Runge Kutta Example

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Here's a rewrite of the stuff I did in class on Tuesday. Suppose we have a point moving in space, and the coordinates of the point are given by $p(t)$. For a one-dimensional spring, the coordinates we're interested in are position, velocity, and acceleration, or

$$p(t) = (x(t), v(t), a(t))$$

For our simple spring, the acceleration is a simple function:

$$a(t) = -2x(t)$$

We also presume to know the derivative at any position:

$$\begin{aligned} p'(p(t)) &= p'(x(t), v(t), a(t)) \\ &= (v(t), a(t), ?) \end{aligned}$$

We don't know the derivative of the acceleration, but it will turn out not to matter.

Now, our system starts at a given time, t_0, and advances by Δt, giving times $t_n = t_0 + n\Delta t$. When we talk about positions or derivatives or vectors at n, it is a simple translation to actual times:

$$\begin{aligned} p(n) &= p(t_0 + n\Delta t) \\ p'(p(n)) &= p'(p(t_0 + n\Delta t)) \end{aligned}$$

For Fourth Order Runge-Kutta integration, we need to calculate four new vectors, and three new points:

$$\begin{aligned} k_1(n) &= p'(p(n))\Delta t \\ p_2(n) &= p(n) + \frac{1}{2}k_1(n) \\ k_2(n) &= p'(p_2(n))\Delta t \\ p_3(n) &= p(n) + \frac{1}{2}k_2(n) \\ k_3(n) &= p'(p_3(n))\Delta t \\ p_4(n) &= p(n) + k_3(n) \\ k_4(n) &= p'(p_4(n))\Delta t \end{aligned}$$

We can now start to fill in our table, with $\Delta t = \frac{1}{2}$ and $a = -2x$.

	N	x	v	a
$p(0)$	0	8	0	-16
$k_1(0)$	0	$0\Delta t$	$-16\Delta t$	?
$k_1(0)$	0	0	-8	?
$p_2(0)$	0	$8 + \frac{1}{2}0$	$0 + \frac{1}{2}(-8)$	?
$p_2(0)$	0	8	-4	-16

At this point we can see why we didn't need the derivative of acceleration. When we calculate a new point, we get it just from $a = -2x$. Continuing:

	N	x	v	a
$p(0)$	0	8	0	-16
$k_1(0)$	0	$0\Delta t$	$-16\Delta t$	?
$k_1(0)$	0	0	-8	?
$p_2(0)$	0	$8 + \frac{1}{2}0$	$0 + \frac{1}{2}(-8)$	?
$p_2(0)$	0	8	-4	-16
$k_2(0)$	0	$-4\Delta t$	$-16\Delta t$	?
$k_2(0)$	0	-2	-8	?
$p_3(0)$	0	$8 + \frac{1}{2}(-2)$	$0 + \frac{1}{2}(-8)$	?
$p_3(0)$	0	7	-4	-14
$k_3(0)$	0	$-4\Delta t$	$-14\Delta t$	?
$k_3(0)$	0	-2	-7	?
$p_4(0)$	0	$8 + (-2)$	$0 + (-7)$	?
$p_4(0)$	0	6	-7	-12
$k_4(0)$	0	$-7\Delta t$	$-12\Delta t$	?
$k_4(0)$	0	$-\frac{7}{2}$	-6	?

Summarizing:

	N	x	v	a
$p(0)$	0	8	0	-16
$k_1(0)$	0	0	-8	?
$k_2(0)$	0	-2	-8	?
$k_3(0)$	0	-2	-7	?
$k_4(0)$	0	$-\frac{7}{2}$	-6	?

Now we can use the update rule for Runge-Kutta:

$$\begin{aligned} p(1) &= p(0) + \frac{k_1(0)}{6} + \frac{k_2(0)}{3} + \frac{k_3(0)}{3} + \frac{k_4(0)}{6} \\ x(1) &= x(0) + \frac{0}{6} + \frac{-2}{3} + \frac{-2}{3} + \frac{7}{(2)(6)} \\ &= 8 + \frac{-4}{3} + \frac{7}{12} \\ &= 8 + \frac{-9}{12} \\ &= 7 + \frac{1}{4} \\ v(1) &= v(0) + \frac{-8}{6} + \frac{-8}{3} + \frac{-7}{3} + \frac{-6}{6} \\ &= \frac{-44}{6} \\ &= -(7 + \frac{1}{3}) \\ a(1) &= -2(7 + \frac{1}{4}) \\ &= -(14 + \frac{1}{2}) \end{aligned}$$

So now we have the next row in our table:

	N	x	v	a
$p(0)$	0	8	0	-16
$p(1)$	1	$7 + \frac{1}{4}$	$-(7 + \frac{1}{3})$	$-(14 + \frac{1}{2})$

and we can start on $p(2)$!