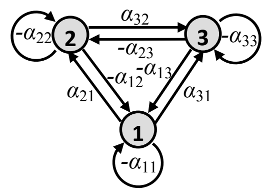
The following set of population dynamic equations represents a fully-connected 3-species food web (the graph (network) is also shown) with a single primary producer (species 1),

$$\frac{dx_1}{dt} = x_1\left(b_1 - d_1 - \alpha_{11}x_1 - \alpha_{12}x_2 - \alpha_{13}x_3\right)$$
$$\frac{dx_2}{dt} = x_2\left(-d_2 + \alpha_{21}x_1 - \alpha_{22}x_2 - \alpha_{23}x_3\right)$$
$$\frac{dx_3}{dt} = x_3\left(-d_3 + \alpha_{31}x_1 + \alpha_{32}x_2 - \alpha_{33}x_3\right)$$



For the i^{th} species, x_i is population density, b_i the intrinsic birth rate (0 for all consumers), d_i the intrinsic death rate, α_{ij} the rate at which an individual of consumer j searches for individuals of resource i, and α_{ii} the intraspecific interference rate (also a search rate). In matrix notation, this system can be represented as

$$\frac{d\mathbf{x}}{dt} = diag\{x_1, x_2, x_3\}(\mathbf{b} - \mathbf{d} + \mathbf{Ax}), \text{ i.e.,}$$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} \left(\begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + \left(\begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ a_{21} & -a_{22} & -a_{23} \\ a_{31} & a_{32} & -a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \right)$$

(1) **a**. Show that a necessary condition for the system to be stable (all species to coexist without going extinct), is det($\mathbf{A}$) $\neq 0$.
b. Calculate expressions for the three equilibrium densities that together constitute the stable interior point. (Hint: Used (a) to minimize the mind-numbing algebra needed to get this.)

(2) **a**. Calculate the jacobian matrix (**J**) of the system.
b. Show that **J** can be represented as $\mathbf{J} = \text{diag}(\hat{x}_i)\mathbf{A}$. (This is convenient because properties of just the interaction matrix **A**, and thus the food web network structure (e.g., see figure above), can be related to the leading eigenvalue of **J**. Recall that this eigenvalue determines whether the system's equilibrium is at least locally stable)

(3) Calculate and compare the necessary coexistence condition (1) above for the alternative scenarios where,
a. the given food web network is actually a food chain where species 3 eats 2 and 2 eats 1 (but 3 no longer eats 1), and
b. species 3 eats 1 and 2 eats 1 (but 3 no longer eats 2) (apparent competition).