Exercises - MaxEnt

1a) $f_0 = 1$ $\qquad f_1 = i$ $\qquad \langle f_0 \rangle\ 1, \quad \langle f_1 \rangle = 2.5$

$$p_i = \frac{e^{-\lambda_0 - \lambda_1 i}}{\sum_{j=1}^{3} e^{-\lambda_0 - \lambda_1 i}} = \frac{e^{-\lambda_1 i}}{\sum_{i=1}^{3} e^{-\lambda_1 i}}$$

using constraint $\langle f_1 \rangle$: $\quad \sum_{i=1}^{3} i p_i = 2.5$

$$\Rightarrow \frac{\sum_{i=1}^{3} i e^{-\lambda_1 i}}{\sum_{i=1}^{3} e^{-\lambda_1 i}} = 2.5$$

$$\Leftrightarrow \frac{e^{-\lambda_1} + 2e^{-2\lambda_1} + 3e^{-3\lambda_1}}{e^{-\lambda_1} + e^{-2\lambda_1} + e^{-3\lambda_1}} = 2.5$$

$e^{-\lambda} \neq 0 \quad \forall \lambda$ and $x := e^{-\lambda}$

$$\Rightarrow 1 + 2x + 3x^2 = 2.5(1 + x + x^2)$$

$$\Leftrightarrow 0.5x^2 - 0.5x - 1.5 = 0$$

$$\Leftrightarrow x^2 - x - 3 = 0$$

$$\Leftrightarrow x_{1,2} = \frac{1 \pm \sqrt{1+12}}{2}$$

as $e^{-\lambda} > 0$, only $x_1 = \frac{1}{2} + \frac{\sqrt{13}}{2}$ makes sense

$$\Rightarrow \lambda = -\ln\left(\frac{1}{2} + \frac{\sqrt{13}}{2}\right) \approx -0.83$$

$$\Rightarrow p_i = \frac{1}{Z} e^{0.83 i}$$

b) as (a) up to... $\quad 1 + 2x + 3x^2 = 2(1 + x + x^2)$

$$\Leftrightarrow x^2 - 1 = 0 \quad \Rightarrow \quad x = 1 \quad \Rightarrow \lambda = 0$$

$$\Rightarrow p_i = \frac{1}{Z} e^{-0 \cdot i} = \frac{1}{Z} = \frac{1}{3}$$

c) $\sum_{i=1}^{3} p_i = 1$ $\quad f_0 = 1$

$\sum_{i=1}^{3} i p_i = \text{mean } \mu$ $\quad f_1(i) = i$

$\sum_{i=1}^{n} (i-\mu)^2 p_i = \text{variance}$ $\quad f_2(i) = (i-\mu)^2$

$$\Rightarrow \quad p_i = \frac{e^{-i\lambda_1 - (i-\mu)^2 \lambda_2}}{Z}$$

d) think about a Gaussian with infinite variance...

2a) state variables: A_0, n_0

constraints: $\sum_{n=1}^{n_0} \pi(n) = 1$ $\quad f_0(n) = 1$

$\sum_{n=1}^{n_0} n \pi(n) = \frac{n_0 A}{A_0}$ $\quad f_1(n) = n$

b) geometric

Counting

1a) [xy|] [x|y] [y|x] [|xy]

b) (2|0) (1|1)

c) [xy |] 4 configurations
[x | y] 12 configurations
$\Rightarrow$ P((2|0|0|0)) = $\frac{1}{4}$, P((1|1|0|0)) = $\frac{3}{4}$

2a) [xx|] [x|x] [|xx]

b) [xx |] 4 configurations
[x | x] 6 configurations
$\Rightarrow$ P((2|0|0|0)) = $\frac{2}{5}$, P((1|1|0|0)) = $\frac{3}{5}$

c) $\left[\begin{array}{c|c} \times & \\ \hline & \end{array}\right]$ starting point

„drop" another individual

$$P\left(\left[\begin{array}{c|c} \times\times & \\ \hline & \end{array}\right]\right) = P(c_1) = \frac{2}{5}$$

$$P\left(\left[\begin{array}{c|c} \times & \times \\ \hline & \end{array}\right]\right) = P(c_2) + P(c_3) + P(c_4)$$

$$= 3 \cdot \frac{1}{5} = \frac{3}{5}$$

Degree Distribution

L = # interaction links $\qquad S$ = # species

$P(d)$ = prob. of species having d incoming / outgoing / total links

constraints: $\sum_{d=0}^{L} P(d) = 1$

$$\sum_{d=0}^{L} d\,P(d) = \frac{L}{S} \quad \text{or} \quad \frac{2L}{S}$$

$\Rightarrow$ MaxEnt prediction: geometric distribution

other constraints: # primary producers / top consumers (with no incoming / outgoing links),

maximum degree (as some functional constraint)

...

Commonality Some ideas can be found in John's book, chapter 11.1.