The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Exercises for Week 1** 2017

There are thirteen teaching weeks. There will be one set of exercises each week from Week 1 through 12, except for Weeks 5 and 11, when quizzes will be held during tutorials. This is the first set of exercises. Preparatory exercises should be attempted before coming to the tutorial and solutions are provided below. Short answers are provided to other selected exercises. **Full solutions to other exercises will be available from the course webpage at the end of the week.** Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) A *geometric vector* $\mathbf{v}$ is a directed line segment in space, described by its length $|\mathbf{v}|$ and direction. Two vectors are *equal* if they have the same magnitude and direction, regardless of their position in space.

(ii) A *scalar* λ is a real number. The *scalar multiple* $\lambda\,\mathbf{v}$ has length $|\lambda||\mathbf{v}|$ and the same direction as $\mathbf{v}$ if λ is positive, and opposite direction if λ is negative.

(iii) If P and Q are points in space then $\overrightarrow{PQ}$ denotes the vector pointing from P to Q. The *position vector* of the point P is the vector $\overrightarrow{OP}$ where O denotes the origin in space.

(iv) A *parallelogram* is a quadrilateral such that two opposite sides are parallel and have the same length (which implies that the other two opposite sides are also parallel and have the same length).

(v) Parallelogram Law of Vector Addition: The *vector sum* $\mathbf{v}+\mathbf{w}$ is represented by the diagonal of the parallelogram formed using sides $\mathbf{v}$ and $\mathbf{w}$.

(vi) Commutative Law of Addition: $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.

(vii) Associative Law of Addition: $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.

(viii) Triangle Inequality: $|\mathbf{v}+\mathbf{w}|\le|\mathbf{v}|+|\mathbf{w}|$.

(ix) The *zero vector* $\mathbf{0}$ has zero length and points in every direction. For every vector $\mathbf{v}$, $\mathbf{0}+\mathbf{v}=\mathbf{v}$ and $0\mathbf{v}=\mathbf{0}$.

(x) The *negative* of $\mathbf{v}$ is $-\mathbf{v}=(-1)\mathbf{v}$ with the same length as $\mathbf{v}$, but pointing in the opposite direction. If P and Q are points then $\overrightarrow{QP}=-\overrightarrow{PQ}$.

(xi) The *vector difference* $\mathbf{v}-\mathbf{w}$ equals $\mathbf{v}+(-\mathbf{w})$ and has the property that $\mathbf{w}+(\mathbf{v}-\mathbf{w})=\mathbf{v}$.

(xii) If $\mathbf{v}$ and $\mathbf{w}$ are vectors and λ and μ are scalars, then

$$\lambda(\mu\mathbf{v})=(\lambda\mu)\mathbf{v}\ ,\quad \lambda(\mathbf{v}+\mathbf{w})=\lambda\mathbf{v}+\lambda\mathbf{w}\ ,\quad (\lambda+\mu)\mathbf{v}=\lambda\mathbf{v}+\mu\mathbf{v}\ ,$$

$$-(-\mathbf{v})=\mathbf{v}\ ,\quad \mathbf{v}-\mathbf{v}=\mathbf{0}\ ,\quad 1\mathbf{v}=\mathbf{v}\ ,\quad (-\lambda)\mathbf{v}=-(\lambda\mathbf{v})\ .$$

Preparatory Exercises:

1. Manipulate an algebraic expression to explain the following phenomenon:

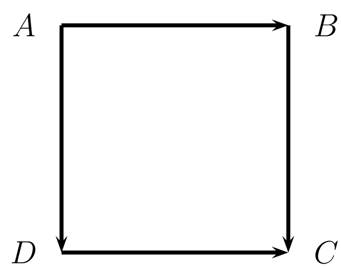
> I am a mind reader. Think of a number from 1 to 21. Double it and add 4. Halve your answer. Take away the number you first started with. You are thinking of the number 2.

2. Find the line obtained by reflecting the line $2x + 3y = 6$ in the line $y = x$. Describe the relationship between the slopes of the original and the reflected lines.

3. Find the line obtained by rotating the line $2x + 3y = 6$ ninety degrees anticlockwise about the origin. Describe the relationship between the slopes of the original and the rotated lines.

4. Find the point of intersection of the lines $2x + 3y = 6$ and $3x + 2y = 6$.

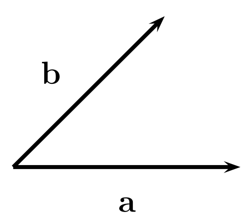
5. The edges of the square $ABCD$ are marked by vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$, $\overrightarrow{DC}$, as shown.

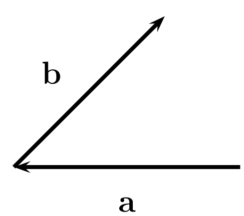


True or false:

(i) $\overrightarrow{AB} = \overrightarrow{BC}$ (ii) $\overrightarrow{AB} = \overrightarrow{CD}$ (iii) $\overrightarrow{AD} = \overrightarrow{BC}$ (iv) $\overrightarrow{AC} = \overrightarrow{BC} + \overrightarrow{DC}$

6. Draw the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ on each diagram.





7. Simplify the following vector expressions.

(i) $3\mathbf{a} + 2\mathbf{b} - 4(\mathbf{b} + \frac{1}{2}\mathbf{a})$ (ii) $-(\mathbf{w} - 6\mathbf{z}) - 2\mathbf{w} + \mathbf{v} - 2\mathbf{z}$

Exercises:

16. Express $2\mathbf{a} - 3\mathbf{b}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, and simplify, when

$$\mathbf{a} = \mathbf{u} + \mathbf{v}\ , \qquad \mathbf{b} = 3\mathbf{u} - 2\mathbf{v}\ .$$

17. Let $ABCDEF$ be a regular hexagon and put

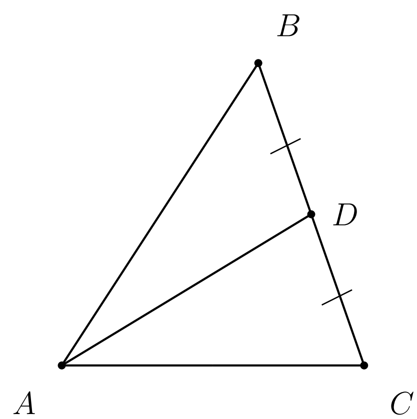
$$\mathbf{a} = \overrightarrow{AB}\ , \qquad \mathbf{b} = \overrightarrow{BC}\ .$$

Find vector expressions in terms of $\mathbf{a}$ and $\mathbf{b}$ for the displacements

$$\overrightarrow{CD}, \quad \overrightarrow{DE}, \quad \overrightarrow{EF}, \quad \overrightarrow{FA}.$$

18. A plane travels 20km in the direction $30°$ north of east and then 10 km southeast. Use trigonometry and your calculator to find the final distance and direction of the aircraft from the starting position.

19. Let D be the midpoint of the side BC of the triangle ABC.



Verify that $\overrightarrow{AD} = \frac{1}{2}\big(\overrightarrow{AB} + \overrightarrow{AC}\big)$.

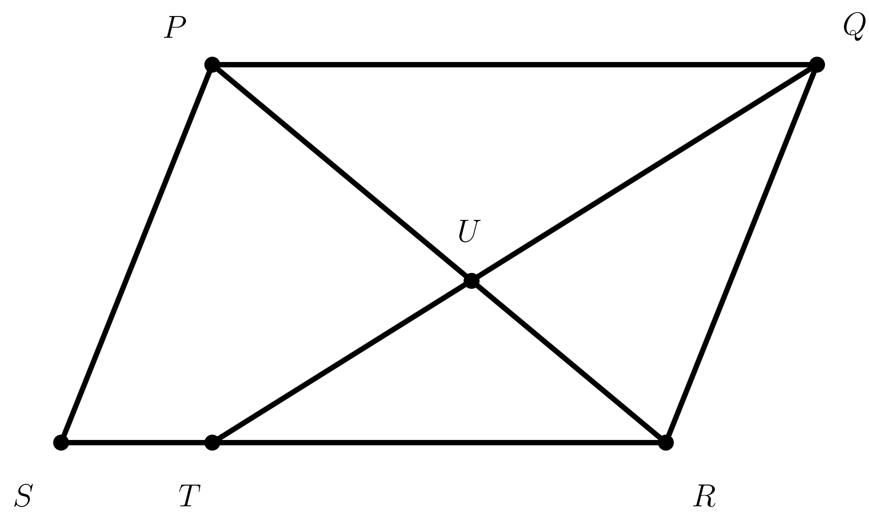
20. Prove that the diagonals of a parallelogram bisect each other.

21.* Prove that the midpoints of the sides of a quadrilateral form a parallelogram.

22.* The *median* of a triangle is the line joining a vertex to the midpoint of the opposite side. Prove that, for any triangle, the medians can be shifted parallel to themselves to form another triangle.

23.* Find the line obtained by rotating the line $ax + by = c$ ninety degrees anticlockwise about the point (x_0, y_0).

24.* Draw a parallelogram $PQRS$. Let T divide the side SR in the ratio $r : s$ and let U be the point of intersection of the diagonal PR with the line QT. Find the ratio in which U divides the diagonal. (As a check, your answer should be such that when $r = 0$ the ratio becomes $1 : 1$, recovering the earlier exercise that the diagonals of a parallelogram bisect each other.)



25.** Manipulate an algebraic expression to explain the following phenomenon:

> I am a mind reader. Think of an integer from 1 to 21. Double it. Take the square root and add a half. Throw away everything to the right of the decimal point, and the decimal point. You are thinking of an integer X. Now consider the following sequence, and move from left to right the same number of steps as the integer you first thought of:
>
> $$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6$$
>
> (For example if you move 10 steps you will reach 4 in the sequence, and after 11 steps you will reach 5.) Call the number you have reached Y. Take Y away from X. You are now thinking of the most important number in mathematics.

Short Answers to Selected Exercises:

1. $\frac{1}{2}(2x+4) - x = 2$

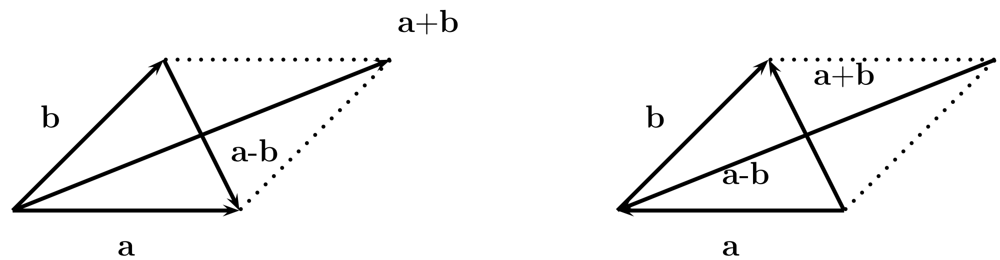
2. $3x + 2y = 6$, and slopes are reciprocals of each other.

3. $-3x + 2y = 6$, and slopes are negative reciprocals of each other.

4. $(6/5, 6/5)$

5. (i) false (ii) false (iii) true (iv) true

6.



7. (i) $\mathbf{a} - 2\mathbf{b}$ (ii) $\mathbf{v} - 3\mathbf{w} + 4\mathbf{z}$

16. $-7\mathbf{u} + 8\mathbf{v}$

17. $\overrightarrow{CD} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{DE} = -\mathbf{a}$, $\overrightarrow{EF} = -\mathbf{b}$, $\overrightarrow{FA} = \mathbf{a} - \mathbf{b}$.

18. final distance 25 km, final direction $7°$ north of east

23. $-bx + ay = c + a(y_0 - x_0) - b(x_0 + y_0)$

24. $r + s : s$