Notes on Integral Calculus and Modelling
1st Instalment

Contents

David Easdown, 17 July 2017

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Notes on Integral Calculus and Modelling

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Introduction

Theme of Course:

Use information about the derivative(s) of a function to explore the function itself.

Some motivating examples:

(not all of the detail you will follow yet)

2.

Example (1): Knowing velocity

$$v = v(t) = \frac{ds}{dt}$$

find or estimate displacement

$$s = s(t).$$

If a car travels at 40 km/hr for 20 sec then it moves

$$40 \times \frac{1000}{3600} \times 20 \approx 222 \text{ metres}$$

Suppose the car accelerates from rest to reach 100 km/hr one minute later:

t sec	0	20	40	60
v km/hr	0	40	60	100

3.

Estimate:

$$\frac{1000}{3600}\big((0\times 20)+(40\times 20)+(60\times 20)\big)$$

$$\leq \quad s \quad \leq \frac{1000}{3600}\big((40\times 20)+(60\times 20)+(100\times 20)\big)$$

i.e. $\frac{20{,}000}{36} \leq s \leq \frac{40{,}000}{36}$

≈ 556 (lower bound) $\qquad \approx 1{,}111$ (upper bound)

If we know more about v we can improve our estimate:

t sec	0	10	20	30	40	50	60
v km/hr	0	10	40	50	60	90	100

4.

New estimate:

$$\frac{1000}{3600}\big((0\times 10)+(10\times 10)+(40\times 10)+(50\times 10)+(60\times 10)+(90\times 10)\big)$$

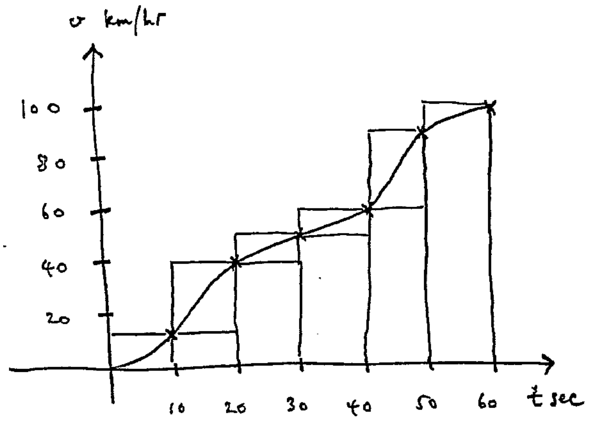
$$\leq \quad s \quad \leq$$

$$\frac{1000}{3600}\big((10\times 10)+(40\times 10)+(50\times 10)+(60\times 10)+(90\times 10)+(100\times 10)\big)$$

i.e. $\frac{25{,}000}{36} \leq s \leq \frac{35{,}000}{36}$

≈ 694 (new lower bound) $\qquad \approx 972$ (new upper bound)

5.



In fact we are using rectangular approximations to

area under velocity curve,

leading to notion of

Riemann sums

↑

ubiquitous in mathematics & applications

6.

Example (2) : Find the escape velocity of a rocket ?

How fast does a rocket need to travel away from the earth's surface to escape the gravitational field ?

We want displacement to become arbitrarily large !!

Newton's Law of Gravitation :

$$F = G\,\frac{m_1 m_2}{x^2}$$

force of attraction between two bodies of masses m_1, m_2, distant x apart

G = gravitation constant

7.

Put v_0 = escape velocity of rocket

m = mass of rocket

M = " " earth

R = radius of earth

$$\boxed{\text{Energy of rocket} = \tfrac{1}{2} m v_0^2}$$

Want this to match work done to escape gravitational field which is the improper integral

$$\boxed{\text{Work} = \int_R^\infty G\,\frac{Mm}{x^2}\,dx}$$

(improper because of ∞)

8.

$$\int_R^\infty \underbrace{G\,\frac{Mm}{x^2}}_{\text{integrand = force}}\,dx$$

$\int$ is called an integral sign, stylized "sum" symbol, comes from Σ for sum

dx is called a differential, an abstraction of

Δx = difference or change in x

Here x represents distance of rocket from centre of earth, which varies from R to ∞.

Solving:

$$\frac{1}{2} m v_0^2 = \int_R^\infty G \frac{Mm}{x^2}\, dx$$

$$= \lim_{D \to \infty} \underbrace{\int_R^D G \frac{Mm}{x^2}\, dx}$$

proper integral !!

$$= GMm \lim_{D \to \infty} \int_R^D x^{-2}\, dx$$

so

$$\frac{1}{2} v_0^2 = GM \lim_{D \to \infty} \left[-x^{-1} \right]_R^D$$

by the Fundamental Theorem of Calculus (see next week) since $\frac{d}{dx}(-x^{-1}) = x^{-2}$

$$= GM \lim_{D \to \infty} \left(-\frac{1}{D} - \left(-\frac{1}{R}\right)\right)$$

$$= \frac{GM}{R}$$

so $$v_0^2 = \frac{2GM}{R}$$

so $$\boxed{v_0 = \sqrt{\frac{2GM}{R}}\,.}$$

Example (3): Taylor series

- reconstructing functions from derivatives at just one point !!!

Recall

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

the Taylor polynomial about $x = a$ of degree n for $y = f(x)$,

built from derivatives and ordinary arithmetic

which approximates $f(x)$ for x near a.

The Taylor series (which can become the function itself !!) is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

goes on forever

e.g. for $f(x) = e^x$ about $x = 0$

$$e^x = e^0 + e^0(x-0) + \dots + \frac{e^0}{n!}(x-0)^n + \dots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}$$

R.H.S. is an infinite polynomial which reproduces itself under differentiation !!!

What is the number e ?

$$\boxed{e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots}$$

Note

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

These hold also when $x \in \mathbb{C}$, so for $i = \sqrt{-1}$, .

$$\boxed{\begin{aligned} \cos(ix) &= \cosh x \\ \sin(ix) &= i \sinh x \end{aligned}}$$

Example (4) : Knowing growth rate

$$\frac{dx}{dt}$$

of a colony of bacteria, find the population size

$$x = x(t) \quad \text{at time } t.$$

Simplest model :

$$\frac{dx}{dt} = kx, \qquad k > 0$$

differential equation

Solving :
$$\frac{dx}{dt} = kx$$

$$\Rightarrow \quad \frac{1}{x}\,dx = k\,dt$$

differentials

$$\Rightarrow \quad \underbrace{\int \frac{1}{x}\,dx} = \int k\,dt = k \underbrace{\int dt}$$

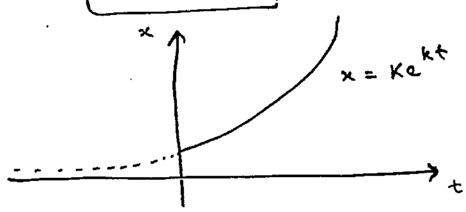
indefinite integrals = antiderivatives

$$\Rightarrow \quad \ln x = kt + C$$

constant of integration

$$\Rightarrow \quad x = e^{\ln x} = e^{kt + C} = e^{C} e^{kt}$$

$$\Rightarrow \quad \boxed{x = K e^{kt}} \qquad (K = e^{C})$$



Note : $\lim_{t \to \infty} x(t) = \infty$,

impossible if resources are limited !

More sophisticated model :

$$\boxed{\frac{dx}{dt} = kx - ax^2} \qquad k, a > 0$$

contribution due to reproduction

retardation due to limit of resources

called the logistic equation.

Solving : $$\frac{dx}{dt} = x(k - ax)$$

$$\Rightarrow \quad \frac{1}{x(k - ax)}\,dx = dt$$

(separating variables)

17.

$$\Rightarrow \int \frac{1}{x(k-ax)}\,dx = \int dt$$

But $\dfrac{1}{x(k-ax)} = \dfrac{1}{kx} + \dfrac{a}{k(k-ax)}$

discovered using <u>method of partial fractions</u> (see later in course)

$$\Rightarrow \int \frac{1}{x(k-ax)}\,dx$$

$$= \frac{1}{k}\int \frac{1}{x}\,dx + \frac{a}{k}\int \frac{1}{k-ax}\,dx$$

$$= \int dt$$

$$\Rightarrow \frac{1}{k}\ln|x| - \frac{1}{k}\ln|k-ax| = t + C$$

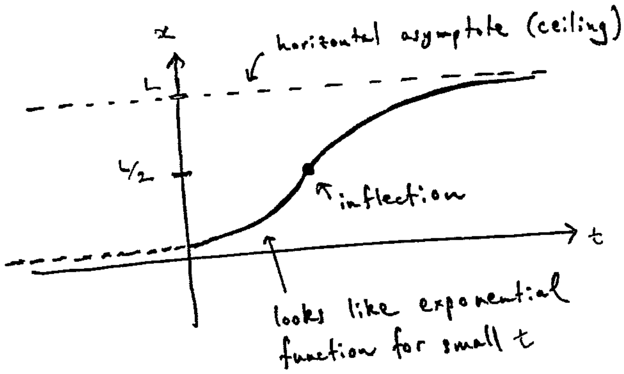
18.

very good exercise to fill this in

$$\Rightarrow \boxed{x = \frac{L}{1+Ke^{-kt}}}$$

for some constants K, L

called the <u>logistic</u> function.



19.

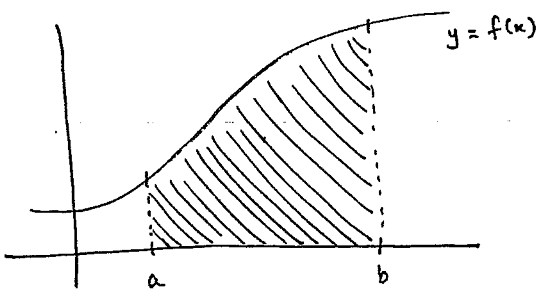
<u>The definite integral and Riemann sums</u>

A <u>definite integral</u> is an expression of the form

$$\int_a^b f(x)\,dx$$

and is intended to represent the <u>area</u> under the curve

$$y = f(x) \quad \text{for} \quad a \le x \le b.$$



20.

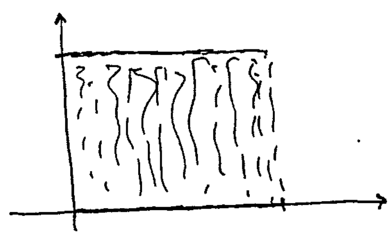
How does one find areas?

Is the notion of area always sensible?

<u>Example</u>: For $x \in [0,1]$ define

$$f(x) = \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$$

where $\mathbb{Q} = \{\text{rational real numbers}\}$.



no sensibly defined "area under graph" !!

Most functions we meet will be continuous, or have at most a finite number of discontinuities. For such functions there is always a well-defined notion of area (though for unbounded functions this will turn out to be very subtle).

Technique of the Greeks:

(1) Approximate a difficult task by an easy one.

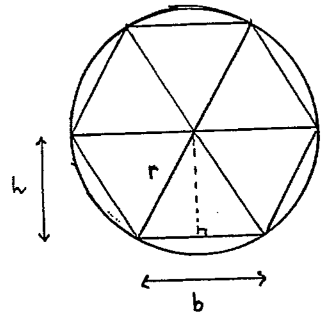
(2) See what happens in the limit.

Example: Given the formula

$$P = 2\pi r$$

for the perimeter P of a circle of radius r, find the area A.

Solution: Divide the circle into n equal segments:



(here $n = 6$)

Area of each segment

$\approx$ area of Δ with height h, base b

$= \frac{1}{2} bh$

Hence $A \approx n(\frac{1}{2}bh) = \dfrac{nbh}{2}$.

The approximation improves as $n \to \infty$. In fact

$$A = \lim_{n\to\infty} \frac{nbh}{2}$$

$$= \frac{1}{2}\left(\lim_{n\to\infty} nb\right)\left(\lim_{n\to\infty} h\right)$$

$$= \tfrac{1}{2} \times P \times r$$

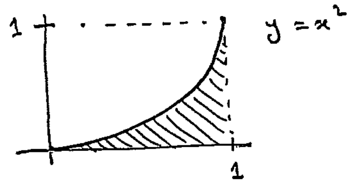
$$= \tfrac{1}{2}(2\pi r) r = \pi r^2.$$

$$\boxed{A = \pi r^2.}$$

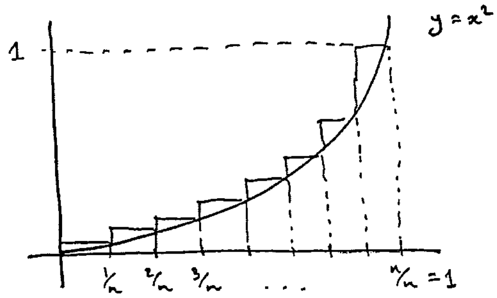
Note: $\dfrac{dA}{dr} = 2\pi r = P$!!

A is an antiderivative of P

Example: Find the area under the parabola $y = x^2$ for $0 \le x \le 1$



Approximate with rectangles:



Area of (upper) rectangles

$= \frac{1}{n}(\frac{1}{n})^2 + \frac{1}{n}(\frac{2}{n})^2 + \frac{1}{n}(\frac{3}{n})^2 + \dots + \frac{1}{n}(\frac{n}{n})^2$

$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \dots + n^2)$.

25.

Here

$$\boxed{A = \lim_{n \to \infty} \frac{1}{n^3} (1^2 + 2^2 + \cdots + n^2)}$$

What is $1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2$?

Telescoping sum method :

$$n^3 = (n^3 - (n-1)^3) + ((n-1)^3 - (n-2)^3)$$
$$+ ((n-2)^3 - (n-3)^3) + \cdots$$
$$+ (2^3 - 1^3) + (1^3 - 0^3)$$
$$= \sum_{i=1}^{n} (i^3 - (i-1)^3)$$
$$= \sum_{i=1}^{n} i^3 - (i^3 - 3i^2 + 3i - 1)$$
$$= \sum_{i=1}^{n} (3i^2 - 3i + 1)$$
$$= 3 \sum_{i=1}^{n} i^2 - 3 \sum_{i=1}^{n} + n$$

26.

Trick of Gauss :

$$\begin{array}{rcl} 1 + 2 + 3 + \cdots + n & = & S \\ n + (n-1) + (n-2) + \cdots + 1 & = & S \\ \hline (n+1) + (n+1) + (n+1) + \cdots + (n+1) & = & 2S \end{array}$$

so $S = \frac{n(n+1)}{2}$

Hence

$$n^3 = 3 \sum_{i=1}^{n} i^2 - 3 \frac{n(n+1)}{2} + n$$

so

$$\sum_{i=1}^{n} i^2 = \frac{1}{3} \left(n^3 + 3 \frac{n(n+1)}{2} - n \right)$$
$$= \frac{1}{3} \left(\frac{2n^3 + 3n^2 + 3n - 2n}{2} \right)$$
$$= \frac{1}{6} (2n^3 + 3n + n)$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}}$$

27.

Hence

$$A = \lim_{n \to \infty} \frac{1}{n^3} \times \frac{n(2n+1)(n+1)}{6}$$
$$= \lim_{n \to \infty} \frac{(2 + 1/n)(1 + 1/n)}{6}$$
$$= \frac{(2+0)(1+0)}{6}$$
$$= 1/3 \quad \text{(known to the Greeks!)}$$

Sneak preview : compare this with the briefer calculation

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = 1/3 - \frac{0}{3} = \frac{1}{3}$$

Fundamental Theorem of Calculus (see below)

28.

Method of Riemann sums :

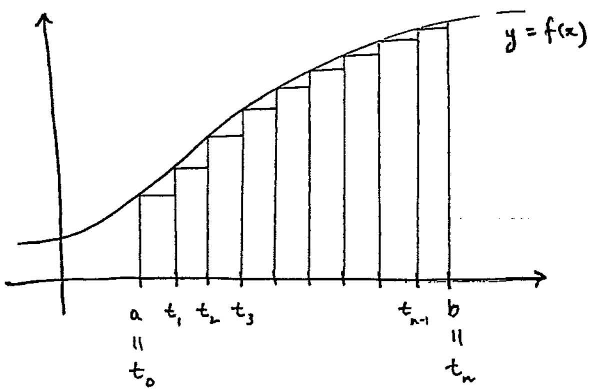
We find the area A under the curve

$$y = f(x) \quad \text{for} \quad a \le x \le b$$

by limiting approximations.

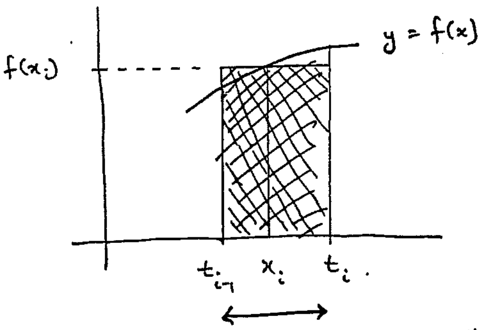
Step (1) : Partition $[a, b]$ into n subintervals

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$



The ith subinterval is $[t_{i-1}, t_i]$

Step (2): For each $i = 1, \dots, n$ choose any $x_i \in [t_{i-1}, t_i]$ and draw a rectangle of height $f(x_i)$



$\Delta x_i = t_i - t_{i-1}$ "change in x"

$\boxed{\text{Area of rectangle} = f(x_i)\,\Delta x_i}$

Step (3): Sum the areas of rectangles for $i = 1, \dots, n$:

$$A \approx f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + \dots + f(x_n)\,\Delta x_n$$

Abbreviate:

$$\boxed{A \approx \sum_{i=1}^{n} f(x_i)\,\Delta x_i}$$

called a Riemann sum

Step (4): Take the limit as $n \to \infty$:

$$\boxed{A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i}$$

Theorem: If $y = f(x)$ is continuous for $a \le x \le b$ then

$$\boxed{A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i}$$

exists, provided the $\Delta x_i \to 0$.

We therefore take this limit to be the definition of area under the curve !!

This is a very subtle result. The value of the limit is independent of the choices made is Steps (1), (2).

(There is no need even for the subintervals to have the same length.)

We write

$$\boxed{A = \int_a^b f(x)\,dx}$$

called the definite integral.

If $f(x_i)$ is always chosen to be the minimum [maximum] value of $f(x)$ over $[t_{i-1}, t_i]$ then we get the lower [upper] Riemann sum

Always

$$\boxed{\text{lower sum} \le \int_a^b f(x)\,dx \le \text{upper sum}}$$

easy to find if f is increasing or decreasing

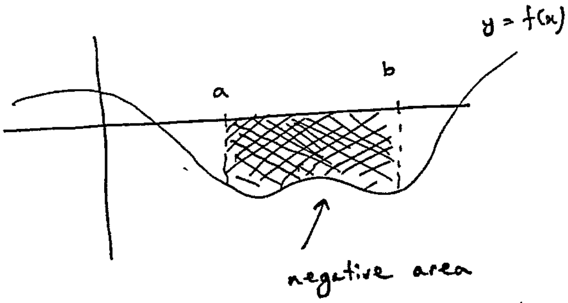
Define

$$\boxed{\int_a^a f(x)\,dx = 0}$$

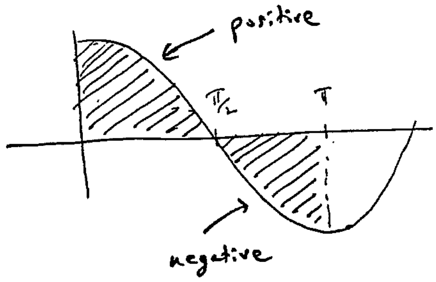
and

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

Note also areas "under" curves are <u>negative</u> if the curve is <u>below</u> the horizontal axis



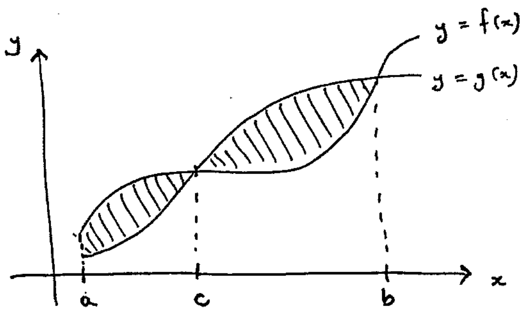
<u>Example</u>: $\int_0^\pi \cos x\,dx = 0$



> To find the absolute value of the area between the curve $y = f(x)$ and the x-axis for $a \le x \le b$, calculate
>
> $$\int_a^b |f(x)|\,dx$$

– this may involve subdividing into intervals on which $f(x)$ is positive or negative

> To find the (absolute) area trapped between curves $y = f(x)$ and $y = g(x)$ for $a \le x \le b$, calculate
>
> $$\int_a^b |f(x) - g(x)|\,dx .$$



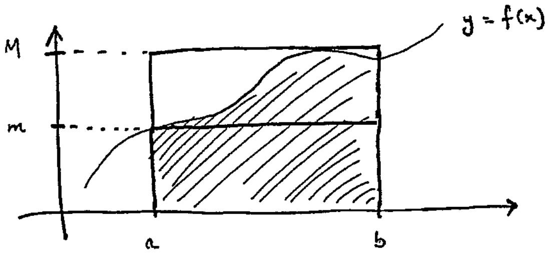
Here $\int_a^b |f(x) - g(x)|\,dx$

$$= \int_a^c g(x) - f(x)\,dx + \int_c^b f(x) - g(x)\,dx .$$

<u>Basic properties of the definite integral</u>:

(1) If m = minimum, M = maximum of $y = f(x)$ for $a \le x \le b$ then

$$\boxed{m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)}$$



(2) If c = constant then

$$\boxed{\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx}$$

"constants come out the front"

37.

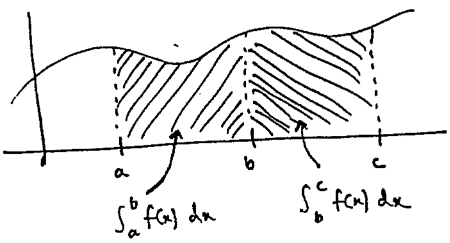
(3) The definite integral is <u>additive</u> :

$$\boxed{\int_a^b f(x) + g(x) \; dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx}$$

(4) Definite integrals may be broken up into subintervals :

$$\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$$

(regardless of the order of a, b, c on the real line)



38.

The Fundamental Theorem of Calculus

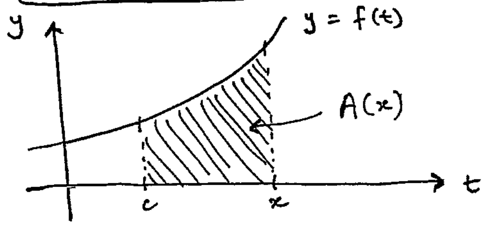
– relationship between <u>areas</u> and <u>derivatives</u> .

Recall, if $A = A(x)$ then

$$\boxed{A'(x) = \frac{dA}{dx} = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}}$$

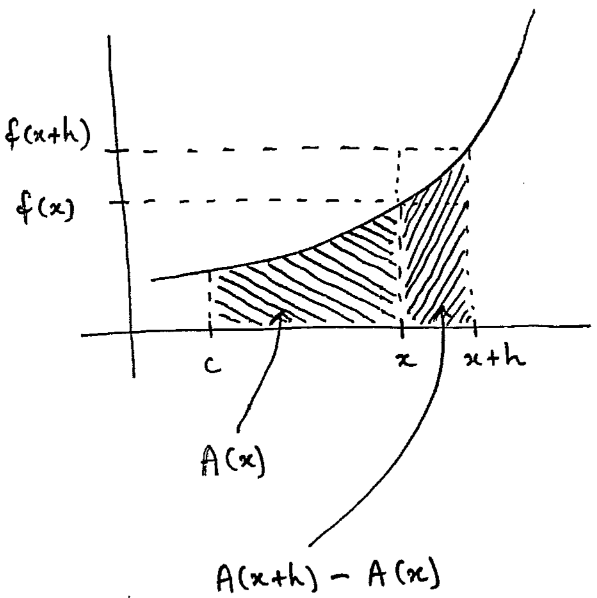
Let $y = f(x)$ be <u>continuous</u>, c constant and define

$$\boxed{\begin{aligned} A(x) &= \text{area under curve between } c \text{ and } x \\ &= \int_c^x f(t)\,dt \end{aligned}}$$



39.

Change x by a small amount h :



40.

In this diagram

$$h\,f(x) \le \underbrace{A(x+h) - A(x)} \le h\,f(x+h)$$

↑ area of smaller rectangle ↑ red shaded area ↑ area of larger rectangle

so

$$f(x) \le \frac{A(x+h) - A(x)}{h} \le f(x+h)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$f(x) = A'(x) = f(x)$$

as $h \to 0$

(using continuity of f for R.H.S.) .

41.

Thus

$$\boxed{A'(x) = f(x)}$$

Useful consequence:

Areas can be found by <u>antidifferentiation</u>

the reverse of differentiation.

Call $F(x)$ an <u>antiderivative</u> of $f(x)$ if

$$F'(x) = f(x)$$

42.

The Fundamental Theorem of Calculus (part 1)

$$A(x) = \int_c^x f(t)\,dt$$

is an antiderivative of $f(x)$

which may also be expressed

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x)$$

43.

<u>FACT</u> (comes from the Mean Value Theorem)

Antiderivatives of a given function differ by a constant

By this we mean

if $F(x)$ is an antiderivative of $f(x)$ then <u>all</u> antiderivatives of $f(x)$ have the form

$$F(x) + C$$

for some constant C.

44.

The Fundamental Theorem of Calculus (part 2)

Suppose that $y = f(x)$ is continuous on $[a, b]$ and that $F'(x) = f(x)$ for $x \in [a, b]$

(that is, F is an <u>antiderivative</u> of f)

Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

45.

Notation : common to write

$$\Big[F(x) \Big]_a^b$$

or

$$F(x) \Big]_a^b$$

or

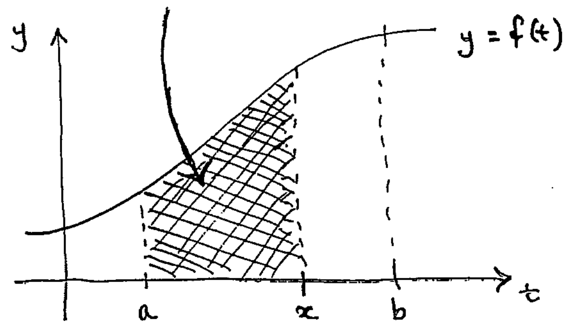
$$F(x) \Big|_a^b$$

for $F(b) - F(a)$.

46.

Proof of the Fundamental Theorem of Calculus (part 2) :

Let $A(x) = \int_a^x f(t)\, dt$.



So $$A(b) = \int_a^b f(t)\,dt$$

$$A(a) = \int_a^a f(t)\,dt = 0 .$$

47.

From Part 1 : $\boxed{A'(x) = f(x)}$.

But $\boxed{\text{antiderivatives differ by a constant.}}$

Hence $A(x) = F(x) + C$ for some constant C .

But $0 = A(a) = F(a) + C$

so $\boxed{C = -F(a)}$.

Thus
$$\int_a^b f(x)\,dx = A(b) = F(b) + C = F(b) - F(a)$$
as required.

48.

Indefinite integrals

Because of the close relationship between definite integrals and antiderivatives, we use the notation

$$\boxed{\underline{\int f(x)\,dx} = F(x) + C}$$

without terminals, called an *indefinite* integral

where $F(x)$ is an antiderivative of $f(x)$.

This equation carries the same information as

$$\boxed{F'(x) = f(x).}$$

We call C the constant of integration.

49.

Recall the Chain Rule :

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}}$$

where $y = y(u)$, $u = u(x)$ are functions.

Put $y = y(u) = \int f(u)\,du$

where f is some function.

Then $\frac{dy}{du} = f(u)$

so $\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = f(u)\,\frac{du}{dx}$.

Thus y is an antiderivative of $f(u)\,\frac{du}{dx}$ with respect to $\underline{\underline{x}}$.

50.

Thus we have derived the formula .

$$\boxed{\int f(u)\,du = \int \underbrace{f(u)\,\frac{du}{dx}}\,dx}$$

look for a substitution $u = u(x)$ which expresses the integrand as $f(u)\,\frac{du}{dx}$

Guiding principle : try to substitute the "complication" away.

51.

Handy notation :

if $\frac{dy}{dx} = g(x)$ then $dy = g(x)\,dx$.

We write

$$\boxed{\int \frac{dx}{g(x)} \quad \text{for} \quad \int \frac{1}{g(x)}\,dx} .$$

52.

Trigonometric substitutions :

The trig identities

$$1 - \sin^2\theta = \cos^2\theta .$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\sec^2\theta - 1 = \tan^2\theta$$

suggest the following useful substitutions :

$x = a\sin\theta$ replaces $a^2 - x^2$ by $a^2\cos^2\theta$
$x = a\tan\theta$ replaces $a^2 + x^2$ by $a^2\sec^2\theta$
$x = a\sec\theta$ replaces $x^2 - a^2$ by $a^2\tan^2\theta$

53.

Volumes of revolution

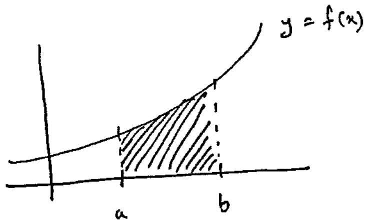
- disc method
- shell method

Consider a region bounded by

x - axis

lines $x = a$, $x = b$

curve $y = f(x) \geq 0$



which sweeps out a volume by rotating about one axis.

54.

What volumes are obtained by rotating this region

(a) about the x-axis ?

(b) " " y-axis ?

Answers :

(a) $\displaystyle\int_a^b \pi \left[f(x)\right]^2 dx$

(by the disc method)

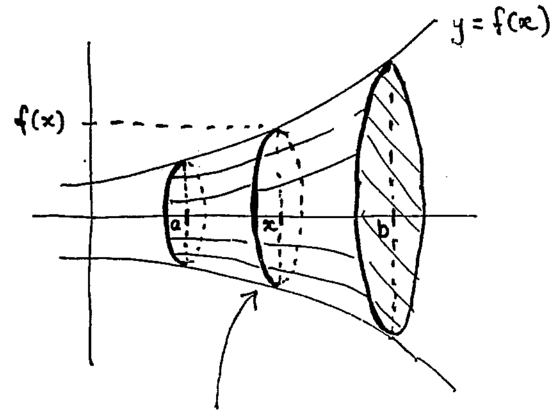
(b) $\displaystyle\int_a^b 2\pi\, x\, f(x)\, dx$

(by the shell method)

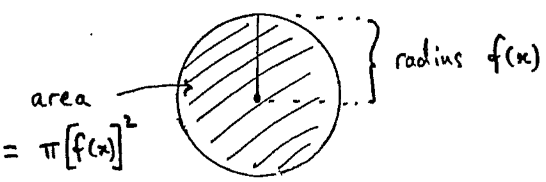
55.

The disc method

Rotate about the x-axis :



cross-section is a circle



56.

Approximate volume of revolution using discs :

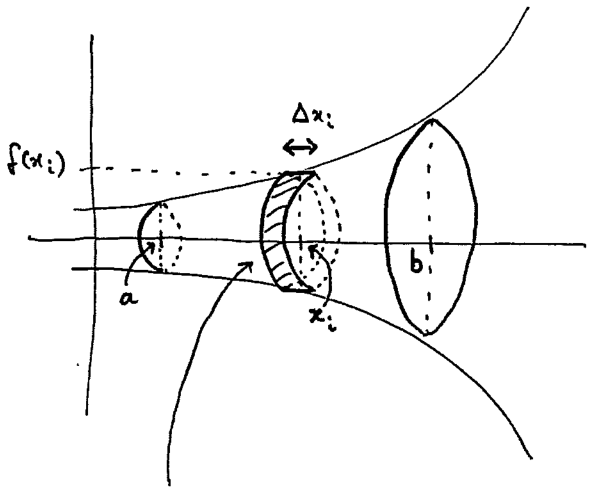
(1) partition $[a,b]$ into n subintervals :

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

(2) for each $i = 1, \ldots, n$ choose $x_i \in [t_{i-1}, t_i]$ and form a disc of

- height (radius) $f(x_i)$
- width $\Delta x_i = t_i - t_{i-1}$

57.



ith disc of

volume $= \pi [f(x_i)]^2 \, \Delta x_i$

58.

Put V = total volume of revolution.

Then

$$V \approx \sum_{i=1}^{n} \pi [f(x_i)]^2 \, \Delta x_i .$$

Passing to the limit:

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi [f(x_i)]^2 \, \Delta x_i$$

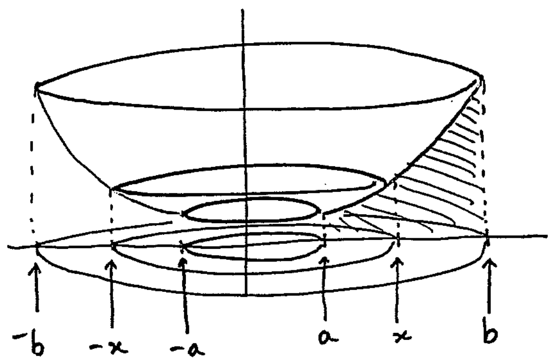
that is

$$\boxed{V = \int_a^b \pi [f(x)]^2 \, dx} .$$

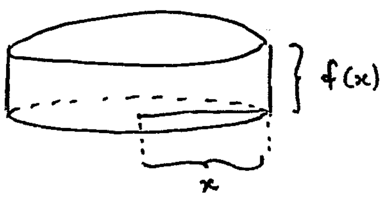
59.

The shell method

Rotate about the y-axis:



Each vertical line sweeps out a cylinder:



60.

> The surface area of a cylinder of height $f(x)$ and radius x is
>
> $$2\pi x \, f(x)$$

To see this, cut the cylinder and open out to a rectangle

of height $f(x)$

width $2\pi x$

with area $2\pi x \, f(x)$.

61.

Approximate volume of revolution using <u>cylindrical shells</u> :

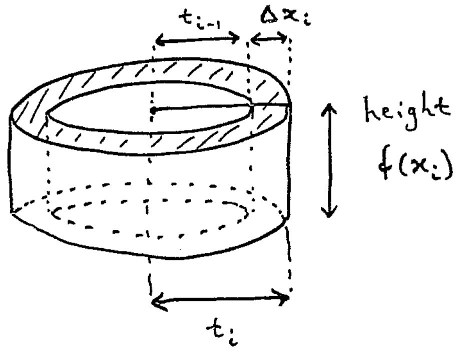
(1) partition $[a, b]$ into n subintervals :

$$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b \; .$$

(2) for each $i = 1, \dots, n$ choose $x_i \in [t_{i-1}, t_i]$ and form a cylindrical shell of

- height $f(x_i)$
- thickness $\Delta x_i = t_i - t_{i-1}$
- radius varying from t_{i-1} to t_i .

62.



Volume of cylindrical shell

$$\begin{aligned}
&= \pi t_i^2 f(x_i) - \pi t_{i-1}^2 f(x_i) \\
&= \pi f(x_i) \left(t_i^2 - t_{i-1}^2 \right) \\
&= \pi f(x_i) (t_i - t_{i-1})(t_i + t_{i-1}) \\
&= \pi f(x_i) \, \Delta x_i \, (t_i + t_{i-1}) \\
&\approx 2\pi \, x_i \, f(x_i) \, \Delta x_i
\end{aligned}$$

since $t_i + t_{i-1} \approx 2 x_i$.

63.

Put V = total volume of revolution about y-axis.

Then

$$V \approx \sum_{i=1}^{n} 2\pi \, x_i \, f(x_i) \, \Delta x_i$$

Passing to the limit :

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \, x_i \, f(x_i) \, \Delta x_i$$

that is,

$$\boxed{V = \int_a^b 2\pi x f(x) \, dx} \; .$$

64.

<u>Length of a curve</u>

Suppose

$$x = x(t)$$
$$y = y(t)$$

are differentiable functions of t for

$$a \le t \le b \; .$$

Then

$$\{ (x, y) \mid a \le t \le b \}$$

is a subset of the plane forming a (<u>parametrized</u>) curve $\mathcal{C}$ (with parameter t).

$$\boxed{\text{What is the length of } \mathcal{C} \text{ ?}}$$

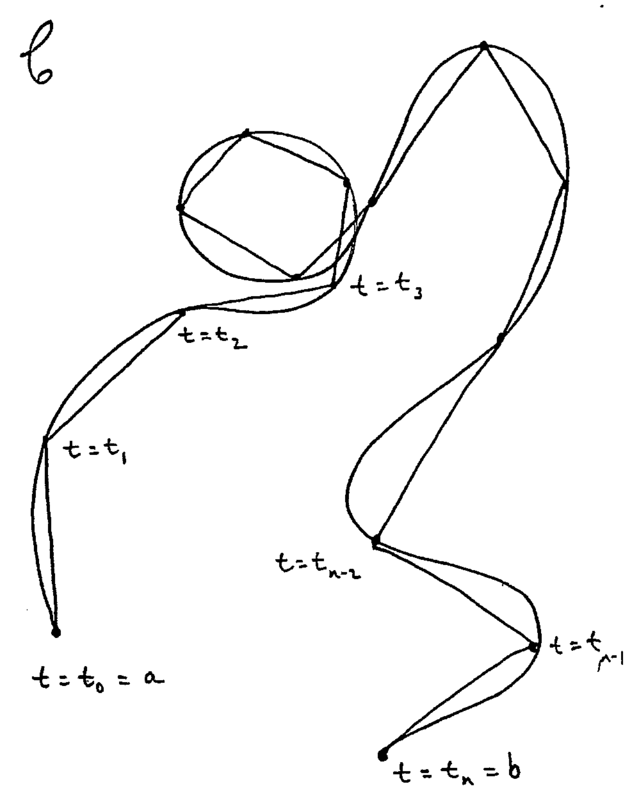
$$\boxed{\text{length of } \mathscr{C} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$

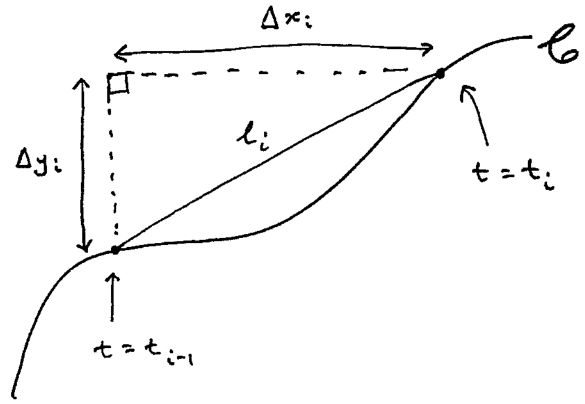
Where does this formula come from?

Answer:

(1) Approximate the curve by straight line segments

(2) See what happens in the limit.





By Pythagoras

$$\ell_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

Put

$$\ell = \text{length of curve } \mathscr{C}$$

so

$$\ell \approx \sum_{i=1}^{n} \ell_i$$

Hence

$$\ell \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{\frac{(\Delta x_i)^2 + (\Delta y_i)^2}{(\Delta t_i)^2}(\Delta t_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\ \Delta t_i.$$

But

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = x'(t)$$

and

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt} = y'(t).$$

Then

$$\ell = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\ \Delta t_i$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

as required.

Important special case:

If $x = t$, $y = f(x)$ then the length of $\mathcal{C}$ is

$$\ell = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$$

Work

Here we only consider forces acting along a straight line.

If a constant force F moves a body d units then the work done is defined to be

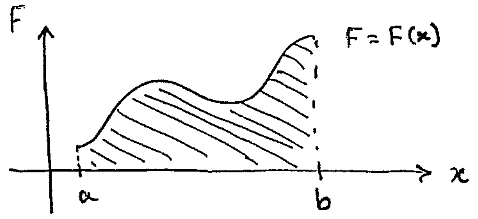
$$W = Fd$$

What happens if the force is allowed to vary?

Suppose a force

$$F = F(x)$$

(which may vary) acts on a particle as it moves along the x-axis from $x = a$ to $x = b$.

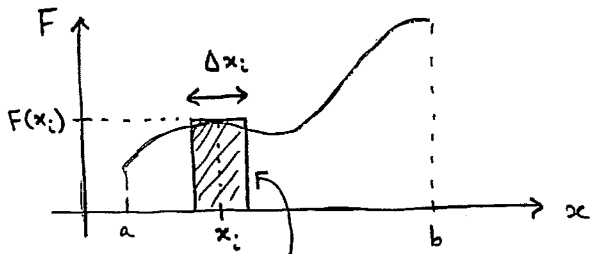


Define the work done by F to be $W = \int_a^b F(x)\,dx$.

Rationale:

$$W = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} F(x_i)\,\Delta x_i}_{\text{usual Riemann sum approximation}}$$

where we think of the force as approximately constant on subintervals.



area $= F(x_i)\,\Delta x_i$
= work by constant force.

__Example__: How much work lifts a mass of 2 kg a distance 5 m ?

__Solution__: Assume constant force

$$F = 2g$$

where $g = 9.8$ newtons (the weight of 1 kg),

so work required is

$$W = F(5) = 10g = 98 \text{ joules}$$

(joule = newton-metre).

(It takes $\approx$ 1 joule to lift an apple 1 m.)

__Example__: (putting a satellite into orbit)

$$\boxed{\text{A satellite of mass } m \text{ kg experiences a force of } F(x) = \frac{mMG}{x^2} \text{ newtons}}$$

where

x = distance to centre of earth

$M = 5.975 \times 10^{24}$ kg (mass of the earth)

$G = 6.6720 \times 10^{-11}$ $N \cdot m^2 / kg^2$ (the __universal gravitation constant__).

$$\boxed{\text{radius of earth} = 6,370,000 \text{ m}}$$

$$\left(\text{so} \quad g = \frac{MG}{(6,370,000)^2} = 9.8 \text{ N} \right).$$

$$\boxed{\text{How much work lifts a 1,000 kg satellite into orbit 10,000 km above earth's surface ?}}$$

__Answer__:

$$W = \int_{6,370,000}^{16,370,000} \frac{1000\, MG}{x^2}\, dx$$

$$= 1000\, MG \left[-\frac{1}{x} \right]_{6,370,000}^{16,370,000}$$

$$= 1000\, MG \left(\frac{-1}{16,370,000} + \frac{1}{6,370,000} \right)$$

$$= 3.823 \times 10^{10} \text{ joules}.$$