THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial for Week 4

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2012

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Topics covered

In lectures last week:

- ☐ Area, volume, length, and surface area.

Objectives

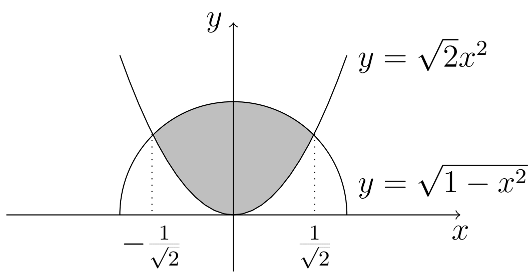
After completing this tutorial sheet you will be able to:

- ☐ Use integration to compute areas.
- ☐ Apply and adapt the disc and shell methods to find volumes.
- ☐ Use hyperbolic changes of variables to compute integrals.
- ☐ Compute lengths of curves.
- ☐ Compute surface areas of solids of revolution.

Preparation questions to do *before* class

1. Sketch the region bounded by the curves $y = \sqrt{1-x^2}$ and $y = \sqrt{2}x^2$. Compute the area of this region.

Solution: A sketch is



The curves intersect when $1 - x^2 = 2x^4$, and so $2(x^2)^2 + (x^2) - 1 = 0$, and so $x^2 = -1, \frac{1}{2}$, and so $x = \pm 1/\sqrt{2}$. Therefore the area is

$$A = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\sqrt{1-x^2} - \sqrt{2}x^2\right) dx = 2\int_0^{1/\sqrt{2}} \sqrt{1-x^2}\, dx - \frac{1}{3}.$$

Making the change of variable $x = \sin\theta$ gives

$$\int_0^{1/\sqrt{2}} \sqrt{1-x^2}\, dx = \int_0^{\pi/4} \cos^2\theta\, d\theta.$$

But $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$, and so

$$\int_0^{1/\sqrt{2}} \sqrt{1-x^2}\,dx = \frac{1}{2}\int_0^{\pi/4} (1+\cos(2\theta))\,d\theta = \frac{\pi}{8}+\frac{1}{4}.$$

Therefore the area is

$$A = 2\left(\frac{\pi}{8}+\frac{1}{4}\right)-\frac{1}{3} = \frac{\pi}{4}+\frac{1}{6}.$$

2. Let S be the sphere of radius R.

(a) Calculate the volume of S using the disc method.

Solution: The volume is twice the volume of the solid obtained by rotating the curve $y=\sqrt{R^2-x^2}$ with $0\le x\le R$ about the x-axis, and therefore

$$V = 2\times\pi\int_0^R (\sqrt{R^2-x^2})^2\,dx = 2\pi\int_0^R (R^2-x^2)\,dx = \frac{4}{3}\pi R^3.$$

(b) Calculate the volume of S using the shell method.

Solution: The volume is twice the volume of the solid obtained by rotating the curve $y=\sqrt{R^2-x^2}$, $0\le x\le R$, around the y-axis. Hence

$$V = 2\times 2\pi\int_0^R x\sqrt{R^2-x^2}\,dx = \frac{4}{3}\pi R^3.$$

(c) Find the surface area of S.

Solution: The surface area is twice the curved surface area of the solid obtained by rotating $y=\sqrt{R^2-x^2}$, $0\le x\le R$, about the x-axis. Hence the surface area of the sphere is

$$S = 2\times 2\pi\int_0^R \sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx = 4\pi\int_0^R R\,dx = 4\pi R^2.$$

Questions to attempt in class

3. Let D be the region bounded by the x-axis, the y-axis, $x=1$, and $y=\cosh x$.

(a) Find the perimeter of D.

Solution: The region D has 4 sides, 3 of which are straight line segments, and one which is the part of the graph $y=\cosh x$ between $x=0$ and $x=1$. Thus the perimeter is

$$P = \underbrace{1+1+\cosh(1)}_{\text{straight line segments}} + \int_0^1 \sqrt{1+\sinh^2 x}\,dx = 2+\cosh(1)+\sinh(1) = 2+e.$$

(b) Find the area of D.

Solution: The area of D is $A = \displaystyle\int_0^1 \cosh x\,dx = \sinh(1).$

(c) Find the volume of the solid obtained by rotating D about the x-axis.

Solution: By the disc method,

$$V = \pi \int_0^1 \cosh^2 x \, dx = \frac{\pi}{2} \int_0^1 (1 + \cosh(2x)) \, dx = \frac{\pi}{2} + \frac{\pi}{4} \sinh(2).$$

(d) Find the volume of the solid obtained by rotating D about the y-axis.

Solution: By the shell method, the volume is

$$\begin{aligned} V &= 2\pi \int_0^1 x \cosh x \, dx \\ &= 2\pi \sinh(1) - 2\pi \int_0^1 \sinh x \, dx \\ &= 2\pi(\sinh(1) - \cosh(1) + 1) = 2\pi(1 - e^{-1}). \end{aligned}$$

(e) Find the surface area of the solid obtained by rotating D about the x-axis.

Solution: The curved surface area is

$$2\pi \int_0^1 f(x)\sqrt{1 + f'(x)^2} \, dx = 2\pi \int_0^1 \cosh^2 x \, dx = \pi + \frac{\pi}{2} \sinh(2).$$

Therefore the total surface area (including the two end caps) is

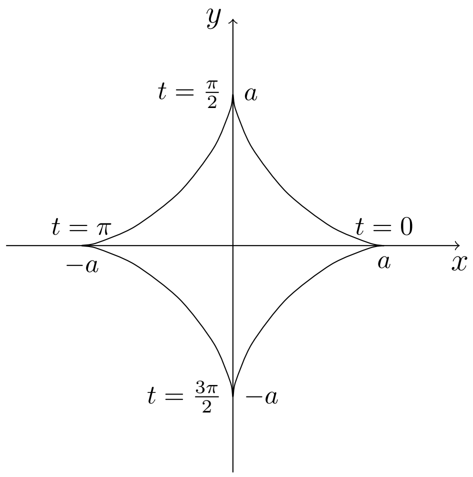
$$A = \pi + \frac{\pi}{2} \sinh(2) + \pi + \pi \cosh^2(1) = \frac{\pi}{2}(e^2 + 3).$$

4. Let $a > 0$ be a constant. Sketch the *hypocycloid*

$$x(t) = a \cos^3 t, \qquad y(t) = a \sin^3 t, \qquad t \in [0, 2\pi],$$

and compute its circumference.

Solution: Here is a sketch:



Using the formula for the length of a parametrised curve, and using the obvious symmetry, the length is

$$\begin{aligned}
L &= 4\int_0^{\pi/2} \sqrt{(-3a\sin t\cos^2 t)^2 + (3a\cos t\sin^2 t)^2}\,dt \\
&= 12a\int_0^{\pi/2} \sqrt{\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)}\,dt \\
&= 6a\int_0^{\pi/2} \sqrt{\sin^2(2t)}\,dt \\
&= 6a\int_0^{\pi/2} |\sin(2t)|\,dt \\
&= 6a\int_0^{\pi/2} \sin(2t)\,dt = 6a.
\end{aligned}$$

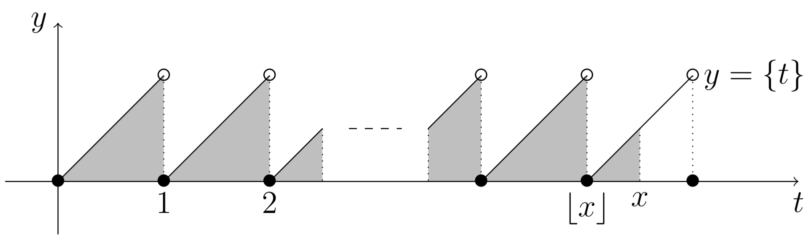
Note that if we integrate over $[0, 2\pi]$ then we need to be rather careful with the absolute value sign:

$$\begin{aligned}
L &= \int_0^{2\pi} |\sin(2t)|\,dt \\
&= \int_0^{\pi/2} \sin(2t)\,dt - \int_{\pi/2}^{\pi} \sin(2t)\,dt + \int_{\pi}^{3\pi/2} \sin(2t)\,dt - \int_{3\pi/2}^{2\pi} \sin(2t)\,dt.
\end{aligned}$$

5. Let $\lfloor x\rfloor$ denote the *integer part of* $x \in \mathbb{R}$. That is: $\lfloor x\rfloor \in \mathbb{Z}$ is the largest integer with the property that $\lfloor x\rfloor \le x$. Let $\{x\} \in [0, 1)$ be the *fractional part of* $x \in \mathbb{R}$. That is, $\{x\} = x - \lfloor x\rfloor$. The function $f(x) = \{x\}$ has a jump discontinuity at each integer. Let $x \ge 0$. Compute the integral

$$\int_0^x \{t\}\,dt \qquad \text{by thinking about areas.}$$

Solution: Here is a sketch:



There are $\lfloor x\rfloor$ triangles with area $\frac{1}{2}$, and 1 triangle with area $\frac{1}{2}\{x\}^2$. Therefore

$$\int_0^x \{t\}\,dt = \frac{1}{2}\lfloor x\rfloor + \frac{1}{2}\{x\}^2.$$

6. Compute the length of the graph $y = \ln x$ for $0 < a \le x \le b$.

Solution: The required length is

$$L = \int_a^b \sqrt{1 + \frac{1}{x^2}}dx = \int_a^b \frac{\sqrt{1+x^2}}{x}\,dx.$$

Setting $x = \sinh t$ gives

$$L = \int_{\sinh^{-1}(a)}^{\sinh^{-1}(b)} \frac{\cosh^2 t}{\sinh t}\,dt = \int_{\sinh^{-1}(a)}^{\sinh^{-1}(b)} \frac{1}{\sinh t}\,dt + \int_{\sinh^{-1}(a)}^{\sinh^{-1}(b)} \sinh t\,dt.$$

The second integral equals

$$\cosh(\sinh^{-1}(b)) - \cosh(\sinh^{-1}(a)) = \sqrt{1+b^2} - \sqrt{1+a^2},$$

and setting $u = \cosh t$ gives

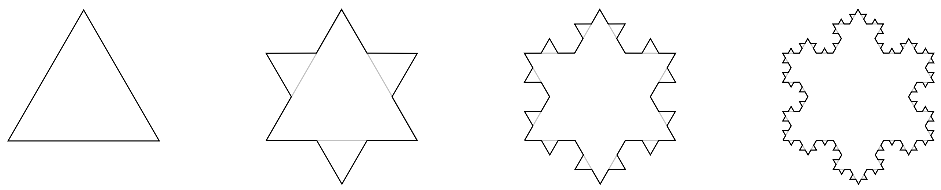
$$\int_{\sinh^{-1}(a)}^{\sinh^{-1}(b)} \frac{1}{\sinh t}\,dt = \int_{\sqrt{1+a^2}}^{\sqrt{1+b^2}} \frac{1}{u^2-1}du = \frac{1}{2}\ln\frac{(\sqrt{1+b^2}-1)(\sqrt{1+a^2}+1)}{(\sqrt{1+b^2}+1)(\sqrt{1+a^2}-1)}.$$

Thus

$$\begin{aligned} L &= \sqrt{1+b^2} - \sqrt{1+a^2} + \frac{1}{2}\ln\frac{(\sqrt{1+b^2}-1)(\sqrt{1+a^2}+1)}{(\sqrt{1+b^2}+1)(\sqrt{1+a^2}-1)} \\ &= \sqrt{1+b^2} - \sqrt{1+a^2} + \ln\frac{a(1+\sqrt{1+a^2})}{b(1+\sqrt{1+b^2})}. \end{aligned}$$

Discussion question

7. The *Koch snowflake* is the curve constructed inductively according to the following picture, where the initial equilateral triangle has side length 1.



At each stage of the construction, each line segment is divided into 3 equal parts and an equilateral triangle is placed on the middle third. The snowflake curve is the "limit curve" of this procedure. What is the area enclosed by the Koch curve? What is the length of the Koch curve? Is this a continuous curve? Do you think it is differentiable?

Solution: Let the sequence of curves in the construction of the snowflake curve be $C_0, C_1, C_2, \ldots$, so that C_0 is the initial equilateral triangle with side length 1. Let A_k be the area enclosed by C_k. We will explicitly compute A_k below, and then take the limit as $k \to \infty$. Before doing this, let us 'abstractly' explain why $\lim_{k\to\infty} A_k$ exists and is finite. This is a nice application of the *Monotone Convergence Theorem* (which we will talk about a few times in lectures). Two things are fairly clear:

(1) The sequence $A_0, A_1, A_2, \ldots$ is monotone increasing: $A_0 \le A_1 \le A_2 \le \cdots$.

(2) The sequence $A_0, A_1, A_2, \ldots$ is bounded above. That is, there is a number $M > 0$ such that $A_k \le M$ for all k. Indeed the curve C_k sits completely inside of a circle of radius $1/\sqrt{3}$, and therefore $A_k \le \pi/3$ for all k.

Now the Monotone Convergence Theorem says that $\lim_{k\to\infty} A_k$ exists, and is at most $\pi/3$. We can be more explicit – we will now compute A_k and the limit.

We first compute the number of straight line segments on C_k. We have the recursion formula

$$(\text{number of line segments on } C_k) = 4 \times (\text{number of line segments on } C_{k-1}),$$

because in the construction of C_k from C_{k-1} each segment of C_{k-1} is broken into 3 segments, and when the new equilateral triangle is placed on the middle segment we cover up the middle segment but add two new segments from the other two sides of the smaller triangle. Thus each straight line segment of C_{k-1} produces 4 segments in C_k. Hence

$$(\#\text{ of segments on } C_k) = 4^k \times (\#\text{ of segments on } C_0) = 3 \times 4^k.$$

We can now derive a recursive formula for the area A_k enclosed by C_k. We have:

$$\begin{aligned} A_k &= A_{k-1} + (\text{area of new triangles}) \\ &= A_{k-1} + (\text{number of new triangles}) \times (\text{area of new triangle}). \end{aligned}$$

The side length of the new triangles is 3^{-k}, and so each new triangle has area

$$\frac{1}{2} 3^{-2k} \sin\frac{\pi}{3} = \frac{\sqrt{3}}{4} 3^{-2k},$$

and the number of new triangles equals the number of line segments on C_{k-1}, which equals $3 \times 4^{k-1}$. Therefore

$$A_k = A_{k-1} + 3 \times 4^{k-1} \times \frac{\sqrt{3}}{4} \times 3^{-2k} = A_{k-1} + \frac{\sqrt{3}}{12}\left(\frac{4}{9}\right)^{k-1}.$$

Repeated use of this recurrence gives

$$A_k = A_0 + \frac{\sqrt{3}}{12}\left(\left(\frac{4}{9}\right)^{k-1} + \left(\frac{4}{9}\right)^{k-2} + \cdots + 1\right),$$

and using the geometric sum formula we get

$$A_k = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20}\left(1 - \left(\frac{4}{9}\right)^k\right) = \frac{2\sqrt{3}}{5} - \frac{3\sqrt{3}}{20}\left(\frac{4}{9}\right)^k.$$

The area A enclosed by the snowflake curve is given by taking the limit as $k \to \infty$, and thus

$$A = \lim_{k\to\infty}\left(\frac{2\sqrt{3}}{5} - \frac{3\sqrt{3}}{20}\left(\frac{4}{9}\right)^k\right) = \frac{2\sqrt{3}}{5}.$$

Now consider the length L_k of the curve C_k. Again we see that the sequence $L_0, L_1, L_2, \ldots$ is monotone increasing. But this time it is not clear that L_k is bounded from above, and so we can't apply the Monotone Convergence Theorem. Let's try to make the calculations directly. The length of the curve C_k is

$$L_k = (\text{number of line segments on } C_k) \times (\text{length of each segment}).$$

The number of line segments is 3×4^k, and the length of each segment is 3^{-k}. Therefore

$$L_k = 3\left(\frac{4}{3}\right)^k,$$

which tends to infinity as $k \to \infty$. So the Koch curve has infinite perimeter, even though it encloses a finite area! (In particular, L_k is **not** bounded from above).

The Koch snowflake curve is continuous everywhere but it is differentiable nowhere. These facts are not trivial to prove – they are things you might want to revisit later in your mathematical lives.

Questions for extra practice

8. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the x-axis, the lines $x = a$ and $x = b$, and the curve $y = \sqrt{1 + x^2}$, $a \leq x \leq b$, where $a \geq 0$.

Solution: We use the shell method. From lectures, the formula for volume is

$$2\pi \int_a^b x f(x)\, dx \;=\; 2\pi \int_a^b x\sqrt{1+x^2}\, dx.$$

Making the substitution $u = 1 + x^2$, we get $\sqrt{1+x^2} = \sqrt{u}$ and $du = 2x\, dx$, and so

$$L \;=\; \pi \int_{1+a^2}^{1+b^2} u^{1/2}\, du \;=\; \left[\frac{2\pi}{3} u^{3/2}\right]_{1+a^2}^{1+b^2} \;=\; \frac{2\pi}{3}\Big\{(1+b^2)^{3/2} - (1+a^2)^{3/2}\Big\}.$$

9. Let T be the solid torus obtained by rotating the circle of centre $(R, 0)$ and radius r about the y-axis (assume that $r \leq R$).

(a) Find the volume of T.

Solution: The circle of radius r and centre $(R, 0)$ is given by the equation $(x - R)^2 + y^2 = r^2$. So the top half of the torus is obtained by rotating the curve $y = \sqrt{r^2 - (x - R)^2}$, $R - r \leq x \leq R + r$, about the y-axis. Hence, using cylindrical shells,

$$\tfrac{1}{2}V \;=\; 2\pi \int_{R-r}^{R+r} x\sqrt{r^2 - (x-R)^2}\, dx \;=\; 2\pi \int_{-r}^{r} (u + R)\sqrt{r^2 - u^2}\, du,$$

where we have made the substitution $u = x - R$, so that $du = dx$. Break the last integral into

$$2\pi \int_{-r}^{r} (u+R)\sqrt{r^2 - u^2}\, du \;=\; 2\pi \int_{-r}^{r} u\sqrt{r^2 - u^2}\, du + 2\pi R \int_{-r}^{r} \sqrt{r^2 - u^2}\, du.$$

The first integral on the right is zero because $u\sqrt{r^2-u^2}$ is an odd function, and we are integrating over an interval which is symmetric with respect to the origin. Hence the volume of the torus is

$$V = 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du = 4\pi R\Big(\frac{\pi r^2}{2}\Big) = 2\pi^2 Rr^2.$$

One can see that this is a plausible answer by imagining the torus straightened out into a cylinder. This cylinder would have a circular base of area πr^2 and a height of $2\pi R$.

You can also make the calculation using the disc method. The details are left as an exercise.

(b) Find the surface area of T. You might like to reposition the circle at $(0, R)$ and rotate about the x-axis.

Solution: Place the circular cross-section on the y-axis at $(0, R)$ so that its equation is $x^2 + (y-R)^2 = r^2$. Then consider separately the surfaces swept out by rotating the upper semicircle and the lower semicircle about the x-axis. The surface area swept out by the upper semicircle is

$$S_1 = 2\pi\int_{-r}^{r}\left\{\frac{Rr}{\sqrt{r^2-x^2}} + r\right\}dx = 4\pi\left[Rr\sin^{-1}\Big(\frac{x}{r}\Big) + rx\right]_0^r = 2\pi^2 Rr + 4\pi r^2.$$

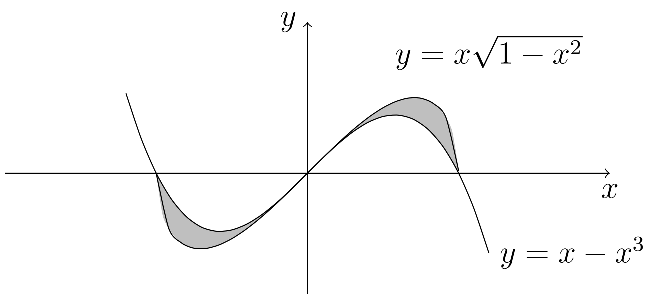
Similarly the surface swept out by the lower semicircle is

$$S_2 = 2\pi\int_{-r}^{r}\left\{\frac{Rr}{\sqrt{r^2-x^2}} - r\right\}dx = 4\pi\left[Rr\sin^{-1}\Big(\frac{x}{r}\Big) - rx\right]_0^r = 2\pi^2 Rr - 4\pi r^2.$$

Therefore the total surface area is $S = S_1 + S_2 = 4\pi^2 Rr$.

10. Sketch the region bounded by the curves $y = x\sqrt{1-x^2}$ and $y = x - x^3$, and find the area of the region. *Note: The area consists of two crescent-shaped pieces.*

Solution: The sketch is as follows (note that the domain of $x\sqrt{1-x^2}$ is $[-1, 1]$):



Both $f(x) = x\sqrt{1-x^2}$ and $g(x) = x - x^3$ are odd functions. So the required area is twice the area between the curves in the first quadrant. For $0 \le x \le 1$, we have

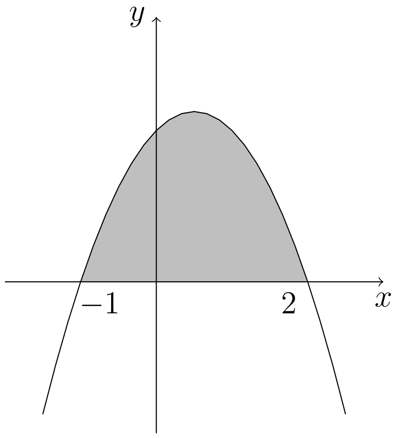
$0 \le x - x^3 = x(1-x^2) \le x\sqrt{1-x^2}$. So the required area is

$$\begin{aligned} A &= 2\int_0^1 (f(x) - g(x))\, dx \\ &= 2\int_0^1 \left(x\sqrt{1-x^2} - (x - x^3) \right) dx \\ &= 2\left(-\frac{1}{3}(1-x^2)^{3/2} - \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 \right) \right) \Bigg|_{x=0}^{x=1} \\ &= \frac{1}{6}. \end{aligned}$$

Note that $\int_{-1}^{1} (f(x) - g(x))\, dx$ gives the wrong answer. Why?

11. Find the interval $[a, b]$ which maximises the value of the integral $\displaystyle\int_a^b (2 + x - x^2)\, dx$.

Solution: We have $2 + x - x^2 = -(x+1)(x-2)$, and so we have the sketch



Thus, interpreting the integral as area (positive if the graph is above the x-axis, and negative if the graph is below the x-axis) it is clear that the interval that maximises the integral is $[-1, 2]$ (giving integral equal to the shaded area). Then the integral equals

$$\int_{-1}^{2} (2 + x - x^2)\, dx = \left(2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Bigg|_{-1}^{2} = \frac{10}{3} - \left(-\frac{7}{6} \right) = \frac{9}{2}.$$

12. Write down two integrals for the length of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (one using the length formula for a graph, and another using the formula for the length of a parametrised curve). Try to compute your integrals.

Solution: The top half of the ellipse if the graph of the function

$$f(x) = b\sqrt{1 - \frac{x^2}{a^2}},$$

and so by the formula for the length of a graph we have

$$L = 4\int_0^a \sqrt{1+f'(x)^2}\,dx = \frac{4}{a}\int_0^a \frac{\sqrt{a^4+(a^2+b^2)x^2}}{\sqrt{a^2-x^2}}\,dx.$$

We can also represent the ellipse parametrically, by $x(t) = a\cos t$, $y(t) = b\sin t$. Then

$$L = 4\int_0^{\pi/2} \sqrt{x'(t)^2+y'(t)^2}\,dt = 4\int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt.$$

Good luck solving these integrals! There appears to be no closed formula in terms of our favourite constants ($\pi, e, \sqrt{2}$, etc).

13. Find the volume of the solid obtained by:

(a) Rotating about the x-axis the region bounded by the curve $y = a\cosh(x/a)$, the x-axis, and the line $x = b$. Here $a, b > 0$.

Solution: Rotating about the x-axis, the volume is found by the disc method to be

$$\begin{aligned} V &= \pi\int_0^b \big(a\cosh(x/a)\big)^2\,dx \\ &= \pi a^2\int_0^b \frac{1+\cosh(2x/a)}{2}\,dx \\ &= \pi a^2\Big[\frac{x}{2}+\frac{a}{4}\sinh(2x/a)\Big]_0^b \\ &= \frac{\pi a^2}{4}\Big(2b + a\sinh(2b/a)\Big). \end{aligned}$$

(b) Rotating about the y-axis the region bounded by the curve $y = x\sqrt{1+x^3}$, the x-axis, and the line $x = 2$.

Solution: By the shell method,

$$V = 2\pi\int_0^2 x^2\sqrt{1+x^3}\,dx = \Big[\frac{4\pi}{9}\big(1+x^3\big)^{3/2}\Big]_0^2 = \frac{4\pi}{9}\big(9^{3/2}-1^{3/2}\big) = \frac{104\pi}{9}.$$

14. Compute the length of:

(a) The curve given by $x = a\cos t$, $y = a\sin t$, $z = bt$ with $0 \le t \le 2\pi$.

Solution: From lectures the length is given by

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt \\ &= \int_0^{2\pi} \sqrt{(-a\sin t)^2+(a\cos t)^2+b^2}\,dt \\ &= \int_0^{2\pi} \sqrt{a^2+b^2}\,dt \\ &= 2\pi\sqrt{a^2+b^2}\,. \end{aligned}$$

(b) The parabola $y = x^2$ between $(0,0)$ and (a, a^2), where $a > 0$.

Solution: From lectures we know that the length is given by

$$L = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^a \sqrt{1 + 4x^2}\, dx.$$

Making the change of variable $x = \frac{1}{2}\sinh t$ gives

$$L = \frac{1}{2}\int_0^{\sinh^{-1}(2a)} \sqrt{1 + \sinh^2 t}\, \cosh t\, dt = \frac{1}{2}\int_0^{\sinh^{-1}(2a)} \cosh^2 t\, dt.$$

The formulas $\cosh^2 t - \sinh^2 t = 1$ and $\cosh^2 t + \sinh^2 t = \cosh(2t)$ imply that $\cosh^2 t = \frac{1}{2}(1 + \cosh(2t))$, and so

$$L = \frac{1}{4}\int_0^{\sinh^{-1}(2a)} (1 + \cosh(2t))\, dt = \frac{1}{4}\sinh^{-1}(2a) + \frac{1}{8}\sinh(2\sinh^{-1}(2a)).$$

Since $\sinh(2y) = 2\sinh y \cosh y = 2\sinh y\sqrt{1 + \sinh^2 y}$ we have

$$L = \frac{1}{4}\sinh^{-1}(2a) + \frac{1}{4}(2a)\sqrt{1 + (2a)^2} = \frac{1}{4}\sinh^{-1}(2a) + \frac{a}{2}\sqrt{1 + 4a^2}.$$

This can be simplified further: If $x = \sinh^{-1} y$ then $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. Rearranging gives $(e^x)^2 - 2y(e^x) - 1 = 0$, and so $e^x = y \pm \sqrt{1 + y^2}$. Since $e^x > 0$ we must take the $+$ sign, and hence $x = \ln(y + \sqrt{1 + y^2})$. It follows that

$$L = \frac{1}{4}\ln(2a + \sqrt{1 + 4a^2}) + \frac{a}{2}\sqrt{1 + 4a^2}.$$

15. Use shells to find the volume of a right circular cone of height h and radius r.

Solution: The line segment joining $(0, h)$ to $(r, 0)$ is part of the line with equation $y = h(1 - x/r)$, and so the volume of the cone is

$$V = 2\pi \int_0^r xh\left(1 - \frac{x}{r}\right) dx = 2\pi h\left[\frac{x^2}{2} - \frac{x^3}{3r}\right]_0^r = \frac{\pi r^2 h}{3}.$$

Challenging questions

16. Find the surface area of the *spheroid* obtained by rotating the half ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $y \geq 0$ and $-a \leq x \leq a$ about the x-axis. Be careful with the two cases $a < b$ and $a > b$.

Solution: Using the formula, the surface area of the spheroid is

$$S = \frac{2\pi b}{a^2}\int_{-a}^{a} \sqrt{a^4 - (a^2 - b^2)x^2}\, dx = \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4 - (a^2 - b^2)x^2}\, dx.$$

To evaluate this integral we need to treat the cases $a > b$ and $a < b$ separately. So suppose first that $a > b$. Then the integral is

$$S = \frac{4\pi b}{R}\int_0^a \sqrt{R^2 - x^2}\, dx, \qquad \text{where} \qquad R = \frac{a^2}{\sqrt{a^2 - b^2}}.$$

The integral is computed by setting $x = R\sin\theta$:

$$\begin{aligned}
\int_0^a \sqrt{R^2 - x^2}\,dx &= R^2 \int_0^{\sin^{-1}(a/R)} \cos^2\theta\,d\theta \\
&= \frac{R^2}{2} \int_0^{\sin^{-1}(a/R)} (1 + \cos 2\theta)\;d\theta \\
&= \frac{R^2}{2} \sin^{-1}\left(\frac{a}{R}\right) + \frac{R^2}{4} \sin\left(2\sin^{-1}(a/R)\right) \\
&= \frac{R^2}{2} \sin^{-1}\left(\frac{a}{R}\right) + \frac{a}{2}\sqrt{R^2 - a^2}.
\end{aligned}$$

Therefore

$$S = 2\pi b R \sin^{-1}\left(\frac{a}{R}\right) + \frac{2\pi ab}{R}\sqrt{R^2 - a^2}.$$

Remembering the formula for R and simplifying gives

$$S = 2\pi b^2 + \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \cos^{-1}\left(\frac{b}{a}\right).$$

If $a < b$ then the integral is

$$S = \frac{4\pi b}{R} \int_0^a \sqrt{R^2 + x^2}\,dx, \qquad \text{where} \qquad R = \frac{a^2}{\sqrt{b^2 - a^2}}.$$

Setting $x = R\sinh\theta$, and making a computation analogous to above, we see that

$$S = 2\pi b R \sinh^{-1}\left(\frac{a}{R}\right) + \frac{2\pi ab}{R}\sqrt{R^2 + a^2},$$

which simplifies to

$$S \;=\; 2\pi b^2 \;+\; \frac{2\pi a^2 b}{\sqrt{b^2 - a^2}} \cosh^{-1}\left(\frac{b}{a}\right).$$

17. A bowl is in the shape of a hemisphere of radius r cm.

(a) If there is water in the bowl with a depth h at the centre of the bowl, what is the volume of this water?

Solution: The bowl can be formed by rotating a suitable circular quadrant about the y-axis. Let the centre be on the y-axis at $(0, r)$. Then the equation of the quadrant is $x^2 + (y - r)^2 = r^2$, the relevant piece running from $(0, 0)$ to (r, r). We wish to use the disc method to calculate volume. Hence we need x expressed as a function of y. The required function is $x = \sqrt{r^2 - (y - r)^2} = \sqrt{2ry - y^2}$ for $0 \le y \le r$. So the volume of the water is the same as the volume obtained by rotating this curve around the y-axis between $y = 0$ and $y = h$. Thus,

$$V \;=\; \pi \int_0^h \left(2ry - y^2\right) dy \;=\; \pi\left(rh^2 - \tfrac{1}{3}h^3\right) \;=\; \tfrac{1}{3}\pi h^2(3r - h).$$

(b) Suppose that water is poured into the bowl at a constant rate of C cubic centimeters per second. At what rate is the water level rising when $h = r/2$?

Solution: By part (a),

$$V = \tfrac{1}{3}\pi h^2(3r - h) = \pi r h^2 - \tfrac{1}{3}\pi h^3.$$

We are given that $dV/dt = C$. So

$$C = \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = (2\pi r h - \pi h^2)\frac{dh}{dt}.$$

Therefore,

$$\frac{dh}{dt} = \frac{C}{\pi h(2r - h)}.$$

When $h = r/2$, the water level is rising at the rate $dh/dt = 4C/(3\pi r^2)$.

18. A *polyhedron* is a closed surface formed by joining a finite number of polygons (faces) edge-to-edge. The polygons need not be regular. Restrict attention to polyhedra that have a well-defined inside and outside. Then the inside together with the boundary forms a solid polyhedron.

(a) Suppose that a particular polyhedron has the property that every face touches a given sphere of radius R tangentially. Prove that the volume V and surface area S of such a polyhedron are related by $V = (1/3)RS$.

Solution: Consider one of the polygonal faces and join every point of that face to the centre of the sphere by a straight line. The solid so formed is a pyramid having the polygon as its base. Since the face touches the sphere, the radius ending at the point of contact is perpendicular to the face. Hence R is the perpendicular height of the pyramid. Let A be the area of the face. Then the volume of the pyramid is $(1/3)AR$.

Next, the totality of such pyramids formed on all the faces fills up the solid polyhedron. All the pyramids have the same perpendicular height, namely, the radius R of the sphere. Hence the total volume is $(1/3)R$ times the total area of the faces. In other words, $V = (1/3)RS$, as required.

(b) By taking a suitable limit, prove that the sphere has the same property and deduce the surface area of the sphere from its volume.

Solution: The sphere itself can be approached arbitrarily closely by polyhedra of the above type in such a way that the number of faces tends to infinity while the largest face (measured by its longest diagonal or edge) tends to infinitesimal size. Since the formula, $V = (1/3)RS$, is exact for all the approximating polyhedra, it holds also for the limiting sphere. Given that $V = (4/3)\pi R^3$, we conclude that $S = 4\pi R^2$.