Quantum Field Theory and Quantum Mechanics

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1 Introduction

The aim of these (work in progress) notes is to use the Standard Model of particle physics to derive all equations in quantum mechanics (and quantum field theory) that we need for our research.

We start by deriving the electroweak Standard Model from the $SU(2)\times U(1)$ symmetry and couple other (standard) assumptions in the quantum field theory. After that, we only want to derive things and make nonrelativistic limits or other approximations in order to derive everything else in quantum mechanics. In particular we show how to derive the Dirac and Schrödinger equations (as a low energy limit). We then show some particular ways to solve those equations, like perturbation theory, scattering theory, ...

The goal is to have a complete theory on about 30 or 40 pages and then lots of examples (arbitrarily long), that use the theory (but do not develop new ideas), so that one can learn how the theory works from the examples. For instance, one can ask "why is there the term $(\mathbf{p}-e\mathbf{A})^2$ in the Schrödinger equation for electromagnetic field, why this and not something else?" or "why is there the $\boldsymbol{\sigma}\cdot\mathbf{B}$ term in the Pauli equation?", to find the answer, one just finds the Pauli equation in the theory and then looks at the derivation, so in this case one quickly finds that it follows from the minimal coupling in QED, e.g. it's the easiest way how electron-foton interaction can be coupled, e.g. the $U(1)$ symmetry. Nice thing about QFT is that one can find really nice geometrical reasons why things are that way and not some other way (just open any advance book on QFT), but the problem is that basically nowhere is some easy (but correct) translation of those results to regular QM, so that everything fits into just couple dozens pages, so that it can serve as a reference.

The advantage of this top-down approach is that it is easy to see where things come from and also to understand exactly what approximations one is using when dealing with any equation in QM. However, as is well-known in physics, to be a good physicist one has to understand all the approaches, e.g. both top-down and bottom-up and all other approaches to QM and QFT, because there are no two approaches that would be 100% equivalent, so one has to use the right approach for the particular problem. So these notes do not aspire to be the right way to teach QM, but rather to serve as a reference to get quickly oriented and to find the equations to start from.

2 Standard Model

2.1 Electroweak Standard Model

Lagrangian with a global $SU(2)\times U(1)$ symmetry:

$$\mathcal{L}=i\bar{L}^{(l)}\gamma_\mu\partial^\mu L^{(l)}+i\bar{l}_R\gamma_\mu\partial^\mu l_R+\tfrac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi-m^2\Phi^*\Phi-\frac{1}{4}\lambda(\Phi^*\Phi)^2-h_e\bar{L}^{(l)}\Phi e_R-\text{h.c.}$$

where $l=e,\mu,\tau$ and $a=1,2$, $l_{L,R}=\frac{1}{2}(1\mp\gamma_5)l$ and

$$L^{(l)}=\begin{pmatrix}\nu_{(l)L}\\ l_L\end{pmatrix}$$

Local $SU(2)\times U(1)$ symmetry:

This consists of two things. First changing the partial derivatives to covariant ones:

$$\partial^\mu \to D^\mu = \partial^\mu - \frac{i}{2} g \tau_k A_k^\mu - \frac{i}{2} g' Y B^\mu$$

and second adding the kinetic terms

$$-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

of the vector gauge particles to the lagrangian.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\Phi = e^{\frac{i}{v}\pi^a(x)\tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

This breaks the gauge invariance. The $\partial^\mu \pi^a$ are going to be added to A_μ^a so we can set $\pi_a = 0$ now.

2.1.1 Higgs Terms

$$\mathcal{L}_{Higgs} = \tfrac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2$$

Plugging in the covariant derivatives and Φ in U-gauge (symmetry breaking):

$$\mathcal{L}_{Higgs} = \frac{1}{2}\Phi^+(\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'YB_\mu)(\overrightarrow{\partial}^\mu + igA^{a\mu}\frac{\tau^a}{2} + ig'YB^\mu)\Phi - \lambda(\Phi^+\Phi - \frac{v^2}{2})^2 =$$

$$= \Phi_U^+(\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'YB_\mu)(\overrightarrow{\partial}^\mu + igA^{a\mu}\frac{\tau^a}{2} + ig'YB\mu)\Phi_U - \lambda(\Phi_U^+\Phi_U - \frac{v^2}{2})^2 =$$

$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 +$$

$$+\frac{1}{8}(v+H)^2 \left(2g^2 \frac{A_\mu^1 + iA_\mu^2}{\sqrt{2}} \frac{A^{1\mu} - iA^{2\mu}}{\sqrt{2}} + (g^2 + 4Y^2 g'^2) \frac{gA_\mu^3 - 2Yg'B_\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \frac{gA^{3\mu} - 2Yg'B^\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \right) =$$

$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \frac{1}{8}(v+H)^2 \left(2g^2 W_\mu^- W^{+\mu} + \frac{g^2}{\cos^2\theta_W} Z_\mu Z^\mu \right) =$$

$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4} g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8\cos^2\theta_W} Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 +$$

$$+\frac{1}{2} v g^2 W_\mu^- W^{+\mu} H + \frac{g^2}{4\cos\theta_W} v Z_\mu Z^\mu H + \frac{1}{4} g^2 W_\mu^- W^{+\mu} H^2 + \frac{g^2}{8\cos\theta_W} Z_\mu Z^\mu H^2$$

Where we put

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2)$$

$$Z_\mu = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}} A_\mu^3 - \frac{2Yg'}{\sqrt{g^2 + 4Y^2 g'^2}} B_\mu$$

we defined θ_W by the relation

$$\cos\theta_W = \frac{g}{\sqrt{g^2+4Y^2g'^2}}$$

so that the expressions simplify a bit, e.g. we now get:

$$\sin\theta_W = \frac{2Yg'}{\sqrt{g^2+4Y^2g'^2}}$$

$$Z_\mu = \cos\theta_W A_\mu^3 - \sin\theta_W B_\mu$$

$$g^2+4Y^2g'^2 = \frac{g^2}{\cos^2\theta_W}$$

2.1.2 Yukawa terms

$$\begin{aligned}\mathcal{L}_{Yukawa} = -h_e\bar{L}\Phi e_R - \text{h.c.} = -h_e\bar{L}\Phi_U e_R - \text{h.c.} =\\ = -\frac{1}{\sqrt{2}}h_e(v+H)(\bar{e}_Le_R+\bar{e}_Re_L) = -\frac{1}{\sqrt{2}}h_e(v+H)\bar{e}e =\\ = -\frac{1}{\sqrt{2}}h_ev\bar{e}e - \frac{1}{\sqrt{2}}h_e\bar{e}eH\end{aligned}$$

The term $\bar{L}\Phi e_R$ is $U(1)$ (hypercharge) invariant, so

$$-Y_L+Y+Y_R=0$$

2.1.3 Leptonic Terms

$$\begin{aligned}\mathcal{L} = i\bar{L}\gamma^\mu\partial_\mu L + i\bar{e}_R\gamma^\mu\partial_\mu e_R \rightarrow\\ \rightarrow i\bar{L}\gamma^\mu(\partial_\mu - igA_\mu^a\frac{\tau^a}{2} - ig'Y_LB_\mu)L + i\bar{e}_R\gamma^\mu(\partial_\mu - ig'Y_RB_\mu)e_R =\\ = i\bar{L}\gamma^\mu\partial_\mu L + i\bar{e}_R\gamma^\mu\partial_\mu e_R + g\bar{L}\gamma^\mu\frac{\tau^a}{2}LA_\mu^a + g'Y_L\bar{L}\gamma^\mu LB_\mu + g'Y_R\bar{e}_R\gamma^\mu e_RB_\mu =\\ = i\bar{L}\gamma^\mu\partial_\mu L + i\bar{e}_R\gamma^\mu\partial_\mu e_R + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_LW_\mu^+ + \text{h.c.}) + \frac{1}{2}g\bar{L}\gamma^\mu\tau^3LA_\mu^3 + g'Y_L\bar{L}\gamma^\mu LB_\mu + g'Y_R\bar{e}_R\gamma^\mu e_RB_\mu =\\ = i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_LW_\mu^+ + \text{h.c.}) + \frac{1}{2}g\bar{\nu}_L\gamma^\mu\nu_LA_\mu^3 - \frac{1}{2}g\bar{e}_L\gamma^\mu e_LA_\mu^3\\ + g'Y_L\bar{\nu}_L\gamma^\mu\nu_LB_\mu + g'Y_L\bar{e}_L\gamma^\mu e_LB_\mu + g'Y_R\bar{e}_R\gamma^\mu e_RB_\mu =\\ = i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_LW_\mu^+ + \text{h.c.})\end{aligned}$$

$$+\left[(\tfrac{1}{2}g\sin\theta_W + Y_Lg'\cos\theta_W)\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2}g\sin\theta_W + Y_Lg'\cos\theta_W)\bar{e}_L\gamma^\mu e_L + Y_Rg'\cos\theta_W\bar{e}_R\gamma^\mu e_R\right]A_\mu$$
$$+\left[(\tfrac{1}{2}g\cos\theta_W - Y_Lg'\sin\theta_W)\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2}g\cos\theta_W - Y_Lg'\sin\theta_W)\bar{e}_L\gamma^\mu e_L - 2Y_Lg'\sin\theta_W\bar{e}_R\gamma^\mu e_R\right]Z_\mu$$

Where we substituted new fields Z_μ and A_μ for the old ones A_μ^3 and B_μ using the relation:

$$Z_\mu = \cos\theta_W A_\mu^3 - \sin\theta_W B_\mu$$

$$A_\mu = \sin\theta_W A_\mu^3 + \cos\theta_W B_\mu$$

The angle θ_W must be the same as in the Higgs sector, so that the field Z_μ is the same. We now need to make the following requirement in order to proceed further:

$$Y = -Y_L$$

This follows for example by requiring that neutrinos have zero charge, i.e. setting $\frac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W = 0$ and substituting for θ_W from the definition (see the Higgs terms), from which one gets $Y = -Y_L$. From $-Y_L + Y + Y_R = 0$ we now get

$$Y_R = 2Y_L$$

it now follows:

$$\frac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W = 0$$

$$-\frac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W = -g\sin\theta_W$$

$$Y_R g'\cos\theta_W = -g\sin\theta_W$$

$$\tan\theta_W = -2Y_L\frac{g'}{g}$$

and the Lagrangian can be further simplified:

$$\mathcal{L} = i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c.})$$

$$-g\sin\theta_W(\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R)A_\mu$$

$$+\frac{g}{\cos\theta_W}\left[\tfrac{1}{2}\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2}+\sin^2\theta_W)\bar{e}_L\gamma^\mu e_L + \sin^2\theta_W\bar{e}_R\gamma^\mu e_R\right]Z_\mu =$$

$$= i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{2\sqrt{2}}(\bar{\nu}\gamma^\mu(1-\gamma_5)eW_\mu^+ + \text{h.c.}) - g\sin\theta_W\bar{e}\gamma^\mu eA_\mu$$

$$+\frac{g}{2\cos\theta_W}\left[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu + \bar{e}\gamma^\mu(-\tfrac{1}{2}+2\sin^2\theta_W+\tfrac{1}{2}\gamma_5)e\right]Z_\mu$$

Where we used the relations $\bar{\nu}_L\gamma^\mu e_L = \frac{1}{2}\bar{\nu}\gamma^\mu(1-\gamma_5)e$ and $\bar{\nu}_R\gamma^\mu e_R = \frac{1}{2}\bar{\nu}\gamma^\mu(1+\gamma_5)e$.

2.1.4 Gauge terms

$$\mathcal{L}_{Gauge} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} =$$

$$= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g\epsilon^{ajk}A^{j\mu}A^{k\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} =$$

$$= -\frac{1}{4}\partial_\mu A^a_\nu\partial^\mu A^{a\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)g\epsilon^{abc}A^{b\mu}A^{c\nu} - \frac{1}{4}g^2\epsilon^{abc}\epsilon^{ajk}A^b_\mu A^c_\nu A^{k\mu}A^{l\nu} =$$

$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - g[(\partial_\mu A^1_\nu - \partial_\nu A^1_\mu)A^{2\mu}A^{3\nu} + \text{cycl. perm. (123)}]$$

$$-\frac{1}{4}g^2[(A^a_\mu A^{a\mu})(A^b_\nu A^{b\nu}) - (A^a_\mu A^a_\nu)(A^{b\mu}A^{b\nu})] =$$

$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - g[A^1_\mu A^2_\nu\overleftrightarrow{\partial}^\mu A^{3\nu} + \text{cycl. perm. (123)}]$$

$$-\frac{1}{4}g^2[(A^a_\mu A^{a\mu})(A^b_\nu A^{b\nu}) - (A^a_\mu A^a_\nu)(A^{b\mu}A^{b\nu})] =$$

$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - ig(W^0_\mu W^-_\nu \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. (0-+)})$$

$$-g^2[\tfrac{1}{2}(W^+_\mu W^{-\mu})^2 - \tfrac{1}{2}(W^+_\mu W^{+\mu})(W^-_\nu W^{-\nu}) + (W^0_\mu W^{0\mu})(W^+_\nu W^{-\nu}) - (W^-_\mu W^+_\nu)(W^{0\mu}W^{0\nu}) =$$

$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \mathcal{L}_{WW\gamma} + L_{WWZ} + L_{WW\gamma\gamma} + L_{WWWW} + L_{WWZZ} + L_{WWZ\gamma}$$

Where $W^0_\mu = A^3_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu$ and:

$$\mathcal{L}_{WW\gamma} = -ig\sin\theta_W(A_\mu W^-_\nu \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. } (A\ W^-\ W^+))$$

$$\mathcal{L}_{WWZ} = -ig\cos\theta_W(Z_\mu W^-_\nu \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. } (Z\ W^-\ W^+))$$

$$\mathcal{L}_{WW\gamma\gamma} = -g^2\sin^2\theta_W(W^-_\mu W^{+\mu}A_\nu A^\nu - W^-_\mu A^\mu W^+_\nu A^\nu)$$

$$\mathcal{L}_{WWWW} = \tfrac{1}{2}g^2(W^-_\mu W^{-\mu}W^+_\nu W^{+\nu} - W^-_\mu W^{+\mu}W^-_\nu W^{+\nu})$$

$$\mathcal{L}_{WWZZ} = -g^2\cos^2\theta_W(W^-_\mu W^{+\mu}Z_\nu Z^\nu - W^-_\mu Z^\mu W^+_\nu Z^\nu)$$

$$\mathcal{L}_{WWZ\gamma} = g^2\sin\theta_W\cos\theta_W(-2W^-_\mu W^{+\mu}A_\nu Z^\nu + W^-_\mu Z^\mu W^+_\nu A^\nu + W^-_\mu A^\mu W^+_\nu Z^\nu)$$

2.1.5 GWS Lagrangian

Plugging everything together we get the GWS Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu H\partial^\mu H - \lambda v^2H^2 + \frac{1}{4}g^2v^2W^-_\mu W^{+\mu} + \frac{g^2v^2}{8\cos^2\theta_W}Z_\mu Z^\mu - \lambda vH^3 - \frac{1}{4}\lambda H^4 +$$

$$+\frac{1}{2}vg^2W^-_\mu W^{+\mu}H + \frac{g^2}{4\cos\theta_W}vZ_\mu Z^\mu H + \frac{1}{4}g^2W^-_\mu W^{+\mu}H^2 + \frac{g^2}{8\cos\theta_W}Z_\mu Z^\mu H^2$$

$$-\frac{1}{\sqrt{2}}h_e v\bar{e}e - \frac{1}{\sqrt{2}}h_e\bar{e}eH$$

$$-\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \mathcal{L}_{WW\gamma} + L_{WWZ} + L_{WW\gamma\gamma} + L_{WWWW} + L_{WWZZ} + L_{WWZ\gamma}$$

$$+i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{2\sqrt{2}}(\bar{\nu}\gamma^\mu(1-\gamma_5)eW^+_\mu + \text{h.c.}) - g\sin\theta_W\bar{e}\gamma^\mu eA_\mu$$

$$+\frac{g}{2\cos\theta_W}\left[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu + \bar{e}\gamma^\mu(-\tfrac{1}{2} + 2\sin^2\theta_W + \tfrac{1}{2}\gamma_5)e\right]Z_\mu$$

$$+(e,\nu_e,h_e \leftrightarrow \mu,\nu_\mu,h_\mu) + (e,\nu_e,h_e \leftrightarrow \tau,\nu_\tau,h_\tau)$$

The free parameters are g, θ_W, v, λ, h_e, h_μ, h_τ.

2.1.6 Particle Masses

The particle masses are deduced from the terms

$$\mathcal{L} = -\frac{1}{2}m_H^2 H^2 + m_W^2 W_\mu^- W^{+\mu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu - m_e \bar{e}e + \cdots$$

comparing to the above:

$$\mathcal{L} = -\lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8\cos^2\theta_W} Z_\mu Z^\mu - \frac{1}{\sqrt{2}} h_e v \bar{e}e + \cdots$$

we get

$$m_W = \tfrac{1}{2}gv$$
$$m_Z = \frac{gv}{2\cos\theta_W} = \frac{m_W}{\cos\theta_W}$$
$$m_H = v\sqrt{2\lambda}$$
$$m_e = \frac{1}{\sqrt{2}}h_e v$$

2.1.7 Quarks

$$\mathcal{L}_{fermion} {+}{=} \sum_{q=d,s,b} i\bar{L}_0^{(q)}\gamma^\mu\partial_\mu L_0^{(q)} + \sum_{q=d,u,s,c,b,t} i\bar{q}_{0R}\gamma^\mu\partial_\mu q_{0R}$$

$$\mathcal{L}_{Yukawa} {+}{=} -\sum_{\substack{q=d,s,b\\ q'=d,s,b}} h_{qq'} i\bar{L}_0^{(q)}\Phi q'_{0R} + \text{h.c.} - \sum_{\substack{q=d,s,b\\ q'=u,c,t}} \tilde{h}_{qq'} i\bar{L}_0^{(q)}\tilde{\Phi} q'_{0R} + \text{h.c.}$$

2.2 QFT

2.2.1 Evolution Operator, S-Matrix Elements

The evolution operator U is defined by the equations:

$$|\phi(t_2)\rangle = U(t_2, t_1)\,|\phi(t_1)\rangle$$
$$i\hbar\frac{\partial U(t, t_1)}{\partial t} = H(t)U(t, t_1)$$
$$U(t_1, t_1) = 1$$

We are interested in calculating the S matrix elements:

$$\langle f|U(-\infty, \infty)|i\rangle = \langle f|S|i\rangle = S_{fi}$$

so we first calculate $U(-\infty, \infty)$. Integrating the equation for the evolution operator:

$$U(t_2, t_1) = U(t_1, t_1) - \frac{i}{\hbar}\int_{t_1}^{t_2} H(t)U(t, t_1)\mathrm{d}t = 1 - \frac{i}{\hbar}\int_{t_1}^{t_2} H(t)U(t, t_1)\mathrm{d}t$$

Now:

$$S = U(-\infty, \infty) = 1 - \frac{i}{\hbar}\int_{-\infty}^{\infty} H(t')U(t', -\infty)\mathrm{d}t' =$$

$$= 1+\left(-\frac{i}{\hbar}\right)\int_{-\infty}^{\infty} H(t')U(t',-\infty)\mathrm{d}t'+\left(-\frac{i}{\hbar}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{t'} H(t')H(t'')U(t'',-\infty)\mathrm{d}t'\mathrm{d}t'' =$$

$$= \cdots = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots T\{H(t_1)H(t_2)\cdots\}\mathrm{d}t_1\mathrm{d}t_2\cdots =$$

$$= T\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty} H(t)\mathrm{d}t\right) = T\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{H}(x)\right)$$

If $\mathcal{L}$ doesn't contain derivatives of the fields, then $\mathcal{H} = -\mathcal{L}$ so:

$$U(-\infty,\infty) = T\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{L}(x)\right)$$

Let's write $S = 1 + iT$ and $|i\rangle = |k_1\cdots k_m\rangle$, $|f\rangle = |p_1\cdots p_n\rangle$. As a first step now, let's investigate a scalar field, e.g. $\mathcal{L} = -\frac{\lambda}{4}\phi^4$ (e.g. a Higgs self interaction term above), we'll look at other fields later:

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \langle p_1\cdots p_n|iT|k_1\cdots k_m\rangle = \frac{1}{\tilde{D}(k_1)\cdots\tilde{D}(k_m)}\frac{1}{\tilde{D}(p_1)\cdots\tilde{D}(p_n)}$$

$$\int\mathrm{d}^4x_1\cdots\mathrm{d}^4x_m e^{-i(k_1x_1+\cdots+k_mx_m)}\int\mathrm{d}^4y_1\cdots\mathrm{d}^4y_n e^{+i(p_1y_1+\cdots+p_ny_n)}G(x_1,\cdots,x_m,y_1,\cdots,y_m)$$

where

$$G(x_1,\cdots,x_n) = \langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle =$$

$$\frac{\langle 0|T\{\phi_I(x_1)\cdots\phi_I(x_n)\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{L}(x)\right)\}|0\rangle}{\langle 0|T\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{L}(x)\right)|0\rangle}$$

This is called the LSZ formula. Now we use the Wick contraction, get some terms like $D_{23}D_{34}$ integrate things out, this will give the delta function and $\tilde{D}(p)$'s and that's it.

Let's see how it goes for $\mathcal{L} = -\frac{\lambda}{4}\phi^4$ for the process $k_1 + k_2 \to p_1 + p_2$:

$$\langle p_1p_2|S|k_1k_2\rangle = \frac{\int\mathrm{d}^4x_1\mathrm{d}^4x_2e^{-i(k_1x_1+k_2x_2)}\int\mathrm{d}^4y_1\mathrm{d}^4y_2e^{-i(p_1y_1+p_2y_2)}}{\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)}$$

$$\frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)\}|0\rangle}{\langle 0|T\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)|0\rangle} =$$

$$= \frac{\int\mathrm{d}^4x_1\mathrm{d}^4x_2e^{-i(k_1x_1+k_2x_2)}\int\mathrm{d}^4y_1\mathrm{d}^4y_2e^{-i(p_1y_1+p_2y_2)}}{\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)}$$

$$\left[\frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\}|0\rangle}{\langle 0|T\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)|0\rangle} + \right.$$

$$+\frac{\left(-\frac{i\lambda}{4\hbar}\right)\int\mathrm{d}^4x\,\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\}|0\rangle}{\langle 0|T\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)|0\rangle}+$$

$$\left.+\frac{\left(-\frac{i\lambda}{4\hbar}\right)^2\int\mathrm{d}^4x\,\mathrm{d}^4y\,\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\phi_I^4(y)\}|0\rangle}{\langle 0|T\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)|0\rangle}+\cdots\right] =$$

$$= \frac{1}{\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)}$$

$$\Big[(2\pi)^4\delta^{(4)}(p_1+p_2)(2\pi)^4\delta^{(4)}(k_1+k_2)\tilde{D}(p_1)\tilde{D}(k_1)+$$

$$(-i\lambda)6(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)+$$

$$(-i\lambda)(\text{disconnected terms with not enough } \tilde{D}(\cdots)\text{s}) + (-i\lambda)^2(\cdots) + \cdots\Big] =$$

$$= (2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\left[6(-i\lambda) + 3(-i\lambda)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{D}(k)\tilde{D}(p1+p2-k) + (-i\lambda)^3(\cdots) + \cdots\right]$$

The denominator cancels with the disconnected terms. We used the Wick contractions (see below for a thorough explanation+derivation):

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\}|0\rangle = D(x_1-x_2)D(y_1-y_2)+D(x_2-y_1)D(x_1-y_2)+D(x_2-y_2)D(x_1-y_1)$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\}|0\rangle = D(x_1-x)D(x_2-x)D(y_1-x)D(y_2-x)+\text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\phi_I^4(y)\}|0\rangle = D(x_1-x)D(x_2-x)D(y_1-y)D(y_2-y)D(x-y)D(x-y)$$
$$+\text{disconnected}$$

Where the "disconnected" terms are $D(x_1-y_1)D(x_2-y_2)D(x-x)D(x-x)$ and similar. When they are integrated over, they do not generate enough $\tilde{D}(p_1)$ propagators to cancel the propagators from the LSZ formula, which will cause the terms to vanish.

For the $\mathcal{L} = \phi^3(x)$ theory, one also needs the following contractions:

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^3(x)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^3(x)\phi_I^3(y)\}|0\rangle = D(x_1-x)D(x_2-x)D(x-y)D(y_1-y)D(y_2-y)$$

Thus it is clear that the only difference from the above is the factor $D(x-y)$ which after integrating changes to $\tilde{D}(p_1+p_2)$ and this ends up in the final result.

One always gets the delta function in the result, so we define the matrix element $\mathcal{M}_{fi}$ by:

$$S_{fi} = (2\pi)^4\delta^{(4)}(p_1+p_2+\cdots-k_1-k_2-\cdots)i\mathcal{M}_{fi}$$

2.2.2 Wick Theorem

As seen above, we need to be able to calculate

$$\langle 0|T\{\phi_I(x_1)\cdots\phi_I(x_n)\}|0\rangle$$

The Wick theorem says, that this is equal to all possible contractions of fields (all fields need to be contracted), where a contraction is defined as:

$$\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle \equiv D(x-y) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\tilde{D}(p)e^{-ip(x-y)}$$

with

$$\tilde{D}(p) = \frac{i}{p^2-m^2+i\epsilon}$$

A few lowest possibilities:

$$\langle 0|T\{\phi_I(x_1)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\}|0\rangle = D_{12}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\}|0\rangle = \text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I(x)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^2(x)\}|0\rangle = \text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^3(x)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^4(x)\}|0\rangle = 4!\, D(x_1-x)D(x_2-x)D(x_3-x)D(x_4-x)+\text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^3(x)\phi_I^3(y)\}|0\rangle =$$

$$= D(x_1-x)D(x_2-x)D(x-y)D(x_3-y)D(x_4-y) + \text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^4(x)\phi_I^4(y)\}|0\rangle =$$

$$= D(x_1-x)D(x_2-x)D(x-y)D(x-y)D(x_3-y)D(x_4-y) + \text{disconnected}$$

For the last two equations, not all possibilities of the connected graphs are listed (and also the combinatorial factor is omitted).

2.2.3 Fermions and Vector Bosons

For fermions:

$$\langle 0|T\{\psi_I(x)\bar{\psi}_I(y)\}|0\rangle \equiv S(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\tilde{S}(p)e^{-ip(x-y)}$$

with

$$\tilde{S}(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon}$$

For vector bosons:

$$\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle \equiv D_{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\tilde{D}_{\mu\nu}(p)e^{-ip(x-y)}$$

with

$$\tilde{D}_{\mu\nu}(p) = i\frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}}{p^2 - m^2 + i\epsilon}$$

For massless bosons:

$$\tilde{D}_{\mu\nu}(p) = i\frac{-g_{\mu\nu}}{p^2 + i\epsilon}$$

2.2.4 Feynman Rules

We can deduce a set of rules, so that one doesn't have to repeat the whole calculation each time. For a scalar field we derived the rules above, for fermion and vector boson fields it's more difficult.

2.2.5 ZZH interaction

Let's calculate the $\mathcal{L}_{ZZH} = \lambda Z_\mu Z^\mu H$ interaction in the SM, where $\lambda = \frac{g^2}{4\cos\theta_W}$. Consider $H(p) \to Z(k) + Z(l)$:

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \langle kl|iT|p\rangle = \frac{\varepsilon_\alpha(k)\varepsilon^\alpha(l)}{\tilde{D}_{\mu\nu}(k)\tilde{D}^{\mu\nu}(l)}\frac{1}{\tilde{D}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(ky_1+ly_2)}\langle 0|T\{Z_\mu(y_1)Z^\mu(y_2)H(x_1)\}|0\rangle =$$

$$= \frac{\varepsilon_\alpha(k)\varepsilon^\alpha(l)}{\tilde{D}_{\mu\nu}(k)\tilde{D}^{\mu\nu}(l)}\frac{1}{\tilde{D}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(ky_1+ly_2)}\int \mathrm{d}^4x i\lambda D_{\alpha\mu}(y_1-x)D^{\alpha\mu}(y_2-x)D(x_1-x) =$$

$$= i\lambda(2\pi)^4\delta^{(4)}(p-k-l)\varepsilon_\alpha(k)\varepsilon^\alpha(l)$$

where we used the fact, that the only nonzero element of the Green function is

$$\int \mathrm{d}^4x\,\langle 0|T\{Z_\alpha(y_1)Z^\alpha(y_2)H(x_1)Z_\mu(x)Z^\mu(x)H(x)\}|0\rangle$$

2.2.6 eeH interaction

Let's calculate the $\mathcal{L}_{eeH} = -\lambda\bar{e}eH$ interaction in the SM, where $\lambda = \frac{h_e}{\sqrt{2}}$. Consider $H(p) \to e^-(k) + e^+(l)$:

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \langle kl|iT|p\rangle = \frac{\bar{u}(k)v(l)}{\tilde{S}(k)\tilde{S}(l)}\frac{1}{\tilde{D}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(ky_1+ly_2)}\langle 0|T\{\bar{e}(y_1)e(y_2)H(x_1)\}|0\rangle =$$

$$= \frac{\bar{u}(k)v(l)}{\tilde{S}(k)\tilde{S}(l)}\frac{1}{\tilde{D}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(ky_1+ly_2)}\int \mathrm{d}^4x(-i\lambda)S(y_1-x)S(y_2-x)D(x_1-x) =$$

$$= (-i\lambda)(2\pi)^4\delta^{(4)}(p-k-l)\bar{u}(k)v(l)$$

where we used the fact, that the only nonzero element of the Green function is

$$\int \mathrm{d}^4x\,\langle 0|T\{\bar{e}(y_1)e(y_2)H(x_1)\bar{e}(x)e(x)H(x)\}|0\rangle$$

2.2.7 ee gamma interaction

Let's calculate the $\mathcal{L}_{ee\gamma} = -\lambda\bar{e}\gamma^\mu e A_\mu$ interaction in the SM, where $\lambda = g\sin\theta_W$. Consider $\gamma(p) \to e^-(k) + e^+(l)$:

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \langle kl|iT|p\rangle = \frac{\bar{u}(k)v(l)}{\tilde{S}(k)\tilde{S}(l)}\frac{\varepsilon_\mu(p)}{\tilde{D}_{\alpha\beta}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(ky_1+ly_2)}\,\langle 0|T\{\bar{e}(y_1)e(y_2)A^\mu(x_1)\}|0\rangle =$$

$$= \frac{\bar{u}(k)v(l)}{\tilde{S}(k)\tilde{S}(l)}\frac{\varepsilon_\mu(p)}{\tilde{D}_{\alpha\beta}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(ky_1+ly_2)}\int \mathrm{d}^4x(-i\lambda)S(y_2-x)\gamma^\mu S(y_1-x)D^\alpha_\mu(x_1-x) =$$

$$= (2\pi)^4\delta^{(4)}(p-k-l)\bar{u}(k)(-i\lambda)\gamma^\mu v(l)\varepsilon_\mu(p)$$

where we used the fact, that the only nonzero element of the Green function is

$$\int \mathrm{d}^4x\,\langle 0|T\{\bar{e}(y_1)e(y_2)A^\alpha(x_1)\bar{e}(x)\gamma^\mu e(x)A_\mu(x)\}|0\rangle =$$

$$= \pm S(y_2-x)\gamma^\mu S(y_1-x)D^\alpha_\mu(x_1-x)$$

2.2.8 eeee interaction

Let's calculate the $\mathcal{L}_{ee\gamma} = -\lambda\bar{e}\gamma^\mu e A_\mu$ interaction in the SM, where $\lambda = g\sin\theta_W$. Consider $e^-(p_1) + e^+(p_2) \to \gamma(q) \to e^-(k_1) + e^+(k_2)$:

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \langle k_1k_2|iT|p_1p_2\rangle = \frac{\bar{u}(k_1)v(k_2)}{\tilde{S}(k_1)\tilde{S}(k_2)}\frac{\bar{v}(p_2)u(p_1)}{\tilde{S}(p_2)\tilde{S}(p_1)}$$

$$\int \mathrm{d}^4x_1\mathrm{d}^4x_2 e^{-i(p_1x_1+p_2x_2)}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(k_1y_1+k_2y_2)}\,\langle 0|T\{\bar{e}(y_1)e(y_2)\bar{e}(x_1)e(x_2)\}|0\rangle =$$

$$= \frac{\bar{u}(k_1)v(k_2)}{\tilde{S}(k_1)\tilde{S}(k_2)}\frac{\bar{v}(p_2)u(p_1)}{\tilde{S}(p_2)\tilde{S}(p_1)}$$

$$\int \mathrm{d}^4x_1\mathrm{d}^4x_2 e^{-i(p_1x_1+p_2x_2)}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(k_1y_1+k_2y_2)}\int \mathrm{d}^4x\int \mathrm{d}^4y$$

$$(-i\lambda)^2 S(x_1-x)\gamma^\mu S(x_2-x)D_{\mu\nu}(x-y)S(y_1-y)\gamma^\nu S(y_2-y) =$$

$$= (2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\bar{v}(p_2)(-i\lambda)\gamma^\mu u(p_1)\tilde{D}_{\mu\nu}(q=p_1+p_2=k_1+k_2)\bar{u}(k_1)(-i\lambda)\gamma^\nu v(k_2)$$

where we used the fact, that the only nonzero element of the Green function is

$$\int \mathrm{d}^4x\int \mathrm{d}^4y\,\langle 0|T\{\bar{e}(y_1)e(y_2)\bar{e}(x_1)e(x_2)\bar{e}(x)\gamma^\mu e(x)A_\mu(x)\bar{e}(y)\gamma^\nu e(y)A_\nu(y)\}|0\rangle =$$

$$= \pm S(x_1-x)\gamma^\mu S(x_2-x)D_{\mu\nu}(x-y)S(y_1-y)\gamma^\nu S(y_2-y)$$

2.3 Low energy theories

2.3.1 Fermi-type theory

This is a low energy ($m_W^2 \gg m_\mu m_e$) model for the EW interactions, that can be derived for example from the muon decay:

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$$

From the SM the relevant Lagrangian is

$$\mathcal{L} = \frac{g}{2\sqrt{2}}(\bar{e}\gamma^\mu(1-\gamma_5)\nu_e W_\mu^-) + \frac{g}{2\sqrt{2}}(\bar{\mu}\gamma^\mu(1-\gamma_5)\nu_\mu W_\mu^-)$$

and one gets the diagram $\mu^- + \bar{\nu}_\mu + \to e^- + \bar{\nu}_e$ and the corresponding matrix element:

$$iM = -i\frac{g^2}{8}[\bar{u}\gamma_\mu(1-\gamma_5)u]\frac{-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_W^2}}{q^2 - m_W^2}[\bar{u}\gamma_\nu(1-\gamma_5)v]$$

which when the momentum transfer q is much less than m_w becomes

$$iM = -i\frac{g^2}{8m_W^2}[\bar{u}\gamma^\mu(1-\gamma_5)u][\bar{u}\gamma_\mu(1-\gamma_5)v]$$

but this element can be derived directly from the Lagrangian:

$$\mathcal{L} = -\frac{G_\mu}{\sqrt{2}}[\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma_5)\psi_\mu][\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_{\nu_e}]$$

with

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

This is the universal V-A theory Lagrangian (after adding the h.c. term).

3 Quantum Mechanics

3.1 From QED to Quantum Mechanics

The QED Lagrangian density is

$$\mathcal{L} = \bar{\psi}(i\hbar c\gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

and

$$D_\mu = \partial_\mu + \frac{i}{\hbar}eA_\mu$$

is the gauge covariant derivative and (e is the elementary charge, which is 1 in atomic units, i.e. the electron has a charge $-e$)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. It's astonishing, that this simple Lagrangian can account for all phenomena from macroscopic scales down to something like 10^{-13} cm. So it's not a surprise that Feynman, Schwinger and Tomonaga received the 1965 Nobel Prize in Physics for such a fantastic achievement.

Plugging this Lagrangian into the Euler-Lagrange equation of motion for a field, we get:

$$(i\hbar c\gamma^\mu D_\mu - mc^2)\psi = 0$$
$$\partial_\nu F^{\nu\mu} = -ec\bar\psi\gamma^\mu\psi$$

The first equation is the Dirac equation in the electromagnetic field and the second equation is a set of Maxwell equations ($\partial_\nu F^{\nu\mu} = -ej^\mu$) with a source $j^\mu = c\bar\psi\gamma^\mu\psi$, which is a 4-current comming from the Dirac equation.

The fields ψ and A^μ are quantized. The first approximation is that we take ψ as a wavefunction, that is, it is a classical 4-component field. It can be shown that this corresponds to taking the tree diagrams in the perturbation theory.

We multiply the Dirac equation by γ^0 from left to get:

$$0 = \gamma^0(i\hbar c\gamma^\mu D_\mu - mc^2)\psi = \gamma^0(i\hbar c\gamma^0(\partial_0 + \frac{i}{\hbar}eA_0) + ic\gamma^i(\partial_i + \frac{i}{\hbar}eA_i) - mc^2)\psi =$$

$$= (i\hbar c\partial_0 + i\hbar c\gamma^0\gamma^i\partial_i - \gamma^0 mc^2 - ceA_0 - ce\gamma^0\gamma^i A_i)\psi$$

and we make the following substitutions (it's just a formalism, nothing more): $\beta = \gamma^0$, $\alpha^i = \gamma^0\gamma^i$, $p_j = i\hbar\partial_j$, $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$ to get

$$(i\hbar\frac{\partial}{\partial t} + c\alpha^i p_i - \beta mc^2 - ceA_0 - ce\alpha^i A_i)\psi = 0\,.$$

or:

$$i\hbar\frac{\partial\psi}{\partial t} = (c\alpha^i(-p_i + eA_i) + \beta mc^2 + ceA_0)\psi\,.$$

This can be written as:

$$i\frac{\partial\psi}{\partial t} = H\psi\,,$$

where the Hamiltonian is given by:

$$H = c\alpha^i(-p_i + eA_i) + \beta mc^2 + ceA_0\,,$$

or introducing the electrostatic potential $\phi = cA_0$ and writing the momentum as a vector (see the appendix for all the details regarding signs):

$$H = c\boldsymbol{\alpha}\cdot(\mathbf{p} - e\mathbf{A}) + \beta mc^2 + e\phi\,.$$

The right hand side of the Maxwell equations is the 4-current, so it's given by:

$$j^\mu = c\bar\psi\gamma^\mu\psi$$

Now we make the substitution $\psi = e^{-imc^2t}\varphi$, which states, that we separate the largest oscillations of the wavefunction and we get

$$j^0 = c\bar\psi\gamma^0\psi = c\psi^\dagger\psi = c\varphi^\dagger\varphi$$

$$j^i = c\bar\psi\gamma^i\psi = c\psi^\dagger\alpha^i\psi = c\varphi^\dagger\alpha^i\varphi$$

3.1.1 Nonrelativistic Limit in the Lagrangian

We use the identity $\frac{\partial}{\partial t}\left(e^{-imc^2t}f(t)\right) = e^{-imc^2t}(-imc^2 + \frac{\partial}{\partial t})f(t)$ to get:

$$L = c^2\partial^\mu\psi^*\partial_\mu\psi - m^2c^4\psi^*\psi = \frac{\partial}{\partial t}\psi^*\frac{\partial}{\partial t}\psi - c^2\partial^i\psi^*\partial_i\psi - m^2c^4\psi^*\psi =$$

$$= (imc^2 + \frac{\partial}{\partial t})\varphi^*(-imc^2 + \frac{\partial}{\partial t})\varphi - c^2\partial^i\varphi^*\partial_i\varphi - m^2c^4\varphi^*\varphi =$$

$$= 2mc^2\left[\frac{1}{2}i(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}) - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi + \frac{1}{2mc^2}\frac{\partial\varphi^*}{\partial t}\frac{\partial\varphi}{\partial t}\right]$$

The constant factor $2mc^2$ in front of the Lagrangian is of course irrelevant, so we drop it and then we take the limit $c \to \infty$ (neglecting the last term) and we get

$$L = \frac{1}{2}i(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}) - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi$$

After integration by parts we arrive at the Lagrangian for the Schrödinger equation:

$$L = i\varphi^*\frac{\partial\varphi}{\partial t} - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi$$

3.1.2 Klein-Gordon Equation

The Dirac equation implies the Klein-Gordon equation:

$$0 = (-i\hbar c\gamma^\mu D_\mu - mc^2)(i\hbar c\gamma^\nu D_\nu - mc^2)\psi = (\hbar^2c^2\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2c^4)\psi =$$

$$= (\hbar^2c^2g^{\mu\nu}D_\mu D_\nu + m^2c^4)\psi = (\hbar^2c^2D^\mu D_\mu + m^2c^4)\psi$$

Note however, the ψ in the true Klein-Gordon equation is just a scalar, but here we get a 4-component spinor. Now:

$$D_\mu D_\nu = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) = \partial_\mu\partial_\nu + ie(A_\mu\partial_\nu + A_\nu\partial_\mu + (\partial_\mu A_\nu)) - e^2A_\mu A_\nu$$

$$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu = ie(\partial_\mu A_\nu) - ie(\partial_\nu A_\mu)$$

We rewrite $D^\mu D_\mu$:

$$D^\mu D_\mu = g^{\mu\nu}D_\mu D_\nu = \partial^\mu\partial_\mu + ie((\partial^\mu A_\mu) + 2A^\mu\partial_\mu) - e^2A^\mu A_\mu =$$

$$= \partial^\mu\partial_\mu + ie((\partial^0A_0) + 2A^0\partial_0 + (\partial^iA_i) + 2A^i\partial_i) - e^2(A^0A_0 + A^iA_i) =$$

$$= \partial^\mu\partial_\mu + i\frac{1}{c^2}\frac{\partial V}{\partial t} + 2i\frac{V}{c^2}\frac{\partial}{\partial t} + ie(\partial^iA_i) + 2ieA^i\partial_i - \frac{V^2}{c^2} - e^2A^iA_i$$

The nonrelativistic limit can also be applied directly to the Klein-Gordon equation:

$$0 = (\hbar^2c^2D^\mu D_\mu + m^2c^4)\psi =$$

$$= \left(\hbar^2c^2\partial^\mu\partial_\mu + i\frac{\partial V}{\partial t} + 2iV\frac{\partial}{\partial t} + i\hbar ec^2(\partial^iA_i) + 2i\hbar ec^2A^i\partial_i - V^2 - e^2c^2A^iA_i + m^2c^4\right)e^{-\frac{i}{\hbar}mc^2t}\varphi =$$

$$= \left(\hbar^2\frac{\partial^2}{\partial t^2} - c^2\hbar^2\nabla^2 + 2iV\frac{\partial}{\partial t} + i\frac{\partial V}{\partial t} + i\hbar ec^2(\partial^iA_i) + 2i\hbar ec^2A^i\partial_i - V^2 - e^2c^2A^iA_i + m^2c^4\right)e^{-\frac{i}{\hbar}mc^2t}\varphi =$$

$$= e^{-\frac{i}{\hbar}mc^2t}\left(\hbar^2(-\frac{i}{\hbar}mc^2+\frac{\partial}{\partial t})^2-\hbar^2c^2\nabla^2+2iV(-\frac{i}{\hbar}mc^2+\frac{\partial}{\partial t})+i\frac{\partial V}{\partial t}+i\hbar ec^2(\partial^i A_i)+2i\hbar ec^2A^i\partial_i-V^2+\right.$$
$$\left.-e^2c^2A^iA_i+m^2c^4\right)\varphi =$$
$$= e^{-\frac{i}{\hbar}mc^2t}\left(-2i\hbar mc^2\frac{\partial}{\partial t}+\hbar^2\frac{\partial^2}{\partial t^2}-c^2\hbar^2\nabla^2+2Vm\frac{c^2}{\hbar}+2iV\frac{\partial}{\partial t}+i\frac{\partial V}{\partial t}+i\hbar ec^2(\partial^i A_i)+2i\hbar ec^2A^i\partial_i-V^2+\right.$$
$$\left.-e^2c^2A^iA_i\right)\varphi =$$
$$= -2mc^2e^{-\frac{i}{\hbar}mc^2t}\left(i\hbar\frac{\partial}{\partial t}+\hbar^2\frac{\nabla^2}{2m}-V-\frac{1}{2mc^2}\frac{\partial^2}{\partial t^2}-\frac{i}{2mc^2}\frac{\partial V}{\partial t}+\frac{V^2}{2mc^2}-\frac{iV}{mc^2}\frac{\partial}{\partial t}+\right.$$
$$\left.-\frac{i\hbar e}{2m}\partial^iA_i-\frac{i\hbar e}{m}A^i\partial_i+\frac{e^2}{2m}A^iA_i\right)\varphi$$

Taking the limit $c\to\infty$ we again recover the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\varphi = \left(-\hbar^2\frac{\nabla^2}{2m}+V+\frac{i\hbar e}{2m}\partial^iA_i+\frac{i\hbar e}{m}A^i\partial_i-\frac{e^2}{2m}A^iA_i\right)\varphi\,,$$

we rewrite the right hand side a little bit:

$$i\hbar\frac{\partial}{\partial t}\varphi = \left(\frac{\hbar^2}{2m}(\partial^i\partial_i+\frac{i}{\hbar}e\partial^iA_i+2\frac{i}{\hbar}eA^i\partial_i-\frac{e^2}{\hbar^2}A^iA_i)+V\right)\varphi\,,$$
$$i\hbar\frac{\partial}{\partial t}\varphi = \left(\frac{\hbar^2}{2m}(\partial^i+\frac{i}{\hbar}eA^i)(\partial_i+\frac{i}{\hbar}eA_i)+V\right)\varphi\,,$$
$$i\hbar\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}\hbar^2D^iD_i+V\right)\varphi\,,$$

Using (see the appendix for details):

$$\hbar^2D^iD_i = -\hbar^2\delta_{ij}D^iD^j = -\hbar^2\left(\frac{i}{\hbar}(\mathbf{p}-e\mathbf{A})\right)^2 = (\mathbf{p}-e\mathbf{A})^2$$

we get the usual form of the Schrödinger equation for the vector potential:

$$i\hbar\frac{\partial}{\partial t}\varphi = \left(\frac{(\mathbf{p}-e\mathbf{A})^2}{2m}+V\right)\varphi\,.$$

A little easier derivation:

$$0 = (\hbar^2c^2D^\mu D_\nu+m^2c^4)\psi =$$
$$= (\hbar^2c^2D^0D_0+\hbar^2c^2D^iD_i+m^2c^4)\psi =$$
$$= 2mc^2\left(\frac{\hbar^2}{2m}D^0D_0+\frac{\hbar^2}{2m}D^iD_i+\tfrac{1}{2}mc^2\right)\psi =$$
$$= 2mc^2\left(\frac{\hbar^2}{2m}\left(\partial^0+\frac{i}{\hbar}eA^0\right)\left(\partial_0+\frac{i}{\hbar}eA_0\right)+\tfrac{1}{2}mc^2+\frac{\hbar^2}{2m}D^iD_i\right)e^{-\frac{i}{\hbar}mc^2t}\varphi =$$
$$= 2mc^2\left(\frac{\hbar^2}{2m}\left(\partial^0+\frac{i}{\hbar}eA^0\right)e^{-\frac{i}{\hbar}mc^2t}\left(\partial_0-\frac{i}{\hbar}mc+\frac{i}{\hbar}eA_0\right)+\tfrac{1}{2}mc^2+\frac{\hbar^2}{2m}D^iD_i\right)\varphi =$$

$$= 2mc^2 e^{-\frac{i}{\hbar}mc^2 t}\left(\frac{\hbar^2}{2m}\left(\partial^0 - \frac{i}{\hbar}mc + \frac{i}{\hbar}eA^0\right)\left(\partial_0 - \frac{i}{\hbar}mc + \frac{i}{\hbar}eA_0\right) + \tfrac{1}{2}mc^2 + \frac{\hbar^2}{2m}D^i D_i\right)\varphi =$$

$$= 2mc^2 e^{-\frac{i}{\hbar}mc^2 t}\left(\frac{\hbar^2}{2m}\partial^0\partial_0 - \tfrac{1}{2}mc^2 - \frac{e^2 A^0 A_0}{2m} + ceA^0 + \frac{\hbar^2}{m}\frac{i}{\hbar}e(\partial^0 A^0 + A^0\partial^0) - i\hbar c\partial_0 + \tfrac{1}{2}mc^2 + \frac{\hbar^2}{2m}D^i D_i\right)\varphi$$

$$= 2mc^2 e^{-\frac{i}{\hbar}mc^2 t}\left(-i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}D^i D_i + ceA^0 + \frac{\hbar^2}{2mc^2}\frac{\partial^2}{\partial t^2} - \frac{e^2\phi^2}{2mc^2} + \frac{ie\hbar}{mc^2}(\frac{\partial}{\partial t}\phi + \phi\frac{\partial}{\partial t})\right)\varphi =$$

$$= 2mc^2 e^{-\frac{i}{\hbar}mc^2 t}\left(-i\hbar\frac{\partial}{\partial t} + \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + e\phi + \frac{\hbar^2}{2mc^2}\frac{\partial^2}{\partial t^2} - \frac{e^2\phi^2}{2mc^2} + \frac{ie\hbar}{mc^2}(\frac{\partial}{\partial t}\phi + \phi\frac{\partial}{\partial t})\right)\varphi$$

and letting $c \to \infty$ we get the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\varphi = \left(\frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + e\phi\right)\varphi$$

3.2 Perturbation Theory

We want to solve the equation:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle = H(t)\left|\psi(t)\right\rangle \tag{1}$$

with $H(t) = H^0 + H^1(t)$, where H^0 is time-independent part whose eigenvalue problem has been solved:

$$H^0\left|n^0\right\rangle = E_n^0\left|n^0\right\rangle$$

and $H^1(t)$ is a small time-dependent perturbation. $\left|n^0\right\rangle$ form a complete basis, so we can express $\left|\psi(t)\right\rangle$ in this basis:

$$\left|\psi(t)\right\rangle = \sum_n d_n(t) e^{-\frac{i}{\hbar}E_n^0 t}\left|n^0\right\rangle \tag{2}$$

Substituting this into (1), we get:

$$\sum_n\left(i\hbar\frac{\mathrm{d}}{\mathrm{d}t}d_n(t) + E_n^0 d_n(t)\right)e^{-\frac{i}{\hbar}E_n^0 t}\left|n^0\right\rangle = \sum_n\left(E_n^0 d_n(t) + H^1 d_n(t)\right)e^{-\frac{i}{\hbar}E_n^0 t}\left|n^0\right\rangle$$

so:

$$\sum_n i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(d_n(t)\right)e^{-\frac{i}{\hbar}E_n^0 t}\left|n^0\right\rangle = \sum_n d_n(t)e^{-\frac{i}{\hbar}E_n^0 t}H^1\left|n^0\right\rangle$$

Choosing some particular state $\left|f^0\right\rangle$ of the H^0 Hamiltonian, we multiply the equation from the left by $\left\langle f^0\right| e^{\frac{i}{\hbar}E_f^0 t}$:

$$\sum_n i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(d_n(t)\right)e^{iw_{fn}t}\left\langle f^0|n^0\right\rangle = \sum_n d_n(t)e^{iw_{fn}t}\left\langle f^0\right|H^1\left|n^0\right\rangle$$

where $w_{fn} = \frac{E_f^0 - E_n^0}{\hbar}$. Using $\left\langle f^0|n^0\right\rangle = \delta_{fn}$:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}d_f(t) = \sum_n d_n(t)e^{iw_{fn}t}\left\langle f^0\right|H^1\left|n^0\right\rangle$$

we integrate from t_1 to t:

$$i\hbar\left((d_f(t) - d_f(t_1)\right) = \sum_n \int_{t_1}^{t} d_n(t')e^{iw_{fn}t'} \langle f^0|H^1(t')|n^0\rangle \, \mathrm{d}t'$$

Let the initial wavefunction at time t_1 be some particular state $|\psi(t_1)\rangle = |i^0\rangle$ of the unperturbed Hamiltonian, then $d_n(t_1) = \delta_{ni}$ and we get:

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\sum_n \int_{t_1}^{t} d_n(t')e^{iw_{fn}t'} \langle f^0|H^1(t')|n^0\rangle \, \mathrm{d}t' \tag{3}$$

This is the equation that we will use for the perturbation theory.

In the zeroth order of the perturbation theory, we set $H^1(t) = 0$ and we get:

$$d_f(t) = \delta_{fi}$$

In the first order of the perturbation theory, we take the solution $d_n(t) = \delta_{ni}$ obtained in the zeroth order and substitute into the right hand side of (3):

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_{t_1}^{t} e^{iw_{fi}t'} \langle f^0|H^1(t')|i^0\rangle \, \mathrm{d}t'$$

In the second order, we take the last solution, substitute into the right hand side of (3) again:

$$d_f(t) = \delta_{fi} + \left(-\frac{i}{\hbar}\right)\int_{t_1}^{t} e^{iw_{fi}t'} \langle f^0|H^1(t')|i^0\rangle \, \mathrm{d}t' +$$

$$+\left(-\frac{i}{\hbar}\right)^2 \sum_n \int_{t_1}^{t} \mathrm{d}t'' \int_{t_1}^{t''} \mathrm{d}t' e^{iw_{fn}t''} \langle f^0|H^1(t'')|n^0\rangle \, e^{iw_{ni}t'} \langle n^0|H^1(t')|i^0\rangle$$

And so on for higher orders of the perturbation theory — more terms will arise on the right hand side of the last formula, so this is our main formula for calculating the $d_n(t)$ coefficients.

3.2.1 Time Independent Perturbation Theory

As a special case, if H^1 doesn't depend on time, the coefficients $d_n(t)$ simplify, so we calculate them in this section explicitly. Let's take

$$H(t) = H^0 + e^{t/\tau}H^1$$

so at the time $t_1 = -\infty$ the Hamiltonian $H(t) = H^0$ is unperturbed and we are interested in the time $t = 0$, when the Hamiltonian becomes $H(t) = H^0 + H^1$ (the coefficients $d_n(t)$ will still depend on the τ variable) and we do the limit $\tau \to \infty$ (this corresponds to smoothly applying the perturbation H^1 at the time negative infinity).

Let's calculate $d_f(0)$:

$$d_f(0) = \delta_{fi} + \left(-\frac{i}{\hbar}\right)\int_{-\infty}^{0} e^{iw_{fi}t'} e^{\frac{t}{\tau}} \mathrm{d}t' \, \langle f^0|H^1|i^0\rangle +$$

$$+\left(-\frac{i}{\hbar}\right)^2\sum_n\int_{-\infty}^0 \mathrm{d}t''\int_{-\infty}^{t''}\mathrm{d}t' e^{iw_{fn}t''}e^{iw_{ni}t'}e^{\frac{t''}{\tau}}e^{\frac{t'}{\tau}}\langle f^0|H^1|n^0\rangle\langle n^0|H^1|i^0\rangle =$$

$$=\delta_{fi}+\left(-\frac{i}{\hbar}\right)\frac{1}{\frac{1}{\tau}+i\omega_{fi}}\langle f^0|H^1|i^0\rangle +$$

$$+\left(-\frac{i}{\hbar}\right)^2\sum_n\frac{1}{\frac{1}{\tau}+i\omega_{ni}}\frac{1}{\frac{2}{\tau}+i\omega_{fn}+i\omega_{ni}}\langle f^0|H^1|n^0\rangle\langle n^0|H^1|i^0\rangle$$

Taking the limit $\tau\to\infty$:

$$d_f(0)=\delta_{fi}+\left(-\frac{1}{\hbar}\right)\frac{1}{\omega_{fi}}\langle f^0|H^1|i^0\rangle +$$

$$+\left(-\frac{1}{\hbar}\right)^2\sum_n\frac{1}{\omega_{ni}}\frac{1}{\omega_{fn}+\omega_{ni}}\langle f^0|H^1|n^0\rangle\langle n^0|H^1|i^0\rangle =$$

$$=\delta_{fi}-\frac{\langle f^0|H^1|i^0\rangle}{E_f^0-E_i^0}+$$

$$+\sum_n\frac{\langle f^0|H^1|n^0\rangle\langle n^0|H^1|i^0\rangle}{(E_n^0-E_i^0)(E_f^0-E_i^0)}$$

Substituting this into (2) evaluated for $t=0$:

$$|\psi(0)\rangle=\sum_n d_n(0)\,|n^0\rangle =$$

$$=|i^0\rangle-\sum_n\frac{|n^0\rangle\langle n^0|H^1|i^0\rangle}{E_n^0-E_i^0}+$$

$$+\sum_{n,m}\frac{|n^0\rangle\langle n^0|H^1|m^0\rangle\langle m^0|H^1|i^0\rangle}{(E_m^0-E_i^0)(E_n^0-E_i^0)}$$

The sum $\sum_n$ is over all $n\neq i$, similarly for the other sum. Let's also calculate the energy:

$$E=\langle\psi(0)|H|\psi(0)\rangle=\langle\psi(0)|H^0+H^1|\psi(0)\rangle =$$

$$\left(\cdots-\sum_{n'\neq i}\frac{\langle i^0|H^1|n'^0\rangle\langle n'^0|}{E_{n'}^0-E_i^0}+\langle i^0|\right)(H^0+H^1)\left(|i^0\rangle-\sum_{n\neq i}\frac{|n^0\rangle\langle n^0|H^1|i^0\rangle}{E_n^0-E_i^0}+\cdots\right)$$

To evaluate this, we use the fact that $\langle i^0|H^0|i^0\rangle=E_i^0$ and $\langle i^0|H^0|n^0\rangle=E_i^0\delta_{ni}$:

$$E=E_i^0+\langle i^0|H^1|i^0\rangle-\sum_{n\neq i}\frac{\langle i^0|H^1|n^0\rangle\langle n^0|H^1|i^0\rangle}{E_n^0-E_i^0}+\cdots=$$

$$=E_i^0+\langle i^0|H^1|i^0\rangle-\sum_{n\neq i}\frac{|\langle n^0|H^1|i^0\rangle|^2}{E_n^0-E_i^0}+\cdots$$

Where we have neglected the higher order terms, so we can identify the corrections to the energy E coming from the particular orders of the perturbation theory:

$$E_i^0=\langle i^0|H^0|i^0\rangle$$

$$E_i^1 = \langle i^0|H^1|i^0\rangle$$

$$E_i^2 = -\sum_{n\neq i} \frac{|\langle n^0|H^1|i^0\rangle|^2}{E_n^0 - E_i^0}$$

3.3 Scattering Theory

The incoming plane wave state is a solution of

$$H_0 |\mathbf{k}\rangle = E_k |\mathbf{k}\rangle$$

with $H_0 = \frac{p^2}{2m}$. E.g.

$$\langle \mathbf{r}|\mathbf{k}\rangle = e^{i\mathbf{r}\cdot\mathbf{k}}$$

$$E_k = \frac{\hbar^2 k^2}{2m}$$

We want to solve:

$$(H_0 + V)|\psi\rangle = E_k |\psi\rangle$$

The solution of this is:

$$|\psi\rangle = |\mathbf{k}\rangle + \frac{1}{E_k - H_0} V|\psi\rangle = |\mathbf{k}\rangle + GV|\psi\rangle$$

where

$$G = \frac{1}{E_k - H_0}$$

is the Green function for the Schrödinger equation. G is not unique, it contains both outgoing and ingoing waves. As shown below, one can distinguish between these two by adding a small $i\epsilon$ into the denominator, that moves the poles of the Green functions above and below the x-axis:

$$G_+ = \frac{1}{E_k - H_0 + i\epsilon}$$

$$G_- = \frac{1}{E_k - H_0 - i\epsilon}$$

Both G_+ and G_- are well-defined and unique. One can calculate both Green functions explicitly:

$$G_+(\mathbf{r},\mathbf{r}') = \langle \mathbf{r}|G_+|\mathbf{r}'\rangle = \langle \mathbf{r}| \frac{1}{E_k - H_0 + i\epsilon} |\mathbf{r}'\rangle =$$

$$= \int \mathrm{d}^3k' \frac{\langle \mathbf{r}|\mathbf{k}'\rangle \langle \mathbf{k}'|\mathbf{r}'\rangle}{E_k - E_{k'} + i\epsilon} = \int \mathrm{d}^3k' \frac{e^{i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{r}')}}{E_k - E_{k'} + i\epsilon} = \frac{2m}{\hbar^2}\int \mathrm{d}^3k' \frac{e^{i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{r}')}}{k^2 - k'^2 + i\epsilon} =$$

$$= \frac{4\pi m}{\hbar^2 i|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{\infty} \mathrm{d}^3k' k' \frac{e^{ik'|\mathbf{r}-\mathbf{r}'|}}{k^2 - k'^2 + i\epsilon} = \frac{4\pi m}{\hbar^2 i|\mathbf{r}-\mathbf{r}'|}(2\pi i)k\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2k} =$$

$$= \frac{4\pi^2 m e^{ik|\mathbf{r}-\mathbf{r}'|}}{\hbar^2|\mathbf{r}-\mathbf{r}'|}$$

$$G_-(\mathbf{r},\mathbf{r}') = \langle \mathbf{r}|G_+|\mathbf{r}'\rangle = \langle \mathbf{r}| \frac{1}{E_k - H_0 - i\epsilon} |\mathbf{r}'\rangle = \cdots = \frac{4\pi^2 m e^{-ik|\mathbf{r}-\mathbf{r}'|}}{\hbar^2|\mathbf{r}-\mathbf{r}'|}$$

Assuming $|\mathbf{r}'| \ll |\mathbf{r}|$, we can taylor expand $|\mathbf{r}-\mathbf{r}'|$:

$$|\mathbf{r}-\mathbf{r}'| = e^{-\mathbf{r}'\cdot\nabla}|\mathbf{r}| = \left(1 - \mathbf{r}'\cdot\nabla + (-\mathbf{r}'\cdot\nabla)^2 + O\left(r'^3\right)\right)|\mathbf{r}| = |\mathbf{r}| - \mathbf{r}'\cdot\nabla|\mathbf{r}| + O\left(r'^2\right) =$$

$$= r - \mathbf{r}'\cdot\hat{\mathbf{r}} + O\left(r'^2\right)$$

and simplify the result even further:

$$G_+(\mathbf{r},\mathbf{r}') = \frac{4\pi^2 m}{\hbar^2}\frac{e^{ikr}}{r}e^{-ik\mathbf{r}'\cdot\hat{\mathbf{r}}}$$

$$G_-(\mathbf{r},\mathbf{r}') = \frac{4\pi^2 m}{\hbar^2}\frac{e^{-ikr}}{r}e^{ik\mathbf{r}'\cdot\hat{\mathbf{r}}}$$

Note: both functions may be divided by the factor $(2\pi)^3$ due to the momentum integration.

Let's get back to the solution to the Schrödinger equation:

$$|\psi\rangle = |\mathbf{k}\rangle + G_+V|\psi\rangle$$

It contains the solution $|\psi\rangle$ on both sides of the equation, so we express it explicitly:

$$|\psi\rangle - G_+V|\psi\rangle = |\mathbf{k}\rangle$$

$$|\psi\rangle = \frac{1}{1-G_+V}|\mathbf{k}\rangle$$

and multiply by V:

$$V|\psi\rangle = \frac{V}{1-G_+V}|\mathbf{k}\rangle = T|\mathbf{k}\rangle$$

where T is the transition matrix:

$$T = \frac{V}{1-G_+V} = V(1 + G_+V + (G_+V)^2 + \cdots) =$$

$$= V + VG_+V + VG_+VG_+V + \cdots =$$

$$= V + V\frac{1}{E_k - H_0 + i\epsilon}V + V\frac{1}{E_k - H_0 + i\epsilon}V\frac{1}{E_k - H_0 + i\epsilon}V + \cdots$$

Then the final solution is:

$$|\psi\rangle = |\mathbf{k}\rangle + G_+V|\psi\rangle = |\mathbf{k}\rangle + G_+T|\mathbf{k}\rangle$$

and in a coordinate representation:

$$\psi(\mathbf{r}) = \langle\mathbf{r}|\psi\rangle = \langle\mathbf{r}|\mathbf{k}\rangle + \langle\mathbf{r}|G_+T|\mathbf{k}\rangle = \langle\mathbf{r}|\mathbf{k}\rangle + \int \mathrm{d}^3r'\,\langle\mathbf{r}|G_+|\mathbf{r}'\rangle\langle\mathbf{r}'|T|\mathbf{k}\rangle =$$

$$= \langle\mathbf{r}|\mathbf{k}\rangle + \int \mathrm{d}^3r'\mathrm{d}^3k'\,\langle\mathbf{r}|G_+|\mathbf{r}'\rangle\langle\mathbf{r}'|\mathbf{k}'\rangle\langle\mathbf{k}'|T|\mathbf{k}\rangle =$$

$$= e^{i\mathbf{k}\cdot\mathbf{r}} + \int \mathrm{d}^3r'\mathrm{d}^3k' G_+(\mathbf{r},\mathbf{r}')e^{i\mathbf{k}'\cdot\mathbf{r}'}\langle\mathbf{k}'|T|\mathbf{k}\rangle$$

Plugging the representation of the Green function for $|\mathbf{r}'| \ll |\mathbf{r}|$ in:

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{4\pi^2 m}{\hbar^2}\frac{e^{ikr}}{r}\int \mathrm{d}^3r'\mathrm{d}^3k' e^{-ik\mathbf{r}'\cdot\hat{\mathbf{r}}}e^{i\mathbf{k}'\cdot\mathbf{r}'}\langle\mathbf{k}'|T|\mathbf{k}\rangle =$$

$$= e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{4\pi^2 m}{\hbar^2}\frac{e^{ikr}}{r}\int \mathrm{d}^3 r' \mathrm{d}^3 k' e^{i\mathbf{r}'\cdot(\mathbf{k}'-k\hat{\mathbf{r}})} \langle \mathbf{k}'|T|\mathbf{k}\rangle =$$

$$= e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{4\pi^2 m}{\hbar^2}\frac{e^{ikr}}{r}\int \mathrm{d}^3 k' \delta(\mathbf{k}'-k\hat{\mathbf{r}}) \langle \mathbf{k}'|T|\mathbf{k}\rangle =$$

$$= e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{4\pi^2 m}{\hbar^2}\frac{e^{ikr}}{r} \langle k\hat{\mathbf{r}}|T|\mathbf{k}\rangle =$$

$$= e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}$$

where the scattering amplitude $f(\theta,\phi)$ is:

$$f(\theta,\phi) = \frac{4\pi^2 m}{\hbar^2}\langle k\hat{\mathbf{r}}|T|\mathbf{k}\rangle = \frac{4\pi^2 m}{\hbar^2}\langle \mathbf{k}'|T|\mathbf{k}\rangle$$

Where $\mathbf{k}' = k\hat{\mathbf{r}}$ is the final momentum.

The differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ is defined as the probability to observe the scattered particle in a given state per solid angle, e.g. the scattered flux per unit of solid angle per incident flux:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{|\mathbf{j}_i|}\frac{\mathrm{d}n}{\mathrm{d}\Omega} = \frac{r^2}{|\mathbf{j}_i|}\frac{\mathrm{d}n}{r^2\mathrm{d}\Omega} = \frac{r^2}{|\mathbf{j}_i|}\frac{\mathrm{d}n}{\mathrm{d}S} = \frac{r^2}{|\mathbf{j}_i|}\mathbf{j}_o\cdot\mathbf{n} = \frac{r^2}{|\mathbf{j}_i|}\mathbf{j}_o\cdot\hat{\mathbf{r}} =$$

$$= \frac{r^2}{\frac{\hbar k}{m}}\frac{\hbar k}{m}\left(\frac{1}{r^2}+\frac{i}{kr^3}\right)|f(\theta,\phi)|^2 = \left(1+\frac{i}{kr}\right)|f(\theta,\phi)|^2 \to |f(\theta,\phi)|^2$$

where we used $|\mathbf{j}_i| = \frac{\hbar k}{m}$ and

$$\mathbf{j}_o\cdot\hat{\mathbf{r}} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)\cdot\hat{\mathbf{r}} = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial}{\partial r}\psi - \psi\frac{\partial}{\partial r}\psi^*\right) =$$

$$= \frac{\hbar}{2mi}\left(f^*(\theta,\phi)\frac{e^{-ikr}}{r}\frac{\partial}{\partial r}\left(f(\theta,\phi)\frac{e^{ikr}}{r}\right) - f(\theta,\phi)\frac{e^{ikr}}{r}\frac{\partial}{\partial r}\left(f^*(\theta,\phi)\frac{e^{-ikr}}{r}\right)\right) =$$

$$= \frac{\hbar k}{m}\left(\frac{1}{r^2}+\frac{i}{kr^3}\right)|f(\theta,\phi)|^2$$

Let's write the explicit formula for the transition matrix:

$$\langle \mathbf{k}'|T|\mathbf{k}\rangle = \int \mathrm{d}^3 r \langle \mathbf{k}'|\mathbf{r}\rangle\langle \mathbf{r}|V|\mathbf{k}\rangle + \int \mathrm{d}^3 r \mathrm{d}^3 r' \langle \mathbf{k}'|\mathbf{r}\rangle\langle \mathbf{r}|VG_+|\mathbf{r}'\rangle\langle \mathbf{r}'|V|\mathbf{k}\rangle + \cdots =$$

$$= \int \mathrm{d}^3 r e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} V(\mathbf{r}) + \int \mathrm{d}^3 r \mathrm{d}^3 r' e^{-i\mathbf{k}'\cdot\mathbf{r}} V(\mathbf{r}) \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} V(\mathbf{r}') e^{i\mathbf{k}\cdot\mathbf{r}'} + \cdots =$$

The Born approximation is just the first term:

$$\langle \mathbf{k}'|T|\mathbf{k}\rangle \approx \int \mathrm{d}^3 r e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} V(\mathbf{r}) = \int \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\, e^{iqr\cos\theta} V(r) r^2 \sin\theta =$$

$$= 4\pi\int_0^\infty rV(r)\sin(qr)\,\mathrm{d}r$$

4 Appendix

4.1 Units and Dimensional Analysis

The evolution operator is dimensionless:

$$U(-\infty,\infty) = T\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{L}(x)\right)$$

So:

$$\left[\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{L}(x)\right] = [\hbar] = M^0$$

where M is an arbitrary mass scale. Length unit is M^{-1}, so then

$$[\mathcal{L}(x)] = M^4$$

For the particular forms of the Lagrangians above we get:

$$[m\bar{e}e] = [m^2 Z_\mu Z^\mu] = [m^2 H^2] = [i\bar{e}\gamma^\mu\partial_\mu e] = [\mathcal{L}] = M^4$$

so $[\bar{e}e] = M^3$, $[Z_\mu Z^\mu] = [H^2] = M^2$ and we get

$$[e] = [\bar{e}] = M^{\frac{3}{2}}$$

$$[Z_\mu] = [Z^\mu] = [H] = [\partial_\mu] = [\partial^\mu] = M^1$$

Example: what is the dimension of G_μ in $\mathcal{L} = -\frac{G_\mu}{\sqrt{2}}[\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma_5)\psi_\mu][\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_{\nu_e}]$? Answer:

$$[\mathcal{L}] = [G_\mu\bar{\psi}\psi\bar{\psi}\psi]$$

$$M^4 = [G_\mu]M^{\frac{3}{2}}M^{\frac{3}{2}}M^{\frac{3}{2}}M^{\frac{3}{2}}$$

$$[G_\mu] = M^{-2}$$

In order to get the above units from the SI units, one has to do the following identification:

$$kg \to M^1$$

$$m \to M^{-1}$$

$$s \to M^{-1}$$

$$A \to M^1$$

The SI units of the above quantities are:

$$[\phi] = \mathrm{V} = \frac{\mathrm{kg\,m^2}}{\mathrm{A\,s^3}} = \mathrm{M}$$

$$[A_\mu] = \frac{[\phi]}{[c]} = \frac{\mathrm{V\,s}}{\mathrm{m}} = \frac{\mathrm{kg\,m}}{\mathrm{A\,s^2}} = \mathrm{M}$$

$$[c] = \frac{\mathrm{m}}{\mathrm{s}} = 1$$

$$[e] = \mathrm{C} = \mathrm{A\,s} = 1$$

$$[\hbar] = \mathrm{J\,s} = \frac{\mathrm{m^2\,kg}}{\mathrm{s}} = 1$$

$$[\partial_\mu] = \frac{1}{\mathrm{m}} = \mathrm{M}$$

$$[F_{\mu\nu}] = [\partial_\mu A_\nu] = \frac{\mathrm{kg}}{\mathrm{A\,s^2}} = \mathrm{M}^2$$

$$[\mathcal{L}] = [F_{\mu\nu}]^2 = \frac{\mathrm{kg}^2}{\mathrm{A^2\,s^4}} = \mathrm{M}^4$$

$$[\psi] = \frac{\mathrm{kg}^{\frac{1}{2}}}{\mathrm{A\,m\,s}} = \mathrm{M}^{\frac{3}{2}}$$

The SI units are useful for checking that the c, e and $\hbar$ constants are at correct places in the expression.

4.2 Tensors in QFT

In general, the covariant and contravariant vectors and tensors work just like in general relativity. We use the metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ (e.g. signature -2, but it's possible to also use the metric with signature +2). The four potential A^μ is given by:

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A}\right) = (A^0, A^1, A^2, A^3)$$

where ϕ is the electrostatic potential. Whenever we write $\mathbf{A}$, the components of it are given by the upper indices, e.g. $\mathbf{A} = (A^1, A^2, A^3)$. The components with lower indices can be calculated using the metric tensor, so it depends on the signature convention:

$$A_\mu = g_{\mu\nu} A^\nu = (A^0, -\mathbf{A}) = (A^0, -A^1, -A^2, -A^3)$$

In our case we got $A_0 = A^0$ and $A_i = -A^i$ (if we used the other signature convention, then the sign of A_0 would differ and A_i would stay the same).

Gradient is defined as:

$$\partial_\mu = (\partial_0, \partial_1, \partial_2, \partial_3) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

the upper indices depend on the signature, e.g. for -2:

$$\partial^\mu = (\partial^0, \partial^1, \partial^2, \partial^3) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

and +2:

$$\partial^\mu = (\partial^0, \partial^1, \partial^2, \partial^3) = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Momentum ($\mathbf{p} = -i\hbar\nabla$) and energy ($E = i\hbar\frac{\partial}{\partial t}$) is combined into 4-momentum as

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right) = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) = i\hbar\left(\partial_0, -\partial_j\right) = i\hbar\left(\partial^0, \partial^j\right) = i\hbar\partial^\mu$$

$$p_\mu = g_{\mu\nu}p^\nu = i\hbar\left(\partial^0, -\partial^j\right) = i\hbar\left(\partial_0, \partial_j\right) = i\hbar\partial_\mu$$

Note: for the signature +2, we would get $p^\mu = -i\hbar\partial^\mu$ and $p_\mu = -i\hbar\partial_\mu$.

For the minimal coupling $D_\mu = \partial_\mu + \frac{i}{\hbar}eA_\mu$ we get:

$$D^0 = \partial^0 + \frac{i}{\hbar}eA^0$$

$$D^j = \partial^j + \frac{i}{\hbar}eA^j = -\frac{i}{\hbar}(i\hbar\partial^j - eA^j) = -\frac{i}{\hbar}(\mathbf{p} - e\mathbf{A})$$

and for the lower indices:

$$D_0 = \partial_0 + \frac{i}{\hbar}eA_0$$

$$D_j = \partial_j + \frac{i}{\hbar}eA_j = -\frac{i}{\hbar}(i\hbar\partial_j - eA_j) = \frac{i}{\hbar}(i\hbar\partial^j - eA^j) = \frac{i}{\hbar}(\mathbf{p} - e\mathbf{A})$$

4.3 Multipole expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{(\mathbf{r}-\mathbf{r}')^2}} = \frac{1}{\sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2}} = \frac{1}{r\sqrt{1 - 2\left(\frac{r'}{r}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}' + \left(\frac{r'}{r}\right)^2}} =$$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') =$$

$$= \frac{1}{r}\left(P_0(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') + P_1(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\frac{r'}{r} + P_2(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\left(\frac{r'}{r}\right)^2 + O\left(\frac{r'^3}{r^3}\right)\right) =$$

$$= \frac{1}{r}\left(1 + \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\frac{r'}{r} + \frac{1}{2}\left(3(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2 - 1\right)\left(\frac{r'}{r}\right)^2 + O\left(\frac{r'^3}{r^3}\right)\right) =$$

$$= \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2r'^2}{2r^5} + O\left(\frac{r'^3}{r^4}\right)$$

We can also use the formula:

$$\sum_m \langle\hat{\mathbf{r}}|lm\rangle\langle lm|\hat{\mathbf{r}}'\rangle = \frac{4\pi}{2l+1}\langle\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'|P_l\rangle$$

and rewrite the expansion using spherical harmonics:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') =$$

$$= \frac{1}{r}\sum_{l,m}\left(\frac{r'}{r}\right)^l\frac{2l+1}{4\pi}\langle\hat{\mathbf{r}}|lm\rangle\langle lm|\hat{\mathbf{r}}'\rangle = \frac{1}{r}\sum_{l,m}\left(\frac{r'}{r}\right)^l\frac{2l+1}{4\pi}Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$