TMA4120 – Øving 12

Vsevolod Karpov

16.1.3:

$$f(z) = z^3 \cosh\left(\frac{1}{z}\right)$$

$$= \sum_{n=0}^{\infty} a_n (z)^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*)^{n+1}} dz^*$$

C: unit circle

$$\Rightarrow a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta(n+1)}} \cdot i e^{i\theta} d\theta$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{4i\theta} \cdot i \cdot \cosh(e^{-i\theta})}{e^{i\theta(n+1)}} d\theta$$

Ugly integral! Instead: $f(z) = z^3 \cdot g(z)$

Expand $g(z) = \cosh\left(\frac{1}{z}\right)$

$$\Rightarrow a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\cosh(e^{-i\theta})}{e^{i\theta(n+1)}} \cdot i e^{i\theta} d\theta$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \cosh(e^{-i\theta}) \cdot e^{-i\theta \cdot n} \cdot i \, d\theta$$

Also ugly.

However: $\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \Rightarrow \cosh\left(\frac{1}{z}\right) =$

$$\sum_{n=0}^{\infty} \frac{z^{-2n}}{(2n)!} \Rightarrow \cosh\left(\frac{1}{z}\right) = \sum_{n=1}^{\infty} \frac{z^{-2(n-1)}}{(2(n-1))!}$$

$$\Rightarrow z^3 \cosh\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{z^{3-2n}}{(2n!)} = z^3 + \frac{z}{2} + \frac{1}{24z} + \frac{1}{720z^3} \dots$$

~~16.1.7~~ Converges for $\underline{\underline{0 < |z| < \infty}}$ 2

Because $\frac{1}{(2n)!}$ as $n \to \infty$ decreases the terms more than z^{-2n} for any z.

16.1.7.

$$f(z) = \frac{\sin(z)}{(z - \frac{1}{4}\pi)^3}, \quad z_0 = \frac{1}{4}\pi$$

$$\sin(z - z_0) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{*2n+1}}{(2n+1)!}, \quad z^* = z - z_0$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(z - \frac{1}{4}\pi)^{2n+1}}{(z - \frac{1}{4}\pi)^3 \cdot (2n+1)!}$$

$$= \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{(z - \frac{1}{4}\pi)^{2n-2}}{(2n+1)!}}}$$

Converges when $\underline{\underline{|z - \frac{1}{4}\pi| > 0}}$

16.1.13:

$$f(z) = \frac{z^8}{1 - z^4}, \quad z_0 = 0$$

Taylor: $f(z^*) = \frac{z^{*2}}{1 - z^*}, \quad z^* = z^4$

$$\Rightarrow f(z^*) = \sum_{n=0}^{\infty} z^{*n} \cdot z^{*2}, \quad |z^*| < 1$$

$$f(z^*) = \frac{-z^*}{1 - \frac{1}{z^*}} \Rightarrow f(z^*) = \sum_{n=0}^{\infty} \left(\frac{1}{z^*}\right)^n \cdot -z^*, \quad |z^*| > 1$$

$\Rightarrow |z| < 1$: ~~$z^8 + z^9 + z^{10}$~~

$$\underline{\underline{z^8 + z^{12} + z^{16} + z^{20} \ldots}}$$

$|z| > 1: \; -z^4 - 1 - z^{-4} - z^{-8} \ldots$

16.2.3

$\tan^2(2z) = f(z)$

~~Fi~~ Find zeros and poles ~~of~~ and orders.

$$f(z) = \left(\frac{\sin(2z)}{\cos(2z)}\right)^2$$

Zeros: $\sin(2z) = 0 \Rightarrow z = \frac{\pi}{2} + k \cdot \frac{\pi}{2} = z_0$

$$f'(z) = 4 \tan(2z) \cdot \left(\frac{\cos(2z)^2 + \sin(2z)^2}{\cos^2(2z)}\right)$$

$$= 4 \frac{\tan(2z)}{\cos^2(2z)}$$

$$f''(z) = \frac{8 \cdot (1 + \tan(2z) \cdot 4 \cdot \sin(2z))}{\cos^4(2z)}$$

$f(z_0) = 0$

$f'(z_0) = 0 \Rightarrow$ zero of first order

$f''(z_0) \neq 0$ at $\frac{\pi}{2} + k \cdot \frac{\pi}{2}$

~~However, if $x = 0 \Rightarrow x^2 = 0$~~

~~and if $\lim_{x \to a} x = \infty \Rightarrow \lim_{x \to a} x^2 = \infty$~~

~~Therefore, poles and zeros for $\tan(2z)$ are also poles and zeros of $\tan^2(2z)$~~

~~$\tan(2z) = \tan(z^*) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!}$~~

~~Pole when $\cos(2z) = 0 \Rightarrow$ Poles at zeros of $\cos(2z)$~~

~~$\cos(2z) = 0 \Rightarrow z = \frac{\pi}{4} + k \cdot \frac{\pi}{2} = z_0$~~

~~$\frac{d\cos(2z)}{dz} = -2\sin(2z), \quad -2\sin(2z_0) \neq 0$~~

~~$\Rightarrow$ zero of ~~tan~~ 1st order at $\frac{\pi}{2} + k \cdot \frac{\pi}{2}$~~

~~simple pole at $\frac{\pi}{4} + k \cdot \frac{\pi}{2}$~~

~~16.2.5~~

Theorem: $\tan^2(2z)$ has poles where $\frac{1}{\tan^2(2z)}$ has zeros.

$$\frac{1}{\tan^2(2z)} = \frac{\cos^2(2z)}{\sin^2(2z)} = g(z), \text{ zero at } z_0 = \frac{\pi}{4} + k \cdot \frac{\pi}{2}$$

$$g'(z) = \frac{-4\cos(2z)}{\sin^2(2z)}$$

$$g''(z) = \frac{8\sin^4(2z) - 8\cos^2(2z) \cdot 3 \cdot \sin(2z)}{\sin^6(2z)}$$

$g'(z_0) = 0$

$g''(z_0) \neq 0$

$\Rightarrow$ zero at $\frac{\pi}{2} + k \cdot \frac{\pi}{2}$

pole at $\frac{\pi}{4} + k \cdot \frac{\pi}{2}$

both of order 2.

16.2.5

$f(z) = g(z)(z-z_0)^n$

$g(z) \neq 0$

$\Rightarrow (f(z))^2 = g(z)^2 (z-z_0)^{2n}$

order

16.2.6 a)

$f^{n-1}(z_0) = 0 \quad , \; f^n(z_0) \neq 0$

$f(z)$ evaluated at z_0 ~~has~~ is zero for all derivatives up to n.

$g(z) = f^1(z)$

$\Rightarrow \quad g^m(z_0) = f^{1+m}(z_0) \neq 0 \; , \; g^{m-1}(z_0) = f^m(z_0) = 0$

$1 + m = n \Rightarrow \underline{\underline{m = n-1}}$

16.2.7

$$f(z) = \frac{1}{(z+2i)^2} - \frac{z}{z-i} + \frac{z+1}{(z-i)^2}$$

Theorem 4: $f(z)$ has ~~poles~~ zeros where $\frac{1}{f(z)}$ has poles and v.v.

Also goes for $\frac{h(z)}{f(z)}$ provided that $h(z_0) \neq 0$

$$f(z) = \frac{1}{(z+2i)^2} + \frac{-z^2 + 2i + z + 1}{(z-i)^2}$$

$$= \frac{1}{l(z)} + \frac{h(z)}{g(z)}$$

First, $\frac{1}{l(z)}$

$l(z) = (z+2i)^2$, $z_0 = -2i \Rightarrow l(z_0) = 0$

$l'(z) = 2(z+2i)$ $\Rightarrow$ $l'(z_0) = 0$

$l''(z) = 2$ $l''(z_0) \neq 0$

$\Rightarrow$ $-2i$ is a pole of degree 2

Now: $\frac{h(z)}{g(z)}$. For now, assume that $h(z) \neq 0$ ~~has not same zeros as $g(z)$.~~ for z_0 being zero for $g(z)$

$g(z) = (z-i)^2$, $z_0 = i$

$g(z_0) = 0$

$g'(z_0) = 0$ $\Rightarrow$ i is a pole of degree 2

$g''(z_0) \neq 0$

$h(z) = -z^2 + zi + z + 1$

$h(i) = 1 - 1 + i + 1 \neq 0$

16.3.1)

$f(z) = \frac{\sin(2z)}{z^6}$, use taylor for $\sin(2z)$

$\Rightarrow f(z) = \frac{1}{z^6} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n+1}}{(2n+1)!}$

$= \frac{1}{z^6} \cdot \left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \frac{(2z)^7}{7!} \dots \right)$

$= \frac{2}{z^5} - \frac{2^3}{3!\,z^3} + \frac{2^5}{5!\,z} - \frac{2^7 z}{7!} \dots$

The principal part, the terms with negative powers: $\frac{2}{z^5} - \frac{2^3}{3!\,z^3} + \frac{2^5}{5!\,z}$

$= \frac{b_1}{z} + \frac{b_3}{z^3} + \frac{b_5}{z^5}$ $\Rightarrow$ pole at $z=0$ of order 5

Residue: $\operatorname*{Res}_{z=z_0} f(z) = b_1 = \frac{z^5}{5!}$

Singularity at $z_0 = 0$ (order 5)

16.3.6.

$$\oint_C \frac{z-23}{z^2-4z-5}\,dz,\quad C: |z-(2+i)| = 3.2$$

$z-23$: no poles are zeros in C.

$z_0^2 - 4z_0 - 5 = 0 \Rightarrow z_0 = 2 \pm 3 = 5 \vee -1$

Both points are inside C.

So, $f(z) = \frac{z-23}{z^2-4z-5}$ has singularities at

$z_0 = 5$ and $z_0 = -1$

Expanding to series:

$$\cancel{f(z) = \frac{(z-23)}{(z+1)} \cdot \sum_{n=0}^{\infty} a_n (z-z_0)^n}$$

$$\cancel{= \frac{(z-23)}{z+1} \cdot -\frac{1}{5} \cdot \frac{1}{1-\frac{z}{5}} = \frac{(z-23)}{(z+1)} \cdot -\frac{1}{5} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{5}\right)^n}$$

$$f(z) = \frac{z-23}{(z+1)(z-5)} = \cancel{\frac{23}{6}} - \frac{18}{6} \cdot \frac{1}{z-5} + \frac{24}{6} \cdot \frac{1}{z+1}$$

$= \frac{-3}{z-5} + \frac{4}{z+1} = \frac{3}{5} \sum_{n=0}^{\infty} \left(\frac{z}{5}\right)^n + 4 \sum_{n=0}^{\infty} (-z)^n$ + principal part.

Principal: $\frac{-3}{z-5} = -\frac{3}{z} \cdot \frac{1}{1-\frac{5}{z}} = -\frac{3}{z} \sum_{n=0}^{\infty} \left(\frac{5}{z}\right)^n$ (I)

$\frac{4}{z+1} = \frac{4}{z} \cdot \frac{1}{1+\frac{1}{z}} = \frac{4}{z} \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n$ (II)

(I) $-\frac{3}{z}\left(1+\frac{5}{z}+\frac{5^2}{z^2}+\frac{5^3}{z^3}\dots\right)=g(z)$

$=-\frac{3}{z}-\frac{15}{z^2}-\frac{3\cdot 5^2}{z^3}-\frac{3\cdot 5^3}{z^4}\dots$

$b_1=-3=\operatorname{Res}_{z=z_0} g(z)$

(II) $\frac{4}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}\dots\right)=l(z)$

$=\frac{4}{z}-\frac{4}{z^2}+\frac{4}{z^3}-\frac{4}{z^4}\dots$

$b_1=4=\operatorname{Res}_{z=z_0} l(z)$

$\Rightarrow \oint f(z)\,dz=2\pi i\cdot(4-3)=\underline{\underline{2\pi i}}$

16.3.9:

$$\oint_C \frac{e^{(-z^2)}}{\sin(4z)}\,dz \quad , C:|z|=1.5$$

e^{-z^2}: no zeros or singularities in C

$\sin(4z)=0 \Rightarrow z=\cancel{\frac{\pi}{8}\pm k\cdot\frac{\pi}{4}}\ \ 0\pm k\cdot\frac{\pi}{4}$

Within C: $-\frac{\pi}{4}$, 0 and $\frac{\pi}{4}$

$$\cancel{\frac{e^{-z^2}}{\sin(4z)}=e^{-z^2}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}}$$

Theorem 4: we need to find the order of poles, so we find the order of zeros in $\sin(4z)=g(z)$

$g(z_0)=0$

$g'(z)=4\cos(4z)\Rightarrow g'(z_0)\neq 0$

$\Rightarrow$ order is 1 (simple poles)

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z-z_0) f(z)$$

$$= \lim_{z \to z_0} \cdot \frac{e^{-z^2}(z-z_0)}{\sin(4z)} = \frac{e^{-z_0^2}}{4\cos(4z_0)}$$

$$z_0 = -\frac{\pi}{4} \Rightarrow Res_1 = e^{-\left(\frac{\pi}{4}\right)^2} / -4$$

$$z_0 = 0 \Rightarrow Res_2 = \frac{e^{-0^2}}{4} = \frac{1}{4}$$

$$z_0 = \frac{\pi}{4} \Rightarrow Res_3 = e^{-\left(\frac{\pi}{4}\right)^2} / -4$$

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i \left(Res_1 + Res_2 + Res_3 \right)$$

$$= \cancel{\frac{\pi i}{2}} \quad 2\pi i \left(\frac{1 - 2e^{-\left(\frac{\pi}{4}\right)^2}}{4} \right) = \pi i \left(\frac{1}{2} - e^{-\left(\frac{\pi}{4}\right)^2} \right)$$

16.4.3

$$\int_0^{2\pi} \frac{\sin^2(\theta)}{5 - 4\cos(\theta)} dz = l(z)$$

$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\Rightarrow \sin^2(\theta) = -\frac{1}{4}\left(z - \frac{1}{z}\right)^2$$

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\Rightarrow l(z) = \oint_C -\frac{1}{4} \cdot \frac{\left(z - \frac{1}{z}\right)^2}{5 - 2\left(z + \frac{1}{z}\right)} \frac{dz}{iz}, \quad C: |z| = 1$$

We find the zeros of $-4(5 - 2(z + \frac{1}{z})) = g(z)$ in order to find the poles.

$g(z) = 0$ if $z = 0.5$ or 2

2 is not inside C

Also, there is a pole at $z=0$

since $(z-\frac{1}{z})^2$ is udifined at $z=0$

pole $-\frac{1}{2}$ has order 1

pole 0 has order 2 (since squared)

But, since we have $\frac{1}{iz}$ in the integrand with pole $z=0$ of order 1, pole 0 for integrand has order $2-1=1$

so: poles: $-\frac{1}{2}$, order 1

0, order 1.

$$\operatorname{Res}_{z=z_0} = b_1 = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0} -\frac{1}{4}\frac{(z-\frac{1}{z})^2}{5-2(z+\frac{1}{z})}\cdot\frac{(z-z_0)}{iz}$$

for $-\frac{1}{2}$: $\operatorname{Res} = -\frac{1}{4}\frac{(-\frac{1}{2}+2)^2}{5i+2i} = -\frac{1}{4}\frac{(\frac{3}{2})^2}{7i}$

for 0: Res =

pole $\frac{1}{2}$ has order 1

pole 0 has order 2 (since squared)

$$\Rightarrow \operatorname{Res}_{z=\frac{1}{2}} = b_1 = \lim_{z\to\frac{1}{2}}(z-\tfrac{1}{2})f(\tfrac{1}{2}) = \cancel{-\frac{1}{4}\frac{(\frac{3}{2})^2}{7i} = \frac{7i}{4}\cdot\frac{3^2}{7^2 2^2}}$$

$$= -\frac{1}{4}\frac{(\frac{1}{2}-2)^2}{3i} = -\frac{1}{4}\frac{(\frac{3}{2})^2}{3i} = \frac{i}{4}\cdot\frac{3^2}{3\cdot 2^2} = \frac{3i}{16}$$

at $z=0$

$$\operatorname{Res} = \lim_{z\to 0}((z)^2 f(z))\frac{d}{dz} = -\frac{5i}{16}$$

$$\Rightarrow \oint_c f(z)\,dz = 2\pi i\left(-\frac{2i}{16}\right) = \frac{\pi}{4}$$

16.4.6

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = 2\pi i \cdot \sum \operatorname{Res} f(z) \cancel{+ \pi i \sum \operatorname{Res} f(z)}$$

$$f(z) = \frac{z^2+1}{z^4+1}, \quad \cancel{\text{poles at } z = \sqrt[4]{1}}$$
$$\cancel{\text{and } z = \sqrt[4]{1}}$$

~~Both of order 1~~

~~Because~~: $z^4+1=0$

$$z^4 = -1$$
$$e^{4i\theta} = -1$$

$\Rightarrow 4\theta = \pm\pi$

$\Rightarrow \theta = \frac{\pi}{4} + k\cdot\frac{\pi}{2} \Rightarrow \theta_1 = \frac{\pi}{4},\ \theta_2 = \frac{3\pi}{4},\ \theta_3 = \frac{5\pi}{4},\ \theta_4 = \frac{7\pi}{4}$

Formula is valid only for poles in upper half plane. Therefore, we only care about θ_1 and θ_2

$\Rightarrow z_{pole} = e^{i\frac{\pi}{4}} \wedge e^{i\frac{3\pi}{4}}$

Both of order 1.

$$\operatorname{Res}_1 = \lim_{z\to z_p} (z-z_p)\cdot\frac{z^2+1}{z^4+1} = \frac{z_p^2+1}{4z_p^3}$$

For $e^{i\frac{\pi}{4}}$: $\operatorname{Res} = \frac{i+1}{4\cdot\frac{1}{\sqrt{2}}(-1+i)} = \frac{(i+1)(-1-i)\cdot\sqrt{2}}{8}$

$= \frac{-i\cdot\sqrt{2}}{4}$

For $e^{i\frac{3\pi}{4}}$: $\operatorname{Res} = \frac{-i+1}{4\cdot\frac{1}{\sqrt{2}}(1+i)} = \frac{(-i+1)(1-i)\cdot\sqrt{2}}{8}$

$= -\frac{i\cdot\sqrt{2}}{4}$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{x^2+1}{x^4+1}\,dx = 2\pi i\left(-\frac{i\sqrt{2}}{4} - \frac{i\sqrt{2}}{4}\right) = \underline{\underline{\sqrt{2}\cdot\pi}}$$

Suplementary T

$$f(z) = \frac{1}{z} + \frac{e^z}{z-1} \quad , z_0 = 1$$

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad , a_n = \frac{1}{n!} f^n(z_0)$$

$$f(z) = g(z) + l(z)$$

$$g(z) = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad (\text{converges when } |z-1| < 1)$$

$$\cancel{g(z) = \frac{1}{z} = \frac{\frac{1}{z}}{\frac{1}{z}} = \sum_{n=0}^{\infty}}$$

$$l(z) = e^z \cdot \frac{1}{z-1} = \frac{1}{z-1} \cdot \cancel{\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}}$$

$l(z)$ has a simple pole at $z_0 = 1$ since $z_0 - 1 = 0$, we expect one negative power.

expand e^z

$$e^z = \sum (z - z_0)^n \cdot a_n \quad , a_n = \frac{1}{n!} \cdot f^n(z_0) = \frac{e}{n!}$$

$$\Rightarrow e^z = \sum \frac{(z - z_0)^n \cdot e}{n!}$$

$$\Rightarrow e^z \cdot \frac{1}{z-1} = \sum_{n=0}^{\infty} \frac{(z - z_0)^{n-1}}{n!} \cdot e \quad , z_0 = 1$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} (-1)^n (z-1)^n + \frac{(z-1)^{n-1}}{n!} \cdot e \qquad |z-1| < 1$$

$L(z)$ converges for all $z \neq 0$

$h(z)$ has another "version" for $|z-1| > 1$

$\frac{1}{z} = \frac{1}{1-x}$, $x = -z+1$

$= -\frac{1}{x}\left(\frac{1}{1-\frac{1}{x}}\right)$, since we are expanding about $z = 1 \Rightarrow x = 0$

$= -\frac{1}{x} \cdot \sum \left(\frac{1}{x}\right)^n = \sum_{n=0}^{\infty} -1 \cdot \left(\frac{1}{1-z}\right)^{n+1}$

So, for $|z-1| > 1$:

$$f(z) = \sum_{n=0}^{\infty} -1\left(\frac{1}{1-z}\right)^{n+1} + \frac{(z-1)^{n-1}}{n!} \cdot e$$

(4) a)

$z_0 = 0$, $f(z) = \frac{1}{z(8z^3-1)} = \frac{1}{z} \cdot \frac{1}{8z^3-1} = \frac{1}{z} \cdot \sum -1 \cdot (8z^3)^n$

$= \sum_{n=0}^{\infty} -1 \cdot 8^n \cdot z^{3n-1} = \sum_{n=0}^{\infty} -1 \cdot 8^n \cdot z^{3n-1}$ (I)

To find convergence: root test

$\lim_{n \to \infty} \sqrt[n]{|-1 \cdot 8^n z^{3n-1}|} = L < 1$

$\lim_{n \to \infty} 8 \cdot z^{\frac{3n-1}{n}} = 8z^3 < 1 \Rightarrow |z| < \sqrt[3]{\frac{1}{8}}$

We can also write

$$\frac{1}{8z^3-1} = \frac{1}{8z^3} \cdot \frac{1}{1-\frac{1}{8z^3}} = \frac{1}{8z^3} \cdot \sum_{n=0}^{\infty}\left(\frac{1}{8z^3}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{8z^3}\right)^{n+1}$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^{n+1} \cdot \left(\frac{1}{z^3}\right)^{n+1} \cdot \frac{1}{z}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^{n+1} \cdot \left(\frac{1}{z}\right)^{3n+4} \quad (II)$$

$$L = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{8}\right)^{n+1} \cdot \left(\frac{1}{z}\right)^{3n+4}} = \lim_{n\to\infty} \left(\frac{1}{8}\right)^{\frac{n+1}{n}} \cdot \left(\frac{1}{z}\right)^{\frac{3n+4}{n}}$$

$$= \left(\frac{1}{8}\right) \cdot \left(\frac{1}{z}\right)^3 < 1$$

$$\Rightarrow |z| > \frac{1}{\sqrt[3]{8}}$$

So: series I with radi $|z| < \frac{1}{\sqrt[3]{8}}$
and II with radi $|z| > \frac{1}{\sqrt[3]{8}}$

b) $\oint_C f(z)\,dz \quad , \quad C: |z| = 1$

$= 2\pi i \cdot \sum \operatorname{Res} f(z)$

$z(8z^3 - 1) = 0 \Rightarrow z = 0 \wedge z = \sqrt[3]{\frac{1}{8}}$ (poles)
all of order 1.

$z = \sqrt[3]{\frac{1}{8}} \Rightarrow r^3 e^{i\theta \cdot 3} = \frac{1}{8} \quad , \quad r = \sqrt[3]{\frac{1}{8}}, \ \theta = k \cdot \frac{2\pi}{3}, \ k \in [0, 2]$

So, four poles: $z = 0$, $z = \frac{1}{2}$, $z = \frac{1}{2} e^{i\frac{2\pi}{3}}$, $z = \frac{1}{2} e^{\frac{4\pi}{3} i}$
all of them simple.

For $z = 0$: $n = 0$ term of series (I) has negative power with $a_0 = -1$

$$\Rightarrow \operatorname*{Res}_{z_0 = 0} = -1$$

$$\operatorname*{Res}_{z=z_0} = \lim_{z \to z_0} \frac{z - z_0}{8z^4 - z} = \frac{1}{4 \cdot 8 z_0^3 - 1}$$

for $z_0 = \frac{1}{2}$, $\underline{\mathrm{Res} = \frac{1}{3}}$

for $z_0 = \frac{1}{2} e^{i\frac{2\pi}{3}}$, $\underline{\mathrm{Res} = \frac{1}{3}}$

for $z_0 = \frac{1}{2} e^{i\frac{4\pi}{3}}$, $\underline{\mathrm{Res} = \frac{1}{3}}$

So: $\underline{\underline{\oint f(z)\,dz = 2\pi i \left(\frac{1}{3} \cdot 3 - 1\right) = 0}}$

$\oint_C \mathrm{Re}(z)\,dz$, $C: |z| < 1$

write $z = e^{i\theta}$, $\frac{dz}{d\theta} = i e^{\theta i}$

$$\Rightarrow \oint_C \mathrm{Re}(z)\,dz = \int_0^{2\pi} \mathrm{Re}(e^{i\theta}) \cdot i e^{\theta i}\,d\theta$$

$$= i \int_0^{2\pi} \cos(\theta) \cdot e^{i\theta}\,d\theta = i \oint_C \cdot \frac{1}{2}\left(z + \frac{1}{z}\right) \cdot z \,\frac{dz}{iz}$$

$= \frac{1}{2} \oint_C z + \frac{1}{z}\,dz$, within C, there is a simple pole at $z_0 = 0$ ($\frac{1}{z_0}$ is undefined)

$$\mathrm{Res} = \lim_{z \to 0} \frac{1}{2}(z - 0) \cdot \left(z + \frac{1}{z}\right) = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} \oint_C z + \frac{1}{z}\,dz = 2\pi i \cdot \mathrm{Res} = \underline{\underline{\pi i}} = \oint_C \mathrm{Re}(z)\,dz$$