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MATH 1905 - Assignment 2

Q1. a) (i) The pairs (x, t) that are possible values of the random ordered pair (X, T) are:

$$x, t \in \mathbb{N}, \quad \text{with } t \geqslant x$$

(ii)
$$\begin{aligned} P(X=x, T=t) &= P(X=x, X+Y=t) \\ &= P(X=x, Y=t-x) \\ &= P(X=x \cap Y=t-x) \\ &= P(X=x) \cdot P(Y=t-x) \quad [X, Y \text{ independent}] \end{aligned}$$

As $X \sim \text{Poisson}(\lambda)$, and $Y \sim \text{Poisson}(\mu)$, we have the following ~~[illegible]~~ distributions.

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$P(Y=y) = \frac{e^{-\mu} \mu^{y}}{y!}$$

As a result, continuing from earlier, we have:

$$\begin{aligned} P(X=x, T=t) &= P(X=x) \cdot P(Y=t-x) \\ &= \frac{e^{-\lambda} \lambda^{x}}{x!} \cdot \frac{e^{-\mu} \mu^{(t-x)}}{(t-x)!} \\ &= e^{-(\lambda+\mu)} \cdot \frac{\lambda^{x} \mu^{(t-x)}}{x!(t-x)!} \\ &= \frac{e^{-(\lambda+\mu)}}{t!} \cdot \frac{t!}{x!(t-x)!} \cdot \lambda^{x} \mu^{(t-x)} \\ &= \frac{e^{-(\lambda+\mu)}}{t!} \cdot \binom{t}{x} \cdot \lambda^{x} \mu^{(t-x)} \end{aligned}$$

b) The PGF of $X \sim \text{Poisson}(\lambda)$ is as follows:

$$\pi_X(s) = E(s^X)$$

$$= \sum_{x=0}^{\infty} s^x P(X = x)$$

$$= \sum_{x=0}^{\infty} s^x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(s\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda s} \quad [\text{Taylor Series for } e^x]$$

$$= e^{-\lambda(1-s)}$$

A similar result can be derived for $Y \sim \text{Poisson}(\mu)$, such that:

$$\pi_Y(s) = e^{-\mu(1-s)}$$

Furthermore, note that X, Y are independent. ~~and~~ Now examining the PGF of $T = X + Y$, we get the results:

$$\pi_T(s) = E(s^T)$$

$$= E(s^{X+Y})$$

$$= E(s^X s^Y)$$

$$= E(s^X) E(s^Y) \quad [X, Y \text{ independent}]$$

$$= e^{-\lambda(1-s)} e^{-\mu(1-s)}$$

$$= e^{-(\lambda+\mu)(1-s)}$$

Therefore $T = X + Y$ is distributed as: $T \sim \text{Poisson}(\lambda + \mu)$

c) (i) The values that x can take to have a positive probability under this conditional distribution of

$$P(X = x \mid T = t)$$

are $x, t \in \mathbb{N}$, $x \leq t$

(ii) The conditional distribution $P(X = x \mid T = t)$, has distribution as follows:

$$P(X = x \mid T = t) = \frac{P(X = x, T = t)}{P(T = t)}$$

$$= \frac{P(X = x) \cdot P(Y = t - x)}{P(T = t)}$$

$$= \frac{\frac{e^{-(\lambda+\mu)}}{t!} \cdot \binom{t}{x} \lambda^{x} \mu^{(t-x)}}{\frac{e^{-(\lambda+\mu)} (\lambda+\mu)^{t}}{t!}}$$

$$= \frac{\binom{t}{x} \lambda^{x} \mu^{(t-x)}}{(\lambda+\mu)^{t}}$$

$$= \frac{1}{(\lambda+\mu)^{t}} \cdot \binom{t}{x} \lambda^{x} \mu^{(t-x)}$$

It is thus clear that the conditional distribution, is distributed as a binomial distribution.

Q2. a) If random variables X and Y satisfy $E(X) = \mu_x$, and $E(Y) = \mu_y$, then we are required to show:

$$Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)] = E(XY) - \mu_x\mu_y$$

$$E[(X-\mu_x)(Y-\mu_y)] = E[XY - \mu_x Y - \mu_y X + \mu_x\mu_y]$$

$$= E(XY) - E(\mu_x Y) - E(\mu_y X) + E(\mu_x\mu_y)$$

$$= E(XY) - \mu_x E(Y) - \mu_y E(X) + \mu_x\mu_y$$

$$= E(XY) - \mu_x\mu_y - \mu_y\mu_x + \mu_x\mu_y$$

$$= E(XY) - \mu_x\mu_y$$

b) (i)

x	0	1	2	3	4	5	6
$P(X=x)$	0.1	0.05	0.3	0.1	0.3	0.05	0.1

$$\therefore E(X) = \sum_{x=0}^{6} x P(X=x)$$

$$= 0\times0.1 + 1\times0.05 + 2\times0.3 + 3\times0.1 + 4\times0.3$$
$$+ 5\times0.05 + 6\times0.1$$

$$= 3$$

(ii) $Y = |X-3|$

y	3	2	1	0	1	2	3
$P(Y=y)$	0.1	0.05	0.3	0.1	0.3	0.05	0.1

y	0	1	2	3
$P(Y=y)$	0.1	0.6	0.1	0.2

(iii) $$E(Y) = \sum_{y=0}^{3} y P(Y=y)$$

$$= 0\times0.1 + 1\times0.6 + 2\times0.1 + 3\times0.2$$
$$= 1.4$$

b) (iv) $XY = X|X-3|$

$= |X||X-3|$

$= |X^2 - 3X|$

xy	0	2	2	0	4	10	18
$P(XY = xy)$	0.1	0.05	0.3	0.1	0.3	0.05	0.1

xy	0	2	4	10	18
$P(XY = xy)$	0.2	0.35	0.3	0.05	0.1

$$\therefore E(XY) = \sum_{xy=0}^{18} xy\, P(XY = xy)$$

$$= 0 \times 0.2 + 2 \times 0.35 + 4 \times 0.3 + 10 \times 0.05 + 18 \times 0.1$$

$$= 4.2$$

(v) $E(X) = \mu_X$, $E(Y) = \mu_Y$

$$\therefore Cov(X, Y) = E(XY) - \mu_X \mu_Y$$

$$= 4.2 - 3(1.4)$$

$$= 4.2 - 4.2$$

$$= 0$$

c) Even though the covariance of X and Y is equal to 0, this does not imply that X and Y are independent. Furthermore, $Y = |X-3|$, and is thus clearly dependent on X, hence X and Y are not independent.

Q3. a) For the following questions, we define the notation R_i as the i-th roll of the die. Furthermore, note that each roll is independent.

$$E(S) = E\left[\sum_{i=1}^{20} R_i\right]$$

$$= 20\,E(R_i) \quad \text{[As each roll has the same expectation]}$$

$$= 20 \times \frac{1}{6}\left[1+2+3+4+5+6\right]$$

$$= 70$$

$$Var(S) = Var\left[\sum_{i=0}^{20} R_i\right]$$

$$= 20\,Var(R_i) \quad \text{[As each roll has the same variance]}$$

$$= 20\left[E(X^2) - [E(X)]^2\right]$$

$$= 20\left[\frac{1}{6}(1+4+9+16+25+36) - \frac{1}{36}(1+2+3+4+5+6)^2\right]$$

$$= 58\frac{1}{3}$$

b) We are required to compute a normal approximation with continuity correction to $P(S \leq 55)$. The continuity correction means we instead compute $P(S < 55.5)$, to improve the normal approximation. The normal distribution is as follows:

$$Z = \frac{X-\mu}{\sigma}$$

In this question, $\mu = 70$, $\sigma = \sqrt{58\frac{1}{3}}$, $X = 55.5$. Using the following R command, we get the following result:

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pnorm(55.5, 70, sqrt(58.33333))
[1] 0.0288 1541
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$\therefore P(S < 55.5) = 0.0288$

c) The absolute error is given by:

$$\text{absolute error} = |0.0288 - 0.0285|$$
$$= 0.0003$$

The relative error is thus calculated as follows:

$$\text{relative error} = \frac{\text{absolute error}}{0.0285}$$
$$= \frac{0.0003}{0.0285}$$
$$= 0.0105263$$
$$= 1.0526\ \%$$