Quantum Field Theory

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1 Derivation from the Standard Model

1.1 Electroweak Standard Model

Lagrangian with a global $SU(2) \times U(1)$ symmetry:

$$L = i\bar{L}^{(l)}\gamma_\mu \partial^\mu L^{(l)} + i\bar{l}_R\gamma_\mu\partial^\mu l_R + \tfrac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2 - h_e\bar{L}^{(l)}\Phi e_R - \text{h.c.}$$

where $l = e, \mu, \tau$ and $a = 1, 2$, $l_{L,R} = \frac{1}{2}(1 \mp \gamma_5)l$ and

$$L^{(l)} = \begin{pmatrix} \nu_{(l)L} \\ l_L \end{pmatrix}$$

Local $SU(2) \times U(1)$ symmetry:

This consists of two things. First changing the partial derivatives to covariant ones:

$$\partial^\mu \to D^\mu = \partial^\mu - \frac{i}{2}g\tau_k A_k^\mu - \frac{i}{2}g'YB^\mu$$

and second adding the kinetic terms

$$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

of the vector gauge particles to the lagrangian.

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\Phi = e^{\frac{i}{v}\pi^a(x)\tau^a} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

This breaks the gauge invariance. The $\partial^\mu \pi^a$ are going to be added to A^a_μ so we can set $\pi_a = 0$ now.

1.1.1 Higgs Terms

$$L_{Higgs} = \tfrac{1}{2}\partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2$$

Plugging in the covariant derivatives and Φ in U-gauge (symmetry breaking):

$$L_{Higgs} = \frac{1}{2}\Phi^+(\overleftarrow{\partial}_\mu + igA^a_\mu\frac{\tau^a}{2} + ig'YB_\mu)(\overrightarrow{\partial}^\mu + igA^{a\mu}\frac{\tau^a}{2} + ig'YB^\mu)\Phi - \lambda(\Phi^+\Phi - \frac{v^2}{2})^2 =$$

$$= \Phi^+_U(\overleftarrow{\partial}_\mu + igA^a_\mu\frac{\tau^a}{2} + ig'YB_\mu)(\overrightarrow{\partial}^\mu + igA^{a\mu}\frac{\tau^a}{2} + ig'YB\mu)\Phi_U - \lambda(\Phi^+_U\Phi_U - \frac{v^2}{2})^2 =$$

$$= \frac{1}{2}\partial_\mu H\partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 +$$

$$+\frac{1}{8}(v+H)^2 \left(2g^2\frac{A^1_\mu + iA^2_\mu}{\sqrt{2}}\frac{A^{1\mu} - iA^{2\mu}}{\sqrt{2}} + (g^2 + 4Y^2g'^2)\frac{gA^3_\mu - 2Yg'B_\mu}{\sqrt{g^2+4Y^2g'^2}}\frac{gA^{3\mu} - 2Yg'B^\mu}{\sqrt{g^2+4Y^2g'^2}}\right) =$$

$$= \frac{1}{2}\partial_\mu H\partial^\mu H - \lambda v^2H^2 - \lambda vH^3 - \frac{1}{4}\lambda H^4 + \frac{1}{8}(v+H)^2\left(2g^2W^-_\mu W^{+\mu} + \frac{g^2}{\cos^2\theta_W}Z_\mu Z^\mu\right) =$$

$$= \frac{1}{2}\partial_\mu H\partial^\mu H - \lambda v^2H^2 + \frac{1}{4}g^2v^2W^-_\mu W^{+\mu} + \frac{g^2v^2}{8\cos^2\theta_W}Z_\mu Z^\mu - \lambda vH^3 - \frac{1}{4}\lambda H^4 +$$

$$+\frac{1}{2}vg^2W^-_\mu W^{+\mu}H + \frac{g^2}{4\cos\theta_W}vZ_\mu Z^\mu H + \frac{1}{4}g^2W^-_\mu W^{+\mu}H^2 + \frac{g^2}{8\cos\theta_W}Z_\mu Z^\mu H^2$$

Where we put

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu)$$

$$Z_\mu = \frac{g}{\sqrt{g^2+4Y^2g'^2}}A^3_\mu - \frac{2Yg'}{\sqrt{g^2+4Y^2g'^2}}B_\mu$$

we defined θ_W by the relation

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + 4Y^2 g'^2}}$$

so that the expressions simplify a bit, e.g. we now get:

$$\sin\theta_W = \frac{2Yg'}{\sqrt{g^2 + 4Y^2 g'^2}}$$

$$Z_\mu = \cos\theta_W A^3_\mu - \sin\theta_W B_\mu$$

$$g^2 + 4Y^2 g'^2 = \frac{g^2}{\cos^2\theta_W}$$

1.1.2 Yukawa terms

$$\begin{aligned} L_{Yukawa} &= -h_e \bar{L}\Phi e_R - \text{h.c.} = -h_e \bar{L}\Phi_U e_R - \text{h.c.} = \\ &= -\frac{1}{\sqrt{2}} h_e (v+H)(\bar{e}_L e_R + \bar{e}_R e_L) = -\frac{1}{\sqrt{2}} h_e (v+H)\bar{e}e = \\ &= -\frac{1}{\sqrt{2}} h_e v \bar{e}e - \frac{1}{\sqrt{2}} h_e \bar{e}eH \end{aligned}$$

The term $\bar{L}\Phi e_R$ is $U(1)$ (hypercharge) invariant, so

$$-Y_L + Y + Y_R = 0$$

1.1.3 Leptonic Terms

$$L = i\bar{L}\gamma^\mu \partial_\mu L + i\bar{e}_R \gamma^\mu \partial_\mu e_R \rightarrow$$

$$\rightarrow i\bar{L}\gamma^\mu(\partial_\mu - igA^a_\mu \frac{\tau^a}{2} - ig' Y_L B_\mu)L + i\bar{e}_R\gamma^\mu(\partial_\mu - ig'Y_R B_\mu)e_R =$$

$$= i\bar{L}\gamma^\mu\partial_\mu L + i\bar{e}_R\gamma^\mu\partial_\mu e_R + g\bar{L}\gamma^\mu \frac{\tau^a}{2} L A^a_\mu + g'Y_L \bar{L}\gamma^\mu L B_\mu + g' Y_R \bar{e}_R \gamma^\mu e_R B_\mu =$$

$$= i\bar{L}\gamma^\mu\partial_\mu L + i\bar{e}_R\gamma^\mu\partial_\mu e_R + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W^+_\mu + \text{h.c.}) + \frac{1}{2} g\bar{L}\gamma^\mu\tau^3 L A^3_\mu + g'Y_L\bar{L}\gamma^\mu L B_\mu + g'Y_R\bar{e}_R\gamma^\mu e_R B_\mu =$$

$$= i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W^+_\mu + \text{h.c.}) + \frac{1}{2}g\bar{\nu}_L\gamma^\mu\nu_L A^3_\mu - \frac{1}{2}g\bar{e}_L\gamma^\mu e_L A^3_\mu$$

$$+g'Y_L\bar{\nu}_L\gamma^\mu\nu_L B_\mu + g'Y_L\bar{e}_L\gamma^\mu e_L B_\mu + g'Y_R\bar{e}_R\gamma^\mu e_R B_\mu =$$

$$= i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L W^+_\mu + \text{h.c.})$$

$$+ \left[(\tfrac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W)\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W)\bar{e}_L\gamma^\mu e_L + Y_R g'\cos\theta_W \bar{e}_R\gamma^\mu e_R\right] A_\mu$$

$$+ \left[(\tfrac{1}{2}g\cos\theta_W - Y_L g'\sin\theta_W)\bar{\nu}_L\gamma^\mu\nu_L + (-\tfrac{1}{2}g\cos\theta_W - Y_L g'\sin\theta_W)\bar{e}_L\gamma^\mu e_L - 2Y_L g'\sin\theta_W \bar{e}_R\gamma^\mu e_R\right] Z_\mu$$

Where we substituted new fields Z_μ and A_μ for the old ones A^3_μ and B_μ using the relation:

$$Z_\mu = \cos\theta_W A^3_\mu - \sin\theta_W B_\mu$$

$$A_\mu = \sin\theta_W A^3_\mu + \cos\theta_W B_\mu$$

The angle θ_W must be the same as in the Higgs sector, so that the field Z_μ is the same. We now need to make the following requirement in order to proceed further:

$$Y = -Y_L$$

This follows for example by requiring that neutrinos have zero charge, i.e. setting $\frac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W = 0$ and substituting for θ_W from the definition (see the Higgs terms), from which one gets $Y = -Y_L$. From $-Y_L + Y + Y_R = 0$ we now get

$$Y_R = 2Y_L$$

it now follows:

$$\frac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W = 0$$

$$-\frac{1}{2}g\sin\theta_W + Y_L g'\cos\theta_W = -g\sin\theta_W$$

$$Y_R g'\cos\theta_W = -g\sin\theta_W$$

$$\tan\theta_W = -2Y_L\frac{g'}{g}$$

and the Lagrangian can be further simplified:

$$L = i\bar\nu_L\gamma^\mu\partial_\mu\nu_L + i\bar e\gamma^\mu\partial_\mu e + \frac{g}{\sqrt{2}}(\bar\nu_L\gamma^\mu e_L W^+_\mu + \text{h.c.})$$

$$-g\sin\theta_W(\bar e_L\gamma^\mu e_L + \bar e_R\gamma^\mu e_R)A_\mu$$

$$+\frac{g}{\cos\theta_W}\left[\tfrac{1}{2}\bar\nu_L\gamma^\mu\nu_L + (-\tfrac{1}{2} + \sin^2\theta_W)\bar e_L\gamma^\mu e_L + \sin^2\theta_W\bar e_R\gamma^\mu e_R\right]Z_\mu =$$

$$= i\bar\nu_L\gamma^\mu\partial_\mu\nu_L + i\bar e\gamma^\mu\partial_\mu e + \frac{g}{2\sqrt{2}}(\bar\nu\gamma^\mu(1-\gamma_5)eW^+_\mu + \text{h.c.}) - g\sin\theta_W\bar e\gamma^\mu e A_\mu$$

$$+\frac{g}{2\cos\theta_W}\left[\bar\nu\gamma^\mu(1-\gamma_5)\nu + \bar e\gamma^\mu(-\tfrac{1}{2} + 2\sin^2\theta_W + \tfrac{1}{2}\gamma_5)e\right]Z_\mu$$

Where we used the relations $\bar\nu_L\gamma^\mu e_L = \frac{1}{2}\bar\nu\gamma^\mu(1-\gamma_5)e$ and $\bar\nu_R\gamma^\mu e_R = \frac{1}{2}\bar\nu\gamma^\mu(1+\gamma_5)e$.

1.1.4 Gauge terms

$$L_{Gauge} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} =$$

$$= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g\epsilon^{ajk}A^{j\mu}A^{k\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} =$$

$$= -\frac{1}{4}\partial_\mu A^a_\nu \partial^\mu A^{a\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)g\epsilon^{abc}A^{b\mu}A^{c\nu} - \frac{1}{4}g^2\epsilon^{abc}\epsilon^{ajk}A^b_\mu A^c_\nu A^{k\mu}A^{l\nu} =$$

$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - g[(\partial_\mu A^1_\nu - \partial_\nu A^1_\mu)A^{2\mu}A^{3\nu} + \text{cycl. perm. } (123)]$$

$$-\frac{1}{4}g^2[(A^a_\mu A^{a\mu})(A^b_\nu A^{b\nu}) - (A^a_\mu A^a_\nu)(A^{b\mu}A^{b\nu})] =$$

$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - g[A^1_\mu A^2_\nu \overleftrightarrow{\partial}^\mu A^{3\nu} + \text{cycl. perm. } (123)]$$

$$-\frac{1}{4}g^2[(A^a_\mu A^{a\mu})(A^b_\nu A^{b\nu}) - (A^a_\mu A^a_\nu)(A^{b\mu}A^{b\nu})] =$$

$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - ig(W^0_\mu W^-_\nu \overleftrightarrow{\partial}^\mu W^{+\nu} + \text{cycl. perm. } (0\text{-}+))$$

$$-g^2[\tfrac{1}{2}(W^+_\mu W^{-\mu})^2 - \tfrac{1}{2}(W^+_\mu W^{+\mu})(W^-_\nu W^{-\nu}) + (W^0_\mu W^{0\mu})(W^+_\nu W^{-\nu}) - (W^-_\mu W^+_\nu)(W^{0\mu}W^{0\nu}) =$$

$$= -\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + [(5.79)\text{ - }(5.84)]$$

1.1.5 GWS Lagrangian

Plugging everything together we get the GWS Lagrangian:

$$L = \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W^-_\mu W^{+\mu} + \frac{g^2 v^2}{8\cos^2\theta_W}Z_\mu Z^\mu - \lambda v H^3 - \frac{1}{4}\lambda H^4 +$$

$$+\frac{1}{2}vg^2 W^-_\mu W^{+\mu}H + \frac{g^2}{4\cos\theta_W}vZ_\mu Z^\mu H + \frac{1}{4}g^2 W^-_\mu W^{+\mu}H^2 + \frac{g^2}{8\cos\theta_W}Z_\mu Z^\mu H^2$$

$$-\frac{1}{\sqrt{2}}h_e v\bar{e}e - \frac{1}{\sqrt{2}}h_e\bar{e}eH$$

$$-\frac{1}{2}W^-_{\mu\nu}W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + [(5.79)\text{ - }(5.84)]$$

$$+i\bar{\nu}_L\gamma^\mu\partial_\mu\nu_L + i\bar{e}\gamma^\mu\partial_\mu e + \frac{g}{2\sqrt{2}}(\bar{\nu}\gamma^\mu(1-\gamma_5)eW^+_\mu + \text{h.c.}) - g\sin\theta_W\bar{e}\gamma^\mu eA_\mu$$

$$+\frac{g}{2\cos\theta_W}\left[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu + \bar{e}\gamma^\mu(-\tfrac{1}{2} + 2\sin^2\theta_W + \tfrac{1}{2}\gamma_5)e\right]Z_\mu$$

$$+(e, \nu_e, h_e \leftrightarrow \mu, \nu_\mu, h_\mu) + (e, \nu_e, h_e \leftrightarrow \tau, \nu_\tau, h_\tau)$$

The free parameters are g, θ_W, v, λ, h_e, h_μ, h_τ.

1.1.6 Particle Masses

The particle masses are deduced from the terms

$$L = -\frac{1}{2}m_H^2 H^2 + m_W^2 W_\mu^- W^{+\mu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu - m_e \bar{e}e + \cdots$$

comparing to the above:

$$L = -\lambda v^2 H^2 + \frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} + \frac{g^2 v^2}{8\cos^2\theta_W} Z_\mu Z^\mu - \frac{1}{\sqrt{2}} h_e v \bar{e}e + \cdots$$

we get

$$\begin{aligned} m_W &= \tfrac{1}{2}gv \\ m_Z &= \frac{gv}{2\cos\theta_W} = \frac{m_W}{\cos\theta} \\ m_H &= v\sqrt{2\lambda} \\ m_e &= \frac{1}{\sqrt{2}}h_e v \end{aligned}$$

1.1.7 Quarks

$$\begin{aligned} L_{fermion} {+}{=} &\sum_{q=d,s,b} i\bar{L}_0^{(q)}\gamma^\mu\partial_\mu L_0^{(q)} + \sum_{q=d,u,s,c,b,t} i\bar{q}_{0R}\gamma^\mu\partial_\mu q_{0R} \\ L_{Yukawa} {+}{=} &-\sum_{\substack{q=d,s,b \\ q'=d,s,b}} h_{qq'} i\bar{L}_0^{(q)}\Phi q'_{0R} + \text{h.c.} - \sum_{\substack{q=d,s,b \\ q'=u,c,t}} \tilde{h}_{qq'} i\bar{L}_0^{(q)}\tilde{\Phi} q'_{0R} + \text{h.c.} \end{aligned}$$

More to be added here...

1.2 Low energy theories

1.2.1 Fermi-type theory

This is a low energy ($m_W^2 \gg m_\mu m_e$) model for the EW interactions, that can be derived for example from the muon decay:

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$$

From the SM the relevant Lagrangian is

$$L = \frac{g}{2\sqrt{2}}(\bar{e}\gamma^\mu(1-\gamma_5)\nu_e W_\mu^-) + \frac{g}{2\sqrt{2}}(\bar{\mu}\gamma^\mu(1-\gamma_5)\nu_\mu W_\mu^-)$$

and one gets the diagram $\mu^- + \bar{\nu}_\mu + \to e^- + \bar{\nu}_e$ and the corresponding matrix element:

$$iM = -i\frac{g^2}{8}[\bar{u}\gamma_\mu(1-\gamma_5)u]\frac{-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_W^2}}{q^2 - m_W^2}[\bar{u}\gamma_\nu(1-\gamma_5)v]$$

which when the momentum transfer q is much less than m_w becomes

$$iM = -i\frac{g^2}{8m_W^2}[\bar{u}\gamma^\mu(1-\gamma_5)u][\bar{u}\gamma_\mu(1-\gamma_5)v]$$

but this element can be derived directly from the Lagrangian:

$$L = -\frac{G_\mu}{\sqrt{2}}[\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma_5)\psi_\mu][\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_{\nu_e}]$$

with

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

This is the universal V-A theory Lagrangian (after adding the h.c. term).

1.2.2 QED

The QED Lagrangian density is

$$L = \bar{\psi}(ic\gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where

$$\psi = (\psi_1\psi_2\psi_3\psi_4)$$

and

$$D_\mu = \partial_\mu + ieA_\mu$$

is the gauge covariant derivative and (e is the elementary charge, which is 1 in atomic units)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. It's astonishing, that this simple Lagrangian can account for all phenomena from macroscopic scales down to something like 10^{-13} cm. So of course Feynman, Schwinger and Tomonaga received the 1965 Nobel Prize in Physics for such a fantastic achievement.

Plugging this Lagrangian into the Euler-Lagrange equation of motion for a field, we get:

$$(ic\gamma^\mu D_\mu - mc^2)\psi = 0$$

$$\partial_\nu F^{\nu\mu} = -ec\bar{\psi}\gamma^\mu\psi$$

The first equation is the Dirac equation in the electromagnetic field and the second equation is a set of Maxwell equations ($\partial_\nu F^{\nu\mu} = -ej^\mu$) with a source $j^\mu = c\bar{\psi}\gamma^\mu\psi$, which is a 4-current comming from the Dirac equation.

The fields ψ and A^μ are quantized. The first approximation is that we take ψ as a wavefunction, that is, it is a classical 4-component field. It can be shown that this corresponds to taking three orders in the perturbation theory.

The first component A_0 of the 4-potential is the electric potential, and because this is the potential that (as we show in a moment) is in the Schrödinger equation, we denote it by V:

$$A_\mu = \left(\frac{V}{ec}, A_1, A_2, A_3\right)$$

So in the non-relativistic limit, the $\frac{V}{e}$ corresponds to the electric potential. We multiply the Dirac equation by γ^0 from left to get:

$$0 = \gamma^0(ic\gamma^\mu D_\mu - mc^2)\psi = \gamma^0(ic\gamma^0(\partial_0 + i\frac{V}{c}) + ic\gamma^i(\partial_i + ieA_i) - mc^2)\psi =$$

$$= (ic\partial_0 + ic\gamma^0\gamma^i\partial_i - \gamma^0 mc^2 - V - ce\gamma^0\gamma^i A_i)\psi$$

and we make the following substitutions (it's just a formalism, nothing more): $\beta = \gamma^0$, $\alpha^i = \gamma^0\gamma^i$, $p_j = -i\partial_j$, $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$ to get

$$(i\frac{\partial}{\partial t} - c\alpha^i p_i - \beta mc^2 - V - ce\alpha^i A_i)\psi = 0\,.$$

This, in most solid state physics texts, is usually written as

$$i\frac{\partial\psi}{\partial t} = H\psi\,,$$

where the Hamiltonian is given by

$$H = c\alpha^i(p_i + eA_i) + \beta mc^2 + V\,.$$

The right hand side of the Maxwell equations is the 4-current, so it's given by:

$$j^\mu = c\bar\psi\gamma^\mu\psi$$

Now we make the substitution $\psi = e^{-imc^2t}\varphi$, which states, that we separate the largest oscillations of the wavefunction and we get

$$j^0 = c\bar\psi\gamma^0\psi = c\psi^\dagger\psi = c\varphi^\dagger\varphi$$

$$j^i = c\bar\psi\gamma^i\psi = c\psi^\dagger\alpha^i\psi = c\varphi^\dagger\alpha^i\varphi$$

The Dirac equation implies the Klein-Gordon equation:

$$(-ic\gamma^\mu D_\mu - mc^2)(ic\gamma^\nu D_\nu - mc^2)\psi = (c^2\gamma^\mu\gamma^\nu D_\mu D_\mu + m^2c^4)\psi =$$

$$= (c^2 D^\mu D_\mu - ic^2[\gamma^\mu, \gamma^\nu]D_\mu D_\nu + m^2c^4)\psi = 0$$

Note however, the ψ in the true Klein-Gordon equation is just a scalar, but here we get a 4-component spinor. Now:

$$D_\mu D_\nu = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) = \partial_\mu\partial_\nu + ie(A_\mu\partial_\nu + A_\nu\partial_\mu + (\partial_\mu A_\nu)) - e^2 A_\mu A_\nu$$

$$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu = ie(\partial_\mu A_\nu) - ie(\partial_\nu A_\mu)$$

We rewrite $D^\mu D_\mu$:

$$D^\mu D_\mu = g^{\mu\nu} D_\mu D_\nu = \partial^\mu \partial_\mu + ie((\partial^\mu A_\mu) + 2A^\mu \partial_\mu) - e^2 A^\mu A_\mu =$$

$$= \partial^\mu \partial_\mu + ie((\partial^0 A_0) + 2A^0 \partial_0 + (\partial^i A_i) + 2A^i \partial_i) - e^2(A^0 A_0 + A^i A_i) =$$

$$= \partial^\mu \partial_\mu + i\frac{1}{c^2}\frac{\partial V}{\partial t} + 2i\frac{V}{c^2}\frac{\partial}{\partial t} + ie(\partial^i A_i) + 2ieA^i\partial_i - \frac{V^2}{c^2} - e^2 A^i A_i$$

We use the identity $\frac{\partial}{\partial t}\left(e^{-imc^2 t} f(t)\right) = e^{-imc^2 t}(-imc^2 + \frac{\partial}{\partial t})f(t)$ to get:

$$L = c^2 \partial^\mu \psi^* \partial_\mu \psi - m^2 c^4 \psi^* \psi = \frac{\partial}{\partial t}\psi^* \frac{\partial}{\partial t}\psi - c^2 \partial^i \psi^* \partial_i \psi - m^2 c^4 \psi^* \psi =$$

$$= (imc^2 + \frac{\partial}{\partial t})\varphi^*(-imc^2 + \frac{\partial}{\partial t})\varphi - c^2 \partial^i \varphi^* \partial_i \varphi - m^2 c^4 \varphi^* \varphi =$$

$$= 2mc^2 \left[\frac{1}{2} i(\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) - \frac{1}{2m}\partial^i \varphi^* \partial_i \varphi + \frac{1}{2mc^2}\frac{\partial \varphi^*}{\partial t}\frac{\partial \varphi}{\partial t}\right]$$

The constant factor $2mc^2$ in front of the Lagrangian is of course irrelevant, so we drop it and then we take the limit $c \to \infty$ (neglecting the last term) and we get

$$L = \frac{1}{2} i(\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) - \frac{1}{2m}\partial^i \varphi^* \partial_i \varphi$$

After integration by parts we arrive at

$$L = i\varphi^* \frac{\partial \varphi}{\partial t} - \frac{1}{2m}\partial^i \varphi^* \partial_i \varphi$$

The nonrelativistic limit can also be applied directly to the Klein-Gordon equation:

$$0 = (c^2 D^\mu D_\mu + m^2 c^4)\psi =$$

$$= \left(c^2 \partial^\mu \partial_\mu + i\frac{\partial V}{\partial t} + 2iV\frac{\partial}{\partial t} + iec^2(\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4\right) e^{-imc^2 t}\varphi =$$

$$= \left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2iV\frac{\partial}{\partial t} + i\frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 - e^2 c^2 A^i A_i + m^2 c^4\right) e^{-imc^2 t}\varphi =$$

$$= e^{-imc^2 t}\Big((-imc^2 + \frac{\partial}{\partial t})^2 - c^2 \nabla^2 + 2iV(-imc^2 + \frac{\partial}{\partial t}) + i\frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 +$$

$$- e^2 c^2 A^i A_i + m^2 c^4\Big)\varphi =$$

$$= e^{-imc^2 t}\Big(-2imc^2 \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 + 2Vmc^2 + 2iV\frac{\partial}{\partial t} + i\frac{\partial V}{\partial t} + iec^2(\partial^i A_i) + 2iec^2 A^i \partial_i - V^2 +$$

$$- e^2 c^2 A^i A_i\Big)\varphi =$$

$$= -2mc^2 e^{-imc^2 t}\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - V - \frac{1}{2mc^2}\frac{\partial^2}{\partial t^2} - \frac{i}{2mc^2}\frac{\partial V}{\partial t} + \frac{V^2}{2mc^2} - \frac{iV}{mc^2}\frac{\partial}{\partial t} + \right.$$

$$\left. - \frac{ie}{2m}\partial^i A_i - \frac{ie}{m}A^i\partial_i + \frac{e^2}{2m}A^i A_i\right)\varphi$$

Taking the limit $c \to \infty$ we again recover the Schrödinger equation:

$$i\frac{\partial}{\partial t}\varphi = \left(-\frac{\nabla^2}{2m} + V + \frac{ie}{2m}\partial^i A_i + \frac{ie}{m}A^i\partial_i - \frac{e^2}{2m}A^i A_i\right)\varphi\,,$$

we rewrite the right hand side a little bit:

$$i\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}(\partial^i\partial_i + ie\partial^i A_i + 2ieA^i\partial_i - e^2 A^i A_i) + V\right)\varphi\,,$$

$$i\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}(\partial^i + ieA^i)(\partial_i + ieA_i) + V\right)\varphi\,,$$

And we get the usual form of the Schrödinger equation for the vector potential $\mathbf{A} = (A_1, A_2, A_3)$:

$$i\frac{\partial}{\partial t}\varphi = \left(-\frac{(\nabla + ie\mathbf{A})^2}{2m} + V\right)\varphi\,.$$