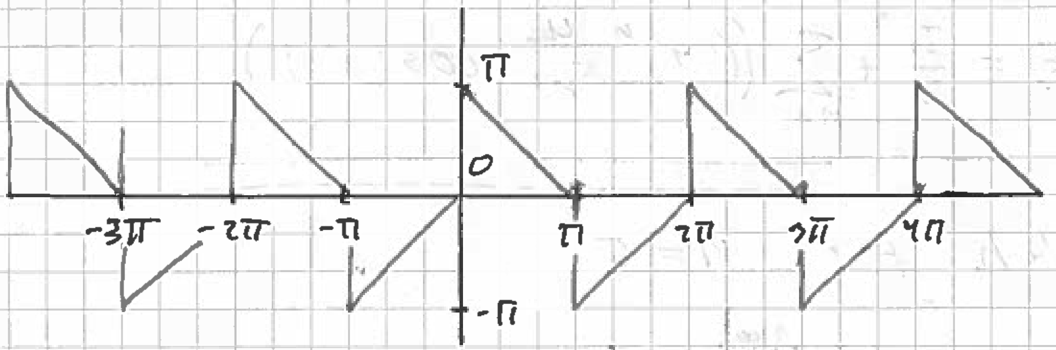
TMA4120

Øving 3

Vsevolod Karpov - vsevolok

11.1.9:

$$f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$$



11.1.14:

$$f(x) = x^2 \quad (-\pi < x < \pi)$$

$$p = 2\pi$$

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cdot \cos(nx) + b_n \cdot \sin(nx) \right)$$

where: $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{6\pi} \left[x^3 \right]_{-\pi}^{\pi} = \frac{2\pi^3}{6\pi} = \underline{\frac{\pi^2}{3}}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cdot \cos(nx) \, dx$$

$$= \frac{2(\pi^2 n^2 - 2)\sin(\pi n) + 4\pi n \cos(\pi n)}{n^3}$$

Since n is always an integer, ~~the~~ $\sin(\pi n) = 0$, while $\cos(\pi n) = 1$ if n is even and $\cos(\pi n) = -1$ if n is odd.

$\Rightarrow a\cancel{lt}_n = \frac{4}{n^2}$ if n is even

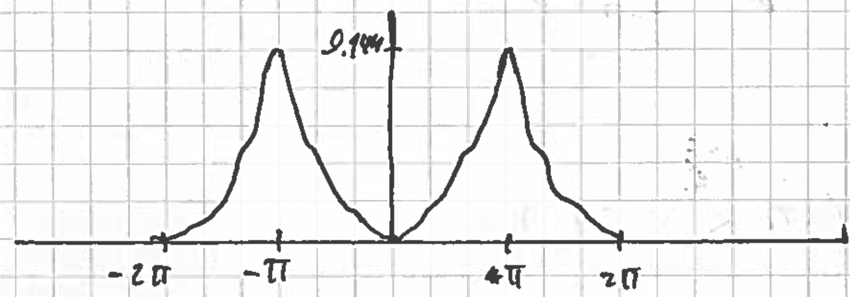
$a\cancel{lan} = \frac{4}{n^2}$ if n is odd.

$\Rightarrow a\cancel{lt}_n(n) = (-1)^n \cdot \frac{4}{n^2}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot \sin(nx)\,dx = \underline{0} \quad (\text{sine is odd})$$

$$\Rightarrow S_f(x) = \frac{\pi^2}{3} + \sum_{n=1} \left((-1)^n \cdot \frac{4}{n^2} \cdot \cos(nx)\right)$$

Sketch for $n = 5$



11.1.19:

$$f(x) = x \cdot u(x)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x \cdot u(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{4\pi}\left[x^2\right]_0^{\pi} = \underline{\frac{\pi}{4}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot u(x) \cdot \cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x \cdot \cos(nx)\,dx$$

$$= \frac{\pi n \sin(\pi n) + \cos(\pi n) - 1}{\pi n^2} = \frac{\cos(\pi n) - 1}{\pi n^2}$$

(since $n \in \mathbb{Z}$)

$$a_n = \frac{(-1)^n - 1}{\pi n^2}$$

Same logic as 11.1.14

~~$b_n = 0$ (sine is odd)~~

$$b_n = \frac{1}{\pi}\int_0^{\pi} x \cdot \sin(nx)\,dx = \frac{\sin(\pi n) - \pi n \cdot \cos(\pi n)}{\pi n^2}$$

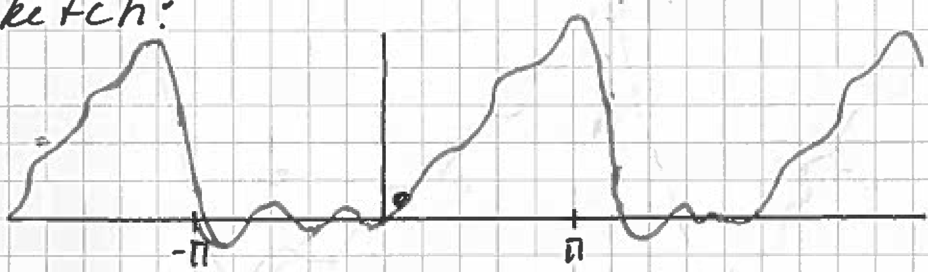
$$= \frac{-\pi n \cdot \cos(\pi n)}{\pi n^2} = \frac{-\cos(\pi n)}{n} = \underline{\frac{-(-1)^n}{n}}$$

$$\Rightarrow S_f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\cos(nx) + \frac{-(-1)^n}{n}\sin(nx)\right)$$

$n = 5 \Rightarrow$

$$S_f(x)_5 = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x)\right)$$
$$+ \sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x)$$
$$+ \frac{1}{5}\sin(5x)$$

Sketch:



11.1.21

Note that $f(x)$ is odd on the interval. This will lead to ~~a_0, a_n and b_n being calculated to be zero~~. We instead write our fuction as $g(x) = -x$ and shift the answer by π:

$$\underline{f(x) = g(x - \pi)}$$

(~~due~~ to tedious calculation)

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$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x = 0$$

$$a_n = -\frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \cos(nx)\,dx = 0 \quad (x \text{ is odd})$$

$$b_n = -\frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin(nx)\,dx = -\frac{1}{\pi}\left(\frac{2\sin(\pi n) - 2\pi n\cos(\pi n)}{n^2}\right)$$

$$n \in \mathbb{Z} \Rightarrow b_n = \frac{2\pi n\cos(\pi n)}{\pi n^2} = \frac{2\cdot\cos(\pi n)}{n}$$

$$= \frac{(-1)^n}{n}\cdot 2$$

$$\Rightarrow S_f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot \sin(n(x-\pi)) \cdot 2$$

Sketch for 5th grade.

11.2.11:

$$f(x) = x^2 \quad (-1 < x < 1), \; p = 2$$

This function is even

$$a_0 = \frac{1}{p}\int_{-1}^{1} x^2\,dx = \frac{1}{6}\left[x^3\right]_{-1}^{1} = \frac{1}{3}$$

$$a_n = \frac{2}{P}\int_{-1}^{1} x^2 \cdot \cos(n\pi x)\,dx = \frac{4\cos(\pi n)}{\pi^2 n^2}$$

$$= \frac{4\cdot(-1)^n}{\pi^2 \cdot n^2}$$

$b_n = 0$ (sine is odd on the interval)

$$\Rightarrow S_f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{4\cdot(-1)^n}{\pi^2\cdot n^2}\cdot\cos(n\pi x)\right)$$

11.2.17:

$$f(x) = \begin{cases} x+1 & : -1 < x < 0 \\ -x+1 & : 0 < x < 1 \end{cases}, \quad p = 2$$

This function is <u>even</u>

We divide the function into two parts ~~to av~~ in order to avoid calculation involving unit step function.

$f_1(x) = x+1 \; : \; -1 < x < 0$
$f_2(x) = -x+1 \; : \; 0 < x < 1$

$S_f(x) = S_{f1}(x) + S_{f2}(x)$

$f_1(x)$:

$$a_0 = \frac{1}{2}\int_{-1}^{0}(x+1)\,dx = \frac{1}{4}$$

$$a_n = \int_{-1}^{0}(x+1)\cdot\cos(n\pi x)\,dx = \frac{1-\cos(\pi n)}{\pi^2\cdot n^2}$$

$$= \frac{1-(-1)^n}{\pi^2\cdot n^2}$$

b_n:

we know that $\int_{-1}^{1} g(x) \cdot \sin(n\pi x)\, dx = 0$, provided that $g(x)$ is <u>even</u> on the interval.

Therefore, we do not calculate b_n for $f_1(x)$ and $f_2(x)$ since their sum will in the end equal zero.

$f_2(x)$:

$$a_0 = \frac{1}{2}\int_0^1 (-x+1)\, dx = \underline{\frac{1}{4}}$$

$$a_n = \int_0^1 (-x+1) \cdot \cos(\pi n x)\, dx = \frac{1-\cos(\pi n)}{\pi^2 \cdot n^2}$$

$$= \frac{1-(-1)^n}{\pi^2 \cdot n^2}$$

$$\Rightarrow S_f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} 2 \cdot \frac{1-(-1)^n}{n^2 \cdot \pi^2} \cdot \cos(\pi n x)$$

11.2.25

$$f(x) = -x + \pi, \quad 0 < x < \pi$$

$p = 2\pi$, $L = \frac{\pi}{\cancel{2}}$

Cosine: $f(x) \sim S_f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(nx)$

Sine: $S_f(x) = a_0 + \sum_{n=1}^{\infty} b_n \cdot \sin(nx)$

Cosine if f is even

Sine if f is odd

Even: $f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ -x + \pi, & 0 < x < \pi \end{cases}$

Odd: $f(x) = \begin{cases} -x - \pi, & -\pi < x < 0 \\ -x + \pi, & 0 < x < \pi \end{cases}$

$p = 2\pi$, $L = \pi$

$\Rightarrow a_0 = \frac{1}{2\pi} \int_{-L}^{L} f(x)\, dx$

For Cosine:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} (x + \pi)\, dx + \frac{1}{2\pi} \int_{0}^{\pi} (-x + \pi)\, dx$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \underline{\frac{\pi}{2}}$$

For sine:

$a_0 = \underline{0}$

For Cosine:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{0} (x + \pi) \cdot \cos(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} (-x + \pi) \cdot \cos(nx)\, dx$$

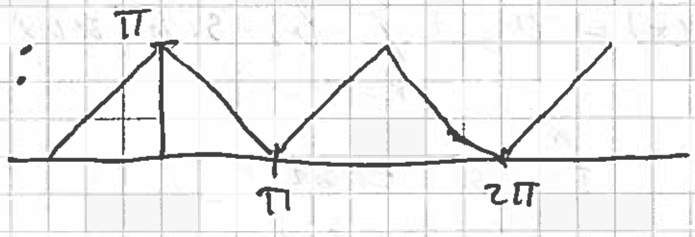
$$= \frac{1 - \cos(\pi n)}{\pi n^2} \cdot 2 = \underline{\frac{2(1 - (-1)^n)}{\pi n^2}}$$

For sine:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cdot\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(-x+\pi)\cdot\sin(nx)\,dx$$

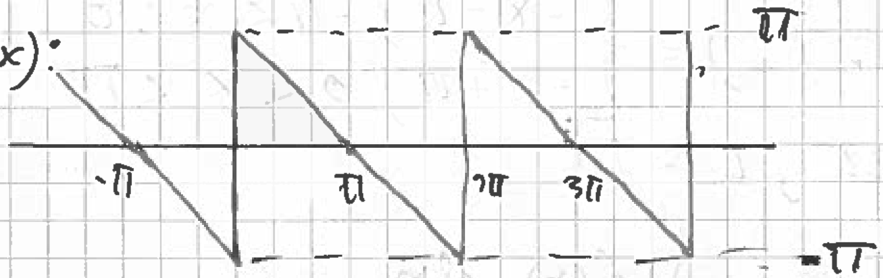
$$= \frac{\pi n - \sin(\pi n)}{\pi n^2}\cdot 2 = \frac{2}{n}$$

$\Rightarrow$ a) $f(x)$:



$$S_f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x)\ldots\right)$$

b) $f(x)$:



$$S_f(x) = 2\left(\sin(x) + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \frac{1}{4}\sin(4x)\ldots\right)$$

13.1.9)

$$z_1 = -2+5i,\quad z_2 = 3-i$$

$$Re(z_1^2) = Re(4-10i-10i-25) = Re(-21-20i)$$

$$= \underline{\underline{-21}}$$

$$(Re(z_1))^2 = (-2)^2 = \underline{\underline{4}}$$

13.1.12 $\frac{z_1}{z_2} = \frac{(-2+5i)}{(3-i)} = \frac{(-2+5i)(3+i)}{10}$

$= \frac{-6+15i-2i-5}{10} = \underline{\underline{-\frac{11}{10} + \frac{13}{10}i}} = z_a$

$\frac{z_1}{z_2} = z_a \Rightarrow \frac{z_2}{z_1} = \frac{1}{z_a} = \frac{\left(-\frac{11}{10} - \frac{13}{10}i\right)}{\frac{11^2+13^2}{10^2}}$

$\underline{\underline{= -\frac{11}{29} - \frac{13}{29}i}}$

13.1.18 $z = x + iy$

$Re\left((1+i)^{16} \cdot z^2\right)$

$1+i = \sqrt{2} \cdot e^{\frac{\pi}{4}i} \Rightarrow (1+i)^{16} = (\sqrt{2})^{16} \cdot e^{\frac{16\pi}{4}i}$

$= 256 \cdot e^{4\pi i} = \underline{256}$

$z^2 = x^2 + 2xyi - y^2$

$\Rightarrow Re\left((1+i)^{16} \cdot z^2\right) = Re\left(256(x^2 + 2xiy - y^2)\right)$

$\underline{\underline{= 256(x^2 - y^2)}}$

13.1.19

$Re\left(z/\bar{z}\right)$ and $Im\left(z/\bar{z}\right)$

$\frac{z}{\bar{z}} = \frac{x+iy}{x-iy} = \frac{(x+iy)^2}{x^2+y^2} = \frac{x^2+2xiy-y^2}{x^2+y^2}$

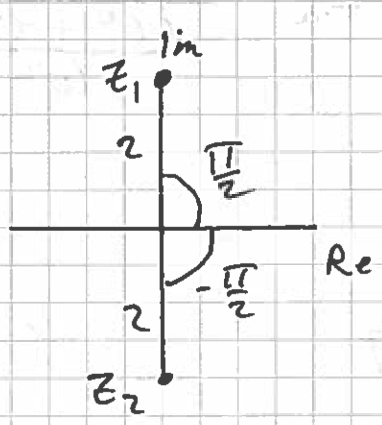
$\underline{\underline{\Rightarrow Re = \frac{x^2-y^2}{x^2+y^2}, \quad Im = \frac{2xy}{x^2+y^2}}}$

13.2.3

$$z_1 = 2i \;,\; z_2 = -2i$$

$$\theta_{z_1} = \frac{\pi}{2} \;,\; \theta_{z_2} = -\frac{\pi}{2} \;,\; r_{z_1} = r_{z_2} = 2$$

$$\Rightarrow z_1 = 2e^{\frac{\pi}{2}i} \;,\; z_2 = 2e^{-\frac{\pi}{2}i} = 2e^{\frac{3\pi}{2}i}$$



13.2.8

$$z = \frac{7+4i}{3-2i} = \frac{(7+4i)(3+2i)}{13} = 1+2i$$

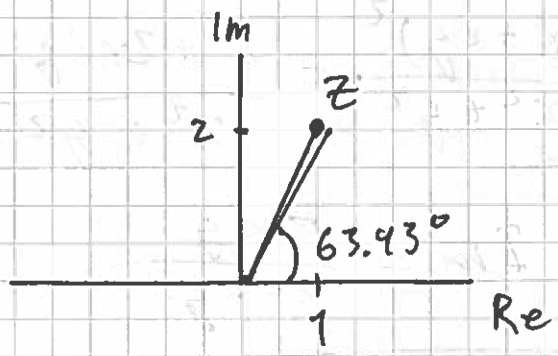
$$= r(\cos(\theta) + i\sin(\theta))$$

$$\Rightarrow r\cdot\cos(\theta) = 1 \;,\; r\cdot\sin(\theta) = 2$$

$$\Rightarrow r = \frac{1}{\cos(\theta)} \;,\; \tan(\theta) = 2$$

$$\Rightarrow \theta = \frac{63.43}{180}\cdot\pi \;,\; r = 2.235$$

$$\Rightarrow z = 2.235 \cdot e^{\frac{63.43}{180}\cdot\pi\cdot i}$$



13.2.21

$$z = \sqrt[3]{1-i}$$

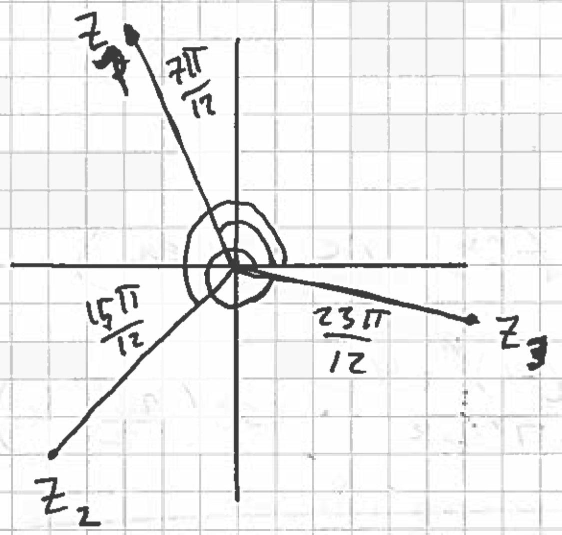
$$1-i = \sqrt{2}\, e^{-\frac{\pi}{4} i} = \sqrt{2}\, e^{\frac{7\pi}{4} i} = g$$

$$\sqrt[3]{g} = \sqrt[6]{2} \cdot e^{\frac{7\pi}{12} i + k \cdot \frac{2\pi}{3} \cdot i} \quad , \quad k \in [0, 2]$$

$$\Rightarrow \quad z_1 = \sqrt[6]{2}\, e^{\frac{7\pi}{12} i}$$

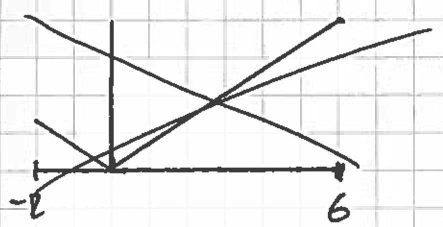
$$z_2 = \sqrt[6]{2}\, e^{\frac{15\pi}{12} i}$$

$$z_3 = \sqrt[6]{2}\, e^{\frac{23\pi}{12} i}$$



Supplementary D.

i) $f(x) = f(-x)$ (even)

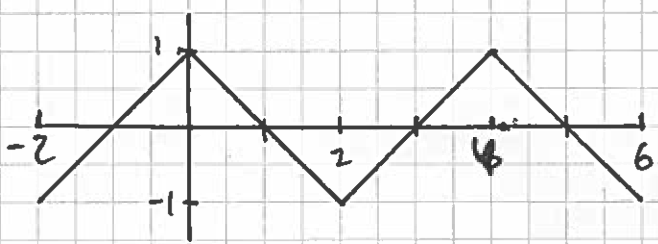


ii) $f(x) = f(x+4)$ $\Rightarrow p = 4$

$\Rightarrow f(-x) = f(-x-4)$

iii) $f(x) = 1 - x$, $0 < x < 2$

$\Rightarrow f(x)$:



$$\Rightarrow a_0 = \frac{1}{4}\int_{-2}^{0}(x+1)\,dx + \frac{1}{4}\int_{0}^{2}(-x+1)\,dx$$

$$= \frac{1}{4}\left[\frac{1}{2}x^2+x\right]_{-2}^{0} + \frac{1}{4}\left[-\frac{1}{2}x^2+x\right]_{0}^{2}$$

$$= -\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2} = 0$$

$$a_n = \frac{1}{2}\int_{-2}^{0}(x+1)\cos\left(n\frac{\pi}{2}x\right)dx + \frac{1}{2}\int_{0}^{2}(-x+1)\cos\left(n\frac{\pi}{2}x\right)dx$$

$$= \frac{-2\cos(\pi n)+2}{\pi^2 n^2} + \frac{-2\cos(\pi n)+2}{\pi^2 n^2}$$

$$= \frac{-4(-1)^n+4}{\pi^2 n^2}$$

$b_n = 0$ (since $f(x)$ is even)

$$\Rightarrow S_f(x) = \sum_{n=1}^{\infty} \frac{-4(-1)^n+4}{\pi^2 n^2}\cdot\cos\left(\frac{n\pi}{2}x\right)$$