limit - polar coordinates.

Using polar coordinates to examine limits as
$(x,y) \to (0,0)$

example.

Consider

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$$

parametrization

In this case the limit exists and equals 0
Notes that the numerator $\to 0$ more quickly
than the denominator.

* Switch to polar coordinates

$x = r\cos\phi$

$y = r\sin\phi$

$x^2+y^2 = r^2$

$$f(x,y) = \frac{x^3-y^3}{x^2+y^2}$$

$$= \frac{r^3\cos^3\phi - r^3\sin^3\phi}{r^2}$$

$$= r(\cos^3\phi - \sin^3\phi)$$

If $|(x,y)| \leq 2r$
(This bound can be lower, but that is not important)

As $(x,y) \to (0,0)$ on any path, the polar radial coordinate
r will $\to 0$, and so will $2r$.

$$0 \leq \frac{x^3-y^3}{x^2+y^2} \leq 2r$$

By the squeeze law the middle tends to 0
as well. $\frac{x^3-y^3}{x^2+y^2} \to 0$ as $(x,y) \to (0,0)$ on every path.

that is

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$$

Many cases where the limit actually exists can be handled by this method or some variants.

e.g. If x^4+y^2 appears on the denominator, change variable $\bar{x}=\sqrt{|x|}$. The new denominator is ~~$\bar{x}^2+\bar{y}^2$~~ $\bar{x}^2+y^2$

Consider

$$\lim_{(x,y)\to(0,0)} \frac{p(x,y)}{(x^2+y^2)^k}$$

where p is a polynomial
In polar coords, the denominator is r^{2k}.
If the numerator has only higher powers of r, the limit will be 0.

If the numerator contains r^{2k}, (possibly higher power as well) [homogeneous polynomial), the limit will depend on the path, and so the ~~z-1)~~ (limit will not exist, If lower powers occurs, the limit will be either infinite (same for all paths) or non existent (different limits on different path)

- It is much easier to prove that a limit does not exist.
To complete such a proof, you need one of the followings
(a) 2 paths, along which different limits are obtained.
(b) 1 path along which no limit exist.
[Note that infinite limits are generally allowed, but they must be $+\infty$ on all paths or $-\infty$ on all paths]

e.g. $\frac{1}{x^2+y^2} \to +\infty$.

as $(x,y) \to (0,0)$ on all paths.

$\ln(x^2+y^2) \to -\infty$.

as $(x,y) \to (0,0)$ on all paths

Cases that depend on the path

Consider

$$f(x,y) = \frac{xy}{x^2+y^2}$$

This has no limit at $(0,0)$ because

y-axis $f(x,0)=0 \to 0$ as $x \to 0$

$y=x$ $f(x,y) = \frac{x^2}{2x^2} = \frac{1}{2} \to \frac{1}{2}$ as $x \to 0$

2 limits on two paths

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$, limits D.N.E

Similarly for

$$\lim_{(x,y)\to(0,0)} \frac{ax^2+bxy+cy^2}{x^2+y^2}.$$

Consider this in polar coordinates.

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$$= \frac{r^2\cos\theta\sin\theta}{r^2}$$

$$= \cos\theta\sin\theta$$

$$= \frac{1}{2}\sin 2\theta.$$

☐ The rays to the origin are <u>level curves</u> of $f(x,y)$.

$y = mx$:

$$f(x, mx) = \frac{mx^2}{x^2+m^2x^2}$$

$$= \frac{m}{1+m^2}$$

$$= \text{constant}.$$

<u>Level curves</u> approaching each other at different height ~~...~~ $\Rightarrow$ no limit (limits depends on path)

The function
$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}.$$
has a removable discontinuity at the origin. We can remove it by extending f to the origin by

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

$$g(x,y) = \frac{\sin(x^2-y^2)}{x^2-y^2}$$

has 2 lines $y = \pm x$ of removeable discontinuity

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2-y^2)}{x^2-y^2}.$$

D.N.E because the path of approach may cross $y=\pm x$, where the function is not defined.

* Remove discontinuity

The extended function.

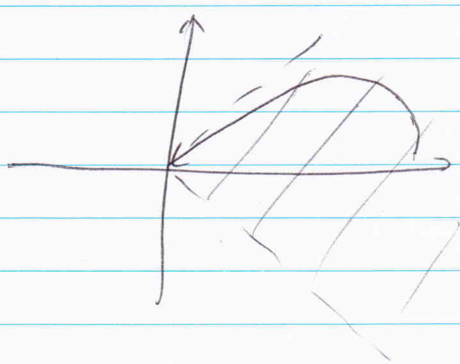
$$g(x,y)=\begin{cases} \dfrac{\sin(x^2-y^2)}{x^2-y^2}. & y\neq \pm x \\ 1 & y=\pm x \end{cases}$$

Consider.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2-y^2}.,$$ "provided $y\neq\pm x$".

The limit does not exist because the function is undefined on the path $y=\pm x$

Suppose we take the limit on paths that stay inside the sector $-x<y<x$, $x>0$.



The limit is 0 on straight paths and paths that are not tangential (or too close) to $y = \pm x$.

Consider the path.

$$y = x - x^5 \qquad x > 0, \quad x \to 0^+$$

$$f(x,y) = \frac{x^3 + y^3}{x^2 - y^2}$$

$$= \frac{x^3 + (x - x^5)^3}{x^2 - (x - x^5)^2}$$

$$= \frac{x^3(1 + (1 + x^4)^3)}{x^6(2 - x^4)}$$

$$= \frac{1}{x^3} \quad \frac{2 + \cdots}{2 + \cdots}$$

$$\sim \frac{1}{x^3}$$

as $x \to 0^+$, the limit on this path is $+\infty$ so the limit still does not exist example.

$$\lim_{(x,y)\to(0,0)} \ln \sin(x^2+y^2)$$

use polar.

$\sin(x^2+y^2) = \sin(r^2)$

$0 < \sin(r^2) < r^2, \quad r>0$

$-\infty < \ln \sin(x^2+y^2) < 2\ln r$

As $(x,y)\to(0,0)$, $r\to 0$ as well.

$2\ln r \to -\infty$

So by the squeeze law.
(applied to infinite limits)

$\lim_{(x,y)\to(0,0)} \ln \sin(x^2+y^2)$ exists and equal $-\infty$.