The University of Sydney
School of Mathematics and Statistics

Tutorial for Week 8

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/
Lecturers: Daniel Daners and David Easdown

Material covered

- ☐ Explicit first order differential equations for y only depending on y'
- ☐ Separation of variables
- ☐ Direction fields
- ☐ Aspects of modelling

Outcomes

After completing this tutorial you should

- ☐ be able to solve the simplest differential equations
- ☐ be able to solve equations by separation of variables
- ☐ be able to sketch direction fields and corresponding solutions of simple differential equations.
- ☐ be able to determine the asymptotic behaviour of solutions by looking at the explicit solution of a differential equation

Summary of essential material

What is a differential equation? A differential equation is an equation, where the unknown is a function. The equation involves that unknown function and some first or higher order derivatives of the unknown function.

Simplest differential equations: Explicit equation

$$\frac{dy}{dx}(x) = g(x) \qquad \text{with solution} \qquad y(x) = \int g(x)\,dx + C \quad \text{(anti-derivative of } g\text{)}.$$

The constant C is determined by an initial condition, for instance $y(0) = a$.

Separable differential equations: These are equations that can be written in the form

$$\frac{dy}{dx} = g\big(y(x)\big)h(x),$$

that is, the derivative y' can be written as a product of a function of y and a function of x. To solve we *separate variables*, putting all x on one side and all y on the other side:

$$\frac{dx}{h(x)} = g(y)dy, \qquad \text{then integrate:} \qquad \int \frac{dx}{h(x)} = \int g(y)dy,$$

and finally solve for y. This also involves an integration constant that is determined by an initial condition. Alternatively one can do a *definite integral* and build initial conditions directly into the calculation.

Direction fields: Assume that $y(x)$ is a solution of the differential equation $y'(x) = f\big(x, y(x)\big)$. The means that the graph $y = y(x)$ has slope $f\big(x, y(x)\big)$. Hence if we plot the slopes $f(x, y)$ at every point (x, y) in the relevant region of the plane, all solutions of the differential equation are tangent to the direction field. We often proceed as follows to sketch the direction field:

- Find *stationary points*, that is, points (x, y) such that $y'(x) = f(x, y) = 0$. If y_0 is such that $f(x, y_0) = 0$ we talk about an *equilibrium point*.
- Find regions where the slope is positive;
- Find regions where the slope is negative;

Note: If the equation is *autonomous*, that is, of the form $y' = f(y)$ (not depending explicitly on x), then the direction field constant in the horizontal direction.

Questions to do before the tutorial

1. Find the general solution of the following differential equations.

(a) $\dfrac{dy}{dx} = 1 + \sin x + \sin^2 x,$ (b) $x^3 \dfrac{dy}{dx} = 2x^2 + 5,\ x > 0,$ (c) $\dfrac{dy}{dx} = \dfrac{1}{\cosh y},$

Questions to complete during the tutorial

2. Find the particular solutions of the differential equations satisfying the given conditions:

(a) $\dfrac{dy}{dx} = 1 - 2x - 3x^2, \quad y(1) = -1.$

(b) $e^{2x}\dfrac{dy}{dx} + 1 = 0, \quad y(x) \to 2 \text{ as } x \to \infty.$

(c) $\dfrac{dy}{dx} = \dfrac{y}{2} + \dfrac{1}{2y}, \quad y(0) = 2.$

3. According to the Gompertz model, the population N of a colony of animals grows according to the differential equation,

$$\frac{dN}{dt} = -\beta N \ln\Big(\frac{N}{M}\Big),$$

where M is the maximum sustainable population size and β is a positive constant.

(a) Sketch the direction field of the differential equation and some possible solutions.

(b) Set $v := \ln(N/M)$. Show that v satisfies the differential equation $v' = -\beta v$.

(c) Solve the differential equation $v' = -\beta v$ and hence find $N(t)$.

(d) Find $\lim_{t\to\infty} N(t)$.

(e) Find the particular solution for which $N(0) = M/2$.

4. Let y be the number of people in a stable economy who have an income of x or more. The economist Vilfredo Pareto (1848–1923) discovered that the rate at which y decreases with increasing x is directly proportional to the number of people with income x or more and inversely proportional to the income x.

(a) Derive a differential equation for $y(x)$.

(b) Find the general solution y in terms of x.

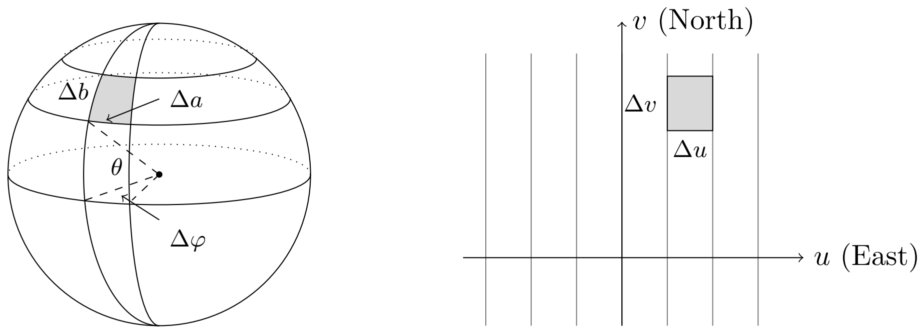
(c) Find the particular solution of the differential equation given that the minimum income is x_0 and the total population is N.

5. Find an equation of the curve that passes through $(1, 1)$ and whose slope at (x, y) is y^2/x^3.

Hint: The curve is tangent to the direction field of the differential equation $y' = y^2/x^3$.

Extra questions for further practice

6. The Mercator map is one of the most frequently used maps of the earth. It displays the earth such that the parallels and meridians form a rectangular grid. If φ denotes longitude and θ denotes latitude the coordinates of the map are therefore given by $u = u(\varphi)$ and $v = v(\theta)$.



Consider a small rectangle on the sphere of side lengths Δa between longitude φ and $\varphi + \Delta\varphi$, and Δb between latitude θ and $\theta + \Delta\theta$, as shown in the figure. That rectangle is mapped onto a rectangle on the plane with edges parallel to the coordinate axes. The spacing of the parallels on the map is such that the north-south distortion of length is the same as the east-west distortion of length on the map, that is,

$$\frac{\Delta u}{\Delta a} \approx \frac{\Delta v}{\Delta b}.$$

Use this condition to derive a differential equation for $v(\theta)$ and solve it. What initial condition should be assumed?

7. Consider a particle of mass m in free fall from height h. Let $x(t)$ be its displacement from the initial position and $v(t) = dx/dt$ its velocity at time t.

(a) If we neglect any friction forces, according to Newton's law, v satisfies the differential equation

$$m\frac{dv}{dt} = -mg.$$

(i) Find the solution with initial condition $v(0) = 0$.

(ii) Find the displacement $x(t)$ with initial condition $x(0) = h$.

(b) Assume now that there is a friction force proportional to the velocity. Then by Newton's law,

$$m\frac{dv}{dt} = -mg - cv$$

for some constant $c > 0$. The negative sign comes from the fact that the force acts in the direction opposite to v.

(i) Find the solution with initial condition $v(0) = 0$.

(ii) Find the terminal speed $v_\infty = |\lim_{t\to\infty} v(t)|$. Express the constant of proportionality c in terms of v_∞ and write down the solution from the previous part.

(iii) Find the displacement $x(t)$ with initial condition $x(0) = h$.

(c) Compute the Taylor polynomials $T_3(t)$ of $x(t)$ for the solutions without and with friction. Verify that for small times they are close to each other.

(d) Denote the solution without friction by $v_n(t)$ and the solution with friction by $v_f(t)$. Show that $|v_n(t)| > |v_f(t)|$ for all $t > 0$.