TMA4120 - Øving 10
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14.3.3

$$f(z) = \frac{z^2}{z^2-1}, \quad C: |z+i| = 1.41$$

$$\cancel{\text{C.R.}}\ \frac{z^2}{z^2-1} = \frac{z^2}{(z-1)(z+1)} = \cancel{\frac{A}{z-1} + \frac{B}{z+1}}$$

$$\cancel{A(z+1) + B(z-1) = z^2}$$

$$\Rightarrow \oint_C f(z)\,dz = \oint_C \frac{g(z)}{z-1}\,dz, \quad g(z) = \frac{z^2}{z+1}$$

C: a circle with centre in $-i$ and $r = 1.41$

Since $z_0 = 1$, ~~path~~ z_0 is <u>not</u> inside C^*, and integral is <u>zero</u>

14.3.12

$$\oint_C \frac{z}{z^2+4z+3}\,dz, \quad C: |z+1| = 2$$

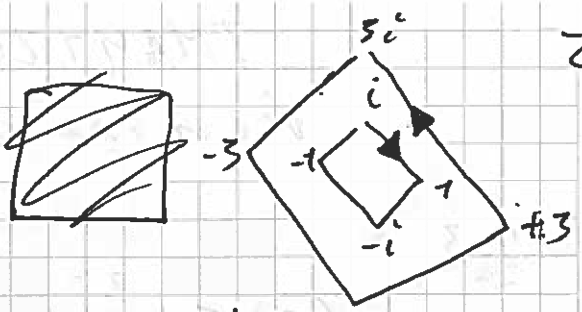
$$\Rightarrow \oint_C \frac{z}{(z+1)(z+3)}\,dz = \oint_C \frac{g(z)}{z+1}\,dz, \quad z_0 = -1$$

$$g(z) = \frac{z}{z+3}$$

$$\Rightarrow \oint_C \frac{g(z)}{z+1}\,dz = 2\pi i f(z_0) = 2\pi i \cdot \left(\frac{-1}{2}\right) = \underline{-\pi i}$$

14.3.18:

$$\oint \frac{\sin(z)}{4z^2-8iz}\,dz \quad . \quad C:$$



$$\oint \frac{\sin(z)}{4z^2-8iz}\,dz = \oint \frac{g(z)}{z-2i}, \quad g(z) = \frac{\sin(z)}{4z}$$

$z_0 = 2i \Rightarrow z_0$ outside of the smaller square
so integral aroud the small square is $\underline{\underline{0}}$

For the bigger square:

$$g(z_0)\cdot 2\pi i \quad . \quad g(z_0) = \frac{\sin(2i)}{8i}$$

$$\Rightarrow \frac{\pi}{4}\cdot \sin(2i) = \frac{\pi}{4}\left(\frac{e^{-2}-e^{2}}{2i}\right) = \underline{\underline{\pi \cdot \frac{e^{-2}-e^{2}}{8i}}}$$

14.4.3: $\oint_C \frac{e^{-z}}{z^n}\,dz$, $n = 1, 2, 3$

Use: $f^{n}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}$

$z_0 = 0 \Rightarrow f^{n}(z_0) = (-1)^n e^{-z_0} = (-1)^n$

$$\Rightarrow \quad \cancel{\frac{2\pi i(-1)^n}{n!}} \qquad \underline{\underline{\frac{2\pi i(-1)^{(n-1)}}{(n-1)!}}}$$

Because $f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^n}$

14.4.4 $\oint_C \frac{e^z\cos(z)}{(z-\pi/4)^3}\,dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{4}\right)$

$f(z) = e^z\cos(z)$

$f'(z) = e^z(\cos(z) - \sin(z))$

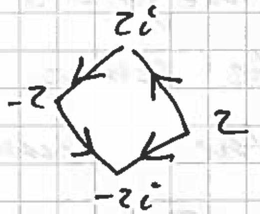
$f''(z) = e^z(\cos(z) - \sin(z)) - e^z(\sin(z) + \cos(z))$

$= -2e^z\sin(z)$

$\Rightarrow f''\left(\frac{\pi}{4}\right) = -2e^{\frac{\pi}{4}}\cdot\frac{1}{\sqrt{2}} = -\sqrt{2}\,e^{\frac{\pi}{4}}$

$\Rightarrow \oint \frac{e^z\cos(z)}{(z-\pi/4)^3} = \underline{\underline{-\sqrt{2}\,\pi i\,e^{\frac{\pi}{4}}}}$

14.4.8.

$\oint \frac{z^3+\sin(z)}{(z-i)^3}\,dz$, C: 

$= \frac{2\pi i}{2!} f''(z_0)$, $z_0 = i$, $f(z) = z^3 + \sin(z)$

$\Rightarrow f'(z) = 3z^2 + \cos(z)$

$f''(z) = 6z - \sin(z)$

$\Rightarrow f''(i) = 6i - \sin(i) = 6i - \left(\frac{e^{-1}-e^{1}}{2}\right)$

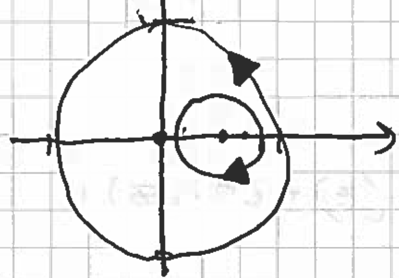
$= 6i - \left(\frac{1-e^2}{e}\right)$

$\Rightarrow \oint \frac{z^3+\sin(z)}{(z-i)^3} = \pi i(6i + \sinh(1))$

$= \underline{\underline{\pi i\sinh(1) - 6\pi}}$

14.4.15 $\oint \frac{\cosh(4z)}{(z-4)^3} dz$; $C: |z| = 6$ c.c.

$|z-3| = 2$ c.



Since $z_0 = 4$, z_0 is <u>inside</u> both circles

if for counterclockwise path integral is $\frac{2\pi i}{2!} f''(4)$, then for <u>clockwise</u> $-\frac{2\pi i}{2!} f''(4)$

So: The integrals for the two paths are the same, with opposite signs and the sum, ergo the answer is: $\underline{\underline{0}}$

15.1.7: $z_n = \frac{(1+i)^{2n}}{2^n} = \left(\frac{(1+i)^2}{2}\right)^n$

$= \left(\frac{(1+2i-1)}{2}\right)^n = i^n$

$\Rightarrow$ <u>Divergent, bounded by $\pm 1, \pm i$</u>

15.1.2:

$z_n = \frac{(1+2i)^n}{n!}$; $\frac{z_{n+1}}{z_n} = \frac{\frac{(1+2i)^n (1+2i)}{n!(n+1)}}{\frac{(1+2i)^n}{n!}} = \frac{1+2i}{n+1}$

Which implies that the series converges since z_{n+1} is smaller than z_n.

Converges to $\underline{\underline{0}}$

15.1.16 $\displaystyle\sum_{n=0}^{\infty} \frac{(20+30i)^n}{n!}$

Ratio test: $\dfrac{z_{n+1}}{z_n} = \dfrac{20+30i}{n+1} = q$

$\lim\limits_{n\to\infty} q = 0 < 1 \Rightarrow$ Converges

15.1.17:
$$\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln(n)}$$

Ratio test: $\dfrac{z_{n+1}}{z_n} = \dfrac{\dfrac{(-i)^n(-i)}{\ln(n+1)}}{\dfrac{(-i)^n}{\ln(n)}} = \dfrac{(-i)\ln(n)}{\ln(n+1)}$

$\Rightarrow \left|\dfrac{(-i)\ln(n)}{\ln(n+1)}\right| = \dfrac{\ln(n)}{\ln(n+1)}$

$\lim\limits_{n\to\infty} \dfrac{\ln(n)}{\ln(n+1)} = 1 \Rightarrow$ no conclusion possible.

Root test: $\lim\limits_{n\to\infty} \sqrt[n]{|z_n|} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt[n]{\ln(n)}} = 1$

$\Rightarrow$ no conclusion!

However, if we use $\dfrac{1}{\ln(n)} > \dfrac{1}{n}$

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{(-i)^n}{n}$, $(-i)^n$ is bounded divergent

$\frac{1}{n}$ is divergent so that series diverges.

Since for $\dfrac{(-i)^n}{\ln(n)}$, each term is bigger than $\dfrac{(-i)^n}{n}$, the original series must also diverge!

15.1.19: $\sum_{n=0}^{\infty} \frac{i^n}{n^2-i}$

Ratio test: $\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{\frac{i^n \cdot i}{(n+1)^2-i}}{\frac{i^n}{n^2-i}}\right| = \left|\frac{i(n^2-i)}{(n+1)^2-i}\right| = q_n$

$\lim_{n\to\infty} q_n = 1 \Rightarrow$ no conclusion

Comparison test: $|a_n| < |b_n|$

$a_n = \frac{i^n}{n^2-i} \Rightarrow |a_n| = \frac{1}{|n^2-i|}$, $|b_n| = \frac{1}{n^2}$, $b_n = \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=0}^{\infty} \frac{i^n}{n^2-i}$ must also converge!

Answer: converges!

15.1.30:

$\cancel{1-q \le}\ \left|\frac{z_{n+1}}{z_n}\right| \le q < 1$

$\Rightarrow 1-q \le -\left|\frac{z_{n+1}}{z_n}\right| + 1$

$\Rightarrow |R_n|(1-q) \le |R_n|\left(1-\left|\frac{z_{n+1}}{z_n}\right|\right)$

$\Rightarrow |R_n| \le \frac{|R_n|}{(1-q)}\left(1-\left|\frac{z_{n+1}}{z_n}\right|\right)$

Rewrite to: $q \ge \frac{|z_{n+1}|}{|R_n|} - 1$

$|R_n| > |z_{n+1}| \Rightarrow \frac{|z_{n+1}|}{|R_n|} < 1 \Rightarrow \frac{|z_{n+1}|}{|R_n|} - 1 < 0$

$\Rightarrow q \ge a < 0$

$q \ge 0 \Rightarrow$ no contradiction, statement must be true!

Error not exceeding 0.05

$\Rightarrow |R_n| \leq 0.05$

$\Rightarrow |z_{n+1}| / (1-q) = 0.05$

$\Rightarrow \left|\frac{n+i}{z^n n}\right| / (1-q) = 0.05$

$\Rightarrow$ Find n that satisfies the condition

$\Rightarrow$ Wolfram it <3 !

15.2.5:

$R = \lim_{n \to \infty} \left|\frac{a_n}{a_{n+1}}\right|$ (I) $\sum a_n z^n$ has Radi of conv R

(II) $\sum a_n z^{2n}$, $\quad z^{2n} = (z - z_0)^{2n}$, $z_0 = 0$

$z^{2n} = (z^2)^n$

$R \,\hat{=}\, |(z - z_0)|$

$= |z|$ (for (I))

For (II) $R = |z^2|$

$\Rightarrow \sqrt{R} = |z|$

15.2.6 $\sum_{n=0}^{\infty} z^n (z-1)^n$

<u>centre: 1</u>

$R = \lim_{n \to \infty} \left|\frac{z^n}{z^n \cdot z}\right| = \frac{1}{z}$

15.2.9

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2^n}(z+i)^{2n}$$

centre : i

Use that convergence radius for $\sum a_n z^{2n}$ is $\sqrt{R}$, wher R is convergence radius for $\sum a_n z^n$

$$\Rightarrow \quad R = \lim_{n\to\infty} \left| \frac{\frac{n(n-1)}{2^n}}{\frac{n(n+1)}{2^n \cdot 2}} \right| = \lim_{n\to\infty} \left| \frac{2(n-1)}{(n+1)} \right| = 2$$

$\Rightarrow$ Radius of conv. $= \sqrt{2}$