What does it mean to test a hypothesis?

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EGAP Learning Days 4: Chile

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Counterfactual Causal Inference

How can we use what we **see** to learn about what we want to **know** ?

			Turnout				
City	Pair	Treatment	Baseline	Outcome	Newspaper	y_1	y_0
Saginaw	1	0	17	16		?	16
Sioux City	1	1	21	22	Sioux City Journal	22	?
Battle Creek	2	0	13	14		?	14
Midland	2	1	12	7	Midland Daily News	7	?
Oxford	3	0	26	23		?	23
Lowell	3	1	25	27	Lowell Sun	27	?
Yakima	4	0	48	58		?	58
Richland	4	1	41	61	Tri-City Herald	61	?

Design and outcomes in the Newspapers Experiment. The Treatment column shows treatment randomized within pair with the newspaper ads as 1 and lack of treatment as 0. The potential outcomes are y_1 for treatment and y_0 for control. Panagopoulos (2006) provides more detail on the design of the experiment.

Counterfactual Causal Inference

How can we use what we **see** to learn about **potential outcomes**?

			Turnout				
City	Pair	Treatment	Baseline	Outcome	Newspaper	y_1	y_0
Saginaw	1	0	17	16		?	16
Sioux City	1	1	21	22	Sioux City Journal	22	?
Battle Creek	2	0	13	14		?	14
Midland	2	1	12	7	Midland Daily News	7	?
Oxford	3	0	26	23		?	23
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Counterfactual Causal Inference

How can we use what we **see** to learn about a causal effect$_i = f(y_{i,1}, y_{i,0})$?

City	Pair	Treatment	Turnout Baseline	Turnout Outcome	Newspaper	y_1	y_0
Saginaw	1	0	17	16		?	16
Sioux City	1	1	21	22	Sioux City Journal	22	?
Battle Creek	2	0	13	14		?	14
Midland	2	1	12	7	Midland Daily News	7	?
Oxford	3	0	26	23		?	23
Lowell	3	1	25	27	Lowell Sun	27	?
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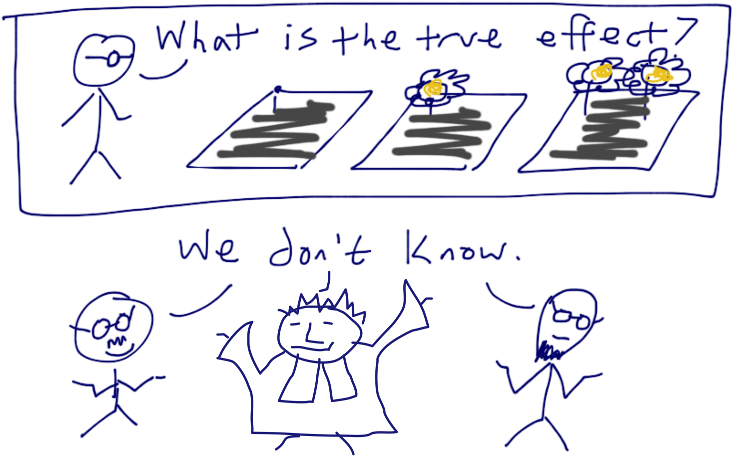
***Small group exercise:* Can you think of more than one way to learn about the counterfactual causal effect of treatment using what we observe?**

What is the true effect of the treatment assignment?



What is the true effect of the treatment assignment?

I don't know the truth, but I can provide a good guess of the average causal effect.

i	Z_i	Y_i	y_{i1}	y_{i0}
A	0	16	?	16
B	1	22	22	?
C	0	7	?	7
D	1	14	14	?
			$\bar{y}_{i1}$	$\bar{y}_{i0}$

$$\widehat{ATE} = \bar{Y}_i \mid Z_i = 1 - \bar{Y}_i \mid Z_i = 0$$

$$= \frac{22+14}{2} - \frac{16+7}{2} = 6.5$$

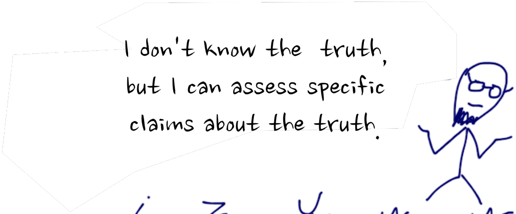
What is the true effect of the treatment assignment?

I dew nut knew thee truth, but, given pryors, I cane predikte itf probabeeleetee.

i	Z_i	Y_i	y_{i1}	y_{i0}
A	0	16		16
B	1	22	22	
C	0	7		7
D	1	14	14	

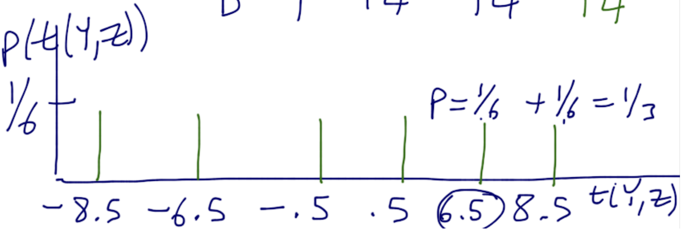
$$P\left(\boxed{}, f(y_{i1} - y_{i0})\right) =$$

What is the true effect of the treatment assignment?



$H_0: y_{i,1} = y_{i,0}$

i	Z_i	Y_i	$y_{i,1}$	$y_{i,0}$
A	0	16	?	16
B	1	22	22	22
C	0	7	?	7
D	1	14	14	14



Testing Fishers Sharp Null Hypothesis of No Effects

Testing Sharp Hypotheses No Effects and No Interference

Aspects of the approach: The tests do not require asymptopia.

What is a p-value? What does a p-value require?

Sharp Hypotheses of Some Effects. Sharp Hypotheses as Causal Models

- The Constant Additive Effect Model

Information arises from both theory and instruments

Summary and Discussion

Testing the Sharp Null of No Effects

```
Z <- c(0,1,0,1) ## Define treatment vector
Y <- c(16,22,7,14) ## Define outcome vector
## There are choose(4,2) ways to assign treatment
## Make a matrix containing all of the ways to assign treatment
Om <- matrix(0,ncol=choose(4,2),nrow=length(Z)) ## First fill with zeros
whotrted <- combn(1:4,2) ## Generate indicators of who is treated
for(i in 1:choose(4,2)){ Om[cbind(whotrted[,i],i)]<-1 }

## Great a function to summarize the hypothesized
## treatment to outcome relationship
meandifftz <- function(y,z){ mean(y[z==1]) - mean(y[z==0]) }
rankdifftz <- function(y,z){ q<-rank(y); mean(q[z==1]) - mean(q[z==0]) }
## Apply the function to all possible experiments
mdist<-apply(Om,2, function(z){ meandifftz(Y,z) })
rdist<-apply(Om,2, function(z){ rankdifftz(Y,z) })
rbind(Om,mdist,rdist)

      [,1] [,2] [,3] [,4] [,5] [,6]
       1.0  1.0  1.0  0.0  0.0  0.0
       1.0  0.0  0.0  1.0  1.0  0.0
       0.0  1.0  0.0  1.0  0.0  1.0
       0.0  0.0  1.0  0.0  1.0  1.0
mdist  8.5 -6.5  0.5 -0.5  6.5 -8.5
rdist  2.0 -1.0  0.0  0.0  1.0 -2.0
```

continued

```
table(mdist)

mdist
-8.5 -6.5 -0.5  0.5  6.5  8.5
   1    1    1    1    1    1

table(rdist)

rdist
-2 -1  0  1  2
 1  1  2  1  1

mobs <- meandifftz(Y,Z) ## observed value of test stat
robs <- rankdifftz(Y,Z) ## observed value of test stat
mean(mdist >= mobs) ## p-value

[1] 0.3333333

mean(rdist >= robs) ## p-value

[1] 0.3333333
```

Faster Method

```
library(RItools,lib.loc="libraries") ## see fishersinference.Rnw to get the dev versi

## Specify design: only 1 block
randomassignment<-simpleRandomSampler(z=Z,b=rep(1,4))

meandiffTZ <- function(ys, z) {
    mean(ys[!(!z)]) - mean(ys[!z])
}

meandifftest<-RItest(y=Y, z=Z, test.stat=meandiffTZ, sampler = randomassignment)
meandifftest

Call:  RItest(y = Y, z = Z, test.stat = meandiffTZ, sampler = randomassignment)



                        Value Pr(>x)
Observed Test Statistic   6.5 0.3333
```

The Newspapers Study

			Turnout				
City	Pair	Treatment	Baseline	Outcome	Newspaper	y_1	y_0
Saginaw	1	0	17	16		?	16
Sioux City	1	1	21	22	Sioux City Journal	22	?
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Design and outcomes in the Newspapers Experiment. The Treatment column shows treatment randomized within pair with the newspaper ads as 1 and lack of treatment as 0. The potential outcomes are y_1 for treatment and y_0 for control. Panagopoulos (2006) provides more detail on the design of the experiment.

The Newspapers Study: $H_0 : y_{i1} = y_{i0}, p = 6/16.$

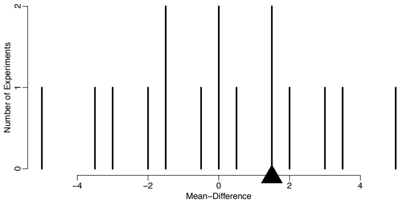
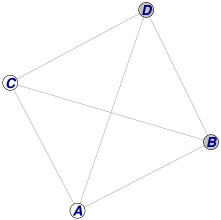


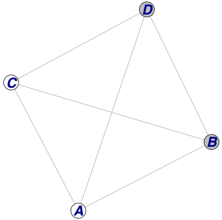
Figure 2: The randomization distribution of the mean-difference test statistic under the null hypothesis of no effects is shown by the tall black lines. The black triangle shows the value of the mean-difference observed in the actually fielded experiment.

Statistical inference for counterfactual quantities with interference?



i	Z_i	Y_i	$y_{i,1100}$	$y_{i,0101}$	$y_{i,1001}$	$y_{i,0110}$	$y_{i,1010}$	$y_{i,0011}$
A	0	16	?	16	?	?	?	?
B	1	22	?	22	?	?	?	?
C	0	7	?	7	?	?	?	?
D	1	14	?	14	?	?	?	?

Statistical inference for counterfactual quantities with interference?



i	Z_i	Y_i	$y_{i,1100}$	$y_{i,0101}$	$y_{i,1001}$	$y_{i,0110}$	$y_{i,1010}$	$y_{i,0011}$	$y_{i,0000} \equiv y_{i,\mathbf{0}}$
A	0	16	?	16	?	?	?	?	16
B	1	22	?	22	?	?	?	?	22
C	0	7	?	7	?	?	?	?	7
D	1	14	?	14	?	?	?	?	14

The sharp null of no effects is a model of no interference:

$H_0 : y_{i,1100} = y_{i,0101} = y_{i,1001} = y_{i,0110} = y_{i,1010} = y_{i,0011} = y_{i,0000}$,
$y_{i,\mathbf{0}} = \mathcal{H}(y_{i,\mathbf{z}}, \mathbf{0}) = y_{i,\mathbf{z}}$, $p = 0.33$.

Introducing the **uniformity trial** $\equiv \mathbf{y}_{i,0000}$ (Rosenbaum, 2007).

Fisherian Tests work when asymptopia is out of reach

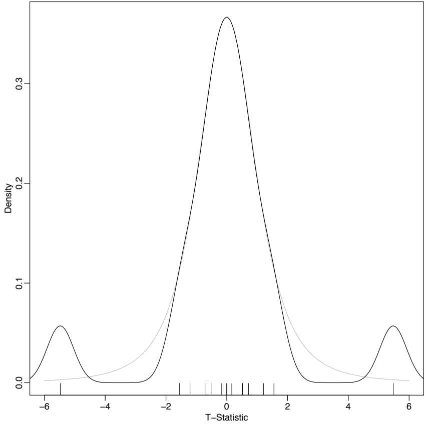


Figure 5: Under the null hypothesis of no effects, t-statistics calculated after repeating the experimental assignment process should be distributed around zero. The exact randomization distribution is tri-modal for the Newspapers study and does not match the t-distribution that we would expect under a larger sample.

Fisherian Tests work when asymptopia is out of reach

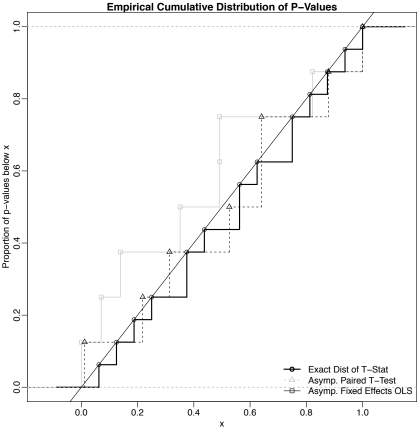


Figure 6: Cumulative distributions of p-values for testing the null hypothesis of no effects. A valid test would be at or below the diagonal line. Tests which would encourage rejection too quickly have points above the line.

What does it mean to test a hypothesis?

What is a p-value?

What do we learn from a hypothesis test?

What do we need to assume to make a hypothesis test (in this simplest case) interpretable?

What range of effects might be surprising?

Hypothesize a model of potential outcomes For $H_0 : y_{i1} = y_{i0} + \tau$, what τ might be surprising?

Map the model to observation via design What would $\tau = 6$ imply for what we observe? If $\tau = 6$ and $Y_i = Z_i y_{i1} + (1 - Z_i) y_{i0}$ then $Y_i - Z_i 6 = y_{i0}$.

Generate the randomization distribution of this hypothesis As before

Summarize information against the hypothesis For hypotheses of $\tau \geq 6$ we have $p \leq .125$ using a mean difference test statistic.

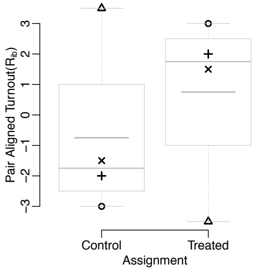


Figure 3: The distribution of outcomes in the control and treatment groups. The values of turnout are “pair aligned” or “pair mean centered” so that we can focus attention on the paired differences rather than on the differences in turnout levels between pairs.

What effects might be least surprising?

The idea of a "best guess" maps onto the Hodges-Lehmann point estimate: the hypotheses for which $E(t(Z,Y) = 0$: ex. the difference of means is zero. Here $\tau = 1.5$ for the mean difference and $\tau = 3.25$ for the rank-based test (which equalizes medians).

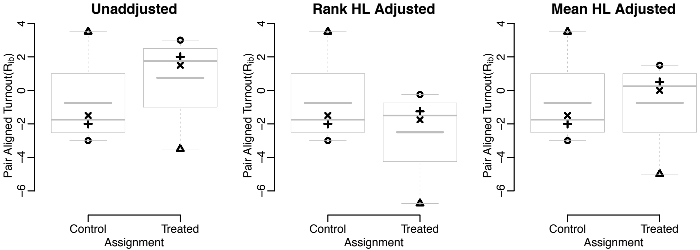


Figure 4: The unadjusted comparison of controls versus treated (left most plot) as compared to the results of adjusting the treatment group based on the Hodges-Lehmann (HL) point estimates derived from two different test statistics (Wilcoxon paired signed ranks and the mean difference). Notice that the control group remains the same and it is the hypotheses which imply changes in the distribution of the treatment group compared to the unadjusted, observed, outcomes.

Theoretical models of potential outcomes can produce sharp hypotheses

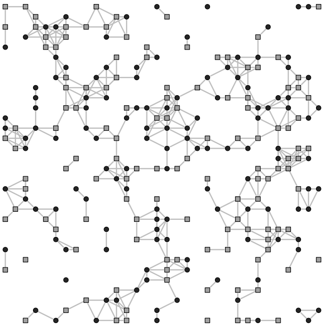


Figure: A simulated data set with 256 units and 512 connections. The $256/2 = 128$ treated units ($Z_i = 1$) are shown as filled circles and an equal number of control units ($Z_i = 0$) are shown as as gray squares.

Theoretical models of potential outcomes can produce sharp hypotheses

$$\mathcal{H}(\mathbf{y_0}, \mathbf{z}, \beta, \tau) = \left[\beta + (1 - z_i)(1 - \beta) \exp\left(-\tau^2 \mathbf{z}^T \mathbf{S}\right)\right] \mathbf{y_0} \tag{1}$$

$$\mathcal{H}(\mathbf{y_z}, \mathbf{0}, \beta, \tau) = \left[\beta + (1 - z_i)(1 - \beta) \exp\left(-\tau^2 \mathbf{z}^T \mathbf{S}\right)\right]^{-1} \mathbf{y_z} \equiv \mathbf{y_0} \tag{2}$$

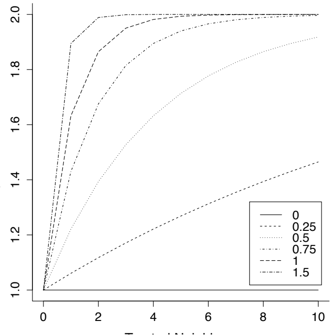


Figure: Growth curve of spillover effects for the expression $\beta + (1 - \beta) \exp\left(-\tau^2 \mathbf{z}^T \mathbf{S}\right)$ as the number of treated neighbors, $\mathbf{z}^T \mathbf{S}$, increases for $\beta = 2$ and a selection of τ values.

Theoretical models of potential outcomes can produce sharp hypotheses

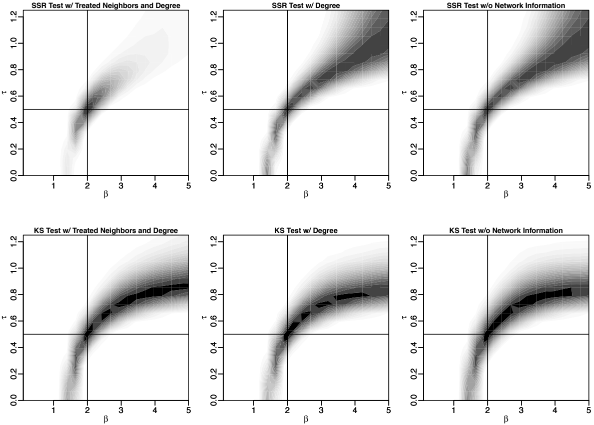


Figure: Proportion of *p*-values less than .05 for randomization tests of joint hypotheses about τ and β. Darker values mean less rejection. Truth is at $\tau = .5, \beta = 2$. All tests reject the truth no more than 5% of the time at $\alpha = .05$. All simulations, not Normal approximations.

A General Fisherian Inference Algorithm

1. Write a model ($\mathcal{H}(\mathbf{y_0}, \mathbf{z}, \theta)$) converting uniformity trial into observed data (i.e. a causal model).
2. Solve for $\mathbf{y_0}$: $\mathcal{H}(\mathbf{y_z}, \mathbf{0}, \theta_0) = \mathbf{y_0}$
3. Select a test statistic that is effect increasing in all relevant dimensions.
4. Compute p-values for substantively meaningful range of θ. Or calculate boundaries of regions.

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Summary and Discussion

Robust test statistics can increase power.

For the simple mean difference test statistic, we have $p = .375$, for a rank sum test (the sum of the ranks of the treated units), we have a $p = .4375$, and for an M-estimator based test (like mean-differences but with weights roughly inversely proportional to the Cook's D influence measure) we have $p = .3125$.

Cook's D	MM–Weights	City	Pair	Treatment	Turnout
0.13	1.00	Saginaw	1	0	16
0.13	1.00	Sioux City	1	1	22
0.48	0.94	Battle Creek	2	0	14
0.48	0.94	Midland	2	1	7
0.04	1.00	Oxford	3	0	23
0.04	1.00	Lowell	3	1	27
0.01	1.00	Yakima	4	0	58
0.01	1.00	Richland	4	1	61

Table 2: Not all cases have the same influence on the mean difference. Cook's D summarizes this influence — Battle Creek and Midland have disproportionate influence. A robust fit (details in the text) downweights Battle Creek and Midland.

Covariance adjusted tests can increase power.

Using the difference pre-vs-post as the outcome (comparing treated pre-vs-post with paired control pre-vs-post), and using the robust test statistic for $H_0: y_{i1} = y_{i0}$ $p = 4/16 = .25$. Using $e_i = (Y_i - Y_{i,t-1}) - (\hat{\beta}_0 + \hat{\beta}_1 \text{pop} + \hat{\beta}_2 \text{num candidates})$ and the robust test statistic to test $H_0: y_{i1} = y_{i0}$ we have $p = 2/16 = .125$.

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Summary and Discussion

Key features of Fisher's approach

Flexible Any scientific model than can generate implications for all units' potential outcomes can, in principle, produce testable parameters.

Design based Requires knowledge of probability of Z not Y or $Y|X$ or $\beta|\gamma$.

Finite Sample Oriented Does not require asymptopia. Can use asymptopia when there for a visit.

Can be slow In between 8 cities and asymptopia is a land of many permutations.

Probably conservative Uses relatively little of the total information we have available about the science.

If you want to know more read Paul Rosenbaum's work The version of Fisher's approach I discuss here is built on work by Paul Rosenbaum. Read his work if you want to learn more.

Panagopoulos, Costas. 2006. “The Impact of Newspaper Advertising on Voter Turnout: Evidence from a Field Experiment.” Paper presented at the MPSA 2006.