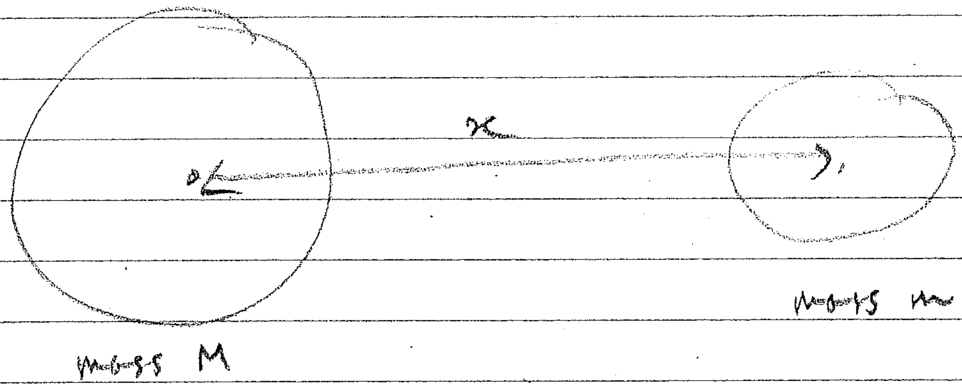
(A)

MATH1903 Lecture 2 Fri 4/8/2017

Example (Newton, 17th century, birth of modern calculus):

We estimate the escape velocity of a rocket.

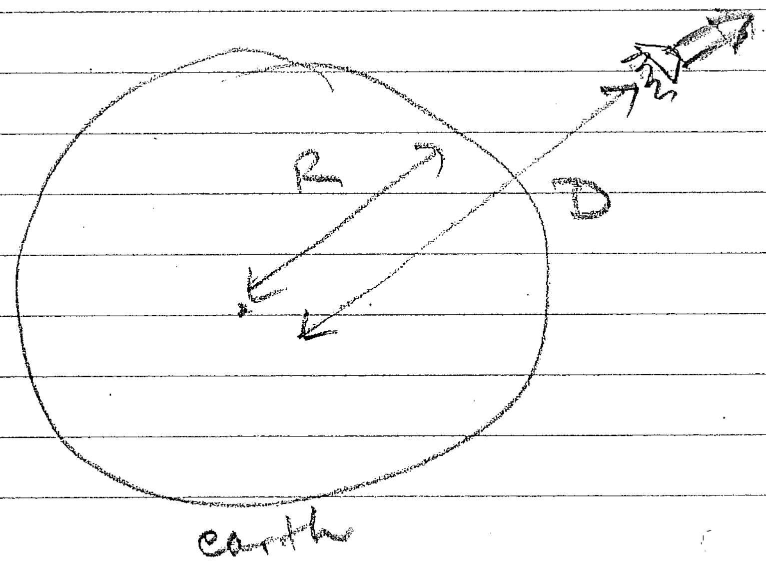


Gravitational attraction:

$$\boxed{F = F(x) = G\,\frac{Mm}{x^2}}$$

constant

Work required to move a rocket from the earth's surface (R km from the centre) to a position D km from the centre?



(B)

M = mass of earth

m = mass of rocket

Work (energy) required is

$$\int_R^D \underbrace{F(x)\,dx}$$

"sum" of all the bits of energy

tiny bit of energy = work expended for tiny movement dx

Want to go arbitrarily far (to the stars!):

$$\int_R^\infty F(x)\,dx = \lim_{D\to\infty} \int_R^D F(x)\,dx$$

called an improper integral

$$= \lim_{D\to\infty} \int_R^D G\frac{Mm}{x^2}\,dx$$

$$= GMm\left(\lim_{D\to\infty} \int_R^D x^{-2}\,dx\right)$$

$$= GMm\left(\lim_{D\to\infty} \left[-x^{-1}\right]_R^D\right)$$ by the Fundamental Theorem of Calculus (next week)

$$= GMm\left(\lim_{D\to\infty} \left(-D^{-1} - (-R^{-1})\right)\right)$$

$$= GMm\left(\lim_{D\to\infty} \frac{1}{R} - \frac{1}{D}\right) = \frac{GMm}{R}$$

Put

v_0 = escape velocity of rocket,

so want the kinetic energy to equal the work done getting the rocket to the stars:

$$\frac{1}{2} m v_0^2 = \frac{GMm}{R}$$

so

$$\frac{1}{2} v_0^2 = \frac{GM}{R}$$ (independent of m!!)

so

$$v_0^2 = \frac{2GM}{R}$$

so

$$\boxed{v_0 = \sqrt{\frac{2GM}{R}} \simeq 11 \text{ km/sec}}$$

<u>The definite integral & Riemann sums</u>

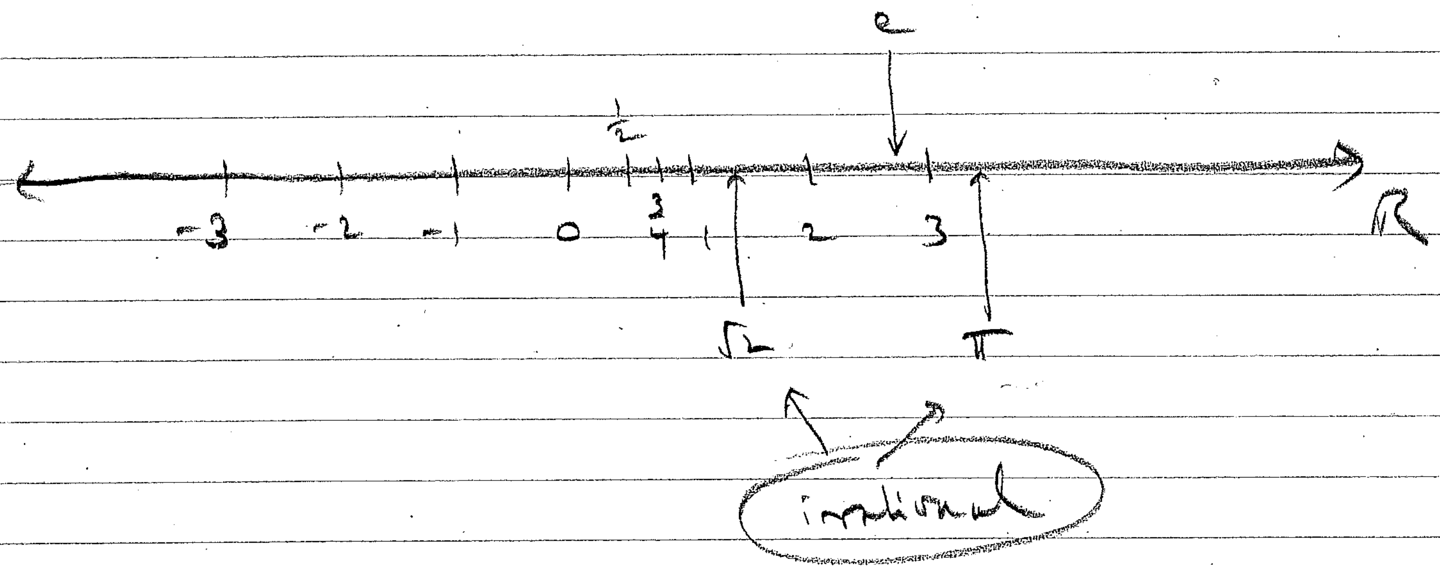
<u>Notes</u>: pp 14-37.

$\boxed{\text{Google: history of integral sign}}$ — due to Leibniz

$\int$ for "sum"

p19 : areas under curves

- intuitively plausible for continuous functions
- not clear in general (p20).



$\mathbb{Q} = \{ a/b \mid a, b \in \mathbb{Z},\ b \neq 0 \}$

$\mathbb{R} \setminus \mathbb{Q} = \{\text{irrationals}\}$.

Both $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$

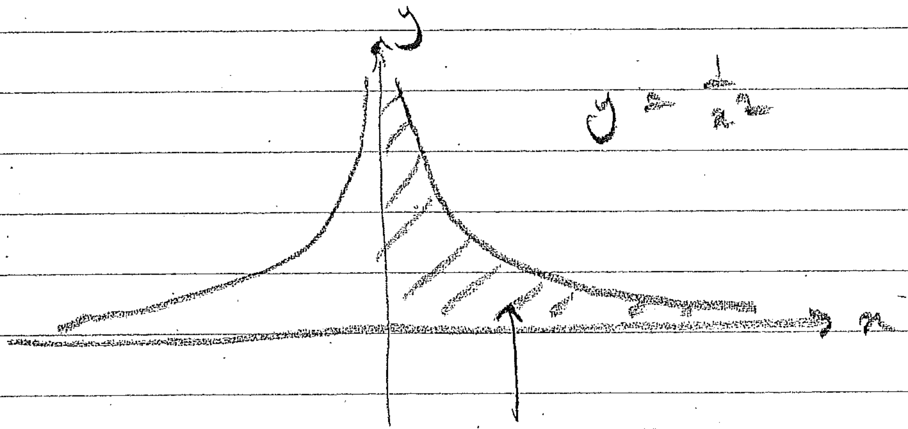
can approximate any real number using $\mathbb{Q}$ or $\mathbb{R}\setminus\mathbb{Q}$ arbitrarily closely

Interesting (and tricky) fact :

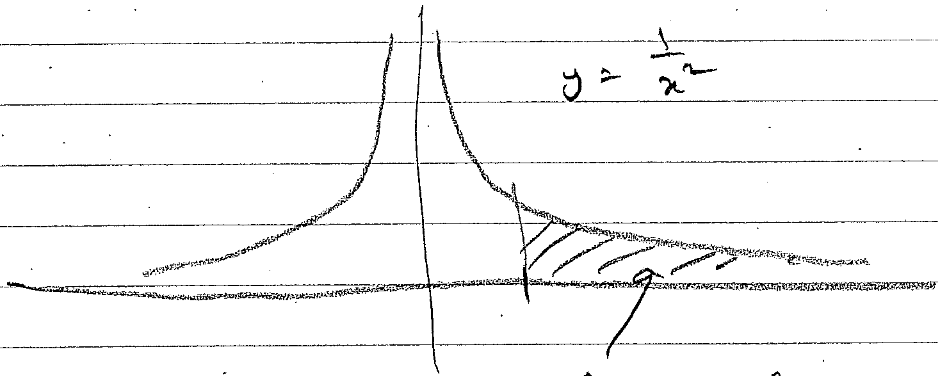
The infinity of $\mathbb{R}\setminus\mathbb{Q}$ is greater than the infinity of $\mathbb{Q}$.

E

Unbounded continuous functions can also have well-defined areas but care required



infinite area



finite area

(related to Newton's thought experiment about escape velocity)

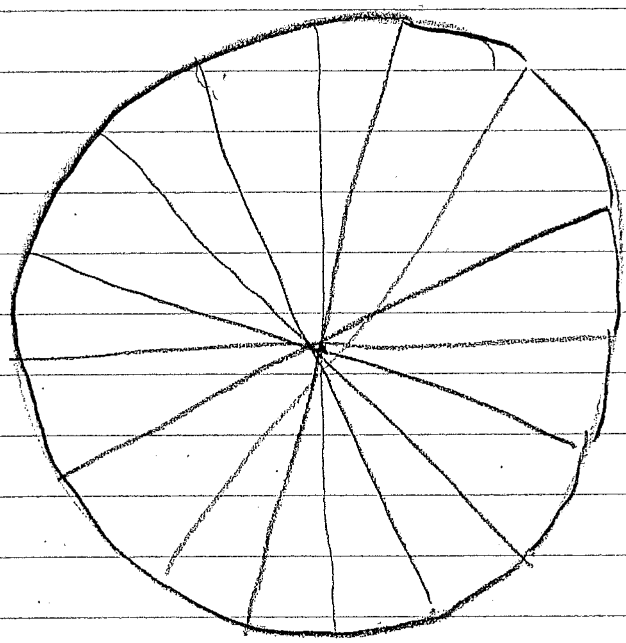
<u>Technique of the Greeks</u> :

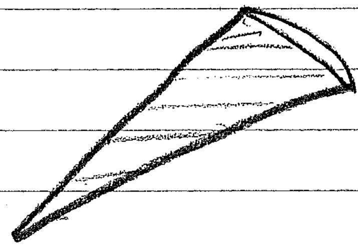
(1) <u>Approximate</u> a difficult task by an easy one.

(2) See what happens in <u>the limit</u>.

F

① Area of a circle:





triangular approximation

pp 22, 23.

② Area under a parabola

pp 24 – 27

<u>Method of Riemann Sums</u> : pp 28 – 31

$$\boxed{\text{Area} = \int_a^b \underline{f(x)\,dx}}$$

definite integral

"sum"

area of an "infinitely thin" rectangle

Properties of definite integrals: pp 32 – 37.