

# 1 Analytic Solution

Typically, in fluid dynamics, the scalar function known as the velocity potential denoted by  $\varphi$ , is what is desired. Once the velocity potential is known, the system is solved because  $\vec{\nabla} \varphi = \mathbf{u}$ , and in the case of a free surface, the deformation of the surface can readily be found from the velocity potential as well. The velocity potential will be found in the coming sub-chapters, and then the energy deposited into the neutron star will be calculated.

## 1.1 Eigenfunctions of the Laplacian

The first step in analytically solving for the velocity potential will be to find the eigenfunctions of Laplace's equation,  $\nabla^2 \varphi = 0$ , since the velocity potential is conservative. Since this is a partial differential equation we will assume the solution is the product of univariate functions,  $\varphi = f(r)g(\theta)h(z)T(t)$ . Expanding the Laplacian for cylindrical systems gives

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \left( \frac{\partial^2 \varphi}{\partial \theta^2} \right) + \frac{\partial^2 \varphi}{\partial z^2} = 0.$$

The temporal component can be divided out since the Laplacian does not involve time, and will be found later on. Substituting in our assumed form, and dividing by  $\varphi$ , yields

$$\frac{1}{f r} \frac{\partial}{\partial r} (r f') + \frac{1}{r^2} \frac{g''}{g} + \frac{h''}{h} = 0. \tag{1}$$

Where primes denote the derivative with respect to the function's only variable. Notice that the function  $h(z)$  has been separated from both  $f(r)$ , and  $g(\theta)$ , and since each function is univariate, it must be the case that the last term is constant,

$$\frac{h''}{h} = k^2.$$

We cleverly force this constant to be positive to satisfy the boundary conditions, namely, that  $h(-\infty) = 0$ . The most general solution to this is of course exponentials,

$$h(z) = Ae^{kz} + Be^{-kz}.$$

Since the velocity potential has to vanish at negative infinity, this is simplified to

$$h(z) = e^{kz}, \tag{2}$$

without loss of generality we can assume the constant is one, since it can be absorbed into  $T(t)$ . By substituting in  $k^2$  and multiplying through by  $r^2$ , (1) becomes

$$\frac{r}{f} \frac{\partial}{\partial r} (rf') + k^2 r^2 + \frac{g''}{g} = 0, \tag{3}$$

and once again,  $g(\theta)$  has been separated from  $f(r)$ , and so the last term must be constant –

$$\frac{g''}{g} = -\mu^2.$$

The constant is forced to be negative since  $g(\theta)$  is expected to be periodic, and not exponential. Of course, the general solution to this differential equation is

$$g(\theta) = A \sin(\mu\theta) + B \cos(\mu\theta).$$

However, our system is cylindrically symmetric; there is no way to differentiate one value of  $\theta$  to another, thus, it is expected that  $\varphi$  has no  $\theta$  dependence, and consequentially,  $g(\theta)$

must be constant, the only way this is satisfied is if  $\mu = 0$ , so that

$$g(\theta) = 1. \quad (4)$$

By substituting  $\mu = 0$  into (3), and multiplying through by  $f$  we obtain

$$r \frac{\partial}{\partial r}(rf') + (k^2 r^2 - 0^2)f = 0,$$

which indeed is Bessel's differential equation of order 0. Therefore,

$$f(r) = AJ_0(kr) + BY_0(kr),$$

but, once again,  $B$  must equal zero since  $Y_0(0)$  is not finite which is unphysical. The radial part thus simplifies to

$$f(r) = J_0(kr). \quad (5)$$

By combining (2), (4), and (5) we find that the velocity potential is

$$\varphi \propto e^{kz}T(t)J_0(kr),$$

where  $k$  is the eigenvalues. Clearly though, the only restriction on  $k$  is that it must be a non-negative real number, all of which are a valid solution to Laplace's equation. Therefore, the total solution must be the sum of all of these, or since  $k$  is valid within an interval, we have the integral form

$$\varphi = \int_0^\infty e^{kz}T(t)J_0(kr)dk. \quad (6)$$

The temporal component can be solved for using equations of fluid dynamics.

## 1.2 Temporal Component of the Velocity Potential

The time component of the velocity potential comes from the specific problem. In our case by linearizing, and assuming the waves have a small amplitude, we have

$$\left( \frac{\partial \varphi}{\partial t} + (g\eta + \Phi) \right) \Big|_{z=0} = 0,$$

where  $\eta$  denotes the deformation of the surface. This comes from the pressure condition at the surface, and in our case, has the addition of the gravitational potential. \*\*\*Fluids book eqn 3.21 3.22\*\*\* By taking the time derivative and using the relation that

$$\frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial z} \tag{7}$$

on the surface \*\*\*, we obtain

$$\left( \frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z} + \frac{\partial \Phi}{\partial t} \right) \Big|_{z=0} = 0. \tag{8}$$

In order to solve this partial differential equation, we must write the gravitational potential as an infinite sum of Bessel functions to match the form of  $\varphi$ . We can do so using the Hankel transform using the fact that it is self-reciprocal,

$$\begin{aligned} \frac{\partial \Phi}{\partial t} \Big|_{z=0} &= \int_0^\infty \left( \mathcal{H} \frac{\partial \Phi}{\partial t} \Big|_{z=0} \right) (k) J_0(kr) k \, dk, \\ &= Gmv^2t \int_0^\infty \left( \mathcal{H} \frac{1}{(r^2 + v^2t^2)^{3/2}} \right) (k) J_0(kr) k \, dk.^\S \end{aligned}$$

Applying the Hankel transform \*\*\* gives

$$\begin{aligned}\left.\frac{\partial\Phi}{\partial t}\right|_{z=0} &= Gmv^2t \int_0^\infty \frac{1}{|vt|} e^{-k|vt|} J_0(kr) k dk, \\ &= Gmv \operatorname{sgn}(t) \int_0^\infty e^{-kv|t|} J_0(kr) k dk,\end{aligned}$$

which we can now substitute into (8) to obtain,

$$\begin{aligned}\int_0^\infty J_0(kr) \ddot{T}(t) dk + g \int_0^\infty k J_0(kr) T(t) dk + Gmv \int_0^\infty \operatorname{sgn}(t) e^{-kv|t|} J_0(kr) k dk &= 0, \\ \text{or, } \int_0^\infty \left[ \frac{\ddot{T}(t)}{k} + gT(t) + Gmv \operatorname{sgn}(t) e^{-kv|t|} \right] J_0(kr) k dk &= 0.\end{aligned}$$

But, this is nothing more than the Hankel transform of the differential equation for  $T$ . By taking the Hankel transform of both sides we can remove the integral, giving an ordinary differential equation for the time component:

$$\frac{\ddot{T}(t)}{k} + gT(t) + Gmv \operatorname{sgn}(t) e^{-kv|t|} = 0.$$

Clearly, the homogeneous solution is  $T(t) = A \cos(\omega_k t) + B \sin(\omega_k t)$  with  $\omega_k^2 = gk$ . The form of this differential equations suggests the particular solution should take the form  $T(t) = C e^{-kv|t|}$ . Substituting this in yields

$$C (k^2 v^2 \operatorname{sgn}^2(t) e^{-kv|t|} + gk e^{-kv|t|}) + Gmvk \operatorname{sgn}(t) e^{-kv|t|} = 0.$$

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<sup>§</sup>Note that  $\int_0^\infty \sqrt{r} (r^2 + v^2 t^2)^{-3/2} dr = \Gamma^2(3/4) (\pi v^3 t^3)^{-1/2} < \infty$  for  $t \neq 0$  which is not concerning since this is true almost everywhere, and so the condition in Theorem \*\*\* is satisfied.

Now, by rearranging we find

$$\begin{aligned} C &= \frac{-Gmvk \operatorname{sgn}(t)}{k^2 v^2 + gk}, \\ &= \frac{Gmv}{g} \frac{-\operatorname{sgn}(t)}{1 + kv^2/g} \end{aligned}$$

as the coefficient, and,

$$T(t) = \frac{Gmv}{g} \frac{1}{1 + kv^2/g} \left( -\operatorname{sgn}(t) e^{-kv|t|} \right) + A \cos(\omega_k t) + B \sin(\omega_k t)$$

as the full time component of the velocity potential. We can now apply the boundary conditions to find  $A$ , and  $B$ . Physically, we expect  $T(t) \in C^1(-\infty, \infty)$ , furthermore, we only expect the sinusoidal terms to contribute at times greater than zero, thus finally, we procure

$$\begin{aligned} T(t) &= \frac{Gmv}{g} \frac{1}{1 + kv^2/g} \left( -\operatorname{sgn}(t) e^{-kv|t|} + 2H(t) \cos(\omega_k t) \right) dk, \text{ and,} \\ \varphi &= \frac{Gmv}{g} \int_0^\infty \frac{J_0(kr) e^{kz}}{1 + kv^2/g} \left( -\operatorname{sgn}(t) e^{-kv|t|} + 2H(t) \cos(\omega_k t) \right) dk. \end{aligned}$$

Another way the velocity potential can be expressed is as the Hankel transform of the time component,  $\varphi = \left( \mathcal{H} e^{kz \frac{T(t;k)}{k}} \right)(r, z, t)$ , which will be of particular importance in Chapter 1.4 for the energy calculation.

### 1.3 Deformation of the Surface

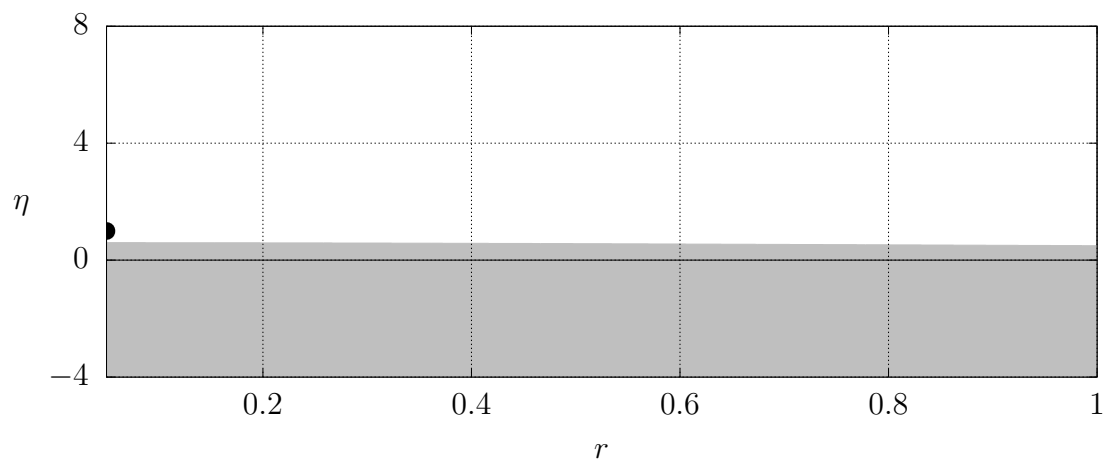
From (7) we can isolate for  $\eta$ :

$$\begin{aligned}
\eta &= \int_0^t \frac{\partial \varphi}{\partial z} \Big|_{z=0} dt \\
&= \int_0^t \frac{Gmv}{g} \int_0^\infty \frac{k J_0(kr)}{1 + kv^2/g} (-\operatorname{sgn}(t) e^{-kv|t|} + 2H(t) \cos(\omega_k t)) dk dt \\
&= \frac{Gmv}{g} \int_0^\infty \frac{k J_0(kr)}{1 + kv^2/g} \left( \int_0^t -\operatorname{sgn}(t) e^{-kv|t|} + 2H(t) \cos(\omega_k t) dt \right) dk \\
&= \frac{Gmv}{g} \int_0^\infty \frac{k J_0(kr)}{1 + kv^2/g} \left( -\operatorname{sgn}(t) \frac{e^{-kv|t|}}{-kv \operatorname{sgn}(t)} + 2H(t) \frac{1}{\omega_k} \sin(\omega_k t) \right) dk \\
&= \frac{Gm}{g} \int_0^\infty \frac{J_0(kr)}{1 + kv^2/g} \left( e^{-kv|t|} + 2H(t) v \sqrt{\frac{k}{g}} \sin(\omega_k t) \right) dk \\
\text{Let } \tilde{T}(t; k) &= \frac{Gm}{g} \frac{1}{1 + kv^2/g} \left( e^{-kv|t|} + 2H(t) v \sqrt{\frac{k}{g}} \sin(\omega_k t) \right)
\end{aligned}$$

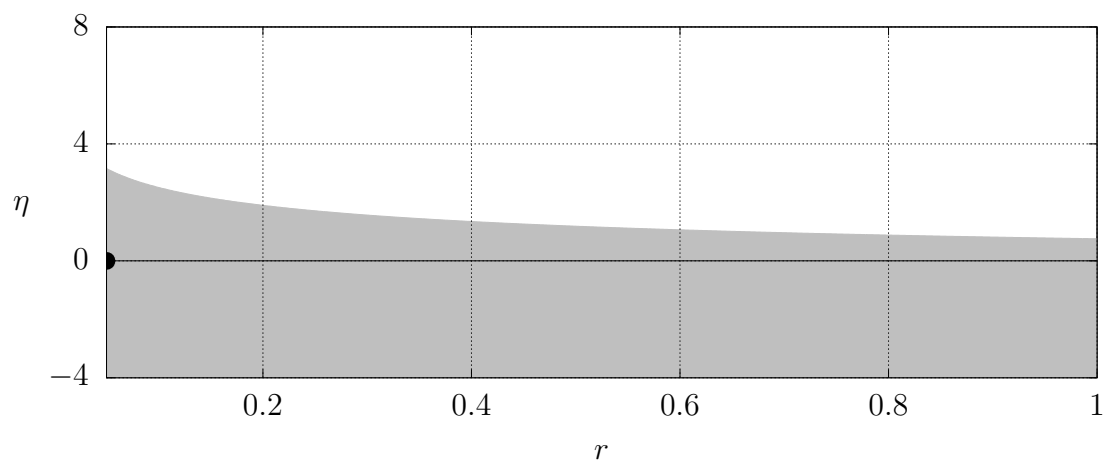
### 1.4 Calculating the Energy Transferred

$$\begin{aligned}
E &= \frac{1}{2} \rho \int \left| \vec{\nabla} \varphi \right|^2 dV + \rho g \int z dV \\
&= \frac{1}{2} \rho \int_0^{2\pi} \int_0^\infty \int_{-\infty}^0 \left| \vec{\nabla} \varphi \right|^2 dz r dr d\theta + \rho g \int_0^{2\pi} \int_0^\infty \int_0^\eta z dz r dr d\theta \\
&= \rho \pi \left( \int_0^\infty \int_{-\infty}^0 \left| \vec{\nabla} \varphi \right|^2 r dz dr + g \int_0^\infty \eta^2 r dr \right) \\
&= \rho \pi \left( \int_0^\infty \int_{-\infty}^0 \left( \frac{\partial \varphi}{\partial r} \right)^2 r dz dr + \int_0^\infty \int_{-\infty}^0 \left( \frac{\partial \varphi}{\partial z} \right)^2 r dz dr + g \int_0^\infty \eta^2 r dr \right)
\end{aligned}$$

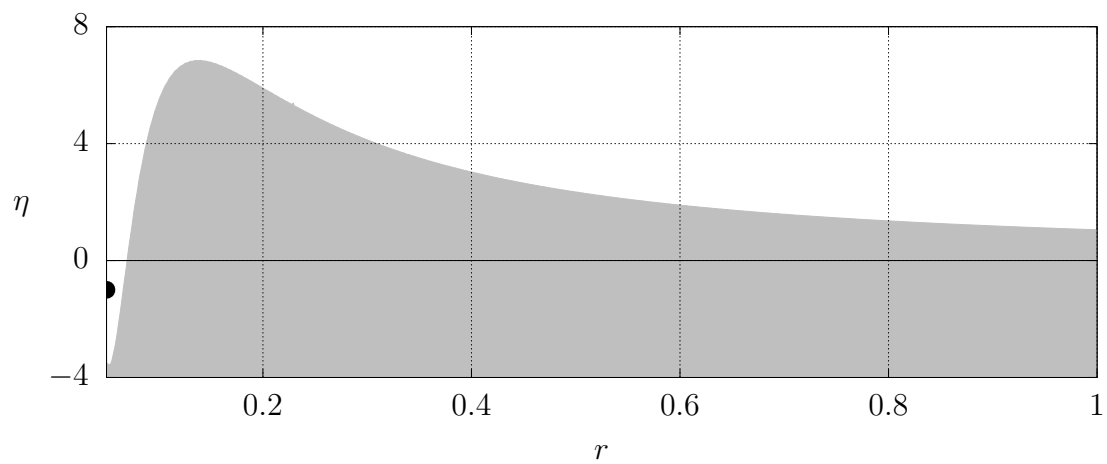
To simplify and tidy this calculation, let  $\clubsuit$ ,  $\spadesuit$ , and  $\diamondsuit$  be the three terms respectively, so



(a)  $t = -1$  s.

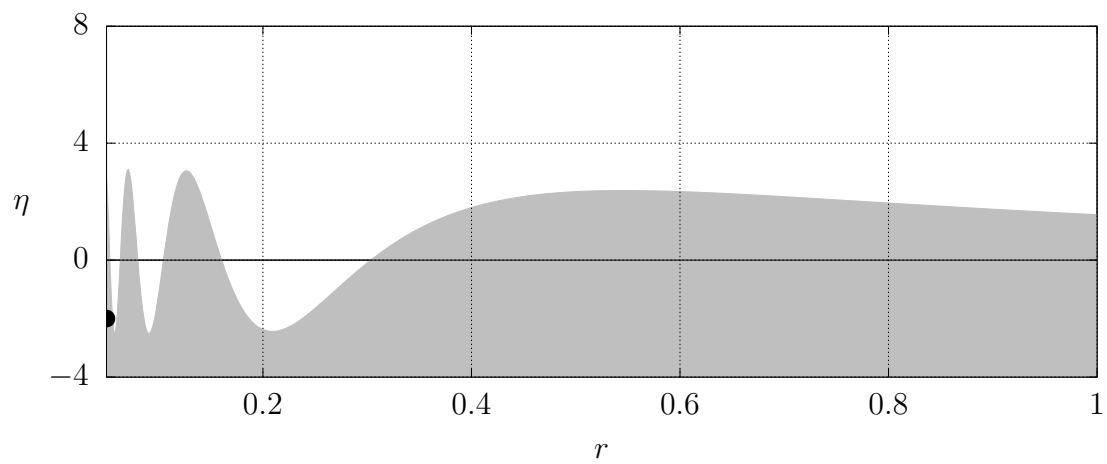


(b)  $t = 0$  s.

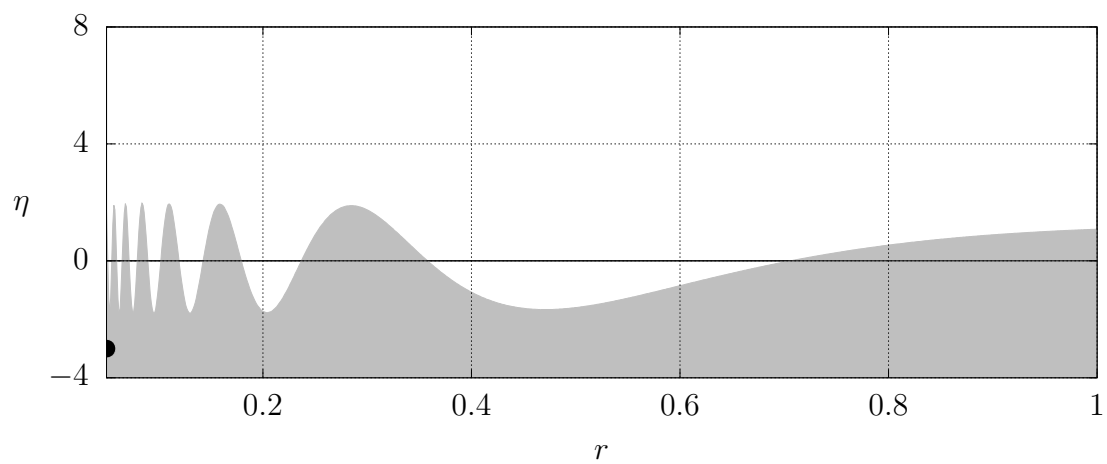


(c)  $t = 1$  s.

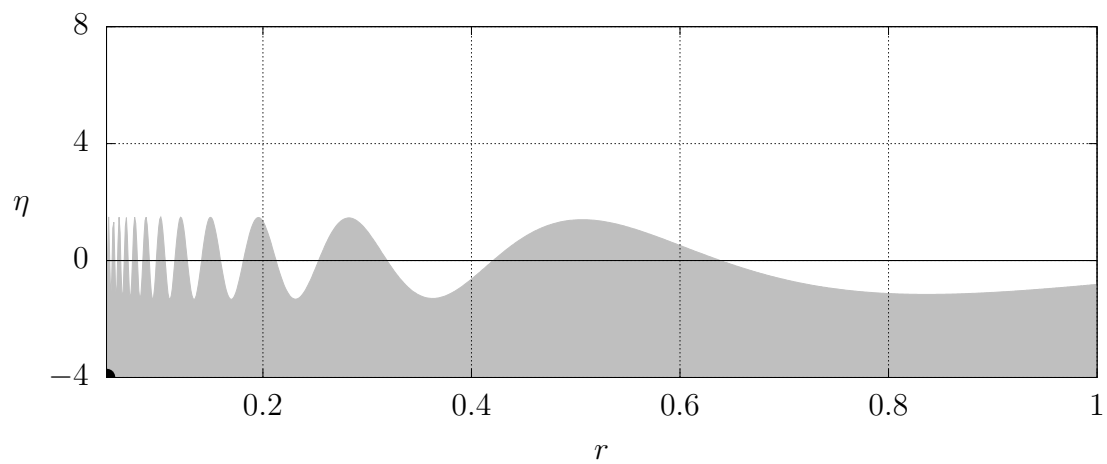




(d)  $t = 2$  s.



(e)  $t = 3$  s.



(f)  $t = 4$  s.

that

$$E = \rho\pi(\clubsuit + \spadesuit + g\diamondsuit).$$

$$\begin{aligned}\clubsuit &= \int_0^\infty \int_{-\infty}^0 \left( \frac{\partial \varphi}{\partial r} \right)^2 r \, dz \, dr \\ &= \int_{-\infty}^0 \int_0^\infty \left( \mathcal{H}_1 \frac{e^{kz} T(t; k)}{k} \right)^2 (r, z, t) r \, dr \, dz\end{aligned}$$

\* \*\*

$$\begin{aligned}&= \int_{-\infty}^0 \int_0^\infty e^{2kz} T^2(t; k) k \, dk \, dz \\ &= \frac{1}{2} \int_0^\infty T^2(t; k) \, dk\end{aligned}$$

$$\begin{aligned}\spadesuit &= \int_0^\infty \int_{-\infty}^0 \left( \frac{\partial \varphi}{\partial z} \right)^2 r \, dz \, dr \\ &= \int_{-\infty}^0 \int_0^\infty \left( \mathcal{H} e^{kz} T(t; k) \right)^2 (r, z, t) r \, dr \, dz\end{aligned}$$

\* \*\*

$$\begin{aligned}&= \int_{-\infty}^0 \int_0^\infty e^{2kz} T^2(t; k) k \, dk \, dz \\ &= \frac{1}{2} \int_0^\infty T^2(t; k) \, dk\end{aligned}$$

$$\begin{aligned}
\blacklozenge &= \int_0^\infty \eta^2 r \, dr \\
&= \int_0^\infty \left( \mathcal{H} \frac{\tilde{T}(t; k)}{k} \right)^2 (r, t) r \, dr
\end{aligned}$$

\* \*\*

$$= \int_0^\infty \frac{\tilde{T}^2(t; k)}{k} dk$$

$$E(t) = \rho\pi \int_0^\infty T^2(t; k) + g \frac{\tilde{T}^2(t; k)}{k} dk$$

$$\begin{aligned}
E = \frac{G^2 m^2 \rho \pi}{g} \int_0^\infty \frac{v^2}{g} &\left( \frac{-\operatorname{sgn}(t) e^{-kv|t|} + 2 \operatorname{H}(t) \cos(\omega_k t)}{1 + kv^2/g} \right)^2 \\
&+ \frac{1}{k} \left( \frac{e^{-kv|t|} + 2 \operatorname{H}(t) v \sqrt{\frac{k}{g}} \sin(\omega_k t)}{1 + kv^2/g} \right)^2 dk
\end{aligned}$$

By taking the long time limit the exponentials decay to zero, and the sine and cosine simplify to 1. The energy then becomes

$$E = 4\pi\rho \frac{G^2 m^2}{g}.$$

\*\*\* Compare with dynamical friction solution

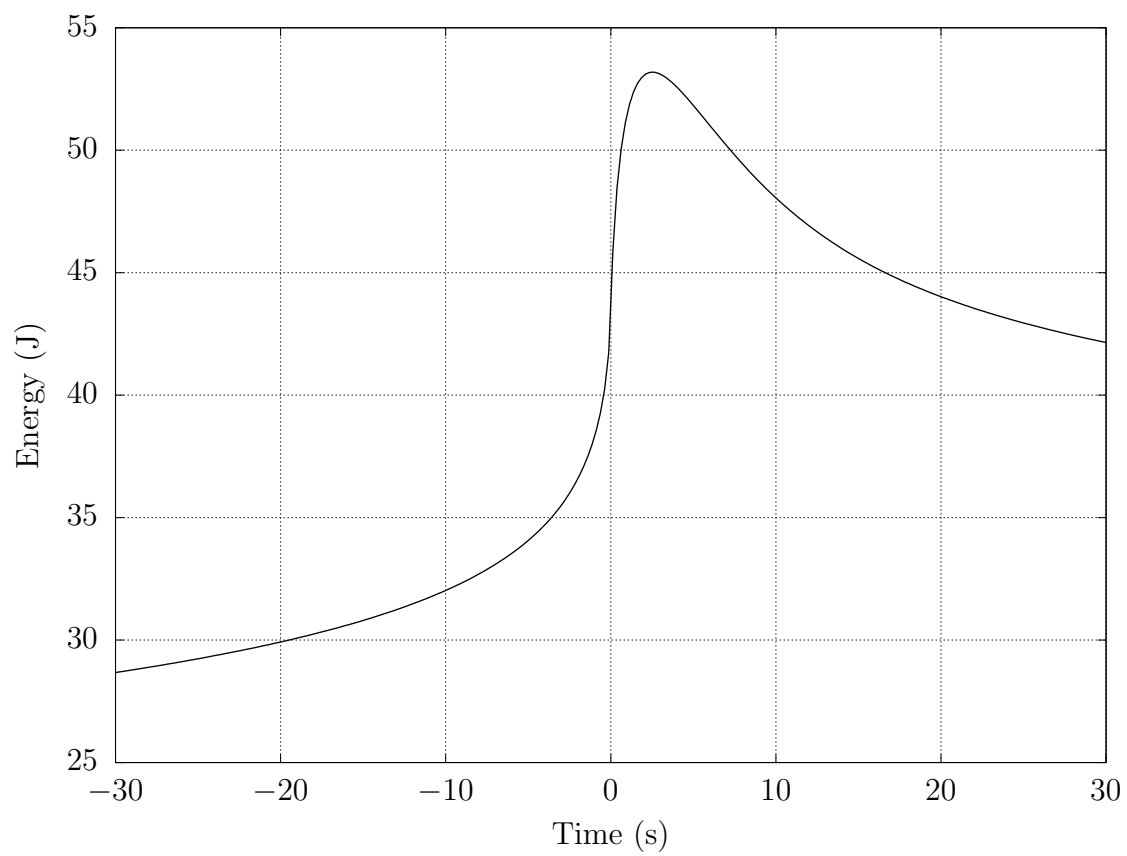


Figure 2: Energy! All variables set to 1.