Quantum Field Theory

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1 Derivation from the Standard Model

1.1 Electroweak Standard Model

Lagrangian with a global $SU(2) \times U(1)$ symmetry:

$$L = i\bar{L}^{(l)}\gamma_\mu\partial^\mu L^{(l)} + i\bar{e}_R\gamma_\mu\partial^\mu e_R + \tfrac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2 - h_e\bar{L}^{(l)}\Phi e_R - \text{h.c.}$$

where $l = e, \mu, \tau$ and $a = 1, 2$.

Local $SU(2) \times U(1)$ symmetry:

This consists of two things. First changing the partial derivatives to covariant ones:

$$\partial^\mu \to D^\mu = \partial^\mu - \frac{i}{2}g\tau_k A_k^\mu - \frac{i}{2}g'YB^\mu$$

and second adding the kinetic terms

$$-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

of the vector gauge particles to the lagrangian.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c$$
$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\Phi = e^{\frac{i}{v}\pi^a(x)\tau^a}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

This breaks the gauge invariance. The $\partial^\mu\pi^a$ are going to be added to A_μ^a so we can set $\pi_a = 0$ now.

1.1.1 Higgs Terms

Φ in U-gauge:

$$L_{Higgs} = \Phi^{+}(\overleftarrow{\partial}_{\mu}+igA^{a}_{\mu}\frac{\tau^{a}}{2}+ig'YB_{\mu})(\overrightarrow{\partial}^{\mu}+igA^{a\mu}\frac{\tau^{a}}{2}+ig'YB^{\mu})\Phi-\lambda(\Phi^{+}\Phi-\frac{v^{2}}{2})^{2} =$$

$$= \Phi_{U}^{+}(\overleftarrow{\partial}_{\mu}+igA^{a}_{\mu}\frac{\tau^{a}}{2}+ig'YB_{\mu})(\overrightarrow{\partial}^{\mu}+igA^{a\mu}\frac{\tau^{a}}{2}+ig'YB\mu)\Phi_{U}-\lambda(\Phi_{U}^{+}\Phi_{U}-\frac{v^{2}}{2})^{2} =$$

$$= \frac{1}{2}\partial_{\mu}H\partial^{\mu}H - \lambda v^{2}H^{2} - \lambda vH^{3} - \frac{1}{4}\lambda H^{4}+$$

$$+\frac{1}{8}(v+H)^{2}\left(2g^{2}\frac{A^{1}_{\mu}+iA^{2}_{\mu}}{\sqrt{2}}\frac{A^{1\mu}-iA^{2\mu}}{\sqrt{2}}+(g^{2}+4Y^{2}g'^{2})\frac{gA^{3}_{\mu}-2Yg'B_{\mu}}{\sqrt{g^{2}+4Y^{2}g'^{2}}}\frac{gA^{3\mu}-2Yg'B^{\mu}}{\sqrt{g^{2}+4Y^{2}g'^{2}}}\right) =$$

$$= \frac{1}{2}\partial_{\mu}H\partial^{\mu}H-\lambda v^{2}H^{2}-\lambda vH^{3}-\frac{1}{4}\lambda H^{4}+\frac{1}{8}(v+H)^{2}\left(2g^{2}W^{-}_{\mu}W^{+\mu}+(g^{2}+4Y^{2}g'^{2})Z_{\mu}Z^{\mu}\right) =$$

$$= \frac{1}{2}\partial_{\mu}H\partial^{\mu}H-\lambda v^{2}H^{2}+\frac{1}{4}g^{2}v^{2}W^{-}_{\mu}W^{+\mu}+\frac{1}{8}(g^{2}+4Y^{2}g'^{2})Z_{\mu}Z^{\mu}-\lambda vH^{3}-\frac{1}{4}\lambda H^{4}+$$

$$+\frac{1}{2}vHg^{2}W^{-}_{\mu}W^{+\mu}+\frac{1}{4}vH(g^{2}+4Y^{2}g'^{2})Z_{\mu}Z^{\mu}+\frac{1}{4}H^{2}g^{2}W^{-}_{\mu}W^{+\mu}+\frac{1}{8}H^{2}(g^{2}+4Y^{2}g'^{2})Z_{\mu}Z^{\mu}$$

Where we put

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(A^{1}_{\mu} \mp iA^{2}_{\mu})$$

$$Z_{\mu} = \frac{g}{\sqrt{g^{2}+4Y^{2}g'^{2}}}A^{3}_{\mu} - \frac{2Yg'}{\sqrt{g^{2}+4Y^{2}g'^{2}}}B_{\mu}$$

By defining θ_W by the relation

$$\cos\theta_{W} = \frac{g}{\sqrt{g^{2}+4Y^{2}g'^{2}}}$$

we can then write:

$$Z_{\mu} = \cos\theta_{W}A^{3}_{\mu} - \sin\theta_{W}B_{\mu}$$

1.1.2 Yukawa terms

$$L_{Yukawa} = -h_{l}\bar{L}\Phi e_{R} + \text{h.c.} = -h_{l}\bar{L}\Phi_{U}e_{R} + \text{h.c.} =$$

$$= -\frac{1}{\sqrt{2}}h_{l}(v+H)(\bar{e}_{L}e_{R}+\bar{e}_{R}e_{L}) = -\frac{1}{\sqrt{2}}h_{l}(v+H)\bar{e}e =$$

$$= -\frac{1}{\sqrt{2}}h_{l}v\bar{e}e - \frac{1}{\sqrt{2}}h_{l}\bar{e}eH$$

1.2 QED

The QED Lagrangian density is

$$L = \bar{\psi}(ic\gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where

$$\psi = (\psi_1\psi_2\psi_3\psi_4)$$

and

$$D_\mu = \partial_\mu + ieA_\mu$$

is the gauge covariant derivative and (e is the elementary charge, which is 1 in atomic units)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. It's astonishing, that this simple Lagrangian can account for all phenomena from macroscopic scales down to something like 10^{-13} cm. So of course Feynman, Schwinger and Tomonaga received the 1965 Nobel Prize in Physics for such a fantastic achievement.

Plugging this Lagrangian into the Euler-Lagrange equation of motion for a field, we get:

$$(ic\gamma^\mu D_\mu - mc^2)\psi = 0$$

$$\partial_\nu F^{\nu\mu} = -ec\bar{\psi}\gamma^\mu\psi$$

The first equation is the Dirac equation in the electromagnetic field and the second equation is a set of Maxwell equations ($\partial_\nu F^{\nu\mu} = -ej^\mu$) with a source $j^\mu = c\bar{\psi}\gamma^\mu\psi$, which is a 4-current comming from the Dirac equation.

The fields ψ and A^μ are quantized. The first approximation is that we take ψ as a wavefunction, that is, it is a classical 4-component field. It can be shown that this corresponds to taking three orders in the perturbation theory.

The first component A_0 of the 4-potential is the electric potential, and because this is the potential that (as we show in a moment) is in the Schrödinger equation, we denote it by V:

$$A_\mu = \left(\frac{V}{ec}, A_1, A_2, A_3\right)$$

So in the non-relativistic limit, the $\frac{V}{e}$ corresponds to the electric potential. We multiply the Dirac equation by γ^0 from left to get:

$$0 = \gamma^0(ic\gamma^\mu D_\mu - mc^2)\psi = \gamma^0(ic\gamma^0(\partial_0 + i\frac{V}{c}) + ic\gamma^i(\partial_i + ieA_i) - mc^2)\psi =$$

$$= (ic\partial_0 + ic\gamma^0\gamma^i\partial_i - \gamma^0 mc^2 - V - ce\gamma^0\gamma^i A_i)\psi$$

and we make the following substitutions (it's just a formalism, nothing more): $\beta = \gamma^0$, $\alpha^i = \gamma^0\gamma^i$, $p_j = -i\partial_j$, $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$ to get

$$(i\frac{\partial}{\partial t} - c\alpha^i p_i - \beta mc^2 - V - ce\alpha^i A_i)\psi = 0\,.$$

This, in most solid state physics texts, is usually written as

$$i\frac{\partial\psi}{\partial t} = H\psi\,,$$

where the Hamiltonian is given by

$$H = c\alpha^i(p_i + eA_i) + \beta mc^2 + V\,.$$

The right hand side of the Maxwell equations is the 4-current, so it's given by:

$$j^\mu = c\bar{\psi}\gamma^\mu\psi$$

Now we make the substitution $\psi = e^{-imc^2t}\varphi$, which states, that we separate the largest oscillations of the wavefunction and we get

$$j^0 = c\bar{\psi}\gamma^0\psi = c\psi^\dagger\psi = c\varphi^\dagger\varphi$$

$$j^i = c\bar{\psi}\gamma^i\psi = c\psi^\dagger\alpha^i\psi = c\varphi^\dagger\alpha^i\varphi$$

The Dirac equation implies the Klein-Gordon equation:

$$(-ic\gamma^\mu D_\mu - mc^2)(ic\gamma^\nu D_\nu - mc^2)\psi = (c^2\gamma^\mu\gamma^\nu D_\mu D_\mu + m^2c^4)\psi =$$

$$= (c^2 D^\mu D_\mu - ic^2[\gamma^\mu,\gamma^\nu]D_\mu D_\nu + m^2c^4)\psi = 0$$

Note however, the ψ in the true Klein-Gordon equation is just a scalar, but here we get a 4-component spinor. Now:

$$D_\mu D_\nu = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) = \partial_\mu\partial_\nu + ie(A_\mu\partial_\nu + A_\nu\partial_\mu + (\partial_\mu A_\nu)) - e^2A_\mu A_\nu$$

$$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu = ie(\partial_\mu A_\nu) - ie(\partial_\nu A_\mu)$$

We rewrite $D^\mu D_\mu$:

$$D^\mu D_\mu = g^{\mu\nu}D_\mu D_\nu = \partial^\mu\partial_\mu + ie((\partial^\mu A_\mu) + 2A^\mu\partial_\mu) - e^2A^\mu A_\mu =$$

$$= \partial^\mu\partial_\mu + ie((\partial^0 A_0) + 2A^0\partial_0 + (\partial^i A_i) + 2A^i\partial_i) - e^2(A^0A_0 + A^iA_i) =$$

$$= \partial^\mu\partial_\mu + i\frac{1}{c^2}\frac{\partial V}{\partial t} + 2i\frac{V}{c^2}\frac{\partial}{\partial t} + ie(\partial^i A_i) + 2ieA^i\partial_i - \frac{V^2}{c^2} - e^2A^iA_i$$

We use the identity $\frac{\partial}{\partial t}\left(e^{-imc^2t}f(t)\right) = e^{-imc^2t}(-imc^2 + \frac{\partial}{\partial t})f(t)$ to get:

$$L = c^2\partial^\mu\psi^*\partial_\mu\psi - m^2c^4\psi^*\psi = \frac{\partial}{\partial t}\psi^*\frac{\partial}{\partial t}\psi - c^2\partial^i\psi^*\partial_i\psi - m^2c^4\psi^*\psi =$$

$$= (imc^2 + \frac{\partial}{\partial t})\varphi^*(-imc^2 + \frac{\partial}{\partial t})\varphi - c^2\partial^i\varphi^*\partial_i\varphi - m^2c^4\varphi^*\varphi =$$

$$= 2mc^2\left[\frac{1}{2}i(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}) - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi + \frac{1}{2mc^2}\frac{\partial\varphi^*}{\partial t}\frac{\partial\varphi}{\partial t}\right]$$

The constant factor $2mc^2$ in front of the Lagrangian is of course irrelevant, so we drop it and then we take the limit $c \to \infty$ (neglecting the last term) and we get

$$L = \frac{1}{2} i (\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

After integration by parts we arrive at

$$L = i \varphi^* \frac{\partial \varphi}{\partial t} - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

The nonrelativistic limit can also be applied directly to the Klein-Gordon equation:

$$0 = (c^2 D^\mu D_\mu + m^2 c^4)\psi =$$

$$= \left(c^2 \partial^\mu \partial_\mu + i \frac{\partial V}{\partial t} + 2iV \frac{\partial}{\partial t} + iec^2(\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi =$$

$$= \left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi =$$

$$= e^{-imc^2 t} \left((-imc^2 + \frac{\partial}{\partial t})^2 - c^2 \nabla^2 + 2iV(-imc^2 + \frac{\partial}{\partial t}) + i \frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 + \right.$$

$$\left. - e^2 c^2 A^i A_i + m^2 c^4 \right) \varphi =$$

$$= e^{-imc^2 t} \left(-2imc^2 \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2Vmc^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 + \right.$$

$$\left. - e^2 c^2 A^i A_i \right) \varphi =$$

$$= -2mc^2 e^{-imc^2 t} \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - V - \frac{1}{2mc^2} \frac{\partial^2}{\partial t^2} - \frac{i}{2mc^2} \frac{\partial V}{\partial t} + \frac{V^2}{2mc^2} - \frac{iV}{mc^2} \frac{\partial}{\partial t} + \right.$$

$$\left. - \frac{ie}{2m} \partial^i A_i - \frac{ie}{m} A^i \partial_i + \frac{e^2}{2m} A^i A_i \right) \varphi$$

Taking the limit $c \to \infty$ we again recover the Schrödinger equation:

$$i \frac{\partial}{\partial t} \varphi = \left(- \frac{\nabla^2}{2m} + V + \frac{ie}{2m} \partial^i A_i + \frac{ie}{m} A^i \partial_i - \frac{e^2}{2m} A^i A_i \right) \varphi \,,$$

we rewrite the right hand side a little bit:

$$i \frac{\partial}{\partial t} \varphi = \left(\frac{1}{2m} (\partial^i \partial_i + ie \partial^i A_i + 2ie A^i \partial_i - e^2 A^i A_i) + V \right) \varphi \,,$$

$$i \frac{\partial}{\partial t} \varphi = \left(\frac{1}{2m} (\partial^i + ie A^i)(\partial_i + ie A_i) + V \right) \varphi \,,$$

And we get the usual form of the Schrödinger equation for the vector potential $\mathbf{A} = (A_1, A_2, A_3)$:

$$i \frac{\partial}{\partial t} \varphi = \left(- \frac{(\nabla + ie\mathbf{A})^2}{2m} + V \right) \varphi \,.$$