(A)

MATH1903 Lecture 11 Thurs 7/9/2017

Representing functions by power series

reference: pp 2.69 – 2.94

- polynomials that "go on forever"

$$\boxed{\begin{aligned}\sum a_k x^k &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \\ &= \lim_{n\to\infty} (a_0 + a_1 x + \cdots + a_n x^n)\end{aligned}}$$

issues about convergence

- technique invented by

Colin Maclaurin (1698 – 1746) — Scottish
Brook Taylor (1685 – 1731) — English

Maclaurin & Taylor series

Taylor's Theorem (1st semester) underlies convergence:

$$f(x) = \underbrace{T_n(x)}_{} + \underbrace{R_n(x)}_{}$$

Taylor polynomial

remainder term (which can often be controlled)

B

$\boxed{\sum a_k x^k \text{ is called a } \underline{\text{Maclaurin series}}}$

$$\boxed{\begin{aligned}\sum a_k(x-c)^k &= a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots \\ &\quad + a_n(x-c)^n + \cdots \\ &\text{is called a } \underline{\text{Taylor series}} \text{ about } x = c\end{aligned}}$$

(so a Maclaurin series is a Taylor series about $x=0$).

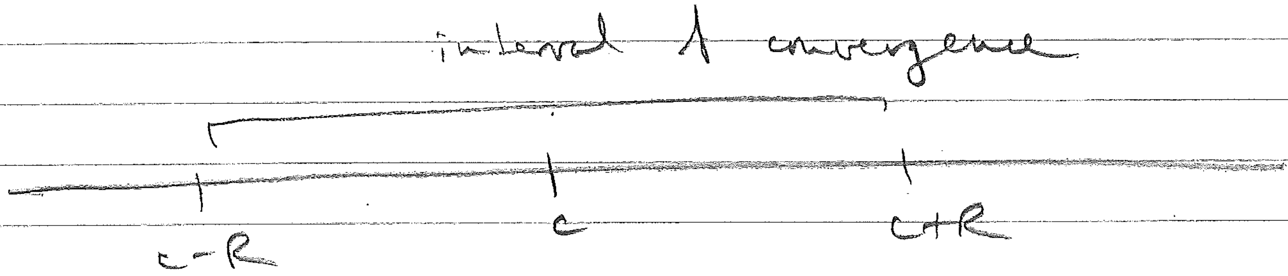
Theorem: All Taylor series $\sum a_k(x-c)^k$ have a radius of convergence R, by which we mean the series converges for all $x \in \mathbb{R}$ such that

$$|x-c| < R,$$

i.e. $c-R < x < c+R$.

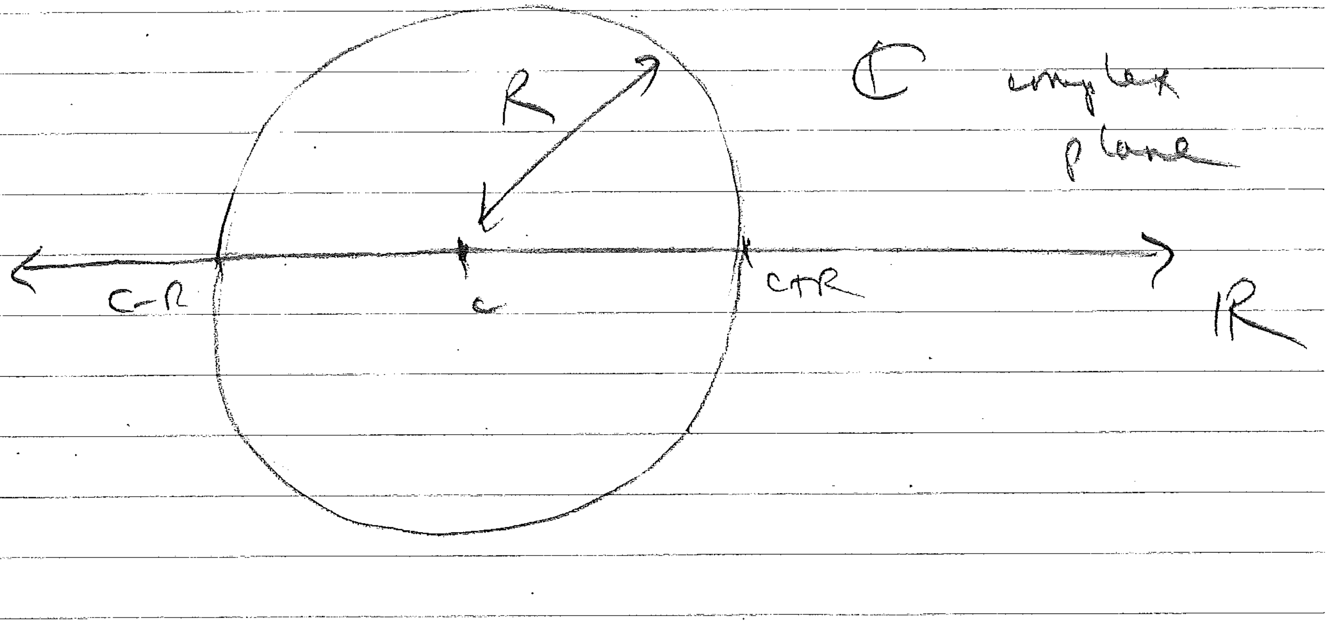
On this interval of convergence, the series defines a differentiable function with derivative having the same radius of convergence.

c



Note: any sort of behaviour can take place on the boundary i.e. when $x = c-R$ or $x = c+R$.

The theorem holds more generally for complex functions yielding a circle of convergence (on the interior), which reduces to an interval if $c \in \mathbb{R}$ and we specialise to real functions:



Recall (from last week):

Ratio Test for convergence: Consider the series

$$\sum_{k=0}^{\infty} b_k = b_0 + b_1 + \cdots + b_n + \cdots$$

Suppose

$$L = \lim_{k \to \infty} \left| \frac{b_{k+1}}{b_k} \right|$$

exists. Then

$$\sum b_k \begin{cases} \text{converges if } L < 1 \\ \text{diverges if } L > 1 \end{cases}$$

(The Ratio Test gives no information if $L = 1$.)

Example: Let $b_k = \frac{1}{k+1}$, so

$$\sum b_k = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is the harmonic series. Then

$$L = \lim_{k \to \infty} \left| \frac{b_{k+1}}{b_k} \right| = \lim_{k \to \infty} \frac{1/k+2}{1/k+1}$$

$$= \lim_{k \to \infty} \frac{k+1}{k+2} = \lim_{k \to \infty} \frac{1 + 1/k}{1 + 2/k} = 1,$$

so the Ratio Test is inconclusive.

In fact, the harmonic series diverges (last week).

If we put

$$b_k = (-1)^k \frac{1}{k+1}$$

then we get the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

and again $L = \lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = 1$,

but this time we have convergence:

$$\boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2}$$

difficult exercise

<u>Example</u>: Let $b_k = x^k$, where $x \in \mathbb{R}$, so

$$\sum b_k = 1 + x + x^2 + \cdots + x^k + \cdots$$

is the geometric series, and

$$L = \lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\right| = \lim_{k\to\infty}|x|.$$

Then $L<1$ iff $|x|<1$, so we get the expected result that the geometric series converges for $|x|<1$.

Example: let $b_k = \frac{x^k}{k!}$, where $x \in \mathbb{R}$, so

$$\sum b_k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

and $$L = \lim_{k \to \infty} \left| \frac{b_{k+1}}{b_k} \right| = \dots = 0 \quad \text{(last week)}$$

so $f(x) = 1 + x + \frac{x^2}{2!} + \dots$ $\underbrace{\text{converges for all } x}$,

and $f(x) = e^x$,

last week

radius of convergence is ∞

Starting with a given differentiable function

$$y = f(x),$$

how does one find a power series that represents it?

$$\boxed{\text{Answer: } f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \quad \text{for all } x \text{ within the interval of convergence about } c}$$

have flexibility to choose c

Reason : for simplicity suppose

$$\boxed{f(x) = \sum_{k=0}^{\infty} a_k x^k}$$

series about $x=0$, and we want to find a_k ?

pp 2·81, 2·82, 2·83 yielding the Maclaurin series

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k}$$

p 2·83 : series for e^x

pp 2·84 – 2·88 : series for $\sin x$, $\cos x$, $\sinh x$, $\cosh x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

– By Ratio Test these converge for all x.

(H)

What about $\ln x$? Observe, for $x > 0$,

$$\ln' x = \frac{1}{x} = \frac{1}{1-(1-x)}$$

$$= 1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots$$

geometric series with radius of convergence 1.

Hence $\ln x = \int \frac{dx}{x} = x + \frac{(1-x)^2}{2}(-1) + \frac{(1-x)^3(-1)}{3} + \frac{(1-x)^4(-1)}{4}$

$$+ \cdots + C$$

$$= C + x - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \frac{(1-x)^4}{4} - \cdots$$

But $0 = \ln(1) = C + 1 - 0 - 0 - 0 - \cdots = C + 1$,

so $C = -1$. Hence

$$\ln x = -1 + x - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \frac{(1-x)^4}{4} - \cdots$$

$$= x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots,$$

a Taylor series expansion about $x = 1$, and the radius of convergence of 1 is preserved (general theory).

On the boundary: $x = 0$: $\ln x = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$ diverges (negative harmonic series)

$x = 2$: $\ln x = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges (alternating harmonic series)

$(= \ln 2)$. (starred exercise)