The University of Sydney

MATH1901/06 Differential Calculus (Advanced)

Semester 1 — Short answers to exam questions — 2007

1. (a) $2\sqrt{x^2+y^2} = 2x+1$ simplifies to the parabola $y^2 = x + \frac{1}{4}$ (focus at origin).

(b) f is surjective because the equation $z - 1/z = w$ for z has at least one root for every $w \in \mathbf{C}$. It is not injective because there are two roots, in general.

(c) Image of AB: straight line segment from $e\,\mathrm{cis}(1)$ to $e^{-1}\,\mathrm{cis}(1)$.
Image of BC: circular arc radius $1/e$ from $e^{-1}\,\mathrm{cis}(1)$ to $e^{-1}\,\mathrm{cis}(-1)$.
Image of CD: straight line segment from $e^{-1}\,\mathrm{cis}(-1)$ to $e\,\mathrm{cis}(-1)$.
Image of DA: circular arc radius e from $e\,\mathrm{cis}(-1)$ to $e\,\mathrm{cis}(1)$.

2. (a) Limit is 1/12. (L'Hôpital, rationalise numerator, or binomial series.)

(b) Limit is 4. (Simplify to $4(x^2+1)/(x^2-4)$.)

(c) Limit is e^3. (Take logs and use l'Hôpital, or write as $e^3 \cdot 2^{-y}(1+e^{-6/y})^y$.)

(d) Limit is 0. (Use polar coordinates.)

3. (a) (i). $h(x) = \cosh^{-1} x$, $x > 1$ (endpoint excluded). Let $y = h(x)$. Then $x = \cosh y$, $dx/dy = \sinh y = \sqrt{x^2-1}$, $dy/dx = h'(x) = 1/\sqrt{x^2-1}$.

(ii). Apply MVT to $h(x)$ on the interval $(\cosh 1, x)$:
$$\frac{h(x) - h(\cosh 1)}{x - \cosh 1} = h'(c) = \frac{1}{\sqrt{c^2-1}}, \quad \text{where } \cosh 1 < c < x.$$
Required inequality follows from $h(\cosh 1) = 1$ and $\sqrt{c^2-1} > \sinh 1$.

(b) Let $\epsilon > 0$ be given. We are free to let $0 < \epsilon < 1$. The given limits imply that there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x)| < \epsilon$ whenever $0 < |x-a| < \delta_1$ and $|g(x)| < \epsilon$ whenever $0 < |x-a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. Then $|f(x)g(x)| < \epsilon^2 < \epsilon$ whenever $0 < |x-a| < \delta$, which implies that $\lim_{x\to a} f(x)g(x) = 0$.

4. (a) (i). $\nabla f(x,y) = e^{x^2}\big((1+2x^2+4xy)\mathbf{i} + 2\mathbf{j}\big)$.

(ii). $dy/dx = -f_x/f_y = -(1+2x^2+4xy)/2$. At $(0,1)$, $dy/dx = -1/2$.

(iii). Direction of greatest slope is $\nabla f(0,1) = \mathbf{i} + 2\mathbf{j}$. Greatest slope is the magnitude $\sqrt{5}$. Horizontal directions $\pm(2\mathbf{i} - \mathbf{j})$.

(b) (i). $\nabla(fg) = (fg)_x\mathbf{i} + (fg)_y\mathbf{j} = (fg_x + gf_x)\mathbf{i} + (fg_y + gf_y)\mathbf{j} = f\nabla g + g\nabla f$.

(ii). $f = f(x)$ and $g = g(y)$ implies $\nabla(fg) = g(y)f'(x)\mathbf{i} + f(x)g'(y)\mathbf{j}$.
$\nabla(fg) = 0$ implies either $f(x) = 0$ or $g(y) = 0$ or $f'(x) = 0 = g'(y)$.

Question **5** on page 2.

5. (a) Domain $\mathbf{R}^2\backslash\{(0,0)\}$, range $[0,1]$.
Level curve $z = 1/2$ is the pair of lines $y = \pm x$, excluding origin.
Level curve $z = 1/5$ is the pair of lines $y = \pm 2x$, excluding origin.

(b) (i). $e^x \;=\; 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!} + R_n(x),\;\; R_n(x) \;=\; \dfrac{e^c\, x^{n+1}}{(n+1)!}$.

(ii). $R_n \;=\; R_n(1) \;=\; \dfrac{e^c}{(n+1)!},\;\; 0 < c < 1.$

$1 < e^c < e < 3$ implies $\dfrac{1}{(n+1)!} < R_n < \dfrac{3}{(n+1)!}$.

(iii). $n!e \;=\; n!\left(1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots + \dfrac{1}{n!}\right) + n!\,R_n \;=\; \text{integer} + n!\,R_n,$

$\dfrac{n!}{(n+1)!} < n!\,R_n < \dfrac{3n!}{(n+1)!}$ simplifies to $\dfrac{1}{n+1} < n!\,R_n < \dfrac{3}{n+1}$.

When $n \geq 2,\;\; 0 < n!\,R_n < 1$. So $n!e$ cannot be an integer.

(iv). If e were rational, it would be p/q for some integers p and q, $q \geq 2$.
Then qe and $q!e$ would be integers. But $q!e$ is never an integer.
It follows that e is irrational.