The University of Sydney

MATH1901/06 Differential Calculus (Advanced)

Semester 1 **Practice Questions for Quiz 2** 2012

Quiz 2 will be held in the tutorials in **Week 12** (Monday 28 May 2012).

The quiz questions will be based on material covered in the lectures during Weeks 6–10, which corresponds to material covered in tutorials in Weeks 7–11.

Topics to be tested include derivatives, critical points, extrema, corner points, vertical tangents, l'Hôpital's rule, Rolle's theorem, mean value theorem, Taylor's formula with remainder, applications of Taylor polynomials (approximations, l'Hôpital-type limits), new Taylor polynomials from old, level curves and limits of functions of two variables, partial derivatives.

The quiz will run for 40 minutes under exam conditions. It is worth 10% of the assessment for MATH1901. Use a pen, not a pencil. You may use a non-programmable calculator. At the start of the quiz, place all materials on the floor except your pen, your calculator and your student ID card face up on your desk. Switch off mobile phones.

Note that a calculator is optional for this quiz. However, it is important that you bring an approved 10-digit calculator to the main exam because one or two exam questions will definitely need a calculator.

Solutions to these problems appear below after Question 10.

The actual quiz questions will be considerably shorter than these practice questions and will not have multiple parts. Some quiz questions will be multiple choice.

1. (a) Calculate the following limits. (Some may be assisted by l'Hôpital's rule, some by Taylor polynomials, some by neither of these methods.)

$$\lim_{x\to 1}\frac{x^3-5x+4}{x^3-4x+3}, \qquad \lim_{x\to 0^+}(\sinh x)^{1/x}, \qquad \lim_{x\to\infty}(\sinh x)^{1/x},$$

$$\lim_{x\to 2}\frac{x^x-4}{2^x-4}, \qquad \lim_{x\to 0^+}x^x, \qquad \lim_{x\to 0^+}\frac{d}{dx}\,x^x,$$

$$\lim_{x\to 0}\frac{(ax-\sin ax)^2}{(1-\cos bx)^3}, \qquad \lim_{x\to 0}\frac{e^{-1/x^2}}{x^n}, \quad n=0,1,2,3,\ \ldots .$$

(b) Show that the following sequences are increasingly rapidly growing:

$$\{n^{100}\},\quad \{n^{\ln n}\},\quad \{e^{\sqrt{n}}\},\quad \{n^{\sqrt{n}}\},\quad \{2^n\},\quad \{n!\},\quad \{n^n\},\quad \{2^{n^2}\}.$$

(Use the notation, $2^n << n!$ or $n! >> 2^n$, etc.)

2. Show from first principles that $(d/dx)\sin x = \cos x$ and $(d/dx)\cos x = -\sin x$. You will need to use the limit $(\sin\theta)/\theta \to 1$ as $\theta \to 0$, proved in lectures.

3. Suppose $f(x)$ is a function that is continuous on $[0,1]$ and differentiable on $(0,1)$. Which of the following statements are TRUE and which are FALSE?

(a) $f(x)$ must have a right derivative at $x = 0$.

(b) If $f'(x)$ tends to a finite limit L as $x \to 0^+$, then $f(x)$ necessarily has a right derivative at $x = 0$ whose value is L.

(c) If $f'(x)$ tends to either $+\infty$ or $-\infty$ as $x \to 0^+$, then the graph of $y = f(x)$ on $[0,1]$ must have a one-sided vertical tangent at $x = 0$.

(d) If $f'(x)$ does not tend to a finite limit as $x \to 0^+$, then $f(x)$ cannot have a right derivative at $x = 0$.

(e) If $f(x)$ satisfies the inequality $ax \le f(x) \le bx$ on $[0,1]$ for particular real numbers a, b, $a < b$, then $f'(x)$ satisfies $a \le f'(x) \le b$ on $(0,1)$.

(f) There exists a point $c \in (0,1)$ such that $f'(c) = f(1) - f(0)$.

(g) The point c in the previous part is unique when it exists.

4. (a) Show that the graph of $y = x^{1/3}$ has a vertical tangent and an inflection at $x = 0$.

(b) Suppose the function $f(x)$ has a zero first derivative and a positive second derivative at $x = x_0$, but is possibly not twice differentiable anywhere else. Prove that $f(x)$ has a local minimum at $x = x_0$.

(c) Show that the graph of $y = \sin^{-1}(1 - x^2)$ has a corner point at $x = 0$.

(d) Find all values of the positive real parameters a and b such that the function,

$$f(x) = \begin{cases} |x|^a \sin(1/|x|^b), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

has a second derivative $f''(0)$ at $x = 0$. When is $f''(x)$ continuous at $x = 0$?

(e) Show that $(1 + 1/x)^x$ is increasing for all $x > 0$.

(f) Without using calculus, show that the sequence $(1 + 1/n)^n$ is increasing as n runs through the positive integers. Show also that the sequence tends to a limit as $n \to \infty$, and to the same limit as $n \to -\infty$. (This is one of several ways to define the transcendental number e.)

5. (a) The function $f(x) = \sin^{-1} x$ has the derivative $f'(x) = (1 - x^2)^{-1/2}$ on the interval $(-1,1)$. Use the binomial series to deduce the Taylor polynomial $T_{2n}(x)$ of order $2n$ for $\sin^{-1} x$ about $x = 0$. (Because the inverse sine is odd, the actual degree of $T_{2n}(x)$ is $2n - 1$.) Express the coefficients in terms of binomial coefficients of index $-1/2$ and also in terms of integer factorials.

(b) Do the same for $g(x) = \tan^{-1} x$.

6. (a) Define the function,

$$G(x) = \begin{cases} \cos\sqrt{x}\,, & x \geq 0, \\ \cosh\sqrt{-x}\,, & x < 0. \end{cases}$$

Show that $G(x)$ is differentiable to all orders at $x = 0$ and give an explicit formula for the nth derivative $G^{(n)}(0)$. [*Hint*. Consider the Taylor polynomials of order $2n$ for $\cos x$ and $\cosh x$ about $x = 0$.]

(b) Do the same for the function, $H(x) = \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(c) In terms of factorials, write down the 50th derivative of $\sin(x^{10})$ at $x = 0$.

7. (a) Show that for a quadratic polynomial, the number c in the Mean Value Theorem (as it is stated in the lecture notes) is always the midpoint of the interval $[a, b]$, $a < b$.

(b) On the other hand, show that c is never the midpoint in the case of a cubic polynomial. Find c in the case when the chord joining the endpoints of the cubic curve segment on $[a, b]$ is tangent to the curve at $x = b$.

8. Let $f : [-1, 3] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 4x^2 + x, & -1 \leq x < 0, \\ 2\sqrt{x}\,, & 0 \leq x < 1, \\ (4x^3 - 21x^2 + 36x - 7)/6, & 1 \leq x \leq 3. \end{cases}$$

Find all the critical points of f on $(-1, 3)$, identify each type of critical point (horizontal tangent, vertical tangent, vertical cusp, corner point, other), and decide which critical points are local extrema. Find also the absolute extrema of f on $[-1, 3]$.

9. (a) Draw a set of level curves (corresponding to equally spaced heights) for the following functions:

(i). $f(x, y) = e^{x^2+y^2}$.

(ii). $g(x, y) = \dfrac{xy}{x^2 + y^2}$.

(iii). $h(x, y) = \left(\sqrt{x} + \sqrt{y}\right)^2$.

(b) Calculate all the first- and second-order partial derivatives of the functions in part (a).

10. Calculate the following two-dimensional limits, or prove that they do not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + 2y^3}{x^2 + y^2}\,, \qquad \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2 + y^2}\,, \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}\,.$$

Solutions:

1. (a) Denote the eight parts of this exercise (i), (ii), $\ldots$, $(viii)$.

(i). The limit,

$$\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3},$$

was done in the Practice Questions for Quiz 1 by dividing out the common factor $x - 1$. It can also be done by l'Hôpital's rule for 0/0-type limits:

$$\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3} = \lim_{x\to 1} \frac{3x^2 - 5}{3x^2 - 4} = \left[\frac{3x^2 - 5}{3x^2 - 4} \right]_{x=1} = 2.$$

(ii). Let $L = \lim_{x\to 0^+} (\sinh x)^{1/x}$. The continuity of the logarithm function allows us to write $\ln L = \lim_{x\to 0^+} (\ln \sinh x)/x$ (even though we will end up on the boundary of the domain of continuity). This is not a l'Hôpital problem. The numerator tends to $-\infty$ and the denominator tends to 0^+. Hence, $\ln L = -\infty$ and the required limit is $L = 0$.

Another way to approach this problem is to use the fact that $(\sinh x)/x \to 1$ as $x \to 0$. Then, for some interval $0 < x < \delta$, we have $0 < \sinh x < 2x$ and $0 < (\sinh x)^{1/x} < (2x)^{1/x}$. Since $(2x)^{1/x} \to 0$ as $x \to 0^+$, it follows that $(\sinh x)^{1/x} \to 0$ as $x \to 0^+$.

The required limit can be said to be of 0^∞ type. This is not an indeterminate form, and all such limits are zero.

(iii). Let $L = \lim_{x\to\infty} (\sinh x)^{1/x}$. This can be done with or without l'Hôpital's rule. Take logarithms and use l'Hôpital's rule for ∞/∞-type limits:

$$\ln L = \lim_{x\to\infty} \frac{\ln \sinh x}{x} = \lim_{x\to\infty} \frac{\coth x}{1} = 1,$$

which implies that $L = e$. Alternatively,

$$(\sinh x)^{1/x} = \left(\frac{e^x - e^{-x}}{2} \right)^{1/x} = \frac{e}{2^{1/x}} \left(1 - e^{-2x}\right)^{1/x} \to \frac{e}{1} (1 - 0)^0 = e,$$

as $x \to \infty$. Either way, $L = e$.

(iv). The limit $\lim_{x\to 2} (x^x - 4)/(2^x - 4)$ is of l'Hôpital 0/0 type. We need the derivatives of x^x and 2^x. Use logarithmic differentiation to differentiate variable powers. Let $y = f(x)^{g(x)}$, on a domain where $f(x) > 0$. Then,

$$\ln y = g(x) \ln f(x), \qquad \frac{y'}{y} = g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)},$$

$$\frac{d}{dx} f(x)^{g(x)} = f(x)^{g(x)} \left(g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right).$$

In particular,

$$\frac{d}{dx} x^x = x^x (1 + \ln x), \qquad \frac{d}{dx} a^x = (\ln a) a^x, \qquad a > 0.$$

Hence, l'Hôpital's rule gives

$$\lim_{x\to 2} \frac{x^x - 4}{2^x - 4} = \lim_{x\to 2} \frac{x^x (1 + \ln x)}{2^x \ln 2} = \frac{2^2 (1 + \ln 2)}{2^2 \ln 2} = \frac{1 + \ln 2}{\ln 2}.$$

(v). Let $L = \lim_{x\to 0^+} x^x$. Then

$$\ln L = \lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0,$$

which implies that the required limit is $\lim\limits_{x\to 0^+} x^x = 1$.

(vi). Since $(d/dx)x^x = x^x(1+\ln x)$, while $x^x \to 1$ and $\ln x \to -\infty$ as $x \to 0^+$, it follows that the required limit is $\lim_{x\to 0^+}(d/dx)x^x = -\infty$. Thus the graph of $y = x^x$ has a one-sided vertical tangent at $x = 0$, approaching $y = 1$ from below.

(vii). The limit,

$$\lim_{x\to 0}\frac{(ax-\sin ax)^2}{(1-\cos bx)^3},$$

can be evaluated in several ways. L'Hôpital's rule certainly works, but it requires six applications. The quickest way is to use Taylor polynomials of suitable degree. The Taylor polynomial of order three or four of $\sin ax$ is $ax-(ax)^3/3!$. The Taylor polynomial of order two or three of $\cos bx$ is $1-(bx)^2/2!$. Hence,

$$\lim_{x\to 0}\frac{(ax-\sin ax)^2}{(1-\cos bx)^3} = \lim_{x\to 0}\frac{(a^3x^3/6)^2}{(b^2x^2/2)^3} = \lim_{x\to 0}\frac{a^6x^6/36}{b^6x^6/8} = \frac{2a^6}{9b^6}.$$

Alternatively, one can first obtain the auxiliary limits,

$$L_1 = \lim_{x\to 0}\frac{ax-\sin ax}{x^3} = \frac{a^3}{6}, \qquad L_2 = \lim_{x\to 0}\frac{1-\cos bx}{x^2} = \frac{b^2}{2},$$

which can be done with either Taylor or l'Hôpital (the second limit also appearing in the Practice Questions for Quiz 1 as a consequence of $(\sin x)/x \to 1$ as $x \to 0$). The required limit is

$$\frac{(L_1)^2}{(L_2)^3} = \frac{a^6/36}{b^6/8} = \frac{2a^6}{9b^6}.$$

($viii$). The case $n = 0$ is the limit $e^{-1/x^2} \to 0$ as $x \to 0$ (two-sided). For $n = 1, 2, 3, \ldots$, l'Hôpital's rule works if we recast the given $0/0$-type limit in ∞/∞ form. (If this is not done, l'Hôpital raises the index n and goes the wrong way.) We find

$$\lim_{x\to 0}\frac{e^{-1/x^2}}{x^n} = \lim_{x\to 0}\frac{x^{-n}}{e^{1/x^2}} = \lim_{x\to 0}\frac{-nx^{-n-1}}{-(2/x^3)e^{1/x^2}} = \frac{n}{2}\lim_{x\to 0}\frac{x^{-n+2}}{e^{1/x^2}}.$$

This proves that the limit is zero in the cases $n = 1, 2$. For $n > 2$, use l'Hôpital repeatedly or argue by mathematical induction. Either way,

$$\lim_{x\to 0}\frac{e^{-1/x^2}}{x^n} = 0, \qquad n = 0, 1, 2, 3, \ldots.$$

Another approach is to restrict attention to the right limit $x \to 0^+$ (the left limit being plus or minus the right limit) and let $x = y^{-1/2}$. Then

$$\lim_{x\to 0^+}\frac{e^{-1/x^2}}{x^n} = \lim_{y\to\infty}\frac{y^{n/2}}{e^y}.$$

Repeated application of l'Hôpital's rule brings this to the limit of $A_n e^{-y}$ or $B_n y^{-1/2}e^{-y}$ as $y \to \infty$, according as n is even or odd. The final limit reached is obviously zero. A third option is to write

$$\lim_{y\to\infty}\frac{y^{n/2}}{e^y} = \lim_{y\to\infty}\left(\frac{y}{e^{2y/n}}\right)^{n/2}.$$

Now a single application of l'Hôpital's rule and the substitution law for limits gives the limit zero.

This exercise implies that the function $f(x) = e^{-1/x^2}$ with $f(0) = 0$ has vanishing derivatives to all orders at $x = 0$. Hence $f(x)$ has a convergent Taylor series about $x = 0$ that converges to a different function (the zero function).

(b) We have seven comparisons to make between eight sequences that diverge monotonically to $+\infty$ (at least from some term onwards). Suppose $\{a_n\}$ is any sequence of real or complex numbers and $\{b_n\}$ is a monotonic sequence of positive real numbers that tends to 0 or $+\infty$. Use the notation $a_n << b_n$ and $b_n >> a_n$ to denote the fact that $|a_n|/b_n \to 0$ as $n \to \infty$. When both sequences are real and diverge to $+\infty$, it is useful to know that $a_n << b_n$ implies $e^{a_n} << e^{b_n}$ and vice versa.

(*i*). $n^{100} = e^{100 \ln n}$ and $n^{\ln n} = e^{(\ln n)^2}$. Since $(\ln n)^2 >> 100 \ln n$, it follows that $n^{\ln n} >> n^{100}$. Similarly $n^{\ln n} >> n^k$ for every fixed $k > 0$ (and trivially for $k \le 0$).

(*ii*). L'Hôpital's rule implies that $\sqrt{n} >> (\ln n)^2$. Hence, $e^{\sqrt{n}} >> n^{\ln n}$.

(*iii*). $n^{\sqrt{n}} = e^{\sqrt{n} \ln n}$. Since $\sqrt{n} \ln n >> \sqrt{n}$, it follows that $n^{\sqrt{n}} >> e^{\sqrt{n}}$.

(*iv*). $2^n = e^{n \ln 2}$. L'Hôpital's rule implies that $n \ln 2 >> \sqrt{n} \ln n$. Hence $2^n >> n^{\sqrt{n}}$.

(*v*). We will show that $n! >> a^n$ for every real a. Without loss of generality, we can take $a > 1$. Let m be a fixed integer $> 2a$ and let $n > m$. Then

$$\begin{aligned} \frac{a^n}{n!} &= \frac{a \cdot a \cdot a \cdots a}{1 \cdot 2 \cdot 3 \cdots n} \\ &= \frac{a^m}{m!} \left(\frac{a}{m+1} \right) \left(\frac{a}{m+2} \right) \cdots \left(\frac{a}{n} \right) \\ &< \frac{a^m}{m!} \left(\frac{1}{2} \right)^{n-m} \\ &= \frac{A_m}{2^n}, \end{aligned}$$

where A_m is a constant depending on a and m. Since $A_m/2^n \to 0$ as $n \to \infty$, it follows that $a^n/n! \to 0$ as $n \to \infty$. Thus $n! >> a^n$. In particular $n! >> 2^n$.

(*vi*).

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} < \frac{1}{n}.$$

Hence $n^n >> n!$. (More generally, $n^n >> a^n n!$ for $0 < a < e$ and $n^n << a^n n!$ for $a \ge e$.)

(*vii*). $n^n = e^{n \ln n}$ and $2^{n^2} = e^{n^2 \ln 2}$. Since $n^2 \ln 2 >> n \ln n$, it follows that $2^{n^2} >> n^n$.

2. The derivatives of $\sin x$ and $\cos x$ are

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \sin x \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) + \cos x \left(\lim_{h \to 0} \frac{\sin h}{h} \right), \\ \frac{d}{dx} \cos x &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \cos x \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) - \sin x \left(\lim_{h \to 0} \frac{\sin h}{h} \right). \end{aligned}$$

The addition theorems for $\sin(x+h)$ and $\cos(x+h)$ are geometric in origin and do not depend on calculus or on complex exponentials. [Exercise: derive the addition theorems from $\sin\theta = \mathrm{opp}/\mathrm{hyp}$ and $\cos\theta = \mathrm{adj}/\mathrm{hyp}$.] The two-sided limit,

$$\lim_{h\to 0} \frac{\sin h}{h} = 1,$$

was proved in lectures by a geometric argument. (It is also proved in the lecture notes, pages 43–44, by a different geometric argument.) In particular, we showed that $\sin h < h < \tan h$ for $0 < h < \pi/2$. An exercise in the Practice Questions for Quiz 1 included the result,

$$\lim_{h\to 0} \frac{1-\cos h}{h^2} = \lim_{h\to 0} \frac{1+\cos h}{2} \lim_{h\to 0} \frac{1-\cos h}{h^2} = \lim_{h\to 0} \frac{1-\cos^2 h}{2h^2} = \lim_{h\to 0} \frac{\sin^2 h}{2h^2} = \frac{1}{2}.$$

Multiplying by $-h$ gives the limit that we want here:

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

Note that it is logically incorrect to use l'Hôpital's rule in these instances, as it assumes results that we are trying to prove. We now have the two auxiliary limits that we need to complete the proof of

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x.$$

3. The seven statements are, in turn, False, True, True, False, False, True, False. The false statements can be dismissed quickly with a counerexample. Let $g_a(x)$, $a > 0$, denote the function $g_a(x) = |x|^a \sin(1/x)$, with $g_a(0) = 0$, which is continuous everywhere. We are given that $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$.

(a) "$f(x)$ must have a right derivative at $x = 0$." This is false because the graph of $y = x^a$, $0 < a < 1$, on $[0,1]$ has a one-sided vertical tangent at the left endpoint $x = 0$. In MATH1901, we do not count infinite derivatives as valid derivatives, but some analysis texts do allow them. So look instead at the function $g_a(x)$ with $0 < a \le 1$, whose difference quotient $(g_a(x) - g_a(0))/x$ behaves rather badly in the limit $x \to 0^+$.

(b) "If $f'(x)$ tends to a finite limit L as $x \to 0^+$, then $f(x)$ necessarily has a right derivative at $x = 0$ whose value is L." This is true and is a consequence of the Mean Value Theorem, whose conditions are satisfied here. On the interval $[0,\delta]$, the MVT states that

$$\frac{f(x) - f(0)}{x} = f'(c)$$

for some c such that $0 < c < x$. If $x \to 0^+$, then c is forced to tend to 0^+ as well (possibly in irregular steps, skipping values). Hence, if $f'(x)$ tends to a limit L as $x \to 0^+$, then $f'(c)$ will tend to the same limit and the difference quotient $(f(x) - f(0))/x$ will also tend to the same limit. In other words, the right derivative $f'_+(0)$ exists and equals L.

(c) "If $f'(x)$ tends to either $+\infty$ or $-\infty$ as $x \to 0^+$, then the graph of $y = f(x)$ on $[0,1]$ must have a one-sided vertical tangent at $x = 0$." This is also true and is a similar consequence of the MVT. If $f'(x) \to +\infty$ as $x \to 0^+$, the difference quotient $(f(x) - f(0))/x$ will also tend to $+\infty$ as $x \to 0^+$. This is precisely what identifies a tangent ray pointing vertically up at the left endpoint $x = 0$ on the graph of $y = f(x)$. Similarly $f'(x) \to -\infty$ as $x \to 0^+$ implies that the graph of $y = f(x)$ has a tangent ray pointing vertically down at $x = 0$. The two cases are illustrated by $f(x) = x^a$ and $f(x) = -x^a$ with $0 < a < 1$.

(d) "If $f'(x)$ does not tend to a finite limit as $x \to 0^+$, then $f(x)$ cannot have a right derivative at $x = 0$." This is false, though perhaps not obviously so. The function $g_a(x)$ with $1 < a \leq 2$ provides a counterexample. When $x > 0$, the product and chain rules give

$$g_a'(x) \;=\; \frac{d}{dx} x^a \sin(1/x) \;=\; a x^{a-1} \sin(1/x) - x^{a-2} \cos(1/x).$$

The second term has unbounded oscillations as $x \to 0^+$ when $1 < a < 2$. When $a = 2$, the oscillations are bounded. Either way, $g_a'(x)$ does not tend to a limit as $x \to 0^+$. However, $g_a'(0)$ exists as a two-sided derivative for all $a > 1$ because

$$\begin{aligned} g_a'(0) \;&=\; \lim_{x\to 0} \frac{g_a(x) - g_a(0)}{x} \\ &=\; \lim_{x\to 0} \frac{|x|^a \sin(1/x) - 0}{x} \\ &=\; \lim_{x\to 0} \pm |x|^{a-1} \sin(1/x) \\ &=\; 0. \end{aligned}$$

The last step is justified by the squeeze lemma when $a > 1$.

This example shows that $f'(x)$ can exist on an interval and have an isolated discontinuity at one point, say x_0. However, regardless of whether or not $f'(x_0)$ exists, part (b) shows that the discontinuity in $f'(x)$ cannot be a removable discontinuity.

(e) "If $f(x)$ satisfies the inequality $ax \leq f(x) \leq bx$ on $[0, 1]$ for particular real numbers a, b, $a < b$, then $f'(x)$ satisfies $a \leq f'(x) \leq b$ on $(0, 1)$." This is false because a differentiable function can have rapid oscillations with small amplitudes. The given bounds on $f(x)$ are satisfied by

$$f(x) \;=\; \tfrac{1}{2}(b + a)x + \tfrac{1}{2}(b - a) g_c(x), \qquad c \geq 1.$$

However, $f'(x)$ has unbounded oscillations when $1 \leq c < 2$ and therefore cannot be contained between a and b.

(f) "There exists a point $c \in (0, 1)$ such that $f'(c) = f(1) - f(0)$." This is true because it is precisely what the MVT states for $f(x)$ on $[0, 1]$.

(g) "The point c in the previous part is unique when it exists." This is false because a differentiable function $f(x)$ can have several turning points or changes of concavity on $(0, 1)$. If any line with slope $f(1) - f(0)$ cuts the graph three or more times, then the MVT on each subinterval provides a distinct value of c. So c is only unique in special circumstances.

4. (a) A direct proof that the graph of $y = f(x) = x^{1/3}$ has a vertical tangent at $x = 0$ can be provided by the difference quotient:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} \;=\; \lim_{x\to 0} \frac{x^{1/3}}{x} \;=\; \lim_{x\to 0} x^{-2/3} \;=\; +\infty.$$

The limit is two-sided, and so the vertical tangent is two-sided and the curve is smooth there (in contrast to the case of a vertical cusp).

In view of the statement in Question **3**(c), a vertical tangent can be identified more quickly by the following two-sided limit:

$$\frac{d}{dx} x^{1/3} \;=\; \frac{1}{3} x^{-2/3} \;\to\; +\infty,$$

as $x \to 0$. (Of course, not all vertical tangents can be captured this way. Consider the case $f(x) = x^{1/3} + x^{6/5} \sin(1/x)$.)

To prove that $x = 0$ is also a point of inflection of $x^{1/3}$, consider $f''(x)$ on either side. For $x \neq 0$,

$$f'(x) = \frac{1}{3}x^{-2/3}, \qquad f''(x) = -\frac{2}{9}x^{-5/3}.$$

Thus $f''(x) < 0$ for $x > 0$ and $f''(x) > 0$ for $x < 0$. Thus $f(x)$ changes concavity from up to down at $x = 0$, which is precisely what identifies $x = 0$ as a point of inflection.

Another way to view this problem is to observe that $f : \mathbf{R} \to \mathbf{R}$, $x \mapsto x^{1/3}$, is bijective and has an inverse function $f^{-1}(x) = x^3$. The horizontal tangent and inflection of x^3 at $x = 0$ implies the vertical tangent and inflection of $x^{1/3}$ at $x = 0$ and vice-versa.

(b) We are given that $f'(x_0) = 0$ and $f''(x_0) > 0$. We can give a quick answer if $f''(x)$ is assumed to be continuous at x_0, for then $f''(x) > 0$ on an interval covering x_0. That would imply that $f(x)$ is concave up, and therefore has a local minimum at a point where the tangent is horizontal. The question asked us not to assume that $f''(x)$ even exists away from x_0. Nevertheless, several conclusions can be drawn from the existence of $f''(x)$ at just the single point x_0. The definition of derivative requires that $f'(x)$ be defined and bounded on an interval covering x_0 and be continuous at x_0 itself. Then $f(x)$ must be continuous on that same interval. The existence of $f''(x_0)$ and the given value $f'(x_0) = 0$ imply

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0}.$$

Given arbitrary $\epsilon > 0$, there exists $\delta > 0$ (depending on ϵ) such that

$$\left| \frac{f'(x)}{x - x_0} - f''(x_0) \right| < \epsilon$$

for all x such that $0 < |x - x_0| < \delta$. For such x,

$$f''(x_0) - \epsilon < \frac{f'(x)}{x - x_0} < f''(x_0) + \epsilon.$$

Since $f''(x_0) > 0$, we can choose ϵ anywhere in the interval $0 < \epsilon < f''(x_0)/2$. (We don't need ϵ to be small in this case.) Then

$$\frac{f'(x)}{x - x_0} > \frac{f''(x_0)}{2}$$

for all x such that $0 < |x - x_0| < \delta$. This result implies

$$f'(x) > \frac{f''(x_0)}{2}(x - x_0) > 0, \qquad x_0 < x < x_0 + \delta,$$

$$f'(x) < \frac{f''(x_0)}{2}(x - x_0) < 0, \qquad x_0 - \delta < x < x_0.$$

These inequalities imply that $f(x)$ is strictly increasing for $x > x_0$ near x_0 and strictly decreasing for $x < x_0$ near x_0. Hence, we have proved that $f(x)$ has a strict local minimum at x_0 under the stated hypotheses. (In case you were wondering what a non-strict local minimum looks like, consider $x^2 \sin^2(1/x)$ near $x = 0$.)

(c) The corner point in the graph of $y = f(x) = \sin^{-1}(1 - x^2)$ can be seen with or without taking a derivative. If we do it with derivatives, we need to know that

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \qquad -1 < x < 1,$$

which can be proved by differentiating the identity $\sin(\sin^{-1} x) = x$ with the chain rule. Then, for $x \neq 0$, the chain rule gives

$$\begin{aligned} f'(x) &= \frac{d}{dx} \sin^{-1}(1-x^2) \\ &= -\frac{2x}{\sqrt{1-(1-x^2)^2}} \\ &= -\frac{2x}{\sqrt{2x^2-x^4}} \\ &= -\frac{2x}{|x|\sqrt{2-x^2}}\,. \end{aligned}$$

Since $f'(x)$ has one-sided limits as $x \to 0^{\pm}$, these are the one-sided derivatives at $x = 0$, according to Question **3**(b). So we get the right and left derivatives,

$$f'_{+}(0) = -\sqrt{2}\,, \qquad f'_{-}(0) = \sqrt{2}\,.$$

Non-equal one-sided derivatives imply that the graph has a corner point at $x = 0$. It is also a local and absolute maximum (which is obvious from the original function). More generally, any continuous even function with a nonzero right derivative at $x = 0$ has a left derivative of opposite sign, and therefore a corner point at $x = 0$.

To see the corner point without taking derivatives, we exploit the limit $(\sin x)/x \to 1$ as $x \to 0$. Use the notation $f(x) \sim g(x)$ as $x \to x_0$ to mean that $f(x)/g(x) \to 1$ as $x \to x_0$. In particular, $\sin x \sim x$ as $x \to 0$, which implies that $\sin^{-1} x \sim x$ as $x \to 0$. We need to express the given inverse sine in terms of an inverse sine of small argument. Apply the trigonometric identity,

$$\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x = \frac{\pi}{2} - \sin^{-1}\sqrt{1-x^2}\,,$$

which is valid for $0 \leq x \leq 1$. We find, for $-1 < x < 1$,

$$\begin{aligned} \sin^{-1}(1-x^2) &= \frac{\pi}{2} - \cos^{-1}(1-x^2) \\ &= \frac{\pi}{2} - \sin^{-1}\sqrt{1-(1-x^2)^2} \\ &= \frac{\pi}{2} - \sin^{-1}\sqrt{2x^2-x^4} \\ &= \frac{\pi}{2} - \sin^{-1}\left(|x|\sqrt{2-x^2}\,\right) \\ &\sim \frac{\pi}{2} - |x|\sqrt{2-x^2} \qquad \text{as } x \to 0 \\ &\sim \frac{\pi}{2} - \sqrt{2}\,|x| \qquad \text{as } x \to 0. \end{aligned}$$

Thus $\sin^{-1}(1-x^2)$ has a corner point of the same character as the absolute value function (upside-down because of the minus sign, steeper on account of the factor $\sqrt{2}$, which is the left derivative at $x = 0$).

(d) We are given the function,

$$f(x) = \begin{cases} |x|^a \sin(1/|x|^b), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad a > 0, \quad b > 0,$$

which is even and continuous at $x = 0$. Before we can examine second derivatives, we need to know when $f(x)$ has a first derivative at $x = 0$. Consider the difference quotient,

$$\frac{f(x) - f(0)}{x} \;=\; \frac{|x|^a}{x}\,\sin(1/|x|^b) \;=\; \pm\,|x|^{a-1}\sin(1/|x|^b).$$

Since this is squeezed between $|x|^{a-1}$ and $-|x|^{a-1}$, the limit $x \to 0$ exists and equals zero whenever $a > 1$. The limit does not exist when $a \le 1$. Thus

$$f'(0) \;=\; 0, \qquad a > 1,$$

and $f'(0)$ does not exist for $a \le 1$. To find when $f'(x)$ is continuous at $x = 0$, apply the product and chain rules when $x > 0$:

$$f'(x) \;=\; \frac{d}{dx}x^a\sin(x^{-b}) \;=\; ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b}).$$

The evenness of $f(x)$ implies $f'(-x) = -f'(x)$. So $f'(x) \to 0$ when $x \to 0$ (two-sided) whenever $a > b+1$, while $f'(x)$ oscillates as $x \to 0$ if $a \le b+1$. Hence $f'(x)$ is continuous at $x = 0$ whenever $a > b+1$.

To get the second derivative $f''(0)$, assume that $a > b+1$ and consider, for $x > 0$, the difference quotient,

$$\frac{f'(x) - f'(0)}{x} \;=\; ax^{a-2}\sin(x^{-b}) - bx^{a-b-2}\cos(x^{-b}).$$

This tends to zero whenever $a > b+2$ and does not tend to a limit when $a \le b+2$. Because $f''(x)$ is even, the limit from the left is the same. Hence,

$$f''(0) \;=\; 0, \qquad a > b+2,$$

and $f''(0)$ does not exist for $a \le b+2$. To find when $f''(x)$ is continuous at $x = 0$, apply the product and chain rules as before with $x > 0$:

$$\begin{aligned} f''(x) \;&=\; \frac{d}{dx}\big\{ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b})\big\} \\ &=\; a(a-1)x^{a-2}\sin(x^{-b}) - b(2a-b-1)x^{a-b-2}\cos(x^{-b}) - b^2x^{a-2b-2}\sin(x^{-b}). \end{aligned}$$

This tends to zero whenever $a > 2b+2$ and does not tend to a limit when $a \le 2b+2$. We conclude that $f''(0)$ exists if and only if $a > b+2$ and $f''(x)$ is continuous at $x = 0$ if and only if $a > 2b+2$.

(e) Let $f(x) = (1+1/x)^x$ and $g(x) = \ln f(x) = x\ln(1+1/x)$ for $x > 0$. Then $f(x)$ is increasing if and only if $g(x)$ is increasing. Differentiate $g(x)$:

$$\begin{aligned} g'(x) \;&=\; \frac{d}{dx}x\ln\Big(1+\frac{1}{x}\Big) \\ &=\; \frac{d}{dx}x\big\{\ln(1+x) - \ln x\big\} \\ &=\; \ln(1+x) - \ln x + x\Big(\frac{1}{1+x} - \frac{1}{x}\Big) \\ &=\; \ln(1+x) - \ln x - \frac{1}{1+x}\,. \end{aligned}$$

Observe that $g'(x) \to 0$ as $x \to \infty$. Since the sign of $g'(x)$ is not immediately obvious, take another derivative:

$$g''(x) \;=\; \frac{1}{1+x} - \frac{1}{x} + \frac{1}{(1+x)^2} \;=\; -\,\frac{1}{x(x+1)^2}\,.$$

Hence $g''(x)$ is negative and $g'(x)$ is decreasing. Since $g'(x)$ decreases to zero in the limit $x \to \infty$, it follows that $g'(x)$ is positive for all $x > 0$. Thence $g(x)$ is increasing and therefore also $f(x)$ is increasing for all $x > 0$. This completes the proof. (L'Hôpital's rule shows that $g(x) \to 1$ and $f(x) \to e$ as $x \to \infty$.)

(f) The sequence under consideration here is $\{f_n\}$, $n = 1, 2, 3, \ldots$, where

$$f_n = \left(1 + \frac{1}{n}\right)^n.$$

This is $f(n)$, where f is the function of a real variable appearing in the previous exercise. Here, we want to prove that f_n is increasing and find its limit by a first-principles approach that does not use any calculus.

Suppose $n \geq 2$ and consider the ratio,

$$\begin{aligned}
\frac{f_n}{f_{n-1}} &= \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n-1}\right)^{-(n-1)} \\
&= \left(\frac{n+1}{n}\right)^n \left(\frac{n-1}{n}\right)^{n-1} \\
&= \frac{n}{n-1} \left(\frac{n^2-1}{n^2}\right)^n \\
&= \frac{n}{n-1} \left(1 - \frac{1}{n^2}\right)^n.
\end{aligned}$$

We want a lower bound on the last expression. The bound that we need can be supplied if we can prove that

$$(1-x)^n > 1 - nx,$$

whenever $0 < x < 1$ and $n = 2, 3, 4, \ldots$. You may have seen this result as an application of the Mean Value Theorem, but that is a calculus theorem, and we want to do this exercise without calculus. (Of course, the result is only interesting on the smaller interval $0 < x < 1/n$ because the right-hand side is negative or zero otherwise.) Let us prove the result by induction on n. Start at $n = 2$:

$$(1-x)^2 = 1 - 2x + x^2 > 1 - 2x,$$

and so the result is true for $n = 2$. Suppose the result is true for $n = k$, where k is a particular integer ≥ 2 and consider the case $n = k + 1$:

$$\begin{aligned}
(1-x)^{k+1} &= (1-x)(1-x)^k \\
&> (1-x)(1-kx) \qquad \text{(by the induction hypothesis)} \\
&= 1 - (k+1)x + kx^2 \\
&> 1 - (k+1)x.
\end{aligned}$$

So truth for $n = k$ implies truth for $n = k + 1$. This completes the proof by mathematical induction that $(1-x)^n > 1 - nx$ whenever $0 < x < 1$ and $n \geq 2$.

Put $x = 1/n^2$ in the result just proved. We get

$$\left(1 - \frac{1}{n^2}\right)^n > 1 - \frac{1}{n},$$

whenever $n \geq 2$. Hence,

$$\frac{f_n}{f_{n-1}} \; > \; \frac{n}{n-1}\left(1 - \frac{1}{n}\right) \; = \; 1.$$

In other words,

$$f_n \; > \; f_{n-1} \qquad n = 2, 3, 4, \, \ldots \, .$$

We have proved that the sequence $\{f_n\}$ is increasing.

The next part of the exercise is to show that the sequence tends to a limit (which we will call "e") and to find that limit. When a sequence is increasing, a necessary and sufficient condition for it to tend to a finite limit is that the terms be bounded above. We can supply an upper bound using the binomial theorem:

$$(a+b)^n \; = \; a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \, \ldots \, + b^n,$$

where the binomial coefficient is given by

$$\binom{n}{k} \; = \; \frac{n(n-1)(n-2) \, \cdots \, (n-k+1)}{k!},$$

$k = 0, 1, 2, \, \ldots \, , n$. (It reduces to 1 when $k = 0$ and $k = n$.) For positive integers n, the binomial theorem is an algebra theorem that can be proved by induction on n. (In contrast, in the case of negative and fractional exponents α instead of n, the binomial theorem becomes an infinite series convergent for $|b| < a$ and can be derived from the Taylor series for $(1+x)^\alpha$ about $x = 0$.) When applied to f_n, the binomial theorem gives

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n \; &= \; \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
&= \; \sum_{k=0}^{n} \frac{n(n-1)(n-2) \, \cdots \, (n-k+1)}{k!} \, \frac{1}{n^k} \\
&< \; \sum_{k=0}^{n} \frac{n \cdot n \cdot n \, \cdots \, n}{k!} \, \frac{1}{n^k} \\
&= \; \sum_{k=0}^{n} \frac{n^k}{k!} \, \frac{1}{n^k} \\
&= \; \sum_{k=0}^{n} \frac{1}{k!} \\
&= \; 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \, \ldots \, + \frac{1}{n!} \\
&< \; 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \, \ldots \, .
\end{aligned}$$

This upper bound is a rapidly convergent infinite series because the factorials $k! = 1 \cdot 2 \cdot 3 \, \cdots \, k$ in the denominators grow rapidly. (The first two terms also follow the pattern because $0! = 1$ and $1! = 1$.) It proves that the limit defining e exists, and that

$$e \; \leq \; 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \, \ldots \, .$$

Of course, the series on the right exactly equals e. To complete the proof of that fact, we need a lower bound for $(1+1/n)^n$. Let m be any integer such that $1 < m < n$. Then

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} > \sum_{k=0}^{m} \binom{n}{k} \frac{1}{n^k}.$$

Suppose $k \geq 1$. A rather rough lower bound on the binomial coefficients, which will be accurate enough for our purposes, is

$$\begin{aligned} \binom{n}{k} \frac{1}{n^k} &= \frac{n(n-1)(n-2)\,\cdots\,(n-k+1)}{k!\,n^k} \\ &\geq \frac{(n-k+1)^k}{k!\,n^k} \\ &= \frac{1}{k!}\left(1-\frac{k-1}{n}\right)^k \\ &\geq \frac{1}{k!}\left(1-\frac{k(k-1)}{n}\right) \\ &> \frac{1}{k!}\left(1-\frac{m^2}{n}\right). \end{aligned}$$

This bound also holds for $k = 0$. Hence, for $1 < m < n$,

$$\left(1+\frac{1}{n}\right)^n > \sum_{k=0}^{m} \frac{1}{k!}\left(1-\frac{m^2}{n}\right).$$

Of course, this bound is negative and therefore useless if $m > \sqrt{n}$. However, it is just what we need if we let $m = [n^{1/3}]$, where $[x]$ denotes the greatest integer $\leq x$. With m so chosen, take the limit as $n \to \infty$ of both the upper and lower bounds on $(1+1/n)^n$. The squeeze law forces both bounds to tend to the same limit:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = 2.71828\,18284\,59045\,23536\ldots.$$

A minor variation of this method produces the everywhere convergent power series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$

and replacing x by z in this power series defines the complex extension of e^z to the entire complex plane. It is possible to set up the theory of exponential functions and logarithms using this power series as the starting point.

Finally, the question asked us to prove that f_n and f_{-n} tend to the same limit as $n \to \infty$. The methods of the previous part show that $(1-1/x)^{-x}$ is decreasing on $(1,\infty)$, and l'Hôpital's rule gives the limit e as $x \to \infty$. (Do this as an exercise.) In keeping with the calculus-free character of the present exercise, we will give a direct proof. There are several ways to do it. First, for $n \geq 2$,

$$\frac{f_n}{f_{-n}} = \frac{(1+1/n)^n}{(1-1/n)^{-n}} = \left(1-\frac{1}{n^2}\right)^n.$$

From the result above,

$$1-\frac{1}{n} < \frac{f_n}{f_{-n}} < 1.$$

Hence $f_n/f_{-n} \to 1$ as $n \to \infty$ by the squeeze law. A better way is to consider the ratio,

$$\frac{f_{-(n+1)}}{f_n} \;=\; \frac{(1-1/(n+1))^{-(n+1)}}{(1+1/n)^n} \;=\; 1+\frac{1}{n}\,, \qquad n \geq 1.$$

This shows immediately that f_n and $f_{-(n+1)}$ tend to the same limit, which, of course, implies that f_n and f_{-n} tend to the same limit, that limit being e.

5. (a) To get the complete Taylor series, or, equivalently, the Taylor polynomial of any order, for $\sin^{-1} x$ about $x = 0$, we use the facts that

$$\frac{d}{dx}\sin^{-1} x \;=\; \frac{1}{\sqrt{1-x^2}}\,, \qquad -1 < x < 1,$$

and that the right-hand side has a binomial series expansion. The inverse trig functions are bijective on the following domains and codomains:

$$\begin{aligned} \sin^{-1} &: [-1,1] \;\to\; [-\pi/2, \pi/2], \\ \cos^{-1} &: [-1,1] \;\to\; [0,\pi], \\ \tan^{-1} &: \mathbf{R} \;\to\; (-\pi/2, \pi/2). \end{aligned}$$

To get the derivative of $\sin^{-1} x$, let $y = \sin^{-1} x$. Then $x = \sin y$ with domain $-\pi/2 \leq y \leq \pi/2$. Its derivative is $dx/dy = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$, where the nonnegative square root is the correct one. Taking reciprocals to get dy/dx and removing the endpoints $x = \pm 1$ from the x-domain where the tangents are vertical, we get $dy/dx = (d/dx)\sin^{-1} x = (1-x^2)^{-1/2}$. [Similarly, $(d/dx)\cos^{-1} x = (d/dx)(\pi/2 - \sin^{-1} x) = -(1-x^2)^{-1/2}$ and $(d/dx)\tan^{-1} x = 1/(1+x^2)$.]

The binomial series for $g(x) = (1+x)^\alpha$, $\alpha \in \mathbf{R}$, is its Taylor series about $x = 0$. We find

$$g(0) = 1, \qquad g'(0) = \alpha, \qquad g''(0) = \alpha(\alpha-1),$$

and, in the general case,

$$g^{(k)}(0) \;=\; \alpha(\alpha-1)(\alpha-2)\,\cdots\,(\alpha-k+1).$$

The Taylor coefficient $g^{(k)}(0)/k!$ is the binomial coefficient,

$$\frac{g^{(k)}(0)}{k!} \;=\; \binom{\alpha}{k} \;=\; \frac{\alpha(\alpha-1)(\alpha-2)\,\cdots\,(\alpha-k+1)}{k!}\,,$$

which is 1 when $k = 0$. The binomial series is

$$(1+x)^\alpha \;=\; \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k,$$

valid at least for $-1 < x < 1$. [*Remarks.* This domain of validity was stated in lectures but not proved. The proof appears as a tutorial exercise in MATH1903. The standard Taylor remainder term works well in the smaller domain $-1/2 < x < 1$. The endpoint -1 is included when $\alpha \geq 0$. The endpoint 1 is included when $\alpha > -1$. When α is a nonnegative integer n, the binomial series terminates as a polynomial of degree n and agrees with the usual binomial theorem of elementary algebra, in which case its domain of validity is $\mathbf{R}$.]

To get the binomial series for $(1-x^2)^{-1/2}$, let $\alpha = -1/2$ and replace x by $-x^2$. We arrive at

$$(1-x^2)^{-1/2} \;=\; \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} x^{2k}, \qquad -1 < x < 1.$$

Since we have not treated the Taylor remainder term for the binomial series properly in lectures, we will have to assume that it is valid to integrate this series term by term. (It is always valid inside the interval of convergence of a power series.) Renaming x to t and integrating both sides with respect to t from 0 to x, we get the Taylor series,

$$\sin^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \binom{-1/2}{k} x^{2k+1}, \qquad -1 \le x \le 1.$$

Interestingly, the validity of this result extends to the endpoints $x = \pm 1$, where the original binomial series diverged to $+\infty$, but the proof will not be given here.

The question asked for the Taylor polynomial of order $2n$, which will be of degree $2n-1$ since only odd powers appear. Truncating the Taylor series at order $2n$ gives

$$\begin{aligned} T_{2n}(x) &= \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{-1/2}{k} x^{2k+1} \\ &= x - \frac{1}{3}\binom{-1/2}{1} x^3 + \frac{1}{5}\binom{-1/2}{2} x^5 - \frac{1}{7}\binom{-1/2}{3} x^7 \\ &\qquad + \ldots + \frac{(-1)^{n-1}}{2n-1}\binom{-1/2}{n-1} x^{2n-1}, \end{aligned}$$

for $f(x) = \sin^{-1} x$ about $x = 0$. The binomial coefficient with index $-1/2$ can be expressed in terms of integer factorials:

$$\begin{aligned} \binom{-1/2}{k} &= \frac{1}{k!}\Big(-\frac{1}{2}\Big)\Big(-\frac{3}{2}\Big)\Big(-\frac{5}{2}\Big) \cdots \Big(-\frac{2k-1}{2}\Big) \\ &= \frac{(-1)^k}{k!} \, \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k} \\ &= \frac{(-1)^k}{2^k\, k!} \, \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-1)(2k)}{2 \cdot 4 \cdot 6 \cdots (2k)} \\ &= \frac{(-1)^k}{2^k\, k!} \, \frac{(2k)!}{2^k\, k!} \\ &= (-1)^k \frac{(2k)!}{2^{2k} (k!)^2} . \end{aligned}$$

This can also be written $(-1)^k (2k-1)!/\{2^{2k-1}\, k!(k-1)!\}$. Hence, the Taylor polynomial for $\sin^{-1} x$ about $x = 0$ becomes

$$T_{2n}(x) = \sum_{k=0}^{n-1} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} x^{2k+1}.$$

All the coefficients are positive. The first five nonzero terms are

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \ldots .$$

(b) The corresponding treatment of the inverse tangent is simpler, and it is possible to give a simple remainder term (not the standard Taylor remainder term). First, let $y = \tan^{-1} x$ with domain $\mathbf{R}$ and range $(-\pi/2, \pi/2)$. Then $x = \tan y$, $dx/dy = \sec^2 y = 1 + \tan^2 y = 1 + x^2$. This shows that

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}, \qquad x \in \mathbf{R}.$$

The Taylor polynomials for $1/(1+x)$ are just finite geometric series that we can sum exactly with a well-known formula, and letting $x \to x^2$ means that we can do the same with $1/(1+x^2)$. In the finite geometric series,

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \;=\; \frac{a(1-r^n)}{1-r}\,, \qquad r \neq 1,$$

let $a = 1$ and $r = -x^2$. We find

$$1 - x^2 + x^4 - x^6 + \ldots + (-1)^{n-1}x^{2n-2} \;=\; \frac{1 + (-1)^{n-1}x^{2n}}{1+x^2}\,,$$

valid for all $x \in \mathbf{R}$. The polynomial on the left is the Taylor polynomial or order $2n-1$ for $1/(1+x^2)$ about $x = 0$. We can rearrange the previous result as

$$\begin{aligned}
\frac{1}{1+x^2} \;&=\; \tilde{T}_{2n-1}(x) + \tilde{R}_{2n-1}(x), \\
\tilde{T}_{2n-1}(x) \;&=\; 1 - x^2 + x^4 - x^6 + \ldots + (-1)^{n-1}x^{2n-2}, \\
\tilde{R}_{2n-1}(x) \;&=\; (-1)^n\,\frac{x^{2n}}{1+x^2}\,,
\end{aligned}$$

valid for all $x \in \mathbf{R}$. (We placed a tilde on T and R because we wish to use these symbols again for the inverse tangent.) If we wish to form the Taylor series for $1/(1+x^2)$ about $x = 0$, then we need to restrict x to the open interval $(-1, 1)$ so that the factor x^{2n} in the remainder term tends to zero as $n \to \infty$.

Term-by-term integration of both sides gives

$$\begin{aligned}
\tan^{-1} x \;&=\; T_{2n}(x) + R_{2n}(x), \\
T_{2n}(x) \;&=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^{n-1}\,\frac{x^{2n-1}}{2n-1}\,, \\
R_{2n}(x) \;&=\; (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt,
\end{aligned}$$

valid for all $x \in \mathbf{R}$. Here, $T_{2n}(x)$ is the required Taylor polynomial of order $2n$ for $\tan^{-1} x$ about $x = 0$. It is a polynomial of degree $2n-1$.

We can stop here since we have answered the question, but we should put a bound on the remainder term and determine the domain of validity of the Taylor series while we are in this position. This topic belongs to the integral theory of Taylor series covered in MATH1903 (we do the differential theory).

The remainder $R_{2n}(x)$ is an odd function of x and has the sign $(-1)^n \operatorname{sgn}(x)$, where $\operatorname{sgn} x$ denotes the sign of x when $x \neq 0$. Since $1/(1+t^2) \leq 1$, with equality only when $t = 0$, we have, for $x \neq 0$,

$$0 \;<\; (-1)^n \operatorname{sgn}(x)\, R_{2n}(x) \;<\; \int_0^{|x|} t^{2n}\,dt \;=\; \frac{|x|^{2n+1}}{2n+1}\,.$$

If we are not interested in the precise sign of the remainder, we can just write

$$0 \;\leq\; \bigl|R_{2n}(x)\bigr| \;\leq\; \frac{|x|^{2n+1}}{2n+1}\,,$$

valid for all $x \in \mathbf{R}$, with equality only when $x = 0$. We see that the remainder term tends to zero as $n \to \infty$ whenever x is in the closed interval $[-1, 1]$. The endpoints are included, even though they were not included in the Taylor series for the derivative $1/(1+x^2)$. So we have proved the Taylor series expansion,

$$\tan^{-1} x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\,, \qquad -1 \le x \le 1.$$

Up until the age of computers, this inverse tangent series was the most popular for high-precision calculations of π. It converges well for small x. John Machin (1706) gave the formula,

$$\frac{\pi}{4} \;=\; 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}\,.$$

Numerous improvements to Machin's formula have been provided since then. Even in recent times, when extremely rapidly converging schemes have been developed to calculate π to billions of decimal places, the inverse tangent series is still a serious contender because it is so easy to implement. In the year 2002, π was calculated to 1.24 trillion decimal places by Yasumasa Kanada and coworkers using the inverse tangent series and the following two Machin-like formulae:

$$\frac{\pi}{4} \;=\; 12\tan^{-1}\frac{1}{49} + 32\tan^{-1}\frac{1}{57} - 5\tan^{-1}\frac{1}{239} + 12\tan^{-1}\frac{1}{110443}\,,$$

$$\frac{\pi}{4} \;=\; 44\tan^{-1}\frac{1}{57} + 7\tan^{-1}\frac{1}{239} - 12\tan^{-1}\frac{1}{682} + 24\tan^{-1}\frac{1}{12943}\,.$$

(The current record as of May 2012 is over ten trillion decimal places.)

6. (a) Let $U_{2n}(x)$ and $V_{2n}(x)$ denote the Taylor polynomials of order $2n$ or $2n+1$ about $x=0$ for $\cos x$ and $\cosh x$, respectively. The standard results are

$$U_{2n}(x)\ \ (\text{for } \cos x) \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ (-1)^n\frac{x^{2n}}{(2n)!}\,,$$

$$V_{2n}(x)\ \ (\text{for } \cosh x) \;=\; 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ \frac{x^{2n}}{(2n)!}\,.$$

They are so closely related because $\cosh x = \cos(ix)$. In the first case, the absolute value of the remainder term has the upper bound $x^{2n+2}/(2n+2)!$. In the second case, a corresponding bound is $x^{2n+2}\cosh x/(2n+2)!$. If x is restricted to a finite (possibly large) interval covering $x=0$, both remainder terms are bounded by $Ax^{2n+2}/(2n+2)!$ for some constant A and both tend to zero as $n \to \infty$ on such an interval.

Replacing x by $\sqrt{x}$ in the first case and x by $\sqrt{-x}$ in the second gives the same polynomial,

$$T_n(x) \;=\; 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots\ (-1)^n\frac{x^n}{(2n)!}\,,$$

of degree n. This is necessarily the two-sided Taylor polynomial of order n about $x=0$ for the function,

$$G(x) \;=\; \begin{cases} \cos\sqrt{x}\,, & x \ge 0, \\ \cosh\sqrt{-x}\,, & x < 0. \end{cases}$$

The remainder $R_n(x)$ has a bound of the form $A|x|^{n+1}/(2n+2)!$ on any finite interval that covers $x=0$, and tends to zero as $n \to \infty$ on any such interval. This means that we can identify the polynomial $T_n(x)$ with

$$T_n(x) \;=\; G(0) + G'(0)x + \frac{G''(0)}{2!}x^2 + \frac{G'''(0)}{3!}x^3 + \ldots + \frac{G^{(n)}(0)}{n!}x^n.$$

In particular,

$$G(0) = 0, \quad G'(0) = -\frac{1}{2}, \quad G''(0) = \frac{1}{12}, \quad G'''(0) = -\frac{1}{120}, \quad \ldots,$$

and the nth (two-sided) derivative is

$$G^{(n)}(0) \;=\; (-1)^n \frac{n!}{(2n)!}.$$

(b) The standard Taylor polynomial of order $2n+1$ or $2n+2$ for $\sin x$ about $x = 0$ is

$$U_{2n+1}(x) \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Its remainder term is bounded by $|x|^{2n+3}/(2n+3)!$, which tends to zero as $n \to \infty$ for each fixed $x \in \mathbf{R}$. Dividing by x gives

$$T_{2n}(x) \;=\; 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots + (-1)^n \frac{x^{2n}}{(2n+1)!},$$

which converges to

$$H(x) \;=\; \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0, \end{cases}$$

in the limit $n \to \infty$ for each fixed $x \in \mathbf{R}$. So we can identify $T_{2n}(x)$ with the Taylor polynomial,

$$T_{2n}(x) \;=\; H(0) + H'(0)x + \frac{H''(0)}{2!}x^2 + \frac{H'''(0)}{3!}x^3 + \ldots + \frac{H^{(2n)}(0)}{(2n)!}x^{2n}.$$

Because $H(x)$ is even, all the odd-order derivatives of $H(x)$ are zero at $x = 0$. Comparing the two expressions for $T_{2n}(x)$, we get

$$H^{(2n)}(0) \;=\; (-1)^n \frac{(2n)!}{(2n+1)!} \;=\; \frac{(-1)^n}{2n+1}.$$

(c) (Pay attention, there could be a quiz question like this!) The Taylor polynomial of order five or six for $\sin x$ about $x = 0$ is

$$U_5(x) \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Replacing x by x^{10} gives

$$T_{50}(x) \;=\; x^{10} - \frac{x^{30}}{3!} + \frac{x^{50}}{5!}.$$

This is the Taylor polynomial of orders 50, 51, 52, ..., 69 for $f(x) = \sin(x^{10})$ about $x = 0$. So we can read off the derivatives,

$$f^{(10)}(0) \;=\; 10!, \qquad f^{(30)}(0) \;=\; -\frac{30!}{3!}, \qquad f^{(50)}(0) \;=\; \frac{50!}{5!}.$$

7. (a) Let $f(x) = Ax^2 + Bx + C$. On the interval $[a, b]$ the Mean Value Theorem states that

$$\frac{f(b) - f(a)}{b - a} \;=\; f'(c)$$

for some c such that $a < c < b$. Let us find c explicitly:

$$\begin{aligned} 2Ac + B \;&=\; \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b - a} \\ &=\; A(a + b) + B, \end{aligned}$$

which implies $c = (a + b)/2$. So c is unique and is the midpoint of the interval $[a, b]$.

(b) Let $f(x) = Ax^3 + Bx^2 + Cx + D$. The MVT gives

$$\begin{aligned} 3Ac^2 + 2Bc + C &= \frac{Ab^3 + Bb^2 + Cb + D - Aa^3 - Ba^2 - Ca - D}{b-a} \\ &= A(a^2 + ab + b^2) + B(a+b) + C, \\ 3Ac^2 + 2Bc &= A(a^2 + ab + b^2) + B(a+b). \end{aligned}$$

Substitute the midpoint $c = (a+b)/2$ and subtract the right-hand side from the left-hand side:

$$\begin{aligned} &3A(a+b)^2/4 + B(a+b) - A(a^2+ab+b^2) - B(a+b) \\ &= (A/4)(3a^2 + 6ab + 3b^2 - 4a^2 - 4ab - 4b^2) \\ &= -A(a-b)^2/4. \end{aligned}$$

Because $A \neq 0$ and $b > a$, this can never vanish. So c can never be the midpoint of $[a, b]$ in the case of a cubic polynomial.

From above, the slope of the chord joining the ends of the cubic arc on $[a, b]$ is $A(a^2 + ab + b^2) + B(a+b) + C$. If this chord is tangent to the cubic curve at $x = b$, then, with $c = b$ above, we get

$$\begin{aligned} 3Ab^2 + 2Bb &= A(a^2 + ab + b^2) + B(a+b), \\ A(a^2 + ab - 2b^2) + B(a-b) &= 0, \\ (a-b)\{A(a+2b) + B\} &= 0, \\ B &= -A(a+2b). \end{aligned}$$

With this constraint on B, the equation above for c becomes

$$\begin{aligned} 3Ac^2 + 2Bc &= A(a^2 + ab + b^2) + B(a+b), \\ 3Ac^2 - 2A(a+2b)c &= A(a^2 + ab + b^2) - A(a+2b)(a+b), \\ 3c^2 - 2(a+2b)c &= -2ab - b^2, \\ (c-b)(3c - 2a - b) &= 0. \end{aligned}$$

The root $c = b$ is out of range because it is an endpoint. The MVT states that c is an interior point. Hence c is unique and takes the value,

$$c = \frac{2a+b}{3}.$$

This is one-third of the way along the interval from a to b.

8. We are given the function,

$$f(x) = \begin{cases} 4x^2 + x, & -1 \le x < 0, \\ 2\sqrt{x}, & 0 \le x < 1, \\ (4x^3 - 21x^2 + 36x - 7)/6, & 1 \le x \le 3. \end{cases}$$

on the interval $[-1, 3]$. The critical points are the interior points where either $f'(x) = 0$ or $f'(x)$ does not exist. All local extrema are included among the critical points. The absolute extrema are included among the local extrema and the endpoints. Since $f(x)$ is defined by three rules, we must check the joins as well as the interiors of each subinterval.

On $[-1,0]$, $f(x) = 4x^2 + x$ and $f'(x) = 8x + 1$. There is a critical point at $x = -1/8$, at which $f(-1/8) = -1/16$ and $f''(-1/8) = 8$. So a local minimum occurs at $x = -1/8$. At $x = 0$, $f(x)$ has a left derivative $f'_-(0) = 1$ and so $f(x)$ is increasing as x approaches 0 from the left.

On $[0,1]$, $f(x) = 2\sqrt{x}$ and $f'(x) = 1/\sqrt{x}$ except for a one-sided vertical tangent at $x = 0$. So $x = 0$ is a critical point of $f(x)$. It is a corner point through which $f(x)$ is increasing. So $x = 0$ is not an extremum of $f(x)$. Since $f(x)$ is strictly increasing on $[0,1]$, there are no critical points in $(0,1)$. The join at $x = 1$ needs to be checked. The left derivative there is $f'_-(1) = 1$.

On $[1,3]$,

$$\begin{aligned} f(x) &= (4x^3 - 21x^2 + 36x - 7)/6, \\ f'(x) &= 2x^2 - 7x + 6 = (x-2)(2x-3), \\ f''(x) &= 4x - 7. \end{aligned}$$

So critical points occur at $x = 3/2$ and $x = 2$. The right derivative at $x = 1$ is $f'_+(1) = 1$, which is the same as the left derivative. So $x = 1$ is not a critical point of $f(x)$. At $x = 3/2$, $f(3/2) = (1/6)(27/2 - 189/4 + 54 - 7) = 53/24$ and $f''(3/2) = -1$. So $f(x)$ has a local maximum at $x = 3/2$. At $x = 2$, $f(2) = (1/6)(32 - 84 + 72 - 7) = 13/6$ and $f''(2) = 1$. So $f(x)$ has a local minimum at $x = 2$. The endpoint values are $f(-1) = 3$ and $f(3) = (108 - 189 + 108 - 7)/6 = 10/3$. So the absolute minimum occurs at $x = -1/8$, where $f(-1/8) = -1/16$, and the absolute maximum occurs at the right endpoint $x = 3$, where $f(3) = 10/3$.

The results so far can be summarised as follows:

$x = -1$: left endpoint, not an extremum;

$x = -1/8$: critical point, horizontal tangent, local and absolute minimum;

$x = 0$: critical point, corner, right vertical tangent, not an extremum;

$x = 1$: smooth join, not a critical point;

$x = 3/2$: critical point, horizontal tangent, local maximum;

$x = 2$: critical point, horizontal tangent, local minimum;

$x = 3$: right endpoint, absolute maximum.

9. (a) We will describe the level curves (typesetting diagrams is hard work). Level curves are horizontal slices of the given surface that have been dropped down to the xy-plane. They are most informative when a set of level curves are drawn that correspond to equally spaced heights (values of z) in the range of the function. The level curve diagram then forms a contour map of the surface.

(i). $f(x,y) = e^{x^2+y^2}$. The range is $z \geq 1$, so a natural choice for level curve heights is the set of positive integers $z = 1, 2, 3, \ldots$. At height z, the equation of the level curve is

$$x^2 + y^2 = \ln z,$$

which is the equation of a circle, centre $(0,0)$, radius $\sqrt{\ln z}$. When $z = 1, 2, 3, \ldots$, the radii are 0, $\sqrt{\ln 2}$, $\sqrt{\ln 3}$, ... These are a family of concentric circles of increasing radii that are rapidly getting closer together as z increases, which indicates that the surface is getting steeper as the height increases.

(ii). $g(x,y) = xy/(x^2+y^2)$. This surface becomes easier to visualize in polar coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$g(x,y) \;=\; \frac{xy}{x^2+y^2} \;=\; \frac{r^2\cos\theta\sin\theta}{r^2} \;=\; \frac{\sin 2\theta}{2}\,.$$

The level curves are curves of constant θ, which are, in general, pairs of straight lines through the origin. The origin itself must be cut from every line, because the function is not defined there. (In any case, level curves cannot cross each other, for otherwise the function would fail the vertical line test and be multi-valued.) The range of $g(x,y)$ is $-1/2 \le z \le 1/2$. To get a reasonable contour map of this surface, pick $z = -0.50$, -0.45, -0.40, ... , 0.35, 0.40, 0.45 and 0.50. The largest z corresponds to 45-degree lines in the first and third quadrants. The smallest z corresponds to 45-degree lines in the second and fourth quadrants. The lines should be drawn closer together near the x and y axes, where the surface is steepest in the transverse direction.

(iii). $h(x,y) = (\sqrt{x}+\sqrt{y}\,)^2$. This function has the range $z \ge 0$. So a natural choice of heights is the nonnegative integers $z = 0, 1, 2, 3, \ldots$. The equation of the level curve at height z is

$$\sqrt{x} + \sqrt{y} \;=\; \sqrt{z}\,,$$

where the square roots are nonnegative. This curve is a parabolic arc in the first quadrant running from $(z,0)$ on the x-axis to $(0,z)$ on the y-axis, and tangent to both axes. The axis of symmetry of the parabola is the line $y = x$ in the first quadrant. The parabola would continue beyond these contact points if we allowed $\sqrt{x}$ and $\sqrt{y}$ to take both signs. The family of parabolic arcs is equally spaced.

The surface itself is part of a tilted circular cone which touches the three coordinate planes in the first octant. Its vertex is at the origin and its axis is the line $x = y = z$. The line $z = x$, $x \ge 0$, in the xz-plane is a generator of the cone, and similarly for the line $z = y$, $y \ge 0$ in the yz-plane. The line $y = x$ in the xy-plane is a generator of the full cone, but does not belong to the part that is the graph of the function $h(x,y)$. The level curves are parts of the horizontal parabolic conic sections parallel to the latter generator.

(b) The partial derivatives of the functions f, g and h in part (a) are routine applications of the product, quotient and chain rules for differentiation. Working will not be shown. We will use subscript notation for partial derivatives, for example, $f_x = \partial f/\partial x$ and $f_{xy} = \partial^2 f/\partial y\partial x$.

(i).

$$f(x,y) \;=\; e^{x^2+y^2}, \qquad f_x \;=\; 2x\,e^{x^2+y^2}, \qquad f_y \;=\; 2y\,e^{x^2+y^2},$$

$$f_{xx} \;=\; (4x^2+2)e^{x^2+y^2}, \qquad f_{xy} \;=\; f_{yx} \;=\; 4xy\,e^{x^2+y^2}, \qquad f_{yy} \;=\; (4y^2+2)e^{x^2+y^2}.$$

(ii).

$$g(x,y) \;=\; \frac{xy}{x^2+y^2}\,, \qquad g_x \;=\; \frac{y(y^2-x^2)}{(x^2+y^2)^2}\,, \qquad g_y \;=\; \frac{x(x^2-y^2)}{(x^2+y^2)^2}\,,$$

$$g_{xx} \;=\; \frac{2xy(x^2-3y^2)}{(x^2+y^2)^3}\,, \qquad g_{xy} \;=\; g_{yx} \;=\; -\,\frac{x^4-6x^2y^2+y^4}{(x^2+y^2)^3}\,, \qquad g_{yy} \;=\; \frac{2xy(y^2-3x^2)}{(x^2+y^2)^3}\,.$$

(iii).

$$h(x,y) \;=\; (\sqrt{x}+\sqrt{y}\,)^2, \qquad h_x \;=\; 1+\frac{\sqrt{y}}{\sqrt{x}}\,, \qquad h_y \;=\; 1+\frac{\sqrt{x}}{\sqrt{y}}\,,$$

$$h_{xx} \;=\; -\,\frac{\sqrt{y}}{2x^{3/2}}\,, \qquad h_{xy} \;=\; h_{yx} \;=\; \frac{1}{2\sqrt{xy}}\,, \qquad h_{yy} \;=\; -\,\frac{\sqrt{x}}{2y^{3/2}}\,.$$

10. Denote the three parts of this exercise (a), (b) and (c).

(a) In the case of the limit,

$$\lim_{(x,y)\to(0,0)} \frac{x^3+2y^3}{x^2+y^2}\,,$$

we would expect the limit to exist and be zero because the numerator tends to zero more rapidly than the denominator. The best way to prove this is with polar coordinates, as suggested by the appearance of x^2+y^2 in the denominator. So let $x = r\cos\theta$ and $y = r\sin\theta$, $r > 0$. Then

$$\frac{x^3+2y^3}{x^2+y^2} = \frac{r^3\cos^3\theta + 2r^3\sin^3\theta}{r^2} = r(\cos^3\theta + 2\sin^3\theta).$$

The bounds $|\cos\theta| \le 1$ and $|\sin\theta| \le 1$ now imply

$$0 \le \left|\frac{x^3+2y^3}{x^2+y^2}\right| \le 3r.$$

(The upper bound can be lowered to $2r$, but that is not important.) On every path that ends at the origin, r will tend to zero, and so $f(x,y)$ will be squeezed between two bounds that both tend to zero. So the required limit exists and equals zero.

We can say essentially the same thing with epsilons and deltas. Let L denote the limit, so that $L = 0$. Let arbitrary $\epsilon > 0$ be given. Then, the inequality just found implies that

$$\left|\frac{x^3+2y^3}{x^2+y^2} - L\right| < \epsilon$$

for every (x,y) satisfying $0 < \sqrt{x^2+y^2} < \delta$, provided we choose $\delta < \epsilon/3$. For example, $\delta = \epsilon/4$ will do the job. Either way, we have proved

$$\lim_{(x,y)\to(0,0)} \frac{x^3+2y^3}{x^2+y^2} = 0.$$

(b) The second limit does not exist because the limit depends on the path of approach to the origin. Let $f(x,y)$ denote $\sin(xy)/(x^2+y^2)$. Consider the limit along the x-axis:

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{\sin 0}{x^2} = 0.$$

The limit along the y-axis is also zero, by symmetry. Consider the limit along the path $y = x$:

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{\sin(x^2)}{2x^2} = \frac{1}{2}\,.$$

This limit differs from the limit along the x-axis. That is all we need to complete the proof that the two-dimensional limit does not exist.

(c) The limit,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}\,,$$

is interesting because it is zero on every straight path that ends at the origin. Nevertheless, the two-dimensional limit does not exist. On the parabola $y = x^2$, the limit is $1/2$. On $y = -x^2$, the limit is $-1/2$. In fact, the parabolas $y = mx^2$, $x \ne 0$, are level curves for this function, having heights between $-1/2$ and $1/2$ inclusive. So level curves corresponding to different heights approach the origin. This implies that different paths to the origin yield different limits.