Mathematical Modeling and Optimization

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1 The Problem

You are building computers from spare parts, and you want to optimize the total value of the computers built. There are two types of computers that you can build: Type X and Type Y. However, you have limited resources, in particular only 3 monitors and 11 processors.

Building one Type X computer requires 1 monitor and 5 processors, while building one Type Y computer requires 1 monitor and 3 processors. Type X computers are worth $1000 each, and Type Y computers are worth $4000 each. In addition, we have the additional constraint that we cannot build more than 3 Type Y computers. How many computers of each type should we build?

For this problem, assume that fractional computers are worthless, and remember that we cannot build negative computers!

2 Formulation

Let x and y be the number of Type X and Type Y computers that you decide to build respectively.

2.1 Objective Function

We are trying to maximize the total value of the computers that we build. Because Type X and Type Y computers are worth $1000 and $4000 respectively, their total value is

$1000x + 4000y$. Therefore, the objective function is:

$$\max 1000x + 4000y$$

2.2 Constraints

We need to write constraints for each of the limited resources, which are monitors and processors. Because Type X and Type Y computers require 1 monitor each, the total number of monitors that we use is $x + y$. We only have 3 monitors, so this is an upper bound for the total number of monitors. Therefore, the monitor constraint is:

$$x + y \leq 3$$

Because Type X and Type Y computers require 5 processors and 3 processors respectively, the total number of processors that we use is $5x + 3y$. We only have 11 processors, so this is an upper bound for the total number of processors. Therefore, the processor constraint is:

$$5x + 3y \leq 11$$

2.3 Decision Variables

The decision variables are x and y, which were defined at the beginning of this section. Since we cannot build negative computers, we need to add the constraints $x \geq 0$ and $y \geq 0$. Also, we cannot build more than 3 of the Type Y computers, so we add the constraint $y \leq 3$. Therefore, we have:

$$x \geq 0$$

$$0 \leq y \leq 3$$

Since we do not want to build fractional computers, we add the integer constraint for x and y:

$$x, y \text{ integer}$$

2.4 Final Formulation

Putting all of the previous parts together, we obtain the full optimization model:

$$
\begin{aligned}
\max \quad & 1000x + 4000y \\
\text{s.t.} \quad & x + y \leq 3 \\
& 5x + 3y \leq 11 \\
& x \geq 0 \\
& 0 \leq y \leq 3 \\
& x, y \text{ integer}
\end{aligned} \tag{1}
$$

To check that we have covered all of the necessary constraints, read through the problem again and see if we have used all of the information that was given to us. Now, we are ready to code up the optimization problem and solve it on the computer!

3 More Examples

3.1 Furniture Company

You are in charge of a manufacturing plant for a furniture company that produces tables, chairs, and desks. In order to maximize revenue, you would like to determine the optimal number of tables, chairs, and desks to produce.

Tables require 6 planks of wood, 2 hours of carpentry work, and 4 hours of machine work to produce. Chairs require 2 planks of wood, 2 hours of carpentry work, and 2 hours of machine work to produce. Desks require 5 planks of wood, 3 hours of carpentry work, and 6 hours of machine work to produce. We have 500 planks of wood, 300 hours of carpentry work, and 450 hours of machine work total that we can use. Tables sell for \$60, chairs sell for \$40, and desks sell for \$100 respectively.

Assuming that we cannot build fractional products, how many pieces of each type of furniture should we produce? Formulate this as an optimization problem, with an objective function and constraints.

3.2 New England Power

The electricity in New England comes from a variety of sources, including nuclear, fossil-fuel, wind, and solar power plants. Every day, the electrical grid is optimized at the central hub, in order to meet demand while minimizing cost. Assume for simplicity that there are only 4 power plants: A, B, C, D which have maximum output of 800, 400, 600, and 350 terawatts (TW) respectively, and we are trying to allocate power between the western and eastern halves of Massachusetts.

One terawatt is 1,000 gigawatts. Power plants A, B, C, D each have a base cost of \$2000, \$1600, \$2800, \$2200 per gigawatt respectively. Based on the geographic location of the power plants, the transmitting cost varies. The transmitting costs for power plants A, B, C, D are \$200, \$600, \$1200, \$100 per gigawatt for Eastern Massachusetts and \$800, \$600, \$200, \$900 per gigawatt for Western Massachusetts respectively. The total cost is the sum of the base costs and the transmitting costs. The demand for tomorrow is 900 terawatts in Eastern Massachusetts, and 700 terawatts in Western Massachusetts. In addition, because power plant A is nuclear, we cannot have more than 25% of our total power supplied by power plant A.

How many gigawatts should we allocate from each power plant to Western and Eastern Massachusetts? Formulate this as an optimization problem, with an objective function and constraints.

(Hint: Define the decision variables AW, BW, CW, DW as the amounts of gigawatts supplied to Western Massachusetts from power plants A, B, C, D respectively. Similarly, define the decision variables AE, BE, CE, DE as the amounts of gigawatts supplied to Eastern Massachusetts.)