THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education, Engineering and Science

MATH1902
Linear Algebra (Advanced)

June 2004

Time Allowed: One and a half hours

Lecturers: R Howlett, DJ Ivers, N Joshi

This examination has three printed components:

(1) An Extended Answer Question Paper (this booklet, Green 80/16A), 4 pages numbered 1 to 4, 5 questions numbered 1 to 5;

(2) A Multiple Choice Question Paper (Yellow 80/16B), 3 pages numbered 1 to 3, 15 questions numbered 1 to 15;

(3) A Multiple Choice Answer Sheet (White 80/16C), 1 page.

Components 2 and 3 MUST NOT be removed from the examination room.

This examination has two sections: **Extended Answer** and **Multiple Choice**.

The **Extended Answer Section** is worth 75% of the total marks for the paper: all questions may be attempted; questions are of equal value; working must be shown.

The **Multiple Choice Section** is worth 25% of the total marks for the paper: all questions may be attempted; questions are of equal value; answers must be coded onto the **Multiple Choice Answer Sheet**.

Calculators will be supplied; no other electronic calculators are permitted.

1. (*i*) **(6 marks).** Let $\mathbf{u} = 2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} + 4\,\mathbf{k}$.

(a) Find $|\mathbf{u}|$ and $\mathbf{u} \cdot \mathbf{v}$.

(b) Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

(c) Find the vector that is the projection of $\mathbf{v}$ in the direction of $\mathbf{u}$. Hence express $\mathbf{v}$ in the form $\mathbf{a} + \mathbf{b}$, where $\mathbf{a}$ is parallel to $\mathbf{u}$ and $\mathbf{b}$ is perpendicular to $\mathbf{u}$.

(*ii*) **(9 marks)** Let ℓ be the line given by the equations $\dfrac{x-1}{2} = \dfrac{y+2}{-3} = \dfrac{z-3}{4}$.

(a) Find the coordinates of a point A on ℓ and find a vector $\mathbf{u}$ that is parallel to ℓ.

(b) Suppose that a plane $\mathscr{P}$ contains the line ℓ and the point $B(3, -2, 1)$. Find the vector $\mathbf{v} = \overrightarrow{AB}$, and then find the Cartesian equation of the plane $\mathscr{P}$.

(c) Write down a system of two equations in the three unknowns a, b, c whose solutions give all planes $ax + by + cz = 1$ in which ℓ lies, and solve the system.

2. (*i*) **(9 marks).** Let $\mathbf{a}$ and $\mathbf{b}$ be vectors, and let $\mathbf{c} = \mathbf{a} + \mathbf{b}$. Let a, b and c be the lengths of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(a) Show that $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = a^2 b^2$.

(b) By expanding $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$, show that $2\mathbf{a} \cdot \mathbf{b} = c^2 - a^2 - b^2$.

(c) Deduce that the area of a triangle with sides of length a, b, c is given by $\frac{1}{4}\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$.

(*ii*) **(6 marks).** Suppose that the origin, O, is a vertex of the parallelepiped $\mathscr{P}$, and let A, B and C be the vertices of $\mathscr{P}$ that are adjacent to O (so that OA, OB and OC are edges of $\mathscr{P}$). Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of A, B and C relative to the origin.

(a) Let A', B', C' and O' be the vertices of $\mathscr{P}$ that are diagonally opposite to A, B, C and O (respectively). Express the position vectors of A', B', C' and O' (relative to the origin) in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and hence find expressions for $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ and $\overrightarrow{OO'}$.

(b) Show that the four diagonals of $\mathscr{P}$ all bisect each other.

3. (*i*) **(5 marks).** Consider the following system of linear equations in the unknowns x, y and z, with parameter p:

$$\begin{aligned} x + (p+2)y + pz &= p + 1 \\ y + (2-p)z &= 1 \\ 2x + 2py + (p^2 + 8p - 23)z &= 3p + 3. \end{aligned}$$

(a) Find the values of p (if any) for which the system has no solution.

(b) Find the values of p (if any) for which there is a unique solution,

(c) Find the general solution whenever there is more than one solution.

(*ii*) **(2 marks)**. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 2 & 2 & -2 \end{pmatrix}$.

(*iii*) **(2 marks)**. Express the matrix A in Part (*ii*) as a product of elementary matrices.

(*iv*) **(6 marks)**. Answer true or false to each of the following, giving a counterexample when the statement is false.

(a) A system of r linear equations in n unknowns has an infinite number of solutions if $r < n$.

(b) A homogeneous system of r linear equations in n unknowns is inconsistent if the number of equations, n, exceeds the number of unknowns, r.

(c) If A and B are 2×2 matrices such that $AB = B$ then either A is the identity matrix or B is the zero matrix.

4. (*i*) **(4 marks)**. Consider the matrix $A = \begin{pmatrix} -3 & 4 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A.

(c) Find the eigenspace of the positive eigenvalue of A.

(*ii*) **(11 marks)**. Let n be a positive integer, and define the function

$$f_n(x_1, x_2, \ldots, x_n) = \det \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & x_3 & \ldots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \ldots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \ldots & x_n^{n-1} \end{pmatrix}.$$

(a) In the case $n = 3$, show that $f_3(x_1, x_2, x_3) = (x_3 - x_2)(x_2 - x_1)(x_3 - x_1)$.

(b) By considering the first column expansion of the determinant, show that

$$f_n(x_1, x_2, \ldots, x_n) = g_0 + g_1 x_1 + g_2 x_1^2 + \cdots + g_{n-1} x_1^{n-1}$$

where each g_i is a function of x_2, x_3, $\ldots$, x_n (not involving x_1), and, in particular, $g_{n-1} = (-1)^{n-1} f_{n-1}(x_2, x_3, \ldots, x_n)$.

(c) Show that the polynomial $g_0 + g_1 x_1 + g_2 x_1^2 + \cdots + g_{n-1} x_1^{n-1}$ appearing in Part (b) has $x_i - x_1$ as a factor, for all values of i from 2 to n.

(d) Evaluate $f(x_1, x_2, \ldots, x_n)$.

5. (*i*) **(4 marks)**. Find all matrices $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $X^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(ii) **(5 marks).**

(a) Show that

$$\begin{pmatrix} x & -1 & 0 & \dots & 0 & 0 \\ 0 & x & -1 & \dots & 0 & 0 \\ 0 & 0 & x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x & -1 \\ c_0 & c_1 & c_2 & \dots & c_{n-2} & x+c_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-2} \\ x^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ p(x) \end{pmatrix},$$

where $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1x + c_0$.

(b) Let $v(\lambda)$ be the following $n \times 1$ column vector:

$$v(\lambda) = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{pmatrix}.$$

Show that if λ is a zero of the polynomial $p(x)$ in Part (a), then $v(\lambda)$ is an eigenvector of the matrix M below, and find the corresponding eigenvalue.

$$M = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \dots & -c_{n-2} & -c_{n-1} \end{pmatrix}.$$

(iii) **(6 marks).** Let A be an $n \times n$ matrix, and let B be the adjoint matrix of $xI - A$.

(a) Show that each entry b_{ij} of B is a polynomial in x of degree at most $n-1$, and deduce that B has the form $B = x^{n-1}B_{n-1} + x^{n-2}B_{n-2} + \cdots + xB_1 + B_0$, where $B_0, B_1, \dots, B_{n-1}$ are $n \times n$ matrices independent of x.

(b) Let $f(x) = \det(xI - A) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$. Use the fact that $(xI - A)B = f(x)I$ for all values of x to show that

$$\begin{aligned} B_{n-1} &= I, \\ B_{n-2} - AB_{n-1} &= a_{n-1}I, \\ B_{n-3} - AB_{n-2} &= a_{n-2}I, \\ &\vdots \\ B_0 - AB_1 &= a_1I, \\ -AB_0 &= a_0I. \end{aligned}$$

(c) Deduce that $A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1A + a_0I = 0$.

THIS IS THE LAST PAGE OF THE EXTENDED ANSWER QUESTION PAPER.