1 Analytic Solution

1.1 Eigenfunctions of the Laplacian

$$\begin{aligned}
\nabla^2\varphi &= 0 \\
\varphi &= f(r)g(\theta)h(z)T(t) \\
\nabla^2 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}\right) + \frac{\partial^2}{\partial z^2} \\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\varphi}{\partial\theta^2}\right) + \frac{\partial^2\varphi}{\partial z^2} &= 0 \\
\frac{1}{fr}\frac{\partial}{\partial r}(rf') + \frac{1}{r^2}\frac{g''}{g} + \frac{h''}{h} &= 0 \\
\frac{h''}{h} &= k^2 \\
h(z) &= Ae^{kz} + Be^{-kz} \\
&= e^{kz} \\
\frac{1}{fr}\frac{\partial}{\partial r}(rf') + \frac{1}{r^2}\frac{g''}{g} + k^2 &= 0 \\
\frac{r}{f}\frac{\partial}{\partial r}(rf') + k^2r^2 + \frac{g''}{g} &= 0 \\
\frac{g''}{g} &= -\mu^2 \\
g(\theta) &= A\sin(\mu\theta) + B\cos(\mu\theta) \\
\mu &= 0 \\
g(\theta) &= 1 \\
r\frac{\partial}{\partial r}(rf') + (k^2r^2 - 0^2)f &= 0 \\
f(r) &= AJ_0(kr) + BY_0(kr) \\
&= AJ_0(kr) \\
\varphi &\propto e^{kz}J_0(kr)T(t) \\
&= \int_0^\infty e^{kz}T(t)J_0(kr)dk
\end{aligned}$$

1.2 Solving the Velocity Potential

We must write the gravitational potential as an infinite sum of Bessel functions to match the form of φ, to do so, we take the Hankel transform,

$$\begin{aligned}
\left.\frac{\partial\Phi}{\partial t}\right|_{z=0} &= \int_0^\infty \left(\mathscr{H}\left.\frac{\partial\Phi}{\partial t}\right|_{z=0}\right)(k)J_0(kr)k\,dk \\
&= Gmv^2t\int_0^\infty \left(\mathscr{H}\frac{1}{(r^2+v^2t^2)^{3/2}}\right)(k)J_0(kr)k\,dk^1 \\
&= Gmv^2t\int_0^\infty \frac{1}{|vt|}e^{-k|vt|}J_0(kr)k\,dk \\
&= Gmv\,\mathrm{sgn}(t)\int_0^\infty e^{-kv|t|}J_0(kr)k\,dk.
\end{aligned}$$

We can now substitute this into the differential equation and find φ,

$$\int_0^\infty J_0(kr)\ddot{T}(t)dk + g\int_0^\infty kJ_0(kr)T(t)dk + Gmv\int_0^\infty \operatorname{sgn}(t)e^{-kv|t|}J_0(kr)k\,dk = 0$$
$$\int_0^\infty \left[\frac{\ddot{T}(t)}{k} + gT(t) + Gmv\operatorname{sgn}(t)e^{-kv|t|}\right]J_0(kr)k\,dk = 0,$$

but, this is nothing more than the Hankel transform of the differential equation for T. By taking the Hankel transform of both sides we can remove the integral,

$$\frac{\ddot{T}(t)}{k} + gT(t) + Gmv\operatorname{sgn}(t)e^{-kv|t|} = 0.$$

Clearly, the homogeneous solution is $T(t) = A\cos(\omega_k t) + B\sin(\omega_k t)$ with $\omega_k^2 = gk$. The form of the differential equations suggests the form $T(t) = Ce^{-kv|t|}$ for the particular solution. Substituting this in yields

$$C\left(k^2v^2\operatorname{sgn}^2(t)e^{-kv|t|} + gke^{-kv|t|}\right) + Gmvk\operatorname{sgn}(t)e^{-kv|t|} = 0,$$

giving

$$\begin{aligned} C &= \frac{-Gmvk\operatorname{sgn}(t)}{k^2v^2 + gk} \\ &= \frac{Gmv}{g}\frac{-\operatorname{sgn}(t)}{1 + kv^2/g} \end{aligned}$$

as the coefficient, and,

$$T(t) = \frac{Gmv}{g}\frac{1}{1 + kv^2/g}\left(-\operatorname{sgn}(t)e^{-kv|t|}\right) + A\cos(\omega_k t) + B\sin(\omega_k t)$$

as the full time component of the velocity potential. We can now apply the boundary conditions to find A, and B. Physically, we expect $T(t) \in C^1(-\infty,\infty)$, furthermore, we only expect the sinusoidal terms to contribute at times greater than zero, thus,

$$\begin{aligned} T(t) &= \frac{Gmv}{g}\frac{1}{1 + kv^2/g}\left(-\operatorname{sgn}(t)e^{-kv|t|} + 2H(t)\cos(\omega_k t)\right)dk, \text{ and,} \\ \varphi &= \frac{Gmv}{g}\int_0^\infty \frac{J_0(kr)e^{kz}}{1 + kv^2/g}\left(-\operatorname{sgn}(t)e^{-kv|t|} + 2H(t)\cos(\omega_k t)\right)dk. \end{aligned}$$

$$\begin{aligned} \varphi &= \int_0^\infty J_0(kr)e^{kz}T(t)dk \\ \left(\frac{\partial\varphi}{\partial t} + (g\eta + \Phi)\right)\bigg|_{z=0} &= 0 \\ \left(\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial z} + \frac{\partial\Phi}{\partial t}\right)\bigg|_{z=0} &= 0 \\ \text{where } \Phi &= \frac{-Gm}{\sqrt{r^2 + (z - vt)^2}} \\ \varphi = \frac{Gmv}{g}\int_0^\infty \frac{J_0(kr)e^{kz}}{1 + kv^2/g}&\left(-\operatorname{sgn}(t)e^{-kv|t|} + 2\,\mathrm{H}(t)\cos(\omega_k t)\right)dk \\ &\text{with } \omega_k^2 = gk \end{aligned}$$

[1]Note that $\int_0^\infty \sqrt{r}(r^2 + v^2t^2)^{-3/2}dr = \Gamma^2(3/4)(\pi v^3t^3)^{-1/2} < \infty$ for $t \neq 0$ which is not concerning since this is true almost everywhere, and so the condition in Theorem ***** is satisfied.