THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 11

MATH1901: Differential Calculus (Advanced) — Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Real valued functions $f(x,y)$ of 2 real variables.
- ☐ Natural domain and corresponding range of $f(x,y)$.
- ☐ The graph of $f(x,y)$.
- ☐ Level curves (more generally, cross-sections).
- ☐ Limits of functions of 2 variables.

Outcomes

After completing this tutorial you should

- ☐ find the natural domain and corresponding range of a function $f(x,y)$;
- ☐ draw level curves, and sketch graphs of functions $f(x,y)$ in simple cases;
- ☐ calculate limits of functions $f(x,y)$, or show that they don't exist.

Summary of essential material

Functions of two variables. Consider a function $f\colon D\to\mathbb{R}$ where the domain D is a subset of $\mathbb{R}$. The set of points

$$\operatorname{graph}(f)=\big\{\big(x,y,f(x,y)\big)\mid(x,y)\in D\big\}\subseteq\mathbb{R}^3$$

is called the *graph* of f and is a surface in $\mathbb{R}^3$. Given $c\in\mathbb{R}$ the set

$$\big\{(x,y)\in D\mid f(x,y)=c\big\}$$

is called the *level set* or *level curve of f at height c*. It is typically a proper curve, but can be empty (if c is not in the image of f) or "degenerate" such as a point or crossing curves.

Limits of functions of two variables. We say that $\lim\limits_{(x,y)\to(a,b)}f(x,y)=\ell$ if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$0<\sqrt{(x-a)^2+(y-b)^2}<\delta\quad\Rightarrow\quad|f(x,y)-\ell|<\varepsilon.$$

Analogues of the one-variable limit laws, including the Squeeze Law, hold for functions of two-variables.
Useful techniques to determine limits

- Use simple inequalities: $|xy|\leq x^2+y^2$, $|\sin a|\leq|a|$, $\log x\leq x-1$ $(x>0)$ and others.
- Convert to polar coordinates: $x=r\cos\theta$, $y=r\sin\theta$, limit to $(0,0)$ is the same as $r\to0$ *independently* of θ.

Useful ways to show that a limit does *not* exist:

- Find two "paths of approach" giving two different limits. Often along the axes, the diagonals or some parabola, depending of the structure of the expression of interest.
- The limit along one particular path does not exist.

Continuity of functions of two variables. A real valued function $f(x,y)$ of two real variables is *continuous* at a point (a,b) in the domain D of f if

$$\lim_{(x,y)\to(a,b)}f(x,y)=f(a,b).$$

Questions to complete during the tutorial

1. The Taylor polynomial of order n for the exponential function e^x is

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Substitute $x = i\theta$ into this polynomial and show that $T_n(i\theta) = C_n(\theta) + iS_n(\theta)$, where $C_n(\theta)$ and $S_n(\theta)$ are the Taylor polynomials of order n of the cosine and sine functions respectively.

Solution: We know that $i^2 = -1, i^3 = -i, i^4 = 1$. If n is even let $m = n/2$ and if n is odd let $m = (n+1)/2$ this means that

$$\begin{aligned} T_n(i\theta) &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \cdots + i^n\frac{\theta^n}{n!} \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + (-1)^m\frac{\theta^{2m}}{(2m)!}\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots + (-1)^{m-1}\frac{\theta^{2m-1}}{(2m-1)!}\right) \\ &= C_n(\theta) + iS_n(\theta) \end{aligned}$$

as claimed. This provides another connection between the real and the complex exponential function and motivation for the exponential notation $e^{i\theta}$.

2. Determine the natural domain and corresponding range of the following functions.

(a) $f(x, y) = \sqrt{x - y}$

Solution: The domain is $D = \{(x, y) \mid y \le x\}$. We claim that the range is $[0, \infty)$. For if $(x, y) \in D$ then $f(x, y) = \sqrt{x - y} \ge 0$, and so range$(f) \subseteq [0, \infty)$. On the other hand, if $a \ge 0$ then $(a^2, 0) \in D$ and $f(a^2, 0) = a$, and so $[0, \infty) \subseteq$ range(f). Hence equality holds.

(b) $f(x, y) = \tan^{-1}(y/x)$

Solution: Since $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$, the domain of $f(x, y)$ is $D = \{(x, y) \mid x \ne 0\}$ (this is all of $\mathbb{R}^2$ except for the y axis). We claim that the image (range) of f is $\mathrm{im}(f) = (-\frac{\pi}{2}, \frac{\pi}{2})$. To see this, note that $\mathrm{im}(f) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$, because $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$. Now note that $f(1, y) = \tan^{-1} y$, and since $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is surjective we see that $(-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq$ range(f).

(c) $f(x, y) = \sqrt{x + y} - \sqrt{x - y}$

Solution: The domain is $D = \{(x, y) \mid x + y \ge 0 \text{ and } x - y \ge 0\} = \{(x, y) \mid -x \le y \le x\}$. These are the points in the first and fourth quadrants on and between the lines $y = x$ and $y = -x$. Observe that on the line $y = x$ in the domain, $f(x, y) = \sqrt{2x}$, which can take all values in $[0, \infty)$. On the line $y = -x$ in the domain, $f(x, y) = -\sqrt{2x}$, which can take all values in $(-\infty, 0]$. Thus the range is $\mathbb{R}$.

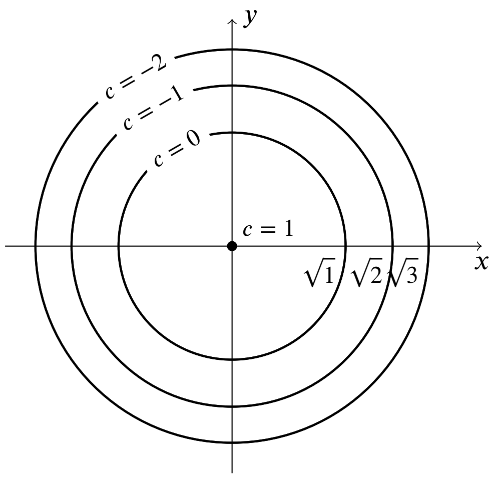
(d) $f(x, y) = \sin^{-1}(x + y)$

Solution: The domain is $D = \{(x, y) \mid -1 \le x + y \le 1\}$. These are the points lying in an infinite strip on and between the lines $y = -1 - x$ and $y = 1 - x$. When $x + y = -1$, $f(x, y) = \sin^{-1}(-1) = -\pi/2$. When $x + y = 1$, $f(x, y) = \sin^{-1}(1) = \pi/2$. The $\sin^{-1}$ function is an increasing function and takes all values between $-\pi/2$ and $\pi/2$ as the value of $x + y$ moves from -1 to 1. Therefore the range is $[-\pi/2, \pi/2]$.

3. Sketch the level curves at heights $c = 0, \pm 1, \pm 2$ for the functions $f(x, y)$:

(a) $1 - x^2 - y^2$

Solution: If $f(x, y) = c$ then $x^2 + y^2 = 1 - c$. If $c = 2$ then the level curve is empty (this is because 2 is not in the range of $f(x, y)$). If $c = 1$ then the level curve consists of the single point $(x, y) = (0, 0)$. For $c = 0, -1, -2$ the level curves are circles with radius $1, \sqrt{2}, \sqrt{3}$ (respectively).

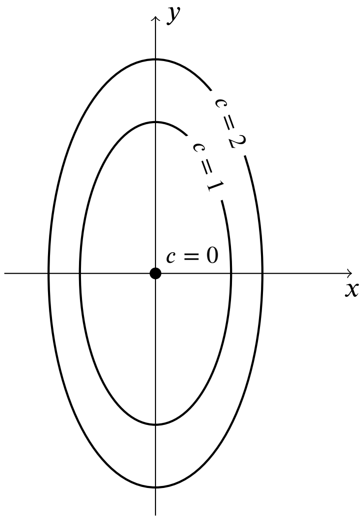


(b) $4x^2 + y^2$

Solution: If $c = 0$ then the level curve is a single point $(x, y) = (0, 0)$, and if $c < 0$ then the level curve is empty. So suppose that $c > 0$. If $f(x, y) = c$ then

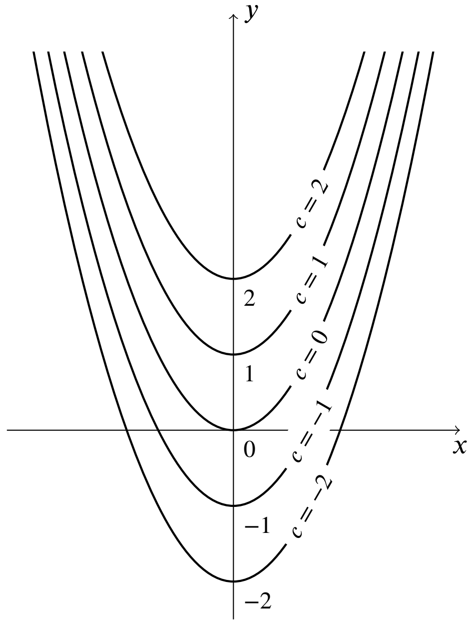
$$\frac{x^2}{(\sqrt{c}/2)^2} + \frac{y^2}{(\sqrt{c})^2} = 1,$$

and this is the equation of an ellipse. The level curves at heights $c = 0, 1, 2$ are shown in the figure.



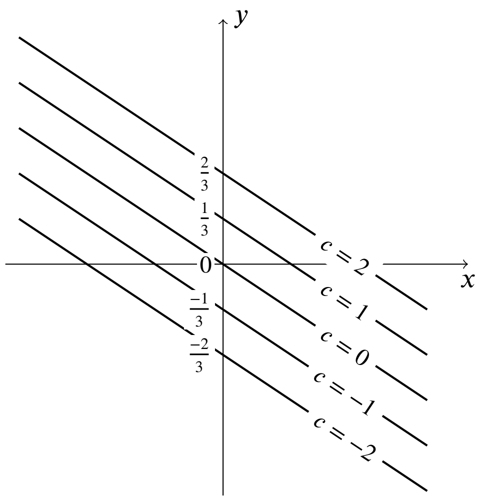
(c) $y - x^2$

Solution: The level curve $f(x, y) = c$ is $y = x^2 + c$. Thus the level curves are parabolas:



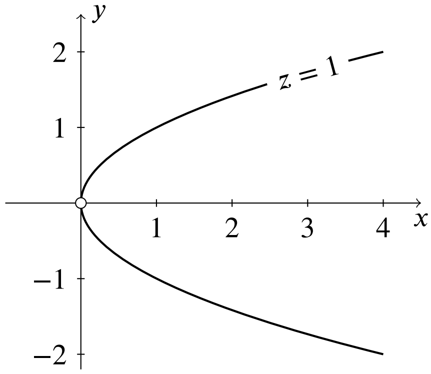
(d) $2x + 3y$

Solution: The level curve $f(x, y) = c$ is the straight line $2x + 3y = c$:



4. Sketch the level curve of height $z = 1$ for the function $f(x, y) = \dfrac{2xy^2}{x^2 + y^4}$.

Solution: The domain is $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$. The level curve is the set of $(x, y) \in D$ such that $f(x, y) = 1$. Thus $2xy^2 = x^2 + y^4$, and so $(y^2 - x)^2 = 0$. Thus $x = y^2$. We need to remember that $(0, 0)$ is not in the domain of the function, therefore the level curve is a parabola with the origin removed:



5. Consider the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}, \text{ defined for } (x, y) \neq (0, 0).$$

Is it possible to define $f(0, 0)$ so that f is continuous at $(0, 0)$? Continuous, as in case of a function of one variable, means that the limit exists and is equal to the function value.

Solution: Using polar coordinates for x and y, (that is, $x = r\cos\theta,\ y = r\sin\theta$), we have

$$\frac{\sin(x^2 + y^2)}{x^2 + y^2} = \frac{\sin r^2}{r^2}.$$

Since $(x, y) \to (0, 0)$ if and only if $r^2 \to 0$, we see that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r^2\to 0} \frac{\sin r^2}{r^2} = 1.$$

Thus we can define $f(0, 0) = 1$ to make f continuous at $(0, 0)$.

6. Find the limit, if it exists, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

Solution: By limit laws, the limit is $(0 + 0 - 5)/(2 - 0) = -\frac{5}{2}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x - y}{x^2 + y^2}$

Solution: Let $f(x, y) = (x - y)/(x^2 + y^2)$. Approaching $(0, 0)$ along $y = 0$ we have

$$f(x, 0) = \frac{x}{x^2} = \frac{1}{x},$$

and the limit of this expression as $x \to 0$ does not exist. Thus $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist.

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2}{x^2 + y^2}$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x(x^2 + y^2)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} x = 0.$$

Extra questions for further practice

7. Find the domains and ranges, and describe the level curves, of the functions defined by:

(a) $\sqrt{4 - x^2 - y^2}$

Solution: The domain is $\{(x, y) \mid x^2 + y^2 \leq 4\}$ (a circle and its interior); range $=$ $[0, 2]$. Level curves are either empty (at height $k < 0$ or $k > 2$), a single point (at height $k = 2$) or circles (at heights k where $0 \leq k < 2$).

(b) $(x - 1)(y + 1)$

Solution: Domain $= \mathbb{R}^2$; range $= \mathbb{R}$; level curves are hyperbolas in "shifted" first and third quadrants at positive heights and hyperbolas in "shifted" second and fourth quadrants at negative heights. At height 0, the level curve is the union of the lines $x = 1$ and $y = -1$.

8. Decide whether or not the following limits exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$

Solution: Approaching along $y = x$ we have

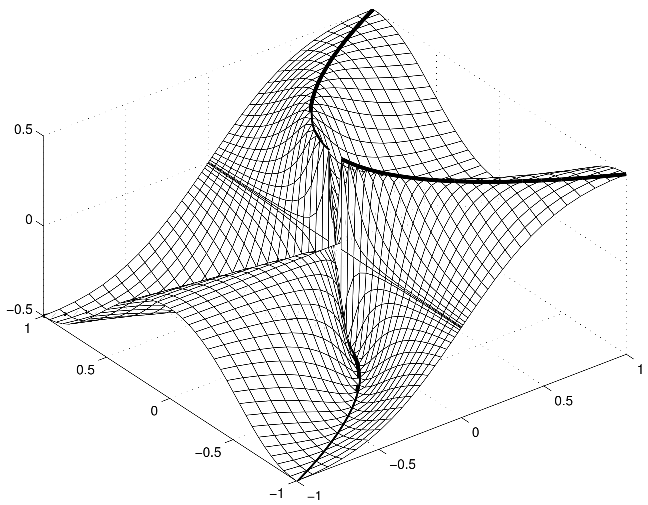
$$f(x, x) = \frac{x^3}{x^2 + x^4} = \frac{x}{1 + x^2} \to 0.$$

Approaching along $x = y^2$ we have

$$f(y^2, y) = \frac{y^4}{2y^4} = \frac{1}{2} \to \frac{1}{2}.$$

Therefore the limit does not exist.

Here is an image of the surface, in which one particular path $x = y^2$ to the origin has been highlighted.



(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} \sin \frac{1}{x^2 + y^4}$

Solution: We use the Squeeze Law:

$$\left| \frac{xy^2}{x^2 + y^2} \sin \frac{1}{x^2 + y^4} \right| \leq \frac{|x| y^2}{x^2 + y^2} \leq \frac{|x|(x^2 + y^2)}{x^2 + y^2} = |x|.$$

Since $\displaystyle\lim_{(x,y)\to(0,0)} |x| = 0$ we have

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} \sin \frac{1}{x^2 + y^4} = 0.$$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$

Solution: let $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Approaching $(0,0)$ along the x-axis (that is, $y = 0$), we have

$$f(x,0) = \frac{x^2}{2x^2} = \frac{1}{2} \to \frac{1}{2},$$

and approaching along the y-axis (that is, $x = 0$), we have

$$f(0,y) = \frac{-y^2}{2y^2} = -\frac{1}{2} \to -\frac{1}{2}.$$

Therefore the limit does not exist.

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$

Solution: We use the Squeeze Law. Using $|x^2 - y^2| \le x^2 + y^2$ (triangle inequality) gives:

$$0 \le \left| \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \right| \le \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2}.$$

Since $\sqrt{x^2 + y^2} \to 0$ we have $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0$.

Challenge questions (optional)

9. Use the ϵ, δ definition of the limit of a function of two variables to show that

$$\lim_{(x,y)\to(1,2)} (x^2 + y) = 3.$$

Solution: We want to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < \sqrt{(x-1)^2 + (y-2)^2} < \delta \implies |x^2 + y - 3| < \epsilon.$$

Note that the set of points (x, y) satisfying the inequality $0 < \sqrt{(x-1)^2 + (y-2)^2} < \delta$ can be interpreted geometrically as the set of points in the interior of a circle with centre $(1, 2)$ and radius δ, without the centre itself.

We examine the difference between $x^2 + y$ and 3 and try to write this in such a way as to incorporate terms in $x - 1$ and $y - 2$.

$$\begin{aligned} |x^2 + y - 3| &= |(x-1)^2 + 2x - 1 + (y-2) + 2 - 3| \\ &= |(x-1)^2 + 2(x-1) + (y-2)| \\ &\le (x-1)^2 + 2|x-1| + |y-2| \end{aligned}$$

To guarantee that $|x^2 + y - 3| < \epsilon$, we need only be sure that each of the three expressions $(x-1)^2$, $2|x-1|$, $|y-2|$ is less than $\epsilon/3$. Now as $\lim_{x\to1}(x-1)^2 = 0$, $\lim_{x\to1} 2|x-1| = 0$ and $\lim_{y\to2}|y-2| = 0$, there exists $\delta_1 > 0$ such that

$$0 < |x-1| < \delta_1 \implies (x-1)^2 < \epsilon/3$$

(for example, $\delta_1 = \sqrt{\epsilon/3}$), there exists $\delta_2 > 0$ such that

$$0 < |x-1| < \delta_2 \implies 2|x-1| < \epsilon/3$$

($\delta_2 = \epsilon/6$), and there exists $\delta_3 > 0$ such that

$$0 < |y-2| < \delta_3 \implies |y-2| < \epsilon/3$$

($\delta_3 = \epsilon/3$). Now choose δ to be the minimum of δ_1, δ_2,, δ_3. Then, whenever (x, y) is a point inside a circle with centre at $(1, 2)$ and radius δ (but not the centre itself), we can be sure that $|x^2 + y - 3| < \epsilon$. That is, given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < \sqrt{(x-1)^2 + (y-2)^2} < \delta \implies |x^2 + y - 3| < \epsilon.$$

This proves the result.