

# Quantum Field Theory

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## 1 Derivation from the Standard Model

### 1.1 Electroweak Standard Model

Lagrangian with a global  $SU(2) \times U(1)$  symmetry:

$$\mathcal{L} = i\bar{L}^{(l)}\gamma_\mu\partial^\mu L^{(l)} + i\bar{l}_R\gamma_\mu\partial^\mu l_R + \frac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2 - h_e\bar{L}^{(l)}\Phi e_R - \text{h.c.}$$

where  $l = e, \mu, \tau$  and  $a = 1, 2$ ,  $l_{L,R} = \frac{1}{2}(1 \mp \gamma_5)l$  and

$$L^{(l)} = \begin{pmatrix} \nu_{(l)L} \\ l_L \end{pmatrix}$$

Local  $SU(2) \times U(1)$  symmetry:

This consists of two things. First changing the partial derivatives to covariant ones:

$$\partial^\mu \rightarrow D^\mu = \partial^\mu - \frac{i}{2}g\tau_k A_k^\mu - \frac{i}{2}g'Y B^\mu$$

and second adding the kinetic terms

$$-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

of the vector gauge particles to the lagrangian.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\Phi = e^{\frac{i}{v}\pi^a(x)\tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

This breaks the gauge invariance. The  $\partial^\mu \pi^a$  are going to be added to  $A_\mu^a$  so we can set  $\pi_a = 0$  now.

### 1.1.1 Higgs Terms

$$\mathcal{L}_{Higgs} = \frac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4}\lambda(\Phi^* \Phi)^2$$

Plugging in the covariant derivatives and  $\Phi$  in U-gauge (symmetry breaking):

$$\begin{aligned} \mathcal{L}_{Higgs} &= \frac{1}{2}\Phi^+ (\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'YB_\mu) (\overrightarrow{\partial}^\mu + igA^{a\mu} \frac{\tau^a}{2} + ig'YB^\mu) \Phi - \lambda(\Phi^+ \Phi - \frac{v^2}{2})^2 = \\ &= \Phi_U^+ (\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'YB_\mu) (\overrightarrow{\partial}^\mu + igA^{a\mu} \frac{\tau^a}{2} + ig'YB^\mu) \Phi_U - \lambda(\Phi_U^+ \Phi_U - \frac{v^2}{2})^2 = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\ &+ \frac{1}{8}(v+H)^2 \left( 2g^2 \frac{A_\mu^1 + iA_\mu^2}{\sqrt{2}} \frac{A^{1\mu} - iA^{2\mu}}{\sqrt{2}} + (g^2 + 4Y^2 g'^2) \frac{gA_\mu^3 - 2Yg'B_\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \frac{gA^{3\mu} - 2Yg'B^\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \right) = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \frac{1}{8}(v+H)^2 \left( 2g^2 W_\mu^- W^{+\mu} + \frac{g^2}{\cos^2 \theta_W} Z_\mu Z^\mu \right) = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8 \cos^2 \theta_W} Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\ &+ \frac{1}{2}vg^2 W_\mu^- W^{+\mu} H + \frac{g^2}{4 \cos \theta_W} v Z_\mu Z^\mu H + \frac{1}{4}g^2 W_\mu^- W^{+\mu} H^2 + \frac{g^2}{8 \cos \theta_W} Z_\mu Z^\mu H^2 \end{aligned}$$

Where we put

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2)$$

$$Z_\mu = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}} A_\mu^3 - \frac{2Yg'}{\sqrt{g^2 + 4Y^2 g'^2}} B_\mu$$

we defined  $\theta_W$  by the relation

$$\cos \theta_W = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}}$$

so that the expressions simplify a bit, e.g. we now get:

$$\sin \theta_W = \frac{2Yg'}{\sqrt{g^2 + 4Y^2 g'^2}}$$

$$Z_\mu = \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu$$

$$g^2 + 4Y^2 g'^2 = \frac{g^2}{\cos^2 \theta_W}$$

### 1.1.2 Yukawa terms

$$\begin{aligned}
\mathcal{L}_{Yukawa} &= -h_e \bar{L} \Phi e_R - \text{h.c.} = -h_e \bar{L} \Phi_U e_R - \text{h.c.} = \\
&= -\frac{1}{\sqrt{2}} h_e (v + H) (\bar{e}_L e_R + \bar{e}_R e_L) = -\frac{1}{\sqrt{2}} h_e (v + H) \bar{e} e = \\
&= -\frac{1}{\sqrt{2}} h_e v \bar{e} e - \frac{1}{\sqrt{2}} h_e \bar{e} e H
\end{aligned}$$

The term  $\bar{L} \Phi e_R$  is  $U(1)$  (hypercharge) invariant, so

$$-Y_L + Y + Y_R = 0$$

### 1.1.3 Leptonic Terms

$$\begin{aligned}
\mathcal{L} &= i \bar{L} \gamma^\mu \partial_\mu L + i \bar{e}_R \gamma^\mu \partial_\mu e_R \rightarrow \\
&\rightarrow i \bar{L} \gamma^\mu (\partial_\mu - i g A_\mu^\tau \frac{\tau^a}{2} - i g' Y_L B_\mu) L + i \bar{e}_R \gamma^\mu (\partial_\mu - i g' Y_R B_\mu) e_R = \\
&= i \bar{L} \gamma^\mu \partial_\mu L + i \bar{e}_R \gamma^\mu \partial_\mu e_R + g \bar{L} \gamma^\mu \frac{\tau^a}{2} L A_\mu^a + g' Y_L \bar{L} \gamma^\mu L B_\mu + g' Y_R \bar{e}_R \gamma^\mu e_R B_\mu = \\
&= i \bar{L} \gamma^\mu \partial_\mu L + i \bar{e}_R \gamma^\mu \partial_\mu e_R + \frac{g}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \text{h.c.}) + \frac{1}{2} g \bar{L} \gamma^\mu \tau^3 L A_\mu^3 + g' Y_L \bar{L} \gamma^\mu L B_\mu + g' Y_R \bar{e}_R \gamma^\mu e_R B_\mu = \\
&= i \bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i \bar{e}_R \gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \text{h.c.}) + \frac{1}{2} g \bar{\nu}_L \gamma^\mu \nu_L A_\mu^3 - \frac{1}{2} g \bar{e}_L \gamma^\mu e_L A_\mu^3 \\
&\quad + g' Y_L \bar{\nu}_L \gamma^\mu \nu_L B_\mu + g' Y_L \bar{e}_L \gamma^\mu e_L B_\mu + g' Y_R \bar{e}_R \gamma^\mu e_R B_\mu = \\
&= i \bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i \bar{e}_R \gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \text{h.c.}) \\
&+ \left[ \left( \frac{1}{2} g \sin \theta_W + Y_L g' \cos \theta_W \right) \bar{\nu}_L \gamma^\mu \nu_L + \left( -\frac{1}{2} g \sin \theta_W + Y_L g' \cos \theta_W \right) \bar{e}_L \gamma^\mu e_L + Y_R g' \cos \theta_W \bar{e}_R \gamma^\mu e_R \right] A_\mu \\
&+ \left[ \left( \frac{1}{2} g \cos \theta_W - Y_L g' \sin \theta_W \right) \bar{\nu}_L \gamma^\mu \nu_L + \left( -\frac{1}{2} g \cos \theta_W - Y_L g' \sin \theta_W \right) \bar{e}_L \gamma^\mu e_L - 2 Y_L g' \sin \theta_W \bar{e}_R \gamma^\mu e_R \right] Z_\mu
\end{aligned}$$

Where we substituted new fields  $Z_\mu$  and  $A_\mu$  for the old ones  $A_\mu^3$  and  $B_\mu$  using the relation:

$$\begin{aligned}
Z_\mu &= \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu \\
A_\mu &= \sin \theta_W A_\mu^3 + \cos \theta_W B_\mu
\end{aligned}$$

The angle  $\theta_W$  must be the same as in the Higgs sector, so that the field  $Z_\mu$  is the same. We now need to make the following requirement in order to proceed further:

$$Y = -Y_L$$

This follows for example by requiring that neutrinos have zero charge, i.e. setting  $\frac{1}{2} g \sin \theta_W + Y_L g' \cos \theta_W = 0$  and substituting for  $\theta_W$  from the definition (see the Higgs terms), from which one gets  $Y = -Y_L$ . From  $-Y_L + Y + Y_R = 0$  we now get

$$Y_R = 2Y_L$$

it now follows:

$$\begin{aligned}
\frac{1}{2} g \sin \theta_W + Y_L g' \cos \theta_W &= 0 \\
-\frac{1}{2} g \sin \theta_W + Y_L g' \cos \theta_W &= -g \sin \theta_W
\end{aligned}$$

$$Y_R g' \cos \theta_W = -g \sin \theta_W$$

$$\tan \theta_W = -2Y_L \frac{g'}{g}$$

and the Lagrangian can be further simplified:

$$\begin{aligned} \mathcal{L} &= i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e} \gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \text{h.c.}) \\ &\quad - g \sin \theta_W (\bar{e}_L \gamma^\mu e_L + \bar{e}_R \gamma^\mu e_R) A_\mu \\ &\quad + \frac{g}{\cos \theta_W} \left[ \frac{1}{2} \bar{\nu}_L \gamma^\mu \nu_L + \left(-\frac{1}{2} + \sin^2 \theta_W\right) \bar{e}_L \gamma^\mu e_L + \sin^2 \theta_W \bar{e}_R \gamma^\mu e_R \right] Z_\mu = \\ &= i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e} \gamma^\mu \partial_\mu e + \frac{g}{2\sqrt{2}} (\bar{\nu} \gamma^\mu (1 - \gamma_5) e W_\mu^+ + \text{h.c.}) - g \sin \theta_W \bar{e} \gamma^\mu e A_\mu \\ &\quad + \frac{g}{2 \cos \theta_W} \left[ \bar{\nu} \gamma^\mu (1 - \gamma_5) \nu + \bar{e} \gamma^\mu \left(-\frac{1}{2} + 2 \sin^2 \theta_W + \frac{1}{2} \gamma_5\right) e \right] Z_\mu \end{aligned}$$

Where we used the relations  $\bar{\nu}_L \gamma^\mu e_L = \frac{1}{2} \bar{\nu} \gamma^\mu (1 - \gamma_5) e$  and  $\bar{\nu}_R \gamma^\mu e_R = \frac{1}{2} \bar{\nu} \gamma^\mu (1 + \gamma_5) e$ .

#### 1.1.4 Gauge terms

$$\begin{aligned} \mathcal{L}_{Gauge} &= -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = \\ &= -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon^{abc} A_\mu^b A_\nu^c) (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + g \epsilon^{ajk} A^{j\mu} A^{k\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = \\ &= -\frac{1}{4} \partial_\mu A_\nu^a \partial^\mu A^{\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g \epsilon^{abc} A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 \epsilon^{abc} \epsilon^{ajk} A_\mu^b A_\nu^c A^{k\mu} A^{j\nu} = \\ &= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g [(\partial_\mu A_\nu^1 - \partial_\nu A_\mu^1) A^{2\mu} A^{3\nu} + \text{cycl. perm. (123)}] \\ &\quad - \frac{1}{4} g^2 [(A_\mu^a A^{a\mu})(A_\nu^b A^{b\nu}) - (A_\mu^a A_\nu^a)(A^{b\mu} A^{b\nu})] = \\ &= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g [A_\mu^1 A_\nu^2 \overleftrightarrow{\partial}^\mu A^{3\nu} + \text{cycl. perm. (123)}] \\ &\quad - \frac{1}{4} g^2 [(A_\mu^a A^{a\mu})(A_\nu^b A^{b\nu}) - (A_\mu^a A_\nu^a)(A^{b\mu} A^{b\nu})] = \\ &= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - ig (W_\mu^0 W_\nu^- \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. (0-+)}) \\ &\quad - g^2 [\frac{1}{2} (W_\mu^+ W^{-\mu})^2 - \frac{1}{2} (W_\mu^+ W^{+\mu})(W_\nu^- W^{-\nu}) + (W_\mu^0 W^{0\mu})(W_\nu^+ W^{-\nu}) - (W_\mu^- W_\nu^+)(W^{0\mu} W^{0\nu})] = \\ &\quad = -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + [(5.79) - (5.84)] \end{aligned}$$

### 1.1.5 GWS Lagrangian

Plugging everything together we get the GWS Lagrangian:

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2} \partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4} g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8 \cos^2 \theta_W} Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4} \lambda H^4 + \\
& + \frac{1}{2} v g^2 W_\mu^- W^{+\mu} H + \frac{g^2}{4 \cos \theta_W} v Z_\mu Z^\mu H + \frac{1}{4} g^2 W_\mu^- W^{+\mu} H^2 + \frac{g^2}{8 \cos \theta_W} Z_\mu Z^\mu H^2 \\
& - \frac{1}{\sqrt{2}} h_e v \bar{e} e - \frac{1}{\sqrt{2}} h_e \bar{e} e H \\
& - \frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + [(5.79) - (5.84)] \\
& + i \bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i \bar{e} \gamma^\mu \partial_\mu e + \frac{g}{2\sqrt{2}} (\bar{\nu} \gamma^\mu (1 - \gamma_5) e W_\mu^+ + \text{h.c.}) - g \sin \theta_W \bar{e} \gamma^\mu e A_\mu \\
& + \frac{g}{2 \cos \theta_W} [\bar{\nu} \gamma^\mu (1 - \gamma_5) \nu + \bar{e} \gamma^\mu (-\frac{1}{2} + 2 \sin^2 \theta_W + \frac{1}{2} \gamma_5) e] Z_\mu \\
& + (e, \nu_e, h_e \leftrightarrow \mu, \nu_\mu, h_\mu) + (e, \nu_e, h_e \leftrightarrow \tau, \nu_\tau, h_\tau)
\end{aligned}$$

The free parameters are  $g, \theta_W, v, \lambda, h_e, h_\mu, h_\tau$ .

### 1.1.6 Particle Masses

The particle masses are deduced from the terms

$$\mathcal{L} = -\frac{1}{2} m_H^2 H^2 + m_W^2 W_\mu^- W^{+\mu} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu - m_e \bar{e} e + \dots$$

comparing to the above:

$$\mathcal{L} = -\lambda v^2 H^2 + \frac{1}{4} g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8 \cos^2 \theta_W} Z_\mu Z^\mu - \frac{1}{\sqrt{2}} h_e v \bar{e} e + \dots$$

we get

$$\begin{aligned}
m_W &= \frac{1}{2} g v \\
m_Z &= \frac{g v}{2 \cos \theta_W} = \frac{m_W}{\cos \theta} \\
m_H &= v \sqrt{2 \lambda} \\
m_e &= \frac{1}{\sqrt{2}} h_e v
\end{aligned}$$

### 1.1.7 Quarks

$$\begin{aligned}
\mathcal{L}_{fermion} = & \sum_{q=d,s,b} i \bar{L}_0^{(q)} \gamma^\mu \partial_\mu L_0^{(q)} + \sum_{q=d,u,s,c,b,t} i \bar{q}_{0R} \gamma^\mu \partial_\mu q_{0R} \\
\mathcal{L}_{Yukawa} = & - \sum_{\substack{q=d,s,b \\ q'=d,s,b}} h_{qq'} i \bar{L}_0^{(q)} \Phi q'_{0R} + \text{h.c.} - \sum_{\substack{q=d,s,b \\ q'=u,c,t}} \tilde{h}_{qq'} i \bar{L}_0^{(q)} \tilde{\Phi} q'_{0R} + \text{h.c.}
\end{aligned}$$

More to be added here...

## 1.2 QFT

### 1.2.1 Evolution Operator, S-Matrix Elements

The evolution operator  $U$  is defined by the equations:

$$\begin{aligned} |\phi(t_2)\rangle &= U(t_2, t_1) |\phi(t_1)\rangle \\ i\hbar \frac{\partial U(t, t_1)}{\partial t} &= H(t)U(t, t_1) \\ U(t_1, t_1) &= 1 \end{aligned}$$

We are interested in calculating the S matrix elements:

$$S_{fi} = \langle f | U(-\infty, \infty) | i \rangle$$

so we first calculate  $U(-\infty, \infty)$ . Integrating:

$$U(t_2, t_1) = U(t_1, t_1) - \frac{i}{\hbar} \int_{t_1}^{t_2} H(t)U(t, t_1)dt = 1 - \frac{i}{\hbar} \int_{t_1}^{t_2} H(t)U(t, t_1)dt$$

Now:

$$\begin{aligned} U(-\infty, \infty) &= 1 - \frac{i}{\hbar} \int_{-\infty}^{\infty} H(t')U(t', -\infty)dt' = \\ &= 1 + \left(-\frac{i}{\hbar}\right) \int_{-\infty}^{\infty} H(t')U(t', -\infty)dt' + \left(-\frac{i}{\hbar}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{t'} H(t')H(t'')U(t'', -\infty)dt'dt'' = \\ &= \dots = \sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots T\{H(t_1)H(t_2)\dots\}dt_1dt_2\dots = \\ &= T \exp\left(-\frac{i}{\hbar} \int_{-\infty}^{\infty} H(t)dt\right) = T \exp\left(-\frac{i}{\hbar} \int_{-\infty}^{\infty} d^4x \mathcal{H}(x)\right) \end{aligned}$$

If  $\mathcal{L}$  doesn't contain derivatives of the fields, then  $\mathcal{H} = -\mathcal{L}$  so:

$$U(-\infty, \infty) = T \exp\left(\frac{i}{\hbar} \int_{-\infty}^{\infty} d^4x \mathcal{L}(x)\right)$$

Let's write  $S = 1 + iT$  and  $|i\rangle = |k_1 \dots k_m\rangle$ ,  $|f\rangle = |p_1 \dots p_n\rangle$ . As a first step now, let's investigate a scalar field, e.g.  $\mathcal{L} = \lambda\phi^4$  (e.g. a Higgs self interaction term above), we'll look at other fields later:

$$\begin{aligned} \langle f | S | i \rangle &= \langle f | iT | i \rangle = \langle p_1 \dots p_n | iT | k_1 \dots k_m \rangle = \frac{1}{\tilde{D}(k_1) \dots \tilde{D}(k_m)} \frac{1}{\tilde{D}(p_1) \dots \tilde{D}(p_n)} \\ &\int d^4x_1 \dots d^4x_m e^{-i(k_1x_1 + \dots + k_mx_m)} \int d^4y_1 \dots d^4y_n e^{+i(p_1y_1 + \dots + p_ny_n)} G(x_1, \dots, x_m, y_1, \dots, y_n) \end{aligned}$$

where

$$\begin{aligned} G(x_1, \dots, x_n) &= \langle 0 | T \{ \phi(x_1) \dots \phi(x_n) \} | 0 \rangle = \\ &\frac{\langle 0 | T \{ \phi_I(x_1) \dots \phi_I(x_n) \exp\left(\frac{i}{\hbar} \int_{-\infty}^{\infty} d^4x \mathcal{L}(x)\right) \} | 0 \rangle}{\langle 0 | T \exp\left(\frac{i}{\hbar} \int_{-\infty}^{\infty} d^4x \mathcal{L}(x)\right) | 0 \rangle} \end{aligned}$$

Now we use the Wick contraction, get some terms like  $D_{23}D_{34}$  integrate things out, this will give the delta function and  $\tilde{D}(p)$ 's and that's it.

## 1.3 Low energy theories

### 1.3.1 Fermi-type theory

This is a low energy ( $m_W^2 \gg m_\mu m_e$ ) model for the EW interactions, that can be derived for example from the muon decay:

$$\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$$

From the SM the relevant Lagrangian is

$$\mathcal{L} = \frac{g}{2\sqrt{2}}(\bar{e}\gamma^\mu(1-\gamma_5)\nu_e W_\mu^-) + \frac{g}{2\sqrt{2}}(\bar{\mu}\gamma^\mu(1-\gamma_5)\nu_\mu W_\mu^-)$$

and one gets the diagram  $\mu^- + \bar{\nu}_\mu \rightarrow e^- + \bar{\nu}_e$  and the corresponding matrix element:

$$iM = -i\frac{g^2}{8}[\bar{u}\gamma_\mu(1-\gamma_5)u]\frac{-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_W^2}}{q^2 - m_W^2}[\bar{u}\gamma_\nu(1-\gamma_5)v]$$

which when the momentum transfer  $q$  is much less than  $m_w$  becomes

$$iM = -i\frac{g^2}{8m_W^2}[\bar{u}\gamma^\mu(1-\gamma_5)u][\bar{u}\gamma_\mu(1-\gamma_5)v]$$

but this element can be derived directly from the Lagrangian:

$$\mathcal{L} = -\frac{G_\mu}{\sqrt{2}}[\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma_5)\psi_\mu][\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_{\nu_e}]$$

with

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

This is the universal V-A theory Lagrangian (after adding the h.c. term).

### 1.3.2 QED

The QED Lagrangian density is

$$\mathcal{L} = \bar{\psi}(ic\gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where

$$\psi = (\psi_1\psi_2\psi_3\psi_4)$$

and

$$D_\mu = \partial_\mu + ieA_\mu$$

is the gauge covariant derivative and ( $e$  is the elementary charge, which is 1 in atomic units)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. It's astonishing, that this simple Lagrangian can account for all phenomena from macroscopic scales down to something like  $10^{-13}$  cm. So of course Feynman, Schwinger and Tomonaga received the 1965 Nobel Prize in Physics for such a fantastic achievement.

Plugging this Lagrangian into the Euler-Lagrange equation of motion for a field, we get:

$$(ic\gamma^\mu D_\mu - mc^2)\psi = 0$$

$$\partial_\nu F^{\nu\mu} = -ec\bar{\psi}\gamma^\mu\psi$$

The first equation is the Dirac equation in the electromagnetic field and the second equation is a set of Maxwell equations ( $\partial_\nu F^{\nu\mu} = -ej^\mu$ ) with a source  $j^\mu = c\bar{\psi}\gamma^\mu\psi$ , which is a 4-current coming from the Dirac equation.

The fields  $\psi$  and  $A^\mu$  are quantized. The first approximation is that we take  $\psi$  as a wavefunction, that is, it is a classical 4-component field. It can be shown that this corresponds to taking three orders in the perturbation theory.

The first component  $A_0$  of the 4-potential is the electric potential, and because this is the potential that (as we show in a moment) is in the Schrödinger equation, we denote it by  $V$ :

$$A_\mu = \left( \frac{V}{ec}, A_1, A_2, A_3 \right)$$

So in the non-relativistic limit, the  $\frac{V}{e}$  corresponds to the electric potential. We multiply the Dirac equation by  $\gamma^0$  from left to get:

$$0 = \gamma^0(ic\gamma^\mu D_\mu - mc^2)\psi = \gamma^0(ic\gamma^0(\partial_0 + i\frac{V}{c}) + ic\gamma^i(\partial_i + ieA_i) - mc^2)\psi =$$

$$= (ic\partial_0 + ic\gamma^0\gamma^i\partial_i - \gamma^0mc^2 - V - ce\gamma^0\gamma^iA_i)\psi$$

and we make the following substitutions (it's just a formalism, nothing more):  $\beta = \gamma^0$ ,  $\alpha^i = \gamma^0\gamma^i$ ,  $p_j = -i\partial_j$ ,  $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$  to get

$$(i\frac{\partial}{\partial t} - c\alpha^ip_i - \beta mc^2 - V - ce\alpha^iA_i)\psi = 0.$$

This, in most solid state physics texts, is usually written as

$$i\frac{\partial\psi}{\partial t} = H\psi,$$

where the Hamiltonian is given by

$$H = c\alpha^i(p_i + eA_i) + \beta mc^2 + V.$$

The right hand side of the Maxwell equations is the 4-current, so it's given by:

$$j^\mu = c\bar{\psi}\gamma^\mu\psi$$

Now we make the substitution  $\psi = e^{-imc^2t}\varphi$ , which states, that we separate the largest oscillations of the wavefunction and we get

$$j^0 = c\bar{\psi}\gamma^0\psi = c\psi^\dagger\psi = c\varphi^\dagger\varphi$$

$$j^i = c\bar{\psi}\gamma^i\psi = c\psi^\dagger\alpha^i\psi = c\varphi^\dagger\alpha^i\varphi$$

The Dirac equation implies the Klein-Gordon equation:

$$(-ic\gamma^\mu D_\mu - mc^2)(ic\gamma^\nu D_\nu - mc^2)\psi = (c^2\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2c^4)\psi =$$



$$= (c^2 D^\mu D_\mu - ic^2 [\gamma^\mu, \gamma^\nu] D_\mu D_\nu + m^2 c^4) \psi = 0$$

Note however, the  $\psi$  in the true Klein-Gordon equation is just a scalar, but here we get a 4-component spinor. Now:

$$D_\mu D_\nu = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) = \partial_\mu \partial_\nu + ie(A_\mu \partial_\nu + A_\nu \partial_\mu + (\partial_\mu A_\nu)) - e^2 A_\mu A_\nu$$

$$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu = ie(\partial_\mu A_\nu) - ie(\partial_\nu A_\mu)$$

We rewrite  $D^\mu D_\mu$ :

$$\begin{aligned} D^\mu D_\mu &= g^{\mu\nu} D_\mu D_\nu = \partial^\mu \partial_\mu + ie((\partial^\mu A_\mu) + 2A^\mu \partial_\mu) - e^2 A^\mu A_\mu = \\ &= \partial^\mu \partial_\mu + ie((\partial^0 A_0) + 2A^0 \partial_0 + (\partial^i A_i) + 2A^i \partial_i) - e^2 (A^0 A_0 + A^i A_i) = \\ &= \partial^\mu \partial_\mu + i \frac{1}{c^2} \frac{\partial V}{\partial t} + 2i \frac{V}{c^2} \frac{\partial}{\partial t} + ie(\partial^i A_i) + 2ieA^i \partial_i - \frac{V^2}{c^2} - e^2 A^i A_i \end{aligned}$$

We use the identity  $\frac{\partial}{\partial t} \left( e^{-imc^2 t} f(t) \right) = e^{-imc^2 t} (-imc^2 + \frac{\partial}{\partial t}) f(t)$  to get:

$$\begin{aligned} L &= c^2 \partial^\mu \psi^* \partial_\mu \psi - m^2 c^4 \psi^* \psi = \frac{\partial}{\partial t} \psi^* \frac{\partial}{\partial t} \psi - c^2 \partial^i \psi^* \partial_i \psi - m^2 c^4 \psi^* \psi = \\ &= (imc^2 + \frac{\partial}{\partial t}) \varphi^* (-imc^2 + \frac{\partial}{\partial t}) \varphi - c^2 \partial^i \varphi^* \partial_i \varphi - m^2 c^4 \varphi^* \varphi = \\ &= 2mc^2 \left[ \frac{1}{2} i \left( \varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t} \right) - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi + \frac{1}{2mc^2} \frac{\partial \varphi^*}{\partial t} \frac{\partial \varphi}{\partial t} \right] \end{aligned}$$

The constant factor  $2mc^2$  in front of the Lagrangian is of course irrelevant, so we drop it and then we take the limit  $c \rightarrow \infty$  (neglecting the last term) and we get

$$L = \frac{1}{2} i \left( \varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t} \right) - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

After integration by parts we arrive at

$$L = i \varphi^* \frac{\partial \varphi}{\partial t} - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

The nonrelativistic limit can also be applied directly to the Klein-Gordon equation:

$$\begin{aligned} 0 &= (c^2 D^\mu D_\mu + m^2 c^4) \psi = \\ &= \left( c^2 \partial^\mu \partial_\mu + i \frac{\partial V}{\partial t} + 2iV \frac{\partial}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi = \\ &= \left( \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi = \\ &= e^{-imc^2 t} \left( (-imc^2 + \frac{\partial}{\partial t})^2 - c^2 \nabla^2 + 2iV(-imc^2 + \frac{\partial}{\partial t}) + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 + \right. \\ &\quad \left. - e^2 c^2 A^i A_i + m^2 c^4 \right) \varphi = \\ &= e^{-imc^2 t} \left( -2imc^2 \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2Vmc^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 + \right. \end{aligned}$$

$$\begin{aligned}
& -e^2 c^2 A^i A_i) \varphi = \\
& = -2mc^2 e^{-imc^2 t} \left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - V - \frac{1}{2mc^2} \frac{\partial^2}{\partial t^2} - \frac{i}{2mc^2} \frac{\partial V}{\partial t} + \frac{V^2}{2mc^2} - \frac{iV}{mc^2} \frac{\partial}{\partial t} + \right. \\
& \quad \left. - \frac{ie}{2m} \partial^i A_i - \frac{ie}{m} A^i \partial_i + \frac{e^2}{2m} A^i A_i \right) \varphi
\end{aligned}$$

Taking the limit  $c \rightarrow \infty$  we again recover the Schrödinger equation:

$$i \frac{\partial}{\partial t} \varphi = \left( -\frac{\nabla^2}{2m} + V + \frac{ie}{2m} \partial^i A_i + \frac{ie}{m} A^i \partial_i - \frac{e^2}{2m} A^i A_i \right) \varphi,$$

we rewrite the right hand side a little bit:

$$i \frac{\partial}{\partial t} \varphi = \left( \frac{1}{2m} (\partial^i \partial_i + ie \partial^i A_i + 2ie A^i \partial_i - e^2 A^i A_i) + V \right) \varphi,$$

$$i \frac{\partial}{\partial t} \varphi = \left( \frac{1}{2m} (\partial^i + ie A^i) (\partial_i + ie A_i) + V \right) \varphi,$$

And we get the usual form of the Schrödinger equation for the vector potential  $\mathbf{A} = (A_1, A_2, A_3)$ :

$$i \frac{\partial}{\partial t} \varphi = \left( -\frac{(\nabla + ie \mathbf{A})^2}{2m} + V \right) \varphi.$$