Firedrake Notes

Supervisors: Dr Pawar, Prof Piggott, Dr Sebille • 2017-18 • Imperial College London

Last Revision: December 8, 2017

Table of Contents

Key Notes to NEVER FORGET

"Integration by Parts"

This comes up constantly in FEM stuff, when expressing problems in variational form. The usual spiel is to say "multiply by a test function v and then integrate by parts", to obtain the desired form. This hides a number of key subtle steps that otherwise look like magic.

First to note is that "integrate by parts" really means "apply (a corollary of) the Divergence Theorem":

Theorem 1.1 (Divergence Theorem)**.** If Ω is a compact subset of $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega = \Gamma$, and if $\boldsymbol{F}$ is a continuously differentiable vevctor field defined on a neighbourhood of Ω then we have:

$$\int_\Omega (\nabla \cdot \boldsymbol{F}) \; d\Omega = \oiint_\Gamma (\boldsymbol{F} \cdot \boldsymbol{n}) \; d\Gamma$$

where $\boldsymbol{n}$ is the outward pointing unit normal field of the boundary Γ.

Corollary 1.1. Replacing $\boldsymbol{F}$ with $\boldsymbol{F}g$ in the theorem, where g is a scalar function, we get:

$$\int_\Omega \boldsymbol{F} \cdot (\nabla g) \; d\Omega + \int_\Omega g(\nabla \cdot \boldsymbol{F}) \; d\Omega = \oiint_\Gamma g\boldsymbol{F} \cdot \boldsymbol{n} \; d\Gamma$$

We can apply this corollary to the LHS (i.e. to the terms involving u) to rewrite it as the sum of a different volume integral and a surface integral, which can often be made to vanish by applying boundary conditions.

Example: Linear Poisson Equation

Let us take an initial easy example of the basic linear Poisson problem:

$$\begin{aligned} (-\Delta u) &= f, \text{ on } \Omega \\ u &= 0, \text{ on } \partial\Omega = \Gamma \end{aligned}$$

We multiply both sides by the test function v and integrate to obtain:

$$\int_\Omega (-\Delta u)v \; d\Omega = \int_\Omega fv \; d\Omega$$

Now we apply the Corollary to the LHS (replacing $\boldsymbol{F}$ with ∇u and g with v) to get:

$$\int_\Omega \nabla u \cdot \nabla v \; d\Omega + \oiint_\Gamma v\nabla u \cdot \boldsymbol{n} \; d\Omega = \int_\Omega (-\Delta u)v \; d\Omega = \int_\Omega fv \; d\Omega$$

The second term in the new LHS is a closed line integral of a grad function and thus equal to the difference of its endpoints, which are the same, hence the term is zero, leaving us with the desired variational form $a(u, v) = L(v)$:

$$\int_\Omega \nabla u \cdot \nabla v \; d\Omega = \int_\Omega fv \; d\Omega$$

Example: Nonlinear Poisson Equation

Let's now look at the following nonlinear Poisson problem:

$$\begin{aligned} -\nabla \cdot \big((1+u)\nabla u\big) &= f, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega = \Gamma \end{aligned}$$

We multiply by the test function v and integrate both sides:

$$\int_\Omega \Big(-\nabla \cdot \big((1+u)\nabla u\big)\Big) v \; d\Omega = \int_\Omega fv \; d\Omega$$

Again we apply the Corollary to the LHS (replacing $\boldsymbol{F}$ with $\big((1+u)\nabla u\big)$ and g with v) to get:

$$\int_\Omega \big((1+u)\nabla u\big) \cdot \nabla v \; d\Omega + \oiint_\Gamma v\Big(\big((1+u)\nabla u\big)\Big) \cdot \boldsymbol{n} \; d\Gamma = \int_\Omega \Big(-\nabla \cdot \big((1+u)\nabla u\big)\Big) v \; d\Omega = \int_\Omega fv \; d\Omega$$

Looking again at the surface integral term on the new LHS, we recall the initial condition $u = 0$ on Γ and thus this term simplifies to:

$$\oiint_\Gamma v\big(\nabla u \cdot \boldsymbol{n}\big) \; d\Gamma = 0 \qquad \text{(closed line integral of a grad function)}$$

Leaving us with the desired variational form $F(u; v) = 0$:

$$\int_\Omega \big((1+u)\nabla u\big) \cdot \nabla v \; d\Omega = \int_\Omega fv \; d\Omega$$

References

- https://en.wikipedia.org/wiki/Divergence_theorem
- https://en.wikipedia.org/wiki/Surface_integral
- http://mathinsight.org/gradient_theorem_line_integrals