CHAPTER **FIVE**

The Binomial Theorem

In the previous chapter we discussed the factoring of a polynomial into irreducible factors, so that it could be written in a form such as

$$P(x) = (x-4)^2(x+1)^3(x^2+x+1).$$

In this chapter we will now study in more detail the individual factors like $(x-4)^2$ and $(x+1)^3$ which appear in such a factorisation. For example, we know already that

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1.$$

The coefficients in the general expansion of $(x+a)^n$ will be investigated through the patterns they form when they are written down in the *Pascal triangle*. These patterns lead to a formula for the coefficients, called the *binomial theorem*, and this formula is the key to further study of the binomial expansion and its coefficients. We will be able to apply much of this work in Chapter Ten on probability, because the systematic counting required there turns out to be closely related to the binomial theorem.

Study Notes: Sections 5A and 5B develop the Pascal triangle and apply it to numerical problems on binomial expansions, first of $(1+x)^n$ and then of $(x+y)^n$. Section 5C introduces the notation $n!$ for factorials in preparation for the binomial theorem itself in Section 5D. Section 5E uses the binomial theorem to find the maximum coefficient and term in a binomial expansion, then Section 5F turns attention to some of the identities relating the binomial coefficients and to the resulting patterns in the Pascal triangle.

The notation ${}^n\mathrm{C}_r$ is introduced in a preliminary manner in the notes of Section 5B, but Exercise 5B has been written so that use of the new notation can be delayed until Exercise 5D.

5 A The Pascal Triangle

This section is restricted to the expansion of $(1+x)^n$ and to the various techniques arising from such expansions. The techniques are based on the Pascal triangle and its basic properties, but the proofs of these properties will be left until Section 5B.

Some Expansions of $(1+x)^n$: Here are the expansions of $(1+x)^n$ for low values of n. The calculations have been carried out using two rows so that like terms can be written above each other in columns. In this way, the process by which the coefficients build up can be followed better.

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$$\begin{aligned}(1+x)^0 &= 1\\ (1+x)^1 &= 1+x\\ (1+x)^2 &= 1(1+x)+x(1+x)\\ &= 1+\ \ x\\ &\quad\ \ +\ \ x+x^2\\ &= 1+2x+x^2\end{aligned}$$

$$\begin{aligned}(1+x)^3 &= 1(1+x)^2+x(1+x)^2\\ &= 1+2x+\ \ x^2\\ &\quad\ \ +\ \ x+2x^2+x^3\\ &= 1+3x+3x^2+x^3\\ (1+x)^4 &= 1(1+x)^3+x(1+x)^3\\ &= 1+3x+3x^2+\ \ x^3\\ &\quad\ \ +\ \ x+3x^2+3x^3+x^4\\ &= 1+4x+6x^2+4x^3+x^4\end{aligned}$$

Notice how the expansion of $(1+x)^2$ has 3 terms, that of $(1+x)^3$ has 4 terms, and so on. In general, the expansion of $(1+x)^n$ has $n+1$ terms, from the constant term in $x^0 = 1$ to the term in x^n. Be careful — this is inclusive counting — there are $n+1$ numbers from 0 to n inclusive.

The Pascal Triangle and the Addition Property: When the coefficients in the expansions of $(1+x)^n$ are arranged in a table, the result is known as the *Pascal triangle*. The table below contains the first five rows of the triangle, copied from the expansions above, plus the next four rows, obtained by continuing these calculations up to $(1+x)^8$.

Coefficient of:

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
0	1								
1	1	1							
2	1	2	1						
3	1	3	$\boxed{3}$	$\boxed{1}$					
4	1	4	6	$\boxed{4}$	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Four properties of this triangle should quickly become obvious. They will be used in this section, and proven formally in the next.

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BASIC PROPERTIES OF THE PASCAL TRIANGLE:
1. Each row starts and ends with 1.
2. Each row is reversible.
3. The sum of each row is 2^n.
4. [The addition property] Every number in the triangle, apart from the 1s, is the sum of the number directly above, and the number above and to the left.

The first three properties should be reasonably obvious after looking at the expansions at the start of the section. The fourth property, called the *addition property*, however, needs attention. Three numbers in the Pascal triangle above have been boxed as an example of this — notice that

$$1+3=4.$$

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The expansions on the first page of this chapter were written with the columns aligned to make this property stand out. For example, $1 + 3 = 4$ arises like this — in the expansion of $(1+x)^4$, the coefficient of x^3 is the sum of the coefficients of x^3 and x^2 in the expansion of $(1+x)^3$.

The whole Pascal triangle can be constructed using these rules, and the first question in the following exercise asks for the first thirteen rows to be calculated.

Using Pascal's Triangle: The following worked exercises illustrate various calculations involving the coefficients of $(1+x)^n$ for low values of n.

WORKED EXERCISE: Use the Pascal triangle to write out the expansions of:

(a) $(1-x)^4$ (b) $(1+2a)^6$ (c) $(1-\frac{2}{3}x)^5$

SOLUTION:

(a) $$\begin{aligned}(1-x)^4 &= 1 + 4(-x) + 6(-x)^2 + 4(-x)^3 + (-x)^4 \\ &= 1 - 4x + 6x^2 - 4x^3 + x^4\end{aligned}$$

(b) $$\begin{aligned}(1+2a)^6 &= 1 + 6(2a) + 15(2a)^2 + 20(2a)^3 + 15(2a)^4 + 6(2a)^5 + (2a)^6 \\ &= 1 + 12a + 60a^2 + 160a^3 + 240a^4 + 192a^5 + 64a^6\end{aligned}$$

(c) $$\begin{aligned}(1-\tfrac{2}{3}x)^5 &= 1 + 5(-\tfrac{2}{3}x) + 10(-\tfrac{2}{3}x)^2 + 10(-\tfrac{2}{3}x)^3 + 5(-\tfrac{2}{3}x)^4 + (-\tfrac{2}{3}x)^5 \\ &= 1 - \tfrac{10}{3}x + \tfrac{40}{9}x^2 - \tfrac{80}{27}x^3 + \tfrac{80}{81}x^4 - \tfrac{32}{243}x^5\end{aligned}$$

WORKED EXERCISE:

(a) Write out the expansion of $\left(1+\dfrac{5}{x}\right)^2$, then write out the first four terms in the expansion of $(1-x)^8$.

(b) Hence find, in the expansion of $\left(1+\dfrac{5}{x}\right)^2(1-x)^8$:

(i) the term independent of x, (ii) the term in x.

SOLUTION:

(a) $$\begin{aligned}\left(1+\frac{5}{x}\right)^2 &= 1 + 10x^{-1} + 25x^{-2} \\ (1-x)^8 &= 1 - 8x + 28x^2 - 56x^3 + \cdots\end{aligned}$$

(b) Hence in the expansion of $\left(1+\dfrac{5}{x}\right)^2(1-x)^8$:

(i) $$\begin{aligned}\text{constant term} &= 1\times 1 \;+\; (10x^{-1})\times(-8x) \;+\; (25x^{-2})\times(28x^2) \\ &= 1 - 80 + 700 \\ &= 621.\end{aligned}$$

(ii) $$\begin{aligned}\text{term in } x &= 1\times(-8x) \;+\; (10x^{-1})\times(28x^2) \;+\; (25x^{-2})\times(-56x^3) \\ &= -8x + 280x - 1400x \\ &= -1128x.\end{aligned}$$

WORKED EXERCISE: By expanding the first few terms of $(1+0{\cdot}02)^8$, find an approximation of $1{\cdot}02^8$ correct to five decimal places.

SOLUTION:

$$\begin{aligned}(1+0{\cdot}02)^8 &= 1 \;+\; 8\times 0{\cdot}02 \;+\; 28\times(0{\cdot}02)^2 \;+\; 56\times(0{\cdot}02)^3 \;+\; 70\times(0{\cdot}02)^4 \;+\; \cdots \\ &= 1 + 0{\cdot}16 + 0{\cdot}0112 + 0{\cdot}000\,448 + 0{\cdot}000\,011\,20 + \cdots \\ &\doteqdot 1{\cdot}171\,66\end{aligned}$$

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WORKED EXERCISE: Find the value of k if, in the expansion of $(1 + 2kx)^6$:
(a) the terms in x^4 and x^3 have coefficients in the ratio $2 : 3$,
(b) the terms in x^2, x^3 and x^4 have coefficients in arithmetic progression.

SOLUTION:
$$\begin{aligned}(1 + 2kx)^6 &= \cdots + 15(2kx)^2 + 20(2kx)^3 + 15(2kx)^4 + \cdots\\ &= \cdots + 60k^2x^2 + 160k^3x^3 + 240k^4x^4 + \cdots\end{aligned}$$

(a) Put $\dfrac{240k^4}{160k^3} = \dfrac{2}{3}$.
Then $\frac{3}{2}k = \frac{2}{3}$
$k = \frac{4}{9}$.

(b) Put $240k^4 - 160k^3 = 160k^3 - 60k^2$.
Then $240k^4 - 320k^3 + 60k^2 = 0$
$12k^4 - 16k^3 + 3k^2 = 0$.
Either $k = 0$, or $12k^2 - 16k + 3 = 0$.
For the quadratic, $\Delta = 256 - 144 = 112 = 16 \times 7$,
so $k = 0$ or $\dfrac{16 + 4\sqrt{7}}{24}$ or $\dfrac{16 - 4\sqrt{7}}{24}$
$= 0$ or $\frac{1}{6}(4 + \sqrt{7})$ or $\frac{1}{6}(4 - \sqrt{7})$.

WORKED EXERCISE: [A harder example] Expand $(1 + x + x^2)^4$ using the Pascal triangle, by writing $1 + x + x^2 = 1 + (x + x^2)$, and writing $x + x^2 = x(1 + x)$.

SOLUTION:
$$\begin{aligned}(1 + x + x^2)^4 &= \Big(1 + \big(x(x + 1)\big)\Big)^4\\ &= 1 + 4x(1 + x) + 6x^2(1 + x)^2 + 4x^3(1 + x)^3 + x^4(1 + x)^4\\ &= 1 + 4x(1 + x) + 6x^2(1 + 2x + x^2) + 4x^3(1 + 3x + 3x^2 + x^3)\\ &\quad + x^4(1 + 4x + 6x^2 + 4x^3 + x^4)\\ &= 1 + (4x + 4x^2) + (6x^2 + 12x^3 + 6x^4) + (4x^3 + 12x^4 + 12x^5 + 4x^6)\\ &\quad + (x^4 + 4x^5 + 6x^6 + 4x^7 + x^8)\\ &= 1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8\end{aligned}$$

Exercise 5A

1. Complete all the rows of Pascal's triangle for $n = 0, 1, 2, 3, \ldots, 12$. Keep this in a prominent place for use in the rest of this chapter.

2. Using Pascal's triangle of binomial coefficients, give the expansions of each of the following:
   (a) $(1 + x)^6$
   (b) $(1 - x)^6$
   (c) $(1 + x)^9$
   (d) $(1 - x)^9$
   (e) $(1 + c)^5$
   (f) $(1 + 2y)^4$
   (g) $\left(1 + \dfrac{x}{3}\right)^7$
   (h) $(1 - 3z)^3$
   (i) $\left(1 - \dfrac{1}{x}\right)^8$
   (j) $\left(1 + \dfrac{2}{x}\right)^5$
   (k) $\left(1 + \dfrac{y}{x}\right)^5$
   (l) $\left(1 + \dfrac{3x}{y}\right)^4$

3. Continue the calculations of the expansions of $(1 + x)^n$ at the beginning of this section, expanding $(1 + x)^5$ and $(1 + x)^6$ in the same manner. Keep your work in columns, so that the addition property of the Pascal triangle is clear.

4. Find the specified term in each of the following expansions.
   (a) For $(1 + x)^{11}$: (i) find the term in x^2, (ii) find the term in x^8.

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(b) For $(1-x)^7$: (i) find the term in x^3, (ii) find the term in x^5.
(c) For $(1+2x)^6$: (i) find the term in x^4, (ii) find the term in x^5.
(d) For $\left(1-\dfrac{3}{x}\right)^4$: (i) find the term in x^{-1}, (ii) find the term in x^{-2}.

5. Sketch on one set of axes:
(a) $y=(1-x)^0$, $y=(1-x)^2$, $y=(1-x)^4$, $y=(1-x)^6$.
(b) $y=(1-x)^1$, $y=(1-x)^3$, $y=(1-x)^5$.

6. Expand $(1+x)^9$ and $(1+x)^{10}$, and show that the sum of the coefficients of the second expansion is twice the sum of the coefficients in the first expansion.

DEVELOPMENT

7. Without expanding, simplify:
(a) $1+3(x-1)+3(x-1)^2+(x-1)^3$
(b) $1-6(x+1)+15(x+1)^2-20(x+1)^3+15(x+1)^4-6(x+1)^5+(x+1)^6$

8. Find the coefficient of x^4 in the expansion of $(1-x)^4+(1-x)^5+(1-x)^6$.

9. Find integers a and b such that:
(a) $(1+\sqrt{3}\,)^5=a+b\sqrt{3}$
(b) $(1-\sqrt{5}\,)^3=a+b\sqrt{5}$
(c) $(1+3\sqrt{2}\,)^4=a+b\sqrt{2}$
(d) $(1-2\sqrt{3}\,)^6=a+b\sqrt{3}$

10. Expand and simplify:
(a) $(1+\sqrt{3}\,)^5+(1-\sqrt{3}\,)^5$
(b) $(1+\sqrt{3}\,)^5-(1-\sqrt{3}\,)^5$

11. Verify by direct expansion, and by taking out the common factor, that:
(a) $(1+x)^4-(1+x)^3=x(1+x)^3$
(b) $(1+x)^7-(1+x)^6=x(1+x)^6$

12. (a) Expand the first few terms of $(1+x)^6$, hence evaluate $1{\cdot}003^6$ to five decimal places.
(b) Similarly, expand $(1-4x)^5$, and hence evaluate $0{\cdot}96^5$ to five decimal places.
(c) Expand $(1+x)^8-(1-x)^8$, and hence evaluate $1{\cdot}002^8-0{\cdot}998^8$ to five decimal places.

13. (a) (i) Expand $(1+x)^4$ as far as the term in x^2.
(ii) Hence find the coefficient of x^2 in the expansion of $(1-5x)(1+x)^4$.
(b) (i) Expand $(1+2x)^5$ as far as the term in x^3.
(ii) Hence find the coefficient of x^3 in the expansion of $(2-3x)(1+2x)^5$.
(c) (i) Expand $(1-3x)^4$ as far as the term in x^3.
(ii) Hence find the coefficient of x^3 in the expansion of $(2+x)^2(1-3x)^4$.

14. Find the coefficient of:
(a) x^3 in $(3-4x)(1+x)^4$
(b) x in $(1+3x+x^2)(1-x)^3$
(c) x^4 in $(5-2x^3)(1+2x)^5$
(d) x^0 in $\left(1-\dfrac{x}{3}\right)^3\left(1+\dfrac{2}{x}\right)^2$
(e) x^5 in $(1+5x)^4(1-3x)^2$
(f) x^3 in $(1+3x)^3(1-x)^7$

15. Determine the value of the term independent of x in the expansion of:
(a) $(1+2x)^4\left(1-\dfrac{1}{x^2}\right)^6$
(b) $\left(1-\dfrac{x}{3}\right)^5\left(1+\dfrac{2}{x}\right)^3$

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16. (a) In the expansion of $(1+x)^6$:

(i) find the term in x^2,

(ii) find the term in x^3,

(iii) find the ratio of the term in x^2 to the term in x^3,

(iv) find the values of (i), (ii) and (iii) when $x = 3$.

(b) In the expansion of $\left(1 + \dfrac{2}{3x}\right)^7$:

(i) find the term in x^{-5},

(ii) find the term in x^{-6},

(iii) find the ratio of the term in x^{-5} to the term in x^{-6},

(iv) find the values of (i), (ii) and (iii) when $x = 2$.

17. (a) When $(1+2x)^5$ is expanded in increasing powers of x, the third and fourth terms in the expansion are equal. Find the value of x.

(b) When $(1+x)^5$, where $x \neq 0$, is expanded in increasing powers of x, the first, second and fourth terms in the expansion form a geometric sequence. Find the value of x.

(c) When $(1+x)^7$ is expanded in increasing powers of x, the fifth, sixth and seventh terms in the expansion form an arithmetic sequence. Find the value of x.

18. (a) Find the coefficients of x^4 and x^5 in the expansion of $(1+kx)^8$. Hence find k if these coefficients are in the ratio $1:4$.

(b) Find the coefficients of x^3 and x^4 in the expansion of $(1+kx)^6$. Hence find k if these coefficients are in the ratio $8:3$.

(c) Find the coefficients of x^5 and x^6 in the expansion of $(1-\frac{3}{4}kx)^9$. Hence find k if these coefficients are equal.

19. Use Pascal's triangle to help evaluate the integrals arising from the following questions.

(a) Find the area bounded by the curve $y = x(1-x)^5$ and the x-axis, where $0 \le x \le 1$.

(b) Find the area bounded by the curve $y = x^4(1-x)^4$ and the x-axis, where $0 \le x \le 1$.

(c) Find the volume of the solid formed when the region between the x-axis and the curve $y = \sqrt{x(1-x)^3}$, for $0 \le x \le 1$, is revolved around the x-axis.

20. If \$$P$ is invested at the compound interest rate R per annum for n years, and interest is compounded annually, the accumulated amount is \$$A$, where $A = P\,(1+R)^n$.

(a) Write down as decimals all terms in the expansion of $(1+0{\cdot}04)^3$.

(b) Hence find the amount to which an investment of \$1000 will grow, if it is invested for 3 years at a rate of 4% per annum, and interest is compounded annually.

21. By writing $(1+x+3x^2)^6$ as $(1+A)^6$, where $A = x+3x^2$, expand $(1+x+3x^2)^6$ as far as the term in x^3. Hence evaluate $(1{\cdot}0103)^6$ to four decimal places.

22. [Patterns in Pascal's triangle] Check the following results using the triangle you constructed in question 1. (These will not be proven until later.)

(a) The sum of the numbers in the row beginning $1, n, \ldots$ is equal to 2^n.

(b) If the second member of a row is a prime number, all the numbers in that row excluding the 1s are divisible by it.

(c) [The hockey stick pattern] Starting at any 1 on the left side of the triangle, go diagonally downwards any number of steps. Then the sum of these numbers is the number directly below the last number. For example, if you start at the 1 on the left hand side of the row 1, 3, 3, 1 and move down the diagonal 1, 4, 10, 20 the total of these numbers, namely 35, is found directly below 20.

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(d) [The powers of 11] If a row is made into a single number by using each element as a digit of the number, the number is a power of 11 (except that after the row 1, 4, 6, 4, 1, the pattern gets confused by carrying).

(e) Find the diagonal and the column containing the triangular numbers, and show that adding adjacent pairs gives the square numbers.

23. [These geometrical results should be related to the numbers in the Pascal triangle.]

(a) Place three points on the circumference of a circle. How many line segments and triangles can be formed using these three points?

(b) Place four points on the circumference of a circle. How many segments, triangles and quadrilaterals can be formed using these four points?

(c) What happens if five points are placed on the circle.

(d) How many pentagons could you form if you placed seven points on the circumference of a circle?

EXTENSION

24. [The Pascal pyramid] By considering the expansion of $(1+x+y)^n$, where $0 \le n \le 4$, calculate the first five layers of the Pascal pyramid.

5 B Further Work with the Pascal Triangle

We pass now to the more general case of the expansion of $(x+y)^n$. Because x and y are both variables, the symmetries of the expansion will be more obvious, and this section offers proofs of the addition property and the other basic patterns in the Pascal triangle.

The Pattern of the Indices in the Expansion of $(x+y)^n$: Here are the expansions of $(x+y)^n$ for low values of n. Again, the calculations have been carried out with like terms written in the same column so that the addition property is clear.

$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$\begin{aligned}(x+y)^2 &= x(x+y)+y(x+y)\\ &= x^2 + xy\\ & + xy + y^2\\ &= x^2 + 2xy + y^2\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= x(x+y)^2 + y(x+y)^2\\ &= x^3 + 2x^2y + xy^2\\ & + x^2y + 2xy^2 + y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

$$\begin{aligned}(x+y)^4 &= x(x+y)^3 + y(x+y)^3\\ &= x^4 + 3x^3y + 3x^2y^2 + xy^3\\ & + x^3y + 3x^2y^2 + 3xy^3 + y^4\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

The pattern for the indices of x and y is straightforward. The expansion of $(x+y)^3$, for example, has four terms, and in each term the indices of x and y are whole numbers adding to 3. Similarly the expansion of $(x+y)^4$ has five terms, and in each term the indices of x and y are whole numbers adding to 4. The useful phrase for this is that $(x+y)^n$ is *homogeneous of degree n in x and y together*.

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THE TERMS OF $(x+y)^n$: The expansion of $(x+y)^n$ has $n+1$ terms, and in each term the indices of x and y are whole numbers adding to n.

That is, the expression $(x+y)^n$ is *homogeneous of degree n in x and y together*, and so also is its expansion.

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The reason for this is clear, and formal proof by induction should not be necessary. In each successive expansion, the terms of the previous expansion are multiplied first by x and then by y, so the sum of the indices goes up by 1. We now know that in general

$$(x+y)^n = *x^n \;+\; *x^{n-1}y \;+\; *x^{n-2}y^2 \;+\; \cdots \;+\; *x^2y^{n-2} \;+\; *xy^{n-1} \;+\; *y^n,$$

where $*$ denotes the different coefficients.

A Symbol for the Coefficients: The coefficients here are, of course, the same as in the previous section, as can be seen by replacing x and y by 1 and x in the expansion of $(x+y)^n$, and we will first deal with the special case of the expansion of $(1+x)^n$. To investigate these coefficients further, we take the approach of giving names to the things we want to study.

3 THE DEFINITION OF ${}^n\mathrm{C}_r$: Define the number ${}^n\mathrm{C}_r$ to be the coefficient of x^r in the expansion of $(1+x)^n$.

The symbol is usually read as 'n choose r', and the notations ${}^n\mathrm{C}_r$ and $\binom{n}{r}$ are both used for these coefficients.

This definition will need some thought. Defining a number as a coefficient in an expansion is standard practice in mathematics, but it seems very strange the first time it is encountered. We can now write out the expansion of $(1+x)^n$.

4 THE EXPANSION OF $(1+x)^n$: Using the notation ${}^n\mathrm{C}_r$ for the coefficients,

$$(1+x)^n = {}^n\mathrm{C}_0 \;+\; {}^n\mathrm{C}_1\,x \;+\; {}^n\mathrm{C}_2\,x^2 \;+\; \cdots \;+\; {}^n\mathrm{C}_n\,x^n.$$

There are $n+1$ terms, and the general term of the expansion is

$$\text{term in } x^r = {}^n\mathrm{C}_r\,x^r.$$

Alternatively, using sigma notation, the expansion can be written as

$$(1+x)^n = \sum_{r=0}^{n} {}^n\mathrm{C}_r\,x^r.$$

WORKED EXERCISE:

(a) Write out the expansion of $(1+x)^2$ and $(1+x)^3$ using ${}^n\mathrm{C}_r$ notation.

(b) Hence give the values of ${}^2\mathrm{C}_0$, ${}^2\mathrm{C}_1$, ${}^2\mathrm{C}_2$ and of ${}^3\mathrm{C}_0$, ${}^3\mathrm{C}_1$, ${}^3\mathrm{C}_2$, ${}^3\mathrm{C}_3$.

SOLUTION:

(a) $(1+x)^2 = {}^2\mathrm{C}_0 + {}^2\mathrm{C}_1\,x + {}^2\mathrm{C}_2\,x^2$

and $(1+x)^3 = {}^3\mathrm{C}_0 + {}^3\mathrm{C}_1\,x + {}^3\mathrm{C}_2\,x^2 + {}^3\mathrm{C}_3\,x^3$

(b) But $(1+x)^2 = 1+2x+x^2$,

so ${}^2\mathrm{C}_0 = 1$, ${}^2\mathrm{C}_1 = 2$ and ${}^2\mathrm{C}_2 = 1$.

Also $(1+x)^3 = 1+3x+3x^2+x^3$,

so ${}^3\mathrm{C}_0 = 1$, ${}^3\mathrm{C}_1 = 3$, ${}^3\mathrm{C}_2 = 3$ and ${}^3\mathrm{C}_3 = 1$.

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The Expansion of $(x + y)^n$: The coefficients in the expansion of $(x + y)^n$ are the same numbers ${}^n\mathrm{C}_r$ as in the expansion of $(1 + x)^n$. Thus we can now write out the expansion of $(x + y)^n$ as well.

5 THE EXPANSION OF $(x + y)^n$: Using the ${}^n\mathrm{C}_r$ notation,

$$(x+y)^n = {}^n\mathrm{C}_0\, x^n \ + \ {}^n\mathrm{C}_1\, x^{n-1}y \ + \ {}^n\mathrm{C}_2\, x^{n-2}y^2 \ + \ \cdots \ + \ {}^n\mathrm{C}_n\, y^n.$$

There are $n + 1$ terms, and the general term of the expansion is

$$\text{term in } x^{n-r}y^r = {}^n\mathrm{C}_r\, x^{n-r}y^r.$$

Alternatively, using sigma notation, the expansion can be written as

$$(x+y)^n = \sum_{r=0}^{n} {}^n\mathrm{C}_r\, x^{n-r}y^r.$$

The Pascal Triangle: The Pascal triangle of the previous section now becomes the table of values of the function ${}^n\mathrm{C}_r$, with the rows indexed by n and the columns by r:

${}^n\mathrm{C}_r$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	$\boxed{4}$	$\boxed{6}$	4	1				
5	1	5	$\boxed{10}$	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

The boxed numbers provide another example of the addition property of the Pascal triangle, and will be discussed further below.

Using the General Expansion: The general expansion of $(x+y)^n$ is applied in the same way as the expansion of $(1 + x)^n$.

WORKED EXERCISE: Use the Pascal triangle to write out the expansions of:

(a) $(2 - 3x)^4$ (b) $(5x + \frac{1}{5}a)^5$

SOLUTION:

(a)
$$\begin{aligned}(2-3x)^4 &= 2^4 \ + \ 4\times 2^3\times(-3x) \ + \ 6\times 2^2\times(-3x)^2 \\ &\qquad + \ 4\times 2\times(-3x)^3 \ + \ (-3x)^4 \\ &= 16 - 96x + 72x^2 - 216x^3 + 81x^4\end{aligned}$$

(b)
$$\begin{aligned}(5x+\tfrac{1}{5}a)^5 &= (5x)^5 \ + \ 5\times(5x)^4\times\tfrac{1}{5}a \ + \ 10\times(5x)^3\times(\tfrac{1}{5}a)^2 \\ &\qquad + \ 10\times(5x)^2\times(\tfrac{1}{5}a)^3 \ + \ 5\times(5x)\times(\tfrac{1}{5}a)^4 + (\tfrac{1}{5}a)^5 \\ &= 3125x^5 + 625ax^4 + 50a^2x^3 + 2a^3x^2 + \tfrac{1}{25}a^4x + \tfrac{1}{3125}a^5\end{aligned}$$

WORKED EXERCISE: Use the Pascal triangle to write out the expansion of $(2x + x^{-2})^6$, leaving the terms unsimplified. Hence find:

(a) the term independent of x, (b) the term in x^{-3}.

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SOLUTION: $(2x + x^{-2})^6 = (2x)^6 + 6 \times (2x)^5 \times (x^{-2}) + 15 \times (2x)^4 \times (x^{-2})^2$
$+ 20 \times (2x)^3 \times (x^{-2})^3 + 15 \times (2x)^2 \times (x^{-2})^4 + 6 \times (2x) \times (x^{-2})^5 + (x^{-2})^6$

(a) Constant term
$= 15 \times (2x)^4 \times (x^{-2})^2$
$= 15 \times 2^4 \times x^4 \times x^{-4}$
$= 240.$

(b) Term in x^{-3}
$= 20 \times (2x)^3 \times (x^{-2})^3$
$= 20 \times 2^3 \times x^3 \times x^{-6}$
$= 160.$

WORKED EXERCISE: Expand $(2 - 3x)^7$ as far as the term in x^2, and hence find the term in x^2 in the expansion of $(5 + x)(2 - 3x)^7$.

SOLUTION: $(2 - 3x)^7 = 2^7 - 7 \times 2^6 \times (3x) + 21 \times 2^5 \times (3x)^2 - \cdots$
$= 128 - 1344\,x + 6048\,x^2 - \cdots .$

Hence the term in x^2 in the expansion of $(5 + x)(2 - 3x)^7$
$= 5 \times 6048\,x^2 - x \times 1344\,x$
$= 28\,896\,x^2.$

Proofs of the First Three Basic Properties: The first three basic properties of the Pascal triangle can now be expressed in ${}^n\mathrm{C}_r$ notation and proven straightforwardly.

6

BASIC PROPERTIES OF THE PASCAL TRIANGLE:

1. Each row starts and ends with 1, that is,
$${}^n\mathrm{C}_0 = {}^n\mathrm{C}_n = 1, \quad \text{for all cardinals } n.$$
2. Each row is reversible, that is,
$${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}, \quad \text{for all cardinals } n \text{ and } r \text{ with } r \le n.$$
3. The sum of each row is 2^n, that is,
$${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n = 2^n, \quad \text{for all cardinals } n.$$

PROOF: Each proof begins with the general expansion
$$(x + y)^n = {}^n\mathrm{C}_0\, x^n + {}^n\mathrm{C}_1\, x^{n-1}y + {}^n\mathrm{C}_2\, x^{n-2}y^2 + \cdots + {}^n\mathrm{C}_n\, y^n.$$

Parts 1 and 3 then proceed by substitution, and part 2 by equating coefficients. These methods are both commonly required for solving problems, and they should be studied carefully.

1. Substituting $x = 1$ and $y = 0$, $(1 + 0)^n = {}^n\mathrm{C}_0 + 0 + \cdots + 0$,
and so as required, $1 = {}^n\mathrm{C}_0$.
Substituting $x = 0$ and $y = 1$, $(0 + 1)^n = 0 + 0 + \cdots + 0 + {}^n\mathrm{C}_n$
and so as required, $1 = {}^n\mathrm{C}_n$.

2. We know that $(x + y)^n = (y + x)^n$.
Now $(x + y)^n = {}^n\mathrm{C}_0\, x^n + {}^n\mathrm{C}_1\, x^{n-1}y + \cdots + {}^n\mathrm{C}_{n-1}\, xy^{n-1} + {}^n\mathrm{C}_n\, y^n$,
and $(y + x)^n = {}^n\mathrm{C}_0\, y^n + {}^n\mathrm{C}_1\, y^{n-1}x + \cdots + {}^n\mathrm{C}_{n-1}\, yx^{n-1} + {}^n\mathrm{C}_n\, x^n$.
Equating coefficients of like terms in the two expansions,
$${}^n\mathrm{C}_0 = {}^n\mathrm{C}_n, \quad {}^n\mathrm{C}_1 = {}^n\mathrm{C}_{n-1}, \quad {}^n\mathrm{C}_2 = {}^n\mathrm{C}_{n-2}, \quad \ldots, \quad {}^n\mathrm{C}_n = {}^n\mathrm{C}_0,$$
and in general ${}^n\mathrm{C}_{n-r} = {}^n\mathrm{C}_r$, for $r = 0, 1, 2, \ldots, n$.

3. Substituting $x = 1$ and $y = 1$, $(1 + 1)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n$,
and so as required, $2^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n$.

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Proof of the Addition Property of the Pascal Triangle: The addition property also needs to be restated in ${}^n\mathrm{C}_r$ notation. In words, it says that every number in the triangle is the sum of the number directly above it, and the number above and to the left of it (apart from the first and the last numbers of each row). The boxed numbers in the Pascal triangle above provide an example of this — they show that

$${}^5\mathrm{C}_2 = {}^4\mathrm{C}_2 + {}^4\mathrm{C}_1 \quad \text{(that is, } 10 = 6 + 4).$$

The general statement, in symbolic form, is therefore:

7 THE ADDITION PROPERTY: If n and r are positive integers with $r \le n$, then

$${}^{n+1}\mathrm{C}_r = {}^n\mathrm{C}_r + {}^n\mathrm{C}_{r-1}, \quad \text{for } 1 \le r \le n.$$

PROOF: The expansions at the start of Sections 5A and 5B were written so that the columns aligned to make the addition property obvious. A formal proof will require examination of the coefficients in the expansion of $(1+x)^{n+1}$. We begin by noticing that

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^n\\ &= (1+x)^n + x(1+x)^n.\end{aligned}$$

On the LHS, the general term in x^r is ${}^{n+1}\mathrm{C}_r\, x^r$.
On the RHS, the term in x^r in the first expression is ${}^n\mathrm{C}_r\, x^r$,
and the term in x^r in the second expression is $x \times {}^n\mathrm{C}_{r-1}\, x^{r-1} = {}^n\mathrm{C}_{r-1}\, x^r$,
so the general term in x^r on the RHS is the sum $({}^n\mathrm{C}_r + {}^n\mathrm{C}_{r-1})x^r$.
Equating coefficients of these two terms proves the result.

Exercise 5B

NOTE: Questions 3 and 4 should be omitted by those wanting to delay the introduction of ${}^n\mathrm{C}_r$ notation until Section 5D.

1. Use Pascal's triangle to expand each of the following:
(a) $(x+y)^4$ (b) $(x-y)^4$ (c) $(r-s)^6$ (d) $(p+q)^{10}$ (e) $(a-b)^9$ (f) $(2x+y)^5$ (g) $(p-2q)^7$ (h) $(3x+2y)^4$ (i) $(a-\frac{1}{2}b)^3$ (j) $(\frac{1}{2}r+\frac{1}{3}s)^5$ (k) $\left(x+\dfrac{1}{x}\right)^6$

2. Use Pascal's triangle to expand each of the following:
(a) $(1+x^2)^4$ (b) $(1-3x^2)^3$ (c) $(x^2+2y^3)^6$ (d) $\left(x-\dfrac{1}{x}\right)^9$ (e) $(\sqrt{x}+\sqrt{y}\,)^7$ (f) $\left(\dfrac{2}{x}+3x^2\right)^5$

3. (a) Expand $(1+x)^4$, and hence write down the values of ${}^4\mathrm{C}_0$, ${}^4\mathrm{C}_1$, ${}^4\mathrm{C}_2$, ${}^4\mathrm{C}_3$ and ${}^4\mathrm{C}_4$.
[NOTE: ${}^n\mathrm{C}_r$ is defined to be the coefficient of x^r in the expansion of $(1+x)^n$.]
(b) Hence find: (i) ${}^4\mathrm{C}_0 + {}^4\mathrm{C}_1 + {}^4\mathrm{C}_2 + {}^4\mathrm{C}_3 + {}^4\mathrm{C}_4$ (ii) ${}^4\mathrm{C}_0 - {}^4\mathrm{C}_1 + {}^4\mathrm{C}_2 - {}^4\mathrm{C}_3 + {}^4\mathrm{C}_4$

4. Use the values of ${}^n\mathrm{C}_r$ from the Pascal triangle in the notes above to find:
(a) ${}^6\mathrm{C}_0 + {}^6\mathrm{C}_2 + {}^6\mathrm{C}_4 + {}^6\mathrm{C}_6$
(b) ${}^6\mathrm{C}_1 + {}^6\mathrm{C}_3 + {}^6\mathrm{C}_5$
(c) ${}^2\mathrm{C}_2 + {}^3\mathrm{C}_2 + {}^4\mathrm{C}_2 + {}^5\mathrm{C}_2$
(d) $({}^5\mathrm{C}_0)^2 + ({}^5\mathrm{C}_1)^2 + ({}^5\mathrm{C}_2)^2 + ({}^5\mathrm{C}_3)^2 + ({}^5\mathrm{C}_4)^2 + ({}^5\mathrm{C}_5)^2$

5. Simplify the following without expanding the brackets:
(a) $y^5 + 5y^4(x-y) + 10y^3(x-y)^2 + 10y^2(x-y)^3 + 5y(x-y)^4 + (x-y)^5$
(b) $a^4 - 4a^3(a-b) + 6a^2(a-b)^2 - 4a(a-b)^3 + (a-b)^4$

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(c) $x^3 + 3x^2(2y - x) + 3x(2y - x)^2 + (2y - x)^3$

(d) $(x+y)^6 - 6(x+y)^5(x-y) + 15(x+y)^4(x-y)^2 - 20(x+y)^3(x-y)^3$
$+ 15(x+y)^2(x-y)^4 - 6(x+y)(x-y)^5 + (x-y)^6$

6. (a) (i) Expand $(4+x)^5$ as far as the term in x^3.
(ii) Hence find the coefficient of x^3 in the expansion of $(3-x)(4+x)^5$.
(b) (i) Expand $(1-2x)^6$ as far as the term in x^4.
(ii) Hence find the coefficient of x^4 in the expansion of $(1-3x)(1-2x)^6$.
(c) (i) Expand $(3-y)^7$ as far as the term in y^4.
(ii) Hence find the coefficient of y^4 in the expansion of $(1-y)^2(3-y)^7$.

DEVELOPMENT

7. (a) Expand and simplify $(x+y)^6 + (x-y)^6$.
(b) Hence (and without a calculator) prove that $5^6 + 5^5 \times 3^3 + 5^3 \times 3^5 + 3^6 = 2^5(2^{12}+1)$.

8. Find the coefficient of:
(a) x^3 in $(2-5x)(x^2-3)^4$
(b) x^5 in $(x^2 - 3x + 11)(4 + x^3)^3$
(c) x^0 in $(3-2x)^2\left(x + \dfrac{2}{x}\right)^5$
(d) x^9 in $(x+2)^3(x-2)^7$

9. (a) (i) Use Pascal's triangle to expand $(x+h)^3$.
(ii) If $f(x) = x^3$, simplify $f(x+h) - f(x)$.
(iii) Hence use the definition $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ to differentiate x^3.
(b) Similarly, differentiate x^5 from first principles.

10. (a) Show that $(3+\sqrt{5}\,)^6 + (3-\sqrt{5}\,)^6 = 20\,608$.
(b) Show that $(2+\sqrt{7}\,)^4 + (2-\sqrt{7}\,)^4$ is rational.
(c) Simplify $(5+\sqrt{2}\,)^5 - (5-\sqrt{2}\,)^5$.
(d) If $(\sqrt{6}+\sqrt{3}\,)^3 - (\sqrt{6}-\sqrt{3}\,)^3 = a\sqrt{3}$, where a is an integer, find the value of a.

11. (a) Show that $\dfrac{1}{(\sqrt{3}-1)^4} + \dfrac{1}{(\sqrt{3}+1)^4} = \dfrac{(\sqrt{3}+1)^4 + (\sqrt{3}-1)^4}{(\sqrt{3}-1)^4(\sqrt{3}+1)^4}$ by putting the LHS over a common denominator. Then simplify the expression using Pascal's triangle.
(b) Similarly, simplify $\dfrac{1}{(\sqrt{7}-\sqrt{5}\,)^5} + \dfrac{1}{(\sqrt{7}+\sqrt{5}\,)^5}$.

12. By starting with $((x+y)+z)^3$, expand $(x+y+z)^3$.

13. Expand $(x+2y)^5$ and hence evaluate: (a) $(1{\cdot}02)^5$ correct to to five decimal places,
(b) $(0{\cdot}98)^5$ correct to to five decimal places, (c) $(2{\cdot}2)^5$ correct to four significant figures.

14. (a) Expand: (i) $\left(x+\dfrac{1}{x}\right)^3$ (ii) $\left(x+\dfrac{1}{x}\right)^5$ (iii) $\left(x+\dfrac{1}{x}\right)^7$
(b) Hence, if $x + \dfrac{1}{x} = 2$, evaluate: (i) $x^3 + \dfrac{1}{x^3}$ (ii) $x^5 + \dfrac{1}{x^5}$ (iii) $x^7 + \dfrac{1}{x^7}$

15. Find the coefficients of x and x^{-3} in the expansion of $\left(3x - \dfrac{a}{x}\right)^5$. Hence find the values of a if these coefficients are in the ratio $2:1$.

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16. The coefficients of the terms in a^3 and a^{-3} in the expansion of $\left(ma + \frac{n}{a^2}\right)^6$ are equal, where m and n are nonzero real numbers. Prove that $m^2 : n^2 = 10 : 3$.

17. (a) Expand $\left(x + \frac{1}{x}\right)^6$. (b) If $U = x + \frac{1}{x}$, express $x^6 + \frac{1}{x^6}$ in the form $U^6 + AU^4 + BU^2 + C$. State the values of A, B and C.

EXTENSION

18. Find the term independent of x in the expansion of $(x + 1 + x^{-1})^4$.

19. [The Sierpinski triangle fractal]

(a) Draw an equilateral triangle of side length 1 unit on a piece of white paper. Join the midpoints of the sides of this triangle to form a smaller triangle. Colour it black. Repeat this process on all white triangles that remain. What do you notice?

(b) Draw up Pascal's triangle in the shape of an equilateral triangle, then colour all the even numbers black and leave the odd numbers white. What do you notice? This pattern will be more evident if you take at least the first 16 rows — perhaps use a computer program to generate 100 rows of Pascal's triangle.

5 C Factorial Notation

So far, we have been using the Pascal triangle to supply the binomial coefficients. There is a formula for ${}^n\mathrm{C}_r$, but it involves taking products like

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040.$$

The notation $7!$, read as 'seven factorial', used here for this product is new. This section will develop familiarity with the notation and some algorithms for handling it, in preparation for the formula for ${}^n\mathrm{C}_r$ in the next section.

The Definition of Factorials: The number 'n factorial', written as $n!$, is the product of all the positive integers from n down to 1:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

But it is better to define $n!$ recursively. This means first defining $0!$, and then saying exactly how to proceed from $(n-1)!$ to $n!$:

$$\begin{cases} 0! = 1, \\ n! = n \times (n-1)!, \quad \text{for } n \geq 1. \end{cases}$$

This form of the definition gives more insight into how to manipulate factorial notation, and it also avoids the dots ... in the first definition.

8

TWO DEFINITIONS OF $n!$ (CALLED n FACTORIAL):

1. For each cardinal n, define $n!$ to be the product of all positive integers from n down to 1:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

2. Define the function $n!$ recursively by

$$\begin{cases} 0! = 1, \\ n! = n \times (n-1)!, \quad \text{for } n \geq 1. \end{cases}$$

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The fact that $0! = 1$ requires some thought. An empty product is 1, because if nothing has yet been multiplied, the register has been set back to 1. In a similar way, an empty sum is 0, because if nothing has yet been added, the register has been set back to 0. In any case, $0!$ is defined to be equal to 1. So using the recursive definition,

$$\begin{array}{lll} 0! = 1 & 4! = 4 \times 3! = 24 & 8! = 8 \times 7! = 40\,320 \\ 1! = 1 \times 0! = 1 & 5! = 5 \times 4! = 120 & 9! = 9 \times 8! = 362\,880 \\ 2! = 2 \times 1! = 2 & 6! = 6 \times 5! = 720 & 10! = 10 \times 9! = 3\,628\,800 \\ 3! = 3 \times 2! = 6 & 7! = 7 \times 6! = 5040 & 11! = 11 \times 10! = 39\,916\,800 \end{array}$$

and so on, increasing very quickly indeed. Calculators have a factorial button labelled $\boxed{x!}$ or $\boxed{n!}$. Use it straight away to convince yourself that at least the *calculator* believes that $0! = 1$. Notice also the error message if n is not a cardinal number — the domain of the function $n!$ is $\mathbf{N} = \{\,0,\ 1,\ 2,\ \ldots\,\}$.

Unrolling Factorials: The recursive definition of $n!$ given above is very useful in calculations. Successive applications of the definition can be thought of as *unrolling* the factorial further and further:

$$\begin{aligned} 8! &= 8 \times 7! \quad \text{(unrolling once)} \\ &= 8 \times 7 \times 6! \quad \text{(unrolling twice)} \\ &= 8 \times 7 \times 6 \times 5! \quad \text{(unrolling three times)} \end{aligned}$$

and so on. This idea is vital when there are fractions involved.

WORKED EXERCISE: Simplify the following using unrolling techniques:

(a) $\dfrac{10!}{7!}$ (b) $\dfrac{(n+2)!}{(n-1)!}$ (c) $\dfrac{n!}{(n-r)!}$

SOLUTION:

(a) $$\begin{aligned} \frac{10!}{7!} &= \frac{10 \times 9 \times 8 \times 7!}{7!} \\ &= 10 \times 9 \times 8 \\ &= 720 \end{aligned}$$

(b) $$\begin{aligned} \frac{(n+2)!}{(n-1)!} &= \frac{(n+2)(n+1)n(n-1)!}{(n-1)!} \\ &= (n+2)(n+1)n \end{aligned}$$

(c) $$\begin{aligned} \frac{n!}{(n-r)!} &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} \\ &= \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r\ \text{factors}} \end{aligned}$$

A Lemma to be Used Later: A *lemma* is a theorem, usually of a technical nature, whose principal purpose is to assist in the proof of a later theorem. The following lemma will be used in the proof of the binomial theorem in the next section. Its proof is not easy, but it is an excellent example of the unrolling technique.

9 A LEMMA ABOUT FACTORIALS: Let n and r be cardinal numbers, with $1 \le r \le n$.

Then $$\frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!} = \frac{(n+1)!}{(n-r+1)!\,r!}\,.$$

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PROOF: A common denominator for the two fractions is required:

$$\begin{aligned}
\text{LHS} &= \frac{n!}{(n-r)! \times r!} + \frac{n!}{(n-r+1)! \times (r-1)!} \\
&= \frac{n!}{(n-r)! \times r \times (r-1)!} + \frac{n!}{(n-r+1) \times (n-r)! \times (r-1)!} \\
&= \frac{(n-r+1) \times n! \; + \; r \times n!}{r \;\times\; (r-1)! \;\times\; (n-r+1) \;\times\; (n-r)!} \\
&= \frac{\big((n-r+1) \; + \; r\big) \times n!}{r! \;\times\; (n-r+1)!} \\
&= \frac{(n+1) \times n!}{r! \times (n-r+1)!} \\
&= \frac{(n+1)!}{r! \times (n-r+1)!} \\
&= \text{RHS}.
\end{aligned}$$

Exercise 5C

1. Evaluate the following:

(a) $3!$ (b) $7!$ (c) $10!$ (d) $1!$ (e) $0!$

(f) $\dfrac{9!}{4!}$ (g) $\dfrac{15!}{14!}$ (h) $\dfrac{8!}{3!}$

(i) $\dfrac{10!}{8! \times 2!}$ (j) $\dfrac{12!}{3! \times 9!}$ (k) $\dfrac{8!}{4! \times 4!}$

(l) $\dfrac{10!}{5! \times 3! \times 2!}$ (m) $\dfrac{15!}{3! \times 5! \times 9!}$ (n) $\dfrac{12!}{2! \times 3! \times 4! \times 5!}$

2. If $f(x) = x^6$, find:

(a) $f'(x)$ (b) $f''(x)$ (c) $f'''(x)$ (d) $f''''(x)$ (e) $f^{(5)}(x)$ (f) $f^{(6)}(x)$ (g) $f^{(7)}(x)$

3. Simplify by unrolling factorials appropriately:

(a) $\dfrac{n!}{(n-1)!}$ (b) $n \times (n-1)!$

(c) $\dfrac{n(n-1)!}{n!}$ (d) $\dfrac{(n+1)!}{(n-1)!}$

(e) $\dfrac{(n+2)!}{n!}$ (f) $\dfrac{(n-2)!}{n!}$

(g) $\dfrac{(n-2)!\,(n-1)!}{n!\,(n-3)!}$ (h) $\dfrac{n!\,(n-1)!}{(n+1)!}$

4. Simplify by taking out a common factor:

(a) $8! \; - \; 7!$ (b) $(n+1)! \; - \; n!$

(c) $8! \; + \; 6!$ (d) $(n+1)! \; + \; (n-1)!$

(e) $9! \; + \; 8! \; + \; 7!$ (f) $(n+1)! \; + \; n! \; + \; (n-1)!$

DEVELOPMENT

5. Write each expression as a single fraction:

(a) $\dfrac{1}{n!} + \dfrac{1}{(n-1)!}$ (b) $\dfrac{1}{n!} - \dfrac{1}{(n+1)!}$ (c) $\dfrac{1}{(n+1)!} - \dfrac{1}{(n-1)!}$

6. (a) If $f(x) = x^n$, find: (i) $f'(x)$ (ii) $f''(x)$ (iii) $f^{(n)}(x)$ (iv) $f^{(k)}(x)$, where $k \le n$.

(b) If $f(x) = \dfrac{1}{x}$, find: (i) $f'(x)$ (ii) $f''(x)$ (iii) $f^{(5)}(x)$ (iv) $f^{(n)}(x)$.

7. (a) Show that $k \times k! = (k+1)! - k!$

(b) Hence by considering each individual term as a difference of two terms, sum the series $1 \times 1! \; + \; 2 \times 2! \; + \; 3 \times 3! \; + \; \cdots \; + \; n \times n!$

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8. Prove that $\dfrac{n!}{r!\,(n-r)!} + \dfrac{n!}{(r-1)!\,(n-r+1)!} = \dfrac{(n+1)!}{r!\,(n-r+1)!}$.

9. (a) Find what power of: (i) 2, (ii) 10, is a divisor of 10!

(b) Find what power of: (i) 2, (ii) 5, (iii) 7, (iv) 13, is a divisor of 100!

10. [A relationship between higher derivatives of polynomials and factorials]

(a) If $f(x) = 11x^3 + 7x^2 + 5x + 3$, show that:

(i) $f(0) = 3 \times 0!$ (ii) $f'(0) = 5 \times 1!$ (iii) $f''(0) = 7 \times 2!$

(iv) $f'''(0) = 11 \times 3!$ (v) $f^{(k)}(0) = 0$, for all $k > 4$.

Hence explain why $f(x)$ can be written $f(x) = \dfrac{f(0)}{0!} + \dfrac{f'(0)\,x}{1!} + \dfrac{f''(0)\,x^2}{2!} + \dfrac{f'''(0)\,x^3}{3!}$.

(b) Show that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is any polynomial, then

$$f^{(k)}(0) = \begin{cases} a_k\, k!\,, & \text{for } k = 0, 1, 2, \ldots, n, \\ 0, & \text{for } k > n, \end{cases}$$

and hence explain why $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0) \times x^k}{k!}$.

11. (a) Evaluate $\dfrac{k}{(k+1)!}$, for $k = 1, 2, 3, 4$ and 5.

(b) Evaluate $\displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!}$, for $n = 1, 2, 3, 4$ and 5.

(c) Make a reasonable guess about the value of $\displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!}$, and prove this result by mathematical induction. Hence find $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{(k+1)!}$.

(d) Prove that $\dfrac{k}{(k+1)!} = \dfrac{1}{k!} - \dfrac{1}{(k+1)!}$, and hence produce an alternative proof of part (c) using a collapsing sequence.

EXTENSION

12. Express using factorial notation:

(a) $30 \times 28 \times 26 \times \cdots \times 2$ (b) $29 \times 27 \times 25 \times \cdots \times 1$ (c) $\dfrac{30 \times 28 \times 26 \times \cdots \times 2}{29 \times 27 \times 25 \times \cdots \times 1}$

13. [Maclaurin series] The infinite power series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0) \times x^k}{k!}$ at the end of question 10(b) can also be generated by a function $f(x)$ whose higher derivatives do not eventually vanish. The resulting series is called the *Maclaurin series* of $f(x)$, and for most straightforward functions, if the Maclaurin series does converge, it converges to the original function $f(x)$.

(a) (i) Find the Maclaurin series expansion of $f(x) = \dfrac{1}{1-x}$.

(ii) For what values of x does this series converge, and what is its limit?

(b) Find the Maclaurin series for $f(x) = \log(1-x)$. Refer to the last question in Exercise 12D of the Year 11 volume for a discussion of the convergence of this series.

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(c) (i) Find the Maclaurin series for $f(x) = \sin x$. Refer to the last question in Exercise 14I of the Year 11 volume to see why this series always converges to $\sin x$.
(ii) Find the Maclaurin series for $f(x) = e^x$. Refer to the last two questions in Exercise 13C of the Year 11 volume to see why this series always converges to e^x.
(iii) Hence find the first four nonzero terms of the Maclaurin series for $e^x \sin x$.

14. [Stirling's formula] The following formula is too difficult to prove at this stage (see question 24 of Exercise 6F for a preliminary lemma), but it is most important because it provides a continuous function that approximates $n!$ for integer values of n:

$$n! \doteqdot \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}, \quad \text{in the sense that the percentage error} \to 0 \text{ as } n \to \infty.$$

Show that the formula has an error of approximately 2·73% for 3! and 0·83% for 10! Find the percentage error for 60!

5 D The Binomial Theorem

There is a rather straightforward formula for the coefficients ${}^n\mathrm{C}_r$. It can be discovered by looking along a typical line of the Pascal triangle to see how each entry can be calculated from the entry to the left.

An Investigation for a Formula for ${}^7\mathrm{C}_r$: Here is the line corresponding to $n = 7$:

${}^7\mathrm{C}_r$	0	1	2	3	4	5	6	7
7	1	7	21	35	35	21	7	1

And here is how to work along the line:

The first entry in the line is 1.
For the entry 7, multiply by $7 = \frac{7}{1}$.
For the entry 21, multiply by $3 = \frac{6}{2}$.
For the entry 35, multiply by $\frac{5}{3}$.
For the entry 35, multiply by $1 = \frac{4}{4}$.
For the entry 21, multiply by $\frac{3}{5}$.
For the entry 7, multiply by $\frac{1}{3} = \frac{2}{6}$.
For the final entry 1, multiply by $\frac{1}{7}$.

Now we can write each entry ${}^7\mathrm{C}_r$ as a product of fractions:

$${}^7\mathrm{C}_0 = 1$$

$${}^7\mathrm{C}_1 = \frac{7}{1} = 7$$

$${}^7\mathrm{C}_2 = \frac{7 \times 6}{1 \times 2} = 21$$

$${}^7\mathrm{C}_3 = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$$

$${}^7\mathrm{C}_4 = \frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4} = 35$$

$${}^7\mathrm{C}_5 = \frac{7 \times 6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4 \times 5} = 21$$

$${}^7\mathrm{C}_6 = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 7$$

$${}^7\mathrm{C}_7 = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 1$$

Statement of the Binomial Theorem: By now one should be reasonably convinced that the general formula for ${}^n\mathrm{C}_r$ is

$${}^n\mathrm{C}_r = \frac{\overbrace{n \times (n-1) \times (n-2) \times (n-3) \times \cdots}^{r \text{ factors}}}{\underbrace{1 \times 2 \times 3 \times 4 \times \cdots}_{r \text{ factors}}}.$$

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This is not yet very elegant. The denominator is $1 \times 2 \times 3 \times 4 \times \cdots \times r = r!$ The numerator is $n \times (n-1) \times \cdots \times (n-r+1)$, which is $\dfrac{n!}{(n-r)!}$, by the unrolling procedures. This gives a very concise formulation of the binomial theorem.

10 **THE BINOMIAL THEOREM:** For all cardinal numbers n,

$$ {}^n\mathrm{C}_r = \frac{n!}{r!\,(n-r)!}\,, \quad \text{for } r = 0,\ 1,\ \ldots,\ n. $$

Alternatively, $\displaystyle {}^n\mathrm{C}_r = \frac{n \times (n-1) \times \cdots \times (n-r+1)}{1 \times 2 \times \cdots \times r}$.

The difficult proof will be given at the end of this section. See the Extension section of Exercise 5E for an alternative and easier proof using differentiation. A further interesting proof by combinatoric methods will be given in Chapter Ten. Notice that the formula for ${}^n\mathrm{C}_r$ remains unchanged when r is replace by $n-r$:

$$ {}^n\mathrm{C}_{n-r} = \frac{n!}{(n-r)!\,\big(n-(n-r)\big)!} = \frac{n!}{(n-r)!\,r!} = {}^n\mathrm{C}_r\,, $$

confirming the symmetry of each row in the Pascal triangle, as proven in Section 5B.

Scientific calculators have a button labelled $\boxed{{}^n\mathrm{C}_r}$ which will find values of ${}^n\mathrm{C}_r$. For low values of n and r, the answers are exact, but for high values they are only approximations.

Examples of the Binomial Theorem: Here are some worked examples using the formula to calculate ${}^n\mathrm{C}_r$ for some values of n and r.

WORKED EXERCISE: Evaluate, using the binomial theorem: (a) ${}^8\mathrm{C}_5$ (b) ${}^n\mathrm{C}_3$

SOLUTION:

(a) $$ \begin{aligned} {}^8\mathrm{C}_5 &= \frac{8!}{3! \times 5!} \\ &= \frac{8 \times 7 \times 6 \times \cancel{5!}}{3 \times 2 \times 1 \times \cancel{5!}} \\ &= 56 \end{aligned} $$

(b) $$ \begin{aligned} {}^n\mathrm{C}_3 &= \frac{n(n-1)(n-2)(n-3)!}{(n-3)! \times 3!} \\ &= \frac{n(n-1)(n-2)}{6} \end{aligned} $$

WORKED EXERCISE: Find ${}^{16}\mathrm{C}_5$, leaving your answer factored into primes.

SOLUTION:

$$ \begin{aligned} {}^{16}\mathrm{C}_5 &= \frac{16 \times 15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4 \times 5} \\ &= 2 \times 14 \times 13 \times 12 \\ &= 2^4 \times 3 \times 7 \times 13 \quad \text{(Check this on the calculator.)} \end{aligned} $$

WORKED EXERCISE:

(a) Find the general term in the expansion of $(2x^2 - x^{-1})^{20}$.

(b) (i) Find the term in x^{34}. (ii) Find the term in x^{-5}.

Give each coefficient as a numeral, and factored into primes.

SOLUTION:

(a) $$ \begin{aligned} \text{General term } &= {}^{20}\mathrm{C}_r \times (2x^2)^{20-r} \times (-x^{-1})^r \\ &= {}^{20}\mathrm{C}_r \times 2^{20-r} \times x^{40-2r} \times (-1)^r \times x^{-r} \\ &= {}^{20}\mathrm{C}_r \times 2^{20-r} \times (-1)^r \times x^{40-3r}. \end{aligned} $$

(b) (i) To obtain the term in x^{34}, $40 - 3r = 34$

$$r = 2.$$

Hence the term in
$$\begin{aligned} x^{34} &= {}^{20}\mathrm{C}_2 \times (2x^2)^{20-2} \times (-x^{-1})^2 \\ &= \frac{20 \times 19}{1 \times 2} \times 2^{18} \times x^{36} \times x^{-2} \\ &= 2^{19} \times 5 \times 19 \times x^{34} \\ &= 49\,807\,360\, x^{34}. \end{aligned}$$

(Check this on the calculator using ${}^{20}\mathrm{C}_2 \times 2^{18}$.)

(ii) To obtain the term in x^{-5}, $40 - 3r = -5$

$$r = 15.$$

Hence the term in
$$\begin{aligned} x^{-5} &= {}^{20}\mathrm{C}_{15} \times (2x^2)^{20-15} \times (-x^{-1})^{15} \\ &= -\frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5} \times 2^5 \times x^{10} \times x^{-15} \\ &= -19 \times 3 \times 17 \times 16 \times 2^5 \times x^{-5} \\ &= -2^9 \times 3 \times 17 \times 19 \times x^{-5} \\ &= -496\,128 x^{-5}. \end{aligned}$$

(Check this on the calculator using ${}^{20}\mathrm{C}_{15} \times 2^5$.)

WORKED EXERCISE: In the expansion of $\left(x + \frac{1}{x}\right)^{40}\left(x - \frac{1}{x}\right)^{40}$, find the term independent of x. Give your answer in the form ${}^n\mathrm{C}_r$, and also as a numeral.

SOLUTION: We can write $\left(x + \frac{1}{x}\right)^{40}\left(x - \frac{1}{x}\right)^{40} = (x^2 - x^{-2})^{40}$.

Hence the term independent of
$$\begin{aligned} x &= {}^{40}\mathrm{C}_{20} \times (x^2)^{20} \times (-x^{-2})^{20} \\ &= {}^{40}\mathrm{C}_{20} \\ &\doteqdot 1{\cdot}378 \times 10^{11} \quad \text{(using the calculator).} \end{aligned}$$

The Values of ${}^n\mathrm{C}_r$ for $r = 0$, 1 and 2: The particular formulae for ${}^n\mathrm{C}_r$ for $n = 0$, 1 and 2 are important enough to be memorised:

$$ {}^n\mathrm{C}_0 = \frac{n!}{0!\,n!} = 1 \qquad {}^n\mathrm{C}_1 = \frac{n!}{1!\,(n-1)!} = n \qquad {}^n\mathrm{C}_2 = \frac{n!}{2!\,(n-2)!} = \frac{n \times (n-1)}{1 \times 2} = \tfrac{1}{2}n(n-1)$$

By the symmetry of the rows, these are also the values of ${}^n\mathrm{C}_n$, ${}^n\mathrm{C}_{n-1}$ and ${}^n\mathrm{C}_{n-2}$.

11 SOME PARTICULAR VALUES OF ${}^n\mathrm{C}_r$: For all cardinals n,

$$\begin{aligned} {}^n\mathrm{C}_0 &= {}^n\mathrm{C}_n &&= 1, \\ {}^n\mathrm{C}_1 &= {}^n\mathrm{C}_{n-1} &&= n, \\ {}^n\mathrm{C}_2 &= {}^n\mathrm{C}_{n-2} &&= \tfrac{1}{2}n(n-1). \end{aligned}$$

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WORKED EXERCISE: Find the value of n if:

(a) ${}^{n}\mathrm{C}_2 = 55$ (b) ${}^{n}\mathrm{C}_2 + {}^{n}\mathrm{C}_1 + {}^{n}\mathrm{C}_0 = 29$

SOLUTION: We know that ${}^{n}\mathrm{C}_0 = 1$ and ${}^{n}\mathrm{C}_1 = n$ and ${}^{n}\mathrm{C}_2 = \frac{1}{2}n(n-1)$.

(a)
$$\begin{aligned} \tfrac{1}{2}n(n-1) &= 55 \\ n^2 - n - 110 &= 0 \\ (n-11)(n+10) &= 0 \end{aligned}$$
Since $n \ge 0$, $\quad n = 11$.

(b)
$$\begin{aligned} {}^{n}\mathrm{C}_2 + {}^{n}\mathrm{C}_1 + {}^{n}\mathrm{C}_0 &= 29 \\ \tfrac{1}{2}n(n-1) + n + 1 &= 29 \\ n^2 - n + 2n + 2 &= 58 \\ n^2 + n - 56 &= 0 \\ (n-7)(n+8) &= 0 \end{aligned}$$
Since $n \ge 0$, $\quad n = 7$.

Proof of the Binomial Theorem: The demanding proof of the binomial theorem uses mathematical induction. The key to the proof is the addition property of the Pascal triangle, proven in Section 5B, because it allows ${}^{k+1}\mathrm{C}_r$ to be expressed as the sum of ${}^{k}\mathrm{C}_r$ and ${}^{k}\mathrm{C}_{r-1}$. Towards the end of part B, the proof uses the technical lemma, proven in the previous section, about adding two fractions involving factorials.

PROOF: The proof is by mathematical induction on the degree n. The 'result' that the proof keeps referring to is the statement that

$${}^{n}\mathrm{C}_r = \frac{n!}{(n-r)!\,r!}\,, \quad \text{for } r = 0, 1, 2, \ldots, n,$$

which says that the formula holds for all values of r from 0 to n. In other words, we shall prove that if any one row of the triangle obeys the theorem, then the next row also obeys it.

A. We prove the result for $n = 0$. In this case, $r = 0$ is the only possible value of r, and so there is only a single formula to prove, namely ${}^{0}\mathrm{C}_0 = \dfrac{0!}{0! \times 0!}$.

Here LHS $= 1$, because of the expansion $(x+y)^0 = 1$.
Also RHS $= 1$, since $0! = 1$ by definition.

B. Suppose that k is a value of n for which the result is true.

That is, $${}^{k}\mathrm{C}_r = \frac{k!}{(k-r)!\,r!}\,, \quad \text{for } r = 0, 1, 2, \ldots, k. \qquad (**)$$

We now prove the result for $n = k+1$.

That is, we prove $${}^{k+1}\mathrm{C}_r = \frac{(k+1)!}{(k-r+1)!\,r!}\,, \quad \text{for } r = 0, 1, 2, \ldots, k+1.$$

The proof of this will be in two parts. The first part confirms that the formula is true for the two ends of the row when $r = 0$ and $r = k+1$, and the second part proves it true for the other values $r = 1, 2, \ldots, k$.

1) When $r = 0$,
$$\begin{aligned} \text{LHS} &= {}^{k+1}\mathrm{C}_0 \\ &= 1, \text{ as proven in Section 5B,} \\ \text{RHS} &= \frac{(k+1)!}{(k+1)! \times 0!} \\ &= 1, \text{ since } 0! = 1. \end{aligned}$$

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When $r = k+1$, $\text{LHS} = {}^{k+1}\mathrm{C}_{k+1}$
$= 1$, again as proven in Section 5B,
$\text{RHS} = \dfrac{(k+1)!}{0! \times (k+1)!}$
$= 1$, since $0! = 1$.

Hence the formula is true for $r = 0$ and $r = k+1$.

2) Now suppose that $r = 1, 2, \ldots, k$.

$\text{LHS} = {}^{k+1}\mathrm{C}_r$
$= {}^{k}\mathrm{C}_r + {}^{k}\mathrm{C}_{r-1}$, by the addition property with $n = k+1$,
$= \dfrac{k!}{(k-r)!\,r!} + \dfrac{k!}{(k-r+1)!\,(r-1)!}$, by the induction hypothesis $(**)$,
$= \dfrac{(k+1)!}{(k-r+1)!\,r!}$, by the lemma in Section 5C,
$= \text{RHS}$.

C. It follows now from A and B by mathematical induction that the result is true for all cardinals n.

Exercise 5D

1. Use the result ${}^n\mathrm{C}_r = \dfrac{n!}{r!\,(n-r)!}$ to evaluate the following. Check your answers for parts (a)–(i) against the Pascal triangle you developed in Exercise 5A.

(a) ${}^5\mathrm{C}_2$ (b) ${}^{10}\mathrm{C}_4$ (c) ${}^6\mathrm{C}_3$ (d) ${}^{13}\mathrm{C}_5$ (e) ${}^{12}\mathrm{C}_7$ (f) ${}^8\mathrm{C}_8$ (g) ${}^9\mathrm{C}_1$ (h) ${}^{11}\mathrm{C}_{10}$ (i) ${}^7\mathrm{C}_3$ (j) $\dfrac{{}^{10}\mathrm{C}_6}{{}^7\mathrm{C}_2}$ (k) $\dfrac{{}^8\mathrm{C}_5}{{}^4\mathrm{C}_3}$ (l) $\dfrac{{}^{12}\mathrm{C}_7}{{}^4\mathrm{C}_2}$

2. (a) Evaluate: (i) ${}^8\mathrm{C}_3$ and ${}^8\mathrm{C}_5$, (ii) ${}^7\mathrm{C}_4$ and ${}^7\mathrm{C}_3$.
(b) If ${}^n\mathrm{C}_3 = {}^n\mathrm{C}_2$, find the value of n.

3. (a) By evaluating the LHS and RHS, verify the following results for $n = 8$ and $r = 3$:
(i) ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$ (ii) ${}^n\mathrm{C}_{r-1} + {}^n\mathrm{C}_r = {}^{n+1}\mathrm{C}_r$
(b) Use these two identities to solve the following equations for n:
(i) ${}^5\mathrm{C}_3 + {}^5\mathrm{C}_4 = {}^n\mathrm{C}_4$ (ii) ${}^n\mathrm{C}_7 + {}^n\mathrm{C}_8 = {}^{11}\mathrm{C}_8$ (iii) ${}^n\mathrm{C}_{10} = {}^n\mathrm{C}_{20}$ (iv) ${}^{12}\mathrm{C}_4 = {}^{12}\mathrm{C}_n$

4. Find the specified terms in each of the following expansions.
(a) For $(2+x)^7$: (i) find the term in x^2, (ii) find the term in x^4.
(b) For $(x+\frac{1}{2}y)^{14}$: (i) find the term in x^9y^5, (ii) find the term in x^5y^9.
(c) For $(\frac{1}{2}x - 3y^2)^{11}$: (i) find the term in $x^{10}y^2$, (ii) find the term in x^5y^{12}.
(d) For $(a - b^{\frac{1}{2}})^{20}$: (i) find the term in $a^3b^{\frac{17}{2}}$, (ii) find the term in a^2b^9.

5. (a) Use the binomial theorem to obtain formulae for:
(i) ${}^n\mathrm{C}_0$ (ii) ${}^n\mathrm{C}_1$ (iii) ${}^n\mathrm{C}_2$ (iv) ${}^n\mathrm{C}_3$
(b) Hence solve each of the following equations for n:
(i) ${}^9\mathrm{C}_2 - {}^n\mathrm{C}_1 = {}^6\mathrm{C}_3$ (ii) ${}^n\mathrm{C}_2 = 36$ (iii) ${}^n\mathrm{C}_2 + {}^6\mathrm{C}_2 = {}^7\mathrm{C}_2$ (iv) ${}^n\mathrm{C}_2 + {}^n\mathrm{C}_1 = 22 - {}^n\mathrm{C}_0$ (v) ${}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 = {}^5\mathrm{C}_2$ (vi) ${}^n\mathrm{C}_3 + {}^n\mathrm{C}_2 = 8\,{}^n\mathrm{C}_1$
(c) Use the formula for ${}^n\mathrm{C}_2$ to show that ${}^n\mathrm{C}_2 + {}^{n+1}\mathrm{C}_2 = n^2$, and verify the result on the third column of the Pascal triangle.

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DEVELOPMENT

6. (a) In the expansion of $(1+x)^{16}$, find the ratio of the term in x^{13} to the term in x^{11}.
(b) Find the ratio of the coefficients of x^{14} and x^5 in the expansion of $(1+x)^{20}$.
(c) In the expansion of $(2+x)^{18}$, find the ratio of the coefficients of x^{10} and x^{16}.

7. Consider the expansion $\left(x^2+\dfrac{1}{x}\right)^9 = \displaystyle\sum_{i=0}^{9} {}^9\mathrm{C}_i\,(x^2)^{9-i}\left(\dfrac{1}{x}\right)^i$.
(a) Show that each term in the expansion of $\left(x^2+\dfrac{1}{x}\right)^9$ can be written as ${}^9\mathrm{C}_i\,x^{18-3i}$.
(b) Hence find the coefficients of: (i) x^3 (ii) x^{-3} (iii) x^0

8. In the expansion of $(2x^3+3x^{-2})^{10}$, the general term is ${}^{10}\mathrm{C}_k\,\left(2x^3\right)^{10-k}\left(3x^{-2}\right)^k$.
(a) Show that this general term can be written as ${}^{10}\mathrm{C}_k\,2^{10-k}\,3^k\,x^{30-5k}$.
(b) Hence find the coefficients of the following terms, giving your answer factored into primes: (i) x^{10} (ii) x^{-5} (iii) x^0

9. (a) Show that the general term in the expansion of $\left(\dfrac{x}{2}-\dfrac{5}{x}\right)^{15}$ can be written as

$${}^{15}\mathrm{C}_j\,(-1)^j\,5^j\,2^{j-15}\,x^{15-2j}.$$

(b) Hence find, without simplifying, the coefficients of: (i) x^{11} (ii) x (iii) x^{-5}

10. Find the term independent of x in each expansion:
(a) $\left(x+\dfrac{3}{x}\right)^8$ (b) $\left(2x^3-\dfrac{1}{x}\right)^{12}$ (c) $(5x^4-2x^{-1})^{10}$ (d) $(ax^{-2}+\tfrac{1}{2}x)^6$

11. Find the coefficient of the power of x specified in each of the following expansions (leave the answer to part (f) unsimplified):
(a) x^{15} in $\left(x^3-\dfrac{2}{x}\right)^9$
(b) the constant term in $\left(\dfrac{1}{2x^3}+x\right)^{20}$
(c) x^{-14} in $(x-3x^{-4})^{11}$
(d) x^7 in $\left(5x^2+\dfrac{1}{2x}\right)^8$
(e) x^{-1} in $(x^{-1}+\tfrac{2}{7}x)^5$
(f) x^{11} in $\left(\dfrac{3x^2}{5}-\dfrac{1}{x}\right)^{19}$

12. Determine the coefficients of the specified terms in each of the following expansions:
(a) For $(3+x)(1-x)^{15}$: (i) find the term in x^4, (ii) find the term in x^{13}.
(b) For $(2-5x+x^2)(1+x)^{11}$: (i) find the term in x^9, (ii) find the term in x^3.
(c) For $(x-3)(x+2)^{15}$: (i) find the term in x^7, (ii) find the term in x^{12}.
(d) For $(1-2x-4x^2)\left(1-\dfrac{3}{x}\right)^9$: (i) find the term in x^0, (ii) find the term in x^{-5}.

13. (a) Find the middle term when the terms in the following expansions are arranged in increasing powers of y.
(i) $(2x-3y)^{10}$ (ii) $(x^{\frac{1}{2}}+y^{\frac{1}{3}})^{12}$ (iii) $(\tfrac{1}{5}x-y^2)^{18}$ (iv) $\left(\dfrac{2}{y}+\dfrac{x}{3}\right)^6$
(b) Find the two middle terms when the terms in the following expansions are arranged in increasing powers of b.
(i) $(a+3b)^5$ (ii) $(\tfrac{1}{2}a-\tfrac{1}{3}b)^{11}$ (iii) $(a^{\frac{1}{3}}+b^{\frac{1}{2}})^{15}$ (iv) $\left(\dfrac{1}{a}-\dfrac{b}{2}\right)^9$

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14. (a) Find x if the terms in x^{10} and x^{11} in the expansion of $(5 + 2x)^{15}$ are equal.
(b) Find x if the terms in x^{13} and x^{14} in the expansion of $(2 - 3x)^{17}$ are equal.

15. (a) Find the coefficient of x in the expansion of $\left(x + \dfrac{1}{x}\right)^5 \left(x - \dfrac{1}{x}\right)^4$.
(b) Find the coefficient of x^2 in the expansion of $\left(x - \dfrac{1}{x}\right)^9 \left(x + \dfrac{1}{x}\right)^5$.
(c) Find the coefficient of y^{-3} in the expansion of $\left(y + \dfrac{1}{y}\right)^{10} \left(y - \dfrac{1}{y}\right)^7$.

16. (a) In the expansion of $(2 + ax + bx^2)(1 + x)^{13}$, the coefficients of x^0, x^1 and x^2 are all equal to 2. Find the values of a and b.
(b) In the expansion of $(1 + x)^n$, the coefficient of x^5 is 1287. Find the value of n by trial and error, and hence find the coefficient of x^{10}.

17. The expression $(1 + ax)^n$ is expanded in increasing powers of x. Find the values of a and n if the first three terms are:
(a) $1 + 28x + 364x^2 + \cdots$ (b) $1 - \frac{10}{3}x + 5x^2 - \cdots$

18. (a) In the expansion of $(2 + 3x)^n$, the coefficients of x^5 and x^6 are in the ratio $4 : 9$. Find the value of n.
(b) In the expansion of $(1 + 3x)^n$, the coefficients of x^8 and x^{10} are in the ratio $1 : 2$. Find the value of n.
(c) The expression $\left(3 + \dfrac{x}{5}\right)^n$ is expanded in increasing powers of x. When $x = 2$, the ratio of the 7th and 8th terms is $35 : 2$. Find the value of n.

19. If n is a positive integer, use the binomial theorem to prove that $(5 + \sqrt{11}\,)^n + (5 - \sqrt{11}\,)^n$ is an integer.

20. Use binomial expansions and the binomial theorem to find the value of:
(a) $(0{\cdot}99)^{13}$ correct to five significant figures,
(b) $(1{\cdot}01)^{11}$ correct to four decimal places,
(c) $(0{\cdot}999)^{15}$ correct to five significant figures.

21. (a) When $(1 + x)^n$ is expanded in increasing powers of x, the ratios of three consecutive coefficients are $9 : 24 : 42$. Find the value of n.
(b) In the expansion of $(1 + x)^n$, the coefficients of x, x^2 and x^3 form an arithmetic progression. Find the value of n.
(c) In the expansion of $(1 + x)^n$, the coefficients of x^4, x^5 and x^6 form an AP.
(i) Explain why $2 \times {}^n\mathrm{C}_5 = {}^n\mathrm{C}_4 + {}^n\mathrm{C}_6$, and hence show that $n^2 - 21n + 98 = 0$.
(ii) Hence find the two possible values of n.

22. By writing it as $\big((1 - x) + x^2\big)^4$, expand $(1 - x + x^2)^4$ in ascending powers of x as far as the term containing x^4.

23. Show that there will always be a term independent of x in the expansion of $\left(x^p + \dfrac{1}{x^{2p}}\right)^{3n}$, where n is a positive integer, and find that term.

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24. (a) Write down the term in x^r in the expansion of $(a - bx)^{12}$.

(b) In the expansion of $(1 + x)(a - bx)^{12}$, the coefficient of x^8 is zero. Find the value of the ratio $\frac{a}{b}$ in simplest form.

25. [Divisibility problems]

(a) Use the binomial theorem to show that $7^n + 2$ is divisible by 3, where n is a positive integer. [HINT: Write $7 = 6 + 1$.]

(b) Use the binomial theorem to show that $5^n + 3$ is divisible by 4, where n is a positive integer.

(c) Suppose that b, c and n are positive integers, and $a = b + c$. Use the binomial expansion of $(b + c)^n$ to show that $a^n - b^{n-1}(b + cn)$ is divisible by c^2. Hence show that $5^{42} - 2^{48}$ is divisible by 9.

26. (a) Use the binomial theorem to expand $(x + h)^n$.

(b) Hence use the definition $f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ to differentiate x^n from first principles.

27. (a) Given that ${}^n\mathrm{C}_r = \dfrac{n!}{r!\,(n - r)!}$, show that $\dfrac{r \times {}^n\mathrm{C}_r}{{}^n\mathrm{C}_{r-1}} = n - r + 1$.

(b) Hence prove that

$$\frac{{}^n\mathrm{C}_1}{{}^n\mathrm{C}_0} + \frac{2 \times {}^n\mathrm{C}_2}{{}^n\mathrm{C}_1} + \frac{3 \times {}^n\mathrm{C}_3}{{}^n\mathrm{C}_2} + \cdots + \frac{n \times {}^n\mathrm{C}_n}{{}^n\mathrm{C}_{n-1}} = \frac{n}{2}\,(n + 1).$$

28. In the expansion of $(1 + 3x + ax^2)^n$, where n is a positive integer, the coefficient of x^2 is 0. Find, in terms of n, the value of: (a) a, (b) the coefficient of x^3.

29. [APs in the Pascal triangle — see the last question in Exercise 5F for the general case.]

(a) In the expansion of $(1 + x)^n$, the coefficients of x^{r-1}, x^r and x^{r+1} form an arithmetic sequence. Prove that $4r^2 - 4rn + n^2 - n - 2 = 0$.

(b) Hence find three consecutive coefficients of the expansion of $(1 + x)^{14}$ which form an arithmetic sequence.

EXTENSION

30. (a) Show that $n\binom{n-1}{1} + n\binom{n-1}{2} + \cdots + n\binom{n-1}{n-2} = n(2^{n-1} - 2)$.

(b) Hence use trial and error to find the smallest positive integer n such that

$$n\binom{n-1}{1} + n\binom{n-1}{2} + \cdots + n\binom{n-1}{n-2} > 15\,000.$$

31. (a) If $r > p + 1$, show that ${}^r\mathrm{C}_p = {}^{r+1}\mathrm{C}_{p+1} - {}^r\mathrm{C}_{p+1}$.

(b) Hence deduce that for $n > p$, ${}^p\mathrm{C}_p + {}^{p+1}\mathrm{C}_p + {}^{p+2}\mathrm{C}_p + \cdots + {}^n\mathrm{C}_p = {}^{n+1}\mathrm{C}_{p+1}$.

(c) What is the significance of this result in the Pascal triangle?

32. (a) Find the value of $\dfrac{{}^n\mathrm{C}_r}{{}^n\mathrm{C}_{r-1}}$.

(b) Evaluate $\dfrac{{}^n\mathrm{C}_1}{{}^n\mathrm{C}_0} + 2\dfrac{{}^n\mathrm{C}_2}{{}^n\mathrm{C}_1} + 3\dfrac{{}^n\mathrm{C}_3}{{}^n\mathrm{C}_2} + \cdots + n\dfrac{{}^n\mathrm{C}_n}{{}^n\mathrm{C}_{n-1}}$.

(c) Prove the following identity, and verify it using the row indexed by $n = 4$:

$$({}^n\mathrm{C}_0 + {}^n\mathrm{C}_1)({}^n\mathrm{C}_1 + {}^n\mathrm{C}_2)\cdots({}^n\mathrm{C}_{n-1} + {}^n\mathrm{C}_n) = {}^n\mathrm{C}_0\,{}^n\mathrm{C}_1\,{}^n\mathrm{C}_2 \cdots {}^n\mathrm{C}_n \times \frac{(n + 1)^n}{n!}.$$

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33. [A more general form of the binomial theorem] The binomial theorem can be written as a power series:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots.$$

In this form, the theorem is true even when n is fractional or negative, provided that $|x| < 1$, in the sense that the infinite series on the RHS converges to $(1+x)^n$.

(a) Prove, using the convergence of geometric series, that the result is true for $n = -1$.

(b) Generate the binomial expansions of:

(i) $\dfrac{1}{1+x}$ (ii) $\dfrac{1}{(1-x)^2}$ (iii) $\dfrac{1}{(1+x)^2}$ (iv) $\sqrt{1+x}$

5 E Greatest Coefficient and Greatest Term

In a typical binomial expansion like

$$(1+2x)^4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4,$$

the coefficients of successive terms rise and then fall. The greatest coefficient is the coefficient of x^3, which is 32.

If we now make a substitution like $x = \frac{3}{4}$, the expansion becomes

$$(1+\tfrac{3}{2})^4 = 1 + 6 + 13\tfrac{1}{2} + 13\tfrac{1}{2} + 5\tfrac{1}{16},$$

and again the terms rise and fall. There are two greatest terms, both $13\frac{1}{2}$.

The purpose of the this section is to develop a systematic method, based on the binomial theorem, of finding these greatest terms and greatest coefficients. These methods, and the results, will have a particular role in some probability questions in Chapter Ten.

A Systematic Method: We will use the binomial theorem to find the ratio of successive coefficients or terms. This ratio will be greater than 1 when the coefficients or terms are increasing, and it will be less than 1 when the coefficients or terms are decreasing.

In the following worked exercise, the general method is applied to the two very simple expansions above — it is, of course, designed to be used with expansions of much higher degree, where writing out all the terms would be impossible. It is not necessary to use sigma notation — all that is needed is the general term — but we shall need the notation in the next section, and it does the job of clearly displaying the general term.

WORKED EXERCISE:

(a) Write the expansion of $(1+2x)^4$ in the form $\displaystyle\sum_{k=0}^{n} t_k\, x^k$. Find the ratio $\dfrac{t_{k+1}}{t_k}$, and hence find the greatest coefficient.

(b) Write the expansion of $(1+2x)^4$ in the form $\displaystyle\sum_{k=0}^{n} T_k$. Find the ratio $\dfrac{T_{k+1}}{T_k}$, and hence find the greatest term if $x = \frac{3}{4}$.

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SOLUTION:

(a) Expanding, $(1+2x)^4 = \sum_{k=0}^{4} {}^4C_k\,(2x)^k = \sum_{k=0}^{4} {}^4C_k\,2^k\,x^k$,

so $(1+2x)^4 = \sum_{k=0}^{4} t_k\,x^k$, where $t_k = {}^4C_k\,2^k$.

Hence
$$\begin{aligned}\frac{t_{k+1}}{t_k} &= \frac{{}^4C_{k+1}\,2^{k+1}}{{}^4C_k\,2^k}\\ &= \frac{4!}{(k+1)!\,(3-k)!} \times \frac{k!\,(4-k)!}{4!} \times 2\\ &= \frac{(4-k)\times 2}{k+1}\\ &= \frac{8-2k}{k+1}.\end{aligned}$$

To find where the coefficients are increasing, we solve $t_{k+1} > t_k$,

that is, $\dfrac{t_{k+1}}{t_k} > 1$ (notice that t_k is positive).

From above, $\dfrac{8-2k}{k+1} > 1$
$$8-2k > k+1$$
$$k < 2\tfrac{1}{3} \quad \text{(remember that } k \text{ is an integer)},$$

so $t_{k+1} > t_k$ for $k = 0$, 1 and 2, and $t_{k+1} < t_k$ for $k = 3$.

Hence $t_0 < t_1 < t_2 < t_3 > t_4$,

and the greatest coefficient is $t_3 = {}^4C_3 \times 2^3 = 32$.

(b) From above, $(1+2x)^4 = \sum_{k=0}^{4} T_k$, where $T_k = {}^4C_k\,2^k\,x^k$.

$$\begin{aligned}\frac{T_{k+1}}{T_k} &= \frac{{}^4C_{k+1}\,2^{k+1}\,x^{k+1}}{{}^4C_k\,2^k\,x^k}\\ &= \frac{(8-2k)x}{k+1}, \quad \text{using the previous working,}\\ &= \frac{3(8-2k)}{4(k+1)}, \quad \text{substituting } x = \tfrac{3}{4},\\ &= \frac{12-3k}{2k+2}, \quad \text{after cancelling the 2s.}\end{aligned}$$

To find where the terms are increasing, we solve $T_{k+1} > T_k$,

that is, $\dfrac{T_{k+1}}{T_k} > 1$ (again, T_k is positive).

From above, $\dfrac{12-3k}{2k+2} > 1$
$$12-3k > 2k+2$$
$$k < 2,$$

so $T_{k+1} > T_k$ for $k = 0$ and 1, $T_{k+1} = T_k$ for $k = 2$, and $T_{k+1} < T_k$ for $k = 3$.

Hence $T_0 < T_1 < T_2 = T_3 > T_4$,

and the greatest terms are $T_2 = {}^4C_2 \times (\tfrac{3}{2})^2 = 13\tfrac{1}{2}$

and $T_3 = {}^4C_3 \times (\tfrac{3}{2})^3 = 13\tfrac{1}{2}$.

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NOTE: Equality of successive terms or coefficients arises when the solution of the inequality is $k >$ some whole number, because then that whole number is the solution of the corresponding equality. There is no real need to make qualifications about this in the working, which is already complicated.

Exercise 5E

1. Let $(2+3x)^{12} = \sum_{k=0}^{12} t_k\, x^k$, where $t_k = {}^{12}\mathrm{C}_k \times 2^{12-k} \times 3^k$ is the coefficient of x^k.

(a) Write down expressions for t_k and t_{k+1}, and show that $\dfrac{t_{k+1}}{t_k} = \dfrac{36-3k}{2k+2}$.

(b) Hence show that t_7 is the greatest coefficient.

(c) Write down the greatest coefficient (and leave it factored).

2. Let $(7+3x)^{25} = \sum_{k=0}^{25} c_k\, x^k$.

(a) Write down expressions for c_k and c_{k+1}, and show that $\dfrac{c_{k+1}}{c_k} = \dfrac{75-3k}{7k+7}$.

(b) Hence show that c_7 is the greatest coefficient.

(c) Write down the greatest coefficient (and leave it factored).

3. Let $(3+4x)^{13} = \sum_{k=0}^{13} T_k$, where $T_k = {}^{13}\mathrm{C}_k \times 3^{13-k} \times (4x)^k$ is the term in x^k.

(a) Write down expressions for T_k and T_{k+1}, and show that $\dfrac{T_{k+1}}{T_k} = \dfrac{4x(13-k)}{3(k+1)}$.

(b) Hence show that when $x = \frac{1}{2}$, T_5 is the greatest term in the expansion.

(c) Write down the greatest term when $x = \frac{1}{2}$ (and leave it factored).

4. Let $(1+5x)^{21} = \sum_{k=0}^{21} T_k$, where T_k is the term in x^k.

(a) Write down expressions for T_k and T_{k+1}, and show that $\dfrac{T_{k+1}}{T_k} = \dfrac{5x(21-k)}{k+1}$.

(b) Hence show that when $x = \frac{3}{5}$, T_{16} is the greatest term in the expansion.

(c) Write down the greatest term when $x = \frac{3}{5}$ (and leave it factored).

5. (a) Let $(5+2x)^{15} = \sum_{k=0}^{15} t_k\, x^k$.

(i) Write down expressions for t_k and t_{k+1}, and show that $\dfrac{t_{k+1}}{t_k} = \dfrac{30-2k}{5k+5}$.

(ii) Hence show that t_4 is the greatest coefficient.

(iii) Write down the greatest coefficient (and leave it factored).

(b) Let $(5+2x)^{15} = \sum_{k=0}^{15} T_k$, where T_k is the term in x_k.

(i) Write down T_k and T_{k+1}, and show that when $x = \frac{5}{3}$, $\dfrac{T_{k+1}}{T_k} = \dfrac{30-2k}{3k+3}$.

(ii) Hence show that when $x = \frac{5}{3}$, T_6 is the greatest term in the expansion.

(iii) Write down the greatest term when $x = \frac{5}{3}$ (and leave it factored).

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DEVELOPMENT

6. For each expansion, find: (i) the greatest coefficient, (ii) the greatest term if $x = \frac{2}{3}$.
(a) $(1 + 4x)^{11}$ (b) $(2 + \frac{3}{4}x)^9$ (c) $(5x + 3)^{12}$ (d) $(5 + 6x)^{11}$

7. For each of the following expansions, find: (i) the coefficient with greatest absolute value, (ii) the term with greatest absolute value if $x = \frac{2}{3}$ and $y = 3$.
(a) $(1 - 7x)^9$ (b) $(7 - 2x)^{14}$ (c) $(x - 2y)^{12}$ (d) $(2x - y)^{15}$

8. For each of the following expansions, find: (i) the greatest coefficient, (ii) the greatest term if $x = \frac{3}{2}$ and $y = \frac{4}{9}$.
(a) $\left(2x^2 + \dfrac{3}{x}\right)^{10}$ (b) $(2x + 3y)^{12}$

9. Show that in the expansion of $(1 + x)^{14}$, where $x = \frac{2}{3}$, two consecutive terms are equal to each other and greater than any other term.

10. Let $(x + y)^n = \displaystyle\sum_{r=0}^{n} T_r$, where T_r is the term in $x^{n-r}y^r$.

(a) Write down expressions for T_r and T_{r+1} and hence show that $\dfrac{T_{r+1}}{T_r} = \dfrac{(n-r)y}{(r+1)x}$.

(b) Consider the expansion of $(a + 3b)^8$, where $a = 2$ and $b = \frac{3}{4}$. By substituting the appropriate values for x, y and n into the expression in (i), show that $\dfrac{T_{r+1}}{T_r} = \dfrac{72 - 9r}{8r + 8}$.
Hence show that T_4 is the numerically greatest term in the expansion.

(c) Find the greatest term in the expansion $(p + q)^{10}$, where $p = q = \frac{1}{2}$.

11. Let $(1 + x)^n = \displaystyle\sum_{r=0}^{n} t_r\, x^r$.

(a) Find the values of n and r if $t_{r+1} = 5t_r$ and $t_{r+4} = 2t_{r+3}$.

(b) Hence find the greatest coefficient in the expansion.

12. Let $(1 + 0{\cdot}01)^{12} = \displaystyle\sum_{r=0}^{12} T_r$, where T_r is the term in $(0{\cdot}01)^r$. Find the first value of r for which the ratio $\dfrac{T_{r+1}}{T_r} < 0{\cdot}005$.

13. Let $(\sin\theta + \cos\theta)^{20} = \displaystyle\sum_{k=0}^{20} T_k$, where T_k is the term in $\sin^{20-k}\theta\cos^k\theta$. Find, to the nearest degree, the first positive angle for which $T_{14} > T_{15}$.

14. (a) Show that in the expansion of $(1 + x)^n$, where n is an even integer, the term with the greatest coefficient is the term in $x^{\frac{1}{2}n}$.

(b) By considering the expansion of $(1 + x)^{2n}$, for any positive integer n, prove that the largest value of ${}^{2n}\mathrm{C}_r$ is ${}^{2n}\mathrm{C}_n$.

EXTENSION

15. [An alternative proof of the binomial theorem by differentiation] Let $f(x) = (1 + x)^n$.

(a) Find the kth derivative of $f(x)$, and show that $f^{(k)}(0) = \dfrac{n!}{(n-k)!}$.

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(b) Expand $f(x)$ using the binomial theorem, and show that $f^{(k)}(0) = k!\,{}^n\mathrm{C}_k$.

(c) By equating these two expressions for $f^{(k)}(0)$, prove that ${}^n\mathrm{C}_k = \dfrac{n!}{k!\,(n-k)!}$.

16. [The Poisson probability distribution] The probability that n car accidents occur at a given set of traffic lights during a year is

$$P_n = \frac{e^{-2\cdot 6} \times 2\cdot 6^n}{n!}, \quad \text{for } n = 0,\ 1,\ 2,\ \ldots.$$

By considering values of n for which $\dfrac{P_{n+1}}{P_n} \geq 1$, determine the most likely number of accidents at this intersection in a given one-year period.

5 F Identities on the Binomial Coefficients

There are a great number of patterns in the Pascal triangle. Some are quite straightforward to recognise and to prove, others are more complicated. They can be very important in any application of the binomial theorem, and many of them will reappear in Chapter Ten on probability. Each pattern in the Pascal triangle is described by an identity on the binomial coefficients ${}^n\mathrm{C}_k$ — these identities have a rather forbidding appearance, and it is important to take the time to interpret each identity as some sort of pattern in the Pascal triangle.

Methods of proof as well as the identities themselves are the subject of this section. Each proof begins with some form of the binomial expansion

$$(x+y)^n = \sum_{k=0}^{n} {}^n\mathrm{C}_k\, x^{n-k}\, y^k.$$

Three approaches to generating identities from this expansion are developed in turn: substitutions, methods from calculus, and equating coefficients.

Here again is the first part of the Pascal triangle. Each identity that is obtained should be interpreted as a pattern in the triangle and verified there, either before or after the proof is completed.

$n\backslash r$	0	1	2	3	4	5	6	7	8		
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

The First Approach — Substitution: Substitutions into the basic binomial expansion or any subsequent development from it will yield identities.

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WORKED EXERCISE: Obtain identities by substituting into the basic expansion:

(a) $x = 1$ and $y = 1$ (b) $x = 1$ and $y = -1$ (c) $x = 1$ and $y = 2$

Then explain what pattern each identity describes in the Pascal triangle.

SOLUTION: We begin with the expansion $(x+y)^n = \sum_{k=0}^{n} {}^n\mathrm{C}_k\, x^{n-k}\, y^k$.

(a) Substituting $x = 1$ and $y = 1$ gives $2^n = \sum_{k=0}^{n} {}^n\mathrm{C}_k$,

that is, ${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n = 2^n$.

In the Pascal triangle, this means that the sum of every row is 2^n.

For example, $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$ (as proven in Section 5A).

(b) Substituting $x = 1$ and $y = -1$ gives $0 = \sum_{k=0}^{n} {}^n\mathrm{C}_k\,(-1)^k$,

that is, ${}^n\mathrm{C}_0 - {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 - \cdots + (-1)^n\, {}^n\mathrm{C}_n = 0$.

This means that the alternating sum of every row is zero.

For odd n, this is trivial: $1 - 5 + 10 - 10 + 5 - 1 = 0$,

but for even n, $1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$.

(c) Substituting $x = 1$ and $y = 2$ gives $3^n = \sum_{k=0}^{n} {}^n\mathrm{C}_k\, 2^k$,

that is, $1 \times {}^n\mathrm{C}_0 + 2 \times {}^n\mathrm{C}_1 + 2^2 \times {}^n\mathrm{C}_2 + \cdots + 2^n \times {}^n\mathrm{C}_n = 3^n$.

Taking as an example the row 1, 4, 6, 4, 1,

$$1 \times 1 + 2 \times 4 + 4 \times 6 + 8 \times 4 + 16 \times 1 = 81 = 3^4.$$

Second Approach — Differentiation and Integration: The basic expansion can be differentiated or integrated before substitutions are made. As always, integration involves finding an unknown constant.

WORKED EXERCISE: Consider the expansion $(1+x)^n = \sum_{k=0}^{n} {}^n\mathrm{C}_k\, x^k$.

(a) Differentiate the expansion, then substitute $x = 1$ to obtain an identity.

(b) Integrate the expansion, then substitute $x = -1$ to obtain an identity.

Then give an example of each identity on the Pascal triangle.

SOLUTION:

(a) Differentiating, $n(1+x)^{n-1} = \sum_{k=0}^{n} k \times {}^n\mathrm{C}_k \times x^{k-1}$.

Substituting $x = 1$, $n \times 2^{n-1} = \sum_{k=0}^{n} k \times {}^n\mathrm{C}_k$,

that is, $0 \times {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + 2 \times {}^n\mathrm{C}_2 + 3 \times {}^n\mathrm{C}_3 + \cdots + n \times {}^n\mathrm{C}_n = 0$.

Taking as an example the row 1, 4, 6, 4, 1,

$$0 \times 1 + 1 \times 4 + 2 \times 6 + 3 \times 4 + 4 \times 1 = 32 = 4 \times 2^3.$$

(b) Integrating, $\dfrac{(1+x)^{n+1}}{n+1} = C + \sum_{k=0}^{n} \dfrac{{}^n\mathrm{C}_k\, x^{k+1}}{k+1}$, for some constant C.

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To find the constant C of integration, substitute $x = 0$,

then $$\frac{1}{n+1} = C + \sum_{k=0}^{n} 0,$$

so $C = \dfrac{1}{n+1}$, and $$\frac{(1+x)^{n+1}}{n+1} = \frac{1}{n+1} + \sum_{k=0}^{n} \frac{{}^n\mathrm{C}_k\, x^{k+1}}{k+1}.$$

Substituting $x = -1$, $$0 = \frac{1}{n+1} + \sum_{k=0}^{n} \frac{{}^n\mathrm{C}_k\, (-1)^{k+1}}{k+1},$$

that is, $$\frac{1}{n+1} - {}^n\mathrm{C}_0 + \frac{{}^n\mathrm{C}_1}{2} - \frac{{}^n\mathrm{C}_2}{3} + \frac{{}^n\mathrm{C}_3}{4} - \cdots + \frac{(-1)^{n+1}\, {}^n\mathrm{C}_n}{n+1} = 0$$

$\boxed{\times(-1)}$ $${}^n\mathrm{C}_0 - \frac{{}^n\mathrm{C}_1}{2} + \frac{{}^n\mathrm{C}_2}{3} - \frac{{}^n\mathrm{C}_3}{4} + \cdots + \frac{(-1)^{n}\, {}^n\mathrm{C}_n}{n+1} = \frac{1}{n+1}.$$

Taking as an example the row 1, 4, 6, 4, 1,

$$\begin{aligned} 1 \times 1 - \tfrac{1}{2} \times 4 + \tfrac{1}{3} \times 6 - \tfrac{1}{4} \times 4 + \tfrac{1}{5} \times 1 &= 1 - 2 + 2 - 1 + \tfrac{1}{5} \\ &= \tfrac{1}{5}. \end{aligned}$$

Third Approach — Equating Coefficients: The third method involves taking two equal expansions and equating coefficients.

WORKED EXERCISE: Taking $\left(x + \dfrac{1}{x}\right)^n \left(x + \dfrac{1}{x}\right)^n = \left(x + \dfrac{1}{x}\right)^{2n}$ and expanding and equating constants, prove that

$$({}^n\mathrm{C}_0)^2 + ({}^n\mathrm{C}_1)^2 + ({}^n\mathrm{C}_2)^2 + \cdots + ({}^n\mathrm{C}_n)^2 = {}^{2n}\mathrm{C}_n.$$

Then interpret the identity on the Pascal triangle.

SOLUTION: The constant term on the RHS is ${}^{2n}\mathrm{C}_n$.

The constant term on the LHS is the sum of the products

$${}^n\mathrm{C}_0 \times {}^n\mathrm{C}_n + {}^n\mathrm{C}_1 \times {}^n\mathrm{C}_{n-1} + {}^n\mathrm{C}_2 \times {}^n\mathrm{C}_{n-2} + \cdots + {}^n\mathrm{C}_n \times {}^n\mathrm{C}_0,$$

and because ${}^n\mathrm{C}_{n-k} = {}^n\mathrm{C}_k$, by the symmetry of the row, this constant term is

$$({}^n\mathrm{C}_0)^2 + ({}^n\mathrm{C}_1)^2 + ({}^n\mathrm{C}_2)^2 + \cdots + ({}^n\mathrm{C}_n)^2.$$

Equating the two constant terms,

$$({}^n\mathrm{C}_0)^2 + ({}^n\mathrm{C}_1)^2 + ({}^n\mathrm{C}_2)^2 + \cdots + ({}^n\mathrm{C}_n)^2 = {}^{2n}\mathrm{C}_n, \text{ as required.}$$

This means that if the entries of any row are squared and added, the sum is the middle entry in the row twice as far down. For example, with the row 1, 3, 3, 1,

$$1^2 + 3^2 + 3^2 + 1^2 = 20 = {}^6\mathrm{C}_3,$$

and with the row 1, 4, 6, 4, 1,

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70 = {}^8\mathrm{C}_4.$$

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Exercise 5F

1. Consider the identity $(1+x)^4 = {}^4C_0 + {}^4C_1\,x + {}^4C_2\,x^2 + {}^4C_3\,x^3 + {}^4C_4\,x^4$. Prove the following, and explain each result in terms of the row indexed by $n=4$ in Pascal's triangle.

(a) By substituting $x=1$, show that ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4$.

(b) (i) By substituting $x=-1$, show that ${}^4C_0 + {}^4C_2 + {}^4C_4 = {}^4C_1 + {}^4C_3$.

(ii) Hence, by using the result of part (a), show that ${}^4C_0 + {}^4C_2 + {}^4C_4 = 2^3$.

(c) (i) Differentiate both sides of the identity.

(ii) By substituting $x=1$, show that ${}^4C_1 + 2\,{}^4C_2 + 3\,{}^4C_3 + 4\,{}^4C_4 = 4\times 2^3$.

(iii) By substituting $x=-1$, show that ${}^4C_1 - 2\,{}^4C_2 + 3\,{}^4C_3 - 4\,{}^4C_4 = 0$.

(d) (i) By integrating both sides of the identity, show that for some constant K,

$$\tfrac{1}{5}(1+x)^5 + K = {}^4C_0\,x + \tfrac{1}{2}\,{}^4C_1\,x^2 + \tfrac{1}{3}\,{}^4C_2\,x^3 + \tfrac{1}{4}\,{}^4C_3\,x^4 + \tfrac{1}{5}\,{}^4C_4\,x^5.$$

(ii) By substituting $x=0$, show that $K = -\tfrac{1}{5}$.

(iii) By substituting $x=-1$ in part (i), show that

$${}^4C_0 - \tfrac{1}{2}\,{}^4C_1 + \tfrac{1}{3}\,{}^4C_2 - \tfrac{1}{4}\,{}^4C_3 + \tfrac{1}{5}\,{}^4C_4 = \tfrac{1}{5}.$$

2. Consider the identity $(1+x)^n = \displaystyle\sum_{k=0}^{n} {}^nC_k\,x^k$. Prove the following, and explain each result in terms of the row indexed by $n=5$ in Pascal's triangle.

(a) By substituting $x=1$, show that $\displaystyle\sum_{k=0}^{n} {}^nC_k = 2^n$.

(b) (i) By substituting $x=-1$, show that ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$

(ii) Hence, by using the result of part (a), show that ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = 2^{n-1}$.

(c) (i) Differentiate both sides of the identity.

(ii) By substituting $x=1$, show that $\displaystyle\sum_{k=1}^{n} k\,{}^nC_k = n\,2^{n-1}$.

(iii) By substituting $x=-1$, show that $\displaystyle\sum_{k=1}^{n} (-1)^{k-1}\,k\,{}^nC_k = 0$.

(d) (i) By integrating both sides of the identity, show that for some constant K,

$$\frac{1}{n+1}(1+x)^{n+1} + K = \sum_{k=0}^{n} {}^nC_k\,\frac{x^{k+1}}{k+1}.$$

(ii) By substituting $x=0$, show that $K = -\dfrac{1}{n+1}$.

(iii) By substituting $x=-1$ in part (i), show that $\displaystyle\sum_{k=0}^{n} (-1)^k\,\frac{{}^nC_k}{k+1} = \frac{1}{n+1}$.

3. This question follows the same steps as question 2.

Consider the identity $(1+x)^{2n} = \displaystyle\sum_{r=0}^{2n} {}^{2n}C_r\,x^r$.

(a) Show that $\displaystyle\sum_{r=0}^{2n} {}^{2n}C_r = 2^{2n}$. (b) Show that ${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \cdots + {}^{2n}C_{2n-1} = 2^{2n-1}$.

Check both results on the Pascal triangle, using $n=3$ and $n=4$.

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(c) By differentiating both sides of the identity, show that:

(i) $\displaystyle\sum_{r=1}^{2n} r\,{}^{2n}\mathrm{C}_r = n\,2^{2n}$ (ii) $\displaystyle\sum_{r=1}^{2n} (-1)^{r-1} r\,{}^{2n}\mathrm{C}_r = 0$

Check both results on the Pascal triangle, using $n = 3$ and $n = 4$.

(d) By integrating both sides of the identity, show that:

(i) $\displaystyle\sum_{r=0}^{2n} {}^{2n}\mathrm{C}_r\,\frac{1}{r+1} = \frac{2^{2n+1}-1}{2n+1}$ [HINT: The constant of integration is $-\dfrac{1}{2n+1}$.]

(ii) $\displaystyle\sum_{r=0}^{2n} (-1)^r\,\frac{{}^{2n}\mathrm{C}_r}{r+1} = \frac{1}{2n+1}$.

Check both results on the Pascal triangle, using $n = 3$ and $n = 4$.

4. (a) By equating the coefficients of x^3 on the RHS and LHS of the identity

$$(1+x)^3(1+x)^9 = (1+x)^{12},$$

show that ${}^3\mathrm{C}_0\,{}^9\mathrm{C}_3 + {}^3\mathrm{C}_1\,{}^9\mathrm{C}_2 + {}^3\mathrm{C}_2\,{}^9\mathrm{C}_1 + {}^3\mathrm{C}_3\,{}^9\mathrm{C}_0 = {}^{12}\mathrm{C}_3$.

(b) By equating the coefficients of x^3 on the RHS and LHS of the identity

$$(1+x)^m(1+x)^n = (1+x)^{m+n},$$

show that ${}^m\mathrm{C}_0\,{}^n\mathrm{C}_3 + {}^m\mathrm{C}_1\,{}^n\mathrm{C}_2 + {}^m\mathrm{C}_2\,{}^n\mathrm{C}_1 + {}^m\mathrm{C}_3\,{}^n\mathrm{C}_0 = {}^{m+n}\mathrm{C}_3$.

DEVELOPMENT

5. (a) Show that ${}^n\mathrm{C}_k = {}^n\mathrm{C}_{n-k}$.

(b) By comparing coefficients of x^{10} on both sides of $(1+x)^{10}(1+x)^{10} = (1+x)^{20}$, show that $\displaystyle\sum_{k=0}^{10} ({}^{10}\mathrm{C}_k)^2 = {}^{20}\mathrm{C}_{10}$.

(c) By comparing coefficients of x^n on both sides of the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$, show that $\displaystyle\sum_{k=0}^{n} ({}^n\mathrm{C}_k)^2 = {}^{2n}\mathrm{C}_n$. Check this identity on the Pascal triangle by adding the squares of the rows indexed by $n = 1, 2, 3, 4, 5$ and 6.

(d) By comparing coefficients of x^{n+1} on both sides of $(1+x)^n(1+x)^n = (1+x)^{2n}$, show that

$${}^n\mathrm{C}_0 \times {}^n\mathrm{C}_1 + {}^n\mathrm{C}_1 \times {}^n\mathrm{C}_2 + {}^n\mathrm{C}_2 \times {}^n\mathrm{C}_3 + \cdots + {}^n\mathrm{C}_{n-1} \times {}^n\mathrm{C}_n = \frac{(2n)!}{(n-1)!\,(n+1)!}.$$

Check this identity on the rows indexed by $n = 3, 4, 5$ and 6 of the Pascal triangle.

(e) Prove that $(1+x)^n\left(1+\dfrac{1}{x}\right)^n = \dfrac{1}{x^n}(1+x)^{2n}$. By equating coefficients of $\dfrac{1}{x}$, give an alternative proof of the result in part (d).

6. (a) By equating coefficients of $\dfrac{1}{x^4}$ in the expansion of $\left(1+\dfrac{1}{x}\right)^4\left(1-\dfrac{1}{x}\right)^4 = \left(1-\dfrac{1}{x^2}\right)^4$, prove that $\left({}^4\mathrm{C}_0\right)^2 - \left({}^4\mathrm{C}_1\right)^2 + \left({}^4\mathrm{C}_2\right)^2 - \left({}^4\mathrm{C}_3\right)^2 + \left({}^4\mathrm{C}_4\right)^2 = {}^4\mathrm{C}_2$.

(b) Generalise this result, and prove it, by considering the expansion of

$$\left(1+\frac{1}{x}\right)^{2n}\left(1-\frac{1}{x}\right)^{2n} = \left(1-\frac{1}{x^2}\right)^{2n}.$$

Check your identity on the Pascal triangle, for $n = 4, 5$ and 6.

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7. (a) By expanding both sides of the identity $(1+x)^{n+4} = (1+x)^n(1+x)^4$, show that

$$\binom{n+4}{r} = \binom{n}{r} + 4\binom{n}{r-1} + 6\binom{n}{r-2} + 4\binom{n}{r-3} + \binom{n}{r-4},$$

and state the necessary restriction on r. Check this identity on the Pascal triangle, using $n = r = 5$ and using $n = 6$ and $r = 4$.

(b) By expanding both sides of the identity $(1+x)^{p+q} = (1+x)^p(1+x)^q$, show that

$$\binom{p+q}{r} = \binom{p}{r} + \binom{p}{r-1}\binom{q}{1} + \binom{p}{r-2}\binom{q}{2} + \cdots + \binom{p}{1}\binom{q}{r-1} + \binom{q}{r},$$

and state the necessary restriction on r.

8. (a) By considering the expansion of $(1+x)^n$, show that:

(i) $\displaystyle\sum_{k=0}^{n} {}^n\mathrm{C}_k = 2^n$ (ii) $\displaystyle\sum_{k=1}^{n} k\,{}^n\mathrm{C}_k = n\,2^{n-1}$

(b) Hence show that $\displaystyle\sum_{k=0}^{n}(k+1)\,{}^n\mathrm{C}_k = 2^{n-1}(n+2)$. Check this identity on the Pascal triangle, using $n = 4$, 5 and 6.

9. (a) Find the coefficient of x^{n+r} in the expansion of $(1+x)^{3n}$.

(b) By writing $(1+x)^{3n}$ as $(1+x)^n(1+x)^{2n}$, prove that for $0 < r \le n$,

$$\binom{n}{0}\binom{2n}{r} + \binom{n}{1}\binom{2n}{r+1} + \binom{n}{2}\binom{2n}{r+2} + \cdots + \binom{n}{n}\binom{2n}{r+n} = \binom{3n}{n+r}.$$

Check this identity on the Pascal triangle, using $n = 4$ and $r = 3$.

10. (a) Evaluate $\displaystyle\int_0^4 (1+x)^n\,dx$.

(b) By expanding $(1+x)^n$ and then integrating, show that $\displaystyle\sum_{r=0}^{n} \frac{4^{r+1}}{r+1}\,{}^n\mathrm{C}_r = \frac{5^{n+1}-1}{n+1}$.

Check this identity on the Pascal triangle, for $n = 3$ and $n = 4$.

11. (a) Show that $x^n(1+x)^n\left(1+\dfrac{1}{x}\right)^n = (1+x)^{2n}$.

(b) Hence prove that $1 + \dbinom{n}{1}^2 + \dbinom{n}{2}^2 + \cdots + \dbinom{n}{n}^2 = \dbinom{2n}{n}$.

12. (a) When the entries of the row 1, 5, 10, 10, 5, 1 indexed by $n = 5$ in Pascal's triangle are multiplied by 0, 1, 2, 3, 4, 5 respectively, the results are 0, 5, 20, 30, 20, 5. Ignoring the zero, this is five times the row 1, 4, 6, 4, 1. Formulate this result algebraically, for $n = 5$ and then for generally n, and prove it using the binomial theorem.

(b) When the entries of the row 1, 5, 10, 10, 5, 1 are divided by 1, 2, 3, 4, 5 and 6 respectively, the result is 1, $2\frac{1}{2}$, $3\frac{1}{3}$, $2\frac{1}{2}$, 1, $\frac{1}{6}$. If you add $\frac{1}{6}$ at the start, this is $\frac{1}{6}$th of the row 1, 6, 15, 20, 15, 6, 1. Formulate this result algebraically, for $n = 5$ and then for general n, and prove it using the binomial theorem.

13. If $(1+x)^n = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$, show that

$$c_0c_2 + c_1c_3 + c_2c_4 + \cdots + c_{n-2}c_n = \frac{(2n)!}{(n+2)!\,(n-2)!}.$$

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14. (a) Consider the row 1, 7, 21, 35, 35, 21, 7, 1 from Pascal's triangle. If a, b, c and d are any four consecutive terms from this row, show that $\dfrac{a}{a+b} + \dfrac{c}{c+d} = \dfrac{2b}{b+c}$.

(b) Choose four consecutive terms from any other row and show that the identity holds.

(c) Prove the identity by letting $a = {}^n\mathrm{C}_{r-1}$, $b = {}^n\mathrm{C}_r$, $c = {}^n\mathrm{C}_{r+1}$ and $d = {}^n\mathrm{C}_{r+2}$. You will need to use the addition property of Pascal's triangle.

15. (a) By using the substitution $u = \sin x$, prove that $\displaystyle\int_0^{\frac{\pi}{2}} (\sin x)^{2k} \cos x\, dx = \frac{1}{2k+1}$, where k is a positive integer.

(b) By writing $\cos^{2n+1} x = \cos^{2n} x \cos x = (1-\sin^2 x)^n \cos x$, show that

$$\cos^{2n+1} x = \sum_{k=0}^{n} {}^n\mathrm{C}_k\, (-1)^k \sin^{2k} x \cos x.$$

(c) Hence, by using part (a), show that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n+1} x\, dx = \sum_{k=0}^{n} \frac{(-1)^k\, {}^n\mathrm{C}_k}{2k+1}$.

(d) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5 x\, dx$.

EXTENSION

16. [The hockey stick pattern] Use induction to prove the result from question 22(c) of Exercise 5A. That is, for any fixed value of n and p, where $n < p$, prove that

$${}^n\mathrm{C}_0 + {}^{n+1}\mathrm{C}_1 + {}^{n+2}\mathrm{C}_2 + \cdots + {}^{n+p}\mathrm{C}_p = {}^{n+p+1}\mathrm{C}_p.$$

You will need to use the addition property of Pascal's triangle.

17. By substituting $x = 1$ into the expansion of $(1+x)^{2n}$, prove that $\displaystyle\sum_{r=0}^{n} {}^{2n}\mathrm{C}_r = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$.

18. (a) Show that $1 + (1+x) + (1+x)^2 + \cdots + (1+x)^n = \dfrac{(1+x)^{n+1} - 1}{x}$.

(b) By considering the coefficient of x^r on both sides of this identity, show that

$${}^n\mathrm{C}_r + {}^{n-1}\mathrm{C}_r + {}^{n-2}\mathrm{C}_r + \cdots + {}^r\mathrm{C}_r = {}^{n+1}\mathrm{C}_{r+1}.$$

19. By considering the identity $(1-x^2)^n = (1+x)^n(1-x)^n$ or otherwise, show that

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n \binom{n}{n}^2$$

is zero, when n is odd, but that when n is even, its value is

$$\frac{(-1)^{\frac{n}{2}}(n+2)(n+4)\cdots(2n)}{2 \times 4 \times \cdots \times n} = \frac{(-1)^{\frac{n}{2}}\, n!}{\left(\left(\frac{n}{2}\right)!\right)^2}.$$

20. [APs in the Pascal triangle]

(a) Show that ${}^n\mathrm{C}_{r-1} = \dfrac{r}{n-r+1}\, {}^n\mathrm{C}_r$ and ${}^n\mathrm{C}_{r+1} = \dfrac{n-r}{r+1}\, {}^n\mathrm{C}_r$.

(b) Show that if ${}^n\mathrm{C}_{r-1}$, ${}^n\mathrm{C}_r$ and ${}^n\mathrm{C}_{r+1}$ form an AP, then

(i) $n + 2 = (n-2r)^2$ is a perfect square, and

(ii) $r = \frac{1}{2}(n - \sqrt{n+2})$ or $r = \frac{1}{2}(n + \sqrt{n+2})$.

(c) Hence find the first three rows of the Pascal triangle in which three consecutive terms form an AP, and identify those terms.

(d) Prove that four consecutive terms in a row of the Pascal triangle cannot form an AP.

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