The University of Sydney

School of Mathematics and Statistics

MATH1903/1907

Integral Calculus and Modelling (Advanced)

November 2012 Lecturers: D Daners, J Parkinson

Time Allowed: **One and a half hours**

Family Name: Solutions

Other Names:

SID: Seat Number:

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 35% of the total examination; there are 20 questions; the questions are of equal value; all questions may be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 65% of the total examination; there are 4 questions; the questions are of equal value; all questions may be attempted; working must be shown.

Approved non-programmable calculators may be used.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Marker's use only	

Extended Answer Section

*There are **four** questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

MARKS

1. (a) Let

$$G(x) = \int_0^x \frac{1}{1+t^3}\,dt.$$

(*i*) Find $\frac{d}{dx}G(x^2)$. **2**

(*ii*) Calculate the integral **3**

$$\int_0^1 xG(x)\,dx.$$

in terms of $G(1)$.

(i) Using the fundamental thm of calculus and the chain rule

$$\frac{d}{dx}G(x^2) = \frac{2x}{1+x^6}$$

(ii) Note that $G'(x) = \frac{1}{1+x^3}$, so integrating by parts

$$\int_0^1 xG(x)\,dx = \frac{x^2}{2}G(x)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^3}\,dx$$

$$= \frac{1}{2}G(1) - \frac{1}{6}\ln(1+x^3)\Big|_0^1$$

$$= \frac{1}{2}G(1) - \frac{1}{6}\ln 2.$$

QUESTION 1 CONTINUES ON THE NEXT PAGE

MARKS

(b) Let D be the region of the plane with $0 \leq x \leq 1$ and $0 \leq y \leq e^x$. Calculate the volume of the solid obtained by revolving D around the y-axis. **2**

Use the shell method and integrate by parts:

$$\text{volume} = 2\pi \int_0^1 x e^x \, dx$$

$$= 2\pi \left(x e^x \Big|_0^1 - \int_0^1 e^x \, dx \right)$$

$$= 2\pi (x-1) e^x \Big|_0^1 = 2\pi e$$

MARKS

(c) Calculate the value of the improper integral 3

$$\int_1^{\infty} \left(\frac{1}{x+2} - \frac{5}{5x+1} \right) dx.$$

$$\int_1^b \frac{1}{x+2} - \frac{5}{5x+1} \, dx$$

$$= \ln(x+2) - \ln(5x+1) \Big|_1^b$$

$$= \ln \frac{x+2}{5x+1} \Big|_1^b = \ln \frac{b+2}{5b+1} - \ln \frac{3}{6}$$

$$= \ln \frac{b+2}{5b+1} - \ln \frac{1}{2}$$

Hence

$$\int_1^{\infty} \frac{1}{x+2} - \frac{5}{5x+1} \, dx = \lim_{b \to \infty} \left(\ln \frac{b+2}{5b+1} - \ln \frac{1}{2} \right)$$

$$= \lim_{b \to \infty} \left(\ln \frac{1 + \frac{2}{b}}{5 + \frac{1}{b}} - \ln \frac{1}{2} \right) = \ln \frac{1}{5} - \ln \frac{1}{2}$$

$$= \underline{\ln \frac{2}{5}} \, .$$

MARKS

2. (a) Calculate the length of the graph $y = \cosh x$ between $x = 0$ and $x = 1$. **2**

$$\int_0^1 \sqrt{1 + |\cosh' x|^2}\, dx = \int_0^1 \sqrt{1 + \sinh^2 x}\, dx$$

$$= \int_0^1 \cosh x \, dx = \sinh x \Big|_0^1 = \sinh 1 .$$

(b) Use a suitable comparison test to prove either convergence, or divergence, of the improper integral **2**

$$\int_1^\infty \frac{\cos x}{x^2+1}\, dx$$

For $x \geq 1$ we have $\left| \frac{\cos x}{1+x^2} \right| \leq \frac{1}{1+x^2} \leq \frac{1}{x^2}$.

As $\int_1^\infty \frac{1}{x^2}\, dx = \lim_{b \to \infty} -\frac{1}{x}\Big|_1^b = \lim_{b \to \infty}\left(-\frac{1}{b} + 1\right)$

$$= 1 < \infty$$

the integral converges.

MARKS

(c) Let $f(x) = \sqrt{1+x}$.

(i) Calculate the second order Taylor polynomial $T_2(x)$ of $f(x)$ centred at 0. **2**

(ii) Use Taylor's Theorem to write down a formula for the second order remainder term $R_2(x) = f(x) - T_2(x)$. Hence show that **2**

$$0 \leq f(x) - T_2(x) \leq \frac{x^3}{16} \qquad \text{for all } x \geq 0.$$

(iii) Hence approximate the integral **2**

$$\int_0^1 \sqrt{1+x^3}\, dx$$

correct to 1 decimal place. (Note the x^3 in the integrand).

(i) $f'(x) = \dfrac{1}{2\sqrt{1+x}}$ $\qquad f'(0) = \dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4(1+x)^{3/2}}$ $\qquad f''(0) = -\dfrac{1}{4}$

Hence

$$T_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8}$$

(ii) $f'''(x) = \dfrac{3}{8(1+x)^{5/2}}$. Hence

$$R_2(x) = \frac{1}{3!}\,\frac{3x^3}{8(1+c)^{5/2}} = \frac{x^3}{16(1+c)^{5/2}}$$

for some c between 0 and x.

As $1+c \geq 1$ if $0 \leq c \leq x$ we get

$$0 \leq \frac{x^3}{16} \leq R_2(x) \quad \text{for all } x \geq 0.$$

(iii) From (ii) we have

$$\int_0^1 T(x^3)\,dx \leq \int_0^1 \sqrt{1+x^3}\,dx \leq \int_0^1 T_2(x^3) + \frac{x^9}{16}\,dx$$

Now

$$\int_0^1 T(x^3)\,dx = \int_0^1 1 + \frac{x^3}{2} - \frac{x^6}{8}\,dx = x + \frac{x^4}{8} - \frac{x^7}{56}\Big|_0^1$$

$$= 1 + \frac{1}{8} - \frac{1}{56} \approx 1.1071$$

and

$$\int_0^1 \frac{x^9}{16}\,dx = \frac{x^{10}}{160}\Big|_0^1 = \frac{1}{160} \approx 0.0063$$

Hence, to one decimal place,

$$\int_0^1 \sqrt{1+x^3}\,dx = 1.1 .$$

MARKS

3. (a) Consider the differential equation

$$u'' + 6u' + 13u = 0$$

(i) Find the general solution of the differential equation **2**

The auxiliary equation is

$$\lambda^2 + 6\lambda + 13 = 0$$

Its solutions are

$$\lambda = -3 \pm \sqrt{9-13} = -3 \pm 2i$$

The general solution is

$$u(t) = e^{-3t}(A\cos 2t + B\sin 2t)$$

MARKS

(*ii*) Find the particular solution of the differential equation satisfying the conditions $u(0) = 0$ and $u'(0) = 1$. **2**

From (i)

$u(0) = A = 0$

$u'(t) = -3e^{-3t}(A\cos 2t + B\sin 2t) + e^{-3t}(-2A\sin 2t + 2B\cos 2t)$

$u'(0) = 1 = 2B$, so $B = \frac{1}{2}$.

Hence the particular solution is

$$u(t) = \frac{1}{2}e^{-3t}\sin 2t.$$

(*iii*) Let $x(t) = u(t)$ and $y(t) = u'(t)$. Derive a first order system of differential equations for $x(t)$ and $y(t)$ which is equivalent to the given second order differential equation. **2**

$x'(t) = u'(t) = y(t)$

$y'(t) = u''(t) = -6u'(t) - 13u(t)$

$\qquad = -6y(t) - 13x(t).$

Hence the system is

$x' = y$

$y' = -13x - 6y.$

Question 3 continues on the next page

MARKS

(b) Find the solution of the differential equation **4**

$$t^2 y'(t) = \frac{4+t}{y(t)}$$

satisfying the initial condition $y(1) = -2$.

Separate variables and integrate:

$$y\,dy = \frac{4+t}{t^2}\,dt = \left(\frac{4}{t^2} + \frac{1}{t}\right) dt$$

$$\frac{y^2}{2} = -\frac{4}{t} + \ln t + C$$

If $t=1$, then $\frac{(-2)^2}{2} = -\frac{4}{1} + \ln 1 + C = C - 4$

Hence $C = 6$ and

$$y^2 = 12 - \frac{8}{t} + 2\ln t$$

$$y = \pm\sqrt{12 - \frac{8}{t} + 2\ln t}$$

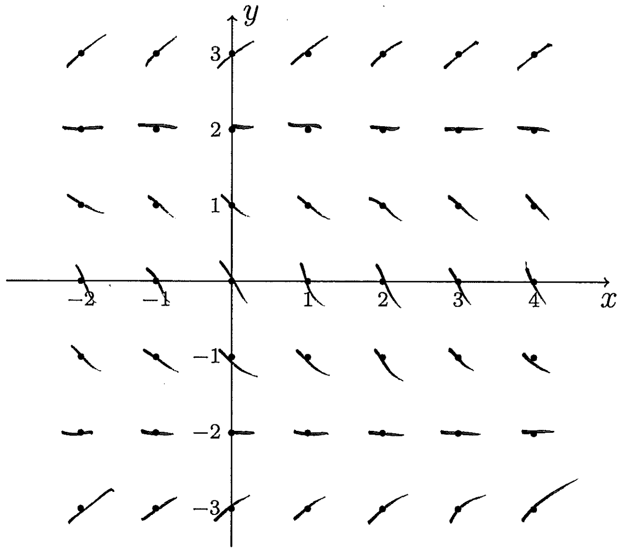
As $y(1) = -2 < 0$ we get

$$y = -\sqrt{12 - \frac{8}{t} + 2\ln t}$$

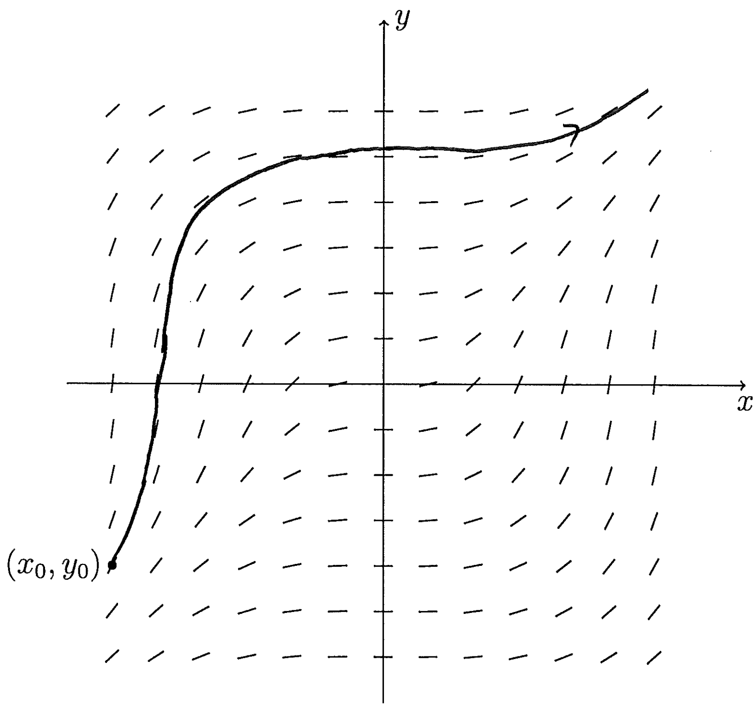
MARKS

4. (a) On the graph below, sketch the direction field of the differential equation **2**

$$y' = y^2 - 4.$$



(b) The following graph shows the direction field of a differential equation. Sketch the solution starting at the given value (x_0, y_0). **1**



Question 4 continues on the next page

MARKS

(c) Find the general solution of the linear inhomogeneous differential equation **3**

$$t^2y' + y = t^3e^{1/t}.$$

Standard form of the equation is

$$y' + \frac{1}{t^2}y = te^{1/t}$$

Integrating factor $e^{\int \frac{1}{t^2}dt} = e^{-\frac{1}{t}}$.

Hence

$$\left(ye^{-\frac{1}{t}}\right)' = te^{1/t}e^{-1/t} = t$$

and so

$$ye^{-\frac{1}{t}} = \int t\,dt = \frac{t^2}{2} + C$$

The general solution therefore is

$$y(t) = \frac{t^2}{2}e^{1/t} + Ce^{1/t}$$

MARKS

(d) Consider the system of differential equations **4**

$$\dot{x} = x + 3y \qquad (1)$$
$$\dot{y} = 4x + 2y \qquad (2)$$

Find the solution of the system with $x(0) = 5$ and $y(0) = 2$.

Eliminate y:

$$\ddot{x} = \dot{x} + 3\dot{y} = \dot{x} + 3(4x + 2y)$$

From (1) $y = \frac{1}{3}(\dot{x} - x)$, so

$$\ddot{x} = \dot{x} + 12x + 2(\dot{x} - x) = 3\dot{x} + 10x.$$

Hence

$$\ddot{x} - 3\dot{x} - 10x = 0$$

Auxiliary equation $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$,

so $\lambda = 5, -2$. Therefore

$$x(t) = Ae^{5t} + Be^{-2t}$$

From (1)

$$y(t) = \frac{1}{3}(\dot{x} - x) = \frac{1}{3}\left(5Ae^{5t} - 2Be^{-2t} - Ae^{5t} - Be^{-2t}\right)$$
$$= \frac{4}{3}Ae^{5t} - Be^{-2t}$$

By the initial conditions $x(0) = 5 = A + B$, $y(0) = 2 = \frac{4}{3}A - B$

Solving the system we get $A = 3$, $B = 2$

The solution with $x(0) = 5$ and $y(0) = 2$ is

$$x(t) = 3e^{5t} + 2e^{-2t}$$
$$y(t) = 4e^{5t} - 2e^{-2t}$$

Alternative solution using linear algebra:

System matrix is $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

Characteristic polynomial

$$\det \begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 12$$

$$= \lambda^2 - 3\lambda + 2 - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$$

Eigenvalues $\lambda = 5, -2$.

Determine eigenvectors

$\lambda = +5: \begin{bmatrix} 1-5 & 3 \\ 4 & 2-5 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix}$ eigenvector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\lambda = 2: \begin{bmatrix} 1+2 & 3 \\ 4 & 2+2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{5t} + B \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

Initial conditions $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = A \begin{bmatrix} 3 \\ 4 \end{bmatrix} + B \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$

Solution is $A = 1$, $B = 2$, so

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{5t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-2t}$$ as before.