# Partition Method: A Block View of Gaussian Elimination


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***Abstract*—This paper presents the fundamental introduction to the partition method of solving a system of linear equations. This method primarily uses two procedures called matrix reduction and the usual back substitution. In particular, we have used the Schur complement of a submatrix to reduce a matrix into smaller submatrices. We discuss the Schur complement and how it is used in the matrix reduction procedure. Later, we establish the equivalence between Gaussian elimination and partition method and show that both of the methods have the same time complexity or operation count. Finally, we have provided a few of the scenarios where the partition method is used to solve a system of linear equations.**



***Keywords*—Partition method, Gaussian elimination, Schur complement, System of linear equations**


## I. Introduction

The system of linear equations appears in many areas of applications in science, engineering and other related fields. They appear in operations research to formulate a dynamic system, in social sciences & economics to describe the relationship between various sectors of the economy, in electrical engineering to find the magnitude of currents running through various components of an electrical circuits etc.

There are many ways to solve such linear equations. In this paper, we introduce an interesting approach to solve such problems which was originally published by Theodore J. Sheskin in 1996.[1]

This method is divided into 2 main procedures called matrix reduction and back substitution. First, the coefficient matrix is reduced into smaller matrices similar to how it is done in Gaussian elimination and then we use the back substitution to find the unknowns, again just like how it is done in Gaussian elimination. The only difference between the above two mentioned methods is that the Gaussian elimination eliminates individual coefficients(or entries) while the partition method uses Schur complement, which is described later in this paper, to eliminate an entire block matrix.

We will see how the partition method is actually equivalent to the Gaussian elimination and is actually a "Block elimination" process rather than "entry elimination" process of Gaussian elimination.

## II. Prerequisites

To understand the matrix reduction procedure of partition method, we first need to know about the Schur complement of a block matrix.

### A. Schur complement

The Schur complement of a block matrix is defined as follows.

If A, B, C, D are matrices of order $p*p$, $p*q$, $q*p$ and $q*q$ respectively and $p, q > 0$ such that

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

so that M is a $(p+q)*(p+q)$ matrix. If $det(D) \neq 0$ i.e., if D is invertible then the Schur complement of block D of the matrix M is defined as follows. [2]

$$M/D := A - BD^{-1}C$$

Similarly, If $det(A) \neq 0$ i.e., if A is invertible then the Schur complement of block A of the matrix M is defined as follows.

$$M/A := D - CA^{-1}B$$

### B. Block Elimination using Schur complement

Similar to how individual coefficients of the coefficient matrix are eliminated in the Gaussian elimination, we can also eliminate a block(or submatrix) inside another matrix. This is illustrated as follows.

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \longrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ -D^{-1}C & I_q \end{bmatrix}$$

which becomes

$$\begin{bmatrix} A - BD^{-1}C & B \\ 0 & D \end{bmatrix}$$

Here $I_p$ and $I_q$ are identity matrices of order $p$ and $q$ respectively. The Schur complement then $M/D := A - BD^{-1}C$ appears in the upper-left p×p block of M. [2]

We can see that now the matrix M becomes an upper triangular matrix as all the entries below the main diagonal become 0(which is denoted by the zero matrix in the lower left corner). This is similar to how in Gaussian elimination, the coefficient matrix becomes an upper triangular at the end of the elimination process.

## III. Partition Method Algorithm

As discussed earlier, the partition method works in two phases.

At the beginning, we create an augmented matrix by placing the right hand vector of the equation as a column in the coefficient matrix at the end of it. Then we partition the matrix into 4 submatrices which are as follows.

- First matrix is formed by the single element at the upper left corner of the matrix. This will be a 1 x 1 matrix. [1]
- Second matrix is formed by the remaining elements of the first row. So, this is actually a vector with only one row. [1]
- Third matrix is formed by the remaining elements of the first column. Again, this is also a vector but with only one column. [1]
- Fourth matrix is formed by all remaining elements that were not part of the above three matrices. In other words, these are the elements that were below the first row and to the right of the first column. The dimensions of this matrix are only 1 less than both the dimensions of the original matrix. [1]

We can see that the first and fourth matrices(or block A and D of matrix M) are square matrices of order 1 and n - 1 respectively. So, the criteria of Schur complement is satisfied and we can use the Schur complement of the fourth matrix to make the coefficient matrix block upper triangular.

Now, we will have a new augmented matrix which is actually just the Schur complement of the fourth block matrix in the original matrix. We will use this new matrix in the same fashion as the previous one. Specifically, we will again divide this reduced matrix into 4 individual block matrices and will calculate the Schur complement of the fourth matrix(the lower right block matrix).

This process is continued till the original matrix is reduced into a single row matrix. At this time the matrix reduction process is over and we have to use back substitution to find the unknowns. Note that we can use the same approach to find the inverse of a matrix as well by solving for various right hand sides which are the columns of an identity matrix.

## IV. Comparison with Gaussian Elimination

### A. Equivalence with Gaussian Elimination

It is intuitive to see that the partition method is similar to the Gaussian elimination in the sense that both of the methods try to solve the system of linear equations by eliminating unknowns. On the one hand, the Gaussian elimination process eliminates the one entry at a time, the partition method eliminates the lower left block matrix to make the coefficient matrix upper triangular. Both of these methods also take a total of $n - 1$ steps to process the coefficient matrix. [1]

More formally, it can be observed that the coefficient matrix after k steps of Gaussian elimination is equal to the Schur complement of the element at position $a_{kk}$, i.e., the element at row k and column k.

So after k steps of both Gaussian elimination and matrix reduction procedure of the partition method, the same operations are performed on the coefficient matrix. The primary difference between these two methods is that Gaussian elimination uses the elementary row operations while the matrix reduction procedure of the partition method uses the Schur complement of the element situated at $a_{kk}$.

### B. Time Complexity

We have already shown above that the Gaussian elimination and matrix reduction procedure of the partition method are equivalent algebraically. Since both the methods use back substitution after processing the matrices, their time complexity is also the same.

More specifically, we can observe that the in the first iteration of Gaussian elimination or matrix reduction we have to perform $(n-1)*(n)$ multiplication and the same number of subtraction. In the second iteration, we have to perform $(n-2)*(n-1)$ multiplication and the same number of subtraction and so on.

Also for back substitution, to get the value for all of the n unknowns, we have to perform $n*(n-1)/2$ multiplication and the same number of subtraction.

If we add the number of all of the operations then it is easy to see that the total count comes out to be of order of $n^3/3$. [1]

### C. Space Complexity

We need to store the first row of the augmented matrix in every step as it will be used in the back substitution process to find the values of unknowns. The size of the first row in first iteration is n(as it is a 1 x n vector), in the second iteration it is (n - 1) and so on till 1. Hence, the space complexity is the order of $n^2$.

## V. Applications of Partition Method

### A. Image Restoration

Partitioning method is used in image restoration of blurred image. It has been a fundamental problem in digital imaging for a long time. The field of image restoration has been of great interest in the scientific field. Image restoration is done by partitioning method and its application in several scientific area including medical imaging and diagnosis, military surveillance, satellite imaging etc.

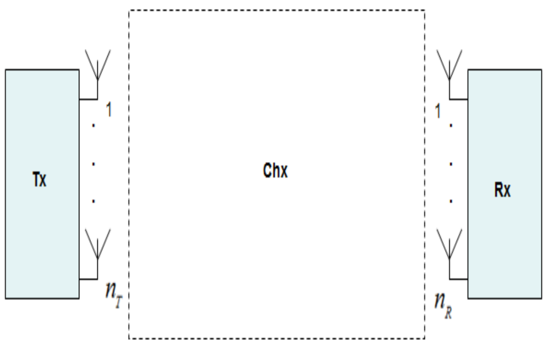
### B. Robotics

In control robotics, non-collocated partial linearization is applied to underactuated mechanical system through inertia-decoupling regulators which employ a pseudoinverse as part of a modified input control law.

### C. Adaptive Processing in Radar

Partition matrix is widely employed in several scientific technology application area .Toeplitz based covariance matrices are use to model structure properties for space-time multivariate adaptive processing in radar.

### D. Wireless Communication System




Tx—transmitior Rx—receiver Chx—channel In the wireless communication system, there are three component in wireless communication system which is formally known as transmitter, channel and receiver. Transmitter transmits the signal and receiver received the signal and signal are pass through channel. Channel are represented in the form of matrix and that matrix are solved by using partitioning method in very easy way.

## Contributors

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## References


[1] Theodore J. Sheskin (1996) A partitioning algorithm for solving systems of linear equations, International Journal of Mathematical Education in Science and Technology, 27:5, 641-648, DOI:10.1080/0020739960270502

[2] Wikimedia Foundation. (2021, September 2). Schur complement. Wikipedia. Retrieved October 13, 2021, from https://en.wikipedia.org/wiki/Schur_complement.