Integrating the Solar System

Applied Math 205 Group Activity

29-Oct-2021

Presented by:

Michael S. Emanuel

Physics of Gravitational Attraction

Newton's Law of Universal Gravitation

- Newton's Law of Universal Gravitation

$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$

$$G = 6.674 \times 10^{-11} \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$$

- Newton's Second Law of Motion $F = ma$
- Calculate the acceleration of body 2 due to body 1
  - The mass m_2 cancels out

$$\boxed{a = -G\frac{m_1}{\|\mathbf{r}\|^2}\hat{\mathbf{r}}}$$

N-Body Problem Formulation

- Consider n point masses of mass m_i, i = 1, 2, … n
- Write the vector $\mathbf{q}_i$ for the position of body i
- Get a coupled second order ODE for the positions

$$\boxed{\frac{d^2\mathbf{q}_i}{dt^2} = G \cdot \sum_{\substack{i=1 \\ j \neq i}}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{\|\mathbf{q}_j - \mathbf{q}_i\|^3}}$$

- This can be solved analytically only in the case that n=2
- A solution also exists for n=3 when one of the particles has zero mass

Conserved Quantities: **p**, **L**, H

- The gravitational equation describes a conservative system
- Important conserved quantities include momentum, angular momentum, and energy

$$\text{Momentum:} \qquad \mathbf{p} = \sum_{i=1}^{n} m_i \mathbf{v}_i$$

$$\text{Angular Momentum:} \quad \mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{q}_i \times \mathbf{v}_i$$

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i \|\mathbf{v}_i\|^2$$

$$U = -G \cdot \sum_{i=1}^{n} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|}$$

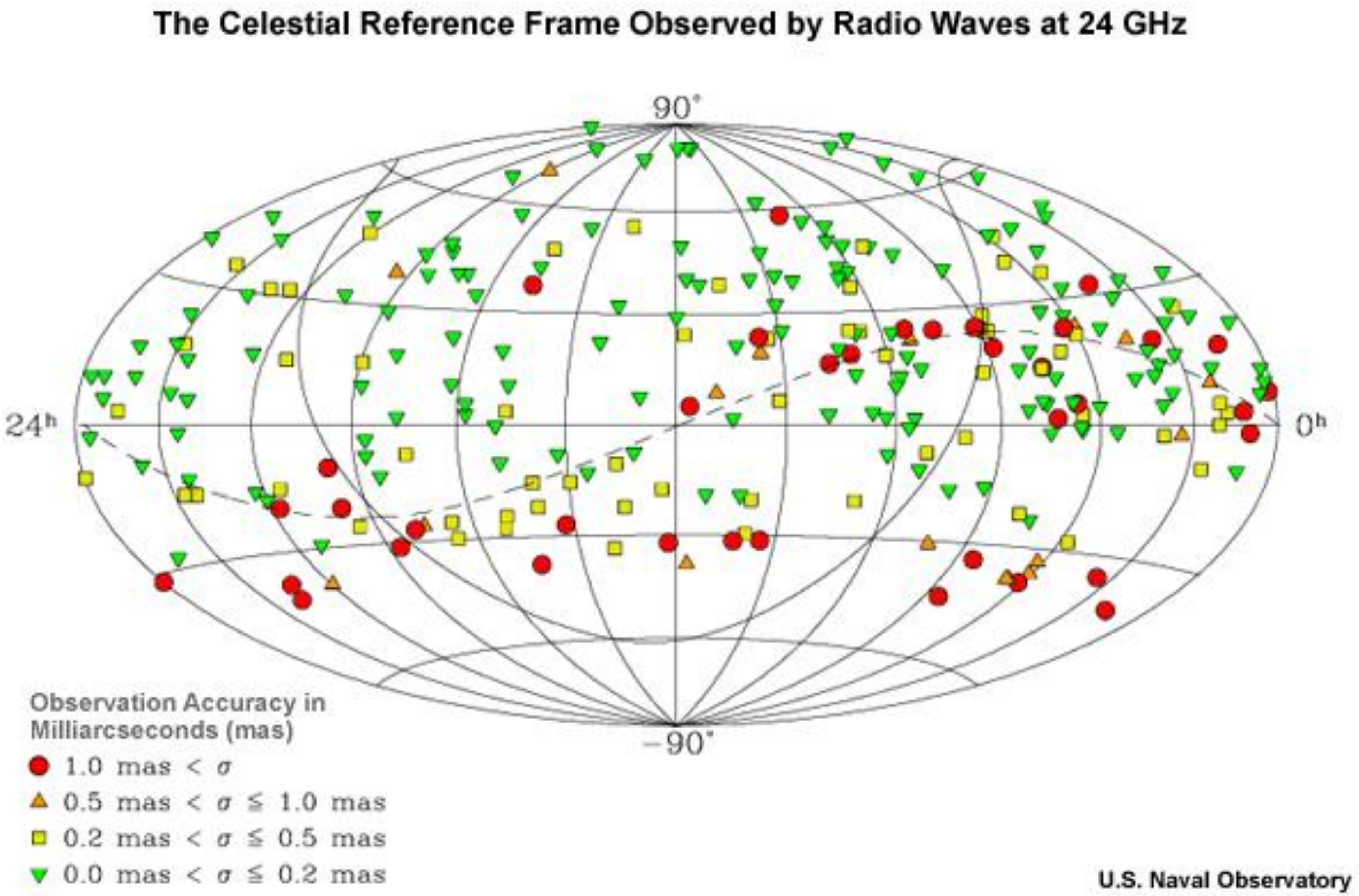
$$H = T + U \quad \text{(total energy)}$$

Choosing an Inertial Frame and Units

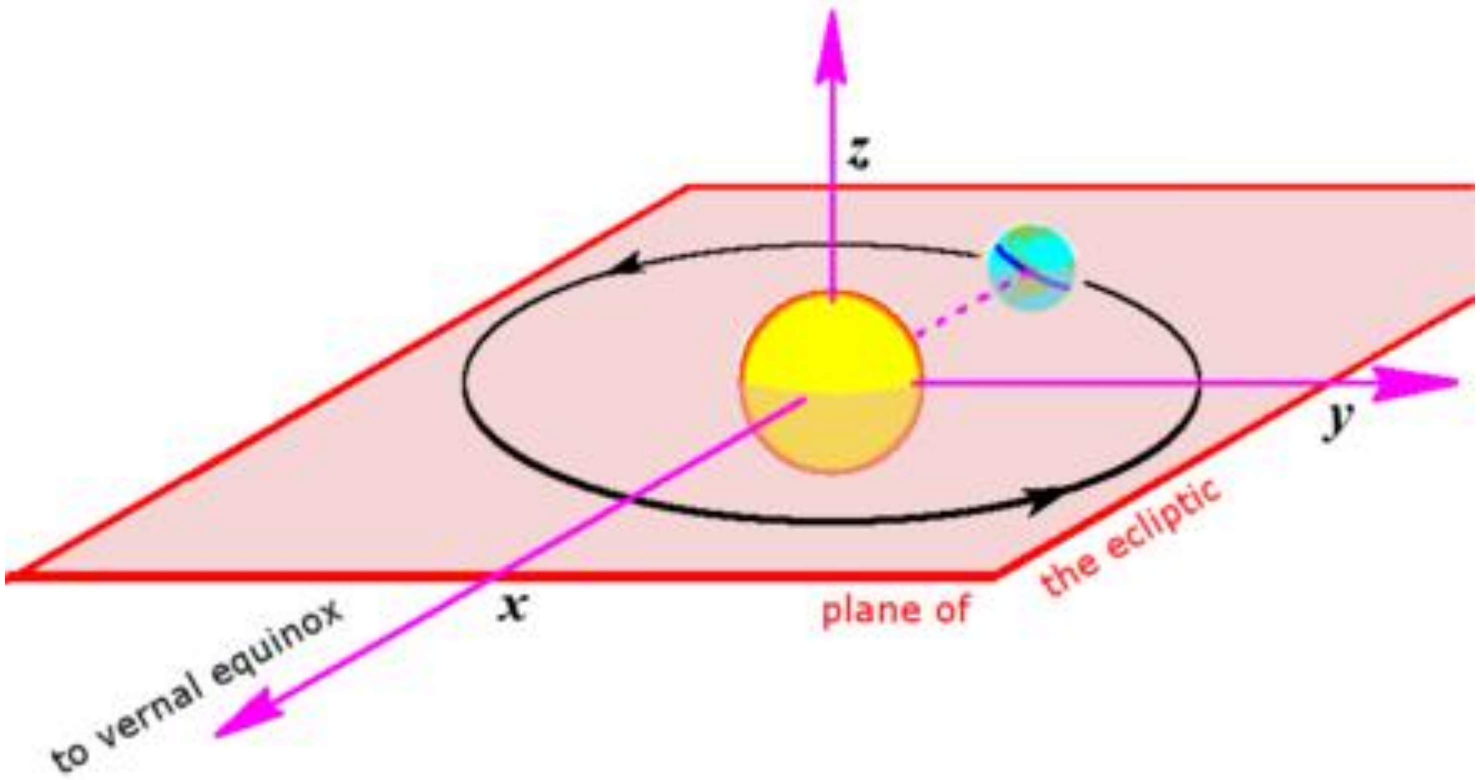
Choosing an Inertial Frame

- Momentum conservation makes it convenient to work in a frame of reference where the total momentum is zero
- This is the frame of the center of mass, or “barycenter”
- The Solar System Barycenter is physically defined, but how do we orient the coordinate axes?
- BME: Barycentric Mean Ecliptic – use Earth’s orbit; intuitive
- ICRF: International Celestial Reference Frame – modern, precise

International Celestial Reference Frame (ICRF)



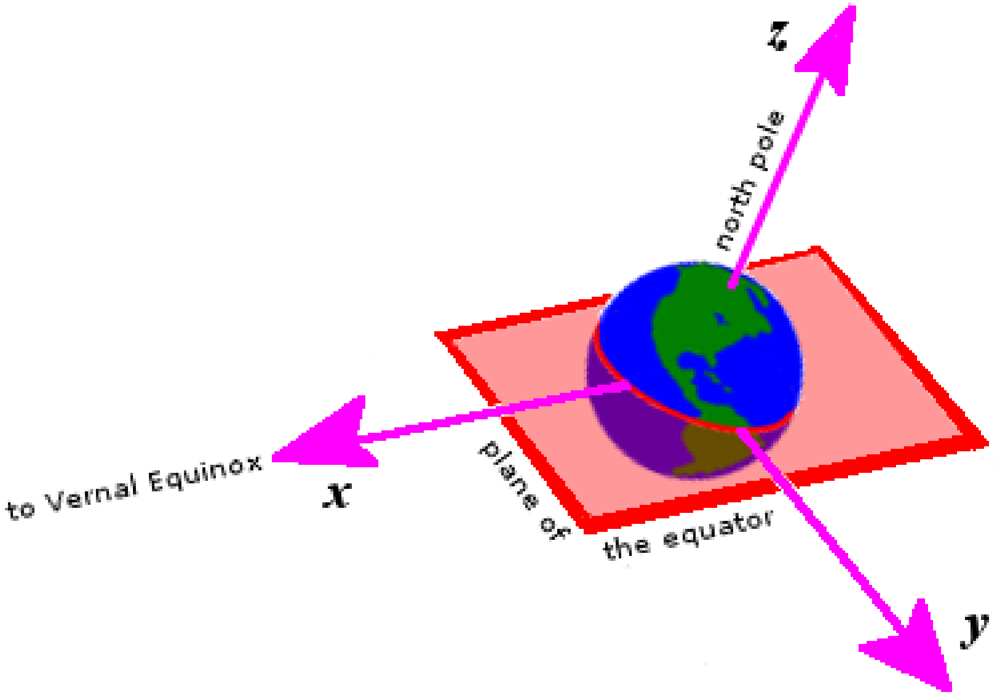
Barycentric Mean Ecliptic Frame

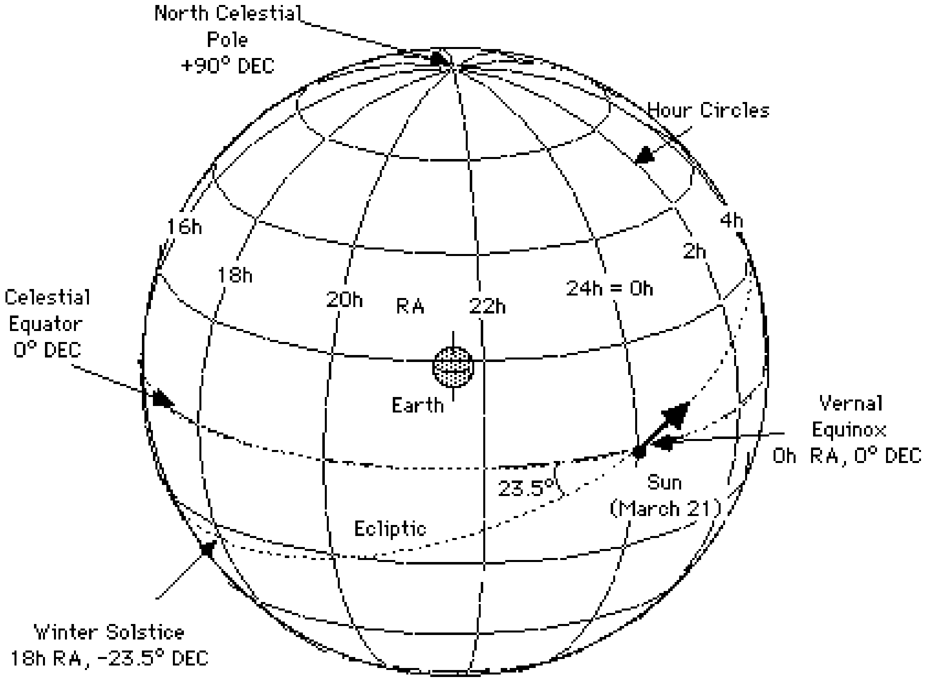


- Convert between ICRF and BME using astropy library:

```
obs_icrs = astropy.SkyCoord(ra=ra, dec=dec, obstime=obstime, frame=ICRS)
obs_ecl = obs_icrs.transform_to(BarycentricMeanEcliptic)
```

Right Ascension and Declination





- Fundamental plane is aligned with Earth's equator
- Intuitive, dates to ancient astronomers
- Two problems: precession (drift) and nutation (wobbles) in direction of North Pole

Recommend Units for Solar System Dynamics

- For dynamical calculations in the Solar System, units must be chosen for **mass**, **length**, and **time**
- SI units are great for physics… most of the time
- But for Solar System problems, the scales are not convenient
- A convenient set of units is based on our Solar System
- Mass: a conventional figure for the mass of the sun $M_{\odot}$
- Length: the astronomical unit; the average distance Sun to Earth
- Time: one **day** (86,400 seconds)

Astronomical Units and Solar Masses

- The **astronomical unit** (au) was historically defined as the mean distance from Sun to Earth
- Modern definition: $\boxed{1 \text{ au} = 149\ 597\ 870\ 700 \text{ meters}}$
- The **solar mass** $M_\odot$ is approximately $1.988\ 48 \pm 0.000\ 07 \times 10^{30}\mathrm{kg}$
- The solar mass parameter $GM_\odot$ is easier to measure

$$\boxed{GM_\odot = 1.323\ 124\ 400\ \times 10^{20}\mathrm{m}^3\mathrm{s}^{-2}}$$

- One **day** is conventionally defined as 24 x 60 x 60 = 86,400 seconds
- The gravitational constant G (per JPL) in units $(M_\odot, \mathrm{au}, \mathrm{day})$ is

$$\boxed{G \cdot M_\odot = 2.959\ 122\ 082\ 855\ 910\ 95 \times 10^{-4}\ \mathrm{au}^3\mathrm{day}^{-2}}$$

Modified Julian Dates (MJDs)

- Astronomy has its own unique conventions for times and dates
  - Need a simple linear scale allowing for easy subtraction of datetimes that are far apart
- A Julian period is the product of three cycles:
  - 28 (solar cycle) x 19 (lunar cycle) x 15 (indiction cycle) = 7980 years (!)
  - This made it possible to correctly assign the year to ancient historical events
- The Julian day: # days since Julian period that began at 12:00 January 1, 4713 BC
- Since this is cumbersomely large, the Modified Julian Date (**MJD)** is defined as

  MJD = JD – 2400000.5
- The **epoch** of the MJD is 0:00 November 17, 1858
- To convert between a JD and a Unix time use

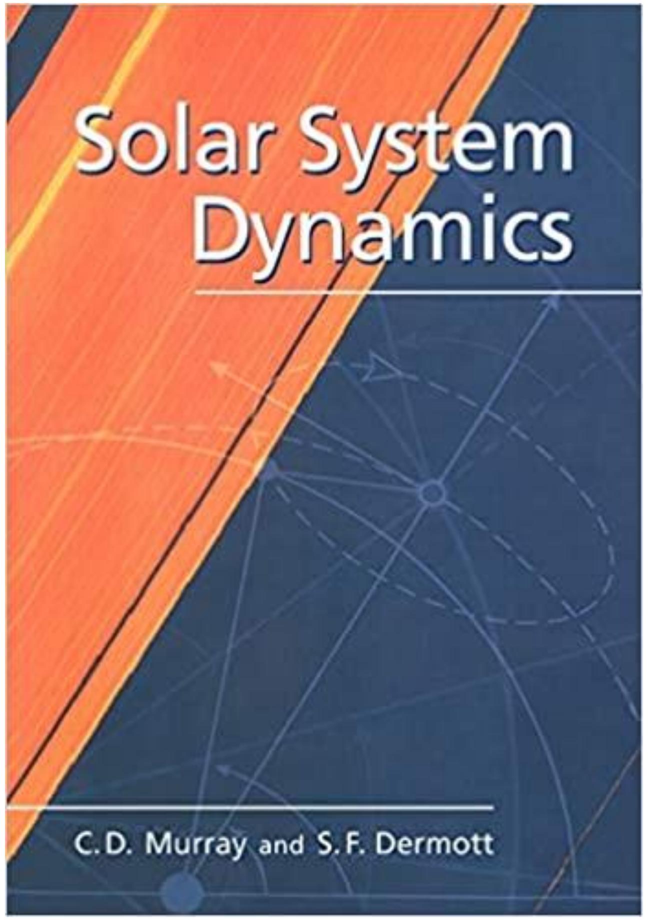
  UnixTime = (JD – 2440587.5) x 86400
- Better idea… NEVER write these formulas yourself! Understand them once, then use a library for actual calculations.
- The MJD of today's date (29-Oct-2021) is 59516

Two Body Problem: Analytic Solution

Kepler’s Laws of Planetary Motion

1. The orbit of a planet is an ellipse with the Sun at one focus
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time
3. The square of a planet’s orbital period is proportional to the cube of the length of the semi-major axis of its orbit

Reference Text: Solar System Dynamics



Aside: Definitive Books > Wikipedia

- Here is a random aside I wanted to share with you
- This is something I learned in graduate school in my forties
  - So it's not entirely obvious
- Wikipedia is amazing-free, convenient, high quality
- But… it is not the most definitive source of knowledge in the world
- If you are doing serious work in a field, spend some time trying to determine whether there are any "classical" or definitive text
- In the case of integrating the solar system, Solar System Dynamics (C.D. Murray and S.F. Dermott) is the acknowledged classic
- You will also see it cited in the rebound software documentation

Relative Motion is Elliptical

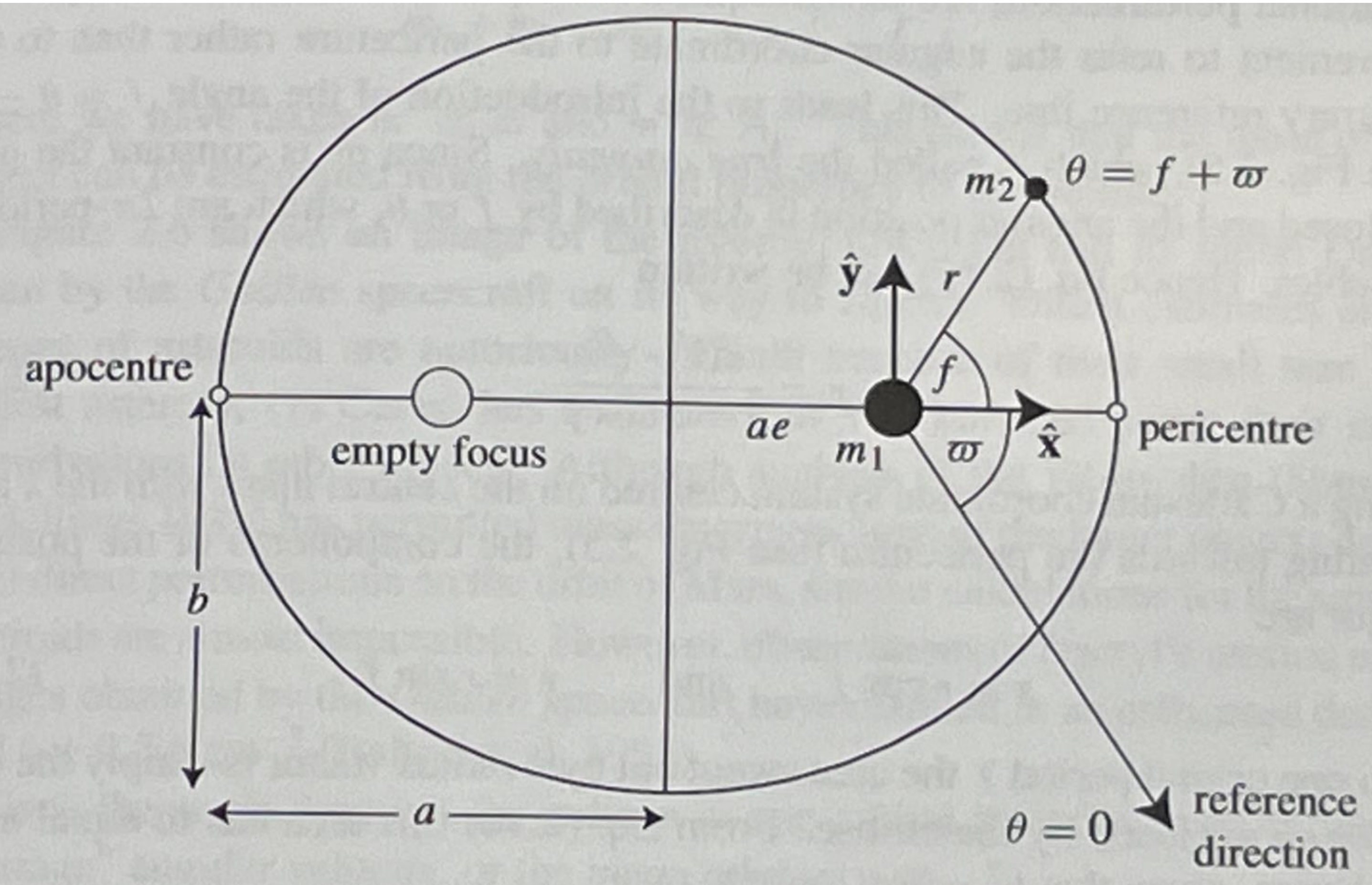


Fig. 2.5. The geometry of the ellipse of semi-major axis a, semi-minor axis b, eccentricity e, and longitude of pericentre ϖ.

Two Body ODE- Cartesian Coordinates

- Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be positions of two bodies with masses m_1 and m_2
- Let $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ be their **relative displacement**
- Let $r = \|\mathbf{r}\|$ be scalar distance between them

$$\boxed{\begin{aligned} \ddot{\mathbf{r}}_1 &= +\frac{Gm_2\mathbf{r}}{r^2} \\ \ddot{\mathbf{r}}_2 &= -\frac{Gm_1\mathbf{r}}{r^2} \end{aligned}}$$

$$\boxed{\mu \equiv G(m_1 + m_2)}$$

$$\boxed{\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu\mathbf{r}}{r^3} = 0 \quad \text{(SSD 2.5)}}$$

- Specific angular momentum is conserved:

$$\boxed{\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = \text{const} \quad \text{(SSD 2.6)}}$$

Two Body ODE – Polar Coordinates

- Switch from Cartesian to polar coordinates (SSD 2.7)

$$\boxed{\begin{aligned}&\mathbf{r} = r\hat{\mathbf{r}}\\&\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}\\&\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \left(\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right)\hat{\theta}\end{aligned}}$$

- Specific angular momentum conservation: $h = r^2\dot{\theta}$
- Kepler's Second Law true for **any** central force:

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}h$$

Two Body ODE – Reciprocal Distance

- Match radial components in SSD 2.5

$$\boxed{\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \quad (\text{SSD } 2.11)}$$

- Introduce substitution $\boxed{u = 1/r}$ and use constant $h = r^2\dot{\theta}$
- Differentiate r with respect to time twice to get SSD 2.12

$$\boxed{r = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -h\frac{du}{d\theta}} \qquad \boxed{\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2}}$$

- Transformed version of SSD 2.11 in terms of u

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \quad (\text{SSD } 2.13)}$$

Two Body ODE – Solution

- Equation SSD 2.13 has an analytical solution!

$$\boxed{u = \frac{\mu}{h^2}[1 + e\cos(\theta - \varpi)] \quad (\mathrm{SSD}\ 2.14)}$$

- Express solution in terms of radial distance r

$$\boxed{r = \frac{p}{1 + e\cos(\theta - \varpi)} \quad (\mathrm{SSD}\ 2.15)} \qquad \boxed{p = \frac{h^2}{\mu} \quad (\mathrm{SSD}\ 2.12)}$$

- *p* is called the **semi-latus rectum**
- *e* is called the **eccentricity**
- ϖ is called the **longitude of pericenter**

Elliptical Orbits for Two Body Solution

- a is the **semi-major axis**
- b is the semi-minor axis

$$b^2 = a^2(1 - e^2) \quad \text{(SSD 2.18)}$$

- Polar equation of ellipse

$$\boxed{r = \frac{a \cdot (1 - e^2)}{1 + e\cos(f)} \quad \text{(SSD 2.20)}}$$

- f is called the **true anomaly**
- Transformation to Cartesian coordinates

$$x = r\cos f \ \text{(SSD 2.21)}$$
$$y = r\sin f$$

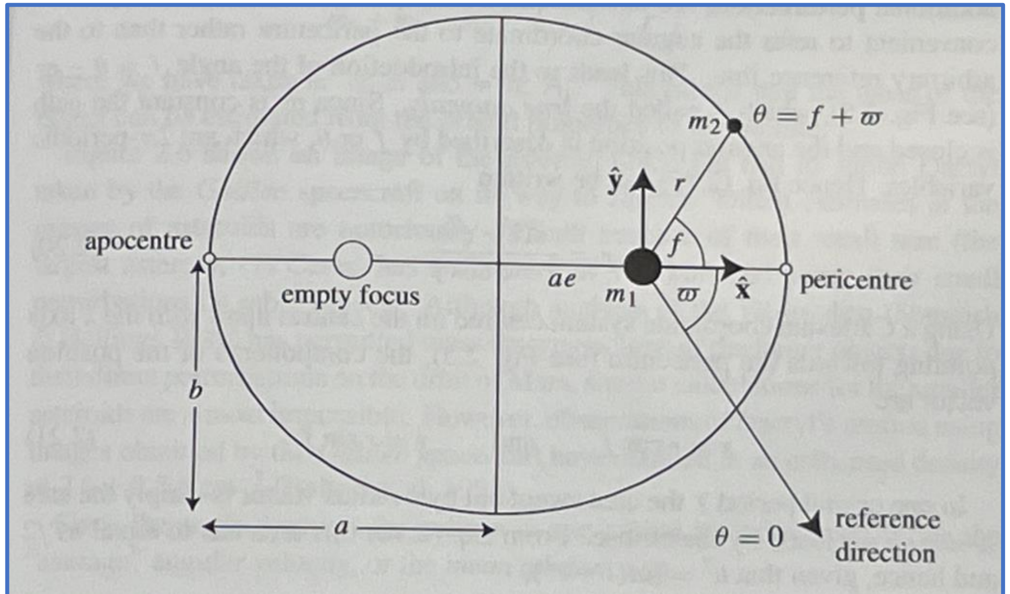


Fig. 2.5. The geometry of the ellipse of semi-major axis a, semi-minor axis b, eccentricity e, and longitude of pericentre ϖ.

- When f=0, r = a(1-e) ; called **pericenter**
- When f=π, r = 1(1+e) ; called **apocenter**
- Fastest at pericenter, slowest at apocenter
- Semi-major axis a sets size of ellipse
- Eccentricity e sets how elongated it is
- f controls position along the orbit

Keplerian Orbital Elements

Mean Motion n and Mean Anomaly M

- The orbital period T is $\boxed{T^2 = \left({4\pi^2}/{\mu}\right) a^3 \quad \text{(SSD 2.22)}}$
- Consistent with Kepler's Third Law!
- Angle θ sweeps out 2π radians per orbital period T
- Define the **mean motion** n as average angular velocity

$$\boxed{n = 2\pi/T \quad \text{(SSD 2.25)}}$$

- n also relates to μ and specific angular momentum h via

$$\boxed{\mu = n^2 a^3 \quad h = na^2\sqrt{1-e^2} = \sqrt{\mu a(1-e^2)} \quad \text{(SSD 2.26)}}$$

- Define the **mean anomaly** M as a fictitious angle that's linear in time

$$\boxed{M = n(t-\tau) \quad \text{(SSD 2.39)}}$$

- τ is called the **time of pericenter passage** and a constant

Eccentric and True Anomaly

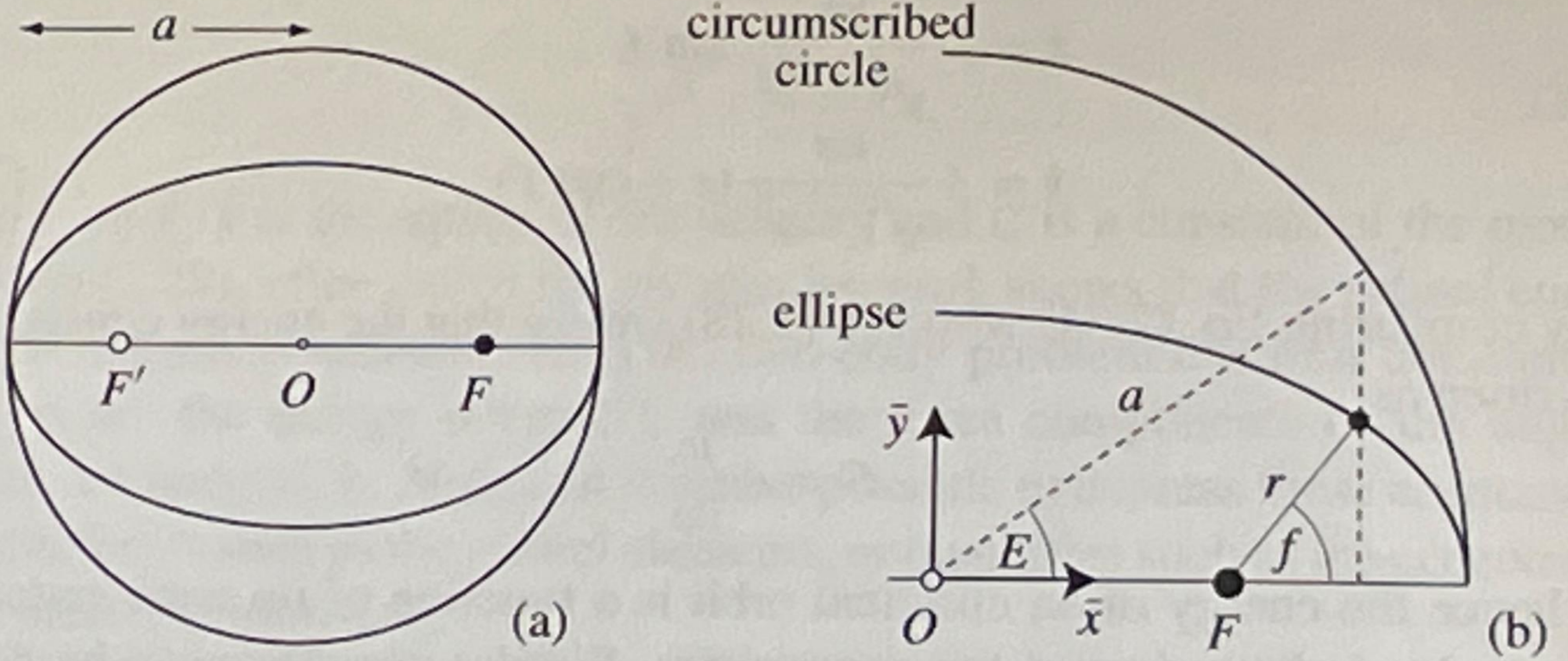


Fig. 2.7. (a) The circumscribed, concentric circle has a radius a equal to the semi-major axis of the ellipse. (b) The relationship between the true anomaly f and the eccentric anomaly E.

Eccentric Anomaly Equations

- Centered ellipse equation

$$\left(\frac{\bar{x}}{a}\right)^2 + \left(\frac{\bar{y}}{b}\right)^2 = 1 \quad \text{(SSD 2.40)}$$

- Cartesian coordinates from a and E

$$x = a(cosE - e) \quad y = a\sqrt{1-e^2}\sin E \quad \text{(SSD 2.41)}$$

- Distance r from a, e and E (Do $x^2 + y^2$)

$$r = a(1 - e\cos E) \quad \text{(SSD 2.42)}$$

- Relationship between E and f – useful!

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right) \quad \text{(SSD 2.46)}$$

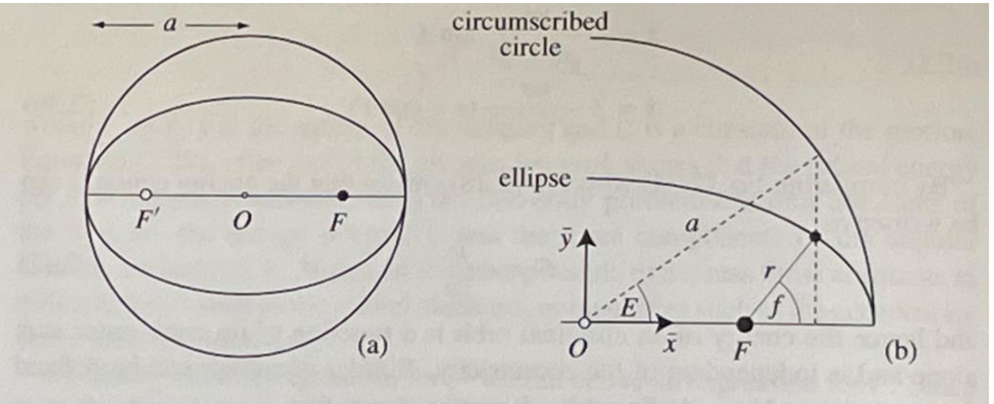


Fig. 2.7. (a) The circumscribed, concentric circle has a radius a equal to the semi-major axis of the ellipse. (b) The relationship between the true anomaly f and the eccentric anomaly E.

Kepler’s Equation: Relating M and E

- Calculate $\dot{x}$ and $\dot{y}$ from SSD 2.41, then compute $\mathbf{r} \times \dot{\mathbf{r}}$
- The magnitude of this is equal to h; use SSD 2.26 and

$$\boxed{\frac{dE}{dt} = \frac{n}{1 - e\cos E} \quad (\text{SSD } 2.50)}$$

- Integrate this and match the boundary condition E=0 at t=τ and

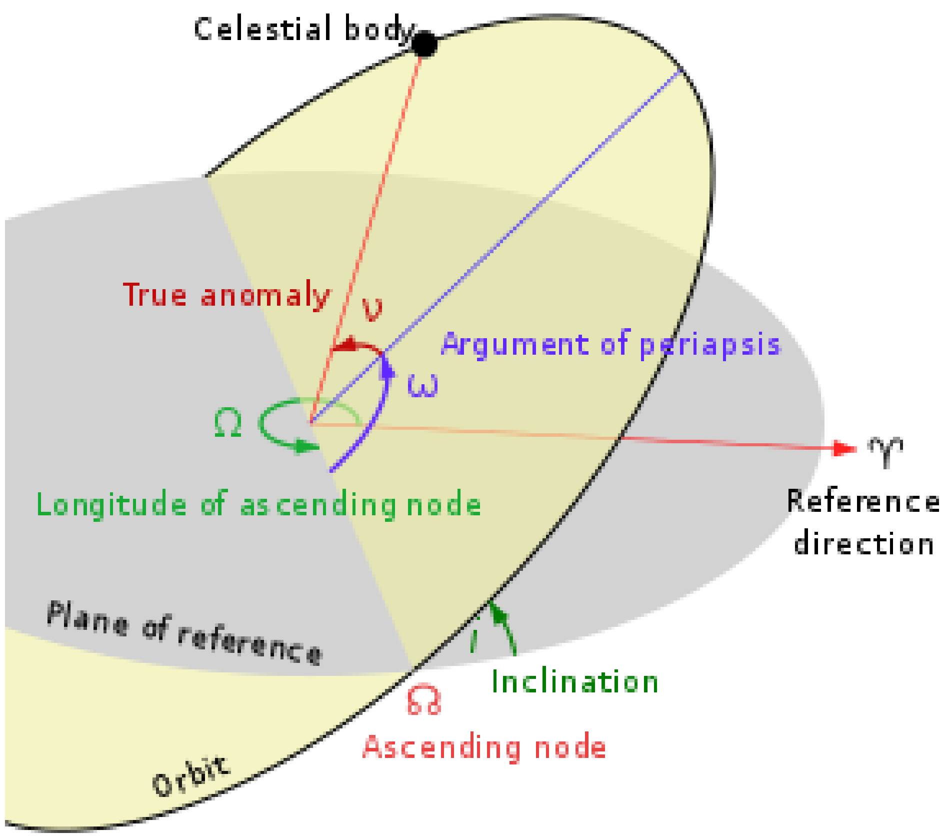
$$\boxed{M = E - e\sin E \quad (\text{SSD } 2.52)}$$

- This is the famous **Kepler’s Equation**
  - Kepler’s Equation cannot be solved analytically for E as a function of M
  - But it can be solved numerically efficiently using Newton-Raphson
- This method allows us to compute two orbits as a function of time!

Orienting the Orbit in Space

- We've perfectly solved the two body problem in its orbital plane
- The orbit is described by 3 elements: a, e, f
- If you only wanted the orbit of one body around the Sun, you could choose the frame so the ecliptic was the XY plane
- But there are many bodies of interest in the Solar System all orbiting in different planes
- We would like a formalism to permit us to orient our planar 2D orbits into 3D space with one preferred frame (e.g. BME)
- This is what the remaining orbital elements i, Ω, ω do

Keplerian Orbital Elements



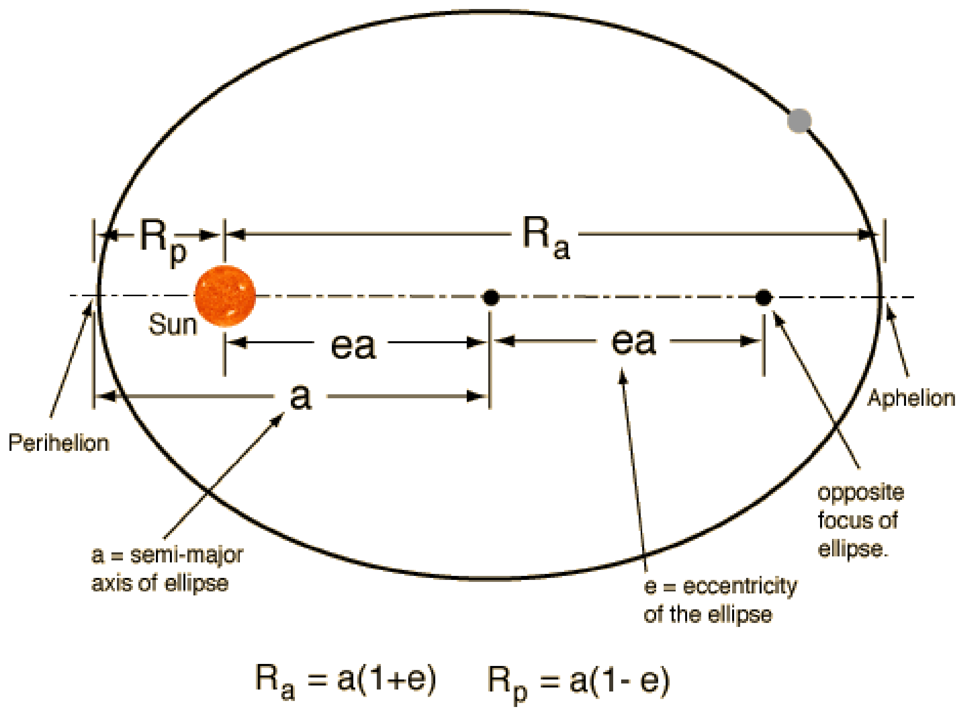


Image Credits: WikiPedia, Cool Cosmos

- Semi-major axis *a* and eccentricity *e* describe the size and shape of the orbital ellipse
- Inclination *i*, ascending node Ω, perihelion ω are angles orienting orbit in the ecliptic plane
- True anomaly *f* is location of the body on its orbital ellipse

Mapping Elements to Cartesian Coordinates

- The three angles define rotation matrices
- The equation mapping from orbital elements to Cartesian coordinates

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = r \begin{bmatrix} \cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos I \\ \sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos I \\ \sin(\omega+f)\sin I \end{bmatrix} \qquad \text{(SSD 2.122)}$$

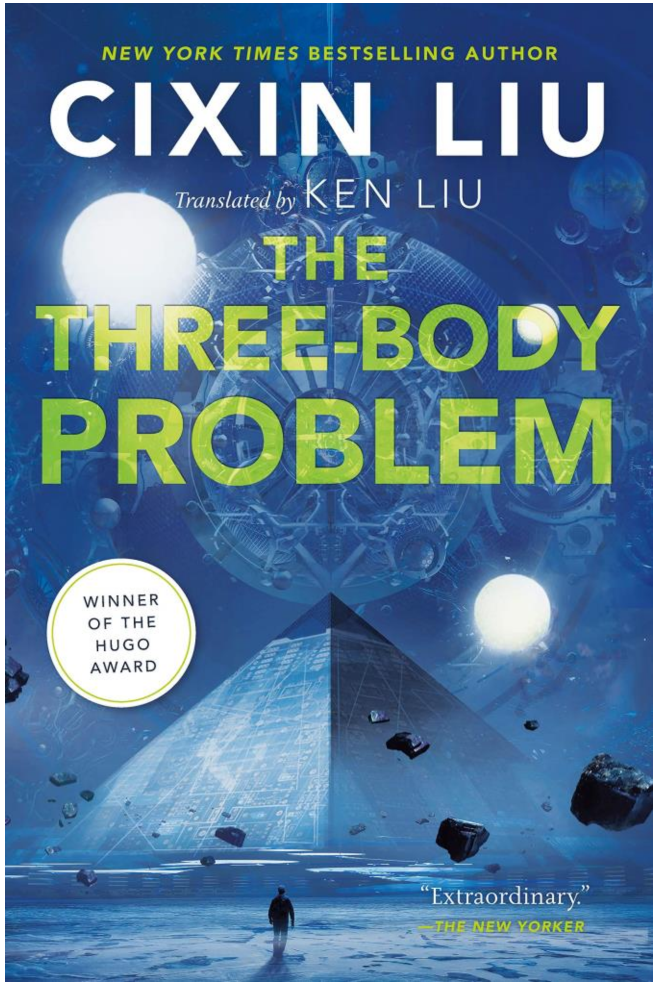
- This can be inverted, but it's a bit messy, and sometimes the answer is not unique (e.g. if e=0)
- The most common scenario is to convert from (a, e, I, Ω, ω, f) to (q_x, q_y, q_z, v_x, v_y, v_z)
- It's also possible to fit Kepler elements from Cartesian vectors

Three Body Solution: Analytic Solution

Just Kidding

- Just kidding! There is no known complete analytical solution to the full Three Body Problem
- There is a beautiful solution to the Restricted Three Body Problem
- That has two massive bodies (think Sun and Jupiter) and solves for the motion of a massless “test particle” (think Earth, approximately)

The Three-Body Problem: Sci Fi



N-Body Problem:
Numerical Solution with Rebound

Reality Check: Numerical Integrators!

- The analytical solution to the two body solution is elegant
- It's historically important and gives useful, fast approximations for the Solar System, but…
- Reality check: if you want Solar System orbits "for real" and not on a school homework problem…
- You are going to do it numerically on a computer!
- While Python odeint is a good general purpose tool…
- Gravitational N-body integrators are a well studied problem with highly optimized packages available

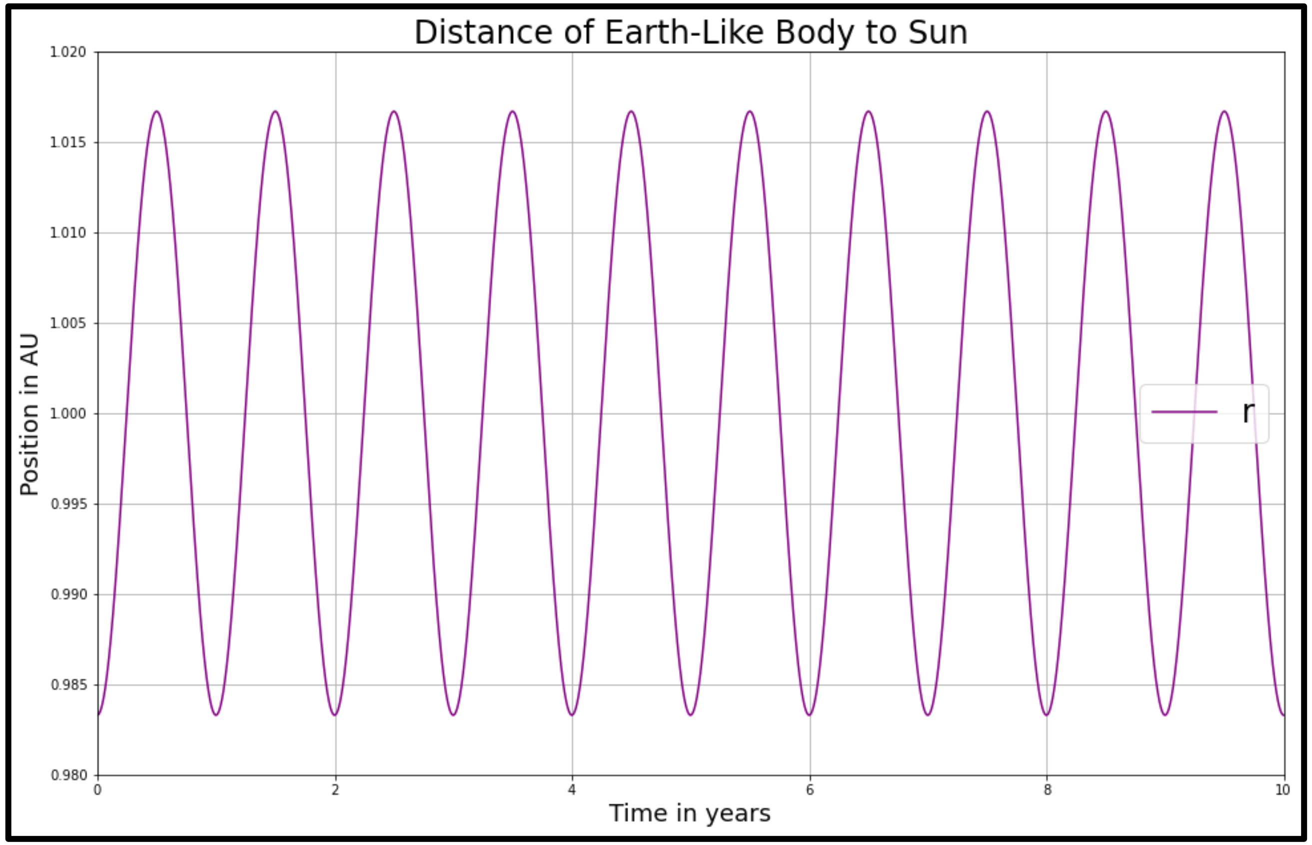
REBOUND Integrator for N-Body Problem

- REBOUND is a modern, open source integrator
  - It numerically solves the gravitational N-body problem
  - IAS15 adaptive integrator uses Gauss-Radau quadrature and a “predictor-corrector” scheme
  - github.com/hannorein/rebound and PyPI
- Horizons: API provided by NASA JPL to obtain state vectors (position and velocity) of objects in the Solar System
  - Considered “gold standard” for initial conditions of an integration

First Simulation with Rebound

- See 01-GetStarted.ipynb
- Key statements:

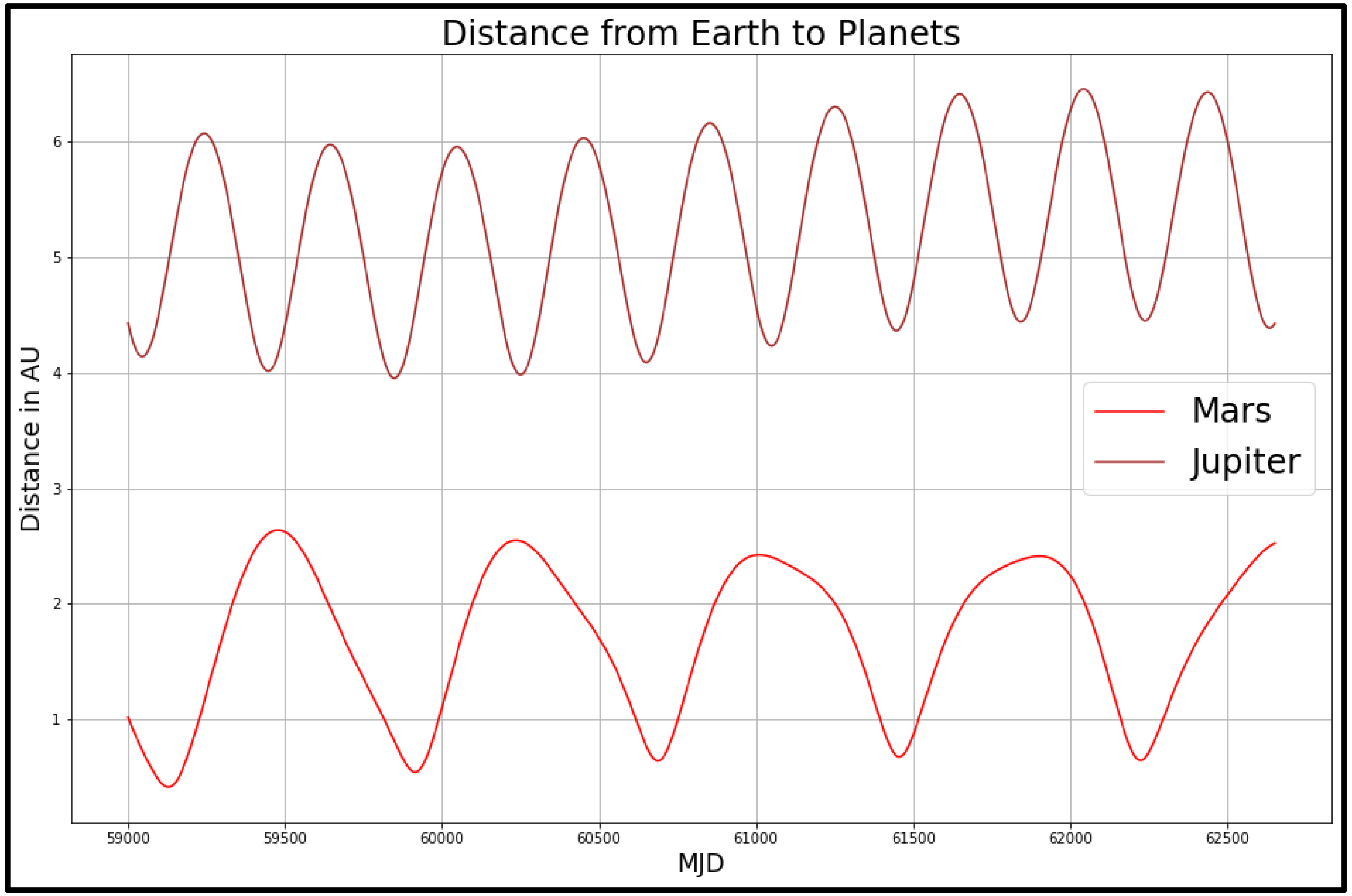
```
import rebound
sim = rebound.Simulation()
sim.units = ('day', 'AU', 'Msun')
sim.add(m=1.0)
sim.add(m=3.003E-6, a=1.0, e=0.0167)
sim.integrate(t, exact_finish_time=1)
sim.serialize_particle_data(xyz=q[i])
```



Getting the Initial Conditions from Horizons

- See 02-Planets.ipynb
- Key statements:

```
body_names = [‘Sun’, ‘Earth’,
‘Moon’, ‘Jupiter Barycenter’…]
date = ‘JD240059000.5’
sim.t = 59000.0
sim.add(body, date=date)
sim.save(‘planets_59000.bin’)
sim = rebound.Simulation(
    ‘planets_59000.bin’)
```

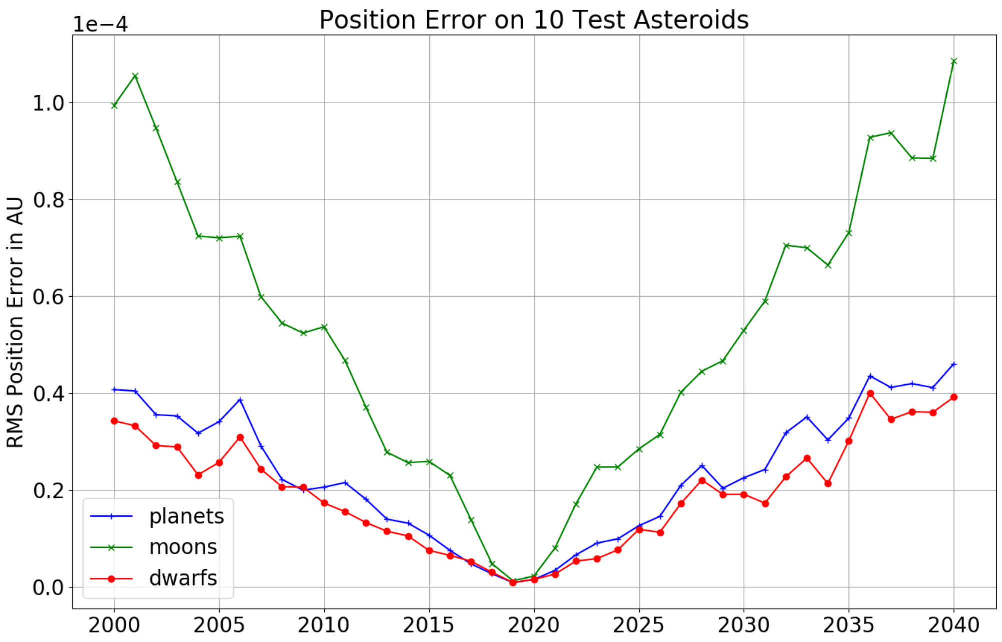


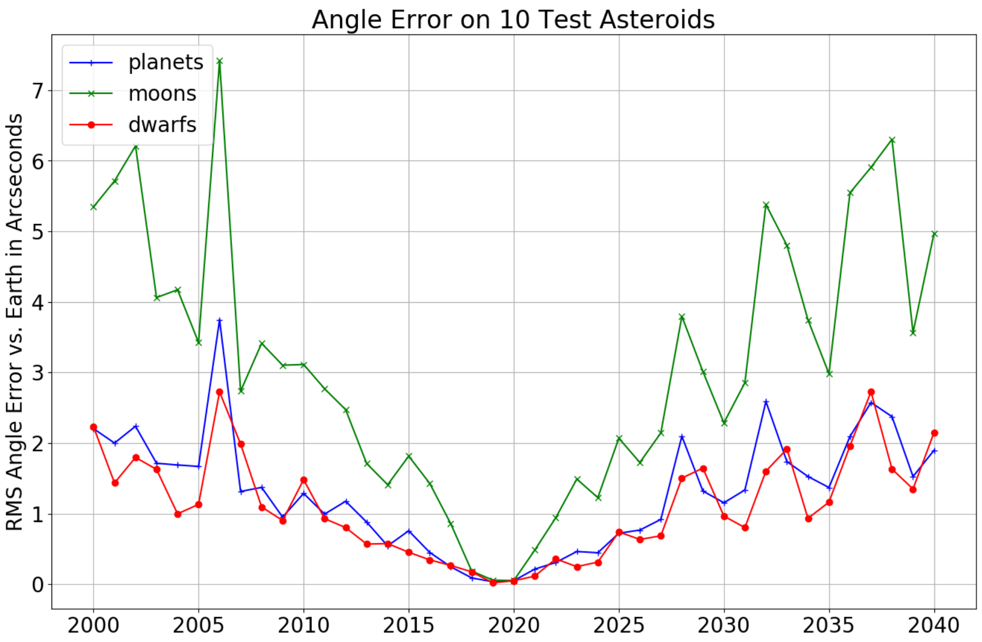
Appendix

Some uses I made of REBOUND in my own research

Validating Integration vs. Horizons

- Integrate three massive body collections and asteroids
- Error metrics: position in (AU) and instantaneous angle from asteroid to Earth (arc seconds)
- Accuracy is excellent!
  - RMS error on planets is 5.4E-6 AU
  - Angle error from asteroids to planets 0.8 arc seconds





Bulk Integration of 733,489 Asteroids

```
# Load all the asteroid elements
ast_elt = load_ast_elt()
```

```
ast_elt
```

	Num	Name	epoch	a	e	inc	Omega	omega	M	H	G	Ref	f
Num													
1	1	Ceres	58600.0	2.769165	0.076009	0.184901	1.401596	1.284522	1.350398	3.34	0.12	JPL 46	1.501306
2	2	Pallas	58600.0	2.772466	0.230337	0.608007	3.020817	5.411373	1.041946	4.13	0.11	JPL 35	1.490912
3	3	Juno	58600.0	2.669150	0.256942	0.226699	2.964490	4.330836	0.609557	5.33	0.32	JPL 108	0.996719
4	4	Vesta	58600.0	2.361418	0.088721	0.124647	1.811840	2.630709	1.673106	3.20	0.32	JPL 34	-4.436417
5	5	Astraea	58600.0	2.574249	0.191095	0.093672	2.470978	6.260280	4.928221	6.85	0.15	JPL 108	-1.738676
...	...	...	...	...	...	...	...	...	...	...	...	...	...
1255499	1255499	2019 QG	58600.0	0.822197	0.237862	0.220677	5.066979	3.770460	0.503214	21.55	0.15	JPL 1	0.807024
1255501	1255501	2019 QL	58600.0	2.722045	0.530676	0.113833	4.741919	2.351059	5.297173	19.21	0.15	JPL 1	-2.082964
1255502	1255502	2019 QQ	58600.0	1.053137	0.389091	0.172121	5.648270	2.028352	3.266522	25.31	0.15	JPL 1	-3.081905
1255513	1255513	6331 P-L	58600.0	2.334803	0.282830	0.141058	6.200287	0.091869	2.609695	18.50	0.15	JPL 8	2.827595
1255514	1255514	6344 P-L	58600.0	2.812944	0.664688	0.081955	3.199363	4.094863	2.738525	20.40	0.15	JPL 17	3.032066

733489 rows × 19 columns

Validate Asteroid Integration vs. Horizons

- Test bulk asteroid integration on first 25 IAU asteroids
- Excellent results! RMS 2.49E-6 AU and 0.45 arc seconds