LSA University of Michigan

ASTRO201: Introduction to Astrophysics

Homework 2

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1. Let the hypothetical star have mass M, and radius R, and density distribution given by,

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right).$$

a) To calculate ρ_c, we first use the formula relating the rate of change of mass $m(r)$ over radius r with density $\rho(r)$,

$$\begin{aligned}
\frac{dm(r)}{dr} &= 4\pi r^2 \rho(r) \\
\therefore dm(r) &= 4\pi r^2 \rho_c \left(1 - \frac{r}{R}\right) dr.
\end{aligned}$$

Integrating from a radius $r = 0$ to a radius of $r = R$,

$$\begin{aligned}
\int_0^R dm(r) &= \int_0^R 4\pi r^2 \rho_c \left(1 - \frac{r}{R}\right) dr \\
m(r)|_0^R &= 4\pi\rho_c \int_0^R \left(r^2 - \frac{r^3}{R}\right) dr \\
m(R) - m(0) &= 4\pi\rho_c \left[\frac{r^3}{3} - \frac{r^4}{4R}\right]_0^R \\
M &= 4\pi\rho_c \left[\frac{R^3}{3} - \frac{R^3}{4}\right] \\
M &= 4\pi\rho_c \left(\frac{R^3}{12}\right) \\
\therefore \rho_c &= \frac{3M}{\pi R^3}.
\end{aligned}$$

b) To determine the gravitational acceleration at $r = \frac{R}{2}$, we must first determine the mass contained inside a radius of $r = \frac{R}{2}$,

$$\begin{aligned}
\frac{dm(r)}{dr} &= 4\pi r^2 \rho(r) \\
&= 4\pi r^2 \rho_c \left(1 - \frac{r}{R}\right) \\
&= 4\pi r^2 \left(\frac{3M}{\pi R^3}\right) \left(1 - \frac{r}{R}\right) \\
&= \frac{12Mr^2}{R^3} \left(1 - \frac{r}{R}\right) \\
\therefore dm(r) &= \frac{12M}{R^3} \left(r^2 - \frac{r^3}{R}\right) dr.
\end{aligned}$$

Integrating from a radius of $r = 0$ to a radius of $r = \dfrac{R}{2}$,

$$\begin{aligned}
\int_0^{\frac{R}{2}} dm(r) &= \int_0^{\frac{R}{2}} \frac{12M}{R^3}\left(r^2 - \frac{r^3}{R}\right) dr \\
m(r)\Big|_0^{\frac{R}{2}} &= \frac{12M}{R^3}\int_0^{\frac{R}{2}} \left(r^2 - \frac{r^3}{R}\right) dr \\
m\left(\frac{R}{2}\right) - m(0) &= \frac{12M}{R^3}\left[\frac{r^3}{3} - \frac{r^4}{4R}\right]_0^{\frac{R}{2}} \\
m\left(\frac{R}{2}\right) &= \frac{12M}{R^3}\left(\frac{R^3}{24} - \frac{R^3}{64}\right) \\
\therefore m\left(\frac{R}{2}\right) &= \frac{5M}{16}.
\end{aligned}$$

Thus, using the acceleration due to gravity formula, we have

$$\begin{aligned}
a_g(r) &= -\frac{GM(r)}{r^2} \\
a_g\left(\frac{R}{2}\right) &= -\frac{GM\left(\frac{R}{2}\right)}{\left(\frac{R}{2}\right)^2} \\
&= -\frac{G\left(\frac{5M}{16}\right)}{\frac{R^2}{4}} \\
\therefore a_g\left(\frac{R}{2}\right) &= -\frac{5GM}{4R^2}
\end{aligned}$$

2. The hydrostatic equilibrium equation is given as

$$-\frac{dP}{dr} = \frac{GM(< r)}{r^2}\rho.$$

a) The term $\dfrac{dP}{dr}$ is the rate of change of the force of pressure with respect to the distance from the centre of the star, r. The term ρ denotes the density of the star as a function of distance from the centre of the star, r. The term $M(< r)$ denotes the amount of mass inside distance of r units from the centre of the star.

b) Inside a star, the inward force of gravity is balanced by the outward force of thermal pressure.

c) Rewriting the hydrostatic equilibrium equation above, with $V(r)$ denoting volume at a distance of r units from the centre,

$$\begin{aligned}
-\frac{dP}{dr} &= \frac{GM(<r)}{r^2}\rho(r) \\
&= \frac{G\rho(r)V(r)}{r^2}\rho(r) \\
&= \frac{4G\pi r^3}{3r^2}\rho(r)^2 \\
&= \frac{4G\pi r}{3}\left(\frac{3M}{\pi R^3}\right)^2\left(1-\frac{r}{R}\right)^2 \\
&= \frac{12GM^2 r}{\pi R^6}\left(1-\frac{2r}{R}+\frac{r^2}{R^2}\right) \\
\therefore -\frac{dP}{dr} &= \frac{12GM^2}{\pi R^6}\left(r-\frac{2r^2}{R}+\frac{r^3}{R^2}\right)
\end{aligned}$$

Integrating the above result, subject to the zero boundary condition $P(R) = 0$, we have

$$\begin{aligned}
-\frac{dP}{dr} &= \frac{12GM^2}{\pi R^6}\left(r-\frac{2r^2}{R}+\frac{r^3}{R^2}\right) \\
\frac{dP}{dr} &= -\frac{12GM^2}{\pi R^6}\left(r-\frac{2r^2}{R}+\frac{r^3}{R^2}\right) \\
\int \frac{dP}{dr} &= \int -\frac{12GM^2}{\pi R^6}\left(r-\frac{2r^2}{R}+\frac{r^3}{R^2}\right) \\
\int dP &= -\frac{12GM^2}{\pi R^6}\int\left(r-\frac{2r^2}{R}+\frac{r^3}{R^2}\right)dr \\
\therefore P(r) &= -\frac{12GM^2}{\pi R^6}\left[\frac{r^2}{2}-\frac{2r^3}{3R}+\frac{r^4}{4R^2}\right]+C \\
\therefore 0 &= -\frac{12GM^2}{\pi R^6}\left[\frac{R^2}{2}-\frac{2R^3}{3R}+\frac{R^4}{4R^2}\right]+C \\
C &= \frac{12GM^2}{\pi R^6}\left[\frac{R^2}{2}-\frac{2R^2}{3}+\frac{R^2}{4}\right] \\
&= \frac{12GM^2}{\pi R^6}\left[\frac{R^2}{12}\right] \\
\therefore C &= \frac{GM^2}{\pi R^4} \\
\therefore P(r) &= -\frac{12GM^2}{\pi R^6}\left[\frac{r^2}{2}-\frac{2r^3}{3R}+\frac{r^4}{4R^2}\right]+\frac{GM^2}{\pi R^4} \\
&= \frac{GM^2}{\pi R^4}\left[1-6\frac{r^2}{R^2}+8\frac{r^3}{R^3}-3\frac{r^4}{R^4}\right] \\
\therefore P(r) &= \frac{GM^2}{\pi R^4}\left[1-6\left(\frac{r}{R}\right)^2+8\left(\frac{r}{R}\right)^3-3\left(\frac{r}{R}\right)^4\right].
\end{aligned}$$

Clearly, $P_c = \dfrac{GM^2}{\pi R^4}$, and $P(r)$ satisfies $P(0) = P_c$.

3. We shall assume for this question, that the Sun's luminosity remains constant for the next 10^{10} years, that the mass of the Sun is entirely protons, and that the Sun generates energy through hydrogen fusion, which has an efficiency rating of 0.7%.

   a) If the Sun converts half of its mass into energy, via hydrogen fusion, the energy produced is

$$\begin{aligned} E &= 0.007 \times \frac{M}{2} \times c^2 \\ &= 0.007 \times \frac{2.0 \times 10^33}{2} \times \left(3.0 \times 10^{10}\right)^2 \\ &= 0.063 \times 10^{53} \\ \therefore E &= 6.3 \times 10^{51}. \end{aligned}$$

   Using the energy output derived above, we can compute the lifetime of the Sun in this scenario using the formula for lifetime,

$$\begin{aligned} \tau &= \frac{E}{L} \\ &= \frac{6.3^51}{4 \times 10^{33}} \\ &= 1.575 \times 10^{18} \\ \therefore \tau &= 1.6 \times 10^{18}. \end{aligned}$$

   In this scenario, the Sun would have a lifetime of 1.6×10^{18} seconds, or 5.0×10^{10} years.

   b) Again, using the assumptions started at the start of the question, if the Sun converts 10% of its mass into energy, via hydrogen fusion, the energy produced is

$$\begin{aligned} E &= 0.007 \times \frac{M}{10} \times c^2 \\ &= 0.007 \times \frac{2.0 \times 10^{33}}{10} \times \left(3.0 \times 10^{10}\right)^2 \\ &= 0.0126 \times 10^{53} \\ \therefore E &= 1.26 \times 10^{51}. \end{aligned}$$

   Using the energy output derived above, we can compute the lifetime of the Sun in this scenario using the formula for lifetime,

$$\begin{aligned} \tau &= \frac{E}{L} \\ &= \frac{1.26 \times 10^{51}}{4 \times 10^{33}} \\ &= 0.315 \times 10^{18} \\ \therefore \tau &= 3.2 \times 10^{17}. \end{aligned}$$

   In this scenario, the Sun would have a lifetime of 3.2×10^{17} seconds, or 10^{10} years.

   c) Due to mass-energy equivalence, and from the assumptions above, the Sun will lose 0.07% of its mass, which is 1.4×10^{30} grams.

d) Using the formula relating mass and luminosity for stars on the main sequence, where Tau Scorpii has a mass of $14.7\,\mathsf{M}_\odot$, the lifetime of Tau Scorpii is,

$$\begin{aligned}
\frac{\tau}{\tau_\odot} &= \left(\frac{M}{M_\odot}\right)^{-2.5} \\
\tau &= \tau_\odot \left(\frac{M}{M_\odot}\right)^{-2.5} \\
&= \left(3.15 \times 10^{17}\right)\left(\frac{14.7\,M_\odot}{M_\odot}\right)^{-2.5} \\
&= 3.802 \times 10^{14} \\
\therefore \tau &= 3.8 \times 10^{14}.
\end{aligned}$$

The lifetime of Tau Scorpii is 3.8^{14} seconds. Using the formula relating energy and lifetime, where Tau Scorpii has a luminosity of $20400\,\mathsf{L}_\odot$, we get an energy output for Tau Scorpii of

$$\begin{aligned}
\tau &= \frac{E}{L} \\
\therefore E &= \tau L \\
&= 3.802 \times 10^{14} \times 20400 \times 4 \times 10^{33} \\
&= 3.1024 \times 10^{52} \\
\therefore E &= 3.1 \times 10^{52}.
\end{aligned}$$

The energy output of Tau Scorpii over its lifetime is 3.1×10^{52}. Now, using the principle of mass-energy conservation,

$$\begin{aligned}
E &= Mc^2 \\
\therefore M &= \frac{E}{c^2} \\
&= \frac{3.1024 \times 10^{52}}{\left(3 \times 10^{10}\right)^2} \\
&= 3.447 \times 10^{31} \\
\therefore M &= 3.4 \times 10^{31}.
\end{aligned}$$

Thus, the mass lost due to energy production in Tau Scorpii is 3.4×10^{31} grams. Using the lifetime of Tau Scorpii as calculated above, 3.8^{14} seconds, Tau Scorpii loses 9.1×10^{16} grams of mass per second. As hydrogen fusion has an efficiency of 0.7%, 1.3×10^{19} grams of hydrogen are being converted to helium in the fusion process per second.

4. From the book, the rate of energy released per second by hydrogen fusion is 4.3×10^{-5} erg/s. Using the luminosity of the Sun as 4×10^{33} erg, we get the number of reactions occuring per second as

$$\begin{aligned}
\#\ \text{reactions/s} &= \frac{4 \times 10^{33}}{4.3 \times 10^{-5}} \\
\therefore \#\ \text{reactions/s} &= 9.3 \times 10^{37}.
\end{aligned}$$

Using the equation given for the reaction, two neutrinos are released per reaction, giving us 1.86×10^{38} neutrinos released per second. Considering a sphere of radius 1AU, the number of neutrinos passing through $1\ \mathsf{cm}^2$ is,

$$\begin{aligned}
\#\ \mathsf{neutrinos/cm}^2\mathsf{/s} &= \frac{\#\ \mathsf{neutrinos/s}}{4\pi r^2} \\
&= \frac{1.86 \times 10^{38}}{4\pi \left(1.496 \times 10^{13}\right)^2} \\
&= 6.6136 \times 10^{10} \\
\therefore \#\ \mathsf{neutrinos/cm}^2\mathsf{/s} &= 6.6 \times 10^{10}.
\end{aligned}$$