Time Series Analysis and Forecasting with Complex Dynamic Models

# Introduction

The theory of complex variables functions is actively used in a variety of disciplines, including modern physics and engineering sciences. It is relatively easy to describe the complex phenomena that are studied in these areas of science using the models and methods of this branch of mathematics. Social sciences, being much more complicated due to the unpredictability of human behaviour, tend to use simpler instruments for modelling complex processes (references). For example, there are only few scientific publications in which methods and models of the theory of complex variables functions are used in economics, and they typically use that instrument for diagnostics or statistical tests (references) rather than proper modelling.

In 2012, Springer published the monograph “Complex-valued Modeling in Economics and Finance” (Svetunkov, 2012), which presented the theory and methodology of modelling using complex variables in economics. Svetunkov (2012) have summarised the main principles of using complex variables in economics and discussed how to estimate some of those models. While, this was the first monograph that discussed the topic, the first paper in this direction was Ben Tamari (1997). The author first introduced Wealth as a complex variable consisting of Output (real part) and Money (imaginary part) and showed how, even with such an elementary representation, interesting new results can be obtained in the area of economics. Unfortunately, this work has gone unnoticed in scientific world. We became aware of this work only in 2016, when Ben Tamari had kindly sent Sergey Svetunkov his paper.

Since 2012, there has been some development in the area of modelling using complex variables, notably a paper by Svetunkov & Kourentzes (2022) on Complex Exponential Smoothing and Kourentez et al. (2019) on an error measure based on the idea of complex numbers. Up until now, the modelling with complex variables has been mainly picked up by academics working in the areas of forecasting and engineering. The latter group has been using complex autoregressions for a couple of decades, modelling and predicting signals. The former group has only started using the principles described in Svetunkov (2012). In other disciplines, complex variables are not used directly for model building. The probable reason for this is the lack of the communication between the disciplines and the inherited inertia of academia.

In a try to speed up the adoption of the new instrument, we have written this monograph, summarising the research that has been done in the area of dynamic models since 2012. We should clarify that the term “dynamic model” used in this monograph refers to models that have a structure that changes over time. The classical example of such a model is ARIMA (AutoRegression Integrated with Moving Average by Box & Jenkins, 1976), which allows producing forecasts for a variety of processes based on the existing historical time series. Another example is a multivariate version of ARMA, Vector ARMA or VARMA, which allows modelling the dynamics of several related processes simultaneously. For instance, in business and economics, these models are used for prices, demand forecasting and for capturing complex interactions between macroeconomic indicators over time.

However, all autoregression models that exist today use real variables. In this monograph we will discuss their complex-valued counterparts, namely complex ARIMA and complex VARMA, and study their properties, showing how to identify their orders and how to estimate these models in practice. But before rushing into the discussion, we will explain the basics of random complex variables and complex-valued statistics, which has been developed in signal processing literature, but has not been used to its full potential.

Chapter 1 discusses the theory of random complex variables, conventional and complex-valued statistics, the complex least squares method, maximum likelihood and complex autocorrelation and partial autocorrelation functions. In Chapter 2, we will move to the discussion of simple dynamic model – complex AR, starting from its properties and slowly moving to its identification and estimation and then to forecasting. After that, in Chapter 3, we will move towards complex MA, again discussing its properties and how one can apply it in practice. The two parts will be united in a cARIMA model in Chapter 4, where we will also discuss seasonal counterpart of the model. Finally, we will move to vector models in Chapter 5, introducing cVAR, cVMA and cVARMA, showing their advantages in comparison with the conventional real valued models. All of this will be supported by examples in R, which will rely heavily on the package “complex”, developed especially for this monograph. This means that anyone can then use the proposed models for purposes of time series analysis and forecasting.

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# Introduction to random complex variables

## Theory of complex variables

This section sets out the basic concepts of the Theory of Complex Variables Functions (TCVF) with some historical overlook of the idea of complex numbers. Hopefully, this will help in understanding the main idea that we will use extensively in the following chapters.

The theory of complex numbers started in 1572 in the small Italian town of Kura. It was in this place and precisely in this year that the manuscript of an Italian mathematician Rafael Bombelli was published. The book was called Algebra. Rafael Bombelli showed in it how to solve the following cubic equation:

(1.1.1)

The root of cubic equations at that time was calculated using the formula of Scipione del Ferro. With regards to the task at hand, finding the root should have been carried out as follows:

(1.1.2)

As can be seen from equation (1.1.2), there are several square roots, and it follows directly from the right hand side of this equation that to calculate the roots of equation (1.1.1), we need to extract the square root from the negative number, -121. This means that it will not be possible to find a solution to this equation in the domain of real numbers.

This does not mean at all that equation (1.1) has no solution. If you depict the problem on a plane, then you can easily make sure that there is a solution. But it is impossible to find it arithmetically using the Scipione del Ferro formula, and precisely because there is a negative number in the radicand expression on the right side (1.2), and the square root of a negative number, as anyone knows from the school curriculum, does not exist.

The mathematicians of that time struggled to solve this problem, but could not come up with anything. And Rafael Bombelli suggested not to pay attention to the minus sign in the radical expression (Bombelli, 12). After all, the number (-121) can be represented as the product of two numbers: (-1 \* 121). Then the square root of this number can be written as.

That is, formula (1.2) will take the form:

. (1.1.3)

Now one can get the solution of the problem, and the root of the cubic equation *x = 4.*

For a long time, Bombelli's approach was considered by scientists as a convenient mathematical trick. And the square root of minus one was called differently, as: "magic unit", "vanishing unit", etc. Finally, the following name was established with respect to it: " an imaginary unit". And numbers that, in addition to the real one, also have an imaginary component, began to be called "complex" numbers, that is, complex ones.

A complex number, as can be seen, is a numeric pair consisting of two parts - real and imaginary:

 , (1.1.4)

where *x* is the real part of a complex number,

*iy -* is the imaginary part of a complex number,

*x* and *y* are real numbers,

*i* - is an imaginary unit that satisfies the equality:

 или . (1.1.5)

With complex numbers one can perform almost all the same actions as with real numbers. But taking into account the character of the of the imaginary unit properties, these actions have an original character, often not inherent in operations in the domain of real numbers.

The main problem that the economist faces when presenting some economic indicator in the form a complex number is the complexity of the economic interpretation of the imaginary part. The main question in this case can be formulated something like this: in real economic life where can imaginary numbers be met in general, and an imaginary unit, in particular? And what is the meaning of an imaginary unit?

The imaginary unit has no meaning - neither economic nor physical. The imaginary unit is a mathematical rule, and that is all. Where in real economic life an economist meets, for example, a decimal logarithm? Nowhere! In the world around us there is neither a decimal logarithm, nor any other logarithms. The decimal (or any other) logarithm is a mathematical rule with which it turned out to be very convenient to solve applied problems, including in economy. In the same way, with the help of an imaginary unit, which, as it has already been mentioned, can be regarded as a mathematical rule, it turned out to be very convenient to solve a whole class of applied problems in different areas of human activity.

With the help of the rules specified by conditions (1.4) and (1.5), it becomes possible to use new mathematical operations, obtain new mathematical results, and form new mathematical models. It should be pointed out right away that in none of the areas of natural science there are processes in which imaginary numbers or an imaginary unit appear.

Imaginary and complex numbers are a mathematical model that may either describe or not describe some real-life phenomena. If scientists make up their minds to use complex variables for real processes modelling, they predetermine the rules, according to which one component of a complex process is always attributed to the real part, and another component to the imaginary part of the complex variable.

In the same way, there are no phenomena in economics that, in their essence, should be attributed to the real or imaginary parts of a complex variable, just as there are no phenomena in which the real and imaginary parts are clearly distinguished. We, like scientists in other fields of science, will set the rules according to which it becomes possible to represent economic phenomena in the form of complex numbers and complex variables. And in the case when such a representation of a complex economic object allows a more accurate description of its properties, we will use models of a complex variable or models of several complex variables instead of real variables models.

It is known that a real number represents a certain segment on the numerical axis, which has a zero point and tends to plus infinity or minus infinity. This real number is characterized by the distance from the zero point to the number. If a number is located on the numerical axis to the left of the zero point, then it will be negative, and if it is located on the numerical axis to the right of the zero point, then it will be positive

Since, unlike a real variable, a complex variable consists of two parts, these are those two parts that determine the complex plane. A complex number is a point on the complex plane (Fig. 1.1). In order to determine unequivocally a given point on the complex plane, it is necessary to use two coordinates - a segment on the axis of the real part and a segment on the axis of the imaginary part.

The graph in Fig. 1.1 shows these two axes, which by definition are orthogonal. Let us make a reservation right away that we have before us the plane of the Cartesian coordinate system, on the axes of which real numbers *x* and *y* are plotted. Simply, the real part of the complex variable will be plotted along the horizontal axis, and its imaginary part along the vertical axis

*x*

*y*

*0*

*φ*

*Z*

*r*

Figure.1.1. Complex number on the complex plane

Any point lying on the complex plane defined by the indicated axes is a complex number, even if this point lies on the axis of real numbers.

In this case, it will be a complex number with a zero imaginary part. Therefore, mathematicians specifically stipulate in advance the field they work in – of real numbers or imaginary ones.

The complex number (1.1.4) can be considered on the plane of the Cartesian coordinate system as a vector that leaves origin of coordinates and ends at the point *(x, y)*. Then any complex number can be represented in polar coordinates using the modulus of the vector and its polar angle:

 (1.1.6)

Here *φ* is the polar angle, *r* is the polar radius (vector length), which is called the modulus of a complex number. The modulus of a complex number is:



The polar angle can also be easily found:

, (1.1.8)

where *k* is an integer number. The polar angle is sometimes referred to as the argument of a complex number. In most cases, the condition *k = 0* is accepted.

The fact that complex numbers can be considered both in Cartesian coordinates and polar coordinates is their significant advantage over other numbers.

Two complex numbers are equal to each other if and only if their real and imaginary parts are equal to each other.

And this means that, for example, the equality

 (1.1.9)

should be considered as a more compact form of two equations of real variables system recording:



Quite often this is also an important advantage – to present compactly complex systems of real variables equations.

And another conclusion results from this - any function of a complex variable can be represented as a two functions system - separately for the real part, separately for the imaginary part. To make it more convenient, the real part of the complex-valued function is denoted as *Ref (z),* while the imaginary part -as Im*f (z).*

The trigonometric form of a complex number recording (1.1.6) is especially convenient for multiplying complex numbers by each other. Suppose, for example, in addition to the complex number (1.6) there is another complex number:

 (1.1.11)

Let us calculate the product of one complex number *z1* by another complex number *z2* using their trigonometric form. Omitting the elementary conclusion, one can present the final result

. (1.1.12)

This equation is known as the Moivre formula. Moivre's formula facilitates greatly such operations with complex numbers as exponentiation or root extraction from a complex number. Indeed, in order to find, for example, the square of a complex number, it is necessary to square its modulus, and multiply the polar angle by two. And in order to raise it to the power of ¼, the modulus should be raised to the power of ¼, and the polar angle multiplied by ¼.

In 1748, L. Euler, in his book "Introduction to the Infinitesimal Analysis ", substantiated the formula that bears his name today, namely (Euler, 1961, pp. 118-119):

. (1.1.13)

Using Euler's formula, any complex number *Z* with modulus *r* and argument φ can be written in the following exponential form:

. (1.1.14)

This form of notation turns out to be even more convenient for performing mathematical operations with complex numbers. Indeed, using (1.1.14), we multiply again the complex number *z1* by another complex number *z2*:

. (1.1.15)

The modulus of a complex number can be represented in exponential form:

. (1.1.16)

This makes it possible to represent the complex number (1.1.4) in the following form:

. (1.1.17)

And this form of notation allows you to calculate the logarithms of a complex number. Indeed, taking into account the fact that the argument of the complex number (1.1.8) is determined up to a term that is a multiple of *2π*, the logarithm of the complex number *z* can be calculated as follows:

 (1.1.18)

That is, the logarithm of a complex number is a periodic function. In practice, the principal value of the logarithm is usually used, taking *k =* 0.

While using the exponential notation of a complex number, one can perform more complex operations with complex numbers, for example, you can raise a complex number to a real power of *a*:

. (1.1.19)

Moreover, when dealing with complex numbers it is possible to perform such a nontrivial action as raising them to an imaginary power *ib*:

, (1.1.20)

Where .

Obviously, now it is also possible to raise a complex number to a complex power:

. (1.1.21)

A complex number in a complex degree, as can be easily seen, is also a complex number, the modulus of which is

, (1.1.22)

and the argument is

. (1.1.23)

It is not necessary to explain for a very long time that in the field of real numbers, similar actions will be extremely difficult. That is, we are convinced that with the help of mathematical operations with complex variables, it is quite simple to get results that are very difficult to obtain in the field of real variables. It is this very fact that predetermined such popularity of complex variables in natural and engineering sciences.

Complex variable function

, (1.1.24)

is quite often referred to as a "complex-valued" one. And we will further use this word to mention that we are dealing with a function of a complex variable

The function (1.1.24) itself is defined as follows: it is a certain law according to which each point *z* from the set of acceptable values ​​is associated with a certain point or a set of points *w*. It is not by chance that it is stipulated here that, according to the rule (1.1.24), not one point but several ones can correspond to one point *z*. Complex-valued functions can be very different

For example, the complex-valued function



each point (*xr + ixi*) is associated with one and only one point *(yr + iyi*). Such functions are called single-valued (or univalent).

While the complex-valued function

. (1.1.25)

will be multiple-valued (multivalent).

Indeed, according to the law (1.1.25), two different complex numbers (*0 + i*) and (0-i) correspond to the same number (*-1 + i0).*

Therefore, when working with complex-valued functions, it is always necessary to understand that they can be multivalent.

Most often, in economy, several indicators (more than two) depend on several factors (more than two). In fact, these are vector regressions. But I would really like to model this dependence using complex variables, that is, to connect by some mathematical equation a set of economic indicators with a set of factors influencing them - to use hypercomplex numbers.

But an attempt to introduce a system of numbers containing three imaginary units has not given any positive results. We succeeded in constructing a system of numbers with four imaginary units. In this case, the so-called system of quaternions is obtained, that is, numbers of the form:

, (1.1.26)

where *a,b,c,d* – real numbers, *i, j, k* – imaginary units.

Actions with quaternions are complex in nature, which does not allow them to be actively used for any practical purposes. Quaternions still remain the area of ​​idealized research, although scientific works of applied nature have appeared, economics being among them Why is it difficult to work with quaternions? Because, for example, the commutativity property of multiplication does not hold in the field of quaternions. This results in curiosities. Thus, for the equation:

(1.1.27)  
there is an infinite number of roots:

 , where.

 (1.1.28)

Therefore, the obvious desire to describe the dependence of some complex indicator, presented in the form of a quaternion (1.1.26), on another economic indicator, presented in the form of another quaternion, is not yet fully feasible.

**1.2. Complex -valued economics**

Any complex number can be considered as a two-dimensional vector. And from (1.1.10) it follows that any complex-valued function can be considered as a function of a two-dimensional vector. Then it is clear that complex variables can be used where two economic indicators reflect different properties of one phenomenon or where the behavior of an object is determined by two factors

That is, where there is a dependence of one two-dimensional vector of economic indicators on another two-dimensional vector of economic indicators (or several two-dimensional indicators), it is very easy to describe such a dependence using models and methods of the complex variable function theory.

Where do such dependencies occur in the economy? Yes, almost at every step, because so many economic indicators are aggregated values that consist of several and, most often, two main values.

For example, gross output *Q* is made up of gross costs *C* and gross profit *G*, which makes it possible to represent the results as a complex variable *(C+iG).*

The capital that is used in production is also an aggregated value – it can be production *Ko* and non-production *K1*. Therefore, it can also be represented as a complex variable *(Ko+iK1).*

The labor used in production is also a generalized quantity. And it is also possible to take into account different effects of different f labor types on the results of production using a complex variable *(Lo+iL1).* Here *Lo* is the labor of production personnel, and *L1* is the labor of non-production personnel.

On stock exchanges, analysts monitor the dynamics of different companies` sales shares. In doing so, they use the aggregated value of transactions volume in shares:

. (1.2.1)

They will receive additional information if they use a complex variable *(pt+iqt),* having previously reduced the price per unit of *pt* shares and the volume of sales of *qt* shares at this price to a single scale and dimension. This possibility is demonstrated in the monograph "Complex-valued Modeling in Economics and Finance" (Svetunkov, 2012).

For the economy as a whole, at the meso or macro level, the volume of production can be divided into goods *P* and services *S* – and again, a complex variable *(P+ iS)* can be used for modeling.

In retail, these can be durable goods *Lo* and non-durable goods *Sh*, and again the sales volumes can be described by using a complex variable *(Lo + iSh).*

Similar examples can be continued further, but in any case, it is already clear that the scope of complex variables application in economics is extensive.

In this case, most often it does not matter which variable is attributed to the real part of the complex variable, and which to the imaginary one, because when forming a vector, such a problem is not worth it.

But when modeling the economy, there are such situations when the order of the variables, when they are formed into a complex number, takes on a certain meaning This may be the case, for example, when modeling the dependence of production results on production resources, that is, when constructing production functions.

In the domain of real variables, production functions most often take the form of power models of the type:

. (1.2.2)

And their difference from each other lies in the way of setting restrictions on the limits of change in degree indicators αand *β*.

Complex-valued analogues of this function can be very diverse, but the form of a complex-valued production function with real coefficients is universal (Svetunkov S.G., Svetunkov I.S., 2008):

. (1.2.3)

Here C stands for gross production costs,

*G* - is gross profit,

*L* - is labor costs,

*K* - capital.

This function can be represented as a system of two real equations:

 or  (1.2.4)

It is obvious that in the range of real numbers, production functions of this kind do not occur.

It can be seen from (1.2.4) that it is precisely this order of assigning real numbers to the real or to the imaginary parts of complex variables that turns out to be important.

First of all, it should be noted that the real and imaginary parts of the complex production result of the model (1.2.3) react differently to the increase in each of the resources. So, with an increase in capital expenditures *K* with a constant expenditure of labor resources *L*, the cosine of the resources angle decreases, and the sine of this angle increases. Then it follows from the first equation of the system (1.2.4) that with the growth of capital, gross production costs decrease, and from the second equation of the system (1.2.4) it follows that gross profit increases. And that is exactly what happens in the real economy. That is, the complex-valued model (1.2.3) describes the details of the production process more accurately than the real model (1.2.2).

But here we have several more properties of the production function that are important from an economic point of view. If we present the complex production result in the exponential form of the record, then we get:

. (1.2.5)

What characterizes the polar angle of a complex production result? It characterizes the profitability of production (the ratio of gross profit to gross costs). That is, it is this order of attribution of gross profit to the imaginary part, and gross costs to the material part that makes economic sense. And since the polar angle of complex production resources characterizes the capital-labor ratio (the ratio of capital to labor), then the complex-valued production function (1.2.3) models the impact of capital-labor ratio on the production profitability. This testifies in favor of the fact that the model (1.2.3) has a bright economic sense.

But that is not all! Since the economic meaning of such equality is obvious

, (1.2.6)

it turns out that the complex-valued model (1.2.3) describes not only the impact of resources on gross costs *C* and gross profit *G*, but also on gross output *Q*.

So, in some cases, the question of which part to attribute a particular economic indicator- to the real or to the imaginary part of a complex number - is important.

In order to use the apparatus of complex variables functions theory in economics when combining two economic indicators into one complex variable, the following conditions must be met, determined by the features of complex numbers:

1. These indicators should be two characteristics of one and the same process or phenomenon, that is, they should reflect different sides of this phenomenon;

2. They should also have the same dimension. In addition, they should have the same scale.

The first condition results from such reasons.

As a result of a complex variable formation from the two real variables, the complex variable is further considered as an independent single variable. Figuratively speaking, it carries information about its two constituent quantities and reflects the influence of each of these components on a certain result. These values must reflect different sides of the same phenomenon, otherwise their combination into one variable loses every meaning. They may be in close functional dependence with each other, or they may have a complex indirect relationship, but the main condition is that they must carry information about some common process for them. This is due to the fact that such characteristics of a complex number as its modulus and argument make sense only when the complex number components reflect the general content.

The second condition, which requires the same dimension of the complex variable components, is determined by the peculiarity of the complex number properties

Indeed, how can the modulus of a complex number be calculated if the real and imaginary parts have different dimensions, for example, rubles and pieces? There is no way to square each of them and add them – $2 cannot be added to pcs2. Similarly, when calculating the polar angle, it is necessary to find the ratio of the imaginary part to the real part, and then find the arctangent of the resulting number. If the real and imaginary parts are of different dimensions, then nothing can be done, because the tangent of the angle is a dimensionless value, it cannot be measured in rubles / pieces.

In economics, a significant part of the indicators can be reduced to monetary units of measurement, for example, labor costs can be determined not in "man-hours", but in the cost of labor remuneration – by the amount of the wages fund at the enterprise or the enterprise subdivision. Therefore, the condition of the same dimension and scale in most of the real economic problems is quite feasible. But in the case when it is impossible to do this, each of the indicators should be reduced to relative dimensionless values in the way that turns out to be the best for the chosen form of the model.

Hereinafter we will assume that all these conditions are met and we can work with economic random complex variables

**1.3. Random variable on the complex plane**

If a researcher has several observations of a complex variable of economic indicators at his disposal, then they can be considered as random variables, since in the economy there are always a lot of random factors.

The complex random variable *Y* is the value

, (1.3.1)

in which *yr* and *yi* are real random variables, and *i* is an imaginary unit.

This complex random variable in a stationary process is a point on the complex plane, and the set of observations of it, which is subject to the influence of random factors, on the complex plane will be a certain scattering cloud.

This complex random variable in a stationary process represents a point on the complex plane, and a number of observations of it, which is affected by random factors, on the complex plane will represent some scattering cloud.

All random processes that are studied by the economists have various features, which can mostly be reduced to some typical situations. These typical situations are well described in mathematical statistics and are called "the law of probability distribution." In order to understand which law of probability distribution a particular process belongs to, it is necessary to calculate its main characteristics and draw an appropriate conclusion from them.

The main characteristic of any random variable, including a complex random variable, is its mathematical expectation.

The mathematical expectation of a complex random variable (1.3.1) is called a complex number

. (1.3.2),

where *mr* is the mathematical expectation of the real part, and *mi* is the mathematical expectation of the imaginary part of a complex random variable.

Since complex random numbers are points on the complex plane, the mathematical expectation of a complex random variable is also a point on the complex plane around which random complex variables are scattered. Moreover, for a normal distribution, such a rule is obvious - the probability that a random complex variable will be closer to its mathematical expectation is greater than the probability that it will be further from it.

The way the points are located on the complex plane is due to the presence or absence of a relationship between the real and imaginary parts of a complex random variable. Let us therefore consider two possible cases:

1) when both parts of a complex random variable are independent of each other and

2) when the real and imaginary parts of a complex random variable are interrelated.

The first case. The real and imaginary parts of a complex random variable are independent of each other. In modern mathematical statistics, this position is the main one and is considered as an axiomatic (*Park*, 2019; *Panchev*, 2013;). It is quite possible that in those branches of modern science where statistics of a complex random variable are used, this is the case. As the review of published works in this area shows, the complex random variable is mainly used in the signal theory (*Schreier, 20*10; *M.Kay*, 2010; *Tuelay,* 2011 etc.) and there the independence of two signals of each other is quite a natural phenomenon. And since modern scientists mainly consider this particular case, we cannot ignore it, although it was previously shown that such cases are meaningless for the economy.

Since the real and imaginary parts of such a complex variable do not depend on each other, then all statistical characteristics of a complex random variable do not depend on each other.

Then the variance of the real part of the complex random variable will be equal to:

, (1.3.3)

and the variance of its imaginary part is:

. (1.3.4)

The total variance of a complex random variable with independent real and imaginary parts will be equal to the sum of its real and imaginary parts variances:

. (1.3.5)

It should be noted that variance is an important characteristic that makes it possible to describe the probability distribution density. We will exclusively consider the normal probability distribution, since most often we must deal with it in practice.

The Gauss formula for the real part of a complex number will look like this:

. (1.3.6)

Similarly, we can write down the probability distribution density formula for the imaginary part of a complex number:

. (1.3.7)

Then, due to the independence of the real and imaginary parts of a complex random variable from each other, its distribution density will be equal to the product of the distribution densities of the real and imaginary parts:

. (1.3.8)

The form of this distribution is shown in Fig. 1.1, where the axes of the horizontal plane are the real and imaginary parts of a complex random variable (complex plane), and its distribution density is plotted horizontally.

*mY*

*yr*

*0*

*f(Y)*

*yi*

*myr*

*myi*

*Figure 1.1. Mathematical expectation of a complex random variable with independent real and imaginary parts*

All points on the complex plane have a different probability of occurrence – the further away from the mathematical expectation of *my*, the less likely they are to appear on the complex plane

All points lying on a straight line with the coordinate *myi* have a different probability of occurrence, and the maximum probability of a random complex variable occurrence falls on the mathematical expectation point *my.* Similarly, all points lying on a straight line with the coordinate *myr* have different probability of occurrence, but the maximum probability of occurrence of a random complex variable on this line also falls on the mathematical expectation point *m*y.

It can be seen from the figure that the cross-section of the surface in Fig. 1.1 with planes parallel to the complex plane, that is, planes of equal probability density, gives different ellipses. In probability theory, these ellipses are called "scattering ellipses", the equation of which in our case is determined by the variance of each of the parts and their mathematical expectation:



(1.3.9)

Ellipses can be easily projected onto the complex plane (Fig. 1.2).

*mY*

*yr*

*0*

*yi*

*myr*

*myi*

*Figure 1.2. Scattering ellipses for the situation of independence of the real and imaginary parts of a complex random variable*

*yi1*

*yi2*

*1*

*2*

*yr1*

*yr2*

In this figure, two points are plotted - 1 and 2, which characterize two different random complex numbers *(yr1; yi1*) and (*yr1; yi1*) . If we calculate the distances from them to the mathematical expectation, we get:

*1*

 and (1.3.10)

It is known that for the discrete case, the variance of a random variable *x* can be written as:

. (1.3.11)

For a complex random variable (1.3.1), the form of the variance record will be as follows:

. (1.3.12).

As it clearly follows from (1.3.12), the variance of a complex random variable with real and imaginary parts independent of each other will be the sum of the squared distances from the random variables lying on the complex plane to their mathematical expectation multiplied by the probabilities of these random variables’ occurrence.

If the independent variances of the real and imaginary parts are equal to each other, then the ellipses of Fig. 2 turn into scattering circles.

The second case. The real and imaginary parts of a complex random variable are dependent on each other.

It would seem that in mathematical statistics it was quite logical to study all possible variants of a complex random variable properties. And if there is a variant of the real and imaginary parts of a complex random variable independence from each other, then there should be the second option - the variant of the real and imaginary parts of a complex random variable dependence on each other. Then all the options will be considered and the scientists will get full knowledge about the subject of research.

But it is just the variant of the real and imaginary parts of a complex random variable dependence on each other that the scientists did not consider in full. To all such cases and statistical characteristics, they, thanks to H. Harter and M. Lum`s example, add the prefix "pseudo" (*Harter, 1955*). And there are historical reasons for this.

Interest in statistical processing of observations of changes in a complex variable arose in the 50s-60s of the twentieth century. For the first time this problem was formulated by R. Wooding, who proposed an approach of a complex random variable representation from the standpoint of a normal distribution (*Wooding, 1956*). This approach was developed in their works by R. Arens (*Arens, 1957*) and I. Reed (*Reed, 1962*). A priori, these publications assumed the independence of the normally distributed real and imaginary parts of a complex random variable from each other, but this was not explicitly mentioned. In 1963, N. Goodman explicitly formulated this assumption (*Goodman, 1963*). Based on it, scientists further formulated the basic concepts and characteristics of a random normally distributed complex variable: mathematical expectation, moments (including the correlation moment), covariance, variance, etc. (*Feller, 1966*).

Modern researchers who use complex random variables in their scientific works always use the option when both parts of it are independent of each other (*Tavares, 2006*). There were, however, the first attempts to comprehend the situation when the real and imaginary parts depend on each other, but scientists considering this option immediately add "pseudo" - pseudo covariance or pseudo variance, etc. *(Picinbono, 1997*; *Kammeyer, 2002*; *Soroush, 2010*; *Tülay*, *2010; Tuelay, 2011* etc.). And they do not go further than calculating pseudo moments, pseudo variances and pseudo covariances.

Therefore, it turned out that we have no ready-made solutions proposed by mathematical statistics for random complex variable whose real and imaginary parts depend on each other. We will have to deal with this issue on our own, checking with the level that modern mathematical statistics offers us in this matter.

Since random variables are considered, the relationship between them will be correlative. Denote by *rri* the coefficient of paired correlation between the real *yr* and the imaginary *yi* part of a complex random variable

The density of the normal distribution of two random interconnected quantities, as is known from the theory of probability and mathematical statistics, taking into account the notation we have adopted, will take the form:

. (1.3.13)

One can make sure that when the pair correlation coefficient *ryryi*is equal to zero, the formula (1.3.13) turns into the formula (1.3.8).

This formula is used in modern mathematical statistics to describe the probabilistic characteristics of normally distributed random complex variables (*Trampitsch, 2013, p. 40*) .

The density of a complex random variable normal distribution with its interconnected parts has in three-dimensional space approximately the same form as shown in Fig. 1.1, but with a slight difference. As can be seen from Fig. 1.1, with the independence of the real and imaginary parts of a complex random variable, the three-dimensional model of the distribution density is symmetric with respect to the lines passing through the mathematical expectation point and parallel to the axes of the complex plane.

And in the case of the dependence of these real and imaginary parts of a complex random variable on each other, the model becomes asymmetric to these lines. It becomes symmetrical to the lines that are not parallel to the axes of the complex plane (Fig. 1.3). In this case, the scattering ellipses also change their position

*mY*

*yr*

*0*

*f(Y)*

*yi*

*myr*

*myi*

*Figure 1.3. Mathematical expectation and probability distribution density with dependent on each other parts of a complex random variable*

It is more convenient to consider not a three-dimensional figure in space, but ellipses of scattering.

They, for the case of the existing dependence between the real and imaginary parts, will have the following form

 (1.3.14)

Figure 1.4 shows one of these scattering ellipses.

And it is characteristic for it that the distances from the points lying on the ellipse to the mathematical expectation *my* are equal

 и  (1.3.15)

*mY*

*yr*

*0*

*yi*

*myr*

*myi*

*Figure 1.4. The scattering ellipse on the complex plane with the relationship between the real and imaginary parts of a complex random variable*

*yi1*

*yi2*

*1*

*2*

*yr1*

*yr2*

And the variance is respectively equal to:

(1.3.16)

Here-  is the probability of a complex random variable occurrence corresponding to (1.3.13).

Since the probabilities in the case of a) the independence of the real and imaginary parts and in case of b) their dependences on each other are of different nature and are differently calculated, this means that the variance in the latter case cannot be calculated as in the case of the parts of a complex random variable independence from each other (1.3.12), that is:



Let us show it.

In Fig. 5, two scattering ellipses corresponding to the same value of the probability distribution density are plotted on the complex plane. The probability of points appearing on the lines of these ellipses is the same. But the blue scattering ellipse corresponds to the situation of the real and imaginary parts of a complex random variable independence from each other, and the red ellipse corresponds to the second variant, when the real and imaginary parts of a complex random variable depend on each other.

*mY*

*yr*

*0*

*yi*

*myr*

*myi*

*Figure 1.5. Scattering ellipses on the complex plane with the interrelation between the real and imaginary parts of a complex random variable (red) and in the case of their independence (blue)*

*yi1*

*yi2*

*1*

*2*

*yr1*

*yr2*

Point 1 lies both on the line of the blue scattering ellipse and on the line of the red scattering ellipse. The probability of this point occurrence is the same both in the case of the real and imaginary parts dependence on each other and in the case of their independence from each other. But point 2 lies only on the line of the red ellipse and is above the blue scattering ellipse. This means that the probability *p* of this point occurrence in the case of the real and imaginary parts of a complex random variable independence from each other is less than the probability *pri* of this point occurrence in the case of the components of a complex random variable dependence on each other:

*p< pri*.

Then:

. (1.3.18)

Since the variance of a complex random variable with the mutual dependence of the real and imaginary parts is not a simple sum of the variances of the real and imaginary parts, its properties should be studied in more detail.

**1.4. Variance of complex-valued variables in modern statistics**

Complex situation with the use of complex-valued variables models is caused by the fact that the branch of mathematical statistics studying random complex-valued variables seems undeveloped so far (*Svetunkov, 2018*). Scientists paid attention to statistical treatment of complex-valued variables dynamics in the 50-60s of the 20th century. R. Wooding (*Wooding, 1956*) was the first who formulated this task and offered the approach to presenting the complex random value based on the normal distribution. R.Arens (*Arens, 1957*) and I.S.Reed (*Reed, 1962*) introduced major concepts and characteristics of the random normally distributed complex-valued variable such as mathematical mean value, moment coefficients (including correlation coefficient), covariance, variance, etc. It was a priori supposed that the situation of independent normally distributed real and imaginary parts was under discussion, and only N.R.Goodman formulated the hypothesis more clearly (*Goodman, 1963*). Since that time scientists have begun began to consider the distribution of complex random value as an aggregate of two independent normally distributed random values – real and imaginary parts. It was W. Feller who systematized these statements (*Feller, 1966*). Today this assumption about the autonomy of real and imaginary parts of the complex-valued random variable serves as a key prerequisite for mathematical statistics of complex-valued random variable.

As the scope of functions of modeling using complex-valued variables in various scientific spheres had widened, scientists faced the necessity to develop the body of mathematical statistics which allowed to do it. Since this function is of interest not only for economists but for the specialists dealing with complex-valued variables in other sciences as well, an adapted least square method (*Tavares, 2007*) was proposed based on the real part of the complex random value variance. However, at this point the development of the statistical tools of the complex random value stopped – no tools for calculating complex-related correlations were offered, no tools for determining the confidence limits or other instruments of statistical treatment of random complex-valued variables were proposed.

Nowadays mathematical statistics deals with variance of independent constituents – real and imaginary parts – rather than variance of the complex-valued variable in general. In this case statistical characteristics of the complex-valued random variable are regarded as real values. To calculate the real characteristics of the complex random value, this value is multiplied by the complex conjugate. This procedure, as it is known, allows to find out the real characteristic of the complex number. Variance of a complex-valued variable is presented as mathematical mean value of squared absolute value of the corresponding centered variable (*Bliss, 2013*; *Panchev*, *2013*; *Tuelay, 2011*):

, (1.4.1)

where , (1.4.2)

. (1.4.3)

Viz:

. (1.4.4)

However, such interpretation of the complex-valued variable imposes the limits to statistical data treatment. Let us illustrate it by determining the correlations between two complex-valued random variables. Actually, we can calculate the pair correlation coefficient of the variables by using the correlation moment and the variance:

. (1.4.5)

Let us take this approach to calculate the correlation between the complex-valued random variables. The correlation moment is represented as a real value using one of the variables in the conjugate form, whereas the variance is calculated as real characteristics in accordance with (1.4.1) (*Miyabe*, 2015; *Panchev*, 2013). It should be noted that the correlation moment calculated as

 (1.4.6)

is not the real number. It is the complex number as when multiplying and grouping the summands, we obtain:

 (1.4.7)

And only in the case when *zX=zY*, the latter summond (1.4.7) with the imaginary part equals zero, and the correlation moment becomes a real number. In all other cases the correlation moment will be of complex type, thus the pair correlation coefficient between two complex-valued random variables will be the complex value. Keeping this in mind, scientists claim that they calculate the absolute value of the correlation moment, i.e., instead of (1.4.6) they use Re(*µXY*) (*Miyabe*, 2015).

However, let us find the pair correlation coefficient by using (1.4.6).

For the sake of simplicity, we assume that the discrete sequence of the complex random value *z* centered in relation to its arithmetic mean is under review, hence:

. (1.4.8)

Sample value of the correlation coefficient (1.4.5) when using the real-valued variance (1.4.1) and the correlation moment (1.4.6) is of the form:

. (1.4.9)

On the other hand, we should keep in mind that the pair correlation coefficient as related to real numbers was suggested in the 90s of the 19th century by K. Pearson to estimate linear interrelation of complex-valued variables. It was defined as geometric mean of regressions *y* by *x* and *x* by *y* (*Pearson, 2013)*:

, (1.4.10)

where the proportionality factors of simple regressions *a1* and *b1* are found by using the least square method.

To find the formula for calculating the sample value of the pair correlation coefficient for the case of the two complex-valued variables using K. Pearson’s approach (1.4.10), we shall handle that variant of the least square method which results from the assumption about real character of the complex-valued variable variance (*Tavares, 2007*). The complex regression coefficient of linear relationship between complex-valued variable *Y* and other complex-valued variable *X* by using this approach to the least square method will be calculated in the following way:

. (1.4.11)

Inverse relationship of complex-valued variable *X* to other complex random variable *Y*, represented in the linear form, has the following formula for calculating the complex regression coefficient found by using the least square method:

. (1.4.12)

By using these formulae of estimating the sample values of proportionality coefficients of regression lines *Y* by *X* and *X* by *Y* in (1.4.10), we obtain the formula for calculating the sample value of the complex pair correlation coefficient:

. (1.4.13)

Now let us compare formula (1.4.9) with formula (1.4.13). This should be one and the same coefficient which is calculated based on the same basic prerequisites. However, if the denominators of formulae (1.4.8) and (1.4.13) coincide, their numerators differ fundamentally from each other. These are different formulae that are used to calculate different coefficients, and these coefficients will give different values for one and the same sequence. This is why it seems unclear: should we use formula (1.4.9) or should we use formula (1.4.13) or none of these formulae can be used? The obtained result is contradictory, and it does not allow scientists to form the body of complex correlations.

Even if we agree with the suggestions made by the followers of the conception of real-value character of the complex-valued variables variance and use their real parts (*Miyabe, 2013*) instead of complex characteristics, the conflict will not be solved.

In reality, for (1.4.9) we will obtain:

, (1.4.14)

and for (1.4.13) we will obtain:

. (1.4.15)

As it can be seen from two results compared, different formulae and a contradictory result are obtained again. This is why P.Schreier and L. Scharf note that so far the research carried out in the area of the correlation of complex-valued random variables has produced deplorable results (*Schreier, 2010)*.

**1.5. Complex-valued variance**

Exactly the same problems arise in the field of statistical hypotheses and other branches of mathematical statistics of complex-valued variables which to certain extent rely upon an important variability measure – variance. Since economists come across the problem of interrelation (direct or indirect) between the factor and the indicator when considering some object or phenomenon, the assumption that the variance of real and imaginary parts are not interrelated is rarely met. Thus, the hypothesis saying that the variance of the complex random value should always be real cannot be taken as a basis in econometrics. On the contrary, the variance of economic complex-valued random variable should be presented as a complex characteristic of the variability of a random complex sequence (*Svetunkov, 2018*).

Then the complex variance of the complex random value can be represented in the following way:

, (1.5.1)

where 

As will readily be observed, variance (1.4.1) is a special case of variance (1.5.1), namely – when vectorial angle *θ* between real part and imaginary part of the complex-valued variable is equal to *θ=πk, k=0,1,2,3,…*, i.e. real part and imaginary part are not interdependent.

How can the assumed hypothesis about the relationship between the real and imaginary parts, and that the variance of the complex-valued variable should be considered as the complex value, help in solving applied econometric problems? To answer this question let us turn back to the calculation of correlations between complex random values using two methods, which resulted in an impasse if we assume that the variance of the complex-valued variable is a real number.

We shall consider all the characteristics of the complex random value as complex numbers. Therefore, we shall not resort to their artificial transformation into real numbers of these characteristics by multiplying the complex number by its conjugate. Let us represent the correlation moment of two random complex-valued variables as a complex number:

 (1.5.2)

If we apply the values of complex variance (1.4.14) and complex correlation moment (1.5.2) to the formula for calculation of pair correlation coefficient (1.4.5), we obtain:

. (1.5.3)

The obtained formula for the calculation of the sample value of the pair correlation coefficient of two random variables (1.4.7) does not coincide with any of the previously derived formulae (1.4.9) and (1.4.13), when all the characteristics were considered to be real ones.

We shall calculate the complex pair correlation coefficient using the second method – as the geometric mean of sample values of the regression coefficients. In order to find this coefficient, let us formulate the criterion of least square method. It is to be recalled that assuming that the variance of the random complex-valued variable is a real value, the least square method means in fact searching for such regression coefficients whereby:

. (1.5.4)

where - are approximation errors.

In case of the complex variance of the complex random value the least square method reduces to searching for other coefficients whereby (*Svetunkov, 2012,* p. 83):

. (1.5.5)

One can pay attention here to the interrelation between criteria (1.5.4) and (1.5.5). With this aim in view let us present the complex approximation error in the exponential form:

. (1.5.6)

Taking it into account, criterion (1.5.4) takes the form:

, (1.5.7)

and criterion (1.5.5) is as follows:

, (1.5.8)

Therefore, the criterion of the least square method (1.5.4) suggested by G.N.Tavares and L.M.Tavares, is a special case of criterion (1.5.5) – when the vectorial angle of the complex approximation error is equal to zero.

Now, using criterion (1.5.5) in relation to the complex regression coefficient of complex number *X* by complex number *Y* denoted as *a,* we will obtain such formula using the least square method with criterion (1.5.5) (*Svetunkov, 2012, p. 103 – 112*):

. (1.5.9)

The complex coefficient of proportionality *b* of the inverse regression can also be calculated by using criterion (1.5.5) as:

. (1.5.10)

Now when plugging these coefficients in the formula for calculation of the pair correlation coefficient (1.4.10), we obtain:

. (1.5.11)

As it can be seen, the same formula of the complex pair correlation coefficient (1.5.3) as in the case of its calculation through the complex correlation moment (1.5.1) is obtained. This means that the obtained result is not contradictory. Both pair correlation coefficient (1.5.3) between two random complex-valued variables calculated through variance and correlation moment and the pair correlation coefficient calculated through the geometric mean of linear regression have one and the same form. This in turn means that our hypothesis about the need to use complex variance and other complex characteristics of the complex-valued variables in statistics of the complex-valued random variables, is confirmed.

Let us turn now to the analysis of the properties of complex variance of complex random value (1.4.14), the use of which in econometrics of the complex-valued random variables has just been justified. To illustrate it, let us write complex variance in the arithmetic form:

. (1.5.12)

Depending on what form the real and imaginary parts have, complex variance can be a complex, real or imaginary value – the variety of its values correspond to the variety of the properties of the complex-valued random variable. In addition, complex variance can be both positive and negative. Let us consider these options and properties of the complex random value sequence, for which these options of complex variance are valid.

Firstly, let us pay attention to the imaginary part of complex variance (1.5.12):

. (1.5.13)

It has a simple meaning – it is double covariance between the real and imaginary parts of the random complex-valued variable. If there is no correlation between variables, the variable covariance is equal to zero. This means that the imaginary part of the complex variance serves the basis to assume on the presence or absence of correlation between the real and imaginary parts of the random complex-valued variable.

The real part of complex variance of the random complex value is also meaningful for the researcher:

. (1.5.14)

As it can be seen, it characterizes the degree of distinction between the variance of the real part of the random complex-valued variable and the variance of the imaginary part of the given variable. This is why in case when both types of variances are equal to each other, the real part of the complex variance is equal to zero. If the variance of the real part of the complex-valued variable is larger than that of the imaginary part of the complex-valued variable, real part (1.5.12) of the complex variance will be positive. Otherwise, it will be negative.

It is noteworthy that justifying of complex character of the complex random value does not refuse the possibility to use variance in real form – it can be applied as an additional characteristic of the process under research, since real variance characterizes the variability measure of the absolute value of complex variance.

**1.6. Properties of the complex coefficient of pair correlation**

Having determined how to calculate correctly the complex coefficient of pair correlation between two random complex variables, one should give an interpretation to the values that it can take.

A linear relationship between two complex variables means that both the real and imaginary parts of one complex variable act as two-factor linear dependencies on the real and imaginary parts of another complex variable. Therefore, if one variable changes non-linearly, then the other variable will change non-linearly, and such a dependence is often difficult to be determined visually. If the dependence under study is not functional, but regression, then the scatter of points on the complex planes causes is even less associated with any dependence. Therefore, a visual analysis of the relationship between complex variables is difficult and the linear relationship of two complex variables can be judged solely by the calculated characteristics. In the domain of real variables, this function is performed by the pair correlation coefficient, and in the domain of complex random variables, by the complex pair correlation coefficient (1.5.11).

The complex coefficient of pair correlation, as it follows from the materials of the previous paragraph, is the geometric mean of two complex regression coefficients

 (1.6.1)

Therefore, for a strictly functional linear complex-valued dependence 

the obvious equality will hold:

, whence: . (1.6.2)

That is, the complex coefficient of pair correlation for a linear functional dependence is equal to:

** (1.6.3)

So, for a linear functional relationship between two complex variables, the modulus of the real part of the complex coefficient of pair correlation

will be equal to one, and its imaginary component will be zero.

This means that the square of the complex coefficient of pair correlation (complex coefficient of determination) for a linear relationship will be always equal to a real unit. But in which cases of linear functional dependence between two complex variables will the correlation coefficient take the values "plus one", and in which cases – "minus one"?

To answer this question, we present the complex proportionality coefficients in exponential and trigonometric forms:

, (1.6.4)

, (1.6.5)

Then their product will be equal to:

. (1.6.6)

Since for a linear functional dependence (1.6.3) holds, that is, the imaginary part of the complex coefficient of pair correlation is zero, hence it clearly follows that in this case the condition

. (1.6.7)

must be met.

We will consider for simplicity the case when *k*=0. Then the complex pair correlation coefficient is defined as the square root of such a product:

. (1.6.8)

Since the modulus of each proportionality coefficient is positive by definition, the equality of the complex coefficient of pair correlation "plus one" or "minus one" is completely determined by the cosine of the angle α. We are interested in the case when a radicand can be like this:

 (1.6.9)

Then it is possible to determine the characteristics of dependence in the case when the complex coefficient of proportionality becomes equal to minus one. From the equality (1.6.7), the cosine of the radicand (1.6.8) can be written as:

,

Now it is easy to determine that the case of interest to us (1.6.9) is determined by the polar angle of the proportionality complex coefficient , lying in the range:

 (1.6.10)

For this case, the real component of the complex proportionality coefficient is always not positive:

 (1.6.11)

and its imaginary part is always not less than the real one:

 (1.6.12)

These conditions are satisfied, for example, by such complex proportionality coefficients:



What is a linear complex-valued relationship with such values?

To answer this question, we present a linear functional complex-valued dependence as a system of two equalities of real and imaginary parts:

, (1.6.13)

. (1.6.14)

According to the conditions (1.6.11) and (1.6.12), the coefficient *a0* is always not positive, and the imaginary part *a1* can take both positive and negative values. Considering the situation when the complex argument increases in the first quadrant of the complex plane, that is, when *xr* and *xi* are continuously growing and are positive numbers, we obtain that in this case the real part of the complex result *Yr* decreases, and the imaginary part *Yi* because of (1.6.12) can both increase and decrease

That is, the values equal to "minus one" are accepted by the complex coefficient of pair correlation only when there is an inverse relationship between the variables – an increase in the values of the argument X results in a decrease in the values of the real part *Yr* of the complex random variable *Y*.

So, the real part of the complex coefficient of pair correlation *rr* indicates the degree of the dependence approximation between random complex variables to a linear dependence and the interpretation of its values is similar to the values interpretation of the pair correlation coefficient in the real numbers’ domain.

The imaginary component of the complex coefficient of pair correlation *ri*, as it clearly follows from (1.6.3), in the case when there is a linear functional dependence between the complex variables, will be zero.

So, if the real part of the complex coefficient of pair correlation modulo is close to one, and its imaginary part modulo is close to zero, then the researcher can claim that the dependence, the presence of which he assumes between two complex random variables, is close to linear.

**1.7. Nonlinear correlations or lack of correlation**

Models of complex variables describe the processes under study differently than models of real variables. Therefore, the complex coefficient of pair correlation should also be expected to have new properties that are not inherent in the coefficient of paired correlation of real variables.

At the very beginning of these new properties` study, let us pay attention to the situation when the real part of the complex coefficient of pair correlation is zero, and its imaginary part is equal in modulus to one:

**. (1.7.1)

This case, as it is easy to see, is the opposite of the just considered case of linear correlation(1.6.3).

For (1.7.1), the following condition must be met:

. (1.7.2)

This in turn means that

. (1.7.3)

What does the obtained equality show, for example, when *k=1*?

It shows that if, for a proportionality complex coefficient of the complex argument regression to a complex result equal to

,

the least-squares method (LSM) , applied to the inverse dependence - of the argument on the result - should give such estimates of the complex coefficient:



Only in this case the condition (1.7.3) will be met.

The same condition will be met if, when estimating the LSM on the set of values of complex random variables, the coefficient of the result dependence on the argument



will correspond to the inverse linear dependence coefficient equal to



that is, a vector opposite in the complex plane towards the first one.

Now we can understand in which case the real part of the complex coefficient of pair correlation will be zero, and its imaginary part modulo will be equal to one.

When finding the regression of a complex argument to a complex result



the proportionality coefficient found by the LSM will simulate some linear sequence  as close as possible to the original series of a random variable.

When using LSM to find on the same data the inverse dependence of the complex argument on the complex result:

,

LSM will give such a complex coefficient  that its use for regression



will simulate a series of points  rotated relative to the original series  by an angle of *π*, that is, changing in the opposite direction.

This is possible only in case if there is a complete absence of linear dependence and in general any dependence between two random complex variables, that is, when the two analyzed random complex variables are absolutely independent of each other.

Thus, we have shown what the values of the complex pair correlation coefficient mean:

- if the complex pair correlation coefficient is *±(1+i0)*, then this indicates the presence of a linear functional relationship between the variables;

- if the complex pair correlation coefficient is *±(0+i)*, then this indicates that there is complete independence of two complex random variables from each other.

However, what has been proved does not at all allow us to assert that the complex pair correlation coefficient lies in the range from ±(*0+i*) to ±(*1+i0*) and in this interval it characterizes a different degree of approximation to a linear complex-valued dependence. The second statement is generally true – the closer the calculated values of the complex pair correlation coefficient are to ±(*1+i0*), the closer the relationship between them is to a linear form. But the first statement about the limits of the change in the complex pair correlation coefficient is not true. The behavior of the complex pair correlation coefficient is much more complicated than of its real counterpart.

In order to understand how the complex pair correlation coefficient can change its values, we will reveal the numerator of this coefficient:

 . (1.7.4)

Let's write it down through covariances (after all, all the original variables are centered relative to their arithmetic averages). Then we get:

 . (1.7.5)

Why are covariances important to us? Because it is known from mathematical statistics that the covariance of two independent random variables is zero.

The denominator of the complex pair correlation coefficient can be written in terms of complex variances, the properties of which have already been well known to us (1.5.2). Then the complex pair correlation coefficient in such a more detailed and convenient for an analysis form will be written as follows:

. (1.7.6)

This very form of the correlation coefficient recording already makes it possible to judge substantively its properties.

When the variances of the imaginary and real parts of each of the original variables are equal to each other, that is:

, (1.7.7)

then we get for the denominator:

. (1.7.8)

If it still happens that the real and imaginary parts of complex variables are independent of each other, then the covariances in the radicand (1.7.8) will be equal to zero. It should be noted that for sample values they will be rarely equal to zero, just like (1.7.7) will be strictly equal to zero for sample values extremely rarely. In the overwhelming majority of cases, both (1.7.7) and (1.7.8) will be close to zero, but not strictly equal to it.

In the real and imaginary parts of the numerator (1.7.6) there are cross covariances, which in the sample case can be even less often equal to zero. Therefore, dividing the numerator by a value close to zero values will result in a situation where the complex coefficient of pair correlation can have high values of both real and imaginary parts, and these values can significantly exceed one in modulus.

Examples of such cases are given in Table 1.1

*Table 1.1.*

*Exceptional cases of the complex coefficient of pair correlation values*

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| № | Graph of the complex variable *X* | Variance of the complex variable *X* | Graph of the complex variable *Y* | Variance of the complex variable *Y* | Complex coefficient of pair correlation |
|  |  | *0-i15,0* |  | *45,1-i59,9* | *33651969-i11231561* |
|  |  | *0-i15,0* |  | *-8,6-i36,4* | *-5544718-i643532* |
|  |  | *-0,928+i4,0* |  | *-8,6-i36,4* | *-0,316-i0,259* |

In the real practice of econometric models’ construction, the situation in which such conditions develop is unlikely, if only because the real and imaginary parts of such random complex variables are dependent on each other - otherwise their formation in the economy is meaningless! Therefore, even if the variances of the real and imaginary parts of each of the considered complex random variables are equal to each other (an extremely rare case for sample values), the denominator (1.7.6) will still be far from zero.

Knowing this, it is possible to interpret the values of the real part of the complex coefficient of pair correlation in this way: there can be a linear correlation between the two complex random variables if the condition is met:

 (1.7.9)

In all other cases, the linear relationship will not be so close and, most likely, the researcher needs to look for another form of dependence between two random complex variables.

**1.8. Correlation function**

Correlation and autocorrelation functions are still actively used in science, including in economics. If there are two time-varying random variables *xt* and *yt*, then they can be interconnected with each other not only at this timepoint *t*. It is quite possible that this relationship has some lag in time and the variable *yt* depends not on *xt*, but on the same variable that was seen on *τ* observations earlier, that is, - *yt* depend*s* on *xt-τ*.

Most often, the lag value *τ* is unknown to the researcher and should be found. This, in particular, is what the correlation function is used for.

If we calculate the pair correlation coefficients *r* between *yt* and *xt*, after that between *yt* and *xt-1*, between *yt* and *xt-2*, etc., then the change in the pair correlation coefficient depending on the lag will be a correlation function in tabular discrete form.

If the researcher studies the correlation not between two variables, but studies the influence of the previous values of the variable on its current values, then he calculates the coefficients of pair correlation between *yt* and yt*-1*, between *yt* and *yt-2*, between *yt* and *yt-3*, etc. The change in the pair correlation coefficient from the time shift of the series is called an autocorrelation function.

It is considered that the lags between variables or autoregression lags are detected by such a backward shift *τ*, at which the modulus of the pair correlation coefficient modulo is not less than *0,8*. Since the pair correlation coefficient can be both positive and negative, it is argued that with a positive coefficient value, the past value of the indicator corresponding to this lag has a positive effect on the current value. In the case when the pair correlation coefficient for some lag is negative, it means the reverse effect of past values with such a lag on the current value of the variable.

We will not analyze the degree of correlation and autocorrelation functions suitability in economic forecasting tasks. We will draw attention to the fact that in the case of using complex economic variables, it also becomes possible to construct correlation and autocorrelation functions, but since the complex coefficient of paired correlation consists of real and imaginary parts, it is necessary to consider complex-valued correlation and autocorrelation functions.

The peculiarity of the complex-valued correlation (autocorrelation) function is that this function has real and imaginary parts. And each of them will represent a dependence on the time shift:

. (1.8.1)

It is convenient to study the correlation function by depicting it graphically - when time shifts are plotted along the abscissa axis, and the corresponding values of the pair correlation coefficients are plotted along the ordinate axis. Such a graphical model is called a "correlogram" or "autocorrelogram", if an autocorrelation function is considered. Fig. 1.6 shows an example graph of such a correlogram.

With regard to the complex correlation function, several options for constructing a correlogram are possible.

1

-1

*r*

*0*

*τ*

1

2

3

4

5

6

7

8

*Figure 1.6. An example of a correlogram for real variables*

The first option is a simple development of the correlogram of real variables, when the graphs of the real and imaginary parts are located on the same graph. In this case, a time shift is plotted along the abscissa axis, and the real Re(*rcXY*) and imaginary Im(*rcXY*) parts of the complex coefficient of pair correlation are located on this graph depending on the values they take for each lag.

In order not to invent the form of such a "parallel" correlogram, two series (2830 and 2831) from the database of the International Institute for Forecasters (*Makridakis, 2000*) were used. These two series of 103 observations each, were combined into a single complex variable. Then, for the resulting series of this complex variable, a complex autocorrelation function was calculated. The results of the calculations are shown in Table 1.2.

*Table 1.2.*

*Example of an autocorrelation function for a complex series composed of series 2830 and 2831*

|  |  |  |
| --- | --- | --- |
| Lag, *τ* | Re(*rcXY*) | Im(*rcXY*) |
| 1 | -0,757 | 0,178 |
| 2 | -0,554 | 0,286 |
| 3 | -0,480 | 0,267 |
| 4 | -0,415 | 0,277 |
| 5 | -0,377 | 0,274 |
| 6 | -0,356 | 0,257 |
| 7 | -0,345 | 0,242 |
| 8 | -0,335 | 0,198 |
| 9 | -0,332 | 0,196 |
| 10 | -0,319 | 0,195 |
| 11 | -0,288 | 0,206 |
| 12 | -0,222 | 0,193 |
| 13 | -0,180 | 0,201 |
| 14 | -0,139 | 0,212 |
| 15 | -0,142 | 0,193 |
| 16 | -0,199 | 0,148 |
| 17 | -0,251 | 0,120 |
| 18 | -0,260 | 0,120 |
| 19 | -0,269 | 0,114 |
| 20 | -0,253 | 0,131 |
| 21 | -0,210 | 0,160 |

Using the data in Table 1.2, it is possible to depict a complex autocorrelogram in such a parallel form (Fig. 1.7).

*Figure 1.7. The first type of autocorrelogram table 1.2*

The autocorrelogram in Figure 1.7 is similar to the autocorrelograms of real variables and its interpretation does not cause any particular difficulties. Here, the imaginary part of the complex correlation coefficient

Im(*rcXY*) is shown in red – it is positive over the entire calculated interval; therefore, it is located above zero, and the blue color shows the correlogram of the real part Re(*rcXY*). It is located below zero, because all of its values happened to be negative.

It is known from the previous paragraph that the real part of the complex coefficient of pair correlation characterizes the approximation degree of the complex-valued dependence to a linear form, and the imaginary part reflects the degree of dispersion around this line. It follows from the analysis of the given complex autocorrelogram, that the series under consideration can be described by the first-order autoregression, but good accuracy cannot be expected from this model, since the real part is less than *0.8*, and the imaginary part is far from zero.

Another form of a complex correlogram is also possible. Since each lag value corresponds to one value of the complex coefficient of pair correlation, consisting of the real and imaginary parts, the correlogram can be considered as the trajectory of a point on the phase plane, and the correlogram itself as a phase portrait of the structure of the series.

Fig.1.8 shows a correlogram on such a phase plane, the axes of which are the real and imaginary parts of the complex correlation coefficient (in this case, autocorrelation).

*Figure 1.8. Autocorrelogram of Table 1.2 as a phase portrait*

Here, the points on the complex plane of the correlogram are connected to each other not just by straight line segments, but by directed segments (arrows), the direction being maintained in strict order - from the first value of the complex coefficient of pair correlation to the last one, as it has been done in the figure.

This is a different autocorrelogram representation, the one which presents the process in a way different from the one in Figure 1.7. Here it is possible to obtain a variety of figures, the interpretation of which being a separate task. Phase portraits of complex autocorrelograms can have a very different shape and be located in different quadrants of the phase plane. For example, you can get a closed figure like a hysteresis loop, which will correspond to the cycle length if the process develops cyclically, for example, as shown in Fig.1.9.

*Figure 1.9. Possible phase portrait of the correlogram*

If we consider a parallel complex correlogram, then the presence of such a loop may not be noticed, as it follows from Fig. 1.10. It shows the same complex correlogram as in the phase plane in Fig. 1.9.

*Figure 1.10. The complex correlogram corresponding to the case in Fig. 9, but presented in a parallel form.*

Since the parallel form of the complex correlogram and its phase portrait give emphasis on different features of the correlation function, these two forms should be used in the analysis of the complex random series under study.

**1.9. Confidence limits of a complex random variable**

We consider the problems and tasks of constructing econometric models exclusively to the conditions of reversible processes - random and normally distributed. And this means that the researcher is dealing with sample values of random variables by which he judges the general population as a whole. Since sample values are being evaluated, it is necessary to determine how much these sample values can be trusted, that is, to assess how close they are to their true value, namely, to the mathematical expectation.

It is clear that if the researcher is faced with the task of studying a simple stationary process of a random real variable, the one which is represented by some sample from the general population, then, assuming the normal distribution of this variable *Yi*, its arithmetic mean should be calculated first:

, (1.9.1)

and, having calculating the variance of the actual observations` deviations from this mean *σ2*, it is possible to determine the interval in which the true value of *Y* is located:

. (1.9.2)

Here *tα is* the value of the Student's *t*-statistics.

As can be seen from (2.8.2), the confidence limits for real one-dimensional variables are a segment on the numerical axis, within which random variables can be located with a given probability.

If, instead of the real case, we consider a complex-valued variable, then the course of reasoning should not, at first glance, be violated – the arithmetic means for a complex random variable are calculated (which is identical to the arithmetic means calculation for the real and imaginary parts separately), variances are determined for them, after which, using the standard approach (1.9.2), the confidence limits are determined. Then the confidence limits of the two components values of a random complex variable should be defined as follows: (1.9.3)

 (1.9.3)

But the meaning of this method of determining the confidence limits of a complex random variable will be revealed if it is written as a system of two conditions for changing the confidence limits separately for the real and imaginary parts:

 (1.9.4)

It clearly follows from this that on the complex plane of the random complex variable, the confidence region will be a rectangle outlined by the sides determined by the confidence limits (1.9.4), and the sides of this rectangle being parallel to the axes of the complex plane. The center of this rectangle, and hence the confidence region in the form of a rectangle, will be a point on the complex plane determined by the coordinates of the arithmetic mean of the complex random variable .

As it has been shown in paragraph 1.3 of this monograph, the confidence region of a complex random variable should be a scattering cloud of acceptable values, and not a rectangle. In addition, this cloud should have the shape of an ellipse, the axes of which are parallel to the axes of the complex plane only if the real and imaginary parts of the complex variable do not depend on each other. And in the case of their dependence on each other, and this is the case that we are considering, the axes of the scattering ellipse will not be parallel to the axes of the complex plane (Fig. 3).

Thus, the procedure for finding confidence limits using the rule (1.9.3), which at first sight seems correct, turns out to be a very rough approximation to reality and it can be used only for the purpose of finding very approximate boundaries of the confidence region.

Therefore, the standard approach, which seems to be so obvious, turns out to be wrong. For scientific and practical research, it is necessary to use a confidence region, which represents an ellipse, inside of which there are those points that enter the confidence region, and outside of which there are points that go beyond the confidence region.

Let us use the scattering ellipse equation. With regard to our problem, the confidence region must be inside this ellipse, that is, the following condition must be met:

. (1.9.5)

Here *sα,n* is some number defining the limits of the confidence region. This number depends on the confidence probability level α and the number of degrees of freedom *n*

For sample values, when instead of mathematical expectations we know their estimate - the arithmetic means and sample variance values, the equation of the confidence region ellipse of the complex random variable will look like this:

 (1.9.6)

We failed to find an analytical relationship between *sα,n* and *tα*, and this will be the task of our further research study. But, since modern computing technology allows to perform numerous simulation experiments as well as computer tests, this approach makes it possible to find a tabular relationship between them. Table 1.4 shows the recommended values of *sα,n* depending on the of confidence probability level *α* and the number of degrees of freedom *n*, which were obtained during such machine experiments.

*Table 1.4.*

*Critical points of distribution sα,n*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Number of degrees of freedom *n* | Significance level *α* | | | |
| 0,10 | 0,05 | 0,02 | 0,01 |
| 1 | 19,908 | 80,645 | 506,256 | 2028,846 |
| 2 | 2,842 | 6,163 | 16,194 | 32,802 |
| 3 | 1,381 | 2,528 | 5,153 | 8,526 |
| 4 | 0,907 | 1,546 | 2,812 | 4,233 |
| 5 | 0,673 | 1,100 | 1,536 | 2,707 |
| 6 | 0,538 | 0,857 | 1,409 | 1,966 |
| 7 | 0,477 | 0,696 | 1,125 | 1,531 |
| 8 | 0,384 | 0,593 | 0,935 | 1,254 |
| 9 | 0,335 | 0,511 | 0,795 | 1,056 |
| 10 | 0,298 | 0,452 | 0,692 | 0,914 |
| 11 | 0,270 | 0,403 | 0,617 | 0,806 |
| 12 | 0,244 | 0,366 | 0,552 | 0,716 |
| 13 | 0,224 | 0,333 | 0,502 | 0,647 |
| 14 | 0,207 | 0,305 | 0,458 | 0,592 |
| 15 | 0,191 | 0,284 | 0,423 | 0,544 |
| 16 | 0,180 | 0,264 | 0,392 | 0,502 |
| 17 | 0,168 | 0,247 | 0,367 | 0,467 |
| 18 | 0,158 | 0,232 | 0,342 | 0,437 |
| 19 | 0,150 | 0,218 | 0,323 | 0,409 |
| 20 | 0,143 | 0,208 | 0,305 | 0,387 |
| 21 | 0,134 | 0,197 | 0,289 | 0,364 |
| 22 | 0,129 | 0,186 | 0,274 | 0,346 |
| 23 | 0,122 | 0,177 | 0,260 | 0,329 |
| 24 | 0,117 | 0,170 | 0,248 | 0,314 |
| 25 | 0,112 | 0,163 | 0,238 | 0,299 |
| 26 | 0,108 | 0,157 | 0,228 | 0,286 |
| 27 | 0,104 | 0,150 | 0,218 | 0,274 |
| 28 | 0,100 | 0,145 | 0,209 | 0,263 |
| 29 | 0,096 | 0,140 | 0,202 | 0,254 |
| 30 | 0,095 | 0,138 | 0,199 | 0,250 |
| 40 | 0,094 | 0,136 | 0,195 | 0,243 |
| 60 | 0,093 | 0,133 | 0,190 | 0,236 |
| 120 | 0,092 | 0,131 | 0,186 | 0,229 |

Let us show how to use this table and the conditions (1.9.6) on a specific example.

We have at our disposal data on the results of daily quotations on the world commodity exchanges of two commodities - *Brent* crude oil and natural gas from January 4,2010 to August 9,2013. Since these two products reflect the situation on the world market of organic fuel, they can be represented as one random complex variable:

. (1.9.7)

where *yrt* is the price for a barrel of *Brent* crude oil, and *yit* is the price of a cubic meter of natural gas.

Since the dimensions and scale of these variables are different, they must be reduced to the same dimension and to the same scale. The easiest way to do this is to divide each value of value series of the price for a barrel of oil by this indicator of the first observation value dated January the 4th, 2010, and each value of value series of the gas price to divide by the first value of the gas price dated January the 4th, 2010. Then dimensionless quantities comparable both to each other and in scale will be obtained.

For the obtained series of more than 900 observations, the arithmetic mean has been found, which is equal to:

. (1.9.8)

For the same series, the variances and their standard deviation (SD) have been calculated, which are equal to: *σr=0,00616* and *σi=0,00427*. The pair correlation coefficient between the real and imaginary parts of the complex random variable has also been calculated, which turned out to be equal to *r=-0,57001.* This, by the way, once again reiterates our conviction that such economic indicators, combined into one complex variable, in no case can be considered as independent of each other and the variance of such a complex random variable should be considered as a complex value.

Now it is possible to substitute these values into the condition (1.9.6) and to determine the region of confidence limits for the values of this series of complex random variable:

. (1.9.9)

Let us find the answer to such a question - whether the number (*1.40+i0.80*) falls into the confidence parameter region with a probability of 0.95?

To answer this question, the specified values of a complex random variable should be substituted in (1.9.9) and the value of the left side of the inequality should be calculated. Let us do this. As a result of calculations, the number s = 2831.366 was obtained. It is significantly higher than the critical value, the one which, as can be seen from Table 1.4, for more than 900 observations and the significance level *α=0,05* is equal to *sα,n=0,131*. Therefore, the specified number *(1,40+i0,80*) goes beyond the range of confidence parameter.

Now let us find the answer to another question – whether the number *(1,26+i0,64)* falls into the range of confidence parameters with the same probability of 0.95?

Substituting the values of this complex number in (1.9.6) and calculating the value of *s* (the left side of the inequality (1.9.6)), we obtain *s=0,018*. It can be seen that the calculated value of the left side of the inequality is less than the critical one and the inequality is satisfied:

*s* < *sα, n=0,131*.

Therefore, the number in question *(1,26+i0,64)* is inside the confidence region.

Thus, the proposed procedure for determining confidence limits for a complex random variable can be used for scientific and practical research in the field of complex-valued econometrics.

It can also be used to estimate the confidence limits of other sample complex variables, for example, complex coefficients of regression models.

We will not consider the issue of estimating the confidence limits for the complex coefficient of pair correlation here, realizing that the approach for estimating the confidence limits of a complex random variable, considered in these paragraphs, is universal, and it can also help to estimate the confidence limits of sample values of the complex coefficient of pair correlation.

**Part two. Complex - valued autoregressions**

In the first part of the monograph, those provisions of mathematical statistics that relate to complex random variables were formed and brought together. These tools are especially diverse when the assumption of the real and imaginary parts independence of this random variable from each other is not fulfilled. Then there is an opportunity to develop a consistent tool base for mathematical statistics of a complex random variable.

Now, being armed with this powerful tool of scientific research, let us turn our attention to the use of this apparatus for the formation of one particular section of a Complex-valued economics –to the development and practical use of complex-valued autoregression models that can be used both for the analysis of the processes under study and for short-term economic forecasting.

**2.1. Autoregressions in economic forecasting**

Historically, the logic behind the formation of autoregressive models was as follows.

The researchers faced two different tasks of economic forecasting:

1) forecasting the indicator for the future as a continuation of some trend over time or as some factor dependence;

2) forecasting of an indicator that does not tend to grow, but fluctuates under the influence of random factors in a certain range.

These two tasks differed in different methods and different forecasting time periods. The first problem was solved with the help of regression-correlation analysis and made it possible to carry out medium and long-term economic forecasts. The second task was solved by calculating different variants of the average value indicator and made it possible to carry out short-term economic forecasts. And today these two tasks represent two independent directions in economic forecasting.

In fact, the task of medium- and long-term economic forecasting is to identify, describe and predict trends and relationships. The task of short-term forecasting is to predict deviations from this general trend or relationships.

The object of our scientific research is precisely the second direction – short-term forecasting.

Since the processes initially studied and predicted in this case had no tendencies to increase or decrease, they began to be called "stationary".

The dominant point of view on how the "stationary process" is understood today is most clearly reflected by such a definition: "In practice, there are often random processes that occur approximately uniformly in time and have the form of continuous random fluctuations around a certain average value, and neither the average amplitude nor the nature of these fluctuations shows significant changes over time. Such random processes are called stationary" (*Wentzel, 2010, p. 479*). Most often, such processes have a normal probability distribution. And since fluctuations in the stationary process occur relative to a "certain average value “, then the natural model of short-term forecasting is this very average value.

The simplest is the arithmetic mean of the last *p* observations:

 (2.1.1)

Here *yt* are the values of a series varying in time *t*. These future values of the series are the object of forecasting. It is logical to assume that the next observed value of the considered series *yt+1* will be close to this arithmetic mean, that is, the arithmetic mean acts as a forecast for the next step of observation:

 (2.1.2)

The main problem faced by forecasters when using the arithmetic mean as a forecasting model was the choice of the averaging period *p*. This turned out to be especially important for small samples. When choosing different averaging periods, different arithmetic means are obtained, which predicted the series in different ways. Therefore, for different series, the researchers experimentally selected different periods of averaging *p*. And after choosing this averaging period and obtaining new values, the average (2.1.1) was recalculated taking into account the new values, but keeping the averaging period *p* unchanged. Since such an average "moved" along the series, it became known as the "moving average".

The next stage in the development of short-term forecasting models was the understanding that the weighted average can be better than the arithmetic mean:

 (2.1.3)

Here, the *bt* weights reflect the different contribution of observations to the predicted value of the indicator. An important requirement is that the sum of these weights is equal to one. Only then can we talk about the average. If this rule is violated, the amount will not be the average. Since it is a weighted moving average, it is denoted by *MA(p)*.

Weights *bτ* can be set in a variety of ways. Scientists used different procedures for this, including weighing forward, backward and in the middle of the averaging segment. By changing these averaging centers, one can get different options for describing the series and change the accuracy of its prediction. To smooth the graph and eliminate the influence of strong random deviations on the overall view of the series, averaging over the sample center can be used. And for economic forecasting, the models whose weights decreased with decreasing indicators in the past turned out to be more accurate, that is, when, in addition to (2.1.3), the following condition was also fulfilled:

 (2.1.4)

In the economic short-term forecasting moving weighted average models began to be successfully used. But in this case, there was one problem that the forecaster faced every time he faced the need to predict a new series – how to choose weights in the moving average (2.1.3) so as to get the best forecast in terms of accuracy? After all, one can set weights as he likes! It was important for scientists to set the weights in such a way that it would become universal. This was the method for calculating the weights of observations as members of a series of an infinite geometric progression (*Brown, 1956; Holt, 1957*):

 (2.1.5)

There is no need to think about what values each weight takes – they are calculated using this formula. Therefore, it is enough just to set the value *α* and all the weights *bτ* are easily calculated.

It is important to set the smoothing constant *α*, so that the sum of the terms of the series (2.1.4) calculated with the help of (2.1.5) is equal to one.

For an infinite geometric progression, this means that the condition must be met:

 (2.1.6)

Hence, the boundaries in which the values of the smoothing constant should lie are clear (Svetunkov,1997):

 (2.1.7)

If we substitute in (2.1.3) the weights of *bt*, which are calculated using rule (2.1.5) under condition (2.1.7), and assume that the series of observations is infinite, then the mean will take the following form:

 (2.1.8)

This forecasting model, called exponential smoothing, turned out to be convenient for calculations due to its simplicity and due to the fact, that by changing the values of the smoothing constant in the range (2.1.7), one can find such a value *α*, at which the model will give the smallest error in short-term economic forecasting.

Historically, forecasters began to use its truncated band instead of the band of values (2.1.7):

 (2.1.9)

It was not clear to them what meaning the model acquires beyond these limits. But if the model works within the limits (2.1.9), then its meaning is clear - the model is adaptive to the current information to varying degrees, and the smoothing constant *α* characterizes the degree of this adaptation - at α =0 the model is absolutely not adaptive to the current information, and at *α* =1 - it fully considers the current information, ignoring the past information.

Quite often, when finding the optimal value of the smoothing constant *α*, scientists and practicing economists were convinced that this optimal value is equal to one. This means that

 (2.1.10)

That is– the current value of the indicator turned out to be the best predictive value. Forecasters began to call this situation the *NAÏVE* model.

In fact, there are almost no situations when the optimal value of the smoothing constant is equal to one (*α=1*). It is just that in such cases the smoothing constant goes beyond (2.1.9) and its optimal value lies in the range

 (2.1.11)

In this range, the exponential smoothing model acquires special properties. In order to understand them, we will present under the conditions (2.1.9) the model (2.1.8) in this form:

 (2.1.12)

It can be seen from this form of the model representation that it works like this: the forecast value is calculated as the previous calculated value of the indicator, adjusted for the current error *εt*. The degree of this correction is determined by the value of the smoothing constant *α*. If it is equal to one, the *NAÏVE* model is obtained

In the infinite set of the smoothing constants (2.1.11), when the smoothing constant is greater than one, the model behaves differently:

 (2.1.13)

That is– not the calculated, but the actual value of the predicted indicator is adjusted for the current forecast error *εt*. The model acquires the property of self-learning.

The success of the exponential smoothing model in the short-term economic forecasting was somewhat obscured by the fact that the sum of the weights (2.1.5) will be equal to one only if the number of observations, and, therefore, the number of weighting coefficients is equal to infinity. That is, strictly speaking, the model (2.1.8) is not an average for small and medium samples. But with a large number of observations, the sum of the series (2.1.5) differs from one by such small values that this disadvantage can be neglected.

The fact that a stationary series can be successfully predicted using a model in which the sum of the weighting coefficients is not equal to one and this model is not some form of average has been known for quite a long time. As far back as 1907, A.A. Markov initiated the formation of an extensive class of stochastic processes with a discrete time component, which were named after him. Markov`s processes describe the following probabilistic state of the process depending on the current state. These were the first autoregressive models that have undergone significant development since then. In order to avoid difficulties with indices, the calculated and predicted value of the indicator is referred to the current time *t* and written as follows:

 (2.1.14)

This model is usually denoted as *AR(p)*, where *p* is the autoregression order.

It is easy to notice that in the case when *p→∞*, and the coefficients *aτ* take the values of the terms of the geometric progression series (2.1.5), the *AR(p)* model turns into an exponential smoothing model. Consequently, the autoregressive model (2.1.14) is a general model of short-term forecasting, the frequent cases of which are both the exponential smoothing model and the *NAÏVE* model.

The actual values of the predicted indicator differ from the calculated values by some error:

 (2.1.15)

It follows from this that this error, in turn, can be considered as a certain time series that does not tend to increase or decrease its indicators, but is related to the predicted indicator. Therefore, a series of *yt* values can be predicted, for example, using the moving average of this error:

 (2.1.16)

The order of averaging *q* of this moving average does not necessarily have to coincide with the order of autoregression *p*, since approximation errors behave differently than the simulated series. This model of the moving average approximation error became known as *MA(q)*. If we remove the restrictions on the equality of the sum of coefficients in the *MA(q)* model to one, then the moving average will cease to be an average, but will become an autoregression. And autoregression is a more general and more accurate forecasting model than a moving average model. Today, a non-sliding weighted average (2.1.16) is used, namely, the autoregression of the error, when the sum of the coefficients *bτ* is not equal to one. But until now, autoregression, in which the errors *εt*, are the factors, continue to be called the *MA(q)* model.

The *MA(q)* model will give different predictive values of the *yt*, indicator than the model (2.1.14). In order to take advantage of each model –*AR(p)* and *MA(q*) - they were combined into one *ARMA (p, q)* model:

 (2.1.17)

A lot of educational and scientific literature is devoted to the study of the properties of these models. It is important for us that this literature shows the relationship between the coefficients *aτ* and *bτ*, as well as the interdependence of the order *p* from the order *q*. Using this property, a procedure for detecting lags and evaluating the coefficients of the model (2.1.17) was developed, which is called the "Box-Jenkins` methodology"*.*

The main difficulty in practical application of the *ARMA(p,q)* model is to determine the autoregression order *p* and the order *q*. In simple cases, studies of autocorrelation and partial autocorrelation functions can be useful. But in cases where the orders of p and *q* in the original series are greater than 3, such studies are ineffective.

Today there are several generally accepted methods of constructing models (2.1.17), which can be divided into two groups:

1) simultaneous evaluation of the coefficients *aτ* and *bτ* of the model (2.1.17), when, gradually increasing the orders of the model *p* and q and calculating one of the information criteria, the best model is selected according to this criterion;

2) the lag of the *AR(p)* model is estimated and the coefficients *aτ* are calculated, after which the errors *εt* are calculated and the *MA(q)* model is constructed on their basis. Then, based on the *MA(q)* model, the coefficients of the *AR(p)* model are adjusted. Since the *AR(p)* model obtained at the second step will not always correspond to the original model, a multi-iterative procedure for "debugging" the model is carried out – this is the practical implementation of the Box-Jenkins methodology.

In practice, not every economic process can be classified as stationary.

Therefore, the *ARMA (p, q)* model cannot always be used directly for economic forecasting. In such non-stationary cases, researchers transform the original series of values so that the resulting series does not increase or decrease. Most often, this is achieved by calculating the finite differences of the original series. The number of this finite difference is denoted as *d* and it is included in the name of the model, which, taking into account such transformations, is called *ARIMA (p, d, q)*.

The basic *ARIMA (p, d, q)*. model today is the basis for the further development of short-term forecasting models that are used to solve various particular problems. Exogenous variables are added to this model and it is denoted as *ARIMAX (p, d, q, b)*, autoregression is made nonlinear and denoted as *NARMA (p, q)*, the seasonality factor is included in the model and this model is called *SARMA (p, q)* etc.

**2.2. Vector autoregressions**

No economic indicator develops autonomously. Its dynamics is influenced by many different factors. There are also cases in economy when some indicators change, influencing each other. Such a joint interdependent dynamics was described for some time by a system of simultaneous equations, and later it began to be replaced by vector regressions.

In the case when the vector of indicators at time *t* is determined by the values of the same vector at previous points in time, it is appropriate to say that such processes are described by vector autoregressions.

Vector autoregression of order *p*, denoted as *VAR(p)*, can be presented in this form (*Lütkepohl, 2005, p.13*):

. (2.2.1)

Here *Yt* is a *k*-dimensional vector of variables;

*A0* - *k*-dimensional vector of coefficients;

*Aτ* – *k* х *k* - dimensional constant real matrices.

In the two-dimensional case, *VAR (1)* will take the form:

. (2.2.2)

*VAR (1)* in the three-dimensional case will look like this:

. (2.2.3)

*VAR (1)* for the four-dimensional case will be:

. (2.2.4)

It is easy to notice that the number of coefficients to be estimated for first-order vector autoregression is equal to *k2*, where *k* is the size of the vector used. And in general, the number of coefficients to be estimated for *VAR(p)* will be equal to (*k2*∙*p*). For example, for *VAR (3)* at *k=5*, it is necessary to estimate (52 3=75) unknown coefficients. That is, it is necessary to solve a system of 75 linear equations with 75 unknown quantities. It is obvious that the solution of such problems in economic practice is exotic today.

If we compare the one-dimensional autoregression model and the vector autoregression model with each other, we can see that the complexity of the autoregression models *AR(p)* is determined by the lag order *p*, and the complexity of the vector autoregression is determined primarily by the size of the vector *k*, and only then by its lag *p*. The theory of vector autoregressions has been developed in general, but these autoregressions have not found wide application in practice, since it is necessary to evaluate a large number of coefficients and in publications devoted to the construction of *VAR(p)* in practice, vectors with a dimension higher than four and a lag greater than two are rarely occurred. Therefore, the dimension of the vector is not indicated anywhere in the model notation, and it is the dimension of the vector, as can be seen from the previous arguments, that has a decisive influence on the model complexity. That is why, when using one or another model of vector autoregression, it is mandatory to indicate the dimension of its vector. Based on these considerations, hereinafter, referring to vector autoregressions, the dimension of the vector in the model designation will be indicated with the symbol *k.* Consequently, by the abbreviation *VAR k(p)* we will denote the autoregression of the order *p* of a *k*-dimensional vector, immediately imagining that when using this model in practice it will be necessary to estimate (*k2*∙*p*) coefficients.

**2.3. Complex-valued autoregression as a competitor to the VAR k (p)**

**model**

When modeling many economic processes, the use of models and methods of the theory of functions of a complex variable turns out to be no worse, and in some cases better, than models of real variables. For example, when modeling production processes, the production functions of complex variables describe these processes in more detail, and in some cases demonstrate greater accuracy in forecasting than the production functions of real variables (*Svetunkov, 2012*).

This gave rise to a desire to test the possibility of using complex-valued economics with regard to the economic forecasting problems, in particular, in relation to the tasks of short-term economic forecasting using autoregression models

In general, the complex autoregression model can be written as follows:

 (2.3.1)

Here *y1t* and *y2t are* real variables predicted at point of time *t*;

*i* is an imaginary unit,*;*

*F* is some complex-valued function;

*τ* is the autoregression lag;

*p* is the autoregression order;

*ɛ1t* and *ɛ2t* are approximation errors of the first and second variables at point of time *t.*

Depending on the type of complex-valued function *F*, complex autoregressions (2.3.1) can be either linear or nonlinear. Nonlinear autoregressive models of real variables are not often found both in practical application and in theoretical research. Therefore, in our study, we will focus our attention on linear autoregressions, and from now on we will understand complex autoregressions to be linear forms of the model (2.3.1), and will denote these models as *CAR(p)*.

Thus, the complex-valued *CAR(p)* autoregressive models under consideration will be generally represented in this form:

 (2.3.2)

where *b0* and *b1* are coefficients (free terms) reflecting the initial value of the complex series;

*a0τ* and *a0τ* are coefficients of proportionality

Usually, when presenting autoregression models, free terms are omitted, since they can be eliminated by centering the original variables with respect to their arithmetic means. Therefore, we will further assume that the coefficients *b0* and *b1* of the *CAR(p)* model are equal to zero.

Then the complex autoregression of the first order *CAR (1)* can be represented either like this:



or in a vector form:

. (2.3.3)

While comparing (2.3.3) with (2.3.2), we can make sure that we are dealing with a special case of 2-dimensional vector autoregression *VAR 2(1)*. At the same time, for the practical application of the *VAR2(1)* model, it is necessary to estimate four unknown coefficients, and for the practical application of the *CAR (1)* model, only two coefficients are needed. This means that the *VAR2(1)* model, with the help of two additional coefficients, will take into account some nuances, and therefore it will describe the initial process more accurately than complex autoregression.

Perhaps this will result in a greater prognostic accuracy of this model compared to the *CAR (1*) model. But the undeniable advantage of the *CAR (1)* model is that in order to apply it in practice, it is necessary to estimate only two coefficients, not four.

Can *CAR(p)* be applied in case when k-dimensional vectors of order greater than three are used? There is such a possibility. For example, for *k=3* and *p*=1, the following *CAR3(1)* model should be used:

 (2.3.4)

Or in a vector form:

. (2.3.5)

In this case, it is necessary to estimate seven unknown coefficients, and in the 3-dimensional model *VAR3(1)* (2.3.3) it is necessary to estimate nine unknown coefficients. The efforts saving in estimating the coefficients is obvious.

This saving is even more obvious if we compare the 4-dimensional vector *VAR4(1)* (2.3.4) and the *CAR4(1)* model. In the classical vector autoregression *VAR4(1)* it is necessary to estimate 16 unknown coefficients. And the *CAR4(1)* model will be written like this:

 (2.3.6)

Whence it can be seen that in order to use this model in practice, it is necessary to estimate 8 coefficients, and not 16 coefficients. This is two times less than that of the model *VAR4(1)*.

If we reduce the *СARk(p)* model to a vector form and compare it with the vector autoregression model, then we can determine the number of coefficients of the *VARk(1)* and *СARk(1)* models of different dimensions *k* and understand their difference. This has been done in Table 2.1.

*Table 2.1.*

*Coefficients of VARk(1) and CARk(1) models for different dimensions of vector k*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *k* | *VARk(p)* | *СARk(p)* | *Number of coefficients* | |
| *VARk(p)* | *СARk(p)* |
| *2* |  |  | *4* | *2* |
| *3* |  |  | *9* | *7* |
| *4* |  |  | *16* | *8* |
| *5* |  |  | *25* | *17* |
| *6* |  |  | *36* | *18* |
| *7* |  |  | *49* | *31* |

The table shows the coefficients of the two models for vectors with dimension up to *k=7*. It makes no sense to continue this table, since the pattern is already obvious – to build a model of complex vector autoregression, it is always necessary to estimate fewer coefficients than for the classical model of vector autoregression. And it is already clear that, for example, to use the *VAR8(1)* model, at *k=8*, it will be necessary to estimate 64 unknown coefficients, while for the *CAR8(1)* model, only 32 unknown coefficients. For the vector autoregressions with an even vector dimension, to construct a complex autoregression, it is necessary to estimate two times less coefficients than for classical vector autoregression. As far as information criteria are used to select complex models, this circumstance can be decisive when choosing a model, since in these criteria the number of model coefficients significantly affects the value of the criterion. Therefore, it can be concluded from Table 2.1 that the two models under consideration - *VARk(p)* and *CARk(p)*- can really compete with each other only if two- or three-dimensional vector autoregressions are used. In these cases, an increase in the number of coefficients of the *VARk(p)* model will be compensated by a decrease in the variance of the approximation error or forecasting error, and the information criterion can make a choice in favor of the *VARk(p)* model. But in all other cases, one should expect that the choice will fall on the *CARk (p)* model.

Let us demonstrate this statement with a concrete example.

We have at our disposal 118 data on exchange rates on the Moscow Interbank Currency Exchange from 20.09.2018 to 16.03.2019. This period of time has been chosen deliberately, because during this period the behavior of currencies can well be attributed to a stationary process. We will use the following exchange rates in Russian rubles:

1) US dollar, *y1t*;

2) euro, *y2t*;

3) pound sterling, *y3t*;

4) Swiss franc, *y4t*.

We will sequentially build models VAR*2(1)* and *CAR (1)* for the first two variables (*y1t*, *y2t*), *VAR3(1)* and *CAR3(1)* for the first three variables

(*y1t*, *y2t*, *y3t*) and *VAR4(1)* and *CAR4(1)* for the four-dimensional vector (*y1t*, *y2t*, *y3t*, *y4t*).

Let us start with a simple two-dimensional case. By estimating the coefficients of the *VAR2(1)* and *CAR2(1)* models on these data using the least squares method, it is possible to obtain the models that, as expected, describe the original data differently. The results of the available data approximation by these models are summarized in Table 2.2.

In this table and in all subsequent tables of this paragraph, the indices *i* and *j* indicate the currency numbers in this order: 1 - US dollar, 2 - euro, 3 - pound sterling, 4 - Swiss franc. Therefore, for example, at the intersection of the line denoted by the number 1 and the column denoted by the number 2, the value of the coefficient *a12* = 0.026 of two-dimensional vector autoregression is given. This coefficient shows the influence of the euro exchange rate at the preceding moment on the dollar exchange rate at the present moment.

The last column of the table shows the variances of approximation error for each model as a whole and for each currency type individually. So, the approximation error of the *VAR2(1)* model was 0.3695 - this is shown in the first line of the table, and the approximation error of the dollar rate by this model turned out to be 0.1661.

The first part of the table shows data on the *VAR2(1)* model, and the second part shows data on the *CAR (1)* model.

*Table 2.2.*

*Comparative results of currency exchange rate modeling by VAR2(1) and CAR (1) models*

|  |  |  |  |
| --- | --- | --- | --- |
| *i*  *j* | Coefficients | | σ2 |
| 1 | 2 |
| **Model *VAR****2****(1)*** | | | ***0,3695*** |
| 1 | 0,9968 | 0,0026 | 0,1661 |
| 2 | 0,1409 | 0,8761 | 0,2034 |
| Model ***СAR(1)*** | | | ***0,3774*** |
| 1 | 0,9996 | –0,0001 | 0,1660 |
| 2 | 0,0001 | 0,9996 | 0,2114 |

A simple comparison of the results shows that, as expected, the *VAR2(1)* model turned out to be generally more accurate than the *CAR (1)* model, but insignificantly: the total variance of vector autoregression was 0.3695, while the total variance of complex autoregression turned out to be higher - 0.3774, which is only 2.15% worse.

The variance of approximation error of the dollar rate using the *VAR2(1)* model is almost equal to the variance of approximation error of the dollar rate when using *CAR (1)*. The difference between them was only one ten thousandth: 0.1661-0.1660. And the complex autoregression model turned out to be more accurate.

The euro exchange rate model *VAR2(1)* described more accurately: the variance of the approximation error is 0.2034, while for the *CAR (1)* model, the variance of approximation error of euro rate turned out to be 0.2114, which is 5% more.

Since information criteria are used to select the best model as a compromise between the desire to choose the most accurate model in the approximation and, at the same time, to choose a simpler model, Bayesian information criteria were calculated:

 (2.3.7)

Here *N is the* number of observations.

The calculated values of the criteria differ from each other

- *BICVAR2(1) =-0,834;*

*- BICCAR (1) =-0,894.*

The information criteria turned out to be negative, since the variances of approximation errors are less than one and the logarithms of these variances become negative: for the first model, this logarithm is equal to

(-0.9957), and for the second model - (-0.9744). Since four coefficients are used in the *VAR2(1)* model, and two coefficients are used in the *CAR (1)* model, the information criterion for complex autoregression turned out to be less and the preference should be given to it, since the increase in the accuracy of the model with its simultaneous complication turned out to be not so significant as to sacrifice the simplicity of the model.

The results for a three-dimensional vector, *k=3*., turned out to be even more significant for the complex autoregression. They are presented in table 2.3.

*Table 2.3.*

*Comparative results of currency exchange rate modeling by VAR3(1) and CAR3(1) models*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| i  j | Coefficients | | | σ2 |
| 1 | 2 | 3 |
| Model ***VAR3(1)*** | | | | ***0,7456*** |
| 1 | 1,0331 | -0,0091 | -0,0177 | 0,1654 |
| 2 | 0,1521 | 0,8968 | -0,0269 | 0,2023 |
| 3 | 0,2111 | -0,1065 | 0,9307 | 0,3779 |
| Model ***СAR3(1)*** | | | | ***0,7545*** |
| 1 | 1,0149 | 0,0083 | -0,0044 | 0,1657 |
| 2 | -0,0083 | 1,0149 | -0,0199 | 0,2110 |
| 3 | 0,2098 | -0,1020 | 0,9277 | 0,3779 |

For the three-dimensional case, the first two variables *y1t* and *y2t* are better described by the *VAR3(1)* model, and the third variable is described equally well by both the *VAR3(1)* model and the С*AR3(1)* model. And in general, the variance of vector autoregression 0.7456 is less than the variances of complex-valued autoregression 0.7545. This difference in approximation accuracy is 1.2%.

But for *VAR3(1)*, the information criterion turned out to be equal to *BIC=0,0703*, and for *CAR3(1)* it was significantly less, namely, *BIC=0,0014.*

That is, according to the information criterion, the model *CAR3(1)* is more preferable than the *VAR3(1)* model and it is the model of complex vector autoregression that the researcher will choose the without hesitation.

In the four-dimensional vector autoregression *VAR4(1)*, the estimates require 16 unknown coefficients, and in the complex–valued autoregression *СAR4(1)* - only 8 unknown coefficients. All necessary calculations were performed and the coefficients of these two models were found. The values of these coefficients and the variances of approximation errors by these two models of the initial series are given in Table. 2.4.

*Table 2.4.*

*Comparative results of currency exchange rate modeling by VAR4(1) and CAR4(1) models*

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| i  j | Coefficients | | | | σ2 |
| 1 | 2 | 3 | 4 |
| Model ***VAR4(1)*** | | | | | ***0,9023*** |
| 1 | 1,0717 | 0,2750 | -0,0627 | -0,3030 | 0,1574 |
| 2 | 0,1701 | 0,9501 | -0,0365 | -0,0660 | 0,2010 |
| 3 | 0,2531 | 0,1768 | 0,8820 | -0,3007 | 0,3713 |
| 4 | 0,1990 | 0,2474 | -0,0842 | 0,6291 | 0,1727 |
| Model **С*AR4(1)*** | | | | | ***0,9497*** |
| 1 | 1,0114 | 0,0254 | -0,0063 | -0,0253 | 0,1669 |
| 2 | -0,0254 | 1,0114 | 0,0253 | -0,0063 | 0,2092 |
| 3 | 0,1009 | 0,0302 | 0,9132 | -0,0453 | 0,3869 |
| 4 | -0,0302 | 0,1009 | 0,0453 | 0,9132 | 0,1866 |

And again, we can make sure that if we use the criterion of the minimum variance of approximation error for to selecting the best forecasting model, then the model *VAR4(1)* is slightly more accurate over all positions than the *СAR4(1)* model. And, in general, the variance of approximation error of the *VAR4(1)* model is less than the approximation error of the *СAR4(1)* model by 5.3%, which seems to be significant. But the information criterion for vector autoregression is *BIC=0,5440*, and for the complex-valued autoregression it is two times less - *BIC=0,2718*. So, if we use the information criterion to choose the best model, then it will unambiguously indicate that the complication of the model with a simultaneous increase in the number of estimated coefficients does not make sense. Complex-valued autoregression should be used, which describes the vector of changing variables maybe a little worse, but it is much simpler and therefore it should be expected to be more stable in the results of short-term forecasting and more accurate.

Interesting results were obtained by Yevgeny Goltsev, who, at my request, built two first-order vector autoregressions with a dimension *k=8*, namely, *VAR8(1)* and *CAR8(1)* using the example of the Moscow Exchange economic conditions indices. 8 industry indices are distinguished on it**[[1]](#footnote-1)**:

1) consumer sector *y1t*;

2) chemistry and petrochemistry *y2t*;

3) finance *y3t*;

4) power industry *y4t*;

5) metals and mining *y5t*;

6) oil and gas *y6t*;

7) telecommunications *y7t*;

8) transport *y8t*.

According to the series of changes in these indices from 01.05.2016 to 20.09.2020, the coefficients of these two models were evaluated - *VAR8(1)* (64 unknown coefficients) and *CAR8(1)* (32 unknown coefficients)

The model *VAR8(1)* looks like this:

 (2.3.8)

It describes the basic data with a standard deviation *σ=430, 27.* The information criterion for it is BIC*=13,65.*

The model of complex vector autoregression CAR*8(1)* has other coefficients:

 (2.3.9)

And it describes the basic data somewhat worse. It has a standard deviation equal to *σ=441,94*. But due to the fact that the latter model estimates twice as few coefficients, its information criterion is less than that of the *VAR8(1)* model, and is *BIC=12,94*.

Therefore, for short-term forecasting purposes, a preference should be given to the complex autoregression model *CAR8(1)*.

Based on the data from 09/27/2020 to 04/25/2021, Yevgeny Goltsev with the help of these models made retro forecasts for one-step of observations and compared the prognostic values with the actual ones. The results of this comparison are shown in Table 2.5.

*Table 2.5.*

*Results of the retro forecast of the economic conditions indices of the models (2.3.8) and (2.3.9)*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| № | Industry Index | Standard error of retro forecast, σ | | Improved accuracy % |
| *VAR8(1)* (2.3.8) | *СAR8(1)* (2.3.9) |
|  | Oil and gas | 331,26 | 267,52 | 21,29 |
|  | Finance | 344,25 | 307,14 | 11,39 |
|  | Metal and mining | 321,45 | 295,21 | 8,51 |
|  | Power industry | 89,63 | 54,09 | 49,46 |
|  | Consumer | 177,91 | 164,85 | 7,62 |
|  | Chemistry and petrochemistry | 331,38 | 372,46 | **-11,67** |
|  | Transport | 45,16 | 36,96 | 19,98 |
|  | Telecom | 64,20 | 36,42 | 55,23 |

Only for one index, namely, the Chemistry and Petrochemistry index of Russia, the retro forecast, with the help of complex autoregression, turned out to be worse than the forecast performed by using vector autoregression. All other indices of the Moscow Exchange are predicted better using the *СAR8(1)* (2.3.9) model, and in the case of Power Industry and Telecom indices, the forecasting accuracy is even twice as high.

We took the simplest and most accessible data as an example. And we found that to predict this series of values, BIC always recommends to use complex-valued autoregression, rather than vector autoregression. This means that in most practical cases complex-valued autoregression will be preferable to vector autoregression. And the greater the dimension of the vectors and the higher the order of autoregression, the lower the value will be taken by the BIC criterion for this model in comparison with vector autoregression.

This means that the complex-valued autoregression model is a competing vector autoregression model, and a significant reduction in the complexity of new models compared to vector autoregression models allows them to be widely used in economic practice.

**2.4. CARMAk(p,q) in economic forecasting**

We have seen that the *CARk(p)* model is a good alternative to the vector autoregression models – they are simpler, contain fewer coefficients that need to be estimated, and they do not significantly lose in approximation accuracy to the *VARk(p)* models. That is why the information criterion for the models considered in the previous paragraph recommends using a complex autoregression model, rather than a vector one.

Earlier it was shown how the *AR(p)* autoregression model was transformed into the ARMA *(p, q)* model. The same logic was used by scientists to form a more complex *VARMAk (p,q)* model based on the vector autoregression model:

. (2.4.1)

Here *U* is the vector of approximation errors which has the same dimension *k* as the vector of indicators *Y*. In the general case, the number *q* of preceding vectors of approximation error *U* is not equal to the number *p* of preceding vectors of predicted indicators *Y*.

At the same time, as in the case of *ARMA(p,q)*, a relationship is assumed between *VARk(p)* and *MAk(q)* (*Lütkepohl, 2005, p. 436*).

In the two-dimensional case, the model *VARMA2(1,1)*, taking into account the notations we have adopted, will be written as follows (*Lütkepohl, 2005, p. 443*):

 (2.4.2)

Concerning the errors of the *εtt* vector U, they are considered to be "white noise" with zero mathematical expectation. In order to use this model in practice, it is necessary to estimate its eight coefficients: 4 coefficients of the component *VAR2(1)* and 4 coefficients of the second component *MA2(1)*.

Considering the vector complex autoregression model as a special case of vector autoregression, one can obtain a similar (2.4.1) *CARMA(p,q)* model. We will not consider the option when the component *MA(q)* is represented as a simple *k*-dimensional vector. We will also consider this component in a complex form. Then, for the two-dimensional case, *CARMA(1,1)* will be written as:

 (2.4.3)

Or, in a complex form:

 (2.4.4)

Here, as can be seen, it is necessary to estimate the values of only 4 unknown coefficients, and not 8 coefficients, as is required for the practical application of the vector autoregression model (2.4.2).

In order to estimate the practical acceptability of the *CARMA(p,q)* model, let us consider a simple case of constructing the *VARMA4(2,1),* model, which will be written as follows:



It is easy to see that for the practical use of this model, it is necessary to estimate 48 unknown coefficients from statistical data. This is not an easy task for practicing economists, although the four-dimensional case of the economic vector seems to be a fairly simple reflection of real economic situations. Therefore, despite the simplicity of the model, few practicing forecasters will dare to build such a model and use it in practice.

But a similar to it model С*ARMA4(2,1)*, in which not only the variables are presented in a complex form, but the error vector is also presented in a complex form, will be written as follows:



In order to use this model in the practice of short-term economic forecasting, it is necessary to estimate only 24 coefficients - two times less.

And this task can be solved by almost any economic forecaster.

The number of coefficients of the *VARMAk(p,q)* model, which should be estimated on the available set of values, depends both on the orders *p* and *q* of the model (2.4.1) and on the dimension *k* of the vector of variables, and it is the dimension of the vector *k* that plays a decisive role in increasing the complexity of the model. The total number of coefficients of such a model can be described by the formula: *k2∙(p+q)*. The number of coefficients of the *СARMAk(p,q)* model will always be less than that of the *VARMAk(p,q)* model. And for an even number of vector dimension *k*, the number of these coefficients will always be two times less and equal to *k2∙(p+q)*/2. For example, 192 unknown coefficients should be estimated for the *VARMA4(3,4)* model, and 86 coefficients for the *CARMA4(3,4*) model.

This means that the higher the dimension of the vector in a vector autoregression, the less chance the vector autoregression has of being better compared to a complex autoregression of the same order, since the information criteria will always recommend models with fewer coefficients number.

Let us define, for example, the condition under which the forecaster is hesitant about which model to prefer - *VARMA4(2,1)* or *CARMA4(2,1)*?

[These models have been given above in full form. The situation of choice is possible only if the information criteria calculated for each model turn out to be equal to each other, that is, it is fulfilled:](https://www.bing.com/translator" \t "_blank)

 (2.4.5)

Hence, we have for the variances:

 (2.4.6)

In the previous paragraph, 118 observations were used for calculations. Let us substitute this number *N=118* в (2.4.6). Then we get that the variance of *CARMA4(2,1)* should be 2.64 times greater than the variance of *VARMA4(2,1)*, that is, the variance of the approximation error of the complex autoregression *CARMA4(2,1)* should exceed the variance of the approximation error of the *VARMA4(2,1)* model by 264%! Obviously, such a situation is unlikely to be encountered in practice.

We draw an unambiguous conclusion from this: *CARMAk(p,q)* models will almost always be preferable to *VARMAk(p,q)* models at *k>3.* But even in the case when *k*=2, the complex autoregression has a good chance of becoming a better model than the vector autoregression model, which is demonstrated by the example in Table 2.2, when simple *VAR2(1)* and *CAR(1)* models were considered without the *MA(q)*. component.

**2.5. Basic properties of complex-valued autoregression**

We will not refer further to vector autoregressions and their variety - complex-valued vector autoregressions. We have shown that the use of complex variables and complex models opens up wide opportunities for practical use in short-term economic forecasting of vector autoregressions

And that is enough. And now we should pay attention to the properties of the complex autoregression model *CAR(p).*

The first order autoregression model of a complex variable can be recorded as follows:

 (2.5.1)

It is relevant to remind here that the first order autoregression model of a real variable is recorded as follows:

. (2.5.2)

And finally, it represents the following function:

. (2.5.3)

The dynamic of variation of this exponential function in time is fully determined by the base coefficient of the function a.

It is easy to show (*Svetunkov, 2012*) that similarly the complex-value auto regression model (2.5.1) can be represented as a power complex-valued function:

 (2.5.4)

It is known that the power complex-value function is periodical and diverges in the form of a spiral with the exponent’s increase if the modulus of the base is higher than one, and converges along the spiral to zero if the complex-value modulus in the power base is lower than one.

The stationarity requirement for such a model is that the model coefficients are within the following limits:

 (2.5.5)

But this is how the theoretical model behaves itself. In practice, a complex approximation error will play a significant role in the behavior of the model – after all, when modeling real processes, the right part of the model (2.5.1) is substituted not with the calculated value, but with the actual value, contaminated by random errors. Therefore, the trajectories of the modeled processes can have a wide variety of forms, just as the most diverse trajectories are modeled by Markov processes, which are first-order autoregression of real variables (2.5.2).

Understanding how *CAR (1)* autoregression will behave itself in the absence of random errors, let u's pay attention to one of its modifications:

. (2.5.6)

In this modification, at each step of the time *t* change, the real and imaginary parts change places. More precisely, each variable refers to either the real or the imaginary part. Indeed, at time *t* , the first variable *y1t* refers to the real part, and the second variable *y2t* refers to the imaginary part: (*y1t+iy2t)*. At the previous moment of time it was vice versa, namely, the second variable *y2t* belonged to the real part, and the first variable *y1t* was in the imaginary part, that is, a complex variable of the following type was used: *(y2,t-1+iy1,t-1)/* In its turn, this variable was preceded by a complex variable, where again the first real variable *y1t*, belonged to the real part, and the second real variable *y2t* belonged to the imaginary part: *(y1, t-2+iy2,t-2)*, etc.

In order to understand how this model will behave in comparison with the original model (2.5.1), we will present both models in exponential form.

The standard model of complex autoregression (2.5.1) in exponential form is written as follows:

 (2.5.7)

Здесь:

, (2.5.8)

, (2.5.9)

, (2.5.10)

. (2.5.11)

(2.5.7) provides an understanding of how the underlying *CAR(p)* model behaves. Since the polar angle of the model changes by the value *φa* at each step *t*:

 (2.5.12),

then, when the polar angle of the coefficients *φa* is positive, the polar angle of the modeled numbers increases and the spiral, along which the numbers are modeled, turns counterclockwise. If, in this case, the modulus of the proportionality coefficient (2.5.9) is greater than one, then the spiral unwinds, and if it is less than one, then the spiral twists.

In the case when the polar angle of the coefficients *φa* is negative, the spiral unwinds (or twists, depending on the value of the coefficient modulus (2.5.9) in the opposite direction clockwise, since with each step the polar angle of the modeled complex indicator decreases by *φa*.

The model (2.5.6) is fundamentally different. And this follows from the exponential form of its notation

 (2.5.13)

where  (2.5.14)

Since the equality holds for polar angles,

, (2.5.15),

it can be noticed that:

 (2.5.16)

If we consider the time behavior of a complex variable on a complex plane at *φa*=0, then the model (2.5.6) describes an original oscillatory process when the polar angle *φt* will take only two values: *φ0* and *φ1*.

That is, we are faced with a model that behaves in an unconventional way. Therefore, it makes sense to study it in more detail.

Let us conduct a study for the simple case when

y*1t+iy2t* = *1+i*, (2.5.17)

and the modulus of the complex coefficient is equal to one, for example, with such coefficients:

*a0+ia1* = *0,7071+i0,7071.* (2.5.18)

In this case, the model generates only two points:

|  |  |
| --- | --- |
| *yr* | *yi* |
| 1,000 | 1,000 |
| 0,000 | 1,414 |
| 1,000 | 1,000 |
| 0,000 | 1,414 |

But as soon as we move from the calculated values to values clogged with a random component varying from -0.1 to +0.1 (that is, up to 10% of the initial level), this model begins to generate a variety of nonlinear and cyclical growth trajectories. By analogy with Markov processes, these processes can be called random wandering processes on the complex plane. The trajectories generated by this process are determined by random errors and may look, for example, as shown in Fig. 2.1.

*Figure 2.1. Random wandering process trajectory for the model (2.5.6) at y1t+iy2t = 1+i и a0+ia1 = 0,7071+i0,7071*

Here, the random wandering process begins at the point with coordinates (1,1) and rushes upward to the right in the first quadrant of the complex plane of variables.

For comparison, Figure 2.2 shows the process of random wandering on the complex plane for the basic autoregression model (2.5.1) with the same initial parameters.

*Figure 2.2. Random wandering process trajectory for the model (2.5.1) at y1t+iy2t = 1+i и a0+ia1 = 0,7071+i0,7071*

The difference between the two models (2.5.1) and (2.5.6) is huge – the first model generates the trajectory of nonlinear oscillatory growth, the second model - oscillates around a circle. The direction of nonlinear cyclical growth, which is generated by the model (2.5.6), is determined by the signs at complex coefficients of proportionality, that is, by the polar angle of the complex proportionality coefficient.

In the case when the modulus of the complex proportionality coefficient of the model (2.5.6) is greater than one, the model (2.5.6) generates an increasing process with an increasing range of oscillations, for example, as shown in Fig. 2.3.

*Figure 2.3. The trajectory of the model (2.5.6) at y1t+iy2t = 1+i and the modulus of the complex coefficient is greater than one - a0+ia1= 0,5500+i0,8930*

If, under the same conditions, the modulus of the complex proportionality coefficient turns out to be less than one, for example, when *a0+ia1 = -0,5-i0,8*, then the model generates damped oscillations around a certain trajectory (Fig. 2.4)

*Figure 2.4. Trajectory of the model (2.5.6) at y1t+iy2t = 1+i and the modulus of the complex coefficient is less than one - a0+ia1 =-0,5-i0,8*

If we simulate the process with such initial values for the standard complex autoregression, a process approaching the zero point along a nonlinear spiral will be obtained (Fig. 2.5).

*Figure2.5. The trajectory of the model (2.5.1) at y1t+iy2t = 1+i and the modulus of the complex coefficient is less than one - a0+ia1 =-0,5-i0,8*

Fig. 2.5 shows that the model starts at the point with coordinates (1,1), and then consistently approaches a zero point. The model (2.5.6) under the same conditions, also starts from a point with coordinates (1,1), and the oscillation range of this model also decreases with an increase in the number of iterations. But the point to which this process tends is not zero, and for the example under consideration it is located in the second quadrant of the complex plane. Depending on the signs of the real and imaginary parts of the complex proportionality coefficient, this point will be located in different quadrants of the complex plane.

Since the model (2.5.6) generates quite different, but necessarily oscillatory trajectories, sometimes with unpredictable ranges of these oscillations, then this model, to distinguish it from other models, we will call the "drunkard model" and we will denote it for short as DCAR*(p)*.

Concluding the paragraph, it should be pointed out that the model (2.5.6) is also a special case of vector autoregression, since in a vector form it can be represented as follows:

. (2.5.19)

Using the original complex-valued autoregression and its variant (2.5.6) together, it is possible to obtain a modification of the *CAR(2)* model with the possibility of modeling other economic processes: increasing, decreasing, oscillatory, linear, etc.

This model will take this form:

. (2.5.20)

**2.6. Varieties of one-dimensional complex autoregressions**

Previously integrated model autoregressions were presented in general form as follows:

 (2.6.1)

Here *y1t* and *y2t* are the real variables predicted at time *t*;

*i* - is an imaginary unit;

*F* - some complex-valued function;

*τ -* an autoregression lag;

*p* - autoregression order;

*ɛ1t* and *ɛ2t* - approximation errors of the first and second variables at time *t*;

In our study, we will not refer to nonlinear models, but emphasize the linear autoregressions, which are denoted as *CAR(p)*:

 (2.6.2)

where *b0* and *b1* are coefficients (free terms), reflecting the initial value of a complex series;

*a0τ* and *a0τ* are proportionality coefficients.

In relation to autoregressive models, it is assumed that the free term can be taken equal to zero. Hereafter we will assume that the coefficients *b0* and *b1* are equal to zero.

Model (2.6.2) is a special case of a vector autoregression for a two-dimensional vector and simulates two variables and their mutual influence.

Is it possible to transform this model to the one-dimensional case when only one variable is modeled and predicted?

There is such a possibility.

Complex variables in economics are formed so that these two variables reflected different sides of the same economic object or phenomenon, for example, labor and capital, which reflect the cost of resources for production. Consequently, in order to use the model (2.6.2) to predict the indicator *y1t*, it is necessary to match it with some accompanying variable y*2t*, so that it is an additional characteristic of the predicted variable.

What are the options for setting the *y2t* variable as an additional characteristic of the *y1t* variable? Several variants can be proposed here, depending on the modeled and predicted variable *y1t*, but in general three basic forms may be acceptable:

1) complex autoregression with error, when y*2t=ɛt*. The error *ɛt* characterizes the features of the dynamics of the series under consideration and the degree of its oscillation, therefore it may well serve as an additional characteristic of the modeled series. With this in mind, the complex variable of the model (2.6.2) will be written as follows: (*yt*+*iɛt*), and the model itself will take this form:

 (2.6.3)

Models of this kind will be denoted as *CARE(p)*

2) time complex autoregression, when *y2t=t*. Here, time acts not only as the series ordering index, but also as its active component. In this case the complex variable takes the form (*yt*+*it*), and the model will be like this:

 (2.6.4)

This model will be denoted as *CTAR(p)*;

3) complex autoregression with increment, when *y2t=*Δ*y1t*.The increment can be equal to zero, it can be positive or negative. It characterizes the trend of changes in the indicator at the last moment of observation and can act as its additional characteristic. The complex autoregression variable (2.6.2) will be written as follows: (*yt*+*i*Δ*yt*). In this case, the model should be written in this form:

 (2.6.5)

Such models of the genus will be denoted as *GCAR(p)*.

The properties of each model, as well as the limits of their application in practice, we will consider in the following paragraphs.

**2.7. Autoregression model with CARE(p)**

The *CARE(p)* model is a complex variable that includes in the real part the predicted indicator *yt*, and the approximation errors *εt*, attributed to its imaginary part. The very idea of such a complex variable was suggested as far back as in 2010 (*Svetunkov I., 2011*), when the complex-valued form of the exponential smoothing model was presented:

 (2.7.1)

Researches have shown that this complex-valued exponential smoothing model provides more accurate economic forecasts compared to exponential smoothing models of real variables. This is explained by the fact that the forecast value of the indicator takes into account not only the previous values of the variable itself, but also current and past approximation errors of a model. This conclusion was confirmed in 2015 (*Svetunkov, Kourentzes, 2015*).

The complex variable in autoregression can be represented:

1) in a simple form as (*yt*+*iεt*), where *εt* is the current approximation error;

2) in the form when not the current approximation error is used, but the previous error: (*yt*+*iε(t-g)*), where *g* – is some error lag. Such model representation assumes that the model learns from the past errors, not from the current ones;

3) in a form that is nonlinear with respect to the approximation error: (*yt*+*if(εt*)), where *f(εt)* – can take the forms specified by the researcher.

In the first case, linear with respect to the approximation error, the *CARE(p)* model will have the following form:

. (2.7.2)

The *CARE(p)* *model* is a complex-valued equality. It can be represented as a system of two equalities of real variables – a separate equality for the real component of the left and right parts (2.7.2) and a separate equality for the imaginary component of the left and right parts (2.7.2). We will get:

. (2.7.3)

Then, for the real part of the complex-valued model (2.7.2) we have:

. (2.7.4)

And for the imaginary part of this model:

. (2.7.5)

Thus, in economic forecasting, in addition to the general model *CARE(p)* (2.7.2), two more independent models that differ from each other can be used: Re*CARE(p)* and Im*CARE(p)* (*Svetunkov, Complex-valued autoregression, 2020*).

The practical application of the model (2.7.4) is obvious - it can be considered as a special case of the well-known *ARMA(p,q)* autoregression model. The *ARMA(p,q)* model has the following form:

. (2.7.6)

It can be noted that the models (2.7.4) and (2.7.6) will coincide in the case when *p=q*. The sign before the second sum of the right parts of these models (in the model (2.7.5) is negative, and in the model (2.7.6) - positive) is determined automatically when evaluating the coefficients of the models, therefore it is not a fundamental difference between the models from each other.

But an important difference between the Re*CARE(p)* model and the *ARMA(p,p)* model is that in (2.7.4) both terms are parts of a single whole and are evaluated simultaneously. And in the model (2.7.6), a relationship between the first part of *AR(p)* and the second part of *MA(q)* is assumed (*Box, Jenkins, 2015*), as a result of which the coefficients *a0τ* determine the values ​of the coefficients *a1τ* and vice versa. This predetermines the need to use an iterative method for finding the coefficients of the model (2.7.6).

One of the difficulties of practical use of *ARMA(p,q)* is that there is still no generally accepted satisfactory solution to the problem of determining the lags of *p* and *q*. To do this, autocorrelation and partial autocorrelation functions for the variables are used. The general principle of constructing the *ARMA(p,q)* model is as follows. Since the variables *εt* are unobservable, the coefficients of the component *AR(p)* are first evaluated. With the help of this model, the residuals of *εt* are calculated. Their analysis allows us to determine the value of the lag *q*. And since the researcher has both the values of the variable *yt*, and the values of the variable *εt*, at his disposal, it makes possible to evaluate all the coefficients of the model *ARMA(p,q)* (2.7.6).The model describes the initial data *yt* with new approximation errors *εt*. Using the original data on the variables *yt* and the new approximation errors ε*t,* the model coefficients are re-evaluated and so goes on until the residuals *εt* cease to change with some predetermined accuracy. In this procedure, many problems result precisely from the definition of lags *p* and *q.*

However, with the development of computational capabilities of modern computers, another procedure is increasingly used in practice, namely, the procedure for automatic evaluation of *ARMA(p,q)* coefficients. For this purpose, by increasing sequentially the lags *p* and *q*, coefficients are calculated for each model (2.7.6), after that, based on the variance of the approximation error and the number of coefficients, an information criterion (*AIC* or *BIC*) is calculated and the best model with the appropriate lags *p* and *q*, is selected by the criterion of the minimum of this information criterion. In this case, the Re*CARE(p)* model, which is chosen in this way, can be considered as a special case of the *ARMA(p,q)* model.

Any model describes a real series with some error:

. (2.7.7)

And the task of estimating the coefficients of this model is most often reduced to minimizing the sum of the squares of this error. For the ReCARE*(p)* model, this error is equal to

. (2.7.8)

The sum of squares (2.7.8) can be minimized using any numerical methods. Therefore, the problem is easily solved.

In the Im*CARE(p)* (2.7.5) model, some estimate of the predicted approximation error is calculated, which is described by the error *μt*:

. (2.7.9)

And in this case, the problem is reduced to a simple solution: the sum of the squares of this remainder should be minimized:

. (2.7.10)

At the same time, we are faced with such a situation:

- if we minimize the sum of squares (2.7.8), then the errors (2.7.9) will be very large;

- if we minimize the sum of squares (2.7.10), then the approximation errors of the original series (2.7.7) will be large.

Since it is the *yt* series that is predicted, preference should be given to Re*CARE(p)* model. The characteristics calculated using the Im*CARE(p)* model can serve as some additional characteristic of the predicted process, but its meaning is not yet obvious, and we will postpone the study of the Im*CARE(p)* model for the future.

The original *CARE(p)* model itself describes the initial series of values with a complex approximation error:

. (2.7.11)

Using a complex LCM for this task, we will obtain such a system of normal equations, the solution of which will allow us to estimate the coefficients of the *CARE(p)* model (*Svetunkov, 2012*):

 (2.7.12)

Since the errors of *εt* are not observed, it is possible to solve this system only with the help of numerical methods. But, in our opinion, there is no need in finding the coefficients of the model in this way – it is quite enough to use the Re*CARE(p)* model.

It is easy to notice that the Re*CARE(p)* model turns into an ordinary autoregression model *AR(p)* if the coefficient *a1τ* becomes equal to zero.

In all other cases, the *ReCARE(p)* model will differ from *AR(p)* and give more accurate forecasts, since it also takes into account previous errors.

Nikolai Pitukhin, for example, compared the accuracy of the retro-forecast of these two models, by changing the order of autoregression pfrom one to ten on the example of a decreasing series of excavators produced in Russia from 1990 to 2018. The results of his calculations are given in Table 2.5. The standard deviation (SD) of the retro-forecast error by one step was chosen as the criterion for the accuracy of the retro-forecast.

*Table 2.5.*

*Comparison of the retro-forecast accuracy of the ReCARE(p) and AR(p)models*

|  |  |  |  |
| --- | --- | --- | --- |
| Autoregression order*, p* | SD for the *AR(p)* model | SD for the *ReCARE(p)*model | *%* of SD reduction |
| 1 | 1021,8 | 703,9 | 31,1 |
| 2 | 693,6 | 562,9 | 18,9 |
| 3 | 608,0 | 510,8 | 16,0 |
| 4 | 582,6 | 524,3 | 10,0 |
| 5 | 595,0 | 444,5 | 25,3 |
| 6 | 564,4 | 367,2 | 34,9 |
| 7 | 569,6 | 399,6 | 29,8 |
| 8 | 511,0 | 312,0 | 39,0 |
| 9 | 512,2 | 204,5 | 60,1 |
| 10 | 515,7 | 357,5 | 30,7 |

As expected, *ReCARE(p)* model has always proved to be more accurate than the *AR(p)* model. The same results were obtained for other data series.

Yulia Selivanova, Galina Siruk and Nazira Shaikhleeva performed calculations comparing the accuracy of the *ReCARE(p)* *and* *AR(p)* retro-forecasts with each other on the database of the International Institute of Forecasters. For this purpose, 60 different dynamic series of different lengths and different indicators were used. Only in 18.2% of cases, the *ReCARE(p)* and *AR(p)* models showed the same accuracy in forecasting.

For all these cases *a1τ ≈0*. But in all other 82% of cases, this coefficient is different from zero, and the more modulo it is greater than zero, the more accurate the Re*CARE(p)* model is compared to *AR(p)*. In 3 cases out of 60, the improvement in the forecast accuracy of the Re*CARE(p)* model compared to *AR(p)* was over 65%.

**2.8. The model with a ReCARE(p,g) shift**

The complex variable in *CARE(p)* can be represented in the form when not the current approximation error is used, but the previous error: (*yt*+*iε(t-g)*), where *g* is some error lag. This representation of the model assumes that the model learns from the past errors rather than the current ones. For such a model we get:

. (2.8.1)

This model will be hereafter denoted as *CARE(p,g)*.

In the case when equality *g=0* is satisfied for the Re*CARE(p,g)* model, it turns into the Re*CARE(p)* model. It follows from this that the Re*CARE(p,g)* model is the most common in this class of models.

Unlike the Re*CARE(p)* model, it is assumed here that the current value of the predicted indicator is influenced by the approximation errors shifted by *g* observations backwards, but the number of value pairs of the indicator *yt* and the error *εt* is equal to each other and is *p* values. That is, in such a model the elements of the cyclical development influence of the series (seasonality) are laid down, which are met quite often. This seasonality is reflected by the repeatability of the influence of the *εt-g* error on the current value of the variable *yt*.

This complex-valued model has real and imaginary parts. The real part of the model (2.8.1) will take the form:

, (2.8.2)

and the imaginary part of the model is as follows:

. (2.8.3)

It is difficult to imagine the practical application of the imaginary part (2.8.3) of this model, but the real part of the model (2.8.2) can be successfully used in practice of short-term economic forecasting

At any *p* and g values of the *ARMA(p,q)* model, it does not become the *ReCARE(p,g).*model. This is the original autoregression model. The model (2.8.2) will coincide with the ARMA*(p,q)* model, only when the double equality is satisfied: *g=0* and *p=q*. In all other cases, the Re*CARE(p,g)* and *ARMA(p,q)* models differ from each other.

Therefore, it makes sense to compare the behavior of the model (2.8.2) and the *ARMA(p,q)* model. And if the new model shows no worse approximation properties or prognostic properties than the *ARMA(p,q)* model, then it can be recommended for short-term economic forecasting purposes.

For the model (2.8.2) Re*CARE(p,g)*, regardless of the shift of *g*, it is necessary to find *2\*p* unknown coefficients, since pairs of complex variables are considered in (2.8.1). The *ARMA(p,q)* model contains *p+q* unknown coefficients. So, in the general case, the best model for the forecasting series, Re*CARE(p,g)*, may have a different number of coefficients than the best forecasting model for the same series *ARMA(p,q)*.

Let us first compare the approximation abilities of the models Re*CARE(p,g)* and *ARMA(p,q)*. We will use for this series № 2830, selected at random from the database of the International Institute of Forecasters. In order for the models to have the same starting conditions, *ɛ1*=0 was adopted for all models. Consistently increasing the order of *p*, and for *ARMA(p,q)* models also *q*, the values of the *AIC* and *BIC* criteria were calculated and the models with minimal values of these criteria were selected. For the ReCARE*(p,g)* model optimization was also carried out for the *g* shift.

The results of calculations are given in Table 2.7. From the set of models, the characteristics of the two best models from the *ARMA (p, q)* class and the two best models from the Re*CARE (p, g)* class are given here.

*Table 2.7.*

*Comparative accuracy of ARMA (p, q) and ReCARE (p, g) models*

|  |  |  |  |
| --- | --- | --- | --- |
| Model type | σ | AIC | BIC |
| Models *ARMA(p,q)* | | | |
| *ARIMA (4,4)* | 100,815 | **9,410** | 9,648 |
| *ARIMA (5,4)* | **100,276** | 9,687 | **9,422** |
| Models Re*CARE (p,g)* | | | |
| Re*CARE (4,2)* | 93,382 | 9,261 | **9,501** |
| Re*CARE (6,3)* | **88,455** | **9,245** | 9,599 |

Let us first consider the first part of the table, which shows the characteristics of the two best models from the *ARMA(p,q)* class. It can be seen that the *ARMA (4,4)* model is the best in its class according to the value of the AIC information criterion. And the model *ARMA (5,4)* is the best on the Bayesian information criterion BIC. Since the standard deviation of the approximation error of this model is less than that of the model *ARMA (4,4)*, then the *ARMA (5,4)* model should be preferred for approximation.

The second part of the table contains characteristics of the two best models in the Re*CARE(p,g)* family - Re*CARE(4,2)* and Re*CARE(6,3)*.

The Re*CARE (4,2)* model is the best according to the BIC information criterion, which is known to be more sensitive to the increase in the number of coefficients. The Re*CARE (4,2)* model contains 4\*2=8 unknown coefficients, and the ReCARE *(6,3)* model contains 6\*2=12 unknown coefficients.

The Akaike Information Criterion (AIC) gives preference to another model - ReCARE *(6,3)*, which, moreover, turned out to be the best among all the models under consideration by the value of the standard deviation (SD) of the approximation error.

If we now compare with each other the best models in each class in terms of approximation accuracy, the Re*CARE (6,3)* model will approximate the original series by 13,3 % more accurately than the *ARMA (5,4)* model. And this is a very serious increase in the accuracy of approximation. Note also that in the RE*CARE (6,3)* model 12 coefficients were estimated, and the in *ARMA (5,4)* model 9 coefficients were estimated. The complication of the model is obvious.

In the example under consideration, the models of the new class demonstrate their preference over the models of the *ARMA(p,q)* class.

The accuracy of the approximation error is an important indicator that testifies in favor of the new model of short-term forecasting, but, as it is known from economic practice, by no means always the best model in approximation is the best one in forecasting. And we are interested in the possibility of using the new model for the purposes of short-term forecasting in economy. That is why, the models of the *ARMA(p,q)* family should be compared with the models of the Re*CARE(p, g)* family in terms of retro-forecasting accuracy, and the information criteria values will be needed only as some additional characteristic of the models.

We will use the same base series of 104 observations. Now let us divide it into two parts – the training and the verification ones. The first 85 observations of the training set of values will be used to calculate the coefficients of the models of the two compared types - *ARMA(p,q)* and Re*CARE(p,g)*. And the last 19 observations of the series will be used as a test set for each of these models. On these latter values, short-term forecasts for one step will be performed using the models estimated on the training set: first on the 86th observation, then on the 87th observation, etc. until the last 104th observation.

The predicted values were compared with the real values, and the retro-forecast error was calculated. Just as in the case of comparing the approximation accuracy of *ARMA(p,q)* and Re*CARE(p,g)* models, we will choose two best models of each type that show the smallest variance of the retro-forecast error on the check set. These results are shown in Table 2.8.

Table 2.8.

*Comparative accuracy of retro-forecast of ARMA(p,q) and ReCARE(p,g) models*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Model type | Results of approximation on the training set | | | SD of retro-forecast error on the check set |
| σ | AIC | BIC |
| Models *ARMA(p,q)* | | | | |
| *ARMA (5,1)* | 121,913 | 8,867 | 9,060 | 117,275 |
| *ARMA (3,1)* | 125,071 | 9,117 | 9,284 | 118,661 |
| Models Re*CARE (p,g)* | | | | |
| Re*CARE (5,4)* | 95,1161 | 8,512 | 8,379 | **111,619** |
| Re*CARE (1,3)* | 126,530 | 9,068 | 9,152 | 116,275 |

It can be seen from the table 2.8 that the best-in-class model, Re*CARE (5,4)*, predicts 19 recent observations with a minimal retro-forecast error, since for it the standard deviation is *σ =111,619*. At the same time, the best *ARMA (5,1)* model in the compared class of *ARMA(p,q)* models gives a higher SD of the retro-forecast error, namely *σ =117,275*. This is 5% more than the Re*CARE (5,4)* model. And a five percent increase in forecast accuracy is a significant value.

It is also interesting that the Re*CARE (5,4)* model turned out to be the best in approximating the past 85 values – its SD is 24% less than that of the best model in the *ARMA(p,q)* class. This is such a serious value that all information criteria give preference to the Re*CARE (5,4)* model, despite the fact that for its use it is necessary to estimate 5\*2=10 unknown coefficients, while the best *ARMA (5,1)* model has only 5+1=6 unknown coefficients.

It is interesting, that for the entire series considered as an example, consisting of 104 observations, the optimal shift *g* turned out to be equal to 3 (Table 2.7), and from the position of retro-forecasting on the last 19 observations of this series, the optimal error shift *g* turned out to be equal to 4 (Table 2.8). As you can see, they are close to each other and this means that the series itself is characterized by some delay in the effect of the error on the current result. Since we do not know what kind of series is represented in the database under the number 2830, we cannot give an economic interpretation to this phenomenon.

So, we see that both from the standpoint of the best approximation of the original series, and from the position of retro-forecasting, the new Re*CARE(p,g)* model turned out to be better than the well-known model *ARMA(p,q)*. Of course, it does not follow from this that the models of this class will always be more accurate than *ARMA(p,q)* models. For a given randomly selected series, it turned out to be better, and for another series it could be worse.

But now we can already conclude that the ReCARE*(p,g)* model can occupy the same place in the tools of short-term economic forecasting as the class of *ARMA(p,q)* models, acting in some cases as a more successful alternative to the models of this class.

**2.9. Models with nonlinear errors**

The *ARMA(p,q)* model has many subspecies and subclasses. One of them is *NARMA(p,q)* - nonlinear autoregressive models. In the models of this subclass, the variable models are nonlinear – exogenous or endogenous - depending on the form of the model.

Our approach assumes the possibility of constructing not only similar nonlinear *NCARE (p,g)* models, but also a new subclass of models in which the nonlinear effect is exerted by the error ε*t*. Since in the original *CARE(p)* model we consider this error as an additional characteristic of the simulated process, this characteristic can take different forms, including nonlinear ones, that is, the original complex variable will be represented as follows: (*y1t*+*if(εt*)). Substituting this variable into the *CARE(p)*, model, we get:

. (2.9.1)

To distinguish this model from other models of the family under consideration, let us name it the *CARNE(p)* model.

And again, we should consider the complex-valued form of the recording (2.9.1) as a more compact form of two equalities system recording - the real and imaginary parts of this model:

 (2.9.2)

The real part of the model is of obvious interest, namely - Re*CARNE(p)*:

. (2.9.3)

Consider the properties of this model.

Since approximation errors have positive, negative and zero values, not every function *f(εt)* can be used in this model, for example, a logarithmic function cannot be used, since the domain of the function definition is the set of all positive numbers.

The simplest representation in the model (2.9.1), and hence in the model (2.9.3), the function from the approximation error is the power function:

. (2.9.4)

Then the model (2.9.3) will take the form:

. (2.9.5)

If in the model (2.9.5) the exponent is α*=1*, then we have the *CARE(p)* model, and in all other cases, models other than it will be obtained. Therefore, the *CARNE(p)* model and its real part (2.9.5) are more general models that include the *CARE(p)* model and its real part Re*CARE(p)*.

The exponent *α* distinguishes the new model and, depending on its values, the properties of the model change.

In the case when the forecaster examines a stationary process, the approximation error *εt* becomes comparable in scale with the values of the simulated series *yt*. Therefore, using as an exponent *α* a series of natural numbers following one n=2, 3, ... results in a sharp increase in the calculated values and the model goes "out of order". This is caused by the nature of autoregression. For this reason, this kind of the task of a model exponent has to be abandoned.

Another way of setting the exponent of the model, when the degree is fractional, can be used, for example:

. (2.9.6)

Here *n* - are the natural series numbers.

One should Immediately note that for even values of the natural series we have a peculiar feature. Since approximation errors (or, as they are sometimes otherwise called, residuals) can be both positive and negative, in the case of even values of n, real (with non-negative errors of *εt)* and imaginary numbers (with negative errors of *εt*) will be generated. The model becomes unusual. We will consider this feature a little later. And now we will pay attention to the model

. (2.9.7)

which has *n=3,5,7,…*

In this model, the approximation error is taken into account to a lesser extent than in the Re*CARE(p)* model, since the root of the *1/n-*th power is calculated from the approximation error, after which this value is substituted into the model. That is, the influence of the approximation error on the simulated result is significantly reduced, and the model becomes less sensitive to it. Sometimes it will be an advantage, sometimes it will be a disadvantage.

Let us compare the of Re*CARE(p)* and Re*CARNE(p)* models for different cases of setting the exponent in (2.9.7) on a practical example. We will use another set of data to compare these models. Earlier, we used series № 2830, and now we use series № 2831 following it in the database. It also has 104 values, as well as the series № 2830, so we do not lose anything from the standpoint of representativeness. The results of the calculations are given in Table 2.9.

Table 2.9.

*Comparative approximation accuracy of the series № 2831 by the ReCARE(p) and ReCARNE(p) models*

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Lag *p* | *Model* Re*CARE(p)* | | *Model* ReCARNE*(p)* (2.9.7) with *1/n* equal to: | | | | | | | |
| 1/3 | | 1/5 | | 1/7 | | 1/9 | |
| σ | BIC | σ | BIC | σ | BIC | σ | BIC | σ | BIC |
| 1 | **693,279** | **13,220** | 784,130 | 13,466 | 792,950 | 13,489 | 801,724 | 13,511 | 792,694 | 13,488 |
| 2 | 725,047 | 13,311 | 766,168 | 13,421 | 766,079 | 13,421 | 764,766 | 13,417 | 764,473 | 13,417 |
| 3 | 721,417 | 13,302 | 733,496 | 13,335 | 720,295 | 13,299 | 730,633 | 13,327 | 734,859 | 13,339 |
| 4 | 722,288 | 13,305 | **712,012** | **13,277** | **706,471** | **13,261** | 715,797 | 13,287 | 719,579 | 13,298 |
| 5 | 724,495 | 13,312 | 743,862 | 13,365 | 714,215 | 13,284 | **703,718** | **13,254** | **699,469** | **13,242** |
| 6 | 727,046 | 13,321 | 722,055 | 13,307 | 721,837 | 13,306 | 727,947 | 13,323 | 732,073 | 13,334 |

Once again, I would like to draw your attention to the fact that at *α* =1, the model Re*CARNE(p)* turns into the model Re*CARE(p)*.

In the table, the best values of indicators for each of the models are highlighted in bold. It can be seen that the Re*CARE (1)* model turned out to be more accurate than other models in the original series approximation, and from the standpoint of the information criterion, it turned out to be the best. However, it can be noted that the model Re*CARNE(5)* is already quite competitive with the model Re*CARE(1).* Moreover, as follows from the results of table 2.4.1, with the growth of *n*, the accuracy of the model also increases (2.9.7). This gives grounds to assume that for the considered series with the growth of *n*, it is possible to construct such a model that will be more accurate than the Re*CARE (1)*. model. And this hypothesis was corroborated, which follows from the data in Table 2.10:

Table 2.10.

*Comparative accuracy of approximation series No. 2831 by the models ReCARE(p) and*

*ReCARNE(p)*

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *Lag, p* | *Model ReCARE(p)* | | *Model ReCARNE(p) (2.9.6) at 1/n equal to:* | | | |
| 1/15 | | 1/17 | |
| σ | BIC | σ | BIC | σ | BIC |
| 1 | **693,279** | **13,220** | 794,627 | 13,493 | 794,895 | 13,494 |
| 2 | 725,047 | 13,311 | 764,194 | 13,416 | 764,672 | 13,417 |
| 3 | 721,417 | 13,302 | 734,135 | 13,337 | 735,058 | 13,339 |
| 4 | 722,288 | 13,305 | 727,184 | 13,319 | 728,339 | 13,322 |
| 5 | 724,495 | 13,312 | **691,711** | **13,219** | **691,820** | **13,220** |
| 6 | 727,046 | 13,321 | 735,895 | 13,345 | 736,197 | 13,346 |

Thus, for the data series under consideration the optimal model both from the position of approximation error and from the position of the information criterion was the model of following type:



It is well known that far from always when the best model from the approximation point of view will perfectly predict the series described.

Therefore, when analyzing the possibility of using a new model with nonlinear errors, it is necessary to check how much it can be used in forecasting.

To verify this possibility, we will use the same series № 2831, which consists of 104 observations. Again, we will divide it into two parts - a training and a testing ones. Based on the first 72 observations of the training set of values, the coefficients of the models of the two types being compared - ARMA*(p,q)* and Re*CARE(p,g)*. - will be calculated. And the last 32 observations of the series will be used as a test set for the retro-forecast accuracy for each of these models. On these last values, short-term forecasts for one step will be performed using the models estimated on the training set of models: first on the 73rd observation, then on the 74th observation, etc. until the last 104th observation. Then, the predicted values are subtracted from the actual values of the test set and the SD errors of the retro-forecast on the test set *σf* are calculated. To understand the properties of the model, the SD of the approximation error σ on the training set was calculated.

The results of these calculations are recorded in Table 2.11.

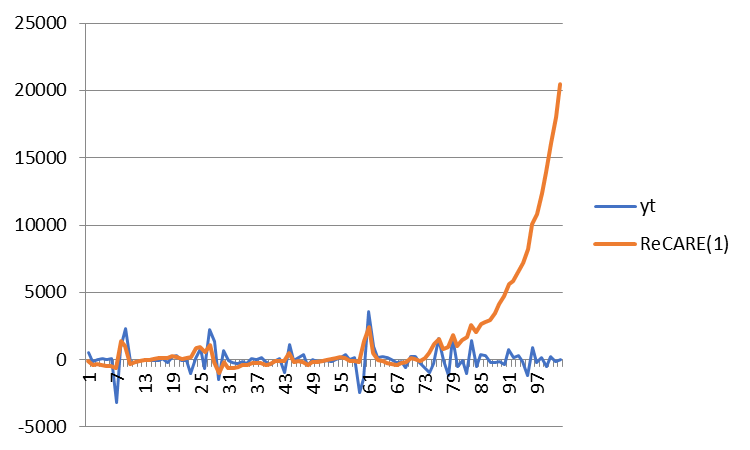
Table 2.11.

*Comparative accuracy of the retro-forecast for the series No. 2831 by the models ReCARE(p) and ReCARNE(p)*

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *Lag, p* | *Model*  *ReCARE(p)* | | *Model ReCARNE(p) (2.9.6) at 1/n equal to:* | | | | | | | |
| 1/3 | | 1/5 | | 1/7 | | 1/9 | |
| *σ* | *σf* | *σ* | *σf* | *σ* | *σf* | *σ* | *σf* | *σ* | *σf* |
| 1 | 734,742 | **6826,821** | 860,526 | **621,865** | 869,646 | 628,912 | 879,571 | **629,687** | 870,685 | 632,682 |
| 2 | 770,670 | **736,621** | 826,828 | 676,411 | 826,313 | 677,804 | 826,571 | 668,526 | 826,451 | 672,004 |
| 3 | 762,279 | 760,198 | 787,891 | 652,227 | 782,001 | **613,011** | 803,485 | 659,061 | 805,725 | **617,513** |
| 4 | 766,992 | 773,879 | 741,090 | 697,681 | 733,624 | 724,195 | 736,586 | 726,294 | 751,129 | 686,260 |
| 5 | 735,628 | 926,495 | 777,610 | 687,691 | 740,946 | 763,679 | 748,335 | 634,481 | 742,979 | 704,784 |
| 6 | 740,710 | 784,822 | 749,599 | 756,271 | 747,825 | 794,749 | 765,297 | 734,374 | 762,374 | 782,512 |

From the results given in Table 2.11, it can be seen that the Re*CARNE(3)* model with an exponent of 1/5 was the best in terms of accurate short-term forecasting for the series in question. At the same time, the results of the retro-forecast using the Re*CARE (1)* model were incredibly bad: the SD of the retro-forecast turned out to be 6826,821, which is eleven times more than the SD of the retro-forecast of the Re*CARE (3)* model with an error rate of 1/5. How this happened can be seen in Figure 2.7.

The model, as you can see from the figure, described the training set up to 72 observations fairly well. But some of the input data volatility, starting from 73 to 79, the model did not "bear" - it went “out of order” and began to lift up



*Fig. 2.7. The initial yt series and the calculated values of the ReCARE (1) model.*

By the way. And the Re*CARNE(5)* model with the exponent for the approximation error equal to 1/15, which showed the best result in Table 2.10, also turned out to be not the best from the standpoint of retro-forecasting on the test set. Table 2.12 shows the retro-forecasting results of this model and the model with an exponent of 1/17 compared to the best Re*CARNE(3)* model with an exponent of 1/5.

*Table 2.12.*

*Comparative accuracy of the retro-forecast for series № 2831 by ReCARNE(p) models*

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *Lag, p* | *Model ReCARNE(p) (2.9.6) at 1/n equal to:* | | | | | |
| 1/5 | | 1/15 | | 1/17 | |
| *σ* | *σf* | *σ* | *σf* | *σ* | *σf* |
| 1 | 869,646 | 628,912 | 871,368 | 684,068 | 872,575 | **645,236** |
| 2 | 826,313 | 677,804 | 826,481 | 674,935 | 826,476 | 674,602 |
| 3 | 782,001 | **613,011** | 795,369 | 661,705 | 796,434 | 666,663 |
| 4 | 733,624 | 724,195 | 762,254 | 720,519 | 757,440 | 737,850 |
| 5 | 740,946 | 763,679 | 739,392 | **658,642** | 737,350 | 676,733 |
| 6 | 747,825 | 794,749 | 767,229 | 784,903 | 764,740 | 805,294 |

Series № 2831 was taken by us at random to demonstrate the properties of the Re*CARNE(p)* model. We did not set the task of finding the best forecast model for this series and performing a short-term economic forecast. And the fact that the model, which is nonlinear by mistake, demonstrates good properties both in approximation and forecasting, indicates the possibility of using these models in practice.

Here attention should be drawn to the fact that the exponent α of the model (2.9.5) can take not only positive fractional values, as shown in (2.9.7), but also negative values:

. (2.9.8)

It is important that here, as in (2.9.7), only odd values are chosen from the natural series n: 1, 3, 5, 7, …

Since the negative degree to which the approximation error is raised means that the error moves to the numerator, it is possible in this case to raise it in the numerator to a negative odd degree:

 (2.9.9)

Thus, in the model (2.9.5), the exponent *α* can take the following values

 (2.9.10)

And, by choosing the best approximation error index from this set of values, one can make additional "tuning" of the perfect Re*CARE(p)* model, turning it into Re*CARNE(p)*, if it turns out to be appropriate.

Similarly, we can specify the model Re*CARE(p,g)* by transforming it into the nonlinear model Re*CARNE(p,g)* with respect to the approximation error. Since there are no difficulties with such a transformation, we will not dwell on it in more detail.

**2.10. ReCARNE(p) model with even numbers in exponents**

Studying the model (2.9.5) as the simplest among the set of possible models of the Re*CARNE(p)* class, we noticed that the exponent can be equal to *α=1/n,* where *n* are odd values of a natural series of numbers. The exclusion of odd values from this series of numbers is explained simply - the approximation error can take negative values, and in the domain of real values, the root of an even degree from a negative number does not exist.

But since we are working just in the field of complex numbers, the root of an even degree of a negative number should not confuse us.

Let us consider the model (2.9.1) again, but this time in relation to the case when the nonlinearity of the approximation error *εt* is determined by raising it to a fractional power:

. (2.10.1)

in which *n=2,4,6, …*

Proceeding from the general scientific method "from simple to complex", we will consider the simplest version of the model (2.10.1) in the case when p*=1* and *α=1/2*. The model (2.10.1) for this case can be written as follows:

. (2.10.2)

or

. (2.10.3)

If the approximation error *εt* is non-negative, then this model repeats the *CARNE(1)* model studied in the previous paragraph.

But in the case when the error *εt* becomes negative, then the radical expression in (2.10.3) becomes negative. And since we are working in the field of complex numbers, this means that the square root of the negative error should be considered as an imaginary number. And this significantly changes both the model itself and its properties.

In the case when ε*t-1*<0, the error can be written as follows:

. (2.10.4)

Then for ε*t-1*<0, we have:

. (2.10.5)

Substituting (2.10.5) into model (2.10.3), we get:

. (2.10.6)

And this equality is possible only when the real and imaginary components of the left part are equal to the real and imaginary components of the right one. With respect to the imaginary parts of equality (2.10.6), this means the following:

. (2.10.7)

This equality holds in two cases: when

. (2.10.8)

The first case where the equality holds



should be recognized as unlikely.

And the second case *a11*=0 is easily feasible, since it occurs in equality (2.10.6) only in the imaginary part of the model. And its real part looks like this:

. (2.10.9)

With this in mind, the real part of the model under consideration for any values of the approximation error will have the following form:

 (2.10.10)

This model describes real values with some approximation error:

. (2.10.11)

Substituting in (2.10.10) the approximation error (2.10.11), we get:

 (2.10.12)

If the forecaster needs to estimate the coefficients of the Re*CARNE(p)* model with exponent *α=1/2* and *p=1*, so that the variance of the approximation error is minimal, then he should minimize the sum of squared deviations (SSD) (2.10.12) over the entire set of observations. It is clear that such a problem cannot be simply solved, since, by changing the coefficients *a01* and *a11* in (2.10.12), we will obtain changing values of the approximation errors. Therefore, to solve the problem, it is necessary to use a program with conditional transitions, which will orient the calculation of the approximation error either to the first or the second part of the system (2.10.12) depending on the sign of the previous approximation error ε*t-1*.

We will postpone the solution of this problem and the study of the resulting model properties for the future.

**2.11. Model CMA(q)**

The CARE(p) model (2.7.2) describes a one-dimensional series with some complex error:

. (2.11.1)

This series is considered by definition as stationary.

One can consider the complex error as some kind of complex "white noise". In this case, the complex variable can be calculated through this “white noise” in the same way as it is done with real processes using the *MA(q)* model, that is:

. (2.11.2)

Where *b0τ* and *b1τ* are some model coefficients;

*μrt* and *μit* are unknown errors at time *t*,

*μr, t-1* and *μi, t-1* are known errors at the previous time t*-1*,

*q* - lag.

This is a complex “moving average” model, which it would be logical to designate by analogy with *MA(q)* as *CMA(q).*

The problem as a whole is reduced to finding such coefficients of the model (2.11.2) in order to minimize the sum of squares:

. (2.11.3)

But this model has real and imaginary parts. Therefore, the task can be simplified if we consider separately the real and imaginary parts of the model (2.11.2). The model (2.11.2) can be represented as a system of two equalities:

. (2.11.4)

The Re*CMA(q)* model describes the original series, and the Im*CMA(q)* model describes the approximation error.

The real part of the model is of interest, since it is the indicator *yt* that is forecasted. But as can be seen from (2.11.4), for this purpose, the component *μi,t-τ*, which is related to the imaginary part of the model, should be also used.

In order to understand how to use Re*CMA(q)* in practice, let us consider the simplest case of this model, namely *CMA (1)*:

. (2.11.5)

The real and imaginary parts of this model will be as follows:

. (2.11.6)

To calculate this model at the first step *t=1*, it is necessary, first of all, to calculate the calculated value of the indicator from the known initial values of the complex error and from some starting values of the coefficients *b01* and *b11*:

. (2.11.7)

The initial errors *μrt* and *μit* can be equated to zero, or they can be a tool for additional model tuning.

Knowing this calculated value of the indicator on the first observation, we can calculate the current error:

. (2.11.8)

It is also the first member of a series of “white noise” values:

. (2.11.9)

Substituting the value of the current error into the imaginary part (2.11.6), it is possible to calculate:

. (2.11.10)

Now we can move on to the second observation *t=2*:

. (2.11.11)

After that the current approximation error is calculated:

. (2.11.12)

Then - the error of the imaginary part:

. (2.11.13)

After that, one should move on to the third observation, the fourth, and so on.

The model coefficients are calculated by numerical methods so that the sum is minimized:

. (2.11.14)

The algorithm for estimating coefficients will not change if we consider the following *CMA (2)* model in terms of complexity:

. (2.11.15)

The real and imaginary parts of this model will be as follows:

. (2.11.16)

And calculations in such a model should start with *t=2*...

Let us compare the Re*CMA(q)* model with the *MA(q)* model described earlier in paragraph 2.1. We will make a comparison using the example of series No. 2831 (Table 2.13).

*Table 2.13.*

*Comparative accuracy of MA(q) and ReCMA(q) models*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Lag *q* | *σ* | *BIC* | *σ* | *BIC* |
| Model Re*CMA (q)* | | *Model MA(q)* | |
| 1 | 797,469 | 13,453 | 804,971 | 13,427 |
| 2 | 738,715 | **13,391** | 744,462 | **13,316** |
| 3 | 714,125 | 13,416 | **712,687** | 13,322 |
| 4 | **708,915** | 13,496 | 712,687 | 13,322 |
| 5 | 712,259 | 13,601 | 713,723 | 13,373 |
| 6 | 724,500 | 13,732 | 723,980 | 13,731 |

As can be seen from the data in Table 2.13, the Re*CMA(p)* model as a whole describes the real series of values more accurately. But at the same time, due to twice as many coefficients, it always loses to the MA(q) model by the information criterion.

The same calculations were made for other series of the database under consideration №№ 2832, 2833 and 2834. Table 2.14 shows these results.

*Table 2.14.*

*Comparative accuracy of MA(q) and ReCMA(q) models on series №№ 2832, 2833 and 2834*

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *№2832* | | | | | *№2833* | | | | | *№2834* | | | | |
| Lag *q* | *σ* | *BIC* | *σ* | *BIC* | Lag *q* | *σ* | *BIC* | *σ* | *BIC* | Lag  *q* | *σ* | *BIC* | *σ* | *BIC* |
| Models Re*CMA (q)* | | Models MA(q) | | Models ReCMA (q) | | Models MA(q) | | Models ReCMA (q) | | Models MA(q) | |  |
| 1 | 2304,48 | **15,58** | 2339,54 | **15,56** | 1 | 1239,34 | **14,34** | 1241,87 | **14,29** | 1 | 38,70 | **7,40** | 38,98 | 7,37 |
| 2 | 2221,37 | 15,59 | 2252,09 | 15,53 | 2 | 1199,78 | 14,36 | 1226,41 | 14,31 | 2 | 37,54 | 7,43 | 37,66 | **7,35** |
| 3 | 2204,29 | 15,67 | 2203,33 | 15,58 | 3 | 1149,80 | 14,37 | 1211,76 | 14,38 | 3 | 35,74 | 7,43 | 37,22 | 7,42 |
| 4 | 2210,72 | 15,77 | 2203,33 | 15,58 | 4 | 1154,82 | 14,47 | 1211,76 | 14,38 | 4 | 35,68 | 7,52 | 37,22 | 7,42 |
| 5 | **2199,119** | 15,856 | 2206,591 | 15,630 | 5 | **1135,112** | 14,533 | 1211,019 | 14,431 | 5 | 35,382 | 7,597 | 37,026 | 7,455 |
| 6 | 2199,60 | 15,95 | **2201,27** | 15,96 | 6 | 1163,99 | 14,68 | **1189,88** | 14,76 | 6 | **33,69** | 7,60 | **36,71** | 7,77 |

And again, we can make sure that the Re*CMA(p)* model for these three series on the whole describes the original series more accurately than the *MA(q)* model, but the information criterion shows that reducing the variance by complicating the model does not make sense in these cases. According to this criterion, the *MA(q)* model is the best for all series.

But this does not mean that the Re*CMA(p)* model does not stand up to competition with the *MA(q)* model. There are also other cases. Here, the table. 2.15 shows the calculations performed on shorter data series of the base used, namely, on series №№ 646, 647, 649.

Table 2.15.

*Comparative accuracy of MA(q) and ReCMA(p) models on series №№ 646, 647 and 649*

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *№646* | | | | | *№647* | | | | | *№649* | | | | |
| Lag *q* | *σ* | *BIC* | *σ* | *BIC* | Lag *q* | *σ* | *BIC* | *σ* | *BIC* | Lag *q* | *σ* | *BIC* | *σ* | *BIC* |
| Models Re*CMA (q)* | | Models MA(q) | | Models Re*CMA (q)* | | Models MA(q) | | Models Re*CMA (q)* | | Models MA(q) | |  |
| 1 | 931,43 | 13,85 | 953,62 | **13,81** | 1 | 499,13 | 12,60 | 646,53 | 13,03 | 1 | 533,80 | 12,73 | 536,97 | **12,66** |
| 2 | 910,32 | 13,98 | 942,18 | 13,87 | 2 | 518,48 | 12,85 | 522,46 | 12,69 | 2 | 485,61 | **12,72** | 539,82 | 12,76 |
| 3 | 947,80 | 14,24 | 960,83 | 14,10 | 3 | 394,37 | 12,49 | 530,77 | 12,91 | 3 | 460,62 | 12,80 | 518,81 | 12,87 |
| 4 | 786,44 | 14,06 | 960,83 | 14,10 | 4 | 355,18 | 12,47 | 530,77 | 12,91 | 4 | 506,41 | 13,18 | 518,81 | 12,87 |
| 5 | 631,72 | 13,82 | 972,55 | 14,22 | 5 | 249,89 | 11,96 | 522,87 | 12,98 | 5 | 428,23 | 13,04 | 519,61 | 12,97 |
| 6 | 539,65 | **13,71** | 882,80 | 14,69 | 6 | **196,42** | **11,69** | **402,41** | 13,12 | 6 | **366,41** | 12,94 | 520,66 | 13,64 |
| 7 | **499,63** | 13,767 | **879,99** | 14,23 |  | 332,40 | 12,95 | 425,78 | **12,78** |  | 374,72 | 13,19 | **504,53** | 13,12 |

The series №№ 2831, 2832, 2833, 2834 contain 103 observations, and series №№ 646,647 and 649 contain almost two and a half times fewer observations, namely 43. For these shorter series, the new Re*CMA(p)* model turned out to be better than the *MA(q)* model by the criterion of the minimum variance of the approximation error. Thus, for series №. 646, the Re*CMA (7)* model gives *σ= 499,63*. The *MA (7)* model, that has the smallest variance of the approximation error in its class for this series, has a significantly higher RMSE: *σ= 879, 99.* The best Re*CMA (6)* model according to the information criterion has *BIC = 13,709*, and the best model according to this criterion *MA (1)* has *BIC=13,805*. At the same time, the Re*CMA (6)* model contains 12 coefficients, and the *MA (1)* model contains only one coefficient, and still, it loses to the new model by the information criterion.

And for series № 647, the Re*CMA(q)* model showed such a significant increase in the approximation accuracy that, according to the information criterion *BIC=11,688*, the Re*CMA(6)* model, which has 12 coefficients, is preferable to the *MA (7)* model, which has seven coefficients, but *BIC=12,78*. Thus, the complication of the model both in the case of series № 646 and in the case of series № 647 is justified.

But the situation is somewhat different for the series № 649. And here the ReCMA *(6)* model shows a significantly better result on the variance of the approximation error. For the best-in-class ReCMA *(6)*,the RMSE of the approximation error is *σ= 366,405*, and for the best-in-class *MA (7)* model, it is *σ= 504,532*. This is 37% higher. But according to the information criterion, the model from the *MA(q)* *class* turns out to be the best for this series, namely the *MA (1)* *model* with *BIC=12,656*.

We have already seen earlier that the best model in approximation is rarely the best one in the forecast. Therefore, it makes sense to compare these models with each other in the retro-forecasting procedure. Let us use all the same series №№ 2831, 2832, 2833, 2834 and №№ 646,647,649 in order to compare the best models in terms of approximation with the best models in terms of retro-forecasting accuracy. To do this, we divide each series into a training set (the first three quarters of the series) and a test set (the last quarter of the series) to perform retro-forecasting using the models whose coefficients are obtained on the training set.

The calculation results are shown below in Table 2.16. The results of how well each model approximated the past on the training set and the value of the information criterion for each model are given here for information.

*Table 2.16.*

*The results of retro-forecasting by MA(q) and ReCMA(q) models on series №№ 646, 647 and 649*

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Lag, *q* | Re*CMA(q)* | | | *MA(q)* | | |
| *σ* approximations | *BIC* approximations | *σ*  retro-forecast | *σ* approximations | *BIC* approximations | *σ*  retro-forecast |
| Series № 646 | | | | | | |
| 1 | *305,775* | *11,684* | **146,500** | *301,157* | *11,531* | 147,881 |
| 2 | 310,427 | 11,964 | 158,189 | 311,373 | 11,726 | 144,495 |
| 3 | 313,664 | 12,249 | 139,897 | 322,211 | 12,065 | 144,668 |
| 4 | 319,498 | 12,564 | 149,048 | 322,211 | 12,065 | 144,133 |
| 5 | 324,324 | 13,478 | 153,934 | 328,469 | 12,251 | 143,926 |
| 6 | 330,421 | 13,237 | 168,974 | 334,209 | 12,441 | 140,877 |
| 7 | 337,085 | 13,608 | 676,426 | 339,045 | 12,495 | **139,402** |
| Series № 647 | | | | | | |
| 1 | 46,353 | *7,911* | 170,074 | 64,240 | 8,441 | 134,532 |
| 2 | 50,796 | 8,344 | 180,711 | 66,101 | 8,626 | 124,634 |
| 3 | 48,810 | 8,528 | 180,711 | **48,445** | **8,276** | 1116,578 |
| 4 | 49,091 | 8,817 | 193,356 | 48,445 | 8,276 | 323,908 |
| 5 | *39,340* | 9,259 | **146,744** | 57,311 | 8,759 | 138,958 |
| 6 | 41,242 | 9,075 | 2008,968 | 52,914 | 8,755 | 201,095 |
| 7 | 42,352 | 9,459 | 5942,129 | 56,690 | 8,918 | **102,269** |
| Series № 649 | | | | | | |
| 1 | 251,252 | *11,291* | **819,683** | 247,018 | *11,135* | 851,776 |
| 2 | 226,646 | 11,335 | 3143,612 | 255,871 | 11,333 | 824,731 |
| 3 | 224,527 | 11,580 | 2354,797 | 258,038 | 11,447 | 818,388 |
| 4 | 256,748 | 12,126 | 828,344 | 258,038 | 11,621 | 815,526 |
| 5 | 221,850 | 12,128 | 2440,103 | *232,361* | 11,559 | 1553,784 |
| 6 | 261,647 | 12,770 | 1119,804 | 267,024 | 11,993 | 816,607 |
| 7 | *216,989* | 12,727 | 3646,674 | 272,583 | 12,059 | **809,938** |

From the results given in Table 2.7.3, it can be noticed that the best approximations of the past of the model are never the best in terms of retro-forecasting accuracy. More than that. For the series №№ 646 and 649, the *MA(q)* models, that were the worst at describing the past are the best at forecasting the future. This is the *MA (7)* pattern for the series № 646 and the *MA (7)* pattern for the series № 649. In none of the cases considered, the *CMA(q)* model was the best in retro-forecasting, although for the last two series it is the best in approximating past values.

Table 2.17 shows the results of retro-forecasting for the four longer series №№ 2831, 2832, 2833, 2834.

*Table 2.17.*

*Retro-forecasting results by MA(q) and ReCMA(q) models on series №№ 2831, 2832,*

*2833 and 2834*

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Lag, *q* | *CMA(q)* | | | *MA(q)* | | |
| *σ* approximation | *BIC* approximation | *σ*  retro-forecast | *σ* approximation | *BIC* approximation | *σ*  retro-forecast |
| Series № 2831 | | | | | | |
| 1 | 855,797 | 13,618 | *663,17* | 867,648 | 13,588 | *691,54* |
| 2 | 752,546 | *13,477* | 889,45 | 760,202 | *13,382* | 869,50 |
| 3 | 727,785 | 13,529 | 3228,30 | *728,229* | 13,416 | 837,98 |
| 4 | 723,616 | 13,639 | 1212,58 | 728,229 | 13,416 | 810,32 |
| 5 | 728,824 | 13,777 | 1242,84 | 730,628 | 13,485 | 818,81 |
| 6 | *702,057* | 13,828 | 2176,33 | 741,244 | 13,937 | 826,50 |
| Series № 2832 | | | | | | |
| 1 | 2223,860 | 15,528 | 2764,53 | 2228,541 | *15,475* | 2692,69 |
| 2 | *2073,960* | *15,505* | 4762,07 | 2214,649 | 15,521 | 2491,48 |
| 3 | 2199,678 | 15,741 | 3073,68 | *2195,724* | 15,624 | 2485,63 |
| 4 | 2216,666 | 15,878 | *2504,00* | 2195,724 | 15,624 | *2186,91* |
| 5 | 2181,016 | 15,969 | 4189,99 | 2198,341 | 15,688 | 2405,83 |
| 6 | 2177,230 | 16,092 | 3615,99 | 2200,897 | 16,114 | 2375,42 |
| Series № 2833 | | | | | | |
| 1 | 1246,867 | *14,371* | *1246,43* | 1246,176 | *14,312* | 1251,02 |
| 2 | 1201,346 | 14,413 | 1508,74 | 1219,172 | 14,327 | 1290,42 |
| 3 | 1080,853 | 14,320 | 3261,05 | 1216,031 | 14,442 | 1239,19 |
| 4 | *1046,354* | 14,376 | 2913,42 | 1216,031 | 14,442 | 1239,80 |
| 5 | 1101,798 | 14,603 | 2468,06 | 1224,087 | 14,517 | 1231,86 |
| 6 | 1128,905 | 14,778 | 5562,29 | *1211,932* | 14,920 | *1197,68* |
| Series № 2834 | | | | | | |
| 1 | 40,196 | 7,501 | 46,68 | 40,721 | 7,470 | *36,33* |
| 2 | 36,884 | 7,446 | 92,30 | 38,604 | *7,422* | 36,70 |
| 3 | 33,565 | *7,376* | 91,13 | 36,813 | 7,447 | 36,27 |
| 4 | 34,993 | 7,580 | 96,36 | 36,813 | 7,447 | 47,27 |
| 5 | 34,795 | 7,693 | 53,04 | 36,700 | 7,503 | 44,54 |
| 6 | *32,689* | 7,695 | *43,04* | *36,564* | 7,919 | 43,71 |

Once again, we can see that the existing *MA(q)* model performed better than the CMA*(q)* model in three of the four series in short-term forecasting on the test set. At the same time, for all the four series, the CMA*(q)* model showed better approximation properties on the training set than the existing *MA (q)* model.

These examples do not mean that for all the forecasted economic series the CMA*(q)* model will be worse for short-term forecasting purposes than the *MA(q)* model. The opposite situation is quite possible. But the complexity of using the CMA*(q)* model, which is not confirmed by an increase in the forecast accuracy, does not allow us to recommend it as a worthy alternative to *MA(q)* in short-term economic forecasting. So far, it is only of theoretical interest.

**2.12. CTAR(p)**

In the general case, the model of complex autoregression was presented in this form:

 (2.12.1)

Earlier we considered the properties of this complex-valued autoregression in the case when the complex variable (*y1t*+*iy2t*) in (2.12.1) was presented in full form, as well as in the case when *y2t=ɛt*. In both cases, we have seen that these new models act as a worthy alternative to the vector autoregressions *VARk(p)* and the *ARMA(p,q)* models popular in short-term forecasting today.

Now we will consider complex autoregression with time when *y2t=t*. In this case, the complex variable takes the form (*y1t*+*it*) and we will denote such a model as *CTAR(p)*. In general form, this model will be written as follows (*Svetunkov,* *Forecasting economic dynamics*, *2020*):

 (2.12.2)

First of all, let us pay attention to the fact that the model (2.12.2) predicts not only the economic indicator *yt*, but also the time *t*. In all autoregression models, time acts as an index of the indicators ordering, but not as an active element taken into account in forecasting. In the model (2.12.2), time comes both as a determining factor and as a dependent variable. This is a new property of the forecasting model and it should be paid attention to.

Consider the linear form of autoregression (2.12.2). It will take the form:

. (2.12.3)

The simplest of this class of models is the *CTAR (1)* linear autoregressive model:

. (2.12.4)

This complex-valued function can be represented as a system of two real equalities – an individual equality for the real part and an individual equality for the imaginary part (2.12.4):

 (2.12.5)

The first equation of the system (2.12.5) is a two-factor regression, the variables of which are the indicator yt*-1* and the time *(t-1)* at which this indicator was observed.

The imaginary part of the model is also a two-factor model, but it is a model of the dependence of the future time on the previous moment of time (*t-1)* and the value of the predicted indicator at the prior moment in time *yt-1*.

Let us first observe the real part of the model:

. (2.12.6)

This model consists of two parts - a simple autoregression and a linear component of the trend.

In order to understand exactly how this model behaves, we will consider its change, starting with some initial values. If the model coefficients (2.12.6) and the value of *y1* are known, then the second value is calculated as follows:

.

The calculated value of the indicator at *t=3* will be equal to:



For *t=4* we obtain:

.

For any *t=T* , this model gives this calculated value:

. (2.12.7)

Thus, the dynamics of the calculated indicator will be exactly non-stationary if *a0*>1 and *a0*>1. Depending on what the values of *a0*, *a1* and *y1* will be*,* various nonlinear non-stationary processes will be simulated. Fig.2.8 shows some trajectories modeled using (2.12.6) at different values of the starting value of the variable and the model coefficients.

*Figure 2.8. Trajectories described by the ReCTAR(1) model at different values of a0, a1, y1.*

Obviously, higher-order *ReCTAR* models will describe more complex non-stationary trajectories.

Since the practice of reducing nonstationary series to a stationary form by calculating finite differences has long been accepted for nonstationary series, and the use of simple autoregressions in this case is very effective, the *ReCTAR(p)* model has no advantages over the *AR(p)* model.

Comparative studies on different data series have shown that when modeling and predicting nonstationary series, *ReCTAR(p)* models are insignificantly more accurate in describing and predicting these nonstationary series compared to *ARI(p,d)* models. But since they are much more complicated than simple autoregressions and have twice the number of coefficients, BIC always recommends giving preference to standard autoregression models.

The imaginary part of the *CTAR (1)* model will look like this:

. (2.12.8)

That is, in general, a non-integer time is predicted.

At first sight, it seems incredible – time, which seems to us an objective reference scale, suddenly becomes, as in Einstein's theory of relativity, nonlinear. In fact, this is not the case. After all, we are talking about a system in which two elements are simultaneously calculated – the indicator itself and the corresponding time *t*, at the onset of which the indicator will take this value. If we consider a complex plane, on the horizontal axis of which the real values of the indicator are located, and on the vertical axis - the imaginary component to which time is assigned in *CTAR(p)*, then any complex number *yt*+*it* will represent a point on this complex plane, and its coordinates will be *yt* and *t*. Therefore, by calculating the complex number *yt*+*it*, we simply determine the location of a point on the plane. There is nothing sacred about it.

We subtract the previous complex number from the left and right parts of this model:

. (2.12.9)

We get:

. (2.12.10)

From here:

. (2.12.11)

Or:

. (2.12.12)

Then:

. (2.12.13)

As a result, we get a difference equation for the forecasted indicator:

. (2.12.14)

Under restrictions:

. (2.12.15)

If the time increments are constant and equal to one, then (2.12.14) and (2.12.15) represent such a system:

 (2.12.16)

Since this model does not work with the initial series of values, but with its first difference, it makes sense to compare the accuracy of approximation and the accuracy of the retro-forecast of this CTAR *(1)* model with the *ARI (1,1)* *model*.

We will use for this the same database of the International Institute of Forecasters, but, for a diversity, other series, namely, №№ 1 - 20. The results of this comparative analysis are given in Table 2.18. These are the series of some annual observations for twenty years. All of them have some dynamics for growth, so working not with the initial series, but with its first differences is justified. The approximation error was calculated for all twenty observations, and the retro-forecast error was calculated as follows. At the first fifteen observations of each series, the coefficients of the compared models were evaluated, and then these models with these coefficients were used to predict per one step of observation over the last five years.

*Table 2.18.*

*Comparative analysis of the approximation and retro-forecasting accuracy by the CTAR (1) and ARI (1,1) models on the series №№ 1 – 20*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Series number | RMSE of  approximation error | | RMSE of  retro-forecast | |
|  | CTAR | AR | CTAR | AR |
| 1 | **140,1696** | 160,19176 | **255,1167** | 268,3755 |
| 2 | 690,9747 | 661,93427 | 687,9618 | 489,8449 |
| 3 | 653,4255 | 608,24993 | **440,8778** | 530,9836 |
| 4 | 669,2756 | 640,83454 | 787,9096 | 549,8271 |
| 5 | **726,7625** | 729,67937 | **445,1941** | 447,5899 |
| 6 | 244,2261 | 221,80027 | **271,5892** | 287,6148 |
| 7 | 489,2181 | 478,40631 | 382,7154 | 273,1848 |
| 8 | 1037,14 | 935,62974 | 2157,841 | 947,078 |
| 9 | 812,0841 | 935,62974 | **812,0841** | 835,1936 |
| 10 | 1434,711 | 1334,5135 | 2734,152 | 2663,634 |
| 11 | **293,7468** | 315,13915 | 1336,271 | 892,5164 |
| 12 | 319,2022 | 288,0936 | 618,0703 | 389,8893 |
| 13 | 325,1974 | 266,94953 | 715,7141 | 470,0367 |
| 14 | 610,0327 | 590,83242 | **247,9228** | 705,9091 |
| 15 | 203,4974 | 172,33917 | 347,9195 | 240,4212 |
| 16 | 563,5613 | 550,77074 | **865,1852** | 928,8341 |
| 17 | 363,9672 | 294,95626 | 686,2616 | 343,5572 |
| 18 | 949,994 | 922,39645 | 2070,852 | 1475,759 |
| 19 | 480,0783 | 427,37885 | 1081,836 | 806,9651 |
| 20 | 925,5194 | 770,8984 | 2364,137 | 1041,976 |

Only in three cases out of twenty does the CTAR *(1)* model approximate the original data better than the *ARI (1,1)* model (series 1, 5, and 11). From the standpoint of the retro-forecast, the results of the comparative analysis turned out to be somewhat better for the new model – in seven cases out of twenty it gives better forecast results per one step.

All this suggests that the new model *CTAR(p)* has the right to exist both in its original form (2.12.3) and in its difference form (2.12.14) - (2.12.15). But it requires additional research.

In the case when  it is possible to obtain from the model (2.12.14) a differential equation that can be used for a variety of purposes for modeling economic dynamics in continuous time. We will not dwell on this issue, since it requires additional research beyond the scope of our work. Let us just note such an opportunity that opens up when using the *CTAR(p)* model in economic forecasting and modeling.

**2.13. Complex-valued autoregression with increment**

The last variety of the basic models of complex-valued autoregressions, which were proposed in paragraph 2.6, is a model in which the predicted indicator is supplemented in an imaginary part by its increase compared to the previous point in time. Such a complex -valued autoregression will have the form:

 (2.13.1)

Models of this kind have been proposed to be denoted as *GCAR(p)*.

Various modifications of the model are possible here, for example, to use not the increment Δ*yt=yt-yt-1*, but, for example, the increment with respect to the level of two observations back Δ*yt=yt-yt-2*, three observations back Δ*yt=yt-yt-3*, etc. One can use the second difference Δ2*yt=*Δ*yt-*Δ*yt-1*, etc. as an additional characteristic instead of the first difference. To study all this variety of possibilities is beyond the scope of this paper, therefore we will consider only the main characteristics of the basic models of the type (2.13.1).

First of all, let us represent this model as a system of two equalities: the real and imaginary parts:

 (2.13.2)

If we consider the real and imaginary parts of the model separately, then, at first sight, they seem to be different from each other. But considered separately, the real and imaginary parts, the coefficients of which are evaluated with the help of LMS, simulate the processes in the same way.

Indeed, let us represent the real part of the model for the simplest case of Re*GCAR(1)*:

 (2.13.3)

This is an autoregression model with increment. It can be used in the case when the forecast series has a pronounced tendency to increase or decrease its indicators. In all other cases, the model will poorly describe the initial series and predict it also poorly.

The imaginary part of the complex-valued autoregression *GCAR (1)*, as follows from (2.13.2), has the following form:

 (2.13.4)

It can be transformed and reduced to this format:

. (2.13.5)

If we use the model (2.13.3) to simulate some series, then using, for example, LMS, two coefficients of this model are found at variables yt*-1* and Δ *yt-1*. When using model (2.13.5) in the simulation of the same process, the coefficients at the same variables are evaluated with the help of LMS, which means that these coefficients will be the same as those for the model (2.13.3). Therefore, when studying the properties of the model (2.13.1), it is sufficient to understand the properties of either its real or its imaginary part, since individually they behave the same way.

Let us focus on the properties of the real part of the model (2.13.2). Due to the fact that in the complex variable, some increase, corresponding to the nature of the modeled process, belongs to the imaginary part, then this increase may not relate to the current time, but to some past observation shifted relative to the current one by gobservations backwards. We will call this model Re*GCAR(p,g)* and write it like this:

. (2.13.6)

The simplest model of this class is the Re*GCAR(1, g)*, which has the form:

 (2.13.7)

The increment Δ*yt-(1+g)* in it refers to the observation (*t-1*) in the case when g*=0*... And in all other cases, when *g=1, 2, 3, 4*, … the increment refers to an observation shifted back by *g=1, 2, 3, 4* units of time, for example, for *g=5*:

 (2.13.8)

It should be noted that purely by appearance the model (2.13.3), that is, the Re*GCAR(1,0)* model, is similar to the *ARIMA(1,1,0)* model, since the latter model has this form:

 (2.13.9)

But when comparing (2.13.3) with (2.13.9), it can be seen that these are still different models. And the point is not only in the sign of the second summand of the right part of these two equalities, but that in the new model the previous value of the series level *yt-1* is corrected by the multiplier *a01*, and in the *ARIMA (1,1,0)* model this level of *yt-1* is not corrected, because it is multiplied by one. Therefore, the *ARIMA (1,1,0)* model is a particular case of the Re*GCAR(1,0)* model. And since the model has the ability to be more finely tuned during modeling to the features of the series due to the coefficient a*01* and the shift g, we should expect more accuracy from the new model than from the *ARIMA (1,1,0)* model. This conclusion extends to the more general case, since all *ARIMA(p,1,0)* models are a particular case of the Re*GCAR(p,0)* model.

First of all, let us see what processes are generated by the real (or imaginary) part of the model.

Since this is a model with increments, it is by definition best suited for modeling and forecasting the series that have a non-linear tendency to increase or decrease their values. Typical processes generated by the ReGCAR*(p,1)* model are shown in Fig. 2.9.

*Figure 2.9. Dynamics of the model (2.13.3) at а) a0=0,1; a1=0,1 and b) a0=-0,1; a1=-0,1*

But, since this is a model of autoregression, it can describe more complex dynamics than the monotonic type dynamics shown in Figure 2.9 at different initial values ​ ​ of the initial series and at different coefficients` values. Examples of complex nonstationary processes that can result from the application of the model in question are shown in Figure 2.10.

*Figure 2.10. Dynamics of the model (2.13.3) at a) a0=-1,5; a1=0,15 and b) a0=-0,2 a1=-1,1*

As can be understood from the above figures, the Re*GCAR(p,1)* model will work well when modeling nonstationary processes and not very well when modeling stationary processes.

Let us use the results of Table 2.17, in which two models were compared - MA*(q)* and ReCMA*(q)* on series №№. 2831, 2832, 2833 and 2834. For these series, the *MA(q)* model showed better results than Re*CMA(q)*. Of course, the *MA(q)* model was used for the series reduced to a stationary form (their first differences), that is, it is the *ARIMA (0,1, p)* model.

Table 2.19 shows the results of using two models Re*GCAR(p)* – and *MA(q)* (the same as *ARIMA (0,1, p)*) on these series.

*Table 2.19*.

*Retro-forecast results by MA(q,1) and ReGCAR(p) models on series №№ 2831,2832, 2833 и 2834*

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Shift, *p* | Re*GCAR(p)* | | | Lag, *q* | *MA(q)* | | |
| *σ*  approximation | *BIC* approximation | *σ*  retro-forecast | *σ*  approximation | *BIC* approximation | *σ*  retro-forecast |
|  | Series № 2831 | | | | | | |
| 1 | 820,606 | 6,837 | 592,440 | 1 | 867,648 | 13,588 | *691,54* |
| 2 | 141,196 | 5,079 | 104,788 | 2 | 760,202 | *13,382* | 869,50 |
| 3 | 138,502 | 5,061 | *102,625* | 3 | *728,229* | 13,416 | 837,98 |
| 4 | 140,335 | 5,076 | 104,951 | 4 | 728,229 | 13,416 | 810,32 |
| 5 | *135,972* | *5,046* | 107,968 | 5 | 730,628 | 13,485 | 818,81 |
| 6 | 139,376 | 5,072 | 104,277 | 6 | 741,244 | 13,937 | 826,50 |
|  | Series № 2832 | | | | | | |
| 1 | *2168,677* | *7,807* | 2421,121 | 1 | 2228,541 | *15,475* | 2692,69 |
| 2 | 2189,222 | 7,820 | 2403,525 | 2 | 2214,649 | 15,521 | 2491,48 |
| 3 | 2210,173 | 7,831 | *2377,044* | 3 | *2195,724* | 15,624 | 2485,63 |
| 4 | 2223,974 | 7,839 | 2409,979 | 4 | 2195,724 | 15,624 | *2186,91* |
| 5 | 2183,648 | 7,822 | 2464,281 | 5 | 2198,341 | 15,688 | 2405,83 |
| 6 | 2212,802 | 7,837 | 2426,490 | 6 | 2200,897 | 16,114 | 2375,42 |
|  | Series № 2833 | | | | | | |
| 1 | *1185,347* | *7,205* | 1357,438 | 1 | 1246,176 | *14,312* | 1251,02 |
| 2 | 1204,340 | 7,222 | 1372,047 | 2 | 1219,172 | 14,327 | 1290,42 |
| 3 | 1198,145 | 7,218 | *1347,143* | 3 | 1216,031 | 14,442 | 1239,19 |
| 4 | 1219,005 | 7,237 | 1388,979 | 4 | 1216,031 | 14,442 | 1239,80 |
| 5 | 1218,210 | 7,238 | 1403,711 | 5 | 1224,087 | 14,517 | 1231,86 |
| 6 | 1217,162 | 7,239 | 1367,331 | 6 | *1211,932* | 14,920 | *1197,68* |
|  | Series № 2834 | | | | | | |
| 1 | 42,139 | 3,866 | 31,602 | 1 | 40,721 | 7,470 | *36,33* |
| 2 | 39,950 | 3,815 | 30,525 | 2 | 38,604 | *7,422* | 36,70 |
| 3 | 39,981 | 3,817 | *30,188* | 3 | 36,813 | 7,447 | 36,27 |
| 4 | 40,468 | 3,830 | 32,261 | 4 | 36,813 | 7,447 | 47,27 |
| 5 | 40,926 | 3,843 | 30,709 | 5 | 36,700 | 7,503 | 44,54 |
| 6 | *38,349* | *3,780* | 31,797 | 6 | *36,564* | 7,919 | 43,71 |

As can be seen, the new Re*GCAR(p)* model for series № 2831 and №2834 works better than the *MA(q)* model. And it works worse for the series № 2832 and № 2833.

Why does the new model better describe and predict exactly two series № 2831 and № 2834? Because these two series have some non-stationary tendency (Fig 2.9).

*Figure 2.9. Dynamics of series № 2831 and № 2834.*

And the other two series do not have such a tendency and can be considered as stationary ones (Fig. 2.10).

*Figure 2.10. Dynamics of the series № 2832 and № 2833.*

If *AR(p)* model is used instead of the *MA(q)* model, then similar results will be obtained – for series that have a non-stationary character, the autoregression model with increments is often more accurate than autoregression models applied to the same series reduced to a stationary form.

As an example, I will give the result of a retro-forecast for series № 2840, which consists of 103 values and has an obviously non-stationary form. Its first difference completely makes this series stationary, so the simple autoregression model *AR(p)* can be applied to the first differences. By changing in this model *p* from 1 to 10, the best model was determined. It turned out to be the *AR (1)* model, the best one both by the information criterion *BIC= 3,934*, and by the standard error of the one-step retro- forecast on the last 36 values of the series *σ=40,512*.

A range of Re*GCAR(g)* models was built on the same non-stationary series with *g* varying from 1 to 10. For this model, the model with a shift by one also turned out to be the best: Re*GCAR(1).* The information criterion for it was equal to *BIC= 3,919*, that is, less than for the *AR (1)* model. And the standard error of the retro-forecast for this model turned out to be less - σ *=* *39,571*.

2.13. Complex-valued autoregression with increment. General form

The last variety of the basic models of complex-valued autoregressions, which were proposed in paragraph 2.6, is a model in which the predicted indicator is supplemented in an imaginary part by its increase compared to the previous point in time. Such a complex -valued autoregression will have the form:

 (2.13.1)

Models of this kind have been proposed to be denoted as *GCAR(p)*.

Various modifications of the model are possible here, for example, to use not the increment Δ*yt=yt-yt-1*, but, for example, the increment with respect to the level of two observations back Δ*yt=yt-yt-2*, three observations back Δ*yt=yt-yt-3*, etc. One can use the second difference Δ2*yt=*Δ*yt-*Δ*yt-1*, etc. as an additional characteristic instead of the first difference. To study all this variety of possibilities is beyond the scope of this paper, therefore we will consider only the main characteristics of the basic models of the type (2.13.1).

First of all, let us represent this model as a system of two equalities: the real and imaginary parts:

 (2.13.2)

If we consider the real and imaginary parts of the model separately, then, at first sight, they seem to be different from each other. But considered separately, the real and imaginary parts, the coefficients of which are evaluated with the help of LMS, simulate the processes in the same way.

Indeed, let us represent the real part of the model for the simplest case of Re*GCAR(1)*:

 (2.13.3)

This is an autoregression model with increment. It can be used in the case when the forecast series has a pronounced tendency to increase or decrease its indicators. In all other cases, the model will poorly describe the initial series and predict it also poorly.

The imaginary part of the complex-valued autoregression *GCAR (1)*, as follows from (2.13.2), has the following form:

 (2.13.4)

It can be transformed and reduced to this format:

. (2.13.5)

If we use the model (2.13.3) to simulate some series, then using, for example, LMS, two coefficients of this model are found at variables *yt-1* and Δ *yt-1*. When using model (2.13.5) in the simulation of the same process, the coefficients at the same variables are evaluated with the help of LMS, which means that these coefficients will be the same as those for the model (2.13.3). Therefore, when studying the properties of the model (2.13.1), it is sufficient to understand the properties of either its real or its imaginary part, since individually they behave the same way.

Let us focus on the properties of the real part of the model (2.13.2). Due to the fact that in the complex variable, some increase, corresponding to the nature of the modeled process, belongs to the imaginary part, then this increase may not relate to the current time, but to some past observation shifted relative to the current one by gobservations backwards. We will call this model Re*GCAR(p,g)* and write it like this:

. (2.13.6)

The simplest model of this class is the Re*GCAR(1, g)*, which has the form:

 (2.13.7)

The increment Δ*yt-(1+g)* in it refers to the observation (*t-1*) in the case when g*=0*... And in all other cases, when *g=1, 2, 3, 4*, … the increment refers to an observation shifted back by *g=1, 2, 3, 4* units of time, for example, for *g=5*:

 (2.13.8)

It should be noted that purely by appearance the model (2.13.3), that is, the Re*GCAR(1,0)* model, is similar to the *ARIMA(1,1,0)* model, since the latter model has this form:

 (2.13.9)

But when comparing (2.13.3) with (2.13.9), it can be seen that these are still different models. And the point is not only in the sign of the second summand of the right part of these two equalities, but that in the new model the previous value of the series level *yt-1* is corrected by the multiplier *a01*, and in the *ARIMA (1,1,0)* model this level of *yt-1* is not corrected, because it is multiplied by one. Therefore, the *ARIMA (1,1,0)* model is a particular case of the Re*GCAR(1,0)* model. And since the model has the ability to be more finely tuned during modeling to the features of the series due to the coefficient a*01* and the shift *g*, we should expect more accuracy from the new model than from the *ARIMA (1,1,0)* model. This conclusion extends to the more general case, since all *ARIMA(p,1,0)* models are a particular case of the Re*GCAR(p,0)* model.

First of all, let us see what processes are generated by the real (or imaginary) part of the model.

Since this is a model with increments, it is by definition best suited for modeling and forecasting the series that have a non-linear tendency to increase or decrease their values. Typical processes generated by the ReGCAR*(p,1)* model are shown in Fig. 2.9.

*Figure 2.9. Dynamics of the model (2.13.3) at а) a0=0,1; a1=0,1 and b) a0=-0,1; a1=-0,1*

But, since this is a model of autoregression, it can describe more complex dynamics than the monotonic type dynamics shown in Figure 2.9 at different initial values ​ ​ of the initial series and at different coefficients` values. Examples of complex nonstationary processes that can result from the application of the model in question are shown in Figure 2.10.

*Figure 2.10. Dynamics of the model (2.13.3) at a) a0=-1,5; a1=0,15 and b) a0=-0,2 a1=-1,1*

As can be understood from the above figures, the Re*GCAR(p,1)* model will work well when modeling nonstationary processes and not very well when modeling stationary processes.

Let us use the results of Table 2.17, in which two models were compared - MA*(q)* and ReCMA*(q)* on series №№. 2831, 2832, 2833 and 2834. For these series, the *MA(q)* model showed better results than Re*CMA(q)*. Of course, the *MA(q)* model was used for the series reduced to a stationary form (their first differences), that is, it is the *ARIMA (0,1, p)* model.

Table 2.19 shows the results of using two models Re*GCAR(p)* – and *MA(q)* (the same as *ARIMA (0,1, p)*) on these series.

*Table 2.19*.

*Retro-forecast results by MA(q,1) and ReGCAR(p) models on series №№ 2831,2832, 2833 и 2834*

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Shift, *p* | Re*GCAR(p)* | | | Lag, *q* | *MA(q)* | | |
| *σ*  approximation | *BIC* approximation | *σ*  retro-forecast | *σ*  approximation | *BIC* approximation | *σ*  retro-forecast |
|  | Series № 2831 | | | | | | |
| 1 | 820,606 | 6,837 | 592,440 | 1 | 867,648 | 13,588 | *691,54* |
| 2 | 141,196 | 5,079 | 104,788 | 2 | 760,202 | *13,382* | 869,50 |
| 3 | 138,502 | 5,061 | *102,625* | 3 | *728,229* | 13,416 | 837,98 |
| 4 | 140,335 | 5,076 | 104,951 | 4 | 728,229 | 13,416 | 810,32 |
| 5 | *135,972* | *5,046* | 107,968 | 5 | 730,628 | 13,485 | 818,81 |
| 6 | 139,376 | 5,072 | 104,277 | 6 | 741,244 | 13,937 | 826,50 |
|  | Series № 2832 | | | | | | |
| 1 | *2168,677* | *7,807* | 2421,121 | 1 | 2228,541 | *15,475* | 2692,69 |
| 2 | 2189,222 | 7,820 | 2403,525 | 2 | 2214,649 | 15,521 | 2491,48 |
| 3 | 2210,173 | 7,831 | *2377,044* | 3 | *2195,724* | 15,624 | 2485,63 |
| 4 | 2223,974 | 7,839 | 2409,979 | 4 | 2195,724 | 15,624 | *2186,91* |
| 5 | 2183,648 | 7,822 | 2464,281 | 5 | 2198,341 | 15,688 | 2405,83 |
| 6 | 2212,802 | 7,837 | 2426,490 | 6 | 2200,897 | 16,114 | 2375,42 |
|  | Series № 2833 | | | | | | |
| 1 | *1185,347* | *7,205* | 1357,438 | 1 | 1246,176 | *14,312* | 1251,02 |
| 2 | 1204,340 | 7,222 | 1372,047 | 2 | 1219,172 | 14,327 | 1290,42 |
| 3 | 1198,145 | 7,218 | *1347,143* | 3 | 1216,031 | 14,442 | 1239,19 |
| 4 | 1219,005 | 7,237 | 1388,979 | 4 | 1216,031 | 14,442 | 1239,80 |
| 5 | 1218,210 | 7,238 | 1403,711 | 5 | 1224,087 | 14,517 | 1231,86 |
| 6 | 1217,162 | 7,239 | 1367,331 | 6 | *1211,932* | 14,920 | *1197,68* |
|  | Series № 2834 | | | | | | |
| 1 | 42,139 | 3,866 | 31,602 | 1 | 40,721 | 7,470 | *36,33* |
| 2 | 39,950 | 3,815 | 30,525 | 2 | 38,604 | *7,422* | 36,70 |
| 3 | 39,981 | 3,817 | *30,188* | 3 | 36,813 | 7,447 | 36,27 |
| 4 | 40,468 | 3,830 | 32,261 | 4 | 36,813 | 7,447 | 47,27 |
| 5 | 40,926 | 3,843 | 30,709 | 5 | 36,700 | 7,503 | 44,54 |
| 6 | *38,349* | *3,780* | 31,797 | 6 | *36,564* | 7,919 | 43,71 |

As can be seen, the new Re*GCAR(p)* model for series № 2831 and №2834 works better than the *MA(q)* model. And it works worse for the series № 2832 and № 2833.

Why does the new model better describe and predict exactly two series № 2831 and № 2834? Because these two series have some non-stationary tendency (Fig 2.9).

*Figure 2.9. Dynamics of series № 2831 and № 2834.*

And the other two series do not have such a tendency and can be considered as stationary ones (Fig. 2.10).

*Figure 2.10. Dynamics of the series № 2832 and № 2833.*

If *AR(p)* model is used instead of the *MA(q)* model, then similar results will be obtained – for series that have a non-stationary character, the autoregression model with increments is often more accurate than autoregression models applied to the same series reduced to a stationary form.

As an example, I will give the result of a retro-forecast for series № 2840, which consists of 103 values and has an obviously non-stationary form. Its first difference completely makes this series stationary, so the simple autoregression model *AR(p)* can be applied to the first differences. By changing in this model *p* from 1 to 10, the best model was determined. It turned out to be the *AR (1)* model, the best one both by the information criterion *BIC= 3,934*, and by the standard error of the one-step retro- forecast on the last 36 values of the series *σ=40,512*.

A range of Re*GCAR(g)* models was built on the same non-stationary series with *g* varying from 1 to 10. For this model, the model with a shift by one also turned out to be the best: Re*GCAR(1).* The information criterion for it was equal to *BIC= 3,919*, that is, less than for the *AR (1)* model. And the standard error of the retro-forecast for this model turned out to be less - σ *=* *39,571*.

**Conclusion**

As soon as the scientists engaged in mathematical statistics abandon the assumption that the real and imaginary parts of a complex random variable are independent of each other, they will immediately have amazing opportunities for extensive scientific research in terms of the development of mathematical statistics of a complex random variable. In the first chapter of the monograph, this has been done - by taking as a basis the statement that the real and imaginary parts of a complex random variable are correlated with each other, we managed to solve several "dead-end" problems of modern mathematical statistics of a complex random variable.

A key role in mathematical statistics of a complex random variable is played by a complex variance – the one some modern scientists call "pseudo-variance". In fact, it is exactly this – complex – that the variance of a complex random variable should be. Such variance has real and imaginary parts.

The monograph reveals that the real part of the complex variance shows how different the variances of the real and imaginary parts are from each other, that is, how much the variation of the real component of a complex random variable differs from the variation of the imaginary part of this variable. And the imaginary part of the complex variance is the covariance between the real and imaginary parts of a complex random variable multiplied by two. It is known that the covariance of two independent random variables is equal to zero. Therefore, in case when the variance of a complex random variable with real and imaginary parts independent of each other is calculated, the complex variance becomes real.

A simple rule for working with a complex random variable follows from this- there is no need to ask whether there is an interrelation between the real and imaginary parts of a complex random variable or not. The calculation of the complex variance and the value of its imaginary part will automatically answer this question. However, it is possible to pre-calculate the value of the complex

coefficient of pair correlation between two complex random variables.

The properties of this complex coefficient of pair correlation have been generally studied in the monograph, but many properties of this coefficient have not been discovered and studied yet. Particularly, studies have not been conducted yet on what the relationship is between this coefficient and the coefficient of pair correlation of two real variables. So far, it is not yet clear how to calculate confidence limits for sampled values of the complex coefficient of pair correlation, etc.

The least-squares method for complex random variables, substantiated in the first chapter of the monograph, allows, according to statistical data to construct complex-valued regressions of any complexity, including complex autoregressions.

But since the coefficients of such models are sampled values, the task of estimating their confidence limits arises. In the first chapter of the monograph, such a method was proposed. The coefficients of the corresponding s-statistics have been proposed as well, those which allow to construct confidence limits. These coefficients were obtained by me not as a result of a thorough analysis of the scattering ellipse of a complex random variable, and not as a result of using a Gaussian probability distribution, but on the basis of computer experiments linking the Student's *t*-statistics of real statistic characteristics with a complex form.

In future these values of s-statistics can be defined more precisely, but today they allow for a satisfactory solution to the task at hand.

After developing the missing sections of mathematical statistics of a complex random variable, it turned out to be possible to proceed to the solution of the main task of this study – the construction of complex-valued autoregressions.

These complex-valued *CAR* autoregressions represent a new class of models that have not been previously studied by other scientists.

As shown in the monograph, all these new autoregression models can successfully compete with autoregression models of real variables. And they should be used together along with existing models, each time, as a result of the competition, choosing the best model.

As for the complex vector autoregressions *CARIMAk (p, d,q)*, I can definitely state that they should be used instead of vector autoregressions *VARIMA(p,d,q)*.

These new vector autoregressions are inferior in approximation accuracy to vector autoregressions of real variables *VARIMAk(p, d,q)* due to the fact that complex vector autoregressions *CARIMAk (p,d,q)* have significantly smaller number of coefficients. But this loss in approximation accuracy is not so significant that these models prove to be ineffective. On the contrary, precisely because the number of coefficients of the new class *CARIMAk(p,d,q)* models is (with an even number of vector elements) twice less than that of the simple vector autoregression *VARIMAk(p,d,q)*, the Bayesian information criterion *BIC* will always be less for complex vector autoregressions than for vector autoregressions in real form.

The monograph only outlines the contours of an approach to the formation and use in practice of *CARIMAk(p,d,q)* models. Their research and practical application will certainly result in emerging new problems facing the researcher, the essence of which is not clear today. But in any case, it can be argued that the *CARIMAk(p, d,q)* models are the future of short-term modeling and forecasting of complex multidimensional stochastic processes.

The properties and advantages of one-dimensional complex-valued models are not as obvious as the advantages of *CARIMAk(p,d,q)* over vector autoregressions of real variables. Nevertheless, the Re*CARE(p,d,g)* models, as was shown in paragraphs 2.7 – 2.10, have very good approximation and forecasting properties. In the cases studied in the monograph, they turned out to be better than the *ARIMA(p,d,q)* models. This, however, does not mean at all that in all other cases these models will be better and more accurate than *ARIMA(p,d,q)*. But the Re*CARE(p,d,g)* models can compete successfully with *ARIMA(p,d,q)*, and therefore already today it is possible to recommend them for use in the tasks of short-term forecasting or simulation of stochastic processes.

Other types of one-dimensional complex-valued autoregressions – *CMA(p,d,g)*, *CTAR*(*p,d,g*) and *GCAR*(*p,d,g*) show good results in some cases and not so good results in others. All of them can be developed and modernized, as a result of which they can take the form of models that can also successfully compete with the best autoregressions models of real variables. Concerning the *CTAR* (*p, d, g*) and *GCAR*(*p,d,g*) models, the direction of such modernization is obvious – the error *εt* is not used in them. Compared to models of the same kind *AR(p)* or *MA(q)*, these new models look very decent. Therefore, taking into account the error ε*t* in the form of *MA(q)* or *CMA(q)* will allow to form and use new classes of *CTARMA* (*p, d, q, g*) and *GCARMA* (*p, d, q, g*) models, which, with the support of such a component, will have both higher approximation properties and more accurate forecasting properties. But the study of this possibility is the subject of further scientific research.

In any case, the tools of modeling stochastic processes and their short-term forecasting are complemented by the use of complex-valued autoregression models. And this means that the researcher can operate with more diverse models than before and obtain more accurate models of random stochastic processes.

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