THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH1903/1907

INTEGRAL CALCULUS AND MODELLING (ADVANCED)

November 2007

LECTURER: D J Galloway

TIME ALLOWED: **One and a half hours**

Name: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 25% of the total examination;
there are 15 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 75% of the total examination;
there are 5 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.
There is a table of integrals after the last question in this booklet.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

MARKS

1. (a) (*i*) Decompose $\dfrac{2x^2 - x + 3}{(x-1)(x^2+2)}$ into partial fractions. 2

(*ii*) Find $\displaystyle\int \frac{2x^2 - x + 3}{(x-1)(x^2+2)}\,dx$ in simplest form. 2

(b) Evaluate $\displaystyle\int_1^{e^{1/3}} x^2 \ln(x)\,dx$ using integration by parts. 4

(c) Evaluate $\displaystyle\int \frac{dx}{\sqrt{9x^2+16}}$ using a suitable hyperbolic substitution, showing all working. 4

2. (a) (*i*) Determine if the region R_∞ between the curves:

$$y = \frac{1}{\sqrt{x}}, \quad y = \frac{\sqrt{x}+1}{x},$$

extending from $x = 1$ to infinity is finite. Explain why the area is infinite or determine its value. 2

(*ii*) Determine if the volume of revolution obtained by rotating the region R_∞ about the x-axis is finite. Explain why the volume is infinite or determine its value. 4

(b) For each of the following differential equations, find the general solution and the particular solution for the given conditions.

(*i*) $\dfrac{dy}{dx} = \dfrac{1 + 7x - xy}{x^2}, \quad x = 1,\ y = 0$ 3

(*ii*) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = 0, \quad x = 0,\ y = 0,\ \dfrac{dy}{dx} = 3$ 3

3. According to the Gompertz model, the population N of a colony of animals grows according to the differential equation,

$$\frac{dN}{dt} = F(N) \equiv \beta N \ln\left(\frac{M}{N}\right),$$

where M is the maximum sustainable population size and β is a positive constant.

(a) Sketch $F(N)$, identifying any fixed points and maxima or minima; which fixed point is stable and which unstable? 3

(b) Find the general solution to the differential equation. 5

(c) Find $\lim\limits_{t\to\infty} N(t)$. 1

(d) Find the particular solution for which $N = M/5$ when $t = 0$. 3

MARKS

4. (a) Consider the inhomogeneous second-order differential equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x),$$

where $a(x), b(x)$ and $f(x)$ are given functions, and suppose that two linearly independent solutions $y_1(x)$ and $y_2(x)$ to the homogeneous problem with $f(x) = 0$ have already been determined. Obtain a formula to solve the inhomogeneous problem by going through the following steps:

(i) Assume the solution has the form

$$y(x) = C_1(x)y_1(x) + C_2(x)y_2(x) ,$$

where $C_1(x)$ and $C_2(x)$ are functions to be determined.. Use the product rule to work out an expression for y', where dash denotes d/dx. Impose the condition

$$C_1'y_1 + C_2'y_2 \equiv 0$$

on C_1 and C_2, and work out an expression for y''. Substitute both expressions into the inhomogeneous equation to obtain the result

$$C_1'y_1' + C_2'y_2' = f(x)$$

(remember that y_1 and y_2 each satisfy the homogeneous equation). **4**

(ii) Solve the above equation together with the imposed condition to find expressions for C_1' and C_2'. Hence show that the general solution to the inhomogeneous problem is

$$y = -y_1(x)\int^x \frac{f(u)y_2(u)}{W(u)}du + y_2(x)\int^x \frac{f(u)y_1(u)}{W(u)}du + Ay_1(x) + By_2(x),$$

4

where $W(x) = y_1y_2' - y_2y_1'$ and A and B are arbitrary constants. (This is called the method of variation of parameters; W cannot vanish as long as y_1 and y_2. are linearly independent.)

(b) Use the method in part (a) to solve the problem

$$y'' - 6y' + 9y = \frac{e^{3x}}{x^2},$$

4

giving first the general solution and then the particular solution satisfying $y(1) = y'(1) = 0$.

MARKS

5. (a) Let $f(x)$ be a continuous and nonincreasing function on $[1, \infty)$ such that $f(k) = 1/k$ for integer values k. Define

$$F(x) = \int_1^x f(t)dt,$$

and let n be a positive integer.
Find expressions for the lower and upper Riemann sums L_n and U_n on $[1, n+1]$ such that

$$L_n \leq F(n+1) \leq U_n,$$

using n equal intervals, and determine

$$\lim_{n\to\infty} (U_n - L_n).$$

4

(b) Let $g(x)$ be the continuous function

$$g(x) = \frac{1}{x}\left(1 + \frac{\sin^2(\pi x)}{\pi x}\right),$$

(which satisfies $g(k) = 1/k$ for all positive integers k).

(i) Compute the derivative of $g(x)$, and use this to show that $g(x)$ is nonincreasing on $[1, \infty)$. 4

(ii) Use the bound $|\sin(\pi x)| \leq 1$ to find a bound $C < 1$ such that

$$0 \leq \int_1^x g(t)dt - \log(x) \leq C.$$

4

Table of Standard Integrals

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

3. $\displaystyle\int e^x\,dx = e^x + C$

4. $\displaystyle\int \sin x\,dx = -\cos x + C$

5. $\displaystyle\int \cos x\,dx = \sin x + C$

6. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$

7. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

8. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

9. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

10. $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

12. $\displaystyle\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + C$

13. $\displaystyle\int \sinh x\,dx = \cosh x + C$

14. $\displaystyle\int \cosh x\,dx = \sinh x + C$

15. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (|x| < a)$

16. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C'$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C' \quad (x > a)$

Linearity: $\displaystyle\int (\lambda f(x) + \mu g(x))\,dx = \lambda\int f(x)\,dx + \mu\int g(x)\,dx$

Integration by substitution: $\displaystyle\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$

Integration by parts: $\displaystyle\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

End of Extended Answer Section

THIS IS THE LAST PAGE OF THE QUESTION PAPER.