THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial for Week 2

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

Lecturers: Daniel Daners and James Parkinson

Topics covered

In lectures last week:

- □ Partitions $P = \{x_0, x_1, \dots, x_n\}$, sample points $x_j^* \in [x_{j-1}, x_j]$, and $\|P\|$.
- □ Riemann sums $\sum_{j=1}^{n} f(x_j^*)\Delta x_j$
- □ Upper and lower Riemann sums U_P and L_P.
- □ Riemann integrability and Riemann integrals.
- □ Continuous functions are Riemann integrable.

Objectives

After completing this tutorial sheet you will be able to:

- □ Work with partitions, and compute some Riemann sums.
- □ Understand the connection between Riemann sums and Riemann integrals.
- □ Be able to work with inequalities involving Riemann sums.
- □ Critically evaluate a mathematical statement.
- □ Develop summation techniques.
- □ Prove theorems about the Riemann integral.

Preparation questions to do *before* class

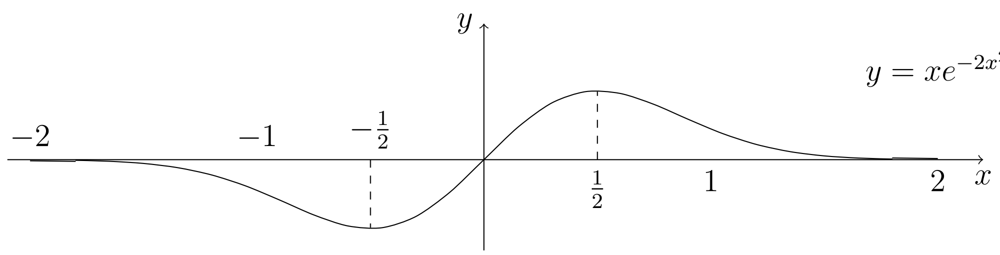
1. Let P be the partition $P = \{-2, 0, 1, 2\}$ of $[-2, 2]$, and let $f(x) = xe^{-2x^2}$.

(a) What is $\|P\|$?

Solution: We have $\Delta x_1 = 2$, $\Delta x_2 = 1$, and $\Delta x_3 = 1$. Therefore $\|P\| = 2$.

(b) Find all local maxima and minima of $f(x)$.

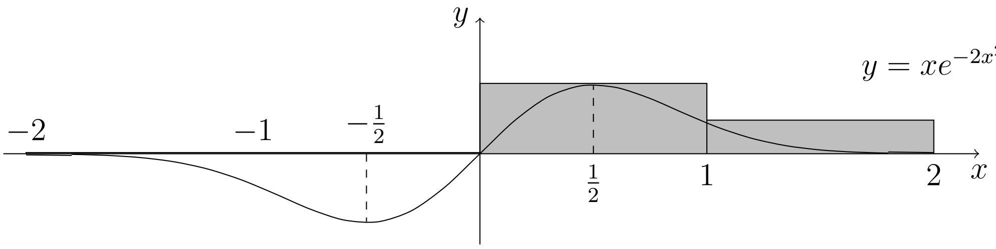
Solution: We have $f'(x) = (1 - 4x^2)e^{-2x^2}$, and so $f'(x) = 0$ when $x = \frac{1}{2}$ or $x = -\frac{1}{2}$. Therefore the sketch of $f(x)$ looks like



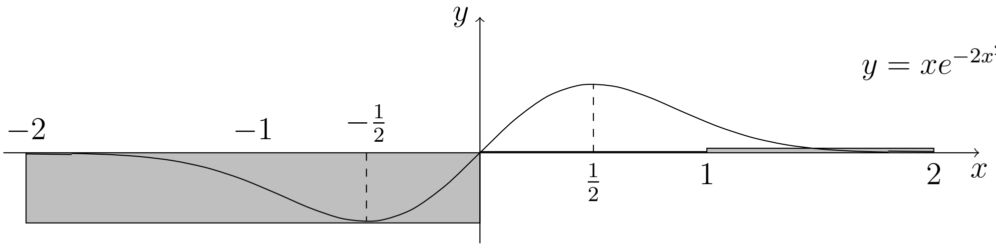
So f has a local maxima at $x = \frac{1}{2}$, and a local minima at $x = -\frac{1}{2}$.

(c) Compute the upper and lower Riemann sums for $f(x)$ on the interval $[-2, 2]$.

Solution: The pictures are



giving $U_P = 0 + \frac{1}{2}e^{-\frac{1}{2}} + e^{-2}$, and



giving $L_P = -e^{-\frac{1}{2}} + 0 + 2e^{-8}$.

2. Let $n \geq 1$ and let $S(n)$ be the sum

$$S(n) = \sum_{j=1}^{n} \big(j^2 - (j-1)^2\big).$$

Compute this sum in 2 ways: One way by realising that the sum 'collapses', and another way by writing $j^2 - (j-1)^2 = 2j - 1$. Hence find a formula for $\sum_{j=1}^{n} j$.

Solution: On the one hand we have

$$S(n) = \big(1^2 - 0^2\big) + \big(2^2 - 1^2\big) + \big(3^2 - 2^2\big) + \cdots + \big(n^2 - (n-1)^2\big) = n^2,$$

and on the other hand we have

$$S(n) = \sum_{j=1}^{n} \big(j^2 - (j^2 - 2j + 1)\big) = 2\left(\sum_{j=1}^{n} j\right) - n.$$

Therefore

$$\sum_{j=1}^{n} j = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

Questions to attempt in class

3. (a) Let $P = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[a, b]$ into n equal parts, and choose $x_j^* = x_j$. Write down the corresponding Riemann sum for $f(x) = x^2$.

Solution: We have $x_j = a + \frac{b-a}{n}j$, and $\Delta x_j = \frac{b-a}{n}$. Thus the Riemann sum is

$$
\begin{aligned}
R_n &= \sum_{j=1}^{n} \left(a + \frac{b-a}{n} j\right)^2 \frac{b-a}{n} \\
&= \frac{(b-a)a^2}{n} \left(\sum_{j=1}^{n} 1\right) + 2\frac{(b-a)^2 a}{n^2} \left(\sum_{j=1}^{n} j\right) + \frac{(b-a)^3}{n^3} \left(\sum_{j=1}^{n} j^2\right) \\
&= (b-a)a^2 + (b-a)^2 a \frac{n+1}{n} + \frac{(b-a)^3}{n^3} \sum_{j=1}^{n} j^2,
\end{aligned}
$$

where we have used the formula from Question 2.

(b) Generalise the technique of Question 2 to find a closed formula for $\sum_{j=1}^{n} j^2$, and hence find a closed formula for the Riemann sum in Question 3(a).

Solution: we consider the collapsing sum:

$$\sum_{j=1}^{n} \left(j^3 - (j-1)^3\right) = n^3.$$

On the other hand we have

$$\sum_{j=1}^{n} \left(j^3 - (j-1)^3\right) = \sum_{j=1}^{n} \left(j^3 - (j^3 - 3j^2 + 3j - 1)\right) = 3\sum_{j=1}^{n} j^2 - 3\sum_{j=1}^{n} j + n.$$

We remember that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$. Thus the value of $\sum_{j=1}^{n} j^2$ is obtained by solving the equation

$$n^3 = 3\sum_{j=1}^{n} j^2 - \frac{3n(n+1)}{2} + n, \qquad \text{giving} \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Therefore our Riemann sum is

$$R_n = (b-a)a^2 + (b-a)^2 a \frac{n+1}{n} + (b-a)^3 \frac{(n+1)(2n+1)}{6n^2}. \tag{1}$$

Remark: It is clear that this technique extends to allow us to compute the sum of higher powers. For example, to compute $\sum_{j=1}^{n} j^3$ we look at the collapsing sum

$$n^4 = \sum_{j=1}^{n} \left(j^4 - (j-1)^4\right) = 4\sum_{j=1}^{n} j^3 - 6\sum_{j=1}^{n} j^2 + 4\sum_{j=1}^{n} j - n.$$

Using the above formulae for $\sum_{j=1}^{n} j$ and $\sum_{j=1}^{n} j^2$ gives $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$.

(c) Compute the limit as $n \to \infty$ in your formula, and explain why the answer is what it is using a theorem from class.

Solution: Taking the limit as $n \to \infty$ in (1) we see that

$$\lim_{n\to\infty} R_n = (b-a)a^2 + (b-a)^2 a + \frac{1}{3}(b-a)^3 = \frac{1}{3}(b^3 - a^3).$$

Of course this is what we expect: $f(x) = x^2$ is continuous on $[a, b]$, and therefore is Riemann integrable on $[a, b]$. Hence the limit of the Riemann sums equals the Riemann integral, and we know that

$$\int_a^b x^2\,dx = \frac{1}{3}x^3\Big|_a^b = \frac{1}{3}(b^3 - a^3).$$

4. Suppose that $f(x)$ is monotonically increasing on $[a, b]$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$ into n subintervals.

(a) Write down expressions for the upper and lower Riemann sums U_P and L_P.

Solution: Since f is monotonically increasing, the maximum of f on $[x_{j-1}, x_j]$ occurs at x_j, and the minimum occurs at x_{j-1}. Therefore

$$U_P = \sum_{j=1}^{n} f(x_j)\Delta x_j \qquad \text{and} \qquad L_P = \sum_{j=1}^{n} f(x_{j-1})\Delta x_j.$$

(b) Show that $U_P - L_P \le (f(b) - f(a))\|P\|$, where $\|P\| = \max\{\Delta x_1, \ldots, \Delta x_n\}$.

Solution: Using $\Delta x_j \le \|P\|$ gives

$$U_P - L_P = \sum_{j=1}^{n} \left(f(x_j) - f(x_{j-1})\right)\Delta x_j \le \|P\| \sum_{j=1}^{n} \left(f(x_j) - f(x_{j-1})\right).$$

This last sum is a collapsing sum, and equals $f(b) - f(a)$.

5. Let $f(x) = x^{-2}$ and let $0 < a < b$.

(a) Let $P = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition of $[a, b]$, and make the clever choice $x_j^* = \sqrt{x_{j-1}x_j}$. Compute the corresponding Riemann sum of f.

Solution: Notice that $x_{j-1}^2 \le x_{j-1}x_j \le x_j^2$, and so $x_{j-1} \le \sqrt{x_{j-1}x_j} \le x_j$. Thus the sample point $x_j^* = \sqrt{x_{j-1}x_j}$ is indeed between x_{j-1} and x_j. The Riemann sum is

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j = \sum_{j=1}^{n} \frac{1}{x_{j-1}x_j}(x_j - x_{j-1}) = \sum_{j=1}^{n}\left(\frac{1}{x_{j-1}} - \frac{1}{x_j}\right) = \frac{1}{a} - \frac{1}{b}.$$

Surprisingly, this value does not depend on n.

(b) Explain how you know that the Riemann integral $\int_a^b f(x)\,dx$ exists. What is its value?

Solution: The function $f(x) = x^{-2}$ is *continuous* on $[a, b]$ (since $0 < a < b$) and so by a theorem in lectures the function is Riemann integrable on $[a, b]$. Therefore if $P_1, P_2, \ldots$ are partitions of $[a, b]$ with $\|P_n\| \to 0$ then $R(f; P_n) \to \int_a^b f(x)\,dx$. So the previous part shows that

$$\int_a^b \frac{1}{x^2}\,dx = \frac{1}{a} - \frac{1}{b}.$$

Discussion question

6. Let $f(x)$ be the function

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

Is f continuous? Is f differentiable at any point? Is f Riemann integrable on $[0, 1]$?

Solution: The quick answer is: f is continuous and differentiable at $x = 0$, and discontinuous everywhere else. For example, to see that f is differentiable at $x = 0$ we notice that

$$\frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h} = \begin{cases} h & \text{if } h \text{ is irrational} \\ 0 & \text{if } h \text{ is rational.} \end{cases}$$

Therefore

$$\left| \frac{f(0+h) - f(0)}{h} \right| \leq |h|,$$

and so by the squeeze law

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 0.$$

Therefore f is differentiable at $x = 0$, and $f'(0) = 0$.

$f(x)$ is not Riemann integrable on $[0, 1]$. To see this, let P be the partition of $[0, 1]$ into n equal parts. Now make two different choices of sample points: One choice with all x_j^* rational, and one with all x_j^* irrational. What is the limit. What is the limit (as $n \to \infty$) of the corresponding Riemann sums? The details are left for you to think about.

Questions for extra practice

7. Let $f(x) = e^x$, and let $P = \{x_0, \dots, x_n\}$ be the partition of $[0, 1]$ into n equal parts. Choose sample points $x_j^* = x_j$.

(a) Compute the Riemann sum $\sum_{j=1}^n f(x_j^*) \Delta x_j$.

Solution: Using the geometric sum formula we compute

$$\sum_{j=1}^n f(x_j^*) \Delta x_j = \sum_{j=1}^n e^{j/n} \frac{1}{n} = (e-1) \frac{n^{-1}}{1 - e^{-n^{-1}}}.$$

(b) Find the limit of your Riemann sum as $n \to \infty$, and explain why your answer is what it is using a theorem from class.

Solution: We have

$$\lim_{n \to \infty} \frac{n^{-1}}{1 - e^{-n^{-1}}} = \lim_{x \to 0} \frac{x}{1 - e^{-x}} = \lim_{x \to 0} \frac{1}{e^{-x}} = 1 \quad \text{(by L'Hôpitals rule)},$$

and so

$$\lim_{n\to\infty}\sum_{j=1}^{n} f(x_j^*)\Delta x_j = e-1.$$

This is no surprise: Since e^x is continuous we know from lectures that the limit of the Riemann sum must equal the definite integral $\int_0^1 e^x dx$, and we know how to compute this integral; it is $e-1$.

8. Let $\alpha\in\mathbb{R}$ and let $0<a<b$. Use the partition $P=\{a, ar, \ldots, ar^n\}$ with $r=\sqrt[n]{b/a}$ to compute the integral $\displaystyle\int_a^b x^\alpha\,dx$ from first principles. (Treat $\alpha=-1$ separately).

Solution: Choose $x_j^*=x_j$. If $\alpha\neq -1$ the geometric sum formula gives

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j = \sum_{j=1}^{n}\left(ar^j\right)^\alpha\left(ar^j-ar^{j-1}\right) = a^{\alpha+1}(r^{\alpha n+n}-1)\frac{1-r^{-1}}{1-r^{-\alpha-1}}.$$

Recalling that $r=\sqrt[n]{b/a}$ we get

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j = \left(b^{\alpha+1}-a^{\alpha+1}\right)\frac{1-\left(\frac{b}{a}\right)^{-\frac{1}{n}}}{1-\left(\frac{b}{a}\right)^{-\frac{\alpha+1}{n}}}.$$

Let $y=(b/a)^{-1/n}$. As $n\to\infty$ we have $y\to 1$. Therefore (by L'Hôpital's rule)

$$\lim_{n\to\infty}\frac{1-\left(\frac{b}{a}\right)^{-\frac{1}{n}}}{1-\left(\frac{b}{a}\right)^{-\frac{\alpha+1}{n}}} = \lim_{y\to 1}\frac{1-y}{1-y^{\alpha+1}} = \lim_{y\to 1}\frac{-1}{-(\alpha+1)y^\alpha} = \frac{1}{\alpha+1}.$$

Thus

$$\sum_{j=1}^{n} f(x_j^*)\Delta x_j \to \frac{1}{\alpha+1}\left(b^{\alpha+1}-a^{\alpha+1}\right).$$

Since x^α is continuous on $[a,b]$ we know that this Riemann sum converges to the definite integral $\displaystyle\int_a^b x^\alpha\,dx$, and so $\displaystyle\int_a^b x^\alpha\,dx = \frac{1}{\alpha+1}(b^{\alpha+1}-a^{\alpha+1})$.

If $\alpha=-1$ then since $r=\sqrt[n]{b/a}$ we have

$$R_n = \sum_{j=1}^{n}\left(1-\frac{1}{r}\right) = n\left(1-(b/a)^{-1/n}\right).$$

Therefore

$$\lim_{n\to\infty} R_n = \lim_{t\to 0^+}\frac{1-(b/a)^{-t}}{t} = \ln b - \ln a,$$

and so

$$\int_a^b \frac{1}{x}\,dx = \ln b-\ln a.$$

9. (a) Use the identity $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$ to prove that

$$2\sin(j\theta)\sin\left(\tfrac{1}{2}\theta\right) = \cos\left((j-\tfrac{1}{2})\theta\right) - \cos\left((j+\tfrac{1}{2})\theta\right).$$

Solution: The formula $2\sin(j\theta)\sin\left(\frac{1}{2}\theta\right) = \cos\left((j-\frac{1}{2})\theta\right) - \cos\left((j+\frac{1}{2})\theta\right)$ is immediate from $\cos(A-B)-\cos(A+B) = 2\sin A\sin B$, on taking $A = j\theta$ and $B = \frac{1}{2}\theta$.

(b) Deduce that

$$\sum_{j=1}^{n} \sin(j\theta) = \frac{\cos\left(\frac{1}{2}\theta\right) - \cos\left((n+\frac{1}{2})\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} \qquad \text{if } \theta \text{ is not a multiple of } 2\pi.$$

Solution: If we write $x_j = \cos\left((j+\frac{1}{2})\theta\right)$, then the above formula can be written, $2\sin(j\theta)\sin\left(\frac{1}{2}\theta\right) = x_{j-1} - x_j$, so that

$$2\sin\left(\tfrac{1}{2}\theta\right)\sum_{j=1}^{n}\sin(j\theta) = \sum_{j=1}^{n}\bigl(x_{j-1} - x_j\bigr).$$

The sum on the right collapses to $x_0 - x_n = \cos\left(\frac{1}{2}\theta\right) - \cos\left((n+\frac{1}{2})\theta\right)$.
If we require that θ is not a multiple of 2π, then $\sin\left(\frac{1}{2}\theta\right) \neq 0$. So we can divide both sides of the formula by $\sin\left(\frac{1}{2}\theta\right)$giving the stated formula for $\sum_{j=1}^{n}\sin(j\theta)$.

(c) Let $a > 0$ and let $\{x_0, \ldots, x_n\}$ be a partition of $[0, a]$ into n subintervals of length a/n. Let $x_j^* = x_j$ for each j. Show that

$$\sum_{j=1}^{n}\sin(x_j^*)\Delta x_j = \frac{a/(2n)}{\sin(a/(2n))}\left[\cos\left(\frac{a}{2n}\right) - \cos\left(a + \frac{a}{2n}\right)\right].$$

Show that this tends to $1 - \cos a$ as $n \to \infty$. Explain this using a theorem.

Solution: If $\{x_0, \ldots, x_n\}$ is the partition of $[0, a]$ into n subintervals of equal length, then $x_j = ja/n$ for each j. So taking $x_j^* = x_j$ for each j, the corresponding Riemann sum is

$$\sum_{j=1}^{n}\sin(x_j^*)\Delta x_j = \sum_{j=1}^{n}\sin\left(\frac{ja}{n}\right)\cdot\frac{a}{n}.$$

This involves the sum $\sum_{j=1}^{n}\sin(j\theta)$ for $\theta = a/n$. Substituting $\theta = a/n$ into our formula above for this sum, we get the stated expression for $\sum_{j=1}^{n}\sin(x_j^*)\Delta x_j$.
As $x \to 0$, we know that $(\sin x)/x$ tends to 1. Applying this to $x = a/(2n)$ we see that

$$\frac{a/(2n)}{\sin(a/(2n))} \to 1 \quad \text{as } n \to \infty.$$

Also, by continuity of the cosine function, we know that $\cos\left(\frac{a}{2n}\right) \to \cos 0 = 1$ and that $\cos\left(a + \frac{a}{2n}\right) \to \cos a$ as $n \to \infty$. So the Riemann sum tends to $1 - \cos a$ as $n \to \infty$. This limit was to be expected because $\sum_{j=1}^{n}\sin(x_j^*)\Delta x_j$ tends to $\int_0^a \sin x\,dx$ as $\|P\| \to 0$, and, by the Fundamental Theorem of Calculus, $\int_0^a \sin x\,dx = -\cos x|_0^a = 1 - \cos a$.

Challenging problems

10. (a) Find a closed formula for the sum $\displaystyle\sum_{j=0}^{n-1} jr^j$, where $r \neq 1$.

Hint: Use the geometric sum formula on $\sum_{j=0}^{n-1} r^j$, and differentiate with respect to r.

Solution: Let

$$S_n(r) = \sum_{j=0}^{n-1} r^j = \frac{1-r^n}{1-r}.$$

Then

$$S_n'(r) = \sum_{j=0}^{n-1} jr^{j-1} = \frac{nr^n - nr^{n-1} + 1 - r^n}{(1-r)^2}.$$

Multiplying through by r gives

$$\sum_{j=0}^{n-1} jr^j = \frac{nr^{n+1} - nr^n + r - r^{n+1}}{(1-r)^2}.$$

(b) Hence find a closed formula for the Riemann sum of $f(x) = x2^x$ over the interval $[0,1]$ using the partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$ into n equal parts, with $x_j^* = x_{j-1}$.

Solution: This Riemann sum has $x_j = \frac{j}{n}$, and $\Delta x_j = \frac{1}{n}$. Therefore the Riemann sum is

$$\begin{aligned} R_n &= \sum_{j=1}^{n} \frac{j-1}{n} 2^{\frac{j-1}{n}} \frac{1}{n} = \frac{1}{n^2} \sum_{j=0}^{n-1} j \left(2^{1/n}\right)^j \\ &= \frac{2^{\frac{n+1}{n}} n - 2n + 2^{\frac{1}{n}} - 2^{\frac{n+1}{n}}}{n^2(1-2^{\frac{1}{n}})^2} = \frac{2}{n(2^{1/n}-1)} - \frac{2^{1/n}}{n^2(2^{1/n}-1)^2}. \end{aligned}$$

(c) Calculate the limit of this Riemann sum as $n \to \infty$.

Solution: It is a bit neater to set $n = \frac{1}{t}$. Then

$$\lim_{n\to\infty} R_n = \lim_{t\to 0^+} \left(\frac{2t}{2^t - 1} - \frac{t^2 2^t}{(2^t-1)^2} \right).$$

By L'Hôpital's Rule we have

$$\lim_{t\to 0^+} \frac{t}{2^t-1} = \lim_{t\to 0^+} \frac{1}{\ln 2 \cdot 2^t} = \frac{1}{\ln 2}.$$

We now notice the shortcut:

$$\lim_{t\to 0^+} \frac{t^2 2^t}{(2^t-1)^2} = \lim_{t\to 0^+} \left[\left(\frac{t}{2^t-1} \right)^2 \cdot 2^t \right] = \frac{1}{(\ln 2)^2}$$

by limit laws (you could have also made a lengthy L'Hôpital's Rule computation). Therefore

$$\lim_{n\to\infty} R_n = \frac{2}{\ln 2} - \frac{1}{(\ln 2)^2} = \frac{2\ln 2 - 1}{(\ln 2)^2}.$$

We can check this answer by integration. Since $f(x) = x2^x$ is continuous, we know that $f(x)$ is Riemann integrable, and therefore the above limit equals the value of the Riemann integral. Using the laws of integration that we know and love, we have

$$\int_0^1 x2^x\,dx = \frac{x2^x}{\ln 2}\bigg|_0^1 - \frac{1}{\ln 2}\int_0^1 2^x\,dx \qquad \text{(integration by parts)}$$
$$= \frac{2}{\ln 2} - \frac{1}{(\ln 2)^2}2^x\bigg|_0^1 = \frac{2\ln 2 - 1}{(\ln 2)^2}.$$

11. Suppose that f is an unbounded positive function on the interval $[a, b]$. Show that f is not Riemann integrable on $[a, b]$.

Hint: Let M be a given (big) number. Show that there is a partition P and sample points x_j^* such that $\|P\|$ small and such that $\sum_{j=1}^n f(x_j^*)\Delta x_j > M$.

Solution: Let P be *any* partition of $[a, b]$. If f were bounded above on each of the n intervals $[x_{j-1}, x_j]$, then it would be bounded above on the whole interval $[a, b]$, contrary to hypothesis. For simplicity of notation, suppose that $f(x)$ is unbounded on $[x_0, x_1]$. Now choose any points $x_j^* \in [x_{j-1}, x_j]$ for $j = 2, 3, \dots, n$. Let

$$\sum_{j=2}^n f(x_j^*)\Delta x_j = K.$$

Because $f(x)$ is unbounded on $[x_0, x_1]$, it is not true that $f(x)(x_1 - x_0) \leq M - K$ for all $x \in [x_0, x_1]$. So there must exist $x \in [x_0, x_1]$ such that $f(x)(x_1 - x_0) > M - K$. Let x_1^* be this x. Then

$$f(x_1^*)(x_1 - x_0) > M - K = M - \sum_{j=2}^n f(x_j^*)\Delta x_j.$$

Thus

$$\sum_{j=1}^n f(x_j^*)\Delta x_j > M.$$

So there is no number A such that the Riemann sums are all close to A whenever $\|P\|$ is small, and hence f is not Riemann integrable on $[a, b]$.

12. Suppose that f is continuous on $[a, b]$. Show that if $f \geq 0$ and $\int_a^b f(x)dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$. What happens if we drop the assumption of continuity?

Hint: Continuity implies that if $f(\alpha) > \epsilon > 0$ for some α, then $f(x) > \epsilon/2$ for all x in some (small) interval containing α.

Solution: If $f(x) \neq 0$ for all $x \in [a, b]$ then there is $\alpha \in [a, b]$ and $\epsilon > 0$ such that $f(\alpha) > \epsilon$. By continuity there is a $\delta > 0$ such that

$$|x - \alpha| \leq \delta \implies |f(x) - f(\alpha)| < \epsilon/2.$$

Therefore for $x \in [\alpha - \delta, \alpha + \delta]$ we have

$$f(x) - f(\alpha) > -\epsilon/2, \qquad \text{and so} \qquad f(x) > f(\alpha) - \epsilon/2 > \epsilon - \epsilon/2 = \epsilon/2.$$

Then (since f is positive) we have

$$\int_a^b f(x)dx \geq \int_{\alpha-\delta}^{\alpha+\delta} f(x)dx \geq \int_{\alpha-\delta}^{\alpha+\delta} (\epsilon/2)dx = \frac{\epsilon}{2} \times 2\delta = \epsilon\delta > 0,$$

a contradiction.

Note 1: We have used quite a few "obvious" properties of integrals in the above string of inequalities - at least they are obvious when you think of the Riemann integral as calculating area. You might like to think about how to prove these properties rigorously from the definition of the Riemann integral. Then again, you might like to not do this.

Note 2: In the proof we have assumed for simplicity that $\alpha \in (a, b)$. There are some obvious modifications if $\alpha = a$ or $\alpha = b$.

Finally, if the assumption that f is continuous is dropped then the statement "$\int_a^b f(x)dx = 0$ and $f \geq 0 \Longrightarrow f(x) = 0$ for all $x \in [a, b]$" fails. For example

$$f(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2} \end{cases} \quad \text{is Riemann integrable on } [0,1] \text{ with } \quad \int_0^1 f(x)dx = 0.$$

13. This question shows how the famous "22/7 approximation" for π is related to integrals. By directly computing the integral, show that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx = \frac{22}{7} - \pi.$$

Deduce that $3.1412 < \pi < 3.1429$. Find better bounds for π using the integral

$$\int_0^1 \frac{x^8(1-x)^8}{1+x^2}\, dx.$$

Solution: Polynomial long division gives

$$\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2},$$

and integrating gives

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx = \frac{1}{7} - \frac{4}{6} + \frac{5}{5} - \frac{4}{3} + 4 - 4\tan^{-1}(1) = \frac{22}{7} - \pi.$$

Now, the integral is obviously positive, and hence

$$0 < \frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx \leq \int_0^1 x^4(1-x)^4\, dx < 0.0016.$$

The stated inequality follows.

The improved approximation is left as an exercise. Note that:

$$\frac{x^8(1-x)^8}{1+x^2} = x^{14} - 8x^{13} + 27x^{12} - 48x^{11} + 43x^{10} - 8x^9 - 15x^8 + 16(x^6 - x^4 + x^2 - 1) + \frac{16}{1+x^2}.$$