TMA4120 - Høst 2016
Øving 5
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11.7.2
$$\int_0^\infty \frac{\sin(\pi w)\sin(xw)}{1-w^2}\,dw = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } x > \pi \end{cases}$$

$$= f(x) = \int_0^\infty A(w)\cos(wx) + B(w)\sin(wx)\,dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)\,dv$$

$$= \frac{1}{\pi}\int_0^\pi \frac{\pi}{2}\sin(v)\cdot\cos(wv)\,dv = \frac{1}{2}\cdot\frac{\cos(\pi w)+1}{1-w^2}$$

$$B(w) = \frac{1}{\pi}\int_0^\pi \frac{\pi}{2}\sin(v)\cdot\sin(wv)\,dv = \frac{1}{2}\cdot\frac{\sin(\pi w)}{1-w^2}$$

$$\Rightarrow f(x) = \frac{1}{2}\int_0^\infty \frac{\cos(\pi w)\cdot\cos(wx) + \cos(wx) + \sin(\pi w)\cdot\sin(wx)}{1-w^2}\,dw$$

But! If we insted choose a sine integral

$$\Rightarrow A(w) = 0$$

$$B(w) = \frac{2}{\pi}\int_{-\infty}^{\infty} f(x)\cdot\sin(wv)\,dv = \frac{\sin(\pi w)}{1-w^2}$$

$$\Rightarrow f(x) = \int_0^\infty B(w)\cdot\sin(wx)\,dw$$

$$= \underline{\underline{\int_0^\infty \frac{\sin(\pi w)\sin(wx)}{1-w^2}\,dw}}$$

11.7.11 $\qquad f(x) = \begin{cases} \sin(x) & , 0 < x < \pi \\ 0 & , x > \pi \end{cases}$ 2

$$f(x) = \int_0^\infty A(w) \cdot \cos(wx)\, dw, \quad A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cdot \cos(wv)\, dv$$

$$\Rightarrow A(w) = \frac{2}{\pi} \int_0^\pi \sin(v) \cdot \cos(wv)\, dv = \frac{\cos(\pi w) + 1}{1 - w^2} \cdot \frac{2}{\pi}$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos(\pi w) + 1}{1 - w^2} \cdot \cos(wx)\, dw$$

11.7.19 $\qquad f(x) = \begin{cases} e^x & ; 0 < x < 1 \\ 0 & ; x > 1 \end{cases}$

$$f(x) = \int_0^\infty B(w) \cdot \sin(wx)\, dw, \quad B(w) = \frac{2}{\pi} \int_0^\infty f(v) \cdot \sin(wv)\, dv$$

$$\Rightarrow B(w) = \frac{2}{\pi} \int_0^1 e^v \sin(wv)\, dv = \frac{2}{\pi} \cdot \frac{w + e(\sin(w) - w\cos(w))}{w^2 + 1}$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^\infty \sin(wx) \cdot \frac{w + e(\sin(w) - w\cos(w))}{1 + w^2}\, dw$$

11.9.4 $\qquad f(x) = \begin{cases} e^{kx} & x < 0 \quad (k > 0) \\ 0 & x > 0 \end{cases}$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-kx} e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-x(k + iw)}}{-(k + iw)} \right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{k + iw} = \frac{1}{\sqrt{2\pi}(k + iw)}$$

11.9.9

$$f(x) = \begin{cases} |x| & ; -1 < x < 1 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-1}^{0} -x\, e^{-iwx}\, dx + \int_{0}^{1} x\, e^{-iwx}\, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[e^{-iwx} \left(\frac{1}{w^2} + \frac{ix}{w} \right) \right]_0^1 - \left[e^{-iwx} \left(\frac{1}{w^2} + \frac{ix}{w} \right) \right]_{-1}^0 \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(e^{-iw} \left(\frac{1}{w^2} + \frac{i}{w} \right) - \frac{1}{w^2} - \frac{1}{w^2} + e^{iw} \left(\frac{1}{w^2} - \frac{i}{w} \right) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \Big((\cos(w) - i \sin(w)) \left(\frac{1}{w^2} + \frac{i}{w} \right)$$

$$+ (\cos(w) + i \sin(w)) \left(\frac{1}{w^2} - \frac{i}{w} \right) - \frac{2}{w^2} \Big)$$

$$= \frac{1}{\sqrt{2\pi}} \Big(\frac{\cos(w)}{w^2} - \frac{i\sin(w)}{w^2} + \frac{i\cos(w)}{w} + \frac{\sin(w)}{w}$$

$$+ \frac{\cos(w)}{w^2} + \frac{i\sin(w)}{w^2} - \frac{i\cos(w)}{w} + \frac{\sin(w)}{w} - \frac{2}{w^2} \Big)$$

$$= \frac{1}{\sqrt{2\pi}} \left(2 \left(\frac{\cos(w) - 1 + w\sin(w)}{w^2} \right) \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\cos(w) - 1 + w \sin(w) \right) / w^2$$

Supplementary H

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0 \end{cases}$$

$$\Rightarrow \hat{f}(w) = \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{x} e^{-iwx}\, dx + \int_{0}^{\infty} e^{-x} e^{-iwx}\, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{e^{x(1-iw)}}{1-iw} \right]_{-\infty}^{0} + \left[\frac{e^{-x(1+iw)}}{-(1+iw)} \right]_{0}^{\infty} \right)$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{1+w^2} = \underline{\underline{\sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+w^2}}}$$

inverse fourier

$$\mathcal{F}^{-1}(\hat{f}(w)) = \int_{-\infty}^{\infty} \hat{f}(w) \cdot e^{iwx}\, dw \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{0}^{\infty} \hat{f}(-w) e^{-iwx} + \hat{f}(w) e^{iwx}\, dw \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{\sqrt{2\pi}} \left(\int_{0}^{\infty} \frac{e^{-iwx} + e^{iwx}}{1+w^2}\, dw \right)$$

$$= \frac{1}{\pi} \cdot \int_{0}^{\infty} \frac{\cos(wx) - i\sin(wx) + \cos(wx) + i\sin(wx)}{1+w^2}\, dw$$

$$= \frac{2}{\pi} \cdot \int_{0}^{\infty} \frac{\cos(wx)}{1+w^2}\, dw = f(x)$$

$$\Rightarrow f(1)\frac{\pi}{2} = \int_{0}^{\infty} \frac{\cos(w)}{1+w^2}\, dw$$

$$\frac{df(x)}{dx} \cdot \frac{\pi}{2} = \frac{d}{dx} \int_{0}^{\infty} \frac{\cos(wx)}{1+w^2}\, dw$$

$$\Rightarrow -f'(x) \cdot \frac{\pi}{2} = \int_{0}^{\infty} \frac{w \sin(wx)}{1+w^2}\, dw$$

$$\Rightarrow -f'(1) \cdot \frac{\pi}{2} = \underline{\underline{\int_{0}^{\infty} \frac{w \sin(w)}{1+w^2}\, dw = \frac{\pi}{2e}}}$$

я ЛЮБЛЮ евгения ♡

Supplementary I

$$f(t) = \cos(t)\, e^{-t^2}$$

$$\Rightarrow f(t) + g(t) = \cos(t)\, e^{-t^2} + g(t)$$

$$g(t) = i\sin(t)\, e^{-t^2}$$

$$\Rightarrow f(t) + g(t) = e^{-t^2} \cdot e^{it} = r(t) \cdot e^{it}$$

$$(r(t) = e^{-t^2})$$

$$\mathcal{F}(r(t)\, e^{it}) = \hat{r}(\omega - 1)$$

$$\hat{r}(\omega) = \mathcal{F}(e^{-t^2}) = \frac{1}{\sqrt{2}}\, e^{-\frac{\omega^2}{4}}$$ (A similar transform is calculated in suppl. J)

$$\Rightarrow \hat{f}(\omega) + \hat{g}(\omega) = \frac{1}{\sqrt{2}}\, e^{-\frac{(\omega-1)^2}{4}}$$

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} i\sin(t)\, e^{-t^2} \cdot e^{-i\omega t}\, dt \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, i\sin(t)\left(\cos(\omega t) - i\sin(\omega t)\right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\left(i\sin(t)\cos(\omega t) + \sin(t)\sin(\omega t)\right) dt$$

$$g'(t) = -i\cos(t)\, e^{-t^2} - i2t\, \cancel{\cos}\, \sin(t)\, e^{-t^2}$$

$$\Rightarrow g'(t) = -i f(t) - \cancel{i}2t\, g(t)$$

$$\Rightarrow \mathcal{F}(g'(t)) = \mathcal{F}(-i f(t)) + \mathcal{F}(-\cancel{i}2t\, g(t))$$

$$\Rightarrow i\omega \hat{g}(\omega) = -i\hat{f}(\omega)\, \cancel{\pm}\, 2\hat{g}'(\omega)$$

$$2\frac{d\hat{g}}{d\omega} = -i\hat{f}(\omega) - i\omega\hat{g}(\omega)$$

$$g''(t) = -i\left(\sin(t)\, e^{-t^2} - 2t\cos(t)\, e^{-t^2}\right) -$$

$g(t) = i\sin(t)e^{-t^2}$, $l(t) = -i\sin(t)e^{-t^2}$

$f(t) = \cos(t)e^{-t^2}$

$\Rightarrow \hat{f}(w) + \hat{l}(w) = \hat{f}(w) - \hat{g}(w)$

$f(t) + l(t) = e^{-t^2} \cdot e^{-it}$

$$\Rightarrow \hat{f}(w) + \hat{l}(w) = \frac{1}{\sqrt{2}} \cdot e^{-\frac{(w+1)^2}{4}}$$

$$\Rightarrow 2\hat{f}(w) = \frac{1}{\sqrt{2}}\left(e^{-\frac{(w+1)^2}{4}} + e^{-\frac{(w-1)^2}{4}}\right)$$

$$\Rightarrow \hat{f}(w) = \frac{1}{\sqrt{8}}\left(e^{-\frac{(w+1)^2}{4}} + e^{-\frac{(w-1)^2}{4}}\right)$$

Supplementary 9

$h(x) = e^{-x^2} * e^{-x^2} = f(x) * f(x)$

$\mathcal{F}(f(x) * f(x)) = \sqrt{2\pi} \cdot \hat{f}(w) \cdot \hat{f}(w)$

First, we find $\hat{f}(w)$:

$f'(x) = -2xe^{-x^2} = -2x \cdot f(x)$

$\Rightarrow \quad f'(x) = -2x \cdot f(x)$

$\Rightarrow \quad \mathcal{F}(f'(x)) = \mathcal{F}(-2x \cdot f(x))$

$\underline{\mathcal{F}(f'(x)) = iw \cdot \hat{f}(w)}$

$\mathcal{F}\{(-ix)^n \cdot f(x)\} = \hat{f}^{(n)}(w)$

$\Rightarrow \mathcal{F}\{x \cdot f(x)\} = \frac{1}{-i} \hat{f}'(w) = i\hat{f}'(w)$

$\Rightarrow \underline{\mathcal{F}(-2x \cdot f(x)) = -2i\hat{f}'(w)}$

$\Rightarrow \quad iw \cdot \hat{f}(w) = -2i\hat{f}'(w)$

$w \cdot \hat{f} = -2\frac{d\hat{f}}{dw}$

$-\frac{1}{2}\int w \, dw = \int \frac{1}{\hat{f}} \, d\hat{f} \Rightarrow -\frac{1}{4}w^2 = \ln(\hat{f}) + C$

$\Rightarrow \hat{f} = e^{-\frac{1}{4}w^2 + C} = C \cdot e^{-\frac{w^2}{4}}$

$C = \hat{f}(0) = a\int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-iw \cdot 0} \, dx = a\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \cdot a$

VERY famous integral

!!! $a = \frac{1}{\sqrt{2\pi}}$

$\Rightarrow \hat{f}(w) = \sqrt{\pi} \cdot e^{-\frac{w^2}{4}} \cdot \frac{1}{\sqrt{2\pi}}$

$\Rightarrow \hat{h}(w) = \cancel{\frac{\pi}{\sqrt{2\pi}}} \cdot e^{-\frac{w^2}{2}} \cdot \cancel{\frac{\pi}{\pi}} \cdot \frac{1}{2} \cdot \sqrt{2\pi}$

$= \frac{1}{2} e^{-\frac{w^2}{2}} \cdot \sqrt{2\pi}$

Now, we find the inverse fourier transform of $\hat{h}(w)$

$$\mathcal{F}^{-1}(\hat{h}(w)) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} e^{iwx} \, dx \cdot \sqrt{2\pi}$$

Also: $\hat{h}'(w) = -\frac{w}{2} e^{-\frac{w^2}{2}} \cdot \sqrt{2\pi}$

$\Rightarrow \hat{h}'(w) = -w \cdot \hat{h}(w)$

$\Rightarrow \mathcal{F}^{-1}(\hat{h}'(w)) = \mathcal{F}^{-1}(-\hat{h}(w) \cdot w)$

$\Rightarrow -ix \cdot f(x) = i \cdot f'(x) \qquad (f(x) = h(x))$

$-x \cdot f = \frac{df}{dx}$

$\Rightarrow -\int x \, dx = \int \frac{1}{f} \, df$

$-\frac{1}{2}x^2 = \ln(f) + c$

$\Rightarrow \underline{f(x) = c \cdot e^{-\frac{x^2}{2}}}$

$f(0) = c = \underset{(x=0)}{\mathcal{F}^{-1}(\hat{h}(w))} = \frac{1}{2} \cdot \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} \, dw = \sqrt{2\pi}$

$\Rightarrow c = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}$

$\Rightarrow \underline{\underline{f(x) = h(x) = \sqrt{\frac{\pi}{2}} \cdot e^{-\frac{x^2}{2}}}}$

Supplementary K

Eugenia ♡

$$f(x) - \int_{-\infty}^{\infty} e^{-\cancel{4}3|x-t|} f(t)\,dt = e^{-3|x|}$$

Yurii ♡

$$f(x) - f(x) * g(x) = g(x)$$

$$\Rightarrow \hat{f}(\omega) - \hat{f}(\omega)\cdot\hat{g}(\omega)\cdot\sqrt{2\pi} = \hat{g}(\omega)$$

$$\Rightarrow \hat{f}(\omega)\left(1 - \hat{g}(\omega)\sqrt{2\pi}\right) = \hat{g}(\omega)$$

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\cdot e^{-i\omega x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} e^{3x} e^{-i\omega x}\,dx + \int_{0}^{\infty} e^{-3x} e^{-i\omega x}\,dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\left[\frac{e^{x(3-i\omega)}}{3-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-x(3+i\omega)}}{-(3+i\omega)}\right]_{0}^{\infty}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{3-i\omega} + \frac{1}{3+i\omega}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{3+i\omega+3-i\omega}{9+\omega^2}\right)$$

$$= \frac{3\cdot 2}{\sqrt{2\pi}}\cdot\frac{1}{9+\omega^2} = \underline{\sqrt{\frac{2}{\pi}}\cdot\frac{3}{9+\omega^2}} = \hat{g}(\omega)$$

$$\hat{f}(\omega) = \frac{\hat{g}(\omega)}{1-\sqrt{2\pi}\,\hat{g}(\omega)} = \frac{\sqrt{\frac{2}{\pi}}\cdot 3}{9+\omega^2-6} = \frac{\sqrt{\frac{2}{\pi}}\cdot 3}{\omega^2+3}$$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty}\frac{3}{\omega^2+3}\cdot e^{+i\omega x}\,d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{3e^{i\omega x}}{\omega^2+3}\,d\omega = \underline{\underline{\sqrt{3}\,e^{-\sqrt{3}|x|}}}$$

The result can be obtained through integration by fractional expansion or simply looking up in the table.

Supplementary L

$$f(x) = \begin{cases} 1 , & |x| < 1 \\ 0 , & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} e^{-x} , & x > 0 \\ 0 ; & \text{otherwise} \end{cases}$$

a) $$\hat{f}(w) = \int_{-\infty}^{\infty} f(x) e^{-iwx} dx \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \int_{-1}^{1} e^{-iwx} dx \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-iw} - e^{iw}}{-iw} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{iw} - e^{-iw}}{iw} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\cos(w) + i\sin(w) - (\cos(w) - i\sin(w))}{iw} \right) = \frac{2}{\sqrt{2\pi}} \frac{\sin(w)}{w}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin(w)}{w}$$

$$\hat{g}(w) = \int_{0}^{\infty} e^{-x(1+iw)} dx \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-x(1+iw)}}{-(1+iw)} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(1+iw)} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1-iw}{1+w^2}$$

b) $$h(x) = f(x) * g(x)$$

$$\Rightarrow \hat{h}(w) = \sqrt{2\pi} \cdot \hat{f}(w) \cdot \hat{g}(w)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(w)(1-iw)}{w \cdot (1+w^2)}$$

$$\Rightarrow \mathcal{F}^{-1}(\hat{h}(w)) = h(x) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \cdot \int_{-\infty}^{\infty} \frac{\sin(w)(1-iw)}{w(1+w^2)} e^{iwx} dw$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(w)(1-iw)}{w(1+w^2)} e^{iwx} dw$$

$$\pi \cdot h(0) = \int_{-\infty}^{\infty} \frac{\sin(w)(1-iw)}{w(1+w^2)} dw$$

$$h(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy = \int_{0}^{1} f(x) g(x-y) dy = \int_{0}^{1} e^{-(x-y)} dy$$

$$= e^{-x} [e^y]_0^1 = e^{-x}(e-1) \Rightarrow \pi \cdot h(0) = \pi(e-1)$$

Uppg L

$$\Rightarrow \pi \cdot h(0) + \int_{-\infty}^{\infty} \frac{i\omega \sin(\omega)}{\omega(1+\omega^2)} d\omega = \int_{-\infty}^{\infty} \frac{\sin(\omega)}{\omega(1+\omega^2)} d\omega$$

$$z_1 = z_2$$

$$\Rightarrow Re(z_1) = Re(z_2)$$

$$\Rightarrow \pi \cdot h(0) = \int_{-\infty}^{\infty} \frac{\sin(\omega)}{\omega(1+\omega^2)} d\omega$$

$$h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy$$

$$= \int_{-1}^{1} 1 \cdot e^{-(x-y)} dy$$

$$g(x-y) = \begin{cases} e^{-(x-y)}, & x-y > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow y < x$$

$$\Rightarrow h(x) = \int_{-1}^{x} e^{-x} e^{y} dy = e^{-x}\left[e^{y}\right]_{-1}^{x} = e^{-x}(e^{x} - e^{-1})$$

$$\Rightarrow h(0) = \frac{e-1}{e} \Rightarrow$$

$$\int_{-\infty}^{\infty} \frac{\sin(\omega)}{\omega(1+\omega^2)} d\omega = \frac{\pi(e-1)}{e}$$

Supplementary M

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

$$\Rightarrow \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-1}^{1} 1 \cdot e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-1}^{1}$$

$$= \cancel{\frac{1}{\sqrt{2\pi}} \left(\frac{e^{-i\omega} - e^{i\omega}}{-i\omega} \right)} = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-i\omega} - e^{i\omega}}{-i\omega} \right)$$

$$= \frac{i\omega}{\sqrt{2\pi}} \cdot \frac{e^{-i\omega} - e^{i\omega}}{\omega^2} = \underline{\underline{\frac{i}{\sqrt{2\pi}} \cdot \frac{e^{-i\omega} - e^{i\omega}}{\omega}}}$$

Note: There is either a mistype in assignment or the $\hat{f}(\omega) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} dx$ normalization is used. However, if we are consistent in the use of normalization, the same result for the inverse will be obtained.

$$\mathcal{F}(f(x) * f(x)) = \sqrt{2\pi} \cdot \hat{f}(\omega) \cdot \hat{f}(\omega) = \frac{-1}{\sqrt{2\pi}} \left(\frac{e^{-i\omega} - e^{i\omega}}{\omega} \right)^2$$

$$= \frac{-1}{\sqrt{2\pi}} \left(\frac{e^{-2i\omega} - 2e^{i\omega} \cdot e^{-i\omega} + e^{2i\omega}}{\omega^2} \right) = \frac{-1}{\sqrt{2\pi}} \left(\frac{e^{-2i\omega} + e^{2i\omega} - 2}{\omega^2} \right)$$

$$\Rightarrow f(x) * f(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(x-2)} + e^{i\omega(x+2)} - 2e^{i\omega x}}{\omega^2} d\omega$$

$$= \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos(\omega(x-2)) + i\sin(\omega(x-2)) + \cos(\omega(x+2)) + i\sin(\omega(x+2))}{\omega^2} - \frac{2(\cos(\omega x) + i\sin(\omega x))}{\omega^2} d\omega$$

$$= z(x) \Rightarrow Re(z(3)) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos(5\omega) - 2\cos(3\omega) + \cos(\omega)}{\omega^2} d\omega$$

$$\Rightarrow -2\pi(f(3) * f(3)) = \int_{-\infty}^{\infty} \frac{\cos(5\omega) - 2\cos(3\omega) + \cos(\omega)}{\omega^2} d\omega$$

$$f(x) = 1 \cdot u(x+1) - 1 \cdot u(x-1) = 0, \text{ if } x = 3$$

$$\cancel{\Rightarrow f(x) * f(x) = \int_{-\infty}^{\infty} (u(y+1) - u(y-1)) \cdot (u(x-y+1) - u(x-y-1)) dy}$$

$$\Rightarrow -2\pi\left(f(3) * f(3)\right) = -2\pi\,(0 * 0)$$

$$= 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos(5w) - 2\cos(3w) + \cos(w)}{w^2}\, dw = 0$$