Let $\{Z_i\}_{i=1}^N$ be some iid samples of a random variable Z. We are interested in estimating $\mathrm{CVaR}_\alpha[Z]$. As long as the distribution P of Z is integrable, we can write

$$\mathrm{CVaR}_\alpha[Z] = \sup_{\xi \in \mathcal{U}} \mathbb{E}_P[\xi Z] \tag{1}$$

where $\mathcal{U} := \{\xi \mid 0 \leq \xi \leq \frac{1}{\alpha},\, \mathbb{E}_P[\xi] = 1\}$. Moreover, the following result is standard.

$$\xi^* := \frac{1}{\alpha}\mathbf{1}_{\{Z \geq \nu_\alpha\}} \in \arg\max_{\xi \in \mathcal{U}} \mathbb{E}_P[\xi Z]$$

where ν_α is the $(1-\alpha)$-quantile of the distribution P.

Using definition (1), the plug-in estimator of $\mathrm{CVaR}_\alpha[Z]$ uses P_n instead of P, where $P_n := \frac{1}{N}\sum_{i=1}^N \delta_{Z_i}$ is the empirical distribution of P. Then,

$$\widehat{\mathrm{CVaR}}_\alpha[Z] := \sup_{\xi \in \widehat{\mathcal{U}}} \mathbb{E}_{P_n}[\xi Z] \tag{2}$$

A density that attains the supremum in (2) is equal to

$$\widehat{\xi}^* := \frac{1}{\alpha}\mathbf{1}_{\{Z \geq \widehat{\nu}_\alpha\}}$$

where $\widehat{\nu}_\alpha$ is the $(1-\alpha)$-quantile of P_n (this is what I meant by the empirical quantile in our discussion on Thursday 23). Then, once $\widehat{\nu}_\alpha$ is computed, we get

$$\begin{aligned}\widehat{\mathrm{CVaR}}_\alpha[Z] &:= \mathbb{E}_{P_n}[\widehat{\xi}^* Z] \\ &= \frac{1}{N}\sum_{i=1}^N Z_i \frac{1}{\alpha}\mathbf{1}_{\{Z_i \geq \widehat{\nu}_\alpha\}}\end{aligned}$$

To summarize, in order to compute an empirical estimate of CVaR,

1. get iid samples $Z_1, \ldots, Z_N$.
2. compute the $(1-\alpha)$-quantile $\widehat{\nu}_\alpha$ of the empirical distribution P_n.
3. compute the empirical estimate $\widehat{\mathrm{CVaR}}_\alpha[Z]$.

Note that it is equivalent to the empirical estimator given in Shapiro's textbook. He writes

$$\widehat{\mathrm{CVaR}}_\alpha[Z] = \inf_{t \in \mathbb{R}} \left\{ t + \frac{1}{\alpha N}\sum_{i=1}^N (Z_i - t)_+ \right\}$$

The t variable that minimizes the previous expression is the empirical $(1-\alpha)$-quantile of P_n:

$$\widehat{\mathrm{CVaR}}_\alpha[Z] = \widehat{\nu}_\alpha + \frac{1}{\alpha N}\sum_{i=1}^N (Z_i - \widehat{\nu}_\alpha)_+$$

Expanding the previous RHS,

$$\begin{aligned}\widehat{\mathrm{CVaR}}_\alpha[Z] &= \widehat{\nu}_\alpha + \frac{1}{\alpha N}\sum_{i=1}^N (Z_i - \widehat{\nu}_\alpha)\mathbf{1}_{\{Z_i \geq \widehat{\nu}_\alpha\}} \\ &= \widehat{\nu}_\alpha + \frac{1}{\alpha N}\sum_{i=1}^N Z_i\mathbf{1}_{\{Z_i \geq \widehat{\nu}_\alpha\}} - \frac{1}{\alpha N}\sum_{i=1}^N \widehat{\nu}_\alpha\mathbf{1}_{\{Z_i \geq \widehat{\nu}_\alpha\}}\end{aligned}$$

But,

$$\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\{Z_i \geq \widehat{\nu}_\alpha\}} = \alpha$$

Therefore,

$$\frac{1}{\alpha N}\sum_{i=1}^N \widehat{\nu}_\alpha\mathbf{1}_{\{Z_i \geq \widehat{\nu}_\alpha\}} = \widehat{\nu}_\alpha$$

And,

$$\widehat{\mathrm{CVaR}}_\alpha[Z] = \frac{1}{\alpha N}\sum_{i=1}^N Z_i\mathbf{1}_{\{Z_i \geq \widehat{\nu}_\alpha\}}$$

This shows that the estimator in Shapiro's textbook is exactly the so-called plug-in estimator.