

# 1 Analytic Solution

## 1.1 Eigenfunctions of the Laplacian

$$\begin{aligned}
\nabla^2 \varphi &= 0 \\
\varphi &= f(r)g(\theta)h(z)T(t) \\
\nabla^2 &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} \right) + \frac{\partial^2}{\partial z^2} \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \left( \frac{\partial^2 \varphi}{\partial \theta^2} \right) + \frac{\partial^2 \varphi}{\partial z^2} &= 0 \\
\frac{1}{fr} \frac{\partial}{\partial r} (rf') + \frac{1}{r^2} \frac{g''}{g} + \frac{h''}{h} &= 0 \\
\frac{h''}{h} &= k^2 \\
h(z) &= Ae^{kz} + Be^{-kz} \\
&= e^{kz} \\
\frac{1}{fr} \frac{\partial}{\partial r} (rf') + \frac{1}{r^2} \frac{g''}{g} + k^2 &= 0 \\
\frac{r}{f} \frac{\partial}{\partial r} (rf') + k^2 r^2 + \frac{g''}{g} &= 0 \\
\frac{g''}{g} &= -\mu^2 \\
g(\theta) &= A \sin(\mu\theta) + B \cos(\mu\theta) \\
\mu &= 0 \\
g(\theta) &= 1 \\
r \frac{\partial}{\partial r} (rf') + (k^2 r^2 - 0^2)f &= 0 \\
f(r) &= AJ_0(kr) + BY_0(kr) \\
&= AJ_0(kr) \\
\varphi &\propto e^{kz} J_0(kr) T(t) \\
&= \int_0^\infty e^{kz} T(t) J_0(kr) dk
\end{aligned}$$

## 1.2 Solving the Velocity Potential

We must write the gravitational potential as an infinite sum of Bessel functions to match the form of  $\varphi$ , to do so, we take the Hankel transform,

$$\begin{aligned}
\left. \frac{\partial \Phi}{\partial t} \right|_{z=0} &= \int_0^\infty \left( \mathcal{H} \left. \frac{\partial \Phi}{\partial t} \right|_{z=0} \right) (k) J_0(kr) k \, dk \\
&= Gmv^2 t \int_0^\infty \left( \mathcal{H} \frac{1}{(r^2 + v^2 t^2)^{3/2}} \right) (k) J_0(kr) k \, dk^1 \\
&= Gmv^2 t \int_0^\infty \frac{1}{|vt|} e^{-k|vt|} J_0(kr) k \, dk \\
&= Gmv \operatorname{sgn}(t) \int_0^\infty e^{-kv|t|} J_0(kr) k \, dk.
\end{aligned}$$

We can now substitute this into the differential equation and find  $\varphi$ ,

$$\int_0^\infty J_0(kr)\ddot{T}(t)dk + g \int_0^\infty kJ_0(kr)T(t)dk + Gmv \int_0^\infty \text{sgn}(t)e^{-kv|t|}J_0(kr)k dk = 0$$

$$\int_0^\infty \left[ \frac{\ddot{T}(t)}{k} + gT(t) + Gmv \text{sgn}(t)e^{-kv|t|} \right] J_0(kr)k dk = 0,$$

but, this is nothing more than the Hankel transform of the differential equation for  $T$ . By taking the Hankel transform of both sides we can remove the integral,

$$\frac{\ddot{T}(t)}{k} + gT(t) + Gmv \text{sgn}(t)e^{-kv|t|} = 0.$$

Clearly, the homogeneous solution is  $T(t) = A \cos(\omega_k t) + B \sin(\omega_k t)$  with  $\omega_k^2 = gk$ . The form of the differential equations suggests the form  $T(t) = Ce^{-kv|t|}$  for the particular solution. Substituting this in yields

$$C \left( k^2 v^2 \text{sgn}^2(t) e^{-kv|t|} + gk e^{-kv|t|} \right) + Gmvk \text{sgn}(t) e^{-kv|t|} = 0,$$

giving

$$C = \frac{-Gmvk \text{sgn}(t)}{k^2 v^2 + gk}$$

$$= \frac{Gmv}{g} \frac{-\text{sgn}(t)}{1 + kv^2/g}$$

as the coefficient, and,

$$T(t) = \frac{Gmv}{g} \frac{1}{1 + kv^2/g} \left( -\text{sgn}(t) e^{-kv|t|} \right) + A \cos(\omega_k t) + B \sin(\omega_k t)$$

as the full time component of the velocity potential. We can now apply the boundary conditions to find  $A$ , and  $B$ . Physically, we expect  $T(t) \in C^1(-\infty, \infty)$ , furthermore, we only expect the sinusoidal terms to contribute at times greater than zero, thus,

$$T(t) = \frac{Gmv}{g} \frac{1}{1 + kv^2/g} \left( -\text{sgn}(t) e^{-kv|t|} + 2H(t) \cos(\omega_k t) \right) dk, \text{ and,}$$

$$\varphi = \frac{Gmv}{g} \int_0^\infty \frac{J_0(kr)e^{kz}}{1 + kv^2/g} \left( -\text{sgn}(t) e^{-kv|t|} + 2H(t) \cos(\omega_k t) \right) dk.$$

$$\varphi = \int_0^\infty J_0(kr)e^{kz}T(t)dk$$

$$\left( \frac{\partial \varphi}{\partial t} + (g\eta + \Phi) \right) \Big|_{z=0} = 0$$

$$\left( \frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z} + \frac{\partial \Phi}{\partial t} \right) \Big|_{z=0} = 0$$

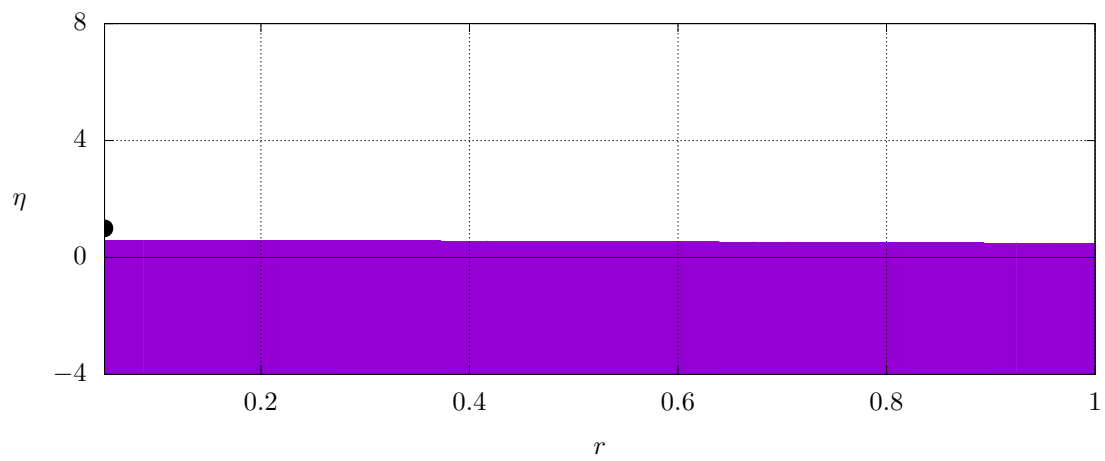
$$\text{where } \Phi = \frac{-Gm}{\sqrt{r^2 + (z - vt)^2}}$$

$$\varphi = \frac{Gmv}{g} \int_0^\infty \frac{J_0(kr)e^{kz}}{1 + kv^2/g} \left( -\text{sgn}(t) e^{-kv|t|} + 2H(t) \cos(\omega_k t) \right) dk$$

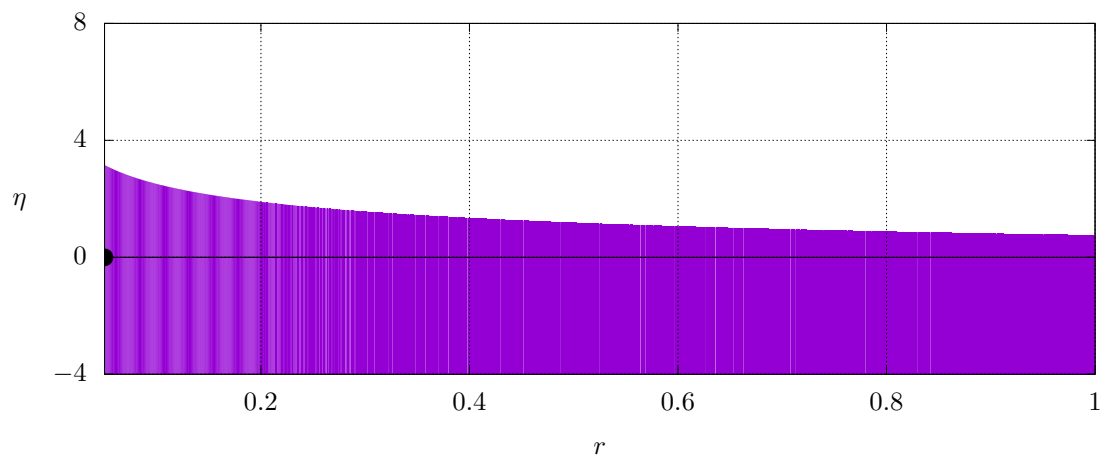
with  $\omega_k^2 = gk$

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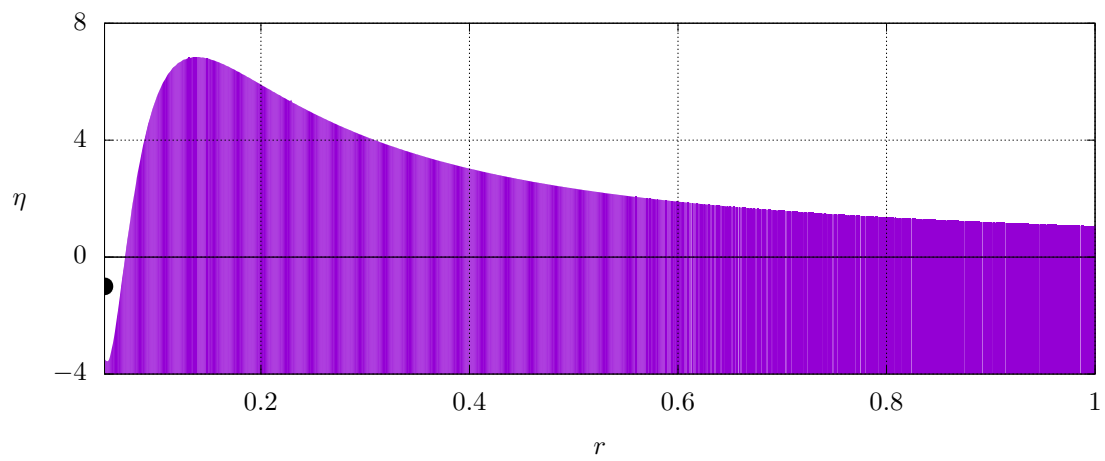
<sup>1</sup>Note that  $\int_0^\infty \sqrt{r}(r^2 + v^2 t^2)^{-3/2} dr = \Gamma^2(3/4)(\pi v^3 t^3)^{-1/2} < \infty$  for  $t \neq 0$  which is not concerning since this is true almost everywhere, and so the condition in Theorem \*\*\*\*\* is satisfied.



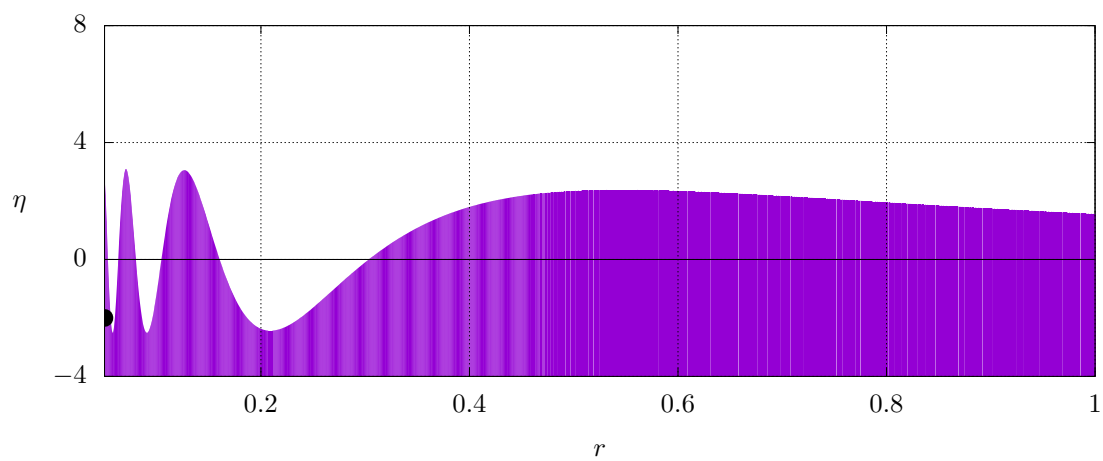
(a)  $t = -1$  s.



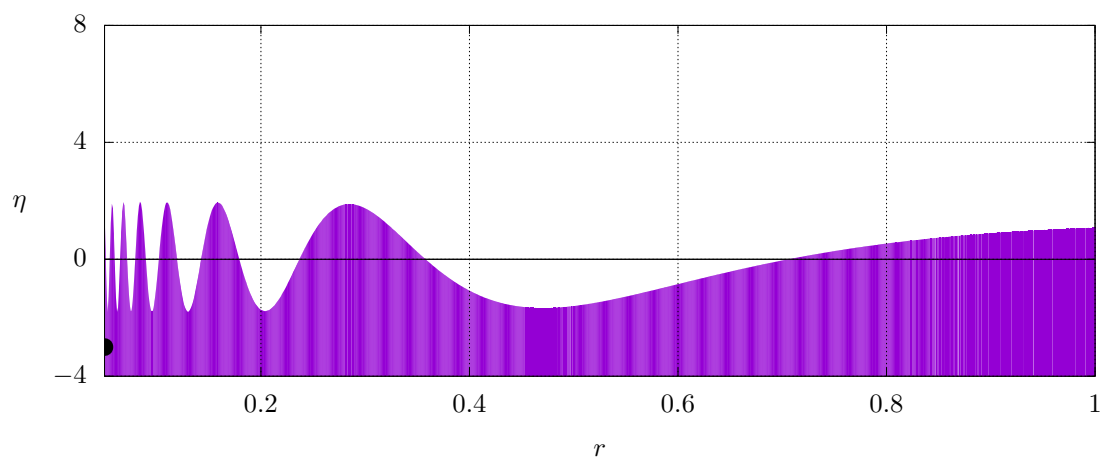
(b)  $t = 0$  s.



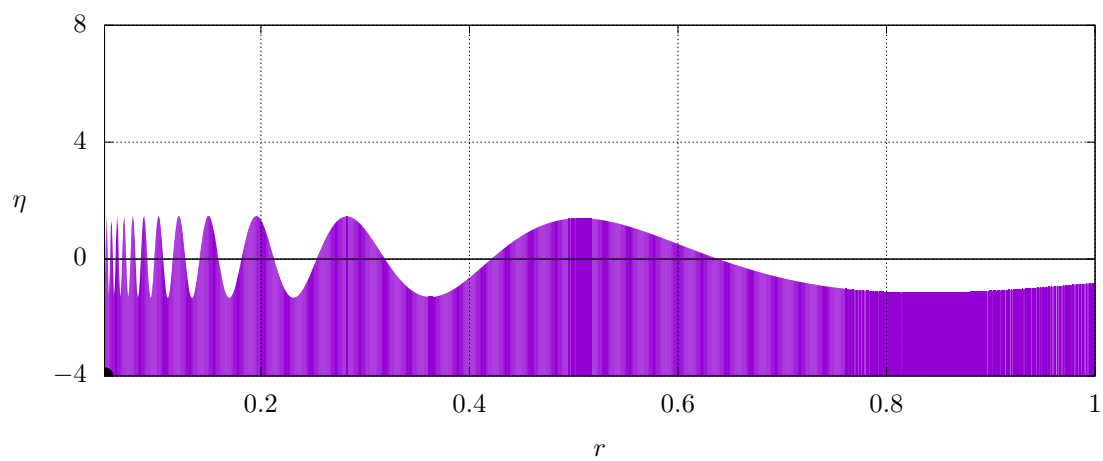
(c)  $t = 1$  s.



(d)  $t = 2$  s.



(e)  $t = 3$  s.



(f)  $t = 4$  s.