TMA4120 - Høst 2016 1

Øving 4

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11.3.15

$r(t) = t(\pi^2 - t^2), \; t \in [-\pi, \pi]$

and $r(t + 2\pi) = r(t)$

$\Rightarrow p = 2\pi$

$y'' + cy' + y = r(t), \quad c > 0$

$r(t) = \sum_{n=1}^{\infty} b_n \cdot \sin(nt)$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t(\pi^2 - t^2) \cdot \sin(nt) \, dt$

$= -\frac{12 n \cos(\pi n)}{n^4} = -\frac{12(-1)^n}{n^3}$

$\Rightarrow r(t) = -12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nt)$

$\Rightarrow y'' + cy' + y = -12 \frac{(-1)^n}{n^3} \sin(nt) \quad , (n = 1, 2, 3 \ldots)$

$y_n = A_n \cos(nt) + B_n \sin(nt)$

$\Rightarrow -A_n n^2 \cos(nt) - B_n n^2 \sin(nt) + c(-A_n n \sin(nt) + B_n n \cos(nt))$
$+ A_n \cos(nt) + B_n \sin(nt) = -12 \frac{(-1)^n}{n^3} \sin(nt)$

$\Rightarrow -A_n n^2 + cB_n n + A_n = 0$

$-B_n n^2 - cA_n n + B_n = -12 \frac{(-1)^n}{n^3}$

$\Rightarrow A_n = \frac{-12(-1)^n c}{n^2 \cdot D_n}, \quad D_n = (-n^4 + n^2(2 - c^2) - 1)$

$B_n = \frac{-12(-1)^n (1 - n^2)}{n^3 \cdot D_n}$

$$\Rightarrow \dot{y} = \sum_{n=1}^{\infty} A_n \cos(nt) + B_n(\sin(nt))$$

$$, A_n = \frac{(-1)^n \cdot 12c}{n^2 D_n} \quad , B_n = \frac{-12\cdot(-1)^n(1-n^2)}{n^3 D_n}$$

$$, D_n = (1-n^2)^2 + n^2c^2$$

NB! The later A_n, B_n & D_n are <u>after</u> minus simplification.

11.3.19

$$E(t) = 200t(\pi^2 - t^2) \,, \quad t \in [-\pi, \pi]$$

$$L \cdot I'(t) + R \cdot I(t) + \frac{1}{c}\int_0^t I(\tau)\,d\tau = E(t)$$

$$\Rightarrow L \cdot I''(t) + R \cdot I'(t) + \frac{1}{c} I(t) = E'(t)$$

$$E'(t) = 200(\pi^2 - t^2) + 200t \cdot -2t$$

$$= \underline{200\pi^2 - 600t^2}$$

$$\underline{L = 1, \ R = 10, \ c = 10^{-1}}$$

$$E'(t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(nt) \,, \quad E'(t) \text{ is even}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 200\pi^2 - 600t^2\, dt = \underline{0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (200\pi^2 - 600t^2) \cdot \cos(nt)\, dt$$

$$= \underline{\frac{-2400 \cdot (-1)^n}{n^2}}$$

$$I_n(t) = A_n \cdot \cos(nt) + B_n \cdot \sin(nt)$$

$$\Rightarrow -A_n n^2 \cos(nt) - B_n n^2 \sin(nt) + 10(-A_n n \sin(nt) + B_n n \cos(nt))$$
$$+ 10(A_n \cos(nt) + B_n \sin(nt))$$

$$\Rightarrow \quad -A_n n^2 + 10 B_n n + 10 A_n = -\frac{2400(-1)^n}{n^2}$$
$$-B_n n^2 - 10 A_n n + 10 B_n = 0$$

$$\Rightarrow A_n = \frac{-2400(-1)^n (10 - n^2)}{n^2 D_n}$$

$$B_n = \frac{-24000(-1)^n}{n \cdot D_n}$$

$$D_n = (10 - n^2)^2 + 100 n^2$$

$$\Rightarrow I(t) = \cancel{A_n \cos(nt)} \sum_{n=1}^{\infty} A_n \cos(nt) + B_n (\sin(nt))$$

$$I_4(t) = \sum_{n=1}^{4} A_n \cos(nt) + B_n \sin(nt)$$

11.4.9 (9th edition)

$$f(x) = x \,, \quad (-\pi < x < \pi)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx} \,, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}$$

$$\Rightarrow c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx = \frac{i(-1)^n}{n}$$

$$\Rightarrow f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{i \cdot e^{inx}}{n} (-1)^n$$

11.4*10.

$$f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i \cdot e^{inx}(-1)^n}{n}$$

$$\frac{i \cdot e^{inx}(-1)^n}{n} = \frac{e^{i\frac{\pi}{2}} \cdot e^{inx}(-1)^n}{n} = \frac{e^{i(\frac{\pi}{2}+nx)}(-1)^n}{n}$$

$$= \frac{(\cos(\frac{\pi}{2}+nx) + i\sin(\frac{\pi}{2}+nx))(-1)^n}{n}$$

$$\Rightarrow Re = \frac{(-1)^n}{n} \cdot \cos(\frac{\pi}{2}+nx)$$

$$\Rightarrow f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} \cdot \cos(\frac{\pi}{2}+nx)$$

11.4*13

$$f(x) = x, \quad 0 < x < 2\pi$$

$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} x \cdot e^{-inx}\,dx = \frac{-1 + e^{-2i\pi n}(1+2i\pi n)}{n^2 \cdot 2\pi}$$

$$= \frac{i}{n} \Rightarrow f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{n} \cdot e^{inx} + a_0$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \underline{\pi}$$

11.4.4 (10th edition)

$$F_N(x) = A_0 + \sum_{n=1}^{N} \left(A_n \cos(nx) + B_n \sin(nx)\right)$$

$$E^* = \int_{-\pi}^{\pi} f^2 \cdot dx - \pi\left(2A_0^2 + \sum_{n=1}^{N}\left(A_n^2 + B_n^2\right)\right)$$

$$f(x) = x^2, \ (-\pi < x < \pi)$$

$f(x)$ is even $\Rightarrow$ $B_n = 0$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{2\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} = \frac{2\pi^3}{6\pi} = \frac{\pi^2}{3}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot \cos(nx) \, dx = \frac{4 \cdot (-1)^n}{n^2}$$

$$\Rightarrow E^* = \int_{-\pi}^{\pi} x^4 \, dx - \pi\left(\frac{2\pi^4}{9} + \sum_{n=1}^{N}\left(\frac{4 \cdot (-1)^n}{n^2}\right)^2\right)$$

$$= \frac{2\pi^5}{5} - \frac{2\pi^5}{9} - \pi \sum_{n=1}^{N}\left(\frac{16}{n^4}\right)$$

$$= \frac{8}{45}\pi^5 - \pi \sum_{n=1}^{N}\left(\frac{16}{n^4}\right)$$

N	E^*
1	4,138
2	0.996
3	0.3758
4	0.1795
5	0.09908
⋮	
50	0.00013

11.4.8

$f(x) = |\sin(x)|$, $-\pi < x < \pi$

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left(2A_0^2 + \sum_{n=1}^{N}(A_n^2 + B_n^2)\right)$$

$f(x)$ is even $\Rightarrow$ $B_n = 0$

We decompose $f(x)$

$$\Rightarrow f(x) \begin{cases} -\sin(x), & -\pi < x < 0 \\ \sin(x), & 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} -\sin(x)\, dx + \frac{1}{2\pi}\int_{0}^{\pi} \sin(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(x)\cdot\cos(nx)\, dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\cdot\cos(nx)\, dx$$

$$a_0 = \frac{2}{\pi} \quad , \quad a_n = \frac{2}{\pi}\left(\frac{(-1)^n + 1}{1 - n^2}\right)$$

$$\int_{-\pi}^{\pi} f^2 dx = \int_{-\pi}^{\pi} \sin^2(x)\, dx = \pi$$

$$\Rightarrow E^* = \pi - \pi\left(\frac{8}{\pi^2} + \sum_{n=2}^{N} \frac{4}{\pi^2}\cdot\frac{2(1 + (-1)^n)}{(1 - n^2)^2}\right)$$

N	E^*
2	0.02922
3	0.02922
4	0.00659
5	0.00659
⋮	
50	0.00000659

Notice that n starts at 2 since $n = 1$ leads to division by zero!

Also, since a_n is 0 for any odd n, the error for odd N is the same as error for previous, even N.

$$\frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}\ldots = \sum_{n=1}^{\infty}\frac{1}{2}\frac{(1-(-1)^n)}{n^2} = \sum_{n=1}^{\infty} c\cdot\frac{(1-(-1)^n)}{n^2}$$

Consider a function:

$$f(x)=\begin{cases}0, & -\pi<x<0\\ K, & 0<x<\pi\end{cases}, \quad p=2\pi$$

The fourrier cooefficients:

$$a_0=\frac{K}{2},\quad a_n=\frac{K}{\pi}\cdot\frac{\sin(\pi n)}{n}=0 \text{ (for all } n\in\mathbb{Z})$$

$$b_n=\frac{K}{\pi}\left(\frac{1-\cos(\pi n)}{n}\right)=\frac{K}{\pi}\left(\frac{1-(-1)^n}{n}\right)$$

Note that $b_n^2=\frac{2K^2}{\pi^2}\left(\frac{1-(-1)^n}{n^2}\right)$

We now find a value of ~~k~~ K so that:

$$\sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} c\cdot\frac{(1-(-1)^n)}{n^2}$$

$$\Rightarrow \frac{2K^2}{\pi^2}=c=\frac{1}{2} \Rightarrow \underline{K=\frac{\pi}{2}}$$

Parsevals identity:

$$2a_0^2+\sum_{n=1}^{\infty} a_n^2+b_n^2=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx$$

In our case:

$$2a_0^2+\sum_{n=1}^{\infty} b_n^2=\frac{1}{\pi}\int_0^{\pi} K^2\,dx=\frac{\pi^2}{4}$$

$$\Rightarrow \frac{\pi^2}{4}-2a_0^2=\sum_{n=1}^{\infty} b_n^2=\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1-(-1)^n}{n^2}\right)=1+\frac{1}{3^2}+\frac{1}{5^2}\ldots$$

$$\frac{\pi^2}{4}-2a_0^2=\frac{\pi^2}{4}-2\frac{K^2}{4}=\frac{\pi^2}{4}-\frac{\pi^2}{8}=\underline{\underline{\frac{\pi^2}{8}}}$$

11.R.15

$$f(x) = e^x, \quad -5 < x < 5$$

$$\Rightarrow a_0 = \frac{e^5 - e^{-5}}{10}$$

$$a_n = \frac{1}{5}\left[\frac{e^x}{1^2 + \left(\frac{n\pi}{5}\right)^2} \cdot \left(\cos\left(\frac{n\pi}{5}x\right) + \frac{n\pi}{5}\sin\left(\frac{n\pi}{5}x\right)\right)\right]_{-5}^{5}$$

$$= \frac{1}{5}\left(\frac{(-1)^n(e^5 - e^{-5})}{1^2 + \left(\frac{n\pi}{5}\right)^2}\right)$$

$$b_n = \frac{1}{5}\left[\frac{e^x}{1^2 + \left(\frac{n\pi}{5}\right)^2}\left(\sin\left(\frac{n\pi}{5}x\right) - \frac{n\pi}{5}\cos\left(\frac{n\pi}{5}x\right)\right)\right]_{-5}^{5}$$

$$= \frac{n\pi}{25}\left(\frac{(-1)^n(e^{-5} - e^5)}{1 + \left(\frac{n\pi}{5}\right)^2}\right)$$

$$\Rightarrow f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\tfrac{\pi}{5}x\right) + b_n \sin\left(n\tfrac{\pi}{5}x\right)$$

cosine terms   sine terms

These can be graphed and are similar to $\cosh(x)$ and $\sinh(x)$.

Ergo: $a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\tfrac{\pi}{5}x\right) = S(\cosh(x))$

Fourier series of $\cosh(x)$

Which will reduce to:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\tfrac{\pi}{5}x\right) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\tfrac{\pi}{5}x\right)$$

And same may be done for $\sinh(x)$ and sine terms. Which means that original assumtion is true.

Also we know ~~tak~~ that $e^x = \sinh(x) + \cosh(x)$

Ergo: sine terms represent sinh, cosine cosh.

11.R.17

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots = \pi/4$$

Consider function

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$

$$\Rightarrow a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2k}{\pi}\left(\frac{1-(-1)^n}{n}\right)$$

set $k = \frac{\pi}{4}$

$$\Rightarrow b_n = \frac{1}{2}\left(\frac{1-(-1)^n}{n}\right)$$

$$\Rightarrow f(x) = \sum \frac{1}{2}\left(\frac{1-(-1)^n}{n}\right) \sin(nx)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sum \frac{1}{2}\left(\frac{1-(-1)^n}{n}\right) \sin\left(n\frac{\pi}{2}\right) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots$$

$$\Rightarrow \underline{\underline{f\left(\frac{\pi}{2}\right) = \frac{\pi}{4} = k}}$$

$$\underline{\underline{f(x) = \begin{cases} -\frac{\pi}{4}, & -\pi < x < 0 \\ \frac{\pi}{4}, & 0 < x < \pi \end{cases}}}$$

Supplementary E

$f(x) = x(\pi - x)$, $0 < x < \pi$

Sine fourier : $f(x) \sim \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{\sin(x(2m+1))}{(2m+1)^3}$

Series we wish to find the sum of:

$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{15^3} + \dots$

Observe that the series of $(2m+1)^3$ for $m \in [0, \infty\rangle$, $m \in \mathbb{Z}$ is:

$1^3 + 3^3 + 5^3 + 7^3 \dots$

So, the right denominator is already there.

Observe that the series $(\sin(x(2m+1))$ for $\underline{x = \frac{\pi}{4}}$, $m \in [0, \infty\rangle$, $m \in \mathbb{Z}$ is

$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \dots$

Which gives us the desired sign behaviour! Ergo $\Rightarrow$

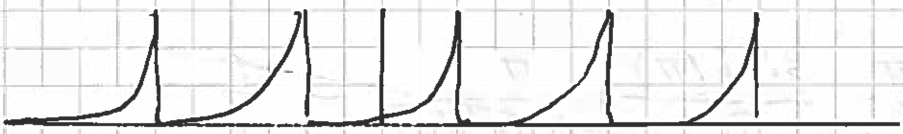
$$f\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{8} = \sum_{m=0}^{\infty} \frac{\sin\left(\frac{\pi}{4}(2m+1)\right)}{(2m+1)^3} = \frac{\sqrt{2}}{2 \cdot 1^3} + \frac{\sqrt{2}}{2 \cdot 3^3} - \frac{\sqrt{2}}{2 \cdot 5^3} - \frac{\sqrt{2}}{2 \cdot 7^3} \dots$$

$$\Rightarrow \frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} \dots = f\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{8} \cdot \frac{2}{\sqrt{2}}$$

$$= \frac{\pi}{4} \cdot \frac{3\pi}{4} \cdot \frac{\pi}{8} \cdot \frac{2}{\sqrt{2}} = \underline{\underline{\frac{\pi^3}{48 \cdot \sqrt{2}}}}$$

Supplementary F

a) $f(x) = e^x \,, \; -\pi < x < \pi$



$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx} \,, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cdot e^{-inx}$$

$$c_n = \left[\frac{i\, e^{(1-in)\cdot x}}{2\pi(n+i)}\right]_{-\pi}^{\pi} = \frac{i}{2\pi(n+i)} \cdot \left(e^{\pi(1-in)} - e^{-\pi(1-in)}\right)$$

$$\Rightarrow f(x) \sim \sum_{n=-\infty}^{\infty} \frac{i\, e^{inx}}{2\pi(n+i)} \cdot \left(e^{\pi(1-in)} - e^{-\pi(1-in)}\right)$$

$$\sim \sum_{n=-\infty}^{\infty} \frac{e^{\pi} - e^{-\pi}}{2\pi} \cdot (-1)^n \cdot \frac{1+in}{1+n^2} \cdot e^{inx}$$

$$= \frac{\sinh(\pi)}{\pi} \cdot \sum_{n=-\infty}^{\infty} (-1)^n \cdot \frac{1+in}{1+n^2} \cdot e^{inx}$$

b) Obtain the real series.

$$f(x) \sim \frac{\sinh(\pi)}{\pi} \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{1+in}{1+n^2} \cdot e^{inx} + \sum_{n=1}^{\infty} (-1)^n \frac{(1-in)}{1+n^2} \cdot e^{-inx}\right)$$

$$= \frac{\sinh(\pi)}{\pi} \cdot \sum_{n=1}^{\infty} (-1)^n \cdot \frac{2(\cos(nx) - n\sin(nx))}{1+n^2} + \frac{\sinh(\pi)}{\pi}$$

Set $x = 0$

$$\Rightarrow f(0) = \frac{2 \cdot \sinh(\pi)}{\pi} \cdot \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}\right) + \frac{\sinh(\pi)}{\pi}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{2 \cdot \sinh(\pi)} - \frac{1}{2}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{2 \cdot \sinh(\pi)} - \frac{1}{2} + \frac{1}{2} = \frac{\pi}{2 \cdot \sinh(\pi)}$$

set $x = \pi$

$$\Rightarrow f(\pi) = \frac{2\sinh(\pi)}{\pi}\left(\left(\sum_{n=1}^{\infty} \frac{1}{1+n^2}\right) + \frac{1}{2}\right)$$

$$\cancel{\Rightarrow \sum_{n=2}^{\infty} = \left(\frac{e^{\pi}}{2} - \frac{\sinh(\pi)}{\pi}\right)\cdot\frac{\pi}{2\cdot\sinh(\pi)} - \frac{1}{2}}$$

$$\cancel{= \frac{e^{\pi}\cdot\pi}{4\cdot\sinh(\pi)} - 1}$$

$$\sum_{n=2}^{\infty} \frac{1}{1+n^2} = \left(\frac{e^{\pi} - e^{-\pi}}{2} - \frac{\sinh(\pi)}{\pi}\right)\cdot\frac{\pi}{2\cdot\sinh(\pi)} - \frac{1}{2}$$

$f(\pi)$

$$= \frac{\sinh(\pi)(\pi - 1)}{\pi}\cdot\frac{\pi}{2\cdot\sinh(\pi)} - \frac{1}{2}$$

$$= 0.57079$$

Supplementary G

$$f(x) = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{(-1)^n 8(\pi^2 n^2 - 6)}{n^4} \cos(nx)$$

$$\Rightarrow \left(f(x) - \frac{\pi^4}{5}\right)/8 = \sum_{n=1}^{\infty} \frac{(-1)^n(\pi^2 n^2 - 6)}{n^4} \cos(nx)$$

$\underline{x = \pi}$ $\Rightarrow \left(f(\pi) - \frac{\pi^4}{5}\right)/8 = \underline{\underline{\sum_{n=1}^{\infty} \frac{\pi^2 n^2 - 6}{n^4}}}$

$\Rightarrow$ i)

Answer: $\left(f(\pi) - \frac{\pi^4}{5}\right)/8 = \underline{\underline{\frac{\pi^4}{10}}}$

ii) In the given fourier series, we know that:

$$b_n = \frac{(-1)^n \cdot 8(\pi^2 n^2 - 6)}{n^4}, \quad a_0 = \frac{\pi^4}{5}, \quad a_n = 0$$

$$\Rightarrow b_n^2 = 64 \cdot \frac{(\pi^4 n^4 - 12\pi^2 n^2 + 36)}{n^8}$$

We wish ~~to~~ to find $\sum_{n=1}^{\infty} \left(\frac{\pi^4 n^4 - 12\pi^2 n^2 + 36}{n^8}\right)$

Parsevals identity:

$$2a_0^2 + \sum_{n=1}^{\infty} b_n^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

$$\Rightarrow \frac{2\pi^8}{25} + 64 \sum_{n=1}^{\infty} \left(\frac{\pi^4 n^4 - 12\pi^2 n^2 + 36}{n^8}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{\pi^4 n^4 - 12\pi^2 n^2 + 36}{n^8}\right) = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} x^8 dx - \frac{2\pi^8}{25}\right)/64$$

$$= \underline{\underline{\frac{\pi^8}{450}}}$$