Problem Sheet 1 - outline solutiosn

1. In the poker hand two pair there are two pairs of cards with each card in the pair matched by value; the fifth card is a different value. What is the probability of two pairs when five cards are drawn randomly. In a full house there is one pair and one triple, what is the probability of getting a full house?

   Solution: There are 13 choose two choices for the two values for the two pairs and for each pair there are four choose two possible cards. For the remaining card there are 11 possible values and four possible suits. Thus, the number of possible pairs is

   $$\begin{pmatrix} 13 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}^2 \times 44 = \frac{13 \times 12}{1 \times 2} \times 36 \times 44 = 123552 \tag{1}$$

   and hence the probability is 123552/2598960=0.0475. For full house, there are 13 possible values for the pair and 12 for the triple; including the choice of suits we have

   $$13 \times 12 \times \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 3744 \tag{2}$$

   and the probability is 0.0014.

2. A student answers a multiple choice question with four options, one of which is correct. 80% of students know the answer, 20% of students guess and choose randomly. If a student gets the answer correct what is the chance they knew the answer.

   Solution: Let K be the event the student knows the right answer and C is the event that the student chooses the correct answer. We want $P(K|C)$. By Bayes's rule

   $$P(K|C) = \frac{P(C|K)P(K)}{P(C)} \tag{3}$$

   Now

   $$P(C) = P(C|K)P(K) + P(C|\bar{K})P(\bar{K}) = 0.8 + 0.25 * 0.2 = 0.85 \tag{4}$$

   and hence

   $$P(K|C) = \frac{0.8}{0.85} = 0.94 \tag{5}$$

3. In the xkcd cartoon above, what is the chance the Bayesian will win his or her bet if the chance the sun has exploded is one in a million? In reality the chance is, of course, much less than one in a million! Show the answer to six decimal places.

   Solution: Let N be the event that the sun has exploded and L be the event the machine says that the sun has exploded. Hence $P(L|N) = 35/36$ whereas $P(L|\bar{N}) = 1/36$ and $P(N) = 10^{-6}$. The Bayesian will win his or her bet is $P(\bar{N}|L)$:

   $$P(\bar{N}|L) = \frac{P(L|\bar{N})P(\bar{N})}{P(L)} = \frac{1/36 \times (1 - 10^{-6})}{1/36 \times (1 - 10^{-6}) + 35/36 \times 10^{-6}} = 0.999965 \tag{6}$$

4. A three-sided dice is rolled three times. X is the sum of the largest two values. Write down the probability distribution for X.

Solution: Well lets write down table and then explain where we got the numbers from

	2	3	4	5	6
p_X	1/27	1/9	7/27	1/3	7/27

So there are 27 possible outcomes of rolling the dice three times. To get $X = 2$ you need to roll 111, to get $X = 3$ you can roll 112, 121 or 211. To get $X = 4$ there are three permutations of 113 and three permutations of 122, along with 222. To get $X = 5$ there are six permutations of 123 along with three permutations of 223. Finally to rolls six there are three permutations of each of 133 and 233, along with 333.

Extra questions

1. When it started in 1987 the Irish lottery has 36 numbers; participants paid 50 Irish pence to buy a combination of six different numbers; they would win if these numbers matched the six drawn. In the last week in May in 1992 a syndicate tried to buy all combinations of numbers. If they had succeeded how many numbers would they have bought?

Solution: Well this is just 36 choose six:

$$\begin{pmatrix} 36 \\ 6 \end{pmatrix} = 1947792 \tag{7}$$

so they would've spend $973,896$ Irish pounds.

2. From a group of three undergraduates and five graduate students, four students are randomly selected to act as TAs. What is the chance there will be exactly two undergraduate TAs?

Solution: So this is another counting exercise, the total ways of selecting four out of eight is

$$\begin{pmatrix} 8 \\ 4 \end{pmatrix} = frac8 \times 7 \times 6 \times 51 \times 2 \times 3 \times 4 = 70 \tag{8}$$

Now the number of ways of choosing two undergraduates out of three is three and the number of ways of picking two graduates out of five is 10. Hence the answer is 3/7.

3. Prove

$$\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix} \tag{9}$$

Solution: This follows from the definition

$$\left(\begin{array}{c} n \\ r \end{array}\right) = \frac{n!}{r!(n-r)!} \tag{10}$$

which stays the same if you swap r and $n-r$.

4. Two events A and B have probabilities $P(A) = 0.2$, $P(B) = 0.3$ and $P(A \cup B) = 0.4$. Find
   a) Find $P(A \cap B)$.
   b) Find $P(\bar{A} \cap \bar{B})$.
   c) Find $P(A|B)$.

   Solution: So
   $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{11}$$
   so
   $$P(A \cap B) = 0.2 + 0.3 - 0.4 = 0.1 \tag{12}$$
   If $P(A \cap B) = 0.1$ then $P(\bar{A} \cap B) = P(B) - P(A \cap B) = 0.2$. Finally
   $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.3} = \frac{1}{3} \tag{13}$$

5. One night in a bar in Las Vegas you meet a dodgy character who tells you that there are two types of slot machine in the Topicana, one that pays out 10% of the time, the other 20%. One sort of machine is blue, the other red. Unfortunately the dodgy character is too drunk to remember which is which. The next day you randomly select a red machine and put in a coin. You lose. Assuming the dodgy character was telling the truth, what is the chance the red machine is the one that pays out more. If you had won instead of losing, what would the chance be?[1]

   Solution: So let J be the event of winning and R be the event that the red machine is the one that pays out more. We want
   $$P(R|!J) = \frac{P(!J|R)P(R)}{P(!J)} \tag{14}$$
   where we are writing $!J$, not J, for $\bar{J}$. Now $P(!J|R) = 0.8$ according to the DC and $P(R) = 0.5$ because we chose between red and blue randomly. Finally
   $$P(!J) = P(!J|R)P(R) + P(!J|B)P(B) = 0.8 \times 0.5 + 0.9 \times 0.5 = 0.85 \tag{15}$$
   and hence
   $$P(R|\bar{J}) = \frac{0.8 \times 0.5}{0.85} \approx 0.47 \tag{16}$$

[1] I stole this problem from `courses.smp.uq.edu.au/MATH3104/`

so, consider the prior $P(R) = 0.5$ we haven't learned much from this lost coin. If we had won you'd have

$$P(R|J) = \frac{P(J|R)P(R)}{P(J)} = \frac{0.2 \times 0.5}{0.15} \approx 0.67 \tag{17}$$

so you'd learn a lot more from a win, this makes sense since it is a rarer event.