1 Operators and Integral Transforms

Definition 1.1 (Operator)**.** *Let A, and B be vector spaces with respective subspaces, X, and Y. An operator $\mathcal{T}$, maps any $x \in X$ to Y, and is denoted by $\mathcal{T}(x)$.*

Common examples of operators are the Sturm-Liouville operator, the Laplacian, or Hamiltonian. Our focus will be on the integral operator, or transform. Let the domain, and co-domain of the transform be $C[a,b]$ and $K : \mathbb{R}^2 \to \mathbb{R}$, then we can define our operator $\mathcal{T} : C[a,b] \to C[a,b]$ as

$$(\mathcal{T}f)(x) = \int_a^b f(y)K(x,y)dy,$$

where K is called the kernel function.

Theorem 1.2. *$\mathcal{T}f$ is continuous iff $\int_a^b |f(y)|dy < \infty$, and $K(x,y)$ is uniformly continuous on $[a,b]$.*

Proof. For all $\epsilon > 0$, choose $\delta : |x - x_0| < \delta$, so that $|K(x,y) - K(x_0,y)| < \epsilon/M$, with $M = \int_a^b |f(y)|dy$. $\mathcal{T}f$ is continuous iff

$$\begin{aligned}
|(\mathcal{T}f)(x) - (\mathcal{T}f)(x_0)| &= \left| \int_a^b K(x,y)f(y)dy - \int_a^b K(x_0,y)f(y)dy \right| \\
&\leq \int_a^b |K(x,y) - K(x_0,y)||f(y)|dy \\
&< \int_a^b \frac{\epsilon}{M}|f(y)|dy \\
&< \epsilon
\end{aligned}$$

□

Corollary 1.3. *The conditions for Theorem 1.2 are satisfied if a and b are finite, as well as if f and K are bounded and continuous.*

Proof. A bounded continuous function is integrable over a compact domain. □

1.1 Hankel Transform

Definition 1.4 (Hankel Transform)**.** *The Hankel transform of a function $f(s)$ is given by*

$$(\mathscr{H}_\nu f)(\sigma) = \int_0^\infty f(s)J_\nu(s\sigma)s\,ds,$$

where J_ν is the Bessel function of the first kind, of order $\nu \geq -\frac{1}{2}$, and σ is a non-negative real variable.

Corollary 1.5 (Inverse Hankel Transform)**.** *The Hankel transform is self-reciprocal, that is, the inverse Hankel transform is also given by Definition 1.4.*

Proof. The Hankel transform is self-reciprocal

$$\begin{aligned}
\iff f(s) &= \int_0^\infty (\mathscr{H}_\nu f)(\sigma)J_\nu(s\sigma)\sigma\,d\sigma \\
\iff f(s) &= \int_0^\infty \int_0^\infty f(s')J_\nu(s\sigma)s'\,ds'\,J_\nu(s\sigma)\sigma\,d\sigma \\
&= \int_0^\infty f(s')s' \int_0^\infty J_\nu(s'\sigma)J_\nu(s\sigma)\sigma\,d\sigma\,ds' \\
&= f(s),
\end{aligned}$$

by the orthogonality of the Bessel functions. □

$$\varphi = \int_0^\infty J_0(kr)e^{kz}T(t)dk$$
$$\left.\left(\frac{\partial\varphi}{\partial t} + (g\eta + \Phi)\right)\right|_{z=0} = 0$$
$$\left.\left(\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial z} + \frac{\partial\Phi}{\partial t}\right)\right|_{z=0} = 0$$
$$\text{where } \Phi = \frac{-Gm}{\sqrt{r^2 + (z - vt)^2}}$$
$$\varphi = \frac{Gmv}{g}\int_0^\infty \frac{J_0(kr)e^{kz}}{1 + kv^2/g}\left(-\operatorname{sgn}(t)e^{-kv|t|} + 2\,\mathrm{H}(t)\cos(\omega_k t)\right)dk$$
$$\text{with } \omega_k^2 = gk$$

We must write the gravitational potential as an infinite sum of Bessel functions,

$$\left.\frac{\partial\Phi}{\partial t}\right|_{z=0} = \int_0^\infty a(k;t)J_0(kr)k\,dk$$
$$a(k;t) = \int_0^\infty \left.\frac{\partial\Phi}{\partial t}\right|_{z=0} J_0(kr)r\,dr \text{ *HT*}$$
$$a(k;t) = Gmv^2tk\int_0^\infty \frac{J_0(kr)r}{(r^2 + v^2t^2)^{3/2}}dr$$
$$= Gmv^2t\frac{1}{|vt|}e^{-k|vt|} \text{ *HT*}$$
$$= Gmv\operatorname{sgn}(t)e^{-kv|t|}$$

$$\int_0^\infty J_0(kr)\ddot{T}(t)dk + g\int_0^\infty kJ_0(kr)T(t)dk + Gmv\int_0^\infty \operatorname{sgn}(t)e^{-kv|t|}J_0(kr)k\,dk = 0$$
$$\int_0^\infty \left[\frac{\ddot{T}(t)}{k} + gT(t) + Gmv\operatorname{sgn}(t)e^{-kv|t|}\right]J_0(kr)k\,dk = 0$$
$$\frac{\ddot{T}(t)}{k} + gT(t) + Gmv\operatorname{sgn}(t)e^{-kv|t|} = \int_0^\infty 0 \times J_0(kr)r\,dr = 0 \text{ *HT*}$$
$$T(t) = A\cos(\omega_k t) + B\sin(\omega_k t) \text{ homogenous solution } t \geq 0$$
$$\text{Assume } T(t) = Ce^{\xi|t|} \text{ for the particular solution}$$
$$\implies C\left(\xi^2\operatorname{sgn}^2(t)e^{\xi|t|} + gke^{\xi|t|}\right) + Gmvk\operatorname{sgn}(t)e^{-kv|t|} = 0$$
$$\implies \xi = -kv, \quad C = \frac{-Gmvk\operatorname{sgn}(t)}{k^2v^2 + gk} = \frac{Gmv}{g}\frac{-\operatorname{sgn}(t)}{1 + kv^2/g}$$
$$\implies T(t) = \frac{Gmv}{g}\frac{1}{1 + kv^2/g}\left(-\operatorname{sgn}(t)e^{-kv|t|} + \mathrm{H}(t)\left(\widetilde{A}\cos(\omega_k t) + \widetilde{B}\sin(\omega_k t)\right)\right)$$
$$T(t) \text{ must be at least of class } C^1 \implies \widetilde{A} = 2, \quad \widetilde{B} = 0$$
$$\implies \varphi = \frac{Gmv}{g}\int_0^\infty \frac{J_0(kr)e^{kz}}{1 + kv^2/g}\left(-\operatorname{sgn}(t)e^{-kv|t|} + 2H(t)\cos(\omega_k t)\right)dk$$