THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Problem Sheet for Week 3

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Roots of complex numbers;
- ☐ Complex exponential function $e^z = e^x(\cos y + i \sin y)$;
- ☐ Functions of a complex variable;
- ☐ Sketching the image of a region.

Outcomes

After completing this tutorial you should

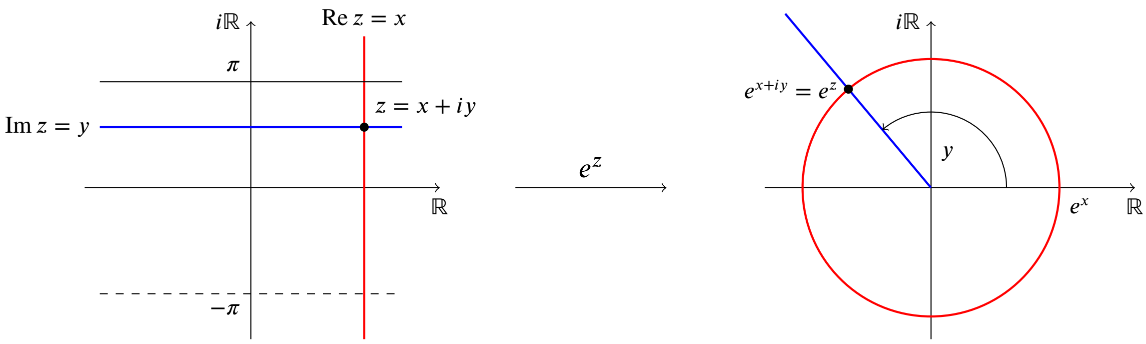
- ☐ find the roots of a complex number;
- ☐ understand the definition of the complex exponential function;
- ☐ manipulate the complex exponential function;
- ☐ solve equations involving the complex exponential function;
- ☐ understand what is meant by a function of a complex variable;
- ☐ sketch the image of a region in the complex plane under a function.

Summary of essential material

The complex exponential function. For any complex number $z = x + iy$, $x, y \in \mathbb{R}$ we define the *complex exponential function* by

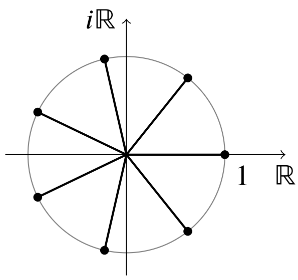
$$e^z := e^x(\cos y + i \sin y),$$

The sketch below shows the mapping properties of the exponential function.



It is an invertible map from $\{z \in \mathbb{C} \mid -\pi < \operatorname{Im} z \leq \pi\}$ onto $\mathbb{C} \setminus \{0\}$. The above mapping properties together with the argument modulus form are useful to determine the images of sets under certain complex functions.

Roots of complex numbers. Given $n \in \mathbb{N}$, the *n-th roots* of a complex number α are the solutions to the equation $z^n = \alpha$. If $\alpha = 1$ we talk about the *roots of unity*. By De Moivre's theorem the roots of unity are equally spaced on the unit circle:



By De Moivre's theorem we have $\left(e^{2\pi ki/n}\right)^n = e^{2\pi ki} = 1$, so $e^{2\pi ki/n}$, $k = 0, \dots, n-1$, are the roots of unity. If $z = re^{i\theta}$ is given in polar form, then again by de Moivre's theorem the n-th roots of z are given by

$$\alpha_k = r^{1/n} e^{\theta i/n + 2\pi ki/n} \qquad k = 0, 1, \dots, n-1.$$

Again the n roots lie on a circel. Its radius in $r^{1/n}$ and they are equally spaced starting from $r^{1/n}e^{i\theta/n}$.

Hints for determining images. Let $f : \mathbb{C} \to \mathbb{C}$ be a function, and let $D \subseteq \mathbb{C}$. The *image of D under f* is

$$\operatorname{im}(D) = \{f(z) \mid z \in D\}.$$

If you need to determine the image of a set under a complex map there are several approaches:

- Write $z = x + iy$ with $x, y \in \mathbb{R}$, then do a computation. This is sometimes useful, but a lot of the time inefficient. The method should only be applied as a last step after using some of the techniques below.
- Use that $z\bar{z} = |z|^2$
- Use geometric properties, in particular $|z_1 - z_2|$ is the distance between z_1 and z_2 on the complex plane.
- Write $z = re^{i\theta}$ in modulus-argument form, in particular if powers of z are involved.

Questions to complete during the tutorial

1. Express the following complex numbers in Cartesian form:

(a) $e^{2\pi i/3}$ (b) $e^{i\frac{\pi}{12}}e^{i\frac{2\pi}{3}}e^{i\frac{\pi}{4}}$

2. Solve the following equations and plot the solutions in the complex plane:

(a) $z^5 = 1$ (b) $z^4 = 8\sqrt{2}(1+i)$

3. Find all solutions of the following equations:

(a) $e^z = i$ (b) $e^z = -10$ (c) $e^z = -1 - i\sqrt{3}$ (d) $e^{2z} = -i$

4. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $z \mapsto z^2$. Sketch the following sets, and then sketch their images under the function f.

(a) $A = \{z \in \mathbb{C} \mid \operatorname{Im}(z) = 2\}$
(b) $B = \{z \in \mathbb{C} \mid \operatorname{Im}(z) = 2\operatorname{Re}(z)\}$
(c) $C = \{z \in \mathbb{C} \mid |z| = 1 \text{ and } \operatorname{Im}(z) \geq 0\}$
(d) $D = \{z \in \mathbb{C} \mid |z| = 1\}$

5. Sketch the following sets and their images under the function $z \mapsto e^z$.

(a) $A = \{z \in \mathbb{C} \mid 0 < \operatorname{Re}(z) < 2 \text{ and } \operatorname{Im}(z) = \frac{\pi}{2}\}$
(b) $B = \{z \in \mathbb{C} \mid \operatorname{Re}(z) = 1 \text{ and } |\operatorname{Im}(z)| < \pi/2\}$
(c) $C = \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0 \text{ and } \pi/3 < \operatorname{Im}(z) < \pi\}$
(d) $D = \{z = (1+i)t \mid t \in \mathbb{R}\}$

6. Find all solutions of the equation $e^{2z} - (1+3i)e^z + i - 2 = 0$.

7. (a) Use the definition of the complex exponential function to show that for all $n \in \mathbb{N}$ and all $\theta \in \mathbb{R}$

$$\sum_{k=-n}^{n} \left(e^{i\theta}\right)^k = 1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos n\theta.$$

(b) Hence, use the formula for a geometric series to show that

$$1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos n\theta = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} \quad \text{whenever } \theta \notin 2\pi\mathbb{Z}.$$

The expression is called the n-th *Dirichlet kernel* and appears in the summation of Fourier series.

Extra questions for further practice

8. Express the following complex numbers in Cartesian form:

(a) $e^{-i\pi}$ (b) $e^{\ln 7+2\pi i}$ (c) $\sin(i\pi)$

9. Solve the following equations and plot the solutions in the complex plane:

(a) $z^3 = -i$ (b) $z^6 = -1$

10. (a) Sketch the set $A = \{z \in \mathbb{C} \mid 1/2 < |z| < 4,\ 0 \leq \operatorname{Arg}(z) \leq \pi/4\}$.

(b) Sketch the image B of A in the w-plane under the function $z \mapsto 1/z$.

11. (a) Show that every complex number $z \in \mathbb{C}$ can be expressed as $z = w + 1/w$ for some $w \in \mathbb{C}$.

(b) Use this substitution to solve the equation $z^3 - 3z - 1 = 0$.

12. Solve, using a completion of squares, the general quadratic equation

$$az^2 + bz + c = 0 \quad \text{with } a, b, c \in \mathbb{C}.$$

In other words, prove the standard formula to solve a quadratic equation.

Challenge questions (optional)

13. From the definition of $e^{i\theta} = \cos\theta + i\sin\theta$ we deduce that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We replace $\theta \in \mathbb{R}$ by any complex number $z \in \mathbb{C}$ and define

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

and call them the complex cosine and sine functions.

(a) Show that when z is real, $\cos z$ and $\sin z$ reduce to the familiar real functions.

(b) Show that $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

(c) Show that $\cos(z + w) = \cos z \cos w - \sin z \sin w$ for all $z, w \in \mathbb{C}$.

(d) Is it true that $|\sin z| \leq 1$ and $|\cos z| \leq 1$ for all $z \in \mathbb{C}$?

14. There is a 'cubic formula' analogous to the much loved quadratic formula, although it is a lot more complicated. In this question you solve the general cubic equation

$$az^3 + bz^2 + cz + d = 0 \quad \text{with } a, b, c, d \in \mathbb{C},$$

generalising the method of Question 11. Here is an outline of the strategy:

(a) Make a substitution of the form $z = u + \alpha$, with α to be determined, to reduce the equation to the form $u^3 + pu - q = 0$.

(b) Now attempt a substitution of the form $u = w + \beta/w$ with a cleverly chosen β to reduce the equation to a quadratic in w^3.

(c) You can now solve this quadratic equation for w, hence back-track to find z.

15. Let n be a given positive integer. A *primitive nth root of unity* is a solution $z = \alpha$ of the equation $z^n = 1$ with the property that the powers $\alpha, \alpha^2, \cdots, \alpha^{n-1}, \alpha^n$ give all of the nth roots of unity. For example, the 4th root of unity $\alpha = i$ is primitive, since $i, i^2 = -1, i^3 = -i, i^4 = 1$ gives us all 4th roots of unity, while the 4th root $\alpha = -1$ is not primitive, since $-1, (-1)^2 = 1, (-1)^3 = -1, (-1)^4 = 1$ fails to give us all of the 4th roots of unity.

(a) Find all primitive 6th roots of unity.

(b) Find all primitive 5th roots of unity.

(c) For which values of k, $0 \leq k \leq n-1$, is $e^{i\frac{2\pi k}{n}}$ a primitive nth root of unity?

16. We introduced the complex numbers by saying something along the lines of: "Append a solution i of the equation $x^2 + 1 = 0$ to the real numbers $\mathbb{R}$". This is a bit mysterious, and you might ask: "What is this magical element i? Where does it live?". Here is a more formal approach. Let $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$ and define an addition operation and a multiplication operation on $\mathbb{R}^2$ by:

$$\begin{aligned} (a, b) + (c, d) &= (a + c, b + d) && \text{for all } (a, b) \in \mathbb{R}^2 \\ (a, b)(c, d) &= (ac - bd, bc + ad) && \text{for all } (a, b) \in \mathbb{R}^2. \end{aligned}$$

Let $\mathbf{0} = (0, 0)$, $\mathbf{1} = (1, 0)$, and $\mathbf{i} = (0, 1)$.

(a) Show that $\mathbf{0} + (a, b) = (a, b)$ for all $(a, b) \in \mathbb{R}^2$.

(b) Show that $\mathbf{1}(a, b) = (a, b)$ for all $(a, b) \in \mathbb{R}^2$.

(c) Show that $\mathbf{i}^2 + \mathbf{1} = \mathbf{0}$.

(d) Explain why $\mathbb{R}^2$ with the above operations is really just $\mathbb{C}$ in disguise. Thus it is possible to introduce $\mathbb{C}$ without ever talking about the 'imaginary' number i.