MLE from a Competing Risks Model of an Exponential Failure and a Lognormal Survival

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1 Introduction.

1.1 Assumptions.

Two usage-measured lifetime random variables are discussed in a fixed-time life test. The items under study are considered to accumulate usage independently and with a different rate. The survival ssspopulation is considered *Lognormally distributed* (with parameters μ and σ), while the failure popoulation - *Exponential* (with parameter λ).

An estimation of the parameter vector $(\lambda,\, \mu,\, \sigma)'$ is sought using all usage data collected (from both survival and failure cases)

- $\eta \sim \textit{Lognormal}$
- $\psi \sim \textit{Exponential}$
- $f_\eta(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} = \frac{1}{x\sigma}\phi\left(\frac{\ln x - \mu}{\sigma}\right)$ is the *probability densitiy function* of the Lognormal distribution, with ϕ being the density function of the Standard Normal Distribution
- $\overline{F}_\eta(x) = 1 - \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} = \overline{\Phi}\left(\frac{\ln x - \mu}{\sigma}\right)$ is its *survival function*, with $\overline{\Phi}$ being the survival function of the Standard Normal Distribution
- $f_\psi(x) = \lambda e^{-\lambda x}$ is the *pdf* of the Exponential distribution
- $\overline{F}_\psi(x) = e^{-\lambda x}$ is its *survival function*
- $N_f = \{i : x_i \ is \ failed\}$ and $n_f = \#N_f$ (i.e. number of failures)
- $N_s = \{i : x_i \ is \ survived\}$ and $n_f = \#N_s$ (i.e. number of survivals)
- n is the total number of units under test ($n = n_f + n_s$)
- $\mathbf{x}$ is the data vector and $\mathbf{x} = \left[\mathbf{x}^f, \mathbf{x}^s\right] = \left[x_1^f, ..., x_{n_f}^f, x_1^s, ..., x_{n_s}^s\right]$, where:
- $\mathbf{x}^f$ is the data for failed items, $\mathbf{x}^f = \left[x_1^f, ..., x_{n_f}^f\right]$
- $\mathbf{x}^s$ is the data for survived items, $\mathbf{x}^s = \left[x_1^s, ..., x_{n_s}^s\right]$
- $\lambda, \mu, \sigma, x > 0$

1.2 Maximum Likelihood Method.

The Likelihood function is given by (since η and ψ are independent):

$$L(\lambda, \mu, \sigma|\mathbf{x}) = \prod_{i \in N_f} \left[f_\psi \left(x_i^f \right) \overline{F}_\eta \left(x_i^f \right) \right] \prod_{i \in N_s} \left[f_\eta \left(x_i^s \right) \overline{F}_\psi \left(x_i^s \right) \right] \tag{1}$$

Now from the above we can derive the Log Likelihood:

$$LogL = L^*(\lambda, \mu, \sigma|\mathbf{x}) = \sum_{i \in N_f} \ln \left[f_\psi (x_i^f) \right] + \sum_{i \in N_f} \ln \left[\overline{F}_\eta (x_i^f) \right] + \sum_{i \in N_s} \ln \left[f_\eta (x_i^s) \right] + \sum_{i \in N_s} \ln \left[\overline{F}_\psi (x_i^s) \right] \tag{2}$$

And if we apply the above notation and rework:

$$L^*(\lambda, \mu, \sigma|\mathbf{x}) = N_f \ln \lambda - \lambda \sum_{i=1}^{n} x_i + \sum_{i \in N_f} \ln \overline{\Phi} \left(\frac{\ln x_i^f - \mu}{\sigma} \right) - \sum_{i \in N_s} \ln \sigma x_i^s + \sum_{i \in N_s} \ln \phi \left(\frac{\ln x_i^s - \mu}{\sigma} \right) \tag{3}$$

The MLE estimators $\left(\hat{\lambda}, \hat{\mu}, \hat{\sigma} \right)$ would be the ones that turn L^* into a minimum, so now we need to compute the *Gradient* vector and *Hessian* matrix in order to find them. Furthermore we would like to show that the estimators have the *consistency* property.

1.3 First order derivatives of L^* w.r.t (λ, μ, σ)

Let's denote: $z_i^{f,s} = \frac{\left(\ln x_i^{f,s} - \mu \right)}{\sigma}$. Then we derive:

$$\frac{\partial z}{\partial \mu} = -\frac{1}{\sigma}, \frac{\partial z}{\partial \sigma} = -\frac{1}{\sigma} z, \frac{\partial \phi}{\partial \sigma} = \frac{1}{\sigma} \phi z^2, \frac{\partial \phi}{\partial \mu} = \frac{1}{\sigma} \phi z \tag{4}$$

$$\frac{\partial L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \lambda} = \frac{n_f}{\lambda} - \sum_{i=1}^{n} x_i \tag{5}$$

$$\frac{\partial L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \mu} = \frac{1}{\sigma} \sum_{i \in N_f} \frac{\phi \left(z_i^f \right)}{\overline{\Phi} \left(z_i^f \right)} + \frac{1}{\sigma} \sum_{i \in N_s} z_i^s \tag{6}$$

$$\frac{\partial L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \sigma} = \frac{1}{\sigma} \sum_{i \in N_f} \frac{\phi \left(z_i^f \right) z_i^f}{\overline{\Phi} \left(z_i^f \right)} + \frac{1}{\sigma} \sum_{i \in N_s} (z_i^s)^2 - \frac{n_s}{\sigma} \tag{7}$$

1.4 Second order derivatives of L^* w.r.t (λ, μ, σ)

Let's denote again: $z_i^{f,s} = \frac{\left(\ln x_i^{f,s} - \mu \right)}{\sigma}$. Then we derive:

$$\frac{\partial^2 L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \lambda^2} = -\frac{n_f}{\lambda^2} \tag{8}$$

$$\frac{\partial^2 L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \mu^2} = \frac{1}{\sigma^2} \sum_{i \in N_f} \left[\frac{\phi \left(z_i^f \right)}{\overline{\Phi} \left(z_i^f \right)} \left(1 + \frac{\phi \left(z_i^f \right)}{\overline{\Phi} \left(z_i^f \right)} \right) z_i^f \right] - \frac{1}{\sigma^2} \tag{9}$$

$$\frac{\partial^2 L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \mu \partial \sigma} = \frac{1}{\sigma^2} \sum_{i \in N_f} \frac{\phi \left(z_i^f \right)}{\overline{\Phi} \left(z_i^f \right)} + \frac{1}{\sigma^2} \sum_{i \in N_f} \left[\frac{\phi \left(z_i^f \right)}{\overline{\Phi} \left(z_i^f \right)} \left(1 + \frac{\phi \left(z_i^f \right)}{\overline{\Phi} \left(z_i^f \right)} \right) z_i^f \right] - \frac{2}{\sigma^2} \sum_{i \in N_s} z_i^s \tag{10}$$

$$\frac{\partial^2 L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \sigma^2} = \frac{1}{\sigma^2} \sum_{i \in N_f} \left[\frac{\phi(z_i) z_i^3}{\overline{\Phi}} - 2 \frac{\phi(z_i) z_i}{\overline{\Phi}} - \frac{\phi^2(z_i) z_i^2}{\overline{\Phi}^2} \right] - \frac{3}{\sigma^2} \sum_{i \in N_s} z_i^2 + \frac{n_s}{\sigma^2} \tag{11}$$

2 Maximum Likelihood Estimators' properties.

2.1 Normality of the estimators.

This can be seen from the results from 4 simulation sets of true value parameters. 150 repetitions for each parameter set. Diagram 1.

2.2 Consistency of the estimators.

Simulations were done with the same parameter sets but in 300 repetitions. Diagram 2.

3 Next steps.

Check the properties of the Variance Covariance matrix, Score, and perform more simulations on other true parameter sets and with more repetitions.