TMA4120 – Høst 2016

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Øving 11

15.3.4) $(1-z)^{-2} = \sum_{n=0}^{\infty} (n+1) z^n$

a) Cauchy product.

$$f(z) = g(z) = \frac{1}{(1-z)} \Rightarrow f(z)g(z) = \frac{1}{(1-z)^2}$$

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$\Rightarrow f(z)g(z) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0) z^n$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 \dots$$

$$= 1 + 2z + 3z^2 \dots$$

$$= (0+1) z^0 + (1+1) z + (2+1) z^2 \dots$$

$$= \underline{\underline{\sum_{n=0}^{\infty} (n+1) z^n}}$$

b) Differentiation.

$$\sum_{n=0}^{\infty} (n+1) z^n = \sum_{n=0}^{\infty} z^{(n+1)} \frac{d}{dz}$$

$$\sum_{n=0}^{\infty} z^{(n+1)} = \sum_{n=1}^{\infty} z^n = z + z^1 + z^2 + z^3 + z^4 \dots$$

$$= \underline{\frac{1}{1-z} - 1} \quad \left(\sum_{n=0}^{\infty} z^n = \frac{1}{1-z},\ z^0 = 1 \Rightarrow \sum_{n=1}^{\infty} z^n = \frac{1}{1-z} - 1 \right.$$

$$\cancel{\frac{1}{1-z} \frac{d}{dz}} \Rightarrow \frac{1}{1-z} - 1 \frac{d}{dz} = \frac{-1}{-(1-z)^2} = \underline{\underline{(1-z)^{-2}}}$$

15.3.7)

$$\sum_{n=1}^{\infty} \frac{n}{5^n}(z+2i)^{2n} = \sum_{n=1}^{\infty} \frac{n}{5^n}\cancel{(z^2+4iz-2)^n}\left((z+2i)^n\right)^2$$

a) $$\cancel{R = \lim_{n\to\infty}\left|\frac{\frac{n}{5^n}(z+2i)^{2n}}{\frac{(n+1)}{5^{n+1}}(z+2i)^{2n}\cdot(z+2i)^2}\right| = \lim_{n\to\infty}\left|\frac{n\cdot 5}{n+1}(z+2i)^2\right| = |5(z+2i)^2|}$$

$$\Rightarrow R^2 = \lim_{n\to\infty}\left|\frac{\frac{n}{5^n}}{\frac{n+1}{5^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{5\cdot n}{n+1}\right| = 5$$

$$\Rightarrow \underline{\underline{R=\sqrt{5}}} \quad \left(\sum a_n z^n \Rightarrow R = r, \sum a_n (z^n)^b \Rightarrow R = \sqrt[b]{r}\right)$$

b) $$f(z) = \sum_{n=0}^{\infty} \frac{n}{5^n} z_1^{2n} \Rightarrow f^{(n)}(z_1) = \sum_{n=0}^{\infty} \frac{n!\,n}{5^n} z_1^n = \sum_{n=0}^{\infty} n!\cdot n\cdot\left(\frac{z_1}{5}\right)^n$$

$$\left|\frac{z_1}{5}\right| < 1 \Rightarrow \left|\frac{z+2i}{5}\right| < 1 \Rightarrow z+2i < 5 \Rightarrow \underline{\underline{R=5}}$$

15.3.16) $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$f(z) = f(-z) \Rightarrow f(z) - f(-z) = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n (-z)^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (z^n - (-z)^n) = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n (-z)^n$$

$$\Rightarrow a_0 + a_1\cdot z + a_2\cdot z^2 + a_3\cdot z^3 \ldots = b_0 + b_1(-z) + b_2 z^2 + b_3(-z)\ldots$$

Theorem 1 $\Rightarrow a_n = b_n$

$a_1\cdot z = b_1(-z) \Rightarrow a_1 = -b_1 \Rightarrow$ only possible if both are zero. Hence, same goes for all odd a_n & b_n

15.4.5

$$f(z) = \frac{1}{8+z^4} = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!} f^n(0)$$

$$\frac{1}{8+z^4} = \frac{1}{8}\left(\frac{1}{1-(-\frac{1}{8}z^4)}\right) = \underline{\underline{\sum \frac{1}{8}\left(-\frac{1}{8}z^4\right)^n}}$$

$$\Rightarrow \underline{\underline{R = \sqrt[4]{8}}} \quad \left(\left|\left(-\frac{1}{8}z^4\right)\right| < 1\right)$$

15.4.8)

$$\sin^2(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!} f^n(0)$$

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$f(z) = \sin^2(z) \Rightarrow f(0) = 0$$
$$f'(z) = \sin(2z) \Rightarrow f'(0) = 0$$
$$f''(z) = 2\cos(2z) \Rightarrow f''(0) = 2$$
$$f'''(z) = -4\sin(2z) \Rightarrow f^{(3)}(0) = 0$$
$$f''''(z) = -8\cos(2z) \Rightarrow f^4(0) = -8$$
$$f'''''(z) = +16\sin(2z) \Rightarrow f^5(0) = 0$$
$$f^{(6)}(z) = 32\cos(2z) \Rightarrow f^6(0) = 32$$

$$\Rightarrow a_n = \frac{1}{n!} \cdot \cancel{(1+(-1)^n)} \cdot \frac{2^{n-1}}{\cancel{2^1}} \cdot \sin\left(\frac{n-1}{2} \cdot \pi\right)$$

$$\Rightarrow \underline{\underline{\sin^2(z) = \sum_{n=1}^{\infty} \frac{2^{(n-1)}}{n!} \sin\left(\frac{n-1}{2} \cdot \pi\right) \cdot z^n}}$$

$$\underline{\underline{R = 1}}$$

15.4.9

$$f(z) = \int_0^z e^{-t^2}\,dt$$

$$f'(z) = e^{-z^2}$$
$$f''(z) = -2z e^{-z^2}$$
$$\cancel{f''(z) = 4z^2 e^{-z^2}}$$

$$f'''(z) = e^{-z^2}(4z^2-2)$$
$$f^{(4)}(z) = -4e^{-z^2}(2z^3-3z)$$
$$f^{(5)}(z) = 4e^{-z^2}(4z^4-12z^2+3)$$

$$\underline{f'(z) = \sum_{n=0}^{\infty} \frac{(-z^2)^n}{n!}}$$ (Disregard! True, but irrelevant)

$$\cancel{\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{(-z^2)^{n-1}}{n!} \cdot n \cdot -2z}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{z^{2n+1}}{2n+1}$$ (Disregard! True but irrelevant)

$$f(z) = \int_0^z \left(1 + \left(\frac{-2}{2!} z^2\right) + \frac{12}{4!} z^4 \ldots\right) dz$$

$$= \underline{\underline{z - \frac{2}{2!} z^3 + \frac{12}{4!} z^5 \ldots}}$$

$$\underline{\underline{R = \infty}}$$

15.4.19

$$f(z) = \frac{1}{1+z}, \quad z_0 = -i$$

$$\Rightarrow f(z) = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n$$ (Disregard)

$$f^{(n)}(z) = \frac{n!}{(1+z)^{n+1}} \cdot (-1)^n$$

$$\Rightarrow f(z) = \sum (z-z_0)^n \cdot \frac{(-1)^n}{(1+z_0)^{n+1}} = \underline{\underline{\sum \frac{(z+i)^n}{(1+i)^{n+1}} (-1)^n}}$$

15.4.19)

$$z_0 = -i$$

$$f(z) = \frac{1}{1+z}$$

$$f'(z) = \frac{-1}{(1+z)^2}$$

$$f''(z) = \frac{2}{(1+z)^3}$$

$$f'''(z) = \frac{-6}{(1+z)^4}$$

$$\Rightarrow f(z) = (z+i)\cdot\frac{1}{1!}\cdot\frac{-1}{(1-i)^2} + (z+i)^2\cdot\frac{1}{2!}\cdot\frac{2}{(1-i)^3}$$

$$+ (z+i)^3\,\frac{1}{3!}\cdot\frac{-6}{(1-i)^4}\ \ldots$$

$$f^n(z) = \frac{n!\,(-1)^n}{(1+z)^{n+1}}$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} (z+i)^n\cdot\frac{(-1)^n}{(1-i)^{n+1}}$$

$$R = \lim_{n\to\infty}\left|\frac{\frac{(-1)^n}{(1-i)^{n+1}}}{\frac{(-1)^n(-1)}{(1-i)^{n+2}}}\right| = \lim_{n\to\infty}\left|\frac{(1-i)}{-1}\right| = \sqrt{2}$$

15.4.24)

$$f(z) = e^{z(z-2)} \quad , z_0 = 1$$

$$f'(z) = (2z-2)e^{z(z-2)}$$

$$f''(z) = 2e^{z(z-2)} + (2z-2)^2 e^{z(z-2)}$$

$$f'''(z) = 2(2z-2)e^{z(z-2)} + 4(2z-2)e^{z(z-2)} + (2z-2)^3 e^{z(z-2)}$$

$$\Rightarrow \cancel{f^n(z) = \sum}$$

$$f^{(4)}(z) = 12e^{z(z-2)} + 6(2z-2)e^{z(z-2)}$$

$$\cancel{\Rightarrow f(z) = \sum (z-1)^n\cdot\frac{1}{n!}\cdot\frac{n!}{(\frac{n}{2})!}\cdot\frac{1}{e} = \sum\frac{(z-1)^n}{e(\frac{n}{2})!}}$$

$$\Rightarrow f(z) = \sum (z-1)^n \cdot \frac{1}{n!} \cdot \frac{n!}{(\frac{n}{2})!} \cdot \frac{1}{e} \cdot \frac{(1+(-1)^n)}{2}$$

$$= \sum_{n=0}^{\infty} (z-1)^n \frac{(1+(-1)^n)}{(\frac{n}{2})! 2e}$$

$$= \frac{1}{e} + (z-1)^2 \frac{1}{e} + (z-1)^4 \frac{1}{2!e} + (z-1)^6 \frac{1}{3!e} \dots$$

$$R = \lim_{n\to\infty} \left| \frac{\frac{(1+(-1)^n)}{(\frac{n}{2})! 2e}}{\frac{(1+(-1)^n(-1))}{(\frac{n+1}{2})! 2e}} \right| = \lim_{n\to\infty} \left| \frac{(1+(-1)^n)\left(\frac{n+1}{2}\right)!}{(\frac{n}{2})!\,(1+(-1)^n(-1))} \right|$$

$\Rightarrow R = \infty$ (or the limit doesn't exist)
not shure!

5.R.14)

$$\sum_{n=1}^{\infty} \frac{n^5}{n!}(z-3i)^{2n} \Rightarrow R^2 = \lim_{n\to\infty} \left| \frac{\frac{n^5}{n!}}{\frac{(n+1)^5}{(n+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^5 (n+1)!}{(n+1)^5 n!} \right| = \left| \lim_{n\to\infty} \frac{n^5}{(n+1)^5} \cdot \lim_{n\to\infty} \frac{(n+1)!}{n!} \right|$$

$$= \cancel{\infty}\, 1 \cdot \infty \Rightarrow R = \infty$$

5.R.18)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi z)^{2n+1} = f(z)$$

$$f'(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n+1) \cdot \pi}{(2n+1)!} (\pi z)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi}{(2n)!} (\pi z)^{2n}$$

~~$\Rightarrow R^2 = \lim$~~ Note! $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{(2n)}$

$$\Rightarrow f'(z) = \pi \cos(\pi z)$$

$$\Rightarrow f(z) = \sin(\pi z) \quad \Rightarrow R = \cancel{1}\ \infty \text{ (ratio test)}$$

5.R.26)

$\cancel{th}\ f(z) = z^5 \quad , z_0 = i$

$f(z) = \sum \cancel{z}^n a_n \quad , a_n = \frac{1}{n!} \cdot f^{(n)}(z_0)$

$(z-z_0)^n$

$f'(z) = 5z^4 , f''(z) = 20z^3 , f'''(z) = 60z^2 , f^{(4)}(z) = 120z \quad f^{(5)}(z) = 120 \ . , f^{(6)}(z) = 0$

$\Rightarrow f(z) = i^5 + (z-i) \cdot 5i^4 + (z-i)^2 \cdot 10i^3 + (z-i)^3 10 i^2 + (z-i)^4 5i + (z-i)^5$

$\cancel{f(z) = \sum_{n=0}^{5} (z-i)^n \cdot \frac{(5-n+1)! \, i^{(5-n)}}{n!} \ (n \leq 5)}$

$\underline{R = \infty}$ (no singularities, ~~function~~ sum must converge everywhere)

$\cancel{R = \lim_{n \to 5} \left| (5-n+1)! \right.}$

Note! For all z, after term $n=5$, all terms are zero, meaning the sum is finite, erg $\underline{R = \infty}$

5.R.29) $f(z) = Ln(z) \quad , z_0 = 3$

$\cancel{f'(z) = \frac{1}{z}}$

$Ln(z) = \ln|z| + i Arg(z)$

$\cancel{f''(z)} =$

$\Rightarrow f'(z) = \frac{1}{z}$

$f''(z) = \frac{-1}{z^2}$

$f'''(z) = \frac{2}{z^3}$

$f''''(z) = \frac{-6}{z^4} \Rightarrow$

$$\cancel{\Rightarrow f(z) = \ln(3) + (z-3)\cdot\frac{1}{3}\,1 - (z-3)^2\cdot\frac{2\cdot 1}{2\cdot 3^2} + (z-3)^3\cdot\frac{2\cdot 2}{3!\cdot 3^3}}$$
$$\cancel{-(z-3)^4\cdot\frac{6}{\ }}$$

$$f(z) = \ln(3)\left(1 + (z-3)\frac{1}{3} - (z-3)^2\frac{1}{2\cdot 3^2} + (z-3)^3\cdot\frac{2}{3!\cdot 3^3}\right.$$
$$\left. -(z-3)^4\cdot\frac{6}{4!\,3^4}\ldots\right)$$

~~No singularities $\Rightarrow R = \infty$~~

$$f^{n}(z) = (-1)^{n+1}\cdot\frac{(n-1)!}{z^n}$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}(z-3)^n\cdot\frac{(-1)^{n+1}}{n!}\cdot\frac{(n-1)!}{3^n}$$
$$= \sum_{n=0}^{\infty}\left(\frac{z-3}{3}\right)^n\cdot\frac{(-1)^{n+1}}{n!}\,\frac{(n-1)!}{1}$$

$$R = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n-1)!}{n!\,3^n}}{\frac{(-1)^{n+2}\,n!}{(n+1)!\,3^n\cdot 3}}\right| = \lim_{n\to\infty}\left|\frac{(n-1)!\,(n+1)!\cdot 3(-1)}{n!\,n!}\right|$$

$$= \underline{\underline{3}}$$