INTERACTIONS OF PRIMORDIAL BLACK HOLES WITH NEUTRON STARS

by

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Into

Our goal is to model and simulate the interaction caused by the head on collision of a PBH ~~as~~ with an NS. Initially, we shall follow "On Tidal Capture of Primordial Black Holes by Neutron Stars" by Defillon etal.

Flat Star

Our first model will be that the NS is a flat and infinitely deep ~~fluid~~. incompressible fluid. As such, we have $\nabla^2 \varphi = 0$. Let us first determine the eigenfunctions of φ. Let us assume a product solution, $\varphi = f(r)g(\theta)h(z)T(t)$, and do separation of variables. However, solving the Laplacian will not give us $T(t)$; this will be found from the boundary conditions as will see. We are in a cylindrically symmetric system, and using cylindrical coordinates, therefore,

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2}\right) + \frac{\partial^2}{\partial z^2}.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r f' g h\right) + \frac{1}{r^2} f g'' h + f g h'' = 0$$

$$\frac{1}{fr}\frac{\partial}{\partial r}\left(r f'\right) + \frac{1}{r^2}\frac{g''}{g} + \frac{h''}{h} = 0$$

we have now separated h from f and g.

$$\Rightarrow \frac{h''}{h} = k^2 \Rightarrow h(z) = Ae^{kz} + Be^{-kz}$$

From our infinite depth assumption $B=0$ since $z \in (0,-\infty)$.

$$\boxed{h(z) = e^{kz}} \quad (\text{wlog } A=1)$$

$$\Rightarrow \frac{1}{rf}\frac{\partial}{\partial r}(rf') + \frac{1}{r^2}\frac{g''}{g} + k^2 = 0$$

$$\frac{r}{f}\frac{\partial}{\partial r}(rf') + k^2r^2 + \frac{g''}{g} = 0$$

g has now been separated from f, $\Rightarrow \frac{g''}{g} = -\mu^2$

$\Rightarrow g = A\sin(\mu\theta) + B\cos(\mu\theta)$. But our system is symmetric about θ; there should be no θ dependence.

$$\Rightarrow \boxed{\mu = 0} \Rightarrow g = 1 \quad (\text{wlog } B=1)$$

$$\Rightarrow \frac{r}{f}\frac{\partial}{\partial r}(rf') + k^2r^2 - \mu^2 = 0$$

$\Rightarrow r\frac{\partial}{\partial r}(rf') + (k^2r^2 - \mu^2)f = 0$ which is Bessel's equation

$\Rightarrow f = AJ_{\mu}(kr) + BY_{\mu}(kr)$. But $\mu=0$, and our solution needs to be well behaved at $r=0 \Rightarrow B=0$

$$\Rightarrow \boxed{f = AJ_0(kr)}$$

~~Fa~~ Thus, $\varphi_K = A_k e^{kz} J_0(kr) T(t,k)$

Assuming our surface waves are small perturbations, we have,

$\left(\frac{\partial \varphi}{\partial t} + g\eta\right)\Big|_{z=0} = 0$, and since, $\frac{\partial \varphi}{\partial z} \approx \frac{\partial \eta}{\partial t}$, we take

$\partial/\partial t$ and substitute.

$$\left(\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial z}\right)\Big|_{z=0} = 0.$$

$$\Rightarrow \left(\cancel{A_k}\cancel{e^{kz}}\cancel{J_0(kr)}T''(t,k) + g\cancel{A_k}k\cancel{e^{kz}}\cancel{J_0(kr)}T(t,k)\right)\Big|_{z=0} = 0$$

$$T''(t,k) + gkT(t,k) = 0$$

$\Rightarrow \boxed{T(t,k) = Ae^{i\omega_k t} + Be^{-i\omega_k t}}$ with $\omega_k^2 = gk$ is the dispersion relation.

The paper takes $A=0$, which I just noticed. I'm not really sure why.

$$\boxed{\varphi \propto e^{kz} J_0(kr) e^{-i\omega_k t}}$$

Gravitational potential from PBH

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The next step is to add the perturbation of the gravitational potential of the PBH.

$$\Phi(r,z,t) = \frac{-Gm}{(r^2+(z+vt)^2)^{1/2}} = \frac{-Gm}{\vec{r}}$$

with $\vec{r}$ as the Euclidean distance between the PBH and a fluid element as a function of t.

we must resolve for $T(t,k)$ using our new boundary condition at the surface:

$$\left(\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial z}\right)_{z=0} = -\frac{\partial \Phi}{\partial t}.$$

The authors suggest writing Φ as a sum of Bessel functions, however, this is not an obvious substitution, since Φ is multivariate and cannot be written as a product. This PDE has proven to be quite difficult to solve. Regardless, the solution is given as,

$$\varphi(r,z,t) = \frac{Gmv}{g}\int_0^\infty \frac{e^{kz} J_0(kr)}{1+kv^2/g}\left[-\mathcal{E}(t)e^{-kv|t|} + 2\Theta(t)\cos(\omega_k t)\right],$$

with $\mathcal{E}(t)$ the sgn function, and $\Theta(t)$ the Heaviside function.

Deformation of Surface

The velocity potential, φ, and the profile of the surface, η, are related by $\frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t}$ at $z=0$. In other words,

$$\eta = \int \frac{\partial \varphi}{\partial z}\Big|_{z=0} dt .$$

$$= \int \frac{6mv}{g} \int_0^\infty \frac{kJ_0(kr)}{1+kv^2/g} \left(-\varepsilon(t) e^{-kv|t|} + 2\Theta(t)\cos(\omega_k t)\right) dk\, dt$$

$$= \frac{6mv}{g} \int_0^\infty \frac{kJ_0(kr)}{1+kv^2/g} \left(\int -\varepsilon(t) e^{-kv|t|} dt + 2\int \Theta(t)\cos(\omega_k t)\, dt\right) dk$$

we shall focus on these two integrals separately.

For the first, notice that $\frac{d|t|}{dt} = \varepsilon(t)$, and therefore, only a simple u-sub is required. For the second $\Theta(t)$ is essentially a constant.

$$= \frac{6mv}{g} \int_0^\infty \frac{kJ_0(kr)}{1+kv^2/g} \left(\frac{e^{-kv|t|}}{kv} + 2\Theta(t)\sin(\omega_k t)\frac{1}{\omega_k}\right) dk$$

$$= \frac{6m}{g} \int_0^\infty \frac{J_0(kr)}{1+kv^2/g} \left(e^{-kv|t|} + 2\Theta(t)\sin(\omega_k t)\frac{kv}{\sqrt{gk}}\right) dk$$

$$= \frac{6m}{g} \int_0^\infty \frac{J_0(kr)}{1+kv^2/g} \left(e^{-kv|t|} + 2\Theta(t)\, v\sqrt{\frac{k}{g}}\sin(\omega_k t)\right) dk.$$

Plotting the Surface waves

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A visualization of $\eta(r,t)$ would be quite handy. Unfortunately, due to it's infinite upper bound, and it's highly oscillatory nature from the sine and Bessel functions, it is quite difficult to evaluate, ~~even numerically~~. Both python and ~~m~~ MATLAB were unable to evaluate the integral, it was unable to converge.

Mathematica, however, was able to evaluate the integral. The fastest evaluation, and convergence resulted ~~with 'Levins Method'~~ when using the 'LevinRule' method. ~~I generated a~~

However, ~~it~~ even Mathematica seemed to have convergence problems for $r < 0.05$ (with $g = \sigma = m = \nu = 1$). Techniques exist for numerically evaluating improper integrals, such as a reparameterization,

$$\int_0^\infty f(x)dx = \cancel{\int_0^1 \frac{f(1-t)}{1}} = \int_0^1 \frac{f\left(\frac{t}{1-t}\right)}{(1-t)^2} dt.$$

But, this does not seem useful for oscillatory functions, since the wavelength becomes infinitesimal at 1. I generated a .dat file with Mathematica, using $g = \sigma = \nu = m = 1$. I evaluated the integral for $0.05 \leq r \leq 1$ in steps of 10^{-3}, and $0 \leq t \leq 5$, in steps of 0.1. It took about 4 hours to finish.

Integrating k from 0 to 10^3.

with the help of a few scripts, the image generation was very fast. I made ~~the~~ plots of r vs. η for each time step using gnuplot. The script set the labels, and limits, and such then looped over each column of the data file, saving each plot with a sequential name, as a .png. Then using ffmpeg, I stitched together each image to make an animation of the surface waves. Furthermore, I wrote another, similar, script, but to export ~~as~~ a plot of integer times, as a .tex file, to use LaTeX for the fonts and typesetting, as well as easy implementation ~~to~~ for the ~~ne~~ write up.

I have started a repository for this project, github.com/bmetherall/Primordial_Black_Holes, the data file, scripts, and animation, ~~is~~ have been pushed.

Solution to the PDE (more or less) Oct 26/16

$$\left(\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial z}\right)\bigg|_{z=0} = -\frac{\partial\Phi}{\partial t}\bigg|_{z=0}$$

$$= -\frac{+Gm}{(r^2+v^2t^2)^{3/2}}\left(\frac{+1}{\cancel{2}}\right)(\cancel{2}v^2t)$$

$$= -\frac{Gmv^2t}{(r^2+v^2t^2)^{3/2}}$$

Let $\tilde{\Phi} = -\frac{Gmv^2t}{(r^2+v^2t^2)^{3/2}}$, we want to write $\tilde{\Phi}$ in the form of $\tilde{\Phi} = \sum_k a(k) J_0(kr)$ to match the form of φ. we can do this, i, since $J_0(kr)$ is complete

ii, Φ satisfys $\nabla^2\Phi = 0$, and so has the same eigenfunctions.

In our case, the sum is over all positive real k.

$$\Rightarrow \tilde{\Phi} = \int_0^\infty a(k) J_0(kr)\,dk$$

$$\cancel{\tilde{\Phi} J_0(kr) r = \int_0^\infty a(k) J_0^2(kr) r\,dk}$$

$$\cancel{\int_0^\infty \tilde{\Phi} J_0(kr) r\,dr = \int_0^\infty a(k) \int_0^\infty J_0^2(kr) r\,dr\,dk}$$

$$\tilde{\Phi}\, J_0(\alpha r) r = \int_0^\infty a(k) J_0(kr) J_0(\alpha r) r\, dk$$

$$\int_0^\infty \tilde{\Phi}\, J_0(\alpha r) r\, dr = \int_0^\infty a(k) \int_0^\infty J_0(kr) J_0(\alpha r) r\, dr\, dk$$

and $\int_0^\infty J_0(kr) J_0(\alpha r) r\, dr = \frac{1}{k}\delta_{k\alpha}$

$$= \int_0^\infty a(k) \frac{1}{k} \delta_{k\alpha}\, dk$$

$$= \frac{a(\alpha)}{\alpha}$$

$\Rightarrow \quad a(k) = k \int_0^\infty \tilde{\Phi}\, J_0(kr) r\, dr$ which Mathematica gives as

$$= -k \frac{Gmtv^2}{\sqrt{t^2v^2}} e^{-k\sqrt{t^2v^2}}$$

Assuming $Re(t^2v^2) > 0$, $Re(k) > 0$, $Im(k) = 0$

Solution on page 17

$$= -Gmvk\,\mathcal{E}(t) e^{-kv|t|}$$

which are all satisfied. with $\mathcal{E}(t)$ as $\mathrm{sign}(t)$.

$$\Rightarrow \quad \tilde{\Phi} = \int_0^\infty -Gmvk\,\mathcal{E}(t) e^{-kv|t|} J_0(kr)\, dk$$

we shall now work on the left hand side:

$$\left(\frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z} \right)\Big|_{z=0} \quad \text{with}$$

$$\varphi = \int_0^\infty e^{kz} J_0(kr) T(t)\, dk$$

$$= \left(\int_0^\infty e^{kz} J_0(kr) T''(t)\,dk + g \int_0^\infty k e^{kz} J_0(kr) T(t)\,dk \right)\Big|_{z=0}$$

$$= \int_0^\infty J_0(kr) T''(t)\,dk + g \int_0^\infty k J_0(kr) T(t)\,dk$$

Since both integrals have the same bounds, and are both wrt k, we can combine them.

$$= \int_0^\infty J_0(kr)\left[T''(t) + gkT(t)\right] dk$$

Then,

$$\int_0^\infty J_0(kr)\left[T''(t) + gkT(t)\right] dk = -\int_0^\infty 6mvk\,\mathcal{E}(t)\, e^{-kv|t|} J_0(kr)\,dk$$

And in a similar manner,

$$\int_0^\infty J_0(kr)\left[T''(t) + gkT(t) + 6mvk\,\mathcal{E}(t) e^{-kv|t|}\right] dk = 0$$

$$\cancel{\int_0^\infty \int_{\alpha/k}^{r} J_0'(kr')\left[T''(t) + gkT(t) + 6mvk\,\mathcal{E}(t) e^{-kv|t|}\right] dr'\,dk = 0}$$

~~with α equal to the first (or any) zero of $J_0(x)$.~~

$$\Leftrightarrow T''(t) + gkT(t) + 6mvk\,\mathcal{E}(t) e^{-kv|t|} = 0$$

I'll prove this later.

without loss of generality, assume $t>0$:

$$T''(t) + kg T(t) + 6 m v k e^{-kvt} \mathcal{E}(t) = 0$$

The homogeneous solution is of course, as shown before, $T(t) = \alpha \cos(\omega_k t) + \beta \sin(\omega_k t)$.

we seek the particular solution, by first assuming $T(t)$ is in the form, $T(t) = Ae^{xt}$:

$$\Rightarrow A x^2 e^{xt} + gk A e^{xt} + 6mvk e^{-kvt} = 0$$

$\Leftrightarrow x = -kv$ by the orthogonality of the exponential function.

$$\Rightarrow A k^2 v^2 e^{-kvt} + gkA e^{-kvt} + 6mvk e^{-kvt} = 0$$

$$\Rightarrow A = \frac{-6mvk}{gk + k^2 v^2}$$

$$= -\frac{6mv}{g} \frac{1}{1 + kv^2/g}$$

$$\Rightarrow T(t) = -\frac{6mv}{g} \frac{1}{1+kv^2/g} \mathcal{E}(t) e^{-kv|t|} + \alpha \cos(\omega_k t) + \beta \sin(\omega_k t).$$

For convenience, which we shall see shortly, ~~we~~ rewrite this as

$$T(t) = \frac{6mv}{g} \frac{1}{1+kv^2/g} \left[-\mathcal{E}(t) e^{-kv|t|} + \tilde{\alpha} \cos(\omega_k t) + \tilde{\beta} \sin(\omega_k t) \right].$$

Physically, we only expect the sinusoidal oscillations to occur at times $t>0$; after the collision. As such, we shall introduce $\Theta(t)$ the Heaviside step fuction,

$$T(t) = \frac{Gmv}{g}\frac{1}{1+kv^2/g}\left[-\mathcal{E}(t)e^{-kv|t|} + \Theta(t)\left(\tilde{\alpha}\cos(\omega_k t) + \tilde{\beta}\sin(\omega_k t)\right)\right].$$

Now, the velocity potential, φ, must be continuous, and smooth everywhere, ~~includ~~ including $t=0$. More specifically, $\lim_{t\to 0^-} T(t) = \lim_{t\to 0^+} T(t)$, and $\lim_{t\to 0^-} T'(t) = \lim_{t\to 0^+} T'(t)$. These conditions will allow us to compute $\tilde{\alpha}$, and $\tilde{\beta}$. Let us start with the first condition:

$$\lim_{t\to 0^-} T(t) = \lim_{t\to 0^+} T(t)$$

$$\lim_{t\to 0}\ \cancel{\frac{Gmv}{g}\frac{1}{1+kv^2/g}}\left[e^{kvt}\right] = \lim_{t\to 0}\ \cancel{\frac{Gmv}{g}\frac{1}{1+kv^2/g}}\left[-e^{-kvt} + \tilde{\alpha}\cos(\omega_k t) + \tilde{\beta}\sin(\omega_k t)\right]$$

$$\Rightarrow \quad 1 = -1 + \tilde{\alpha}$$

$$\Rightarrow \quad \boxed{\tilde{\alpha} = 2}$$

And now the second condition:

$$\lim_{t\to 0^-} T'(t) = \lim_{t\to 0^+} T'(t)$$

$$\lim_{t\to 0} \cancel{\frac{6mv}{g}} \frac{1}{1+kv^2/g}\left[e^{kvt}(kv)\right] = \lim_{t\to 0} \cancel{\frac{6mv}{g}} \frac{1}{1+kv^2/g}\left[e^{kvt}(kv) - \tilde{\alpha}\,\omega_k \sin(\omega_k t) + \tilde{\beta}\,\omega_k \cos(\omega_k t)\right]$$

$$\Rightarrow \quad kv = kv + \omega_k \tilde{\beta}$$

$$\Rightarrow \quad \boxed{\tilde{\beta} = 0}$$

$$\Rightarrow T(t) = \frac{6mv}{g}\frac{1}{1+kv^2/g}\left[-\mathcal{E}(t)e^{-kv|t|} + 2\Theta(t)\cos(\omega_k t)\right].$$

$$\Rightarrow \varphi = \frac{6mv}{g}\int_0^\infty \frac{e^{kz} J_0(kr)}{1+kv^2/g}\left[-\mathcal{E}(t)e^{-kv|t|} + 2\Theta(t)\cos(\omega_k t)\right].$$

Hawking Radiation and the Evaporation of Black Holes

Oct 27/16

The temperature of Hawking Radiation is given as

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} .$$

Stefan-Boltzmann power law:

$$P = A \varepsilon \sigma T^4 \qquad (\text{take } \varepsilon = 1 \text{ for BH}).$$

$$\Rightarrow P = A \sigma T_H^4$$

$$= 4\pi r_s^2 \sigma T_H^4$$

$$= 4\pi \left(\frac{2GM}{c^2}\right)^2 \left(\frac{\pi^2 k_B^4}{60\hbar^3 c^2}\right)\left(\frac{\hbar c^3}{8\pi G M k_B}\right)^4$$

$$= 4\cancel{\pi}\, 4 \frac{G^2 M^2}{c^4} \frac{\cancel{\pi^2} \cancel{k_B^4}}{60 \cancel{\hbar^3} c^2} \frac{\hbar^4 c^{12}}{8^4 \pi^4 G^4 M^4 \cancel{k_B^4}}$$

$$= \frac{16}{60 \cdot 8^4} \frac{\hbar}{\pi} \frac{c^6}{G^2 M^2}$$

$$\boxed{P = \frac{\hbar c^6}{15360\pi G^2 M^2}}$$

Let $\Xi = \dfrac{\hbar c^6}{15360\pi G^2}$

$$= \frac{\Xi}{M^2}$$

But $P = \dfrac{-dE}{dt}$

$$= -c^2 \frac{dM}{dt}$$

$$\frac{\Xi}{M^2} = -c^2 \frac{dM}{dt}$$

$$-\frac{\Xi}{c^2} dt = M^2 dM$$

As the BH evaporates, M goes from M_0 to 0, and t goes from 0 to t_{ev}.

$$-\frac{\Xi}{c^2} \int_0^{t_{ev}} dt = \int_{M_0}^{0} M^2 dM$$

$$t_{ev} = \frac{c^2}{\Xi} \frac{M_0^3}{3}$$

$$\boxed{t_{ev} = \frac{5120\pi G^2 M_0^3}{\hbar c^4}}$$

The age of the universe is more or less 4.3×10^{17} s

$$M_{crit} = \left(\frac{t_{universe} \hbar c^4}{5120\pi G^2} \right)^{1/3}$$

would be the lower bound for a PBH created in the early universe.

$$\boxed{M_{crit} \approx 1.7 \times 10^{14} \text{ g.}}$$

Solving for η, the deformation of the surface. oct 27/16

Now that we have φ, we can solve for η using the fact that

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=0} = \frac{\partial \eta}{\partial t}.$$ essentially,

$$\eta = \int \left.\frac{\partial \varphi}{\partial z}\right|_{z=0} dt$$

$$= \int \frac{6mv}{g} \int_0^\infty \frac{k e^{kz} J_0(kr)}{1+kv^2/g} \left[-\varepsilon(t) e^{-kv|t|} + 2\Theta(t)\cos(\omega_k t)\right] dk \Big|_{z=0} dt$$

$$= \frac{6mv}{g} \int_0^\infty \frac{k J_0(kr)}{1+kv^2/g} \left[\int -\varepsilon(t) e^{-kv|t|} + 2\Theta(t)\cos(\omega_k t)\, dt\right] dk$$

$\varepsilon(t)$ and $\Theta(t)$ can be treated as constants since their derivatives are zero.

$$= \frac{6mv}{g} \int_0^\infty \frac{k J_0(kr)}{1+kv^2/g} \left[\frac{1}{kv} e^{-kv|t|} + \frac{2\Theta(t)}{\omega_k}\sin(\omega_k t)\right] dk$$

$$\eta = \frac{6m}{g} \int_0^\infty \frac{J_0(kr)}{1+kv^2/g} \left[e^{-kv|t|} + 2\Theta(t)\sqrt{\frac{k}{g}}\sin(\omega_k t)\right] dk$$

Expansion of Φ in $J_0(kr)$ Oct 27/16

Earlier we needed to write Φ in $\left.\frac{\partial \Phi}{\partial t}\right|_{z=0}$ interms of $J_0(kr)$. To do so we needed to ~~solve~~ evaluate

$$\int_0^\infty - \left.\frac{\partial \Phi}{\partial t}\right|_{z=0} J_0(kr)\, r\, dr.$$

$$= \int_0^\infty \frac{-6mv^2t}{(r^2+v^2t^2)^{3/2}} J_0(kr)\, r\, dr$$

From dlmf.nist.gov/10.22#E46 states

$$\int_0^\infty \frac{t^{\nu+1} J_\nu(at)}{(t^2+b^2)^{\mu+1}}\, dt = \frac{a^\mu b^{\nu-\mu}}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(ab)$$

In our case:

$t=r$,
$\nu=0$,
$a=k$,
$b=vt$,
$\mu=1/2$.

$$\Rightarrow \; = -6mv^2t \frac{k^{1/2}(vt)^{-1/2}}{2^{1/2}\Gamma(3/2)} K_{-1/2}(kvt)$$

$$= -6mv^2t \frac{\sqrt{k}}{\sqrt{vt}} \frac{1}{\sqrt{2}} \frac{2}{\sqrt{\pi}} K_{-1/2}(kvt)$$

~~From the de~~

$$K_{-1/2}(kvt) = K_{1/2}(kvt)$$

$$= \sqrt{\pi/2}\, \frac{e^{-|kvt|}}{\sqrt{kvt}}$$

$$= -6mv^2t \frac{\sqrt{\cancel{k}}}{\sqrt{\cancel{v}t}} \frac{1}{\cancel{\sqrt{2}}} \frac{\cancel{2}}{\cancel{\sqrt{\pi}}} \frac{\cancel{\sqrt{\pi}}}{\cancel{\sqrt{2}}} \frac{e^{-|kvt|}}{\sqrt{\cancel{k}\cancel{v}t}}$$

$$= -6mv \frac{t}{\sqrt{t^2}} e^{-kv|t|}$$

$$= -6mv\ \varepsilon(t) e^{-kv|t|}$$

Calculation of the Energy

Nov 9 /2016

we can calculate the energy by taking the sum of the kinetic and the potential. It is as we would expect, with $\vec{v} = \nabla\varphi$, and for the z component of potential, we are only interested in the deviation from the ~~upertar~~ unperturbed fluid, that is $0 < z \leq \eta$.

$$E = \frac{1}{2}\rho \int |\nabla\varphi|^2 dx^3 + \rho g \int z \, dx^3 .$$

$$= \frac{1}{2}\rho \int_0^{2\pi}\int_0^{\infty}\int_{-\infty}^{0} |\nabla\varphi|^2 dz\, r dr d\theta + \rho g \int_0^{2\pi}\int_0^{\infty}\int_0^{\eta} z\, dz\, r dr d\theta$$

$$= \frac{1}{\cancel{2}}\rho \cancel{2}\pi \int_0^{\infty}\int_{-\infty}^{0} |\nabla\varphi|^2 r dz dr + \rho g \cancel{2}\pi \frac{1}{\cancel{2}} \int_0^{\infty} \eta^2 r dr$$

$$= \rho\pi \left[\int_0^{\infty}\int_{-\infty}^{0} |\nabla\varphi|^2 r dz dr + g \int_0^{\infty} \eta^2 r dr \right] .$$

As we shall see, this is a very involved calculation, and we will need to split this into three pieces.

Firstly, $|\nabla\varphi|^2 = \left(\frac{\partial\varphi}{\partial r}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2$

$$= \rho\pi \left[\int_0^{\infty}\int_{-\infty}^{0} \left(\frac{\partial\varphi}{\partial r}\right)^2 r dz dr + \int_0^{\infty}\int_{-\infty}^{0} \left(\frac{\partial\varphi}{\partial z}\right)^2 r dz dr + g\int_0^{\infty} \eta^2 r dr \right]$$

Let α, β, γ denote these terms respectively.

Calculation of α and β

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$$\alpha = \int_0^\infty \int_{-\infty}^0 \left(\frac{\partial\varphi}{\partial r}\right)^2 r\,dz\,dr$$

Due to the nature of φ, $\left(\frac{\partial\varphi}{\partial r}\right)^2$ is quite lengthy itself.

$$= \int_0^\infty \int_{-\infty}^0 \int_0^\infty \frac{Gmv}{g} \frac{e^{kz}(-kJ_1(kr))}{1+kv^2/g}\left[\varphi^T(t,k)\right] dk \int_0^\infty \frac{Gmv}{g}\frac{e^{k'z}(-k'J_1(k'r))}{1+k'v^2/g}\varphi^T_{(t,k')}dk'\,r\,dz\,dr$$

by letting φ^T be the time component of φ.

$$= \int_0^\infty \int_{-\infty}^0 \int_0^\infty \int_0^\infty \frac{G^2m^2v^2}{g^2}\frac{e^{kz+k'z}}{(1+kv^2/g)(1+k'v^2/g)}\varphi^T(t,k)\varphi^T(t,k')\,kk'J_1(kr)J_1(k'r)\,r\,dk\,dk'\,dz\,dr$$

$$= \int_{-\infty}^0 \int_0^\infty \frac{G^2m^2v^2}{g^2}\frac{e^{2kz}}{(1+kv^2/g)^2}\varphi^{T^2}(t,k)\,k\,dk\,dz$$

$$= \int_0^\infty \frac{G^2m^2v^2}{g^2}\frac{e^{2kz}}{(1+kv^2/g)^2}\varphi^{T^2}(t,k)\,k\,\frac{1}{2k}\,dk\,\Big|_{z=-\infty}^{0}$$

$$= \frac{G^2m^2v^2}{2g^2}\int_0^\infty \frac{\varphi^{T^2}(t,k)\,dk}{(1+kv^2/g)^2}.$$

I claim that $\alpha=\beta$ due to the fact that $\frac{\partial\varphi}{\partial z}$ and $\frac{\partial\varphi}{\partial r}$ both pick up a k, and that J_0, or J_1 does not change the final answer.

Calculation of γ

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$$\gamma = \int_0^\infty \eta^2 r\,dr$$

$$= \int_0^\infty \frac{\sigma m}{g}\int_0^\infty \frac{J_0(kr)}{1+kv^2/g}\,{}_\eta T(t,k)\,dk\,\frac{\sigma m}{g}\int_0^\infty \frac{J_0(k'r)}{1+k'v^2/g}\,{}_\eta T(t,k')\,dk'\,r\,dr$$

$$= \frac{\sigma^2 m^2}{g^2}\int_0^\infty\int_0^\infty\int_0^\infty \frac{J_0(kr)J_0(k'r)\,r\,dr}{(1+kv^2/g)(1+k'v^2/g)}\,{}_\eta T(t,k)\,{}_\eta T(t,k')\,dk\,dk'$$

$$= \frac{\sigma^2 m^2}{g^2}\int_0^\infty \frac{1}{k}\,{}_\eta T^2(t,k)\,\frac{1}{(1+kv^2/g)^2}\,dk$$

Putting It All Together

Nov 9/16

$$E = \rho\pi(\alpha + \beta + g\gamma)$$

$$= \rho\pi\left(\frac{\sigma^2 m^2 v^2}{g^2}\int_0^\infty \frac{{}_\varphi T^2(t,k)\,dk}{(1+kv^2/g)^2} + \frac{\sigma^2 m^2}{g}\int_0^\infty \frac{{}_\eta T^2(t,k)\,dk}{k(1+kv^2/g)^2}\right)$$

$$= \rho\pi\,\frac{\sigma^2 m^2}{g}\left(\frac{v^2}{g}\int_0^\infty \frac{\left(-\epsilon(t)e^{-kv|t|} + 2\Theta(t)\cos(\omega_k t)\right)^2}{(1+kv^2/g)^2}\,dk + \int_0^\infty \frac{\left(e^{-kv|t|} + 2\Theta(t)\,v\sqrt{\frac{k}{g}}\sin(\omega_k t)\right)^2 dk}{k\,(1+kv^2/g)^2}\right)$$

we shall take $\lim_{t\to\infty}$ since we expect E to be maximum at $t=\infty$, we can justify this by $\frac{dE}{dt}=0$.

[illegible]

$$= \rho\pi \frac{G^2 m^2}{g}\left[\int_0^\infty \frac{v^2}{g} \frac{2^2\cos(\omega_k t)^2}{(1+kv^2/g)^2} + \frac{2^2 \sin(\omega_k t)^2}{k(1+kv^2/g)^2} v^2 \frac{k}{g}\, dk\right]$$

$$= \rho\pi \frac{G^2 m^2 v^2}{g^2}\left(\int_0^\infty \frac{4(\cos^2(\omega_k t)+\sin^2(\omega_k t))}{(1+kv^2/g)^2}\, dk\right)$$

$$= 4\pi\rho \frac{G^2 m^2 v^2}{g^2} \int_0^\infty \frac{dk}{(1+kv^2/g)^2}$$

$$E = 4\pi\rho \frac{G^2 m^2}{g}$$

If we apply this result to the case of a spherical star,

$$g = \frac{GM}{R^2} \qquad M = \frac{4}{3}\pi R^3 \rho$$

$$\Rightarrow \quad = \frac{G}{\cancel{R^2}} \frac{4}{3}\pi \cancel{R^3} \rho$$

$$\Rightarrow E = \cancel{4\pi\rho} \frac{G^{\cancel{2}} m^2\, 3}{\cancel{G}\cancel{4}\cancel{\pi} R \cancel{\rho}}$$

$$= \frac{3Gm^2}{R}$$

which agrees with the dynamical friction approach.