CHAPTER **THIRTEEN**

The Exponential Function

Exponential functions and logarithmic functions are mutual inverses, so it is now fairly straightforward to develop the calculus of exponential functions from the calculus of the logarithmic function in the previous chapter. Again the natural base to use is e, and the chapter is essentially a study of the characteristic properties of $y = e^x$.

Understanding the exponential function $y = e^x$ is one of the main goals of this course. Functions involving e^x are essential for modelling some of the most common situations in the natural world, such as population growth, radioactive decay, the dying away of a note on the piano, inflation and depreciation — the phrase 'growing exponentially' has now entered the common vocabulary. Calculus is essential for the study of e^x, in fact the very definition of e involves calculus. This is quite unlike the study of linear and quadratic functions in earlier chapters, where algebraic and geometric techniques predominate.

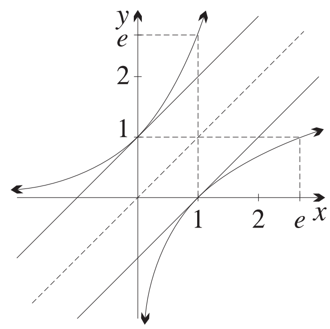
STUDY NOTES: Once again, a thorough algebraic and graphical understanding that exponential and logarithmic functions are inverses is fundamental. Drawing tangents on a graph-paper sketch of $y = e^x$, or on some software package version of it, should be used to reinforce the key understanding that at each point on this curve, gradient is equal to height. More complicated versions of natural growth, such as Newton's law of cooling, have been left until later, but they could be developed now.

13 A The Exponential Function and its Derivative

As with logarithmic functions, the most natural base to use for exponential functions in calculus is e, and the function $y = e^x$ is therefore called *the exponential function* to distinguish it from exponential functions like $y = 2^x$ which have other bases. Here are the tables of values and graphs of the mutually inverse functions $y = e^x$ and $y = \log x$.

x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2
$\log x$	-2	-1	0	1	2

x	-2	-1	0	1	2
e^x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2



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The graphs are mutual reflections in the diagonal line $y = x$. Furthermore, the tangent to $y = \log x$ at its x-intercept is reflected into the tangent to $y = e^x$ at its y-intercept — both tangents have gradient 1 and are parallel to $y = x$.

Differentiating the Exponential Function: The most significant thing about the exponential function is that its derivative is equal to itself.

1 THE EXPONENTIAL FUNCTION IS ITS OWN DERIVATIVE: $\dfrac{d}{dx}\, e^x = e^x$

PROOF: Let $y = e^x$.

Then $x = \log y$,

so $\dfrac{dx}{dy} = \dfrac{1}{y}$, as established in the last chapter.

Hence $\dfrac{dy}{dx} = y$, since $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ are reciprocals of each other,

$= e^x$.

Gradient Equals Height: The geometrical interpretation of this result is a striking relationship between the gradient and the height of the curve $y = e^x$ at each point on it.

2 GRADIENT EQUALS HEIGHT: At each point on the curve $y = e^x$, the gradient of the curve is equal to the height above the x-axis.

In particular, the gradient at the y-intercept is exactly 1 (which was already clear because the curves $y = e^x$ and $y = \log x$ are mutual reflections in $y = x$).

The Standard Forms for Differentiation: These are the standard forms for differentiation. The second follows by the chain rule with $u = ax + b$, and the third restates the chain rule.

3 STANDARD FORMS FOR DIFFERENTIATION:

A. $\dfrac{d}{dx}\, e^x = e^x$

B. $\dfrac{d}{dx}\, e^{ax+b} = a\, e^{ax+b}$

C. $\dfrac{d}{dx}\, e^u = e^u\, \dfrac{du}{dx}$ OR $\dfrac{d}{dx}\, e^{f(x)} = f'(x)\, e^{f(x)}$

WORKED EXERCISE: Find the derivatives of: (a) $e^{\frac{1}{2}(9-x)}$ (b) e^{x^2} (c) $x^3\, e^x$

SOLUTION: (a) $\dfrac{d}{dx}\, e^{\frac{1}{2}(9-x)} = -\frac{1}{2}\, e^{\frac{1}{2}(9-x)}$, by the second standard form.

(b) Let $y = e^{x^2}$, then by the chain rule with $u = x^2$,

$\dfrac{dy}{dx} = 2x\, e^{x^2}$.

(c) Let $y = x^3\, e^x$, then by the product rule with $u = x^3$ and $v = e^x$,

$\dfrac{dy}{dx} = 3x^2\, e^x + x^3\, e^x$

$= x^2\, e^x\,(3 + x)$.

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Exponential Functions with Other Bases: Any other exponential function $y = a^x$, with base a different from e, can be expressed as a base e exponential function $y = e^{kx}$. Using the identity $a = e^{\log a}$,

$$\begin{aligned} a^x &= (e^{\log a})^x \\ &= e^{x \log a}, \quad \text{by the index law } (e^m)^x = e^{mx}. \end{aligned}$$

Thus a^x has been expressed in the form e^{kx}, where $k = \log a$ is a constant. In this form, the function can be differentiated:

$$\begin{aligned} \frac{d}{dx}\, a^x &= e^{x \log a} \log a, \quad \text{using the second standard form above,} \\ &= a^x \log a, \quad \text{since } e^{x \log a} = a^x. \end{aligned}$$

Either the process can be reproduced or the results remembered.

4 OTHER BASES: $a^x = e^{x \log a}$ and $\dfrac{d}{dx}\, a^x = a^x \log a$

A Characterisation of the Exponential Function: The derivative of the exponential function $f(x) = e^x$ is $f'(x) = e^x$ — the same function. This property characterises the exponential function amongst all other exponential functions.

5 THEOREM: The exponential function $y = e^x$ is the only exponential function whose derivative is equal to itself.

PROOF: Let $f(x) = a^x$ be any other exponential function.
Then $f'(x) = a^x \log a$,
and so $f'(x)$ and $f(x)$ are identical if and only if
$\log a = 1$,
that is, $a = e$.

NOTE: The fact that the derivative is equal to the function is the fundamental reason why the exponential function is so important in mathematics. The study of natural growth and decay in Section 13E will be based on this property.

Extension — An Alternative Way to Differentiate Powers of Other Bases: The function a^x can also be differentiated using the log laws and implicit differentiation.

Let $y = a^x$.
Then $\log y = \log a^x$, taking logs of both sides,
so $\log y = x \log a$, by the log laws,
$\dfrac{1}{y}\dfrac{dy}{dx} = \log a$, by implicit differentiation,
$\boxed{\times y}$ $\dfrac{dy}{dx} = a^x \log a$.

WORKED EXERCISE:

(a) Express $y = 3 \times 2^x$ as a multiple of a power of e.

(b) Hence find the derivative of $y = 3 \times 2^x$.

(c) [Extension] Differentiate $y = 3 \times 2^x$ by the alternative method of taking logs of both sides and differentiating implicitly.

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SOLUTION:

(a) $y = 3 \times 2^x$
$= 3 \times (e^{\log 2})^x$
$= 3 \times e^{x \log 2}$

(b) $\dfrac{dy}{dx} = 3 \log 2 \times e^{x \log 2}$
$= 3 \log 2 \times 2^x$

(c) Alternatively, taking logs of both sides,
$\log y = \log 3 + x \log 2,$
so using implicit differentiation,
$$\frac{1}{y}\frac{dy}{dx} = \log 2,$$
$$\boxed{\times\, y} \quad \frac{dy}{dx} = y \log 2$$
$$= 3 \log 2 \times 2^x .$$

Exercise 13A

1. Differentiate:
(a) e^{2x} (b) e^{-3x} (c) $-e^{5x}$ (d) $e^{\frac{1}{2}x}$ (e) e^{ax} (f) e^{-kx} (g) $-e^{-\pi x}$ (h) $e^{-\frac{1}{c}x}$

2. Find the derivative of:
(a) e^{2x-1} (b) e^{1-x} (c) $3e^{-3x+4}$ (d) $2e^{\frac{1}{2}x+4}$ (e) e^{px+q} (f) $e^{2x} - e^{-3x}$ (g) $\dfrac{e^x - e^{-x}}{2}$ (h) $\dfrac{e^{ax}}{a} + \dfrac{e^{-bx}}{b}$

3. Write as a power of e and then differentiate:
(a) $(e^x)^2$ (b) $\dfrac{1}{e^{3x}}$ (c) $\sqrt{e^x}$ (d) $\dfrac{1}{\left(\sqrt{e^x}\right)^3}$

4. Use the product and chain rules as appropriate to differentiate:
(a) e^{x^2} (b) e^{1-x^2} (c) e^{x^2+2x} (d) e^{6+x-x^2} (e) $\frac{1}{2}e^{3x^2-2x+1}$ (f) xe^x (g) xe^{-x} (h) $(x-1)e^x$ (i) $(x+1)e^{3x-4}$ (j) xe^{x^2+1} (k) $(x^2+5x-5)e^x$ (l) $(x^2-x^3)e^{-x}$

5. (a) Find the gradient of the curve $y = e^{5x}$ at the point $A(a, e^{5a})$, and show that the gradient is 5 times the height.
(b) Find the gradient of the curve $y = be^{-3x}$ at the point $A(a, be^{-3a})$, and show that the gradient is -3 times the height.
(c) Find the gradient of the curve $y = be^{kx}$ at the point $A(a, be^{ka})$, and show that the gradient is k times the height.

DEVELOPMENT

6. Use the product, quotient, chain and log rules as appropriate to differentiate:
(a) $(x^2 - x)e^{2x-1}$ (b) $\log(1 - e^x)$ (c) $\log(e^x + x)$ (d) $e^x \log x$ (e) $\log\left(e^x + e^{-x}\right)$ (f) $\dfrac{e^x}{x}$ (g) $\dfrac{x+1}{e^x}$ (h) $\dfrac{e^x+1}{e^x-1}$ (i) $\dfrac{e^x}{(x+1)^2}$ (j) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

7. Use the identity $a = e^{\log a}$ to write each expression as a power of e. Thus differentiate it.
(a) 2^x (b) 10^x (c) π^x (d) a^x (e) 2^{3x-1} (f) 5^{2-x} (g) a^{x-2} (h) a^{bx+c} (i) $x2^x$ (j) 3^{x^3-3x}

8. Simplify then differentiate:
(a) $e^{\log x}$ (b) $e^{-\log x}$ (c) $\log\left(e^{-x}\right)$ (d) $\log\left(e^{3x-7}\right)$

9. (a) Show that Ae^0, Ae^{-1}, Ae^{-2}, ... forms a GP, and find its limiting sum.
(b) Show that for the AP $u_1 = a$, $u_2 = a + d$, $u_3 = a + 2d$, ..., the sequence e^{u_1}, e^{u_2}, e^{u_3}, ... is a GP.
(c) Conversely, show that for the GP $u_1 = a$, $u_2 = ar$, $u_3 = ar^2$, ..., the sequence $\log u_1$, $\log u_2$, $\log u_3$, ... is an AP.

10. (a) Show that both $y = e^{-x}$ and $y = xe^{-x}$ are solutions of $y'' + 2y' + y = 0$.
(b) Show that $y = e^{-3x} + x - 1$ is a solution of the differential equation $y'' + 2y' - 3y = 5 - 3x$.
(c) Find the values of λ that make $y = 5e^{\lambda x}$ a solution of:
(i) $y'' + 3y' - 10y = 0$ (ii) $y'' + y' - y = 0$

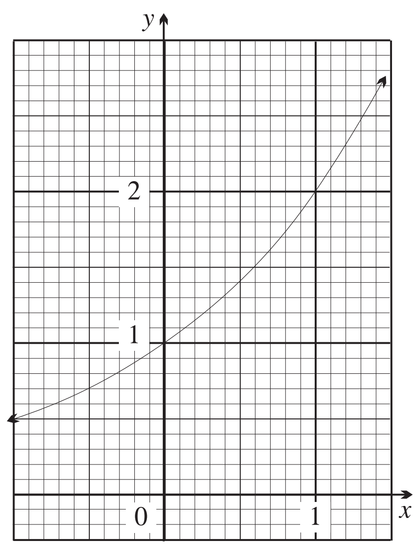
11. (a) Show that $y = Ae^{kx}$ is a solution of: (i) $y' = ky$ (ii) $y'' - k^2 y = 0$
(b) Show that $y = Ae^{kx} + C$ is a solution of $\dfrac{dy}{dx} = k(y - C)$.
(c) Show that $y = (Ax + B)e^{3x}$ is a solution of $y'' - 6y' + 9y = 0$.

12. Differentiate: (a) $e^{\frac{1}{x}}$ (b) $e^{x + \log x}$ (c) $3^{-\frac{1}{x}}$

13. (a) Find the x-coordinates of the stationary points of $y = x\,e^{-x^2}$.
(b) Determine where the second derivative of $y = x^2\,e^x$ changes sign.
(c) Let $f(x) = x\,e^x$ and $g(x) = x\,e^{-x}$. Find $f'(x)\,g'(x)$.

14. We define two new functions $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.
(a) Show that each of these functions is the derivative of the other.
(b) Show that both functions are solutions of the differential equation $y'' - y = 0$.
(c) Show that $\cosh^2 x - \sinh^2 x = 1$.

15. (a) Make a copy of the graph of $y = 2^x$ on the right and on it draw the secant from $x = 0$ to $x = 1$. Also draw the tangent at $x = 0$.
(b) Compare the secant and tangent and hence explain why the gradient of the tangent is less than 1. Measure both gradients to confirm this.
(c) Differentiate $y = 2^x$ by writing $y = e^{x \log 2}$, and hence show that $y' = \log 2$ at $x = 0$. Compare this result with your answer to the previous part.
(d) Show that the gradient of a secant from $x = 0$ to $x = h$ is given by $\dfrac{2^h - 1}{h}$. Use a calculator to evaluate this quantity for smaller and smaller values of h. What is the value of $\lim\limits_{h \to 0} \dfrac{2^h - 1}{h}$?
(e) Differentiate $y = 2^x$ by first principles, and use the previous part to help evaluate the limit.



16. Given that $y = e^x$, it is clear that $x = \log y$. Differentiate the latter equation and hence prove that $\dfrac{dy}{dx} = e^x$.

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17. Differentiate, either by expressing as a power of e, or by taking logs of both sides and using implicit differentiation:
(a) $y = x^x$ (b) $y = x^{2-x}$ (c) $y = x^{\log x}$ (d) $y = x^{\frac{1}{\log x}}$

EXTENSION

18. (a) Find the possible values of λ that make $y = e^{\lambda x}$ a solution of $ay'' + by' + cy = 0$.
(b) When will there be no real solution for λ?

19. (a) Prove that a function of the form $y = Ae^x$, where A is a constant, is the only function that is its own derivative. Proceed by the method of proof by contradiction as follows:
(i) Assume that there exists another function $f(x)$ that has this property and is not a multiple of e^x. Show that $f'(x) - f(x) = 0$.
(ii) Let $g(x) = e^{-x}f(x)$. Show that $g'(x) = 0$.
(iii) Explain why $g(x)$ is constant, and complete the proof.
(b) Show by direct differentiation that if $f(x) = x\,e^{x-\log x}$, then $f(x) = f'(x)$. Explain this result in the light of what was proven in part (a).

20. (a) In Exercise 12B of the previous chapter on logarithms, the result $\displaystyle\lim_{u\to 1}\frac{\log u}{u-1} = 1$ was proven. Use the substitution $h = \log u$ to show that $\displaystyle\lim_{h\to 0}\frac{e^h - 1}{h} = 1$.
(b) Differentiate $y = e^x$ from first principles and use the result $y' = e^x$ to show that $\displaystyle\lim_{h\to 0}\frac{e^h - 1}{h} = 1$.

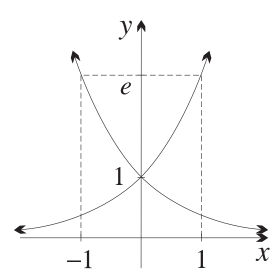
13 B Applications of Differentiation

The derivatives of algebraic functions, logarithmic functions and exponential functions are now known, and the usual applications of differentiation are possible, in particular the sketching of curves whose equations involve e^x.

The Graphs of e^x and e^{-x}: The graphs of $y = e^x$ and $y = e^{-x}$ are the essential graphs for this section. Since x is replaced by $-x$, these two graphs are reflections of each other in the y-axis:

x	-2	-1	0	1	2
e^x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

x	-2	-1	0	1	2
e^{-x}	e^2	e	1	$\frac{1}{e}$	$\frac{1}{e^2}$



The two curves cross at $(0, 1)$. The gradient of $y = e^x$ at $(0, 1)$ is 1, and so the gradient of $y = e^{-x}$ at $(0, 1)$ must be -1. This means that the curves are perpendicular at their point of intersection.

NOTE: The function $y = e^{-x}$ is as important as $y = e^x$ in applications, or even more important. It describes a great many physical situations where a quantity 'dies away exponentially', like the dying away of the sound of a plucked string.

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Two More Special Limits — e^x Dominates x: Just as x dominates $\log x$, so we can show that e^x dominates x. Curve sketching involving exponential functions often requires this idea, which is expressed by the following limits.

6 THE FUNCTION e^x DOMINATES THE FUNCTION x:

$$\lim_{x \to -\infty} x\,e^x = 0 \qquad \text{and} \qquad \lim_{x \to \infty} x\,e^{-x} = 0$$

More colourfully, 'in a battle between x and e^x, e^x always wins'.

PROOF:

A. We know already from Section 12C that $\lim_{u \to 0^+} u \log u = 0$.

Substitute $u = e^x$, so $\log u = x$,

then $u \to 0^+$ as $x \to -\infty$, so that $\lim_{x \to -\infty} e^x\, x = 0$.

B. Similarly, we proved in Section 12C that $\lim_{u \to \infty} \dfrac{\log u}{u} = 0$.

Using the same substitution, $u \to \infty$ as $x \to \infty$,

so $\lim_{x \to \infty} \dfrac{x}{e^x} = 0$.

The more general result is that e^x dominates x^k, for all $k > 0$, but the proof is left to a question in the following exercises.

7 THE FUNCTION e^x DOMINATES THE FUNCTION x^k, FOR $k > 0$:

$$\lim_{x \to -\infty} x^k\,e^x = 0 \qquad \text{and} \qquad \lim_{x \to \infty} x^k\,e^{-x} = 0$$

An Example of Curve Sketching: The application of the standard curve sketching menu to the sketch of $y = x^2\,e^x$ will illustrate the use of these limits.

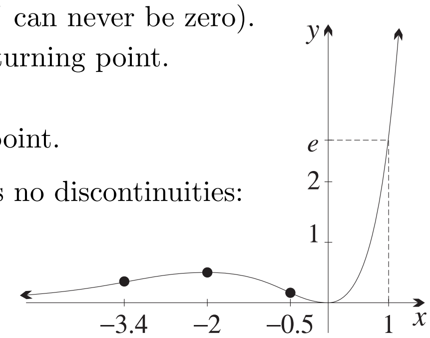
1. The domain is the whole real number line.
2. $f(-x) = x^2\,e^{-x}$, which is neither $f(x)$ nor $-f(x)$, so the function is neither even nor odd.
3. The only zero is $x = 0$. Also, y is positive for all $x \neq 0$.
4. $\lim_{x \to -\infty} y = 0$, since e^x dominates x^2. Also, $y \to \infty$ as $x \to \infty$.
5. Differentiating twice by the product rule,

$$\begin{aligned} f'(x) &= 2x\,e^x + x^2\,e^x \\ &= x(x+2)\,e^x, \end{aligned} \qquad \begin{aligned} f''(x) &= (2x+2)\,e^x + (x^2+2x)\,e^x \\ &= (x^2+4x+2)\,e^x, \end{aligned}$$

so $f'(x) = 0$ when $x = 0$ and $x = -2$ (notice that e^x can never be zero).
Since $f''(0) = 2 > 0$, the point $(0, 0)$ is a minimum turning point.
Also, $f''(-2) = -2e^{-2} < 0$,
so $(-2, 4e^{-2}) \doteqdot (-2, 0{\cdot}541)$ is a maximum turning point.

6. $f''(x)$ has zeroes at $-2-\sqrt{2}$ and $-2+\sqrt{2}$, and has no discontinuities:

x	-4	$-2-\sqrt{2}$	-2	$-2+\sqrt{2}$	0
$f''(x)$	$2e^{-4}$	0	$-2e^{-2}$	0	2
	$\smile$	$\cdot$	$\frown$	$\cdot$	$\smile$



so there are inflexions at $\left(-2-\sqrt{2}\,,\ (6+4\sqrt{2})e^{-2-\sqrt{2}}\right) \doteqdot (-3{\cdot}414, 0{\cdot}384)$,

and also at $\left(-2+\sqrt{2}\,,\ (6-4\sqrt{2})e^{-2+\sqrt{2}}\right) \doteqdot (-0{\cdot}586, 0{\cdot}191)$.

Exercise 13B

1. (a) Copy and complete these tables of values of the functions $y = e^x$ and $y = \log x$.

x	-2	-1	0	1	2
e^x					

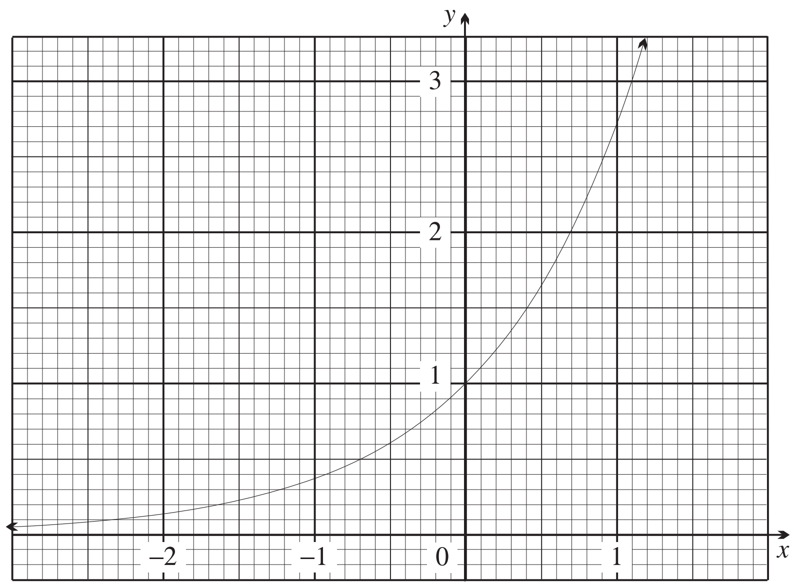
x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log x$					

(b) Sketch both curves on the one set of axes, choosing appropriate scales on the axes.

(c) Add the line $y = x$ to your graph. What transformation transforms the graph of $y = e^x$ into the graph of $y = \log x$?

(d) What are the domain and range of $y = e^x$ and $y = \log x$?

2.



(a) Copy the graph of $y = e^x$ and on it draw the tangent at $x = 0$, extending the tangent down to the x-axis.

(b) Measure the gradient of this tangent and confirm that it is equal to the height of the exponential graph at the point of contact.

(c) Repeat for the tangents at $x = -2, -1$ and 1.

(d) What do you notice about the x-intercepts of the tangents?

3. Use your knowledge of transformations to help sketch the graphs of the given functions:

(a) $y = e^{x-1}$
(b) $y = e^{-x}$
(c) $y = -e^x$
(d) $y = e^{x-2}$
(e) $y = 1 - e^x$
(f) $y = e^{-x} - 1$
(g) $y = e^{\frac{1}{2}x}$
(h) $y = e^{-|x|}$

4. (a) Find the equation of the tangent to $y = e^x$ at its y-intercept.

(b) Show that the tangent to $y = x - e^x$ at $x = 1$ passes through the origin.

5. (a) Find the equation of the normal to $y = e^{-x}$ at the point $P(-1, e)$.

(b) Find the x and y intercepts of the normal.

(c) Find the area of the triangle whose vertices lie at the intercepts and the origin.

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6. (a) Find the first and second derivatives for the curve $y = x - e^x$.
(b) Deduce that the curve is concave down for all values of x.
(c) What is its maximum value? (d) Sketch the curve and write down its range.

7. Consider the curve $y = x\,e^x$.
(a) Show that $y' = (1+x)e^x$ and $y'' = (2+x)e^x$.
(b) Show that there is one stationary point, and determine its nature.
(c) Find the coordinates of the lone point of inflexion.
(d) Examine the behaviour of y as $x \to -\infty$.
(e) Hence sketch the curve, and then write down its range.
(f) Hence also sketch $y = -x\,e^{-x}$ by recognising the simple transformation.

8. (a) Given that $y = e^{-\frac{1}{2}x^2}$, find y' and y''.
(b) Show that this curve has a maximum turning point at its y-intercept, and has two points of inflexion.
(c) Examine the behaviour of y as $x \to -\infty$ and $x \to \infty$.
(d) Sketch the graph and write down its range.

DEVELOPMENT

9. (a) Given that $y = (1-x)e^x$, find y' and y''.
(b) Show that this curve has a maximum turning point at its y-intercept, and an inflexion point at $(-1, 2e^{-1})$.
(c) Sketch the graph and write down its range.

10. Find the x-intercept of the tangent to $y = (1-x)e^x$ at $x = -1$.

11. [Another characterisation of $y = e^x$ — it is the only exponential function whose gradient at its y-intercept is 1.]
(a) Prove that $y = a^x$ has derivative $y = a^x \log a$.
(b) Prove that $y = a^x$ has gradient 1 at its y-intercept if and only if $a = e$.
(c) Prove that $y = Aa^x$ has gradient 1 at its y-intercept if and only if $a = e^{1/A}$.

12. The line $y = mx$ is tangent to the curve $y = \dfrac{e^x}{x}$. Show that the point of contact is $A(2, \frac{1}{2}e^2)$ by showing that the gradient of OA is equal to the gradient of the tangent to the curve at A.

13. (a) Find the equation of the tangent to $y = e^x$ at $x = t$.
(b) Hence show that the x-intercept of this tangent is $t - 1$. Does this agree with your answers to question 2(d)?

14. (a) Show that $y = e - x^2\,e^x$ has an x-intercept at $x = 1$.
(b) Show that the curve has two turning points and classify them.
(c) Examine the behaviour as $x \to \infty$ and deduce that it also has two inflexion points.
(d) Sketch the curve and write down its range.

15. (a) Find the intercepts of $y = (1+x)^2 e^{-x}$.
(b) Show that the curve has two turning points and classify them.
(c) Examine the behaviour of y as $x \to \infty$ and hence deduce that it also has two inflexions.
(d) Sketch the curve and write down its range.

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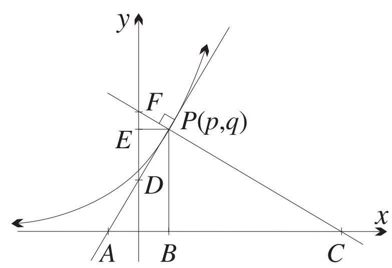
16. (a) Classify the stationary points of $y = x\,e^{-x^2}$.
(b) Locate the three inflexion points, sketch the curve and write down its range.

17. (a) Find the equation of the normal to $y = e^{-x^2}$ at the point where $x = t$.
(b) Determine the x-intercept of the normal.
(c) Hence find the values of t for which the normal passes through the origin.

18. On the graph of $y = e^x$ are drawn the tangent and normal to the curve at the point $P(p, q)$.
(a) Find the coordinates of each of the points A, B, C, D, E and F in terms of p and q.
(b) Hence show: (i) $AB = 1$ (ii) $BC = q^2$ (iii) $DE = pq$ (iv) $EF = \frac{p}{q}$
(c) What is the area of: (i) $\triangle ACP$? (ii) $\triangle DFP$?



19. Find the x-coordinates of the stationary points of $y = x\,e^{-|x|}$ by considering positive and negative values of x separately.

20. Show that $y = (x^2 + 3x + 2)e^x$ has an inflexion point at one of its x-intercepts. Sketch the curve and label all important features. Do not find the y-coordinates of the stationary points.

21. (a) What is the natural domain of $y = \dfrac{e^x}{x}$?
(b) Show that the curve has a local minimum at $(1, e)$ but no inflexion points.
(c) Sketch the curve and state its range.

22. (a) What is the natural domain of $y = e^{\frac{1}{x}}$?
(b) Carefully determine the behaviour of y and y' as $x \to -\infty$, $x \to 0$ and $x \to \infty$.
(c) Deduce that there must be an inflexion point and find it.
(d) Sketch the curve and give its range.

23. Follow the steps in the previous question in order to sketch the graph of $y = x\,e^{\frac{1}{x}}$.

EXTENSION

24. If the positive base a of $y = a^x$ and $y = \log_a x$ is small enough, then the two curves will intersect. What base must be chosen so that the two are tangent at the point of contact? Proceed as follows:
(a) Rewrite both equations with base e, and let $k = \log a$.
(b) Explain why the gradient of the tangent at the point of contact must be 1.
(c) Use the last part to obtain two equations for the gradient.
(d) Solve these simultaneously to find k, and hence write down the base a.

25. [Here are two proofs of the more general dominance result mentioned in the theory above.]
(a) One of the results from the previous chapter is $\lim\limits_{u \to 0} u^{\frac{1}{k}} \log u = 0$. Substitute $u = e^x$ and hence prove that $\lim\limits_{x \to -\infty} x^k\,e^x = 0$.
(b) Given that $\lim\limits_{u \to -\infty} u\,e^u = 0$, substitute $u = x/k$ and hence prove that $\lim\limits_{x \to -\infty} x^k\,e^x = 0$.

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26. One of the fundamental functions in the study of statistics is $y = \int_0^x e^{-\frac{1}{2}t^2}\,dt$.

(a) What is the y-intercept of this function? (b) Use the fundamental theorem of calculus to find y' and show that there are no stationary points.
(c) Show that y' is even. What can be said about y?
(d) Show that there is a point of inflexion at the y-intercept.
(e) Investigate y' as $x \to \infty$. Is it possible to deduce from this alone whether y has any horizontal asymptote? (f) It can be shown (with more advanced techniques) that y has horizontal asymptote $y = \sqrt{\frac{\pi}{2}}$. Now sketch the function and state its range.

13 C Integration of the Exponential Function

The standard forms for differentiation can be reversed to provide standard forms for integration.

8

STANDARD FORMS FOR INTEGRATION:

A. $\int e^x\,dx = e^x + C$

B. $\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + C$

C. $\int e^u\,\frac{du}{dx}\,dx = e^u + C$ OR $\int f'(x)\,e^{f(x)}\,dx = e^{f(x)} + C$

WORKED EXERCISE: Find: (a) $\int_0^2 e^x\,dx$ (b) $\int_2^3 e^{5-2x}\,dx$ (c) $\int_{-1}^1 x\,e^{x^2}\,dx$

SOLUTION:

(a)
$$\begin{aligned}\int_0^2 e^x\,dx &= \Big[e^x\Big]_0^2 \\ &= e^2 - e^0 \\ &= e^2 - 1\end{aligned}$$

(b)
$$\begin{aligned}\int_2^3 e^{5-2x}\,dx &= -\tfrac{1}{2}\Big[e^{5-2x}\Big]_2^3 \\ &= -\tfrac{1}{2}(e^{-1} - e) \\ &= \frac{e^2-1}{2e}\end{aligned}$$

(c)
$$\begin{aligned}\int_{-1}^1 x\,e^{x^2}\,dx &= \tfrac{1}{2}\int_{-1}^1 2x\,e^{x^2}\,dx, \\ &= \tfrac{1}{2}\Big[e^{x^2}\Big]_{-1}^1 \\ &= \tfrac{1}{2}(e - e) \\ &= 0, \text{ since the integrand is odd.}\end{aligned}$$

Let $u = x^2$.
Then $\dfrac{du}{dx} = 2x$.
$\int e^u\,\dfrac{du}{dx}\,dx = e^u$

WORKED EXERCISE: Find $f(x)$ and $f(1)$, if $f'(x) = 1 + 2e^{-x}$ and $f(0) = 1$.

SOLUTION: $f'(x) = 1 + 2e^{-x}$
so $f(x) = x - 2e^{-x} + C$, for some constant C.
Since $f(0) = 1$, $1 = 0 - 2\,e^0 + C$
so $C = 3$ and $f(x) = x - 2e^{-x} + 3$.
Hence $f(1) = 1 - 2e^{-1} + 3$
$= 4 - \dfrac{2}{e}$.

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Integrals of Exponential Functions with Other Bases: Since $\dfrac{d}{dx}\,a^x = a^x \log a$, it follows (omitting the constant of integration) that:

9 OTHER BASES: $\displaystyle\int a^x\,dx = \frac{a^x}{\log a}$, for all positive bases $a \neq 1$.

Either this result or the process of obtaining it should be learnt. The primitive can also be obtained by expressing a^x as a power of e:

$$\begin{aligned}\int a^x\,dx &= \int (e^{\log a})^x\,dx\\ &= \int e^{x\log a}\,dx\\ &= \frac{1}{\log a}\,e^{x\log a},\quad\text{since } \int e^{kx} = \frac{1}{k}\,e^{kx},\\ &= \frac{1}{\log a}\,a^x,\quad\text{since } e^{x\log a} = a^x.\end{aligned}$$

NOTE: The formulae for differentiation and integration of a^x both involve $\log a$:

$$\frac{d}{dx}\,a^x = a^x\,\log a \qquad\text{and}\qquad \int a^x\,dx = \frac{a^x}{\log a}.$$

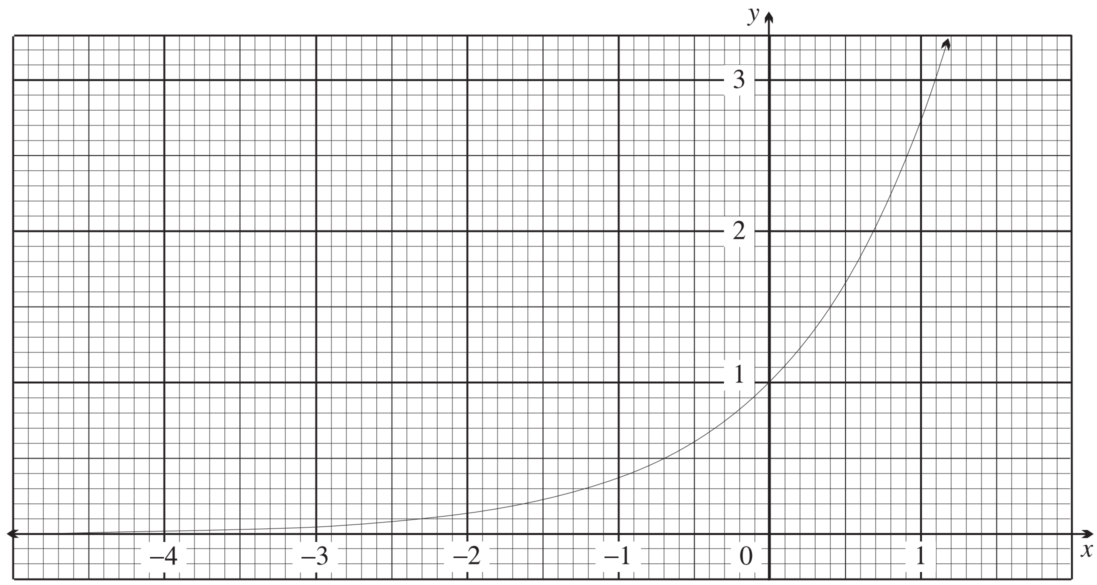
Since $\log a = 1$ when $a = e$, the formulae are simplest when the base is e, which confirms that e is the appropriate base to use for the calculus of exponential functions.

Exercise **13C**

1. (a) Evaluate these definite integrals, then approximate them to two decimal places:

(i) $\displaystyle\int_0^1 e^x\,dx$ (ii) $\displaystyle\int_{-1}^0 e^x\,dx$ (iii) $\displaystyle\int_{-2}^0 e^x\,dx$ (iv) $\displaystyle\int_{-3}^0 e^x\,dx$

(b)



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The graph above shows that $y = e^x$ from $x = -5$ to $x = 1$, with a scale of 10 divisions to 1 unit, so that 100 little squares equal 1 square unit. By counting squares under the curve from $x = 0$ to $x = 1$, find an estimate for $\int_0^1 e^x\,dx$, and compare it with the approximation obtained in part (a).

(c) Count squares to the left of the y-axis to obtain estimates of:

(i) $\int_{-1}^0 e^x\,dx$, (ii) $\int_{-2}^0 e^x\,dx$, (iii) $\int_{-3}^0 e^x\,dx$,

and compare the results with the approximations obtained in part (a).

(d) Continue counting squares to the left of $x = -3$, and estimate the total area under the curve to the left of the y-axis.

2. Find the following indefinite integrals:

(a) $\int e^{2x}\,dx$ (e) $\int e^{4x-2}\,dx$ (i) $\int e^{3-x}\,dx$ (m) $\int \pi e^{3x-\sqrt{2}}\,dx$

(b) $\int e^{\frac{1}{3}x}\,dx$ (f) $\int e^{x-1}\,dx$ (j) $\int e^{7-2x}\,dx$ (n) $\int ae^{b-ax}\,dx$

(c) $\int e^{4x+5}\,dx$ (g) $\int 6e^{3x+2}\,dx$ (k) $\int e^{\pi x-1}\,dx$ (o) $\int ae^{bx+c}\,dx$

(d) $\int e^{1+3x}\,dx$ (h) $\int 2e^{2x-1}\,dx$ (l) $\int \frac{1}{2}e^{1-ex}\,dx$ (p) $\int -\pi e^{a-\pi x}\,dx$

3. Evaluate the following definite integrals:

(a) $\int_1^2 e^x\,dx$ (c) $\int_0^2 e^{x-1}\,dx$ (e) $\int_{\log 2}^{\log 3} e^x\,dx$ (g) $\int_{-\pi}^0 e^{1-\frac{1}{\pi}x}\,dx$

(b) $\int_{-1}^3 e^{-x}\,dx$ (d) $\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{3-2x}\,dx$ (f) $\int_{\log 4}^{\log 6} e^{-x}\,dx$ (h) $\int_a^{2a} e^{bx}\,dx$

4. Use the index laws to simplify each integrand and hence find the indefinite integral of:

(a) $\int \frac{1}{(e^x)^2}\,dx$ (b) $\int \frac{e^x+1}{e^x}\,dx$ (c) $\int \sqrt{e^x}\,dx$ (d) $\int \frac{e^x - e^{-x}}{\sqrt{e^x}}\,dx$

5. Use the identity $a = e^{\log a}$ to express each of the following as a power of e, and hence find its primitive.

(a) 2^x (b) 3^x (c) 5^{-x} (d) π^x

6. (a) Find y as a function of x if $y' = e^{x-1}$ and $y = 1$ when $x = 1$. What is the y-intercept of this curve?

(b) The gradient of a curve is given by $y' = e^{2-x}$ and the curve passes through the point $(0, 1)$. What is the equation of this curve? What is its horizontal asymptote?

(c) Find y, given that $y' = 2^{-x}$, and $y = \frac{1}{2\log 2}$ when $x = 1$.

(d) It is known that $f'(x) = e^x + \dfrac{1}{e}$ and that $f(-1) = -1$. Find $f(0)$.

7. Use the standard form $\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$, or $\int e^u \dfrac{du}{dx}\,dx = e^u + C$, to integrate:

(a) $2xe^{x^2+3}$ (c) $(3x+2)e^{3x^2+4x+1}$

(b) $(10x-2)e^{5x^2-2x}$ (d) $(x^2-2x)e^{x^3-3x^2}$

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DEVELOPMENT

8. Find a primitive of each function:

(a) $(6x^2 - 8x + 6)e^{x^3 - 2x^2 + 3x - 5}$

(b) $\dfrac{1}{x} + e^{3x}$

(c) $\dfrac{1}{\sqrt{x}} - xe^{-x^2}$

(d) $(e^x + 1)^2$

(e) $(e^x - e^{-x})^2$

(f) $\dfrac{e^{\frac{1}{x}}}{x^2}$

(g) $\sqrt{x}\, e^{x\sqrt{x}}$

(h) $e^{2\log x}$

(i) $\log(e^{2kx})$

9. (a) Differentiate xe^x. (b) Hence find $\displaystyle\int_0^2 xe^x\, dx$.

10. (a) Differentiate $e^x + e^{-x}$. (b) Hence find $\displaystyle\int_0^2 \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$.

11. Find the indefinite integral $\displaystyle\int \frac{e^x}{e^x + 1}\, dx$.

12. Use the identity $a = e^{\log a}$ to help find a primitive of:

(a) $2x + \dfrac{2}{x} + 2^x$

(b) $a^x + ax$

(c) $2(x+1)3^{x^2+2x}$

13. (a) Differentiate $y = x^2\, e^{-x^2}$.

(b) Hence show that $\displaystyle\int x^3 e^{-x^2}\, dx = \frac{-e^{-x^2}}{2}(x^2+1) + C$, and calculate $\displaystyle\int_1^2 x^3\, e^{-x^2}\, dx$.

14. (a) Show that $x\, e^x = e^{x + \log x}$.

(b) Hence differentiate $x\, e^x$ without using the product rule.

15. (a) The gradient of a certain curve is $y' = -\sqrt{x}\, e^{-x\sqrt{x}}$. Given that its y-intercept is 1, determine the equation of this curve.

(b) Another curve has gradient $3^{-x}\log 3$ and an horizontal asymptote of $y = 2$. Find the equation of the curve and its y-intercept.

16. (a) Find $\dfrac{d}{dx} e^{ax}$ and hence write down $\displaystyle\int e^{ax}\, dx$.

(b) Show that $\displaystyle\int e^{ax+b}\, dx = e^b \int e^{ax}\, dx$.

(c) Hence confirm the standard form $\displaystyle\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + C$.

17. Find $\displaystyle\int_0^1 2^{\log x}\, dx$. [HINT: Given that $e^{\log x} = x$, what is $2^{\log x}$?]

EXTENSION

18. Show that $\displaystyle\int \frac{e^x + 1}{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}}\, dx = 2\, e^{\frac{1}{2}x} + C$.

19. The intention of this question is to outline a reasonably rigorous proof of the famous result that e^x can be written as the limit of the power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

where $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

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(a) For any positive number R, we know that $1 < e^t < e^R$ for $0 < t < R$, because e^x is an increasing function. Integrate this inequality over the interval $t = 0$ to $t = x$, where $0 < x < R$, and hence show that $x < e^x - 1 < e^R\, x$.

(b) Change the variable to t, giving $t < e^t - 1 < e^R\, t$. Then integrate this new inequality from $t = 0$ to $t = x$, and hence show that $\dfrac{x^2}{2!} < e^x - 1 - x < \dfrac{e^R\, x^2}{2!}$.

(c) Do this process twice more, and prove that:

(i) $\dfrac{x^3}{3!} < e^x - 1 - x - \dfrac{x^2}{2!} < \dfrac{e^R\, x^3}{3!}$

(ii) $\dfrac{x^4}{4!} < e^x - 1 - x - \dfrac{x^2}{2!} - \dfrac{x^3}{3!} < \dfrac{e^R\, x^4}{4!}$

(d) Now use induction to prove that $\dfrac{x^{n+1}}{(n+1)!} < e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^n}{n!} < \dfrac{e^R\, x^{n+1}}{(n+1)!}$.

(e) Show that as $n \to \infty$, the left and right expressions converge to zero. Hence prove that the infinite power series converges to e^x for $x > 0$. [HINT: Let k be the smallest integer greater than x and show that for $n > k$, each term in the sequence is less than the corresponding term of a geometric sequence with ratio $\frac{x}{k}$.]

(f) Prove that the power series also converges to e^x for $x < 0$.

20. (a) Use the power series in the previous question to show that

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots .$$

(b) Find α, the value of the right-hand side, to four decimal places when $x = 0{\cdot}5$.

(c) Let $u = e^{0\cdot5}$. Show that $u^2 - 2\alpha u + 1 = 0$.

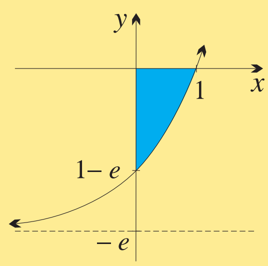
(d) Solve this quadratic equation and hence estimate both $e^{0\cdot5}$ and $e^{-0\cdot5}$ to two decimal places. Compare your answers with the values obtained directly from the calculator.

13 D Applications of Integration

The normal applications of integration to areas, volumes, and primitives are now available with functions involving algebraic, logarithmic and exponential functions.

WORKED EXERCISE: Sketch the graph of $y = e^x - e$, then find the area between the curve, the x-axis and the y-axis.

SOLUTION: Move the graph of $y = e^x$ down e units.



The x-intercept is $x = 1$, because then $y = e^1 - e = 0$.

$$\begin{aligned}\int_0^1 (e^x - e)\,dx &= \Big[e^x - ex\Big]_0^1\\ &= (e - e) - (1 - 0)\\ &= -1 \quad \text{(negative, being below the } x\text{-axis)}\end{aligned}$$

so the required area is 1 square unit.

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WORKED EXERCISE: Find the volume generated when the area between the curve $y = 3e^{1-2x}$ and the ordinates $x = -1$ and $x = 1$ is rotated about the x-axis.

SOLUTION:
$$\begin{aligned}\text{Volume} &= \int_{-1}^{1} \pi y^2\, dx \\ &= \pi \int_{-1}^{1} 9e^{2-4x}\, dx, \quad \text{since } y^2 = 9e^{2-4x}, \\ &= -\tfrac{9}{4}\pi \Big[e^{2-4x}\Big]_{-1}^{1} \\ &= -\tfrac{9}{4}\pi(e^{-2} - e^{6}) \\ &= \frac{9\pi(e^8 - 1)}{4e^2} \text{ cubic units.}\end{aligned}$$

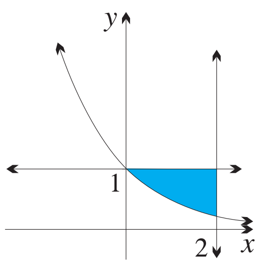
Exercise 13D

1. Find the area between $y = e^x$ and the x-axis for:
(a) $-1 \le x \le 0$ (b) $1 \le x \le 3$ (c) $\log 2 \le x \le \log 5$ (d) $\log \frac{1}{2} \le x \le \log 4$

2. Find the area under the graph $y = e^{-x} + 1$ between $x = 0$ and $x = 2$.

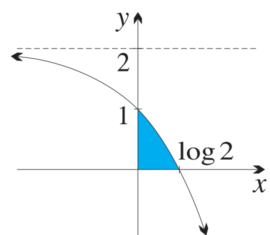
3. (a) Use the trapezoidal rule with five function values to estimate the area under the curve $y = e^{-x^2}$ between $x = 0$ and $x = 4$. Give your answer to four decimal places.
(b) Use Simpson's rule with five function values to estimate the area in part (a).

4. (a)



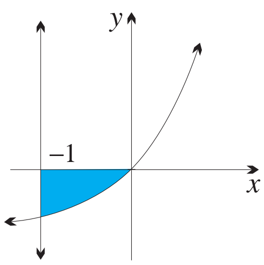
Find the area of the region bounded by the curve $y = e^{-x}$ and the lines $x = 2$ and $y = 1$.

(b)



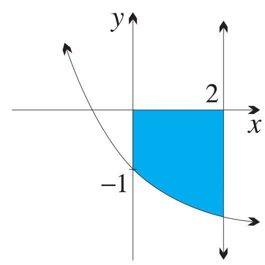
Find the area of the region in the first quadrant bounded by $y = 2 - e^x$ and the coordinate axes.

(c)



Find the area between the x-axis, the curve $y = e^x - 1$, and the line $x = -1$.

(d)



What is the area bounded by $x = 2$, $y = e^{-x} - 2$, the x-axis and the y-axis?

5. Change the subject of $y = \log x$ to x, and hence find the area between $y = \log x$ and the y-axis, and between $y = 0$ and $y = 1$.

6. Sketch a graph of $y = e^x$ and $y = x + 1$, and shade the area between these curves, $x = 0$ and $x = 1$. Then write down the area of this region as an integral and evaluate it.

7. The region under $y = e^x$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Write down the volume of the resulting solid as an integral, and evaluate it.

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DEVELOPMENT

8. (a) Show that the curves $y = x^2$ and $y = e^{x+1}$ intersect at $x = -1$.
(b) Hence sketch the region in the second quadrant between these two curves and the y-axis, and find its area.

9. Sketch the region between the graphs of $y = e^x$ and $y = x$, between the y-axis and $x = 2$, then find its area.

10. Sketch the region bounded by the x-axis, the lines $y = x$ and $x = 2$ and the curve $y = e^{1-x}$. Find its area.

11. The shape of a metal stud is created by rotating the curve $y = e^x - e^{-x}$ about the x-axis between $x = 0$ and $x = \frac{1}{2}$. Find its volume.

12. A horn is generated by rotating the curve $y = 1 + e^{-x}$ about the x-axis between $x = 1$ and $x = 3$. Find its volume to three decimal places.

13. A certain bulb and capillary tube are generated when the curve $y = \sqrt{2x}\, e^{-\frac{1}{2}x^2}$ is rotated about the x-axis between $x = 0$ cm and $x = 2$ cm. Find the capacity of liquid the apparatus could hold. Give your answer to four significant figures.

14. Find the intercepts of the curve $y = 8 - 2^x$ and hence find the area of the region bounded by this curve and the coordinate axes.

15. Consider the two curves $y = ae^x$ and $y = be^{-x}$, where $0 < a < b$.
(a) Find the point of intersection of these two curves.
(b) Hence find the area of the region bounded by these two curves and the y-axis.

16. A rubber seal has the shape generated by rotating the region under $y = \log x$ between $x = 1$ and $x = 2$ about the y-axis. Find the volume of the washer.

17. A sheet of plywood cut out by a jigsaw occupies the region bounded by $y = x$, the x-axis and $y = x - e^x$ between $x = -1$ and $x = 1$, all units being in metres. If the cost of the plywood is \$15 for cutting plus \$8 per square metre, what is the total cost of the sheet to the nearest cent?

18. (a) (i) Find $\displaystyle\int_0^N e^{-x}\,dx$. (ii) Take the limit of part (i) as $N \to \infty$. [This is an amazing result: a region that extends to infinity has a *finite* area! There are many other such examples of unbounded regions with finite areas.]
(b) Similarly find the area of the region in the second quadrant under the curve $y = e^x$, and compare your answer with that to the previous part.
(c) (i) Likewise evaluate $\displaystyle\int_0^N 2xe^{-x^2}\,dx$. (ii) Show that $y = xe^{-x^2}$ is odd.
(iii) Hence show that the total area between the curve $y = xe^{-x^2}$ and the x-axis is 1.

EXTENSION

19. (a) Determine $\displaystyle\int_\delta^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. (b) What happens to the integral as $\delta \to 0^+$?

20. (a) If the graph of the function $y = ce^{-bx} - a$ has intercepts $y(0) = 1$ and $y(1) = 0$, show that $y = \dfrac{e^{b-bx} - 1}{e^b - 1}$.
(b) Calculate the area in the first quadrant cut off by the above function.
(c) Use a calculator to evaluate this area for smaller and smaller values of b and hence predict the limit as $b \to 0$. Hence describe the shape of the area in this limit.

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21. (a) Differentiate $x\,e^{-x}$ and hence find $\displaystyle\int_0^N x\,e^{-x}\,dx$.

(b) Determine the limit of this integral as $N \to \infty$.

(c) Differentiate $x^2\,e^{-x}$, and hence find $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$.

13 E Natural Growth and Decay

The exponential function is its own derivative, meaning that at each point on the curve, the gradient is equal to the height. This property is the reason why it occurs so often in the modelling of natural phenomena.

Consider a growing population, of people in some country, or rabbits on an island, or bacteria in a laboratory culture. Regard the population P as a function of time t. The rate at which the population is growing at any time is proportional to the value of the population at that time. That is, the gradient of the population graph is proportional to the height of the graph:

$$\frac{dP}{dt} = kP, \quad \text{where } k \text{ is a constant of proportionality.}$$

Such a situation is called *natural growth*, and a population growing in this way is said to obey the *law of natural growth*. To model this situation, we need a function whose rate of change is proportional rather than equal to the quantity.

The Natural Growth Theorem: The following theorem gives the complete answer.

10 NATURAL GROWTH: Suppose that the rate of change of y is proportional to y:

$$\frac{dy}{dt} = ky, \quad \text{where } k \text{ is a constant of proportionality.}$$

Then $y = y_0\,e^{kt}$, where y_0 is the value of y at time $t = 0$.

PROOF:

A. The function $y = y_0\,e^{kt}$ certainly satisfies $\dfrac{dy}{dt} = ky$, since

$$\begin{aligned}\frac{dy}{dt} &= y_0\,\frac{d}{dt}\,e^{kt}\\ &= y_0 k\,e^{kt}\\ &= ky.\end{aligned}$$

Also, substituting $t = 0$ gives $y = y_0$, as required.

B. The proof of the converse is harder (and would not be required).

Suppose that a function y of t satisfies $\dfrac{dy}{dt} = ky$.

Let $u = y\,e^{-kt}$.

Then $\begin{aligned}\frac{du}{dt} &= \frac{dy}{dt}\,e^{-kt} - ky\,e^{-kt}, \quad \text{by the product rule,}\\ &= ky\,e^{-kt} - ky\,e^{-kt}\\ &= 0.\end{aligned}$

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Hence $\quad u = C$, for some constant C,

$$ye^{-kt} = C$$

$\boxed{\times\, e^{kt}}$ $\quad y = C\,e^{kt}$,

and substituting $t = 0$ shows that $C = y_0$, as required.

NOTE: Questions often require a proof that a given function is a solution of a differential equation, by substitution of the function into the differential equation, as in Part A above. The 3 Unit course, however, would not require a proof that there are no other solutions, as in Part B.

Problems Involving Natural Growth: If only the differential equation is given, one can use the natural growth theorem to write down the solution with no further working. Usually, the constant k should be calculated from given values of P at different times, then the approximate value of k can be held in the memory of the calculator.

WORKED EXERCISE: The rabbit population P on an island was estimated to be 1000 at the start of 1995 and 3000 at the start of 2000.

(a) Assuming natural growth, find P as a function of the time t years after the start of 1995, and sketch the graph.

(b) How many rabbits are there at the start of 2003 (answer to the nearest 10 rabbits)?

(c) When will the population be 10 000 (answer to the nearest month)?

(d) Find the rate of growth (to the nearest 10 rabbits per year):

(i) when there are 8000 rabbits,

(ii) at the start of 1997.

SOLUTION:

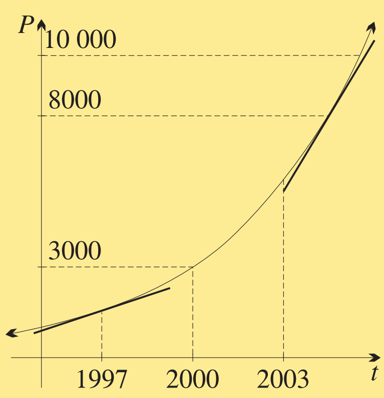
(a) With natural growth, $\dfrac{dP}{dt} = kP$, for some constant $k > 0$,

so by the theorem, $P = 1000\,e^{kt}$, since $P = 1000$ when $t = 0$.

When $t = 5$, $P = 3000$, so $3000 = 1000\,e^{5k}$

$$e^{5k} = 3$$
$$5k = \log 3$$
$$k = \tfrac{1}{5}\log 3$$

(approximate k and store it in the memory).



(b) When $t = 8$, $P = 1000\,e^{8k}$

$$\doteqdot 5800 \text{ rabbits.}$$

(c) Substituting $P = 10\,000$, $10\,000 = 1000\,e^{kt}$

$$e^{kt} = 10$$
$$kt = \log 10$$
$$t = \frac{\log 10}{k}$$
$$\doteqdot 10 \text{ years and 6 months,}$$

so the population will reach 10 000 about 6 months into 2005.

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(d) (i) Substituting $P = 8000$ into $\dfrac{dP}{dt} = kP$, $\dfrac{dP}{dt} = 8000k$

$\doteqdot 1760$ rabbits per year.

(ii) Differentiating, $\dfrac{dP}{dt} = 1000k\,e^{kt}$,

so at the start of 1997, when $t = 2$, $\dfrac{dP}{dt} = 1000k\,e^{2k}$

$\doteqdot 340$ rabbits per year.

WORKED EXERCISE: The price P of a pair of shoes rises with inflation so that

$$\frac{dP}{dt} = kP, \quad \text{for some constant } k > 0,$$

where t is the time in years since records were kept.

(a) Show that $P = P_0\,e^{kt}$, where P_0 is the price at time zero, satisfies the given differential equation.

(b) If the price doubles every ten years, find k, sketch the curve, and find how long it takes for the price to rise tenfold.

SOLUTION:

(a) Substituting $P = P_0\,e^{kt}$ into the differential equation $\dfrac{dP}{dt} = kP$,

$$\begin{aligned} \text{LHS} &= \frac{dP}{dt} \\ &= P_0\,ke^{kt}, \end{aligned} \qquad \begin{aligned} \text{RHS} &= kP \\ &= kP_0\,e^{kt} \\ &= \text{LHS}. \end{aligned}$$

Also, when $t = 0$, $P = P_0\,e^0 = P_0 \times 1$, so P_0 is the price at time zero.

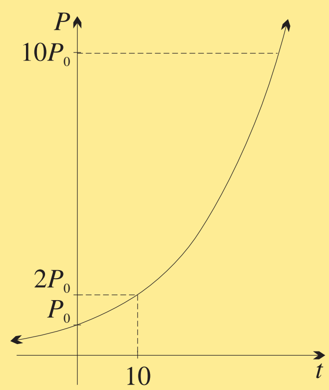
(b) Substituting $P = 2P_0$ when $t = 10$,

$$\begin{aligned} 2P_0 &= P_0\,e^{10k} \\ e^{10k} &= 2 \\ 10k &= \log 2 \\ k &= \tfrac{1}{10}\log 2 \end{aligned}$$

(approximate k and store it in the memory).

Now substituting $P = 10P_0$,

$$\begin{aligned} 10P_0 &= P_0\,e^{kt} \\ e^{kt} &= 10 \\ kt &= \log 10 \\ t &= \frac{\log 10}{k} \\ &\doteqdot 33{\cdot}219, \end{aligned}$$



so it takes about 33·2 years for the price of the shoes to rise tenfold.

Natural Decay: By this same method, we can deal with situations in which some quantity is decreasing at a rate proportional to the quantity itself. Radioactive substances, for example, decay in this manner. Let M be the mass of the substance, regarded as a function of time t. Because M is decreasing, the derivative $\dfrac{dM}{dt}$ is negative, and so

$$\frac{dM}{dt} = -kM, \quad \text{where } k \text{ is a positive constant.}$$

Then applying the theorem, $M = M_0\,e^{-kt}$, where M_0 is the mass at time $t = 0$.

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11 NATURAL DECAY: In situations of natural decay, let the constant of proportionality be $-k$, where k is a positive constant.

It is perfectly acceptable to omit the minus sign so that k is a negative constant, but the arithmetic of logarithms is easier if the minus sign is built in and k is a positive constant.

WORKED EXERCISE: A paddock has been contaminated with strontium-90, which has a half-life of 28 years (meaning that exactly half of any quantity of the isotope will decay in 28 years).

(a) Find the mass of strontium-90 as a function of time, and sketch the graph.

(b) Find what proportion of the radioactivity will remain after 100 years (answer correct to the nearest 0·1%).

(c) How long will it take for the radioactivity to drop to 0·001% of its original value (answer correct to the nearest year)?

SOLUTION:

(a) Let M be the quantity of the isotope at time t years.

Then $\dfrac{dM}{dt} = -kM$, for some positive constant k of proportionality,

so $M = M_0\, e^{-kt}$, where M_0 is the quantity present at time $t = 0$.

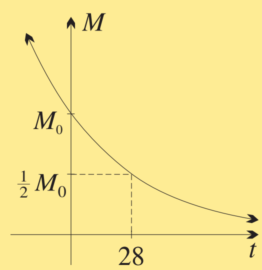
When $t = 28$, $M = \frac{1}{2}M_0$, so

$$\begin{aligned}\tfrac{1}{2}M_0 &= M_0\, e^{-28k}\\ e^{-28k} &= \tfrac{1}{2}.\end{aligned}$$

Taking reciprocals,

$$\begin{aligned}e^{28k} &= 2\\ 28k &= \log 2\\ k &= \tfrac{1}{28}\log 2\end{aligned}$$

(approximate k and store it in the memory).



(b) When $t = 100$,

$$\begin{aligned}M &= M_0\, e^{-100k}\\ &\doteqdot 0{\cdot}084M_0,\end{aligned}$$

and so the radioactivity has dropped to about 8·4% of its original value.

(c) When $M = 10^{-5}M_0$,

$$\begin{aligned}10^{-5}M_0 &= M_0\, e^{-kt}\\ e^{-kt} &= 10^{-5}\\ -kt &= -5\log 10\\ t &= \frac{5\log 10}{k}\\ &\doteqdot 465 \text{ years.}\end{aligned}$$

Exercise 13E

1. Given that $y = 5\, e^{\frac{1}{2}t}$, show that $\dfrac{dy}{dt} = \frac{1}{2}y$. Use these equations to solve the following.

(a) Find y to three significant figures when: (i) $t = 2$ (ii) $t = -3$ (iii) $t = 4{\cdot}5$

(b) Find t to two decimal places when: (i) $y = 10$ (ii) $y = 1$ (iii) $y = 30$

(c) Find the exact value of $\dfrac{dy}{dt}$ when: (i) $y = 8$ (ii) $y = 11$ (iii) $y = \frac{1}{3}$

(d) Find $\dfrac{dy}{dt}$ to three decimal places when: (i) $t = 7$ (ii) $t = 3{\cdot}194$ (iii) $t = \log 3$

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2. In the following questions, give exact answers where appropriate or else approximate to four significant figures.

(a) Given that $y = 3\,e^{2t}$, show that $\dfrac{dy}{dt} = 2y$. Find y when $t = 1{\cdot}5$.

(b) Given that $y = e^{-t}$, show that $\dfrac{dy}{dt} = -y$. Find t when $y = \frac{1}{2}$.

(c) Given that $y = 10\,e^{-\frac{1}{2}t}$, show that $\dfrac{dy}{dt} = -\frac{1}{2}y$. Find $\dfrac{dy}{dt}$ when $y = 6$.

(d) Given that $y = \sqrt{2}\,e^{\sqrt{2}\,t}$, show that $\dfrac{dy}{dt} = \sqrt{2}y$. Find $\dfrac{dy}{dt}$ when $t = \frac{1}{\sqrt{2}}$.

3. The population P of a city rose from 1 million at the beginning of 1975 to 2·5 million at the beginning of 1985. Assuming natural growth, $P = 10^6 \times e^{kt}$ where t is the time in years since the beginning of 1975.

(a) Find the value of the positive constant k, and sketch the curve.

(b) What was the population of the city at the beginning of 1998, to three significant figures?

(c) In what year is the population 10 million?

(d) Find the rate $\dfrac{dP}{dt}$ at which the population is increasing at the beginning of that year. Give your answer to the nearest thousand.

4. Ten kilograms of sugar is gradually dissolved in a vat of water. After t hours, the amount of undissolved sugar remaining is given by $S = 10\,e^{-kt}$.

(a) Calculate k, given that $S = 3{\cdot}2$ when $t = 4$, and sketch the graph.

(b) At what time will there be 1 kg of sugar remaining?

(c) How fast is the sugar dissolving after 1 hour? Give your answer in units of kilograms per hour to the nearest half kilogram.

5. The value V of a factory machine depreciated with time t years such that $\dfrac{dV}{dt} = -kV$, for some constant $k > 0$.

(a) Show that $V = V_0\,e^{-kt}$ satisfies the given differential equation.

(b) The initial value of a particular item of machinery is \$15 000. Explain why $V_0 = 15\,000$.

(c) In the first year the machine depreciates in value by 30%. Find the value of the constant k.

(d) The company that bought the machine writes off any machine when it has depreciated to 5% of its initial value. How many years does this take? Round your answer up to the nearest whole year.

6. In an experiment in which bacteria are grown on a petri dish, it is found that the area A in cm^2 covered by the bacteria increases from $0{\cdot}5\,\text{cm}^2$ to $1\,\text{cm}^2$ in a period of 3 hours.

(a) Assuming that the area covered obeys the law of natural growth, show that A at time t hours after the initial observation is given by $A = \frac{1}{2}\,e^{kt}$.

(b) Find the value of the positive constant k.

(c) What area of the petri dish will be covered after 7 hours? Answer to three significant figures.

(d) If the diameter of the petri dish is 10 cm, how long will it take for the bacteria to cover the dish? Answer to the nearest 10 minutes.

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7. When a liquid is placed in a refrigerator kept at 0° C, the rate at which it cools is proportional to its temperature h at time t, thus $\dfrac{dh}{dt} = -kh$ where k is a positive constant.

(a) Show that $h = h_0\, e^{-kt}$ is a solution of the differential equation.
(b) Find h_0, given that the liquid is initially at 100° C.
(c) After 5 minutes the temperature has dropped to 40° C. Find the value of k.
(d) Find the temperature of the liquid after 15 minutes.

8. The height H of a wave decays away so that $H = H_0\, e^{-\frac{1}{3}t}$, where H_0 is the initial height of the wave. Giving your answer to the nearest whole percent, what percentage of the initial height is the height of the wave when: (a) $t = 1$? (b) $t = 3$? (c) $t = 8$?

DEVELOPMENT

9. A quantity Q of radium at time t years is given by $Q = Q_0\, e^{-kt}$, where k is a positive constant and Q_0 is the amount of radium at time $t = 0$.
(a) Given that $Q = \frac{1}{2}Q_0$ when $t = 1690$ years, calculate k.
(b) After how many years does only 20% of the initial amount of radium remain? Give your answer to the nearest year.

10. The most efficient way of boiling water is to add heat at a rate proportional to the temperature H of the water, thus $\dfrac{dH}{dt} = kH$.

(a) Show $H = H_0\, e^{kt}$ is a solution of the differential equation.
(b) The water is initially at room temperature 20° C, and after $1\frac{1}{2}$ minutes reaches 30° C. Find the value of k.
(c) Find how long, to the nearest second, it takes for the water to reach boiling point.

11. The amount A in grams of carbon-14 isotope in a dead tree trunk after t years is given by $A = A_0\, e^{-kt}$, where A_0 and k are positive constants.

(a) Show that A satisfies the equation $\dfrac{dA}{dt} = -kA$.
(b) The amount of isotope is halved every 5750 years. Find the value of k.
(c) For a certain dead tree trunk the amount of isotope is only 15% of the original amount in the living tree. How long ago did the tree die (answer to the nearest 1000 years)?

12. Current research into Alzheimer's disease suggests that the rate of loss of percentage brain function is proportional to the percentage brain function already lost. That is, if L is the percentage brain function lost, then $\dfrac{dL}{dt} = kL$, for some constant $k > 0$.

(a) Two years ago a patient was initially diagnosed with Alzheimer's disease, with a 15% loss of brain function. This year the patient was diagnosed with 20% loss of brain function. Show that $L = 15\, e^{kt}$, where $k = \frac{1}{2}\log\frac{4}{3}$.
(b) The nearby care centre will admit patients to 24-hour nursing care when a patient reaches 60% loss of brain function. In how many more years will that be? Answer to the nearest year.

13. A wet porous substance loses moisture at a rate that is proportional to the moisture content M, that is $\dfrac{dM}{dt} = -kM$, where $k > 0$. On a particular day a wet towel on a clothes-line loses half its moisture in the first $1\frac{1}{2}$ hours.
(a) Show that $M = M_0\, e^{-kt}$ is a solution of the differential equation.

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(b) Find the value of k.

(c) How long, in total, will it take for the towel to become 99% dry? Answer to the nearest hour.

14. Air pressure P in millibars is a function of the altitude a in metres, with $\dfrac{dP}{da} = -\mu P$. The pressure at sea level is 1013·25 millibars.

(a) Show that $P = 1013{\cdot}25\, e^{-\mu a}$ is a solution to this problem.

(b) One reference book quotes the pressure at 1500 metres to be 845·6 millibars. Find the value of μ for the data in that book.

(c) Another reference book quotes the pressure at 6000 metres to be half that at sea level. Find the value of μ in this case.

(d) Are the data in the two books consistent?

(e) Assuming the first book to be correct:

(i) What is the pressure at 4000 metres?

(ii) What is the pressure 1 km down a mine shaft?

(iii) At what altitude is the pressure 100 millibars?

15. A certain radioactive isotope decays at such a rate that after 68 minutes only a quarter of the initial amount remains.

(a) Find the half-life of this isotope.

(b) What proportion of the initial amount will remain after 3 hours? Give your answer as a percentage to one decimal place.

16. The half-life of naturally occuring radioactive carbon C^{14} is 5730 years. In a living organism, the rate of C^{14} decay is typically constant at around 15·3 disintegrations per minute. When the organism dies the level of C^{14} decays away. Let C be the amount of radioactive carbon measured in disintegrations per minute at time t years.

(a) Assuming natural decay, $C = C_0\, e^{-kt}$. Find the values of k and C_0.

(b) In an archæological dig, bones are found which exhibit 11 disintegrations per minute. How old are the bones, to the nearest year?

(c) Carbon dating is only deemed reliable for ages between 1000 and 10 000 years. In another archæological expedition, artefacts are found in a cave which exhibit 2·25 disintegrations per minute.

(i) How old do these artefacts seem to be, to the nearest year?

(ii) Is this figure reliable or should other tests be carried out to confirm the age of the artefacts?

17. In 1980 the population of Bedsworth was $B = 25\,000$ and the population of Yarra was $Y = 12\,500$. That year the mine was closed in Bedsworth and the population began falling, while the population of Yarra continued to grow, so that

$$B = 25\,000 \times e^{-pt} \quad \text{and} \quad Y = 12\,500 \times e^{qt}.$$

(a) Ten years later it was found the populations of the two towns were $B = 20\,000$ and $Y = 15\,000$. Find the values of p and q.

(b) In what year were the populations of the two towns equal?

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EXTENSION

18. Measurements of the radioactivity of isotopes of certain elements are used to determine the *radiometric age* t of a sample, which is an estimate of its actual age. The rate of decay is proportional to the amount P of the parent isotope, thus $\dfrac{dP}{dt} = -\lambda P$.

(a) If the sample originally had an amount P_0 of the parent isotope, show that $P = P_0\, e^{-\lambda t}$ satisfies the given differential equation.

(b) In a sample that has not been contaminated by outside sources, the sum of the amount P of parent isotope and amount D of daughter isotope must equal P_0, thus $P_0 = P + D$. The ratio $\dfrac{D}{P}$ is easily measured; find a formula for the radiometric age in terms of this ratio.

(c) In a sample of crystal mica, the ratio of daughter strontium Sr^{87} to parent rubidium Rb^{87} is $0{\cdot}74\%$. The half-life of Rb^{87} is 47×10^9 years.

(i) Find the value of λ to three significant figures.

(ii) Find the radiometric age of the sample to the nearest million years.

19. [Here we introduce the technique of *integrating factors* for solving differential equations. The quantity e^{kx} is known as the integrating factor.]

(a) Given that $z = e^{kx} y$, show that $\dfrac{dz}{dx} = e^{kx}\left(\dfrac{dy}{dx} + ky\right)$.

(b) If $\dfrac{dy}{dx} + ky = 0$, use part (a) to show that $z = A$, where A is a constant.

(c) Hence show that $y = A\, e^{-kx}$.

20. [Here we justify guessing a solution of the form $y = e^{kt}$ as a solution to the second-order differential equation $y'' + ay' + by = 0$.]

(a) Show that if λ is one solution of $k^2 + ak + b = 0$, then $\mu = -(\lambda + a)$ is the other solution.

(b) In the next parts, assume that $\lambda \neq \mu$. Let $z = y' - \lambda y$. Find z', and hence show that $y'' = z' + \lambda z + \lambda^2 y$.

(c) Given that $y'' + ay' + by = 0$, show that $z' - \mu z = 0$.

(d) Solve the above equation for z, and hence show that $y' - \lambda y = A\, e^{\mu t}$, where A is a constant.

(e) Multiply the equation for y by $e^{-\lambda t}$, and hence show that $y = B\, e^{\mu t} + C\, e^{\lambda t}$, where B and C are constants.

(f) How would the above change if $\lambda = \mu$?

GO Online Multiple Choice Quiz

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