Ex 3-1

$x \in \mathbb{R}^2 \quad w \in \mathbb{R}^2 \qquad L = MSE(\hat{y}, y)$

$\hat{y} = \sigma(w^T x + p_0) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$= \sigma(w_1 x_1 + w_2 x_2 + p_0)$ Assume $p_1 = w_1 x_1$

$p_2 = w_2 x_2$

$= \sigma(p_0 + p_1 + p_2)$

$$\begin{tikzcd}
 & & p_0 \arrow[dr] & & & \\
w_1 \arrow[r] & p_1 = w_1 x_1 \arrow[rr] & & z = p_0 + p_1 + p_2 \arrow[r, Rightarrow] & \hat{y} = \sigma(z) \arrow[r] & L = MSE(\hat{y}, y) \\
x_1 \arrow[ur] & & & & y \arrow[ur] & \\
w_2 \arrow[r] & p_2 = w_2 x_2 \arrow[uurr] & & & & \\
x_2 \arrow[ur] & & & & &
\end{tikzcd}$$

Ex 3-2

(1) $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} 4/5 \\ -7/5 \end{pmatrix} \quad p_0 = \frac{3}{5}, \quad y = 1. \quad L = (\hat{y} - y)^2$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 4/5 \\ -7/5 \end{pmatrix}$

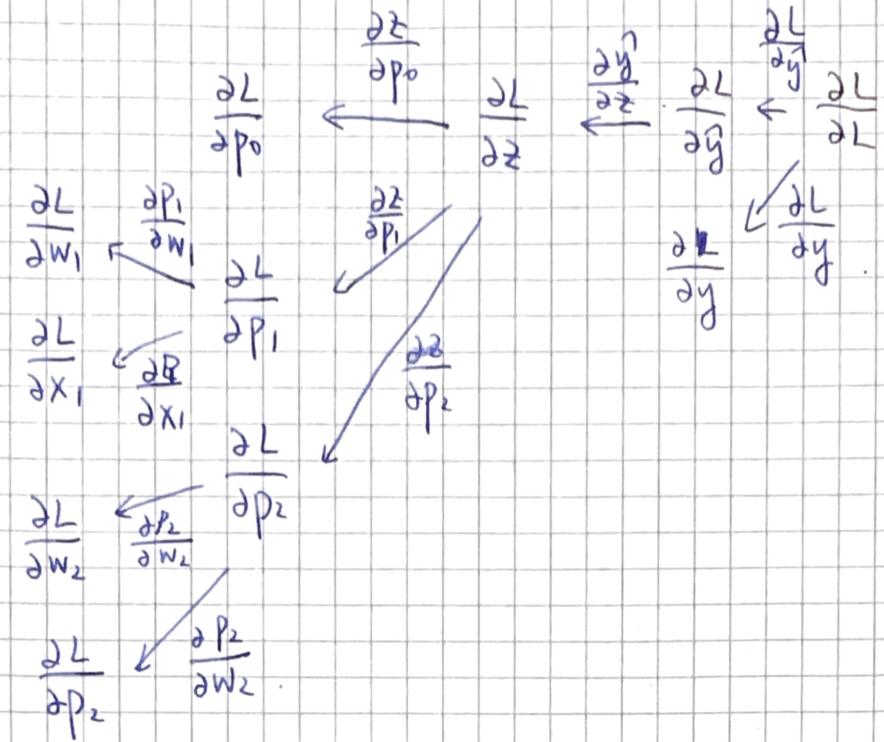
$p_0 = 3/5. \quad p_1 = w_1 x_1 = 4/5. \quad p_2 = w_2 x_2 = -7/5.$

$z = p_0 + p_1 + p_2 = 0.$

$\hat{y} = \sigma(0) = \frac{1}{1 + e^0} = \frac{1}{2}.$

$L = (\frac{1}{2} - 1) = \frac{1}{4}.$

(2)



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(3) $\frac{\partial L}{\partial L} = 1$ $\qquad \frac{\partial L}{\partial \hat{y}} = \frac{\partial (\hat{y}-y)^2}{\partial \hat{y}} = 2(\hat{y}-y) = 2\left(\frac{1}{2}-1\right) = -1$

$\frac{\partial L}{\partial y} = -2(\hat{y}-y) = -2\left(\frac{1}{2}-1\right) = 1$ $\qquad \hat{y} = \sigma(z) = \frac{1}{1+e^{-z}}$

$= \sigma(1-\sigma)$

$\cancel{\frac{\partial \hat{y}}{\partial z} = \sigma(1-\sigma) = 0 \qquad \frac{\partial L}{\partial z} = \frac{\partial \hat{y}}{\partial z} \cdot \frac{\partial L}{\partial \hat{y}} = 0 \cdot (-1) = 0}$

$\sigma(0) = \frac{1}{2}$

$\frac{\partial \hat{y}}{\partial z} = \sigma(1-\sigma) = \frac{1}{2} \cdot \left(1-\frac{1}{2}\right) = \frac{1}{4}$ $\qquad \frac{\partial L}{\partial z} = \frac{1}{4} \times (-1) = -\frac{1}{4}$

$\frac{\partial z}{\partial p_0} = 1$ $\qquad \frac{\partial L}{\partial p_0} = \frac{\partial z}{\partial p_0} \cdot \frac{\partial L}{\partial z} = 1 \cdot \left(-\frac{1}{4}\right) = -\frac{1}{4}$ $\qquad z = p_0 + w_1 x_1 + w_2 x_2$

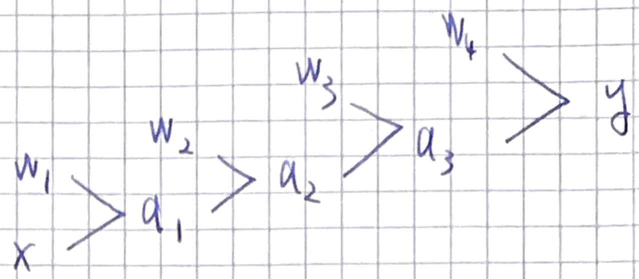
$\frac{\partial z}{\partial p_1} = 1$ $\qquad \frac{\partial L}{\partial p_1} = -\frac{1}{4}$ $\qquad \frac{\partial p_1}{\partial w_1} = x_1 = 1$ $\qquad \frac{\partial L}{\partial w_1} = \frac{\partial L}{\partial p_1} \cdot \frac{\partial p_1}{\partial w_1} = -\frac{1}{4}$

$\frac{\partial z}{\partial p_2} = 1$ $\qquad \frac{\partial L}{\partial p_2} = -\frac{1}{4}$ $\qquad \frac{\partial p_2}{\partial w_2} = x_2 = 1$ $\qquad \frac{\partial L}{\partial w_2} = -\frac{1}{4}$

$\frac{\partial p_1}{\partial x_1} = w_1 = \frac{4}{5}$ $\qquad \frac{\partial L}{\partial x_1} = \frac{\partial p_1}{\partial x_1} \frac{\partial L}{\partial p_1} = \frac{4}{5} \cdot -$ $\qquad \frac{\partial p_2}{\partial x_2} = w_2 = -\frac{7}{5}$

$\frac{\partial L}{\partial x_2} = \frac{\partial p_2}{\partial x_2} \frac{\partial L}{\partial p_2} = -\frac{7}{5}$

Ex 3-4



(1) $x \in \mathbb{R}$

$a_1 = w_1 x$

$a_2 = w_2 a_1$

$a_3 = w_3 a_2$

$a_4 = w_4 a_3$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial a_3} \cdot \frac{\partial a_3}{\partial a_2} \frac{\partial a_2}{\partial a_1} \frac{\partial a_1}{\partial x}$$

$$= \prod_{i=1}^{4} \frac{\partial a_i}{\partial a_{i-1}} = \prod_{i=1}^{4} \frac{\partial \sigma(w_i a_{i-1})}{\partial a_{i-1}}$$

因 $\dfrac{\partial \sigma(w_i a_{i-1})}{\partial a_{i-1}} = w_i \dfrac{\partial \sigma(z_i)}{\partial z_i}$ $\quad z_i = w_i a_{i-1}$

$\therefore \dfrac{\partial y}{\partial x} = \prod_{i=1}^{4} w_i \dfrac{\partial \sigma(z_i)}{\partial z_i}$ $\quad z_i = w_i a_{i-1}$

$a_0 = x$

$a_4 = y$

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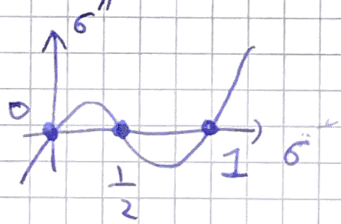
(2) $\sigma' = \sigma(1-\sigma) \in (0,1) \Leftarrow \sigma(z) = \dfrac{1}{1+e^{-z}} \in (0,1)$

$\therefore \sigma \nearrow$ monotonous

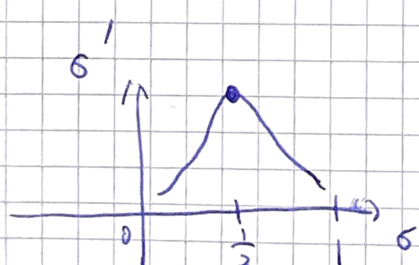
$$\sigma'' = \sigma'(1-\sigma) + \sigma(-\sigma') = \sigma' - \sigma'\sigma - \sigma'\sigma = \sigma'(1-2\sigma)$$

$$= \sigma(1-\sigma)(1-2\sigma) = 2\sigma^3 - 3\sigma^2 + \sigma \to \infty$$



$\because \sigma \in (0,1)$

$\therefore \begin{cases} \sigma'' > 0 & \text{if } \sigma < \frac{1}{2} \\ \sigma'' < 0 & \text{if } \sigma > \frac{1}{2} \end{cases}$



$\sigma'' = 0$ if and only if $\sigma = \frac{1}{2}$

$\sigma'_{max} = \sigma(1-\sigma) = \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$

(3)

$$\frac{\partial y}{\partial x} = \prod_{i=1}^{4} w_i \frac{\partial \sigma(z_i)}{\partial z_i}.$$

In General: $\frac{\partial y}{\partial x} = \prod_{i=1}^{n} w_i \frac{\partial \sigma(z_i)}{\partial z_i}$

$z_i = w_i a_{i-1}$.

Then $\frac{\partial y}{\partial x} = \prod_{i=1}^{n} w_i \frac{\partial \sigma(z_i)}{\partial z_i} \leq \prod_{i=1}^{n} \left(\frac{1}{4} w_i\right)$

If $w \in N(0,1)$ then $w \in (0,1)$

Thus $\frac{\partial y}{\partial x} \leq \prod_{i=1}^{n}\left(\frac{1}{4} w_i\right) < \prod_{i=1}^{n}\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^n$ for most w_i.

$\lim_{n \to \infty} \left(\frac{1}{4}\right)^n = 0$. vanished!

Same for (tanh): $\frac{\partial \tanh(z)}{\partial z} \leq \frac{1}{2}$.

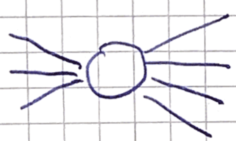
Solution: Xavier (Glorot) weight init.

$w \in N(0, \frac{1}{n_{in}})$ n_{in} : number of inputs.

mean $(w) = 0$

$Var(w) = \frac{1}{n_{in}}$

$n_{in} = 3$ $n_{out} = 4$.



Glorot: $var(w) = \frac{2}{n_{in} + n_{out}}$

Assumption: - non-linearity.

- X: same distribution.

$Var(z_{i+1}) = Var(z_i)$.

$z_j^{(i+1)} = \sum_k w \cdot z_k^{(i)}$.

$Var(z^{(i+1)}) = Var\left(\sum_k w z_k^{(i)}\right)$

$\Rightarrow Var(z^{(i+1)}) = n_{in}\, Var(W)\, Var($