + \dfrac{b}{d}$. Thus,

$$\begin{aligned} f_A(x) &= \frac{a}{d}x + \frac{b}{d} \\ \frac{a}{d}x &= f_A(x) - \frac{b}{d} \\ \therefore x &= \frac{d}{a}f_A(x) - \frac{b}{a} \end{aligned}$$

Clearly, f_A is surjective, as it is surjective for each of the cases, which correspond to the piecewise branches of the definition of f_A. Thus, f_A is bijective.

(d) Show that the set $G = \{f_A \mid A \in GL_2(\mathbb{R})\}$ forms a subgroup of the group $\mathscr{B}(\mathbb{R}P^1)$ of all bijections on $\mathbb{R}P^1$.

Consider $f_A \in G$. f_A is a bijection on $\mathbb{R}P^1$, so $f_A \in \mathscr{B}(\mathbb{R}P^1)$. Thus, G is a non-empty subset of $\mathscr{B}(\mathbb{R}P^1)$. Now, by the Subgroup Lemma, we first consider closure under composition. Let $f_A, f_B \in G$ such that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and $B = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$, where $A, B \in GL_2(\mathbb{R})$. By definition, we have

$$\begin{aligned} f_A(x) &= \frac{ax+b}{cx+d}, \\ f_B(x) &= \frac{ex+f}{gx+h}. \end{aligned}$$

Thus, consdiering the composition yields

$$\begin{aligned} (f_A \circ f_B)(x) &= f_A(f_B(x)) \\ &= f_A\left(\frac{ex+f}{gx+h}\right) \\ &= \frac{a\left(\frac{ex+f}{gx+h}\right) + b}{c\left(\frac{ex+f}{gx+h}\right) + d} \\ &= \frac{\left(\frac{aex+af+bgx+bh}{gx+h}\right)}{\left(\frac{cex+cf+dgx+dh}{gx+h}\right)} \\ &= \frac{(ae+bg)x + (af+bh)}{(ce+dg)x + (cf+dh)} \\ &= f_C(x), \end{aligned}$$

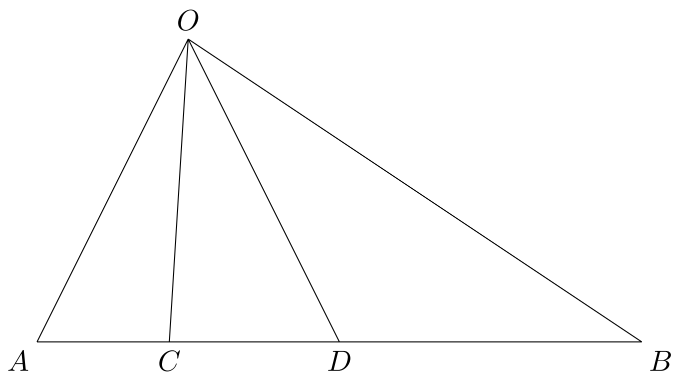
where $C = \left(\begin{smallmatrix} ae+bg & af+bh \\ ce+g & cf+dh \end{smallmatrix}\right)$. Clearly, $C = AB$, so $f_A \circ f_B = f_{AB}$. As $A, B \in GL_2(\mathbb{R})$, then clearly $AB \in GL_2(\mathbb{R})$. Furthermore, as f_A, f_B are bijections, so too is the composition $f_A \circ f_B = f_{AB}$. Thus, $f_{AB} \in G$.

Consider now the second part of the Subgroup Lemma, closure under inverse. Let $f_A \in G$ such that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Consider now the inverse function $f_A^{-1}(x)$, given by

$$\begin{aligned} f_A^{-1}(x) &= \frac{dx - b}{-cx + a} \\ &= f_D(x), \end{aligned}$$

where $D = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Consider $A^{-1} = \dfrac{1}{ad - bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Thus, $D = (ad - bc)A^{-1}$, and as $A \in GL_2(\mathbb{R})$, then $D \in GL_2(\mathbb{R})$. Furthermore, as f_A is a bijection, then so too is f_A^{-1}, thus f_D is a bijection, and so $f_D \in G$. As a result, $G \leq \mathscr{B}(\mathbb{R}P^1)$.

2. The cross-ratio of points A, B, C, D on line l is defined as $(A, B; C, D) = \frac{AC}{BC} \div \frac{AD}{BD}$. If lines a, b, c, d are concurrent and suppose $A, A' \in a$, $B, B' \in b$, $C \in c$, $D \in d$ satisfy $\overline{CA} \perp a$, $\overline{CB} \perp b$, $\overline{DA'} \perp a$, and $\overline{DB'} \perp b$, then the cross-ratio of lines a, b, c, d (or more precisely, the cross-ratio in which lines c, d divide lines a, b) is defined as $(a, b; c, d) = \frac{AC}{BC} \div \frac{A'D}{B'D}$ ($\frac{AC}{BC}$ is called the ratio of division of lines a, b by c, and $\frac{A'D}{B'D}$ the ratio of division of lines a, b by d.) Now consider the following configuration of points A, B, C, D and line l where O is off l.



(a) Show that $(A, B; C, D) = \frac{\sin(\angle AOC)\sin(\angle BOD)}{\sin(\angle BOC)\sin(\angle AOD)}$. (This results in the fact that a cross-ratio is unchanged by a projective transformation.)

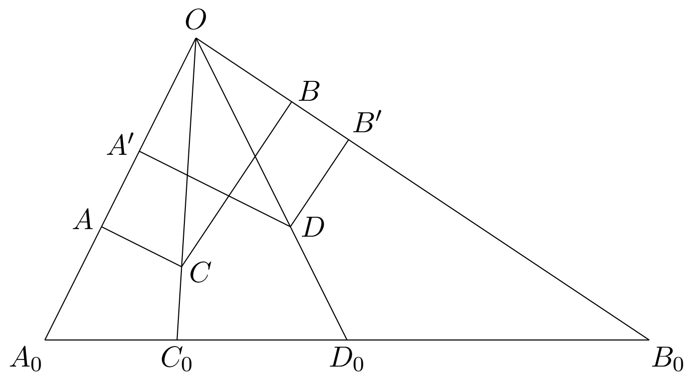
Using the Sine Rule in the given figure, we get the following results.

$$\begin{aligned}
\frac{\sin(\angle AOC)}{AC} &= \frac{\sin(\angle OAC)}{OC} \implies AC = \frac{OC\sin(\angle AOC)}{\sin(\angle OAC)} \\
\frac{\sin(\angle BOD)}{BD} &= \frac{\sin(\angle OBD)}{OD} \implies BD = \frac{OD\sin(\angle BOD)}{\sin(\angle OBD)} \\
\frac{\sin(\angle BOC)}{BC} &= \frac{\sin(\angle OBD)}{OC} \implies BC = \frac{OC\sin(\angle BOC)}{\sin(\angle OBD)} \\
\frac{\sin(\angle AOD)}{AD} &= \frac{\sin(\angle OAC)}{OD} \implies AD = \frac{OD\sin(\angle AOD)}{\sin(\angle OAC)}
\end{aligned}$$

Using these results, we can rewrite the cross-ratio, $(A, B; C, D)$ as

$$
\begin{aligned}
(A, B; C, D) &= \frac{AC}{BC} \times \frac{BD}{AD} \\
&= \frac{\left(\frac{OC\sin(\angle AOC)}{\sin(\angle OAC)}\right)}{\left(\frac{OC\sin(\angle BOC)}{\sin(\angle OBD)}\right)} \times \frac{\left(\frac{OD\sin(\angle BOD)}{\sin(\angle OBD)}\right)}{\left(\frac{OD\sin(\angle AOD)}{\sin(\angle OAC)}\right)} \\
&= \frac{\sin(\angle AOC)\sin(\angle BOD)}{\sin(\angle BOC)\sin(\angle AOD)}
\end{aligned}
$$

(b) If $a = l(O, A)$, $b = l(O, B)$, $c = l(O, C)$, and $d = l(O, D)$, show that $(a, b; c, d) = (A, B; C, D)$. Using the configuration from above, we will relabel the previously defined points A, B, C, D as A_0, B_0, C_0, D_0 (to avoid confusion), thus $a = l(O, A_0)$, $b = l(O, B_0)$, $c = l(O, C_0)$, and $d = l(O, D_0)$. Using the definitions given, our configuration becomes as follows.



From the definition, we have $(a, b; c, d) = \frac{AC}{BC} \times \frac{B'D}{A'D}$. Furthermore, from part (a), we have the result $(A_0, B_0; C_0, D_0) = \frac{\sin(\angle A_0OC_0)\sin(\angle B_0OD_0)}{\sin(\angle B_0OC_0)\sin(\angle A_0OD_0)}$. As we have right angled triangles, we get the following results from trigonometry.

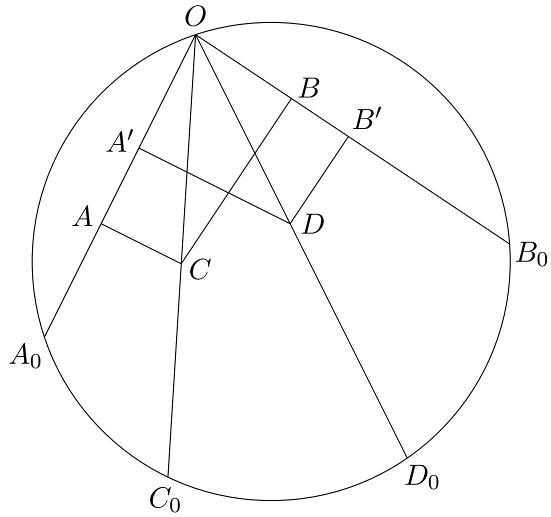
$$
\begin{aligned}
\sin(\angle A_0OC_0) &= \frac{AC}{OC}, & \sin(\angle B_0OD_0) &= \frac{B'D}{OD} \\
\sin(\angle B_0OC_0) &= \frac{BC}{OD}, & \sin(\angle A_0OD_0) &= \frac{A'D}{OC}
\end{aligned}
$$

Thus, we can rewrite the result from part (a) as

$$
\begin{aligned}
(A_0, B_0; C_0, D_0) &= \frac{\sin(\angle A_0OC_0)\sin(\angle B_0OD_0)}{\sin(\angle B_0OC_0)\sin(\angle A_0OD_0)} \\
&= \frac{\left(\frac{AC}{OC} \times \frac{B'D}{OD}\right)}{\left(\frac{BC}{OC} \times \frac{A'D}{OD}\right)} \\
&= \frac{AC}{BC} \times \frac{B'D}{A'D} \\
&= (a, b; c, d)
\end{aligned}
$$

(c) Let A, B, C, D be distinct points on a (non-degenerate) conic section $\mathscr{C}$. If O is another point on $\mathscr{C}$ and define lines a, b, c, d as in (b), show that the cross-ratio $(a, b; c, d)$ does not depend on the point O. (You may use the facts that every (non-degenerate) conic section is projectively equivalent to a circle and that a cross-ratio is unchanged by a projective transformation.)

As every non-degenerate conic section is projectively equivalent to a circle, and the cross-ratio $(a, b; c, d)$ is unchanged by a projective transformation, we only need to consider the case where $\mathscr{C}$ is a circle. So, using the same notation as in part (b), we have the following configuration.



Consider the point O' on $\mathscr{C}$, distinct from the point O. Each angle in the cross-ratio formula derived in part (b) stands on an arc defined by two points from A_0, B_0, C_0, D_0. As arcs subtend equal angles at all points on the circumference, and considering the distinct points O, O', we get

$$\angle A_0OC_0 = \angle A_0O'C_0, \ \angle B_0OD_0 = \angle B_0O'D_0, \ \angle B_0OC_0 = \angle B_0O'C_0, \ \angle A_0OD_0 = \angle A_0O'D_0$$

$$\therefore (a, b; c, d) = \frac{\sin(\angle A_0OC_0)\sin(\angle B_0OD_0)}{\sin(\angle B_0OC_0)\sin(\angle A_0OD_0)} = \frac{\sin(\angle A_0O'C_0)\sin(\angle B_0O'D_0)}{\sin(\angle B_0O'C_0)\sin(\angle A_0O'D_0)}.$$

Clearly, $(a, b; c, d)$ does not depend on the point O.

This assignment is completely my own work except where acknowledged
signed: *date:*