

Foundations of Software Fall 2015

Week 3

Review (and more details)

Recall: Simple Arithmetic Expressions

The set \mathcal{T} of terms is defined by the following abstract grammar:

$t ::=$	<i>terms</i>
true	constant true
false	constant false
if t then t else t	conditional
0	constant zero
succ t	successor
pred t	predecessor
iszero t	zero test

Recall: Inference Rule Notation

More explicitly: The set \mathcal{T} is the *smallest* set *closed* under the following rules.

$$\begin{array}{c} \text{true} \in \mathcal{T} \qquad \text{false} \in \mathcal{T} \qquad 0 \in \mathcal{T} \\ \hline \frac{t_1 \in \mathcal{T}}{\text{succ } t_1 \in \mathcal{T}} \qquad \frac{t_1 \in \mathcal{T}}{\text{pred } t_1 \in \mathcal{T}} \qquad \frac{t_1 \in \mathcal{T}}{\text{iszero } t_1 \in \mathcal{T}} \\ \hline \frac{t_1 \in \mathcal{T} \quad t_2 \in \mathcal{T} \quad t_3 \in \mathcal{T}}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in \mathcal{T}} \end{array}$$

Generating Functions

Each of these rules can be thought of as a *generating function* that, given some elements from \mathcal{T} , generates some other element of \mathcal{T} . Saying that \mathcal{T} is closed under these rules means that \mathcal{T} cannot be made any bigger using these generating functions — it already contains everything “justified by its members.”

$$\begin{array}{c} \text{true} \in \mathcal{T} \\ \hline \text{succ } t_1 \in \mathcal{T} \\ \\ \text{false} \in \mathcal{T} \\ \hline \text{pred } t_1 \in \mathcal{T} \\ \\ 0 \in \mathcal{T} \\ \hline \text{iszero } t_1 \in \mathcal{T} \\ \\ \frac{t_1 \in \mathcal{T} \quad t_2 \in \mathcal{T} \quad t_3 \in \mathcal{T}}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \in \mathcal{T}} \end{array}$$

Let's write these generating functions explicitly.

$$\begin{aligned} F_1(U) &= \{\text{true}\} \\ F_2(U) &= \{\text{false}\} \\ F_3(U) &= \{0\} \\ F_4(U) &= \{\text{succ } t_1 \mid t_1 \in U\} \\ F_5(U) &= \{\text{pred } t_1 \mid t_1 \in U\} \\ F_6(U) &= \{\text{iszero } t_1 \mid t_1 \in U\} \\ F_7(U) &= \{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid t_1, t_2, t_3 \in U\} \end{aligned}$$

Each one takes a set of terms U as input and produces a set of “terms justified by U ” as output.

If we now define a generating function for the whole set of inference rules (by combining the generating functions for the individual rules),

$$F(U) = F_1(U) \cup F_2(U) \cup F_3(U) \cup F_4(U) \cup F_5(U) \cup F_6(U) \cup F_7(U)$$

then we can restate the previous definition of the set of terms \mathcal{T} like this:

Definition:

- ▶ A set U is said to be “closed under F ” (or “ F -closed”) if $F(U) \subseteq U$.
- ▶ The set of terms \mathcal{T} is the smallest F -closed set. (I.e., if \mathcal{O} is another set such that $F(\mathcal{O}) \subseteq \mathcal{O}$, then $\mathcal{T} \subseteq \mathcal{O}$.)

Our alternate definition of the set of terms can also be stated using the generating function F :

$$\begin{aligned} S_0 &= \emptyset \\ S_{i+1} &= F(S_i) \end{aligned}$$

$$S = \bigcup_i S_i$$

Compare this definition of S with the one we saw last time:

$$\begin{aligned} S_0 &= \emptyset \\ S_{i+1} &= \{\text{true, false, 0}\} \\ &\cup \{\text{succ } t_1, \text{pred } t_1, \text{iszero } t_1 \mid t_1 \in S_i\} \\ &\cup \{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid t_1, t_2, t_3 \in S_i\} \end{aligned}$$

$$S = \bigcup_i S_i$$

We have “pulled out” F and given it a name.

Note that our two definitions of terms characterize the same set from different directions:

- ▶ “from above,” as the intersection of all F -closed sets;
- ▶ “from below,” as the limit (union) of a series of sets that start from \emptyset and get “closer and closer to being F -closed.”

Proposition 3.2.6 in the book shows that these two definitions actually define the same set.

Warning: Hard hats on for the next slide!

Structural Induction

The principle of structural induction on terms can also be re-stated using generating functions:

Suppose T is the smallest F -closed set.

If, for each set U ,

from the assumption “ $P(u)$ holds for every $u \in U$ ”

we can show “ $P(v)$ holds for any $v \in F(U)$,”

then $P(t)$ holds for all $t \in T$.

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Why?

Structural Induction

Why? Because:

- ▶ We assumed that T was the *smallest* F -closed set, i.e., that $T \subseteq O$ for any other F -closed set O .

- ▶ But showing

for each set U ,
given $P(u)$ for all $u \in U$
we can show $P(v)$ for all $v \in F(U)$

amounts to showing that “the set of all terms satisfying P ” (call it O) is itself an F -closed set.

- ▶ Since $T \subseteq O$, every element of T satisfies P .

Structural Induction

Compare this with the structural induction principle for terms from last lecture:

If, for each term s ,
given $P(r)$ for all immediate subterms r of s
we can show $P(s)$,
then $P(t)$ holds for all t .

Recall, from the definition of \mathcal{S} , it is clear that, if a term t is in \mathcal{S}_i , then all of its immediate subterms must be in \mathcal{S}_{i-1} , i.e., they must have strictly smaller depths. Therefore:

If, for each term s ,
given $P(r)$ for all immediate subterms r of s
we can show $P(s)$,
then $P(t)$ holds for all t .

Slightly more explicit proof:

- ▶ Assume that for each term s , given $P(r)$ for all immediate subterms of s , we can show $P(s)$.
- ▶ Then show, by induction on i , that $P(t)$ holds for all terms t with depth i .
- ▶ Therefore, $P(t)$ holds for all t .

Operational Semantics and Reasoning

Recall: Abstract Machines

An *abstract machine* consists of:

- ▶ a set of *states*
- ▶ a *transition relation* on states, written \longrightarrow

For the simple languages we are considering at the moment, the term being evaluated is the whole state of the abstract machine.

Recall: Syntax for Booleans

Terms and values

$t ::=$
 true
 false
 if t then t else t

terms
 constant true
 constant false
 conditional

$v ::=$
 true
 false

values
 true value
 false value

Recall: Operational Semantics for Booleans

The evaluation relation $t \longrightarrow t'$ is the smallest relation closed under the following rules:

if true then t_2 else $t_3 \longrightarrow t_2$ (E-IFTRUE)

if false then t_2 else $t_3 \longrightarrow t_3$ (E-IFFALSE)

$$\frac{t_1 \longrightarrow t'_1}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \longrightarrow \text{if } t'_1 \text{ then } t_2 \text{ else } t_3} \text{ (E-IF)}$$

Derivations

We can record the “justification” for a particular pair of terms that are in the evaluation relation in the form of a tree.

(on the board)

Terminology:

- ▶ These trees are called *derivation trees* (or just *derivations*).
- ▶ The final statement in a derivation is its *conclusion*.
- ▶ We say that the derivation is a *witness* for its conclusion (or a *proof* of its conclusion) — it records all the reasoning steps that justify the conclusion.

Observation

Lemma: Suppose we are given a derivation tree \mathcal{D} witnessing the pair (t, t') in the evaluation relation. Then either

1. the final rule used in \mathcal{D} is E-IFTRUE and we have $t = \text{if true then } t_2 \text{ else } t_3$ and $t' = t_2$, for some t_2 and t_3 , or
2. the final rule used in \mathcal{D} is E-IFFALSE and we have $t = \text{if false then } t_2 \text{ else } t_3$ and $t' = t_3$, for some t_2 and t_3 , or
3. the final rule used in \mathcal{D} is E-IF and we have $t = \text{if } t_1 \text{ then } t_2 \text{ else } t_3$ and $t' = \text{if } t'_1 \text{ then } t_2 \text{ else } t_3$, for some t_1, t'_1, t_2 , and t_3 ; moreover, the immediate subderivation of \mathcal{D} witnesses $(t_1, t'_1) \in \longrightarrow$.

Induction on Derivations

We can now write proofs about evaluation “by induction on derivation trees.”

Given an arbitrary derivation \mathcal{D} with conclusion $t \longrightarrow t'$, we assume the desired result for its immediate sub-derivation (if any) and proceed by a case analysis (using the previous lemma) of the final evaluation rule used in constructing the derivation tree.

E.g....

Induction on Derivations — Example

Theorem: If $t \longrightarrow t'$, i.e., if $(t, t') \in \longrightarrow$, then $\text{size}(t) > \text{size}(t')$.

Proof: By induction on a derivation \mathcal{D} of $t \longrightarrow t'$.

1. Suppose the final rule used in \mathcal{D} is E-IFTRUE, with $t = \text{if true then } t_2 \text{ else } t_3$ and $t' = t_2$. Then the result is immediate from the definition of *size*.
2. Suppose the final rule used in \mathcal{D} is E-IFFALSE, with $t = \text{if false then } t_2 \text{ else } t_3$ and $t' = t_3$. Then the result is again immediate from the definition of *size*.
3. Suppose the final rule used in \mathcal{D} is E-IF, with $t = \text{if } t_1 \text{ then } t_2 \text{ else } t_3$ and $t' = \text{if } t'_1 \text{ then } t_2 \text{ else } t_3$, where $(t_1, t'_1) \in \longrightarrow$ is witnessed by a derivation \mathcal{D}_1 . By the induction hypothesis, $\text{size}(t_1) > \text{size}(t'_1)$. But then, by the definition of *size*, we have $\text{size}(t) > \text{size}(t')$.

Normal forms

A *normal form* is a term that cannot be evaluated any further — i.e., a term t is a normal form (or “is in normal form”) if there is no t' such that $t \longrightarrow t'$.

A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a “result” of evaluation.

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Recall that we intended the set of *values* (the boolean constants `true` and `false`) to be exactly the possible “results of evaluation.” Did we get this definition right?

Values = normal forms

Theorem: A term t is a value iff it is in normal form.

Proof:

The \Rightarrow direction is immediate from the definition of the evaluation relation.

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For the \Leftarrow direction, it is convenient to prove the contrapositive:

If t is *not* a value, then it is *not* a normal form.

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For the \Leftarrow direction, it is convenient to prove the contrapositive: If t is *not* a value, then it is *not* a normal form. The argument goes by induction on t .

Note, first, that t must have the form `if t_1 then t_2 else t_3` (otherwise it would be a value). If t_1 is `true` or `false`, then rule E-IFTRUE or E-IFFALSE applies to t , and we are done.

Otherwise, t_1 is not a value and so, by the induction hypothesis, there is some t'_1 such that $t_1 \rightarrow t'_1$. But then rule E-IF yields

`if t_1 then t_2 else $t_3 \rightarrow$ if t'_1 then t_2 else t_3`

i.e., t is not in normal form.

Numbers

New syntactic forms

$t ::= \dots$
`0`
`succ t`
`pred t`
`iszero t`

terms
constant zero
successor
predecessor
zero test

$v ::= \dots$
`nv`

values
numeric value

$nv ::=$
`0`
`succ nv`

numeric values
zero value
successor value

New evaluation rules

$t \rightarrow t'$

$$\frac{t_1 \rightarrow t'_1}{\text{succ } t_1 \rightarrow \text{succ } t'_1} \quad (\text{E-SUCC})$$

$$\text{pred } 0 \rightarrow 0 \quad (\text{E-PREDZERO})$$

$$\text{pred } (\text{succ } nv_1) \rightarrow nv_1 \quad (\text{E-PREDSUCC})$$

$$\frac{t_1 \rightarrow t'_1}{\text{pred } t_1 \rightarrow \text{pred } t'_1} \quad (\text{E-PRED})$$

$$\text{iszero } 0 \rightarrow \text{true} \quad (\text{E-ISZEROZERO})$$

$$\text{iszero } (\text{succ } nv_1) \rightarrow \text{false} \quad (\text{E-ISZEROSUCC})$$

$$\frac{t_1 \rightarrow t'_1}{\text{iszero } t_1 \rightarrow \text{iszero } t'_1} \quad (\text{E-ISZERO})$$

Values are normal forms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?

Values are normal forms, but we have stuck terms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?

No: some terms are *stuck*.

Formally, a stuck term is one that is a normal form but not a value. What are some examples?

Stuck terms model run-time errors.

Multi-step evaluation.

The *multi-step evaluation* relation, \longrightarrow^* , is the reflexive, transitive closure of single-step evaluation.

I.e., it is the smallest relation closed under the following rules:

$$\frac{t \longrightarrow t'}{t \longrightarrow^* t'}$$
$$t \longrightarrow^* t$$
$$\frac{t \longrightarrow^* t' \quad t' \longrightarrow^* t''}{t \longrightarrow^* t''}$$

Termination of evaluation

Theorem: For every t there is some normal form t' such that $t \longrightarrow^* t'$.

Proof:

Termination of evaluation

Theorem: For every t there is some normal form t' such that $t \longrightarrow^* t'$.

Proof:

- ▶ First, recall that single-step evaluation strictly reduces the size of the term:

$$\text{if } t \longrightarrow t', \text{ then } \text{size}(t) > \text{size}(t')$$

- ▶ Now, assume (for a contradiction) that

$$t_0, t_1, t_2, t_3, t_4, \dots$$

is an infinite-length sequence such that

$$t_0 \longrightarrow t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow \dots$$

- ▶ Then

$$\text{size}(t_0) > \text{size}(t_1) > \text{size}(t_2) > \text{size}(t_3) > \dots$$

- ▶ But such a sequence cannot exist — contradiction!

Termination Proofs

Most termination proofs have the same basic form:

Theorem: *The relation $R \subseteq X \times X$ is terminating — i.e., there are no infinite sequences x_0, x_1, x_2 , etc. such that $(x_i, x_{i+1}) \in R$ for each i .*

Proof:

1. Choose
 - ▶ a well-founded set $(W, <)$ — i.e., a set W with a partial order $<$ such that there are no infinite descending chains $w_0 > w_1 > w_2 > \dots$ in W
 - ▶ a function f from X to W
2. Show $f(x) > f(y)$ for all $(x, y) \in R$
3. Conclude that there are no infinite sequences x_0, x_1, x_2 , etc. such that $(x_i, x_{i+1}) \in R$ for each i , since, if there were, we could construct an infinite descending chain in W .

The Lambda Calculus

The lambda-calculus

- ▶ If our previous language of arithmetic expressions was the simplest nontrivial programming language, then the lambda-calculus is the simplest *interesting* programming language...
 - ▶ Turing complete
 - ▶ higher order (functions as data)
- ▶ Indeed, in the lambda-calculus, *all* computation happens by means of function abstraction and application.
- ▶ The *e. coli* of programming language research
- ▶ The foundation of many real-world programming language designs (including ML, Haskell, Scheme, Lisp, ...)

Intuitions

Suppose we want to describe a function that adds three to any number we pass it. We might write

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That is, “plus3 x is succ (succ (succ x)).”

Q: What is plus3 itself?

A: plus3 is the function that, given x, yields succ (succ (succ x)).

```
plus3 = λx. succ (succ (succ x))
```

This function exists independent of the name plus3.

λx. t is written “fun x → t” in OCaml and “x ⇒ t” in Scala.

So plus3 (succ 0) is just a convenient shorthand for “the function that, given x, yields succ (succ (succ x)), applied to succ 0.”

```
plus3 (succ 0)
=
(λx. succ (succ (succ x))) (succ 0)
```

Abstractions over Functions

Consider the λ -abstraction

$$g = \lambda f. f (f (\text{succ } 0))$$

Note that the parameter variable f is used in the *function* position in the body of g . Terms like g are called *higher-order* functions. If we apply g to an argument like plus3 , the “substitution rule” yields a nontrivial computation:

```
g plus3
=  (\lambda f. f (f (\text{succ } 0))) (\lambda x. \text{succ } (\text{succ } (\text{succ } x)))
i.e. (\lambda x. \text{succ } (\text{succ } (\text{succ } x)))
      ((\lambda x. \text{succ } (\text{succ } (\text{succ } x))) (\text{succ } 0))
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```

Abstractions Returning Functions

Consider the following variant of g :

$$\text{double} = \lambda f. \lambda y. f (f y)$$

I.e., double is the function that, when applied to a function f , yields a *function* that, when applied to an argument y , yields $f (f y)$.

Example

```
double plus3 0
=  (\lambda f. \lambda y. f (f y))
      (\lambda x. \text{succ } (\text{succ } (\text{succ } x)))
      0
i.e. (\lambda y. (\lambda x. \text{succ } (\text{succ } (\text{succ } x)))
      ((\lambda x. \text{succ } (\text{succ } (\text{succ } x))) y))
      0
i.e. (\lambda x. \text{succ } (\text{succ } (\text{succ } x)))
      ((\lambda x. \text{succ } (\text{succ } (\text{succ } x))) 0)
i.e. (\lambda x. \text{succ } (\text{succ } (\text{succ } x)))
      (\text{succ } (\text{succ } (\text{succ } 0)))
i.e. \text{succ } (\text{succ } (\text{succ } (\text{succ } (\text{succ } (\text{succ } 0)))))
```

The Pure Lambda-Calculus

As the preceding examples suggest, once we have λ -abstraction and application, we can throw away all the other language primitives and still have left a rich and powerful programming language.

In this language — the “pure lambda-calculus” — *everything* is a function.

- ▶ Variables always denote functions
- ▶ Functions always take other functions as parameters
- ▶ The result of a function is always a function

Formalities

Syntax

$t ::=$	x	terms
	$\lambda x. t$	variable
	$t \ t$	abstraction
		application

Terminology:

- terms in the pure λ -calculus are often called λ -terms
- terms of the form $\lambda x. t$ are called λ -abstractions or just *abstractions*

Syntactic conventions

Since λ -calculus provides only one-argument functions, all multi-argument functions must be written in curried style.

The following conventions make the linear forms of terms easier to read and write:

- Application associates to the left
E.g., $t \ u \ v$ means $(t \ u) \ v$, not $t \ (u \ v)$
- Bodies of λ -abstractions extend as far to the right as possible
E.g., $\lambda x. \lambda y. x \ y$ means $\lambda x. (\lambda y. x \ y)$, not $\lambda x. (\lambda y. x) \ y$

Scope

The λ -abstraction term $\lambda x. t$ binds the variable x .

The *scope* of this binding is the *body* t .

Occurrences of x inside t are said to be *bound* by the abstraction.

Occurrences of x that are *not* within the scope of an abstraction binding x are said to be *free*.

Test:

$\lambda x. \lambda y. x \ y \ z$

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Test:

$$\begin{array}{l} \lambda x. \lambda y. x y z \\ \lambda x. (\lambda y. z y) y \end{array}$$

Values

$$v ::= \lambda x. t$$

values
abstraction value

Operational Semantics

Computation rule:

$$(\lambda x. t_{12}) v_2 \longrightarrow [x \mapsto v_2] t_{12} \quad (\text{E-APPABS})$$

Notation: $[x \mapsto v_2] t_{12}$ is “the term that results from substituting free occurrences of x in t_{12} with v_2 .”

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Congruence rules:

$$\frac{t_1 \longrightarrow t'_1}{t_1 t_2 \longrightarrow t'_1 t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{v_1 t_2 \longrightarrow v_1 t'_2} \quad (\text{E-APP2})$$

Terminology

A term of the form $(\lambda x. t) v$ — that is, a λ -abstraction applied to a *value* — is called a *redex* (short for “reducible expression”).

Alternative evaluation strategies

Strictly speaking, the language we have defined is called the *pure, call-by-value lambda-calculus*.

The evaluation strategy we have chosen — *call by value* — reflects standard conventions found in most mainstream languages.

Some other common ones:

- ▶ Call by name (cf. Haskell)
- ▶ Normal order (leftmost/outermost)
- ▶ Full (non-deterministic) beta-reduction

Classical Lambda Calculus

Full beta reduction

The classical lambda calculus allows full beta reduction.

- ▶ The argument of a β -reduction to be an arbitrary term, not just a value.
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Congruence rules:

$$\frac{t_1 \longrightarrow t'_1}{t_1 t_2 \longrightarrow t'_1 t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{t_1 t_2 \longrightarrow t_1 t'_2} \quad (\text{E-APP2})$$

$$\frac{t \longrightarrow t'}{\lambda x. t \longrightarrow \lambda x. t'} \quad (\text{E-ABS})$$

Substitution revisited

Remember: $[x \mapsto v_2]t_{12}$ is "the term that results from substituting free occurrences of x in t_{12} with v_2 ."

This is trickier than it looks!

For example:

$$\begin{aligned} & (\lambda x. (\lambda y. x)) y \\ \longrightarrow & [x \mapsto y]\lambda y. x \\ = & ??? \end{aligned}$$

Substitution revisited

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For example:

$$\begin{aligned} & (\lambda x. (\lambda y. x)) y \\ \longrightarrow & [x \mapsto y]\lambda y. x \\ = & ??? \end{aligned}$$

Solution:

need to rename bound variables before performing the substitution.

$$\begin{aligned} & (\lambda x. (\lambda y. x)) y \\ = & (\lambda x. (\lambda z. x)) y \\ \longrightarrow & [x \mapsto y]\lambda z. x \\ = & \lambda z. y \end{aligned}$$

Alpha conversion

Renaming bound variables is formalized as α -conversion.
Conversion rule:

$$\frac{y \notin \text{fv}(t)}{\lambda x. t =_{\alpha} \lambda y. [x \mapsto y]t} \quad (\alpha)$$

Equivalence rules:

$$\frac{t_1 =_{\alpha} t_2}{t_2 =_{\alpha} t_1} \quad (\alpha\text{-SYMM})$$

$$\frac{t_1 =_{\alpha} t_2 \quad t_2 =_{\alpha} t_3}{t_1 =_{\alpha} t_3} \quad (\alpha\text{-TRANS})$$

Congruence rules: the usual ones.

Confluence

Full β -reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

Confluence

Full β -reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

The answer is no; this is a consequence of the following

Theorem [Church-Rosser]

Let t, t_1, t_2 be terms such that $t \rightarrow^* t_1$ and $t \rightarrow^* t_2$. Then there exists a term t_3 such that $t_1 \rightarrow^* t_3$ and $t_2 \rightarrow^* t_3$.

Programming in the Lambda-Calculus

Multiple arguments

Consider the function `double`, which returns a function as an argument.

```
double = λf. λy. f (f y)
```

This idiom — a λ -abstraction that does nothing but immediately yield another abstraction — is very common in the λ -calculus.

In general, $\lambda x. \lambda y. t$ is a function that, given a value v for x , yields a function that, given a value u for y , yields t with v in place of x and u in place of y .

That is, $\lambda x. \lambda y. t$ is a two-argument function.

(Recall the discussion of *currying* in OCaml.)

The “Church Booleans”

```
tru  = λt. λf. t
fls  = λt. λf. f
```

```
tru v w
= (λt. λf. t) v w by definition
→ (λf. v) w      reducing the underlined redex
→ v              reducing the underlined redex
```

```
fls v w
= (λt. λf. f) v w by definition
→ (λf. f) w      reducing the underlined redex
→ w              reducing the underlined redex
```

Functions on Booleans

```
not = λb. b fls tru
```

That is, `not` is a function that, given a boolean value v , returns `fls` if v is `tru` and `tru` if v is `fls`.

Functions on Booleans

```
and = λb. λc. b c fls
```

That is, `and` is a function that, given two boolean values v and w , returns w if v is `tru` and `fls` if v is `fls`.
Thus `and v w` yields `tru` if both v and w are `tru` and `fls` if either v or w is `fls`.

Pairs

```
pair = λf.λs.λb. b f s
fst = λp. p tru
snd = λp. p fls
```

That is, `pair v w` is a function that, when applied to a boolean value `b`, applies `b` to `v` and `w`.

By the definition of booleans, this application yields `v` if `b` is `tru` and `w` if `b` is `fls`, so the first and second projection functions `fst` and `snd` can be implemented simply by supplying the appropriate boolean.

Example

```
fst (pair v w)
= fst ((λf. λs. λb. b f s) v w)  by definition
→ fst ((λs. λb. b v s) w)      reducing
→ fst (λb. b v w)              reducing
= (λp. p tru) (λb. b v w)       by definition
→ (λb. b v w) tru              reducing
→ tru v w                      reducing
→* v                           as before.
```

Church numerals

Idea: represent the number `n` by a function that “repeats some action `n` times.”

```
c0 = λs. λz. z
c1 = λs. λz. s z
c2 = λs. λz. s (s z)
c3 = λs. λz. s (s (s z))
```

That is, each number `n` is represented by a term `cn` that takes two arguments, `s` and `z` (for “successor” and “zero”), and applies `s`, `n` times, to `z`.

Functions on Church Numerals

Successor:

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$scc = \lambda n. \lambda s. \lambda z. s (n s z)$

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$plus = \lambda m. \lambda n. \lambda s. \lambda z. m s (n s z)$

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Multiplication:

Functions on Church Numerals

Successor:

$scc = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$

Addition:

$plus = \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z)$

Multiplication:

$times = \lambda m. \lambda n. m \ (plus \ n) \ c_0$

Functions on Church Numerals

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Zero test:

Functions on Church Numerals

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$iszro = \lambda m. m \ (\lambda x. fls) \ tru$

Functions on Church Numerals

Successor:

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Multiplication:

$times = \lambda m. \lambda n. m \ (plus \ n) \ c_0$

Zero test:

$iszro = \lambda m. m \ (\lambda x. fls) \ tru$

What about predecessor?

Predecessor

```
zz = pair c0 c0

ss = λp. pair (snd p) (scc (snd p))

prd = λm. fst (m ss zz)
```

Recursion in the Lambda-Calculus

Recursion and divergence

Recursion and divergence are intertwined, so we need to consider divergent terms.

$$\text{omega} = (\lambda x. x x) (\lambda x. x x)$$

Note that `omega` evaluates in one step to itself!
So evaluation of `omega` never reaches a normal form: it *diverges*.

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So evaluation of `omega` never reaches a normal form: it *diverges*.

Being able to write a divergent computation does not seem very useful in itself. However, there are variants of `omega` that are very useful...

Recall: Normal forms

- ▶ A *normal form* is a term that cannot take an evaluation step.
- ▶ A *stuck* term is a normal form that is not a value.

Does every term evaluate to a normal form?

No, ω is not in normal form.

Recall: Normal forms

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No, ω is not in normal form.

But are there any stuck terms in the pure λ -calculus?

Towards recursion: Iterated application

Suppose f is some λ -abstraction, and consider the following variant of ω :

$$Y_f = (\lambda x. f (x x)) (\lambda x. f (x x))$$

Towards recursion: Iterated application

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Now the “pattern of divergence” becomes more interesting:

$$\begin{aligned} Y_f &= \\ & \underline{(\lambda x. f (x x)) (\lambda x. f (x x))} \\ & \longrightarrow \\ & f (\underline{(\lambda x. f (x x)) (\lambda x. f (x x))}) \\ & \longrightarrow \\ & f (f (\underline{(\lambda x. f (x x)) (\lambda x. f (x x))})) \\ & \longrightarrow \\ & f (f (f (\underline{(\lambda x. f (x x)) (\lambda x. f (x x))}))) \\ & \longrightarrow \\ & \dots \end{aligned}$$

Y_f is still not very useful, since (like ω), all it does is diverge.
Is there any way we could “slow it down”?

Delaying divergence

$\text{poisonpill} = \lambda y. \omega$

Note that poisonpill is a value — it will only diverge when we actually apply it to an argument. This means that we can safely pass it as an argument to other functions, return it as a result from functions, etc.

$$\begin{aligned} & \frac{(\lambda p. \text{fst} (\text{pair } p \text{ fls}) \text{tru}) \text{poisonpill}}{\longrightarrow} \\ & \text{fst} (\text{pair } \text{poisonpill} \text{ fls}) \text{tru} \\ & \longrightarrow^* \\ & \frac{\text{poisonpill } \text{tru}}{\longrightarrow} \\ & \omega \\ & \longrightarrow \\ & \dots \end{aligned}$$

A delayed variant of ω

Here is a variant of ω in which the delay and divergence are a bit more tightly intertwined:

$$\omega_{\text{gav}} = \lambda y. (\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) y$$

Note that ω_{gav} is a normal form. However, if we apply it to any argument v , it diverges:

$$\begin{aligned} & \omega_{\text{gav}} v \\ & = \\ & \frac{(\lambda y. (\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) y) v}{\longrightarrow} \\ & \frac{(\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) v}{\longrightarrow} \\ & (\lambda y. (\lambda x. (\lambda y. x \ x \ y)) (\lambda x. (\lambda y. x \ x \ y)) y) v \\ & = \\ & \omega_{\text{gav}} v \end{aligned}$$

Another delayed variant

Suppose f is a function. Define

$$z_f = \lambda y. (\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) y$$

This term combines the “added f ” from Y_f with the “delayed divergence” of ω_{gav} .

If we now apply z_f to an argument v , something interesting happens:

$$\begin{aligned}
 & z_f \ v \\
 &= \\
 & \frac{(\lambda y. (\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ y) \ v}{\rightarrow} \\
 & \frac{(\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ v}{\rightarrow} \\
 & f (\lambda y. (\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ y) \ v \\
 &= \\
 & f \ z_f \ v
 \end{aligned}$$

Since z_f and v are both values, the next computation step will be the reduction of $f \ z_f$ — that is, before we “diverge,” f gets to do some computation.
Now we are getting somewhere.

Recursion

Let

```
f = λfct.
    λn.
      if n=0 then 1
      else n * (fct (pred n))
```

f looks just the ordinary factorial function, except that, in place of a recursive call in the last time, it calls the function fct , which is passed as a parameter.

N.b.: for brevity, this example uses “real” numbers and booleans, infix syntax, etc. It can easily be translated into the pure lambda-calculus (using Church numerals, etc.).

We can use z to “tie the knot” in the definition of f and obtain a real recursive factorial function:

$$\begin{aligned}
 & z_f \ 3 \\
 & \rightarrow^* \\
 & f \ z_f \ 3 \\
 &= \\
 & (\lambda fct. \lambda n. \dots) \ z_f \ 3 \\
 & \rightarrow \rightarrow \\
 & \text{if } 3=0 \text{ then } 1 \text{ else } 3 * (z_f \ (\text{pred } 3)) \\
 & \rightarrow^* \\
 & 3 * (z_f \ (\text{pred } 3)) \\
 & \rightarrow \\
 & 3 * (z_f \ 2) \\
 & \rightarrow^* \\
 & 3 * (f \ z_f \ 2) \\
 & \dots
 \end{aligned}$$

A Generic z

If we define

$$z = \lambda f. z_f$$

i.e.,

$$z = \lambda f. \lambda y. (\lambda x. f (\lambda y. x \ x \ y)) (\lambda x. f (\lambda y. x \ x \ y)) \ y$$

then we can obtain the behavior of z_f for any f we like, simply by applying z to f .

$$z \ f \ \rightarrow \ z_f$$

For example:

```
fact    =    z ( λfct.
                λn.
                  if n=0 then 1
                  else n * (fct (pred n)) )
```

Technical Note

The term **z** here is essentially the same as the **fix** discussed the book.

```
z =
  λf. λy. (λx. f (λy. x x y)) (λx. f (λy. x x y)) y
```

```
fix =
  λf. (λx. f (λy. x x y)) (λx. f (λy. x x y))
```

z is hopefully slightly easier to understand, since it has the property that $z\ f\ v \longrightarrow^* f\ (z\ f)\ v$, which **fix** does not (quite) share.