Problem Sheet 2 - outline solutions

Questions

Four questions, each worth two marks with two marks for attendance.

1. The illusionist Derren Brown famously flipped a coin on camera so that it landed heads ten times in a row; he claimed that this was because of his mind powers, in fact it was because of his patience, he simply kept trying the trick again and again until it worked. It took him nine hours. What is the probability of a coin landing heads ten times in a row? If you flip a coin ten times what is the probability of getting five heads and five tails?

   Solution: The chance of 10 heads is
   $$p(10) = 0.5^{10} = 0.0009765625 \tag{1}$$
   whereas the chance of five heads is
   $$p(5) = \left(\begin{array}{c} 10 \\ 5 \end{array}\right) 0.5^{10} = 0.24609375 \tag{2}$$

2. A fisher catches on average a fish every 25 minutes. What is the probability that they catche no fish in an hour?

   Solution: If he catches a fish every 25 minutes then his rate for an hour is 60/25=2.4 so
   $$p(0) = e^{-2.4} \approx 0.09 \tag{3}$$

3. The distribution of tree heights in a christmas tree forest is
   $$p(h) = \begin{cases} 0.3 & 0 \leq h < 2 \\ 0.2 & 2 \leq h < 4 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$
   What is the mean height of trees in the forest?

   Solution: So
   $$\mu = \langle H \rangle = \int_{-\infty}^{\infty} hf(h) = \int_0^2 0.3hdh + \int_2^4 0.2hdh \tag{5}$$
   and
   $$\int_0^2 0.3hdh = 0.3\frac{h^2}{2}\bigg|_0^2 = 0.6 \tag{6}$$
   and
   $$\int_2^4 0.2hdh = 0.2\frac{h^2}{2}\bigg|_2^4 = 0.2(8-2) = 1.2 \tag{7}$$
   so $\mu = 1.8$.

4. Like the binomial distribution the geometric probability distribution is related to a series of independent trials where each trial has probability p of success and $q = 1-p$ of failure. The geometric probability $p(r)$ is the probability that the rth trial is the first success. It is

$$p(r) = q^{r-1}p \tag{8}$$

It can be shown that

$$\sum_{r=1}^{\infty} p(r) = 1 \tag{9}$$

as it must be. You can assume that here. What is the mean of the geometric probability? As a hint, this is done much as for the binomial expansion.

Solution: So we have

$$1 = \sum_{r=1}^{\infty} q^{r-1}p \tag{10}$$

If we differenciate both sides by p we get

$$0 = \sum_{r=1}^{\infty} q^{r-1} - \sum_{r=1}^{\infty} (r-1)q^{r-2}p \tag{11}$$

or

$$0 = \frac{1}{p}\sum_{r=1}^{\infty} q^{r-1}p - \sum_{r=1}^{\infty} (r-1)q^{r-2}p \tag{12}$$

In the second term set $s = r-1$ to get

$$\sum_{r=1}^{\infty} (r-1)q^{r-2}p = \sum_{s=0}^{\infty} sq^{s-1}p = \sum_{s=1}^{\infty} sq^{s-1}p \tag{13}$$

where we are able to change drop the $s = 0$ term in the sum because it gives zero. Now back to the original equation:

$$0 = \frac{1}{p}\sum_{r=1}^{\infty} q^{r-1}p - \sum_{s=1}^{\infty} sq^{s-1}p \tag{14}$$

Now note that the first sum gives one and the second gives μ so

$$\mu = \frac{1}{p} \tag{15}$$

We weren't asked to prove

$$\sum_{r=1}^{\infty} p(r) = 1 \tag{16}$$

but in case you are interested it is discussed here

$$\sum_{r} q^{r-1}p = 1 \tag{17}$$

For this we use the expansion

$$\frac{1}{1-q} = 1 + q + q^2 + \dots \tag{18}$$

you can think of this as coming from the binomial expansion of $(1-q)^{-1}$ using the generalized binomial formula discovered by Newton

$$(1+x)^n = \sum_r \frac{n(n-1)\dots(n-r+1)}{r!} x^r \tag{19}$$

with $n = -1$; it can also be derived as the Taylor expansion. Either way we have

$$\frac{1}{1-q} = \sum_{r=0}^{\infty} q^r = \sum_{r=1}^{\infty} q^{r-1} \tag{20}$$

and since $1 - q = p$ this gives the result.

Extra questions

These are for you to do on your own, not for handing up. Solutions will be included in the solutions section. I haven't added these question yet, but they will be added to the online version of this problem sheet over the next couple of days.