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THE MATRIX EXPONENTIAL*

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Abstract. There are many different methods to calculate the exponential of a matrix: series methods, differential equations methods, polynomial methods, matrix decomposition methods, and splitting methods, none of which is entirely satisfactory from either a theoretical or a computational point of view. How then should the matrix exponential be introduced in an elementary differential equations course, for engineering students for example, with a minimum of mathematical prerequisites? In this note, a method is given that uses the students' knowledge of homogeneous linear differential equations with constant coefficients and the Cayley–Hamilton theorem. The method is not new; what is new is the approach.

Key words. matrix exponential, systems of differential equations

AMS subject classification. 34A30

1. Introduction. Just as the scalar exponential function e^{at} can be represented by the power series

$$e^{at} = 1 + at + \frac{1}{2!}a^2t^2 + \cdots + \frac{1}{k!}a^kt^k + \cdots$$

given a constant $n \times n$ matrix A, the corresponding series

$$I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots$$

can be shown to converge entrywise to an $n \times n$ matrix, the *matrix exponential function*, denoted by e^{At}.

It is easily seen that

$$e^0 = e^{A\,0} = I$$

and

$$\frac{d}{dt}\, e^{At} = A\, e^{At}.$$

Therefore, using the matrix exponential function, the solution to the system of homogeneous linear first-order differential equations with constant coefficients

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x}, \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

can be written as $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0$.

Most of the recent differential equations textbooks use this method to solve the initial value problem stated above. They also, however, insist on considering the cases where the matrix A has distinct or repeated eigenvalues separately, and then, in the second case, they go on to consider the subcases where A is or is not diagonalizable. In doing this, they must introduce the eigenvalue problem, generalized eigenvectors, and the Jordan canonical form.

It is not necessary to introduce all of these new concepts. At the time when systems of differential equations are introduced, the students have already learned how to solve the *scalar* nth order homogeneous linear differential equation with constant coefficients

$$x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1x' + c_0x = 0.$$

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In fact, using reduction of order and a simple induction argument, they can obtain a rigorous proof of the existence and uniqueness of solutions to the initial value problem for the scalar equation.

The students can then apply their knowledge of the form of the solution to the scalar equation to find e^{At} as soon as they find the eigenvalues of the matrix A. The only prerequisite from linear algebra, aside from being able to find the eigenvalues of A, is the Cayley–Hamilton theorem, which can be covered at the discretion of the instructor. This method is outlined in Problems I.9 (#5^s) on p. 65 in [2], where it is noted that $p(\lambda)$ can be any polynomial for which $p(A) = 0$, not just the characteristic polynomial of A.

2. Preliminaries. Before proceeding with the examples, we need the following two results. The first theorem guarantees the existence of a unique solution to an initial value problem for a matrix differential equation, while the second theorem gives a method for constructing the matrix exponential from the solutions to certain *scalar* initial value problems.

THEOREM 1. *Let A be a constant $n \times n$ matrix with characteristic polynomial*

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0;$$

then $\Phi(t) = e^{At}$ is the unique solution to the nth order matrix differential equation

$$(*) \qquad \Phi^{(n)} + c_{n-1}\Phi^{(n-1)} + \cdots + c_1\Phi' + c_0\Phi = 0,$$

satisfying the initial conditions

$$(**) \qquad \Phi(0) = I,\ \Phi'(0) = A,\ \Phi''(0) = A^2, \ldots, \Phi^{(n-1)}(0) = A^{n-1}.$$

Proof. Let Φ_1 and Φ_2 be solutions to the nth order matrix differential equation $(*)$ and the initial conditions $(**)$, and let $\Phi = \Phi_1 - \Phi_2$; then Φ satisfies $(*)$ with initial conditions $\Phi(0) = \Phi'(0) = \cdots = \Phi^{(n-1)}(0) = 0$. Therefore, each entry of Φ satisfies the *scalar* initial value problem

$$x^{(n)} + c_{n-1}\,x^{(n-1)} + \cdots + c_1\,x' + c_0\,x = 0,$$
$$x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0$$

with solution $x(t) \equiv 0$, and so $\Phi(t) = 0$ for all $t \in \mathbb{R}$, that is, $\Phi_1 = \Phi_2$.

Now let A be a constant $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0.$$

If $\Phi(t) = e^{At}$, then

$$\Phi'(t) = A\,e^{At},\ \Phi''(t) = A^2\,e^{At}, \ldots, \Phi^{(n)}(t) = A^n\,e^{At},$$

so that

$$\begin{aligned}\Phi^{(n)}(t) + c_{n-1}\Phi^{(n-1)}(t) &+ \cdots + c_1\Phi'(t) + c_0\Phi(t)\\ &= \left(A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I\right)e^{At}\\ &= p(A)\,e^{At} = 0\end{aligned}$$

by the Cayley–Hamilton theorem. Also, $\Phi(0) = I$, $\Phi'(0) = A, \ldots, \Phi^{(n-1)}(0) = A^{n-1}$, and therefore $\Phi(t) = e^{At}$ is the unique solution to the initial value problem

$$\Phi^{(n)} + c_{n-1}\Phi^{(n-1)} + \cdots + c_1\Phi' + c_0\Phi = 0,$$
$$\Phi(0) = I, \Phi'(0) = A, \Phi''(0) = A^2, \ldots, \Phi^{(n-1)}(0) = A^{n-1}.$$

This completes the proof. $\square$

THEOREM 2. *Let A be a constant $n \times n$ matrix with characteristic polynomial*

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0;$$

then

$$e^{At} = x_1(t)\, I + x_2(t)\, A + x_3(t)\, A^2 + \cdots + x_n(t)\, A^{n-1},$$

where the $x_k(t)$, $1 \le k \le n$, are the solutions to the nth order scalar differential equation

$$x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1 x' + c_0 x = 0,$$

satisfying the following initial conditions:

$$\left.\begin{array}{r} x_1(0) = 1 \\ x_1'(0) = 0 \\ \vdots \\ x_1^{(n-1)}(0) = 0 \end{array}\right\} \qquad \left.\begin{array}{r} x_2(0) = 0 \\ x_2'(0) = 1 \\ \vdots \\ x_2^{(n-1)}(0) = 0 \end{array}\right\} \qquad \cdots \qquad \left.\begin{array}{r} x_n(0) = 0 \\ x_n'(0) = 0 \\ \vdots \\ x_n^{(n-1)}(0) = 1 \end{array}\right\}.$$

Proof. Let A be a constant $n \times n$ matrix, and let

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

be its characteristic polynomial.

Define

$$\Phi(t) = x_1(t)\, I + x_2(t)\, A + x_3(t)\, A^2 + \cdots + x_n(t)\, A^{n-1},$$

where the $x_k(t)$, $1 \le k \le n$, are the unique solutions to the nth order scalar differential equation

$$x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1 x' + c_0 x = 0,$$

satisfying the initial conditions stated in the theorem. Then

$$\begin{aligned} \Phi^{(n)} + c_{n-1}\Phi^{(n-1)} &+ \cdots + c_1\Phi' + c_0\Phi \\ &= \left(x_1^{(n)} + c_{n-1}x_1^{(n-1)} + \cdots + c_1 x_1' + c_0 x_1\right) I \\ &+ \left(x_2^{(n)} + c_{n-1}x_2^{(n-1)} + \cdots + c_1 x_2' + c_0 x_2\right) A \\ &\qquad \vdots \\ &+ \left(x_n^{(n)} + c_{n-1}x_n^{(n-1)} + \cdots + c_1 x_n' + c_0 x_n\right) A^{n-1} \\ &= 0\, I + 0\, A + \cdots + 0\, A^{n-1} = 0. \end{aligned}$$

Thus,

$$\Phi^{(n)}(t) + c_{n-1}\, \Phi^{(n-1)}(t) + \cdots + c_1\, \Phi'(t) + c_0\, \Phi(t) = 0$$

for all $t \in \mathbb{R}$, and

$$\begin{aligned} \Phi(0) &= x_1(0)I + x_2(0)A + \cdots + x_n(0)A^{n-1} = I \\ \Phi'(0) &= x_1'(0)I + x_2'(0)A + \cdots + x_n'(0)A^{n-1} = A \\ &\vdots \\ \Phi^{(n-1)}(0) &= x_1^{(n-1)}(0)I + x_2^{(n-1)}(0)A + \cdots + x_n^{(n-1)}(0)A^{n-1} = A^{n-1}. \end{aligned}$$

Therefore

$$\Phi(t) = x_1(t)\, I + x_2(t)\, A + \cdots + x_n(t)\, A^{n-1}$$

satisfies the initial value problem

$$\Phi^{(n)} + c_{n-1}\Phi^{(n-1)} + \cdots + c_1\Phi' + c_0\Phi = 0,$$
$$\Phi(0) = I,\ \Phi'(0) = A,\ \Phi''(0) = A^2, \ldots, \Phi^{(n-1)}(0) = A^{n-1}.$$

The uniqueness of the solution then gives

$$e^{At} = x_1(t)\, I + x_2(t)\, A + \cdots + x_n(t)\, A^{n-1}$$

for all $t \in \mathbb{R}$. $\square$

3. Examples. As noted earlier, we assume that it is known that if λ is a root of the characteristic polynomial of A, with multiplicity m, then its contribution to the general solution of the scalar nth order equation is of the form

$$\left(a_0 + a_1\, t + a_2\, t^2 + \cdots + a_{m-1}\, t^{m-1}\right) e^{\lambda t}.$$

Using this fact we give two examples. In each example the matrix A has a repeated eigenvalue. In the first example, A is diagonalizable, while in the second example, A is not diagonalizable.

The only work required of the student in each of the examples involves solving initial value problems for a scalar nth order homogeneous linear differential equation with constant coefficients. The point to be noted here is that it is no more difficult to solve the second example than it is to solve the first, even though in the second example the matrix A does not have a complete set of eigenvectors. Thus, there is no need to consider these cases separately.

Example 1. Given the system of equations

$$\frac{dx}{dt} = 2x + 0y + 1z,$$
$$\frac{dy}{dt} = 0x + 2y + 0z,$$
$$\frac{dz}{dt} = 0x + 0y + 3z,$$

we can use the previous theorem to calculate e^{At}, where the coefficient matrix A is given by

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

as follows. The eigenvalues of A are $\lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 3$, so that 2 is an eigenvalue of multiplicity 2.

The general solution to $x''' + c_2x'' + c_1x' + c_0x = 0$ is given by

$$x(t) = a_1te^{2t} + a_2e^{2t} + a_3e^{3t},$$

and from this we find the following:

(i) the solution satisfying the initial conditions $x(0) = 1,\ x'(0) = 0,\ x''(0) = 0$ is

$$x_1(t) = -6te^{2t} - 3e^{2t} + 4e^{3t},$$

(ii) while the solution satisfying the initial conditions $x(0) = 0,\ x'(0) = 1,\ x''(0) = 0$ is

$$x_2(t) = 5te^{2t} + 4e^{2t} - 4e^{3t};$$

(iii) finally, the solution satisfying the initial conditions $x(0) = 0,\ x'(0) = 0,\ x''(0) = 1$ is

$$x_3(t) = -te^{2t} - e^{2t} + e^{3t}.$$

Also,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

and

$$A^2 = \begin{bmatrix} 4 & 0 & 5 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix};$$

therefore

$$e^{At} = (-6te^{2t} - 3e^{2t} + 4e^{3t}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (5te^{2t} + 4e^{2t} - 4e^{3t}) \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$+(-te^{2t} - e^{2t} + e^{3t}) \begin{bmatrix} 4 & 0 & 5 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

and thus

$$e^{At} = \begin{bmatrix} e^{2t} & 0 & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

Example 2. In this example we consider the system of equations

$$\frac{dx}{dt} = \lambda x + 1y + 0z,$$
$$\frac{dy}{dt} = 0x + \lambda y + 1z,$$
$$\frac{dz}{dt} = 0x + 0y + \lambda z.$$

The coefficient matrix is now

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

and the eigenvalues of A are $\lambda_1 = \lambda$, $\lambda_2 = \lambda$, $\lambda_3 = \lambda$, so that λ is an eigenvalue of multiplicity 3.

The general solution to $x''' + c_2 x'' + c_1 x' + c_0 x = 0$ is given by

$$x(t) = a_1 e^{\lambda t} + a_2 t e^{\lambda t} + a_3 t^2 e^{\lambda t},$$

and therefore

(i) the solution satisfying the initial conditions $x(0) = 1,\ x'(0) = 0,\ x''(0) = 0$ is

$$x_1(t) = \left(1 - \lambda t + \frac{\lambda^2 t^2}{2}\right) e^{\lambda t},$$

(ii) while the solution satisfying the initial conditions $x(0) = 0,\ x'(0) = 1,\ x''(0) = 0$ is

$$x_2(t) = (t - \lambda t^2) e^{\lambda t};$$

(iii) finally, the solution satisfying the initial conditions $x(0) = 0,\ x'(0) = 0,\ x''(0) = 1$ is

$$x_3(t) = \frac{t^2}{2} e^{\lambda t}.$$

Also,

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix},$$

and

$$A^2 = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix};$$

therefore

$$e^{At} = \left(1 - \lambda t + \frac{\lambda^2 t^2}{2}\right) e^{\lambda t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (t - \lambda t^2) e^{\lambda t} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$
$$+ \frac{t^2}{2} e^{\lambda t} \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}$$

and so

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} e^{\lambda t}.$$

For more information concerning other methods for calculating the matrix exponential, the interested reader is referred to the 1978 survey paper by Moler and Van Loan [1].

REFERENCES

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