PRACTICAL WORK 4

The calculation of the basic reproductive number

For measles, the duration of the infectious period is about a week. Infected children tend to make 2-3 infectious contacts per day. Can you approximate the basic reproductive number?

Based on your calculation, what fraction of the population needs to be vaccinated in order to prevent epidemics?

*Children normally will have had their first measles vaccination by their second birthday and a booster before their 5^{th} birthday. In the first months of their lives, they will carry protection from infection through maternal antibodies. What would be the effect of dropping the first vaccination and relying entirely on the second? What fraction of children would you need to immunise at age 5 to prevent a measles epidemic?

Investigate the dynamics of the SIR model

$$\frac{dS}{dt} = \mu(N-S) - \beta S\frac{I}{N}$$
$$\frac{dP}{dt} = \beta S\frac{I}{N} - \gamma I - \mu I$$
$$\frac{dR}{dt} = \gamma I - \mu R$$

Calculate the equilibrium and assess the stability of the equilibrium. Can the model be unstable? Check your prediction with XPP. The model is in the file `sir.ode`. Set b=1000 and look at the dynamics. Is there any cyclic behaviour?

Investigate the periodically forced SIR model

The model we will use for this is, again, given by SIR model, but the transmission rate now varies periodically, to reflect seasonal variation in the transmission rates, as would be caused by school holidays. The total variables in the model express the fractions of the population that are in different classes. Therefore S+I+R=1. Because of this, it suffices to describe the dynamics of S and I.

$$\frac{dS}{dt} = \mu(1-S) - \beta(t)SI$$
$$\frac{dI}{dt} = \beta(t)SI - \gamma I - \mu I$$

Where $\beta(t) = \mathrm{b}(1 + \mathrm{amp} * \sin(\mathrm{wt}))$

In the file `frcdsir.ode` the system above is implemented. We generated the sine function through adding two extra parameters to the model in the file, but that is a technical fix (see Ermentrout 2002) In the model implemented in this file the number of infected individuals is calculated is the ln(I) (see exercise below about how to do this). This was done to make visualisation of small numbers easier, and it facilitates the numerics. The parameter w is set to 6.28… (=2π, chosen to make sure the variation in transmission follows an annual cycle, mu=0.0133 and gamma=50. This parameter combination was chosen to describe the epidemiology of measles.

Open the model in XPP and check that the parameter b is set to 1000 (corresponding to an R0 of approx. 20). First plot the nr of lnI against time. What do you see? Is there an obvious period to the oscillation? If you can't really see output well, reduce the amount of time you plot the output for. Compare your results to the observed dynamics of measles in the pre-vaccination era.

This is not the only possible output that the model can generate. Change the value of b to 500. What do you see now? Extend the simulation if need to make sure the system settles at a cycle (it will for these parameters and initial conditions.) What is the period of the solution you have now found?

Next, let's try to construct a bifurcation diagram. Either restart the program (or set b=1000), and plot the results in the phase plane (S vs I). Run the model forward for sufficiently long time, you will need to use *Initialconds* and then (*L)ast* for this, until your output settles on a limit cycle. You will need to *Erase* your diagram from time to time to make sure you can actually see the output. Make sure that the dynamics have properly settled on the period 2 cycle. Before being able to take this to AUTO, you will need to tell the programme what the period of this cycle is. Go to *n(U)merics* select *Total* and set this to the period of the cycle, which we know is 2 in this case. Press Esc to leave this menu and run *Initialconds* and then (*G)o* once more. This last step is important otherwise the new period will not be conveyed to AUTO.

Go to *File* and then *Auto*. Select *Run*, and then Periodic. Once this is done, continue the curve backwards. You will now see a curve that is a continuation over the period 2 cycle. There are some funny kinks in it. To make sense of this, investigate the curve at various points. *Grab* a value, then go to the XPP window, and investigate the phase portraits.

After you have done this, use grab and select a green limit cycle of which you know the period is 1 for a high value of b, around 1500. Now go through the following sequence: First use *Grab* then go to *File* and select *Clear Grab*. This clears the value from AUTO, but retains it for the XPP window. Go to your XPP window Go to *n(U)merics* select *Total* and set this to the period of the cycle: 1. Press Esc to leave this menu and run *Initialconds* and then (*G)o* once more. Go to *File* and then *Auto*. Select *Run*, and then Periodic. A missing bit of the diagram disappears. Investigate the diagram and try to make sense of it.

Finding and continuing a period 3 cycle in the forced SIR model.

In Earn et al. (2000) cycles of different length are shown. Can we find these? The answer is, not easily, because these cycles do not connect to the diagram that we already have. But they are there and we can see them. To make the period 3 cycle visible, go through the following sequence. Use *Grab* and select a green limit cycle for a value of b close to 380. (see lecture for explanation of why this value) of which you. Use *Grab* then go to *File* and select *Clear Grab*. This clears the value from AUTO, but retains it for the XPP window. Go to your XPP window Go to *n(U)merics* select *Total* and set this to 3. Press Esc to leave this menu and run *Initialconds* and then (*G)o* once more. You will still see the period one cycle appearing. To find the period 3 cycle you will need to be lucky enough to find out. Either just try your luck with different initial conditions. If you are not lucky set the initial conditions (use ICs) to S=0.13, lnI= -10, U=1, V=0. Press *Initialconds* and then (*G)o,* followed by *Initialconds* and then (*L)ast* a few times. You now should see the phase portrait of the 3 cycle. How do you know it is a 3 cycle?

Go to *File* and then *Auto*. Select *Run*, and then Periodic. The period 3 cycle now appears in your diagram. If you find the diagram too messy, change the axes to either *Average*, or *High*. Use *Grab* to go through the result, and notice special TYpes, and look at the values of the multipliers.

In the paper by Krylova and Earn (2013) there is further analysis of the cycles of forced epidemic models. In the supplementary material to the paper there is a detailed guide to how they produces their figures using XPPAUT. The supplementary material is at
http://rsif.royalsocietypublishing.org/content/royinterface/suppl/2013/05/09/rsif.2013.0098.DC1/rsif20130098supp2.pdf

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