Quantum Field Theory

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1 Derivation from the Standard Model

1.1 Electroweak Standard Model

Lagrangian with a global $SU(2) \times U(1)$ symmetry:

$$L = i\bar{L}^{(l)}\gamma_\mu\partial^\mu L^{(l)} + i\bar{l}_R\gamma_\mu\partial^\mu l_R + \tfrac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2 - h_e\bar{L}^{(l)}\Phi e_R - \text{h.c.}$$

where $l = e, \mu, \tau$ and $a = 1, 2$, $l_{L,R} = \frac{1}{2}(1 \mp \gamma_5)l$ and

$$L^{(l)} = \begin{pmatrix} \nu_{(l)L} \\ l_L \end{pmatrix}$$

Local $SU(2) \times U(1)$ symmetry:

This consists of two things. First changing the partial derivatives to covariant ones:

$$\partial^\mu \to D^\mu = \partial^\mu - \frac{i}{2}g\tau_k A_k^\mu - \frac{i}{2}g'YB^\mu$$

and second adding the kinetic terms

$$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

of the vector gauge particles to the lagrangian.

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\Phi = e^{\frac{i}{v}\pi^a(x)\tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

This breaks the gauge invariance. The $\partial^\mu \pi^a$ are going to be added to A^a_μ so we can set $\pi_a = 0$ now.

1.1.1 Higgs Terms

$$L_{Higgs} = \tfrac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2$$

Plugging in the covariant derivatives and Φ in U-gauge (symmetry breaking):

$$L_{Higgs} = \frac{1}{2}\Phi^+(\overleftarrow{\partial}_\mu + igA^a_\mu \frac{\tau^a}{2} + ig'YB_\mu)(\overrightarrow{\partial}^\mu + igA^{a\mu}\frac{\tau^a}{2} + ig'YB^\mu)\Phi - \lambda(\Phi^+\Phi - \frac{v^2}{2})^2 =$$

$$= \Phi_U^+(\overleftarrow{\partial}_\mu + igA^a_\mu \frac{\tau^a}{2} + ig'YB_\mu)(\overrightarrow{\partial}^\mu + igA^{a\mu}\frac{\tau^a}{2} + ig'YB\mu)\Phi_U - \lambda(\Phi_U^+\Phi_U - \frac{v^2}{2})^2 =$$

$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 +$$

$$+\frac{1}{8}(v+H)^2 \left(2g^2 \frac{A^1_\mu + iA^2_\mu}{\sqrt{2}} \frac{A^{1\mu} - iA^{2\mu}}{\sqrt{2}} + (g^2 + 4Y^2 g'^2) \frac{gA^3_\mu - 2Yg'B_\mu}{\sqrt{g^2 + 4Y^2g'^2}} \frac{gA^{3\mu} - 2Yg'B^\mu}{\sqrt{g^2+4Y^2g'^2}} \right) =$$

$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \frac{1}{8}(v+H)^2 \left(2g^2 W^-_\mu W^{+\mu} + (g^2 + 4Y^2g'^2)Z_\mu Z^\mu\right) =$$

$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4}g^2v^2 W^-_\mu W^{+\mu} + \frac{1}{8}(g^2 + 4Y^2g'^2)Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 +$$

$$+\frac{1}{2}vHg^2W^-_\mu W^{+\mu} + \frac{1}{4}vH(g^2+4Y^2g'^2)Z_\mu Z^\mu + \frac{1}{4}H^2g^2W^-_\mu W^{+\mu} + \frac{1}{8}H^2(g^2+4Y^2g'^2)Z_\mu Z^\mu$$

Where we put

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu)$$

$$Z_\mu = \frac{g}{\sqrt{g^2+4Y^2g'^2}}A^3_\mu - \frac{2Yg'}{\sqrt{g^2+4Y^2g'^2}}B_\mu$$

Now we define θ_W by the relation

$$\cos\theta_W = \frac{g}{\sqrt{g^2+4Y^2g'^2}}$$

so that we can then write:

$$Z_\mu = \cos\theta_W A^3_\mu - \sin\theta_W B_\mu$$

1.1.2 Yukawa terms

$$L_{Yukawa} = -h_l \bar{L} \Phi e_R - \text{h.c.} = -h_l \bar{L} \Phi_U e_R - \text{h.c.} =$$

$$= -\frac{1}{\sqrt{2}} h_l (v+H)(\bar{e}_L e_R + \bar{e}_R e_L) = -\frac{1}{\sqrt{2}} h_l (v+H) \bar{e} e =$$

$$= -\frac{1}{\sqrt{2}} h_l v \bar{e} e - \frac{1}{\sqrt{2}} h_l \bar{e} e H$$

1.1.3 Gauge terms

$$L_{Gauge} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} =$$

$$= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g\epsilon^{ajk} A^{j\mu} A^{k\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} =$$

$$= -\frac{1}{4}\partial_\mu A^a_\nu \partial^\mu A^{a\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) g\epsilon^{abc} A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 \epsilon^{abc}\epsilon^{ajk} A^b_\mu A^c_\nu A^{k\mu} A^{l\nu} =$$

$$= -\frac{1}{2} W^-_{\mu\nu} W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g[(\partial_\mu A^1_\nu - \partial_\nu A^1_\mu) A^{2\mu} A^{3\nu} + \text{cycl. perm. (123)}]$$

$$-\frac{1}{4} g^2 [(A^a_\mu A^{a\mu})(A^b_\nu A^{b\nu}) - (A^a_\mu A^a_\nu)(A^{b\mu} A^{b\nu})] =$$

$$= -\frac{1}{2} W^-_{\mu\nu} W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g[A^1_\mu A^2_\nu \overleftrightarrow{\partial}^\mu A^{3\nu} + \text{cycl. perm. (123)}]$$

$$-\frac{1}{4} g^2 [(A^a_\mu A^{a\mu})(A^b_\nu A^{b\nu}) - (A^a_\mu A^a_\nu)(A^{b\mu} A^{b\nu})] =$$

$$= -\frac{1}{2} W^-_{\mu\nu} W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - ig(W^0_\mu W^-_\nu \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. (0-+)})$$

$$-g^2[\frac{1}{2}(W^+_\mu W^{-\mu})^2 - \text{... (5.77)}] =$$

$$= -\frac{1}{2} W^-_{\mu\nu} W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + [\text{(5.79) - (5.84)}]$$

1.1.4 Leptonic Terms

$$L = i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R \gamma^\mu \partial_\mu e_R \rightarrow$$

$$\rightarrow i\bar{L}\gamma^\mu(\partial_\mu - igA^a_\mu \frac{\tau^a}{2} - ig'Y_L B_\mu)L + i\bar{e}_R\gamma^\mu(\partial_\mu - ig'Y_R B_\mu)e_R =$$

$$= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R \gamma^\mu \partial_\mu e_R + g\bar{L}\gamma^\mu \frac{\tau^a}{2} L A^a_\mu + g'Y_L \bar{L}\gamma^\mu L B_\mu + g'Y_R \bar{e}_R \gamma^\mu e_R B_\mu =$$

$$= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R \gamma^\mu \partial_\mu e_R + \frac{g}{\sqrt{2}}(\bar{\nu}_L \gamma^\mu e_L W^+_\mu + \text{h.c}) + \frac{1}{2} g\bar{L}\gamma^\mu \tau^3 L A^3_\mu + g'Y_L \bar{L}\gamma^\mu L B_\mu + g'Y_R \bar{e}_R \gamma^\mu e_R B_\mu =$$

$$= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R \gamma^\mu \partial_\mu e_R + \frac{g}{\sqrt{2}}(\bar{\nu}_L \gamma^\mu e_L W^+_\mu + \text{h.c}) + \frac{1}{2} g\bar{\nu}_L \gamma^\mu \nu_L A^3_\mu - \frac{1}{2} g\bar{e}_L \gamma^\mu e_L A^3_\mu$$

$$+ g'Y_L \bar{\nu}_L \gamma^\mu \nu_L B_\mu + g'Y_L \bar{e}_L \gamma^\mu e_L B_\mu + g'Y_R \bar{e}_R \gamma^\mu e_R B_\mu =$$

$$= i\bar{L}\gamma^\mu\partial_\mu L + i\bar{e}_R\gamma^\mu\partial_\mu e_R + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c})$$

$$+ \left[(\tfrac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W)\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W)\bar{e}_L\gamma^\mu e_L + Y_R g'\cos\theta_W\bar{e}_R\gamma^\mu e_R\right] A_\mu$$

$$+ \left[(\tfrac{1}{2}g\cos\theta_W - Y_L g'\sin\theta_W)\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2}g\cos\theta_W - Y_L g'\sin\theta_W)\bar{e}_L\gamma^\mu e_L - 2Y_L g'\sin\theta_W\bar{e}_R\gamma^\mu e_R\right] Z_\mu$$

We now need to make some natural requirements in order to move further. The neutrino fields should be absent from the interaction term with A_μ and the right-handed and left-handed components of the electron field should interact with the A_μ with an equal strength:

$$\tfrac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W = 0$$

$$-\tfrac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W = Y_R g'\cos\theta_W$$

from which it follows

$$Y_R = 2Y_L$$

$$\tan\theta_W = -2Y_L\frac{g'}{g}$$

then the Lagrangian can be further simplified:

$$L = i\bar{L}\gamma^\mu\partial_\mu L + i\bar{e}_R\gamma^\mu\partial_\mu e_R + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c})$$

$$-g\sin\theta_W(\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R)A_\mu$$

$$+\frac{g}{\cos\theta_W}\left[\tfrac{1}{2}\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2} + \sin^2\theta_W)\bar{e}_L\gamma^\mu e_L + \sin^2\theta_W\bar{e}_R\gamma^\mu e_R\right] Z_\mu$$

1.2 QED

The QED Lagrangian density is

$$L = \bar{\psi}(ic\gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where

$$\psi = (\psi_1\psi_2\psi_3\psi_4)$$

and

$$D_\mu = \partial_\mu + ieA_\mu$$

is the gauge covariant derivative and (e is the elementary charge, which is 1 in atomic units)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. It's astonishing, that this simple Lagrangian can account for all phenomena from macroscopic scales down to something like 10^{-13} cm. So of course Feynman, Schwinger and Tomonaga received the 1965 Nobel Prize in Physics for such a fantastic achievement.

Plugging this Lagrangian into the Euler-Lagrange equation of motion for a field, we get:

$$(ic\gamma^\mu D_\mu - mc^2)\psi = 0$$

$$\partial_\nu F^{\nu\mu} = -ec\bar\psi\gamma^\mu\psi$$

The first equation is the Dirac equation in the electromagnetic field and the second equation is a set of Maxwell equations ($\partial_\nu F^{\nu\mu} = -ej^\mu$) with a source $j^\mu = c\bar\psi\gamma^\mu\psi$, which is a 4-current comming from the Dirac equation.

The fields ψ and A^μ are quantized. The first approximation is that we take ψ as a wavefunction, that is, it is a classical 4-component field. It can be shown that this corresponds to taking three orders in the perturbation theory.

The first component A_0 of the 4-potential is the electric potential, and because this is the potential that (as we show in a moment) is in the Schrödinger equation, we denote it by V:

$$A_\mu = \left(\frac{V}{ec}, A_1, A_2, A_3\right)$$

So in the non-relativistic limit, the $\frac{V}{e}$ corresponds to the electric potential. We multiply the Dirac equation by γ^0 from left to get:

$$0 = \gamma^0(ic\gamma^\mu D_\mu - mc^2)\psi = \gamma^0(ic\gamma^0(\partial_0 + i\frac{V}{c}) + ic\gamma^i(\partial_i + ieA_i) - mc^2)\psi =$$

$$= (ic\partial_0 + ic\gamma^0\gamma^i\partial_i - \gamma^0 mc^2 - V - ce\gamma^0\gamma^i A_i)\psi$$

and we make the following substitutions (it's just a formalism, nothing more): $\beta = \gamma^0$, $\alpha^i = \gamma^0\gamma^i$, $p_j = -i\partial_j$, $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$ to get

$$(i\frac{\partial}{\partial t} - c\alpha^i p_i - \beta mc^2 - V - ce\alpha^i A_i)\psi = 0\,.$$

This, in most solid state physics texts, is usually written as

$$i\frac{\partial\psi}{\partial t} = H\psi\,,$$

where the Hamiltonian is given by

$$H = c\alpha^i(p_i + eA_i) + \beta mc^2 + V\,.$$

The right hand side of the Maxwell equations is the 4-current, so it's given by:

$$j^\mu = c\bar\psi\gamma^\mu\psi$$

Now we make the substitution $\psi = e^{-imc^2 t}\varphi$, which states, that we separate the largest oscillations of the wavefunction and we get

$$j^0 = c\bar\psi\gamma^0\psi = c\psi^\dagger\psi = c\varphi^\dagger\varphi$$

$$j^i = c\bar{\psi}\gamma^i\psi = c\psi^\dagger\alpha^i\psi = c\varphi^\dagger\alpha^i\varphi$$

The Dirac equation implies the Klein-Gordon equation:

$$(-ic\gamma^\mu D_\mu - mc^2)(ic\gamma^\nu D_\nu - mc^2)\psi = (c^2\gamma^\mu\gamma^\nu D_\mu D_\mu + m^2c^4)\psi =$$

$$= (c^2 D^\mu D_\mu - ic^2[\gamma^\mu, \gamma^\nu]D_\mu D_\nu + m^2c^4)\psi = 0$$

Note however, the ψ in the true Klein-Gordon equation is just a scalar, but here we get a 4-component spinor. Now:

$$D_\mu D_\nu = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) = \partial_\mu\partial_\nu + ie(A_\mu\partial_\nu + A_\nu\partial_\mu + (\partial_\mu A_\nu)) - e^2 A_\mu A_\nu$$

$$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu = ie(\partial_\mu A_\nu) - ie(\partial_\nu A_\mu)$$

We rewrite $D^\mu D_\mu$:

$$D^\mu D_\mu = g^{\mu\nu} D_\mu D_\nu = \partial^\mu\partial_\mu + ie((\partial^\mu A_\mu) + 2A^\mu\partial_\mu) - e^2 A^\mu A_\mu =$$

$$= \partial^\mu\partial_\mu + ie((\partial^0 A_0) + 2A^0\partial_0 + (\partial^i A_i) + 2A^i\partial_i) - e^2(A^0A_0 + A^iA_i) =$$

$$= \partial^\mu\partial_\mu + i\frac{1}{c^2}\frac{\partial V}{\partial t} + 2i\frac{V}{c^2}\frac{\partial}{\partial t} + ie(\partial^i A_i) + 2ieA^i\partial_i - \frac{V^2}{c^2} - e^2A^iA_i$$

We use the identity $\frac{\partial}{\partial t}\left(e^{-imc^2t}f(t)\right) = e^{-imc^2t}(-imc^2 + \frac{\partial}{\partial t})f(t)$ to get:

$$L = c^2\partial^\mu\psi^*\partial_\mu\psi - m^2c^4\psi^*\psi = \frac{\partial}{\partial t}\psi^*\frac{\partial}{\partial t}\psi - c^2\partial^i\psi^*\partial_i\psi - m^2c^4\psi^*\psi =$$

$$= (imc^2 + \frac{\partial}{\partial t})\varphi^*(-imc^2 + \frac{\partial}{\partial t})\varphi - c^2\partial^i\varphi^*\partial_i\varphi - m^2c^4\varphi^*\varphi =$$

$$= 2mc^2\left[\frac{1}{2}i(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}) - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi + \frac{1}{2mc^2}\frac{\partial\varphi^*}{\partial t}\frac{\partial\varphi}{\partial t}\right]$$

The constant factor $2mc^2$ in front of the Lagrangian is of course irrelevant, so we drop it and then we take the limit $c \to \infty$ (neglecting the last term) and we get

$$L = \frac{1}{2}i(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}) - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi$$

After integration by parts we arrive at

$$L = i\varphi^*\frac{\partial\varphi}{\partial t} - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi$$

The nonrelativistic limit can also be applied directly to the Klein-Gordon equation:

$$0 = (c^2D^\mu D_\mu + m^2c^4)\psi =$$

$$= \left(c^2\partial^\mu\partial_\mu + i\frac{\partial V}{\partial t} + 2iV\frac{\partial}{\partial t} + iec^2(\partial^i A_i) + 2iec^2A^i\partial_i - V^2 - e^2c^2A^iA_i + m^2c^4\right)e^{-imc^2t}\varphi =$$

$$= \left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2 + 2iV\frac{\partial}{\partial t} + i\frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2 A^i\partial_i - V^2 - e^2c^2A^iA_i + m^2c^4 \right) e^{-imc^2t}\varphi =$$

$$= e^{-imc^2t} \Big((-imc^2 + \frac{\partial}{\partial t})^2 - c^2\nabla^2 + 2iV(-imc^2 + \frac{\partial}{\partial t}) + i\frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2A^i\partial_i - V^2 +$$
$$-e^2c^2A^iA_i + m^2c^4 \big) \varphi =$$

$$= e^{-imc^2t} \Big(-2imc^2\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - c^2\nabla^2 + 2Vmc^2 + 2iV\frac{\partial}{\partial t} + i\frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2A^i\partial_i - V^2 +$$
$$-e^2c^2A^iA_i \big) \varphi =$$

$$= -2mc^2e^{-imc^2t} \Big(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - V - \frac{1}{2mc^2}\frac{\partial^2}{\partial t^2} - \frac{i}{2mc^2}\frac{\partial V}{\partial t} + \frac{V^2}{2mc^2} - \frac{iV}{mc^2}\frac{\partial}{\partial t} +$$
$$- \frac{ie}{2m}\partial^i A_i - \frac{ie}{m}A^i\partial_i + \frac{e^2}{2m}A^iA_i \Big) \varphi$$

Taking the limit $c \to \infty$ we again recover the Schrödinger equation:

$$i\frac{\partial}{\partial t}\varphi = \left(-\frac{\nabla^2}{2m} + V + \frac{ie}{2m}\partial^i A_i + \frac{ie}{m}A^i\partial_i - \frac{e^2}{2m}A^iA_i \right) \varphi \, ,$$

we rewrite the right hand side a little bit:

$$i\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}(\partial^i\partial_i + ie\partial^i A_i + 2ieA^i\partial_i - e^2A^iA_i) + V \right) \varphi \, ,$$

$$i\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}(\partial^i + ieA^i)(\partial_i + ieA_i) + V \right) \varphi \, ,$$

And we get the usual form of the Schrödinger equation for the vector potential $\mathbf{A} = (A_1, A_2, A_3)$:

$$i\frac{\partial}{\partial t}\varphi = \left(-\frac{(\nabla + ie\mathbf{A})^2}{2m} + V \right) \varphi \, .$$