Quantum Field Theory

June 8, 2008

Contents

1 Derivation from the Standard Model **1**

1.1 Electroweak Standard Model 1

1.1.1 Higgs Terms 2

1.1.2 Yukawa terms 3

1.1.3 Leptonic Terms 3

1.1.4 Gauge terms 4

1.1.5 GWS Lagrangian 5

1.1.6 Particle Masses 6

1.1.7 Dimensional Analysis 6

1.1.8 Quarks 7

1.2 QFT 7

1.2.1 Evolution Operator, S-Matrix Elements 7

1.2.2 Wick Theorem 9

1.2.3 Fermions and Vector Bosons 10

1.2.4 Feynman Rules 10

1.2.5 ZZH interaction 10

1.2.6 eeH interaction 11

1.3 Low energy theories 11

1.3.1 Fermi-type theory 11

1.3.2 QED 12

1 Derivation from the Standard Model

1.1 Electroweak Standard Model

Lagrangian with a global $SU(2) \times U(1)$ symmetry:

$$\mathcal{L} = i\bar{L}^{(l)}\gamma_\mu\partial^\mu L^{(l)} + i\bar{l}_R\gamma_\mu\partial^\mu l_R + \tfrac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2 - h_e\bar{L}^{(l)}\Phi e_R - \text{h.c.}$$

where $l = e, \mu, \tau$ and $a = 1, 2$, $l_{L,R} = \frac{1}{2}(1 \mp \gamma_5)l$ and

$$L^{(l)} = \begin{pmatrix} \nu_{(l)L} \\ l_L \end{pmatrix}$$

Local $SU(2) \times U(1)$ symmetry:

This consists of two things. First changing the partial derivatives to covariant ones:

$$\partial^\mu \to D^\mu = \partial^\mu - \frac{i}{2} g \tau_k A_k^\mu - \frac{i}{2} g' Y B^\mu$$

and second adding the kinetic terms

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

of the vector gauge particles to the lagrangian.

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\Phi = e^{\frac{i}{v}\pi^a(x)\tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

This breaks the gauge invariance. The $\partial^\mu \pi^a$ are going to be added to A^a_μ so we can set $\pi_a = 0$ now.

1.1.1 Higgs Terms

$$\mathcal{L}_{Higgs} = \tfrac{1}{2} \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4} \lambda (\Phi^* \Phi)^2$$

Plugging in the covariant derivatives and Φ in U-gauge (symmetry breaking):

$$\mathcal{L}_{Higgs} = \frac{1}{2} \Phi^+ (\overleftarrow{\partial}_\mu + igA^a_\mu \frac{\tau^a}{2} + ig'YB_\mu)(\overrightarrow{\partial}^\mu + igA^{a\mu}\frac{\tau^a}{2} + ig'YB^\mu)\Phi - \lambda(\Phi^+\Phi - \frac{v^2}{2})^2 =$$

$$= \Phi_U^+ (\overleftarrow{\partial}_\mu + igA^a_\mu \frac{\tau^a}{2} + ig'YB_\mu)(\overrightarrow{\partial}^\mu + igA^{a\mu}\frac{\tau^a}{2} + ig'YB\mu)\Phi_U - \lambda(\Phi_U^+\Phi_U - \frac{v^2}{2})^2 =$$

$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 +$$

$$+ \frac{1}{8}(v+H)^2 \left(2g^2 \frac{A^1_\mu + iA^2_\mu}{\sqrt{2}} \frac{A^{1\mu} - iA^{2\mu}}{\sqrt{2}} + (g^2 + 4Y^2 g'^2) \frac{gA^3_\mu - 2Yg'B_\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \frac{gA^{3\mu} - 2Yg'B^\mu}{\sqrt{g^2 + 4Y^2 g'^2}} \right) =$$

$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \frac{1}{8}(v+H)^2 \left(2g^2 W^-_\mu W^{+\mu} + \frac{g^2}{\cos^2\theta_W} Z_\mu Z^\mu \right) =$$

$$= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4} g^2 v^2 W^-_\mu W^{+\mu} + \frac{g^2 v^2}{8\cos^2\theta_W} Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 +$$

$$+ \frac{1}{2} v g^2 W^-_\mu W^{+\mu} H + \frac{g^2}{4\cos\theta_W} v Z_\mu Z^\mu H + \frac{1}{4} g^2 W^-_\mu W^{+\mu} H^2 + \frac{g^2}{8\cos\theta_W} Z_\mu Z^\mu H^2$$

Where we put

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu)$$

$$Z_\mu = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}} A^3_\mu - \frac{2Yg'}{\sqrt{g^2 + 4Y^2 g'^2}} B_\mu$$

we defined θ_W by the relation

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}}$$

so that the expressions simplify a bit, e.g. we now get:

$$\sin\theta_W = \frac{2Yg'}{\sqrt{g^2 + 4Y^2 g'^2}}$$

$$Z_\mu = \cos\theta_W A^3_\mu - \sin\theta_W B_\mu$$

$$g^2 + 4Y^2 g'^2 = \frac{g^2}{\cos^2\theta_W}$$

1.1.2 Yukawa terms

$$\mathcal{L}_{Yukawa} = -h_e \bar{L}\Phi e_R - \text{h.c.} = -h_e \bar{L}\Phi_U e_R - \text{h.c.} =$$
$$= -\frac{1}{\sqrt{2}} h_e (v+H)(\bar{e}_L e_R + \bar{e}_R e_L) = -\frac{1}{\sqrt{2}} h_e (v+H)\bar{e}e =$$
$$= -\frac{1}{\sqrt{2}} h_e v \bar{e}e - \frac{1}{\sqrt{2}} h_e \bar{e}eH$$

The term $\bar{L}\Phi e_R$ is $U(1)$ (hypercharge) invariant, so

$$-Y_L + Y + Y_R = 0$$

1.1.3 Leptonic Terms

$$\mathcal{L} = i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R \gamma^\mu \partial_\mu e_R \rightarrow$$
$$\rightarrow i\bar{L}\gamma^\mu(\partial_\mu - igA^a_\mu \frac{\tau^a}{2} - ig'Y_L B_\mu)L + i\bar{e}_R\gamma^\mu(\partial_\mu - ig'Y_R B_\mu)e_R =$$
$$= i\bar{L}\gamma^\mu\partial_\mu L + i\bar{e}_R\gamma^\mu\partial_\mu e_R + g\bar{L}\gamma^\mu \frac{\tau^a}{2} L A^a_\mu + g'Y_L \bar{L}\gamma^\mu L B_\mu + g'Y_R \bar{e}_R \gamma^\mu e_R B_\mu =$$
$$= i\bar{L}\gamma^\mu\partial_\mu L + i\bar{e}_R\gamma^\mu\partial_\mu e_R + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W^+_\mu + \text{h.c.}) + \frac{1}{2} g\bar{L}\gamma^\mu\tau^3 L A^3_\mu + g'Y_L\bar{L}\gamma^\mu L B_\mu + g'Y_R \bar{e}_R\gamma^\mu e_R B_\mu =$$
$$= i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W^+_\mu + \text{h.c.}) + \frac{1}{2}g\bar{\nu}_L\gamma^\mu\nu_L A^3_\mu - \frac{1}{2}g\bar{e}_L\gamma^\mu e_L A^3_\mu$$
$$+ g'Y_L\bar{\nu}_L\gamma^\mu\nu_L B_\mu + g'Y_L\bar{e}_L\gamma^\mu e_L B_\mu + g'Y_R\bar{e}_R\gamma^\mu e_R B_\mu =$$
$$= i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W^+_\mu + \text{h.c.})$$
$$+ \left[(\tfrac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W)\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W)\bar{e}_L\gamma^\mu e_L + Y_R g'\cos\theta_W \bar{e}_R\gamma^\mu e_R\right] A_\mu$$
$$+ \left[(\tfrac{1}{2}g\cos\theta_W - Y_L g'\sin\theta_W)\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2}g\cos\theta_W - Y_L g'\sin\theta_W)\bar{e}_L\gamma^\mu e_L - 2Y_L g'\sin\theta_W \bar{e}_R\gamma^\mu e_R\right] Z_\mu$$

Where we substituted new fields Z_μ and A_μ for the old ones A^3_μ and B_μ using the relation:

$$Z_\mu = \cos\theta_W A^3_\mu - \sin\theta_W B_\mu$$

$$A_\mu = \sin\theta_W A^3_\mu + \cos\theta_W B_\mu$$

The angle θ_W must be the same as in the Higgs sector, so that the field Z_μ is the same. We now need to make the following requirement in order to proceed further:

$$Y = -Y_L$$

This follows for example by requiring that neutrinos have zero charge, i.e. setting $\frac{1}{2} g \sin\theta_W + Y_L g' \cos\theta_W = 0$ and substituting for θ_W from the definition (see the Higgs terms), from which one gets $Y = -Y_L$. From $-Y_L + Y + Y_R = 0$ we now get

$$Y_R = 2Y_L$$

it now follows:

$$\begin{aligned}
\tfrac{1}{2} g \sin\theta_W + Y_L g' \cos\theta_W &= 0 \\
-\tfrac{1}{2} g \sin\theta_W + Y_L g' \cos\theta_W &= -g \sin\theta_W \\
Y_R g' \cos\theta_W &= -g \sin\theta_W \\
\tan\theta_W &= -2Y_L \frac{g'}{g}
\end{aligned}$$

and the Lagrangian can be further simplified:

$$\mathcal{L} = i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e}\gamma^\mu \partial_\mu e + \frac{g}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \text{h.c.})$$

$$-g \sin\theta_W (\bar{e}_L \gamma^\mu e_L + \bar{e}_R \gamma^\mu e_R) A_\mu$$

$$+\frac{g}{\cos\theta_W} \left[\tfrac{1}{2} \bar{\nu}_L \gamma^\mu \nu_L + (-\tfrac{1}{2} + \sin^2\theta_W) \bar{e}_L \gamma^\mu e_L + \sin^2\theta_W \bar{e}_R \gamma^\mu e_R\right] Z_\mu =$$

$$= i\bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i\bar{e}\gamma^\mu \partial_\mu e + \frac{g}{2\sqrt{2}} (\bar{\nu}\gamma^\mu (1-\gamma_5) e W_\mu^+ + \text{h.c.}) - g \sin\theta_W \bar{e}\gamma^\mu e A_\mu$$

$$+\frac{g}{2\cos\theta_W} \left[\bar{\nu}\gamma^\mu (1-\gamma_5)\nu + \bar{e}\gamma^\mu (-\tfrac{1}{2} + 2\sin^2\theta_W + \tfrac{1}{2}\gamma_5) e\right] Z_\mu$$

Where we used the relations $\bar{\nu}_L \gamma^\mu e_L = \frac{1}{2} \bar{\nu}\gamma^\mu (1-\gamma_5) e$ and $\bar{\nu}_R \gamma^\mu e_R = \frac{1}{2} \bar{\nu}\gamma^\mu (1+\gamma_5) e$.

1.1.4 Gauge terms

$$\mathcal{L}_{Gauge} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} =$$

$$= -\frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g\epsilon^{ajk} A^{j\mu} A^{k\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} =$$

$$= -\frac{1}{4} \partial_\mu A^a_\nu \partial^\mu A^{a\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) g\epsilon^{abc} A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 \epsilon^{abc} \epsilon^{ajk} A^b_\mu A^c_\nu A^{k\mu} A^{l\nu} =$$

$$= -\frac{1}{2} W^-_{\mu\nu} W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g[(\partial_\mu A^1_\nu - \partial_\nu A^1_\mu) A^{2\mu} A^{3\nu} + \text{cycl. perm. (123)}]$$

$$-\frac{1}{4} g^2 [(A^a_\mu A^{a\mu})(A^b_\nu A^{b\nu}) - (A^a_\mu A^a_\nu)(A^{b\mu} A^{b\nu})] =$$

$$= -\frac{1}{2} W^-_{\mu\nu} W^{+\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - g[A^1_\mu A^2_\nu \overleftrightarrow{\partial}^\mu A^{3\nu} + \text{cycl. perm. (123)}]$$

$$-\frac{1}{4} g^2 [(A^a_\mu A^{a\mu})(A^b_\nu A^{b\nu}) - (A^a_\mu A^a_\nu)(A^{b\mu} A^{b\nu})] =$$

$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - ig(W^0_\mu W^-_\nu \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. (0-+)})$$
$$-g^2[\tfrac{1}{2}(W^+_\mu W^{-\mu})^2 - \tfrac{1}{2}(W^+_\mu W^{+\mu})(W^-_\nu W^{-\nu}) + (W^0_\mu W^{0\mu})(W^+_\nu W^{-\nu}) - (W^-_\mu W^+_\nu)(W^{0\mu}W^{0\nu}) =$$
$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \mathcal{L}_{WW\gamma} + L_{WWZ} + L_{WW\gamma\gamma} + L_{WWWW} + L_{WWZZ} + L_{WWZ\gamma}$$

Where $W^0_\mu = A^3_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu$ and:

$$\mathcal{L}_{WW\gamma} = -ig\sin\theta_W(A_\mu W^-_\nu \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. } (A\ W^-\ W^+))$$
$$\mathcal{L}_{WWZ} = -ig\cos\theta_W(Z_\mu W^-_\nu \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. } (Z\ W^-\ W^+))$$
$$\mathcal{L}_{WW\gamma\gamma} = -g^2\sin^2\theta_W(W^-_\mu W^{+\mu}A_\nu A^\nu - W^-_\mu A^\mu W^+_\nu A^\nu)$$
$$\mathcal{L}_{WWWW} = \tfrac{1}{2}g^2(W^-_\mu W^{-\mu}W^+_\nu W^{+\nu} - W^-_\mu W^{+\mu}W^-_\nu W^{+\nu})$$
$$\mathcal{L}_{WWZZ} = -g^2\cos^2\theta_W(W^-_\mu W^{+\mu}Z_\nu Z^\nu - W^-_\mu Z^\mu W^+_\nu Z^\nu)$$
$$\mathcal{L}_{WWZ\gamma} = g^2\sin\theta_W\cos\theta_W(-2W^-_\mu W^{+\mu}A_\nu Z^\nu + W^-_\mu Z^\mu W^+_\nu A^\nu + W^-_\mu A^\mu W^+_\nu Z^\nu)$$

1.1.5 GWS Lagrangian

Plugging everything together we get the GWS Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu H\partial^\mu H - \lambda v^2H^2 + \frac{1}{4}g^2v^2W^-_\mu W^{+\mu} + \frac{g^2v^2}{8\cos^2\theta_W}Z_\mu Z^\mu - \lambda vH^3 - \frac{1}{4}\lambda H^4 +$$
$$+\frac{1}{2}vg^2W^-_\mu W^{+\mu}H + \frac{g^2}{4\cos\theta_W}vZ_\mu Z^\mu H + \frac{1}{4}g^2W^-_\mu W^{+\mu}H^2 + \frac{g^2}{8\cos\theta_W}Z_\mu Z^\mu H^2$$
$$-\frac{1}{\sqrt{2}}h_e v\bar{e}e - \frac{1}{\sqrt{2}}h_e\bar{e}eH$$
$$-\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \mathcal{L}_{WW\gamma} + L_{WWZ} + L_{WW\gamma\gamma} + L_{WWWW} + L_{WWZZ} + L_{WWZ\gamma}$$
$$+i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{2\sqrt{2}}(\bar{\nu}\gamma^\mu(1-\gamma_5)eW^+_\mu + \text{h.c.}) - g\sin\theta_W\bar{e}\gamma^\mu eA_\mu$$
$$+\frac{g}{2\cos\theta_W}\left[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu + \bar{e}\gamma^\mu(-\tfrac{1}{2} + 2\sin^2\theta_W + \tfrac{1}{2}\gamma_5)e\right]Z_\mu$$
$$+(e,\nu_e,h_e \leftrightarrow \mu,\nu_\mu,h_\mu) + (e,\nu_e,h_e \leftrightarrow \tau,\nu_\tau,h_\tau)$$

The free parameters are g, θ_W, v, λ, h_e, h_μ, h_τ.

1.1.6 Particle Masses

The particle masses are deduced from the terms

$$\mathcal{L} = -\frac{1}{2}m_H^2 H^2 + m_W^2 W_\mu^- W^{+\mu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu - m_e \bar{e}e + \cdots$$

comparing to the above:

$$\mathcal{L} = -\lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8\cos^2\theta_W} Z_\mu Z^\mu - \frac{1}{\sqrt{2}} h_e v \bar{e}e + \cdots$$

we get

$$\begin{aligned} m_W &= \tfrac{1}{2}gv \\ m_Z &= \frac{gv}{2\cos\theta_W} = \frac{m_W}{\cos\theta_W} \\ m_H &= v\sqrt{2\lambda} \\ m_e &= \frac{1}{\sqrt{2}}h_e v \end{aligned}$$

1.1.7 Dimensional Analysis

The evolution operator is dimensionless:

$$U(-\infty, \infty) = T\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}^4 x \mathcal{L}(x)\right)$$

So:

$$\left[\int_{-\infty}^{\infty} \mathrm{d}^4 x \mathcal{L}(x)\right] = [\hbar] = M^0$$

where M is an arbitrary mass scale. Length unit is M^{-1}, so then

$$[\mathcal{L}(x)] = M^4$$

For the particular forms of the Lagrangians above we get:

$$[m\bar{e}e] = [m^2 Z_\mu Z^\mu] = [m^2 H^2] = [i\bar{e}\gamma^\mu \partial_\mu e] = [\mathcal{L}] = M^4$$

so $[\bar{e}e] = M^3$, $[Z_\mu Z^\mu] = [H^2] = M^2$ and we get

$$[e] = [\bar{e}] = M^{\frac{3}{2}}$$

$$[Z_\mu] = [Z^\mu] = [H] = [\partial_\mu] = [\partial^\mu] = M^1$$

Example: what is the dimension of G_μ in $\mathcal{L} = -\frac{G_\mu}{\sqrt{2}}[\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma_5)\psi_\mu][\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_{\nu_e}]$? Answer:

$$\begin{aligned} [\mathcal{L}] &= [G_\mu \bar{\psi}\psi\bar{\psi}\psi] \\ M^4 &= [G_\mu] M^{\frac{3}{2}} M^{\frac{3}{2}} M^{\frac{3}{2}} M^{\frac{3}{2}} \\ [G_\mu] &= M^{-2} \end{aligned}$$

1.1.8 Quarks

$$\mathcal{L}_{fermion} += \sum_{q=d,s,b} i\bar{L}_0^{(q)}\gamma^\mu\partial_\mu L_0^{(q)} + \sum_{q=d,u,s,c,b,t} i\bar{q}_{0R}\gamma^\mu\partial_\mu q_{0R}$$

$$\mathcal{L}_{Yukawa} += -\sum_{\substack{q=d,s,b\\ q'=d,s,b}} h_{qq'} i\bar{L}_0^{(q)}\Phi q'_{0R} + \text{h.c.} - \sum_{\substack{q=d,s,b\\ q'=u,c,t}} \tilde{h}_{qq'} i\bar{L}_0^{(q)}\tilde{\Phi} q'_{0R} + \text{h.c.}$$

More to be added here...

1.2 QFT

1.2.1 Evolution Operator, S-Matrix Elements

The evolution operator U is defined by the equations:

$$|\phi(t_2)\rangle = U(t_2,t_1)\,|\phi(t_1)\rangle$$

$$i\hbar\frac{\partial U(t,t_1)}{\partial t} = H(t)U(t,t_1)$$

$$U(t_1,t_1) = 1$$

We are interested in calculating the S matrix elements:

$$\langle f|U(-\infty,\infty)|i\rangle = \langle f|S|i\rangle = S_{fi}$$

so we first calculate $U(-\infty,\infty)$. Integrating the equation for the evolution operator:

$$U(t_2,t_1) = U(t_1,t_1) - \frac{i}{\hbar}\int_{t_1}^{t_2} H(t)U(t,t_1)\mathrm{d}t = 1 - \frac{i}{\hbar}\int_{t_1}^{t_2} H(t)U(t,t_1)\mathrm{d}t$$

Now:

$$S = U(-\infty,\infty) = 1 - \frac{i}{\hbar}\int_{-\infty}^{\infty} H(t')U(t',-\infty)\mathrm{d}t' =$$

$$= 1+\left(-\frac{i}{\hbar}\right)\int_{-\infty}^{\infty} H(t')U(t',-\infty)\mathrm{d}t' + \left(-\frac{i}{\hbar}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{t'} H(t')H(t'')U(t'',-\infty)\mathrm{d}t'\mathrm{d}t'' =$$

$$= \cdots = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n\frac{1}{n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots T\{H(t_1)H(t_2)\cdots\}\mathrm{d}t_1\mathrm{d}t_2\cdots =$$

$$= T\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty} H(t)\mathrm{d}t\right) = T\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{H}(x)\right)$$

If $\mathcal{L}$ doesn't contain derivatives of the fields, then $\mathcal{H} = -\mathcal{L}$ so:

$$U(-\infty,\infty) = T\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{L}(x)\right)$$

Let's write $S = 1 + iT$ and $|i\rangle = |k_1\cdots k_m\rangle$, $|f\rangle = |p_1\cdots p_n\rangle$. As a first step now, let's investigate a scalar field, e.g. $\mathcal{L} = -\frac{\lambda}{4}\phi^4$ (e.g. a Higgs self interaction term above), we'll look at other fields later:

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \langle p_1\cdots p_n|iT|k_1\cdots k_m\rangle = \frac{1}{\tilde{D}(k_1)\cdots\tilde{D}(k_m)}\frac{1}{\tilde{D}(p_1)\cdots\tilde{D}(p_n)}$$

$$\int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_m e^{-i(k_1x_1+\cdots+k_mx_m)} \int \mathrm{d}^4y_1 \cdots \mathrm{d}^4y_n e^{+i(p_1y_1+\cdots+p_ny_n)} G(x_1,\cdots,x_m,y_1,\cdots,y_m)$$

where

$$G(x_1,\cdots,x_n) = \langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle =$$

$$\frac{\langle 0|T\{\phi_I(x_1)\cdots\phi_I(x_n)\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{L}(x)\right)\}|0\rangle}{\langle 0|T\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}^4x\mathcal{L}(x)\right)|0\rangle}$$

This is called the LSZ formula. Now we use the Wick contraction, get some terms like $D_{23}D_{34}$ integrate things out, this will give the delta function and $\tilde{D}(p)$'s and that's it.

Let's see how it goes for $\mathcal{L} = -\frac{\lambda}{4}\phi^4$ for the process $k_1 + k_2 \to p_1 + p_2$:

$$\langle p_1p_2|S|k_1k_2\rangle = \frac{\int \mathrm{d}^4x_1\mathrm{d}^4x_2 e^{-i(k_1x_1+k_2x_2)} \int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{-i(p_1y_1+p_2y_2)}}{\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)}$$

$$\frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)\}|0\rangle}{\langle 0|T\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)|0\rangle} =$$

$$= \frac{\int \mathrm{d}^4x_1\mathrm{d}^4x_2 e^{-i(k_1x_1+k_2x_2)} \int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{-i(p_1y_1+p_2y_2)}}{\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)}$$

$$\Bigg[\frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\}|0\rangle}{\langle 0|T\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)|0\rangle} +$$

$$+\frac{\left(-\frac{i\lambda}{4\hbar}\right)\int\mathrm{d}^4x\,\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\}|0\rangle}{\langle 0|T\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)|0\rangle} +$$

$$+\frac{\left(-\frac{i\lambda}{4\hbar}\right)^2\int\mathrm{d}^4x\,\mathrm{d}^4y\,\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\phi_I^4(y)\}|0\rangle}{\langle 0|T\exp\left(-\frac{i\lambda}{4\hbar}\int\mathrm{d}^4x\phi_I^4(x)\right)|0\rangle} + \cdots\Bigg] =$$

$$= \frac{1}{\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)}$$

$$\Big[(2\pi)^4\delta^{(4)}(p_1+p_2)(2\pi)^4\delta^{(4)}(k_1+k_2)\tilde{D}(p_1)\tilde{D}(k_1)+$$

$$(-i\lambda)6(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\tilde{D}(k_1)\tilde{D}(k_2)\tilde{D}(p_1)\tilde{D}(p_2)+$$

$$(-i\lambda)(\text{disconnected terms with not enough }\tilde{D}(\cdots)\text{s}) + (-i\lambda)^2(\cdots) + \cdots\Big] =$$

$$= (2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\left[6(-i\lambda) + 3(-i\lambda)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{D}(k)\tilde{D}(p1+p2-k) + (-i\lambda)^3(\cdots) + \cdots\right]$$

The denominator cancels with the disconnected terms. We used the Wick contractions (see below for a thorough explanation+derivation):

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\}|0\rangle = D(x_1-x_2)D(y_1-y_2)+D(x_2-y_1)D(x_1-y_2)+D(x_2-y_2)D(x_1-y_1)$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\}|0\rangle = D(x_1-x)D(x_2-x)D(y_1-x)D(y_2-x)+\text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^4(x)\phi_I^4(y)\}|0\rangle = D(x_1-x)D(x_2-x)D(y_1-y)D(y_2-y)D(x-y)D(x-y)$$

$$+\text{disconnected}$$

Where the "disconnected" terms are $D(x_1-y_1)D(x_2-y_2)D(x-x)D(x-x)$ and similar. When they are integrated over, they do not generate enough $\tilde{D}(p_1)$ propagators to cancel the propagators from the LSZ formula, which will cause the terms to vanish.

For the $\mathcal{L}=\phi^3(x)$ theory, one also needs the following contractions:

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^3(x)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)\phi_I^3(x)\phi_I^3(y)\}|0\rangle = D(x_1-x)D(x_2-x)D(x-y)D(y_1-y)D(y_2-y)$$

Thus it is clear that the only difference from the above is the factor $D(x-y)$ which after integrating changes to $\tilde{D}(p_1+p_2)$ and this ends up in the final result.

One always gets the delta function in the result, so we define the matrix element $\mathcal{M}_{fi}$ by:

$$S_{fi} = (2\pi)^4\delta^{(4)}(p_1+p_2+\cdots-k_1-k_2-\cdots)i\mathcal{M}_{fi}$$

1.2.2 Wick Theorem

As seen above, we need to be able to calculate

$$\langle 0|T\{\phi_I(x_1)\cdots\phi_I(x_n)\}|0\rangle$$

The Wick theorem says, that this is equal to all possible contractions of fields (all fields need to be contracted), where a contraction is defined as:

$$\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle \equiv D(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\tilde{D}(p)e^{-ip(x-y)}$$

with

$$\tilde{D}(p) = \frac{i}{p^2-m^2+i\epsilon}$$

A few lowest possibilities:

$$\langle 0|T\{\phi_I(x_1)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\}|0\rangle = D_{12}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\}|0\rangle = \text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I(x)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^2(x)\}|0\rangle = \text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^3(x)\}|0\rangle = 0$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^4(x)\}|0\rangle = 4!\,D(x_1-x)D(x_2-x)D(x_3-x)D(x_4-x)+\text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^3(x)\phi_I^3(y)\}|0\rangle =$$

$$= D(x_1-x)D(x_2-x)D(x-y)D(x_3-y)D(x_4-y)+\text{disconnected}$$

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\phi_I^4(x)\phi_I^4(y)\}|0\rangle =$$

$$= D(x_1-x)D(x_2-x)D(x-y)D(x-y)D(x_3-y)D(x_4-y)+\text{disconnected}$$

For the last two equations, not all possibilities of the connected graphs are listed (and also the combinatorial factor is omitted).

1.2.3 Fermions and Vector Bosons

For fermions:

$$\langle 0|T\{\psi_I(x)\bar{\psi}_I(y)\}|0\rangle \equiv S(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\tilde{S}(p)e^{-ip(x-y)}$$

with

$$\tilde{S}(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon}$$

For vector bosons:

$$\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle \equiv D_{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\tilde{D}_{\mu\nu}(p)e^{-ip(x-y)}$$

with

$$\tilde{D}_{\mu\nu}(p) = i\frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}}{p^2 - m^2 + i\epsilon}$$

For massless bosons:

$$\tilde{D}_{\mu\nu}(p) = i\frac{-g_{\mu\nu}}{p^2 + i\epsilon}$$

1.2.4 Feynman Rules

We can deduce a set of rules, so that one doesn't have to repeat the whole calculation each time. For a scalar field we derived the rules above, for fermion and vector boson fields it's more difficult.

1.2.5 ZZH interaction

Let's calculate the $\mathcal{L}_{ZZH} = \lambda Z_\mu Z^\mu H$ interaction in the SM, where $\lambda = \frac{g^2}{4\cos\theta_W}$. Consider $H(p) \to Z(k) + Z(l)$:

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \langle kl|iT|p\rangle = \frac{\varepsilon_\alpha(k)\varepsilon^\alpha(l)}{\tilde{D}_{\mu\nu}(k)\tilde{D}^{\mu\nu}(l)}\frac{1}{\tilde{D}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(ky_1+ly_2)}\langle 0|T\{Z_\mu(y_1)Z^\mu(y_2)H(x_1)\}|0\rangle =$$

$$= \frac{\varepsilon_\alpha(k)\varepsilon^\alpha(l)}{\tilde{D}_{\mu\nu}(k)\tilde{D}^{\mu\nu}(l)}\frac{1}{\tilde{D}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1}\int \mathrm{d}^4y_1\mathrm{d}^4y_2 e^{+i(ky_1+ly_2)}\int \mathrm{d}^4x i\lambda D_{\alpha\mu}(y_1-x)D^{\alpha\mu}(y_2-x)D(x_1-x) =$$

$$= i\lambda(2\pi)^4\delta^{(4)}(p-k-l)\varepsilon_\alpha(k)\varepsilon^\alpha(l)$$

where we used the fact, that the only nonzero element of the Green function is

$$\int \mathrm{d}^4x\, \langle 0|T\{Z_\alpha(y_1)Z^\alpha(y_2)H(x_1)Z_\mu(x)Z^\mu(x)H(x)\}|0\rangle$$

1.2.6 eeH interaction

Let's calculate the $\mathcal{L}_{eeH} = -\lambda \bar{e} e H$ interaction in the SM, where $\lambda = \frac{h_e}{\sqrt{2}}$. Consider $H(p) \to e^-(k) + e^+(l)$:

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \langle kl|iT|p\rangle = \frac{\bar{u}(k)v(l)}{\tilde{S}(k)\tilde{S}(l)}\frac{1}{\tilde{D}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1} \int \mathrm{d}^4y_1 \mathrm{d}^4y_2 e^{+i(ky_1+ly_2)} \langle 0|T\bar{e}(y_1)e(y_2)H(x_1)\}|0\rangle =$$

$$= \frac{\bar{u}(k)v(l)}{\tilde{S}(k)\tilde{S}(l)}\frac{1}{\tilde{D}(p)}$$

$$\int \mathrm{d}^4x_1 e^{-ipx_1} \int \mathrm{d}^4y_1 \mathrm{d}^4y_2 e^{+i(ky_1+ly_2)} \int \mathrm{d}^4x(-i\lambda)S(y_1-x)S(y_2-x)D(x_1-x) =$$

$$= (-i\lambda)(2\pi)^4\delta^{(4)}(p-k-l)\bar{u}(k)v(l)$$

where we used the fact, that the only nonzero element of the Green function is

$$\int \mathrm{d}^4x \, \langle 0|T\{\bar{e}(y_1)e(y_2)H(x_1)\bar{e}(x)e(x)H(x)\}|0\rangle$$

1.3 Low energy theories

1.3.1 Fermi-type theory

This is a low energy ($m_W^2 \gg m_\mu m_e$) model for the EW interactions, that can be derived for example from the muon decay:

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$$

From the SM the relevant Lagrangian is

$$\mathcal{L} = \frac{g}{2\sqrt{2}}(\bar{e}\gamma^\mu(1-\gamma_5)\nu_e W_\mu^-) + \frac{g}{2\sqrt{2}}(\bar{\mu}\gamma^\mu(1-\gamma_5)\nu_\mu W_\mu^-)$$

and one gets the diagram $\mu^- + \bar{\nu}_\mu + \to e^- + \bar{\nu}_e$ and the corresponding matrix element:

$$iM = -i\frac{g^2}{8}[\bar{u}\gamma_\mu(1-\gamma_5)u]\frac{-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_W^2}}{q^2 - m_W^2}[\bar{u}\gamma_\nu(1-\gamma_5)v]$$

which when the momentum transfer q is much less than m_w becomes

$$iM = -i\frac{g^2}{8m_W^2}[\bar{u}\gamma^\mu(1-\gamma_5)u][\bar{u}\gamma_\mu(1-\gamma_5)v]$$

but this element can be derived directly from the Lagrangian:

$$\mathcal{L} = -\frac{G_\mu}{\sqrt{2}}[\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma_5)\psi_\mu][\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_{\nu_e}]$$

with

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

This is the universal V-A theory Lagrangian (after adding the h.c. term).

1.3.2 QED

The QED Lagrangian density is

$$\mathcal{L} = \bar{\psi}(ic\gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where

$$\psi = (\psi_1\psi_2\psi_3\psi_4)$$

and

$$D_\mu = \partial_\mu + ieA_\mu$$

is the gauge covariant derivative and (e is the elementary charge, which is 1 in atomic units)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. It's astonishing, that this simple Lagrangian can account for all phenomena from macroscopic scales down to something like 10^{-13} cm. So of course Feynman, Schwinger and Tomonaga received the 1965 Nobel Prize in Physics for such a fantastic achievement.

Plugging this Lagrangian into the Euler-Lagrange equation of motion for a field, we get:

$$(ic\gamma^\mu D_\mu - mc^2)\psi = 0$$

$$\partial_\nu F^{\nu\mu} = -ec\bar{\psi}\gamma^\mu\psi$$

The first equation is the Dirac equation in the electromagnetic field and the second equation is a set of Maxwell equations ($\partial_\nu F^{\nu\mu} = -ej^\mu$) with a source $j^\mu = c\bar{\psi}\gamma^\mu\psi$, which is a 4-current comming from the Dirac equation.

The fields ψ and A^μ are quantized. The first approximation is that we take ψ as a wavefunction, that is, it is a classical 4-component field. It can be shown that this corresponds to taking three orders in the perturbation theory.

The first component A_0 of the 4-potential is the electric potential, and because this is the potential that (as we show in a moment) is in the Schrödinger equation, we denote it by V:

$$A_\mu = \left(\frac{V}{ec}, A_1, A_2, A_3\right)$$

So in the non-relativistic limit, the $\frac{V}{e}$ corresponds to the electric potential. We multiply the Dirac equation by γ^0 from left to get:

$$0 = \gamma^0(ic\gamma^\mu D_\mu - mc^2)\psi = \gamma^0(ic\gamma^0(\partial_0 + i\frac{V}{c}) + ic\gamma^i(\partial_i + ieA_i) - mc^2)\psi =$$

$$= (ic\partial_0 + ic\gamma^0\gamma^i\partial_i - \gamma^0 mc^2 - V - ce\gamma^0\gamma^i A_i)\psi$$

and we make the following substitutions (it's just a formalism, nothing more): $\beta = \gamma^0$, $\alpha^i = \gamma^0\gamma^i$, $p_j = -i\partial_j$, $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$ to get

$$(i\frac{\partial}{\partial t} - c\alpha^i p_i - \beta mc^2 - V - ce\alpha^i A_i)\psi = 0\,.$$

This, in most solid state physics texts, is usually written as

$$i\frac{\partial\psi}{\partial t} = H\psi\,,$$

where the Hamiltonian is given by

$$H = c\alpha^i(p_i + eA_i) + \beta mc^2 + V\,.$$

The right hand side of the Maxwell equations is the 4-current, so it's given by:

$$j^\mu = c\bar\psi\gamma^\mu\psi$$

Now we make the substitution $\psi = e^{-imc^2t}\varphi$, which states, that we separate the largest oscillations of the wavefunction and we get

$$j^0 = c\bar\psi\gamma^0\psi = c\psi^\dagger\psi = c\varphi^\dagger\varphi$$

$$j^i = c\bar\psi\gamma^i\psi = c\psi^\dagger\alpha^i\psi = c\varphi^\dagger\alpha^i\varphi$$

The Dirac equation implies the Klein-Gordon equation:

$$(-ic\gamma^\mu D_\mu - mc^2)(ic\gamma^\nu D_\nu - mc^2)\psi = (c^2\gamma^\mu\gamma^\nu D_\mu D_\mu + m^2c^4)\psi =$$

$$= (c^2 D^\mu D_\mu - ic^2[\gamma^\mu, \gamma^\nu]D_\mu D_\nu + m^2c^4)\psi = 0$$

Note however, the ψ in the true Klein-Gordon equation is just a scalar, but here we get a 4-component spinor. Now:

$$D_\mu D_\nu = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) = \partial_\mu\partial_\nu + ie(A_\mu\partial_\nu + A_\nu\partial_\mu + (\partial_\mu A_\nu)) - e^2A_\mu A_\nu$$

$$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu = ie(\partial_\mu A_\nu) - ie(\partial_\nu A_\mu)$$

We rewrite $D^\mu D_\mu$:

$$D^\mu D_\mu = g^{\mu\nu}D_\mu D_\nu = \partial^\mu\partial_\mu + ie((\partial^\mu A_\mu) + 2A^\mu\partial_\mu) - e^2A^\mu A_\mu =$$

$$= \partial^\mu\partial_\mu + ie((\partial^0A_0) + 2A^0\partial_0 + (\partial^iA_i) + 2A^i\partial_i) - e^2(A^0A_0 + A^iA_i) =$$

$$= \partial^\mu\partial_\mu + i\frac{1}{c^2}\frac{\partial V}{\partial t} + 2i\frac{V}{c^2}\frac{\partial}{\partial t} + ie(\partial^iA_i) + 2ieA^i\partial_i - \frac{V^2}{c^2} - e^2A^iA_i$$

We use the identity $\frac{\partial}{\partial t}\left(e^{-imc^2t}f(t)\right) = e^{-imc^2t}(-imc^2 + \frac{\partial}{\partial t})f(t)$ to get:

$$L = c^2\partial^\mu\psi^*\partial_\mu\psi - m^2c^4\psi^*\psi = \frac{\partial}{\partial t}\psi^*\frac{\partial}{\partial t}\psi - c^2\partial^i\psi^*\partial_i\psi - m^2c^4\psi^*\psi =$$

$$= (imc^2 + \frac{\partial}{\partial t})\varphi^*(-imc^2 + \frac{\partial}{\partial t})\varphi - c^2\partial^i\varphi^*\partial_i\varphi - m^2c^4\varphi^*\varphi =$$

$$= 2mc^2\left[\frac{1}{2}i(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}) - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi + \frac{1}{2mc^2}\frac{\partial\varphi^*}{\partial t}\frac{\partial\varphi}{\partial t}\right]$$

The constant factor $2mc^2$ in front of the Lagrangian is of course irrelevant, so we drop it and then we take the limit $c \to \infty$ (neglecting the last term) and we get

$$L = \frac{1}{2}i(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}) - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi$$

After integration by parts we arrive at

$$L = i\varphi^*\frac{\partial\varphi}{\partial t} - \frac{1}{2m}\partial^i\varphi^*\partial_i\varphi$$

The nonrelativistic limit can also be applied directly to the Klein-Gordon equation:

$$0 = (c^2 D^\mu D_\mu + m^2c^4)\psi =$$

$$= \left(c^2\partial^\mu\partial_\mu + i\frac{\partial V}{\partial t} + 2iV\frac{\partial}{\partial t} + iec^2(\partial^i A_i) + 2iec^2A^i\partial_i - V^2 - e^2c^2A^iA_i + m^2c^4\right)e^{-imc^2t}\varphi =$$

$$= \left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2 + 2iV\frac{\partial}{\partial t} + i\frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2A^i\partial_i - V^2 - e^2c^2A^iA_i + m^2c^4\right)e^{-imc^2t}\varphi =$$

$$= e^{-imc^2t}\left((-imc^2 + \frac{\partial}{\partial t})^2 - c^2\nabla^2 + 2iV(-imc^2 + \frac{\partial}{\partial t}) + i\frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2A^i\partial_i - V^2 + \right.$$
$$\left. -e^2c^2A^iA_i + m^2c^4\right)\varphi =$$

$$= e^{-imc^2t}\left(-2imc^2\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - c^2\nabla^2 + 2Vmc^2 + 2iV\frac{\partial}{\partial t} + i\frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2A^i\partial_i - V^2 + \right.$$
$$\left. -e^2c^2A^iA_i\right)\varphi =$$

$$= -2mc^2e^{-imc^2t}\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - V - \frac{1}{2mc^2}\frac{\partial^2}{\partial t^2} - \frac{i}{2mc^2}\frac{\partial V}{\partial t} + \frac{V^2}{2mc^2} - \frac{iV}{mc^2}\frac{\partial}{\partial t} + \right.$$
$$\left. -\frac{ie}{2m}\partial^i A_i - \frac{ie}{m}A^i\partial_i + \frac{e^2}{2m}A^iA_i\right)\varphi$$

Taking the limit $c \to \infty$ we again recover the Schrödinger equation:

$$i\frac{\partial}{\partial t}\varphi = \left(-\frac{\nabla^2}{2m} + V + \frac{ie}{2m}\partial^i A_i + \frac{ie}{m}A^i\partial_i - \frac{e^2}{2m}A^iA_i\right)\varphi\,,$$

we rewrite the right hand side a little bit:

$$i\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}(\partial^i\partial_i + ie\partial^i A_i + 2ieA^i\partial_i - e^2A^iA_i) + V\right)\varphi\,,$$

$$i\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}(\partial^i + ieA^i)(\partial_i + ieA_i) + V\right)\varphi\,,$$

And we get the usual form of the Schrödinger equation for the vector potential $\mathbf{A} = (A_1, A_2, A_3)$:

$$i\frac{\partial}{\partial t}\varphi = \left(-\frac{(\nabla + ie\mathbf{A})^2}{2m} + V\right)\varphi\,.$$