TMA4120 - Høst 2016

Øving 6

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12.1.3: $u_{tt} = c^2 u_{xx}$

$u = \cos(4t)\sin(2x)$

$\Rightarrow u_{tt} = -16\cos(4t)\sin(2x)$

$u_{xx} = -4\cos(4t)\sin(2x)$

$\Rightarrow u_{tt} = u_{xx} \cdot c^2, \quad c^2 = 4$

12.1.9: $u_t = c^2 u_{xx}$

$u = e^{-\pi^2 t}\cos(25x)$

$\Rightarrow u_t = -\pi^2 e^{-\pi^2 t}\cos(25x)$

$u_{xx} = e^{-\pi^2 t} \cdot (-625)\cos(25x)$

$\Rightarrow u_t = c^2 u_{xx}, \quad c^2 = \frac{625}{\pi^2}$

12.1.15: $u(x,y) = a\ln(x^2+y^2) + b$

Laplace eq: $u_{xx} + u_{yy} = 0$

$u_x = \frac{a}{x^2+y^2} \cdot 2x \Rightarrow u_{xx} = \frac{2a}{x^2+y^2} + 2ax \cdot \frac{1}{-(x^2+y^2)^2} \cdot 2x$

$= \frac{-2a(x^2+y^2) + 4ax^2}{-(x^2+y^2)^2} = \underline{\frac{2a(x^2-y^2)}{-(x^2+y^2)^2}}$

$u_{yy} = \frac{-2a(x^2+y^2)+4ay^2}{-(x^2+y^2)^2} = \frac{2a(y^2-x^2)}{-(x^2+y^2)^2}$

$u_{yy} + u_{xx} = \frac{2ay^2 - 2ax^2 + 2ax^2 - 2ay^2}{-(x^2+y^2)^2} = \underline{0}$

Boundry conditions:

I: $x^2+y^2 = 1 \Rightarrow u = 110$

II $x^2+y^2 = 100 \Rightarrow u = 0$

$\ln(1) = 0 \Rightarrow \underline{\underline{110 = b}}$

$$0 = a \cdot \ln(100) + b = a\ln(100) + 110$$

$$\Rightarrow \underline{\underline{a = \frac{-110}{\ln(100)}}}$$

12.3.1

Wave eq. sol. $u_n(x,t) = (B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)) \sin \frac{n\pi}{L} x$

where $\lambda_n = \frac{cn\pi}{L} = cp$

fundamental mode: $n = 1 \Rightarrow \lambda = \frac{c\pi}{L}$

$\Rightarrow$ The longer the string ($L \to \infty$), the smaller is the fud. frequency and v. v.

p = mass per unit length

$cp = \sqrt{\frac{T}{p}} \cdot p$, $c^2 = \frac{T}{p}$ $\Rightarrow \lambda_n = \cancel{\sqrt{Tp}}\ \sqrt{\frac{T}{p}} \cdot \frac{n\pi}{L}$

$\Rightarrow$ The bigger the mass, the ~~bigger~~ smaller the frequency and v. v.

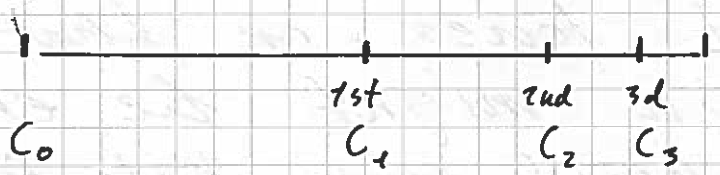
If we double the tension, the frequency is increased by $\sqrt{2}$.

The controbass is a low pitch instrument compared to the violin. So, the length of the strings needs to be bigger in order to produce a lower freq. ergo: a bigger instrument.

However, contrabass strings are typically thicker and more massive than violin strings.

So even without increased length the instrument should have a lower pitch on all its strings.

But, we know that the progression of overtones relative to the length of the string is not linear.



Octaves of different order relative to length of the string.

If an increase of one octave is desired, the string needs to be reduced by half (pressed in the middle).

This means that the "distance" between two tones on a heavy string is larger than on a light string.

So, the contrabass is larger NOT because the string needs to be longer in order to produce a lower pitch since the increased mass of the string already produces a lower pitch. The instrument is larger because if the string is longer, more distinct tones can be produced on the same string.

Basically, if you put a thick contrabass string on a violin and set the tension so that the "loose" (finger not pressing the string anywhere) string produces the appropriate frequency, you will only be able to produce 4-5 distinct tones on that string, which might be enough to play "music" from Katy Perry but not enough to play Beethoven.

12.3.7

$L = 1, \quad c^2 = 1, \quad k = 0.01$

$f(x) = kx(1-x)$

$\Rightarrow u(x,0) = kx(1-x)$

$u_t(x,0) = 0$

$u_{tt} = c^2 u_{xx}, \quad u(0,t) = 0, \quad u(L,t) = 0$

$u_{tt} = F(x)G''(t), \quad u_{xx} = F''(x)G(t)$

$\Rightarrow \frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = K_0$

$\Rightarrow G''(t) - c^2 K_0 G(t) = 0$

$F''(x) - K_0 F(x) = 0$

$K_0 = -p^2$

$\Rightarrow F''(x) + p^2 F(x) = 0$

$\Rightarrow F(x) = A\cos(px) + B\sin(px)$

$F(0) = 0 \Rightarrow \underline{A = 0}, \quad F(L) = 0 = B\sin(pL)$

$\Rightarrow \sin(pL) = 0 \Rightarrow pL = n\pi \Rightarrow \underline{p = \frac{n\pi}{L}}, \; n \in \mathbb{Z}$

$B = 1 \Rightarrow \underline{F_n(x) = \sin\left(\frac{n\pi}{L}x\right)}$

$G''(t) + c^2 p^2 G(t) = 0, \quad \lambda_n = cp = \frac{cn\pi}{L}$

$\Rightarrow G_n(t) = B_n\cos(\lambda_n t) + D_n\sin(\lambda_n t)$

$\Rightarrow u_n(x,t) = (B_n\cos(\lambda_n t) + D_n\sin(\lambda_n t)) \cdot \sin\left(\frac{n\pi}{L}x\right)$

$\Rightarrow \underline{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)}$

$u(x,0) = kx(1-x) \Rightarrow \sum_{n=1}^{\infty} u_n(x,0) = kx(1-x)$

$\Rightarrow kx(1-x) = \sum_{n=1}^{\infty} B_n \cancel{\cos(n\pi 0)} \cdot \sin(n\pi x)$

$\Rightarrow B_n = 2\int_0^1 kx(1-x) \cdot \sin(n\pi x)\,dx$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\cdot \sin\left(n\frac{\pi}{L}x\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{cn\pi}{L}t\right)\cdot \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}(x-ct)\right)\cdot\frac{1}{2} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}(x+ct)\right)\cdot\frac{1}{2}$$

Observe that this is a sum of two fourier sine series of functions $\frac{1}{2}f(x-ct)$ and $\frac{1}{2}f(x+ct)$ on a period $[0, L]$

$$\cancel{\Rightarrow u(x,t) = \frac{1}{2}\left(k(x+ct)(1-(x+ct)) + k(x-ct)(1-(x-ct))\right)}$$

Since we obtained a sine series the function $f^*(x)$ must be odd on period $[-L, L]$

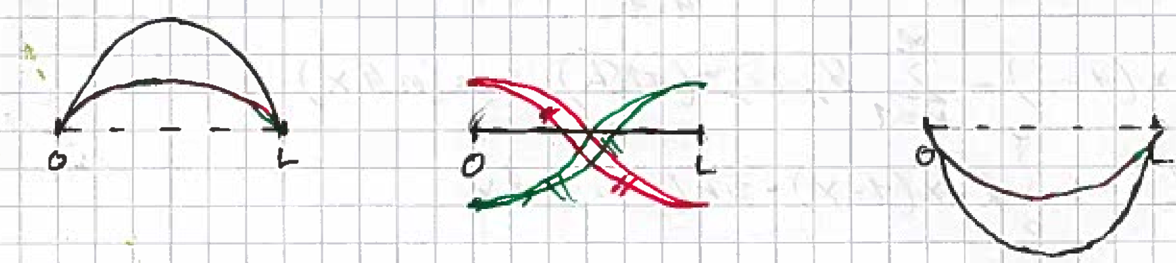
$$\Rightarrow f^*(x) = \begin{cases} -f(x+L), & -L < x < 0 \\ f(x), & 0 < x < L \end{cases}$$

$$\text{So} \Rightarrow \underline{\underline{u(x,t) = \frac{1}{2}\left(f^*(x-ct) + f^*(x+ct)\right)}}$$

$f(x)$ is a second degree polynomial so its periodic odd extension $f^*(x)$ is a sine wave with period ~~2π~~ $2L$

So the solution is a sum of two identical sine waves traveling in opposite directions as $t \to \infty$.

$t=0$ $\quad$ $t = \frac{\pi}{2}$ $\quad$ $t = \pi$



Ergo: an oscilating string with amplitude $\frac{K}{4}$ and frequency $\frac{1}{2\pi}$ Hz

12.3.15:

$$u_{tt} = -c^2 u_{xxxx}$$

$$u = F(x)G(t) \Rightarrow F(x)G''(t) = -c^2 G(t) F^{(4)}(x)$$

$$\Rightarrow \frac{F^4(x)}{F(x)} = \frac{G''(t)}{-c^2 G(t)} = \cancel{\frac{R''(x)}{c}} \text{ const}$$

since changing ~~the~~ one of variables only affects one function but not the other!

const $= \beta^4$

$$\Rightarrow F^4(x) - \beta^4 F(x) = 0$$

$$G''(t) + c^2\beta^4 G(t) = 0$$

$$F_1(x) = A\cos(\beta x) + B\sin(\beta x)$$

$$\Rightarrow F_1^4(x) = \beta^4 (A\cos(\beta x) + B \sin(\beta x))$$

$$\Rightarrow F_1^4(x) - \beta^4 F_1(x) = 0$$

$$F_2(x) = \cancel{A e^{\beta x} + B e^{\beta x}} (C\cosh(\beta x) + D\sinh(\beta x)$$

$$\Rightarrow F_2^4(x) = \beta^4 (\cosh(\beta x)\cdot C + D\cdot \sinh(\beta x))$$

$$\Rightarrow F_2^4(x) - \beta^4 F_2^4(x) = 0$$

$$\Rightarrow \underline{F(x) = F_1(x) + F_2(x)} = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x)$$

$$G(t) = a\cos(\beta^2 c t) + b\sin(\beta^2 c t) \Rightarrow$$

$$G''(t) = -c^2\beta^4 \cdot G(t) \Rightarrow \underline{\underline{G''(t) + c^2\beta^2 G(t) = 0}}$$

12.3.16 8

$$u(0,t) = u(L,t) = 0$$
$$u_{xx}(0,t) = u_{xx}(L,t) = 0$$

$$\Rightarrow F(0) = 0, \quad F(L) = 0$$
$$F''(0) = 0, \quad F''(L) = 0$$

$$F(x) = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\overset{\sinh}{\cancel{\cosh}}(\beta x)$$
$$F''(x) = -\beta^2 A\cos(\beta x) - \beta^2 B\sin(\beta x) + C\beta^2\cosh(\beta x) + D\beta^2\sinh(\beta x)$$

$$F(0) = A + C = 0 \Rightarrow A = -C$$
$$F''(0) = -\beta^2 A + C\beta^2 = 0 \Rightarrow A = C \Rightarrow A = C = 0$$

$$F(L) = \cancel{\cosh(} B\sin(\beta L) + D\sinh(\beta L)$$
$$F''(L) = -\beta^2\sin(L\beta) B + \beta^2 D\sinh(\beta L)$$

$$\Rightarrow \sin(L\beta) = -\sin(L\beta)$$
$$\Rightarrow L\beta = n\pi, \; n \in \mathbb{Z} \Rightarrow \beta = \frac{n\pi}{L}$$

$$\Rightarrow u(x,t) = F(x)\cdot G(t), \quad \text{set } B = 1, \cancel{D=1}$$

$$\Rightarrow u_n(x,t) = \left(\sin\left(\frac{n\pi}{L}x\right) + D\sinh\left(\frac{n\pi}{L}x\right)\right)\cdot G_n(t)$$

$$G_n(t) = a_n\cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right) + b_n\overset{\sin}{\cancel{\cos}}\left(c\left(\frac{n\pi}{L}\right)^2 t\right)$$

However: $F(L) = 0 = F''(L)$

$$\Rightarrow -\beta^2\sin(n\pi) + \beta^2 D\sinh(n\pi) = \sin(n\pi) + D\sinh(n\pi)$$
$$\Rightarrow \beta^2 D = D$$
$$\Rightarrow \frac{2\pi}{L} D = D$$
$$\Rightarrow D = 0$$
$$\Rightarrow u_n(x,t) = \sin\left(\frac{n\pi}{L}x\right)\cdot G_n(t)$$

12.3.17

$$u(x,0) = f(x) = x(L-x)$$

$$\cancel{\Rightarrow F(x) G(0) = x(L-x)}$$

$$\cancel{G(t) = a\cos(c\beta^2 t) + b\sin(c\beta^2 t)}$$

$$\cancel{\Rightarrow G(0) = a}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \cancel{\cos(c\beta^2 t)} \cdot F(x) = f(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \cancel{\cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right)} \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi}{L}x\right) = f(x)$$

$$u_t(x,0) = 0$$

$$\Rightarrow b_n = 0 \Rightarrow G_n(t) = a_n \cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right)$$

$$\Rightarrow a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Because $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$

$$\Rightarrow a_n = -\frac{L^2 \cdot 2}{\pi^3 n^3}(\pi n \sin(\pi n) + 2\cos(\pi n) - 2)$$

$$= -\frac{L^2 \cdot 2}{\pi^3 n^3}(2((-1)^n - 1))$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} -\frac{L^2 \cdot 4}{\pi^3 \cdot n^3}((-1)^n - 1) \cdot \cos\left(c\left(\frac{n\pi}{L}\right)^2 t\right) \cdot \sin\left(\frac{n\pi}{L}x\right)$$

$$= \frac{8L^2}{\pi^3}\left(\cos\left(\left(\frac{\pi}{L}\right)^2 ct\right)\sin\left(\frac{\pi}{L}x\right) + \frac{1}{3^3}\cos\left(\left(\frac{3\pi}{L}\right)^2 ct\right)\sin\left(\frac{3\pi}{L}x\right)\dots\right)$$

12.6.5: $f(x) = \sin(0.1\pi x) = u(x,0)$

$\cancel{u(x,0) = u(x,L) = 0}$

$u(0,t) = u(L,t) = 0$

Insulation $\Rightarrow u_t(x,0) = 0$

Heat eq: $u_t = c^2 u_{xx}$

$\Rightarrow F(x)G'(t) = c^2 G(t) F''(x)$

$\Rightarrow \frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k = -p^2$

$\Rightarrow G'(t) + p^2c^2 G(t) = 0$

$F''(x) + p^2 F(x) = 0$

$\Rightarrow F(x) = A\cos(px) + B\sin(px)$

$F(0) = \underline{0 = A}$

$\Rightarrow F(L) = 0 = B\sin(pL)$, $B = 1 \Rightarrow pL = n\pi,\ n \in \mathbb{Z}$

$\Rightarrow \underline{p = \frac{n\pi}{L}}$

$\Rightarrow F(x) = \sin\left(\frac{n\pi}{L}x\right)$

$p^2c^2 = \lambda_n^2$, $\lambda_n = \frac{cn\pi}{L}$

$\Rightarrow G_n'(t) + \lambda_n^2 G_n(t) = 0$

$\Rightarrow G_n(t) = B_n \cdot e^{-\lambda_n^2 t}$

$\Rightarrow \underline{u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi}{L}x\right)}$

$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = \sin(0.1\pi x)$

$\Rightarrow B_n = \frac{2}{L}\int_0^L \sin(0.1\pi x)\sin\left(\frac{n\pi}{L}x\right)dx$, $\cancel{L = 0.1}$

$\cancel{\Rightarrow B_n = \frac{20}{4}\int_0^{0.1} \sin(0.1\pi x)\sin(10n\pi x)\,dx}$

$\cancel{= 20\left[\frac{\sin(x(0.1\pi - 10\pi n))}{2(0.1\pi - 10\pi n)} - \frac{\sin(x(0.1\pi + 10\pi n))}{2(0.1\pi + 10\pi n)}\right]_0^{0.1}}$

$L = 10\,cm$

$$\Rightarrow B_n = 0.2 \int_0^{10} \sin(0.1\pi x)\sin(0.1 n\pi x)\,dx$$

$$= 0.2 \left[\frac{\sin(x(0.1\pi(1-n)))}{0.2\pi(1-n)} - \frac{\sin(0.1x\pi(1+n))}{0.2\pi(1+n)} \right]_0^{10}$$

$$= \underline{\frac{\sin(\pi(1-n))}{\pi(1-n)} - \frac{\sin(\pi(1+n))}{\pi(1+n)}}$$

For $n > 1$, $n \in \mathbb{Z}$, B_n is 0

For $n = 1$, $-\frac{\sin(\pi(1+n))}{\pi(1+n)} = 0$

While $\lim_{n\to 1} \frac{\sin(\pi(1-n))}{\pi(1-n)} = \lim_{n\to 1} \frac{-\pi\cos(\pi(1-n))}{-\pi}$ (L'Hopital)

$= 1$

So, $B_1 = \underline{1}$, $B_2 = B_3 = \ldots = \underline{0}$

Ergo:

$$u(x,t) = e^{-\lambda^2 t}\sin\left(\frac{\pi}{L}x\right)$$

$\lambda^2 = c^2\pi^2/L^2$, $c^2 = (K/(\sigma p))$

k = thermal cond. = 1.04 cal/(cm sec °C)

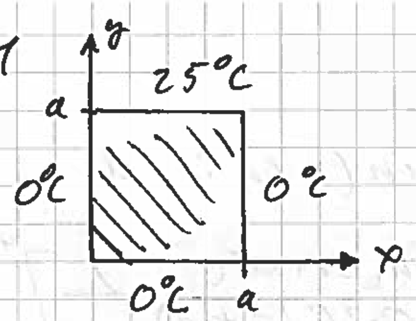
σ = specific heat. = 0.056 cal/(g °C)

p = density = 10,6 g/cm³

$$\Rightarrow u(x,t) = \cancel{e^{0.5t}\sin(0.1\pi x)}$$

$$\underline{\underline{= e^{-1.752\frac{\pi^2}{200}} \cdot \sin(0.1\pi x)}}$$

17.6.21



$a = 24$

12

$u_t = c^2(u_{xx} + u_{yy})$

$u(x,y) = G(y) \cdot F(x)$

Steady state solution => $u_t = 0$ (temperature doesnt change)

Equation reduces to $0 = c^2(u_{xx} + u_{yy})$

$\Rightarrow c^2(F''(x)G(y) + F(x)G''(y)) = 0$

$\Rightarrow \dfrac{F''(x)}{F(x)} = -\dfrac{G''(y)}{G(y)} = -k$ (~~if k = 0 or~~

$\Rightarrow F''(x) + kF(x) = 0$

$G''(y) - kG(y) = 0$

$\Rightarrow F(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$ (try solution)

$\Rightarrow F''(x) + kF(x) = 0$ (fits!)

Boundry conditions: $F(0) = 0$, $F(a) = 0$, $G(0) = 0$, $G(a) = 25$

$\Rightarrow \underline{0 = A}$, $F(a) = 0 = B\sin(\sqrt{k}\,a)$

$\Rightarrow \sin(\sqrt{k}a) = 0 \Rightarrow \sqrt{k}a = n\pi,\ n \in \mathbb{Z}$

$\Rightarrow \underline{k = \left(\frac{n\pi}{a}\right)^2}$, set $B = 1$!

$\Rightarrow G_n''(y) - \left(\frac{n\pi}{a}\right)^2 G_n(y) = 0$

$G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$ (try solution)

$\Rightarrow G_n''(y) - \left(\frac{n\pi}{a}\right)^2 G_n(y) = 0$ (fits)

$G_n(0) = 0 = A_n + B_n \Rightarrow A_n = -B_n$

$G_n(a) = 25 = A_n e^{n\pi} + B_n e^{-n\pi}$

$\Rightarrow \underline{25 = A_n(e^{n\pi} - e^{-n\pi})}$

$\Rightarrow \underline{G_n(y) = A_n \cdot 2 \cdot \sinh(n\pi y/a)}$

Set $D_n = 2A_n \Rightarrow$

$u_n(x,y) = \sum_{n=1}^{\infty} G_n(y)F_n(y) = \sum_{n=1}^{\infty} D_n \sinh(n\pi y/a)\cdot \sin(n\frac{\pi x}{a})$

$G_n(a) = 25$

$\Rightarrow u(x,a) = 25 = \sum_{n=1}^{\infty} D_n \sinh(n\pi)\cdot \sin(\frac{n\pi x}{a})$

$=$ set $D_n \sinh(n\pi) = R_n$

$\Rightarrow 25 = \sum_{n=1}^{\infty} R_n \sin(\frac{n\pi x}{a}) = u(x,a)$

So R_n is fourier coef. of $f(x) = 25$

$\Rightarrow R_n = \frac{2}{a}\int_0^a 25 \cdot \sin(\frac{n\pi x}{a})\,dx$

$= \frac{2}{a}\cdot 25 \left[\left(\frac{n\pi}{a}\right)^{-1} \cdot -\cos\left(\frac{n\pi x}{a}\right)\right]_0^a$

$\cancel{= \frac{50n\pi}{a^2}(-\cos(n\pi)+1) = \frac{50n\pi}{a^2}\cdot(1-(-1)^n)}$

$\Rightarrow D_n = \cancel{\frac{50n\pi\cdot(1-(-1)^n)}{a^2\sinh(n\pi)}}$

$= \frac{2\cdot 25}{n\pi}(1-(-1)^n)$

$\Rightarrow D_n = \frac{50(1-(-1)^n)}{n\pi\sinh(n\pi)}$

$\Rightarrow \underline{u(x,y) = \sum_{n=1}^{\infty} D_n \sinh\left(\frac{n\pi y}{a}\right)\cdot \sin\left(\frac{n\pi x}{a}\right)}$

Notice that D_n is $\underline{0}$ for even n and $\underline{2}$ for odd n, we can rewrite to: $(a=24)$

$$\underline{u(x,y) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh((2n-1)\pi)}\cdot\sinh\left(\frac{(2n-1)\pi y}{24}\right)\cdot\sin\left(\frac{(2n-1)\pi x}{24}\right)}$$

Supplementary O:

a) If u_1 & u_2 are sols to (*), their sum $(u_1 + u_2)$ and $(u_1 + cu_2$, $c = -1 \Rightarrow u_1 + cu_2 = u_1 - u_2)$ must also be a solution to (*).

Boundry conditions for (*):

$u(0,t) = a$, $u(1,t) = b$

$\Rightarrow u_1(0,t) = u_2(0,t) = a$

$u_1(1,t) = u_2(1,t) = b$

Assume there exist a function $g(x,t)$ such that: $g(0,t) = g(1,t) = \underline{1}$

then:

$$u_1(0,t) - ag(0,t) = 0 = u_3(0,t)$$
$$u_1(1,t) - bg(1,t) = 0 = u_3(1,t)$$
$$u_2(0,t) - ag(0,t) = 0 = u_4(0,t)$$
$$u_2(1,t) - bg(1,t) = 0 = u_4(1,t)$$

u_3 and u_4 satisfy the boundry conditions of (**), by principle of superposition (which <u>holds</u> for both problems since they are <u>linear</u>) u_1 and u_2 must also be solutions to (**) as well as their <u>sums.</u>

b) $v(x,t) = u(x,t) - (a + (b-a)x)$
$= u(x,t) - g(x,t)$

From a) it follows that if $v(x,t)$ is a sol to (**), it is also a sol to (*).
$u(x,t)$ is a sol., so we have to prove that $g(x,t)$ is a sol. to (**) by proving that it is a sol to (*)

$g_t(x,t) = 0$, $g_{xx}(x,t) = 0$

$\Rightarrow \underline{\underline{g_t - g_{xx} = 0}}$

Boundry: $g(0,t) = (a + (b-a)\cdot 0) = \underline{\underline{a}}$
$g(1,t) = (a + (b-a)\cdot 1) = \underline{\underline{b}}$

c) $a = -1$, $b = 1$
Boundry: $u(x,0) = \sin(\pi x)$, $x \in [0,1]$
$u(0,t) = -1$, $u(1,t) = 1$

$\Rightarrow u_t - u_{xx} = 0 \Rightarrow u_t = u_{xx}$

$\Rightarrow G'(t)F(x) = G(t)F''(t)$

$\Rightarrow \dfrac{G'(t)}{G(t)} = \dfrac{F''(x)}{F(x)} = k$

$\Rightarrow G'(t) - G(t) = 0$,
$F''(x) - kF(x) = 0$

$\Rightarrow F(x) = Ae^{\sqrt{k}t} + Be^{-\sqrt{k}t}$

$F(0) = -1 \Rightarrow A + B = -1 \Rightarrow B = -1 - A$

$\Rightarrow F(1) = Ae^{\sqrt{k}} - e^{-\sqrt{k}} - Ae^{-\sqrt{k}} = e^{-\sqrt{k}} \neq 1$

Unless $K=0$ which is a trivial solution we are not interested in

Which means that k must be negative: $K = -p^2$

from b)

$$V(x,t) = u(x,t) - (a + (b-a)x)$$

$\Rightarrow$ $u(x,t) = V(x,t) + (a + (b-a)x)$

Where $V(x,t)$ is a sol. to $(**)$.

$V(x,t)$:

$$V(0,t) = 0 \;,\; V(1,t) = 0$$

$\Rightarrow$ $F(0) = 0 \Rightarrow A = 0 \qquad (F(x) = A\cos(px) + B\sin(px))$

$F(1) = 0 \Rightarrow B\sin(p) = 0$

$\Rightarrow \underline{p = n\pi} \;,\; B = 1$

$\Rightarrow F(x) = \sin(px) = \sin(n\pi x)$

$$G'(t) + p^2 G(t) = 0$$

$\Rightarrow G_n(t) = A_n e^{-p^2 t}$

$\Rightarrow V(x,t) = \sum_{n=1}^{\infty} A_n e^{-p^2 t} \cdot \sin(px) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \cdot \sin(n\pi x)$

$V(x,0) = \sin(\pi x)$

$\Rightarrow \sin(\pi x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$

$\Rightarrow A_n = 2\int_0^1 \sin(\pi x)\sin(n\pi x)\,dx = 0$

$\Rightarrow u(x,t) = (a + (b-a)x)$

$a = -1,\; b = 1$

$\Rightarrow \underline{\underline{u(x,t) = -1 + 2x}}$

Not certain, please comment!

Supplementary N

$4 z_0 : |e^{z_0}| = 5$

$\Rightarrow z_0 = a + b i \Rightarrow |e^{a+bi}| = 5$

$\Rightarrow |e^{a} \cdot e^{bi}| = 5 \Rightarrow e^{a} = 5 \Rightarrow a = \ln(5)$

$\Rightarrow |e^{2z_0 + 3i}| = |e^{2a + 2bi + 3i}|$

$= |e^{2a} \cdot e^{i(3+2b)}| = e^{2a} = \underline{\underline{25}}$

Supplementary O

a) Because of completly different boundary conditions, sum of solutions to (*) is only a solution to only (*) and not to (**). $u_1 - u_2 + u_1 + u_2 = 2u_1$

u_1 is a solution, so its multiplum is a solution, therefore, if $g_1 + g_2 = 2u_1$ is a solution, g_1 and g_2 must also be solutions

$g_1 = u_1 - u_2$, $g_2 = u_1 + u_2$

So: $u_1 - u_2$ and $u_1 + u_2$ are solutions to (*). (NOT to (**))

Superposition principle holds for both problems, but since we are not given any solutions for (**), no statements can be made regarding the nature of solutions.

because both are linear