MLE from a Competing Risks Model of an Exponential Failure and a Lognormal Survival

Nikola Chochkov, student number 542064 MSc Statistics, Humboldt University Berlin

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1 Introduction.

1.1 Assumptions.

Two usage-measured lifetime random variables are discussed in a fixed-time life test. The items under study are considered to accumulate usage independently and with a different rate. The survival population is considered *Lognormally distributed* (with parameters μ and σ), while the failure popoulation - *Exponential* (with parameter λ).

An estimation of the parameter vector $(\lambda, \mu, \sigma)'$ is sought using all usage data collected (from both survival and failure cases)

- $\eta \sim Lognormal$
- $\psi \sim Exponential$
- $f_\eta(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} = \frac{1}{x\sigma}\phi\left(\frac{\ln x - \mu}{\sigma}\right)$ is the *probability densitiy function* of the Lognormal distribution, with ϕ being the density function of the Standard Normal Distribution
- $\overline{F}_\eta(x) = 1 - \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} = \overline{\Phi}\left(\frac{\ln x - \mu}{\sigma}\right)$ is its *survival function*, with $\overline{\Phi}$ being the survival function of the Standard Normal Distribution
- $f_\psi(x) = \lambda e^{-\lambda x}$ is the *pdf* of the Exponential distribution
- $\overline{F}_\psi(x) = e^{-\lambda x}$ is its *survival function*
- $N_f = \{i : x_i \ is \ failed\}$ and $n_f = \#N_f$ (i.e. number of failures)
- $N_s = \{i : x_i \ is \ survived\}$ and $n_f = \#N_s$ (i.e. number of survivals)
- n is the total number of units under test ($n = n_f + n_s$)
- $\mathbf{x}$ is the data vector and $\mathbf{x} = \left[\mathbf{x}^f, \mathbf{x}^s\right] = \left[x_1^f, ..., x_{n_f}^f, x_1^s, ..., x_{n_s}^s\right]$, where:
- $\mathbf{x}^f$ is the data for failed items, $\mathbf{x}^f = \left[x_1^f, ..., x_{n_f}^f\right]$
- $\mathbf{x}^s$ is the data for survived items, $\mathbf{x}^s = \left[x_1^s, ..., x_{n_s}^s\right]$
- $\lambda, \mu, \sigma, x > 0$

1.2 Maximum Likelihood Method.

The Likelihood function is given by (since η and ψ are independent):

$$\delta(\lambda, \mu, \sigma) \sim P[\psi < \eta | Fail] * P[\eta < \psi | Surv] = f_\psi(x|F)\overline{F}_\eta(x|F) * f_\eta(x|S)\overline{F}_\psi(x|S) \tag{1}$$

$$L(\lambda, \mu, \sigma|\mathbf{x}) = \prod_{i \in N_f} \left[f_\psi\left(x_i^f\right) \overline{F}_\eta\left(x_i^f\right) \right] \prod_{i \in N_s} \left[f_\eta\left(x_i^s\right) \overline{F}_\psi\left(x_i^s\right) \right] \tag{2}$$

Now from the above we can derive the Log Likelihood:

$$LogL = L^*(\lambda, \mu, \sigma|\mathbf{x}) = \sum_{i \in N_f} \ln\left[f_\psi(x_i^f)\right] + \sum_{i \in N_f} \ln\left[\overline{F}_\eta(x_i^f)\right] + \sum_{i \in N_s} \ln\left[f_\eta(x_i^s)\right] + \sum_{i \in N_s} \ln\left[\overline{F}_\psi(x_i^s)\right] \tag{3}$$

And if we apply the above notation and rework:

$$L^*(\lambda, \mu, \sigma|\mathbf{x}) = N_f \ln\lambda - \lambda \sum_{i=1}^{n} x_i + \sum_{i \in N_f} \ln\overline{\Phi}\left(\frac{\ln x_i^f - \mu}{\sigma}\right) - \sum_{i \in N_s} \ln \sigma x_i^s + \sum_{i \in N_s} \ln\phi\left(\frac{\ln x_i^s - \mu}{\sigma}\right) \tag{4}$$

The MLE estimators $\left(\hat{\lambda}, \hat{\mu}, \hat{\sigma}\right)$ would be the ones that turn L^* into a maximum, so now we need to compute the *Gradient* vector and *Hessian* matrix in order to find them. Furthermore we would like to show that the estimators have the *consistency* property.

1.3 First order derivatives of L^* w.r.t (λ, μ, σ)

Let's denote: $z_i^{f,s} = \frac{\left(\ln x_i^{f,s} - \mu\right)}{\sigma}$. Then we derive:

$$\frac{\partial z}{\partial \mu} = -\frac{1}{\sigma}, \frac{\partial z}{\partial \sigma} = -\frac{1}{\sigma}z, \frac{\partial \phi}{\partial \sigma} = \frac{1}{\sigma}\phi z^2, \frac{\partial \phi}{\partial \mu} = \frac{1}{\sigma}\phi z \tag{5}$$

$$\frac{\partial L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \lambda} = \frac{n_f}{\lambda} - \sum_{i=1}^{n} x_i \tag{6}$$

$$\frac{\partial L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \mu} = \frac{1}{\sigma} \sum_{i \in N_f} \frac{\phi\left(z_i^f\right)}{\overline{\Phi}\left(z_i^f\right)} + \frac{1}{\sigma} \sum_{i \in N_s} z_i^s \tag{7}$$

$$\frac{\partial L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \sigma} = \frac{1}{\sigma} \sum_{i \in N_f} \frac{\phi\left(z_i^f\right) z_i^f}{\overline{\Phi}\left(z_i^f\right)} + \frac{1}{\sigma} \sum_{i \in N_s} (z_i^s)^2 - \frac{n_s}{\sigma} \tag{8}$$

1.4 Second order derivatives of L^* w.r.t (λ, μ, σ)

Let's denote again: $z_i^{f,s} = \frac{\left(\ln x_i^{f,s} - \mu\right)}{\sigma}$. Then we derive:

$$\frac{\partial^2 L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \lambda^2} = -\frac{n_f}{\lambda^2} \tag{9}$$

$$\frac{\partial^2 L^*(\lambda, \mu, \sigma|\mathbf{x})}{\partial \mu^2} = \frac{1}{\sigma^2} \sum_{i \in N_f} \left[\frac{\phi\left(z_i^f\right)}{\overline{\Phi}\left(z_i^f\right)} \left(1 + \frac{\phi\left(z_i^f\right)}{\overline{\Phi}\left(z_i^f\right)}\right) z_i^f \right] - \frac{1}{\sigma^2} \tag{10}$$