THE UNIVERSITY OF SYDNEY
FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH1902
LINEAR ALGEBRA (ADVANCED)

June/July 2007 LECTURER: A Molev

TIME ALLOWED: **One and a half hours**

Name: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 25% of the total examination;
there are 15 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 75% of the total examination;
there are 5 questions; the questions are of equal value;
all questions may be attempted;
working must be shown.

Calculators will be supplied; no other calculators are permitted.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

1. **(10 marks)**. Let π be the plane given by the equation $3x - y - 2z = -3$.

   (a) Find parametric scalar equations for the line ℓ which passes through the point $A(1, 0, -4)$ and which is perpendicular to π.

   (b) Find the coordinates of the intersection point B of ℓ and π.

   (c) Hence calculate the distance from A to π.

   (d) Find the Cartesian equation of the plane π' through the point $A(1, 0, -4)$ which is parallel to the plane π.

   (e) Find all values of c for which the plane $3x - y + cz = -3$ is perpendicular to the plane π.

2. (a) **(6 marks)**. The line m is given by the equations $\dfrac{x}{2} = \dfrac{y+7}{3} = \dfrac{z-6}{-11}$ and the plane ρ is given by the equation $5x - 2y + z = 9$.

   (*i*) Show that m and ρ are not parallel.

   (*ii*) Use vector product to find a nonzero vector which is perpendicular to m and parallel to ρ.

   (*iii*) Find the Cartesian equation of a plane which is perpendicular to the plane ρ and contains the line m.

   (b) **(4 marks)**.

   (*i*) Given that the volume of a pyramid is found as one third of the product of the area of the base and the height, show that the volume of a tetrahedron $ABCD$ can be given by $V = \dfrac{1}{6}\left|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})\right|$.

   (*ii*) Calculate the volume of the tetrahedron $ABCD$ with $A(1, 2, 3)$, $B(-1, 0, 5)$, $C(0, 3, 1)$ and $D(2, 2, 2)$.

3. (10 marks). For the system of linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 2 \\ 2x_1 + x_2 + 2x_3 + x_4 = 2 \\ 2x_1 + x_2 + 2x_3 + 2x_4 = 4 \\ 3x_1 + 2x_2 + 4x_3 + 2x_4 = 4 \end{cases}$$

(a) Write down the augmented coefficient matrix.

(b) Use elementary row operations to bring the augmented coefficient matrix into the reduced row echelon form.

(c) Write down the general solution of the system.

(d) Find all values of the parameters a, b, c and d such that every solution of the above system is also a solution of another system of linear equations given by

$$\begin{cases} a\,x_1 + b\,x_2 + c\,x_3 + d\,x_4 = 0 \\ b\,x_1 + a\,x_2 + c\,x_3 = 0 \\ d\,x_3 + a\,x_4 = 2. \end{cases}$$

4. (10 marks).

(a) Formulate the *definition* of an eigenvalue and an eigenvector of a square matrix A.

(b) Prove that a scalar λ is an eigenvalue of a square matrix if and only if λ is a root of its characteristic polynomial.

(c) Consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 4 & 3 & 1 \end{bmatrix}.$$

Without calculating all eigenvalues, explain why $\lambda = 0$ is an eigenvalue of A.

(d) Determine all eigenvalues of the matrix A.

(e) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of a certain matrix B with the respective eigenvalues λ and μ such that $\lambda \neq \mu$. Is it possible that $\mathbf{u} + \mathbf{v}$ is also an eigenvector of B? Justify your answer.

5. **(10 marks).**

(a) Find a lower triangular $n \times n$ matrix A with non-negative entries satisfying the condition $A\,A^T = C$, where C is the $n \times n$ matrix given by

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & & \cdots & \\ & \cdots & & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}.$$

(b) Hence or otherwise calculate the determinant of C.

(c) Find the eigenvalues and corresponding eigenspaces of the matrix A found in part (a).

End of Extended Answer Section