TMA4120 - Høst 2016
Øving 7
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1

12.4.13

$$u_{xx} + 5u_{xy} + 4u_{yy} = 0$$

$A = 1$, $B = \frac{5}{2}$, $C = 4$

$AC - B^2 = 4 - \frac{25}{4} = \frac{16-25}{4} < 0 \Rightarrow$ <u>Wave equation</u>

<u>Hyperbolic</u>

Transformation:

$$Ay'^2 - 2By' + C = 0$$

$$\Rightarrow y'^2 - 5y' + 4 = 0$$

$$\Rightarrow y' = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2} = \underline{1 \wedge 4}$$

$$\Rightarrow \cancel{\Phi(x,y) = 1,\ \Psi(x,y) = 4}$$

$$\Rightarrow y = x + c,\quad y = 4x + c$$

$$\Phi(x,y) = c = y - x$$

$$\Psi(x,y) = c = y - 4x$$

$$u = f_1(\Phi) + f_2(\Psi)$$

$$\underline{\underline{u = f_1(y - x) + f_2(y - 4x)}} \qquad y = ct$$

12.7.2

$u(x,0) = f(x)$, $f(x) = 1$ if $|x| < a$ and 0 otherwise

$u_t = c^2 u_{xx}$

$$u(x,t) = \int_0^\infty [A(p)\cos(px) + B(p)\sin(px)] e^{-c^2p^2t}\, dp$$

$$u(x,0) = f(x) = \int_0^\infty [A(p)\cos(px) + B(p)\sin(px)]\, dp$$

where $A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(pv)\, dv$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(pv)\, dv$$

$$\Rightarrow u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left(\int_0^\infty e^{-c^2p^2t}\cos(px-pv)\,dp\right) dv$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cdot \frac{\sqrt{\pi}}{c\cdot\sqrt{t}\,2} e^{-\left(\frac{x-v}{2c\sqrt{t}}\right)^2} dv$$

$$\Rightarrow u(x,t) = \frac{1}{2c\sqrt{\pi t}}\cdot\int_{-\infty}^{\infty} f(v)\cdot e^{-\left(\frac{x-v}{2c\sqrt{t}}\right)^2} dv$$

The integral becomes: $\int_{-a}^{a} e^{-\left(\frac{x-v}{2c\sqrt{t}}\right)^2} dv$

$$\cancel{= e^{-\frac{1}{4c^2t}}\cdot\int_{-a}^{a} e^{-(x^2-2xv+v^2)} dv = e^{-\frac{1}{}}}$$

$$= e^{-\left(\frac{x^2}{4c^2t}\right)}\cdot\int_{-a}^{a} e^{\frac{2xv-v^2}{4c^2t}} dv$$

$$\Rightarrow u(x,t) = \frac{e^{-\left(\frac{x^2}{4c^2t}\right)}}{2c\sqrt{\pi t}}\cdot\int_{-a}^{a} e^{\frac{2xv-v^2}{4c^2t}} dv$$

12.7.13:

$f(x) = 1, \quad x > 0 \qquad erf(\infty) = 1$
$f(x) = 0, \quad x < 0$

$$(12): \; u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2cz\sqrt{t})\, e^{-z^2} dz$$

where $z = (v - x)/2c\sqrt{t}$

$f(x) = 1, \; x > 0$

$\Rightarrow f(x + 2cz\sqrt{t}) = 1, \quad x + 2cz\sqrt{t} > 0$

Set: $x = -2c\sqrt{t}z$ in order to obtain the minimum boundry for integral.

$\Rightarrow z_{min} = -x/2c\sqrt{t}$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/2c\sqrt{t}}^{\infty} e^{-z^2} dz$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz - \frac{1}{\sqrt{\pi}} \int_0^{x/2c\sqrt{t}} e^{z^2} dz$$

$$= \frac{1}{2} erf(\infty) - \frac{1}{2} erf(-x/2c\sqrt{t})$$

$$z^2 = -w^2 \Rightarrow \frac{1}{\sqrt{\pi}} \int_0^{-x/2c\sqrt{t}} e^{-w^2} dw = \underline{erf(-x/2c\sqrt{t})}$$

$$\Rightarrow \underline{\underline{u(x,t) = \frac{1}{2} - \frac{1}{2} erf(-x/2c\sqrt{t})}}$$

Supplementary P

a) $f(x) = \pi x - x^2 \quad , 0 \leq x \leq \pi$

Fourier sine:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\cancel{\pi}}{\cancel{L}} x) , \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin(nx)\,dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[\pi\left(\frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx)\right) - \left(\frac{2}{n^2}x\sin(nx) + \frac{2-n^2x^2}{n^3}\cos(nx)\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{n}(-1)^n - \left(\frac{2-n^2\pi^2}{n^3}(-1)^n\right) + \frac{2}{n^3}\right)$$

$$= \frac{2}{\pi}\left(\frac{-(-1)^n n^2 \pi - 2 + n^2\pi^2(-1)^n + 2}{n^3}\right) = \frac{2}{\pi}\left(\frac{\pi(-1)^n(\pi n - 1)}{n^2}\right)$$

b) $u(x,t) = G(t)F(x)$

$$\Rightarrow F(x)G'(t) = G(t)F''(x) - 2G(t)F'(x)$$

$$\Rightarrow \frac{G'(t)}{G(t)} = \frac{F''(x) - 2F'(x)}{F(x)} = k$$

$$\Rightarrow F''(x) - 2F'(x) = kF(x)$$

Assume $F(x) = Ke^x \sin(nx)$

$$\Rightarrow F'(x) = Ke^x\sin(nx) + nKe^x\cos(nx)$$

$$\Rightarrow F''(x) = F'(x) + nKe^x\cos(nx) - n^2Ke^x\sin(nx)$$

$$F''(x) - 2F'(x) = nKe^x\cos(nx) - nKe^x\sin(nx) - F'(x)$$

$$= e^x\sin(nx)(-n^2K - K) = k^2 e^x \sin(nx)$$

~~$\Rightarrow -n^2 - 1 = 1$~~

~~$\Rightarrow n^2 = -2$~~

~~$\Rightarrow n = \sqrt{2}i$, not an~~ integer.

$\Rightarrow -n^2 - 1 = k$

Solution fits!

c) $u(0,t) = 0 = u(\pi, t)$

$\Rightarrow C \cdot 1 \cdot 0 = C \cdot e^{\pi} \cdot \sin(n\pi)$, $C = K$

$\Rightarrow \sin(n\pi) = 0 \Rightarrow n \in \mathbb{Z}$

$\Rightarrow F_n(x) = C_n e^x \sin(nx)$, $C_n = -n^2 - 1$

$G'(t) = \cancel{\ } K\, G(t)$,

$\Rightarrow G(t) = A e^{\cancel{\ } K \cancel{\ } t}$

$\Rightarrow G_n(t) = A\, e^{(-n^2-1)\cancel{\ } t}$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} F_n(x)\, G_n(t) = \cancel{\sum_{n=1}^{\infty} A\, e^{-n^2} \cdot C_n \sin(nx)}$

$= \cancel{B_n e^{-n^2} \sin(nx)} = \underline{\sum_{n=1}^{\infty} B_n e^{(-n^2-1)t} \cdot \sin(nx) \cdot e^x}$

$u(x,0) = e^x f(x) = \sum_{n=1}^{\infty} B_n \sin(nx) \cdot e^x$ $\qquad \underline{(A \cdot C_n = B_n)}$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} B_n \sin(nx)$

$\Rightarrow B_n$ = fourier coef. from a)

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n e^{(-n^2-1)t} \sin(nx)\, e^x$.

Supplementary Q

$z_0 = 1$

i) $f(z) = z \cdot Re(z) = f(x,y) = (x+iy)\cdot x$

$= x^2 + iyx = u(x,y) + iv(x,y)$

R.C. conditions: $V_x = -u_y$, $V_y = u_x$

$V_x = y$, $V_y = x$, $u_x = 2x$, $u_y = 0$

$\cancel{V_x \neq -u_y \, , \, V_y \neq u_x}$

$\Rightarrow$ ~~Not analytic~~

$z_0 = 1 \Rightarrow x_0 = 1, y_0 = 0$

$\Rightarrow V_{x0} = 0$, $V_{y0} = 1$

$u_{x0} = 2$, $u_{y0} = 0$

$u_{x0} \neq V_{y0}$

$\Rightarrow$ Not Analytic

Alternatively:

$$f'(z) = Re(z) + z \cdot \frac{d\,Re(z)}{dz}$$

$$\frac{d\,Re(z)}{dz} = \lim_{(c,d)\to 0} \frac{Re(a+ib+c+id) - Re(a+ib)}{c+id}$$

$$= \lim_{(c,d)\to 0} \frac{c}{c+id}$$

, set $d=0$ (moving along real line)

$\Rightarrow \lim = 1$

set $c = 0$

$\Rightarrow \lim = 0$

Since limits are different, derivative doesn't exist, ergo, not Analytic

ii) $f(z) = z^2$

$f'(z) = 2z$, $f'(z_0) = 2$

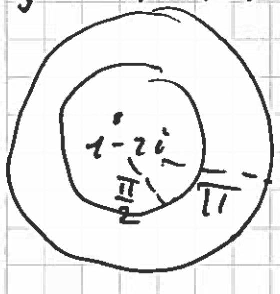
$\Rightarrow$ Analytic

iii) $f(z) = \frac{1}{z}$, $f'(z) = -\frac{1}{z^2} \Rightarrow f'(z_0) = -1$

$\Rightarrow$ Analytic

13.3.3: $\frac{\pi}{2} < |z - 1 + 2i| < \pi$

A circular ring with centre in $1 - 2i$, inner radi $\frac{\pi}{2}$ and outer radi π.



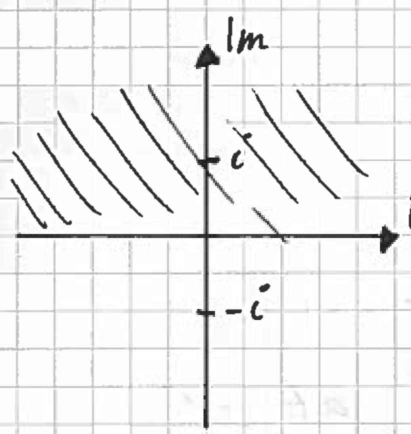
Open set

13.3.7:

$\text{Re}(z) \leq -1$

Closed set

13.3.8: $|z + i| \geq |z - i|$



All points that are closer to i than $-i$ or same distance

Closed set

13.3.10 $f(z) = 5z^2 - 12z + 3 + 2i$, $z_0 = 4 - 3i$

$\Rightarrow f(z_0) = 5(16 - 24i - 9) - 48 + 36i + 3 + 2i$

$\Rightarrow \text{Re}(f(z_0)) = 5 \cdot 16 - 5 \cdot 9 - 48 + 3 = \underline{\underline{-10}}$

$\Rightarrow \text{Im}(f(z_0)) = -5 \cdot 24 + 36 + 2 = \underline{\underline{-82}}$

$\Rightarrow f(z_0) = \underline{\underline{-10 - 82i}}$

13.3.11 $f(z) = \frac{1}{1+z}$, $z_0 = 1-i$

$\Rightarrow f(z_0) = \frac{1}{2-i} = \frac{2+i}{2^2+1^2} = \frac{2+i}{5} \Rightarrow Re = \frac{2}{5}, Im = \frac{1}{5}$

$f(z_0) = \frac{2}{5} + i\frac{1}{5}$

13.3.14 $f(z) = Re\, z^2/|z|$

$z = a+ib \Rightarrow z^2 = a^2+2abi-b^2$

$|z| = \sqrt{a^2+b^2}$

$\Rightarrow f(z) = \frac{a^2-b^2}{\sqrt{a^2+b^2}}$ set $a = r\cos(\theta)$
$b = r\sin(\theta)$

$\Rightarrow f = \frac{r^2(\cos(\theta)-\sin(\theta))}{r} = r(\cos(\theta)-\sin(\theta))$

$r \to 0 \Rightarrow f \to 0$, $f(0) = 0 \Rightarrow$ continuous at $z = 0$

13.3.21

$f(z) = i(1-z)^n$

$\Rightarrow f'(z) = n \cdot i \cdot (1-z)^{n-1} \cdot -1$

$\Rightarrow f'(0) = n \cdot i \cdot 1^{n-1} \cdot -1$

13.3.23 $f(z) = z^3/(z-i)^3$ at $-i$

$\Rightarrow f'(z) = \frac{3z^2(z-i)^3 - 3z^3(z-i)^2}{(z-i)^6}$

$= \frac{3z^2(z-i-z)}{(z-i)^4} = \frac{-i \cdot 3z^2}{(z-i)^4}$

$\Rightarrow f'(-i) = \frac{i \cdot 3}{(-2i)^4} = \frac{3i}{16}$