THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education, Engineering and Science

MATH1902

Linear Algebra (Advanced)

June/July 2006

Lecturer: A Molev

Time Allowed: **One and a half hours**

Name: ....

SID: ....   Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 25% of the total examination;
there are 15 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 75% of the total examination;
there are 5 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book provided.
Ask for extra books if you need them.

1. (a) **(8 marks)**. Let ℓ be the line given by the equations $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-2}{-1}$.

   (*i*) Find the parametric scalar equations of the line ℓ' through the point $B(3,3,4)$ which is parallel to ℓ.

   (*ii*) Find the Cartesian equation of the plane $\mathcal{P}$ through the point $B(3,3,4)$ which is perpendicular to the line ℓ'.

   (*iii*) Find the coordinates of the intersection point A of ℓ and $\mathcal{P}$.

   (*iv*) Calculate the distance between the lines ℓ and ℓ'.

(b) **(7 marks)**. The lines ℓ and m are given by the respective equations $\mathbf{r} = \mathbf{i} - 3\,\mathbf{j} - 3\,\mathbf{k} + t(-\mathbf{i} + \mathbf{k})$ and $\mathbf{r} = -2\,\mathbf{i} + 3\,\mathbf{j} + s(4\,\mathbf{i} - \mathbf{j} - \mathbf{k})$, where t and s are parameters.

   (*i*) Find out whether ℓ and m are parallel, have a common point or they are skew lines.

   (*ii*) Use vector product to find a vector which is perpendicular to both ℓ and m.

   (*iii*) Let ℓ' and m' be the lines through the origin which are parallel to ℓ and m, respectively. Find the (acute) angle between ℓ' and m'.

2. (a) **(6 marks)**. Let A, B and C be the points $(3,-1,6)$, $(-1,2,2)$ and $(1,0,1)$ respectively, and let O be the origin. A plane containing O and C intersects the edge AB of the tetrahedron $OABC$ at a point P such that $AP : PB = \alpha : (1-\alpha)$ for some $0 < \alpha < 1$.

   (*i*) Find the coordinates of P in terms of α.

   (*ii*) Find the area of the triangle OCP in terms of α.

(b) **(9 marks)**. Let C be a point on the Cartesian plane with position vector $\mathbf{c}$ with respect to the origin. The point P with position vector $\mathbf{r} = \overrightarrow{OP}$ lies on the circle with centre C and radius a if and only if $|\mathbf{r}-\mathbf{c}| = a$. Thus $|\mathbf{r}-\mathbf{c}| = a$ is the equation of the circle.

   (*i*) Show that the equation of the circle can be written in the alternative form $|\mathbf{r}|^2 - 2\mathbf{c}\cdot\mathbf{r} + |\mathbf{c}|^2 - a^2 = 0$.

   (*ii*) Given a point B on the circle with position vector $\mathbf{b} = \overrightarrow{OB}$ show that the line $\mathbf{r} = \mathbf{b} + t\mathbf{d}$ through B in the direction $\mathbf{d}$ meets the circle at B and B' corresponding to the values $t = 0$ and $t = -2\mathbf{d}\cdot(\mathbf{b}-\mathbf{c})/|\mathbf{d}|^2$, respectively.

   (*iii*) Let M be an arbitrary point outside the circle on the line $\mathbf{r} = \mathbf{b} + t\mathbf{d}$ with the position vector $\mathbf{m} = \mathbf{b} + t_0\mathbf{d}$. Calculate the scalar product $\overrightarrow{MB}\cdot\overrightarrow{MB'}$ and write the result as an expression involving $\mathbf{m}$, $\mathbf{c}$ and a only.

3. (a) **(8 marks).** Let p be a real number. Given the column vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 3 \\ -1 \\ -5 \\ -3 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ p^2 + p \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ -3 \\ -7 \\ p - 4 \end{bmatrix},$$

we would like to represent $\mathbf{v}$ as a linear combination $\mathbf{v} = x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} + x_4\mathbf{d}$.

(*i*) Find all values of p for which this problem has a unique solution and write down the unique linear combination.

(*ii*) Find all values of p for which such representation of $\mathbf{v}$ is impossible.

(*iii*) Find all values of p for which this problem has infinitely many solutions and write down all the linear combinations for each such value of p.

(b) **(7 marks).** Answer true of false to each of the following, giving a counterexample when the statement is false.

(*i*) Suppose that a system of linear equations is written in a matrix form as $A\mathbf{x} = \mathbf{b}$. If A has a right inverse C, then $\mathbf{x} = C\mathbf{b}$ is the unique solution of the system.

(*ii*) If A has a left inverse B, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(*iii*) The relation $(AB)^2 = A^2B^2$ for 2×2 matrices A and B can only hold if $AB = BA$.

4. (a) **(10 marks).** Consider the matrix

$$A = \begin{bmatrix} 3 & 4 & 0 \\ 3 & 7 & 0 \\ -4 & 4 & 7 \end{bmatrix}.$$

(*i*) Find the eigenvalues of A.

(*ii*) Use the preceding part to explain why the matrix $B = -6I + A$ must be invertible.

(*iii*) Calculate the adjoint matrix $\mathrm{adj}(B)$ where $B = -6I + A$.

(*iv*) Use (*iii*) to calculate B^{-1}.

(b) **(5 marks).** Let A be a 2×2 matrix with distinct eigenvalues λ_1 and λ_2, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the corresponding eigenvectors. Suppose that B is another 2×2 matrix such that $AB = BA$. Show that if $B\mathbf{v}_1 = \alpha\mathbf{v}_2$ for some constant α then $\alpha = 0$.

5. (a) **(9 marks)**. An $n \times n$ matrix A is called *skew-symmetric* if $A^T = -A$, where A^T is the transpose of A.

(*i*) Prove that if A is skew-symmetric and n is odd then $\det A = 0$.

(*ii*) Hence prove that any skew-symmetric matrix of odd size has a zero eigenvalue.

(*iii*) Find the eigenspace of the zero eigenvalue of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix}.$$

(b) **(6 marks)**. Let A be the $n \times n$ matrix whose only nonzero entries are $a_{12} = a_{23} = a_{34} = \cdots = a_{n-1,n} = a_{n1} = 1$. Calculate the characteristic polynomial of A.

End of Extended Answer Section