

Quantum Field Theory

June 1, 2008

Contents

1	Derivation from the Standard Model	1
1.1	Electroweak Standard Model	1
1.1.1	Higgs Terms	2
1.1.2	Yukawa terms	3
1.1.3	Gauge terms	3
1.1.4	Leptonic Terms	3
1.1.5	GWS Lagrangian	5
1.2	QED	5

1 Derivation from the Standard Model

1.1 Electroweak Standard Model

Lagrangian with a global $SU(2) \times U(1)$ symmetry:

$$L = i\bar{L}^{(l)}\gamma_\mu\partial^\mu L^{(l)} + i\bar{l}_R\gamma_\mu\partial^\mu l_R + \frac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2 - h_e\bar{L}^{(l)}\Phi e_R - \text{h.c.}$$

where $l = e, \mu, \tau$ and $a = 1, 2$, $l_{L,R} = \frac{1}{2}(1 \mp \gamma_5)l$ and

$$L^{(l)} = \begin{pmatrix} \nu_{(l)L} \\ l_L \end{pmatrix}$$

Local $SU(2) \times U(1)$ symmetry:

This consists of two things. First changing the partial derivatives to covariant ones:

$$\partial^\mu \rightarrow D^\mu = \partial^\mu - \frac{i}{2}g\tau_k A_k^\mu - \frac{i}{2}g'YB^\mu$$

and second adding the kinetic terms

$$-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

of the vector gauge particles to the lagrangian.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\Phi = e^{\frac{i}{v}\pi^a(x)\tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

This breaks the gauge invariance. The $\partial^\mu \pi^a$ are going to be added to A_μ^a so we can set $\pi_a = 0$ now.

1.1.1 Higgs Terms

$$L_{Higgs} = \frac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4}\lambda(\Phi^* \Phi)^2$$

Plugging in the covariant derivatives and Φ in U-gauge (symmetry breaking):

$$\begin{aligned} L_{Higgs} &= \frac{1}{2}\Phi^+ (\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'Y B_\mu) (\overrightarrow{\partial}^\mu + igA^{a\mu} \frac{\tau^a}{2} + ig'Y B^\mu) \Phi - \lambda(\Phi^+ \Phi - \frac{v^2}{2})^2 = \\ &= \Phi_U^+ (\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'Y B_\mu) (\overrightarrow{\partial}^\mu + igA^{a\mu} \frac{\tau^a}{2} + ig'Y B^\mu) \Phi_U - \lambda(\Phi_U^+ \Phi_U - \frac{v^2}{2})^2 = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\ &+ \frac{1}{8}(v+H)^2 \left(2g^2 \frac{A_\mu^1 + iA_\mu^2}{\sqrt{2}} \frac{A^{1\mu} - iA^{2\mu}}{\sqrt{2}} + (g^2 + 4Y^2 g'^2) \frac{gA_\mu^3 - 2Yg'B_\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \frac{gA^{3\mu} - 2Yg'B^\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \right) = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \frac{1}{8}(v+H)^2 \left(2g^2 W_\mu^- W^{+\mu} + \frac{g^2}{\cos^2 \theta_W} Z_\mu Z^\mu \right) = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8 \cos^2 \theta_W} Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\ &+ \frac{1}{2}v g^2 W_\mu^- W^{+\mu} H + \frac{g^2}{4 \cos \theta_W} v Z_\mu Z^\mu H + \frac{1}{4}g^2 W_\mu^- W^{+\mu} H^2 + \frac{g^2}{8 \cos \theta_W} Z_\mu Z^\mu H^2 \end{aligned}$$

Where we put

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2)$$

$$Z_\mu = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}} A_\mu^3 - \frac{2Yg'}{\sqrt{g^2 + 4Y^2 g'^2}} B_\mu$$

we defined θ_W by the relation

$$\cos \theta_W = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}}$$

so that the expressions simplify a bit, e.g.:

$$Z_\mu = \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu$$

$$g^2 + 4Y^2 g'^2 = \frac{g^2}{\cos^2 \theta_W}$$

1.1.2 Yukawa terms

$$\begin{aligned}
L_{Yukawa} &= -h_l \bar{L} \Phi e_R - \text{h.c.} = -h_l \bar{L} \Phi_U e_R - \text{h.c.} = \\
&= -\frac{1}{\sqrt{2}} h_l (v + H) (\bar{e}_L e_R + \bar{e}_R e_L) = -\frac{1}{\sqrt{2}} h_l (v + H) \bar{e} e = \\
&= -\frac{1}{\sqrt{2}} h_l v \bar{e} e - \frac{1}{\sqrt{2}} h_l \bar{e} e H
\end{aligned}$$

1.1.3 Gauge terms

$$\begin{aligned}
L_{Gauge} &= -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = \\
&= -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c) (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g\epsilon^{ajk} A^{j\mu} A^{k\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = \\
&= -\frac{1}{4} \partial_\mu A_\nu^a \partial^\mu A^{a\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g\epsilon^{abc} A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 \epsilon^{abc} \epsilon^{ajk} A_\mu^b A_\nu^c A^{k\mu} A^{l\nu} = \\
&= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g[(\partial_\mu A_\nu^1 - \partial_\nu A_\mu^1) A^{2\mu} A^{3\nu} + \text{cycl. perm. (123)}] \\
&\quad - \frac{1}{4} g^2 [(A_\mu^a A^{a\mu})(A_\nu^b A^{b\nu}) - (A_\mu^a A_\nu^a)(A^{b\mu} A^{b\nu})] = \\
&= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g[A_\mu^1 A_\nu^2 \overleftrightarrow{\partial}^\mu A^{3\nu} + \text{cycl. perm. (123)}] \\
&\quad - \frac{1}{4} g^2 [(A_\mu^a A^{a\mu})(A_\nu^b A^{b\nu}) - (A_\mu^a A_\nu^a)(A^{b\mu} A^{b\nu})] = \\
&= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - ig(W_\mu^0 W_\nu^- \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. (0-+)}) \\
&= -g^2 [\frac{1}{2} (W_\mu^+ W^{-\mu})^2 - \frac{1}{2} (W_\mu^+ W^{+\mu})(W_\nu^- W^{-\nu}) + (W_\mu^0 W^{0\mu})(W_\nu^+ W^{-\nu}) - (W_\mu^- W_\nu^+)(W^{0\mu} W^{0\nu})] = \\
&= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + [(5.79) - (5.84)]
\end{aligned}$$

1.1.4 Leptonic Terms

$$\begin{aligned}
L &= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R \gamma^\mu \partial_\mu e_R \rightarrow \\
&\rightarrow i\bar{L}\gamma^\mu (\partial_\mu - igA_\mu^a \frac{\tau^a}{2} - ig'Y_L B_\mu) L + i\bar{e}_R \gamma^\mu (\partial_\mu - ig'Y_R B_\mu) e_R = \\
&= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R \gamma^\mu \partial_\mu e_R + g\bar{L}\gamma^\mu \frac{\tau^a}{2} L A_\mu^a + g'Y_L \bar{L}\gamma^\mu L B_\mu + g'Y_R \bar{e}_R \gamma^\mu e_R B_\mu = \\
&= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R \gamma^\mu \partial_\mu e_R + \frac{g}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \text{h.c.}) + \frac{1}{2} g\bar{L}\gamma^\mu \tau^3 L A_\mu^3 + g'Y_L \bar{L}\gamma^\mu L B_\mu + g'Y_R \bar{e}_R \gamma^\mu e_R B_\mu = \\
&= i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e} \gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \text{h.c.}) + \frac{1}{2} g\bar{\nu}_L \gamma^\mu \nu_L A_\mu^3 - \frac{1}{2} g\bar{e}_L \gamma^\mu e_L A_\mu^3 \\
&\quad + g'Y_L \bar{\nu}_L \gamma^\mu \nu_L B_\mu + g'Y_L \bar{e}_L \gamma^\mu e_L B_\mu + g'Y_R \bar{e}_R \gamma^\mu e_R B_\mu =
\end{aligned}$$

$$\begin{aligned}
&= i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c}) \\
&+ \left[\left(\frac{1}{2}g\sin\theta_W + Y_L g' \cos\theta_W \right) \bar{\nu}_L\gamma^\mu\nu_L + \left(-\frac{1}{2}g\sin\theta_W + Y_L g' \cos\theta_W \right) \bar{e}_L\gamma^\mu e_L + Y_R g' \cos\theta_W \bar{e}_R\gamma^\mu e_R \right] A_\mu \\
&+ \left[\left(\frac{1}{2}g\cos\theta_W - Y_L g' \sin\theta_W \right) \bar{\nu}_L\gamma^\mu\nu_L + \left(-\frac{1}{2}g\cos\theta_W - Y_L g' \sin\theta_W \right) \bar{e}_L\gamma^\mu e_L - 2Y_L g' \sin\theta_W \bar{e}_R\gamma^\mu e_R \right] Z_\mu
\end{aligned}$$

Where we substituted new fields Z_μ and A_μ for the old ones A_μ^3 and B_μ using the relation:

$$\begin{aligned}
Z_\mu &= \cos\theta_W A_\mu^3 - \sin\theta_W B_\mu \\
A_\mu &= \sin\theta_W A_\mu^3 + \cos\theta_W B_\mu
\end{aligned}$$

The angle θ_W is arbitrary so far. We now need to make some natural requirements in order to move further. The neutrino fields should be absent from the interaction term with A_μ and the right-handed and left-handed components of the electron field should interact with the A_μ with an equal strength:

$$\begin{aligned}
\frac{1}{2}g\sin\theta_W + Y_L g' \cos\theta_W &= 0 \\
-\frac{1}{2}g\sin\theta_W + Y_L g' \cos\theta_W &= Y_R g' \cos\theta_W
\end{aligned}$$

from which it follows

$$\begin{aligned}
Y_R &= 2Y_L \\
\tan\theta_W &= -2Y_L \frac{g'}{g} \\
\cos\theta_W &= \frac{g}{\sqrt{g^2 + 4Y_L^2 g'^2}}
\end{aligned}$$

then the Lagrangian can be further simplified:

$$\begin{aligned}
L &= i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c}) \\
&- g\sin\theta_W(\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R)A_\mu \\
&+ \frac{g}{\cos\theta_W} \left[\frac{1}{2}\bar{\nu}_L\gamma^\mu\nu_L + \left(-\frac{1}{2} + \sin^2\theta_W \right) \bar{e}_L\gamma^\mu e_L + \sin^2\theta_W \bar{e}_R\gamma^\mu e_R \right] Z_\mu = \\
&= i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{2\sqrt{2}}(\bar{\nu}\gamma^\mu(1-\gamma_5)e W_\mu^+ + \text{h.c}) - g\sin\theta_W \bar{e}\gamma^\mu e A_\mu \\
&+ \frac{g}{2\cos\theta_W} \left[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu + \bar{e}\gamma^\mu\left(-\frac{1}{2} + 2\sin^2\theta_W + \frac{1}{2}\gamma_5\right)e \right] Z_\mu
\end{aligned}$$

Where we used the relations $\bar{\nu}_L\gamma^\mu e_L = \frac{1}{2}\bar{\nu}\gamma^\mu(1-\gamma_5)e$ and $\bar{\nu}_R\gamma^\mu e_R = \frac{1}{2}\bar{\nu}\gamma^\mu(1+\gamma_5)e$.

1.1.5 GWS Lagrangian

We need to set

$$Y = Y_L$$

otherwise the fields from the Higgs terms and the leptonic terms will not be the same. Now θ_W is the same in both set of terms. Plugging everything together we get the GWS Lagrangian:

$$\begin{aligned} L = & \frac{1}{2} \partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4} g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8 \cos^2 \theta_W} Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4} \lambda H^4 + \\ & + \frac{1}{2} v g^2 W_\mu^- W^{+\mu} H + \frac{g^2}{4 \cos \theta_W} v Z_\mu Z^\mu H + \frac{1}{4} g^2 W_\mu^- W^{+\mu} H^2 + \frac{g^2}{8 \cos \theta_W} Z_\mu Z^\mu H^2 \\ & - \frac{1}{\sqrt{2}} h_l v \bar{e} e - \frac{1}{\sqrt{2}} h_l \bar{e} e H \\ & - \frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + [(5.79) - (5.84)] \\ & + i \bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i \bar{e} \gamma^\mu \partial_\mu e + \frac{g}{2\sqrt{2}} (\bar{\nu} \gamma^\mu (1 - \gamma_5) e W_\mu^+ + \text{h.c.}) - g \sin \theta_W \bar{e} \gamma^\mu e A_\mu \\ & + \frac{g}{2 \cos \theta_W} [\bar{\nu} \gamma^\mu (1 - \gamma_5) \nu + \bar{e} \gamma^\mu (-\frac{1}{2} + 2 \sin^2 \theta_W + \frac{1}{2} \gamma_5) e] Z_\mu \end{aligned}$$

1.2 QED

The QED Lagrangian density is

$$L = \bar{\psi} (i c \gamma^\mu D_\mu - m c^2) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$\psi = (\psi_1 \psi_2 \psi_3 \psi_4)$$

and

$$D_\mu = \partial_\mu + i e A_\mu$$

is the gauge covariant derivative and (e is the elementary charge, which is 1 in atomic units)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. It's astonishing, that this simple Lagrangian can account for all phenomena from macroscopic scales down to something like 10^{-13} cm. So of course Feynman, Schwinger and Tomonaga received the 1965 Nobel Prize in Physics for such a fantastic achievement.

Plugging this Lagrangian into the Euler-Lagrange equation of motion for a field, we get:

$$(i c \gamma^\mu D_\mu - m c^2) \psi = 0$$

$$\partial_\nu F^{\nu\mu} = -e c \bar{\psi} \gamma^\mu \psi$$

The first equation is the Dirac equation in the electromagnetic field and the second equation is a set of Maxwell equations ($\partial_\nu F^{\nu\mu} = -ej^\mu$) with a source $j^\mu = c\bar{\psi}\gamma^\mu\psi$, which is a 4-current coming from the Dirac equation.

The fields ψ and A^μ are quantized. The first approximation is that we take ψ as a wavefunction, that is, it is a classical 4-component field. It can be shown that this corresponds to taking three orders in the perturbation theory.

The first component A_0 of the 4-potential is the electric potential, and because this is the potential that (as we show in a moment) is in the Schrödinger equation, we denote it by V :

$$A_\mu = \left(\frac{V}{ec}, A_1, A_2, A_3 \right)$$

So in the non-relativistic limit, the $\frac{V}{e}$ corresponds to the electric potential. We multiply the Dirac equation by γ^0 from left to get:

$$\begin{aligned} 0 &= \gamma^0(ic\gamma^\mu D_\mu - mc^2)\psi = \gamma^0(ic\gamma^0(\partial_0 + i\frac{V}{c}) + ic\gamma^i(\partial_i + ieA_i) - mc^2)\psi = \\ &= (ic\partial_0 + ic\gamma^0\gamma^i\partial_i - \gamma^0mc^2 - V - ce\gamma^0\gamma^iA_i)\psi \end{aligned}$$

and we make the following substitutions (it's just a formalism, nothing more): $\beta = \gamma^0$, $\alpha^i = \gamma^0\gamma^i$, $p_j = -i\partial_j$, $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$ to get

$$(i\frac{\partial}{\partial t} - c\alpha^i p_i - \beta mc^2 - V - ce\alpha^i A_i)\psi = 0.$$

This, in most solid state physics texts, is usually written as

$$i\frac{\partial\psi}{\partial t} = H\psi,$$

where the Hamiltonian is given by

$$H = c\alpha^i(p_i + eA_i) + \beta mc^2 + V.$$

The right hand side of the Maxwell equations is the 4-current, so it's given by:

$$j^\mu = c\bar{\psi}\gamma^\mu\psi$$

Now we make the substitution $\psi = e^{-imc^2 t}\varphi$, which states, that we separate the largest oscillations of the wavefunction and we get

$$\begin{aligned} j^0 &= c\bar{\psi}\gamma^0\psi = c\psi^\dagger\psi = c\varphi^\dagger\varphi \\ j^i &= c\bar{\psi}\gamma^i\psi = c\psi^\dagger\alpha^i\psi = c\varphi^\dagger\alpha^i\varphi \end{aligned}$$

The Dirac equation implies the Klein-Gordon equation:

$$(-ic\gamma^\mu D_\mu - mc^2)(ic\gamma^\nu D_\nu - mc^2)\psi = (c^2\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2c^4)\psi =$$

$$= (c^2 D^\mu D_\mu - ic^2 [\gamma^\mu, \gamma^\nu] D_\mu D_\nu + m^2 c^4) \psi = 0$$

Note however, the ψ in the true Klein-Gordon equation is just a scalar, but here we get a 4-component spinor. Now:

$$D_\mu D_\nu = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) = \partial_\mu \partial_\nu + ie(A_\mu \partial_\nu + A_\nu \partial_\mu + (\partial_\mu A_\nu)) - e^2 A_\mu A_\nu$$

$$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu = ie(\partial_\mu A_\nu) - ie(\partial_\nu A_\mu)$$

We rewrite $D^\mu D_\mu$:

$$D^\mu D_\mu = g^{\mu\nu} D_\mu D_\nu = \partial^\mu \partial_\mu + ie((\partial^\mu A_\mu) + 2A^\mu \partial_\mu) - e^2 A^\mu A_\mu =$$

$$= \partial^\mu \partial_\mu + ie((\partial^0 A_0) + 2A^0 \partial_0 + (\partial^i A_i) + 2A^i \partial_i) - e^2 (A^0 A_0 + A^i A_i) =$$

$$= \partial^\mu \partial_\mu + i \frac{1}{c^2} \frac{\partial V}{\partial t} + 2i \frac{V}{c^2} \frac{\partial}{\partial t} + ie(\partial^i A_i) + 2ieA^i \partial_i - \frac{V^2}{c^2} - e^2 A^i A_i$$

We use the identity $\frac{\partial}{\partial t} (e^{-imc^2 t} f(t)) = e^{-imc^2 t} (-imc^2 + \frac{\partial}{\partial t}) f(t)$ to get:

$$L = c^2 \partial^\mu \psi^* \partial_\mu \psi - m^2 c^4 \psi^* \psi = \frac{\partial}{\partial t} \psi^* \frac{\partial}{\partial t} \psi - c^2 \partial^i \psi^* \partial_i \psi - m^2 c^4 \psi^* \psi =$$

$$= (imc^2 + \frac{\partial}{\partial t}) \varphi^* (-imc^2 + \frac{\partial}{\partial t}) \varphi - c^2 \partial^i \varphi^* \partial_i \varphi - m^2 c^4 \varphi^* \varphi =$$

$$= 2mc^2 \left[\frac{1}{2} i (\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi + \frac{1}{2mc^2} \frac{\partial \varphi^*}{\partial t} \frac{\partial \varphi}{\partial t} \right]$$

The constant factor $2mc^2$ in front of the Lagrangian is of course irrelevant, so we drop it and then we take the limit $c \rightarrow \infty$ (neglecting the last term) and we get

$$L = \frac{1}{2} i (\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

After integration by parts we arrive at

$$L = i \varphi^* \frac{\partial \varphi}{\partial t} - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

The nonrelativistic limit can also be applied directly to the Klein-Gordon equation:

$$0 = (c^2 D^\mu D_\mu + m^2 c^4) \psi =$$

$$= \left(c^2 \partial^\mu \partial_\mu + i \frac{\partial V}{\partial t} + 2iV \frac{\partial}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi =$$

$$= \left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi =$$

$$= e^{-imc^2 t} \left((-imc^2 + \frac{\partial}{\partial t})^2 - c^2 \nabla^2 + 2iV (-imc^2 + \frac{\partial}{\partial t}) + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2ie c^2 A^i \partial_i - V^2 + \right.$$

$$\begin{aligned}
& -e^2 c^2 A^i A_i + m^2 c^4) \varphi = \\
& = e^{-imc^2 t} \left(-2imc^2 \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2Vmc^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 + \right. \\
& \quad \left. -e^2 c^2 A^i A_i \right) \varphi = \\
& = -2mc^2 e^{-imc^2 t} \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - V - \frac{1}{2mc^2} \frac{\partial^2}{\partial t^2} - \frac{i}{2mc^2} \frac{\partial V}{\partial t} + \frac{V^2}{2mc^2} - \frac{iV}{mc^2} \frac{\partial}{\partial t} + \right. \\
& \quad \left. -\frac{ie}{2m} \partial^i A_i - \frac{ie}{m} A^i \partial_i + \frac{e^2}{2m} A^i A_i \right) \varphi
\end{aligned}$$

Taking the limit $c \rightarrow \infty$ we again recover the Schrödinger equation:

$$i \frac{\partial}{\partial t} \varphi = \left(-\frac{\nabla^2}{2m} + V + \frac{ie}{2m} \partial^i A_i + \frac{ie}{m} A^i \partial_i - \frac{e^2}{2m} A^i A_i \right) \varphi,$$

we rewrite the right hand side a little bit:

$$\begin{aligned}
i \frac{\partial}{\partial t} \varphi &= \left(\frac{1}{2m} (\partial^i \partial_i + ie \partial^i A_i + 2ie A^i \partial_i - e^2 A^i A_i) + V \right) \varphi, \\
i \frac{\partial}{\partial t} \varphi &= \left(\frac{1}{2m} (\partial^i + ie A^i) (\partial_i + ie A_i) + V \right) \varphi,
\end{aligned}$$

And we get the usual form of the Schrödinger equation for the vector potential $\mathbf{A} = (A_1, A_2, A_3)$:

$$i \frac{\partial}{\partial t} \varphi = \left(-\frac{(\nabla + ie \mathbf{A})^2}{2m} + V \right) \varphi.$$