

### Tutorial for Week 13

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MATH1903: Integral Calculus and Modelling (Advanced)

Semester 1, 2012

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Web Page: <http://www.maths.usyd.edu.au/u/UG/JM/MATH1903/>

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#### Material covered

- (1) systems of linear differential equations
- (2) matrix exponentials (optional)

#### Outcomes

After completing this tutorial you should

- (1) be able to solve systems of two linear differential equations either by reducing them to a second order equation, or by using the eigenvalues and eigenvectors of the system matrix.

#### Questions to do before the tutorial

- 1. Reduce the first order system

$$x' = 7x - 5y \quad y' = 4x - 5y$$

to a second order equation for either  $x$  or  $y$ , and then find the general solution of the system.

#### Questions to complete during the tutorial

- 2. Find the general solution of the system of differential equations

$$\frac{dx}{dt} = 5x - 3y, \quad \frac{dy}{dt} = 2x$$

by differentiating the first equation and then using the second equation to form a linear second order differential equation for  $x(t)$ . Then find the particular solutions satisfying the initial conditions  $x(0) = 2$  and  $y(0) = 1$ .

- 3. Find the general solution of the system

$$\dot{x} = 2x - y \quad \dot{y} = x + 2y$$

by reducing the system to a second order equation.

- 4. Consider the pair of differential equations

$$\frac{dx}{dt} = 7x - 2y, \quad \frac{dy}{dt} = 2x + 3y.$$

- (a) Obtain a second-order differential equation for  $x(t)$  and find its general solution.
- (b) Find the associated general solution for  $y(t)$ .
- (c) Show that, if  $x(0) > y(0)$ , then  $x(t)$  and  $y(t)$  increase without limit as  $t \rightarrow \infty$ . Conversely, if  $x(0) < y(0)$ , show that  $x(t)$  and  $y(t)$  decrease without limit as  $t \rightarrow \infty$ .
- (d) Find the particular solution for  $x(t)$  and  $y(t)$  satisfying the initial conditions  $x(0) = 2$ ,  $y(0) = 1$ .
- (e) Confirm explicitly that the expressions you have obtained for  $x(t)$  and  $y(t)$  obey the first order equations given in part (a).

## Extra questions for further practice

5. Find the general solution of the pair of differential equations,

$$\frac{dx}{dt} = 5x - 3y, \quad \frac{dy}{dt} = 2x,$$

by differentiating the first equation and then using the second to obtain a linear second-order differential equation for  $x(t)$ . (Your solution should have *two* arbitrary constants of integration.) Find the particular solution satisfying the initial conditions  $x = 2$ ,  $y = 1$  when  $t = 0$ .

6. The following is a model for an arms race between two superpowers  $X$  and  $Y$ . Denote the level of preparation of  $X$  for war by  $x(t)$  and that of  $Y$  by  $y(t)$ , where  $t$  represents time. The model consists of the equations,

$$\frac{dx}{dt} = -ax + by, \quad \frac{dy}{dt} = cx - dy,$$

where  $a$ ,  $b$ ,  $c$  and  $d$  are positive constants.

- (a) Eliminate  $y$  to obtain a second-order homogeneous linear differential equation for  $x(t)$ .
  - (b) Write down the auxiliary equation of this differential equation, and its general solution in terms of its roots  $\lambda_1$  and  $\lambda_2$ . Show that if  $ad > bc$  then both roots of the auxiliary equation must be negative. What does this suggest about the likelihood of war?
  - (c) Suppose  $a = d = 1$ ,  $b = c = 3$ , and that  $x = 5$ ,  $y = 1$  at the initial time  $t = 0$ . Find the particular solution for  $x$  and  $y$ . What can you conclude about the likelihood of war in this case?
  - (d) Briefly discuss the assumptions underlying the model. Do you think these assumptions are realistic?
7. An electrical circuit comprises a capacitor, a resistor, and an inductor connected in series to a voltage generator with a prescribed voltage  $V(t)$ . In terms of the electric charge  $Q(t)$  and the current  $I(t) = dQ/dt$ , the voltage drops across the components are  $Q/C$  for a capacitance  $C$ ,  $IR$  for a resistance  $R$ , and  $LdI/dt$  for an inductance  $L$ . The total voltage drop is equal to that supplied by the generator,

$$\frac{Q}{C} + IR + L\frac{dI}{dt} = V(t).$$

- (a) Using  $I = dQ/dt$ , derive the equation,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}.$$

- (b) Find the general solution of the homogeneous equation, distinguishing three possible types of behaviour, depending on the values of  $R^2 - 4L/C$ . Show that, provided  $L, R > 0$ , all solutions die out as  $t \rightarrow \infty$ .
- (c) Find a particular solution for an AC voltage source  $V(t) = V_0 \cos(\omega_0 t)$ .

## Challenge questions (optional)

In lectures we showed that solutions to a linear systems of differential equations can be obtained by using methods of linear algebra, in particular eigenvalues and eigenvectors. In vector notation we saw that a system takes the form

$$\frac{d}{dt}\mathbf{u}(t) = M\mathbf{u}(t)$$

with  $M$  being a matrix. Such an equation looks like the standard equation  $x' = mx$  in one dimension. Its solution is

$$x(t) = Ce^{mt}.$$

We can try a similar idea for systems and write a matrix exponential

$$\mathbf{u}(t) = e^{Mt}.$$

But what is a matrix exponential? We can define it by means of the *exponential series*

$$e^{Mt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} M^k.$$

It turns out that this series converges. In the following exercises you find examples how to compute these exponentials using your knowledge from first semester linear algebra MATH1902.

8. The system of differential equations from Question 2 can alternatively be written in vector form

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

If we set  $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$  and denote the matrix by  $M$  the above can be written as  $\mathbf{u}' = M\mathbf{u}$ . The solution with initial condition  $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \mathbf{u}_0$  can be written as the matrix exponential  $\mathbf{u}(t) = e^{Mt}\mathbf{u}_0$ . In practice, computing the matrix exponential is often unnecessary, and the general solution is sufficient. If desired the matrix exponential can be obtained from that.

- Compute the eigenvalues and eigenvectors  $\mathbf{v}_1, \mathbf{v}_2$  of the matrix  $M$  by first finding the roots  $\lambda_1, \lambda_2$  of the characteristic polynomial  $\det(M - \lambda I)$ .
- From the previous part  $Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2$  is the general solution of the system of equations. Determine the solution with initial values  $x(0) = 2$  and  $y(0) = 1$ . Compare the result with that in Question 2.
- Here we want to describe a way to get the matrix exponential  $e^{tM}$ . Form the matrix  $X(t) := [e^{\lambda_1 t}\mathbf{v}_1 \quad e^{\lambda_2 t}\mathbf{v}_2]$  with columns given by  $e^{\lambda_1 t}\mathbf{v}_1$  and  $e^{\lambda_2 t}\mathbf{v}_2$ .
  - Show that  $\frac{d}{dt}X(t) = MX(t)$ , where differentiation of matrices is done on each entry.
  - Assuming that the solution of  $\mathbf{u}' = M\mathbf{u}$  with given initial condition  $\mathbf{u}(0) = \mathbf{u}_0$  is unique, show that  $e^{Mt} = X(t)X(0)^{-1}$ . (For the uniqueness, see Question 10.)
- Use the above method to compute  $e^{Mt}$ .

9. Consider the system of differential equations  $x' = -x - y$ ,  $y' = x - 3y$ . Denote by  $M = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$  the matrix associated with that system.

- Show that  $M$  has a double eigenvalue  $\lambda = -2$  and one independent eigenvector  $\mathbf{v}$  only.
- To find a second solution try a solution of the form  $(\mathbf{w} + t\mathbf{v})e^{-2t}$ , where  $\mathbf{w}$  is to be determined and  $\mathbf{v}$  is the eigenvector.
- Use the idea from Question 2(c) to compute  $e^{Mt}$ .

10. We show the uniqueness of the solution of  $\mathbf{u}' = M\mathbf{u}$ , where  $M$  is a matrix.

- From the definition of the matrix exponential, show that  $Me^{-tM} = e^{tM}M$ . Differentiate  $e^{tM}e^{tM}$  and hence show that  $e^{-tM}e^{tM} = I$  is constant.
- Suppose that  $X(t)$  is a solution. Differentiate  $e^{-tM}X(t)$  and hence show it is constant. Deduce that the solution is unique and that  $e^{tM} = X(t)X(0)^{-1}$ .