THE UNIVERSITY OF SYDNEY

MATH1903 INTEGRAL CALCULUS AND MODELLING (ADVANCED)

Semester 2 **Solutions to Exercises for Week 4** 2017

1. A polynomial function p has a rule of the form $p(x) = a_0 + a_1x + \ldots + a_nx^n$ for some nonnegative integer n and constants $a_0, \ldots, a_n$. The special cases $p(x) = a_0$ and $p(x) = x$ are easily verified to be continuous, and a general polynomial function is built from these two using addition and multiplication of functions. It is a theorem that the class of continuous functions is closed under addition and multiplication, and it follows immediately that all polynomial functions are continuous.

2. The Intermediate Value Theorem states that if f is a continuous real-valued function defined on an interval $[a, b]$ and k is a real number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = k$.

3. We have $f(x) = x^2-2$, so that $f(1.4) = 1.96-2 = -0.04$ and $f(1.5) = 2.25-2 = 0.25$, verifying that $f(1.4) < 0 < f(1.5)$. By the Intermediate Value Theorem, there exists $c \in [1.4, 1.5]$ such that $f(c) = c^2 - 2 = 0$, so that $c^2 = 2$. Thus c is a square root of 2 (a number whose square is 2), and c lies between 1.4 and 1.5.

4. We have $f(x) = x^n - a$, so that $f(0) = -a < 0$ and $f(a) = a^n - a = a(a^{n-1} - 1) > 0$, verifying that $f(0) < 0 < f(a)$. By the Intermediate Value Theorem, there exists $c \in [0, a]$ such that $f(c) = c^n - a = 0$, so that $c^n = a$. Thus c is an nth root of a (a number whose nth power is a and in fact lies between 0 and a).

5. Suppose by way of contradiction that $\alpha = \sqrt[3]{5}$ is rational, so $\alpha = a/b$ for some positive integers a and b with no common prime divisors. Hence $5 = \alpha^3 = a^3/b^3$, so that $5b^3 = a^3$. But 5 divides the left-hand side, so must also divide the right-hand side, of this integer equation. It follows that 5 must divide a (by uniqueness of prime decompositions of positive integers, which is a non-trivial fact about integers that we may take for granted). Hence $a = 5k$ for some positive integer k, so that

$$5b^3 \;=\; a^3 \;= 5^3k^3 \,,$$

yielding $b^3 = 5^2k^3$. In particular, 5 divides both sides of this new integer equation, and it follows that 5 divides b. This contradicts that a and b have no common prime divisors. Hence α is irrational.

6. Consider the square area function $S = S(x) = x^2$. If we make a small change Δx in the side length x of a square and distribute it evenly on all sides of the square, then the change in area ΔS is approximately four times the old side length x multiplied by only half of Δx. More precisely,

$$\Delta S \;=\; 4 \times x \times \frac{\Delta x}{2} \;+\; 4 \times \left(\frac{\Delta x}{2}\right)^2 \;=\; 2x\Delta x + \left(\Delta x\right)^2 ,$$

so that, as expected,

$$\frac{dS}{dx} = \lim_{\Delta x\to 0}\frac{\Delta S}{\Delta x} = \lim_{\Delta x\to 0}\frac{2x\Delta x+\left(\Delta x\right)^2}{\Delta x} = \lim_{\Delta x\to 0} 2x+\Delta x = 2x\,.$$

There is in fact no heuristic inconsistency with part (i), because if we differentiate the circle area function $A=\pi r^2=\pi\left(\frac{D}{2}\right)^2=\pi\frac{D^2}{4}$ with respect to diameter, rather than radius, we get $dA/dD=2\pi D/4=\pi D/2=\pi r$, which again is half the perimeter.

Consider the cube volume function $C=C(x)=x^3$. If we make a small change Δx in the side length x of a cube and distribute it evenly on all sides of the cube, then the change in volume ΔC is approximately six times the square of the old side length x multiplied by only half of Δx (corresponding to the six faces). More precisely,

$$\Delta C = 6\times x^2\times\frac{\Delta x}{2} + 8\times\left(\frac{\Delta x}{2}\right)^3 = 3x^2\Delta x+\left(\Delta x\right)^3\,,$$

so that, as expected,

$$\frac{dC}{dx} = \lim_{\Delta x\to 0}\frac{\Delta C}{\Delta x} = \lim_{\Delta x\to 0}\frac{3x^2\Delta x+\left(\Delta x\right)^3}{\Delta x} = \lim_{\Delta x\to 0} 3x^2+\left(\Delta x\right)^2 = 3x^2\,.$$

There is in fact no heuristic inconsistency with part (ii), because if we differentiate the sphere volume function $V=\frac{4}{3}\pi r^3=\frac{4}{3}\pi\left(\frac{D}{2}\right)^3=\pi\frac{D^3}{6}$ with respect to diameter, rather than radius, we get $dV/dD=\pi D^2/2=2\pi r^2$, which again is half the surface area.

7. The curves intersect when $1-x^2=2x^4$, so that $2x^4+x^2-1=(2x^2-1)(x^2+1)=0$, yielding $x=\pm 1/\sqrt{2}$. Therefore the area is

$$\begin{aligned} A &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left(\sqrt{1-x^2}-\sqrt{2}\,x^2\right)\,dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\sqrt{1-x^2}\,dx-\sqrt{2}\left[\frac{x^3}{3}\right]_{-1/\sqrt{2}}^{1/\sqrt{2}} \\ &= 2\int_0^{1/\sqrt{2}}\sqrt{1-x^2}\,dx-\frac{1}{3}\,. \end{aligned}$$

Making the change of variable $x=\sin\theta$ gives

$$\int_0^{1/\sqrt{2}}\sqrt{1-x^2}\,dx = \int_0^{\pi/4}\cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/4}\left(1+\cos(2\theta)\right)\,d\theta = \frac{\pi}{8}+\frac{1}{4}\,.$$

Therefore the area is

$$A = 2\left(\frac{\pi}{8}+\frac{1}{4}\right)-\frac{1}{3} = \frac{\pi}{4}+\frac{1}{6}\,.$$

8. The line segment joining $(0,h)$ to $(r,0)$ is part of the line with equation $y=h(1-x/r)$. Using the shell method, by rotating about the y-axis, the volume of the cone is

$$V = \int_0^r 2\pi xy\,dx = 2\pi\int_0^r xh\left(1-\frac{x}{r}\right)\,dx = 2\pi h\left[\frac{x^2}{2}-\frac{x^3}{3r}\right]_0^r = \frac{\pi r^2 h}{3}\,.$$

Using the disc method, also rotating about the y-axis, and noting that $x=r(1-y/h)$, the volume is

$$V = \int_0^h \pi x^2\,dy = \pi r^2\int_0^h\left(1-\frac{2y}{h}+\frac{y^2}{h^2}\right)\,dy = \pi r^2\left[y-\frac{y^2}{h}+\frac{y^3}{3h^2}\right]_0^h = \frac{\pi r^2 h}{3}\,.$$

9. Note that if $y = \cosh x$ then $y' = \sinh x$, so that

$$\sqrt{1 + (y')^2} \;=\; \sqrt{1 + \sinh^2 x} \;=\; \cosh x\,.$$

Hence, for $-1 \le x \le 1$, the length of the catenary is

$$\int_{-1}^{1} \sqrt{1 + (y')^2}\,dx \;=\; \int_{-1}^{1} \cosh x\,dx \;=\; [\sinh x]_{-1}^{1} \;=\; 2\sinh 1 \;=\; e - e^{-1}\,.$$

10. The circle of radius r and centre $(R, 0)$ has equation $(x-R)^2+y^2 = r^2$. So the top half of the torus is obtained by rotating the curve $y = \sqrt{r^2 - (x-R)^2}$, $R-r \le x \le R+r$, about the y-axis. Hence,

$$\tfrac{1}{2}V \;=\; 2\pi \int_{R-r}^{R+r} x\sqrt{r^2 - (x-R)^2}\,dx \;=\; 2\pi \int_{-r}^{r} (u+R)\sqrt{r^2 - u^2}\,du,$$

where we have made the substitution $u = x - R$, so that $du = dx$, and we have made the appropriate change to the limits of integration. Break the last integral into

$$2\pi \int_{-r}^{r} (u+R)\sqrt{r^2-u^2}\,du \;=\; 2\pi \int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 2\pi R \int_{-r}^{r} \sqrt{r^2-u^2}\,du.$$

The first integral on the right is zero because $u\sqrt{r^2-u^2}$ is an odd function, and we are integrating over an interval which is symmetric with respect to the origin. Hence the volume of the torus is

$$V \;=\; 4\pi R \int_{-r}^{r} \sqrt{r^2-u^2}\,du \;=\; 4\pi R\Big(\frac{\pi r^2}{2}\Big) \;=\; 2\pi^2 R r^2.$$

One can see that this is a plausible answer by imagining the torus straightened out into a cylinder. This cylinder would have a circular base of area πr^2 and a height of $2\pi R$.

11. Revolving the circle with equation $(x - R)^2 + y^2 = r^2$ about the y-axis leads to two calculations, depending on whether you take the positive or negative square root when expressing x in terms of y. We do both calculations simultaneously. We have

$$x = R \pm \sqrt{r^2 - y^2} \quad \text{and} \quad dx/dy = \mp y/\sqrt{r^2 - y^2}\,.$$

Note that dx/dy becomes unbounded, so the calculations become improper integrals (though you will stumble on the correct answer even if you don't take this into account!). The surface areas of the "outer" and "inner" halves are given by

$$\begin{aligned}
\lim_{b\to r^-} \int_{-b}^{b} 2\pi x\sqrt{1 + (dx/dy)^2}\,dy \;&=\; 2\pi \lim_{b\to r^-} \int_{-b}^{b} (R \pm \sqrt{r^2-y^2})\sqrt{1 + \frac{y^2}{r^2-y^2}}\;dy \\
&=\; 2\pi \lim_{b\to r^-} \int_{-b}^{b} (R \pm \sqrt{r^2-y^2})\sqrt{\frac{r^2}{r^2-y^2}}\;dy \;=\; 2\pi r \lim_{b\to r^-} \int_{-b}^{b} \frac{R}{\sqrt{r^2-y^2}} \pm 1\;dy \\
&=\; 2\pi r \lim_{b\to r^-} \Big[R\sin^{-1}\frac{y}{r} \;\pm y\Big]_{-b}^{b} \;=\; 2\pi r \lim_{b\to r^-} \Big(2R\sin^{-1}\frac{b}{r} \;\pm 2b\Big) \\
&=\; 2\pi r(\pi R \pm 2r) \;=\; 2\pi^2 R r \pm 4\pi r^2\,.
\end{aligned}$$

Adding these "outer" and "inner" halves gives the total surface area of the torus to be

$$A \;=\; 4\pi^2 Rr\;.$$

This coincides with the surface area of a cylinder of length $2\pi R$ and radius r. It is interesting to notice that as one "bends" such a cylinder back into a torus, the "inner contraction" and "outer expansion" exactly balance each other out and contribute $\pm$ the surface area $4\pi r^2$ of a sphere of radius r (independent of the length R).

12. (i) The volume is, by the disc method,

$$V \;=\; \int_0^2 \pi y^2\,dx \;=\; \pi\int_0^2 x^2\,dx \;=\; \pi\left[\frac{x^3}{3}\right]_0^2 \;=\; \frac{8\pi}{3}\;.$$

(ii) The volume is, by the shell method,

$$V \;=\; \int_0^2 2\pi xy\,dx \;=\; 2\pi\int_0^2 x^2\,dx \;=\; 2\pi\left[\frac{x^3}{3}\right]_0^2 \;=\; \frac{16\pi}{3}\;.$$

(iii) The volume is, by the shell method,

$$V \;=\; \int_0^2 2\pi(4-x)y\,dx \;=\; 2\pi\int_0^2 (4x-x^2)\,dx \;=\; 2\pi\left[2x^2-\frac{x^3}{3}\right]_0^2 \;=\; \frac{32\pi}{3}\;.$$

(iv) By symmetry, the answer should be the same as part (ii). Directly, by the shell method, the volume is

$$V \;=\; \int_0^2 2\pi(2-y)(2-x)\,dy \;=\; 2\pi\int_0^2 (2-y)^2\,dy \;=\; 2\pi\left[4y-2y^2+\frac{y^3}{3}\right]_0^2 \;=\; \frac{16\pi}{3}\;.$$

13. Using the shell method, the volume of the top of the bagel is

$$V \;=\; 2\pi\int_1^2 x(3x-x^2-2)\,dx \;=\; 2\pi\left[x^3-\tfrac{1}{4}x^4-x^2\right]_1^2 \;=\; 2\pi\left[0-\left(-\tfrac{1}{4}\right)\right] \;=\; \frac{\pi}{2}\,.$$

14. Observe that, in the first quadrant, $dy/dx = -\frac{\sqrt{1-x^{2/3}}}{x^{1/3}}$, so the length of the curve in that quadrant is

$$\int_0^1 \sqrt{1+(dy/dx)^2}\,dx \;=\; \int_0^1 \sqrt{1+\frac{1-x^{2/3}}{x^{2/3}}}\;dx$$
$$=\; \int_0^1 x^{-1/3}\,dx \;=\; \lim_{a\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_a^1 \;=\; \frac{3}{2}\;.$$

By symmetry, the length of the curve then is 4 times the length in the first quadrant, which totals 6.

15. The length of one arch is

$$\int_0^{2\pi} \sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\;\;d\theta \;=\; \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2+r^2\sin^2\theta}\;\;d\theta$$
$$=\; r\int_0^{2\pi}\sqrt{2-2\cos\theta}\;d\theta \;=\; r\int_0^{2\pi}\sqrt{4\,\frac{1-\cos\theta}{2}}\;d\theta = \;2r\int_0^{2\pi}\sqrt{\sin^2\frac{\theta}{2}}\;\;d\theta$$
$$=\; 2r\int_0^{2\pi}\sin\frac{\theta}{2}\;\;d\theta =\; 2r\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} \;=\; 8r\;.$$

16. By analogy with the plane version, the arc length for a curve in space given parametrically by differentiable functions $x = x(t)$, $y = y(t)$, $z = z(t)$ of a parameter t such that $a \le t \le b$ should be

$$L = \int_a^b \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2 + \big[z'(t)\big]^2}\; dt\,.$$

The length of the given spiral then should be

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}\, dt \\ &= \int_0^{2\pi} \sqrt{a^2+b^2}\, dt = 2\pi\sqrt{a^2+b^2}\,. \end{aligned}$$

17. Observe first that $\left(\frac{2}{2}\right)^1 < a$, so $m_1 \ge 2$, so $a_1 \ge \frac{2}{2} = 1$. For $k \in \mathbb{Z}^+$ observe that $a_k = \frac{m_k}{2^k}$, where

$$\left(\frac{2m_k}{2^{k+1}}\right)^n = \left(\frac{m_k}{2^k}\right)^n \le a\,,$$

so $2m_k \le m_{k+1}$, giving

$$a_{k+1} = \frac{m_{k+1}}{2^{k+1}} \ge \frac{2m_k}{2^{k+1}} = \frac{m_k}{2^k} = a_k\,.$$

Also, since $a_k \ge 1$, $a_k \le a_k^n \le a$. Thus $\{a_i\}_{i=1}^{\infty}$ is a nondecreasing sequence bounded above, so has a least upper bound L. We prove $L^n = a$ by contradiction.

Suppose $L^n < a$ and put $\epsilon = a - L^n > 0$. Put

$$K = \binom{n}{1}L^{n-1} + \binom{n}{2}L^{n-2} + \cdots + \binom{n}{n-1}L + 1$$

where $\binom{n}{1}$, $\binom{n}{2}$, etc are binomial coefficients. Choose $\delta \in (0,1)$ such that $\delta < \epsilon/K$. Then

$$\begin{aligned} (L+\delta)^n &= L^n + \binom{n}{1}L^{n-1}\delta + \binom{n}{2}L^{n-2}\delta^2 + \cdots + \binom{n}{n-1}L\delta^{n-1} + \delta^n \\ &= L^n + \delta\left[\binom{n}{1}L^{n-1} + \binom{n}{2}L^{n-2}\delta + \cdots + \binom{n}{n-1}L\delta^{n-2} + \delta^{n-1}\right] \\ &< L^n + \delta K \qquad (\text{since } \delta < 1) \\ &< L^n + \epsilon = a\,. \end{aligned}$$

But $\frac{1}{2^\ell} < \delta$ for some ℓ, and $a_\ell = \frac{m_\ell}{2^\ell} \le L$, so

$$\frac{m_\ell + 1}{2^\ell} \le L + \frac{1}{2^\ell} < L + \delta\,,$$

giving

$$\left(\frac{m_\ell+1}{2^\ell}\right)^n < (L+\delta)^n < a\,,$$

contradicting maximality of m_ℓ.

Now suppose $L^n > a$ and put $\epsilon = L^n - a > 0$. Choose K and δ as before. Then

$$\begin{aligned}(L-\delta)^n &= L^n - \binom{n}{1}L^{n-1}\delta + \binom{n}{2}L^{n-2}\delta^2 - \cdots + (-1)^n\delta^n \\ &\geq L^n - \binom{n}{1}L^{n-1}\delta - \binom{n}{2}L^{n-2}\delta^2 - \cdots - \delta^n \\ &\geq L^n - \delta K \; > \; L^n - \epsilon \; = \; a\ .\end{aligned}$$

But L is the least upper bound of $\{a_i\}_{i=1}^{\infty}$, so there exists a positive integer k such that $a_k > L - \delta$. Hence $a_k^n > (L-\delta)^n \geq a$, contradicting that $a_k^n = (m_k/2^k)^n \leq a$.

This proves $L^n = a$.