

Quantum Field Theory

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Contents

1	Derivation from the Standard Model	1
1.1	Electroweak Standard Model	1
1.1.1	Higgs Terms	2
1.1.2	Yukawa terms	3
1.1.3	Leptonic Terms	3
1.1.4	Gauge terms	4
1.1.5	GWS Lagrangian	5
1.1.6	Particle Masses	5
1.1.7	Dimensional Analysis	6
1.1.8	Quarks	6
1.2	QFT	6
1.2.1	Evolution Operator, S-Matrix Elements	6
1.2.2	Wick Theorem	9
1.2.3	Fermions and Vector Bosons	9
1.2.4	Feynman Rules	10
1.2.5	ZZH interaction	10
1.3	Low energy theories	10
1.3.1	Fermi-type theory	10
1.3.2	QED	11

1 Derivation from the Standard Model

1.1 Electroweak Standard Model

Lagrangian with a global $SU(2) \times U(1)$ symmetry:

$$\mathcal{L} = i\bar{L}^{(l)}\gamma_\mu\partial^\mu L^{(l)} + i\bar{l}_R\gamma_\mu\partial^\mu l_R + \frac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2 - h_e\bar{L}^{(l)}\Phi e_R - \text{h.c.}$$

where $l = e, \mu, \tau$ and $a = 1, 2$, $l_{L,R} = \frac{1}{2}(1 \mp \gamma_5)l$ and

$$L^{(l)} = \begin{pmatrix} \nu_{(l)L} \\ l_L \end{pmatrix}$$

Local $SU(2) \times U(1)$ symmetry:

This consists of two things. First changing the partial derivatives to covariant ones:

$$\partial^\mu \rightarrow D^\mu = \partial^\mu - \frac{i}{2}g\tau_k A_k^\mu - \frac{i}{2}g'YB^\mu$$

and second adding the kinetic terms

$$-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

of the vector gauge particles to the lagrangian.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\Phi = e^{\frac{i}{v}\pi^a(x)\tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

This breaks the gauge invariance. The $\partial^\mu \pi^a$ are going to be added to A_μ^a so we can set $\pi_a = 0$ now.

1.1.1 Higgs Terms

$$\mathcal{L}_{Higgs} = \frac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4}\lambda(\Phi^* \Phi)^2$$

Plugging in the covariant derivatives and Φ in U-gauge (symmetry breaking):

$$\begin{aligned} \mathcal{L}_{Higgs} &= \frac{1}{2}\Phi^+ (\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'YB_\mu) (\overrightarrow{\partial}^\mu + igA^{a\mu} \frac{\tau^a}{2} + ig'YB^\mu) \Phi - \lambda(\Phi^+ \Phi - \frac{v^2}{2})^2 = \\ &= \Phi_U^+ (\overleftarrow{\partial}_\mu + igA_\mu^a \frac{\tau^a}{2} + ig'YB_\mu) (\overrightarrow{\partial}^\mu + igA^{a\mu} \frac{\tau^a}{2} + ig'YB^\mu) \Phi_U - \lambda(\Phi_U^+ \Phi_U - \frac{v^2}{2})^2 = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\ &+ \frac{1}{8}(v+H)^2 \left(2g^2 \frac{A_\mu^1 + iA_\mu^2}{\sqrt{2}} \frac{A^{1\mu} - iA^{2\mu}}{\sqrt{2}} + (g^2 + 4Y^2 g'^2) \frac{gA_\mu^3 - 2Yg'B_\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \frac{gA^{3\mu} - 2Yg'B^\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \right) = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \frac{1}{8}(v+H)^2 \left(2g^2 W_\mu^- W^{+\mu} + \frac{g^2}{\cos^2 \theta_W} Z_\mu Z^\mu \right) = \\ &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8 \cos^2 \theta_W} Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\ &+ \frac{1}{2}v g^2 W_\mu^- W^{+\mu} H + \frac{g^2}{4 \cos \theta_W} v Z_\mu Z^\mu H + \frac{1}{4}g^2 W_\mu^- W^{+\mu} H^2 + \frac{g^2}{8 \cos \theta_W} Z_\mu Z^\mu H^2 \end{aligned}$$

Where we put

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2)$$

$$Z_\mu = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}} A_\mu^3 - \frac{2Yg'}{\sqrt{g^2 + 4Y^2 g'^2}} B_\mu$$

we defined θ_W by the relation

$$\cos \theta_W = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}}$$

so that the expressions simplify a bit, e.g. we now get:

$$\begin{aligned}\sin \theta_W &= \frac{2Yg'}{\sqrt{g^2 + 4Y^2g'^2}} \\ Z_\mu &= \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu \\ g^2 + 4Y^2g'^2 &= \frac{g^2}{\cos^2 \theta_W}\end{aligned}$$

1.1.2 Yukawa terms

$$\begin{aligned}\mathcal{L}_{Yukawa} &= -h_e \bar{L} \Phi e_R - \text{h.c.} = -h_e \bar{L} \Phi_U e_R - \text{h.c.} = \\ &= -\frac{1}{\sqrt{2}} h_e (v + H) (\bar{e}_L e_R + \bar{e}_R e_L) = -\frac{1}{\sqrt{2}} h_e (v + H) \bar{e} e = \\ &= -\frac{1}{\sqrt{2}} h_e v \bar{e} e - \frac{1}{\sqrt{2}} h_e \bar{e} e H\end{aligned}$$

The term $\bar{L} \Phi e_R$ is $U(1)$ (hypercharge) invariant, so

$$-Y_L + Y + Y_R = 0$$

1.1.3 Leptonic Terms

$$\begin{aligned}\mathcal{L} &= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R\gamma^\mu \partial_\mu e_R \rightarrow \\ &\rightarrow i\bar{L}\gamma^\mu (\partial_\mu - igA_\mu^a \frac{\tau^a}{2} - ig'Y_L B_\mu) L + i\bar{e}_R\gamma^\mu (\partial_\mu - ig'Y_R B_\mu) e_R = \\ &= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R\gamma^\mu \partial_\mu e_R + g\bar{L}\gamma^\mu \frac{\tau^a}{2} L A_\mu^a + g'Y_L \bar{L}\gamma^\mu L B_\mu + g'Y_R \bar{e}_R\gamma^\mu e_R B_\mu = \\ &= i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R\gamma^\mu \partial_\mu e_R + \frac{g}{\sqrt{2}} (\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c.}) + \frac{1}{2} g\bar{L}\gamma^\mu \tau^3 L A_\mu^3 + g'Y_L \bar{L}\gamma^\mu L B_\mu + g'Y_R \bar{e}_R\gamma^\mu e_R B_\mu = \\ &= i\bar{\nu}_L\gamma^\mu \partial_\mu \nu_L + i\bar{e}\gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}} (\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c.}) + \frac{1}{2} g\bar{\nu}_L\gamma^\mu \nu_L A_\mu^3 - \frac{1}{2} g\bar{e}_L\gamma^\mu e_L A_\mu^3 \\ &\quad + g'Y_L \bar{\nu}_L\gamma^\mu \nu_L B_\mu + g'Y_L \bar{e}_L\gamma^\mu e_L B_\mu + g'Y_R \bar{e}_R\gamma^\mu e_R B_\mu = \\ &= i\bar{\nu}_L\gamma^\mu \partial_\mu \nu_L + i\bar{e}\gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}} (\bar{\nu}_L\gamma^\mu e_L W_\mu^+ + \text{h.c.}) \\ &+ \left[\left(\frac{1}{2} g \sin \theta_W + Y_L g' \cos \theta_W \right) \bar{\nu}_L\gamma^\mu \nu_L + \left(-\frac{1}{2} g \sin \theta_W + Y_L g' \cos \theta_W \right) \bar{e}_L\gamma^\mu e_L + Y_R g' \cos \theta_W \bar{e}_R\gamma^\mu e_R \right] A_\mu \\ &+ \left[\left(\frac{1}{2} g \cos \theta_W - Y_L g' \sin \theta_W \right) \bar{\nu}_L\gamma^\mu \nu_L + \left(-\frac{1}{2} g \cos \theta_W - Y_L g' \sin \theta_W \right) \bar{e}_L\gamma^\mu e_L - 2Y_L g' \sin \theta_W \bar{e}_R\gamma^\mu e_R \right] Z_\mu\end{aligned}$$

Where we substituted new fields Z_μ and A_μ for the old ones A_μ^3 and B_μ using the relation:

$$\begin{aligned}Z_\mu &= \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu \\ A_\mu &= \sin \theta_W A_\mu^3 + \cos \theta_W B_\mu\end{aligned}$$

The angle θ_W must be the same as in the Higgs sector, so that the field Z_μ is the same. We now need to make the following requirement in order to proceed further:

$$Y = -Y_L$$

This follows for example by requiring that neutrinos have zero charge, i.e. setting $\frac{1}{2}g \sin \theta_W + Y_L g' \cos \theta_W = 0$ and substituting for θ_W from the definition (see the Higgs terms), from which one gets $Y = -Y_L$. From $-Y_L + Y + Y_R = 0$ we now get

$$Y_R = 2Y_L$$

it now follows:

$$\begin{aligned} \frac{1}{2}g \sin \theta_W + Y_L g' \cos \theta_W &= 0 \\ -\frac{1}{2}g \sin \theta_W + Y_L g' \cos \theta_W &= -g \sin \theta_W \\ Y_R g' \cos \theta_W &= -g \sin \theta_W \\ \tan \theta_W &= -2Y_L \frac{g'}{g} \end{aligned}$$

and the Lagrangian can be further simplified:

$$\begin{aligned} \mathcal{L} &= i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e} \gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \text{h.c.}) \\ &\quad -g \sin \theta_W (\bar{e}_L \gamma^\mu e_L + \bar{e}_R \gamma^\mu e_R) A_\mu \\ &\quad + \frac{g}{\cos \theta_W} \left[\frac{1}{2} \bar{\nu}_L \gamma^\mu \nu_L + \left(-\frac{1}{2} + \sin^2 \theta_W\right) \bar{e}_L \gamma^\mu e_L + \sin^2 \theta_W \bar{e}_R \gamma^\mu e_R \right] Z_\mu = \\ &= i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e} \gamma^\mu \partial_\mu e + \frac{g}{2\sqrt{2}}(\bar{\nu} \gamma^\mu (1 - \gamma_5) e W_\mu^+ + \text{h.c.}) - g \sin \theta_W \bar{e} \gamma^\mu e A_\mu \\ &\quad + \frac{g}{2 \cos \theta_W} \left[\bar{\nu} \gamma^\mu (1 - \gamma_5) \nu + \bar{e} \gamma^\mu \left(-\frac{1}{2} + 2 \sin^2 \theta_W + \frac{1}{2} \gamma_5\right) e \right] Z_\mu \end{aligned}$$

Where we used the relations $\bar{\nu}_L \gamma^\mu e_L = \frac{1}{2} \bar{\nu} \gamma^\mu (1 - \gamma_5) e$ and $\bar{\nu}_R \gamma^\mu e_R = \frac{1}{2} \bar{\nu} \gamma^\mu (1 + \gamma_5) e$.

1.1.4 Gauge terms

$$\begin{aligned} \mathcal{L}_{Gauge} &= -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = \\ &= -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon^{abc} A_\mu^b A_\nu^c) (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g \epsilon^{ajk} A^{j\mu} A^{k\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = \\ &= -\frac{1}{4} \partial_\mu A_\nu^a \partial^\mu A^{a\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g \epsilon^{abc} A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 \epsilon^{abc} \epsilon^{ajk} A_\mu^b A_\nu^c A^{k\mu} A^{l\nu} = \\ &= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g [(\partial_\mu A_\nu^1 - \partial_\nu A_\mu^1) A^{2\mu} A^{3\nu} + \text{cycl. perm. (123)}] \\ &\quad - \frac{1}{4} g^2 [(A_\mu^a A^{a\mu})(A_\nu^b A^{b\nu}) - (A_\mu^a A_\nu^a)(A^{b\mu} A^{b\nu})] = \\ &= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g [A_\mu^1 A_\nu^2 \overleftrightarrow{\partial}^\mu A^{3\nu} + \text{cycl. perm. (123)}] \\ &\quad - \frac{1}{4} g^2 [(A_\mu^a A^{a\mu})(A_\nu^b A^{b\nu}) - (A_\mu^a A_\nu^a)(A^{b\mu} A^{b\nu})] = \\ &= -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - ig (W_\mu^0 W_\nu^- \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. (0-+)}) \\ &\quad - g^2 [\frac{1}{2} (W_\mu^+ W^{-\mu})^2 - \frac{1}{2} (W_\mu^+ W^{+\mu})(W_\nu^- W^{-\nu}) + (W_\mu^0 W^{0\mu})(W_\nu^+ W^{-\nu}) - (W_\mu^- W_\nu^+)(W^{0\mu} W^{0\nu})] = \end{aligned}$$

$$= -\frac{1}{2}W_{\mu\nu}^-W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \mathcal{L}_{WW\gamma} + \mathcal{L}_{WWZ} + \mathcal{L}_{WW\gamma\gamma} + \mathcal{L}_{WWWW} + \mathcal{L}_{WWZZ} + \mathcal{L}_{WWZ\gamma}$$

Where $W_\mu^0 = A_\mu^3 = \cos\theta_W Z_\mu + \sin\theta_W A_\mu$ and:

$$\begin{aligned}\mathcal{L}_{WW\gamma} &= -ig \sin\theta_W (A_\mu W_\nu^- \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. } (A W^- W^+)) \\ \mathcal{L}_{WWZ} &= -ig \cos\theta_W (Z_\mu W_\nu^- \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. } (Z W^- W^+)) \\ \mathcal{L}_{WW\gamma\gamma} &= -g^2 \sin^2\theta_W (W_\mu^- W^{+\mu} A_\nu A^\nu - W_\mu^- A^\mu W_\nu^+ A^\nu) \\ \mathcal{L}_{WWWW} &= \frac{1}{2}g^2 (W_\mu^- W^{-\mu} W_\nu^+ W^{+\nu} - W_\mu^- W^{+\mu} W_\nu^- W^{+\nu}) \\ \mathcal{L}_{WWZZ} &= -g^2 \cos^2\theta_W (W_\mu^- W^{+\mu} Z_\nu Z^\nu - W_\mu^- Z^\mu W_\nu^+ Z^\nu) \\ \mathcal{L}_{WWZ\gamma} &= g^2 \sin\theta_W \cos\theta_W (-2W_\mu^- W^{+\mu} A_\nu Z^\nu + W_\mu^- Z^\mu W_\nu^+ A^\nu + W_\mu^- A^\mu W_\nu^+ Z^\nu)\end{aligned}$$

1.1.5 GWS Lagrangian

Plugging everything together we get the GWS Lagrangian:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8 \cos^2\theta_W} Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \\ &+ \frac{1}{2}v g^2 W_\mu^- W^{+\mu} H + \frac{g^2}{4 \cos\theta_W} v Z_\mu Z^\mu H + \frac{1}{4}g^2 W_\mu^- W^{+\mu} H^2 + \frac{g^2}{8 \cos\theta_W} Z_\mu Z^\mu H^2 \\ &\quad - \frac{1}{\sqrt{2}}h_e v \bar{e}e - \frac{1}{\sqrt{2}}h_e \bar{e}e H \\ &- \frac{1}{2}W_{\mu\nu}^-W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \mathcal{L}_{WW\gamma} + \mathcal{L}_{WWZ} + \mathcal{L}_{WW\gamma\gamma} + \mathcal{L}_{WWWW} + \mathcal{L}_{WWZZ} + \mathcal{L}_{WWZ\gamma} \\ &\quad + i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e} \gamma^\mu \partial_\mu e + \frac{g}{2\sqrt{2}}(\bar{\nu} \gamma^\mu (1 - \gamma_5) e W_\mu^+ + \text{h.c.}) - g \sin\theta_W \bar{e} \gamma^\mu e A_\mu \\ &\quad + \frac{g}{2 \cos\theta_W} [\bar{\nu} \gamma^\mu (1 - \gamma_5) \nu + \bar{e} \gamma^\mu (-\frac{1}{2} + 2 \sin^2\theta_W + \frac{1}{2}\gamma_5) e] Z_\mu \\ &\quad + (e, \nu_e, h_e \leftrightarrow \mu, \nu_\mu, h_\mu) + (e, \nu_e, h_e \leftrightarrow \tau, \nu_\tau, h_\tau)\end{aligned}$$

The free parameters are $g, \theta_W, v, \lambda, h_e, h_\mu, h_\tau$.

1.1.6 Particle Masses

The particle masses are deduced from the terms

$$\mathcal{L} = -\frac{1}{2}m_H^2 H^2 + m_W^2 W_\mu^- W^{+\mu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu - m_e \bar{e}e + \dots$$

comparing to the above:

$$\mathcal{L} = -\lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8 \cos^2\theta_W} Z_\mu Z^\mu - \frac{1}{\sqrt{2}}h_e v \bar{e}e + \dots$$

we get

$$\begin{aligned}m_W &= \frac{1}{2}gv \\ m_Z &= \frac{gv}{2 \cos\theta_W} = \frac{m_W}{\cos\theta_W} \\ m_H &= v\sqrt{2\lambda} \\ m_e &= \frac{1}{\sqrt{2}}h_e v\end{aligned}$$

1.1.7 Dimensional Analysis

The evolution operator is dimensionless:

$$U(-\infty, \infty) = T \exp \left(\frac{i}{\hbar} \int_{-\infty}^{\infty} d^4x \mathcal{L}(x) \right)$$

So:

$$\left[\int_{-\infty}^{\infty} d^4x \mathcal{L}(x) \right] = [\hbar] = M^0$$

where M is an arbitrary mass scale. Length unit is M^{-1} , so then

$$[\mathcal{L}(x)] = M^4$$

For the particular forms of the Lagrangians above we get:

$$[m\bar{e}e] = [m^2 Z_\mu Z^\mu] = [m^2 H^2] = [i\bar{e}\gamma^\mu \partial_\mu e] = [\mathcal{L}] = M^4$$

so $[\bar{e}e] = M^3$, $[Z_\mu Z^\mu] = [H^2] = M^2$ and we get

$$[e] = [\bar{e}] = M^{\frac{3}{2}}$$

$$[Z_\mu] = [Z^\mu] = [H] = [\partial_\mu] = [\partial^\mu] = M^1$$

Example: what is the dimension of G_μ in $\mathcal{L} = -\frac{G_\mu}{\sqrt{2}} [\bar{\psi}_{\nu_\mu} \gamma^\mu (1-\gamma_5) \psi_\mu] [\bar{\psi}_e \gamma^\mu (1-\gamma_5) \psi_{\nu_e}]$? Answer:

$$[\mathcal{L}] = [G_\mu \bar{\psi} \psi \bar{\psi} \psi]$$

$$M^4 = [G_\mu] M^{\frac{3}{2}} M^{\frac{3}{2}} M^{\frac{3}{2}} M^{\frac{3}{2}}$$

$$[G_\mu] = M^{-2}$$

1.1.8 Quarks

$$\begin{aligned} \mathcal{L}_{fermion} = & \sum_{q=d,s,b} i\bar{L}_0^{(q)} \gamma^\mu \partial_\mu L_0^{(q)} + \sum_{q=d,u,s,c,b,t} i\bar{q}_{0R} \gamma^\mu \partial_\mu q_{0R} \\ \mathcal{L}_{Yukawa} = & - \sum_{\substack{q=d,s,b \\ q'=d,s,b}} h_{qq'} i\bar{L}_0^{(q)} \Phi q'_{0R} + \text{h.c.} - \sum_{\substack{q=d,s,b \\ q'=u,c,t}} \tilde{h}_{qq'} i\bar{L}_0^{(q)} \tilde{\Phi} q'_{0R} + \text{h.c.} \end{aligned}$$

More to be added here...

1.2 QFT

1.2.1 Evolution Operator, S-Matrix Elements

The evolution operator U is defined by the equations:

$$|\phi(t_2)\rangle = U(t_2, t_1) |\phi(t_1)\rangle$$

$$i\hbar \frac{\partial U(t, t_1)}{\partial t} = H(t) U(t, t_1)$$

$$U(t_1, t_1) = 1$$

We are interested in calculating the S matrix elements:

$$\langle f|U(-\infty, \infty)|i\rangle = \langle f|S|i\rangle = S_{fi}$$

so we first calculate $U(-\infty, \infty)$. Integrating the equation for the evolution operator:

$$U(t_2, t_1) = U(t_1, t_1) - \frac{i}{\hbar} \int_{t_1}^{t_2} H(t)U(t, t_1)dt = 1 - \frac{i}{\hbar} \int_{t_1}^{t_2} H(t)U(t, t_1)dt$$

Now:

$$\begin{aligned} S &= U(-\infty, \infty) = 1 - \frac{i}{\hbar} \int_{-\infty}^{\infty} H(t')U(t', -\infty)dt' = \\ &= 1 + \left(-\frac{i}{\hbar}\right) \int_{-\infty}^{\infty} H(t')U(t', -\infty)dt' + \left(-\frac{i}{\hbar}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{t'} H(t')H(t'')U(t'', -\infty)dt'dt'' = \\ &= \dots = \sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots T\{H(t_1)H(t_2)\dots\}dt_1dt_2\dots = \\ &= T \exp\left(-\frac{i}{\hbar} \int_{-\infty}^{\infty} H(t)dt\right) = T \exp\left(-\frac{i}{\hbar} \int_{-\infty}^{\infty} d^4x \mathcal{H}(x)\right) \end{aligned}$$

If \mathcal{L} doesn't contain derivatives of the fields, then $\mathcal{H} = -\mathcal{L}$ so:

$$U(-\infty, \infty) = T \exp\left(\frac{i}{\hbar} \int_{-\infty}^{\infty} d^4x \mathcal{L}(x)\right)$$

Let's write $S = 1 + iT$ and $|i\rangle = |k_1 \dots k_m\rangle$, $|f\rangle = |p_1 \dots p_n\rangle$. As a first step now, let's investigate a scalar field, e.g. $\mathcal{L} = -\frac{\lambda}{4}\phi^4$ (e.g. a Higgs self interaction term above), we'll look at other fields later:

$$\begin{aligned} \langle f|S|i\rangle &= \langle f|iT|i\rangle = \langle p_1 \dots p_n | iT | k_1 \dots k_m \rangle = \frac{1}{\tilde{D}(k_1) \dots \tilde{D}(k_m)} \frac{1}{\tilde{D}(p_1) \dots \tilde{D}(p_n)} \\ &\int d^4x_1 \dots d^4x_m e^{-i(k_1x_1 + \dots + k_mx_m)} \int d^4y_1 \dots d^4y_n e^{+i(p_1y_1 + \dots + p_ny_n)} G(x_1, \dots, x_m, y_1, \dots, y_m) \end{aligned}$$

where

$$\begin{aligned} G(x_1, \dots, x_n) &= \langle 0 | T \{ \phi(x_1) \dots \phi(x_n) \} | 0 \rangle = \\ &\frac{\langle 0 | T \{ \phi_I(x_1) \dots \phi_I(x_n) \exp\left(\frac{i}{\hbar} \int_{-\infty}^{\infty} d^4x \mathcal{L}(x)\right) \} | 0 \rangle}{\langle 0 | T \exp\left(\frac{i}{\hbar} \int_{-\infty}^{\infty} d^4x \mathcal{L}(x)\right) | 0 \rangle} \end{aligned}$$

This is called the LSZ formula. Now we use the Wick contraction, get some terms like $D_{23}D_{34}$ integrate things out, this will give the delta function and $\tilde{D}(p)$'s and that's it.

Let's see how it goes for $\mathcal{L} = -\frac{\lambda}{4}\phi^4$ for the process $k_1 + k_2 \rightarrow p_1 + p_2$:

$$\begin{aligned} \langle p_1 p_2 | S | k_1 k_2 \rangle &= \frac{\int d^4x_1 d^4x_2 e^{-i(k_1x_1 + k_2x_2)} \int d^4y_1 d^4y_2 e^{-i(p_1y_1 + p_2y_2)}}{\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)} \\ &\frac{\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \phi_I(y_1) \phi_I(y_2) \exp\left(-\frac{i\lambda}{4\hbar} \int d^4x \phi_I^4(x)\right) \} | 0 \rangle}{\langle 0 | T \exp\left(-\frac{i\lambda}{4\hbar} \int d^4x \phi_I^4(x)\right) | 0 \rangle} = \end{aligned}$$

$$\begin{aligned}
&= \frac{\int d^4x_1 d^4x_2 e^{-i(k_1x_1+k_2x_2)} \int d^4y_1 d^4y_2 e^{-i(p_1y_1+p_2y_2)}}{\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)} \\
&\quad \left[\frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\}|0\rangle}{\langle 0|T \exp\left(-\frac{i\lambda}{4\hbar} \int d^4x \phi_I^4(x)\right)|0\rangle} + \right. \\
&\quad + \frac{\left(-\frac{i\lambda}{4\hbar}\right) \int d^4x \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\}|0\rangle}{\langle 0|T \exp\left(-\frac{i\lambda}{4\hbar} \int d^4x \phi_I^4(x)\right)|0\rangle} + \\
&\quad \left. + \frac{\left(-\frac{i\lambda}{4\hbar}\right)^2 \int d^4x d^4y \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\phi_I^4(y)\}|0\rangle}{\langle 0|T \exp\left(-\frac{i\lambda}{4\hbar} \int d^4x \phi_I^4(x)\right)|0\rangle} + \dots \right] = \\
&= \frac{1}{\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)} \\
&\quad \left[(2\pi)^4 \delta^{(4)}(p_1+p_2) (2\pi)^4 \delta^{(4)}(k_1+k_2) \tilde{D}(p_1)\tilde{D}(k_1) + \right. \\
&\quad (-i\lambda) 6(2\pi)^4 \delta^{(4)}(p_1+p_2-k_1-k_2) \tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2) + \\
&\quad \left. (-i\lambda)(\text{disconnected terms with not enough } \tilde{D}(\dots)\text{s}) + (-i\lambda)^2(\dots) + \dots \right] = \\
&= (2\pi)^4 \delta^{(4)}(p_1+p_2-k_1-k_2) \left[6(-i\lambda) + 3(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \tilde{D}(k)\tilde{D}(p_1+p_2-k) + (-i\lambda)^3(\dots) + \dots \right]
\end{aligned}$$

The denominator cancels with the disconnected terms. We used the Wick contractions (see below for a thorough explanation+derivation):

$$\begin{aligned}
\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\}|0\rangle &= D(x_1-x_2)D(y_1-y_2) + D(x_2-y_1)D(x_1-y_2) + D(x_2-y_2)D(x_1-y_1) \\
\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\}|0\rangle &= D(x_1-x)D(x_2-x)D(y_1-x)D(y_2-x) + \text{disconnected} \\
\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\phi_I^4(y)\}|0\rangle &= D(x_1-x)D(x_2-x)D(y_1-y)D(y_2-y)D(x-y)D(x-y) \\
&\quad + \text{disconnected}
\end{aligned}$$

Where the "disconnected" terms are $D(x_1-y_1)D(x_2-y_2)D(x-x)D(x-x)$ and similar. When they are integrated over, they do not generate enough $\tilde{D}(p_1)$ propagators to cancel the propagators from the LSZ formula, which will cause the terms to vanish.

For the $\mathcal{L} = \phi^3(x)$ theory, one also needs the following contractions:

$$\begin{aligned}
\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^3(x)\}|0\rangle &= 0 \\
\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^3(x)\phi_I^3(y)\}|0\rangle &= D(x_1-x)D(x_2-x)D(x-y)D(y_1-y)D(y_2-y)
\end{aligned}$$

Thus it is clear that the only difference from the above is the factor $D(x-y)$ which after integrating changes to $\tilde{D}(p_1+p_2)$ and this ends up in the final result.

One always gets the delta function in the result, so we define the matrix element \mathcal{M}_{fi} by:

$$S_{fi} = (2\pi)^4 \delta^{(4)}(p_1+p_2+\dots-k_1-k_2-\dots) i\mathcal{M}_{fi}$$

1.2.2 Wick Theorem

As seen above, we need to be able to calculate

$$\langle 0|T\{\phi_I(x_1)\cdots\phi_I(x_n)\}|0\rangle$$

The Wick theorem says, that this is equal to all possible contractions of fields (all fields need to be contracted), where a contraction is defined as:

$$\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle \equiv D(x-y) = \int \frac{d^4p}{(2\pi)^4} \tilde{D}(p)e^{-ip(x-y)}$$

with

$$\tilde{D}(p) = \frac{i}{p^2 - m^2 + i\epsilon}$$

A few lowest possibilities:

$$\begin{aligned} \langle 0|T\{\phi_I(x_1)\}|0\rangle &= 0 \\ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\}|0\rangle &= D_{12} \\ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\}|0\rangle &= 0 \\ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\}|0\rangle &= \text{disconnected} \\ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I(x)\}|0\rangle &= 0 \\ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^2(x)\}|0\rangle &= \text{disconnected} \\ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^3(x)\}|0\rangle &= 0 \\ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^4(x)\}|0\rangle &= 4! D(x_1-x)D(x_2-x)D(x_3-x)D(x_4-x) + \text{disconnected} \\ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^3(x)\phi_I^3(y)\}|0\rangle &= \\ &= D(x_1-x)D(x_2-x)D(x-y)D(x_3-y)D(x_4-y) + \text{disconnected} \\ \langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^4(x)\phi_I^4(y)\}|0\rangle &= \\ &= D(x_1-x)D(x_2-x)D(x-y)D(x-y)D(x_3-y)D(x_4-y) + \text{disconnected} \end{aligned}$$

For the last two equations, not all possibilities of the connected graphs are listed (and also the combinatorial factor is omitted).

1.2.3 Fermions and Vector Bosons

For fermions:

$$\langle 0|T\{\psi_I(x)\bar{\psi}_I(y)\}|0\rangle \equiv S(x-y) = \int \frac{d^4p}{(2\pi)^4} \tilde{S}(p)e^{-ip(x-y)}$$

with

$$\tilde{S}(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

For vector bosons:

$$\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle \equiv D_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \tilde{D}_{\mu\nu}(p)e^{-ip(x-y)}$$

with

$$\tilde{D}_{\mu\nu}(p) = i \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}}{p^2 - m^2 + i\epsilon}$$

For massless bosons:

$$\tilde{D}_{\mu\nu}(p) = i \frac{-g_{\mu\nu}}{p^2 + i\epsilon}$$

1.2.4 Feynman Rules

We can deduce a set of rules, so that one doesn't have to repeat the whole calculation each time. For a scalar field we derived the rules above, for fermion and vector boson fields it's more difficult.

1.2.5 ZZH interaction

Let's calculate the $\mathcal{L}_{ZZH} = \lambda Z_\mu Z^\mu H$ interaction in the SM, where $\lambda = \frac{g^2}{4 \cos \theta_W}$. Consider $H(p) \rightarrow Z(k) + Z(l)$:

$$\begin{aligned} \langle f|S|i \rangle &= \langle f|iT|i \rangle = \langle kl|iT|p \rangle = \frac{\varepsilon_\alpha(k)\varepsilon^\alpha(l)}{\tilde{D}_{\mu\nu}(k) \cdots \tilde{D}^{\mu\nu}(l)} \frac{1}{\tilde{D}(p)} \\ &\int d^4x_1 e^{-ipx_1} \int d^4y_1 d^4y_2 e^{+i(ky_1 + ly_2)} \langle 0|T\{Z_\mu(y_1)Z^\mu(y_2)H(x_1)\}|0 \rangle = \\ &= \frac{\varepsilon_\alpha(k)\varepsilon^\alpha(l)}{\tilde{D}_{\mu\nu}(k) \cdots \tilde{D}^{\mu\nu}(l)} \frac{1}{\tilde{D}(p)} \\ &\int d^4x_1 e^{-ipx_1} \int d^4y_1 d^4y_2 e^{+i(ky_1 + ly_2)} \int d^4x i\lambda D_{\alpha\mu}(x_2-x) D^{\alpha\mu}(x_3-x) D(x_1-x) = \\ &= i\lambda(2\pi)^4 \delta^{(4)}(p-k-l) \varepsilon_\alpha(k)\varepsilon^\alpha(l) \end{aligned}$$

where we used the fact, that the only nonzero element of the Green function is

$$\int d^4x \langle 0|T\{Z_\mu(y_1)Z^\mu(y_2)H(x_1)Z_\mu(x)Z^\mu(x)H(x)\}|0 \rangle$$

1.3 Low energy theories

1.3.1 Fermi-type theory

This is a low energy ($m_W^2 \gg m_\mu m_e$) model for the EW interactions, that can be derived for example from the muon decay:

$$\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$$

From the SM the relevant Lagrangian is

$$\mathcal{L} = \frac{g}{2\sqrt{2}}(\bar{e}\gamma^\mu(1-\gamma_5)\nu_e W_\mu^-) + \frac{g}{2\sqrt{2}}(\bar{\mu}\gamma^\mu(1-\gamma_5)\nu_\mu W_\mu^-)$$

and one gets the diagram $\mu^- + \bar{\nu}_\mu \rightarrow e^- + \bar{\nu}_e$ and the corresponding matrix element:

$$iM = -i\frac{g^2}{8}[\bar{u}\gamma_\mu(1-\gamma_5)u] \frac{-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_W^2}}{q^2 - m_W^2}[\bar{u}\gamma_\nu(1-\gamma_5)v]$$

which when the momentum transfer q is much less than m_w becomes

$$iM = -i\frac{g^2}{8m_W^2}[\bar{u}\gamma^\mu(1-\gamma_5)u][\bar{u}\gamma_\mu(1-\gamma_5)v]$$

but this element can be derived directly from the Lagrangian:

$$\mathcal{L} = -\frac{G_\mu}{\sqrt{2}}[\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma_5)\psi_\mu][\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_{\nu_e}]$$

with

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

This is the universal V-A theory Lagrangian (after adding the h.c. term).

1.3.2 QED

The QED Lagrangian density is

$$\mathcal{L} = \bar{\psi}(ic\gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where

$$\psi = (\psi_1\psi_2\psi_3\psi_4)$$

and

$$D_\mu = \partial_\mu + ieA_\mu$$

is the gauge covariant derivative and (e is the elementary charge, which is 1 in atomic units)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. It's astonishing, that this simple Lagrangian can account for all phenomena from macroscopic scales down to something like 10^{-13} cm. So of course Feynman, Schwinger and Tomonaga received the 1965 Nobel Prize in Physics for such a fantastic achievement.

Plugging this Lagrangian into the Euler-Lagrange equation of motion for a field, we get:

$$\begin{aligned} (ic\gamma^\mu D_\mu - mc^2)\psi &= 0 \\ \partial_\nu F^{\nu\mu} &= -ec\bar{\psi}\gamma^\mu\psi \end{aligned}$$

The first equation is the Dirac equation in the electromagnetic field and the second equation is a set of Maxwell equations ($\partial_\nu F^{\nu\mu} = -ej^\mu$) with a source $j^\mu = c\bar{\psi}\gamma^\mu\psi$, which is a 4-current coming from the Dirac equation.

The fields ψ and A^μ are quantized. The first approximation is that we take ψ as a wavefunction, that is, it is a classical 4-component field. It can be shown that this corresponds to taking three orders in the perturbation theory.

The first component A_0 of the 4-potential is the electric potential, and because this is the potential that (as we show in a moment) is in the Schrödinger equation, we denote it by V :

$$A_\mu = \left(\frac{V}{ec}, A_1, A_2, A_3 \right)$$

So in the non-relativistic limit, the $\frac{V}{e}$ corresponds to the electric potential. We multiply the Dirac equation by γ^0 from left to get:

$$\begin{aligned} 0 &= \gamma^0(ic\gamma^\mu D_\mu - mc^2)\psi = \gamma^0(ic\gamma^0(\partial_0 + i\frac{V}{c}) + ic\gamma^i(\partial_i + ieA_i) - mc^2)\psi = \\ &= (ic\partial_0 + ic\gamma^0\gamma^i\partial_i - \gamma^0mc^2 - V - ce\gamma^0\gamma^iA_i)\psi \end{aligned}$$

and we make the following substitutions (it's just a formalism, nothing more):
 $\beta = \gamma^0$, $\alpha^i = \gamma^0 \gamma^i$, $p_j = -i\partial_j$, $\partial_0 = \frac{1}{c} \frac{\partial}{\partial t}$ to get

$$(i \frac{\partial}{\partial t} - c\alpha^i p_i - \beta mc^2 - V - ce\alpha^i A_i)\psi = 0.$$

This, in most solid state physics texts, is usually written as

$$i \frac{\partial \psi}{\partial t} = H\psi,$$

where the Hamiltonian is given by

$$H = c\alpha^i(p_i + eA_i) + \beta mc^2 + V.$$

The right hand side of the Maxwell equations is the 4-current, so it's given by:

$$j^\mu = c\bar{\psi}\gamma^\mu\psi$$

Now we make the substitution $\psi = e^{-imc^2 t}\varphi$, which states, that we separate the largest oscillations of the wavefunction and we get

$$\begin{aligned} j^0 &= c\bar{\psi}\gamma^0\psi = c\psi^\dagger\psi = c\varphi^\dagger\varphi \\ j^i &= c\bar{\psi}\gamma^i\psi = c\psi^\dagger\alpha^i\psi = c\varphi^\dagger\alpha^i\varphi \end{aligned}$$

The Dirac equation implies the Klein-Gordon equation:

$$\begin{aligned} (-ic\gamma^\mu D_\mu - mc^2)(ic\gamma^\nu D_\nu - mc^2)\psi &= (c^2\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2 c^4)\psi = \\ &= (c^2 D^\mu D_\mu - ic^2[\gamma^\mu, \gamma^\nu]D_\mu D_\nu + m^2 c^4)\psi = 0 \end{aligned}$$

Note however, the ψ in the true Klein-Gordon equation is just a scalar, but here we get a 4-component spinor. Now:

$$\begin{aligned} D_\mu D_\nu &= (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) = \partial_\mu\partial_\nu + ie(A_\mu\partial_\nu + A_\nu\partial_\mu + (\partial_\mu A_\nu)) - e^2 A_\mu A_\nu \\ [D_\mu, D_\nu] &= D_\mu D_\nu - D_\nu D_\mu = ie(\partial_\mu A_\nu) - ie(\partial_\nu A_\mu) \end{aligned}$$

We rewrite $D^\mu D_\mu$:

$$\begin{aligned} D^\mu D_\mu &= g^{\mu\nu} D_\mu D_\nu = \partial^\mu\partial_\mu + ie((\partial^\mu A_\mu) + 2A^\mu\partial_\mu) - e^2 A^\mu A_\mu = \\ &= \partial^\mu\partial_\mu + ie((\partial^0 A_0) + 2A^0\partial_0 + (\partial^i A_i) + 2A^i\partial_i) - e^2(A^0 A_0 + A^i A_i) = \\ &= \partial^\mu\partial_\mu + i\frac{1}{c^2}\frac{\partial V}{\partial t} + 2i\frac{V}{c^2}\frac{\partial}{\partial t} + ie(\partial^i A_i) + 2ieA^i\partial_i - \frac{V^2}{c^2} - e^2 A^i A_i \end{aligned}$$

We use the identity $\frac{\partial}{\partial t} \left(e^{-imc^2 t} f(t) \right) = e^{-imc^2 t} (-imc^2 + \frac{\partial}{\partial t}) f(t)$ to get:

$$\begin{aligned} L &= c^2 \partial^\mu \psi^* \partial_\mu \psi - m^2 c^4 \psi^* \psi = \frac{\partial}{\partial t} \psi^* \frac{\partial}{\partial t} \psi - c^2 \partial^i \psi^* \partial_i \psi - m^2 c^4 \psi^* \psi = \\ &= (imc^2 + \frac{\partial}{\partial t}) \varphi^* (-imc^2 + \frac{\partial}{\partial t}) \varphi - c^2 \partial^i \varphi^* \partial_i \varphi - m^2 c^4 \varphi^* \varphi = \end{aligned}$$

$$= 2mc^2 \left[\frac{1}{2} i (\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi + \frac{1}{2mc^2} \frac{\partial \varphi^*}{\partial t} \frac{\partial \varphi}{\partial t} \right]$$

The constant factor $2mc^2$ in front of the Lagrangian is of course irrelevant, so we drop it and then we take the limit $c \rightarrow \infty$ (neglecting the last term) and we get

$$L = \frac{1}{2} i (\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

After integration by parts we arrive at

$$L = i \varphi^* \frac{\partial \varphi}{\partial t} - \frac{1}{2m} \partial^i \varphi^* \partial_i \varphi$$

The nonrelativistic limit can also be applied directly to the Klein-Gordon equation:

$$\begin{aligned} 0 &= (c^2 D^\mu D_\mu + m^2 c^4) \psi = \\ &= \left(c^2 \partial^\mu \partial_\mu + i \frac{\partial V}{\partial t} + 2iV \frac{\partial}{\partial t} + iec^2 (\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi = \\ &= \left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4 \right) e^{-imc^2 t} \varphi = \\ &= e^{-imc^2 t} \left((-imc^2 + \frac{\partial}{\partial t})^2 - c^2 \nabla^2 + 2iV(-imc^2 + \frac{\partial}{\partial t}) + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 + \right. \\ &\quad \left. - e^2 c^2 A^i A_i + m^2 c^4 \right) \varphi = \\ &= e^{-imc^2 t} \left(-2imc^2 \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2Vmc^2 + 2iV \frac{\partial}{\partial t} + i \frac{\partial V}{\partial t} + iec^2 (\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 + \right. \\ &\quad \left. - e^2 c^2 A^i A_i \right) \varphi = \\ &= -2mc^2 e^{-imc^2 t} \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - V - \frac{1}{2mc^2} \frac{\partial^2}{\partial t^2} - \frac{i}{2mc^2} \frac{\partial V}{\partial t} + \frac{V^2}{2mc^2} - \frac{iV}{mc^2} \frac{\partial}{\partial t} + \right. \\ &\quad \left. - \frac{ie}{2m} \partial^i A_i - \frac{ie}{m} A^i \partial_i + \frac{e^2}{2m} A^i A_i \right) \varphi \end{aligned}$$

Taking the limit $c \rightarrow \infty$ we again recover the Schrödinger equation:

$$i \frac{\partial}{\partial t} \varphi = \left(-\frac{\nabla^2}{2m} + V + \frac{ie}{2m} \partial^i A_i + \frac{ie}{m} A^i \partial_i - \frac{e^2}{2m} A^i A_i \right) \varphi,$$

we rewrite the right hand side a little bit:

$$i \frac{\partial}{\partial t} \varphi = \left(\frac{1}{2m} (\partial^i \partial_i + ie \partial^i A_i + 2ie A^i \partial_i - e^2 A^i A_i) + V \right) \varphi,$$

$$i \frac{\partial}{\partial t} \varphi = \left(\frac{1}{2m} (\partial^i + ie A^i) (\partial_i + ie A_i) + V \right) \varphi,$$

And we get the usual form of the Schrödinger equation for the vector potential

$\mathbf{A} = (A_1, A_2, A_3)$:

$$i \frac{\partial}{\partial t} \varphi = \left(-\frac{(\nabla + ie \mathbf{A})^2}{2m} + V \right) \varphi.$$