CHAPTER **ONE**

Complex Numbers

Chapter Overview: One of the significant properties of the real numbers is that any of the four arithmetic operations of addition, subtraction, multiplication and division can be applied to any pair of real numbers, with the exception that division by zero is undefined. As a result, every linear equation

$$ax + b = 0 \qquad \text{where} \qquad a \neq 0$$

can be solved.

The situation is not so satisfactory when quadratic equations are considered. There are some quadratic equations that can be solved, but others, like

$$x^2 + 2x + 3 = 0\,,$$

have no real solution. This apparent inconsistency, that some quadratics have a solution whilst others do not, can be resolved by the introduction of a new type of number, the complex number.

But there is more to complex numbers than just solving quadratic equations. In this chapter the reader is shown an application to geometry and later in the course, complex numbers will be used in the study of polynomials. These new numbers have many applications beyond this course, such as in evaluating certain integrals and in solving problems in electrical engineering. Complex numbers also provide links between seemingly unrelated quantities and areas of mathematics. Here is a stunning example. The four most significant real numbers encountered so far are 0, 1, e and π. Although the proof is beyond the scope of this course, these four are connected in a remarkably simple equation involving the special complex number i, namely

$$e^{i\pi} + 1 = 0\,.$$

1A The Arithmetic of Complex Numbers

Introducing A New Type of Number: We begin by examining the roots of various quadratic equations. For convenience in presenting the new work, we will solely use the method of completing the square.

Suppose that initially we restrict our attention to those quadratic equations with rational solutions such as the equation $x^2 - 4x - 12 = 0$. Completing the square:

$(x-2)^2 = 16$

so $\quad x-2 = 4 \text{ or } -4$

which leads to the two roots

$\alpha = 6 \text{ and } \beta = -2\,.$

Note that $\alpha + \beta = 4$ and $\alpha\beta = -12$.

Repeating this process for a number of quadratics with rational solutions, it soon becomes evident that if $ax^2 + bx + c = 0$ has solutions α and β then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\,.$$

Further investigation reveals that there are some quadratic equations which do not have rational solutions, such as $x^2 - 4x - 1 = 0$. Completing the square:

$$(x-2)^2 = 5\,.$$

Herein lies a problem since there is no rational number whose square is 5 . We seek to overcome this problem by introducing a new type of number, in this case the irrational number $\sqrt{5}$ which has the property that $\left(\sqrt{5}\right)^2 = 5$. Assuming that the normal rules of algebra apply to this new number, we further note that $\left(-\sqrt{5}\right)^2 = \left(\sqrt{5}\right)^2 = 5$, so that 5 has two square roots, namely $\sqrt{5}$ and $-\sqrt{5}$. We hope that the introduction of this new type of number makes sense of our calculations and proceed with the solution. Thus

$x - 2 = \sqrt{5} \text{ or } -\sqrt{5}$

which leads to the two roots

$\alpha = 2 + \sqrt{5} \text{ and } \beta = 2 - \sqrt{5}\,.$

Note that $\alpha + \beta = 4$ and $\alpha\beta = -1$.

Repeating this process for a number of quadratics with irrational solutions, it soon becomes evident that if $ax^2 + bx + c = 0$ has irrational roots α and β then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\,.$$

Since this is consistent with the quadratic equations with rational solutions, it seems that the introduction of surds into our number system is valid. Indeed we have used surds since Year 9 and are now quite comfortable manipulating them.

Yet further investigation reveals that there are some quadratic equations which have neither rational nor irrational solutions, such as $x^2-4x+5 = 0$. Completing the square yields:

$$(x-2)^2 = -1\,.$$

Again there is a problem since there is no known number whose square is -1 . Just as before, we seek to overcome this problem by introducing a new type of number. In this case we introduce the so called imaginary number i which has the property that $i^2 = -1$. Assuming that the normal rules of algebra apply to this new number, we further note that $(-i)^2 = i^2 = -1$, so that -1 has two square roots, namely i and $-i$. We hope that the introduction of this new type of number makes sense of our calculations and proceed with the solution. Thus

$x - 2 = i \text{ or } -i$

which leads to the two roots

$$\alpha = 2 + i \text{ and } \beta = 2 - i\,.$$

Note that $\alpha + \beta = 4$ and

$$\begin{aligned} \alpha\beta &= (2-i)(2+i) \\ &= 2^2 - i^2 \quad \text{(difference of two squares)} \\ &= 4 + 1 \\ &= 5\,. \end{aligned}$$

Repeating this process for a number of quadratics with solutions which involve the imaginary number i, it soon becomes evident that if $ax^2 + bx + c = 0$ has solutions α and β then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\,.$$

Since this is consistent with all previously encountered quadratic equations, it seems reasonable to include the imaginary number i in our number system.

A New Number in Arithmetic: We will introduce the imaginary number i into our system of numbers, which has the special property that $i^2 = -1$. We will treat the number i as if it were an algebraic pronumeral when it is combined with real numbers using the four arithmetic operations of addition, subtraction, multiplication and division.

1 A NEW NUMBER: The new number i has the special property that

$$i^2 = -1\,.$$

It may be used like a pronumeral with real numbers in addition, subtraction, multiplication and division.

It is instructive to write out the first four positive powers of i. They are:

$$i^1 = i \qquad i^2 = -1 \text{ (by definition)} \qquad \begin{aligned} i^3 &= i^2 \times i \\ &= -1 \times i \\ &= -i \end{aligned} \qquad \begin{aligned} i^4 &= i^3 \times i \\ &= -i \times i \\ &= 1 \end{aligned}$$

Writing out the next four powers of i, we see that this sequence repeats.

$$\begin{aligned} i^5 &= i^4 \times i \\ &= 1 \times i \\ &= i \end{aligned} \qquad \begin{aligned} i^6 &= i^4 \times i^2 \\ &= 1 \times (-1) \\ &= -1 \end{aligned} \qquad \begin{aligned} i^7 &= i^4 \times i^3 \\ &= 1 \times (-i) \\ &= -i \end{aligned} \qquad \begin{aligned} i^8 &= \left(i^4\right)^2 \\ &= 1 \end{aligned}$$

It should be clear from these calculations that the sequence continues to cycle. In general we only need to look at the remainder after the index has been divided by 4 in order to determine the result.

2 POWERS OF THE IMAGINARY NUMBER: A power of i may take only one of four possible values. If k is an integer, then these values are:

$$i^{4k} = 1\,, \quad i^{4k+1} = i\,, \quad i^{4k+2} = -1\,, \quad i^{4k+3} = -i\,.$$

WORKED EXERCISE: Simplify: (a) i^{23} (b) $i^7 + i^9$

SOLUTION:

(a) Since $23 = 4 \times 5 + 3$

$$i^{23} = -i$$

(b)

$$\begin{aligned} i^7 + i^9 &= -i + i \\ &= 0 \end{aligned}$$

Complex Numbers: Since we have included i in our number system and since it is to be treated as a pronumeral, our number system must now include the real numbers plus new quantities like

$$2i\,, \qquad -7i\,, \qquad 5+4i \qquad \text{and} \qquad \sqrt{6}-3i\,.$$

The set which includes all such quantities as well as the real numbers is given the symbol $\mathbf{C}$. Each quantity in $\mathbf{C}$ is called a *complex number*. Thus 5, $2i$ and $\sqrt{6}-3i$ are all examples of complex numbers. In the first case, 5 is also a real number, and the real numbers form a special subset of the complex numbers. The number $2i$ is an example of another special subset of the complex numbers. This set consists of all the real multiples of i, which are called *imaginary numbers*. Thus $-7i$ is another example of an imaginary number.

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TWO NEW TYPES OF NUMBERS: Let a and b be real numbers.
COMPLEX NUMBERS: Numbers of the form $a+ib$ are called *complex numbers*. The set of all complex numbers is given the symbol $\mathbf{C}$.
IMAGINARY NUMBERS: Numbers of the form ib, that is the complex numbers for which $a=0$, are called *imaginary numbers*.

Again noting that i is treated as a pronumeral, the addition, subtraction and multiplication of complex numbers presents no problem.

$$(2-3i)+(5+7i)=7+4i\,, \qquad (7+2i)-(5-3i)=2+5i\,,$$
$$3(-5+7i)=-15+21i\,, \qquad \sqrt{3}(2+i\sqrt{3})=2\sqrt{3}+3i\,.$$

In some cases of multiplication we will also need to use binomial expansion and the property that $i^2=-1$.

$$\begin{aligned}(2-3i)(5+7i)&=10-i-21i^2\\&=10-i+21\\&=31-i\end{aligned} \qquad \begin{aligned}(4+3i)^2&=16+24i+9i^2\\&=16+24i-9\\&=7+24i\end{aligned}$$
$$\begin{aligned}(3-2i)^2&=9-12i+4i^2\\&=9-12i-4\\&=5-12i\end{aligned} \qquad \begin{aligned}(2+5i)(2-5i)&=4-25i^2\\&=4+25\\&=29\end{aligned}$$

The last three examples above demonstrate the expansions of $(x+iy)^2$, $(x-iy)^2$ and $(x+iy)(x-iy)$ for real values of x and y. Note that in the final example, the result is the sum of two squares and is a real number. This will always be the case, regardless of the values of x and y.

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THE SUM OF TWO SQUARES: Let x and y be real numbers, then

$$(x+iy)(x-iy)=x^2+y^2$$

which is always a real number.

Complex Conjugates: The last result is significant and will be used frequently. Clearly the pair of numbers $x+iy$ and $x-iy$ are special, and consequently they are given a special description. We say that the numbers $x+iy$ and $x-iy$ are *complex conjugates*. Thus the complex conjugate of $3+2i$ is $3-2i$. Similarly the conjugate of $7-5i$ is $7+5i$.

In order to indicate that the conjugate is required, we write the complex number with a bar above it. Thus:

$$\overline{2+i} = 2 - i \qquad \overline{-3i} = 3i$$

$$\overline{-1+4i} = -1 - 4i \qquad \overline{-3-5i} = -3 + 5i$$

5 COMPLEX CONJUGATES: Let x and y be real numbers, then the two complex numbers $x + iy$ and $x - iy$ are called complex conjugates.

A: The conjugate of $x + iy$ is $\overline{x+iy} = x - iy$.

B: The conjugate of $x - iy$ is $\overline{x-iy} = x + iy$.

Division: Just like real numbers, division by zero is undefined. Dividing a complex number by any other real number presents no problem. As with rational numbers, fractions should be simplified wherever possible by cancelling out common factors.

$$\frac{6+8i}{2} = 3 + 4i \qquad \frac{-2-6i}{3} = -\tfrac{2}{3} - 2i$$

$$\frac{\sqrt{2}-2i}{\sqrt{2}} = 1 - i\sqrt{2} \qquad \frac{-12+21i}{15} = \frac{-4+7i}{5} \text{ or } -\tfrac{4}{5} + \tfrac{7}{5}\,i$$

There is a potential problem if one complex number is divided by another, such as in $\frac{2+i}{3-i}$. As it stands, it is not clear that this sort of quantity is even allowed in our new number system, since it is not in the standard form, $x + iy$.

The problem is resolved by taking a similar approach to that used to deal with surds in the denominator. The process here is called *realising the denominator*. Thus if the divisor is an imaginary number then simply multiply the fraction by i/i, as in the following two examples.

$$\begin{aligned} \frac{1}{4i} &= \frac{1}{4i} \times \frac{i}{i} \\ &= \frac{i}{4i^2} \\ &= -\tfrac{1}{4}\,i \end{aligned} \qquad \begin{aligned} \frac{1+2i}{3i} &= \frac{1+2i}{3i} \times \frac{i}{i} \\ &= \frac{i+2i^2}{3i^2} \\ &= \frac{2-i}{3} \end{aligned}$$

If on the other hand the denominator is a complex number then the method is to multiply top and bottom by its conjugate, as demonstrated here.

$$\begin{aligned} \frac{5}{2+i} &= \frac{5}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{5(2-i)}{4+1} \\ &= 2 - i \end{aligned} \qquad \begin{aligned} \frac{5+2i}{3-4i} &= \frac{5+2i}{3-4i} \times \frac{3+4i}{3+4i} \\ &= \frac{15+26i-8}{9+16} \\ &= \frac{7+26i}{25} \end{aligned}$$

6 REALISING THE DENOMINATOR: There are two cases.

A: If the denominator is an imaginary number, multiply top and bottom by i.

B: If the denominator is complex, multiply top and bottom by its conjugate.

We should now be satisfied that the complex numbers form a valid number system since we have seen on the previous pages that the four basic arithmetic operations of addition, subtraction, multiplication and division all behave in a sensible way.

A Convention for Pronumerals: It is often necessary in developing the theory of complex numbers to perform algebraic manipulations with unknown complex numbers. In order to help distinguish between real and complex variables, the convention that we shall use in this text is that the pronumerals x, y, a and b will represent real numbers and the pronumerals z and w will represent complex numbers. Thus we will often write $z = x + iy$, where it is understood that x and y are real whilst z is complex.

Real and Imaginary Parts: Given the complex number $z = x + iy$, the real part of z is the real number x, and the imaginary part of z is the real number y. It is convenient to define two new functions of the complex variable z for these two quantities. Thus

$$\mathrm{Re}(z) = x \qquad \text{and} \qquad \mathrm{Im}(z) = y$$

from which it is clear that

$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)\,.$$

WORKED EXERCISE: Determine $\mathrm{Re}(z^2 - iz)$, where $z = 3 - i$.

SOLUTION: Expanding the quadratic in z first,

$$\begin{aligned} z^2 - iz &= (3-i)^2 - i(3-i) \\ &= 8 - 6i - 3i - 1 \\ &= 7 - 9i\,, \end{aligned}$$

so $\qquad \mathrm{Re}(z^2 - iz) = 7\,.$

If two complex numbers z and w are equal, by analogy with surds, we expect that $\mathrm{Re}(z) = \mathrm{Re}(w)$ and $\mathrm{Im}(z) = \mathrm{Im}(w)$. This is in fact the case.

7 THEOREM — EQUALITY OF COMPLEX NUMBERS: If two complex numbers z and w are equal then $\mathrm{Re}(z) = \mathrm{Re}(w)$ and $\mathrm{Im}(z) = \mathrm{Im}(w)$.

PROOF: Let $z = x + iy$ and $w = a + ib$, and suppose that $z = w$ then

$$x + iy = a + ib\,.$$

Rearranging $\qquad i(y - b) = a - x \qquad (**)$

and if $y - b \neq 0$ then $\qquad i = \dfrac{a - x}{y - b}$, which is a real number.

This contradicts i being an imaginary number. Thus $y - b = 0$ and hence $y = b$, whence by equation $(**)$ $x = a$, and the proof is complete.

The careful reader will have noticed that the definitions of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ given above are not in terms of the variable z. Both of these functions can be expressed in terms of z by first writing down z and its conjugate.

$$\begin{aligned} z &= x + iy \\ \overline{z} &= x - iy \end{aligned}$$

Thus we have a pair of simultaneous equations which can be solved for x and y to obtain:

$$\mathrm{Re}(z) = \tfrac{1}{2}(z + \overline{z}) \qquad \text{and} \qquad \mathrm{Im}(z) = \tfrac{1}{2i}(z - \overline{z})\,.$$

The Arithmetic of Conjugates: Since taking the complex conjugate of z simply changes the sign of the imaginary part, when it is applied twice in succession the end result leaves z unchanged. Thus

$$\overline{(\overline{z})} = \overline{\left(\overline{x+iy}\right)} = \overline{x-iy} = x+iy = z\,.$$

Another important property of taking conjugates is that it commutes with the four basic arithmetic operations. For example, with addition,

$$\begin{aligned} \overline{(3+i)+(2-4i)} &= \overline{5-3i} \\ &= 5+3i\,, \\ \text{and} \qquad \overline{3+i}+\overline{2-4i} &= 3-i+2+4i \\ &= 5+3i\,. \\ \text{Thus} \qquad \overline{(3+i)+(2-4i)} &= \overline{3+i}+\overline{2-4i}\,. \end{aligned}$$

Notice that it does not matter whether the addition is done first or second, the result is the same. Here is an example with multiplication.

$$\begin{aligned} \overline{(3+i)(2-4i)} &= \overline{10-10i} \\ &= 10+10i\,, \\ \text{and} \qquad \overline{3+i}\times\overline{2-4i} &= (3-i)(2+4i) \\ &= 10+10i\,. \\ \text{Thus} \qquad \overline{(3+i)(2-4i)} &= \overline{3+i}\times\overline{2-4i}\,. \end{aligned}$$

Again notice that it does not matter whether the multiplication is done first or second, the result is the same. This is always the case for addition, subtraction, multiplication and division.

8 THE ARITHMETIC OF CONJUGATES: The taking of complex conjugates is commutative with addition, subtraction, multiplication and division.

(a) $\overline{w+z} = \overline{w}+\overline{z}$ (c) $\overline{wz} = \overline{w}\times\overline{z}$

(b) $\overline{w-z} = \overline{w}-\overline{z}$ (d) $\overline{w\div z} = \overline{w}\div\overline{z}$

The proof of these results is left as a question in the exercise. There are two special cases of these results. To get the conjugate of a negative, put $w=0$ into (b).

$$\begin{aligned} \overline{(-z)} &= \overline{0-z} \\ &= \overline{0}-\overline{z} \\ \text{thus} \qquad \overline{(-z)} &= -\overline{z}\,. \end{aligned}$$

For the conjugate of a reciprocal, put $w=1$ in (d) to get

$$\begin{aligned} \overline{z^{-1}} &= \overline{1\div z} \\ &= \overline{1}\div\overline{z} \\ &= 1\div\overline{z} \\ \text{thus} \qquad \overline{z^{-1}} &= (\overline{z})^{-1}\,. \end{aligned}$$

Integer Powers: The careful reader will have noted that several of the examples used above involve powers of a complex number despite the fact that the meaning of z^n has not yet been properly defined. If the index n is a positive integer then the meaning of z^n is analogous to the real number definition. Thus

$$z^n = \underbrace{z\times z\times\ldots\times z}_{n \text{ factors}}$$

or, the recursive definition,

$$z^1 = z\,,$$
$$z^n = z \times z^{n-1} \qquad \text{for} \qquad n > 1\,.$$

Just like the real numbers, if $z = 0$ then z^0 is undefined. For all other complex numbers, $z^0 = 1$. Again continuing the analogy with the real numbers, a negative integer power yields a reciprocal. Thus if n is a positive integer then

$$z^{-n} = \frac{1}{z^n}\,, \qquad z \neq 0\,.$$

As with other division by complex numbers, the denominator is usually realised by multiplying by the conjugate. The case when $n = 1$ occurs frequently and should be learnt.

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}}$$

Indices which are not integers will not be considered in this text.

Exercise **1A**

1. Use the rule given in Box 2 to simplify:

(a) i^2 (b) i^4 (c) i^7 (d) i^{13} (e) i^{29} (f) i^{2010} (g) $i^3 + i^4 + i^5$ (h) $i^7 + i^{16} + i^{21} + i^{22}$

2. Evaluate:

(a) $\overline{2i}$ (b) $\overline{3+i}$ (c) $\overline{1-i}$ (d) $\overline{5-3i}$ (e) $\overline{-3+2i}$

3. Express in the form $a + ib$, where a and b are real:

(a) $(7+3i)+(5-5i)$ (b) $(-8+6i)+(2-4i)$ (c) $(4-2i)-(3-7i)$ (d) $(3-5i)-(-4+6i)$

4. Express in the form $x + iy$, where x and y are real:

(a) $(4+5i)i$ (b) $(1+2i)(3-i)$ (c) $(3+2i)(4-i)$ (d) $(-7+5i)(8-6i)$ (e) $(5+i)^2$ (f) $(2-3i)^2$ (g) $(2+i)^3$ (h) $(1-i)^4$ (i) $(3-i)^4$

5. Use the rule for the sums of two squares given in Box 4 to simplify:

(a) $(1+2i)(1-2i)$ (b) $(4+i)(4-i)$ (c) $(5+2i)(5-2i)$ (d) $(-4-7i)(-4+7i)$

6. Express in the form $x + iy$, where x and y are real:

(a) $\dfrac{1}{i}$ (b) $\dfrac{2+i}{i}$ (c) $\dfrac{5-i}{1-i}$ (d) $\dfrac{6-7i}{4+i}$ (e) $\dfrac{-11+13i}{5+2i}$ (f) $\dfrac{(1+i)^2}{3-i}$

7. Let $z = 1 + 2i$ and $w = 3 - i$. Find, in the form $x + iy$:

(a) $\overline{(iz)}$ (b) $w + \overline{z}$ (c) $2z + iw$ (d) $\operatorname{Im}(5i - z)$ (e) z^2

8. Let $z = 8 + i$ and $w = 2 - 3i$. Find, in the form $x + iy$:

(a) $\overline{z} - w$ (b) $\operatorname{Im}(3iz + 2w)$ (c) zw (d) $65 \div z$ (e) $z \div w$

9. Let $z = 2 - i$ and $w = -5 - 12i$. Find, in the form $x + iy$:

(a) $-zw$ (b) $(1+i)\overline{z} - w$ (c) $\dfrac{10}{\overline{z}}$ (d) $\dfrac{w}{2-3i}$ (e) $\operatorname{Re}\Big((1+4i)z\Big)$

DEVELOPMENT

10. By equating real and imaginary parts, find the real values of x and y given that:

(a) $(x+yi)(2-3i) = -13i$

(b) $(1+4i)(x+yi) = 6+7i$

(c) $(1+i)x + (2-3i)y = 10$

(d) $x(1+2i) + y(2-i) = 4+5i$

(e) $\dfrac{x}{2+i} + \dfrac{y}{2+3i} = 4+i$

11. Express in the form $x + iy$, where x and y are real:

(a) $\dfrac{1}{1+i} + \dfrac{2}{1+2i}$

(b) $\dfrac{1+i\sqrt{3}}{2} + \dfrac{2}{1+i\sqrt{3}}$

(c) $\dfrac{3+2i}{2-5i} + \dfrac{3-2i}{2+5i}$

(d) $\dfrac{-8+5i}{-2-4i} - \dfrac{3+8i}{1+2i}$

12. Given that $z = x + iy$ and $w = a + ib$, where a, b, x and y are real, prove that:

(a) $\overline{z+w} = \overline{z} + \overline{w}$

(b) $\overline{z-w} = \overline{z} - \overline{w}$

(c) $\overline{zw} = \overline{z}\,\overline{w}$

(d) $\overline{z^2} = (\overline{z})^2$

(e) $\overline{\left(\dfrac{1}{z}\right)} = \dfrac{1}{\overline{z}}$, $z \neq 0$

(f) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$, $w \neq 0$

13. Let $z = a + ib$, where a and b are real and non-zero. Prove that:

(a) $z + \overline{z}$ is real,

(b) $z - \overline{z}$ is imaginary,

(c) $z^2 + (\overline{z})^2$ is real,

(d) $z\overline{z}$ is real and positive.

14. Let $z = a + ib$, where a and b are real. If $\dfrac{z}{z-i}$ is real, show that z is imaginary.

15. Prove that if $z^2 = (\overline{z})^2$ then z is either real or imaginary but not complex.

16. If $z = x + iy$, where x and y are real, express in the form $a + ib$, where a and b are written in terms of x and y:

(a) z^{-1} (b) z^{-2} (c) $\dfrac{z-1}{z+1}$

EXTENSION

17. If both $z + w$ and zw are real, prove that either $z = \overline{w}$ or $\operatorname{Im}(z) = \operatorname{Im}(w) = 0$.

18. Given that $z = 2(\cos\theta + i\sin\theta)$, show that $\operatorname{Re}\left(\dfrac{1}{1-z}\right) = \dfrac{1 - 2\cos\theta}{5 - 4\cos\theta}$.

19. Show that $\dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$.

20. If $z = \cos\theta + i\sin\theta$, show that $\dfrac{2}{1+z} = 1 - it$, where $t = \tan\frac{\theta}{2}$.

1B Quadratic Equations

Now that the arithmetic of complex numbers has been satisfactorily developed, it is appropriate to return to the original problem of solving quadratic equations. To reflect the fact that the solutions may be complex, the variable z will be used.

Quadratic Equations with Real Coefficients: The simplest quadratic equations are the perfect square

$$z^2 = 0$$

for which $z = 0\,,$

and the difference of two squares

$$z^2 - \lambda^2 = 0$$

for which $z = -\lambda$ or $\lambda\,.$

It is now also possible to solve equations involving the sum of two squares, using the result of Box 4 in Section 1A.

Given $z^2 + \lambda^2 = 0$

$$(z + i\lambda)(z - i\lambda) = 0 \qquad \text{(the sum of two squares)}$$

so $z = -i\lambda$ or $i\lambda\,.$

Thus there are three possible cases for a simple quadratic equation: a perfect square, the difference of two squares, or the sum of two squares.

WORKED EXERCISE: Find the two imaginary solutions of $z^2 + 10 = 0$.

SOLUTION: Factoring the sum of two squares

$$(z + i\sqrt{10})(z - i\sqrt{10}) = 0$$

so $z = -i\sqrt{10}$ or $i\sqrt{10}$

For more general quadratic equations, it is simply a matter of completing the square in z to obtain one of the same three situations: a perfect square, the difference of two squares, or the sum of two squares.

WORKED EXERCISE: Find the complex solutions of $z^2 + 6z + 25 = 0$.

SOLUTION: Completing the square:

$$(z + 3)^2 + 16 = 0$$

so $(z + 3 + 4i)(z + 3 - 4i) = 0$ (sum of two squares)

thus $z = -3 - 4i$ or $-3 + 4i\,.$

Notice that the sum of two squares situation always yields two roots which are complex conjugates.

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QUADRATIC EQUATIONS WITH REAL COEFFICIENTS: Complete the square in z to obtain one of the following situations:

A. A PERFECT SQUARE: There is only one real root.
B. THE DIFFERENCE OF TWO SQUARES: There are two real roots.
C. THE SUM OF TWO SQUARES: There are two conjugate complex roots.

There are several of ways of proving the assertion that complex solutions to quadratic equations with real coefficients must occur as conjugate pairs. The approach presented here will later be extended to encompass all polynomials with real coefficients.

PROOF: Let $Q(z) = az^2 + bz + c$, where a, b and c are real numbers. Suppose that the equation $Q(z) = 0$ has at least one complex solution $z = w$, then

$$aw^2 + bw + c = 0\,.$$

Take the conjugate of both sides of this equation to get

$$\overline{aw^2 + bw + c} = \overline{0}\,.$$

Now the conjugate of a real number is the same real number. Further, as noted in Box 9, taking a conjugate is commutative with addition and multiplication. Thus the last equation becomes

$$a(\overline{w})^2 + b(\overline{w}) + c = 0$$

that is $$Q(\overline{w}) = 0\,.$$

Hence if $z = w$ is one complex root of $Q(z) = 0$ then it follows that $z = \overline{w}$ is the other root of the equation, and the proof is complete.

Worked Exercise: Find a quadratic equation with real coefficients given that one of the roots is $w = 5 - i$.

Solution: The other root must be $\overline{w} = 5 + i$ so

$$\big(z - (5 - i)\big)\big(z - (5 + i)\big) = 0$$

or $$\big((z - 5) + i\big)\big((z - 5) - i\big) = 0$$

thus $$(z - 5)^2 + 1 = 0\,.$$

Finally $$z^2 - 10z + 26 = 0\,.$$

In general, the quadratic equation with real coefficients which has a complex root $z = \alpha$ is

$$z^2 - 2\,\mathrm{Re}(\alpha)\,z + \alpha\,\overline{\alpha} = 0\,,$$

as can be observed in the three Worked Exercises above. The proof is quite straight forward, and is one of the questions in the exercise.

10 Real quadratic equations with complex roots: The quadratic equation with real coefficients which has a complex root $z = \alpha$ is

$$z^2 - 2\,\mathrm{Re}(\alpha)\,z + \alpha\,\overline{\alpha} = 0\,.$$

The Quadratic Method: Many readers will know the quadratic formula as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

There is a problem with this formula when complex numbers are involved. When applied to real numbers, the symbol $\sqrt{}$ means the positive square root, but it is unclear what "positive" means when applied to complex numbers. It might be tempting to say that i is positive and $-i$ is negative, but then what is to be said about numbers like $(-1 + i)$ or $(1 - i)$? In short, it does not make sense to speak of positive and negative complex numbers, and so the positive square root has no meaning. Thus it is not appropriate to blindly use the quadratic formula to solve an equation with complex roots.

Recall that the quadratic formula arose from applying the method of completing the square. Let us review this process.

Given $\qquad az^2 + bz + c = 0\,,$

$$z^2 + \tfrac{b}{a}z = -\tfrac{c}{a}$$

so $\qquad \left(z + \tfrac{b}{2a}\right)^2 = \tfrac{\Delta}{(2a)^2}\,, \qquad$ where $\Delta = b^2 - 4ac\,.$

Now suppose there exists a number λ, possibly complex, such that $\Delta = \lambda^2$.

Then $\qquad \left(z + \tfrac{b}{2a}\right)^2 - \left(\tfrac{\lambda}{2a}\right)^2 = 0$

whence $\quad \left(z + \tfrac{b+\lambda}{2a}\right)\left(z + \tfrac{b-\lambda}{2a}\right) = 0 \qquad$ (the difference of two squares)

and so $\qquad z = \tfrac{-b-\lambda}{2a}$ or $\tfrac{-b+\lambda}{2a}\,.$

Thus if we can find a number λ, possibly complex, where $\lambda^2 = \Delta$, then we can write down the solution to the quadratic equation using the last line above. If the quadratic formula is to be applied then this is the method that should always be used.

11 THE QUADRATIC METHOD: Use the following steps to solve $az^2 + bz + c = 0\,.$

1. First find $\Delta = b^2 - 4ac\,.$
2. Next find a number λ, possibly complex, such that $\lambda^2 = \Delta\,.$
3. Finally, the roots are $z = \dfrac{-b-\lambda}{2a}$ or $\dfrac{-b+\lambda}{2a}\,.$

WORKED EXERCISE: Solve $z^2 + 2z + 6 = 0\,.$

SOLUTION:

$$\begin{aligned} \Delta &= 2^2 - 4 \times 1 \times 6 \\ &= -20 \\ &= \left(2i\sqrt{5}\right)^2\,, \end{aligned}$$

hence

$$\begin{aligned} z &= \frac{-2 - 2i\sqrt{5}}{2} \text{ or } \frac{-2 + 2i\sqrt{5}}{2} \\ &= -1 - i\sqrt{5} \text{ or } -1 + i\sqrt{5}\,. \end{aligned}$$

Complex Square Roots: Before extending the above work to the case of a quadratic equation with complex coefficients, it is necessary to develop methods for finding the square roots of complex numbers.

The first thing to notice is that, just like real numbers, every complex number has two square roots. The proof is quite straight forward. Suppose that the complex number z is a square root of another complex number w then

$$z^2 = w\,.$$

Further

$$\begin{aligned} (-z)^2 &= z^2 \\ &= w\,. \end{aligned}$$

Hence w has a second square root which is the opposite of the first, namely $(-z)$. Thus for example $-2i$ has two opposite square roots, $(1 - i)$ and $(-1 + i)$. This is not really very surprising since all real numbers (other than zero) have two opposite square roots. For example, 9 has square roots 3 and -3, whilst -5 has square roots $i\sqrt{5}$ and $-i\sqrt{5}$. The proof that there are no more than two square roots is left as an exercise.

Complex Square Roots and Pythagoras: At this point in the course, the method is to equate the real and imaginary parts of $z^2 = w$ in order to obtain a pair of simultaneous equations.

Given $\quad (x+iy)^2 = a + ib\,,\quad$ where x, y, a and b are real,

$$x^2 - y^2 + 2ixy = a + ib\,.$$

Equating real and imaginary parts yields

$$x^2 - y^2 = a$$

and $$xy = \tfrac{1}{2}\,b\,.$$

In simple cases this pair of equations should be solved by inspection, as in the following example.

WORKED EXERCISE: Find the square roots of $7 + 24i$.

SOLUTION: Let $\quad (x+iy)^2 = 7 + 24i\,,\quad$ where x and y are real,

then $\quad (x^2 - y^2) + 2ixy = 7 + 24i\,.$

Equating real and imatinary parts yields the simultaneous equations

$$x^2 - y^2 = 7$$

and $$xy = 12\,.$$

These equations can be solved by inspecting the factors of 12. Thus $x = 4$ and $y = 3$, or $x = -4$ and $y = -3$. Hence the square roots of $7+24i$ are the opposites

$$4 + 3i \quad \text{and} \quad -4 - 3i\,.$$

Some readers will have noticed in the above example that 7 and 24 are the first two numbers of the Pythagorean triad 7, 24, 25. This is no coincidence. It is often the case that if b is even and the numbers $|a|$, $|b|$ and $\sqrt{a^2+b^2}$ form a Pythagorean triad then the resulting equations for x and y can simply be solved by inspecting the factors of $\frac{1}{2}\,b$.

12 COMPLEX SQUARE ROOTS AND PYTHAGORAS: Given $(x+iy)^2 = a + ib$, equate the real and imaginary parts to get the simultaneous equations

$$x^2 - y^2 = a$$
$$xy = \tfrac{1}{2}\,b\,.$$

If b is even and the numbers $|a|$, $|b|$ and $\sqrt{a^2+b^2}$ form a Pythagorean triad then the these equations can often be solved by inspecting the factors of $\frac{1}{2}\,b$.

Quadratic Equations with Complex Coefficients: We are now ready to solve simple quadratic equations with complex coefficients. All that is needed is to combine the above method for finding the roots of a complex number with either the method of completing the square or the quadratic formula method.

WORKED EXERCISE: Solve $z^2 - (2+6i)z + (-5+2i) = 0$ by completing the square.

SOLUTION: Rearranging

$$z^2 - 2(1+3i)z = 5 - 2i$$

so
$$\begin{aligned}(z-(1+3i))^2 &= (1+3i)^2 + 5 - 2i\\ &= -8 + 6i + 5 - 2i\,,\end{aligned}$$

thus $(z-(1+3i))^2 = -3+4i\,.$
Let $(x+iy)^2 = -3+4i$
then $x^2 - y^2 = -3$
and $xy = 2$
so by inspection one solution is $x = 1$ and $y = 2\,.$
Hence $(z-(1+3i))^2 = (1+2i)^2$
and thus $z = (1+3i)+(1+2i)$ or $(1+3i)-(1+2i)$
that is $z = 2+5i$ or $i\,.$

Harder Complex Square Roots: In most cases the simultaneous equations given in Box 12 cannot be solved by inspection. Fortunately there is an identity that can be used to help easily solve these equations. Recall that if $(x+iy)^2 = a+ib$

then $x^2 - y^2 = a$ (1)
and $2xy = b\,.$ (2)

Squaring these and adding:

$$\begin{aligned} a^2 + b^2 &= (x^2-y^2)^2 + (2xy)^2 \\ &= (x^2)^2 + 2x^2y^2 + (y^2)^2 \\ &= (x^2+y^2)^2\,. \end{aligned}$$

Hence $x^2 + y^2 = \sqrt{a^2+b^2}$ (3)

Equations (1) and (3) now form a very simple pair of simultaneous equations to solve, and equation (2) is used to determine whether x and y have the same or opposite sign. To be explicit, if b is positive then x and y have the same sign, and if b is negative then they have opposite signs.

13 SQUARE ROOTS OF A COMPLEX NUMBER: The complex number $a+ib$ has a square root $x+iy$ where x and y are solutions of the pair of simultaneous equations

$$x^2 - y^2 = a$$
$$x^2 + y^2 = \sqrt{a^2+b^2}$$

with the same sign if b is positive, and opposite sign if b is negative.

WORKED EXERCISE: Determine the two square roots of $-4+2i\,.$

SOLUTION: Let $(x+iy)^2 = -4+2i\,.$ Since $\mathrm{Im}(-4+2i) > 0$, x and y have the same sign. Further, $a^2+b^2 = 20$, so we solve

$x^2 - y^2 = -4$ (1)
and $x^2 + y^2 = 2\sqrt{5}$ (2)

Adding (1) and (2) yields

$$2x^2 = -4 + 2\sqrt{5}$$

so $x = -\sqrt{-2+\sqrt{5}}$ or $\sqrt{-2+\sqrt{5}}\,.$

Subtracting (1) from (2) yields

$$2y^2 = 4 + 2\sqrt{5}$$

so $\qquad y = -\sqrt{2+\sqrt{5}}$ or $\sqrt{2+\sqrt{5}}\,.$

Hence $\qquad x + iy = -\sqrt{-2+\sqrt{5}} - i\sqrt{2+\sqrt{5}}$ or $\sqrt{-2+\sqrt{5}} + i\sqrt{2+\sqrt{5}}\,.$

In fact, the result in Box 13 can be used to develop a formula for the square roots of any complex number, which is derived in one of the Exercise questions. However that formula is not part of the course and should not be memorised.

Harder Quadratic Equations: With the aid of Box 13, it is now possible to solve all quadratic equations, including those with complex discriminants whose square roots cannot be found by inspection.

WORKED EXERCISE: [A HARD EXAMPLE] Solve $z^2 + (4-2i)z + 1 = 0$ by using the quadratic formula method.

SOLUTION:
$$\begin{aligned}\Delta &= (4-2i)^2 - 4\\ &= 12 - 16i - 4\\ &= 8 - 16i\,.\end{aligned}$$

Let $\qquad (x+iy)^2 = 8 - 16i.$

Now $\text{Im}(8-16i) < 0$ so x and y have opposite sign, with

$$x^2 - y^2 = 8 \qquad (1)$$

and $\qquad x^2 + y^2 = \sqrt{8^2 + 16^2}$

or $\qquad x^2 + y^2 = 8\sqrt{5} \qquad (2)$

Adding and subtracting equtions (1) and (2) yields

$$\begin{aligned}2x^2 &= 8 + 8\sqrt{5}\\ x^2 &= 4(1+\sqrt{5})\end{aligned} \qquad \text{and} \qquad \begin{aligned}2y^2 &= -8 + 8\sqrt{5}\\ y^2 &= 4(-1+\sqrt{5})\,.\end{aligned}$$

Thus $\qquad \Delta = \left(2\sqrt{1+\sqrt{5}} - 2i\,\sqrt{-1+\sqrt{5}}\right)^2$

and so $\qquad z = \tfrac{1}{2}\left(-4 + 2i + 2\sqrt{1+\sqrt{5}} - 2i\,\sqrt{-1+\sqrt{5}}\right)$

$\qquad\qquad$ or $\tfrac{1}{2}\left(-4 + 2i - 2\sqrt{1+\sqrt{5}} + 2i\,\sqrt{-1+\sqrt{5}}\right)$

that is $\qquad z = \left(\left(-2 + \sqrt{1+\sqrt{5}}\right) + i\left(1 - \sqrt{-1+\sqrt{5}}\right)\right)$

$\qquad\qquad$ or $\left(\left(-2 - \sqrt{1+\sqrt{5}}\right) + i\left(1 + \sqrt{-1+\sqrt{5}}\right)\right)$

Exercise **1B**

1. Solve for z:

(a) $z^2 + 9 = 0$ $\qquad$ (c) $z^2 + 2z + 5 = 0$ $\qquad$ (e) $16z^2 - 16z + 5 = 0$

(b) $(z-2)^2 + 16 = 0$ $\qquad$ (d) $z^2 - 6z + 10 = 0$ $\qquad$ (f) $4z^2 + 12z + 25 = 0$

2. Write as a product of two complex linear factors:

(a) $z^2 + 36$ $\qquad$ (c) $z^2 - 2z + 10$ $\qquad$ (e) $z^2 - 6z + 14$

(b) $z^2 + 8$ $\qquad$ (d) $z^2 + 4z + 5$ $\qquad$ (f) $z^2 + z + 1$

3. Form a quadratic equation with real coefficients given that one root is:
(a) $i\sqrt{2}$ (b) $1-i$ (c) $-1+2i$ (d) $2-i\sqrt{3}$

4. In each case, find the two square roots of the given number by the inspection method.
(a) $2i$ (c) $-8-6i$ (e) $-5+12i$ (g) $-15-8i$
(b) $3+4i$ (d) $35+12i$ (f) $24-10i$ (h) $9-40i$

DEVELOPMENT

5. (a) Find the two square roots of $-3-4i$.
(b) Hence solve $z^2-3z+(3+i)=0$.

6. (a) Find the two square roots of $-8+6i$.
(b) Hence solve $z^2-(7-i)z+(14-5i)=0$.

7. Solve for z:
(a) $z^2-z+(1+i)=0$ (d) $(1+i)z^2+z-5=0$
(b) $z^2+3z+(4+6i)=0$ (e) $z^2+(2+i)z-13(1-i)=0$
(c) $z^2-6z+(9-2i)=0$ (f) $iz^2-2(1+i)z+10=0$

8. (a) Find the value of w if i is a root of the equation $z^2+wz+(1+i)=0$.
(b) Find the real numbers a and b given that $3-2i$ is a root of the equation $z^2+az+b=0$.
(c) Given that $1-2i$ is a root of the equation $z^2-(3+i)z+k=0$, find k and the other root of the equation.

9. Find the two complex numbers z satisfying $z\,\overline{z}=5$ and $\dfrac{z}{\overline{z}}=\dfrac{1}{5}(3+4i)$.

10. (a) Solve $z^2-2z\cos\theta+1=0$ for z by completing the square.
(b) Rearranging the equation in part (a) gives $\cos\theta=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$. Confirm this result for each of the solutions to part (a) by substitution.

11. By first factoring the sum or difference of two cubes, solve for z:
(a) $z^3=-1$ (b) $z^3+i=0$

12. Consider the quadratic equation $az^2+bz+c=0$, where a, b and c are real and $b^2-4ac<0$. Suppose that ω is one of the complex roots of the equation.
(a) Explain why $a\omega^2+b\omega+c=0$.
(b) By taking the conjugate of both sides of the result in (a), and using the properties of conjugates, show that $a\left(\overline{\omega}\right)^2+b\,\overline{\omega}+c=0$.
(c) What have you just proved about the two complex roots of the equation?

13. Suppose that $z=\alpha$ is a complex solution to a quadratic equation with real coefficients.
(a) Which other number is also a solution of this quadratic equation?
(b) Hence prove that the quadratic equation is $z^2-2\,\mathrm{Re}(\alpha)z+\alpha\,\overline{\alpha}=0$.

14. Let $(x+iy)^2=a+ib$, then we have $x^2-y^2=a$ and $2xy=b$.
(a) For the moment, assume that both a and b are positive.
(i) Sketch the graphs of these two equations on the same number plane.
(ii) What feature of your sketch indicates that there are two square roots of $a+ib$?
(b) Investigate how the sketch changes when either a or b or both are negative or zero.

EXTENSION

15. Use the results of Box 13 to find the two square roots of:
(a) $-i$ (b) $-6+8i$ (c) $2+2i\sqrt{3}$ (d) $10-24i$ (e) $2-4i$

16. Find the discriminant and its square roots, and hence solve:
(a) $z^2+(4+2i)z+(1+2i)=0$
(b) $z^2-2(1+i)z+(2+6i)=0$
(c) $z^2+2(1-i\sqrt{3})z+2+2i\sqrt{3}=0$
(d) $z^2+(1-i)z+(i-1)=0$

17. Let α and β be the two complex roots of $z^3=1$. Show that:
(a) $\beta=\overline{\alpha}$, (b) $\alpha^2=\beta$ and $\beta^2=\alpha$, (c) $1+\alpha+\alpha^2=0$,
(d) the sum of the first n terms of the series $1+\alpha+\alpha^2+\alpha^3+\ldots$ is either 0, 1 or $-\alpha^2$, depending on the remainder when n is divided by 3.

18. Let a, b and c be real with $b^2-4ac<0$, and suppose that the quadratic equation $az^2+bz+c=0$ has complex solutions $\alpha=x+iy$ and $\beta=u+iv$.
(a) By considering the sum and product of the roots, show that

$$\mathrm{Im}(\alpha+\beta)=0 \quad\text{and}\quad \mathrm{Im}(\alpha\beta)=0\,.$$

(b) Hence show that $\alpha=\overline{\beta}$.

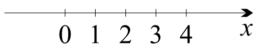
19. Let $(x+iy)^2=a+ib$, where $b\neq 0$. Use the result of Box 13 to prove the formula:

$$x+iy=\pm\left(\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)}+i\,\frac{b}{|b|}\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}\right).$$

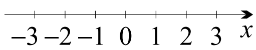
Explain the significance of the term $b/|b|$ in this formula.

1C The Argand Diagram

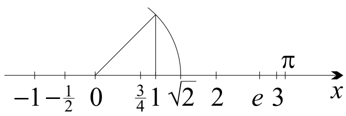
Mathematics requires a knowledge of numbers, and in our course of study at school our understanding of numbers has been enhanced by being able to plot them on a number line, to visualise their properties and relationships. Initially there were the natural numbers, shown at discrete intervals on the number line.



When negative numbers were included to create the integers, the number line was extended to the left of the origin to show these new numbers.



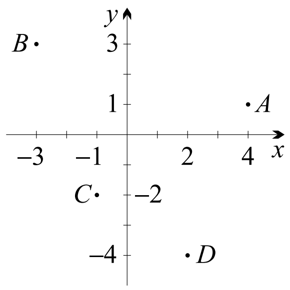
Next came the rationals, the fractions which exist in the spaces between integers. Eventually we became aware of strange numbers called irrationals which fit in the "gaps" that are somehow left between rationals. Some irrational numbers like $\sqrt{2}$ can be constructed geometrically, but others like e and π can only be approximated to so many decimal places. The construction for $\sqrt{2}$ is shown here along with the positions of $-\frac{1}{2}$, $\frac{3}{4}$, e and π.



The number line is now full, the reals have filled it up, and there is no space left for any new objects like complex numbers. Further, since complex numbers come in two parts, real and imaginary, there is no satisfactory way of representing them on a number line. A two dimensional representation is needed.

The Complex Number Plane: Keeping to things that are familiar, the number plane would seem to be a convenient way to represent complex numbers. More formally, for each complex number $z = x + iy$ there corresponds a point $Z(x, y)$ in the Cartesian plane. Equally, given any point $W(a, b)$ in the real number plane, the associated complex number is $w = a + ib$.

Thus in the diagram on the right, the complex numbers $4 + i$ and $-3 + 3i$ are represented by the points A and B respectively. The points C and D represent the complex numbers $-1 - 2i$ and $2 - 4i$. Several different names are used to describe a coordinate plane that is used to represent complex numbers. One name is the *Argand diagram*, after the French mathematician Jean-Robert Argand, born in Geneva in 1768. The terms *complex number plane* or z-plane are also used.



14 THE ARGAND DIAGRAM: The complex number $z = x + iy$ is associated with the point $Z(x, y)$ in the real number plane. A complex number may be represented by a point, and a point may be represented by a complex number.

As a convenient abbreviation, the point $Z(x, y)$ will sometimes be simply referred to as the point z in the Argand diagram. It is important to remember that the complex number plane is just a real number plane which is conveniently used to display complex numbers. By the nature of this representation, if two complex numbers are equal then they represent the same point. The converse is also true.

The Real and Imaginary Axes: If $\text{Im}(z) = 0$, that is $z = x + 0i$, then z is a real number and the corresponding point $Z(x, 0)$ in the Argand diagram lies on the horizontal axis. Thus the horizontal axis is called the *real axis*.

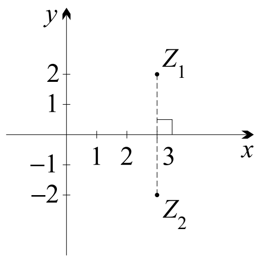
Likewise, if $\text{Re}(z) = 0$, that is $z = 0 + iy$, then z is an imaginary number and the corresponding point $Z(0, y)$ in the Argand diagram lies on the vertical axis. Thus the vertical axis is called the *imaginary axis*.

Some Simple Geometry: Now that the complex plane has been introduced, it is immediately possible to observe the geometry of some simple complex number operations. In particular we will consider the geometry of conjugates, opposites, and multiplication by i.

Let $\quad z = x + iy \quad$ then the conjugate is

$$\overline{z} = x - iy\,,$$

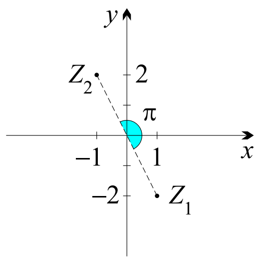
that is, y has been replaced by $-y$. This was encountered in the work on graphs and is known to be a reflection in the real axis. This is clearly evident in the example of $z_1 = 3 + 2i$ and $z_2 = 3 - 2i = \overline{z_1}$ shown on the right.



15 THE GEOMETRY OF CONJUGATES: The points representing z and $\overline{z}$ in the Argand diagram are reflections of each other in the real axis.

Let $\quad z = x + iy \quad$ then the opposite is

$$-z = -x - iy\,.$$

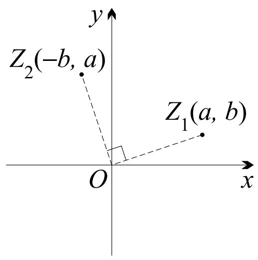


In this case, x and y have been replaced by $-x$ and $-y$ respectively. Thus the result is obtained by reflecting the point in both axes in succession. Alternatively, it is a rotation by π about the origin. The diagram on the right with $z_1 = 1 - 2i$ and $z_2 = -1 + 2i = -z_1$ demonstrates this rotation.

16 THE GEOMETRY OF OPPOSITES: The points representing z and $-z$ in the Argand diagram are rotations of each other by π about the origin.

Let $\quad z_1 = a + ib \quad$ then $z_2 = i\,z_1$ is given by

$$z_2 = -b + ia\,.$$



Consider the corresponding points Z_1 and Z_2 shown in the Argand diagram on the right, where neither a nor b is zero. The product of the gradients of OZ_1 and OZ_2 is

$$\frac{b}{a} \times \frac{a}{-b} = -1\,.$$

Hence OZ_2 is perpendicular to OZ_1 and the conclusion is that multiplication by i is equivalent to an anticlockwise rotation by $\frac{\pi}{2}$ about the origin. The situation is the same whenever z_1 is real or imaginary, but not zero, and the proof is left as an exercise.

17 THE GEOMETRY OF MULTIPLICATION BY i: The point representing iz is the result of rotating the point representing z by $\frac{\pi}{2}$ anticlockwise about the origin.

Notice that multiplication by i twice in succession yields a rotation in the Argand diagram of $2 \times 90° = 180°$. This is consistent with the geometry of opposites, since $i(iz) = i^2 z = -z$.

WORKED EXERCISE: Let $z = x + iy$. Determine $i\,\overline{z}$ and hence give a geometric interpretation of the result.

SOLUTION:

$$\begin{aligned} i\,\overline{z} &= i(x - iy) \\ &= y + ix\,. \end{aligned}$$

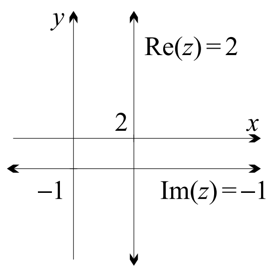
This is just z with x and y swapped. Thus it is a reflection in the line $y = x$.

Simple Locus Problems:

So far we have concentrated our attention on individual points in the complex plane. Often an equation in z will correspond to a whole collection of points, that is a locus. In the simple cases dealt with here, the equation of that locus can be found by putting $z = x + iy$.

WORKED EXERCISE: Graph the following loci:
(a) $\mathrm{Re}(z) = 2$, (b) $\mathrm{Im}(z) = -1$.

SOLUTION: The two loci are:
(a) the vertical line $x = 2$, and
(b) the horizontal line $y = -1$,
as shown in the diagram on the right.
Note that these two lines intersect at $z = 2 - i$.



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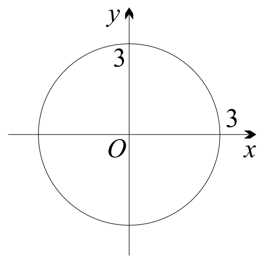
VERTICAL AND HORIZONTAL LINES: In the Argand diagram: • the equation $\mathrm{Re}(z) = a$ is the vertical line $x = a$ • the equation $\mathrm{Im}(z) = b$ is the horizontal line $y = b$ • these two lines intersect at $z = a + ib$.

WORKED EXERCISE: Let the point P in the complex plane represent the number $z = x + iy$. Given that $z\,\overline{z} = 9$, find the locus of P and sketch it.



SOLUTION: The given equation becomes

$$(x + iy)(x - iy) = 9$$

so $$x^2 + y^2 = 3^2$$

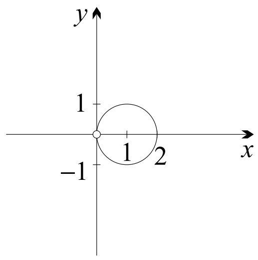
that is a circle with centre the origin and radius 3.

In some examples it is best to manipulate the given equation in z first, and then substitute $x+iy$. It is also important to note any restrictions on z before starting. Both of these points feature in the following example.

WORKED EXERCISE: Find and describe the locus of z in the Argand diagram given

$$\frac{1}{z} + \frac{1}{\overline{z}} = 1\,.$$

SOLUTION: Note that in the given equation $z \neq 0$, since the LHS is undefined there. Multiply both sides by the lowest common denominator to get



$$\overline{z} + z = z\,\overline{z}$$

so $$2x = x^2 + y^2$$

or $$0 = x^2 - 2x + y^2$$

thus $$1 = (x - 1)^2 + y^2$$

that is, the circle with radius 1 and centre $(1, 0)$, excluding the origin.

Exercise **1C**

1. Write down the coordinates of the point in the complex plane that represents:

(a) 2 (b) i (c) $-3 + 5i$ (d) $\overline{2 + 2i}$ (e) $-5(1 + i)$ (f) $(2 + i)i$

2. Write down the complex number that is represented by the point:

(a) $(-3, 0)$ (b) $(0, 3)$ (c) $(7, -5)$ (d) (a, b)

3. Let $z = 1 + 3i$, and let A, B, C and D be the points representing z, iz, i^2z and i^3z respectively.

(a) Plot the points A, B, C and D in the complex plane.

(b) What type of special quadrilateral is $ABCD$?

(c) What appears to be the geometric effect of multiplying a complex number by i?

4. Let $z = 3 + i$ and $w = 1 + 2i$. Plot the points representing each group of complex numbers on separate Argand diagrams.

(a) $z,\ iz,\ -z,\ -iz$ (c) $z,\ \overline{z},\ w,\ \overline{w}$ (e) $z,\ w,\ z - w$
(b) $w,\ iw,\ -w,\ -iw$ (d) $z,\ w,\ z + w$ (f) $z,\ w,\ w - z$

5. Graph the following loci:

(a) $\mathrm{Re}(z) = -3$ (c) $\mathrm{Im}(z) < 1$ (e) $\mathrm{Re}(z) = \mathrm{Im}(z)$ (g) $\mathrm{Re}(z) \le 2\,\mathrm{Im}(z)$
(b) $\mathrm{Im}(z) = 2$ (d) $\mathrm{Re}(z) \ge -2$ (f) $2\,\mathrm{Re}(z) = \mathrm{Im}(z)$ (h) $\mathrm{Re}(z) > -\,\mathrm{Im}(z)$

DEVELOPMENT

6. Let the point P represent the complex number $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$, and let the points Q, R, S and T, represent $\overline{z}$, $-z$, iz and $\frac{1}{z}$ respectively. Plot all these points on an Argand diagram.

7. Show that the point representing $-\overline{z}$ is a reflection of the point representing z in the y-axis.

8. Consider the points represented by the complex numbers z, $\overline{z}$, $-z$ and $-\overline{z}$.
Show that these points form a rectangle by using:

(a) coordinate geometry to show that the diagonals are equal and bisect each other,

(b) the geometry of conjugates and opposites.

9. In the text it was proven that when z is complex, iz is a rotation by $\frac{\pi}{2}$ about the origin. Prove the same result when z is: (a) real, (b) imaginary.

10. The numbers $z = a + ib$ and $w = iz$ are plotted in the complex plane at A and B respectively.

(a) By considering the gradients, show that $OA \perp OB$.

(b) Use the distance formula to show that $OA = OB$.

(c) What type of triangle is $\triangle OAB$?

11. The point P in the complex plane represents the number z. Find and describe the locus of P given that

$$\frac{1}{z} - \frac{1}{\overline{z}} = i\,.$$

12. The complex number z is represented by the point C in the Argand diagram. Find and describe the locus of C if

$$\mathrm{Re}\left(\frac{z-6}{z}\right) = 0\,.$$

13. Show that $(z-2)\,\overline{(z-2)} = 9$ represents a circle in the Argand diagram.

14. Find and describe the locus of points in the Argand diagram which correspond to

$$z\,\overline{z} = \Big(\mathrm{Re}(z - 1 + 3i)\Big)^2\,.$$

EXTENSION

15. Let the point H represent z in the complex plane. Draw the loci of H if:

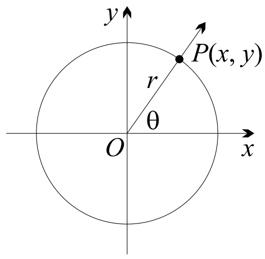
(a) $\mathrm{Im}\left(z^2\right) = 2c^2$ (b) $\mathrm{Re}\left(z^2\right) = c^2$

16. Show that the point representing $-i\overline{z}$ is a reflection of the point representing z in $y = -x$.

17. Show that $\frac{1}{z}$ is a reflection and enlargement of z.

1D Modulus-Argument Form

The Modulus and Argument of a Complex Number:



Recall that in the study of trigonometry it was found that the location of a point P could be expressed either in terms of its horizontal and vertical positions, x and y, or in terms of its distance $OP = r$ from the origin and the angle θ that the ray OP makes with the positive x-axis. The situation is shown in the number plane on the right.

In the complex number plane the distance r is called the *modulus* of z, and owing to its geometric definition as a distance it is written as $|z|$. On squaring:

$$\begin{aligned} |z|^2 &= r^2 \\ &= x^2 + y^2 \\ &= (x+iy)(x-iy) \quad \text{(sum of two squares)} \end{aligned}$$

hence $|z|^2 = z\overline{z}\,.$

The angle θ is called the *argument* of z, and is written $\theta = \arg(z)$. Just as with trigonometry, θ can take infinitely many values for the same point P, but the convention in this course is to choose the value for which $-\pi < \arg(z) \le \pi$. Note the strict inequality on the left hand side, and the use of radian measure.

19 MODULUS AND ARGUMENT: Let P represent the complex number $z = x + iy$ in the Argand diagram, with origin O.
- The *modulus* of z is the distance $|z| = r = OP$. Note that $|z|^2 = z\overline{z}$.
- The *argument* of z is the angle $\arg(z) = \theta$ that the ray OP makes with the positive real axis. By convention, we choose the value of θ for which $-\pi < \theta \le \pi\,.$

From the trigonometric definitions it is clear that

$$x = r\cos\theta \qquad (1)$$

and $$y = r\sin\theta\,, \qquad (2)$$

from which it follows that

$$z = r\cos\theta + ir\sin\theta\,.$$

Notice that the modulus r is a common factor in this last expression and it is more commonly written as

$$z = r(\cos\theta + i\sin\theta)$$

or $$z = r\operatorname{cis}\theta \qquad \text{for short.}$$

In order to contrast the two ways of writing a complex number, $z = x + iy$ is called *real-imaginary* or *Cartesian* form whilst $z = r(\cos\theta + i\sin\theta)$ is called *modulus-argument* form, or *mod-arg* form for short. Equations (1) and (2) above serve to link the two forms.

WORKED EXERCISE: Express each complex number in real-imaginary form.

(a) $z = 4\operatorname{cis}\pi$ (b) $z = 2\operatorname{cis}\frac{\pi}{6}$ (c) $z = \operatorname{cis}\frac{2\pi}{3}$

SOLUTION:

(a) $z = 4\cos\pi + 4i\sin\pi$
$\quad = -4$

(b) $z = 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6}$
$\quad = \sqrt{3} + i$

(c) $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$
$\quad = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$

WORKED EXERCISE: Express each complex number in mod-arg form. In part (c) give $\arg(z)$ correct to two decimal places.

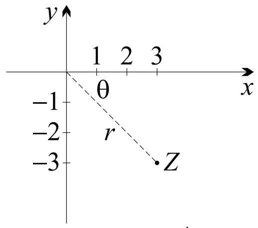
(a) $z = 5i$ (b) $z = 3 - 3i$ (c) $z = -4 - 3i$

SOLUTION: In each case let $z = r\,\mathrm{cis}\,\theta$ with Z the point in the Argand diagram.

(a) In this case Z is on the positive imaginary axis so

$r = 5$ and $\theta = \frac{\pi}{2}$,

hence $z = 5\,\mathrm{cis}\,\frac{\pi}{2}$.



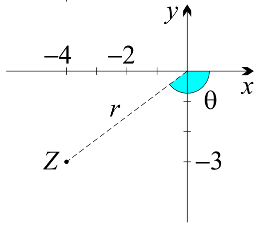
(b) $r^2 = 3^2 + 3^2$

so $r = 3\sqrt{2}$.

Now $\cos\theta = \dfrac{1}{\sqrt{2}}$ and Z is in the fourth quadrant

so $\theta = -\frac{\pi}{4}$,

hence $z = 3\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$.



(c) $r = 5$ (Pythagorian triad)

Now $\cos\theta = -\frac{4}{5}$ and Z is in the third quadrant

so $\theta = -\pi + \cos^{-1}\frac{4}{5} \doteqdot -2{\cdot}50$ radians,

hence $z \doteqdot 5\,\mathrm{cis}(-2{\cdot}50)$.

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FORMS OF A COMPLEX NUMBER:
- $x + iy$ is called the *real-imaginary* form or *Cartesian* form of z.
- $r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$ is called the *modulus-argument* form of z.
- The equations relating the two forms are:

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

Some Simple Algebra: As we shall see in the remainder of this chapter, the use of mod-arg form is a powerful tool, both in simplifying much algebra and in providing geometric interpretations. The first thing to notice is that $|0| = 0$ but that $\arg(0)$ is undefined. This is because $0 = 0\,\mathrm{cis}\,\theta$ for all values of θ.

The second thing to notice is that $|\,\mathrm{cis}\,\theta| = 1$. The geometry of the situation makes the result obvious since if $z = \cos\theta + i\sin\theta$ then the point $Z(\cos\theta, \sin\theta)$ lies on the unit circle. Hence $|z| = OZ = 1$. Here is an algebraic derivation of the same result.

$$\begin{aligned} |\cos\theta + i\sin\theta|^2 &= \cos^2\theta + \sin^2\theta \\ &= 1, \end{aligned}$$

hence $|\,\mathrm{cis}\,\theta| = 1$.

This identity has immediate applications in quadratic equations.

WORKED EXERCISE: Find a quadratic equation with real coefficients given that one root is $z = \mathrm{cis}\,\theta$.

SOLUTION: The other root must be $\overline{\mathrm{cis}\,\theta}$. Thus the quadratic equation is

$$(z - \mathrm{cis}\,\theta)(z - \overline{\mathrm{cis}\,\theta}) = 0$$

or $z^2 - (\mathrm{cis}\,\theta + \overline{\mathrm{cis}\,\theta})z + \mathrm{cis}\,\theta \times \overline{\mathrm{cis}\,\theta} = 0$

that is $z^2 - 2\,\mathrm{Re}(\mathrm{cis}\,\theta)z + |\,\mathrm{cis}\,\theta|^2 = 0$

thus $z^2 - 2z\cos\theta + 1 = 0$.

The Product of Two Complex Numbers: The modulus-argument form of the product of two numbers is a particularly important result. Let $w = a \operatorname{cis}\theta$ and $z = b \operatorname{cis}\phi$, with $a \neq 0$ and $b \neq 0$, then

$$\begin{aligned} wz &= a(\cos\theta + i\sin\theta) \times b(\cos\phi + i\sin\phi) \\ &= ab\Big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\Big) \\ &= ab\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big)\,, \end{aligned}$$

that is $\quad a\operatorname{cis}\theta \times b\operatorname{cis}\phi = ab\operatorname{cis}(\theta+\phi)\,.$

Thus $\qquad |wz| = ab \qquad$ and $\qquad \arg(wz) = \theta + \phi\,.$

This yields the following two significant results:

$$|wz| = |w|\,|z|$$

and

$$\arg(wz) = \arg(w) + \arg(z)\,.$$

WORKED EXERCISE: Let $w = \sqrt{3} + i$ and $z = 1 + i$.

(a) Evaluate wz in real-imaginary form.

(b) Express w and z in mod-arg form and hence evaluate wz in mod-arg form.

(c) Hence find the exact value of $\cos\frac{5\pi}{12}$.

SOLUTION:

(a)
$$\begin{aligned} wz &= (\sqrt{3}+i)(1+i) \\ &= (\sqrt{3}-1) + i(\sqrt{3}+1)\,. \end{aligned}$$

(b) Now
$$\begin{aligned} w &= 2\operatorname{cis}\tfrac{\pi}{6} \\ z &= \sqrt{2}\operatorname{cis}\tfrac{\pi}{4}\,, \end{aligned}$$
hence
$$\begin{aligned} wz &= 2\sqrt{2}\operatorname{cis}(\tfrac{\pi}{6}+\tfrac{\pi}{4}) \\ &= 2\sqrt{2}\operatorname{cis}\tfrac{5\pi}{12}\,. \end{aligned}$$

(c) Equating the real parts of parts (a) and (b) yields

$$2\sqrt{2}\cos\tfrac{5\pi}{12} = \sqrt{3}-1$$

hence
$$\cos\tfrac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}\,.$$

21 THE PRODUCT OF TWO COMPLEX NUMBERS: Let w and z be two complex numbers.

- The modulus of the product is the product of the moduli, that is:
$$|wz| = |w|\,|z|\,.$$
- The argument of the product is the sum of the arguments, that is:
$$\arg(zw) = \arg(w) + \arg(z) \qquad \text{(provided } w \neq 0 \text{ and } z \neq 0\,.)$$

Some Simple Geometry Again: It is instructive to re-examine the geometry of conjugates, opposites and multiplication by i using mod-arg form. Beginning with the conjugate, recall that the geometrical interpretation is a reflection in the real axis. Thus the modulus should be unchanged, and the argument should be opposite.

Let $z = r\operatorname{cis}\theta$ then
$$\begin{aligned} \overline{z} &= r\cos\theta - ir\sin\theta \\ &= r\cos(-\theta) + ir\sin(-\theta) \end{aligned}$$

$$= r\,\mathrm{cis}(-\theta)$$

hence $\qquad |\overline{z}| = |z|$

and $\qquad \arg(\overline{z}) = -\arg(z)$

that is, the modulus is unchanged and the angle is opposite, as expected.

The cases of opposites and multiplication by i are more simply dealt with. Recall that these operations represented rotations in the complex plane by π and $\frac{\pi}{2}$ respectively. Thus, again, the modulus should be the same, and the argument should be increased appropriately. Looking at opposites first:

$$|-z| = |(-1)\times z| = |-1|\times|z| = |z|\,,$$

and $\qquad \arg(-z) = \arg(-1\times z) = \arg(-1) + \arg(z) = \pi + \arg(z)\,.$

That is, the moduli of opposites are equal and the arguments differ by π.

Similarly $\qquad |iz| = |i|\,|z| = |z|\,,$

and $\qquad \arg(iz) = \arg(i) + \arg(z) = \frac{\pi}{2} + \arg(z)\,.$

That is, the moduli of z and iz are equal and the arguments differ by $\frac{\pi}{2}$. In both cases the results are exactly as expected.

The Geometry of Multiplication and Division: Aside from the special cases above, the general geometry of multiplication and division is evident in the results of Box 21. The product of the moduli indicates an enlargement with centre the origin, and the sum of the arguments represents an anticlockwise rotation about the origin.

Consider these two transformations individually and let $w = r\,\mathrm{cis}\,\theta$. When $\theta = 0$ the product wz reduces to $wz = rz$, which is an enlargement without any rotation. Thus both z and rz lie on the same ray.

When $|w| = r = 1$ the product wz becomes $wz = z\,\mathrm{cis}\,\theta$. Using Box 21:

$$\begin{aligned} |wz| &= |w||z| \\ &= |z| \end{aligned}$$

and
$$\begin{aligned} \arg(wz) &= \arg w + \arg z \\ &= \theta + \arg z\,. \end{aligned}$$

This is simply a rotation without any enlargement. Thus z and $z\,\mathrm{cis}\,\theta$ both lie on a circle of radius $|z|$. The following example serves to demonstrate the situation.

WORKED EXERCISE: Let $w = \frac{1}{5}(3+4i)$ and $z = 1+i$. (a) Show that $|w| = 1$.

(b) Evaluate wz and hence confirm that $|wz| = |z|$.

(c) Plot z and wz on the Argand diagram.

(d) What is the angle subtended by these two points at the origin, correct to two decimal places.

SOLUTION:

(a)
$$\begin{aligned} |w|^2 &= (\tfrac{3}{5})^2 + (\tfrac{4}{5})^2 \\ &= 1 \end{aligned}$$

hence $\qquad |w| = 1\,.$

(b)
$$wz = \tfrac{1}{5}(-1+7i)$$

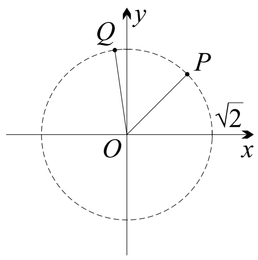
so $\qquad |wz|^2 = \dfrac{1+49}{25}$

$$= 2$$

hence $|wz| = \sqrt{2}$

$$= |z| \,.$$

(c) The number z is shown at P and wz is at Q.



(d) Let $\angle POQ = \theta = \arg(w)$, then

$$\cos\theta = \tfrac{3}{5} \qquad \text{and} \qquad \sin\theta = \tfrac{4}{5}$$

whence θ is acute and

$$\theta = \cos^{-1}\tfrac{3}{5}$$
$$\fallingdotseq 0{\cdot}93 \text{ radians}$$

22 THE GEOMETRY OF MULTIPLICATION: Let $w = r \operatorname{cis}\theta$ then the complex number wz is the result of a rotation of z by θ about the origin and an enlargement of z by factor r with centre the origin.

The corresponding explanation for division is obtained by first writing

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2} = z \times \frac{1}{r}\operatorname{cis}(-\theta)\,.$$

Thus dividing by w yields an enlargement by factor $\frac{1}{r}$ and a rotation about the origin of $-\theta$. In the special case where $z = 1$ we get the reciprocal of w with

$$\frac{1}{w} = \frac{1}{r}\operatorname{cis}(-\theta)\,,$$

whence $|w^{-1}| = |w|^{-1}$ and $\arg(w^{-1}) = -\arg(w)\,.$

Exercise **1D**

1. Express each complex number in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$.

(a) $2i$ (b) -4 (c) $1 + i$ (d) $\sqrt{3} - i$ (e) $-1 + \sqrt{3}\,i$ (f) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

2. Repeat the previous question for each of these complex numbers, writing θ in radians correct to two decimal places.

(a) $3 + 4i$ (b) $12 - 5i$ (c) $-2 + i$ (d) $-1 - 3i$

3. Express in the form $a + ib$, where a and b are real:

(a) $3\operatorname{cis}0$ (b) $5\operatorname{cis}\left(-\frac{\pi}{2}\right)$ (c) $4\operatorname{cis}\frac{\pi}{4}$ (d) $6\operatorname{cis}\left(-\frac{\pi}{6}\right)$ (e) $2\operatorname{cis}\frac{3\pi}{4}$ (f) $2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

4. Given that $z = 1 - i$, express in mod-arg form:

(a) z (b) $\overline{z}$ (c) $-z$ (d) iz (e) z^2 (f) $(\overline{z})^{-1}$

5. Simplify each expression, leaving your answer in mod-arg form:

(a) $5\operatorname{cis}\frac{\pi}{12} \times 2\operatorname{cis}\frac{\pi}{4}$ (b) $3\operatorname{cis}\theta \times 3\operatorname{cis}2\theta$ (c) $6\operatorname{cis}\frac{\pi}{2} \div 3\operatorname{cis}\frac{\pi}{6}$ (d) $\dfrac{3\operatorname{cis}5\alpha}{2\operatorname{cis}4\alpha}$ (e) $\left(4\operatorname{cis}\frac{\pi}{5}\right)^2$ (f) $\left(2\operatorname{cis}\frac{2\pi}{7}\right)^3$

DEVELOPMENT

6. In the complex plane, mark a point K to represent a complex number $z = r\operatorname{cis}\theta$ where $1 < r < 2$ and $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. Hence indicate clearly the points M, N, P, Q, and R representing $\overline{z}$, $-z$, $2z$, iz and $\frac{1}{z}$ respectively.

7. Let z be a non-zero complex number such that $0 < \arg z < \frac{\pi}{2}$. Indicate points A, B, C and D in the complex plane representing the complex numbers z, $-iz$, $(2\operatorname{cis}\frac{\pi}{3})z$ and $\left(\frac{1}{2}\operatorname{cis}(-\frac{\pi}{4})\right)z$.

8. Replace z with $z \div w$ in Box 21 to prove that for $z \neq 0$ and $w \neq 0$:

(a) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ (b) $\arg\left(\dfrac{z}{w}\right) = \arg z - \arg w$

9. Given that $z_1 = \sqrt{3} + i$ and $z_2 = 2\sqrt{2} + 2\sqrt{2}\,i$,

(a) write z_1 and z_2 in mod-arg form, (b) hence write $z_1 z_2$ and $\dfrac{z_2}{z_1}$ in mod-arg form.

10. Repeat the previous question for $z_1 = -\sqrt{3} + i$ and $z_2 = -1 - i$.

11. (a) Express $\dfrac{1 + i\sqrt{3}}{1 + i}$ in real-imaginary form.

(b) Write $1 + i$ and $1 + i\sqrt{3}$ in mod-arg form and hence express $\dfrac{1 + i\sqrt{3}}{1 + i}$ in mod-arg form.

(c) Hence find $\cos\frac{\pi}{12}$ in surd form.

12. Let $z = (\sqrt{3} + 1) + (\sqrt{3} - 1)i$.

(a) By writing $\frac{\pi}{12}$ as $\frac{\pi}{3} - \frac{\pi}{4}$, show that $\tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$.

(b) Hence write z in mod-arg form.

13. Let $z_1 = 1 + 5i$ and $z_2 = 3 + 2i$, and let $z = \dfrac{z_1}{z_2}$.

(a) Find $|z|$ without finding z.

(b) Find $\tan(\tan^{-1} 5 - \tan^{-1}\frac{2}{3})$, and hence find $\arg z$ without finding z.

(c) Hence write z in the form $x + iy$, where x and y are real.

14. Show that for any non-zero complex number $z = r\operatorname{cis}\theta$:

(a) $z\overline{z} = |z|^2$, (b) $\arg(z^2) = 2\arg(z)$, (c) if $|z| = 1$ then $\overline{z} = z^{-1}$.

15. Let z be any non-zero complex number. By considering $\arg(|z|^2)$, use the result in part (a) of the previous question to prove that $\arg\overline{z} = -\arg z$.

16. The complex number z satisfies the equation $|z - 1| = 1$. Square both sides and hence show that $|z|^2 = 2\operatorname{Re}(z)$.

17. If z is a complex number and $|2z - 1| = |z - 2|$, prove that $|z| = 1$.

18. Let $z = \cos\theta + i\sin\theta$. Determine z^2 in two different ways and hence show that:

(a) $\cos 2\theta = \cos^2\theta - \sin^2\theta$ (b) $\sin 2\theta = 2\sin\theta\cos\theta$

19. Let $z = \operatorname{cis}\theta$ and $w = \operatorname{cis}\phi$, that is $|z| = |w| = 1$. Evaluate $z + w$ in mod-arg form and hence show that $\arg(z + w) = \frac{1}{2}(\arg z + \arg w)$.

20. Let $z = 1 + \cos\theta + i\sin\theta$.

(a) Show that $|z| = 2\cos\frac{\theta}{2}$ and $\arg z = \frac{\theta}{2}$. (b) Hence show that $z^{-1} = \frac{1}{2} - \frac{1}{2}i\tan\frac{\theta}{2}$.

EXTENSION

21. [CIRCLE GEOMETRY] The three complex numbers z_0, z_1 and z_2 are related to each other by the equations $z_2 = z_0 + i\lambda z_0$ and $z_2 = z_1 - i\lambda z_1$, where λ is real.

(a) Show that $|z_2 - z_0| = |z_2 - z_1|$. (b) Show that $|z_0| = |z_1|$.

(c) Use circle geometry to describe the situation in the Argand diagram.

22. (a) Prove that $\mathrm{Re}(z) \le |z|$. Under what circumstances are they equal?

(b) Prove that $|z+w| \le |z| + |w|$. Begin by writing $|z+w|^2 = (z+w)\overline{(z+w)}$.

23. (a) Let $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$ be any two complex numbers. Prove that:

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(b) Let z_1, z_2, z_3, …, z_n be complex numbers. Prove by induction that for integers $n \ge 2$:

(i) $|z_1 z_2 z_3 \ldots z_n| = \prod_{i=1}^{n} |z_i|$ (ii) $\arg(z_1 z_2 z_3 \ldots z_n) = \sum_{i=1}^{n} \arg(z_i)$

1E Vectors and the Complex Plane

The goemetry of multiplication and division became evident with the introduction of the modulus-argument form in the previous section. Since the arguments are added or subtracted, it is clear that a rotation is involved. Since the moduli are multiplied or divided, it is clear that an enlargement is involved.

So far, the observed geometry of addition and subtraction has been limited. A better understanding of these two operations is desirable and can be achieved by yet another representation of complex numbers, as vectors.

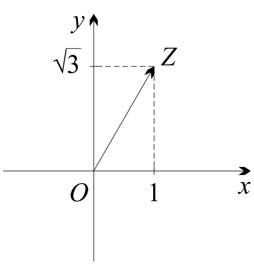
Vectors: A vector has two characteristics, a magnitude and a direction. Thus the instruction on a pirate treasure map “walk 40 paces east” is an example of a displacement vector. The magnitude is “40 paces” and the direction is “east”. A train travelling from Sydney to Perth across the Nullabor at 120 km/h is an example of a velocity vector. The magnitude is 120 km/h and the direction is west.

On the number plane, a vector is represented by an arrow, or more formally a directed line segment. The length of the arrow indicates the magnitude of the vector and the direction of the arrow is the direction of the vector. In particular, in the Argand diagram we will use an arrow joining two points to represent the vector from one complex number to another. When naming a vector, the two letter name of the line segment is used with an arrow above it to indicate the direction, as in the following two examples.

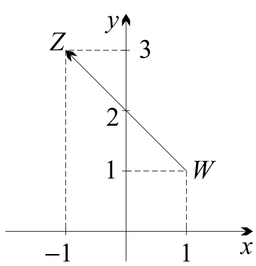
WORKED EXERCISE: In the Argand diagram, draw the vectors which represent:
(a) $\overrightarrow{OZ}$ where $z = 1 + i\sqrt{3}$, (b) $\overrightarrow{WZ}$ where $w = 1 + i$ and $z = -1 + 3i$.

SOLUTION:

(a)



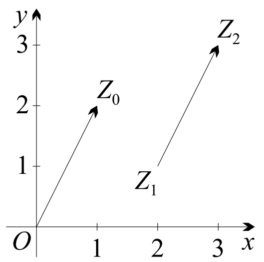
(b)



It should be clear from part (a) of the above exercise that the magnitude of the vector $\overrightarrow{OZ}$ is $|z| = 2$ and the direction is $\arg(z) = \frac{\pi}{3}$. By analogy with shifting in the number plane, the magnitude of the vector $\overrightarrow{WZ}$ in part (b) is $|z - w| = 2\sqrt{2}$ and the direction is $\arg(z - w) = \frac{3\pi}{4}$ radians.

23 VECTORS: The vector $\overrightarrow{WZ}$ from w to z in the Argand diagram has magnitude $|z - w|$ and direction $\arg(z - w)$.

If a vector is translated in the number plane, its length and direction do not change. In the diagram on the right Z_0, Z_1 and Z_2 represent the complex numbers $1+2i$, $2+i$ and $3+3i$ respectively. If the vector $\overrightarrow{OZ_0}$ is shifted so that its tail is at Z_1 then its head is at Z_2. Since the vectors $\overrightarrow{OZ_1}$ and $\overrightarrow{Z_1Z_2}$ have the same magnitude and direction, we say that they are equal and write

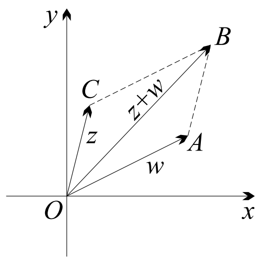


$$\overrightarrow{OZ_0} = \overrightarrow{Z_1Z_2}\ .$$

24 EQUAL VECTORS: Two vectors are said to be equal if they have the same magnitude and direction.

It may be tempting to say that two vectors are equal if they have the same magnitude and are parallel, but this is wrong. In the above example, the vectors $\overrightarrow{OZ_1}$ and $\overrightarrow{Z_2Z_1}$ are parallel but not equal since they have the opposite direction.

Addition and Subtraction: Consider the three points A, B and C which represent the complex numbers w, $w + z$ and z. The direction of $\overrightarrow{AB}$ is



$$\arg\big((w+z) - w\big) = \arg(z)\,,$$

thus $AB||OC$. Likewise, the direction of $\overrightarrow{CB}$ is

$$\arg\big((w+z) - z\big) = \arg(w)\,,$$

thus $CB||OA$. Hence $OABC$ is a parallelogram, from which it follows that the opposite sides are equal in length. That is the corresponding vectors have the same magnitude. We now have two pairs of vectors with the same magnitude and direction, so

$$\overrightarrow{OC} = \overrightarrow{AB}$$

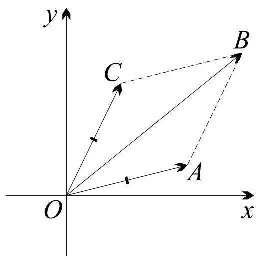
and $$\overrightarrow{OA} = \overrightarrow{CB}\ .$$

Thus we observe that in order to add two complex numbers geometrically, we simply construct the parallelogram $OABC$ from the vectors $\overrightarrow{OA}$ and $\overrightarrow{OC}$, with the sum being the diagonal $\overrightarrow{OB}$. This result is most useful in solving certain algebraic problems geometrically.

WORKED EXERCISE: Given two non-zero complex numbers w and z with equal moduli, show that $\arg(w + z) = \frac{1}{2}\big(\arg(w) + \arg(z)\big)$.

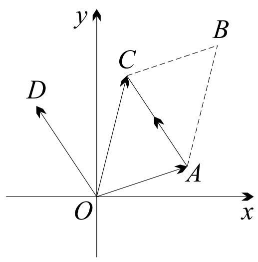
SOLUTION: Consider the points $OABC$ in the z-plane representing the complex numbers 0, w, $w+z$ and z respectively. Now $OABC$ is a parallelogram. Further $OA = OC$, since $|w| = |z|$. Thus in fact $OABC$ is a rhombus. Since the diagonal OB of the rhombus bisects the angle at the vertex O, it follows that

$$\arg(w+z) = \tfrac{1}{2}\big(\arg(w) + \arg(z)\big).$$



25 THE GEOMETRY OF ADDITION: Let the points O, A and C represent the complex numbers 0, w and z. Construct the parallelogram $OABC$. The diagonal vector $\overrightarrow{OB}$ represents the complex number $z+w$.

Given that one diagonal of the parallelogram represents the sum of two complex numbers, it is logical to ask what the other diagonal represents. The vector $\overrightarrow{AC}$ is from w to z, thus it has magnitude $|z-w|$ and its direction is $\arg(z-w)$. That is, it represents the complex number $z-w$. The position D of this point in the Argand diagram is determined by translating the vector $\overrightarrow{AC}$ so that its tail is at the origin, as shown on the right.



WORKED EXERCISE: The points $OABC$ represent the complex numbers 0, w, $w+z$ and z. Given that $z-w = i(z+w)$, explain why $OABC$ is a square.

SOLUTION: Firstly $OABC$ is a parallelogram, where $\overrightarrow{OB}$ represents $z+w$ and $\overrightarrow{AC}$ represents $z-w$. Since $z-w = i(z+w)$ it follows that

$$\begin{aligned} \arg(z-w) &= \arg(i(z+w)) \\ &= \arg(i) + \arg(z+w) \\ &= \tfrac{\pi}{2} + \arg(z+w), \end{aligned}$$

and

$$\begin{aligned} |z-w| &= |i(z+w)| \\ &= |i| \times |z+w| \\ &= |z+w|. \end{aligned}$$

Thus the diagonals OB and AC are at right angles to each other and have the same length. Hence $OABC$ is both a rhombus and a rectangle, that is, a square.

26 THE GEOMETRY OF SUBTRACTION: Let $OABC$ represent the complex numbers 0, w, $w+z$ and z. Then the diagonal $\overrightarrow{AC}$ represents the complex number $z-w$.

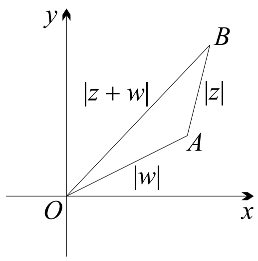
The Triangle Inequality: An important identity encountered with the absolute value of real numbers is the triangle inequality

$$\Big||x| - |y|\Big| \le |x+y| \le |x| + |y|.$$

Given that the absolute value of a real number is analogous to the modulus of a complex number, it is not surprising that the same result holds for the modulus of complex numbers, that is

$$\big||z|-|w|\big| \le |z+w| \le |z|+|w|\,.$$

This result can be explained in terms of the geometry of the addition of complex numbers. Consider only the points O, A and B as defined previously and shown in the diagram on the right. Recall that the three vectors $\overrightarrow{OA}$, $\overrightarrow{AB}$ and $\overrightarrow{OB}$ represent the complex numbers w, z and $z+w$ respectively. Hence the three moduli $|w|$, $|z|$ and $|z+w|$ are the lengths of the sides of $\triangle OAB$.



It is a well known result of Euclidean geometry that the length of one side of a triangle must be less than or equal to the sum of the other two, thus

$$|z+w| \le |z|+|w|\,,$$

with equality when O, A and B are collinear. Similarly the length of one side is greater than or equal to the difference of the other two, thus

$$\big||z|-|w|\big| \le |z+w|\,,$$

with equality again when the points are collinear. Combining these two yields

$$\big||z|-|w|\big| \le |z+w| \le |z|+|w|\,,$$

and replacing w with $-w$ throughout gives

$$\big||z|-|w|\big| \le |z-w| \le |z|+|w|\,.$$

These inequalities are called the *triangle inequalities*, after their geometric origins.

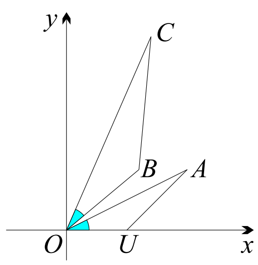
27 THE TRIANGLE INEQUALITIES: For all complex numbers z and w,

- $\big||z|-|w|\big| \le |z+w| \le |z|+|w|$
- $\big||z|-|w|\big| \le |z-w| \le |z|+|w|$

Multiplication and Division: The geometry of these two operations has already been satisfactorily explained as a rotation and enlargement. This interpretation is further demonstrated by the following example.

The diagram below shows the points O, U, A, B and C which correspond to the complex numbers 0, 1, w, z and wz. In $\triangle UOA$ and $\triangle BOC$,

$$\begin{aligned}\angle BOC &= \arg(wz)-\arg(z)\\ &= \arg(w)+\arg(z)-\arg(z)\\ &= \arg(w)\\ &= \angle UOA\,,\end{aligned}$$



and
$$\begin{aligned}\frac{OC}{OB} &= \frac{|wz|}{|z|}\\ &= \frac{|w||z|}{|z|}\\ &= |w|\\ &= \frac{OA}{OU}\,.\end{aligned}$$

Hence $\triangle BOC ||| \triangle UOA$ (SAS)

Note that the similarity ratio is $OB : OU = |z| : 1$.

This provides us with a novel way of constructing the point C for any given complex numbers w and z. First construct $\triangle UOA$, then use the base OB to construct the similar triangle $\triangle BOC$ by applying the similarity ratio $|z| : 1$.

Other than being an application of similar triangles, this construction method is not particularly enlightening, and it is rarely used. The geometry of the situation should always be remembered as a rotation of w by $\arg(z)$ and an enlargement by factor $|z| : 1$.

Two Special Cases: A vector approach is very helpful in analysing the geometry in two special cases of division. Let z_1, z_2, z_3 and z_4 be four complex numbers corresponding to the points A, B, C and D, and let

$$\frac{z_2 - z_1}{z_3 - z_4} = \lambda\,.$$

Suppose that λ is real, then

$$z_2 - z_1 = \lambda(z_3 - z_4)\,,$$

that is, one vector is a multiple of the other. Hence both vectors have the same direction if $\lambda > 0$ (but differ in length) and opposite direction if $\lambda < 0$. In either case the lines AB and CD are parallel.

In the case where λ is imaginary, the two vectors are perpendicular, since multiplication by i is equivalent to a rotation by $\frac{\pi}{2}$. Hence $AB \perp CD$. If $\lambda < 0$ then the rotation is in the clockwise direction.

WORKED EXERCISE: Let z_1 and z_2 be any two complex numbers representing the points A and B in the complex plane. Consider the complex number z given by the equation $\dfrac{z - z_1}{z_2 - z_1} = t$ where t is real. Let the point C represent z.

(a) Show that A, B and C are collinear.

(b) Hence show that C divides AB in the ratio $t : 1 - t$.

SOLUTION:

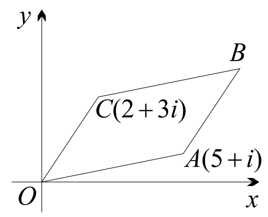
(a) First note that $\overrightarrow{AB}$ represents $z_2 - z_1$ and that $\overrightarrow{AC}$ represents $z - z_1$. Since $\dfrac{z - z_1}{z_2 - z_1}$ is real it follows that AB and AC are parallel. Further since A is common to both lines, it follows that A, B and C are collinear.

(b) If $t < 0$ then $\overrightarrow{AC}$ has the opposite direction to $\overrightarrow{AB}$ and the order of the points is CAB. If $t = 0$ then A and C coincide. If $0 < t < 1$ then both $\overrightarrow{AC}$ and $\overrightarrow{AB}$ have the same direction and $\overrightarrow{AB}$ has the greater magnitude. Hence the order of the points is ACB. If $t = 1$ then C and B coincide. If $t > 1$ then the vectors again have the same direction but $\overrightarrow{AC}$ has the greater magnitude, hence the order is ABC. In all these cases the ratio of the magnitudes is

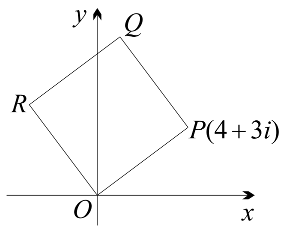
$$\begin{aligned} AC : AB &= |z - z_1| : |z_2 - z_1| \\ &= |t| : 1\,. \end{aligned}$$

Hence in all cases the point C divides AB in the ratio $t : 1 - t$.

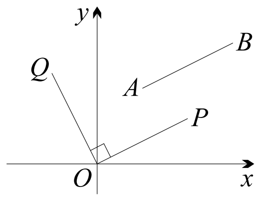
Exercise 1E



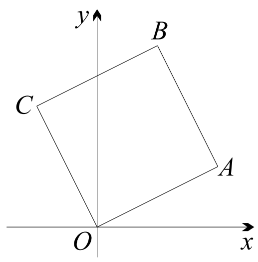
1. In the diagram on the right, $OABC$ is a parallelogram. The points A and C represent $5+i$ and $2+3i$ respectively. Find the complex numbers represented by:
(a) the vector OB,
(b) the vector AC,
(c) the vector CA.



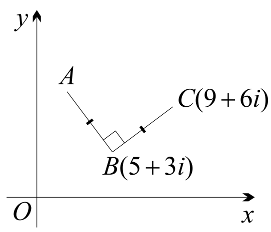
2. In the diagram on the right, $OPQR$ is a square. The point P represents $4+3i$. Find the complex numbers represented by:
(a) the point R,
(b) the point Q,
(c) the vector QR,
(d) the vector PR.



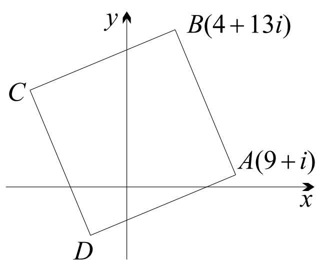
3. In the diagram on the right, intervals AB, OP and OQ are equal in length, OP is parallel to AB and $\angle POQ = \frac{\pi}{2}$. If A and B represent the complex numbers $3+5i$ and $9+8i$ respectively, find the complex number which is represented by Q.



4. In the diagram on the right, $OABC$ is a square. The point A represents the complex number $2+i$.
(a) Find the numbers represented by B and C.
(b) If the square is rotated 45° anticlockwise about O to give $OA'B'C'$, find the number represented by B'.

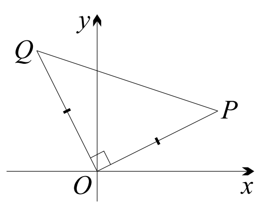


5. In the diagram on the right, $AB = BC$ and $\angle ABC = 90^\circ$. The points B and C represent $5+3i$ and $9+6i$ respectively. Find the complex numbers represented by:
(a) the vector BC,
(b) the vector BA,
(c) the point A.



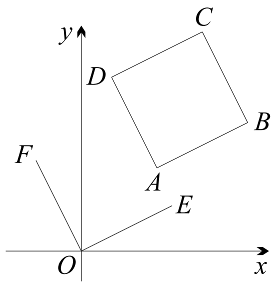
6. The diagram on the right shows a square $ABCD$ in the complex plane. The vertices A and B represent the complex numbers $9+i$ and $4+13i$ respectively. Find the complex numbers that correspond to:
(a) the vector AB,
(b) the vertex D.

DEVELOPMENT



7. In the diagram on the right, the points P and Q correspond to the complex numbers z and w respectively. The triangle OPQ is isosceles and the angle POQ is a right angle.
Prove that $z^2 + w^2 = 0$.

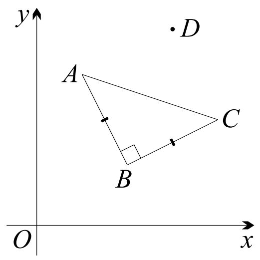
8. In the Argand diagram on the right, $ABCD$ is a square, and OE and OF are parallel and equal in length to AB and AD respectively. The vertices A and B correspond to the complex numbers w_1 and w_2 respectively. What complex numbers correspond to the points E, F, C and D?



9. In the diagram on the right, the vertices of a triangle ABC are represented by the complex numbers z_1, z_2 and z_3 respectively. The triangle is isosceles, and right-angled at B.

(a) Explain why $(z_1 - z_2)^2 + (z_3 - z_2)^2 = 0$.

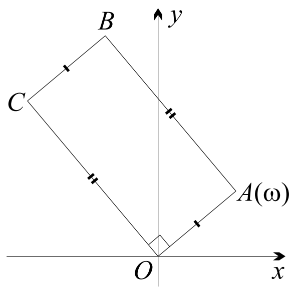
(b) Suppose that D is the point such that $ABCD$ is a square. Find, in terms of z_1, z_2 and z_3, the complex number that the point D represents.



10. In the Argand diagram on the right, $OABC$ is a rectangle, with $OC = 2OA$. The vertex A corresponds to the complex number ω.

(a) What complex number corresponds to the vertex C?

(b) What complex number corresponds to the point of intersection D of the diagonals OB and AC?



11. The vertices of an equilateral triangle are equidistant from the origin. One of its vertices is at $1 + \sqrt{3}i$. Find the complex numbers represented by the other two vertices.
[HINT: What is the angle subtended by the vertices at the origin?]

12. Given $z = 3 + 4i$, find the two possible values of w so that the points representing 0, z and w form a right-angled isosceles triangle whose right-angle is at the point representing:

(a) 0 (b) z (c) w

13. If $z_1 = 4 - i$ and $z_2 = 2i$, find in each case the two possible values of z_3 so that the points representing z_1, z_2 and z_3 form an isosceles right-angled triangle whose right-angle is at:

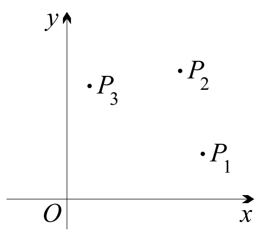
(a) z_1 (b) z_2 (c) z_3

14. Given that $z_1 = 1 + i$, $z_2 = 2 + 6i$ and $z_3 = -1 + 7i$, find the three possible values of z_4 so that the points representing z_1, z_2, z_3 and z_4 form a parallelogram.

15. A triangle in the Argand diagram has vertices at the points representing the complex numbers z_1, z_2 and z_3. If $\dfrac{z_2 - z_1}{z_3 - z_1} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, show that the triangle is equilateral.

16. In an Argand diagram, O is the origin, and the points P and Q represent the complex numbers z_1 and z_2 respectively. The triangle OPQ is equilateral.
Prove that ${z_1}^2 + {z_2}^2 = z_1 z_2$.

17. In the diagram on the right, the points P_1, P_2 and P_3 represent the complex numbers z_1, z_2 and z_3 respectively. If $\dfrac{z_2}{z_1} = \dfrac{z_3}{z_2}$, show that OP_2 bisects $\angle P_1OP_3$.



18. If z_1 and z_2 are complex numbers such that $|z_1| = |z_2|$, show that:

$$\arg(z_1 z_2) = \arg\left((z_1 + z_2)^2\right) .$$

19. Let $z_1 = 2i$ and $z_2 = 1 + \sqrt{3}\,i$.

(a) Express z_1 and z_2 in mod-arg form.

(b) Plot in the complex plane the points P, Q, R and S representing z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ respectively.

(c) Find the exact values of: (i) $\arg(z_1 + z_2)$ (ii) $\arg(z_1 - z_2)$

20. Suppose that the complex number z has modulus one, and that $0 < \arg z < \frac{\pi}{2}$.
Prove that $2\arg(z + 1) = \arg z$.

21. The vertices of the quadrilateral $ABCD$ in the complex plane represent the complex numbers z_1, z_2, z_3 and z_4 respectively.

(a) If $z_1 - z_2 = z_4 - z_3$, show that the quadrilateral $ABCD$ is a parallelogram.

(b) If $z_1 - z_2 = z_4 - z_3$ and $z_1 - z_3 = i(z_4 - z_2)$, show that $ABCD$ is a square.

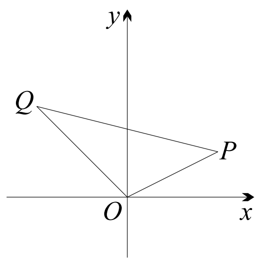
22. (a) Prove that for any complex number z, $|z|^2 = z\overline{z}$.

(b) Hence prove that for any complex numbers z_1 and z_2:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

(c) Explain this result geometrically.

23. In the diagram on the right, the points P and Q represent the complex numbers z and w respectively.



(a) Explain why $|z - w| \le |z| + |w|$.

(b) Indicate on the diagram the point R representing $z + w$.

(c) What type of quadrilateral is $OPRQ$?

(d) If $|z - w| = |z + w|$, what can be said about the complex number $\dfrac{w}{z}$?

24. (a) Prove that the points z_1, z_2 and z_3 are collinear if $\dfrac{z_3 - z_1}{z_2 - z_1}$ is real.

(b) Hence show that the points representing $5 + 8i$, $13 + 20i$ and $19 + 29i$ are collinear.

EXTENSION

25. The complex numbers ω_1 and ω_2 have modulus 1, and arguments α_1 and α_2 respectively, where $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2}$.
Show that $\arg(\omega_1 - \omega_2) = \frac{1}{2}(\alpha_1 + \alpha_2 - \pi)$.

26. [CIRCLE GEOMETRY] It is known that $\arg\left(\dfrac{z_4 - z_1}{z_2 - z_1}\right) + \arg\left(\dfrac{z_2 - z_3}{z_4 - z_3}\right) = \pi$. Explain why the points representing these complex numbers are concyclic.

27. [CIRCLE GEOMETRY] The points representing the complex numbers 0, z_1, z_2 and z_3 are concyclic. Prove that the points representing $\dfrac{1}{z_1}$, $\dfrac{1}{z_2}$ and $\dfrac{1}{z_3}$ are collinear.

[Hint: Show that $\dfrac{z_2^{-1} - z_1^{-1}}{z_3^{-1} - z_1^{-1}}$ is real.]

1F Locus Problems

In many situations a set of equations or conditions on a variable complex number z yields a locus of points in the Argand diagram which is a familiar geometric object, such as a line or a circle. The main aim of this section is to provide a geometric description for a locus specified algebraically. Therefore the examples in this text have been grouped by the various geometries.

There are two basic approaches to identifying a locus, algebraic or geometric. The advantage of the algebraic approach is that most readers will already be proficient at manipulating equations in x and y. Unfortunately the geometry of the situation may be obscured by the algebra. The advantage of the geometric approach is that it will often provide a very elegant solution to the problem, but may also require a keen insight. Both methods should be practised, with the aim to become proficient at the geometric approach.

Straight Lines: Some simple straight lines have already been encountered in 1C, such as the vertical line $\text{Re}(z) = a$. Here are some other examples and their geometric interpretations.

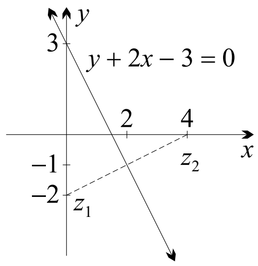
In coordinate geometry, given the coordinates of two points A and B, the task of finding the equation of the perpendicular bisector of AB is a lengthy one. The equivalent complex equation is remarkably simple.

WORKED EXERCISE: Let $z_1 = 4$ and $z_2 = -2i$, and let the variable point z satisfy the equation $|z - z_1| = |z - z_2|$.

(a) Put $z = x + iy$ and hence show that z lies on the straight line $y + 2x - 3 = 0$.

(b) Describe this line geometrically in terms of z_1 and z_2.

SOLUTION: (a) Substitute the values of z_1 and z_2, then square to get



$$|z - 4|^2 = |z + 2i|^2$$

or $\quad (x-4)^2 + y^2 = x^2 + (y+2)^2$

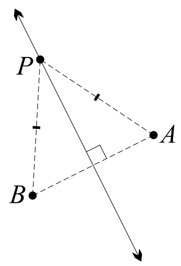
whence $\quad (x-4)^2 - x^2 = (y+2)^2 - y^2$

thus $\quad -4(2x-4) = 2(2y+2)$

so $\quad 4 - 2x = y + 1$

hence $\quad y + 2x - 3 = 0\,.$

(b) Let z, z_1 and z_2 be the points P, A and B in the Argand diagram. Since the modulus is a distance, the given equation yields



$$PA = PB$$

Thus the locus of P is the set of all points equidistant from A and B, that is, the perpendicular bisector of AB.

28 THE PERPENDICULAR BISECTOR OF AN INTERVAL: Let z_1 and z_2 be the fixed points A and B in the Argand diagram, and let z be a variable point P. If

$$|z - z_1| = |z - z_2|$$

then the locus of P is the perpendicular bisector of AB.

WORKED EXERCISE: Let $z_0 = a + ib$ be the fixed point T and let $z = x + iy$ be a variable point P in the complex plane. It is known that $z - z_0 = ikz_0$, where k is a real number.

(a) It is also known that as k varies, the locus of P is a straight line. Find the equation of that straight line in terms of x and y.

(b) What is the geometry of the situation?

SOLUTION: (a) The given equation expands to

$$x + iy - (a + ib) = ik(a + ib)\,.$$

Equating real and imaginary parts yields

$$x - a = -kb$$

and

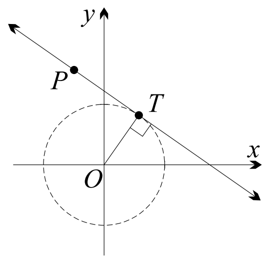
$$y - b = ka\,.$$

Eliminating k from this pair of equations, we get

$$b(y - b) = -a(x - a)$$

or

$$ax + by = a^2 + b^2$$



(b) Some readers will recognise this equation as the tangent to a circle. This geometry is confirmed by examining the given equation more closely.

Since multiplication by i represents a rotation of $\frac{\pi}{2}$, it follows that for $k \neq 0$ the vector $\overrightarrow{TP}$ is perpendicular to $\overrightarrow{OT}$. That is, P lies on a line perpendicular to OT. Further, when $k = 0$, $z = z_0$, so this line passes through T. That is, PT is the tangent to the circle with radius OT, as shown above.

Rays: The horizontal and vertical lines in 1C and the first example above demonstrate some of the geometry of the recently introduced functions $\text{Re}(z)$, $\text{Im}(z)$ and $|z|$. The new function $\arg(z)$ describes a ray in the z-plane.

WORKED EXERCISE: The complex number z satisfies the equation $\arg(z) = \frac{\pi}{3}$.

(a) Let $|z| = r$. Write z in modulus-argument form.

(b) Plot z when $r = 1, 2, 3, 4$, and observe that z lies on a ray.

(c) Explain why the origin is not part of this locus.

(d) Use shifting to sketch $\arg(z - 2 - i) = \frac{\pi}{3}$.

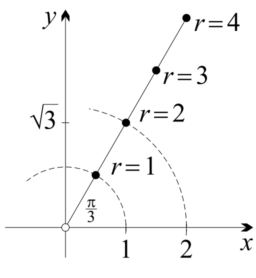
SOLUTION:

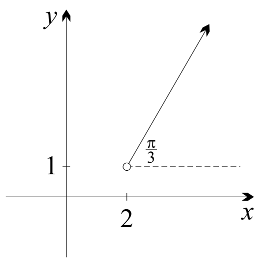
(a) $z = r(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$.

(b) See the first graph on the right.

(c) $\arg(0)$ is undefined so the origin is not included.

(d) $\arg(z - 2 - i) = \arg(z - (2 + i))$ so the ray has been shifted to the point $2 + i$, as shown on the right.





29 RAYS IN THE ARGAND DIAGRAM:

- The equation $\arg(z) = \theta$ represents the ray which makes an angle θ with the positive real axis, omitting the origin.
- The locus $\arg(z - z_0) = \theta$ is the result of shifting the above ray from the origin to the point z_0.

Conics Sections: The conics, that is the circle, parabola, ellipse and hyperbola, may each be written as equations of complex variables. The geometric definitions of each conic in terms of distance or the ratio of two distances is often the key. Here are three examples.

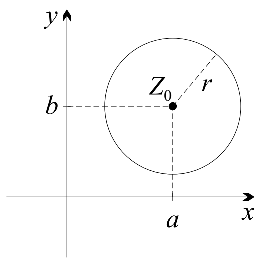
WORKED EXERCISE: Consider the locus specified by the equation $|z-z_0| = r$, for some fixed complex number $z_0 = a + ib$ and positive real number r.

(a) Explain why this represents a circle. State the centre and radius.

(b) Confirm your answer to part a) by putting $z = x+iy$ and finding the cartesian equation.

(c) Expand $|z-1|^2$ in terms of z and $\overline{z}$. Hence determine the locus specified by $|z|^2 = z + \overline{z}$.

SOLUTION: (a) The equation tells us that the distance between z and z_0 is fixed. This is the geometric definition of a circle. The centre is z_0 and the radius is r.



(b) Begin by squaring both sides:

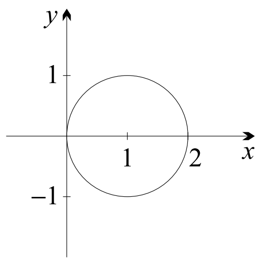
$$|z-z_0|^2 = r^2$$

so $\quad |(x-a)+i(y-b)|^2 = r^2$

thus $\quad (x-a)^2 + (y-b)^2 = r^2\,.$

(c) Noting that $|w|^2 = w\,\overline{w}$, we may write

$$\begin{aligned} |z-1|^2 &= (z-1)\,\overline{(z-1)} \\ &= (z-1)(\overline{z}-1) \\ &= |z|^2 - (z+\overline{z}) + 1\,. \end{aligned}$$



Since we are told that $|z|^2 = z + \overline{z}$, it follows that

$$|z-1|^2 = 1\,,$$

that is, the circle with centre $z = 1$ and radius 1.

30	CIRCLES IN THE ARGAND DIAGRAM: Let z_0 be the fixed point C in the Argand diagram, and let z a variable point P. If $\qquad |z-z_0| = r$ then the locus of P is the circle with centre C and radius r.

WORKED EXERCISE: Let S be the fixed point $1+2i$ and z be the variable point P in the Argand diagram.

(a) Describe the locus of P given that $|z-(1+2i)| = \mathrm{Im}(z)$.

(b) Confirm your answer algebraically by putting $z = x+iy$.

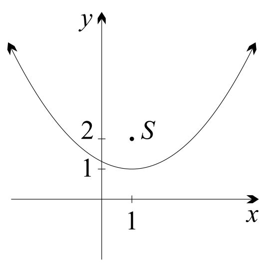
SOLUTION: (a) The quantity $|z-(1+2i)|$ is the distance PS, and $\mathrm{Im}(z)$ is the distance from P to the real axis. Thus P is equidistant from a point and a line. That is, the locus of P is a parabola. The focus is S and the directrix is the real axis, hence the vertex is $1+i$ and the focal length is 1.

(b) Squaring both sides of the given equation

$$(x-1)^2 + (y-2)^2 = y^2$$

so

$$\begin{aligned} (x-1)^2 &= y^2 - (y-2)^2 \\ &= 4y - 4 \end{aligned}$$



thus $\quad (x-1)^2 = 4(y-1)\,.$

This is the equation of a parabola with vertex $1+i$ and focal length 1, as before.

WORKED EXERCISE: Given that $z^2 - \overline{z^2} = 4i$, put $z = x + iy$ and hence determine the corresponding locus in the complex plane.

SOLUTION: The left hand side of the given equation is $2i\,\mathrm{Im}(z^2)$ so

$$2i\,\mathrm{Im}(z^2) = 4i$$

or $\quad \mathrm{Im}(z^2) = 2$

thus $\quad 2xy = 2\,.$

Hence the locus is the rectangular hyperbola

$$xy = 1\,.$$

Regions: In many instances a locus divides the plane into two or more regions. In simple cases a region is defined by the corresponding inequation. Two or more inequations will result in the union or intersection of the regions. Some common examples follow.

WORKED EXERCISE: Sketch the following loci:

(a) $1 \le \mathrm{Re}(z) \le 3$

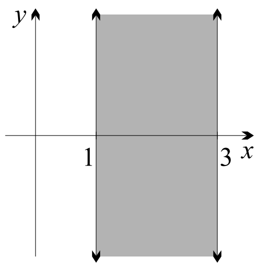
(b) $|z| > |z+2-2i|$

(c) $|z-2+i| < 1$

(d) $0 \le \arg(z) \le \frac{\pi}{4}$

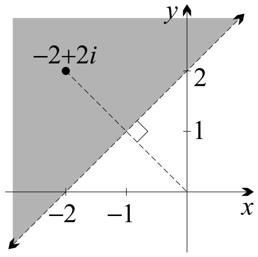
SOLUTION: The first three can be easily explained geometrically.

(a)



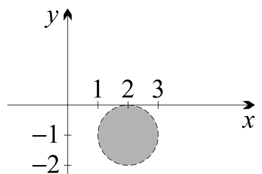
This is $1 \le x \le 3$, the vertical strip between $x = 1$ and $x = 3$.

(b)



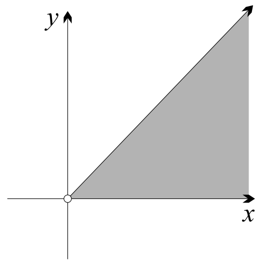
The perpendicular bisector of the segment from 0 to $-2+2i$ is the boundary, and is not included. The region includes the point $-2+2i$, since the RHS of the inequality is zero there.

(c)



The boundary curve is the circle with radius 1 and centre $2-i$, which is not included. The region includes the centre of the circle since the LHS of the inequality is zero there.

(d)



Put $z = r\,\mathrm{cis}\,\theta$ to get $0 \le \theta \le \frac{\pi}{4}$, which defines a wedge excluding the origin, since $\arg(0)$ is undefined.

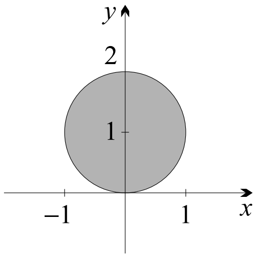
WORKED EXERCISE:

(a) Sketch the regions (i) $|z-i| \le 1$ and (ii) $-\frac{\pi}{6} < \arg(z+1-i) < \frac{\pi}{6}$.

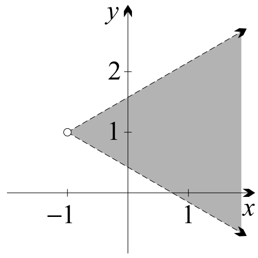
(b) Hence sketch (i) the union and (ii) the intersection of these regions.

SOLUTION:

(a) (i)

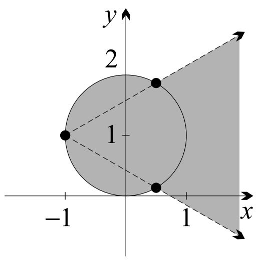


(ii)

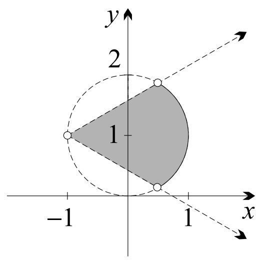


(b) The boundaries intersect at $-1+i$ and, from trigonometry, they interesect again at $\frac{1}{2}+i\,(1+\frac{\sqrt{3}}{2})$ and $\frac{1}{2}+i\,(1-\frac{\sqrt{3}}{2})$. Here are the graphs.

(i)



(ii)



Loci and Circle Geometry: Many of the circle geometry theorems encountered in the Mathematics Extension 1 course may be expressed in terms of the locus of a complex number. One significant result is included here, with other examples to be found in the exercise.

WORKED EXERCISE: Let $z_1 = 3+i$ and $z_2 = 1-i$.

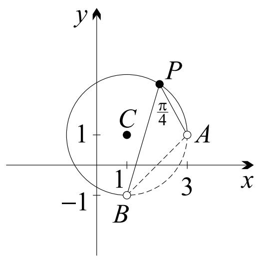
(a) Describe and sketch the locus of z, where $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \frac{\pi}{4}$.

(b) What happens to the locus if z_1 and z_2 are swapped?

SOLUTION:

(a) Let z_1, z_2 and z represent the points A, B and P respectively. The given equation indicates that the angle between the vectors AP and BP is fixed, that is the angle at P subtended by AB is $\frac{\pi}{4}$. Using the converse of the angles in the same segment theorem, it follows that P must lie on the arc of a circle with chord AB. Since

$$\angle APB = \tfrac{\pi}{4} < \tfrac{\pi}{2}$$



it is a major arc. Since $\arg(z-z_1) > \arg(z-z_2)$ the arc is taken anticlockwise from A to B. Lastly, since $\arg(0)$ is undefined, the endpoints of the arc are not included. It simply remains to find the centre and radius of this circle. Let C be the centre of the circle then

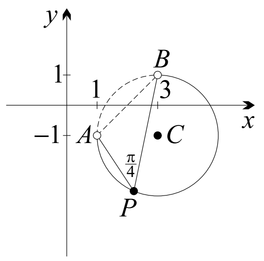
$\angle ACB = \frac{\pi}{2}$ (Angles at the centre and circumference)

whence $\angle CAB = \frac{\pi}{4}$ (base angles of isosceles triangle.)

Since $\arg(z_1 - z_2) = \frac{\pi}{4}$

it follows that AC is horizontal and BC is vertical. Thus $C = 1+i$ is the centre of the circle and $AC = 2$ is its radius.

(b) Since the points A and B have been swapped the horizontal line AC and the vertical line BC intersect at a different point, namely the new centre $C = 3-i$. Effectively the locus has been rotated by π about the mid-point of AB.



Exercise **1F**

1. Sketch these straight lines by using the result of Box 28.

(a) $|z+3| = |z-5|$ (b) $|z-i| = |z+1|$ (c) $|z+2-2i| = |z|$ (d) $|z-i| = |z-4+i|$

2. Graph the rays specified in the following equations.

(a) $\arg(z-4) = \frac{3\pi}{4}$ (b) $\arg(z+1) = \frac{\pi}{4}$ (c) $\arg(z-1-i\sqrt{3}) = \frac{\pi}{3}$

3. Use Box 30 to sketch these circles.

(a) $|z+1-i| = 1$ (b) $|z-3-2i| = 2$ (c) $|z-1+i| = \sqrt{2}$

4. In each case determine the locus of the corresponding boundary equations and hence sketch the indicated region.

(a) $|z-8i| \ge |z-4|$
(b) $|z-2+i| \le |z-4+i|$
(c) $|z+1-i| \ge |z-3+i|$
(d) $0 \le \arg(z) \le \frac{3\pi}{4}$
(e) $-\frac{\pi}{3} < \arg(z) < \frac{\pi}{6}$
(f) $-\frac{\pi}{4} \le \arg(z+2+i) < \frac{\pi}{4}$
(g) $|z| > 2$
(h) $|z+2i| \le 1$
(i) $1 < |z-2+i| \le 2$

DEVELOPMENT

5. In each case sketch (i) the intersection and (ii) the union of the given pair of regions.

(a) $|z-2+i| \le 2$, $\mathrm{Im}(z) \ge 0$
(b) $0 \le \mathrm{Re}(z) \le 2$, $|z-1+i| \le 2$
(c) $|z-\overline{z}| < 2$, $|z-1| \ge 1$
(d) $\mathrm{Re}(z) \le 4$, $|z-4+5i| \le 3$
(e) $|z-1-i| \le 2$, $0 \le \arg(z-1-i) \le \frac{\pi}{4}$
(f) $|z| \le 1$, $0 \le \arg(z+1) \le \frac{\pi}{4}$
(g) $|z+1-2i| \le 3$, $-\frac{\pi}{3} \le \arg z \le \frac{\pi}{4}$
(h) $|z-3-i| \le 5$, $|z+1| \le |z-1|$

6. Put $z = x+iy$ to help sketch these hyperbolae.

(a) $z^2 - (\overline{z})^2 = 16i$ (b) $z^2 - (\overline{z})^2 = 12i$

7. In each case the given equation is that of a parabola. (i) Show this algebraically by putting $z = x+iy$, and hence draw each parabola. (ii) Use the fact that the functions $|z|$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ represent distances to help determine the focus and directrix.

(a) $|z-3i| = \mathrm{Im}(z)$ (b) $|z+2| = -\mathrm{Re}(z)$ (c) $|z| = \mathrm{Re}(z+2)$ (d) $|z-i| = \mathrm{Im}(z+1)$

8. By putting $z = x+iy$ or otherwise, determine the locus specified by:

(a) $\mathrm{Im}(z) = |z|$ (b) $\mathrm{Re}\left(1-\frac{4}{z}\right) = 0$ (c) $\mathrm{Re}\left(1-\frac{1}{z}\right) = 0$

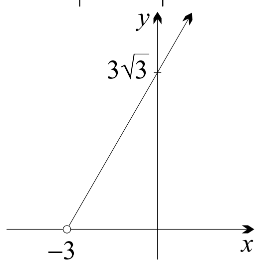
9. Determine the arcs specified by the following equations. Sketch each one, showing the centre and radius of the associated circle.

(a) $\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{2}$
(b) $\arg\left(\frac{z-1+i}{z-1-i}\right) = \frac{\pi}{2}$
(c) $\arg\left(\frac{z-i}{z+i}\right) = \frac{\pi}{4}$
(d) $\arg\left(\frac{z+1}{z-3}\right) = \frac{\pi}{3}$
(e) $\arg\left(\frac{z-2i}{z+2i}\right) = \frac{\pi}{6}$
(f) $\arg\left(\frac{z}{z+4}\right) = \frac{3\pi}{4}$

10. A complex number z satisfies $\arg z = \frac{\pi}{3}$.

(a) Use a diagram to show that $|z-2i| \geq 1$. (b) For which value of z is $|z - 2i| = 1$?

11. The diagram on the right show the locus of a variable point P which represents the complex number z.



(a) Write down an equation for this locus.

(b) Find the modulus and argument of z at the point where $|z|$ takes its minimum value.

(c) Hence find z in Cartesian form when $|z|$ takes its least value.

12. (a) A complex number z satisfies $|z-1| = 2$. Draw a diagram and hence find the greatest and least possible values of $|z|$.

(b) If z is a complex number such that $\text{Re}(z) \leq 2$ and $|z - 3| = 2$, show with the aid of a diagram that $1 \leq |z| \leq \sqrt{7}$.

13. (a) The variable complex number z satisfies $|z - 2 - i| = 1$. Use a diagram to find the maximum and minimum values of: (i) $|z|$, (ii) $|z - 3i|$.

(b) A complex number z satisfies $|z| = 3$. Use a sketch to find the greatest and least values of $|z + 5 - i|$.

(c) The variable complex number z satisfies $|z - z_0| = r$. Use a similar approach to parts (a) and (b) to find the maximum and minimum values of: (i) $|z|$, (ii) $|z-z_1|$.

(d) Confirm your answers to the previous parts by using the triangle inequality.

14. (a) A complex number z satisfies $|z - 2| = 1$.

(i) Sketch the locus of z. (ii) Show that $-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{6}$.

(b) The complex number z is such that $|z| = 1$. Use your answers to part (a) to explain why $-\frac{\pi}{6} \leq \arg(z + 2) \leq \frac{\pi}{6}$.

15. The variable complex number w satisfies $|w| = 10$ and $0 \leq \arg w \leq \frac{\pi}{2}$. The variable complex number z is given by $z = 3 + 4i + w$.

(a) Sketch the locus of z.

(b) Use your diagram to determine the maximum value of $|z|$.

(c) What is the value of z for which this maximum occurs?

16. (a) Show that the circle equation $|z - z_0| = r$ is equivalent to

$$z\,\overline{z} - (z\,\overline{z_0} + \overline{z}\,z_0) + z_0\,\overline{z_0} - r^2 = 0\,.$$

[HINT: Square both sides of $|z - z_0| = r$ and use the result $|w|^2 = w\,\overline{w}$.]

(b) Use the result of part (a) to help identify these circles.

(i) $z\overline{z}+2(z+\overline{z}) = 0$ (ii) $z\overline{z}-(1+i)\overline{z}-(1-i)z+1 = 0$ (iii) $\dfrac{1}{z} + \dfrac{1}{\overline{z}} = 1$

17. Find the locus of z if the value of $\dfrac{z-1}{z-i}$ is: (a) real, (b) imaginary.

18. Sketch the locus of z given that: (a) $\arg(z+i) = \arg(z-1)$, (b) $\arg(z+i) = \arg(z-1)+\pi$.

19. Suppose that $|z| = 1$ and that $\arg z = 2\theta$, where $0 < \theta < \frac{\pi}{2}$. Show that:

(a) $|z^2 - z| = |z - 1|$ (b) $\arg(z^2 - z) = \frac{\pi}{2} + 3\theta$

EXTENSION

20. It is known that the locus specified by $\arg\left(\dfrac{z-z_1}{z-z_2}\right)=\alpha$ is an arc taken anticlockwise from z_1 to z_2. What else can be said about the locus when:

(a) $\alpha=0$ (b) $0<\alpha<\frac{\pi}{2}$ (c) $\alpha=\frac{\pi}{2}$ (d) $\frac{\pi}{2}<\alpha<\pi$ (e) $\alpha=\pi$

21. Put $z=x+iy$ to help sketch the hyperbola $z^2+(\overline{z})^2=2$.

22. [CIRCLE GEOMETRY] Let z_1 and z_2 be two fixed points in the Argand diagram, and for simplicity suppose that $0<\arg(z_2-z_1)<\pi$. The variable point z satisfies the equation

$$\arg\left(\frac{z-z_2}{z-z_1}\right)=\arg(z_2-z_1)\,.$$

(a) Use a theorem in circle geometry to help determine the locus of z.

(b) Investigate the situation if the restriction on $\arg(z_2-z_1)$ is removed.

23. [CONICS] Let a and e be two positive real numbers, with $0<e<1$. Describe the locus of the points z in the complex plane which satisfy $|z-ae|+|z+ae|=2a$.

24. [VERY DIFFICULT] Suppose that $k|z-z_1|=\ell|z-z_2|$, where $k\neq\ell$ and both are positive real numbers.

(a) Show that the locus of z in the Argand diagram is a circle with centre $\dfrac{k^2z_1-\ell^2z_2}{k^2-\ell^2}$ and radius $\dfrac{kl|z_2-z_1|}{|k^2-\ell^2|}$,

(i) by letting $z=x+iy$, (ii) by geometric means.

(b) What happens in the limit as k approaches ℓ?

Chapter One

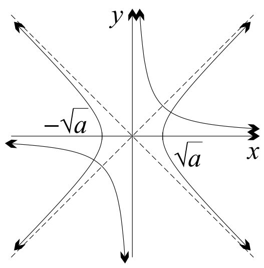
Exercise **1A** (Page 8)

1(a) -1 **(b)** 1 **(c)** $-i$ **(d)** i
(e) i **(f)** -1 **(g)** 1 **(h)** 0
2(a) $-2i$ **(b)** $3-i$ **(c)** $1+i$ **(d)** $5+3i$ **(e)** $-3-2i$
3(a) $12-2i$ **(b)** $-6+2i$ **(c)** $1+5i$ **(d)** $7-11i$
4(a) $-5+4i$ **(b)** $5+5i$ **(c)** $14+5i$ **(d)** $-26+82i$
(e) $24+10i$ **(f)** $-5-12i$ **(g)** $2+11i$ **(h)** -4
(i) $28-96i$
5(a) 5 **(b)** 17 **(c)** 29 **(d)** 65
6(a) $-i$ **(b)** $1-2i$ **(c)** $3+2i$ **(d)** $1-2i$ **(e)** $-1+3i$
(f) $-\frac{1}{5}+\frac{3}{5}i$
7(a) $-2-i$ **(b)** $4-3i$ **(c)** $3+7i$ **(d)** 3 **(e)** $-3+4i$
8(a) $6+2i$ **(b)** 18 **(c)** $19-22i$ **(d)** $8-i$ **(e)** $1+2i$
9(a) $22+19i$ **(b)** $6+15i$ **(c)** $4-2i$ **(d)** $2-3i$
(e) 6
10(a) $x=3$ and $y=-2$ **(b)** $x=2$ and $y=-1$
(c) $x=6$ and $y=2$ **(d)** $x=\frac{14}{5}$ and $y=\frac{3}{5}$
(e) $x=\frac{35}{2}$ and $y=-\frac{39}{2}$
11(a) $\frac{9}{10}-\frac{13}{10}i$ **(b)** 1 **(c)** $-\frac{8}{29}$ **(d)** $-4-\frac{5}{2}i$
16(a) $\frac{x-iy}{x^2+y^2}$ **(b)** $\frac{x^2-y^2-2ixy}{(x^2+y^2)^2}$ **(c)** $\frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$

Exercise **1B** (Page 15)

1(a) $z=\pm 3i$ **(b)** $z=2\pm 4i$ **(c)** $z=-1\pm 2i$
(d) $z=3\pm i$ **(e)** $z=\frac{1}{2}\pm\frac{1}{4}i$ **(f)** $z=-\frac{3}{2}\pm 2i$
2(a) $(z-6i)(z+6i)$ **(b)** $(z-2\sqrt{2}\,i)(z+2\sqrt{2}\,i)$
(c) $(z-1-3i)(z-1+3i)$ **(d)** $(z+2-i)(z+2+i)$
(e) $(z-3+\sqrt{5}\,i)(z-3-\sqrt{5}\,i)$ **(f)** $(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i)$
3(a) $z^2+2=0$ **(b)** $z^2-2z+2=0$ **(c)** $z^2+2z+5=0$ **(d)** $z^2-4z+7=0$
4(a) $\pm(1+i)$ **(b)** $\pm(2+i)$ **(c)** $\pm(-1+3i)$ **(d)** $\pm(6+i)$ **(e)** $\pm(2+3i)$ **(f)** $\pm(5-i)$ **(g)** $\pm(1-4i)$
(h) $\pm(5-4i)$
5(a) $\pm(1-2i)$ **(b)** $z=2-i$ or $1+i$
6(a) $\pm(1+3i)$ **(b)** $z=4+i$ or $3-2i$
7(a) $z=1-i$ or i **(b)** $z=-3+2i$ or $-2i$ **(c)** $z=4+i$ or $2-i$ **(d)** $z=-2+i$ or $\frac{1}{2}(3-i)$ **(e)** $z=-5+i$ or $3-2i$ **(f)** $z=3+i$ or $-1-3i$
8(a) $w=-1$ **(b)** $a=-6$ and $b=13$ **(c)** $k=8-i$ and the other root is $2+3i$.
9 $z=\pm(2+i)$
10(a) $\cos\theta+i\sin\theta$ or $\cos\theta-i\sin\theta$
11(a) $z=-1$ or $\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ **(b)** $z=i$ or $\pm\frac{\sqrt{3}}{2}-\frac{1}{2}i$
12(a) $x=\omega$ satisfies the equation. **(c)** They are complex conjugates.

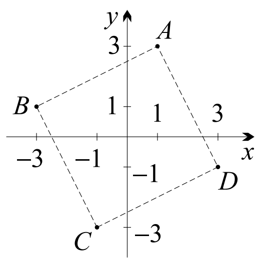
13(a) $\overline{\alpha}$
14(a)(i)



15(a) $\pm\frac{1}{\sqrt{2}}(1-i)$ **(b)** $\pm\sqrt{2}(1+2i)$ **(c)** $\pm(1+i\sqrt{3})$
(d) $\pm\sqrt{2}(3-2i)$
(e) $\pm\left(\sqrt{\sqrt{5}+1}-i\sqrt{\sqrt{5}-1}\right)$
16(a) $-2-i\pm\left(\sqrt{\sqrt{2}+1}+i\sqrt{\sqrt{2}-1}\right)$
(b) $1+i\pm\left(\sqrt{\sqrt{5}-1}-i\sqrt{\sqrt{5}+1}\right)$
(c) $-1+i\sqrt{3}\pm\left(\sqrt{2}-i\sqrt{6}\right)$
(d) $\frac{1}{2}\left(1-i\pm\left(\sqrt{\sqrt{13}+2}-i\sqrt{\sqrt{13}-2}\right)\right)$

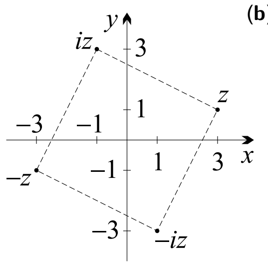
Exercise **1C** (Page 20)

1(a) $(2,0)$ **(b)** $(0,1)$ **(c)** $(-3,5)$ **(d)** $(2,-2)$
(e) $(-5,-5)$ **(f)** $(-1,2)$
2(a) -3 **(b)** $3i$ **(c)** $7-5i$ **(d)** $a+bi$
3(a)

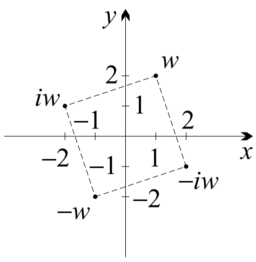


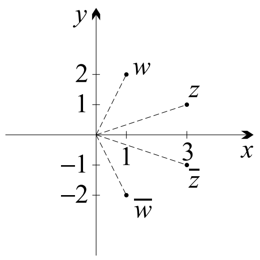
(b) A square. **(c)** An anticlockwise rotation of 90° about the origin.

4(a)



(b)



(c)


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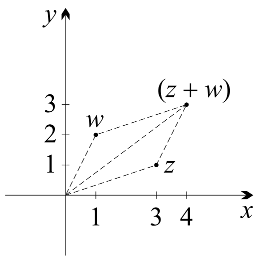

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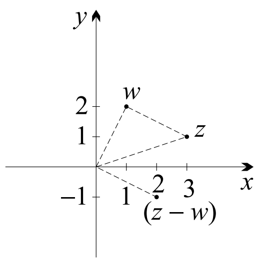

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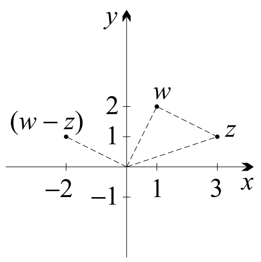
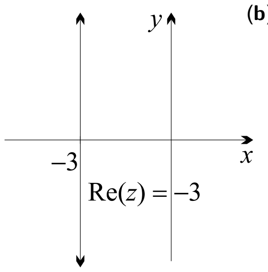


5(a)


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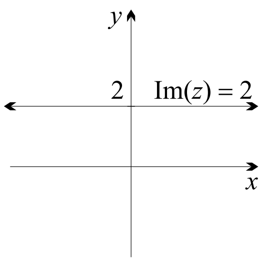

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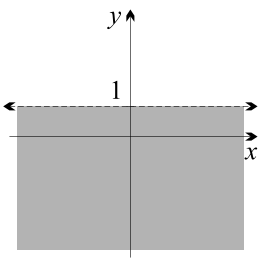

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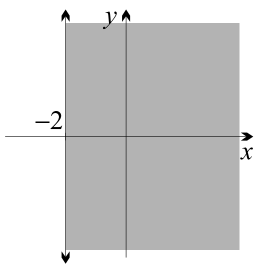

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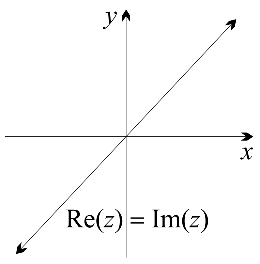

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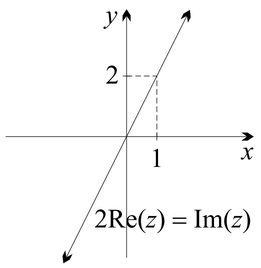
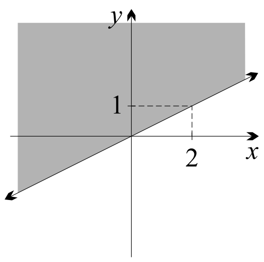


(g)


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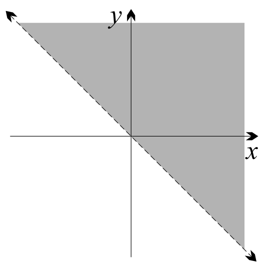

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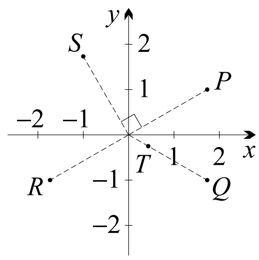
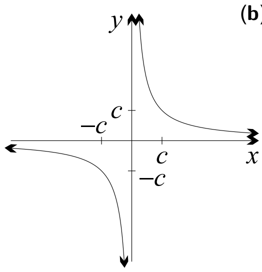


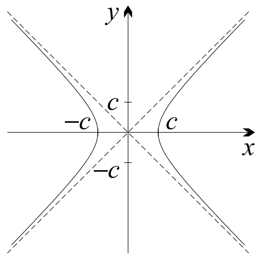
10(c) right-isosceles

11 It is the circle centre $(0, -1)$ with radius 1, omitting the origin.

12 It is the circle centre $(3, 0)$ with radius, omitting the origin.

14 It is a parabola with focus the origin and directrix $x = 1$.

15(a)


(b)


Exercise **1D** (Page 26)

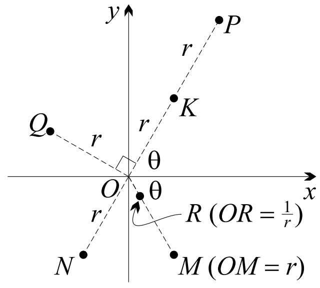
1(a) $2\operatorname{cis}\frac{\pi}{2}$ **(b)** $4\operatorname{cis}\pi$ **(c)** $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ **(d)** $2\operatorname{cis}\left(-\frac{\pi}{6}\right)$
(e) $2\operatorname{cis}\frac{2\pi}{3}$ **(f)** $\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

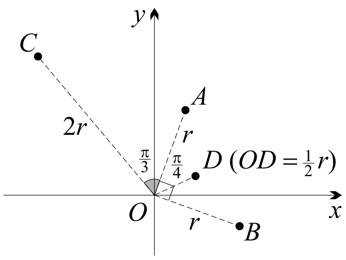
2(a) $5\operatorname{cis}(0{\cdot}93)$ **(b)** $13\operatorname{cis}(-0{\cdot}39)$
(c) $\sqrt{5}\operatorname{cis}(2{\cdot}68)$ **(d)** $\sqrt{10}\operatorname{cis}(-1{\cdot}89)$

3(a) 3 **(b)** $-5i$ **(c)** $2\sqrt{2} + 2\sqrt{2}\,i$ **(d)** $3\sqrt{3} - 3i$
(e) $-\sqrt{2} + \sqrt{2}\,i$ **(f)** $-1 - \sqrt{3}\,i$

4(a) $\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **(b)** $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ **(c)** $\sqrt{2}\operatorname{cis}\frac{3\pi}{4}$
(d) $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ **(e)** $2\operatorname{cis}\left(-\frac{\pi}{2}\right)$ **(f)** $\frac{1}{\sqrt{2}}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

5(a) $10\operatorname{cis}\frac{\pi}{3}$ **(b)** $9\operatorname{cis}3\theta$ **(c)** $2\operatorname{cis}\frac{\pi}{3}$ **(d)** $\frac{3}{2}\operatorname{cis}\alpha$
(e) $16\operatorname{cis}\frac{2\pi}{5}$ **(f)** $8\operatorname{cis}\frac{6\pi}{7}$

6


7


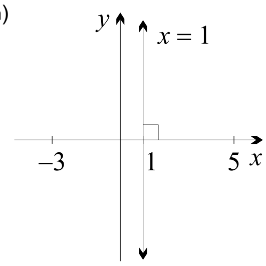
9(a) $z_1 = 2\operatorname{cis}\frac{\pi}{6}$ and $z_2 = 4\operatorname{cis}\frac{\pi}{4}$ **(b)** $z_1z_2 = 8\operatorname{cis}\frac{5\pi}{12}$ and $\frac{z_2}{z_1} = 2\operatorname{cis}\frac{\pi}{12}$
10 $z_1 = 2\operatorname{cis}\frac{5\pi}{6}$, $z_2 = \sqrt{2}\operatorname{cis}(-\frac{3\pi}{4})$,
$z_1z_2 = 2\sqrt{2}\operatorname{cis}\frac{\pi}{12}$ and $\frac{z_2}{z_1} = \frac{\sqrt{2}}{2}\operatorname{cis}\frac{5\pi}{12}$
11(a) $\frac{1}{2}\Big((\sqrt{3}+1)+i(\sqrt{3}-1)\Big)$ **(b)** $\sqrt{2}\operatorname{cis}\frac{\pi}{12}$
(c) $\frac{1}{2\sqrt{2}}(\sqrt{3}+1)$
12(b) $2\sqrt{2}\operatorname{cis}\frac{\pi}{12}$
13(a) $\sqrt{2}$ **(b)** $\frac{\pi}{4}$ **(c)** $1+i$
19 $z+w = 2\cos\left(\frac{\theta-\phi}{2}\right)\operatorname{cis}\left(\frac{\theta+\phi}{2}\right)$
21(c) The tangents at z_0 and z_1 to the circle with centre the origin meet at z_2.
22(a) When $\operatorname{Im}(z) = 0$.

Exercise 1E (Page 32)

1(a) $7+4i$ **(b)** $-3+2i$ **(c)** $3-2i$
2(a) $-3+4i$ **(b)** $1+7i$ **(c)** $-4-3i$ **(d)** $-7+i$
3 $-3+6i$
4(a) B represents $1+3i$, C represents $-1+2i$
(b) $-\sqrt{2}+2\sqrt{2}\,i$
5(a) $4+3i$ **(b)** $-3+4i$ **(c)** $2+7i$
6(a) $-5+12i$ **(b)** $-3-4i$
8 E represents w_2-w_1, F represents $i(w_2-w_1)$, C represents $w_2+i(w_2-w_1)$ and D represents $w_1+i(w_2-w_1)$.
9(a) Vectors BA and BC represent z_1-z_2 and z_3-z_2 respectively, and BA is the anticlockwise rotation of BC through 90° about B. So $z_1-z_2 = i(z_3-z_2)$. Squaring both sides gives the result.
(b) $z_1-z_2+z_3$
10(a) $2\omega i$ **(b)** $\frac{1}{2}\omega(1+2i)$
11 -2 and $1-\sqrt{3}\,i$
12(a) $w = -4+3i$ or $4-3i$ **(b)** $w = -1+7i$ or $7+i$ **(c)** $w = \frac{1}{2}(7+i)$ or $\frac{1}{2}(-1+7i)$
13(a) $1-5i$, $7+3i$ **(b)** $3+6i$, $-3-2i$ **(c)** $\frac{7}{2}+\frac{5}{2}i$, $\frac{1}{2}-\frac{3}{2}i$
14 $-2+2i$, $12i$, 4
19(a) $z_1 = 2\operatorname{cis}\frac{\pi}{2}$, $z_2 = 2\operatorname{cis}\frac{\pi}{3}$ **(c)(i)** $\frac{5\pi}{12}$ **(ii)** $\frac{11\pi}{12}$

22(c) The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.
23(c) parallelogram **(d)** $\arg\frac{w}{z} = \frac{\pi}{2}$, so $\frac{w}{z}$ is purely imaginary.
26 Use the converse of the opposite angles of a cyclic quadrilateral.

Exercise 1F (Page 41)

1(a)


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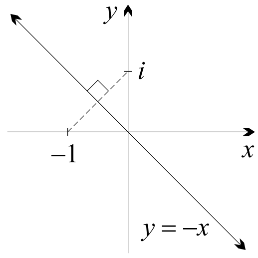

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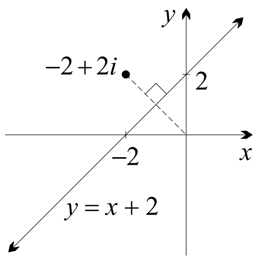
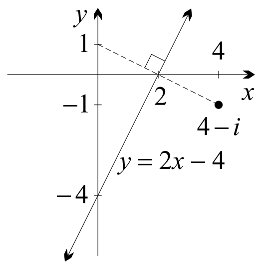
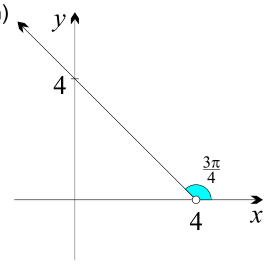


(d)


2(a)


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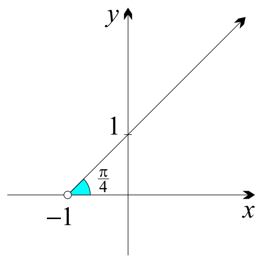

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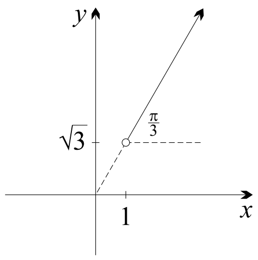
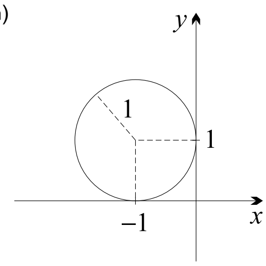


3(a)


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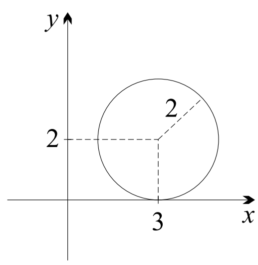

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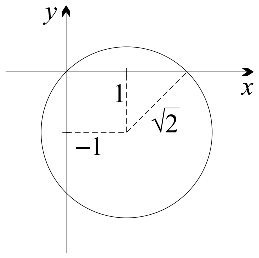
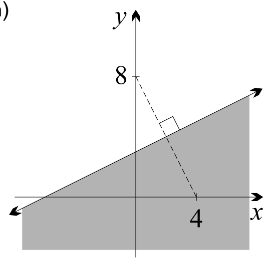


4(a)


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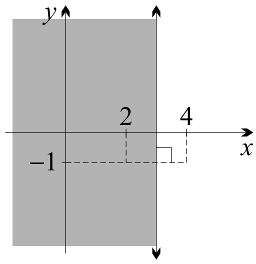

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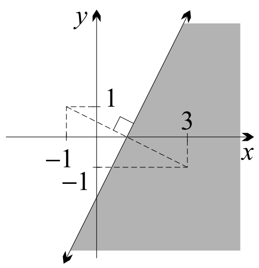

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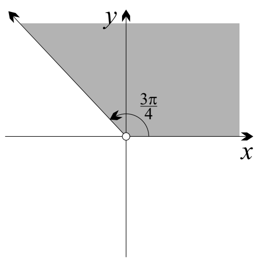

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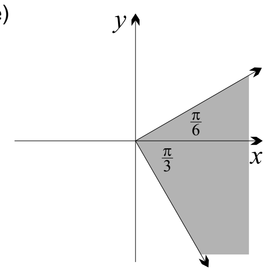

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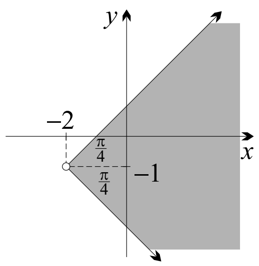

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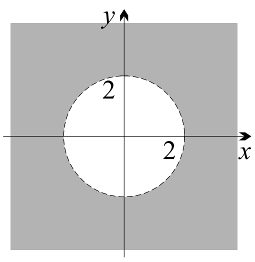

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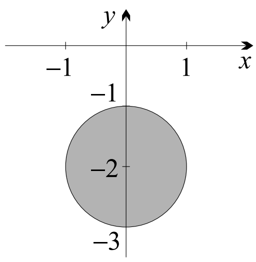

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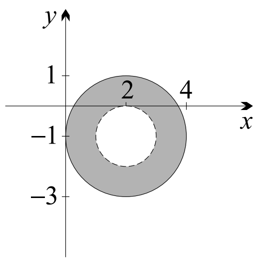
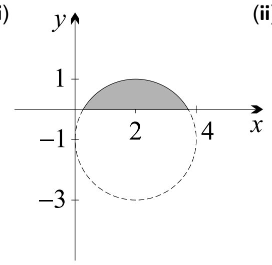


5(a)(i)


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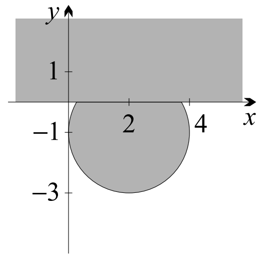

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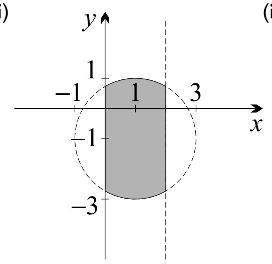

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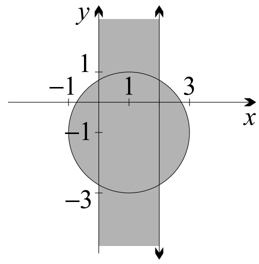

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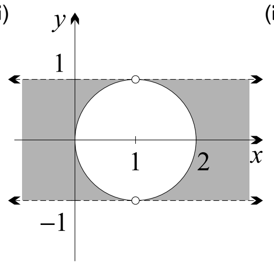

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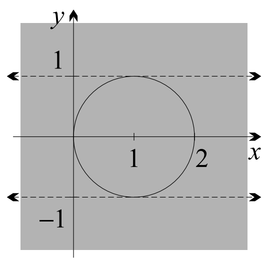

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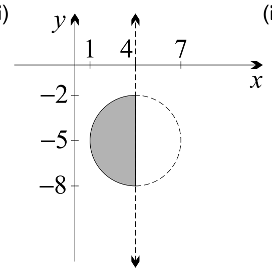

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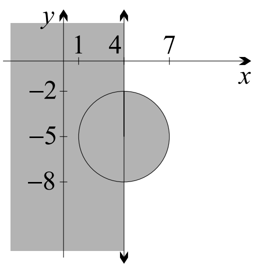

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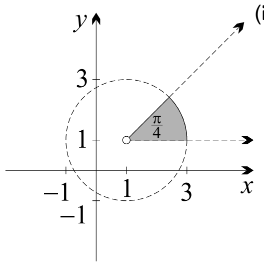

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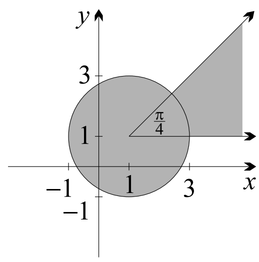

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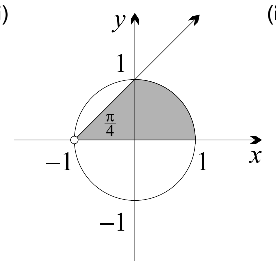

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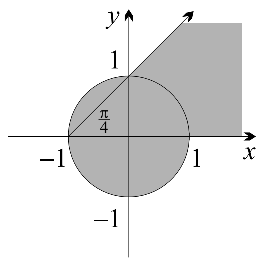
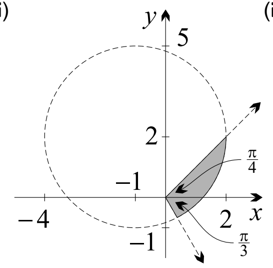
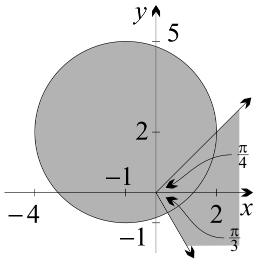


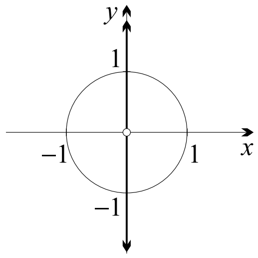
(g)(i)



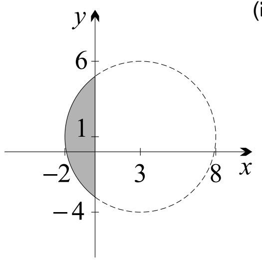
(ii)



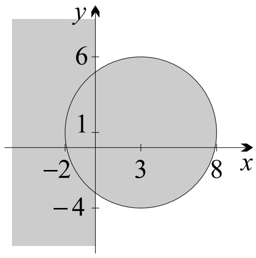
(c)



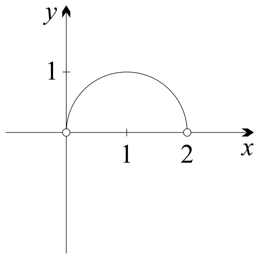
(h)(i)



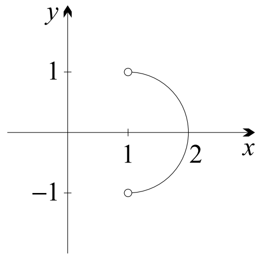
(ii)



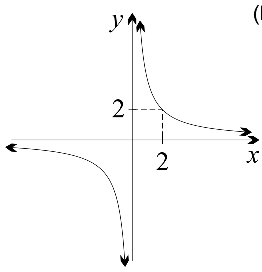
9(a)



(b)



6(a)



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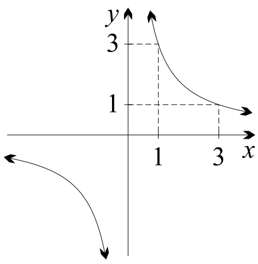

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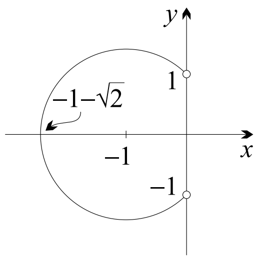
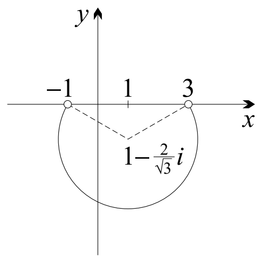
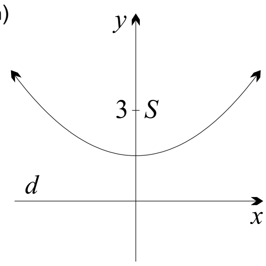


(d)



7(a)



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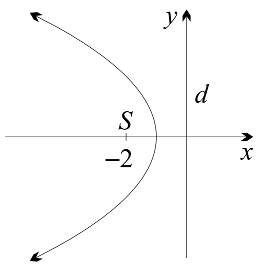
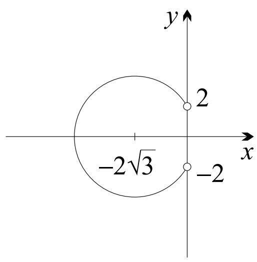
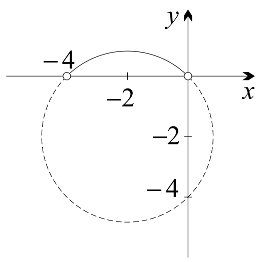


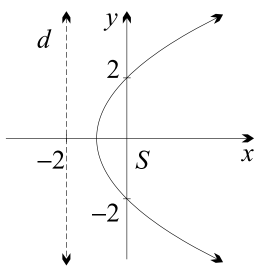
(e)



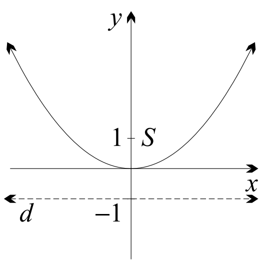
(f)



(c)



(d)

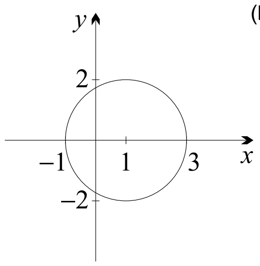


10(b) $\sqrt{3}\operatorname{cis}\frac{\pi}{3} = \frac{\sqrt{3}}{2}(1 + i\sqrt{3}\,)$

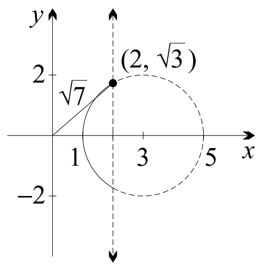
11(a) $\arg(z + 3) = \frac{\pi}{3}$ **(b)** $|z| = \frac{3\sqrt{3}}{2}$, $\arg z = \frac{5\pi}{6}$

(c) $-\frac{9}{4} + \frac{3\sqrt{3}}{4}i$

12(a)

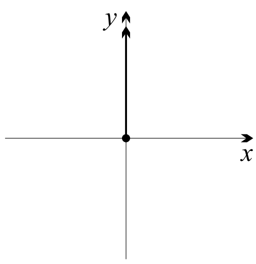


(b)

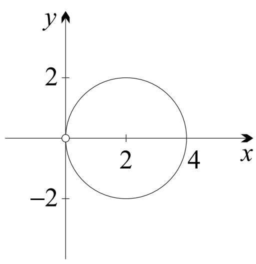


3 and 1

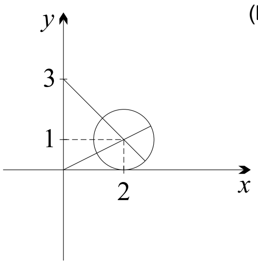
8(a)



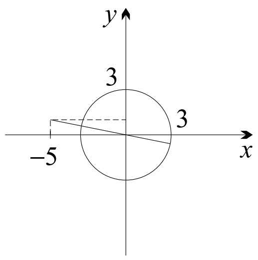
(b)



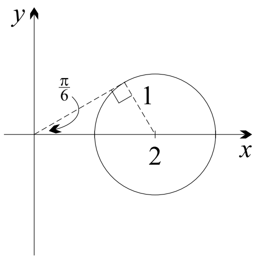
13(a)



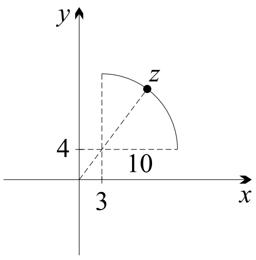
(b)



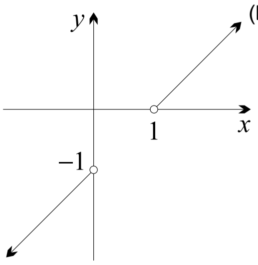
(i) $\sqrt{5}+1$ and $\sqrt{5}-1$ $\sqrt{26}+3$ and $\sqrt{26}-3$
(ii) $2\sqrt{2}+1$ and $2\sqrt{2}-1$
(c)(i) $\Big||z_0|-r\Big| \le |z| \le |z_0|+r$
(ii) $\Big||z_0-z_1|-r\Big| \le |z-z_1| \le |z_0-z_1|+r$
14(a)(i)



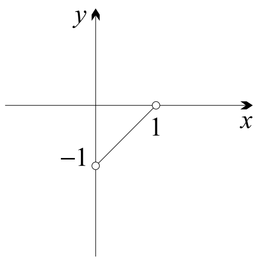
(b) This is simply part (a) shifted left by 2.
15(a)



(b) 15 **(c)** $9+12i$
16(b)(i) $|z+2|=2$, centre -2, radius 2
(ii) $|z-(1+i)|=1$, centre $1+i$, radius 1
(iii) $|z-1|=1$, centre 1, radius 1
17(a) The line through 1 and i, omitting i.
(b) The circle with diameter joining 1 and i, omitting these two points.
18(a)

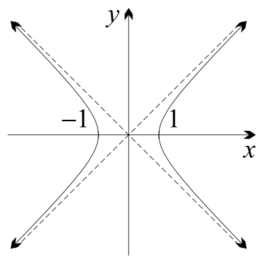


(b)



20(a) straight line external to z_1 and z_2 **(b)** major arc **(c)** semi-circle **(d)** minor arc **(e)** straight line between z_1 and z_2

21



22(a) Angle in the alternate segment theorem: it is the arc taken anticlockwise from z_2 to z_1 of the circle tangent to $y=\operatorname{Im}(z_1)$ and through z_2.
23 The ellipse with eccentricity e,
semi-major axis a and semi-minor axis b,
where $b^2=a^2(1-e^2)$.
24(b) The locus is the perpendicular bisector of the line joining z_1 and z_2.

