1 Operators and Integral Transforms

Definition 1.1 (Operator)**.** *Let A, and B be vector spaces with respective subspaces, X, and Y. An operator $\mathcal{T}$, maps any $x \in X$ to Y, and is denoted by $\mathcal{T}(x)$.*

Common examples of operators are the Sturm-Liouville operator, the Laplacian, or Hamiltonian. Our focus will be on the integral operator, or transform. Let the domain, and co-domain of the transform be $C[a,b]$ and $K : \mathbb{R}^2 \to \mathbb{R}$, then we can define our operator $\mathcal{T} : C[a,b] \to C[a,b]$ as

$$(\mathcal{T}f)(x) = \int_a^b f(y)K(x,y)dy,$$

where K is called the kernel function.

Theorem 1.2. *$\mathcal{T}f$ is continuous if $\int_a^b |f(y)|dy < \infty$, and $K(x,y)$ is uniformly continuous on $[a,b]$.*

Proof. For all $\varepsilon > 0$, choose $\delta : |x - x_0| < \delta$, so that $|K(x,y) - K(x_0,y)| < \varepsilon/M$, with $M = \int_a^b |f(y)|dy$. Then,

$$\begin{aligned}
|(\mathcal{T}f)(x) - (\mathcal{T}f)(x_0)| &= \left| \int_a^b K(x,y)f(y)dy - \int_a^b K(x_0,y)f(y)dy \right| \\
&\leq \int_a^b |K(x,y) - K(x_0,y)||f(y)|dy \\
&< \int_a^b \frac{\varepsilon}{M}|f(y)|dy \\
&< \varepsilon,
\end{aligned}$$

and $\mathcal{T}f$ is continuous. □

The conditions for Theorem 1.2 are automatically satisfied if a and b are finite, in addition to K being bounded and continuous. A bounded continuous function over a compact domain is uniformly continuous and integrable.

1.1 Hankel Transform

Definition 1.3 (Hankel Transform)**.** *The Hankel transform of a function $f(s)$ is given by*

$$(\mathscr{H}_\nu f)(\sigma) = \int_0^\infty f(s)J_\nu(s\sigma)s\,ds,$$

where J_ν is the Bessel function of the first kind, of order $\nu \geq -\frac{1}{2}$, and σ is a non-negative real variable.

Notice though, that the kernel function of the Hankel transform is not $J_\nu(s\sigma)s$, but in fact $J_\nu(s\sigma)\sqrt{s}$, the other $\sqrt{s}$ factor gets absorbed into f. This is to ensure uniform continuity of the kernel, and the continuity of the transform. As a consequence, we have the modified condition that $\int_0^\infty \sqrt{s}|f(s)|ds < \infty$. **Include in appendix?

Corollary 1.4 (Inverse Hankel Transform)**.** *The Hankel transform is self-reciprocal, that is, the inverse Hankel transform is also given by Definition 1.3.*

Proof. The Hankel transform is self-reciprocal

$$\begin{aligned}
\iff f(s) &= \int_0^\infty (\mathscr{H}_\nu f)(\sigma)J_\nu(s\sigma)\sigma\,d\sigma \\
\iff &= \int_0^\infty \int_0^\infty f(s')J_\nu(s\sigma)s'\,ds'\,J_\nu(s\sigma)\sigma\,d\sigma \\
&= \int_0^\infty f(s')s' \int_0^\infty J_\nu(s'\sigma)J_\nu(s\sigma)\sigma\,d\sigma\,ds' \\
&= f(s),
\end{aligned}$$

by the orthogonality of the Bessel functions. □