1 Operators and Integral Transforms

Definition 1.1 (Uniform Continuity)**.** *A function $f : X \to Y$ is uniformly continuous if $\forall \varepsilon > 0$, $\exists\, \delta > 0$: $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$, $\forall x_0 \in X$.*

Theorem 1.2. *If a function $f : X \to \mathbb{R}$ is continuous with X compact, f is uniformly continuous on X.*

Proof. Since f is continuous

$$\forall \frac{\varepsilon}{2} > 0, \exists\, \delta_i > 0 : |x - x_i| < \frac{1}{2}\delta_i \Rightarrow |f(x) - f(x_i)| < \frac{1}{2}\varepsilon \,\forall\, x_i \in X.$$

Then, let $\delta = \min\{\frac{1}{2}\delta_i\}$, and suppose, $|x - x_0| < \delta$, then, $\exists i : |x - x_i| < \delta_i$, and

$$|x_0 - x_i| \leq |x_0 - x| + |x - x_i| < \delta + \frac{1}{2}\delta_i \leq \delta_i,$$
$$|f(x_0) - f(x)| \leq |f(x_0) - f(x_i)| + |f(x_i) - f(x)| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

by the triangle inequality. □

Definition 1.3 (Operator)**.** *Let A, and B be vector spaces with respective subspaces, X, and Y. An operator $\mathcal{T}$, maps any $x \in X$ to Y, and is denoted by $\mathcal{T}(x)$.*

Common examples of operators are the Sturm-Liouville operator, the Laplacian, or Hamiltonian. Our focus will be on the integral operator, or transform. Let the domain, and co-domain of the transform be $C[a, b]$ and $K : \mathbb{R}^2 \to \mathbb{R}$, then we can define our operator $\mathcal{T} : C[a, b] \to C[a, b]$ as

$$(\mathcal{T} f)(x) = \int_a^b f(y)K(x, y)dy,$$

where K is called the kernel function.

Theorem 1.4. *$\mathcal{T} f$ is continuous if $\int_a^b |f(y)|dy < \infty$, and $K(x, y)$ is uniformly continuous on $[a, b]$.*

Proof. For all $\varepsilon > 0$, choose $\delta : |x - x_0| < \delta$, so that $|K(x, y) - K(x_0, y)| < \varepsilon/M$, with $M = \int_a^b |f(y)|dy$. Then,

$$\begin{aligned}
|(\mathcal{T} f)(x) - (\mathcal{T} f)(x_0)| &= \left| \int_a^b K(x, y)f(y)dy - \int_a^b K(x_0, y)f(y)dy \right| \\
&\leq \int_a^b |K(x, y) - K(x_0, y)||f(y)|dy \\
&< \int_a^b \frac{\varepsilon}{M}|f(y)|dy \\
&< \varepsilon,
\end{aligned}$$

and $\mathcal{T} f$ is continuous. □

The conditions for Theorem 1.4 are automatically satisfied by Theorem 1.2 if a and b are finite, and K is bounded and continuous.

1.1 Hankel Transform

Definition 1.5 (Hankel Transform). *The Hankel transform of a function $f(s)$ is given by*

$$(\mathscr{H}_\nu f)(\sigma) = \int_0^\infty f(s) J_\nu(s\sigma) s \, ds,$$

where J_ν is the Bessel function of the first kind, of order $\nu \geq -\frac{1}{2}$, and σ is a non-negative real variable.

Corollary 1.6 (Inverse Hankel Transform). *The Hankel transform is self-reciprocal, that is, the inverse Hankel transform is also given by Definition 1.5.*

Proof. The Hankel transform is self-reciprocal

$$\begin{aligned}
\iff f(s) &= \int_0^\infty (\mathscr{H}_\nu f)(\sigma) J_\nu(s\sigma)\sigma \, d\sigma \\
\iff &= \int_0^\infty \int_0^\infty f(s') J_\nu(s\sigma) s' \, ds' \, J_\nu(s\sigma)\sigma \, d\sigma \\
&= \int_0^\infty f(s') s' \int_0^\infty J_\nu(s'\sigma) J_\nu(s\sigma)\sigma \, d\sigma \, ds' \\
&= f(s),
\end{aligned}$$

by the orthogonality of the Bessel functions. □

However, only the order zero Hankel transform will be used in the analysis, so the ν will be omitted and assumed zero.

Lemma 1.7. *$\sqrt{s} J_0(s\sigma)$ is uniformly continuous on $[0, \infty)$.*

Proof. Let N be a sufficiently large number. By Theorem 1.2, $\sqrt{s} J_0(s\sigma)$ is uniformly continuous on $[0, N]$. And on the interval (N, ∞),

$$J_0(s\sigma) \approx \sqrt{\frac{2}{\pi s\sigma}} \cos\left(s\sigma - \frac{\pi}{4}\right)$$

asymptotically, and therefore $\sqrt{s} J_0(s\sigma)$ is uniformly continuous on $[0, \infty)$ since cosine is uniformly continuous everywhere. □

Notice that the kernel function of the Hankel transform is not $J_0(s\sigma)s$, but in fact $J_0(s\sigma)\sqrt{s}$, the other $\sqrt{s}$ factor gets absorbed into f. This is to ensure uniform continuity of the kernel (Lemma 1.7), and the continuity of the transform. As a consequence, we have the modified condition that $\int_0^\infty \sqrt{s}|f(s)| ds < \infty$.