

# Quantum Field Theory

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## 1 Derivation from the Standard Model

### 1.1 Electroweak Standard Model

Lagrangian with a global  $SU(2) \times U(1)$  symmetry:

$$L = i\bar{L}^{(l)}\gamma_\mu\partial^\mu L^{(l)} + i\bar{l}_R\gamma_\mu\partial^\mu l_R + \frac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2 - h_e\bar{L}^{(l)}\Phi e_R - \text{h.c.}$$

where  $l = e, \mu, \tau$  and  $a = 1, 2$ ,  $l_{L,R} = \frac{1}{2}(1 \mp \gamma_5)l$  and

$$L^{(l)} = \begin{pmatrix} \nu_{(l)L} \\ l_L \end{pmatrix}$$

Local  $SU(2) \times U(1)$  symmetry:

This consists of two things. First changing the partial derivatives to covariant ones:

$$\partial^\mu \rightarrow D^\mu = \partial^\mu - \frac{i}{2}g\tau_k A_k^\mu - \frac{i}{2}g'YB^\mu$$

and second adding the kinetic terms

$$-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

of the vector gauge particles to the lagrangian.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\Phi = e^{\frac{i}{v}\pi^a(x)\tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

This breaks the gauge invariance. The  $\partial^\mu \pi^a$  are going to be added to  $A_\mu^a$  so we can set  $\pi_a = 0$  now.

### 1.1.1 Higgs Terms

$$L_{Higgs} = \frac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4}\lambda(\Phi^* \Phi)^2$$

Plugging in the covariant derivatives and  $\Phi$  in U-gauge (symmetry breaking):

$$\begin{aligned} L_{Higgs} &= \frac{1}{2}\Phi^+ (\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'YB_\mu) (\overrightarrow{\partial}^\mu + igA^{a\mu} \frac{\tau^a}{2} + ig'YB^\mu) \Phi - \lambda(\Phi^+ \Phi - \frac{v^2}{2})^2 = \\ &= \Phi_U^\dagger (\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'YB_\mu) (\overrightarrow{\partial}^\mu + igA^{a\mu} \frac{\tau^a}{2} + ig'YB^\mu) \Phi_U - \lambda(\Phi_U^\dagger \Phi_U - \frac{v^2}{2})^2 = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\ &+ \frac{1}{8}(v+H)^2 \left( 2g^2 \frac{A_\mu^1 + iA_\mu^2}{\sqrt{2}} \frac{A^{1\mu} - iA^{2\mu}}{\sqrt{2}} + (g^2 + 4Y^2 g'^2) \frac{gA_\mu^3 - 2Yg'B_\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \frac{gA^{3\mu} - 2Yg'B^\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \right) = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \frac{1}{8}(v+H)^2 \left( 2g^2 W_\mu^- W^{+\mu} + \frac{g^2}{\cos^2 \theta_W} Z_\mu Z^\mu \right) = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8 \cos^2 \theta_W} Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\ &+ \frac{1}{2}vg^2 W_\mu^- W^{+\mu} H + \frac{g^2}{4 \cos \theta_W} v Z_\mu Z^\mu H + \frac{1}{4}g^2 W_\mu^- W^{+\mu} H^2 + \frac{g^2}{8 \cos \theta_W} Z_\mu Z^\mu H^2 \end{aligned}$$

Where we put

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2)$$

$$Z_\mu = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}} A_\mu^3 - \frac{2Yg'}{\sqrt{g^2 + 4Y^2 g'^2}} B_\mu$$

we defined  $\theta_W$  by the relation

$$\cos \theta_W = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}}$$

so that the expressions simplify a bit, e.g. we now get:

$$\begin{aligned}\sin \theta_W &= \frac{2Yg'}{\sqrt{g^2 + 4Y^2g'^2}} \\ Z_\mu &= \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu \\ g^2 + 4Y^2g'^2 &= \frac{g^2}{\cos^2 \theta_W}\end{aligned}$$

### 1.1.2 Yukawa terms

$$\begin{aligned}L_{Yukawa} &= -h_e \bar{L} \Phi e_R - \text{h.c.} = -h_e \bar{L} \Phi_U e_R - \text{h.c.} = \\ &= -\frac{1}{\sqrt{2}} h_e (v + H) (\bar{e}_L e_R + \bar{e}_R e_L) = -\frac{1}{\sqrt{2}} h_e (v + H) \bar{e} e = \\ &= -\frac{1}{\sqrt{2}} h_e v \bar{e} e - \frac{1}{\sqrt{2}} h_e \bar{e} e H\end{aligned}$$

The term  $\bar{L} \Phi e_R$  is  $U(1)$  (hypercharge) invariant, so

$$-Y_L + Y + Y_R = 0$$

### 1.1.3 Leptonic Terms

$$\begin{aligned}L &= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R\gamma^\mu \partial_\mu e_R \rightarrow \\ &\rightarrow i\bar{L}\gamma^\mu (\partial_\mu - igA_\mu^\tau \frac{\tau^a}{2} - ig'Y_L B_\mu)L + i\bar{e}_R\gamma^\mu (\partial_\mu - ig'Y_R B_\mu)e_R = \\ &= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R\gamma^\mu \partial_\mu e_R + g\bar{L}\gamma^\mu \frac{\tau^a}{2} LA_\mu^a + g'Y_L \bar{L}\gamma^\mu LB_\mu + g'Y_R \bar{e}_R\gamma^\mu e_R B_\mu = \\ &= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R\gamma^\mu \partial_\mu e_R + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c.}) + \frac{1}{2}g\bar{L}\gamma^\mu \tau^3 LA_\mu^3 + g'Y_L \bar{L}\gamma^\mu LB_\mu + g'Y_R \bar{e}_R\gamma^\mu e_R B_\mu = \\ &= i\bar{\nu}_L\gamma^\mu \partial_\mu \nu_L + i\bar{e}\gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c.}) + \frac{1}{2}g\bar{\nu}_L\gamma^\mu \nu_L A_\mu^3 - \frac{1}{2}g\bar{e}_L\gamma^\mu e_L A_\mu^3 \\ &\quad + g'Y_L \bar{\nu}_L\gamma^\mu \nu_L B_\mu + g'Y_L \bar{e}_L\gamma^\mu e_L B_\mu + g'Y_R \bar{e}_R\gamma^\mu e_R B_\mu = \\ &= i\bar{\nu}_L\gamma^\mu \partial_\mu \nu_L + i\bar{e}\gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c.}) \\ &+ \left[ \left( \frac{1}{2}g \sin \theta_W + Y_L g' \cos \theta_W \right) \bar{\nu}_L\gamma^\mu \nu_L + \left( -\frac{1}{2}g \sin \theta_W + Y_L g' \cos \theta_W \right) \bar{e}_L\gamma^\mu e_L + Y_R g' \cos \theta_W \bar{e}_R\gamma^\mu e_R \right] A_\mu \\ &+ \left[ \left( \frac{1}{2}g \cos \theta_W - Y_L g' \sin \theta_W \right) \bar{\nu}_L\gamma^\mu \nu_L + \left( -\frac{1}{2}g \cos \theta_W - Y_L g' \sin \theta_W \right) \bar{e}_L\gamma^\mu e_L - 2Y_L g' \sin \theta_W \bar{e}_R\gamma^\mu e_R \right] Z_\mu\end{aligned}$$

Where we substituted new fields  $Z_\mu$  and  $A_\mu$  for the old ones  $A_\mu^3$  and  $B_\mu$  using the relation:

$$\begin{aligned}Z_\mu &= \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu \\ A_\mu &= \sin \theta_W A_\mu^3 + \cos \theta_W B_\mu\end{aligned}$$

The angle  $\theta_W$  must be the same as in the Higgs sector, so that the field  $Z_\mu$  is the same. We now need to make the following requirement in order to proceed further:

$$Y = -Y_L$$

This follows for example by requiring that neutrinos have zero charge, i.e. setting  $\frac{1}{2}g \sin \theta_W + Y_L g' \cos \theta_W = 0$  and substituting for  $\theta_W$  from the definition (see the Higgs terms), from which one gets  $Y = -Y_L$ . From  $-Y_L + Y + Y_R = 0$  we now get

$$Y_R = 2Y_L$$

it now follows:

$$\begin{aligned} \frac{1}{2}g \sin \theta_W + Y_L g' \cos \theta_W &= 0 \\ -\frac{1}{2}g \sin \theta_W + Y_L g' \cos \theta_W &= -g \sin \theta_W \\ Y_R g' \cos \theta_W &= -g \sin \theta_W \\ \tan \theta_W &= -2Y_L \frac{g'}{g} \end{aligned}$$

and the Lagrangian can be further simplified:

$$\begin{aligned} L &= i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e} \gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \text{h.c.}) \\ &\quad -g \sin \theta_W (\bar{e}_L \gamma^\mu e_L + \bar{e}_R \gamma^\mu e_R) A_\mu \\ &\quad + \frac{g}{\cos \theta_W} \left[ \frac{1}{2} \bar{\nu}_L \gamma^\mu \nu_L + \left(-\frac{1}{2} + \sin^2 \theta_W\right) \bar{e}_L \gamma^\mu e_L + \sin^2 \theta_W \bar{e}_R \gamma^\mu e_R \right] Z_\mu = \\ &= i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e} \gamma^\mu \partial_\mu e + \frac{g}{2\sqrt{2}}(\bar{\nu} \gamma^\mu (1 - \gamma_5) e W_\mu^+ + \text{h.c.}) - g \sin \theta_W \bar{e} \gamma^\mu e A_\mu \\ &\quad + \frac{g}{2 \cos \theta_W} \left[ \bar{\nu} \gamma^\mu (1 - \gamma_5) \nu + \bar{e} \gamma^\mu \left(-\frac{1}{2} + 2 \sin^2 \theta_W + \frac{1}{2} \gamma_5\right) e \right] Z_\mu \end{aligned}$$

Where we used the relations  $\bar{\nu}_L \gamma^\mu e_L = \frac{1}{2} \bar{\nu} \gamma^\mu (1 - \gamma_5) e$  and  $\bar{\nu}_R \gamma^\mu e_R = \frac{1}{2} \bar{\nu} \gamma^\mu (1 + \gamma_5) e$ .

#### 1.1.4 Gauge terms

$$\begin{aligned} L_{Gauge} &= -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = \\ &= -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon^{abc} A_\mu^b A_\nu^c) (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g \epsilon^{ajk} A^{j\mu} A^{k\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = \\ &= -\frac{1}{4} \partial_\mu A_\nu^a \partial^\mu A^{a\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g \epsilon^{abc} A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 \epsilon^{abc} \epsilon^{ajk} A_\mu^b A_\nu^c A^{k\mu} A^{l\nu} = \\ &= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g [(\partial_\mu A_\nu^1 - \partial_\nu A_\mu^1) A^{2\mu} A^{3\nu} + \text{cycl. perm. (123)}] \\ &\quad - \frac{1}{4} g^2 [(A_\mu^a A^{a\mu})(A_\nu^b A^{b\nu}) - (A_\mu^a A_\nu^a)(A^{b\mu} A^{b\nu})] = \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}W_{\mu\nu}^-W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - g[A_\mu^1A_\nu^2\overleftrightarrow{\partial}^\mu A^{3\nu} + \text{cycl. perm. (123)}] \\
&\quad - \frac{1}{4}g^2[(A_\mu^aA^{a\mu})(A_\nu^bA^{b\nu}) - (A_\mu^aA_\nu^a)(A^{b\mu}A^{b\nu})] = \\
&= -\frac{1}{2}W_{\mu\nu}^-W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - ig(W_\mu^0W_\nu^-\overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. (0-+)}) \\
&- g^2[\frac{1}{2}(W_\mu^+W^{-\mu})^2 - \frac{1}{2}(W_\mu^+W^{+\mu})(W_\nu^-W^{-\nu}) + (W_\mu^0W^{0\mu})(W_\nu^+W^{-\nu}) - (W_\mu^-W_\nu^+)(W^{0\mu}W^{0\nu})] = \\
&\quad = -\frac{1}{2}W_{\mu\nu}^-W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + [(5.79) - (5.84)]
\end{aligned}$$

### 1.1.5 GWS Lagrangian

Plugging everything together we get the GWS Lagrangian:

$$\begin{aligned}
L = & \frac{1}{2}\partial_\mu H\partial^\mu H - \lambda v^2 H^2 + \frac{1}{4}g^2v^2W_\mu^-W^{+\mu} + \frac{g^2v^2}{8\cos^2\theta_W}Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\
& + \frac{1}{2}vg^2W_\mu^-W^{+\mu}H + \frac{g^2}{4\cos\theta_W}vZ_\mu Z^\mu H + \frac{1}{4}g^2W_\mu^-W^{+\mu}H^2 + \frac{g^2}{8\cos\theta_W}Z_\mu Z^\mu H^2 \\
& \quad - \frac{1}{\sqrt{2}}h_e v \bar{e}e - \frac{1}{\sqrt{2}}h_e \bar{e}e H \\
& \quad - \frac{1}{2}W_{\mu\nu}^-W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + [(5.79) - (5.84)] \\
& + i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{2\sqrt{2}}(\bar{\nu}\gamma^\mu(1-\gamma_5)eW_\mu^+ + \text{h.c.}) - g\sin\theta_W\bar{e}\gamma^\mu e A_\mu \\
& + \frac{g}{2\cos\theta_W}[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu + \bar{e}\gamma^\mu(-\frac{1}{2} + 2\sin^2\theta_W + \frac{1}{2}\gamma_5)e]Z_\mu \\
& \quad + (e, \nu_e, h_e \leftrightarrow \mu, \nu_\mu, h_\mu) + (e, \nu_e, h_e \leftrightarrow \tau, \nu_\tau, h_\tau)
\end{aligned}$$

The free parameters are  $g, \theta_W, v, \lambda, h_e, h_\mu, h_\tau$ .

### 1.1.6 Particle Masses

The particle masses are deduced from the terms

$$L = -\frac{1}{2}m_H^2H^2 + m_W^2W_\mu^-W^{+\mu} + \frac{1}{2}m_Z^2Z_\mu Z^\mu - m_e\bar{e}e + \dots$$

comparing to the above:

$$L = -\lambda v^2 H^2 + \frac{1}{4}g^2v^2W_\mu^-W^{+\mu} + \frac{g^2v^2}{8\cos^2\theta_W}Z_\mu Z^\mu - \frac{1}{\sqrt{2}}h_e v \bar{e}e + \dots$$

we get

$$\begin{aligned}
m_W &= \frac{1}{2}gv \\
m_Z &= \frac{gv}{2\cos\theta_W} = \frac{m_W}{\cos\theta} \\
m_H &= v\sqrt{2\lambda} \\
m_e &= \frac{1}{\sqrt{2}}h_e v
\end{aligned}$$

### 1.1.7 Quarks

$$L_{fermion} += \sum_{q=d,s,b} i\bar{L}_0^{(q)} \gamma^\mu \partial_\mu L_0^{(q)} + \sum_{q=d,u,s,c,b,t} i\bar{q}_{0R} \gamma^\mu \partial_\mu q_{0R}$$

$$L_{Yukawa} += - \sum_{\substack{q=d,s,b \\ q'=d,s,b}} h_{qq'} i\bar{L}_0^{(q)} \Phi q'_{0R} + \text{h.c.} - \sum_{\substack{q=d,s,b \\ q'=u,c,t}} \tilde{h}_{qq'} i\bar{L}_0^{(q)} \tilde{\Phi} q'_{0R} + \text{h.c.}$$

More to be added here...

## 1.2 QED

The QED Lagrangian density is

$$L = \bar{\psi}(ic\gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where

$$\psi = (\psi_1\psi_2\psi_3\psi_4)$$

and

$$D_\mu = \partial_\mu + ieA_\mu$$

is the gauge covariant derivative and ( $e$  is the elementary charge, which is 1 in atomic units)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. It's astonishing, that this simple Lagrangian can account for all phenomena from macroscopic scales down to something like  $10^{-13}$  cm. So of course Feynman, Schwinger and Tomonaga received the 1965 Nobel Prize in Physics for such a fantastic achievement.

Plugging this Lagrangian into the Euler-Lagrange equation of motion for a field, we get:

$$(ic\gamma^\mu D_\mu - mc^2)\psi = 0$$

$$\partial_\nu F^{\nu\mu} = -ec\bar{\psi}\gamma^\mu\psi$$

The first equation is the Dirac equation in the electromagnetic field and the second equation is a set of Maxwell equations ( $\partial_\nu F^{\nu\mu} = -ej^\mu$ ) with a source  $j^\mu = c\bar{\psi}\gamma^\mu\psi$ , which is a 4-current coming from the Dirac equation.

The fields  $\psi$  and  $A^\mu$  are quantized. The first approximation is that we take  $\psi$  as a wavefunction, that is, it is a classical 4-component field. It can be shown that this corresponds to taking three orders in the perturbation theory.

The first component  $A_0$  of the 4-potential is the electric potential, and because this is the potential that (as we show in a moment) is in the Schrödinger equation, we denote it by  $V$ :

$$A_\mu = \left( \frac{V}{ec}, A_1, A_2, A_3 \right)$$

So in the non-relativistic limit, the  $\frac{V}{e}$  corresponds to the electric potential. We multiply the Dirac equation by  $\gamma^0$  from left to get:

$$\begin{aligned} 0 &= \gamma^0(ic\gamma^\mu D_\mu - mc^2)\psi = \gamma^0(ic\gamma^0(\partial_0 + i\frac{V}{c}) + ic\gamma^i(\partial_i + ieA_i) - mc^2)\psi = \\ &= (ic\partial_0 + ic\gamma^0\gamma^i\partial_i - \gamma^0mc^2 - V - ce\gamma^0\gamma^iA_i)\psi \end{aligned}$$

and we make the following substitutions (it's just a formalism, nothing more):  $\beta = \gamma^0$ ,  $\alpha^i = \gamma^0\gamma^i$ ,  $p_j = -i\partial_j$ ,  $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$  to get

$$(i\frac{\partial}{\partial t} - c\alpha^i p_i - \beta mc^2 - V - ce\alpha^i A_i)\psi = 0.$$

This, in most solid state physics texts, is usually written as

$$i\frac{\partial\psi}{\partial t} = H\psi,$$

where the Hamiltonian is given by

$$H = c\alpha^i(p_i + eA_i) + \beta mc^2 + V.$$

The right hand side of the Maxwell equations is the 4-current, so it's given by:

$$j^\mu = c\bar{\psi}\gamma^\mu\psi$$

Now we make the substitution  $\psi = e^{-imc^2 t}\varphi$ , which states, that we separate the largest oscillations of the wavefunction and we get

$$\begin{aligned} j^0 &= c\bar{\psi}\gamma^0\psi = c\psi^\dagger\psi = c\varphi^\dagger\varphi \\ j^i &= c\bar{\psi}\gamma^i\psi = c\psi^\dagger\alpha^i\psi = c\varphi^\dagger\alpha^i\varphi \end{aligned}$$

The Dirac equation implies the Klein-Gordon equation:

$$\begin{aligned} (-ic\gamma^\mu D_\mu - mc^2)(ic\gamma^\nu D_\nu - mc^2)\psi &= (c^2\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2c^4)\psi = \\ &= (c^2 D^\mu D_\mu - ic^2[\gamma^\mu, \gamma^\nu]D_\mu D_\nu + m^2c^4)\psi = 0 \end{aligned}$$

Note however, the  $\psi$  in the true Klein-Gordon equation is just a scalar, but here we get a 4-component spinor. Now:

$$\begin{aligned} D_\mu D_\nu &= (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) = \partial_\mu\partial_\nu + ie(A_\mu\partial_\nu + A_\nu\partial_\mu + (\partial_\mu A_\nu)) - e^2 A_\mu A_\nu \\ [D_\mu, D_\nu] &= D_\mu D_\nu - D_\nu D_\mu = ie(\partial_\mu A_\nu) - ie(\partial_\nu A_\mu) \end{aligned}$$

We rewrite  $D^\mu D_\mu$ :

$$\begin{aligned} D^\mu D_\mu &= g^{\mu\nu} D_\mu D_\nu = \partial^\mu\partial_\mu + ie((\partial^\mu A_\mu) + 2A^\mu\partial_\mu) - e^2 A^\mu A_\mu = \\ &= \partial^\mu\partial_\mu + ie((\partial^0 A_0) + 2A^0\partial_0 + (\partial^i A_i) + 2A^i\partial_i) - e^2(A^0 A_0 + A^i A_i) = \end{aligned}$$

$$= \partial^\mu \partial_\mu + i \frac{1}{c^2} \frac{\partial V}{\partial t} + 2i \frac{V}{c^2} \frac{\partial}{\partial t} + ie(\partial^i A_i) + 2ieA^i \partial_i - \frac{V^2}{c^2} - e^2 A^i A_i$$

We use the identity  $\frac{\partial}{\partial t} \left( e^{-imc^2 t} f(t) \right) = e^{-imc^2 t} (-imc^2 + \frac{\partial}{\partial t}) f(t)$  to get:

$$\begin{aligned} L &= c^2 \partial^\mu \psi^* \partial_\mu \psi - m^2 c^4 \psi^* \psi = \frac{\partial}{\partial t} \psi^* \frac{\partial}{\partial t} \psi - c^2 \partial^i \psi^* \partial_i \psi - m^2 c^4 \psi^* \psi = \\ &= (imc^2 + \frac{\partial}{\partial t}) \varphi^* (-imc^2 + \frac{\partial}{\partial t}) \varphi - c^2 \partial^i \varphi^* \partial_i \varphi - m^2 c^4 \varphi^* \varphi = \\ &= 2mc^2 \left[ \frac{1}{2} i (\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi + \frac{1}{2mc^2} \frac{\partial \varphi^*}{\partial t} \frac{\partial \varphi}{\partial t} \right] \end{aligned}$$

The constant factor  $2mc^2$  in front of the Lagrangian is of course irrelevant, so we drop it and then we take the limit  $c \rightarrow \infty$  (neglecting the last term) and we get

$$L = \frac{1}{2} i (\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

After integration by parts we arrive at

$$L = i \varphi^* \frac{\partial \varphi}{\partial t} - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

The nonrelativistic limit can also be applied directly to the Klein-Gordon equation:

$$\begin{aligned} 0 &= (c^2 D^\mu D_\mu + m^2 c^4) \psi = \\ &= \left( c^2 \partial^\mu \partial_\mu + i \frac{\partial V}{\partial t} + 2iV \frac{\partial}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi = \\ &= \left( \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi = \\ &= e^{-imc^2 t} \left( (-imc^2 + \frac{\partial}{\partial t})^2 - c^2 \nabla^2 + 2iV (-imc^2 + \frac{\partial}{\partial t}) + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 + \right. \\ &\quad \left. - e^2 c^2 A^i A_i + m^2 c^4 \right) \varphi = \\ &= e^{-imc^2 t} \left( -2imc^2 \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2Vmc^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 + \right. \\ &\quad \left. - e^2 c^2 A^i A_i \right) \varphi = \\ &= -2mc^2 e^{-imc^2 t} \left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - V - \frac{1}{2mc^2} \frac{\partial^2}{\partial t^2} - \frac{i}{2mc^2} \frac{\partial V}{\partial t} + \frac{V^2}{2mc^2} - \frac{iV}{mc^2} \frac{\partial}{\partial t} + \right. \\ &\quad \left. - \frac{ie}{2m} \partial^i A_i - \frac{ie}{m} A^i \partial_i + \frac{e^2}{2m} A^i A_i \right) \varphi \end{aligned}$$



Taking the limit  $c \rightarrow \infty$  we again recover the Schrödinger equation:

$$i\frac{\partial}{\partial t}\varphi = \left(-\frac{\nabla^2}{2m} + V + \frac{ie}{2m}\partial^i A_i + \frac{ie}{m}A^i\partial_i - \frac{e^2}{2m}A^i A_i\right)\varphi,$$

we rewrite the right hand side a little bit:

$$i\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}(\partial^i\partial_i + ie\partial^i A_i + 2ieA^i\partial_i - e^2 A^i A_i) + V\right)\varphi,$$

$$i\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}(\partial^i + ieA^i)(\partial_i + ieA_i) + V\right)\varphi,$$

And we get the usual form of the Schrödinger equation for the vector potential  $\mathbf{A} = (A_1, A_2, A_3)$ :

$$i\frac{\partial}{\partial t}\varphi = \left(-\frac{(\nabla + ie\mathbf{A})^2}{2m} + V\right)\varphi.$$