TMA 4120 - Høst 2016

Øving 9

Vsevolod Karpov - vsevolok

17.1.5) $w = z^2$

$x = 1, 2, 3, 4$ $\qquad z^2 = (x+iy)^2$

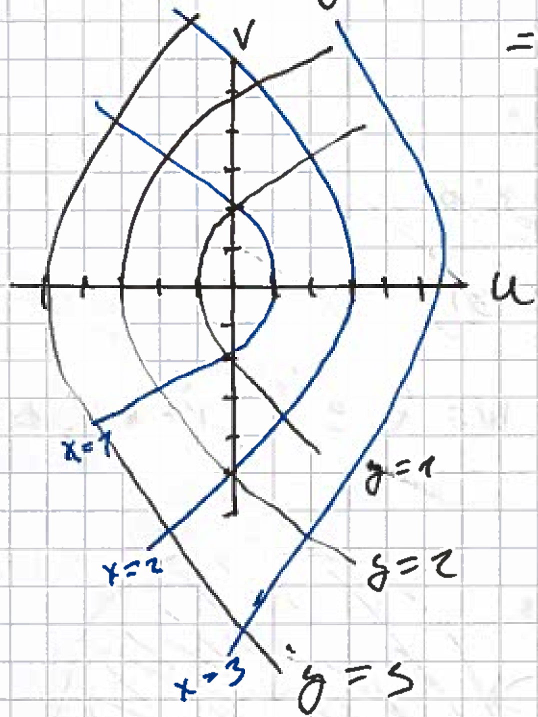
$y = 1, 2, 3, 4$ $\qquad = x^2 + 2ixy - y^2$

$\Rightarrow Re(w) = x^2 - y^2 \Rightarrow u = c^2 - y^2,\ v = 2cy$

$Im(w) = 2xy$ $\qquad \Rightarrow v^2 = 4c^2y^2$

$\Rightarrow v^2 = 4c^2(c^2 - u)$ $\qquad \boxed{c = x_0,\ k = y_0}$

$v^2 = 4k^2(k^2 + u)$



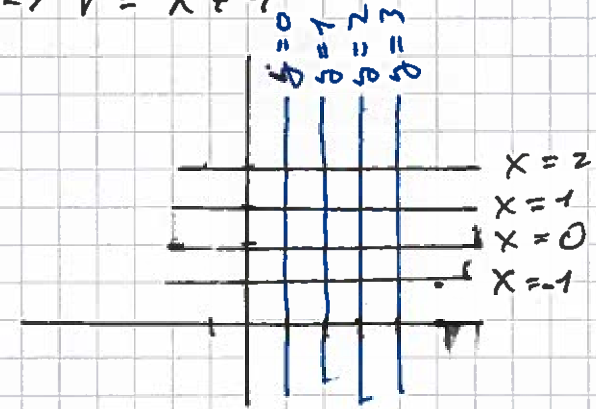
17.1.8) $w = z + 2 + i = x + i(y+1) + 2$

$Re(w) = u = x + 2$ $\qquad x = c \Rightarrow u = c + 2$

$Im(w) = v = y + 1$ $\qquad y = k \Rightarrow v = k + 1$

y translated by $+1$

x translated by $+2$



17.1.11 $\quad 1 < |z| < 3 \;,\; 0 < \operatorname{Arg} z < \frac{\pi}{2} \;,\; w = z^{3}$ $\quad$ Z

$z = x + iy \;,\; z^3 = (x+iy)^2(x+iy)$

$= (x^2 + ixy - y^2)(x+iy) = x^3 + ix^2y - xy^2 + iyx^2 - xy^2 - iy^3$

$= x^3 - 2xy^2 + i(2x^2y - y^3)$

$\Rightarrow u = x^3 - 2xy^2$

$v = 2x^2y - y^3$

$x = c \Rightarrow u = c(c^2 - 2y^2)$

$v = y(2c^2 - y^2)$

Easier with polar:

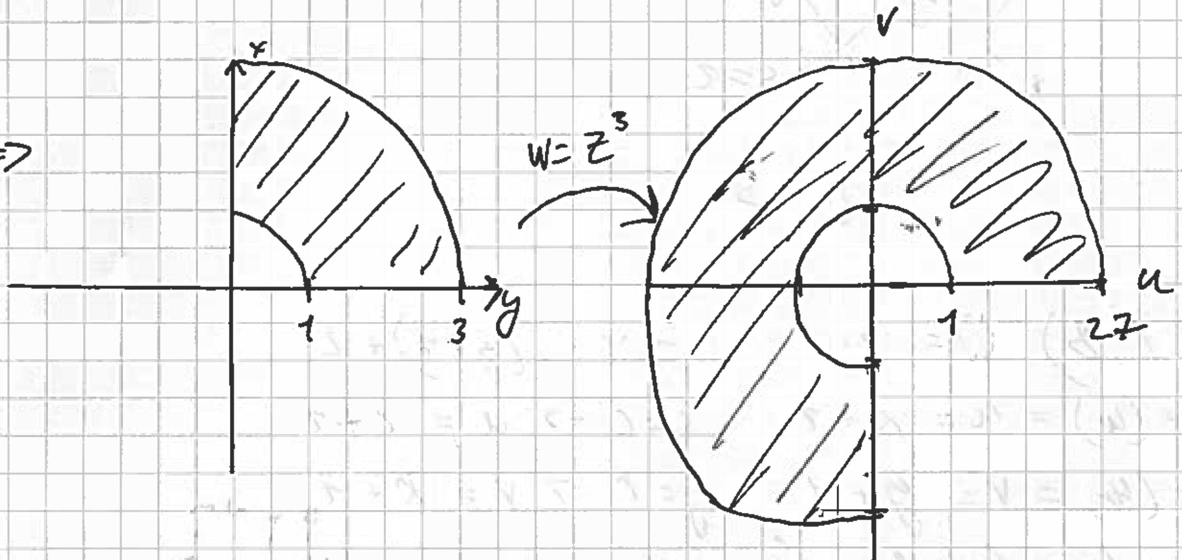
$z = re^{i\theta} \Rightarrow z^3 = r^3 e^{3i\theta}$

$w = r^3(\cos(3\theta) + i\sin(3\theta))$

$\Rightarrow u = r^3\cos(3\theta) \quad \Rightarrow W = R \cdot e^{i\phi} \;,\; R = r^3 \;,\; \phi = 3\theta$

$v = r^3 \sin(3\theta)$

$\Rightarrow$



17.1.13) $x \geq 1$, $w = \frac{1}{z}$

$$w = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \Rightarrow u = \frac{x}{x^2+y^2}$$

$$v = \frac{-y}{x^2+y^2}$$

$$u = \frac{c}{c^2+y^2}$$

$$\Rightarrow y^2 = \frac{c - c^2 \cdot u}{u}$$

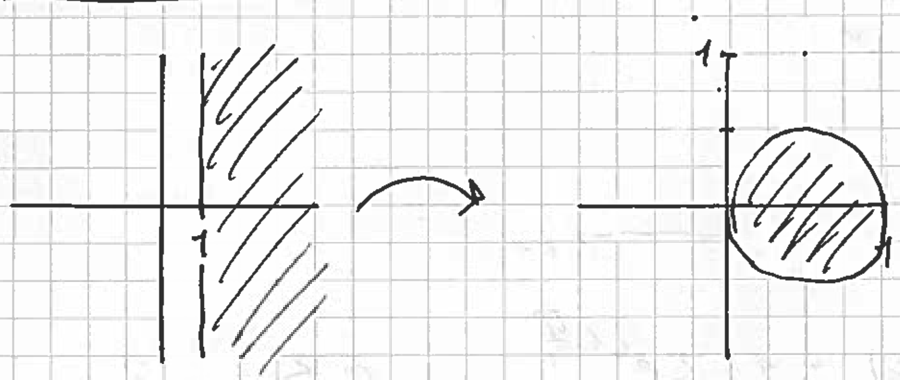
$$\Rightarrow v = \frac{-\sqrt{\frac{c}{u} - c^2}}{c^2 + \frac{c - c^2 u}{u}} = \frac{-u\sqrt{\frac{c}{u} - c^2}}{c}$$

$$\Rightarrow v = -\sqrt{\frac{u}{c} - u^2} \Rightarrow \underline{v^2 = \frac{u}{c} - u^2} \quad (c = x_0)$$

$$v = \frac{-k}{x^2 + k^2} \Rightarrow x^2 = \frac{-k - k^2 v}{v}$$

$$\Rightarrow u = \frac{\sqrt{-\frac{k}{v} - k^2}}{\frac{-k - k^2 v}{v} + k^2} = \frac{\sqrt{-\frac{k}{v} - k^2}}{-\frac{k}{v}}$$

$$\Rightarrow \underline{u^2 = -\frac{v}{k} - v^2}$$



17.1.15)

$w = z^{\cancel{4}3}$

Theorem: $w = f(z)$, where $f(z)$ is analytic is conformal except for critical points where derivative is zero

$\Rightarrow w'(z) = 0 = 3z^2$

$z^2 = 0 \Rightarrow \underline{\underline{z = 0}}$ (at origin)

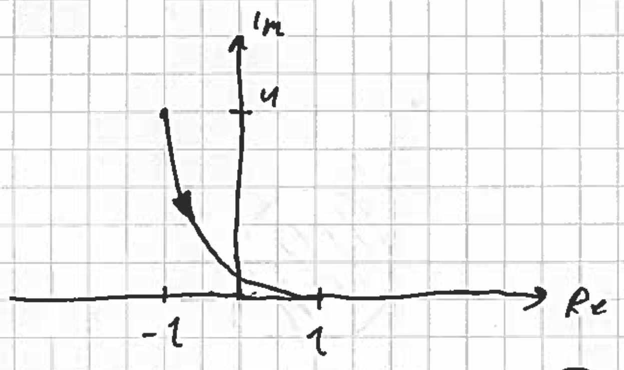
14.1.4)

$z(t) = t + (1-t^{*})^2 i \quad , \quad -1 \leq t \leq 1$

$x = t \, , \; y = (1-t^{*})^2$

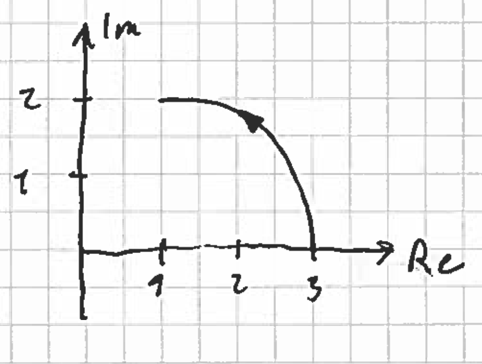
$\Rightarrow y = (1-x)^2 = 1 - 2x + x^2$

$(x,y) \in [(-1,4), (1,0)]$

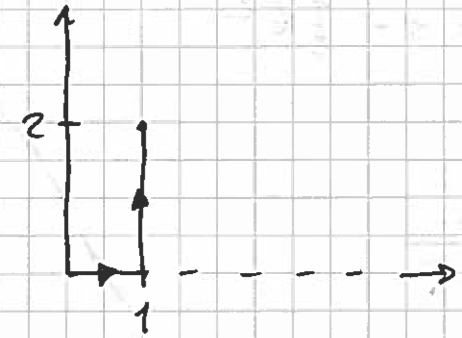


14.1.7) $z(t) = 1 + 2e^{it\frac{\pi}{4}} \quad , \quad 0 \leq t \leq 2$

A quarter of a circle with centre in $(1,0)$ and $r = 2$



14.1.12)



first segment: $z(t) = t \quad 0 \le t \le 1$

second segment: $z(t) = it \quad 1 \le t \le 2$

Together:

$$z(t) = \cancel{t \cdot at}\ t - t \cdot u(t-1) + it(t-1)$$

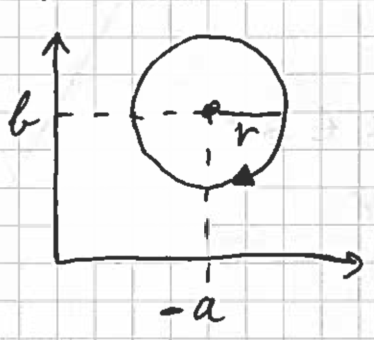
$$\underline{\underline{(t - t \cdot u(t-1), t(t-1))}}$$

14.1.17)

$|z + a - ib| = r$, clockwise

$|z + a - ib| = |z - (-a + ib)|$

A circle with centre in $(-a + ib)$ and radius r:



$$\Rightarrow p(t): \underline{\underline{(r\cos(t) - a, -r\sin(t) + b)}}$$

14.1.21)

$\int_C Re(z)\,dz$, from $1+i$ to $5+5i$

$C: (t, t)$, $t \in [1, 5]$

Function not analytic so the first method cant be used!

$$\int_C Re(z)dz = \int_1^5 Re(t+it)\cdot(1+i)dt$$

$$= \int_1^5 (1+i)\cdot t\,dt = (1+i)\left[\frac{1}{2}t^2\right]_1^5 = (1+i)(12)$$

$$= \underline{\underline{12+12i}}$$

14.1.23)

$\int_C e^z dz$ C: shortest from $\pi/2i$ to πi

Analytic, can use first method

$f(z) = e^z$, $F'(z) = f(z) \Rightarrow F(z) = e^z$

$$\Rightarrow \int_C e^z dz = e^{\pi i} - e^{\pi/2i} = -e^{\frac{-i\pi}{2}} + e^{\pi i} = \underline{\underline{-1+i}}$$

14.1.35)

$\left|\int_C f(z)\right| \leq ML$, $L = |C|$, $M = |f(z)|_{max}$

$f(z) = Re(z)$, $C = t(1+i)$, $t \in [1, 5]$

$|C| = \int_1^5 \sqrt{1+1^2} = \sqrt{4^2+4^2} = \underline{\sqrt{2}\cdot 4}$

$|f(z)|_{max}$: When $t = 5 \Rightarrow f(z_0) = Re(5+5i) = 5$

$$\Rightarrow \underline{\underline{ML = \sqrt{2}\cdot 20}}$$

14.2.14)

$$f(z) = \frac{1}{z}$$

$C: e^{it}, \quad 0 \leq t \leq 2\pi$

$$\Rightarrow \int_C f(z)\,dz = \int_0^{2\pi} e^{-it} \cdot i \cdot e^{it}\,dt = \int_0^{2\pi} i\,dt = \underline{\underline{2\pi i}}$$

function not analytic! Cauchy's theorem does not apply!

14.2.22

Re(z) is not analytic! C.'s theorem does not apply!

$C_1: e^{it}, \quad 0 \leq t \leq \pi$

$C_2: t, \quad -1 \leq t \leq 1$

$$\oint_C \mathrm{Re}(z)\,dz = \int_{C_1} \mathrm{Re}(z)\,dz + \int_{C_2} \mathrm{Re}(z)\,dz$$

$$\int_{C_1} \mathrm{Re}(z)\,dz = \int_0^{\pi} \cos(t) \cdot i \cdot e^{it}\,dt = i \cdot \left[-e^{it} \cdot (i\cos(t) + \sin(t)) \right]_0^{\pi}$$

$$= \underline{0}$$

$$\int_{C_2} \mathrm{Re}(z)\,dz = \int_{-1}^{1} t\,dt = \left[\frac{1}{2} t^2 \right]_{-1}^{1} = 0$$

$$\Rightarrow \oint_C \mathrm{Re}(z)\,dz = \underline{\underline{0}}$$

Note! We are getting zero because of symmetry and function being the real part, NOT because of Cauchy's theorem.

14.2.27) $\oint_C \frac{\cos(z)}{z} dz$

C: $|z| = 1$ counterclockwise and $|z| = 3$ clockwise

Doubly connected domain

$$\oint_C \frac{\cos(z)}{z} dz = \oint_{C_1} \frac{\cos(z)}{z} dz + \oint_{C_2} \frac{\cos(z)}{z} dz$$

Because path C_1 and C_2 have opposite direction, C.'s theorem for multiply connected domains doesnt apply.

$C_1: e^{it}, \quad t \le 2\pi$

$C_2: 3e^{it}, \quad 0 \le t \le -2\pi$

But: $f(z) = \frac{\cos(z)}{z} \Rightarrow f'(z) = -\frac{z\sin(z) + \cos(z)}{z^2}$

So our function is analytic on both C_1 and C_2 and therefore:

$$\oint_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz = 0$$

$$\Rightarrow \oint_C \frac{\cos(z)}{z} dz = 0$$

Cauchy's integral theorem applies.