Experiment Design for Computer Sciences (01CH740)

Topic 02 - Point and Interval Indicators

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Part I - Intro

What is today's class about?

Topics

- Main topic: Indicators;
  - A first look at data from experiments;
  - Point Indicators: What our data tells us about the world?
  - Interval Indicators: How sure are we of our data?
  - Some statistical concepts;
- Some example code;
- Feedback from the last class;

Don't forget to bring your questions to the office hour!

Part II.a: Indicators Basic Concepts

Lecture Outline

In the first lecture, we talked about what is science, and how experiments (carefully designed experiments) are important in science.

Starting from this lecture, we will talk about how we use statistics to understand the data that we gather from experiments, and how we can draw conclusions about them.

Topics for today:

- Experimental Data: Population, Observation and Sample;
- Point indicators: How we obtain information about the population from samples;
- Interval indicators: Indicators with quality info!

Lecture Outline

Data and "Experiment Data"

When we talk about "data" in Computer Science, the first thing that comes to mind is "information that we feed to a program". **For example**: images, network logs, user databases, etc.

In this course, we are talking about "Experiment Data", which should be understood as "Data about the result of an experiment". **For example**: How long did the experiment take? What is the success rate of my program?

In fact, we can use the tecniques of this lecture for the first kind of data too! But to make things simpler, let's concentrate on the second kind of data.

Example of Data Collection

Using Experiment Data to characterize a system

How can we use experiment data to learn more about the world?

Tsukuba University is famous for many olympic athletes. Let's say I want to investigate **WHY**. After thinking a bit, I come up with two questions:

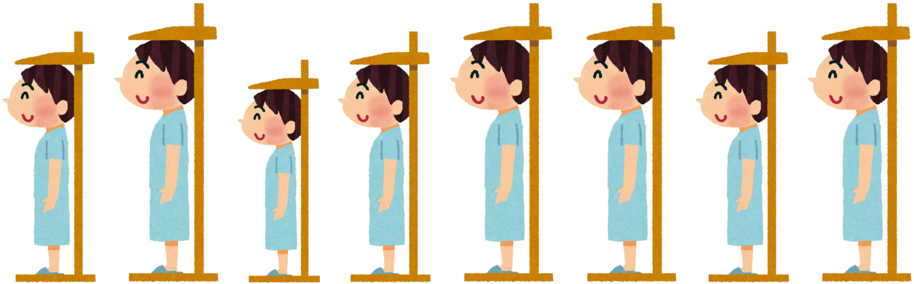
- Is the olympic performance related with the body of the students?
- Are students in Tsukuba stronger / taller / healthier than other unis?

The 1st question seems hard to answer, but maybe the 2nd is easier?

Imagine I take one student from Tsukuba and measure their height. Can this information help me answer the second question?

Using experiment data to characterize a system

From the height of only one student, I cannot really learn anything about the height of the students of the university **in general**. A better approach would be to measure the height of several students.



Now that I have the height of many students, what can I say about the height of the students in the university, **in general**?

Population and Sample

This example introduces us to some important concepts in statistics: Population, Observation, and Sample.

- **Population**: This is a set of objects that we want to learn more about, using experiments. It can be a real set (all students of the university), or a theoretical set (all possible results of running a program).
- **Observation**: This is one element from the population. One student from the university, or one execution of the program. One data point from an experiment.
- **Sample**: This is a set of observations. A subset of the population.

Our goal, when we analyse data from an experiment, is to **"learn something about the population, by examining the observations in the sample"**.

Population and Sample

Learn something about the population, by examining the observations in the sample

Population

You want to know the proportion of colorful balls in a pool (you like the red ones). Because we don't know exactly how many there are, we need to **make an estimation.**

Sample

To learn the proportion of red balls, we pick a number of balls, and **estimate that the proportion of red balls in the pool is equal to the proportion in my hand.**

Population, Model and Parameters

What is a model?

A model is a description of the population, focused on the scientific questions that we want to make.

- The balls are distributed evenly in the pool, so I can take my sample from any part.
- The height of the students in the university can be represented by a **normal curve**, with mean μ and standard deviation *sd*.
- The SIR infection model says that **susceptible** people become **infected**, and then **Recovering**, so if we can learn the number of people in each group and the transition probability, we can predict the progress of a disease.

The goal of many experiments is to use data to **estimate the parameters of a model.**

Population, Model and Parameters

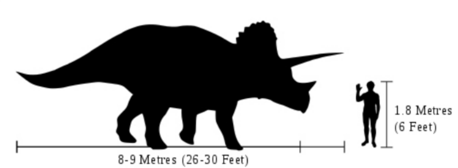
Example of model parameters

By analysing the sample data obtained from an experiment, we can estimate values for the parameters of the model.

The **true value** of the parameters in the model (what we don't know) is usually called θ. The values estimated from the experiment data is named $\hat{\theta}$. Note the difference!

The **model** is determined during the experiment design phase.

Every model is wrong, but some models are useful!



Parmeter	Average Size	Maximal Size
Length	7.5–8.5 meters (25–28 ft)	9 meters (30 ft)
Height	2–2.9 meters (6.6–9.5 ft)	3 meters (9.8 ft)
Mass	6,000–8,500 kilograms (13,200–18,700 lb)	10–12 tonnes (22,000–26,000 lb)
Skull Length	2–2.2 meters (6.6–7.2 ft)	2.5–2.8 meters (8.2– 9.2 ft)
Brow Horns Length (horn core)	70–100 centimetres (28–39 in)	115 centimetres (45 in)
Brow Horns Length	97–130 centimetres (38–51 in)	130–150 centimetres (51–59 in)

Sample Data, Statistics, and Parameters

Statistics are functions on the data

A statistic (singular) is a function, that takes experiment data as its input.

We estimate the value of the parameter of a model, by calculating a statistic, based on data we obtained from the sample. For example:

- How long does it take to run a program? Run the program many times (sample), and calculate the mean execution time (statistic).
- How effective is a drug? We give the drug to sick patients (sample), and count how many patients get better after two days (statistic);
- Is it better to add more perceptrons to a neural network? We run the network with different sizes (sample), and calculate the correlation between size and accuracy (statistic).

Point and Interval Indicators

In this lecture, we focus on two types of statistics: **Point Estimators** and **Interval Estimators**.

- **Point Estimators**: estimate one value for a parameter from a sample;
- **Interval Estimators**: estimate a range of values for a parameter;

Statistics are Random Variables

The estimated value depends on the population and the sample

- The value of an statistic is calculated from the sample, that we obtained from the experiment.
- The sample, in turn, is a subset of the population, which is what we want to study.

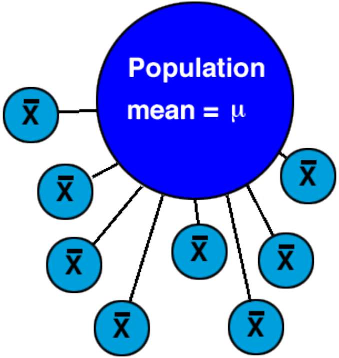
- The implication is that **every time we do a new experiment, the value calculated by an statistic will be a little bit different.**

Don't Worry!

These differences are expected and, if your experiment design is good, can be controlled. Just remember that a value you calculate from a sample is not necessarily **the truth**^{tm}

Statistics are Random Variables

What is measured by statistical tests?



A statistic can show different values, depending on the sample obtained. So it is useful to treat statistics as Random Variables.

As a Random Variable, any statistic has a **sampling distribution**. The sampling distribution is a model that describes what values we expect to obtain from a statistic, when we perform an experiment.

In general, statistical tests work by estimating the sampling distribution, and using the sampling distribution to understand the population under study. (We will study statistical tests in the next lecture).

Indicators, Part II - Point Estimators

Definition of Point Indicator

A *point estimator* is a statistic which provides the value of maximum plausibility for an (unknown) population parameter θ.

Consider a random variable X distributed according to a given $f(X|\theta)$.

Consider also a random sample from this variable: $x = \{x_1, x_2, \ldots, x_n\}$;

A given function $\hat{\Theta} = h(x)$ is called a *point estimator* of the parameter θ, and a value returned by this function for a given sample is referred to as a *point estimate* $\hat{\theta}$ of the parameter.

Examples

Point estimation problems arise frequently in all areas of science and engineering, whenever there is a need for estimating, e.g.,:

- a population mean, μ;
- a population variance, σ^2;
- a population proportion, p;
- the difference in the means of two populations, $\mu_1 - \mu_2$;
- etc..

In each case there are multiple ways of performing the estimation task, and the decision about which estimators to use is based on the mathematical properties of each statistic.

Errors and Biases

Note that we are being very careful to always use the word **estimate** when we talk about statistics. Why is that?

In all the examples that we mentioned, if we are unlucky[1], we could obtain an estimate that is very different from the true value of the population:

Bad statistics example

To estimate the height of the students of a school, we pick 10 students, and we measure the height of the youngest one.

- **Error**: The difference between an estimate and the true value of a population's parameter;
- **Bias**: The property of a statistic that systematically produces wrong estimates;

[1] or careless, or malicious

Unbiased estimators

A good estimator should consistently generate estimates that lie close to the real value of the parameter θ.

A given estimator $\hat{\Theta}$ is said to be *unbiased* for parameter θ if:

$$E\left[\hat{\Theta}\right] = \theta$$

or, equivalently:

$$E\left[\hat{\Theta}\right] - \theta = 0$$

The difference $E\left[\hat{\Theta}\right] - \theta$ is referred to as the *bias* of a given estimator.

Unbiased estimators

For example, the usual estimators for mean is an unbiased estimator;

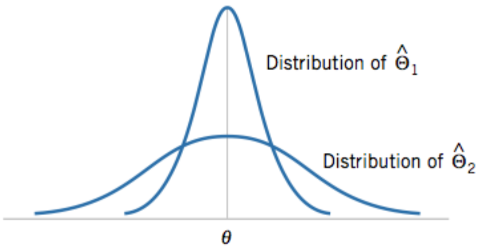
Let $x_1, \ldots, x_n$ be a random sample from a given population X, characterized by its mean μ and variance σ^2. In this situation, it is possible to show that:

$$E\left[\bar{x}\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \mu$$

(Remember that the expected value of one observation is the mean value of the population)

Unbiased estimators

Fora a population parameter θ, it is usually possible to define more than one unbiased estimator. The variances of these estimators may, however, be different



In these cases, we usually want to obtain the unbiased estimator of minimal variance. This is generally called the *minimal-variance unbiased estimator* (MVUE).

MVUE are generally chosen as estimators due to their ability of generating estimates $\hat{\theta}$ that are (relatively) close to the real value of θ.

Standard error of a point estimator

Remember that because a point estimator is a random variable, it has an associated distribution and error. For example, the standard error of an estimator $\hat{\Theta}$ is

$$\sigma_{\hat{\Theta}} = \sqrt{Var\left[\hat{\Theta}\right]}$$

However, we can't know this directly. We can **estimate** the standard error of the estimator from the data in the sample. In this case, we refer to it as the *estimated standard error*, $\hat{\sigma}_{\hat{\Theta}}$ (the notations $s_{\hat{\Theta}}$ and $se(\hat{\Theta})$ are also common).

Standard error of a point estimator

Examples

Assuming a random variable X from a gaussian distribution, and a sample error s, we can calculate the standard errors of several common point indicators[2]

$$\hat{\sigma}_{\bar{X}} = \frac{s}{\sqrt{n}}$$

$$\hat{\sigma}_{S^2} = s^2\sqrt{\frac{2}{n-1}}$$

$$\hat{\sigma}_S = \frac{s}{\sqrt{2(n-1)}} + O\left(\frac{1}{n\sqrt{n}}\right) \approx \frac{s}{\sqrt{2(n-1)}}$$

[2]See Ahn and Fessler (2003), *Standard Errors of Mean, Variance, and Standard Deviation Estimators*: `https://git.io/v5Z5v`

Point Estimator Use Case

Consider an operation to produce coaxial cables[3]. The mean resistance of the production is 50Ω, with a standard deviation of 2Ω (Population mean, and population deviation).

Let's assume that the resistance value of the produced cables are distributed follow a <u>normal distribution ($X \sim \mathcal{N}(\mu = 50, \sigma^2 = 4)$)</u>

[3] Example inspired on `https://www.sas.com/resources/whitepaper/wp_4430.pdf`

Point estimator use case

Suppose that we take a random sample of 25 cables is taken from this production process (an experiment, to measure if the process is correct, for example).

The sample mean of the the observations is:

$$\bar{x} = \frac{1}{25}\sum_{i=1}^{25} x_i$$

The sample mean follows a normal distribution, with $E[\bar{x}] = \mu = 50\Omega$[4] and $\sigma_{\bar{x}} = \sqrt{\sigma^2/25} = 0.4\Omega$. The error depends on the sample size.

[4] since the sample mean is an unbiased estimator

The Central Limit Theorem

In the previous example, the production operation followed a normal distribution. But even for a population with an arbitrary distribution, the sampling distribution of its mean tends to be approximately normal.

More generally, let $x_1, \ldots, x_n$ be a sequence of **independent and identically distributed** (**iid**) random variables, with mean μ and finite variance σ^2. Then:

$$z_n = \frac{\sum_{i=1}^{n} (x_i) - n\mu}{\sqrt{n\sigma^2}} = \frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}}$$

is distributed asymptotically as a standard Normal variable, that is, $z_n \sim \mathcal{N}(0, 1)$.

The Central Limit Theorem

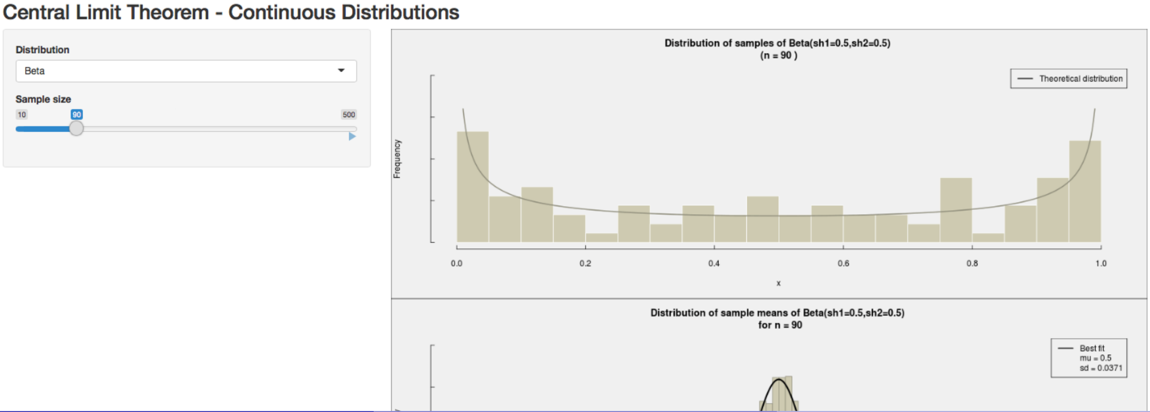
This result is known as the *Central Limit Theorem*[5], and is one of the most useful properties for statistical inference. The CLT allows the use of techniques based on the Normal distribution, even when the population under study is not normal.

For “well-behaved” distributions (continuous, symmetrical, unimodal - the usual bell-shaped pdf we all know and love) even small sample sizes are commonly enough to justify invoking the CLT and using parametric techniques.

[5] For more details on the CLT, see `https://www.encyclopediaofmath.org/index.php/Central_limit_theorem`

Sampling Distributions
The Central Limit Theorem

For an interactive demonstration of the CLT, download the files in `https://git.io/vnPj8` and run on RStudio.



Indicators, Part III - Interval Estimators

Statistical Intervals

Statistical intervals are important in quantifying the uncertainty associated to a given estimate;

As an example, let's recap the coaxial cables example: *a coaxial cable manufacturing operation produces cables with a target resistance of* 50Ω *and a standard deviation of* 2Ω*. Assume that the resistance values can be well modeled by a normal distribution*.

Let us now suppose that a sample mean of $n = 25$ observations of resistance yields $\bar{x} = 48$. Given the sampling variability, it is very likely that this value is not exactly the true value of μ, but we are so far unable quantify how much uncertainty there is in this estimate.

Definition

Statistical intervals define regions that are likely to contain the true value of an estimated parameter.

More formally, it is generally possible to quantify the level of uncertainty associated with the estimation, thereby allowing the derivation of sound conclusions at predefined levels of certainty.

Three of the most common types of interval are:

- Confidence Intervals;
- Tolerance Intervals;
- Prediction Intervals;

How to Interpret a Confidence Interval

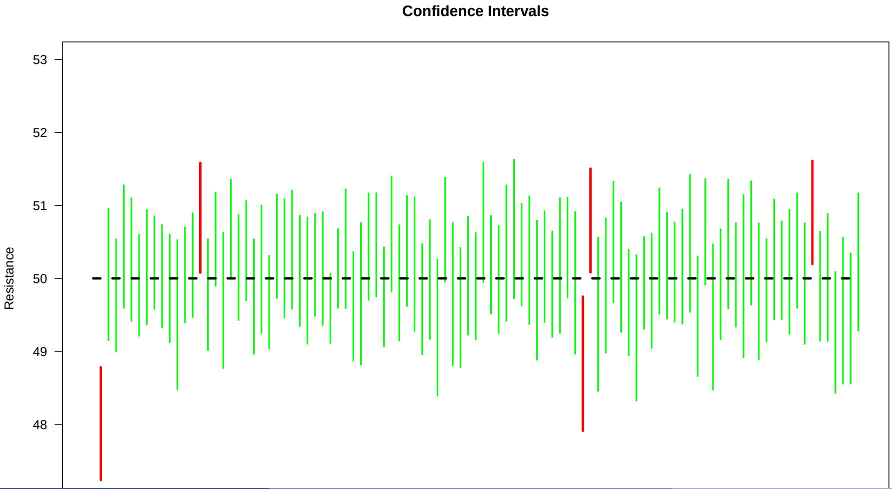
Confidence intervals quantify the degree of uncertainty associated with the estimation of population parameters such as the mean or the variance.

Can be defined as "*the interval that contains the true value of a given population parameter with a confidence level of* $100(1-\alpha)$";

Another useful definition is to think about confidence intervals in terms of confidence *in the method*: "The method used to derive the interval has a hit rate of 95%" - i.e., the interval generated has a 95% chance of 'capturing' the true population parameter."

Confidence Intervals

Example: 100 $CI_{.95}$ for a sample of 25 observations



CI on the Mean of a Normal Variable

The two-sided $CI_{(1-\alpha)}$ for the mean of a normal population with known variance σ^2 is given by:

$$\bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $(1-\alpha)$ is the confidence level and z_x is the x-quantile of the standard normal distribution.

For the more usual case with an unknown variance,

$$\bar{x} + t_{\alpha/2}^{(n-1)}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{1-\alpha/2}^{(n-1)}\frac{s}{\sqrt{n}}$$

where $t_x^{(n-1)}$ is the x-quantile of the t distribution with $n-1$ degrees of freedom.

CI on the Variance and Standard Deviation of a Normal Variable

A two-sided confidence interval on the variance of a normal variable can be easily calculated:

$$\frac{(n-1)s^2}{\chi^{2\,(n-1)}_{1-\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^{2\,(n-1)}_{\alpha/2}}$$

where $\chi^{2\,(n-1)}_{x}$ represents the x-quantile of the χ^2 distribution with $n-1$ degrees of freedom. For the standard deviation one simply needs to take the squared root of the confidence limits.

Wrapping up

Statistical intervals quantify the uncertainty associated with different aspects of estimation;

Reporting intervals is always better than point estimates, as it provides the necessary information to quantify the location and uncertainty of your estimated values;

The correct interpretation is a little tricky (although not very difficult)[7], but it is essential in order to derive the correct conclusions based on the statistical interval of interest.

[7] See the table at the end of `https://git.io/v5ZFh`

Part III - Outro

Summary

Descriptive Statistics

Experiment Data can be used to estimate facts about the world:

- Point estimators: Sample Means, Variance, Correlation, etc.
  - Give us specific information about the model we want to study
  - "What is the average height of a student?"
  - **An estimator is not the real value!**
- Interval estimators: Confidence Interval, IQR, etc.
  - Give us more information than point estimators.
  - "How certain should I be about this point estimator".
  - Size of interval estimator depends on the number of samples.

Recommended Reading

- *D.W. Stockburger*, The Sampling Distribution. In: Introductory Statistics: Concepts, Models, and Applications - `http://psychstat3.missouristate.edu/Documents/IntroBook3/sbk17.htm`
- *J.G. Ramírez*, Statistical Intervals: Confidence, Prediction, Enclosure: `https://git.io/v5ZFh`
- Crash Course Statistics Playlist, in particular videos #3 to #7: `https://www.youtube.com/playlist?list=PL8dPuuaLjXtNM_Y-bUAhblSAdWRnmBUcr`

Programming in R

The material for this week includes some coding examples. These examples are written in the **R** language.

Although we will have an R tutorial in the future, you can read the following material to get yourself acquainted with R:

- R for beginners: `https://cran.r-project.org/doc/contrib/Paradis-rdebuts_en.pdf`
- Rstudio: `https://rstudio.com`

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