1 Analytic Solution

Typically, in fluid dynamics, the scalar function known as the velocity potential denoted by φ, is what is desired. Once the velocity potential is known, the system is solved because $\vec{\nabla}\varphi = \mathbf{u}$, and in the case of a free surface, the deformation of the surface can readily be found from the velocity potential as well. The velocity potential will be found in the coming sub-chapters, and then the energy deposited into the neutron star will be calculated.

1.1 Eigenfunctions of the Laplacian

The first step in analytically solving for the velocity potential will be to find the eigenfunctions of Laplace's equation, $\nabla^2\varphi = 0$, since the velocity potential is conservative. Since this is a partial differential equation we will assume the solution is the product of univariate functions, $\varphi = f(r)g(\theta)h(z)T(t)$. Expanding the Laplacian for cylindrical systems gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\varphi}{\partial\theta^2}\right) + \frac{\partial^2\varphi}{\partial z^2} = 0.$$

The temporal component can be divided out since the Laplacian does not involve time, and will be found later on. Substituting in our assumed form, and dividing by φ, yields

$$\frac{1}{fr}\frac{\partial}{\partial r}(rf') + \frac{1}{r^2}\frac{g''}{g} + \frac{h''}{h} = 0. \tag{1}$$

Where primes denote the derivative with respect to the function's only variable. Notice that the function $h(z)$ has been separated from both $f(r)$, and $g(\theta)$, and since each function is univariate, it must be the case that the last term is constant,

$$\frac{h''}{h} = k^2.$$

We cleverly force this constant to be positive to satisfy the boundary conditions, namely, that $h(-\infty) = 0$. The most general solution to this is of course exponentials,

$$h(z) = Ae^{kz} + Be^{-kz}.$$

Since the velocity potential has to vanish at negative infinity, this is simplified to

$$h(z) = e^{kz}, \tag{2}$$

without loss of generality we can assume the constant is one, since it can be absorbed into $T(t)$. By substituting in k^2 and multiplying through by r^2, (1) becomes

$$\frac{r}{f}\frac{\partial}{\partial r}(rf') + k^2r^2 + \frac{g''}{g} = 0, \tag{3}$$

and once again, $g(\theta)$ has been separated from $f(r)$, and so the last term must be constant –

$$\frac{g''}{g} = -\mu^2.$$

The constant is forced to be negative since $g(\theta)$ is expected to be periodic, and not exponential. Of course, the general solution to this differential equation is

$$g(\theta) = A\sin(\mu\theta) + B\cos(\mu\theta).$$

However, our system is cylindrically symmetric; there is no way to differentiate one value of θ to another, thus, it is expected that φ has no θ dependence, and consequentially, $g(\theta)$

must be constant, the only way this is satisfied is if $\mu = 0$, so that

$$g(\theta) = 1. \tag{4}$$

By substituting $\mu = 0$ into (3), and multiplying through by f we obtain

$$r\frac{\partial}{\partial r}(rf') + (k^2r^2 - 0^2)f = 0,$$

which indeed is Bessel's differential equation of order 0. Therefore,

$$f(r) = AJ_0(kr) + BY_0(kr),$$

but, once again, B must equal zero since $Y_0(0)$ is not finite which is unphysical. The radial part thus simplifies to

$$f(r) = J_0(kr). \tag{5}$$

By combining (2), (4), and (5) we find that the velocity potential is

$$\varphi \propto e^{kz}T(t)J_0(kr),$$

where k is the eigenvalues. Clearly though, the only restriction on k is that it must be a non-negative real number, all of which are a valid solution to Laplace's equation. Therefore, the total solution must be the sum of all of these, or since k is valid within an interval, we have the integral form

$$\varphi = \int_0^\infty e^{kz}T(t)J_0(kr)dk. \tag{6}$$

The temporal component can be solved for using equations of fluid dynamics.

1.2 Temporal Component of the Velocity Potential

The time component of the velocity potential comes from the specific problem. In our case by linearizing, and assuming the waves have a small amplitude, we have

$$\left(\frac{\partial \varphi}{\partial t} + (g\eta + \Phi) \right) \bigg|_{z=0} = 0,$$

where η denotes the deformation of the surface. This comes from the pressure condition at the surface, and in our case, has the addition of the gravitational potential. ***Fluids book eqn 3.21 3.22*** By taking the time derivative and using the relation that

$$\frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial z} \tag{7}$$

on the surface ***, we obtain

$$\left(\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial z} + \frac{\partial \Phi}{\partial t} \right) \bigg|_{z=0} = 0. \tag{8}$$

In order to solve this partial differential equation, we must write the gravitational potential as an infinite sum of Bessel functions to match the form of φ. To do so, we take the Hankel transform using the fact that it is self-reciprocal, then

$$\begin{aligned}
\left.\frac{\partial \Phi}{\partial t}\right|_{z=0} &= \int_0^\infty \left(\mathscr{H} \left.\frac{\partial \Phi}{\partial t}\right|_{z=0}\right)(k) J_0(kr) k\, dk, \\
&= Gmv^2 t \int_0^\infty \left(\mathscr{H} \frac{1}{(r^2+v^2t^2)^{3/2}}\right)(k) J_0(kr) k\, dk, ^1 \\
&= Gmv^2 t \int_0^\infty \frac{1}{|vt|} e^{-k|vt|} J_0(kr) k\, dk, *** \\
&= Gmv \operatorname{sgn}(t) \int_0^\infty e^{-kv|t|} J_0(kr) k\, dk.
\end{aligned}$$

Now we can substitute this into (8) to obtain,

$$\begin{aligned}
\int_0^\infty J_0(kr)\ddot{T}(t) dk + g\int_0^\infty kJ_0(kr)T(t)dk + Gmv\int_0^\infty \operatorname{sgn}(t) e^{-kv|t|} J_0(kr) k\, dk = 0, \\
\text{or,} \quad \int_0^\infty \left[\frac{\ddot{T}(t)}{k} + gT(t) + Gmv \operatorname{sgn}(t) e^{-kv|t|}\right] J_0(kr) k\, dk = 0.
\end{aligned}$$

But, this is nothing more than the Hankel transform of the differential equation for T. By taking the Hankel transform of both sides we can remove the integral, giving an ordinary differential equation for the time component:

$$\frac{\ddot{T}(t)}{k} + gT(t) + Gmv \operatorname{sgn}(t) e^{-kv|t|} = 0.$$

Clearly, the homogeneous solution is $T(t) = A\cos(\omega_k t) + B\sin(\omega_k t)$ with $\omega_k^2 = gk$. The form of this differential equations suggests the particular solution should take the form $T(t) = Ce^{-kv|t|}$. Substituting this in yields

$$C\left(k^2v^2 \operatorname{sgn}^2(t) e^{-kv|t|} + gke^{-kv|t|}\right) + Gmvk \operatorname{sgn}(t) e^{-kv|t|} = 0.$$

[1]Note that $\int_0^\infty \sqrt{r}(r^2+v^2t^2)^{-3/2} dr = \Gamma^2(3/4)(\pi v^3 t^3)^{-1/2} < \infty$ for $t \neq 0$ which is not concerning since this is true almost everywhere, and so the condition in Theorem *** is satisfied.

Now, by rearranging we find

$$
\begin{aligned}
C &= \frac{-Gmvk\,\mathrm{sgn}(t)}{k^2v^2+gk}, \\
&= \frac{Gmv}{g}\frac{-\,\mathrm{sgn}(t)}{1+kv^2/g}
\end{aligned}
$$

as the coefficient, and,

$$
T(t) = \frac{Gmv}{g}\frac{1}{1+kv^2/g}\left(-\,\mathrm{sgn}(t)e^{-kv|t|}\right) + A\cos(\omega_k t) + B\sin(\omega_k t)
$$

as the full time component of the velocity potential. We can now apply the boundary conditions to find A, and B. Physically, we expect $T(t) \in C^1(-\infty,\infty)$, furthermore, we only expect the sinusoidal terms to contribute at times greater than zero, thus finally, we procure

$$
\begin{aligned}
T(t) &= \frac{Gmv}{g}\frac{1}{1+kv^2/g}\left(-\,\mathrm{sgn}(t)e^{-kv|t|} + 2H(t)\cos(\omega_k t)\right)dk, \text{ and,} \\
\varphi &= \frac{Gmv}{g}\int_0^\infty \frac{J_0(kr)e^{kz}}{1+kv^2/g}\left(-\,\mathrm{sgn}(t)e^{-kv|t|} + 2H(t)\cos(\omega_k t)\right)dk.
\end{aligned}
$$

Another way the velocity potential can be expressed is as the Hankel transform of the time component, $\varphi = \left(\mathscr{H}\, e^{kz}\frac{T(t;k)}{k}\right)(r,z,t)$, which will be of particular importance in Chapter 1.4

1.3 Deformation of the Surface

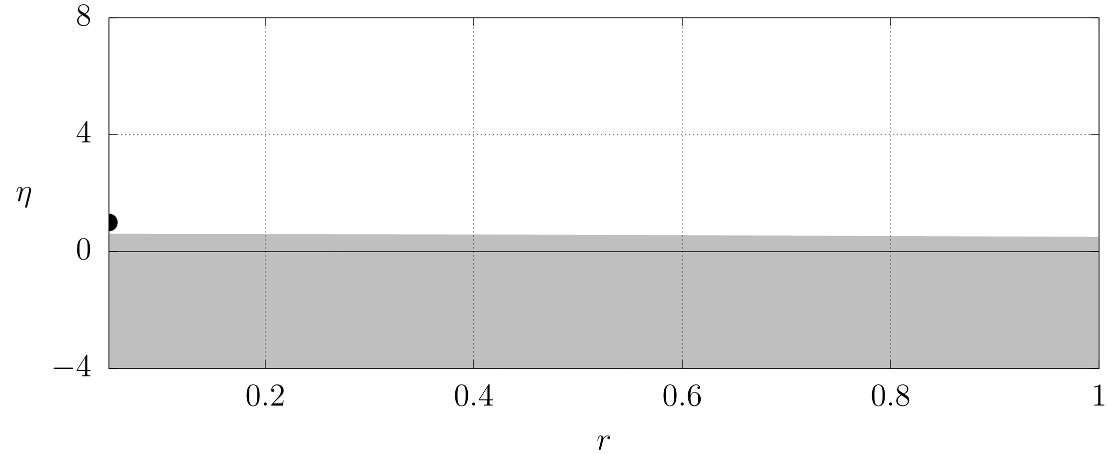
From (7) we can isolate for η:

$$
\begin{aligned}
\eta &= \int_0^t \left.\frac{\partial \varphi}{\partial z}\right|_{z=0} dt \\
&= \int_0^t \frac{Gmv}{g} \int_0^\infty \frac{kJ_0(kr)}{1+kv^2/g} \left(-\operatorname{sgn}(t)e^{-kv|t|} + 2H(t)\cos(\omega_k t)\right) dk\, dt \\
&= \frac{Gmv}{g} \int_0^\infty \frac{kJ_0(kr)}{1+kv^2/g} \left(\int_0^t -\operatorname{sgn}(t)e^{-kv|t|} + 2H(t)\cos(\omega_k t)\, dt\right) dk \\
&= \frac{Gmv}{g} \int_0^\infty \frac{kJ_0(kr)}{1+kv^2/g} \left(-\operatorname{sgn}(t)\frac{e^{-kv|t|}}{-kv\operatorname{sgn}(t)} + 2H(t)\frac{1}{\omega_k}\sin(\omega_k t)\right) dk \\
&= \frac{Gm}{g} \int_0^\infty \frac{J_0(kr)}{1+kv^2/g} \left(e^{-kv|t|} + 2H(t)v\sqrt{\frac{k}{g}}\sin(\omega_k t)\right) dk \\
\text{Let } \widetilde{T}(t;k) &= \frac{Gm}{g}\frac{1}{1+kv^2/g}\left(e^{-kv|t|} + 2H(t)v\sqrt{\frac{k}{g}}\sin(\omega_k t)\right)
\end{aligned}
$$

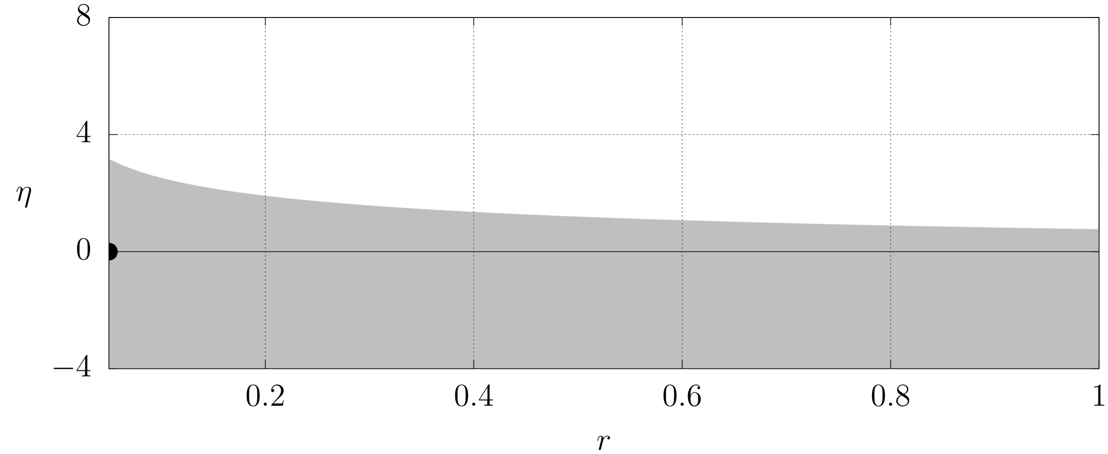
1.4 Calculating the Energy Transferred

$$
\begin{aligned}
E &= \frac{1}{2}\rho \int \left|\vec{\nabla}\varphi\right|^2 dV + \rho g \int z\, dV \\
&= \frac{1}{2}\rho \int_0^{2\pi}\int_0^\infty\int_{-\infty}^0 \left|\vec{\nabla}\varphi\right|^2 dz\, r\, dr\, d\theta + \rho g \int_0^{2\pi}\int_0^\infty\int_0^\eta z\, dz\, r\, dr\, d\theta \\
&= \rho\pi \left(\int_0^\infty\int_{-\infty}^0 \left|\vec{\nabla}\varphi\right|^2 r\, dz\, dr + g\int_0^\infty \eta^2 r\, dr\right) \\
&= \rho\pi \left(\int_0^\infty\int_{-\infty}^0 \left(\frac{\partial\varphi}{\partial r}\right)^2 r\, dz\, dr + \int_0^\infty\int_{-\infty}^0 \left(\frac{\partial\varphi}{\partial z}\right)^2 r\, dz\, dr + g\int_0^\infty \eta^2 r\, dr\right)
\end{aligned}
$$

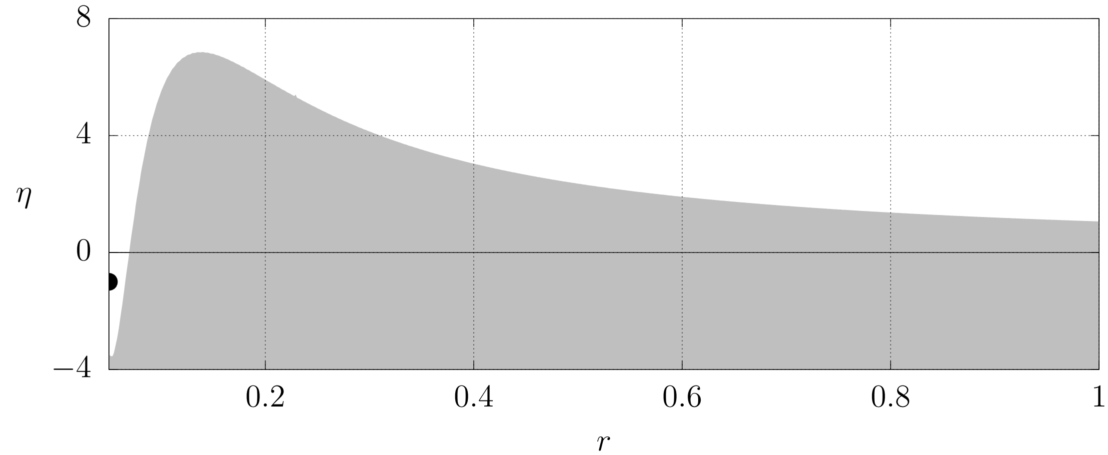
To simplify and tidy this calculation, let ♣, ♠, and ♦ be the three terms respectively, so



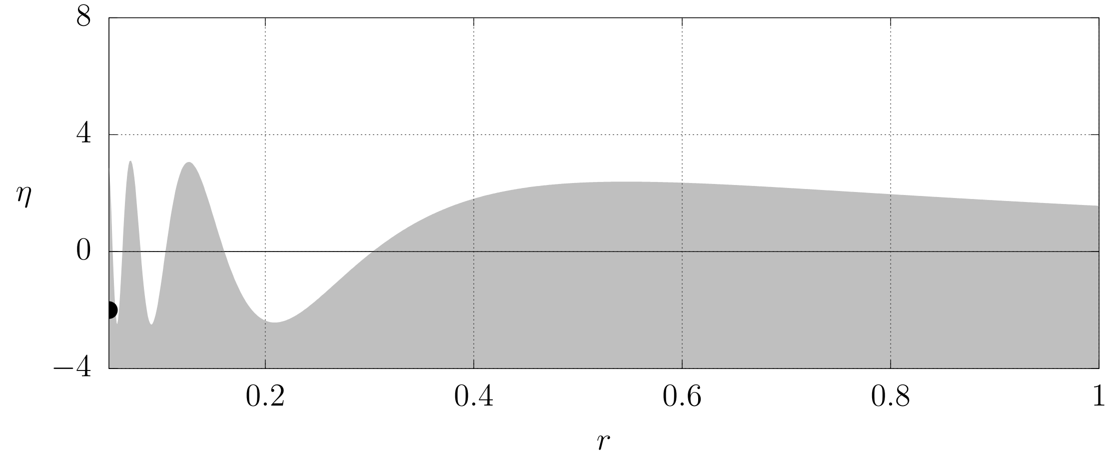
(a) $t = -1$ s.



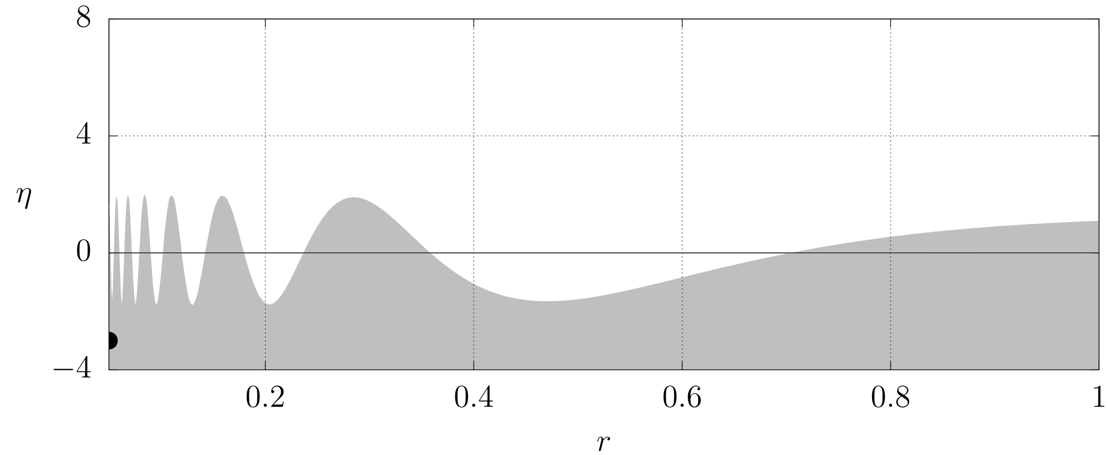
(b) $t = 0$ s.



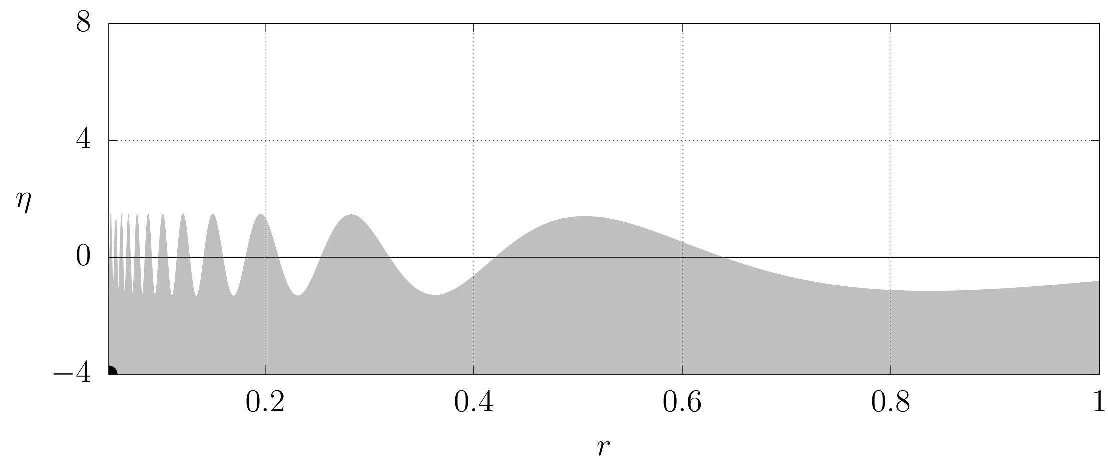
(c) $t = 1$ s.



(d) $t = 2$ s.



(e) $t = 3$ s.



(f) $t = 4$ s.

that

$$E = \rho\pi(\clubsuit + \spadesuit + g\blacklozenge).$$

$$\begin{aligned}
\clubsuit &= \int_0^\infty \int_{-\infty}^0 \left(\frac{\partial\varphi}{\partial r}\right)^2 r\,dz\,dr \\
&= \int_{-\infty}^0 \int_0^\infty \left(\mathscr{H}_1\,\frac{e^{kz}T(t;k)}{k}\right)^2 (r,z,t) r\,dr\,dz \\
&* ** \\
&= \int_{-\infty}^0 \int_0^\infty e^{2kz} T^2(t;k)\,k\,dk\,dz \\
&= \frac{1}{2}\int_0^\infty T^2(t;k)\,dk
\end{aligned}$$

$$\begin{aligned}
\spadesuit &= \int_0^\infty \int_{-\infty}^0 \left(\frac{\partial\varphi}{\partial z}\right)^2 r\,dz\,dr \\
&= \int_{-\infty}^0 \int_0^\infty \left(\mathscr{H}\; e^{kz}T(t;k)\right)^2 (r,z,t) r\,dr\,dz \\
&* ** \\
&= \int_{-\infty}^0 \int_0^\infty e^{2kz} T^2(t;k)\,k\,dk\,dz \\
&= \frac{1}{2}\int_0^\infty T^2(t;k)\,dk
\end{aligned}$$

$$
\begin{aligned}
\blacklozenge &= \int_0^\infty \eta^2 r \, dr \\
&= \int_0^\infty \left(\mathscr{H} \, \frac{\widetilde{T}(t;k)}{k} \right)^2 (r,t) \, r \, dr
\end{aligned}
$$

* **

$$
= \int_0^\infty \frac{\widetilde{T}^2(t;k)}{k} \, dk
$$

$$
E(t) = \rho\pi \int_0^\infty T^2(t;k) + g\frac{\widetilde{T}^2(t;k)}{k} \, dk
$$

$$
\begin{aligned}
E = \frac{G^2 m^2 \rho\pi}{g} \int_0^\infty \frac{v^2}{g} & \left(\frac{-\operatorname{sgn}(t) e^{-kv|t|} + 2\,\mathrm{H}(t)\cos(\omega_k t)}{1 + kv^2/g} \right)^2 \\
& + \frac{1}{k} \left(\frac{e^{-kv|t|} + 2\,\mathrm{H}(t) v \sqrt{\frac{k}{g}} \sin(\omega_k t)}{1 + kv^2/g} \right)^2 \, dk
\end{aligned}
$$

By taking the long time limit the exponentials decay to zero, and the sine and cosine simplify to 1. The energy then becomes

$$
E = 4\pi\rho \frac{G^2 m^2}{g}.
$$

*** Compare with dynamical friction solution

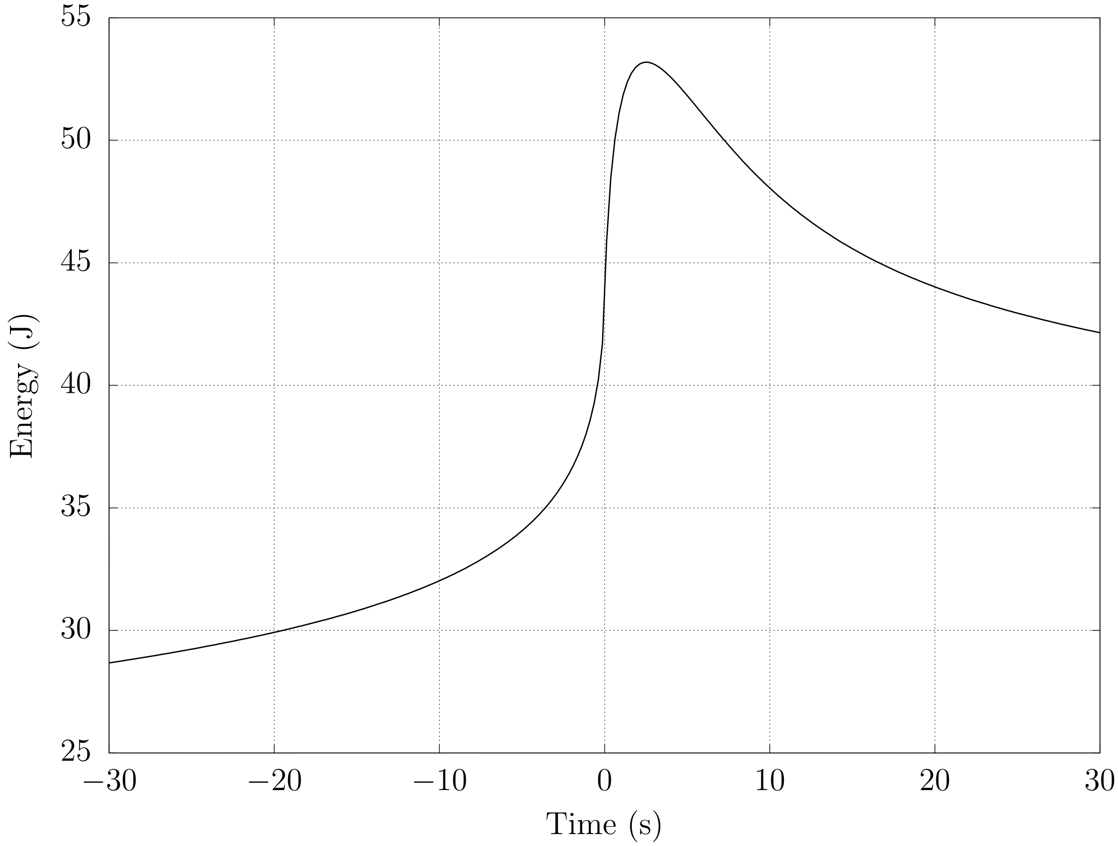


Figure 2: Energy! All variables set to 1.