Computer Lab 4
Contacting Rigid Bodies

Mikica Kocic
miko0008@student.umu.se

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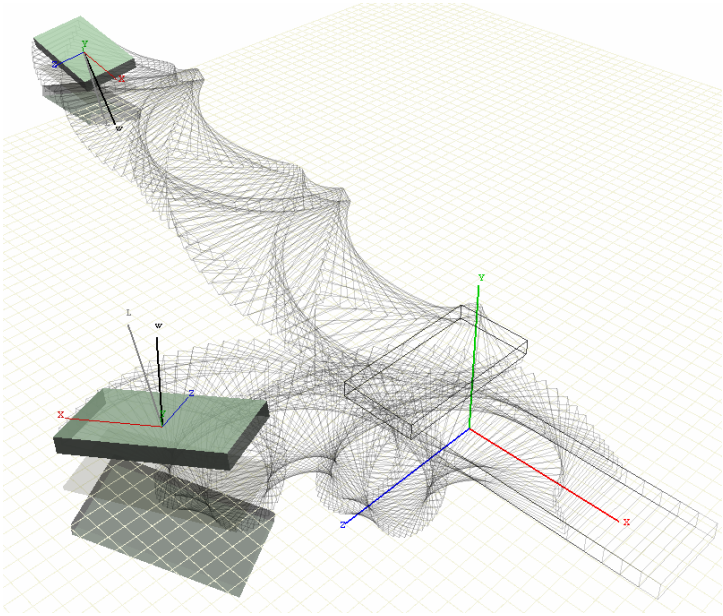


Mark what exercises you have done in this lab		*Points (by supervisor)*
Basic level (mandatory)	×	
Extra exercise #1: effects of dissipation and friction	×	
Extra exercise #2: adding a static ground plane, walls or a staircase geometry	×	
Extra exercise #3: piling objects	×	
Extra exercise #4: 3D rigid bodies simulation framework	×	
Extra exercise #5: collisions between different geometries (sphere, cuboid, half-space, plane)	×	
The report is brief and to the point but still coherent and well structured.	☺	
Late report		
Total points		

Tutors: **Martin Servin** and **John Nordberg**

1 Background

The purpose of this computer lab was to simulate collisions between rigid bodies using either the impulse method or the penalty method, and to validate the simulation by studying the conservation of energy, linear momentum and angular momentum. Advanced level tasks were to improve the simulation by including frictional contacts, more objects, more complicated collision geometries, articulated bodies using constraints and user control over the objects.

2 Implementation and Simulation

2.1 Theory

An overall algorithm for a simulation as well the semi-implicit Euler algorithm for solving rigid body equations of motion, are based on Section 2.3 *The Simulation Algorithm* and Subsection 7.3.3 *Time-Integration in 3D* in *Notes on Discrete Mechanics*, respectively.

On the other hand, rigid-body simulation algorithms, where linear and angular momenta are state variables instead of linear and angular velocities, are based on equations of motion found in Chapter 3 *Rigid Body Motion* in [1] and Section 2.11 *Rigid Body Equations of Motion* in [5].

Rigid body collision *detection* algorithms are based on Chapter 5 *Basic Primitive Tests* in [3] and Chapter 13 *Generating Contacts* in [4], whereas rigid body collision *response* algorithms are based on impulse transfer method found in Section 5.2.2 *Collision Response for Colliding Contact* in [1], Chapter 14 *Collision Resolution* in [4] and Section 8 *Colliding Contact* in [5].

Finally, usage of quaternions for representing spatial rotations and derivatives of time-varying quaternions is based on Chapter 10 *Quaternions* in [1].

2.2 Implementation

The resulting rigid body simulation framework, World of Rigid Bodies or WoRB, is implemented in C++ and it uses GLUT and OpenGL API for animations [2]. The WoRB framework can be built either as a stand-alone simulation test-bed application, a MATLAB mex file or as a C++ library.

Quaternions

In this implementation, there are no vectors and matrices as separate entities; the implementation rather uses quaternions to represent all variables, including state variables like position, orientation, linear or angular momentum. Rotation and translation matrices are kept as q-tensors, which are in fact 4-tuples of quaternions. Quaternions and q-tensors are encapsulated as C++ classes and defined in `Quaternion.h` and `QTensor.h` modules, respectively. The theoretical framework for these classes is implemented as '`Quat.m`' Mathematica package and given in Appendix B, with the accompanying notebook given in Appendix C.

Number of Spatial Dimensions

The original request was to implement algorithms for colliding rigid bodies in 2D. In this implementation, however, equations of motion and collisions are always simulated in 3D. Simulations in 2D are considered as a special case where rectangular rigid bodies are represented by very thin plates (cuboids with negligible thicknesses) with motion restricted to the horizontal (*Oxz*) plane only.

Rigid Body Motion Equations

The theoretical framework for rigid body motion algorithms is implemented as Mathematica notebook and given in Appendix A.

Program Code

The WoRB core implementation consists of ~7k lines of code & comments that are distributed in around 20 modules. The WoRB code metrics is presented in Table 2.2-1.

Table 2.2-1: The WoRB Code Metrics

Module Name	Type	LOC	COM	MVG	L_C	M_C
Const	(S)	30	38	6	0.789	0.158
Quaternion	(S)	193	125	61	1.544	0.488
SpatialVector	(S)	8	8	0	–	–
QTensor	(S)	357	157	46	2.274	0.293
Geometry	(S)	88	31	41	2.839	1.323
Cuboid	(S)	425	253	95	1.680	0.375
Sphere	(S)	100	71	19	1.408	0.268
TruePlane	(S)	11	7	0	–	–
HalfSpace	(S)	11	8	0	–	–
RigidBody	(S)	191	190	18	1.005	0.095
RigidBodies	(S)	43	20	11	2.150	0.550
Collision	(S)	278	235	34	1.183	0.145
CollisionResolver	(S)	197	147	40	1.340	0.272
WoRB	(S)	120	42	10	2.857	0.238
WorldOfRigidBodies	(S)	95	65	8	1.462	0.123
WoRB_TestBed	(G)	717	363	93	1.975	0.256
Ball	(G)	51	18	8	2.833	0.444
Box	(G)	56	16	13	3.500	0.812
GLOrthoScreen	(G)	26	17	0	1.529	–
GLTransform	(G)	23	25	1	0.920	–
GLUT_Framework	(G)	107	31	20	3.452	0.645
GLUT_Renderer	(G)	16	17	0	–	–
WoRB_MexFunction	(M)	158	70	16	2.257	0.229
Mex::Scalar	(M)	52	12	12	4.333	1.000
Mex::String	(M)	49	20	10	2.450	0.500
Mex::Logical	(M)	54	14	14	3.857	1.000
Mex::Matrix	(M)	167	74	50	2.257	0.676

where

- **Type**: (**S**) = simulation related module, (**G**) = animation related module (uses OpenGL), and (**M**) = MATLAB mex-file related module.
- **LOC** = Lines of Code
  Number of non-blank, non-comment lines of source code counted by the analyser.
- **COM** = Lines of Comments
  Number of lines of comment identified by the analyser
- **MVG** = McCabe's Cyclomatic Complexity
  A measure of the decision complexity of the functions which make up the program.
- **L_C** = Lines of code per line of comment
  Indicates density of comments with respect to textual size of program
- **M_C** = Cyclomatic Complexity per line of comment
  Indicates density of comments with respect to logical complexity of program

Program Documentation

The program documentation consists of ~2.5k lines of comment that follows javadoc convention. The resulting documentation, generated from these comments using Doxygen tool, can be found in `doc` subfolder with `doc/index.html` as the main page (opening `README.html` in the top folder will redirect to this page).

The other useful page is 'Files' (`doc/files.html`) which shows the source file list with their brief descriptions. The good point to start browsing the WoRB code are `Main.cpp` and `mexFunction.cpp` files, which both contain main program entry points; the first for a stand-alone application and the second for a MATLAB mex function. Their respective include dependency graphs are shown on Figure 2.2–1.

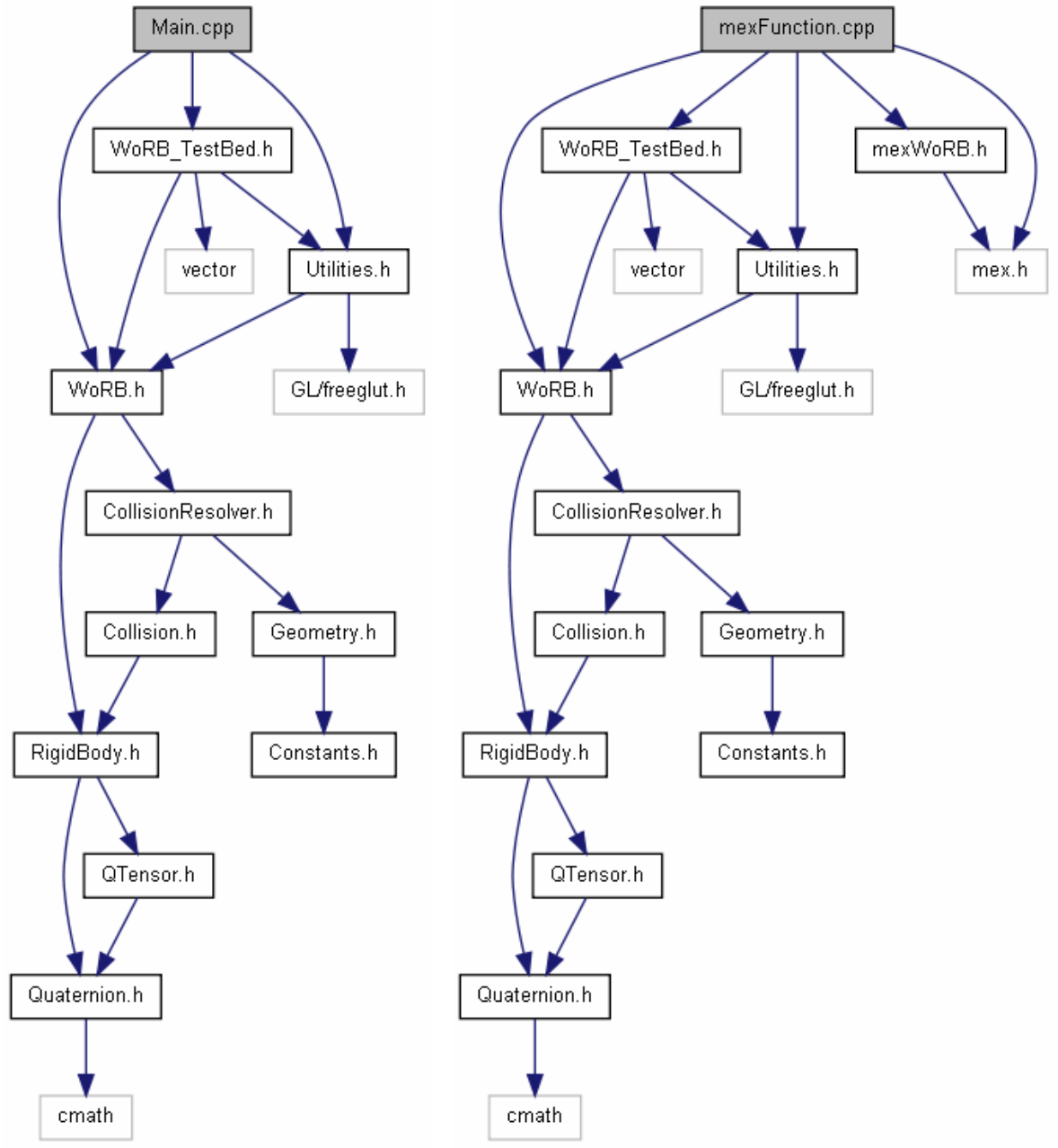


Figure 2.2–1: Include dependency graphs for `Main.cpp` (left) and `mexFunction.cpp` (right)

2.3 Compiling the Code

The code can be compiled either for stand-alone execution, or to be used as a mex function in MATLAB.

The animation part requires OpenGL libraries, which are usually distributed with an operating system, and the freeglut (GL Utility Toolkit) that can be downloaded from http://freeglut.sourceforge.net/

The default distribution contains `mexw32` and `mexw64` binaries compiled with MS VS 2010 compiler. To rebuild binaries on Windows, use VS solution in folder `vs2010`; the accompanying freeglut 2.8.0 development library and DLL files for win32 and win64 platforms can be found in `freeglut` folder.

To rebuild binaries on Linux, use `Makefile`; just type `make` on the command prompt.

To rebuild binaries under the MATLAB, just execute `make` m-file.

Note: On Linux, if you have non-standard location for freeglut, i.e. if it is not installed as a system package, you will need to configure `GLUT_INC` and `GLUT_LIB` variables in `Makefile` and `make.m`.

Here is a sample log showing the compilation process to get the stand-alone application on Linux:

```
[user@gea WoRB]$ make rebuild
 [RM   ]  obj/*.o
 [C++  ]  obj/Constants.o
 [C++  ]  obj/WoRB.o
 [C++  ]  obj/CollisionDetection.o
 [C++  ]  obj/ImpulseMethod.o
 [C++  ]  obj/PositionProjections.o
 [C++  ]  obj/Platform.o
 [C++  ]  obj/Utilities.o
 [C++  ]  obj/WoRB_TestBed.o
 [C++  ]  obj/Main.o
 [LD   ]  bin/WoRB
[user@gea WoRB]$
```

Here is a sample log showing the compilation process to get mex-file on Windows:

```
>> make rebuild
mex -O -outdir obj -c src\Constants.cpp
mex -O -outdir obj -c src\CollisionDetection.cpp
mex -O -outdir obj -c src\ImpulseMethod.cpp
mex -O -outdir obj -c src\PositionProjections.cpp
mex -O -outdir obj -c src\WoRB.cpp
mex -O -outdir obj -c src\Platform.cpp
mex -O -I./freeglut/include -outdir obj -c src\Utilities.cpp
mex -O -I./freeglut/include -outdir obj -c src\WoRB_TestBed.cpp
mex -O -I./freeglut/include -outdir obj -c src\mexFunction.cpp
mex  -output bin\WoRB.mexw32 ...
 obj\mexFunction.obj ...
 obj\Constants.obj ...
 obj\CollisionDetection.obj ...
 obj\ImpulseMethod.obj ...
 obj\PositionProjections.obj ...
 obj\WoRB.obj ...
 obj\Platform.obj ...
 obj\Utilities.obj ...
 obj\WoRB_TestBed.obj ...
 ./freeglut/lib/freeglut.lib
>>
```

2.4 Running Stand-Alone Simulations

To run a stand-alone simulation, just start the `WoRB` executable. The gui should appear:

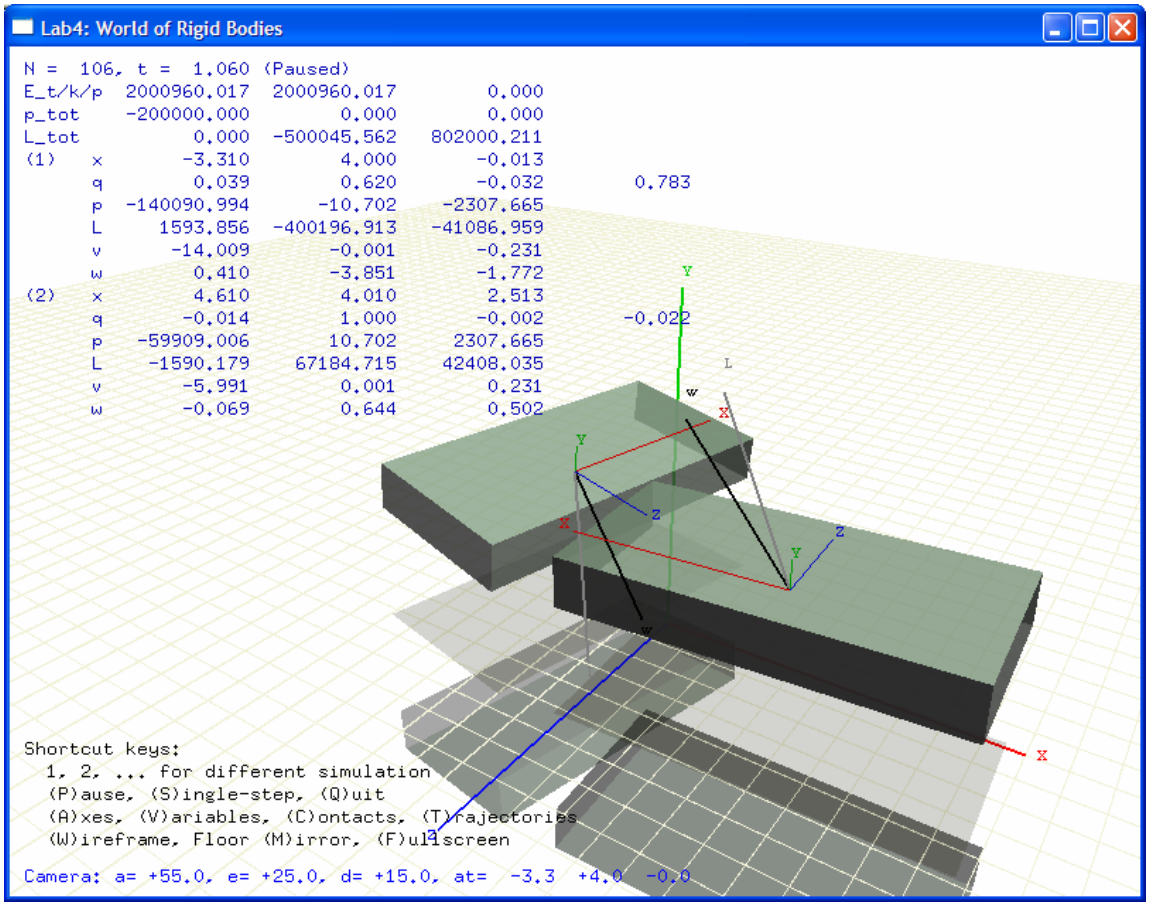


Figure 2.4–1: Snapshot of the WoRB simulation window.

The application window is logically divided into three areas:

central area	Contains the simulated system with the ground floor, bodies and trajectories
top-left area	Contains local time, total energy levels and momenta of the system, as well the state variables of up to 4 bodies. The body state variables are displayed only while simulation is paused or single-stepped.
bottom-left area	Contains camera view position with the short help about shortcut keys above

One can use the following short-cut keys to control the application:

`1, 2, …`	Start different predefined simulation
`A`	Toggle displaying axes (and angular velocity and momentum) of the bodies
`C`	Toggle displaying contact normals during collisions
`F`	Toggle full-screen mode
`H`	Toggle displaying help and camera position
`M`	Toggle displaying rigid bodies mirrored in the ground floor
`P` or `<space>`	Pause/continue simulation (toggle simulation execution)
`S` or `<enter>`	Simulation single-step; keep holding <enter> for slowed-down simulation
`T`	Toggle displaying/tracking trajectories of the bodies
`W`	Toggle displaying objects as wire-frames vs. solid bodies
`F1, F2 …`	Follow the body #1, #2 … with the camera (while it moves)
`F11`	Point the camera towards the coordinate origin, at 55 degrees from *Oxz* side
`F12`	Point the camera towards the coordinate origin, at 90 degrees from above

2.5 Running MATLAB Simulations

To run MATLAB lab4 simulation scripts, you need to have `bin` folder (holding mex-file) in your MATLAB search path.

To ensure this, be sure to execute `make` at the beginning of the simulation session. This will guarantee both that you have the most recent executables, as well that you have `bin` folder in the search path. The other way is to manually add `bin` to search path using `addpath` command.

To start a lab4 simulation, just call `lab4` function with the simulation config-id as the arugment. The gui simulation window will appear, similar to the one shown on the previous Figure 2.4–1.

For example, to start lab4 simulation with config-id '2c' issue command

```
>> lab4 2c
```

Executing lab4 without arguments will start a default 'demo' simulation of two colliding bodies.

The simulation parameters (together with simulated bodies in the system) are configured (constructed) by the `lab4_defs` function, according to the selected config-id. For the full list (and descriptions) of possible configuration config-ids, see `lab4_defs.m`.

3 Results

3.1 Basic Level Tasks

The default simulation parameters were L = 10 m, m = 0.1 kg and v = 1 m/s, without any dissipation or friction. Since the simulation was performed in 3D, it was assumed that bodies had negligible thickness of d = 0.01 m.

The resulting trajectories of the rigid bodies during the collision are shown on the top on Figure 3.1–1 where the point of contact, at the time of the collision, is highlighted on the bottom image.

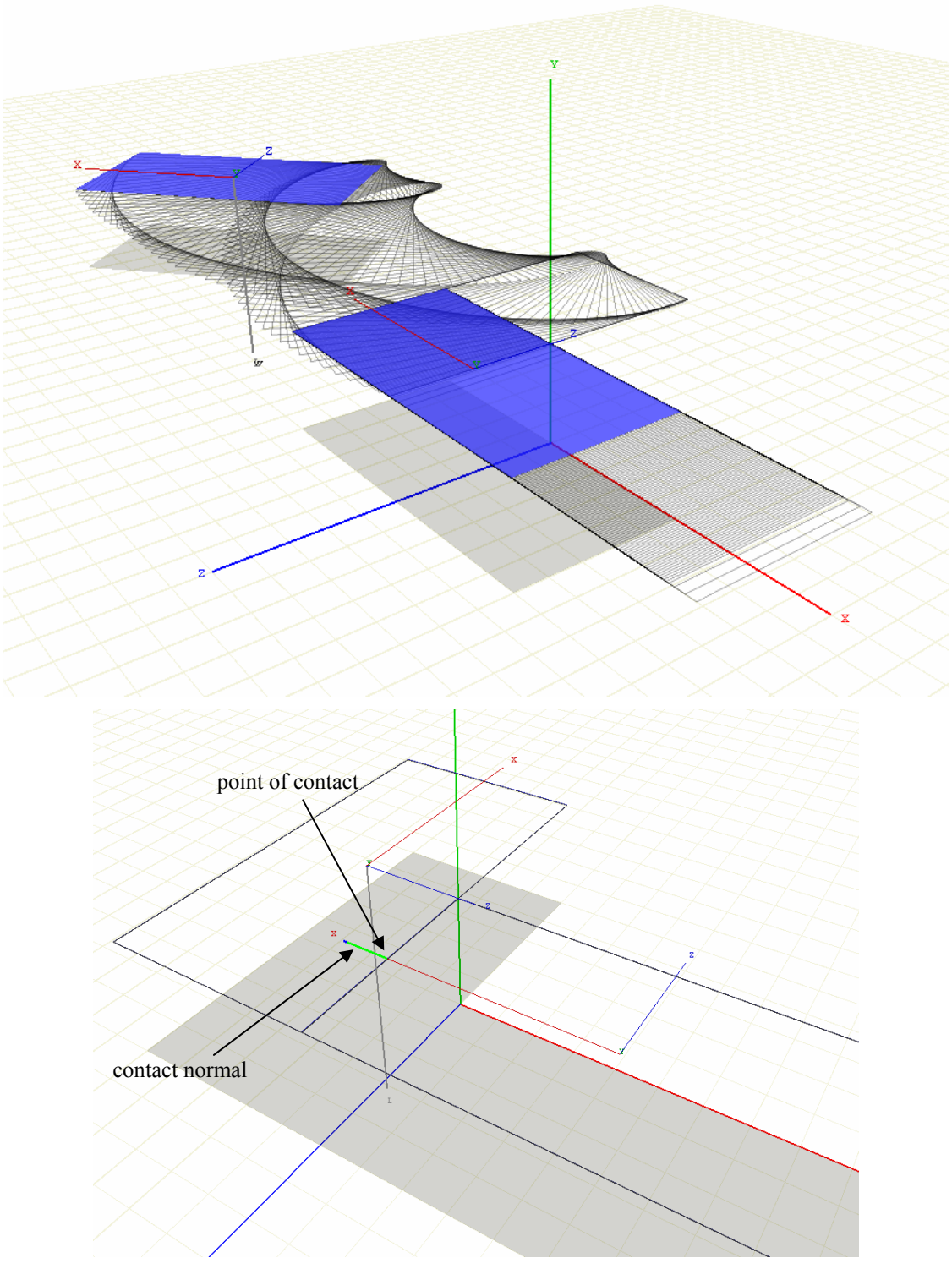


Figure 3.1–1: The resulting trajectories of the rigid bodies during the collision (config-id '1a')

The corresponding change in the linear and angular momentum and velocities of the bodies are shown in Table 3.1-1. (Note that the angular momentum of the bodies, as shown in the table, is their total angular momentum consisting of the center of mass angular momentum summed with the angular momentum induced by the angular velocity.)

The total linear momentum, angular momentum and energy levels of the system over the time are also shown on Figure 3.1–2.

From data in Table 3.1-1 and Figure 3.1–2, it can be concluded that the total linear momentum, angular momentum and kinetic energy are all preserved during the collision.

Table 3.1-1: Changed physical quantities during the collision (config-id '1a').

Quantity		**Before collision**			**After collision *)**		
		body 1	**body 2**	**system**	**body 1**	**body 2**	**system**
Kinetic energy	E_k / J	0.000	0.050	0.050	0047	0.003	0.050
Linear momentum	p_x / (kg·m·s^{-1})	0.000	−0.100	−0.100	−0.077	−0.023	−0.100
	p_y / (kg·m·s^{-1})	0.000	0.000	0.000	0.000	0.000	0.000
	p_z / (kg·m·s^{-1})	0.000	0.000	0.000	0.000	0.000	0.000
Angular momentum	L_x / (kg·m^2·s^{-1})	0.000	0.000	0.000	0.000	0.000	0.000
	L_y / (kg·m^2·s^{-1})	0.000	0.000	−0.250	−0.192	-0.058	−0.250
	L_z / (kg·m^2·s^{-1})	0.000	0.300	0.300	0.000	0.300	0.300
Velocity	v_x / (m·s^{-1})	0.000	−1.000	n/a	−0.769	−0.231	n/a
	v_y / (m·s^{-1})	0.000	0.000	n/a	0.000	0.000	n/a
	v_z / (m·s^{-1})	0.000	0.000	n/a	0.000	0.000	n/a
Angular velocity	ω_x / s^{-1}	0.000	0.000	n/a	0.000	0.000	n/a
	ω_y / s^{-1}	0.000	0.000	n/a	−0.185	0.000	n/a
	ω_z / s^{-1}	0.000	0.000	n/a	0.000	0.000	n/a

*) Quantities changed during the collision are highlighted in grey.

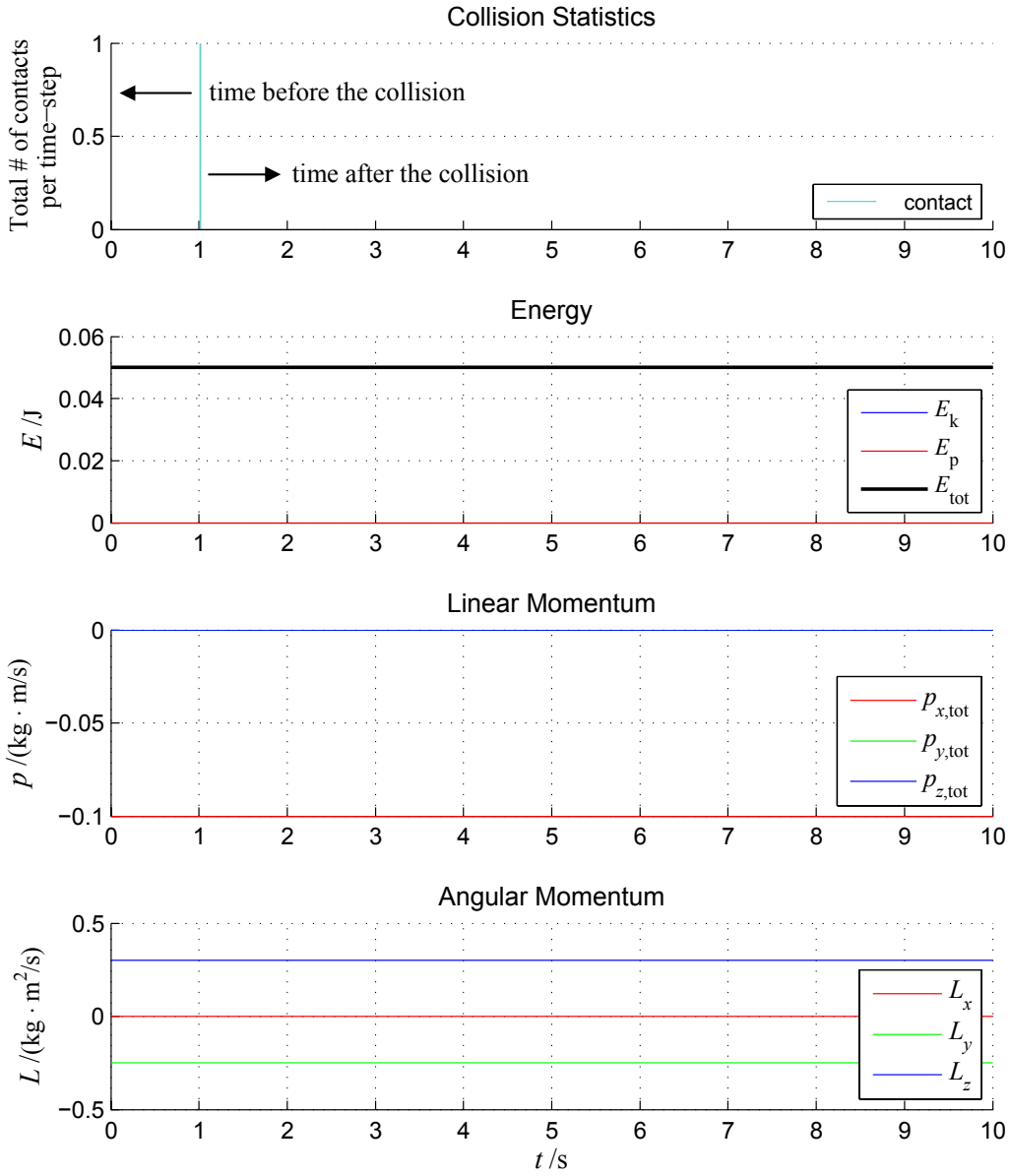


Figure 3.1–2: The total energy, linear momentum and angular momentum of the system over time (config-id '1a').

Note that the resulting trajectories of the rigid bodies are quite sensitive to detected contact point.

If the *x*-axis of an oncoming cuboid is not completely normal to the *z*-aligned edge of a stationary cuboid during the collision, i.e. if, for example, orientation of the oncoming rigid body is ±0.01 % around *y*-axis, then one or the other vertex of the oncoming cuboid will be detected as the contact point, and the resulting trajectories will look as on Figure 3.1–3.

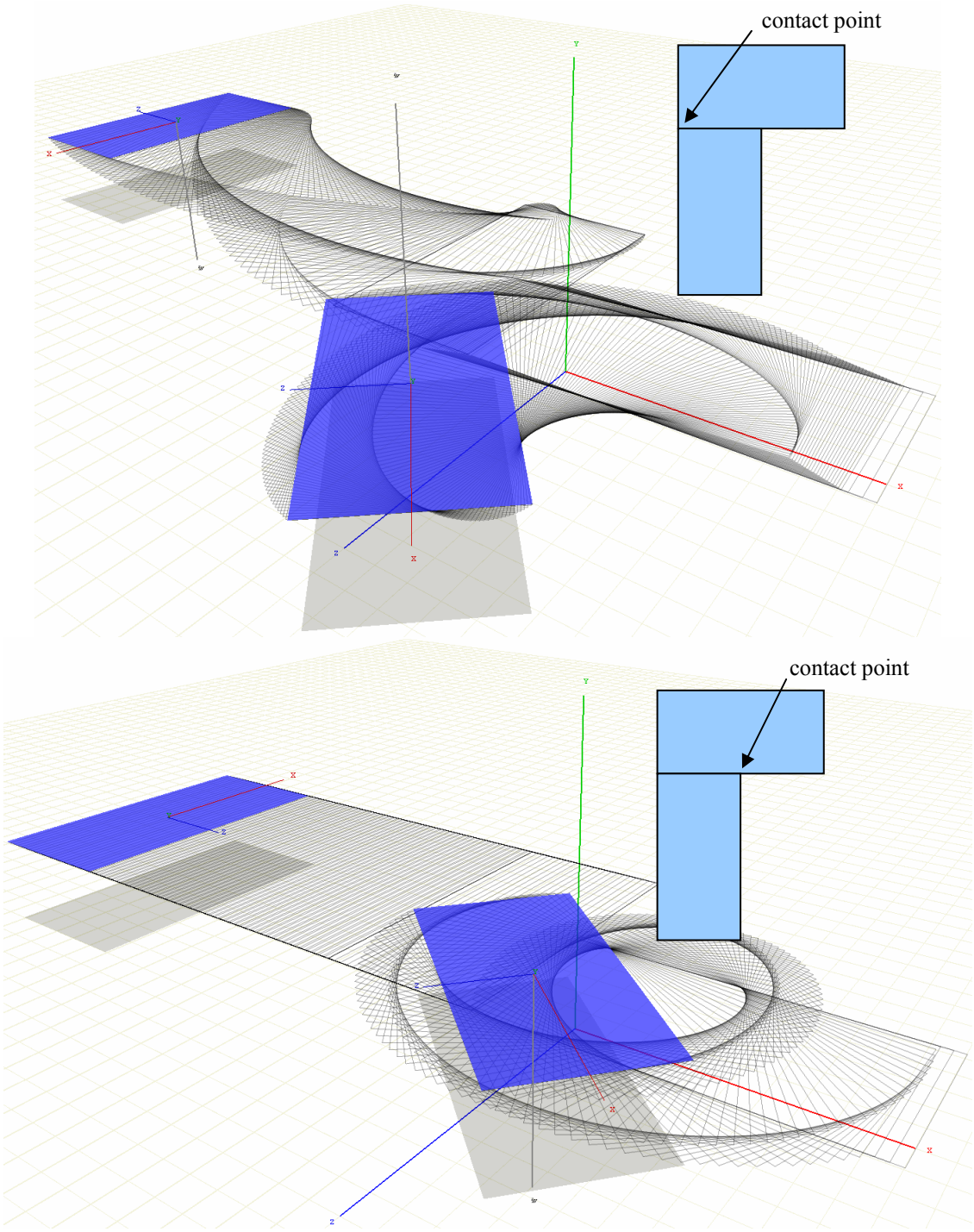


Figure 3.1–3: Config-ids ‘1b’ and ‘1c’ – The resulting trajectories of the rigid bodies during the collisions with slightly disturbed rotation around *y*-axis (config-ids ‘1b’ and ‘1c’)

The snapshot of a more realistic simulation, where cuboid bodies each has thickness $d = 1$ m and mass $m = 10^4$ kg, is shown on Figure 3.1–4. Initial velocity of the oncoming body is $v = 10$ m/s. The total linear momentum, angular momentum and kinetic energy levels of the same system over time are shown on Figure 3.1–5.

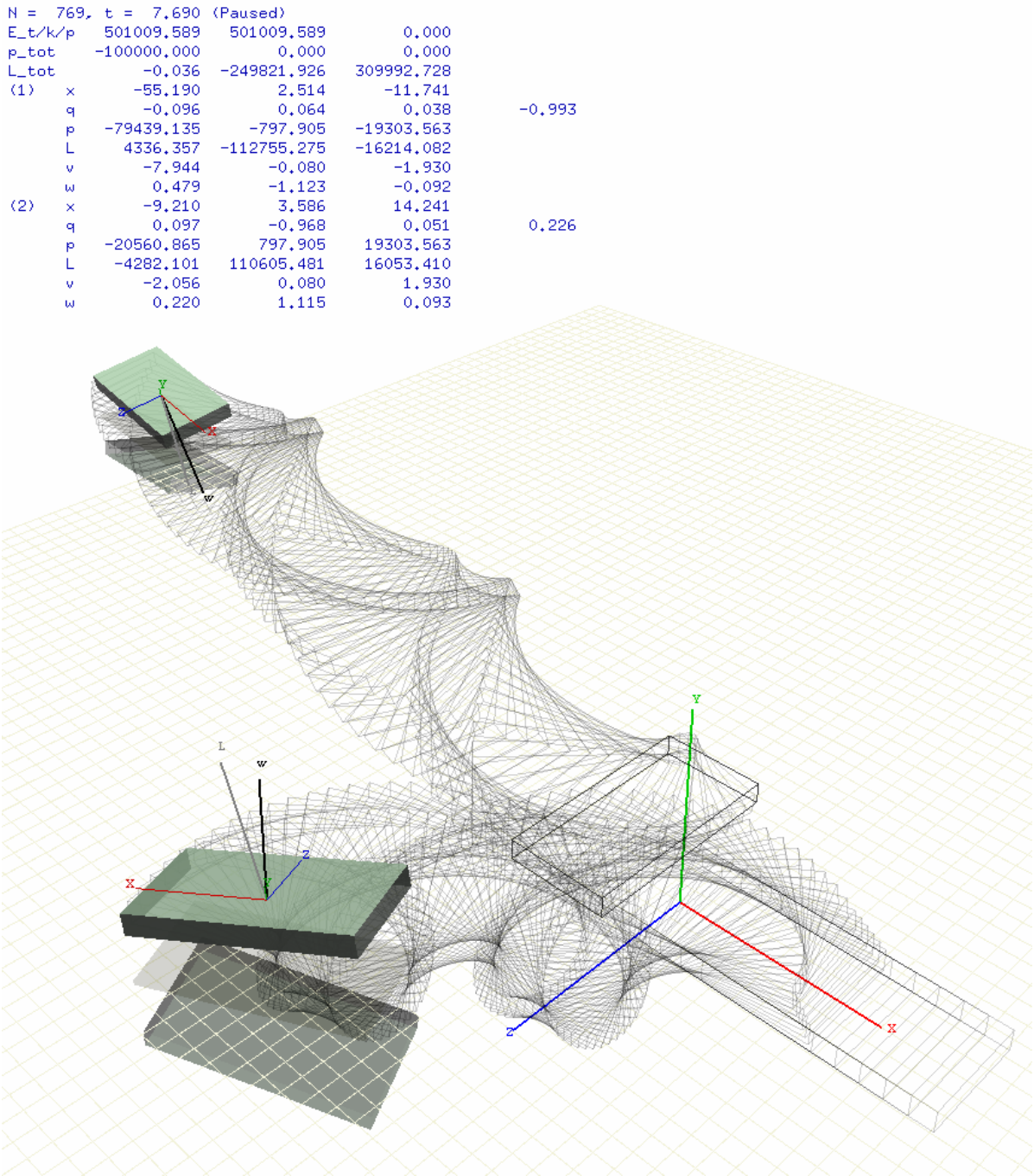


Figure 3.1–4: The resulting trajectories of the rigid bodies during a more realistic collision (config-id '2d').

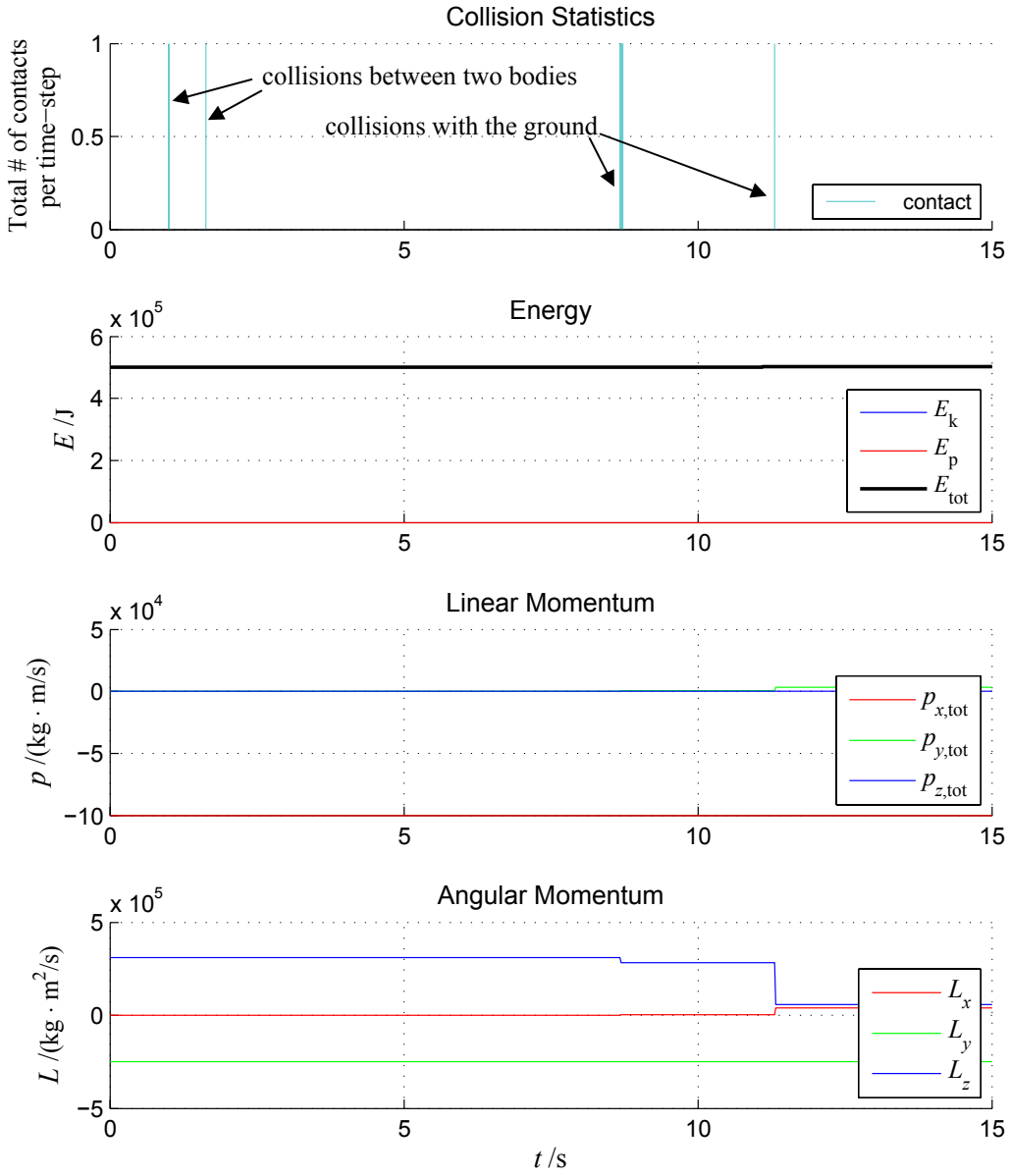


Figure 3.1–5: The total energy, linear momentum and angular momentum of the system over time during a more realistic collision (config-id '2d').

In this case it can be concluded that the total linear momentum, angular momentum and kinetic energy are preserved during collisions between bodies, while only the kinetic energy is preserved during collisions between bodies and the ground plane.

The reason why linear and angular momenta are not preserved during collisions with stationary objects (like the ground) is, first, stationary objects are assumed to have *infinite* masses and second, stationary objects are not allowed to move. If stationary objects were allowed to move and if they masses were finite (but very large), the total momenta of the system would be preserved.

This is demonstrated on Figure 3.1–6, which shows the simulation results of the collision between an oncoming cuboid having mass 10^3 kg hitting a stationary cuboid with mass 10^9 kg. Here, it can be concluded from Figure 3.1–7, that the total linear momentum, angular momentum and kinetic energy of the system are all preserved; the collision also results in an unnoticeable (but finite) change in linear and angular velocities of the stationary body (see state variables on Figure 3.1–6).

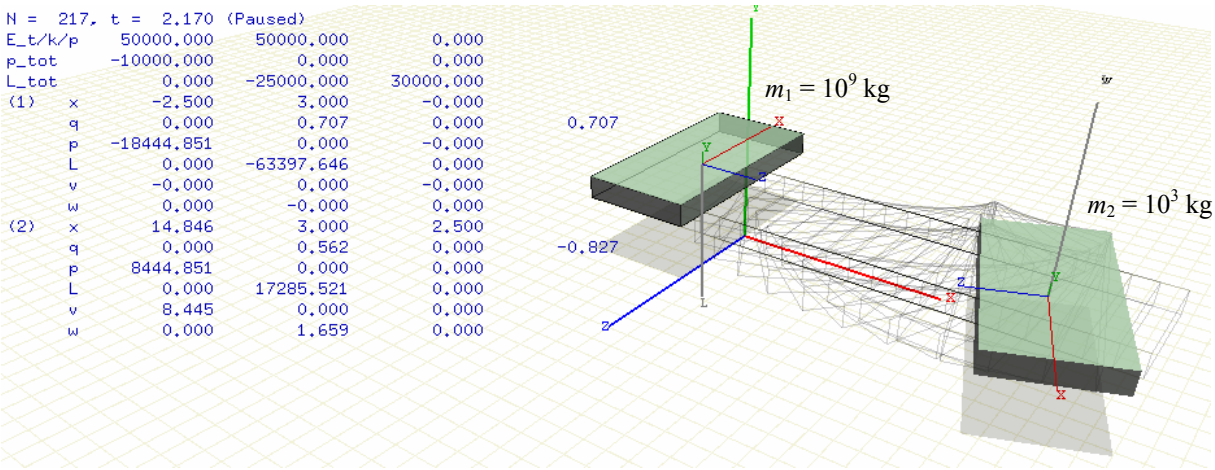


Figure 3.1–6: The resulting trajectories of two colliding rigid bodies, where the stationary body has a finite but very large mass (config-id '2e'). The stationary body does not (noticeably) move while the oncoming rigid body bounces off.

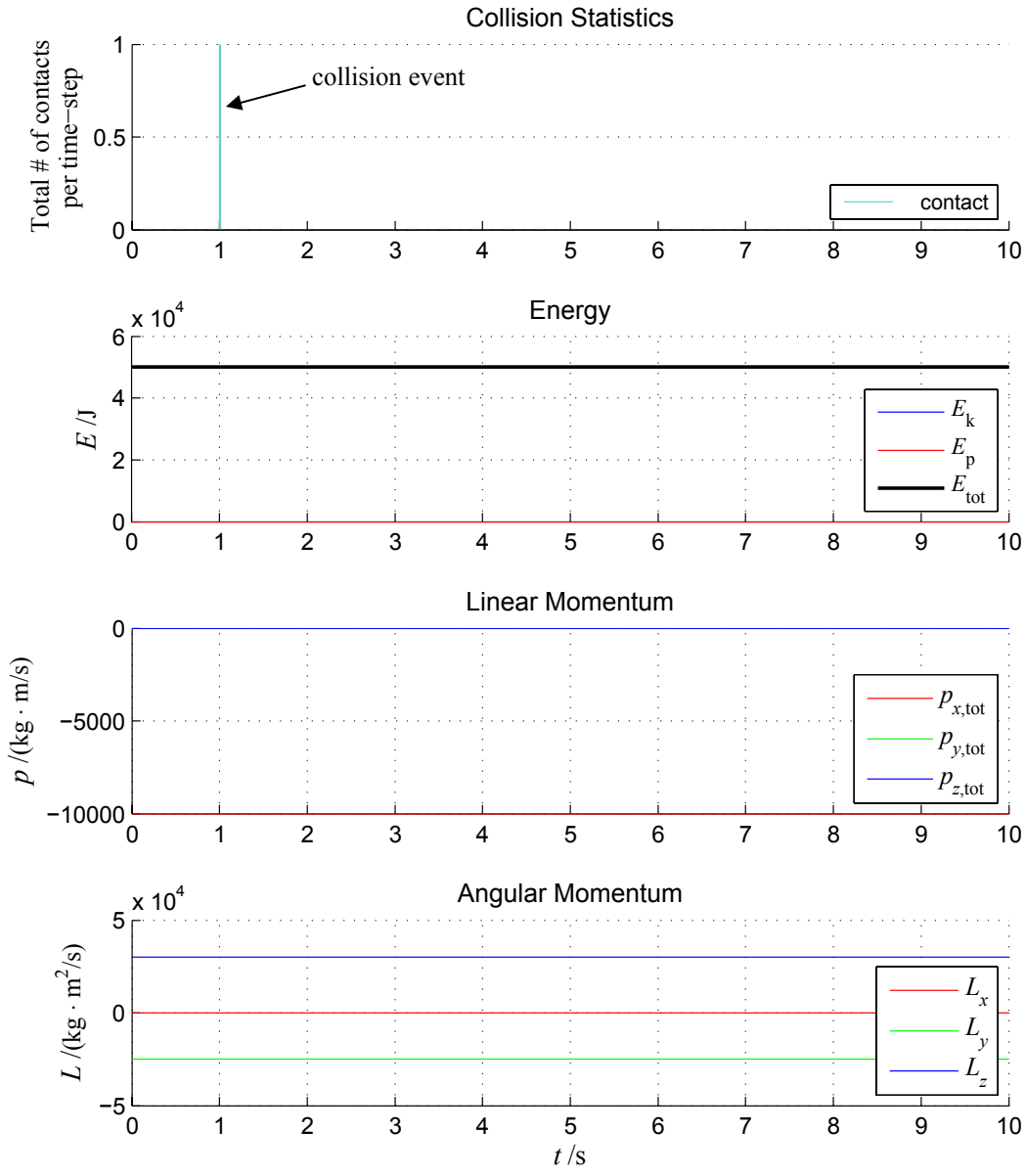


Figure 3.1–7: The total energy, linear momentum and angular momentum of the system over time, where the stationary body has a finite but very large mass (config-id '2e').

3.2 Advanced Level Tasks

Effects of Dissipation and Friction

Snapshots of different simulations that demonstrate effects of dissipation and kinetic friction are shown on Figure 3.2–2 (with dissipation, without friction), Figure 3.2–3 (without dissipation, with kinetic friction) and Figure 3.2–4 (with dissipation, with kinetic friction). The snapshot of the reference simulation is shown Figure 3.2–1 (without dissipation and friction). In all these simulations, two cuboid rigid bodies, having equal masses 10^4 kg, are colliding in the presence of the gravity field.

From Figure 3.2–2, it can be concluded that, with dissipation but without kinetic friction, bodies on ground will glide ad infinitum.

From Figure 3.2–3, it can be concluded that, with friction but without dissipation during the collision with the ground, bodies will bounce for ever, until they lose energy from kinetic friction when occasionally colliding with the horizontal plane tangentially.

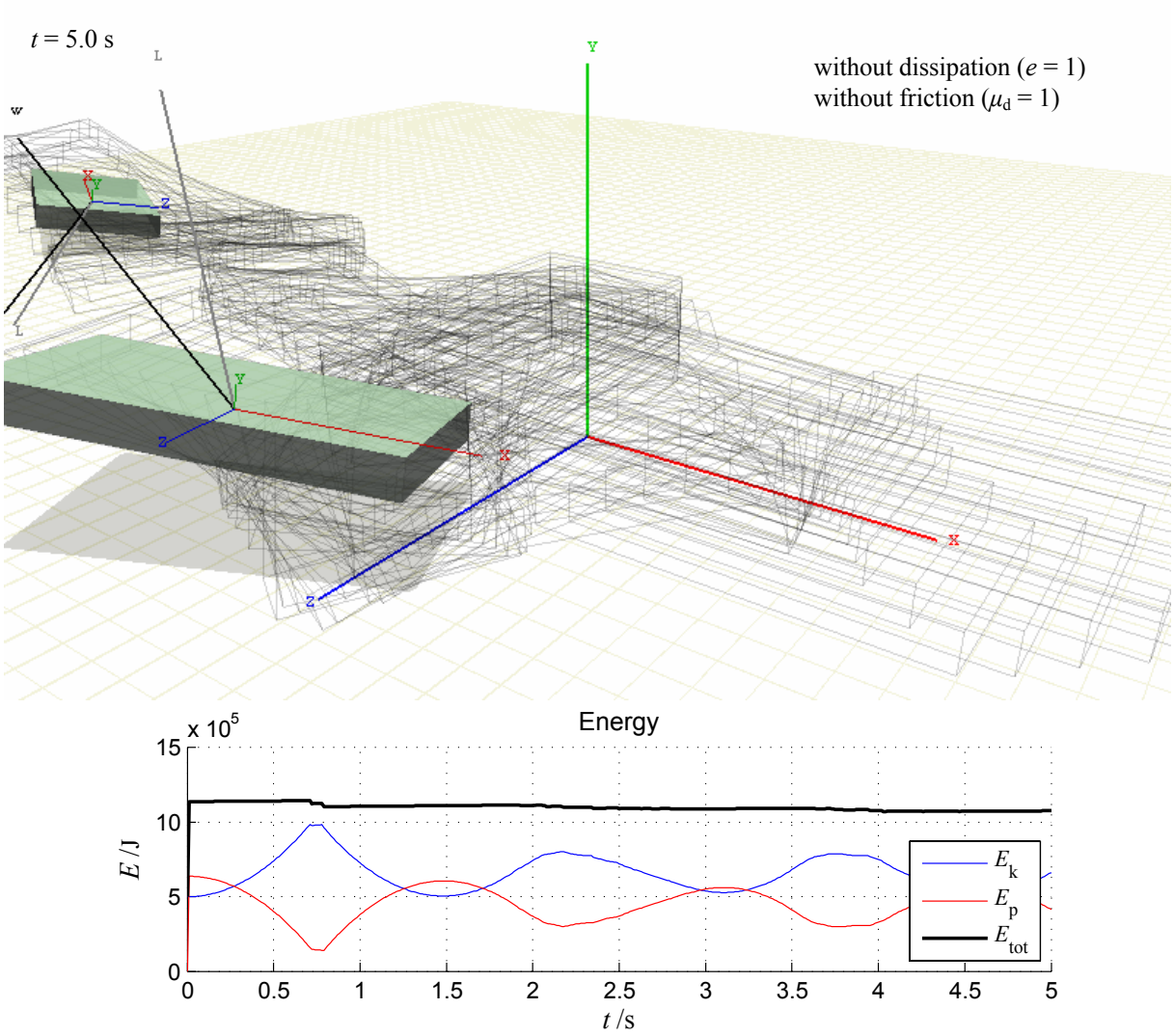


Figure 3.2–1: The snapshot of two colliding cuboids in gravity field with full restitution and without friction (config-id '3a'). Bodies will continue to bounce ad infinitum.

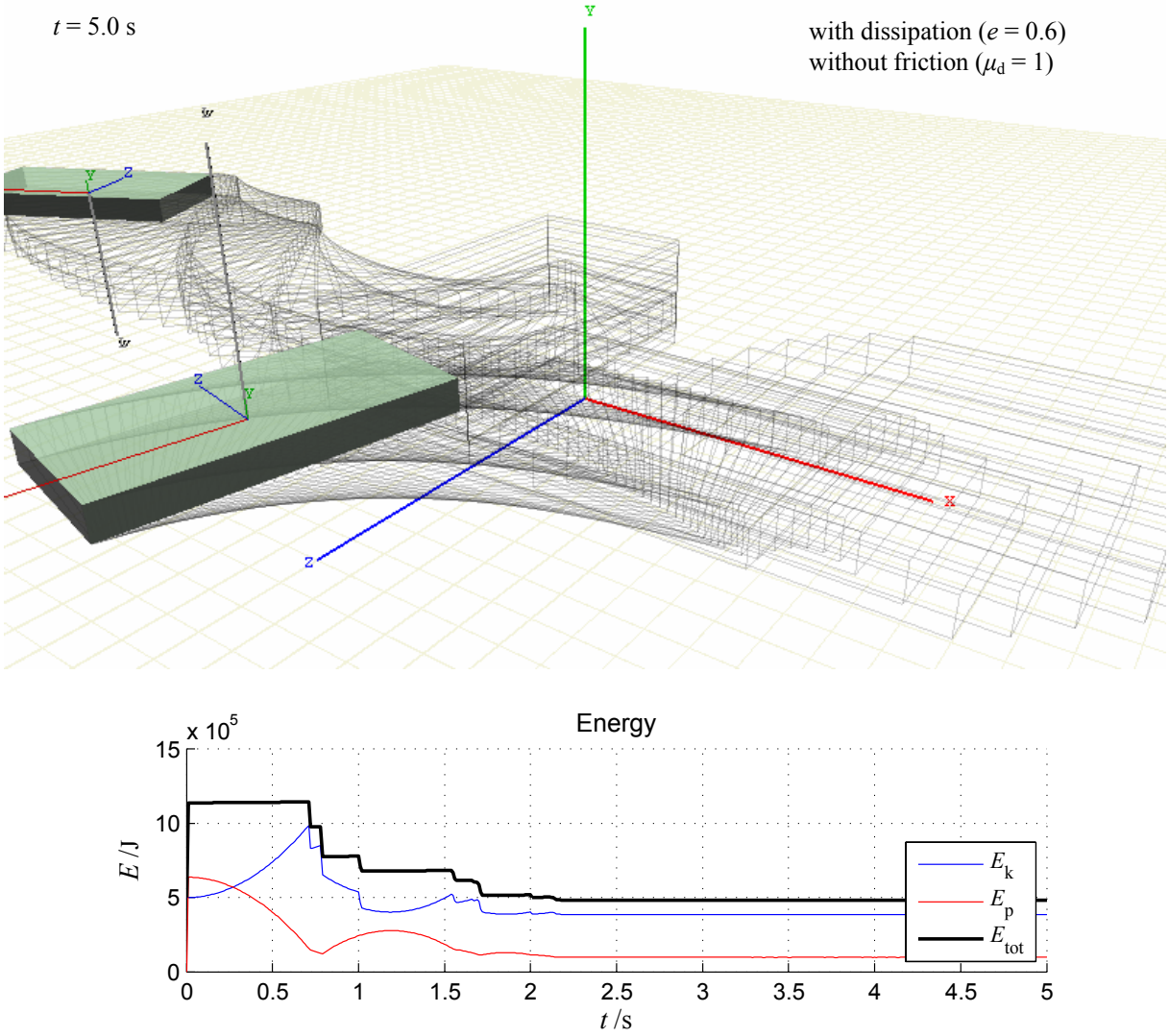


Figure 3.2–2: The snapshot of two colliding cuboids in gravity field with dissipation but without friction (config-id '3b'). Bodies will continue to glide for ever (blue line).

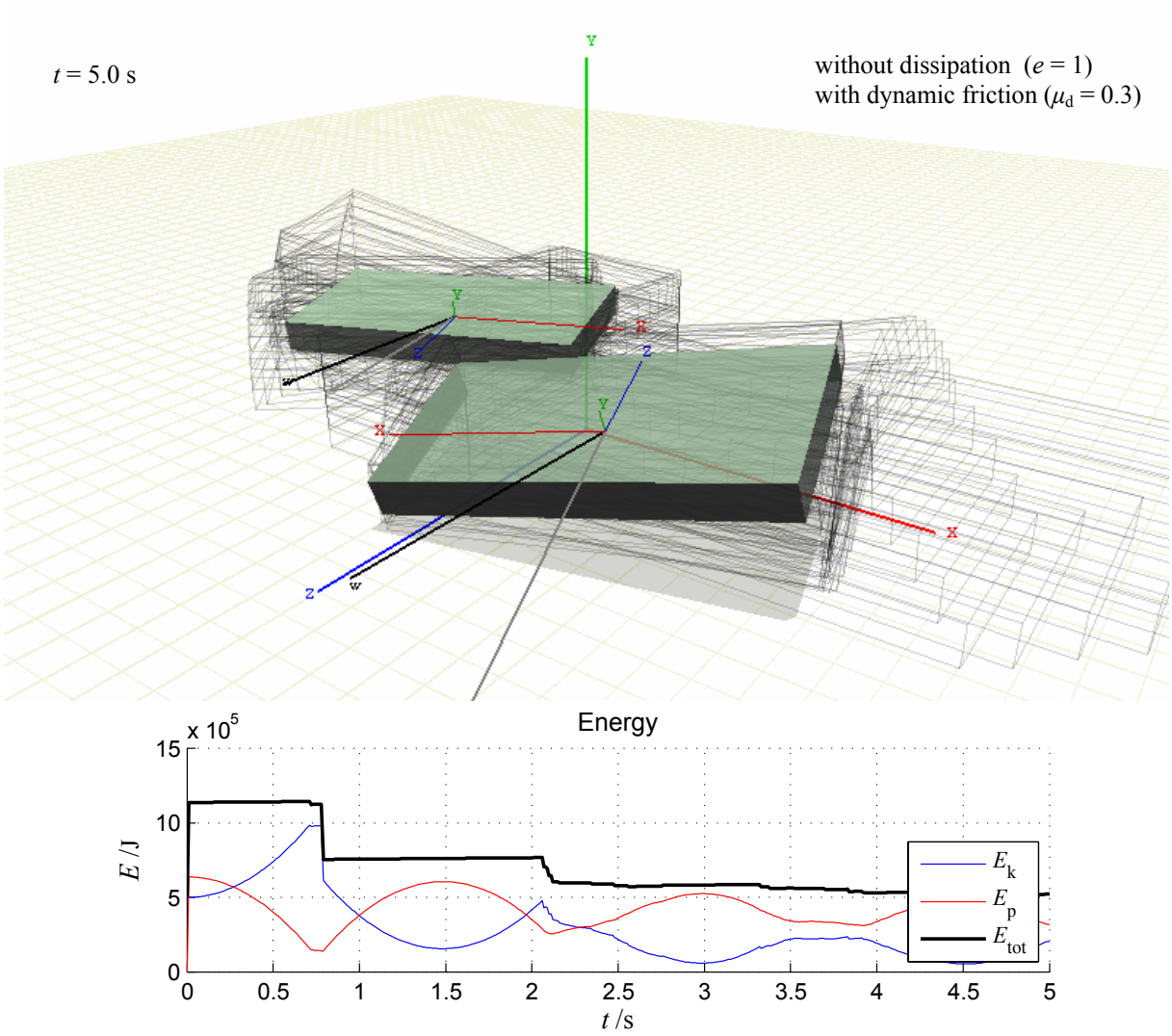


Figure 3.2–3: The snapshot of two colliding cuboids in gravity field without dissipation but with kinetic friction (config-id '3c'). Bodies will continue to bounce until they lose energy from kinetic friction during tangential collisions with the ground plane.

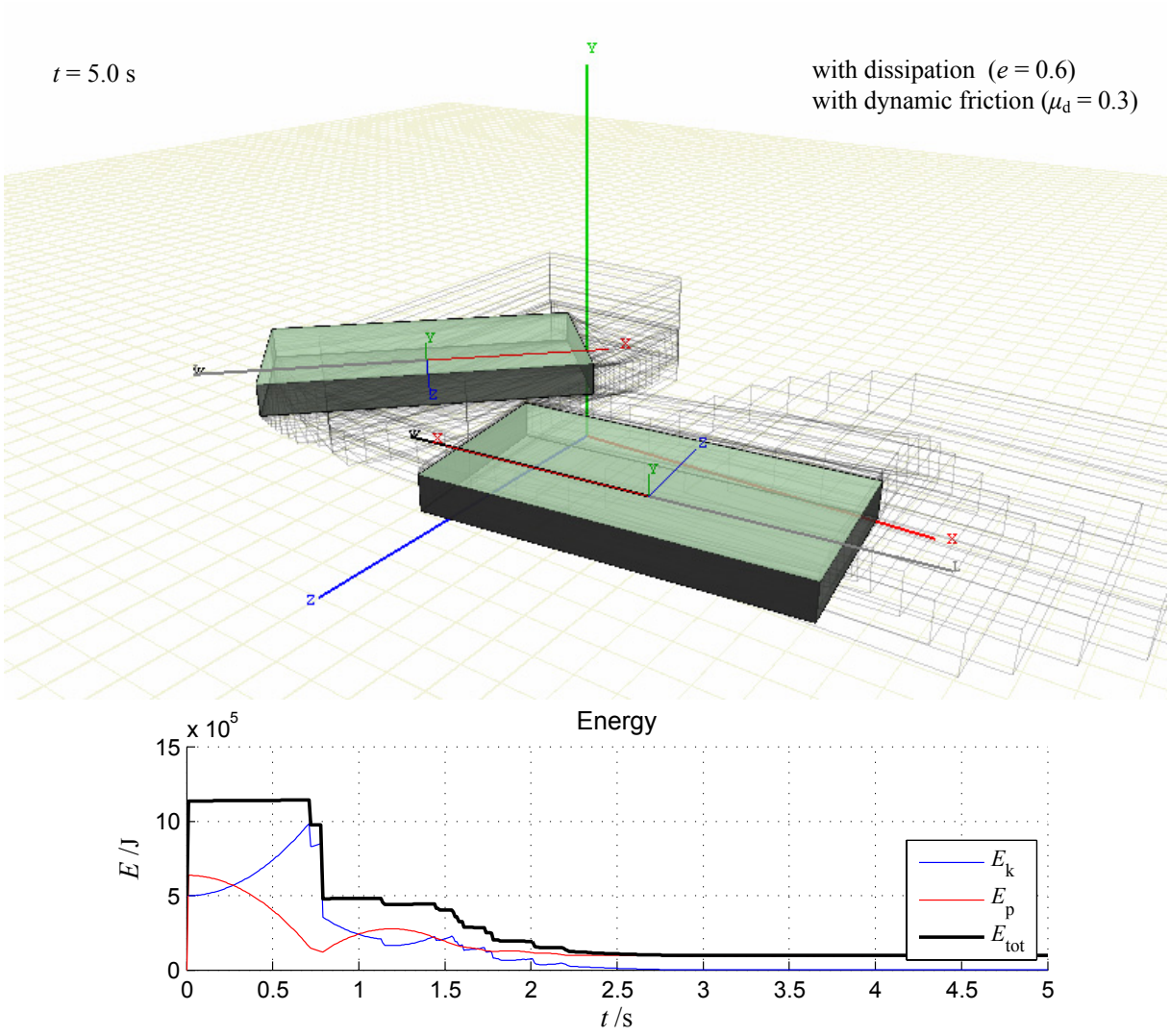


Figure 3.2–4: The snapshot of two colliding cuboids in gravity field with both dissipation and kinetic friction (config-id '3d'). Bodies lose their kinetic energy after ~2.5 s.

Collisions with Static Ground Plane

The snapshot of the simulation of random cuboid rigid bodies, falling and colliding with the ground plane as well with each other, are shown on Figure 3.2–5.

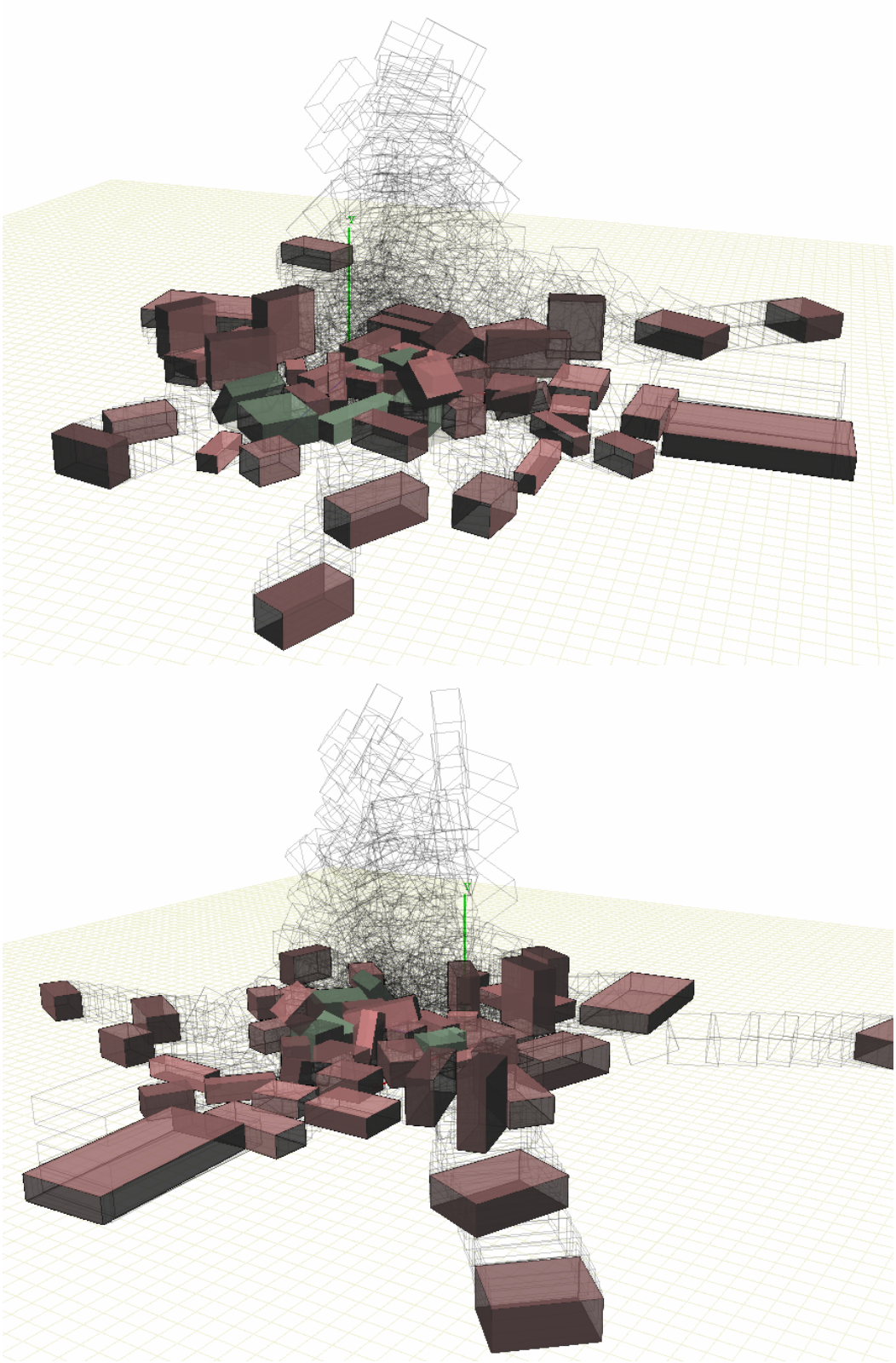


Figure 3.2–5: Snapshots from two different angles of the same simulation where cuboids are falling onto the ground in the gravity field (config-id '4a'). Inactive objects (objects not-moving at the snapshot time) are highlighted in light-red.

Collisions with Staircase

The snapshots of the simulation of 50 random cuboid rigid bodies each having mass 10 kg, falling down the staircase of 10 m height, are shown on Figure 3.2–6.

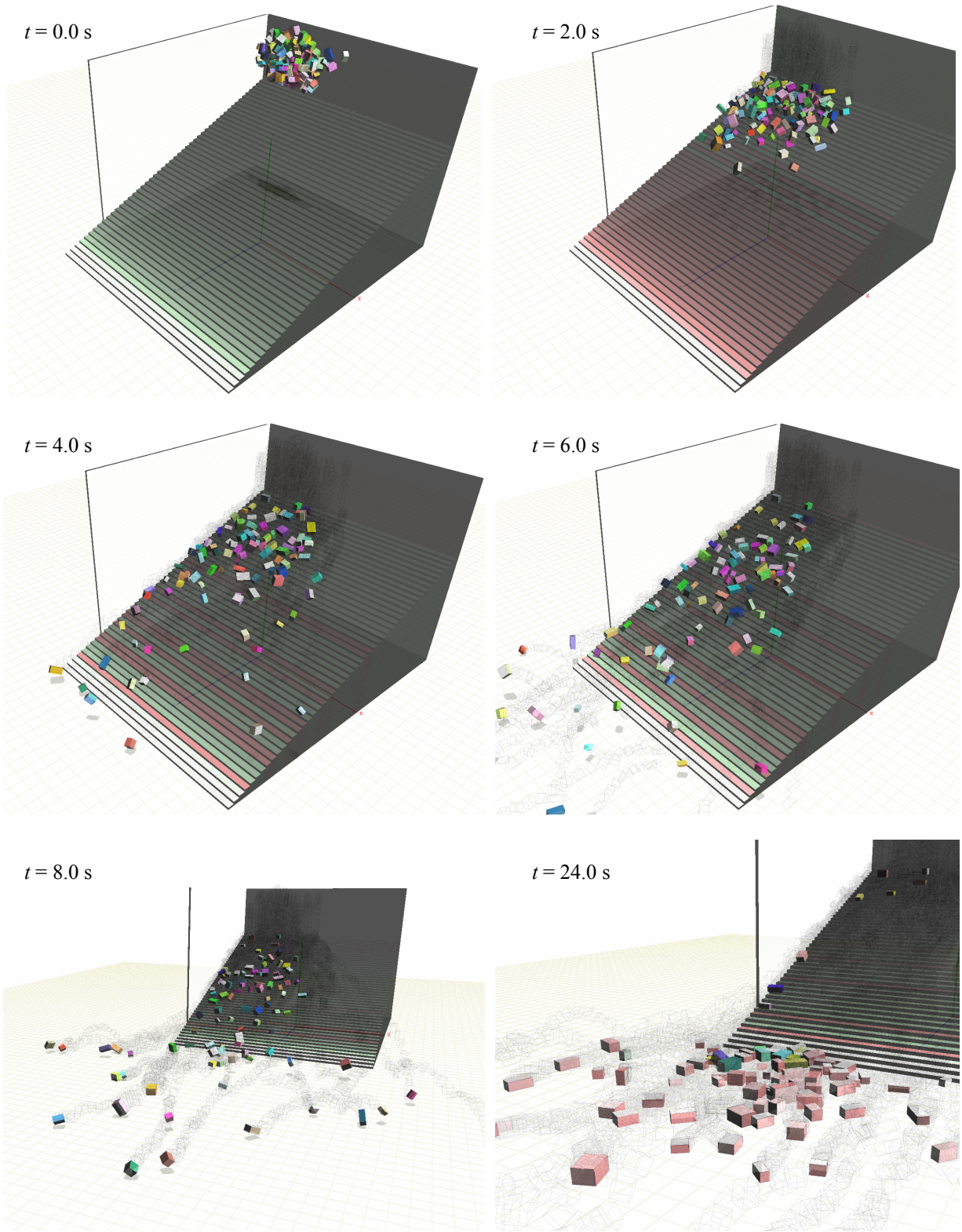


Figure 3.2–6: Snapshots from different angles of the simulation where random cuboids are falling onto a staircase in the gravity field (config-id '4b'). Inactive objects (objects not-moving at the snapshot time) are highlighted in light-red.

Object Piling

The stability of the impulse method was studied by piling cuboid rigid bodies (blocks) in different configurations and by varying some of the reference simulation parameters. As the measure of method stability was taken stability of the vertical position of the center of mass of the top-most block (i.e. the pile height). The reference simulation parameters are given in Table 3.2-2 and particular simulation configuration IDs for different observed simulations are summarized in Table 3.2-2.

Table 3.2-1: The reference simulation parameters

Number of blocks:	$N_b = 100$
Block mass:	$m = 100$ kg
Block dimensions:	$h = 0.4$ m height, $d = 2$ m width and depth
Coefficient of restitution:	$e = 0.8$
Position projections relaxation:	$r = 0.2$
Integrator time-step length:	$h = 0.01$ s

Table 3.2-2: Observed piling blocks configurations

config-id	N_b	e	r	h / s	*Description*
5a	**100**	**0.80**	**0.2**	**0.010**	**Reference simulation**
5b	50	0.80	0.2	0.010	Varying number of blocks
5c	200	0.80	0.2	0.010	
5d	100	0.70	0.2	0.010	Varying restitution
5e	100	0.99	0.2	0.010	
5f	100	0.80	0.1	0.010	Varying relaxation
5g	100	0.80	0.2	0.010	
5h	100	0.80	0.2	0.020	Varying time-step
5i	100	0.80	0.2	0.001	

Snapshots of simulations for different number of blocks in a pile are shown on Figure 3.2–7.

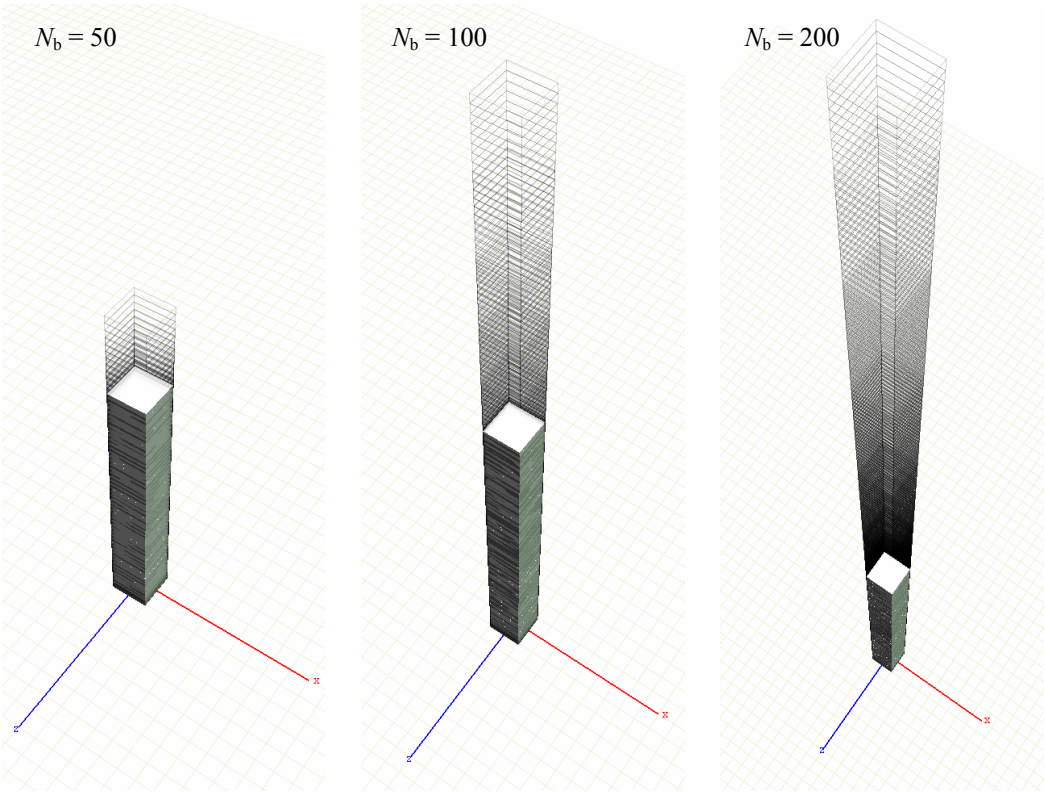


Figure 3.2–7: Snapshots of piling blocks simulation for different number of blocks at $t = 10$ s (config-ids, from left to right: '5b', '5a' and '5c').

Pile height stability for different number of blocks is shown on Figure 3.2–8. It can be observed that when number of piled blocks reaches a certain amount, compression ratio of the pile stabilizes around a certain value. For low number of blocks in the pile, this ratio is minor. By increasing number of blocks in the pile, the compression ratio reaches its maximum which is dependent on the number of blocks in pile. This is summarized in the following table:

Table 3.2-3: Relationship between pile height compression and number of objects in the pile

Number of objects	25	50	100	200
Pile height / %	97	80	50	32

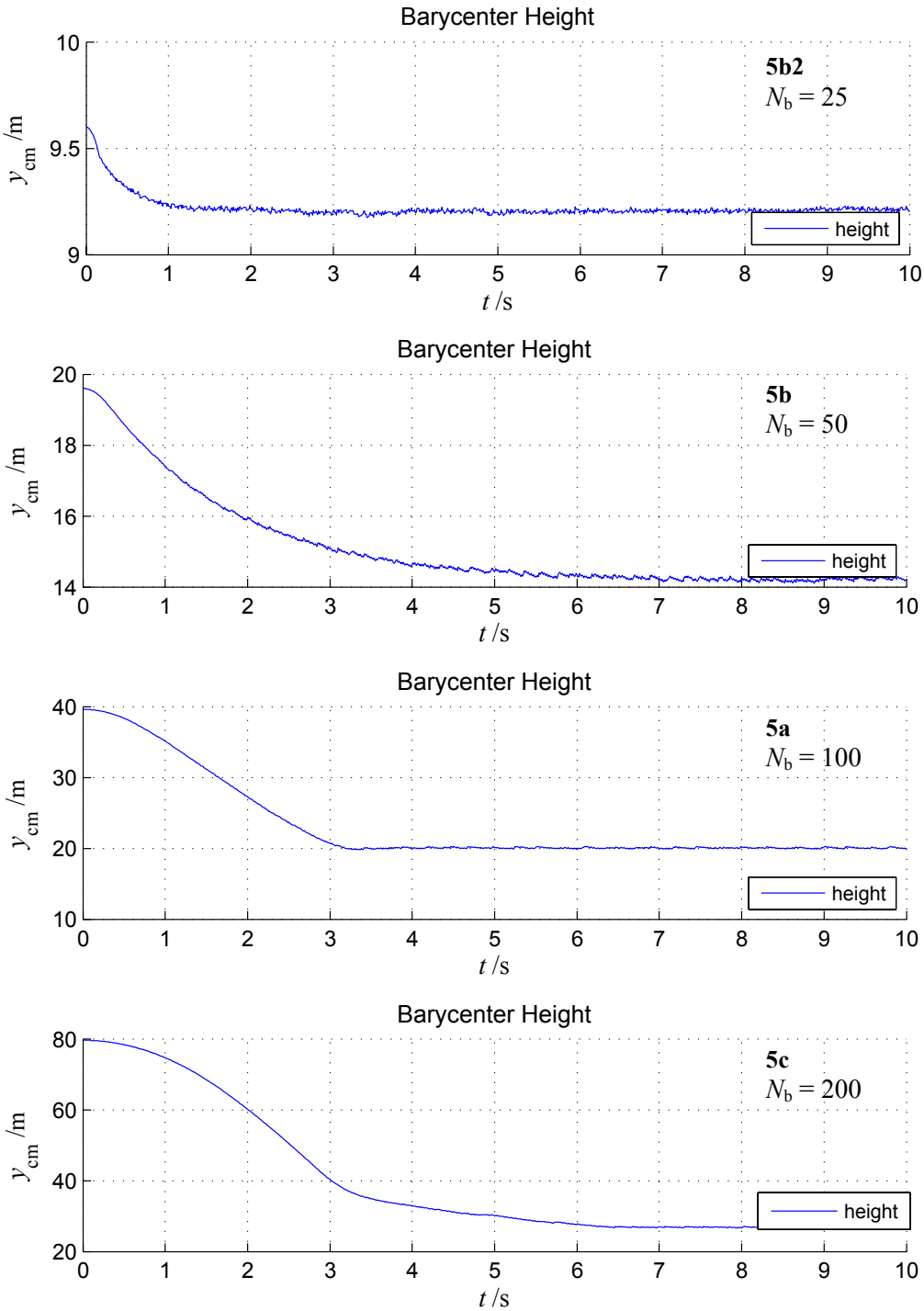


Figure 3.2–8: Pile height over time for different number of blocks (config-ids: '5b2', '5b', '5a' and '5c').

Pile height stability for different coefficients of restitution is shown on Figure 3.2–9. It can be concluded that stability of the impulse method increases when coefficient of restitution decreases.

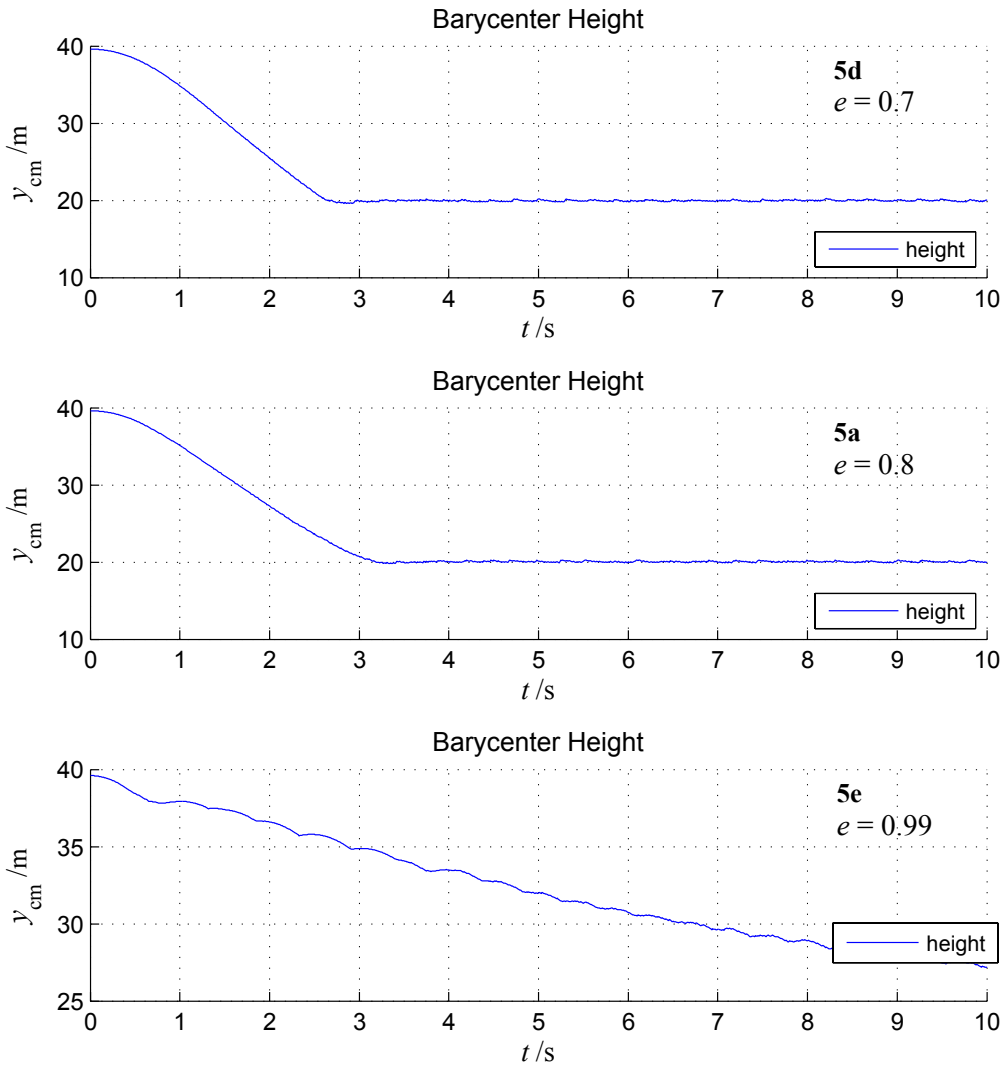


Figure 3.2–9: Pile height over time for different coefficient of restitution (config-ids: '5d', '5a' and '5e').

Pile height stability for different position projections relaxation coefficient is shown on Figure 3.2–10. It can be concluded that stability of the impulse method increases when position projections relaxation coefficient increases.

> The position projections relaxation coefficient $r = 0$ means that position projections are applied in full extent, while the coefficient $r = 1$ means that position projections are not applied at all. Note that implemented collision response algorithm requires $r < 1$.

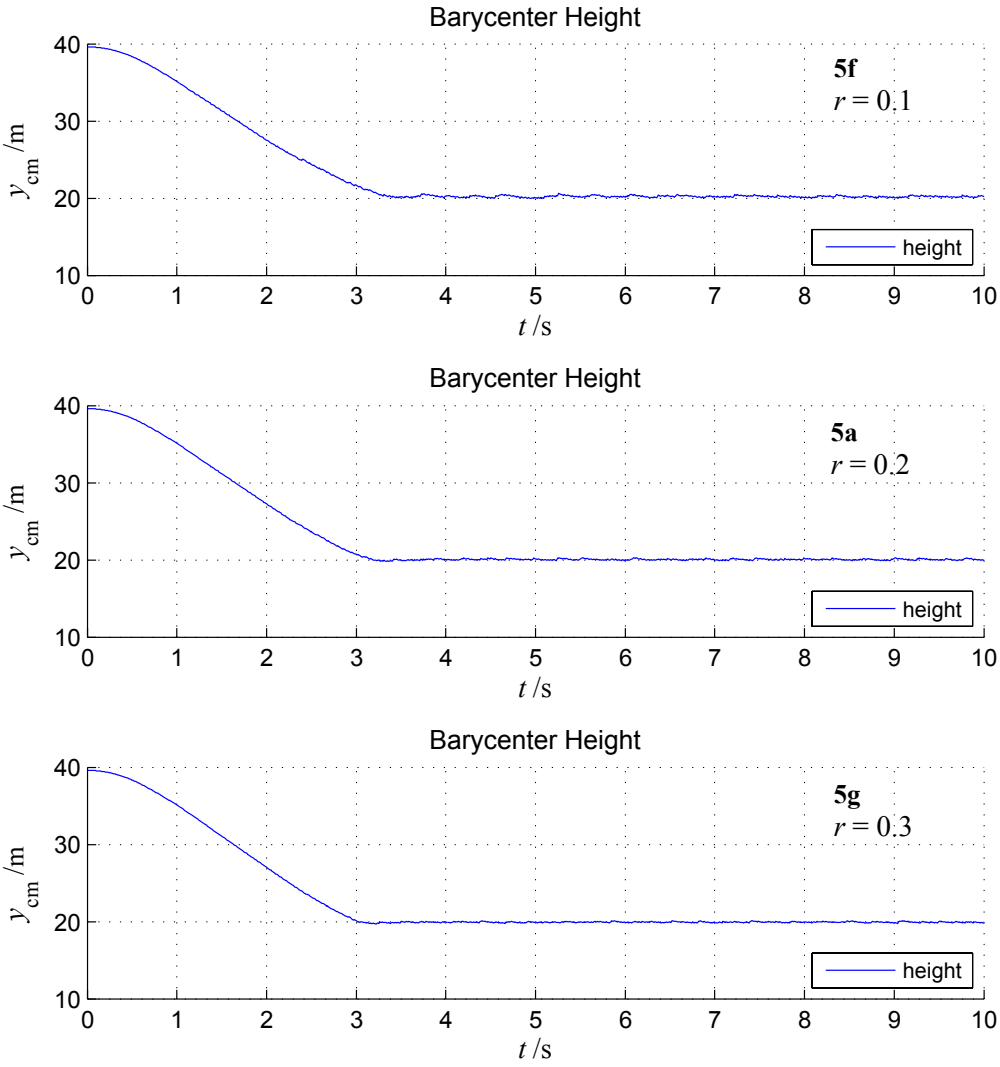


Figure 3.2–10: Pile height over time for different position projections relaxation coefficients (config-ids: '5f', '5a' and '5g').

Pile height stability for different integrator time-step lengths is shown on Figure 3.2–11. It can be concluded that stability of the impulse method increases when integration time-step decreases.

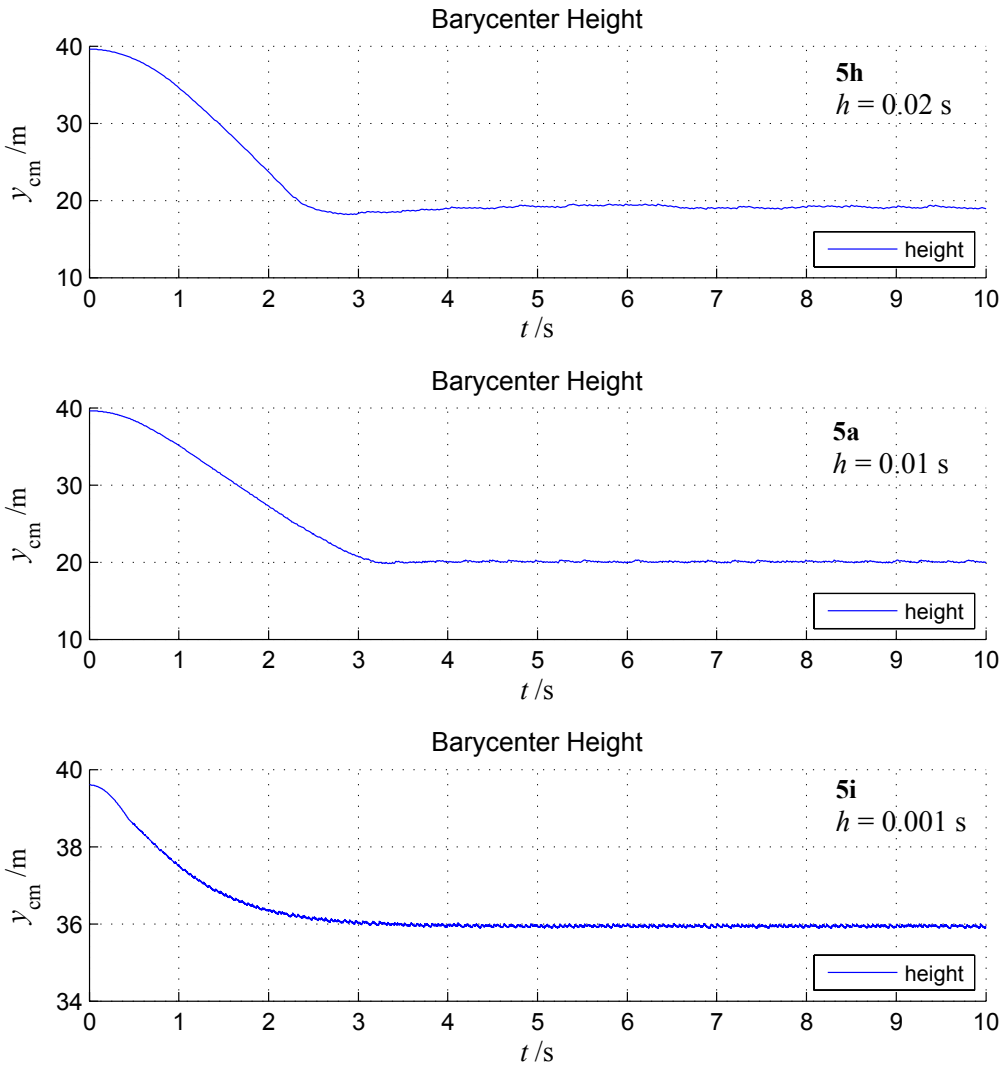


Figure 3.2–11: Pile height over time for different integrator time-step lengths (config-ids: '5h', '5a' and '5i').

4 Extensions

One of the consequences of the selected approach to use quaternions to represent all physical quantities is that the WoRB framework could be extended to simulations in special relativity [6]. In such extension, position quaternion will include local time for example, and linear momentum and velocity quaternion will include total energy and Lorentz boost, respectively. Still, there are few obstacles to overcome in such endeavor; firstly, forces are not invariant in special relativity, meaning that Newton's third law and our rigid body equations of motion do not hold, so the interactions should be restated in terms of e.g. electromagnetic field, and secondly, quaternions should be allowed to have complex components (thus becoming biquaternions) which would require modifications to `Quaternion` and `QTensor` classes.

5 References

[1] Eberly, David H. – *Game Physics*, Morgan Kaufmann publ., 2004; Contrib. by Shoemake, Ken

[2] Eberly, David H. – *3D Game Engine Architecture*, Morgan Kaufmann publ., 2005

[3] Ericson, Christer – *Real-Time Collision Detection*, Morgan Kauffmann publ., 2005

[4] Millington, Ian – *Game Physics Engine Development*, Morgan Kaufmann publ., 2007

[5] Baraff, David – *Physically Based Modeling, Rigid Body Simulation*, SIGGRAPH, 1997

[6] Silberstein, Ludwik – *Quaternionic Form of Relativity*, Phil. Mag. 23:790-809, 1912

Appendix A - Rigid Body Motion in 3D

Mikica Kocic, miko0008@student.umu.se
The Physics of Virtual Environments, 2012-04-25

Definitions

```
Get[ "Quat.m", Path -> { NotebookDirectory [] } ];
```

Parameters

Frame of reference

The flag **frameOfRef** indicates whether angular velocity ω, angular momentum **L** and moment of inertia tensor $\mathcal{J}$ are given with coordinates either in **inertial** (also called **space** or **world**) frame of reference or in **non-inertial** (also called **body-fixed** or **rotating**) frame of reference.

```
frameOfRef := BodyFixed;
```

```
Inertial  =. (* inertial (or world) frame of reference *)
BodyFixed =. (* non-inertial (or rotating) frame of reference *)
```

Other physical quantities like position, velocity, forces or torque, are always given in inertial frame of reference.

Gyroscopic effects

The flag **gyroEffects** indicates whether to *include* or *ignore* gyroscopic effect when calculating the time derivative of the angular momentum.

```
gyroEffects := Include;
```

```
Include =. (* include gyroscopic effects *)
Ignore  =. (* don't account gyroscopic effects *)
```

Physical constants

Gravitational acceleration $\mathbf{g}_n$, kg m s^{-2}

```
gn = { 0, 0, -9.81 };
```

Characteristic dimension ℓ of the system, m

```
ℓ = 10;
```

▼ Rigid body parameters

Rigid body is defined by its mass m, a graphics complex `{body$v,body$i}` with centroid coordinates of the body vertices in the body-fixed frame of reference, and principal moment of inertia tensor $\mathcal{J}_0$ and its inverse $\mathcal{J}_{0,\text{inv}} == \mathcal{J}_0^{-1}$.
For more info about graphics complex, see: http://reference.wolfram.com/mathematica/ref/Graphics-Complex.html

```
setupBodyShape[ mass_, { width_, height_, depth_ }, type_ ] :=
 Module [
  { faces },

  (* Body mass, kg *)
  m = mass;

  (* Body dimensions, m *)
  { a, b, c } = { width, height, depth };

  (* Get graphic complex of the body *)
  { faces } = PolyhedronData[ type, "Faces"];

  (* Rescale graphic complex according to specified dimensions *)
  body$v = ( # {a, b, c}) & /@ faces[[1]];
  body$i = faces[[2]];

  (* Moment of inertia and its inverse *)
  𝒥0 = m/12 ( b^2 + c^2   0          0
              0           a^2 + c^2  0
              0           0          a^2 + b^2 ) // N;
  𝒥0inv = Inverse[ 𝒥0 ];
 ]
```

Now define initial rigid body (a cube having 1 kg mass)

```
setupBodyShape[ 1, { 1, 1, 1}, "Cuboid" ];
```

Initial state variables: position, orientation, linear and angular velocity. All variables, except angular velocity, are given in inertial (world) frame of reference. Angular velocity frame of reference depends on the `frameOfRef` flag.

```
X0 = { 0, 0, 0 };
Q0 = Q[ 0, 0, 0, 0 ];
V0 = { 0, 0, 0 };
ω0 = { 0, 0, 0 };
```

▼ Integration parameters

Integration method, either **`rk4$stepper`** or **`semiImplicitEuler$stepper`**

```
odeIntegrator := rk4$stepper
```

Time-step length, s

```
h = 0.01;
```

Final time, s

```
tf = 4;
```

- **Runge-Kutta 4th order method** (`rk4$stepper`)

  Classical implementation of the 4th order Runge-Kutta integrator. See http://mathworld.wolfram.com/Runge-KuttaMethod.html

```
rk4$stepper[ y_, h_, f_ ] := Module[ {k},

  k1 = h f[ y ];
  k2 = h f[ y + 1/2 k1 ];
  k3 = h f[ y + 1/2 k2 ];
  k4 = h f[ y + k3 ];

  y + 1/6 ( k1 + 2 k2 + 2 k3 + k4 )  // N
 ]
```

- **Semi-implicit Euler method** (`semiImplicitEuler$stepper`)

  The implementation bellow is demonstrative but very inefficient since it calls ODE function $\boldsymbol{f}$ twice. The implemented algorithm also requires that the state vector $\mathbf{y}$ has a special structure: 1) y_1 contains a time variable, 2) the time variable is followed by the ordinary state variables in the bottom-half and 3) the ordinary state variables are followed by their time derivatives in the top-half, i.e. the $\mathbf{y}$ should look like $\mathbf{y} = \{t, \mathbf{x}, \dot{\mathbf{x}}\}$ and its derivative (the ODE function $\boldsymbol{f}$) shoold return $\dot{\mathbf{y}} = \{1, \dot{\mathbf{x}}, \ddot{\mathbf{x}}\}$.

```
semiImplicitEuler$stepper[ y_, h_, f_ ] := Module[ { n, χ, χdot, y2 },

  n = 1 + ( Length[y] - 1 ) / 2; (* Split y into bottom- and top-halves *)

  χdot = h f[ y ];  (* Solve velocities only *)
  χdot[[1;;n]] = 0;    (* disregarding position and time solutions. *)
  y2 = y + χdot;

  χ = h f[ y2 ];    (* Now, solve position and time only *)
  χ[[n+1;;]] = 0;      (* disregarding velocity solution. *)

  y + χ + χdot
 ]
```

■ Animation Functions

▼ ςT: Transforms coordinates from body-fixed to inertial frame of reference

Transformations from non-inertial body-fixed (rotational) frame to intertial (world) frame of reference are based on global orientation quaternion q (variable `Q`) and position vector $\mathbf{x}$: (variablle `X`)

```
ςT[ { x_, y_, z_ } ] := X + Im[ Q ** Q[0, x, y, z] ** Q* ]
```

```
ςT[ {} ] := {};
```

```
ςT[ points_ ?MatrixQ ] := ςT /@ points
```

```
ςT[ points__ ?VectorQ ] := ςT /@ { Sequence[ points ] }
```

▼ getAnimationData: Gets coordinates of graphic primitives to be animated

Transforms body shape, orientation vectors and angular components into the inertial frame of reference

```
getAnimationData [] := {
  (* 1 *) ϛT[ body$v ], (* body shape *)
  (* 2 *) body$i,
  (* 3 *) ϛT[ {0, 0, 0}, {a / 2, 0, 0} ], (* e1 axis *)
  (* 4 *) ϛT[ {0, 0, 0}, {0, b / 2, 0} ], (* e2 axis *)
  (* 5 *) ϛT[ {0, 0, 0}, {0, 0, c / 2} ], (* e3 axis *)
  (* 6 *) If[ frameOfRef === Inertial, { {0, 0, 0}, ω / ωScale },
       (* else in body-fixed *) ϛT[ {0, 0, 0}, ω / ωScale ] ],
  (* 7 *) If[ frameOfRef === Inertial, { {0, 0, 0}, L / LScale },
       (* else in body-fixed *) ϛT[ {0, 0, 0}, L / LScale ] ],
  (* 8 *) ωTrack (* angular velocity trajectory *)
 }
```

▼ showAnimation: Renders 3D objects from animation data

Displays 3D graphics dynamically retrived by `getAnimationData` function.

```
showAnimation [] := Show[
  (* rigid body *)
  Graphics3D[{ Yellow, Opacity[.2],
    GraphicsComplex[ Dynamic[aData[[1]]], Dynamic[aData[[2]]] ] }],
  (* ê1 axis *)
  Graphics3D[{ Red, Thick, Line[Dynamic[ aData[[3]] ]] }],
  (* ê2 axis *)
  Graphics3D[{ Green, Thick, Line[Dynamic[ aData[[4]] ]] }],
  (* ê3 axis *)
  Graphics3D[{ Blue, Thick, Line[Dynamic[ aData[[5]] ]] }],
  (* Angular velocity ω *)
  Graphics3D[{ Black, Thick, Line[Dynamic[ aData[[6]] ]] }],
  (* Angular momentum L *)
  Graphics3D[{ Gray, Thick, Line[Dynamic[ aData[[7]] ]] }],
  (* Angular velocity ω trajectory *)
  Graphics3D[{ Pink, Thick, Line[Dynamic[ aData[[8]] ]] }],
  (* Axes and plot range *)
  Boxed → True,
  (* Axes→True,AxesLabel→{ "x /m","y /m","z /m" },
  LabelStyle→Directive[ FontSize→12 ], *)
  ViewPoint → Front, ImageSize → Scaled[ 0.9 ],
  PlotRange → Dynamic@{ {-ℓ, ℓ}, {-ℓ, ℓ}, {-ℓ, ℓ} }
 ]
```

▼ snapshotVars: Gets a verbose snapshot of state variables

Dumps all variables for debugging purposes. Usage: evaluate expression `Dynamic@snapshotVars[]` to get real-time update.

```
snapshotVars[] := TableForm[{
   {"Parameters", ⋯, ⋯},
   {"  Frame of Reference: " <> ToString@frameOfRef, ⋯, ⋯},
   {"  Gyroscopic Effects: " <> ToString@gyroEffects , ⋯, ⋯},
   {"Integrator", ⋯, ⋯},
   {Switch[odeIntegrator,
     rk4$stepper, "  Runge-Kutta 4",
     semiImplicitEuler$stepper, "  Semi-Implicit Euler",
     _, "?"] , ⋯, ⋯},
   {"  h", "=", h},
   {"Energy", ⋯, ⋯},
   {"  Eₖ", "=", Ek // N},
   {"  Eₚ", "=", Ep // N},
   {"  E_tot", "=", Etot // N},
   {"Linear momentum, velocity and position", ⋯, ⋯},
   {"  |p⃗|", "=", Norm[P] // N},
   {"   p⃗", "=", P // N},
   {"  |v⃗|", "=", Norm[V] // N},
   {"   v⃗", "=", V // N},
   {"   x⃗", "=", X // N},
   {"Angular momentum, velocity and orientation", ⋯, ⋯},
   {"  |L⃗|", "=", Norm[L] // N},
   {"   L⃗", "=", L // N},
   {"  |ω⃗|", "=", Norm[ω] // N},
   {"   ω⃗", "=", ω // N},
   {"  |q|", "=", Abs[Q] // N},
   {"   q", "=", Q // QForm // N},
   {"Moment of inertia", ⋯, ⋯},
   {"  ‖𝒥‖", "=", Det[𝒥] // N}
  }, TableDepth → 2]
```

■ Equations of Motion

▼ rigidBodyEquations: Gets time derivatives of state variables

Equations of motion are depend on the chosen frame of reference and wheter gyroscopic effects are neglected or not. Thus, the moment of inertia $\mathcal{L}$, torque τ, angular momentum $\mathbf{L}$ and orientation time derivative $\dot{q}$ are given as piece-wise functions depending on the flags `frameOfRef` and `gyroEffects`.

```
rigidBodyEquations[{ t_, X_, Q_, P_, L_ }] := Module[
  { Pdot, Ldot, Xdot, Qdot, F, Fext, τ, R, 𝒥inv, ω },

  (* Calculate total force *)
  Fext = m g n; (* Sum up all external forces *)

  (* Calculate linear momentum time derivative *)
  Pdot = Fext + Fint;

  (* Moment of inertia *)
  R = { RotationMatrix[ Q ]   frameOfRef === Inertial ;

  𝒥inv = { R.𝒥0inv.R^T   frameOfRef === Inertial
         { 𝒥0inv         frameOfRef === BodyFixed ;

  (* Derive angular velocity from angular momentum *)
  ω = 𝒥inv.L;

  (* Calculate total torque, depending on frame of reference *)
  τ = { τint                  frameOfRef === Inertial
      { Im[ Q* ** τint ** Q ]  frameOfRef === BodyFixed ;

  (* Calculate angular momentum time derivative *)
          { τ + ω × L   frameOfRef === Inertial ∧ gyroEffects === Ignore
          { τ           frameOfRef === Inertial ∧ gyroEffects === Include
  Ldot = { τ           frameOfRef === BodyFixed ∧ gyroEffects === Ignore ;
          { τ - ω × L   frameOfRef === BodyFixed ∧ gyroEffects === Include

  (* Calculate position time derivative (velocity) *)
  Xdot = m^-1 P;

  (* Calculate orientation time derivative *)
          { 1/2 ω ** Q   frameOfRef === Inertial
  Qdot = {                                      ;
          { 1/2 Q ** ω   frameOfRef === BodyFixed

  (* Return time derivatives of t, X, Q, P and L *)
  { 1, Xdot, Qdot, Pdot, Ldot }
]
```

■ ODE Solver

▼ solverInit: Initializes state variables and computes derived quantities

```
solverInit[] := Module[
  {R, scalef},

  (* Stop any running simulations *)
  runSimulation = False;

  (* Initial time *)
  t = 0;

  (* Initial position and orientation, in inertial frame *)
  X = X0;
  Q = Sign[Q0]; (* Normalize orientation to a versor *)

  (* Moment of inertia, frame dependent *)
  R = {RotationMatrix[Q]   frameOfRef === Inertial ;
  𝒥 = {R.𝒥0.R^T   frameOfRef === Inertial
       𝒥0         frameOfRef === BodyFixed ;
  𝒥inv = {R.𝒥0inv.R^T   frameOfRef === Inertial
          𝒥0inv         frameOfRef === BodyFixed ;

  (* Velocity and linear momentum, in inertial frame *)
  V = V0; (* Velocity *)
  P = m V; (* Linear momentum *)

  (* Initial angular velocity is always in body-fixed frame *)
  ω = {Im[Q ** ω0 ** Q*]   frameOfRef === Inertial
       ω0                  frameOfRef === BodyFixed ;
  (* Angular momentum, frame dependent *)
  L = 𝒥.ω; (* Angular momentum *)

  (* Internal forces and torque *)
  Fint = {0, 0, 0};
  τint = {0, 0, 0};

  (* Derived quantities *)
  Ek = 1/2 P.V + 1/2 L.ω; (* Kinetic energy *)
  Ep = -m 𝑔n.X; (* Potential energy *)
  Etot = Ek + Ep; (* Total energy *)

  (* Keep track of angular velocity *)
  ωTrack = {};

  (* Angular velocity and angular momentum scale factors *)
  scalef = 0.8 Max[Abs[body$v], 0.9ℓ];
  ωScale = Norm[ω] / scalef // N; If[ωScale == 0, ωScale = 1];
  LScale = Norm[L] / scalef // N; If[LScale == 0, LScale = 1];

  (* Update animation data *)
  aData = getAnimationData[];
]
```

▼ solverStep: Solves equations and recalculates derived quantities

Function calls repeatedly chosen ODE integrator, normalizes orientation qaternion to a versor and calculates derived quantities (like energy). (It saves also head of the angular velocity vector in an array to visualize precession and nutation.)

```
solverStep[ count_: 1 ] := Module[
  { R },

  Do[ If[ ¬ runSimulation, Break[] ];

      (* Solve equations using specified integrator *)
      { t, X, Q, P, L } = odeIntegrator[ { t, X, Q, P, L },
            h, rigidBodyEquations ];

      Q = Sign[ Q ];  (* Keep orientation as versor *)

      (* Update moment of inertia, but only if in inertial frame *)
      R = { RotationMatrix[ Q ]   frameOfRef === Inertial ;

      𝒥 = { R.𝒥0.R^T   frameOfRef === Inertial
          { 𝒥0         frameOfRef === BodyFixed ;

      𝒥inv = { R.𝒥0inv.R^T   frameOfRef === Inertial
             { 𝒥0inv         frameOfRef === BodyFixed ;

      (* Calculate derived quantities *)
      V = m^-1 P;    (* Linear velocity from linear momentum *)
      ω = 𝒥inv.L;  (* Angular velocity from angular momentum *)
      Ek = 1/2 P.V + 1/2 L.ω; (* Kinetic energy *)
      Ep = -m g n.X; (* Potential energy *)
      Etot = Ek + Ep; (* Total energy *)

      (* Keep track of angular velocity *)
      AppendTo[ ωTrack,
           If[ frameOfRef === Inertial, ω, ςT[ ω ] ] / ωScale
       ],
   {count}
  ];

  (* Update animation data *)
  aData = getAnimationData [];
 ]
```

▼ solverRun: Runs simulation until stopped

Creates the animation cell (if it does not exist) and evaluates `solverStep` function until `runSimulation` flag is reset to `False`.

```
solverRun[ locateCell_: False ] := Module[
  { nb = EvaluationNotebook[], noAnimationCell },

  (* Locate and evaulate cell containing solverRun[] *)
  If[ locateCell,
   NotebookFind[ nb, "RUNSIMULATION", Next, CellTags, AutoScroll → False ];
   SelectionEvaluateCreateCell[ nb ];
   Return []
  ];

  (* Recreate animation cell, if it does not exist *)
  If[ $Failed === NotebookFind[ nb, "ANIMATION", Next, CellTags, AutoScroll → False ],
       CellPrint[ ExpressionCell[ showAnimation [], CellTags → "ANIMATION" ] ];
       NotebookFind[ nb, "ANIMATION", Next, CellTags, AutoScroll → False ];
       SetOptions[ NotebookSelection[ nb ], CellAutoOverwrite → False ]
  ];

  NotebookFind[ nb, "RUNSIMULATION", Next, CellTags, AutoScroll → False ];
  SelectionMove[ nb, After, CellContents ];

  (* Run simulation, until cancelled *)
  runSimulation = True;
  While[ runSimulation ⋀ t < tf, solverStep [] ];
  runSimulation = False;
 ]
```

■ Initialize Simulation

Default parameters are conviniently modified here before running simulation.

```
ℓ = 3.5; gn = { 0, 0, 0 };  tf = 4;

setupBodyShape[ 10, { 4, 5, 2}, "Cuboid" ];

X0 = { 0, 0, 0 }; (* in inertial frame *)
Q0 = N@ToQ$AngleAxis[ 0.1, { 1, 0, 0.5 } ]; (* in inertial frame *)
V0 = { 0, 0, 0 }; (* in inertial frame *)
ω0 = { -1, -3, 2 }; (* in body-fixed (!) frame *)

solverInit [];
```

Simulation

Frame: Body–fixed , Gyroscopic effects: Include

Integrator: Runge–Kutta 4 , h = 0.01 s

Initialize | Single Step | Run Simulation | Cancel

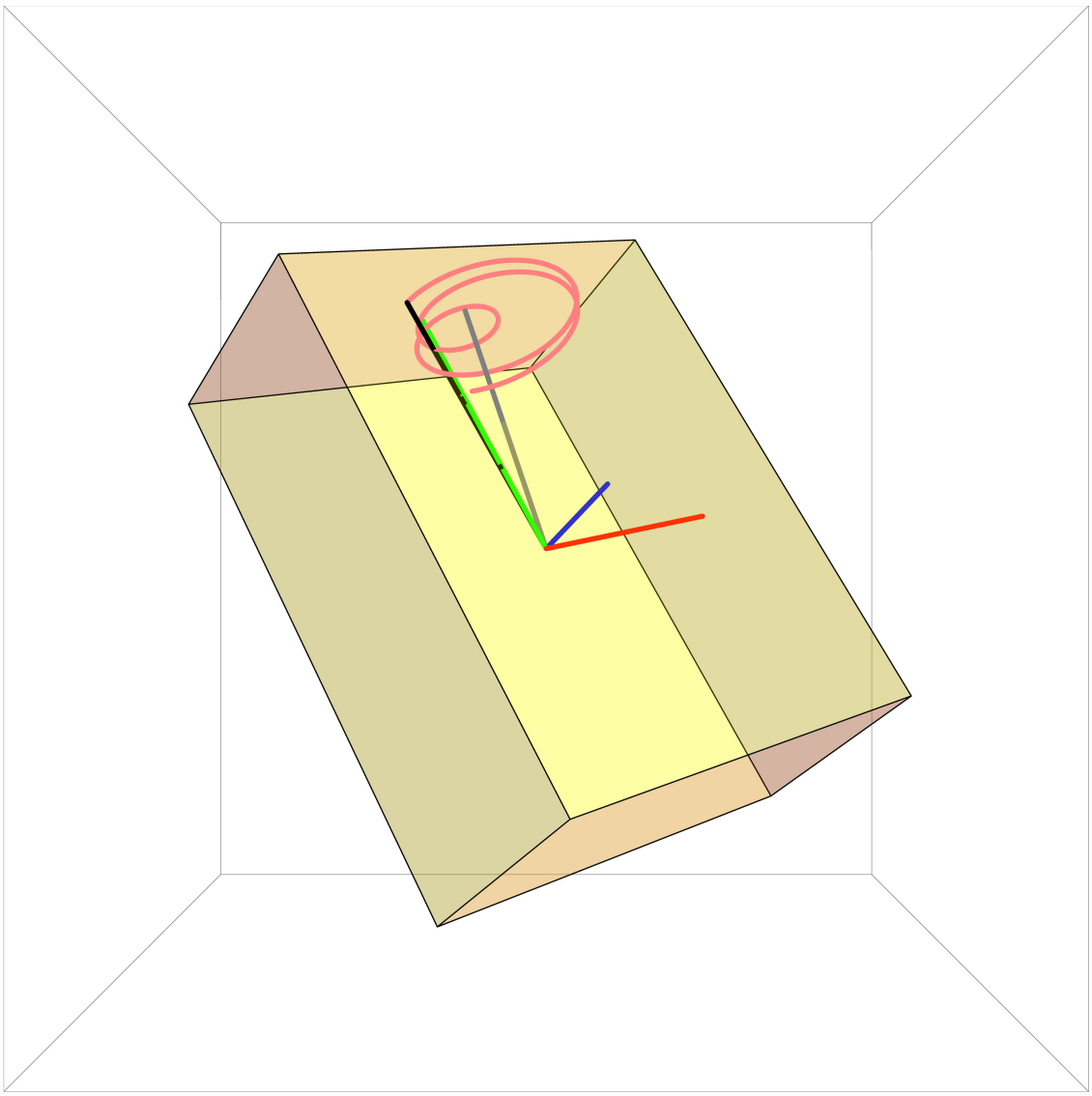
Stopped.

t = 4.080 s, E_{tot} = 155.417 J

$|\vec{\omega}_{world}|$ = { -1.6, -1.6, 2.9} s^{-1}

$|\vec{L}_{world}|$ = { -21.5, -57.7, 62.9} $kg\,m^2\,s^{-1}$

```
solverRun []
```



Appendix B - Quaternions

Package for symbolic calculations with quaternions

```
(*
   Quaternions package implements Hamilton's quaternion algebra.
   This package mimics original Mathematica Quaternions.m, however,
   it fixes various bugs and also allows symbolic transformations
   of quaternions to more extent than Mathematica's original.
   Author:  Mikica B Kocic
   Version: 0.4, 2012-04-22
*)
```

Begin Package

```
$Pre =.
```

```
BeginPackage[ "Quat`" ];
```

```
Unprotect[ Quat`Q ];
```

```
ClearAll[ "Quat`Private`*" ];
ClearAll[ "Quat`*" ];
```

Exported Symbols

```
Q::usage = "Q[w,x,y,z] represents the quaternion with real part w \
(also called scalar part) and imaginary part {x,y,z} (also called vector part)";
```

```
QQ::usage = "QQ[q] gives True if q is a quaternion, \
and False otherwise.";
```

```
ToQ::usage = "ToQ[expr] transforms expr into a quaternion object if at all possible.";
```

```
ScalarQ::usage = "ScalarQ[q] gives True if q is a scalar, and False otherwise.";
```

```
ToList::usage = "ToList[q] gives components of q in a list";
```

```
ToVector::usage = "ToVector[q] gives q as column vector matrix";
```

```
ToMatrix::usage = "ToMatrix[q] gives matrix representation of quaternion q";
```

```
Abs$Im::usage = "Abs$Im[q] gives the absolute value of the vector quaternion part of q.";
```

```
AdjustedSign$Im::usage = "AdjustedSign$Im[q] gives the Sign of the vector part of q, \
adjusted so its first non-zero part is positive.";
```

```
NonCommutativeMultiply::usage = "Implements non-commuatative quaternion multiplication.";
```

```
ToQ$AngleAxis::usage = "ToQ$AngleAxis[θ,{ux,uy,uz}] transforms axis u and angle θ into quat
```

```
ToQ::invargs = "ToQ: failed to construct Q from argument:\n args = `1`"
```

```
RotationMatrix4::usage = "RotationMatrix4[q] gives the 4×4 rotation matrix for a \
counterclockwise 3D rotation around the quaterion q"
```

```
QForm::usage = "QForm[q] prints with the elements of matrix or quaternion \
arranged in a regular array."
```

```
Assert::usage = "QForm[q] prints with the elements of matrix or quaternion \
arranged in a regular array."
```

Private Section

```
Begin[ "Quat`Private`" ];
```

Operators

```
Subscript[ q_ ?QQ, n_Integer /; 1 ≤ n ≤ 4 ] := q[[n]]
```

```
AngleBracket[ q_ ?QQ ] := Re[q]
```

```
OverVector[ q_ ?QQ ] := q[[2;;4]] /. Q → List
```

```
OverHat[ q_ ?QQ ] := q / Abs[ q ]
```

```
SuperStar[ q_ ?QQ ] := Conjugate[ q ]
```

```
BracketingBar[ q_ ?QQ ] := Abs[ q ]
```

Utility Functions

Strip Output Label of Form Info

```
SetAttributes[ TimeIt, HoldAll ];
```

```
TimeIt[ expr_ ] :=
Module[
  { result = expr, out, form },
  If[ TrueQ[ MemberQ[ $OutputForms, Head[result] ] ],
    (* Then *) out = First[result]; form = "//" <> ToString[ Head[result] ],
    (* Else *) out = result; form = ""
  ];
  If[ out =!= Null,
    CellPrint[
      ExpressionCell[ result, "Output",
        CellLabelAutoDelete → True,
        CellLabel → StringJoin[ "Out[", ToString[$Line], "]:="]
      ]
    ];
    Unprotect[ Out ];
    Out[ $Line ] = out;
    Protect[ Out ];
    out (* needed for % *)
  ];
];
```

▼ Assert truth of a logical statement (i.e. prove a theorem)

```
Assert[ statement_, params__: Null ] :=
Module[
  { result },
  (* Try to proove the statement... *)
  result = statement;
  If[ result,
    (* is true *)  Return@ Style[ "⊢ True ∎", Blue ],
    (* is false *) Return@ Style[ "False ☹", Red, Large, Bold ],
    (* is neither true or false *) Null (* continue... *)
  ];
  (* ... now, try even harder to proove the statement *)
  result = Simplify[ statement,params ];
  If[ result,
    (* is true *)  Return@ Style[ "⊢ True (after Simplify) ∎", Blue ],
    (* is false *) Return@ Style[ "False (after Simplify) ☹", Red, Large, Bold ],
    (* is neither true or false *) Null (* continue... *)
  ];
  (* ... now, try even harder to proove the statement *)
  result = FullSimplify[ statement, params ];
  If[ result,
    (* is true *)  Style[ "⊢ True (after FullSimplify) ∎", Blue ],
    (* is false *) Style[ "False (after FullSimplify) ☹", Red, Large, Bold ],
    (* is neither true or false *)
        Style[ "Indeterminate ☹", Darker[Red], Large, Bold ]
  ]
]
```

■ Transformation Rules

▼ Scalar test

```
ScalarQ[ x_ ] := AtomQ[ x ] ⋁ Length[ x ] === 0 ⋁
  ( Length[ x ] =!= 0 ⋀ ¬ListQ[ x ] ⋀ Head[ x ] =!= Q ⋀ Head[ x ] =!= Complex )
```

▼ Quaternion test

```
QQ[ q_ ] :=
  Head[ q ] === Q ⋀ Length[ q ] === 4 ⋀
  And @@ ( ScalarQ /@ q )
```

▼ Transform expressions into Q objects

```
ToQ[ w_ ?ScalarQ, { x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ } ] :=
    Q[ w, x, y, z ]
```

```
ToQ[ q_ ] := q /. {
    Q[ w_, x_, y_, z_ ] ↦ Q[ w, x, y, z ],
    { x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ }  ↦ Q[ 0, x, y, z ],
    { w_ ?ScalarQ, x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ } ↦ Q[ w, x, y, z ],
    { {w_ ?ScalarQ}, {x_ ?ScalarQ}, {y_ ?ScalarQ}, {z_ ?ScalarQ} } ↦ Q[ w, x, y, z ],
    Complex[ x_, y_ ] ↦ Q[ x, y, 0, 0 ],
    Plus[ x_, Times[ Complex[ 0, 1 ], y_ ] ] ↦ Q[ x, y, 0, 0 ],
    Times[ Complex[ 0, 1 ], x_ ] ↦ Q[ 0, x, 0,  0 ],
    Times[ Complex[ 0, x_ ], y_ ] ↦ Q[ 0, x y, 0, 0 ],
    x_ ?ScalarQ ↦ Q[ x, 0, 0, 0],
    (* unknown list *)
    list_ ↦ Module[ {}, Message[ ToQ::invargs, list ]; Abort[] ]
} // QSimplify
```

Axis and angle into Q object

```
ToQ$AngleAxis[ θ_, { ux_, uy_, uz_ } ] := Q[
    Cos[θ/2], ux Sin[θ/2], uy Sin[θ/2], uz Sin[θ/2]
]
```

To list

```
ToList[ q_ ?QQ ] := ( q /. Q→ List )
```

Vector

```
ToVector[ q_ ?QQ ] := Transpose@{ q /. Q→ List }
```

To matrix representation

```
ToMatrix[ Q[ w_, x_, y_, z_ ] ] := {
   {  w,  x,  y,  z },
   { -x,  w, -z,  y },
   { -y,  z,  w, -x },
   { -z, -y,  x,  w }
}
```

To matrix form

```
Q /: MatrixForm[ q: Q[ __ ?ScalarQ ] ] := MatrixForm@ ToList@ q
```

To paritioned matrix form (with divider lines)

```
QForm[ Q_ ?MatrixQ ] := DisplayForm@
  RowBox[{
    "(", "",
    GridBox[ Release[ Q ],
      RowSpacings → 1, ColumnSpacings → 1,
      RowAlignments → Baseline,
      ColumnAlignments → Center,
      GridBoxDividers → {
        "Columns" → { False, GrayLevel[0.84] },
        "Rows" → { False, GrayLevel[0.84] }
      }
   ],
   "", ")"
 }]
```

```
QForm[ q_ ?QQ ] := QForm[ { Release[ ToList@ q ] } ]
```

To rotation matrix, Shoemake's form

```
Q /: RotationMatrix[ Q[ w_, x_, y_, z_ ] ] :=
{
   {  1 - 2( y^2 + z^2 ),    2 x y - 2 w z,      2 x z + 2 w y  },
   {  2 x y + 2 w z,      1 - 2( x^2 + z^2 ),  2 y z - 2 w x  },
   {  2 x z - 2 w y ,     2 w x+2 y z,         1 - 2( x^2 + y^2 ) }
}
```

```
Q /: RotationMatrix4[ Q[ w_, x_, y_, z_ ] ] :=
{
   {  1,  0,                  0,                  0                 },
   {  0,  1 - 2( y^2 + z^2 ),  2 x y - 2 w z,     2 x z + 2 w y     },
   {  0,  2 x y + 2 w z,      1 - 2( x^2 + z^2 ),  2 y z - 2 w x     },
   {  0,  2 x z - 2 w y ,     2 w x + 2 y z,      1 - 2( x^2 + y^2 ) }
}
```

▼ Conjugate

```
Q /: Conjugate[ Q[ w_, x_, y_, z_ ] ]:=
    Q[ w, -x, -y, -z ]
```

▼ Squared Norm

```
Q /: SqNorm[ q:  Q[ __ ?ScalarQ ] ] :=
    Plus @@ ( ( List @@ q )^2 )
```

▼ Norm

```
Q /: Norm[ q: Q[ __ ?ScalarQ ] ]:=
    Sqrt[ SqNorm[ q ] ]
```

▼ Abs

```
Q /: Abs[ q:  Q[ __ ?ScalarQ ] ] :=
    Sqrt[ SqNorm[ q ] ]
```

▼ Round

```
Q /: Round[ q: Q[ __ ?ScalarQ ] ]:=
Module[
  {
    cent = Round /@ q;
    mid = ( Floor /@ q ) + Q[ 1/2, 1/2, 1/2, 1/2 ];
  },
  If[ SqNorm[ q - cent ] <= SqNorm[ q - mid ], cent, mid ]
]
```

▼ Real Part

```
Q /: Re[ q: Q[ __ ?ScalarQ ] ] :=
    q[[1]]
```

▼ Sign (returns Versor)

```
Q /: Sign[ q: Q[ __ ?ScalarQ ] ] :=
    Q[ 1, 0, 0, 0 ] /; Abs[ q ] == 0
```

```
Q /: Sign[ q: Q[ __ ?ScalarQ ] ] :=
    q / Abs[ q ]
```

▼ Imaginary Part

```
Q /: Im[ q: Q[ __ ?ScalarQ ] ] :=
    { q[[2]], q[[3]], q[[4]] }
```

```
AdjustedSign$Im[ q_Complex | q_ ?ScalarQ ] := ⅈ
```

```
AdjustedSign$Im[ Q[ w_ ?ScalarQ, x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ ] ] :=
Which[
    x != 0,
        Sign[ x ] * Sign[ Q[ 0, x, y, z ] ],
    y != 0,
        Sign[ y ] * Sign[ Q[ 0, x, y, z ] ],
    z != 0,
        Sign[ z ] * Sign[ Q[ 0, x, y, z ] ],
    True,
        i (* ?abort? *)
]
```

```
Abs$Im[ Q[ w_ ?ScalarQ, x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ ] ]:=
    Sqrt[x^2 + y^2 + z^2]
```

```
Abs$Im[ x_ ? NumericQ ] :=
    Im[ x ]
```

```
Sign$Im[ Q[ w_ ?ScalarQ, x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ ] ] :=
    Sign[ Q[ 0, x, y, z ] ]
```

▼ Addition

```
Q /: Q[ w1_, x1_, y1_, z1_ ] + Q[ w2_, x2_, y2_, z2_ ] :=
    Q[ w1 + w2, x1 + x2, y1 + y2, z1 + z2 ] // QSimplify
```

```
Q /: Complex[ re_, im_ ] + Q[ w_, x_, y_, z_ ] :=
    Q[ w + re, x + im, y, z ] // QSimplify
```

```
Q /: λ_ ?ScalarQ + Q[ w_, x_, y_, z_ ] :=
    Q[ w + λ, x, y, z ]
```

▼ Multiplication

```
Q /: λ_ ?ScalarQ * Q[ w_, x_, y_, z_ ]:=
    Q[ λ w, λ x, λ y, λ z ]
```

▼ Non-commutative multiplication

```
Q /: Q[ w1_, x1_, y1_, z1_ ] ** Q[ w2_, x2_, y2_, z2_ ]:=
    Q[
        w1 w2 - x1 x2 - y1 y2 - z1 z2,
        w1 x2 + x1 w2 + y1 z2 - z1 y2,
        w1 y2 - x1 z2 + y1 w2 + z1 x2,
        w1 z2 + x1 y2 - y1 x2 + z1 w2
    ] // QSimplify
```

▼ Non-commutative multiplication with complex numbers

```
Unprotect[ NonCommutativeMultiply ]
(*SetAttributes[ NonCommutativeMultiply, Listable ]*)
```

```
a_ ?ScalarQ ** b_ ?QQ := a * b
```

```
a_ ?QQ ** b_ ?ScalarQ := a * b
```

```
{ x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ } ** q_ ?QQ :=
    Q[ 0, x, y, z ] ** q
```

```
q_ ?QQ ** { x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ } :=
    q ** Q[ 0, x, y, z ]
```

```
{ w_ ?ScalarQ, x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ } ** q_ ?QQ :=
    Q[ w, x, y, z ] ** q
```

```
q_ ?QQ ** { w_ ?ScalarQ, x_ ?ScalarQ, y_ ?ScalarQ, z_ ?ScalarQ } :=
    q ** Q[ w, x, y, z ]
```

```
{ {w_ ?ScalarQ}, {x_ ?ScalarQ}, {y_ ?ScalarQ}, {z_ ?ScalarQ} } ** q_ ?QQ :=
    Q[ w, x, y, z ] ** q
```

```
q_ ?QQ ** { {w_ ?ScalarQ}, {x_ ?ScalarQ}, {y_ ?ScalarQ}, {z_ ?ScalarQ} } :=
    q ** Q[ w, x, y, z ]
```

```
( ( a_: 1 ) * Complex[ b_, c_ ] ) ** ( ( x_: 1 ) * Complex[ y_, z_ ] ) :=
    a b x y - a c x z + ( a c x y + a b x z ) i
```

```
( x_ + y_ ) ** a_ := ( x ** a ) + ( y ** a )
a_ ** (x_ + y_) := ( a ** x ) + ( a ** y )
```

```
Protect[ NonCommutativeMultiply ]
```

▼ Math Functions

```
extfunc[ func_, a_, b_ ] :=
    Re[b] + Abs$Im[b] * AdjustedSign$Im[a]
```

```
Block[
    { extend, $Output = {} },

    extend[ foo_ ]:= (
        Unprotect[ foo ];
        Q /: foo[ a: Q[ __ ?ScalarQ ] ] :=
           extfunc[ foo, a, foo[ Re[a] + Abs$Im[a] * i ] ];
        Protect[ foo ];
    );

    extend /@ {
        Log,
        Cos, Sin, Tan, Sec, Csc, Cot,
        ArcCos, ArcSin, ArcTan, ArcSec, ArcCsc, ArcCot,
        Cosh, Sinh, Tanh, Sech, Csch, Coth,
        ArcCosh, ArcSinh, ArcTanh, ArcSech, ArcCsch, ArcCoth
    };
]
```

▼ Exp e^q

```
Q /: Exp[ q: Q[ __ ?ScalarQ ] ] :=
    Exp[ Re[ q] ] *
    ( Cos[ Abs$Im[ q ] ] + Sin[ Abs$Im[ q ] ] * Sign$Im[ q] ) // QSimplify
```

▼ Power

```
Q /: Power[ q: Q[ __ ?ScalarQ ], 0 ] :=
    1
```

```
Q /: Power[ q: Q[ __ ?ScalarQ ], 1 ] :=
    q
```

```
Q /: Power[ q: Q[ __ ?ScalarQ ], -1 ] :=
    Conjugate[q] ** ( 1 / SqNorm[q] )
```

```
Q /: Power[ q: Q[ __ ?ScalarQ ], n_ ] :=
    q ** Power[ q, n - 1 ] /; n > 1 ⋀ n ∈ Integers
```

```
Q /: Power[ q: Q[ __ ?ScalarQ ], n_ ] :=
    Power[ 1/q, -n ] /; n < 0 ⋀ n != -1
```

```
Q /: Power[ q: Q[ __ ?ScalarQ ], n_ ] :=
Module[
  {
    μ = Abs[ q ], (* modulus of {q1,q2,q3,q4} *)
    re = Re[ q ], (* scalar part re = {q1} *)
    im = Abs$Im[ q ], (* modulus of vector part im = {q2,q3,q4} *)
    θ, (* Angle between vector im and scalar re *)
    ϕ = 0, (* Angle between q2 and vector part {q2,q3,q4} *)
    γ = 0 (* Angle between q3 and subvector {q3,q4} *)
  },

  θ = If[ re =!= 0,
        (* Then *) ArcTan[ im / re ],
        (* Else *) π/2
      ];

  If[ im =!= 0,
  (* Then *)
     ϕ = ArcCos[ q[[2]] / im ];
     γ = If[ Sin[ϕ] =!= 0,
         (* Then *)
            ArcCos[ q[[3]] / ( im Sin[ϕ] ) ] Sign[ q[[4]] ],
            ( π/2 ) Sign[ q[[4]] ]
         ]
  ];

  Q[
     μ^n Cos[ n θ ] ,
     μ^n Sin[ n θ ] Cos[ϕ] ,
     μ^n Sin[ n θ ] Sin[ϕ] Cos[γ],
     μ^n Sin[ n θ ] Sin[ϕ] Sin[γ]
  ]  // QSimplify

] /; ScalarQ[n] ⋀ n > 0
```

▼ Power e^q

```
Q /: Power[ E, q: Q[ __ ?ScalarQ ] ] :=
    Exp[ q ]
```

▼ Sqrt

```
Q /: Sqrt[ q: Q[ __ ?ScalarQ ] ] :=
    Power[ q, 1/2 ]
```

▼ Right Divide

```
Q /: Divide[ left: Q[ __ ?ScalarQ ], right : Q[ __ ?ScalarQ ] ] :=
    left ** ( Conjugate[right] ** 1/SqNorm[right] )
```

▼ Simplification

```
Q /: QSimplify[ q: Q[ __ ?ScalarQ ] ] :=
  Simplify[ TrigExpand /@ q ]
```

```
QSimplify[ q_ ] := q
```

End Package

```
$Pre = TimeIt;
```

```
End[];
```

```
EndPackage[];
```

Appendix C - Rotation of Moment of Inertia Tensor using Quaternions

Mikica Kocic, miko0008@student.umu.se
The Physics of Virtual Environments, 2012-04-22

There is a wonderful connection between complex numbers (and quaternions as their extension) and geometry in which, translations correspond to additions, rotations and scaling to multiplications and reflections to conjugations.

However, the beauty of using quaternions for spatial rotations is shadowed by an expression like

$$\begin{pmatrix} 1-2\left(y^2+z^2\right) & 2xy-2wz & 2wy+2xz \\ 2xy+2wz & 1-2\left(x^2+z^2\right) & 2yz-2wx \\ 2xz-2wy & 2wx+2yz & 1-2\left(x^2+y^2\right) \end{pmatrix}\cdot$$

$$\mathcal{J}\cdot\begin{pmatrix} 1-2\left(y^2+z^2\right) & 2xy-2wz & 2wy+2xz \\ 2xy+2wz & 1-2\left(x^2+z^2\right) & 2yz-2wx \\ 2xz-2wy & 2wx+2yz & 1-2\left(x^2+y^2\right) \end{pmatrix}^{\mathrm{T}}$$

which actually represents such simple transformation as rotation of moment of inertia tensor using quaternions [2].

The purpose of this document is to express spatial rotations of a moment of inertia tensor in pure quaternionic form and to highlight an algebraical meaning behind such rotations (read: to expell the evil of linear algebra from rotation equations :)

■ Conventions and Definitions

Load quaternions package (see `Quat.nb` for annotated source code)

```
Get[ "Quat.m", Path -> { NotebookDirectory [] } ];
```

▼ Conventions

In this document, matrices, vectors and matrix representation of quaternions are shown in uppercase font (roman-bold font in text) and quaternions and other numbers are shown in lowercase font (italic font in text). Since the symbol `I` in *Mathematica* is used for imaginary unit, the selected symbol for moment of inertia is uppercase script J i.e. $\mathcal{J}$. The identity matrix is suffixed with number of dimensions, i.e. `I4` represents identity matrix in $\mathbb{R}^4$.

Matrices 4×4 originated from quaternions may be geometrically partitioned into scalar (real), vector (pure imaginary) and cross-product parts [3]. Such partitions are visualized using row and column divider lines throughout this document:

$$\left(\begin{array}{c|ccc} \textit{scalar} & \square & \textit{vector} & \square \\ \hline \square & \square & \square & \square \\ \textit{vector}^{\mathrm{T}} & \square & \textit{cross product} & \square \\ \square & \square & \square & \square \end{array}\right)$$

▼ General symbols and variables used in theorems

Hamilton's quaternions q, q_1 and q_2 with real number components (but not with complex number components as biquaternions; see definition of default `$Assumptions` bellow)

```
q = Q[ w, x, y, z ];
```

```
q1 = Q[ w1, x1, y1, z1 ];
```

```
q2 = Q[ w2, x2, y2, z2 ];
```

Identity matrix `I4` in $\mathbb{R}^4$

```
I4 = IdentityMatrix[4];
```

Common $\mathbb{R}^4$ matrices **A** and **B**

```
A = ( Aww | Awx  Awy  Awz
      Axw | Axx  Axy  Axz
      Ayw | Ayx  Ayy  Ayz
      Azw | Azx  Azy  Azz );
```

```
B = ( Bww | Bwx  Bwy  Bwz
      Bxw | Bxx  Bxy  Bxz
      Byw | Byx  Byy  Byz
      Bzw | Bzx  Bzy  Bzz );
```

Symmetrical $\mathbb{R}^4$ matrix representing moment of inertia tensor $\mathcal{J}$

```
𝒥 = ( 1 |  0    0    0
      0 | Ixx  Ixy  Ixz
      0 | Ixy  Iyy  Iyz
      0 | Ixz  Iyz  Izz );
```

Default assumption used for all asserted statements and expressions in this notebook is that previously defined entities q, q_1, q_2, **A**, **B** and $\mathcal{J}$ are on field of $\mathbb{R}$, i.e. that their components are elements in $\mathbb{R}$

```
$Assumptions = Flatten@{ ToList /@ { q, q1, q2 }, A, B, 𝒥} ∈ Reals;
```

■ Multiplications

The ordirary notation for multiplication in *Mathematica* is

A . **B**	multiplication between matrices ("dot" multiplication)
a × **b**	cross-product between 3D vectors
p ** q	quaternion multiplication (both in `Quat.m` and *Mathematica* Quaternions package)

Beside the ordinary notation for multiplication, this document also introduces the following notation based on the center dot (·) operator.

A · **B**	:= multiplication between matrices
$p \cdot q$	:= quaternion multiplication
$q \cdot$ **A**	:= column-wise quaternion-matrix multiplication

Note that the center dot operator in *Mathematica* is without built-in meaning.

▼ Matrix multiplication

Define center dot operator **A** · **B** as matrix multiplication (otherwise, *Mathematica* uses plain dot "." for matrix multiplications).

```
CenterDot[ m__ ?MatrixQ ] := Dot[ m ]
```

▼ Quaternion multiplication

Mathematica uses non-commutative multiply (**) operator as symbol for quaternion multiplication. The `Quat.m` package and this document follow this convention. Here we define also the center dot operator as symbol for quaternion multiplication, just for convinience and esthetics

```
CenterDot[ q__ ?QQ ] := NonCommutativeMultiply[ q ]
```

▼ Column-wise quaternion-matrix multiplication

Let us now define multiplication between quaternion q and matrix $\mathbf{M}$, where it is assumed that matrix is a row vector of quaternions in columns, i.e. $\mathbf{M} = (\, q_1 \quad q_2 \quad \ldots \quad q_n \,)$, so that quaternion multiplication is done between q and q_j in every column j.

$$q \cdot \left(\begin{array}{c|ccc} q_{11} & q_{12} & & q_{1n} \\ \hline q_{21} & q_{22} & & q_{2n} \\ q_{31} & q_{32} & \cdots & q_{3n} \\ q_{41} & q_{42} & & q_{4n} \end{array}\right) := \overset{\begin{array}{ccccc} q\,q_1 & & q\,q_j & & q\,q_n \\ \downarrow & & \downarrow & & \downarrow \end{array}}{\left(\begin{array}{c|ccc} q_{11} & q_{12} & & q_{1n} \\ \hline q_{21} & q_{22} & & q_{2n} \\ q_{31} & q_{32} & \cdots & q_{3n} \\ q_{41} & q_{42} & & q_{4n} \end{array}\right)}$$

Since the quaternion multiplication is not commutative, there are several cases to distinguish:

a) Left multiplication $q \cdot \mathbf{M}$, where quaternion in each column in $\mathbf{M}$ is multiplied by quaternion q from the left:

```
CenterDot[ q_?QQ, mat_?MatrixQ ] := Transpose@
  Table[
   ToList[ q · ToQ@mat⟦All,j⟧ ], {j, 1, Dimensions[mat]⟦2⟧}
  ]
```

b) Right multiplication $\mathbf{M} \cdot q$, where quaternion in each column in $\mathbf{M}$ is multiplied by quaternion q from the right:

```
CenterDot[ mat_?MatrixQ, q_?QQ ] := Transpose@
  Table[
   ToList[ ToQ@mat⟦All,j⟧ · q ], {j, 1, Dimensions[mat]⟦2⟧}
  ]
```

c) Compound multiplication from both sides $p \cdot \mathbf{M} \cdot q$, where multiplication of each column in M by quaternion p from the left and q from the right:

```
CenterDot[ q1_?QQ, mat_?MatrixQ, q2_?QQ ] := Transpose@
  Table[
   ToList[ q1 · ToQ@mat⟦All,j⟧ · q2 ], {j, 1, Dimensions[mat]⟦2⟧}
  ]
```

The multiplication between quaternion and matrix is associative, i.e. it holds

$$q_1 \cdot \mathbf{M} \cdot q_2 == q_1 \cdot (\mathbf{M} \cdot q_2) == (q_1 \cdot \mathbf{M}) \cdot q_2$$

which is proved as theorem in the following sections.

Note that for a single column matrix $\mathbf{M}$, the multiplication between quaternion and matrix falls back to an ordinary quaternion multiplication.

▼ Row-wise quaternion-matrix multiplication

Row-wise quaternion-matrix multiplication can be defined transposing matrices and using column-wise multiplication

$$\left(q_1 \cdot \mathbf{M}^{\mathsf{T}} \cdot q_2\right)^{\mathsf{T}}$$

▼ Conj[] matrix operator

Conj[] operator conjugates only the vector part of the matrix, but not the scalar and the cross product parts:

```
Conj[ q_?MatrixQ ] :=
```

$$\left(\begin{array}{c|ccc} q_{[\![1,1]\!]} & -q_{[\![1,2]\!]} & -q_{[\![1,3]\!]} & -q_{[\![1,4]\!]} \\ \hline -q_{[\![2,1]\!]} & q_{[\![2,2]\!]} & q_{[\![2,3]\!]} & q_{[\![2,4]\!]} \\ -q_{[\![3,1]\!]} & q_{[\![3,2]\!]} & q_{[\![3,3]\!]} & q_{[\![3,4]\!]} \\ -q_{[\![4,1]\!]} & q_{[\![4,2]\!]} & q_{[\![4,3]\!]} & q_{[\![4,4]\!]} \end{array}\right)$$

Note that Conj[] is **not** a conjugate of a quaternion! The conjugate of a quaternion corresponds to the plain transpose of the matrix.

TODO: compare Conj[] to extracting vector part extraction operations.

Background

Basic idea behind quaternion-matrix multiplication comes from an expression

```
Assert[
 Q[1, 0, 0, 0] == q · Q[1, 0, 0, 0] · q*,
 Assumptions → |q|^2 == 1
]
```

```
⊢ True (after Simplify) ∎
```

for which it is assumed to be also valid when transforming identity matrix I4 that is equvalent representation of quaternion *Q*[1, 0, 0, 0] in matrix form, i.e. that there should be kind of multiplication where

```
Assert[
 (1  0  0  0)^T == q · (1  0  0  0)^T · q*,
 Assumptions → |q|^2 == 1
]
```

```
⊢ True (after Simplify) ∎
```

Theorems

Quaternion-matrix multiplication

Quaternion multiplication associativity (sanity-check)

```
q1 · q · q2 == q1 · (q · q2) == (q1 · q) · q2 // Assert
```

```
⊢ True ∎
```

Quaternion-matrix multiplication associativity

```
q1 · A · q2 == q1 · (A · q2) == (q1 · A) · q2 // Assert
```

```
⊢ True ∎
```

```
q · A · q* == (q · A) · q* // Assert
```

```
⊢ True ∎
```

```
q · A · q* == q · (A · q*) // Assert
```

```
⊢ True ∎
```

```
(q · I4)^T == q* · I4 // Assert
```

```
⊢ True ∎
```

```
(I4 · q)^T == I4 · q* // Assert
```

```
⊢ True ∎
```

```
(q · I4) · A == q · A // Assert
```

```
⊢ True ∎
```

```
(I4 · q) · A == A · q // Assert
```

```
⊢ True ∎
```

```
(q1 · I4) · (I4 · q2) == q1 · I4 · q2 // Assert
```

⊢ True ∎

```
(I4 · q2) · ( (q1 · I4) · A ) == (q1 · A) · q2 == q1 · A · q2 // Assert
```

⊢ True (after Simplify) ∎

```
(I4 · q2) · (q1 · I4) · A == q1 · A · q2 ⋀
  (I4 · q2) · (q1 · I4) · A == q1 · A · q2 ⋀
  (q1 · I4) · (I4 · q2) · A == q1 · A · q2 ⋀
  (q1 · I4 · q2) · A == q1 · A · q2  // Assert
```

⊢ True (after Simplify) ∎

Conjugations of quaternion-matrix multiplications

```
Conj[ q · A ] == Conj[A] · q*  // Assert
```

⊢ True ∎

```
Conj[ A · q ] == q* · Conj[A] // Assert
```

⊢ True ∎

```
Conj[ q1 · A · q2] == Conj[ q1 · (A · q2)] == Conj[A · q2] · q1* == q2* · Conj[A] · q1* // Assert
```

⊢ True (after Simplify) ∎

```
Conj[ q · A · q*] == q · Conj[A] · q* // Assert
```

⊢ True (after Simplify) ∎

▼ Left and right rotation matrices

Because quaternion multiplication is bilinear, it can be expressed in matrix form, and in two different ways ([1]). Multiplication on the left qp gives $\mathbf{L}_q \cdot \mathbf{p}$ where p is now treated as a 4-dimensional column vector $\mathbf{p}$ and multiplication on the right pq gives $\mathbf{p} \cdot \mathbf{R}_q$.

Let us define $\mathbf{L}_q$ and $\mathbf{R}_q$ as quaternion-multiplication with identity matrix.

```
ClearAll[ L, R, Q ]
```

```
L_q_ := q · I4
R_q_ := I4 · q
Q_q_ := L_q · R_q*
```

Proof that quaternion multiplication can be decomposed into L&R matrix parts and that $\mathbf{L}_q \, \mathbf{R}_{q^*} \, \mathbf{p}$ corresponds to $q\,p\,q^*$

```
Assert[
 L_q · R_q* · ToVector[q1] == ToVector[ q · q1 · q* ],
 Assumptions → |q|^2 == 1
]
```

⊢ True (after Simplify) ∎

▼ Properties of L, R and Q matrices

```
L_q* == (L_q)^T  // Assert
```

⊢ True ∎

```
R_q* == (R_q)^T  // Assert
```

⊢ True ∎

```
(q · Rq*) · A == q · (Rq* · A) == (Lq · A) · q* // Assert
```

⊢ True (after Simplify) ∎

```
Qq == Lq · Rq* == (Rq*ᵀ · Lqᵀ)ᵀ == (Rq · Lq*)ᵀ == Rq* · Lq // Assert
```

⊢ True ∎

```
Qq* == Rq · Lq* == Lq* · Rq // Assert
```

⊢ True ∎

```
Qq · Qq* == |q|^4 I4 // Assert
```

⊢ True (after Simplify) ∎

```
Conj[Lq · A] · q* == Conj[q · (Lq · A)] // Assert
```

⊢ True (after Simplify) ∎

```
Conj[Lq · A] · q* == Conj[Lq · A] · q* // Assert
```

⊢ True ∎

▼ L & R matrices components

```
Grid@{
  {"Lq", "Lq*", "Rq", "Rq*"},
  {Lq // QForm, Lq* // QForm, Rq // QForm, Rq* // QForm}
 }
```

L_q

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{pmatrix}$$

L_{q^*}

$$\begin{pmatrix} w & x & y & z \\ -x & w & z & -y \\ -y & -z & w & x \\ -z & y & -x & w \end{pmatrix}$$

R_q

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & z & -y \\ y & -z & w & x \\ z & y & -x & w \end{pmatrix}$$

R_{q^*}

$$\begin{pmatrix} w & x & y & z \\ -x & w & -z & y \\ -y & z & w & -x \\ -z & -y & x & w \end{pmatrix}$$

▼ Rotation matrix Q components

```
Grid@{
  {"Qq == Lq·Rq* == Rq*·Lq", "Qq* == Lq*·Rq == Rq·Lq*"},
  {Simplify[Qq, Assumptions → |q|^2 == 1] // QForm,
   Simplify[Qq*, Assumptions → |q|^2 == 1] // QForm}
 }
```

$Q_q == L_q \cdot R_{q^*} == R_{q^*} \cdot L_q$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2y^2-2z^2 & 2xy-2wz & 2(wy+xz) \\ 0 & 2(xy+wz) & 1-2x^2-2z^2 & -2wx+2yz \\ 0 & -2wy+2xz & 2(wx+yz) & 1-2x^2-2y^2 \end{pmatrix}$$

$Q_{q^*} == L_{q^*} \cdot R_q == R_q \cdot L_{q^*}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2y^2-2z^2 & 2(xy+wz) & -2wy+2xz \\ 0 & 2xy-2wz & 1-2x^2-2z^2 & 2(wx+yz) \\ 0 & 2(wy+xz) & -2wx+2yz & 1-2x^2-2y^2 \end{pmatrix}$$

▼ Shoemake's rotation matrix from quaternion

Classical Shoemake's form of the rotation matrix from a quaternion is given as (see [2])

```
RotationMatrix4[q] // QForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2(y^2+z^2) & 2xy-2wz & 2wy+2xz \\ 0 & 2xy+2wz & 1-2(x^2+z^2) & -2wx+2yz \\ 0 & -2wy+2xz & 2wx+2yz & 1-2(x^2+y^2) \end{pmatrix}$$

Scalar, vector and cross-product parts of L&R matrices

Scalar part

$$\frac{1}{2}\left(L_q + L_{q^*}\right) == \frac{1}{2}\left(R_q + R_{q^*}\right) == \left(\begin{array}{c|ccc} w & 0 & 0 & 0 \\ \hline 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{array}\right) \text{// Assert}$$

⊢ True ∎

Vector part

$$\frac{1}{2}\left(L_{q^*} - R_q\right) == \frac{1}{2}\left(R_{q^*} - L_q\right) == \left(\begin{array}{c|ccc} 0 & x & y & z \\ \hline -x & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ -z & 0 & 0 & 0 \end{array}\right) \text{// Assert}$$

⊢ True ∎

$$\frac{1}{2}\left(L_q - R_{q^*}\right) == \frac{1}{2}\left(R_q - L_{q^*}\right) == \left(\begin{array}{c|ccc} 0 & x & y & z \\ \hline -x & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ -z & 0 & 0 & 0 \end{array}\right)^{\mathsf{T}} \text{// Assert}$$

⊢ True ∎

Cross product part

$$\frac{1}{2}\left(L_q - R_q\right) == \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -z & y \\ 0 & z & 0 & -x \\ 0 & -y & x & 0 \end{array}\right) \text{// Assert}$$

⊢ True ∎

$$\frac{1}{2}\left(L_{q^*} - R_{q^*}\right) == \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -z & y \\ 0 & z & 0 & -x \\ 0 & -y & x & 0 \end{array}\right)^{\mathsf{T}} \text{// Assert}$$

⊢ True ∎

$\mathbf{Q}\cdot\mathbf{Q} + 2\,\mathbf{Q}\cdot$(vector part) form.
Note that vector part is pure imaginary part of a quaternion.

```
ToQ@0 == q^2 + 2 q · ToQ@Im[q*] - |q|^2 // Assert
```

⊢ True ∎

$$\text{Assert}\Big[\; Q_q == L_q \cdot L_q + 2\,L_q \cdot \left(\begin{array}{c|ccc} 0 & x & y & z \\ \hline -x & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ -z & 0 & 0 & 0 \end{array}\right)$$

$$== L_q \cdot L_q + 2\,L_q \cdot \left(\frac{1}{2}\left(R_{q^*} - L_q\right)\right)$$

$$== L_q \cdot R_{q^*}$$

$$\Big]$$

⊢ True (after Simplify) ∎

$$\frac{1}{2}\,L_{q^{-1}} \cdot \left(L_q \cdot L_q - Q_q\right) == \frac{1}{2}\left(L_q - R_{q^*}\right) \text{// Assert}$$

⊢ True (after Simplify) ∎

Equality of rotation matrix from quaternion and Euler-Rodrigues' (tensor) formula

Euler-Rodrigues' formula for the rotation matrix corresponding to a rotation by an angle θ about a fixed axis specified by the unit vector **u**.

Reference: http://mathworld.wolfram.com/RodriguesRotationFormula.html

```
EulerRodrigues[θ_, {x_, y_, z_}] :=
 Cos[θ] {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}} + (1 - Cos[θ]) {{x x, y x, z x}, {x y, y y, z y}, {x z, y z, z z}} + Sin[θ] {{0, -z, y}, {z, 0, -x}, {-y, x, 0}}
```

Prove that rotation matrix from quaternion (as defined in Quat.m package) is equivalent to Euler-Rodrigues' formula.

```
Block[{x, y, z, θ, q},
 q = ToQ$AngleAxis[θ, {x, y, z}];
 Assert[
  RotationMatrix[q] == EulerRodrigues[θ, {x, y, z}],
  Assumptions →
   -π ≤ θ ≤ π ⋀ x^2 + y^2 + z^2 == 1 ⋀ {x, y, z, θ} ∈ Reals
 ]
]
```

⊢ True (after Simplify) ∎

▼ Rotation matrix theorems

```
RM = RotationMatrix4[q];
RM // QForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2\left(y^2+z^2\right) & 2xy-2wz & 2wy+2xz \\ 0 & 2xy+2wz & 1-2\left(x^2+z^2\right) & -2wx+2yz \\ 0 & -2wy+2xz & 2wx+2yz & 1-2\left(x^2+y^2\right) \end{pmatrix}$$

```
Assert[
 RM == Qq == Lq · Rq* ⋀
  RM^T == Qq* == Rq*^T · Lq^T,
 Assumptions → |q|^2 == 1
]
```

⊢ True (after Simplify) ∎

```
Assert[
 RM · ToVector[q1]
       == Lq · Rq* · ToVector[q1]
       == ToVector[q · q1 · q*],
 Assumptions → |q|^2 == 1
]
```

⊢ True (after Simplify) ∎

```
Assert[
 RM · A · RM^T
       == Q_q · A · Q_q*
       == L_q · R_q* · A · L_q* · R_q
       == ( q · ( q · A · q* )^T · q* )^T,
 Assumptions → |q|^2 == 1
]
```

⊢ True (after Simplify) ∎

```
Assert[
 I4
       == (q · I4 · q*) · I4 · (q · I4 · q*)^T
       == (L_q · R_q*) · I4 · (L_q · R_q*)^T
       == L_q · R_q* · I4 · R_q*^T · L_q^T
       == L_q · (R_q* · I4 · R_q*^T) · L_q^T,
 Assumptions → |q|^2 == 1
]
```

⊢ True (after Simplify) ∎

```
Assert[
 RM · A · RM^T
       == (q · I4 · q*) · A · (q · I4 · q*)^T
       == (L_q · R_q*) · A · (L_q · R_q*)^T
       == L_q · R_q* · A · R_q*^T · L_q^T
       == L_q · (R_q* · A · R_q*^T) · L_q^T,
 Assumptions → |q|^2 == 1
]
```

⊢ True (after Simplify) ∎

Proof that both $\mathbf{Q}_q$ and $\mathbf{R}_M$ are orthogonal matrices.

```
Assert[
 Q_q · Q_q^T
       == (L_q · R_q*) · (L_q · R_q*)^T
       == L_q · R_q* · (R_q*)^T · (L_q)^T
       == L_q · R_q* · R_q · L_q*
       == I4,
 Assumptions → |q|^2 == 1
]
```

⊢ True (after Simplify) ∎

```
Assert[
 Q_q · Q_q^T
       == Q_q · Q_q*
       == RM · RM^T
       == I4,
 Assumptions → |q|^2 == 1
]
```

⊢ True (after Simplify) ∎

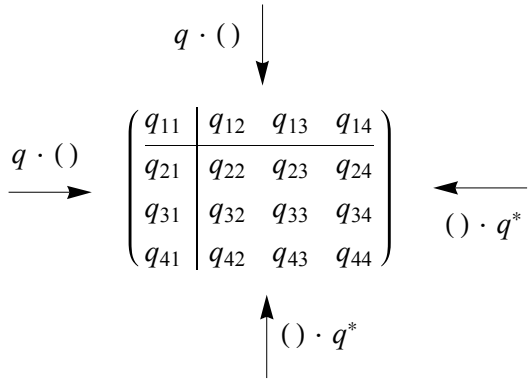
▼ Quaternionic meaning of R(*q*)·A·R(*q*)^T

The hint how quaternion multplication affects (moment of inertia or any other) tensor **A** can be seen from the earlier proven theorem

```
Assert[
 RM · A · RM^T == ( q · ( q · A · q* )^T · q* )^T,
 Assumptions → |q|^2 == 1
]
```

⊢ True (after Simplify) ∎

which shows that tensor **A** is rotated, at first, by rotating quaternions across each column $\mathbf{A}_c = q \cdot \mathbf{A} \cdot q^*$, and at second, by rotating quaternions across each row $\mathbf{A}_{rc} = \left(q \cdot (\mathbf{A}_c)^T \cdot q^*\right)^T$.



The meaning of the compound multiplications across columns and rows is extracting and affecting individual components of quaternions embedded from the previous rotations of the tensor. The moment of inertia tensor can be always reduced to its principal axes in diagonal form. In matrix representation of a quaternion, the diagonal of the matrix is constant and contains the scalar (real) part of the quaternion. (For any quaternion q there exist quaternion p such that qp is scalar.) The principal moment of inertia, on the other hand, has on its diagonal three different scalars in general case, which gives origin to three different quaternions that are embedded and shuffled across-columns/rows during rotations.

■ References

[1] Eberly, David H. with a contrib. by Shoemake, Ken - *Game Physics*, Morgan Kaufmann, 2004, Chapter 10 "Quaternions", pp. 507-544, http://books.google.se/books?id=a9SzFHPJ0mwC

[2] Shoemake, Ken - *Uniform random rotations*, in: D. Kirk (Ed.), *Graphics Gems III*, Academic Press, London, 1992, pp. 124-132, http://books.google.se/books?id=xmW_u3mQLmQC

[3] Shoemake, Ken - *Quaternions*, Department of Computer and Information Science, University of Pennsylvania, 2005-10-07, downloaded from C+MS course CS171 at Caltech: http://courses.cms.caltech.edu/cs171/quatut.pdf