

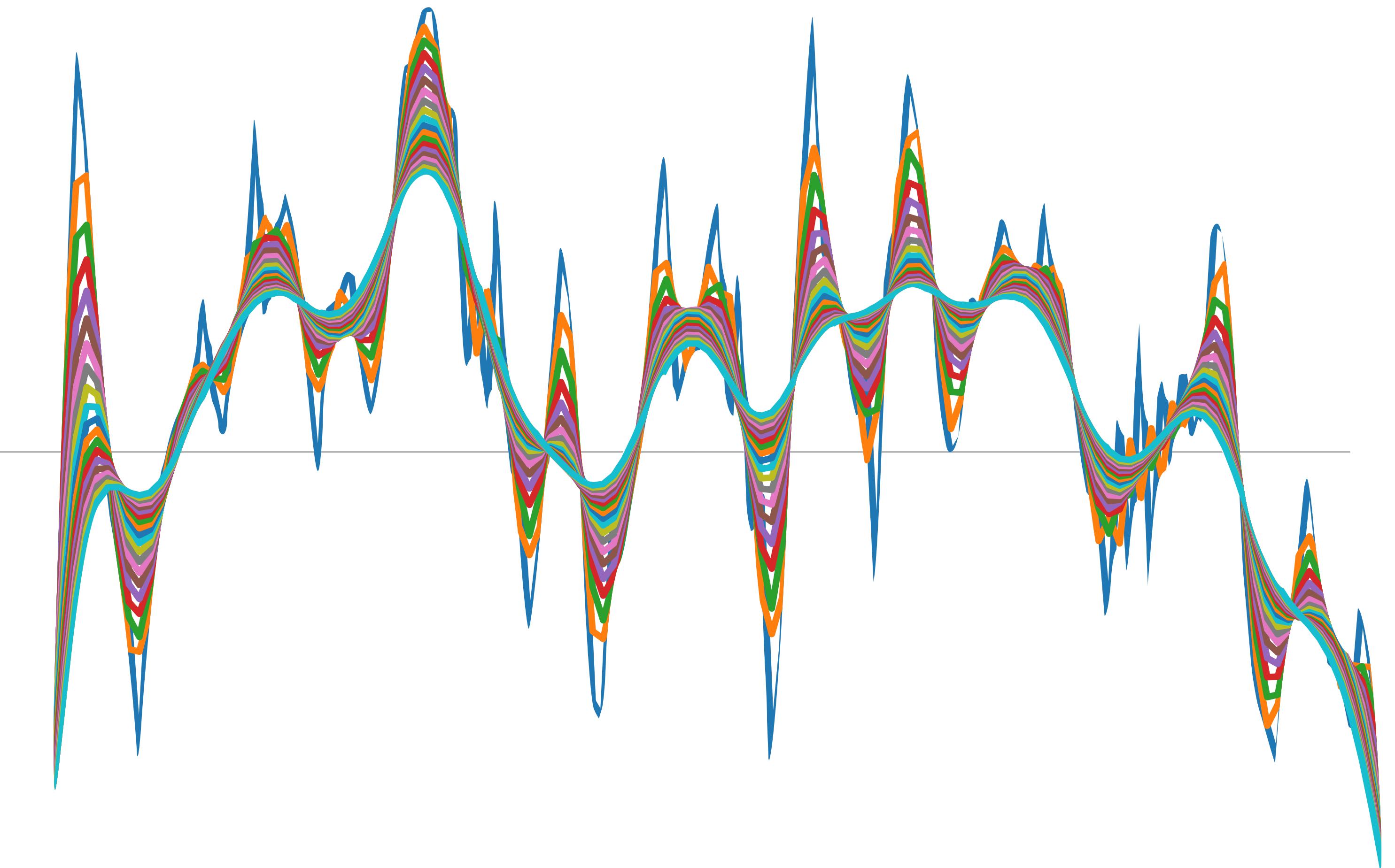
Multigrid Methods

The Basics

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What are we trying to do...

- Solve problems of the form

$$Ax = b$$

- Solve this problem **iteratively**:

$$x_1 \leftarrow x_0 + v$$

- Solve this problem **inexpensively**:

- The update should be “good”
- Finding the update should be “cheap”

Objectives – high level

- Construct a multigrid method for **your** problem
- Interpret the effectiveness of a multigrid method
- Identify *why* a method works — or — *why* a method does not
- Recognize different forms of multigrid, their pitfalls and their uses

Objectives – this session

- **Create** a two level multigrid method
- **Illustrate** the main components of a multigrid method
- **Calculate** the effectiveness of a multigrid method
- **Highlight** some limitations of multigrid
- **Outline** some approaches to fixing multigrid (and why)
- **Identify** key pieces in moving to unstructured problems

A cautionary example

Guess and look for
and update

$$x_1 = x_0 + \text{update}$$

Updating with the
error would be **ideal**

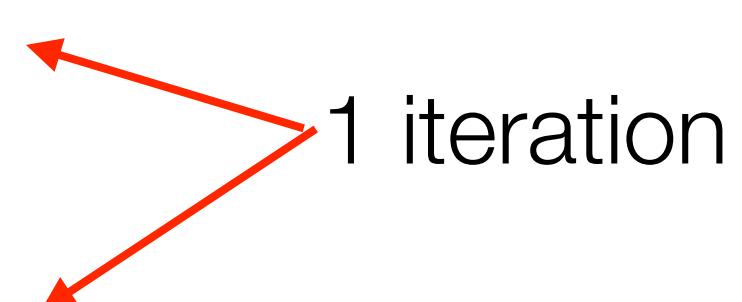
Or in another form

$$x_1 = x_0 + A^{-1}r_0$$

Not practical so...

$$x_1 = x_0 + D^{-1}r_0$$

1 iteration



x^* solution to $Ax = b$

$$e_0 = x^* - x_0 \quad \text{error}$$

$$\begin{aligned} r_0 &= b - Ax_0 \\ &= Ae_0 \end{aligned} \quad \text{residual}$$

$$Ae_0 = r_0 \quad \text{error equation}$$

Jacobi ~ 1 SpMV

A reminder – projection methods

- Take a guess

$$x_0$$

- Look for an update that is the “best”:

$$x_1 \leftarrow x_0 + u$$

- Minimize over a smaller space

$$\min_{u \in \text{span}\{V\}} \|x^* - x_1\|$$

- Then $u = Vy$

$$V^T V y = V^T e_0$$

- So the update looks like

$$x_1 = x_0 + V(V^T V)^{-1} V^T e_0$$

A reminder – projection methods

- Instead, look at the A-norm:

$$\min_{u \in \text{span}\{V\}} \|x^* - x_1\|_A$$

- Then $u = V\bar{y}$

$$V^T A V \bar{y} = V^T A e_0$$

$$V^T A V \bar{y} = V^T r_0$$

- So that

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

A reminder – projection methods

$$x_1 = x_0 + u$$

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

- What about the error

$$x^* - x_1 = x^* - x_0 - V(V^T A V)^{-1} V^T r_0$$

$$e_1 = e_0 - V(V^T A V)^{-1} V^T A e_0$$

$$= \left(I - V(V^T A V)^{-1} V^T A \right) e_0$$

A-orthogonal
projection onto the
range of V

Model Problem

- A model problem

$$\begin{aligned}-u_{xx} &= f \\ u(0) &= u(1) = 0\end{aligned}$$

- Finite differences

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i \quad i = 1, \dots, n \quad u_0 = u_{n+1} = 0$$

- A model matrix problem

$$A = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

Model Problem

- A special matrix:

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix}$$

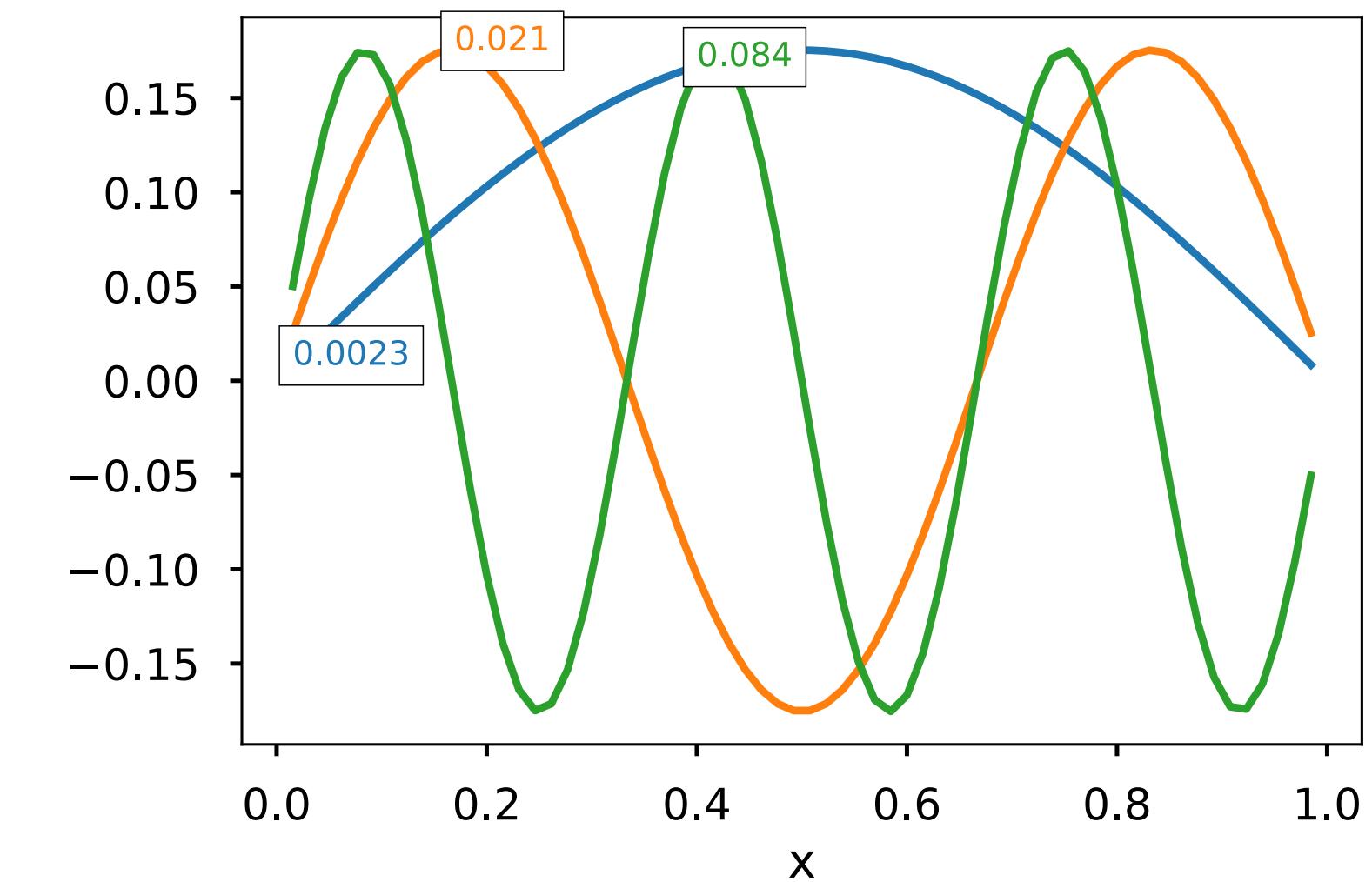
The eigenvalues

range from (0,4]

(or from h^2 to 4)

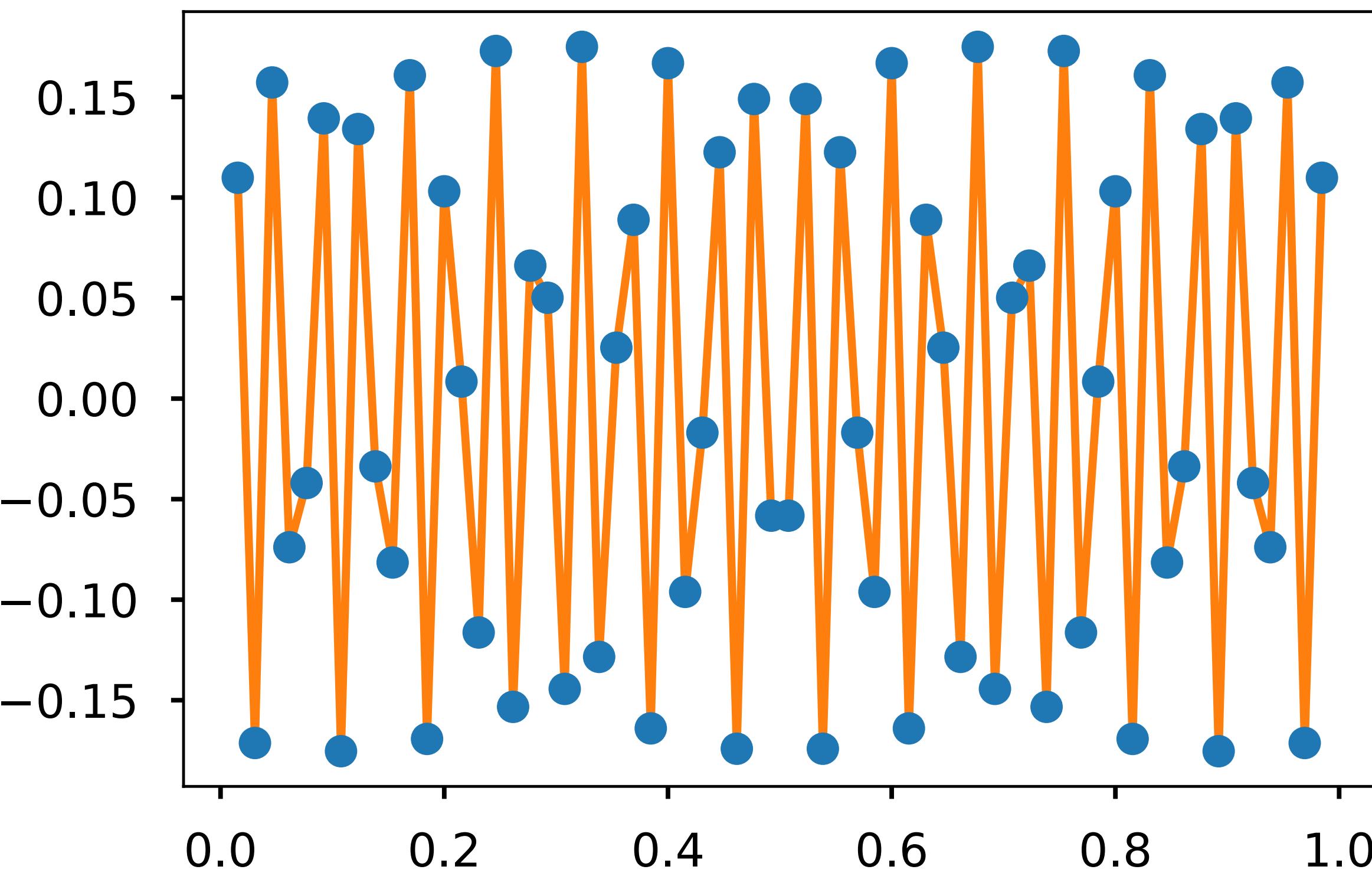
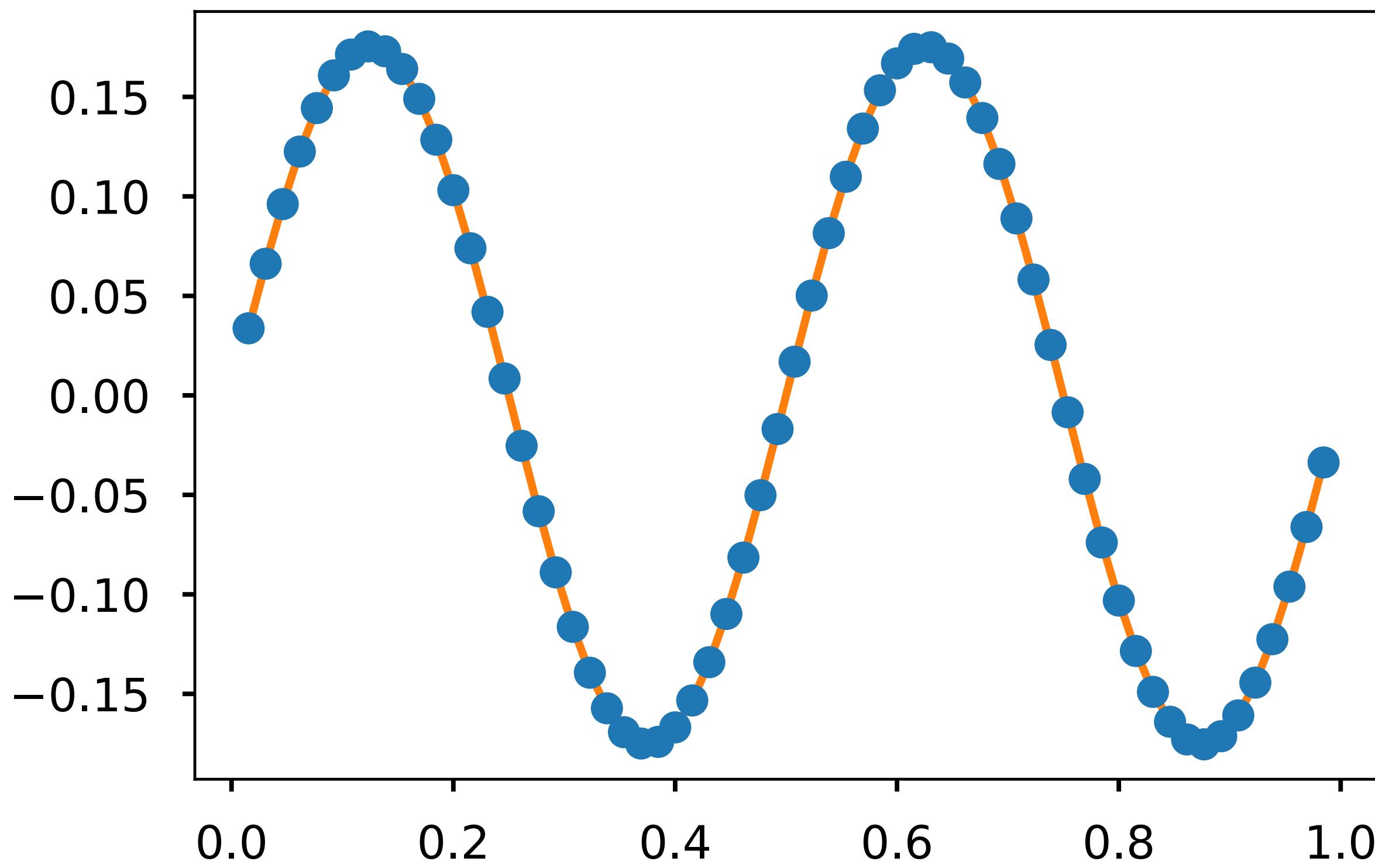
The eigenvectors are Fourier modes:

$$(v_k)_j = \sin\left(\frac{(j+1)*k\pi}{n+1}\right)$$



Smooth Mode ~ low Fourier Modes

- We will talk a lot about “smoothness” – how much variation there is **per** grid point.



First relaxation scheme: Jacobi

- The discretization at point i :

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i$$

- Solving for the variable at this point (eliminating the residual):

$$u_i \leftarrow \frac{1}{2} (u_{i-1} + u_{i+1} + h^2 f_i)$$

- In matrix form:

$$u \leftarrow (I - D^{-1} A)u + h^2 D^{-1} f$$

$$u \leftarrow u + D^{-1} r$$

- And the error:

$$e \leftarrow Ge$$

$$G = I - D^{-1} A$$

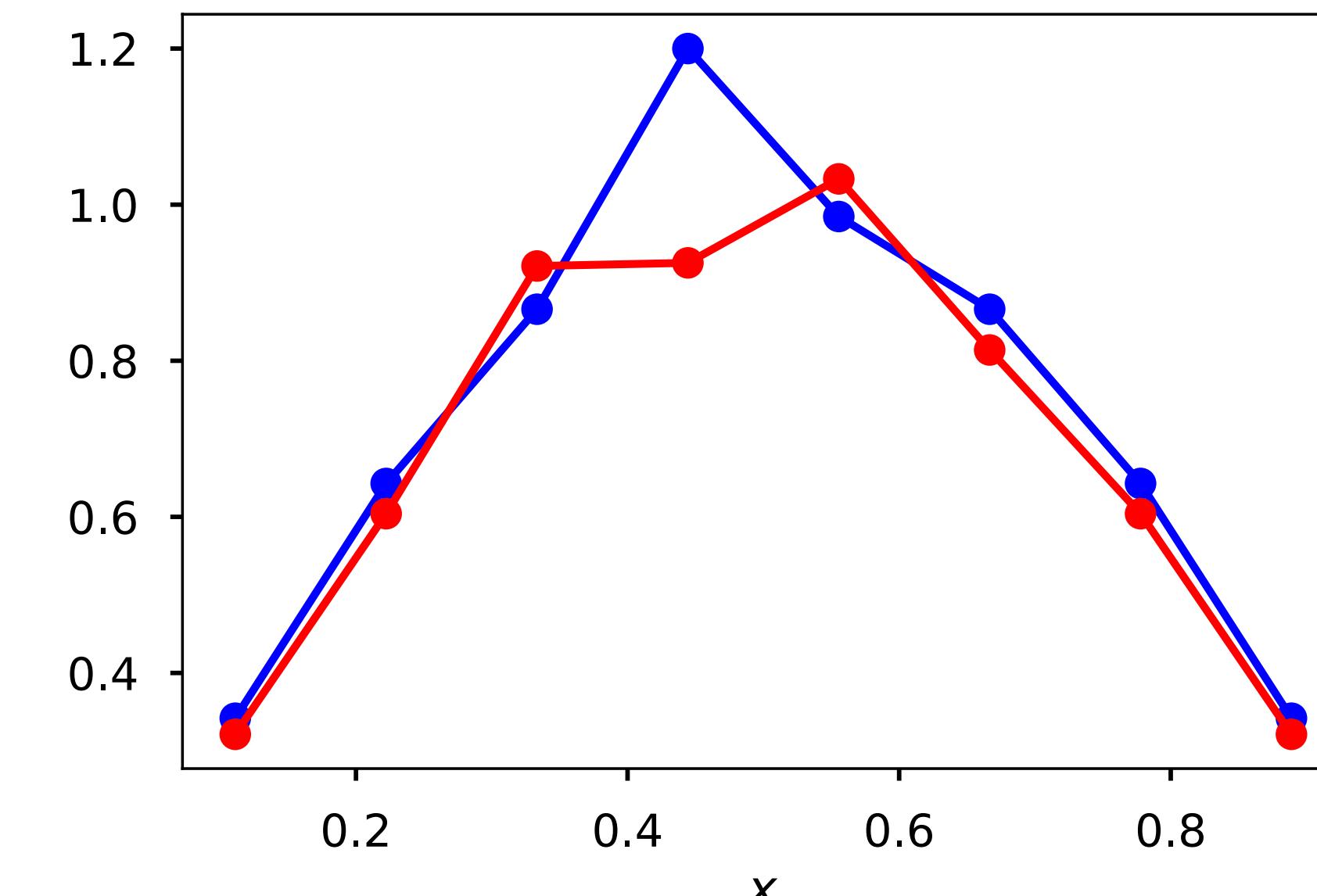
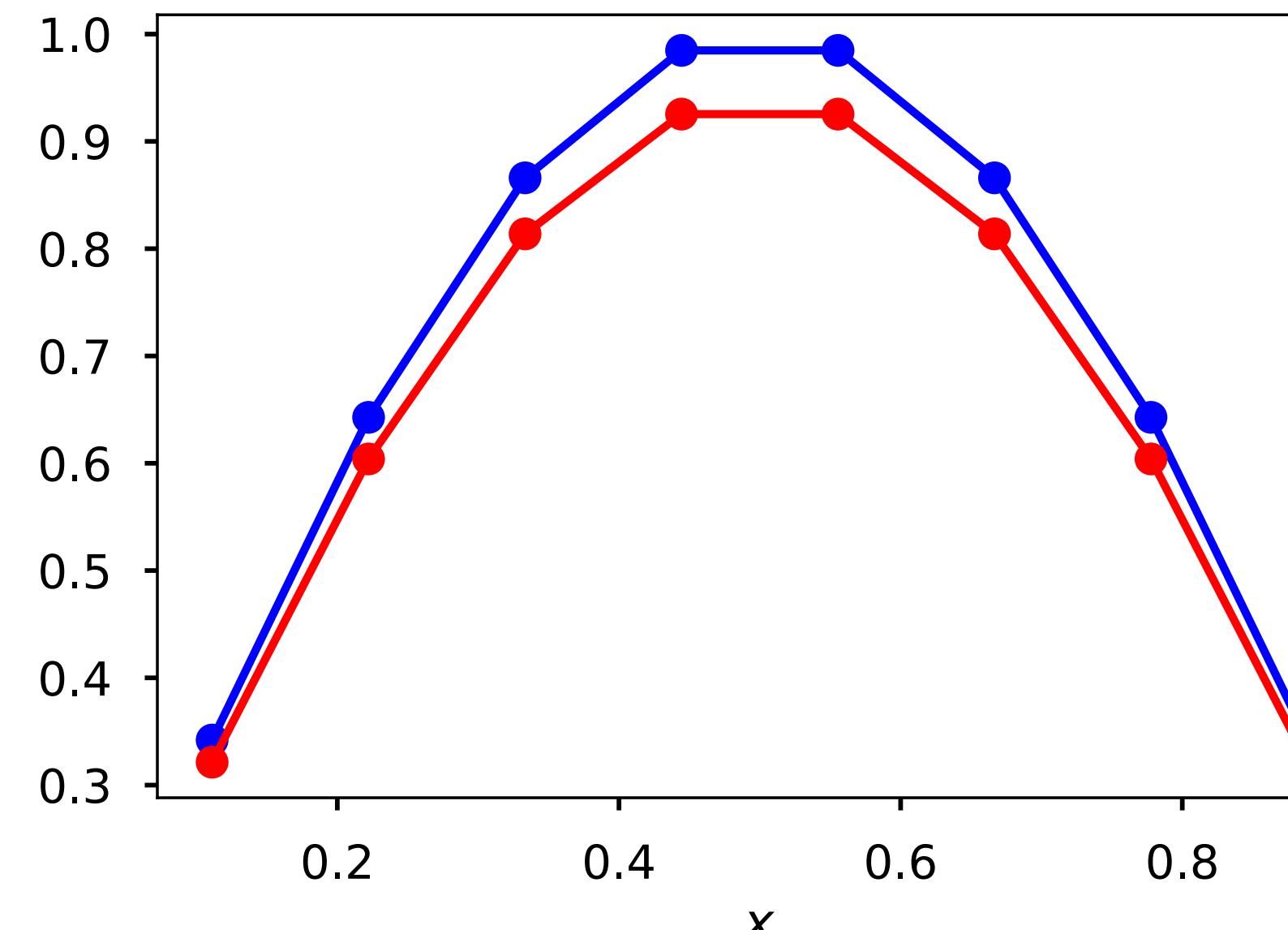
The “error propagation operator”

What does Jacobi do to error?

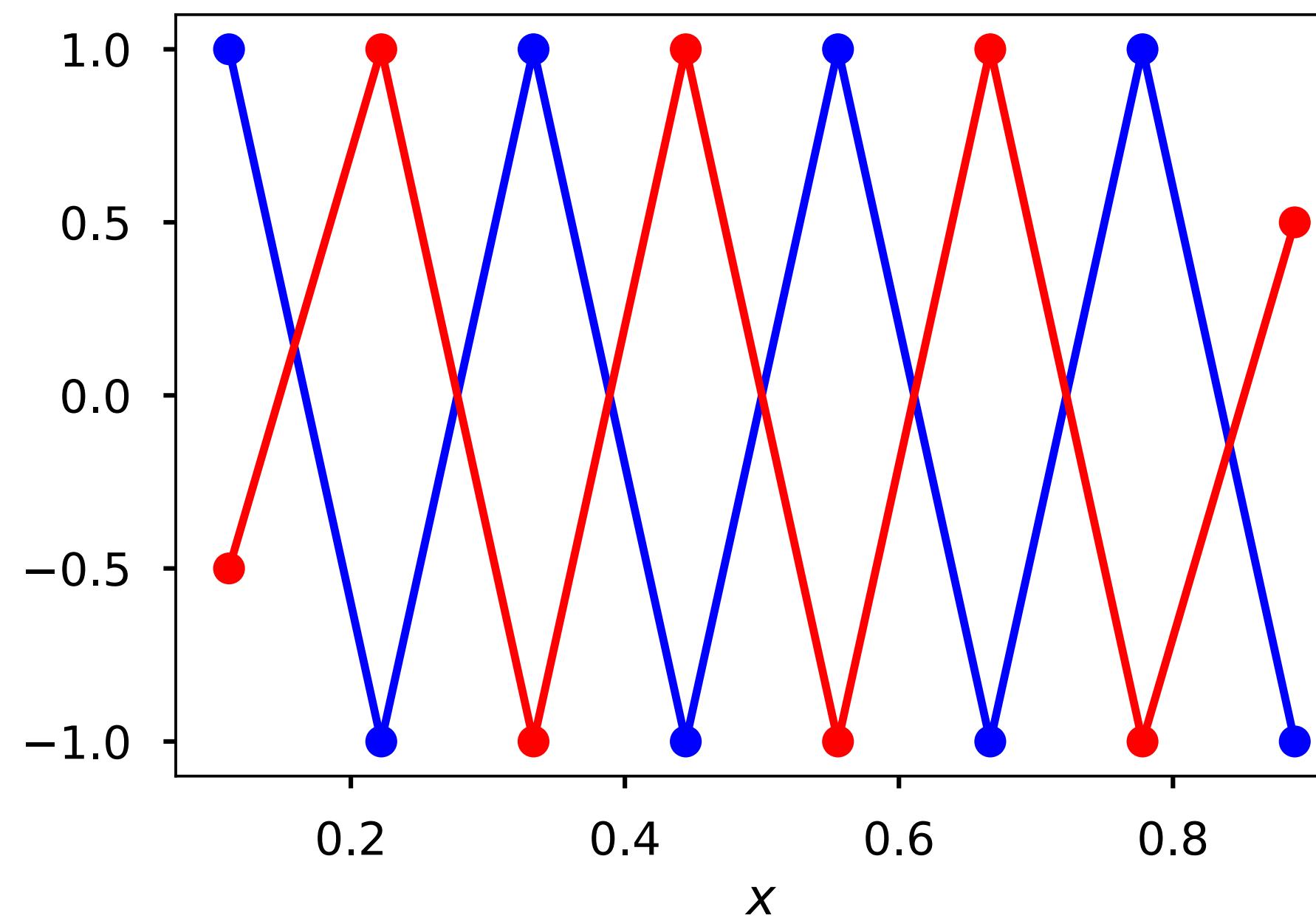
$$e^{new} \leftarrow (I - D^{-1}A)e^{old}$$

$$e_i^{new} \leftarrow \frac{1}{2} (e_{i-1}^{old} + e_{i+1}^{old})$$

- It averages...
- Consider smooth and oscillatory error:



But...

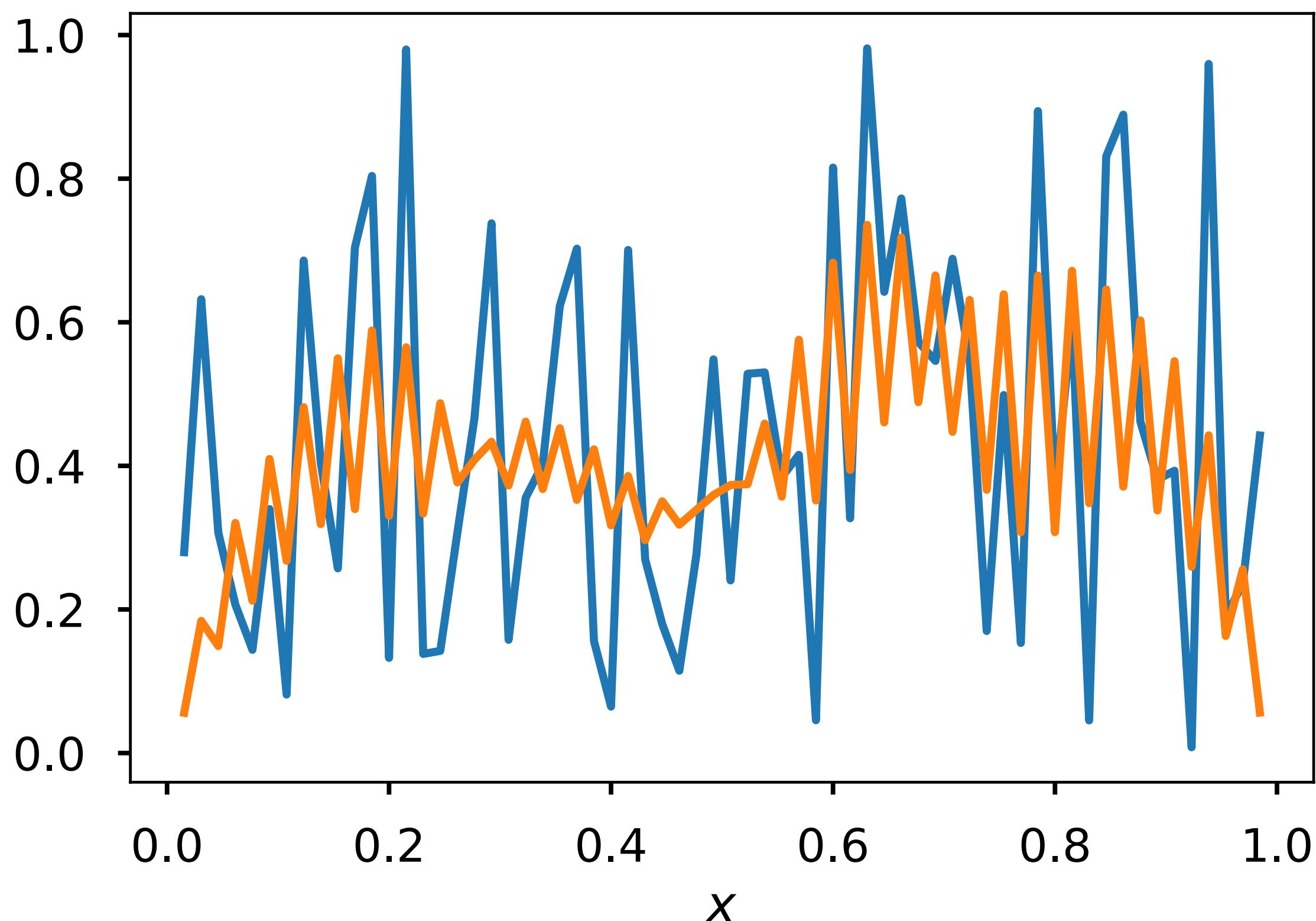


- Jacobi “averages out” certain error very quickly.
- But stagnates on very smooth error or very oscillatory error

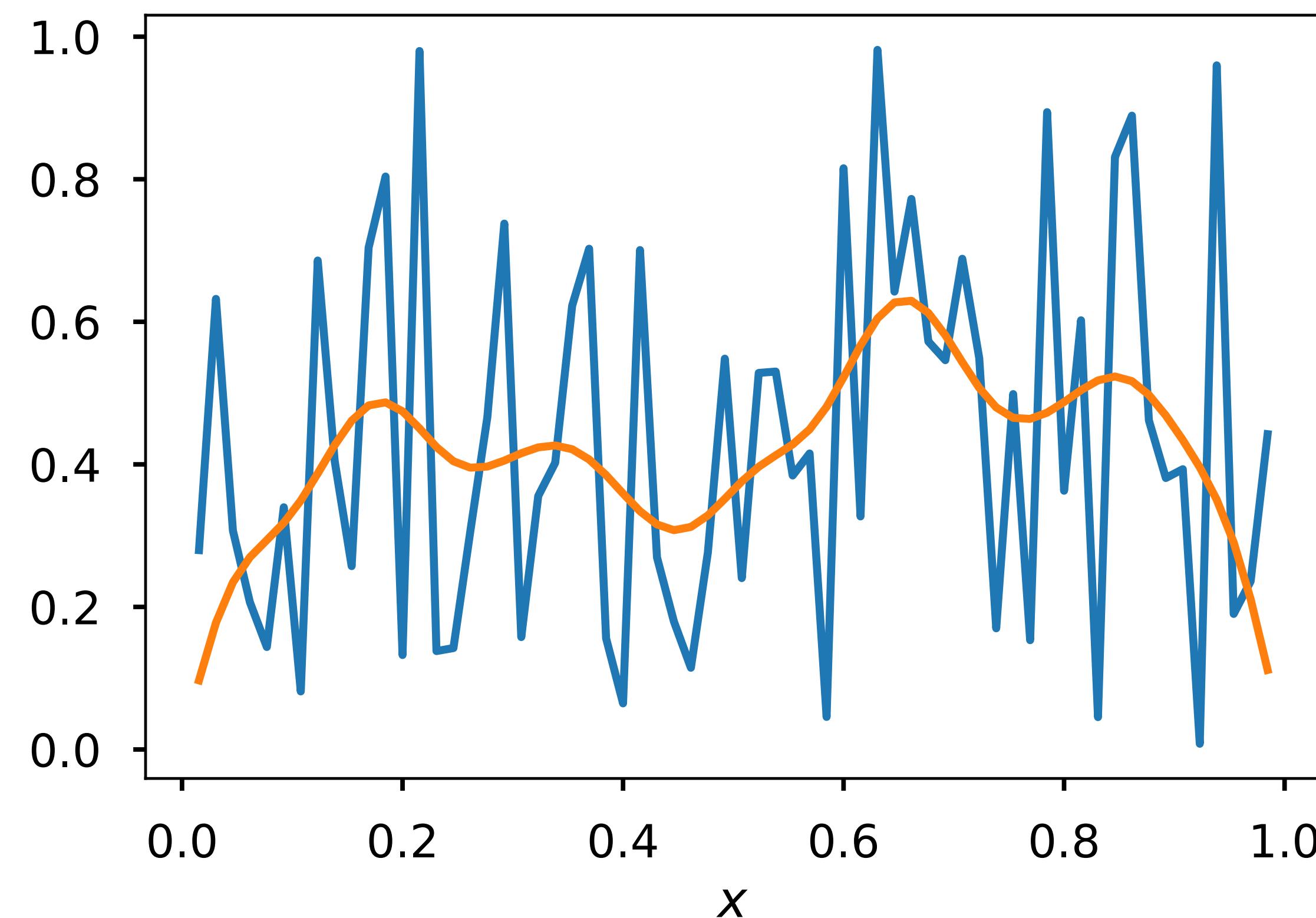
From Jacobi to weighted-Jacobi

- Random initial guess (random error):

$$u \leftarrow u + D^{-1}r$$



$$u \leftarrow u + (2/3)D^{-1}r$$



Weighted Jacobi

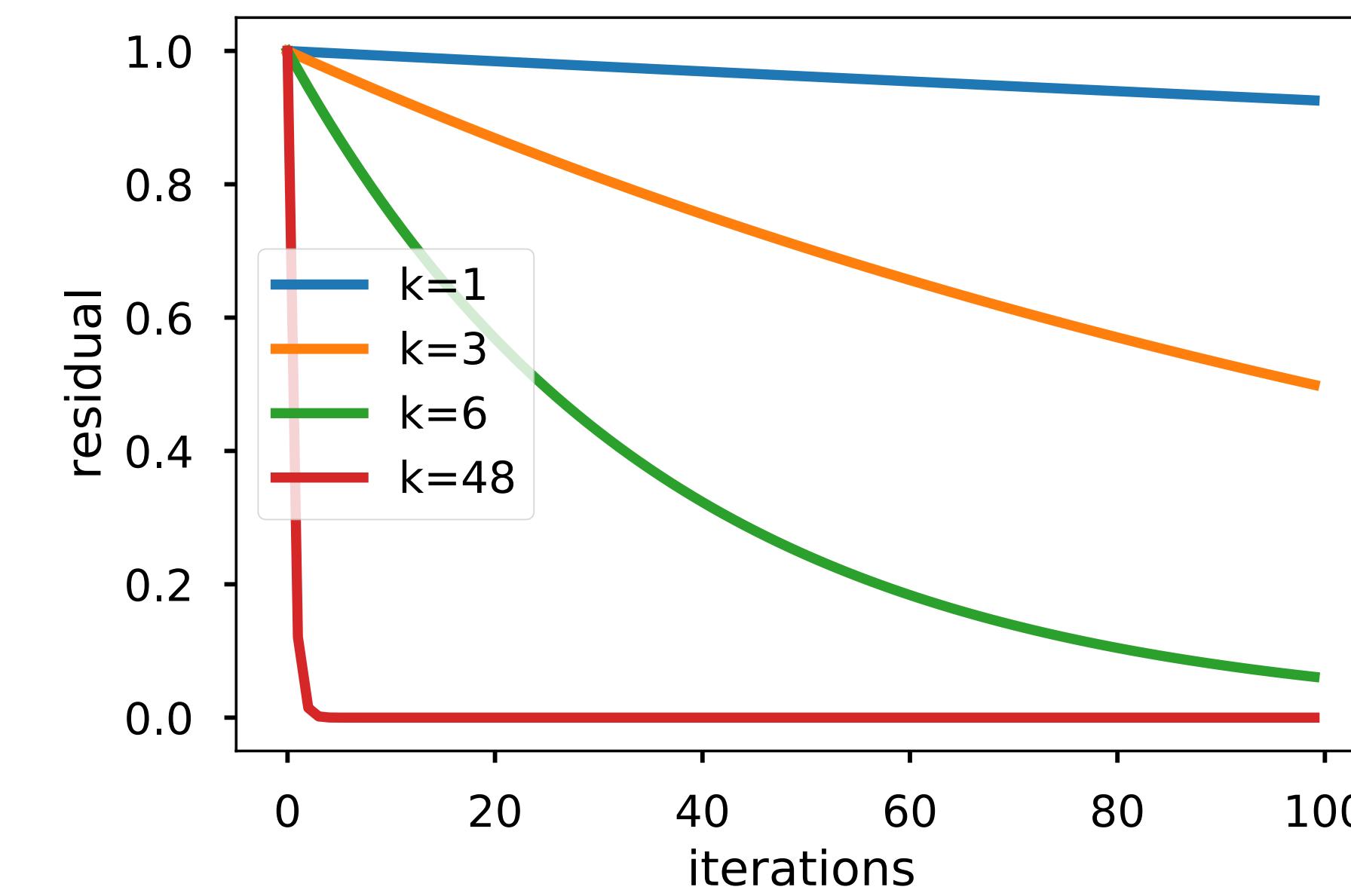
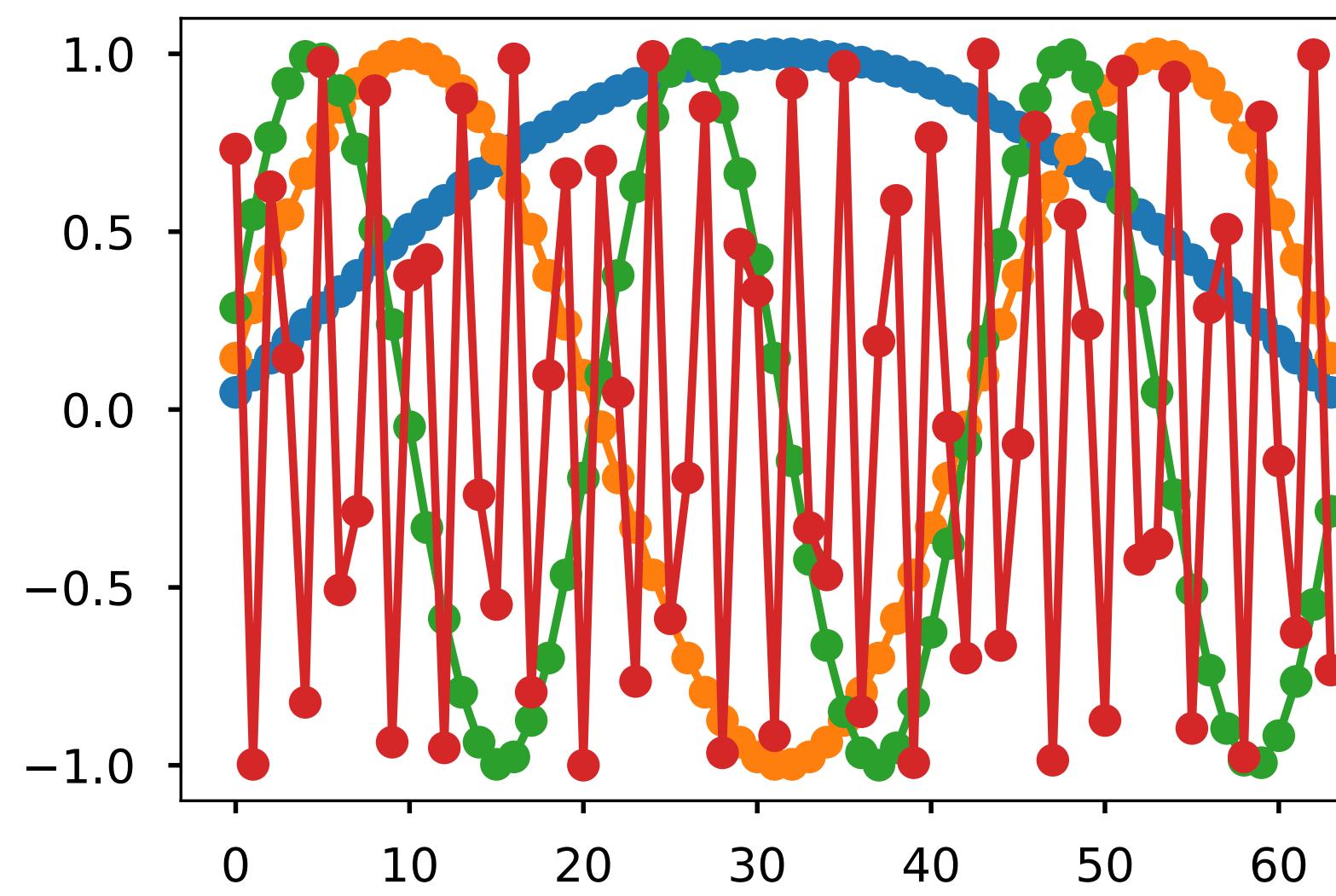
- Weighted Jacobi

$$u \leftarrow u + (2/3)D^{-1}r$$

- What does this do to other modes?
 - Are smooth modes damped slowly?
 - What about oscillatory modes?
- Why did we pick 2/3 – that seemed like a lucky guess?!

Weighted Jacobi

- If we picked 4 modes, 1, 3, 6, and 48
- Then smooth modes still dampen less quickly than higher ones



Error Propagation

- Let's consider an initial error

$$e_0 = \sum_{k=1}^n c_k v_k$$

- And the weight Jacobi iteration matrix

$$e \leftarrow (I - \omega D^{-1} A)e = Ge$$

- From ν iterations we have

$$\begin{aligned} G^\nu e_0 &= \sum_{k=1}^n c_k G^\nu v_k \\ &= \sum_{k=1}^n c_k \lambda_k^\nu v_k \end{aligned}$$

- As a result, mode **k** is reduced by the magnitude of λ_k in every pass

Fundamental Theorem of Iteration

$$G = I - M^{-1}A$$

- Convergent ($G^n \rightarrow 0$) if and only if $\rho(G) \leq 1$
- Suppose A is s.p.d.
$$\frac{\|e_n\|}{\|e_0\|} \leq \|G^n\| \leq \|G\|^n \approx 10^{-d}$$
- How many iterations do we need to guarantee the reduction of the error by d digits?

$$n \approx -\frac{d}{\log_{10} \rho(G)}$$

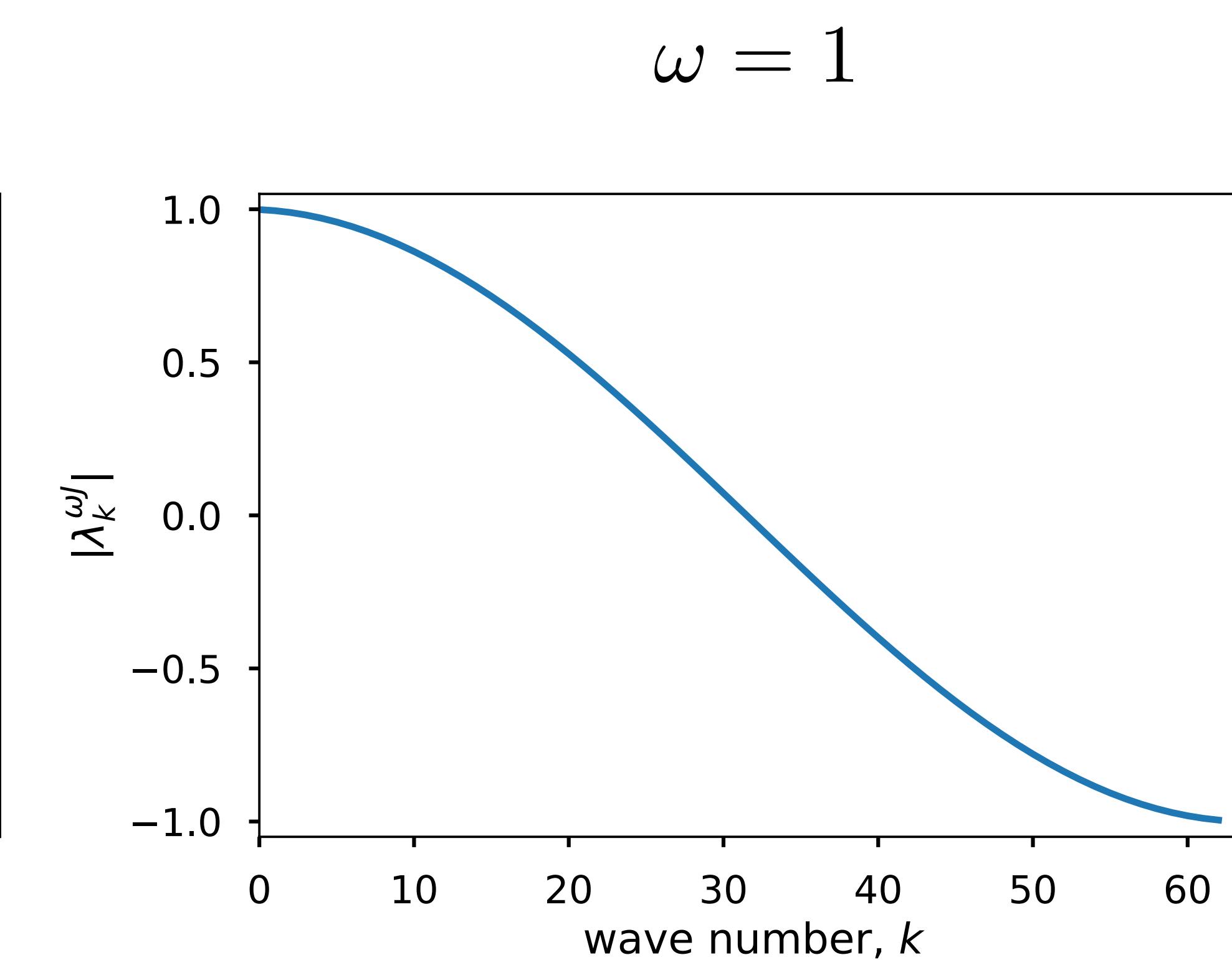
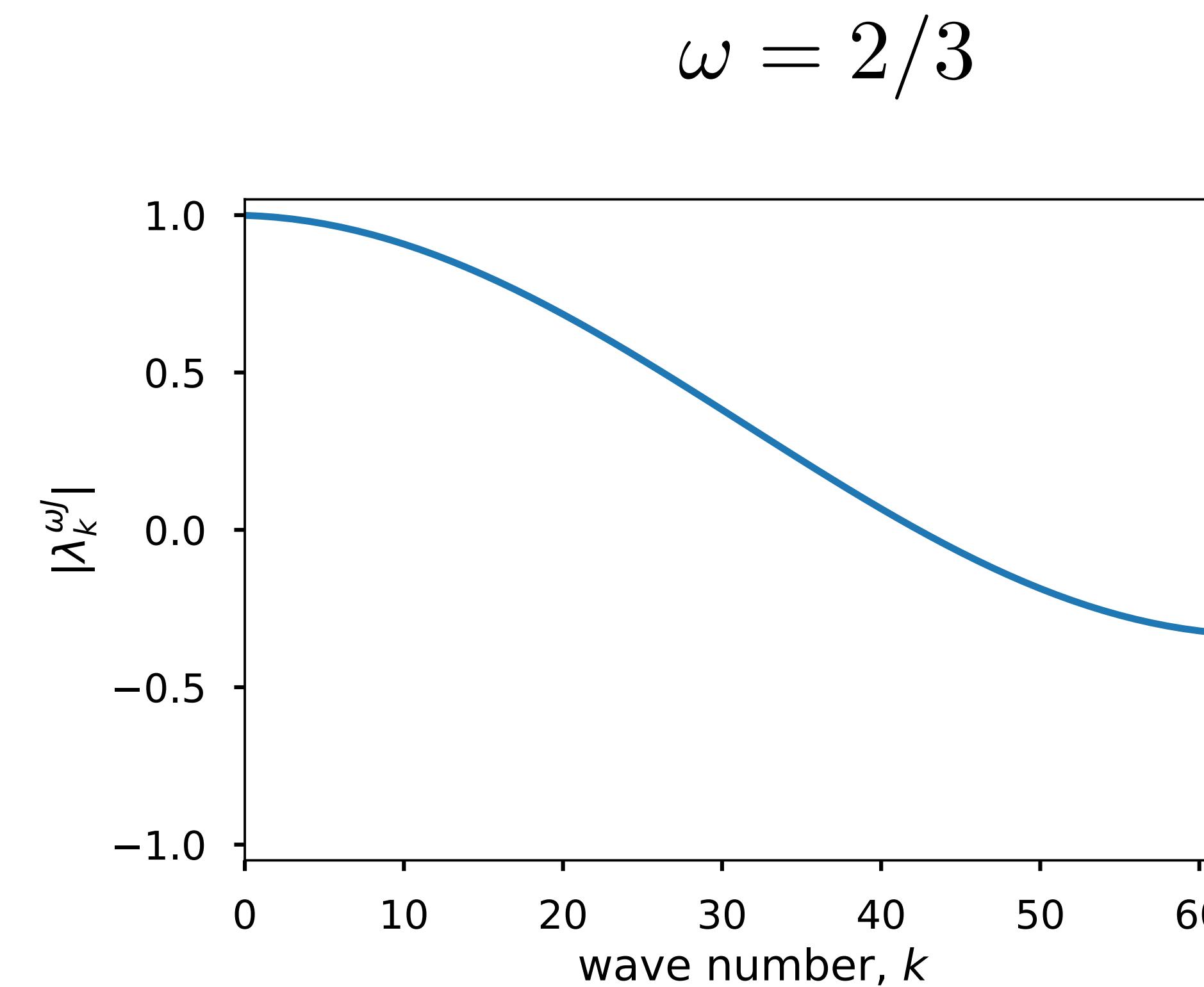
Relaxation

- Convergence **factor** $\|G\|$ or $\rho(G)$
- Convergence **rate** $-\log_{10} \rho(G)$
- For Jacobi:
$$G = I - D^{-1}A \longrightarrow \lambda_k = 1 - \frac{1}{2} \cdot 4 \cdot \sin^2 \left(\frac{k\pi}{2(n+1)} \right)$$
$$G = I - (2/3)D^{-1}A \longrightarrow \lambda_k = 1 - \frac{2}{3} \cdot \frac{1}{2} \cdot 4 \cdot \sin^2 \left(\frac{k\pi}{2(n+1)} \right)$$

The spectral radius is the same, but...

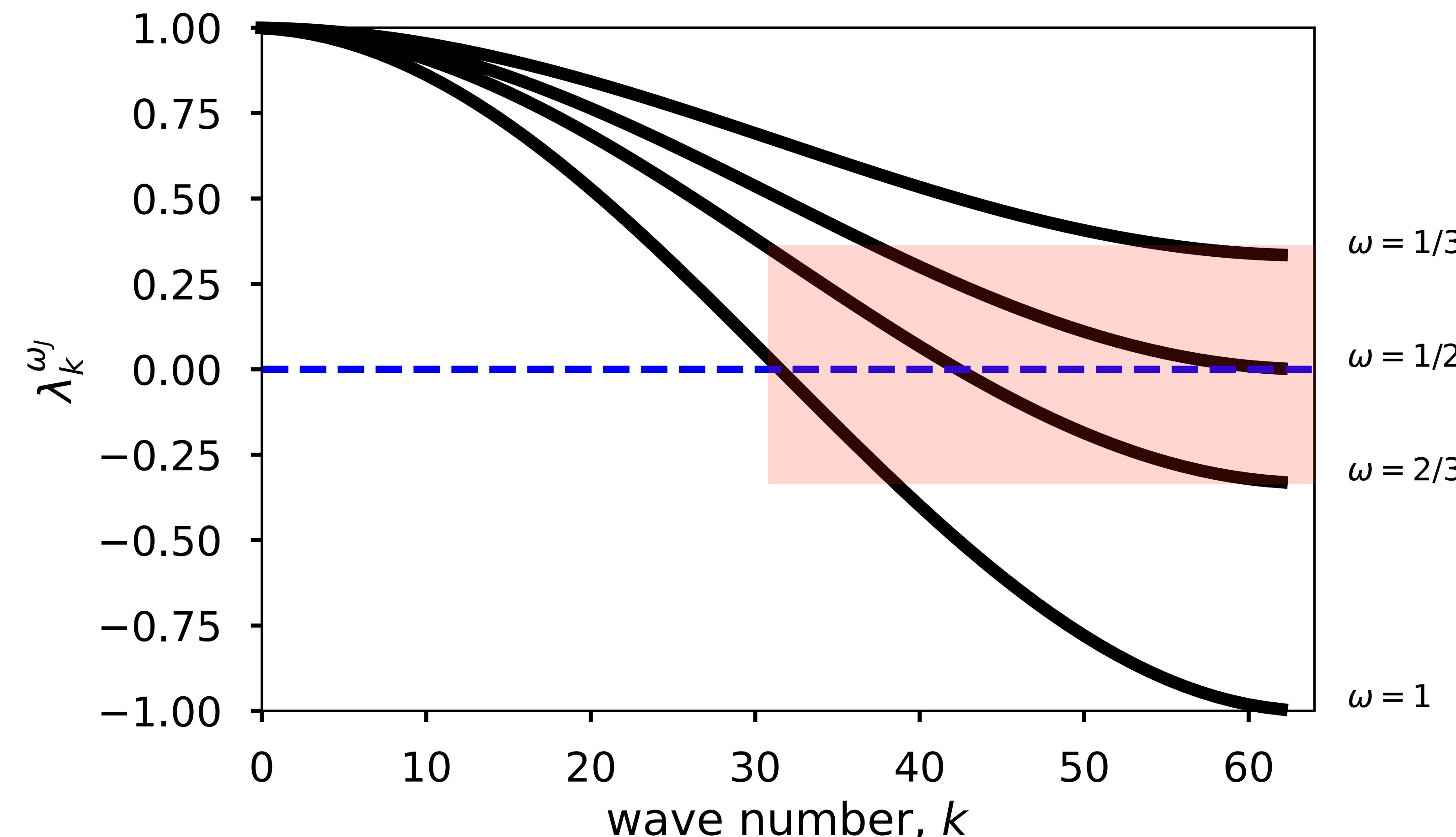
Weighted Jacobi

- If we look at the spectrum:
 - Weighted Jacobi dampens modes that are highly oscillatory



Weighted Jacobi

- Selecting 2/3 balances the reduction in error in the **high** modes



The multigrid smoothing factor

- The **smoothing factor** of relaxation method G is the maximum magnitude of the upper half of the spectrum:

$$\max_{k \in [n/2, n]} |\lambda_k^G|$$

- A common feature:

Oscillatory modes are quick to converge
Smooth modes are slow to converge

Multigrid Step #1: pick a smoother

- For $\omega = 2/3$

$$|\lambda_{n/2}| = |\lambda_n| = 1/3$$

- For $\omega = 1$

$$|\lambda_{n/2}| = |\lambda_n| = 1$$

- Jacobi is not a smoother (weighted *is*)

An important observation on “smoothness”

- So far, we've mainly looked at

$$Au = 0$$

- In general we need to consider

$$Au = f$$

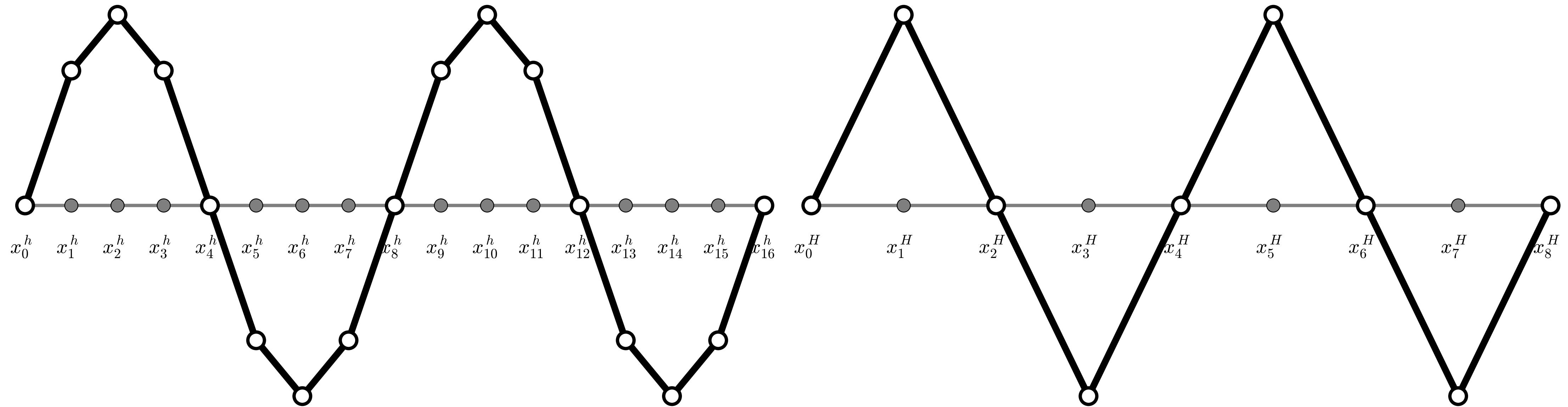
- If we **smooth** with

$$u \leftarrow u + \omega D^{-1}r$$

then the **error** is smooth, not (necessarily the solution).

Coarse Grids

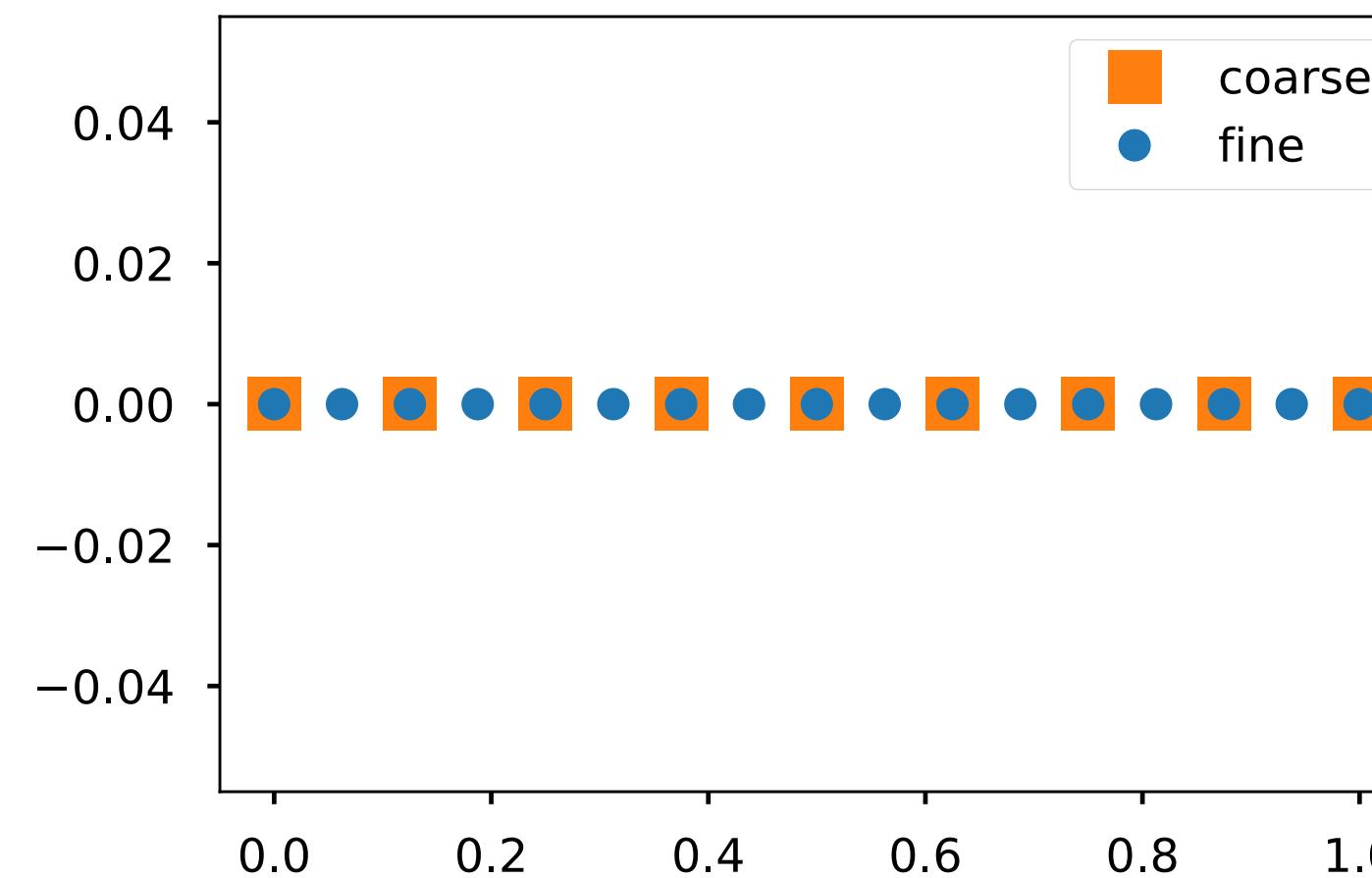
- Smooth modes look like oscillatory modes when sampled on a coarse grid
- 4-mode of 15 **versus** 4-mode of 7



This looks like a “smooth” mode

This looks like an “oscillatory” mode

Coarse modes

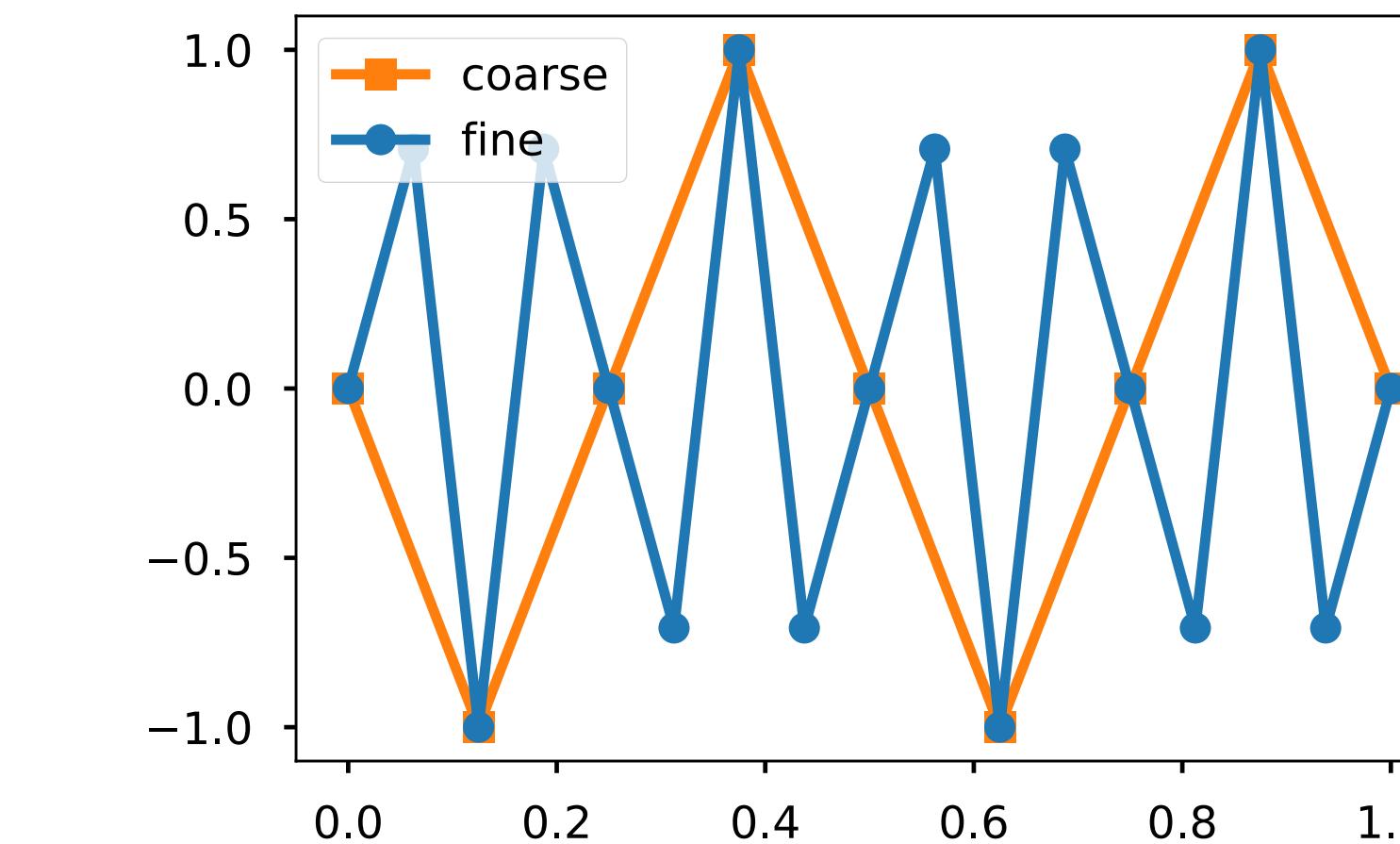


$$\begin{aligned}
 (v_k)_j &= \sin \frac{j k \pi}{n+1} \\
 (v_k)_{2j} &= \sin \frac{2 j k \pi}{n+1} \\
 &= \sin \frac{j k \pi}{(n+1)/2} \\
 &= (\hat{v}_k)_j
 \end{aligned}$$

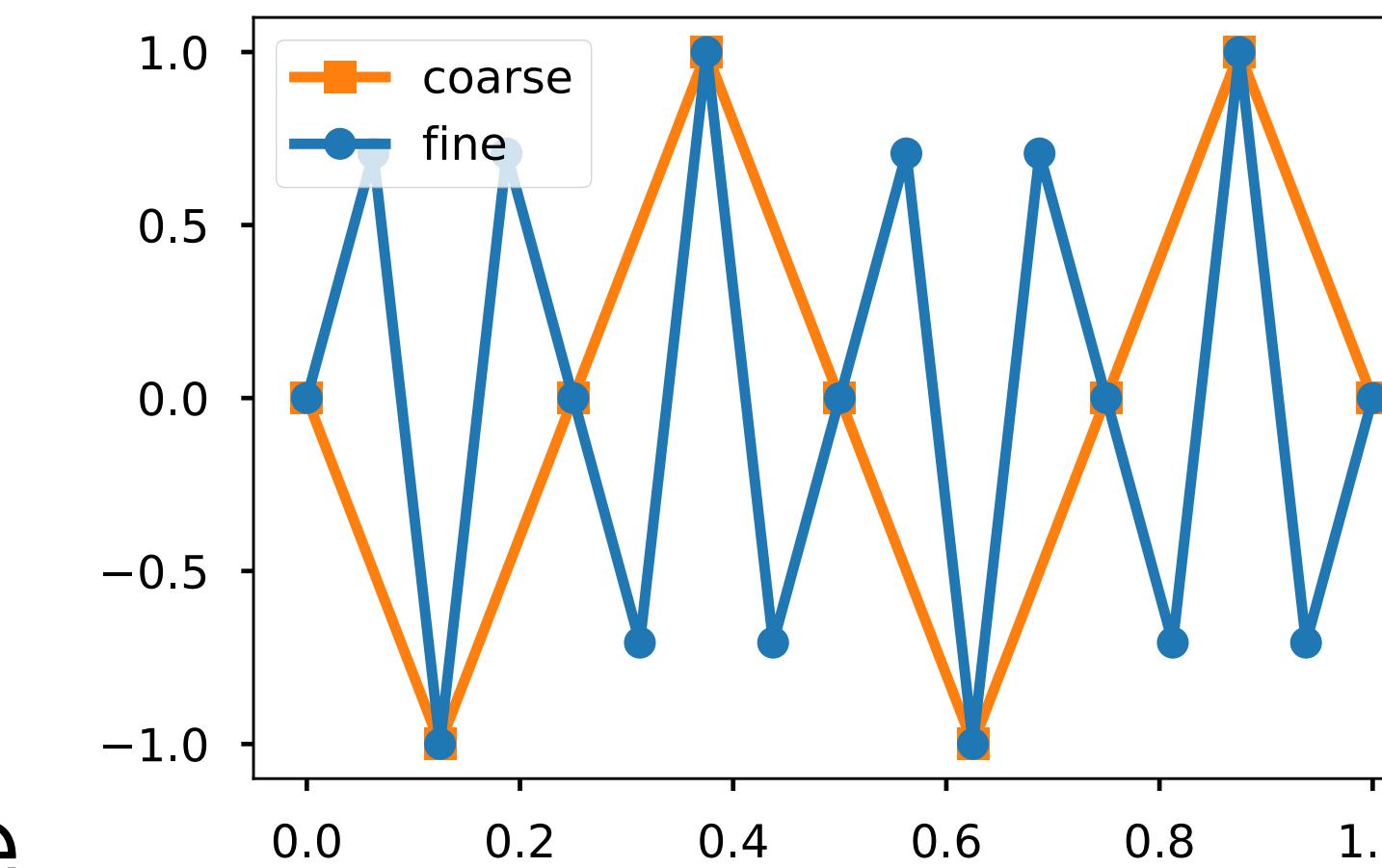
Fine mode

Fine mode
(every other)

Coarse mode



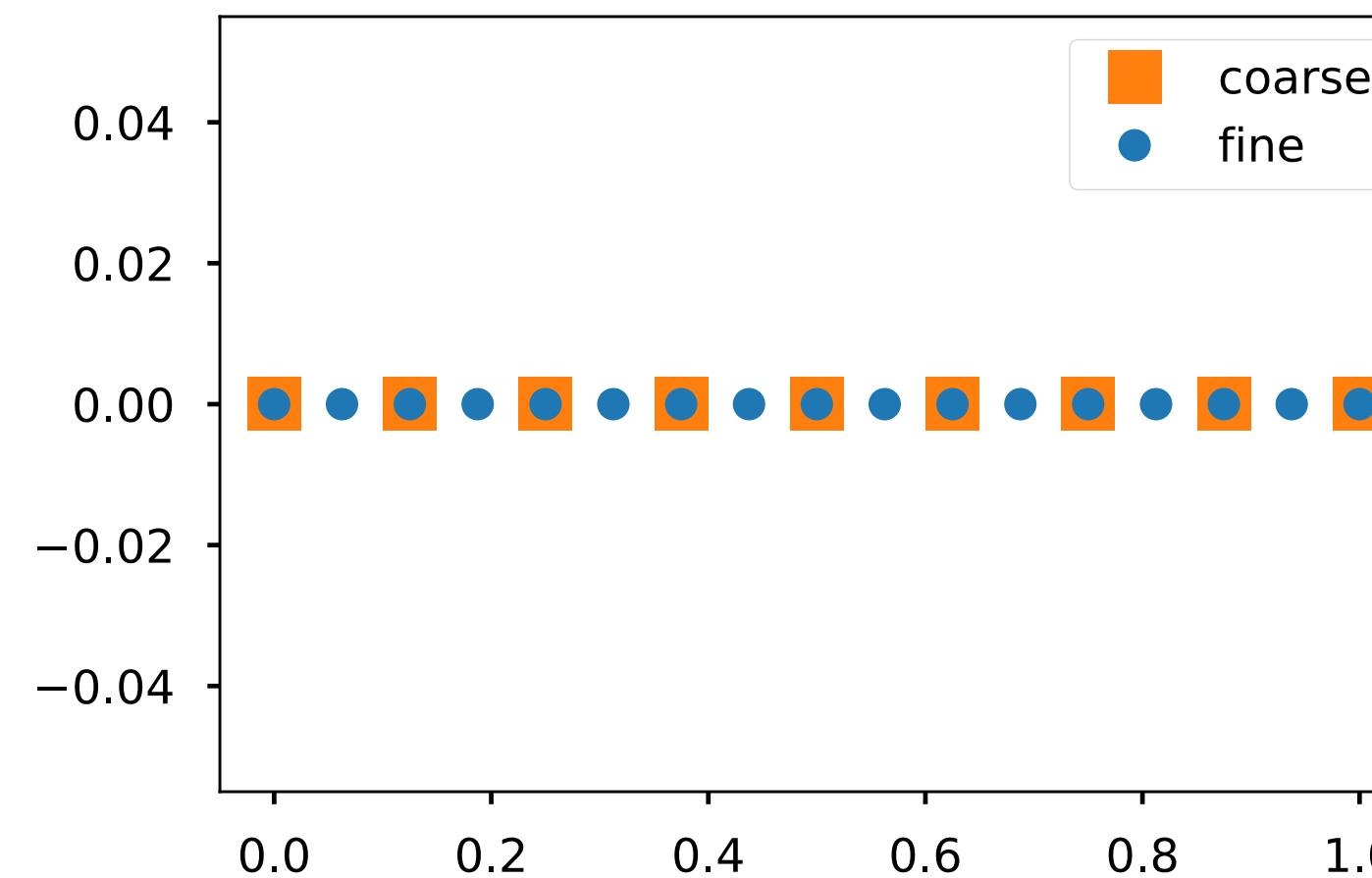
mode 4 of 15



mode 12 of 15

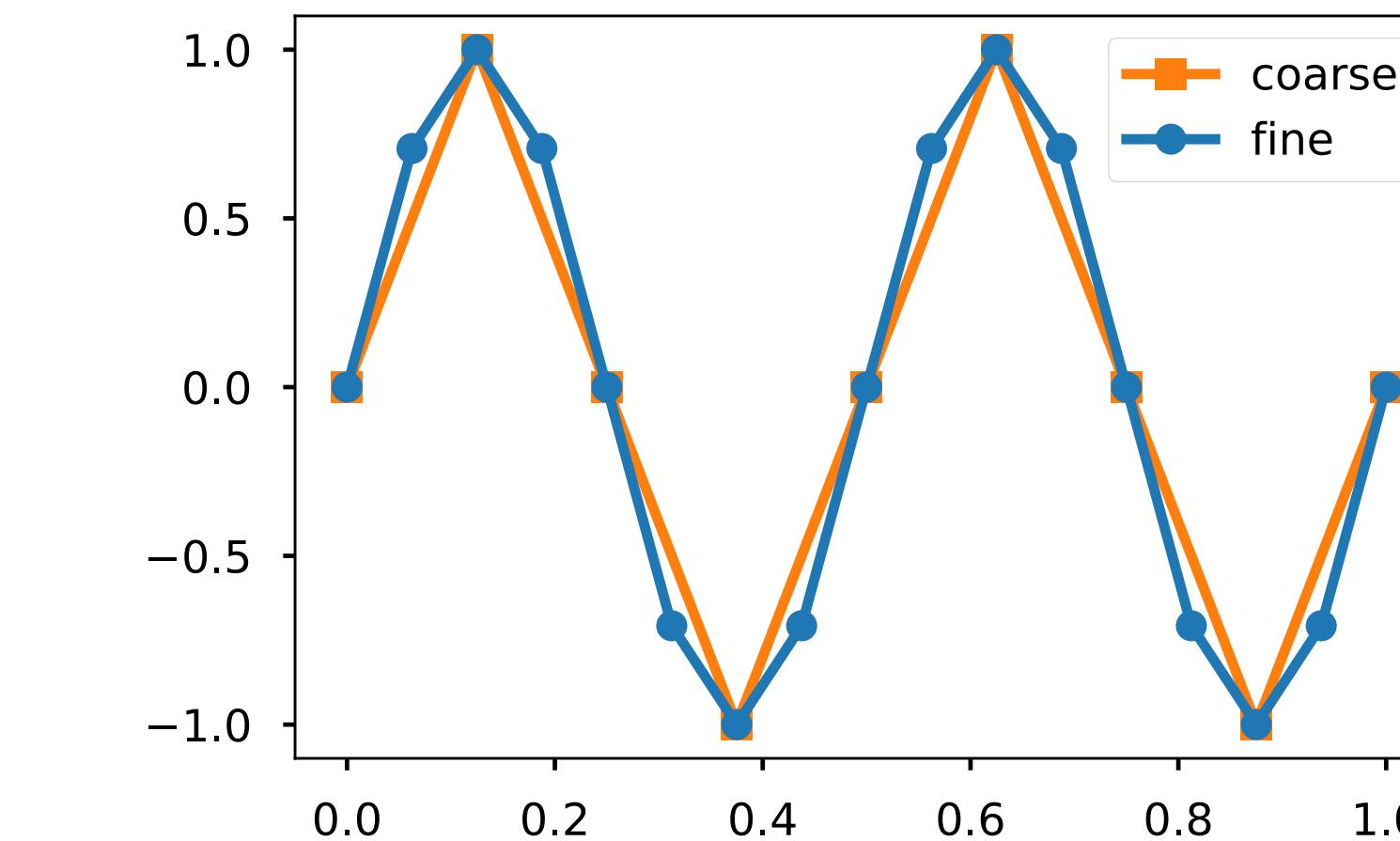
For low modes, k-modes are preserved

Coarse modes

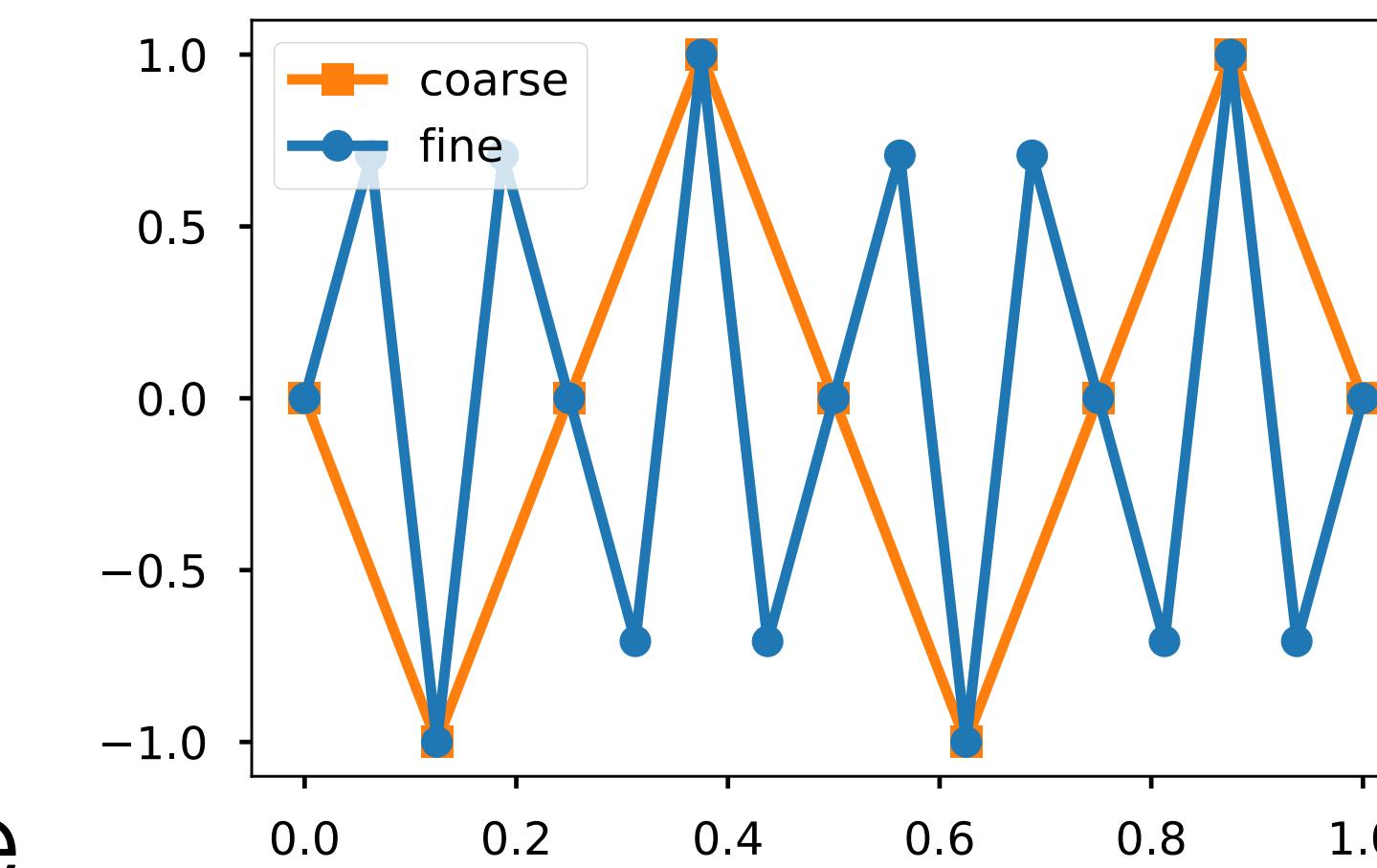


$$\begin{aligned}
 (v_k)_{2j} &= \sin \frac{2jk\pi}{n+1} \\
 &= -\sin \frac{2j(n-k)\pi}{n+1} \quad \text{Fine mode} \\
 &= -\sin \frac{j(n-k)\pi}{(n+1)/2} \quad \text{(every other)} \\
 &= -(\hat{v}_{n-k})_j
 \end{aligned}$$

Fine mode
(every other)
Coarse mode



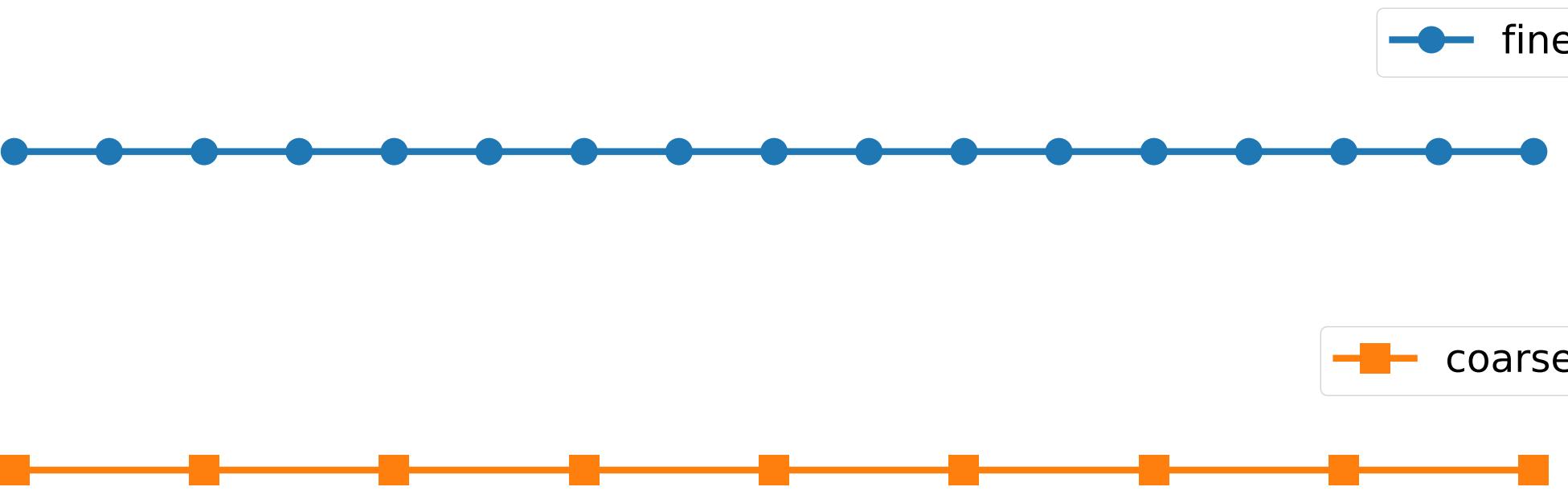
mode 4 of 15



mode 12 of 15

For high modes, k-modes are aliased

Questions to resolve...



- How to transfer between **fine** and **coarse**?
- What do we do “solve” on a coarse grid?

What to use to transfer...

- Return to our projection problem:

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

- If we have a smoothing property, then smoothing a few times will leave smooth error.
- This is A -orthogonal projection onto V
- Let's construct V from say piecewise cont. linears

Interpolation

- Consider coarse grid

$$\Omega^{2h}$$

- and fine grid

$$\Omega^h$$

- Construct an operator

$$P : \Omega^{2h} \rightarrow \Omega^h$$

- Such that

$$Pv^{2h}$$

is continuous and piecewise linear

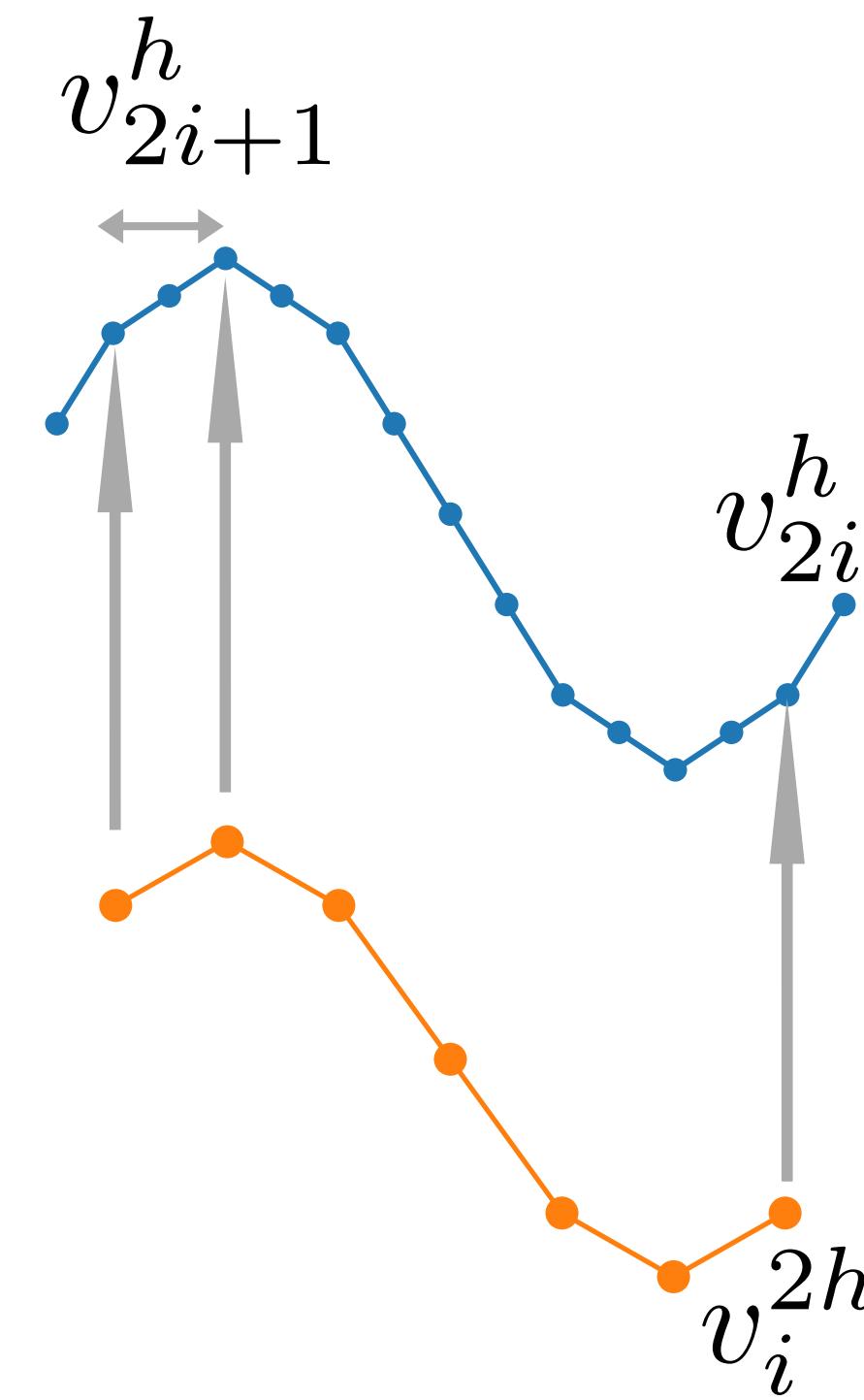
Interpolation

$$v_{2i}^h = v_i^{2h}$$

$$v_{2i+1}^h = \frac{1}{2}(v_i^{2h} + v_{i+1}^{2h})$$

Injection

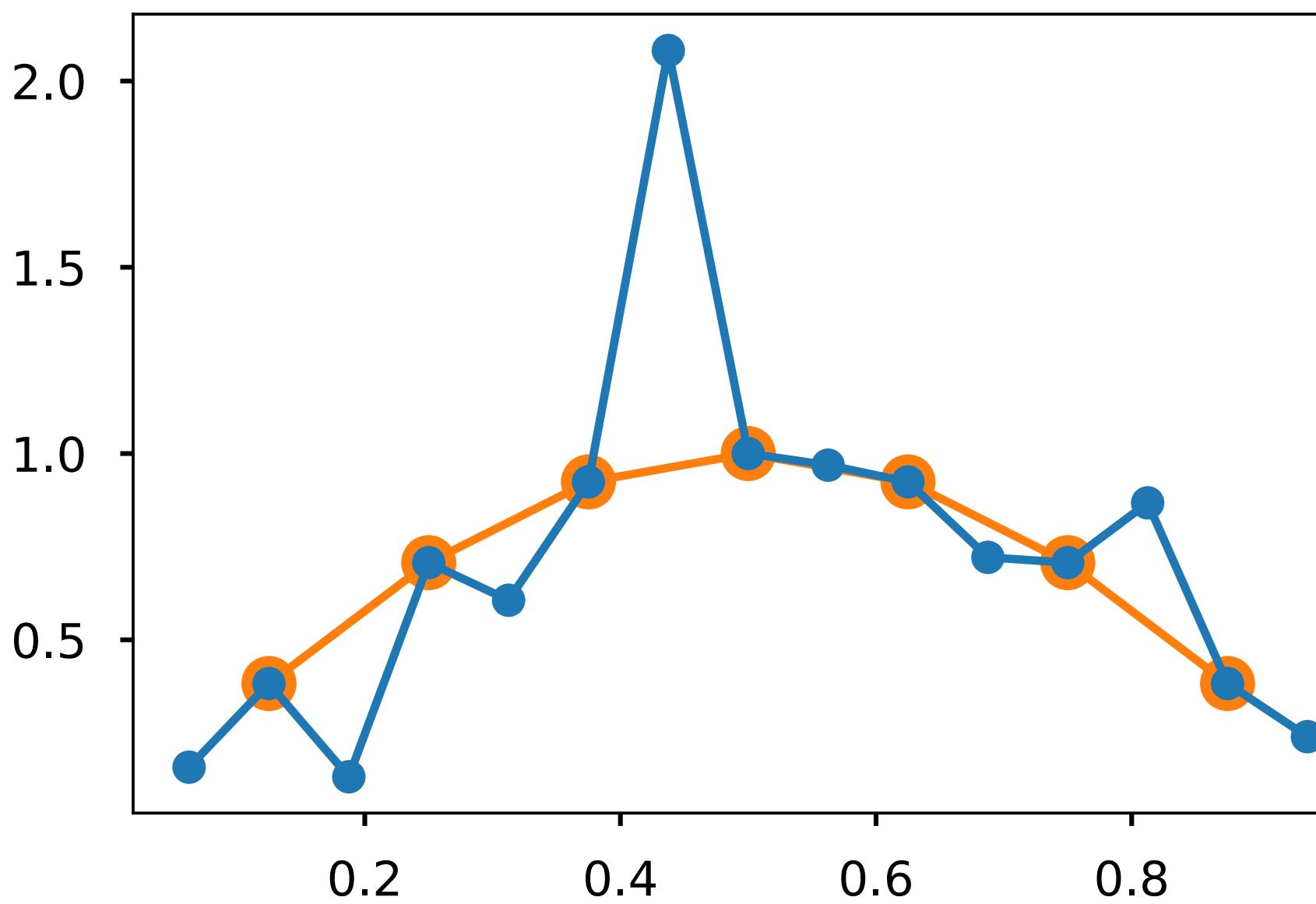
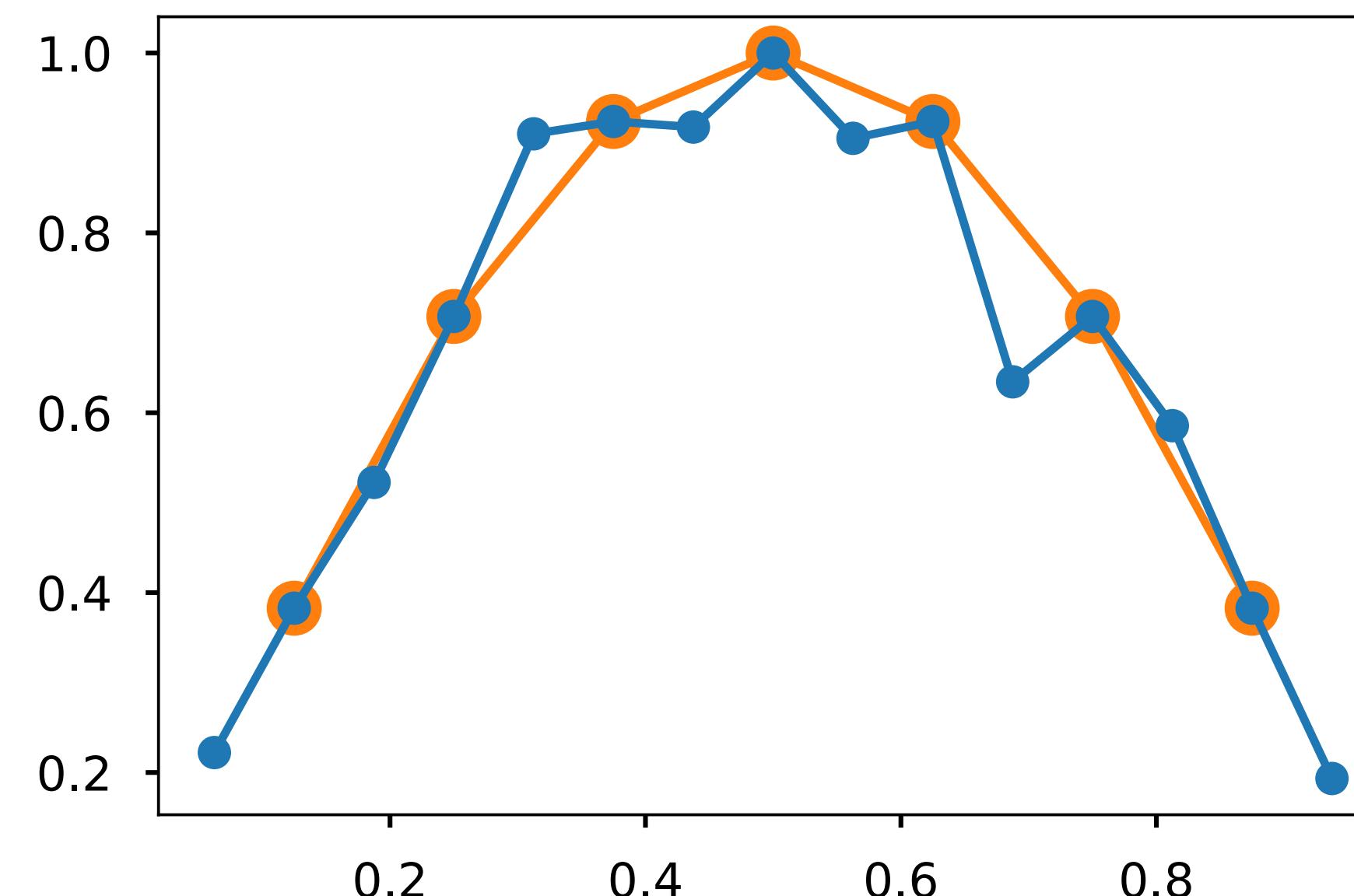
Average
(linear interp)



- Values at points common to both grids are reused (injected)

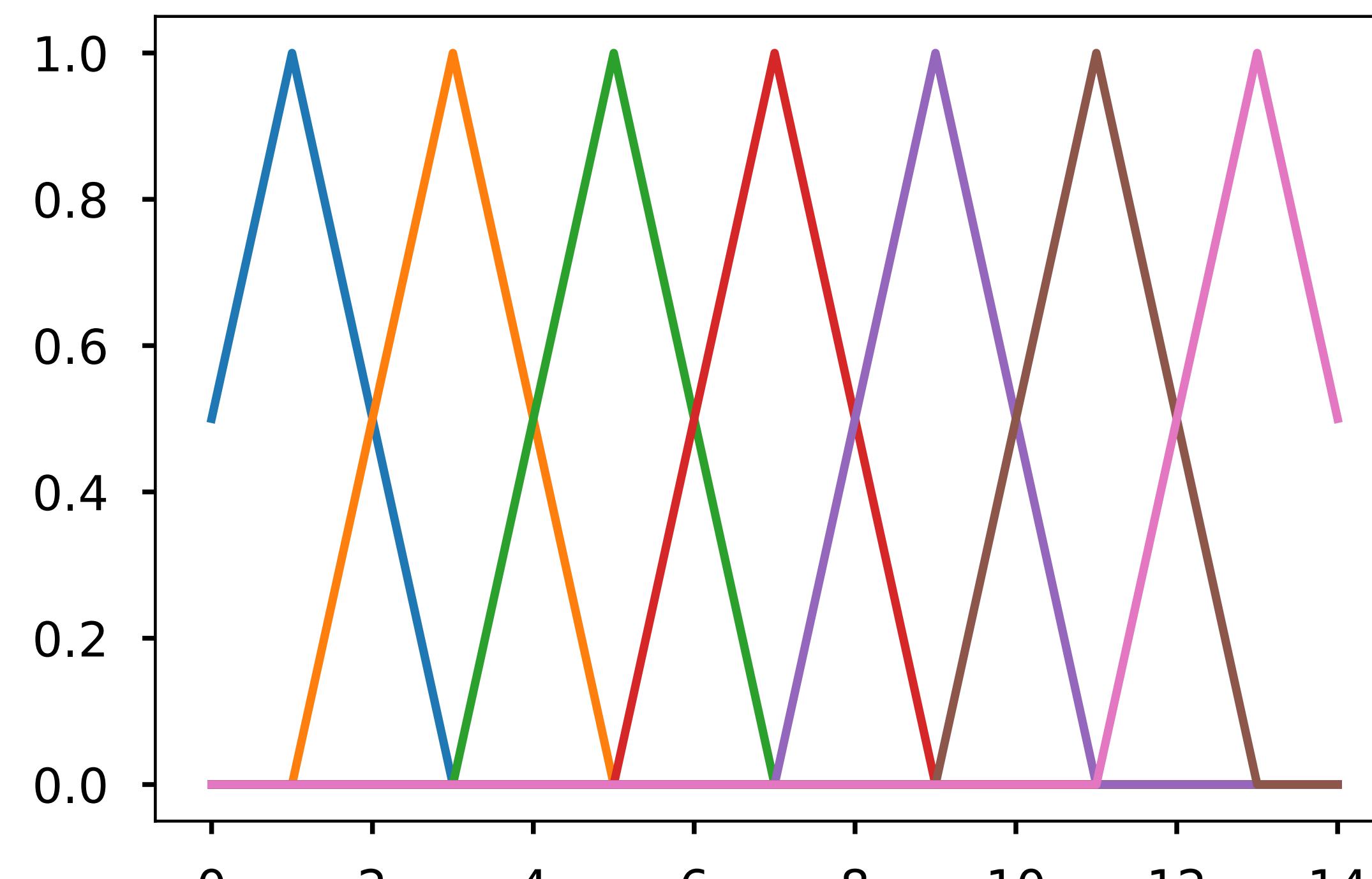
Interpolation

- Things that are smooth are interpolated well (smooth error!)
- Things that are not smooth are not interpolated well (un-smoothed error!)



In matrix form

$$P = \frac{1}{2} \begin{bmatrix} 1. & & & & & \\ & 1. & & & & \\ & & 1. & 1. & & \\ & & & 2. & & \\ & & & & 1. & 1. \\ & & & & & 2. \\ & & & & & & 1. \\ & & & & & & & 2. \\ & & & & & & & & 1. \\ & & & & & & & & & 2. \\ & & & & & & & & & & 1. \\ & & & & & & & & & & & 1. \\ & & & & & & & & & & & & 2. \\ & & & & & & & & & & & & & 1. \\ & & & & & & & & & & & & & & 1. \\ & & & & & & & & & & & & & & & 2. \\ & & & & & & & & & & & & & & & & 1. \end{bmatrix}$$



- The columns of P are basis functions (right)
- The fine grid vectors, Pv , are linear combinations of these functions
- Notice: P is full rank!

The coarse grid operator

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

(sub)space defined by P

- If V is defined by $\text{span}\{P\}$ – or just P ,
- Then V^T defines **restriction** as P^T
- And the **coarse level operator** is defined by $P^T A P$

Two level method

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

↓

$$x_1 = x_0 + P(P^T A P)^{-1} P^T r_0$$

- Given
- Smooth a few times
- Form residual
- Restrict the residual
- Solve the coarse problem
- Interpolate the approx error
- Correct the initial guess

$$\begin{aligned} & x_0 \\ & x_0 \leftarrow x_0 + \omega D^{-1} A r_0 \\ & r_0 = b - A x_0 \\ & P^T r_0 \\ & P^T A P \hat{e}_c = P^T r_0 \\ & P \hat{e}_c \\ & x_0 + P \hat{e}_0 \end{aligned}$$

$$A_c \hat{e}_c = r_c$$

- What should this be?
- How many times
- What does this mean?
- What are we interpolating?
- Are we done?

Notes on variants

- An alternative to restriction, is injection
- Or a weighted transpose of linear interpolation (to restrict constants exactly)
- An alternative to $A_c = P^T A_h P$ is to rediscretize $A_c = A_{2h}$

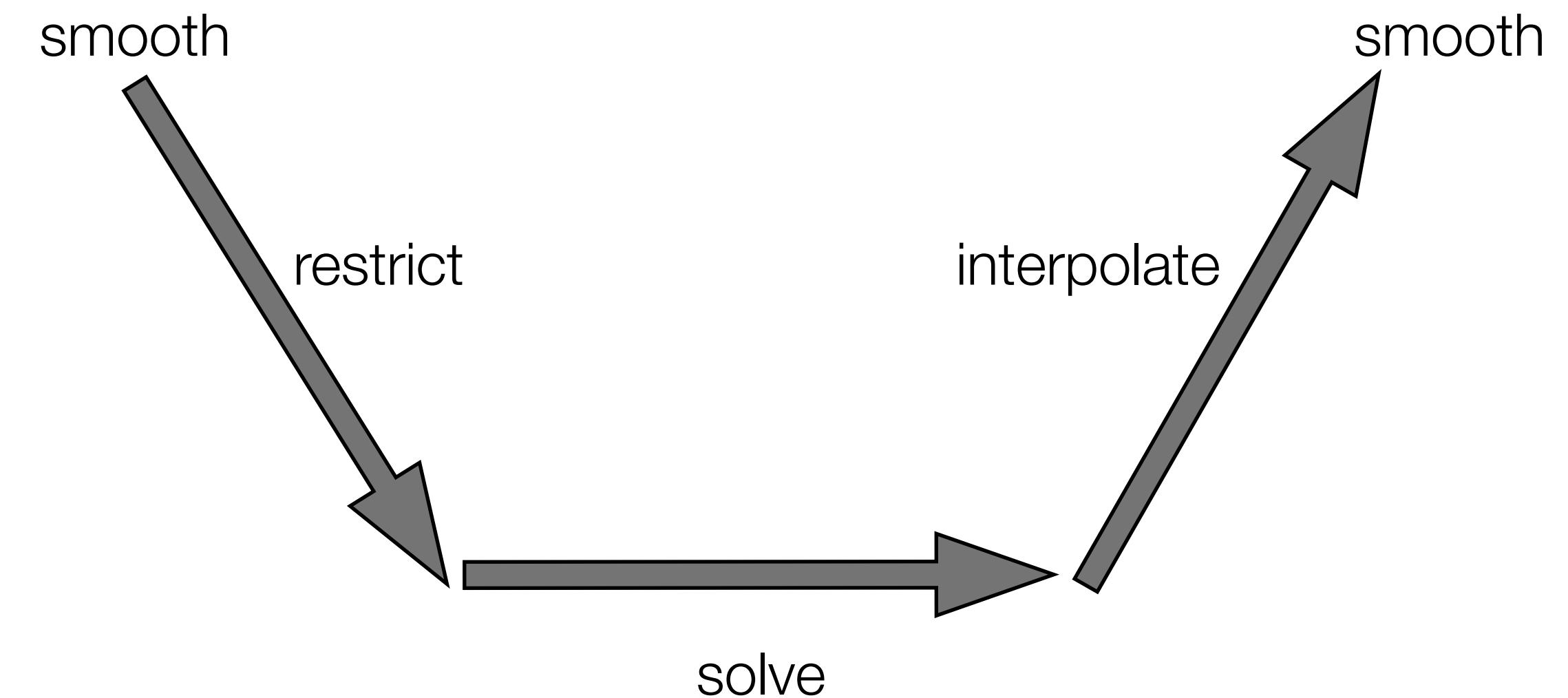
$$\begin{bmatrix} 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & & \\ & & 0 & 1 & 0 & \\ & & & 0 & 1 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 1 & 0 \\ & & & & & & & 0 & 1 \end{bmatrix}$$
$$\frac{1}{4} \begin{bmatrix} 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 2 & 1 \\ & & & & & & & 1 & 2 \end{bmatrix}$$

$$x_1 \leftarrow x_0 + P A_{2h}^{-1} R r_0$$

Algorithm: two-level multigrid

Input: initial guess

1. Smooth ν_{pre} times on $Au = f$
2. Compute $r = f - Au$
3. Compute $r_c = Rr$
4. Solve $A_c e_c = r_c$
5. Interpolate $\hat{e} = Pe_c$
6. Correct $u \leftarrow u + \hat{e}$
7. Smooth ν_{post} times on $Au = f$



A two-level “V” cycle

How Accurate is Multigrid?

- Consider the exact solution to the PDE u^*

$$-u'' = f$$

- The exact solution to the **discrete** problem u_h^*

$$Au = b$$

- The approximate **discrete** solution $u_h \approx u_h^*$

- Define

$$u^* - u_h^* \quad \text{Discretization error}$$

$$u_h^* - u_h \quad \text{Algebraic error}$$

How Accurate is Multigrid?

- Would like the error bounded

$$\begin{aligned}\|u^* - u_h\| &\leq \|u^* - u_h^*\| + \|u_h^* - u^h\| \\ &\leq \varepsilon\end{aligned}$$

- To achieve this,
 - force the discretization (grid space) so that
$$\|u^* - u_h^*\| \leq ch^2 \leq \frac{\varepsilon}{2}$$
 - and the algebraic error (convergence) up to the discretization error
$$\|u_h^* - u^h\| \leq \frac{\varepsilon}{2}$$

Multigrid convergence

- Convergence factor of a cycle – the factor by which the error (residual) is reduced (in some norm) in each iteration

$$\gamma$$

...assume this is independent of n

- Wish to have m cycles such that

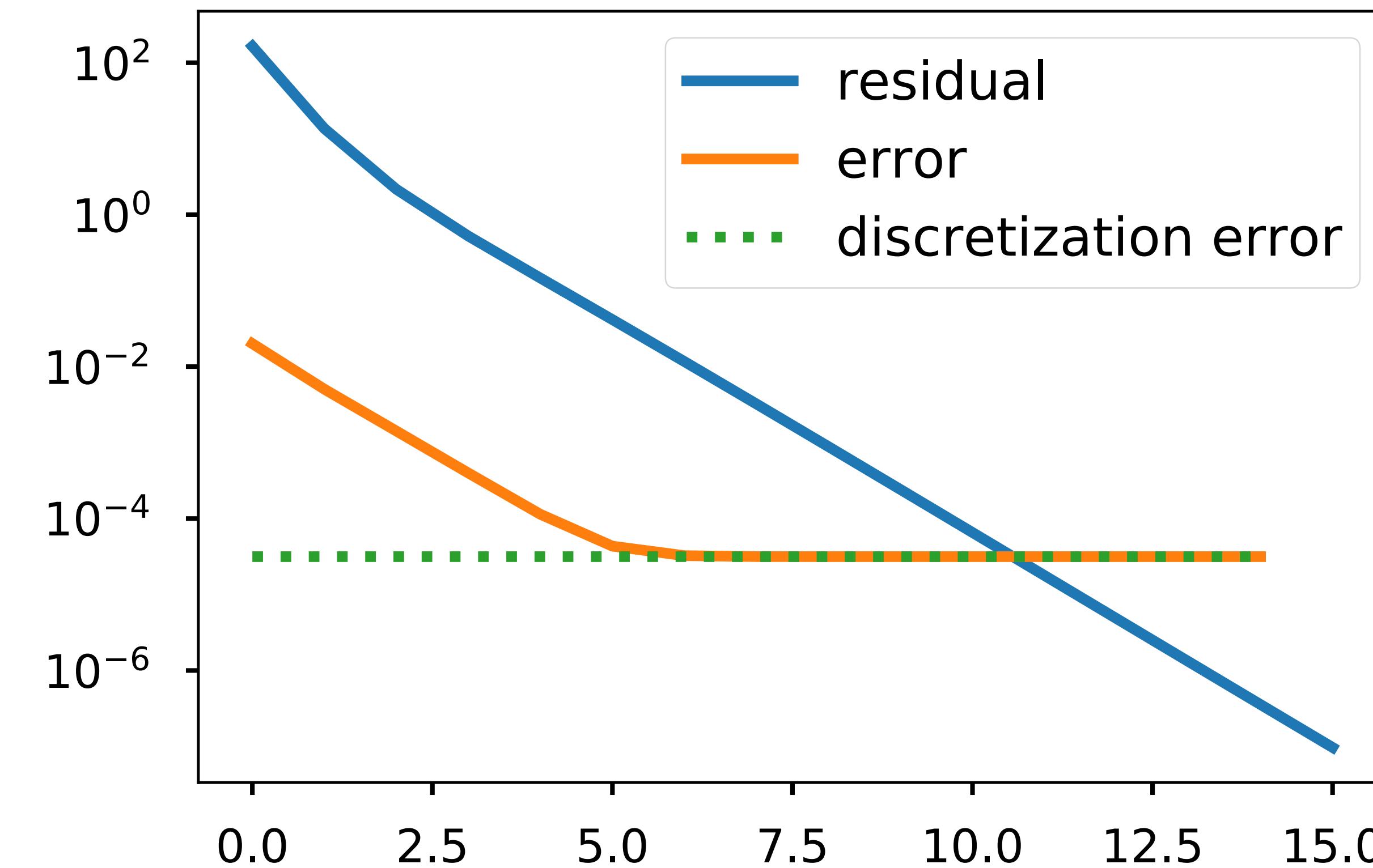
$$\gamma^m \sim \mathcal{O}(n^{-2}) \leq \frac{\varepsilon}{2}$$

- Then we need

$$m \sim \mathcal{O}(\log n)$$

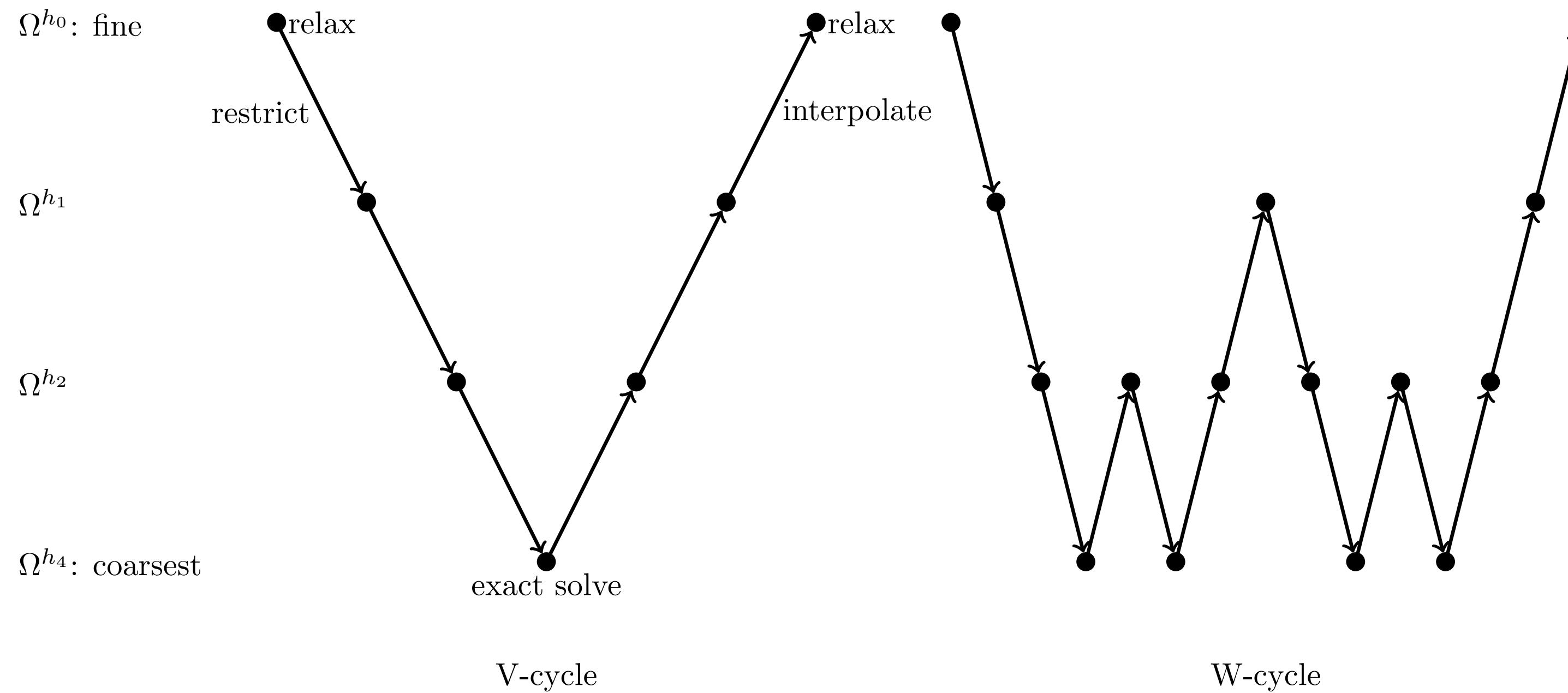
cycles

Algebraic error



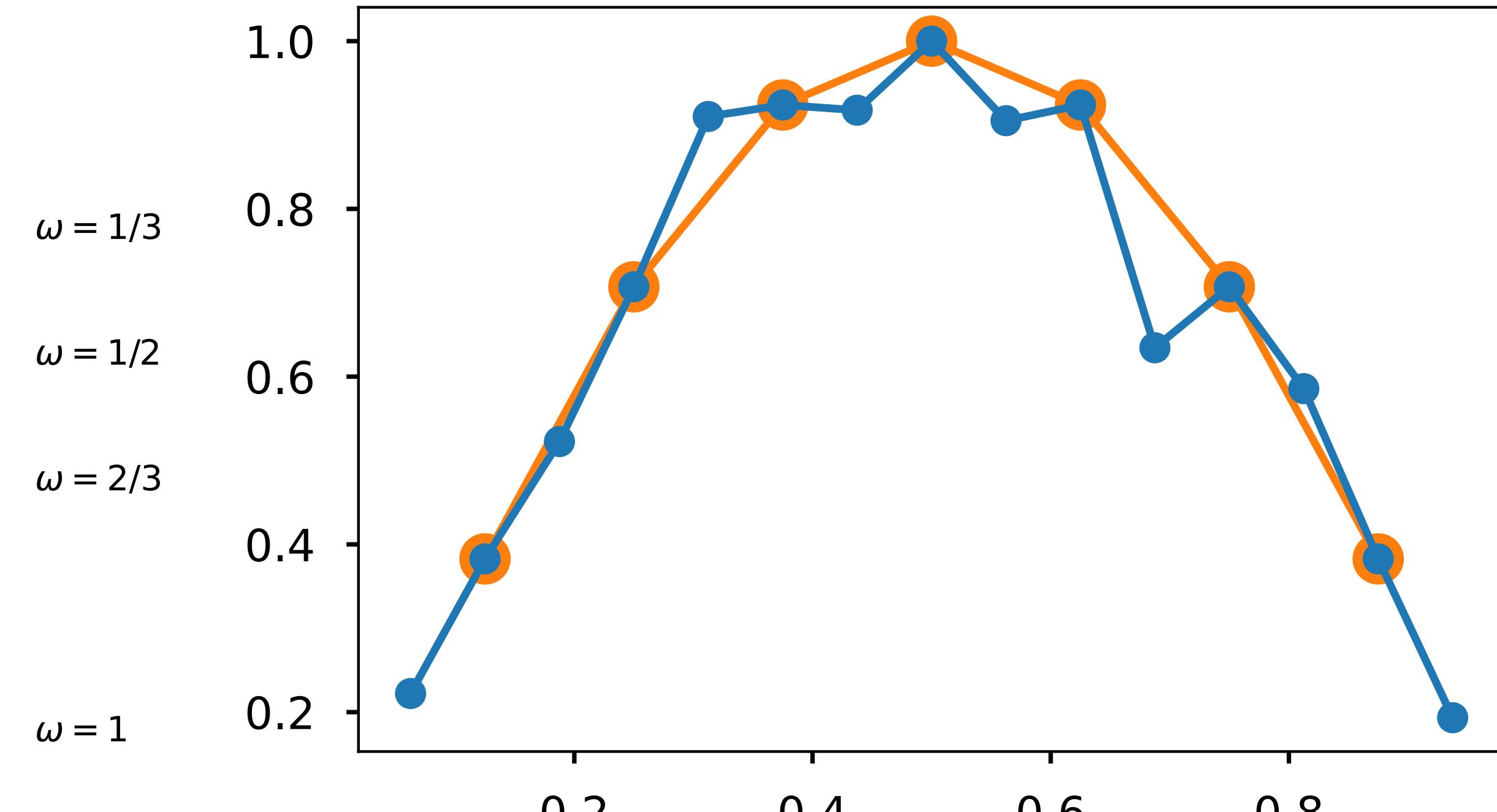
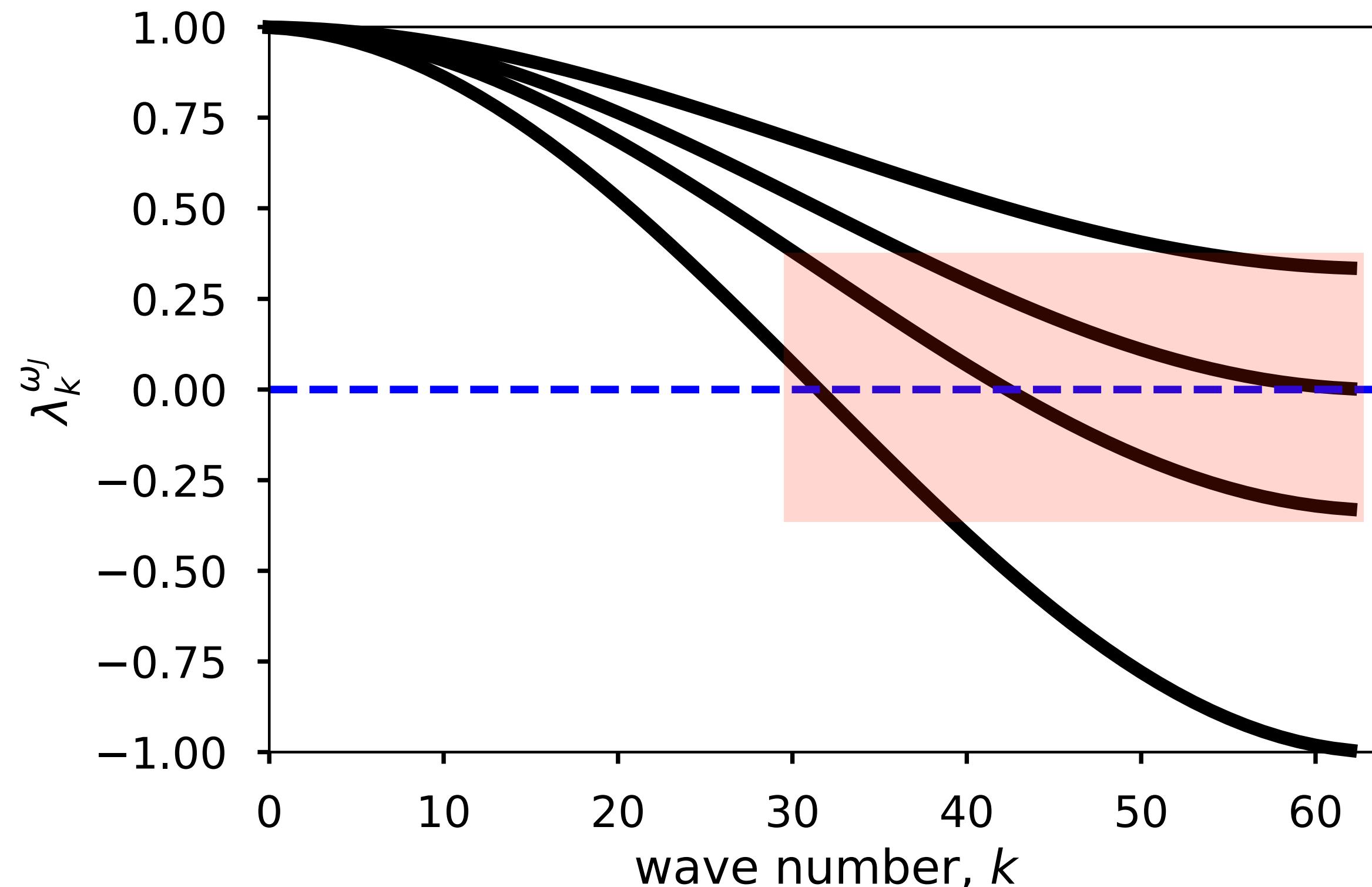
- The total error is limited by the discretization error

The Multigrid V-Cycle and W-Cycle



- Two-grid cycle can expose issues with coarser interpolation
- W-Cycle can account for inadequate coarser level solves
- **Exact solve?** Usually a pseudo-inverse

Recap...

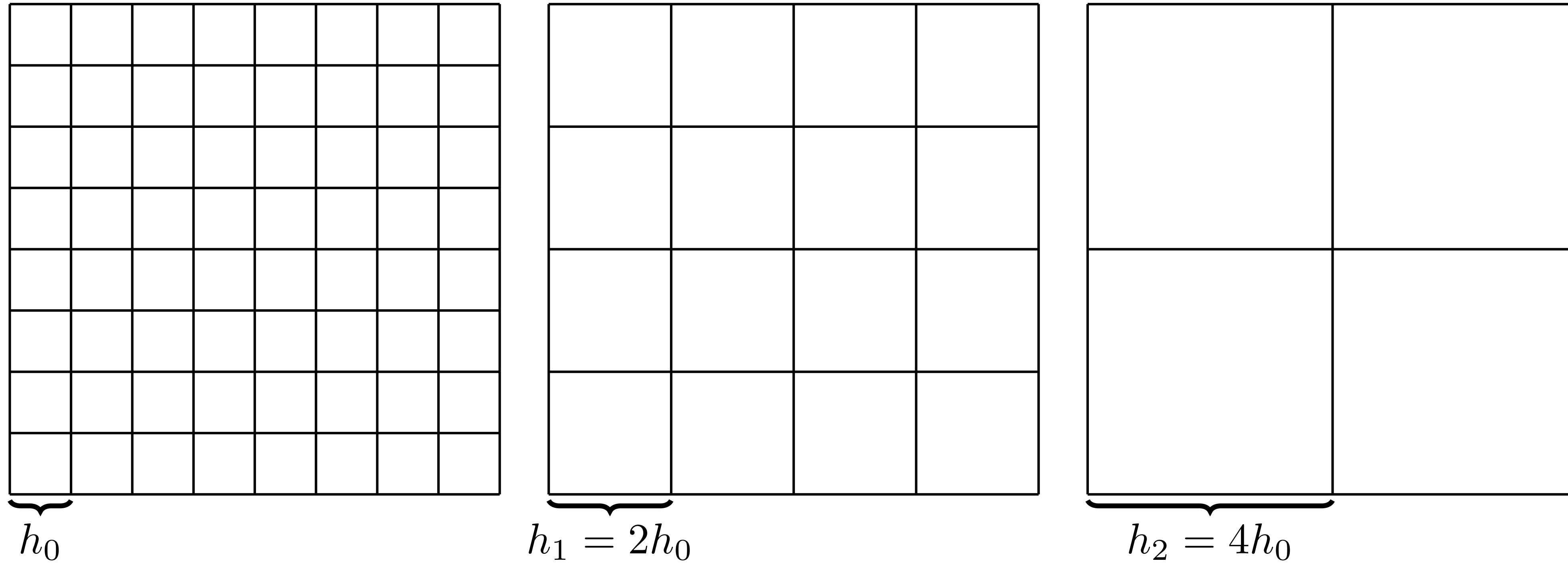


- Smoothing:
Reduce high frequency error

$$e_1 = e_0 - P(P^T A P)^{-1} P^T A G e$$

- Coarse-grid correction:
Reduce smooth-ish things in the range of interpolation

Multigrid in 2D



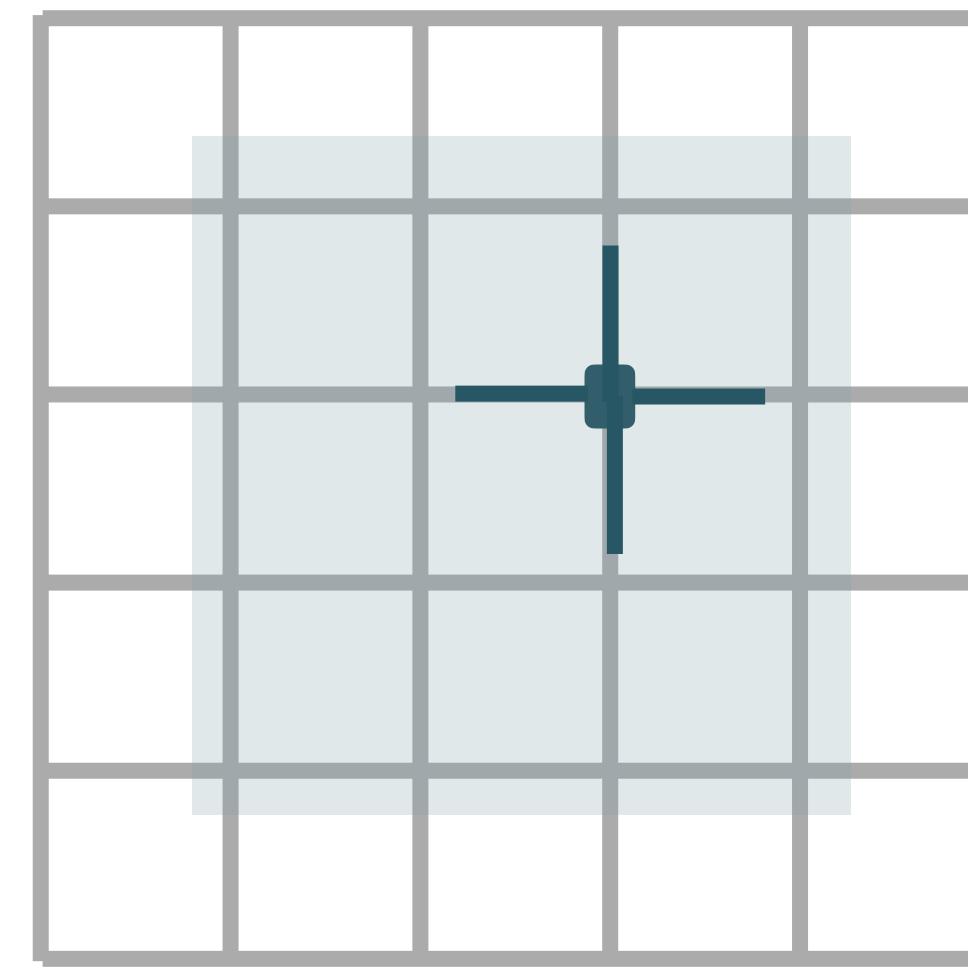
- Again, assume we have a series of uniform grids
- Relaxation remains the same (what is ω ?)

Multigrid in 2D

- Model problem

$$\begin{aligned} -u_{xx} - u_{yy} &= f \\ u &= 0 \quad \text{on boundary} \end{aligned}$$

- Results in the stencil / matrix



$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

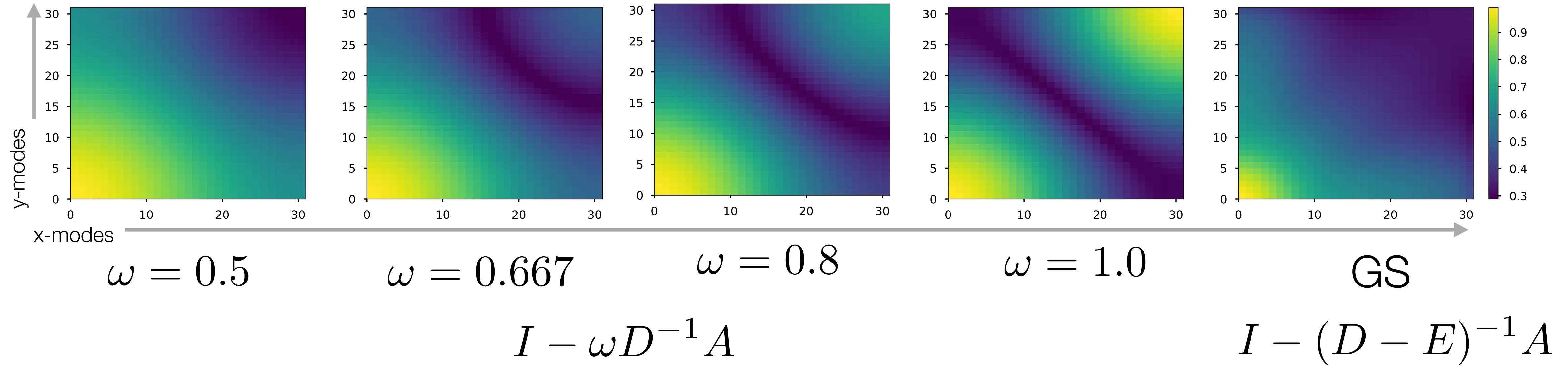


$$\frac{1}{h^2} \begin{bmatrix} 4 & -1 & & & -1 & & & \\ -1 & 4 & -1 & & & & & \\ & -1 & 4 & -1 & & & & \\ & & -1 & 4 & -1 & & & \\ & & & -1 & 4 & -1 & & \\ & & & & -1 & 4 & -1 & \\ & & & & & -1 & 4 & -1 \\ & & & & & & -1 & 4 \\ & & & & & & & -1 \\ & & & & & & & & \ddots \end{bmatrix}$$

Relaxation in 2D

$$\sin\left(\frac{k_x i \pi}{n+1}\right) \sin\left(\frac{k_y j \pi}{n+1}\right)$$

Convergence factor
over 10 sweeps



- weighted Jacobi: Same issue – need to select a parameter
- Gauss-Seidel improved
- Red-Black Gauss-Seidel and other schemes even more effective

Interpolation in 2D

- Bilinear interpolation, tensor of 1D interpolation

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

- Example $3 \times 3 \rightarrow 7 \times 7$

$$\begin{bmatrix} 0.5 \\ 1 \\ 0.5 & 0.5 \\ 1 \\ 0.5 & 0.5 \\ 1 \\ 0.5 \end{bmatrix} \otimes \begin{bmatrix} 0.5 & & \\ & 1 & \\ 0.5 & 0.5 & \\ & 1 & \\ 0.5 & 0.5 & \\ & 1 & \\ 0.5 & & \end{bmatrix}$$



$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

A few observations so far: **One**

- Let's consider a V(1,1) cycle – weighted Jacobi, etc.
- The **error** propagation for this looks like

$$e_1 = \frac{G(I - P(P^T A P)^{-1} P^T A) G e_0}{M}$$

$$G = I - \omega D^{-1} A$$

$$M e_k \leftarrow 0?$$

- One thing we can do, is consider bounds on each operation.

A few observations so far: **One**

- Take the operator

$$M = G(I - P(P^T A P)^{-1} P^T A)G$$

- And makes some bounds

$$\|I - P(P^T A P)^{-1} P^T A\| \|G\|^2$$

The diagram shows the expression $\|I - P(P^T A P)^{-1} P^T A\| \|G\|^2$ decomposed into two parts by horizontal lines. The first part, $\|I - P(P^T A P)^{-1} P^T A\|$, is highlighted in a blue box and labeled "approximation property". The second part, $\|G\|^2$, is highlighted in a green box and labeled "smoothing property".

- General $\|G\| \leq 1$

1D over $[n/2, n]$:

$$\|G\| \leq \frac{1}{3}$$

2D over $[n/2, n]$:

$$\|G\| \leq \frac{3}{5}$$

A few observations so far: **One**

$$\|I - P(P^T A P)^{-1} P^T A\| \|G\|^2$$

approximation property smoothing property

- General $\|G\| \leq 1$

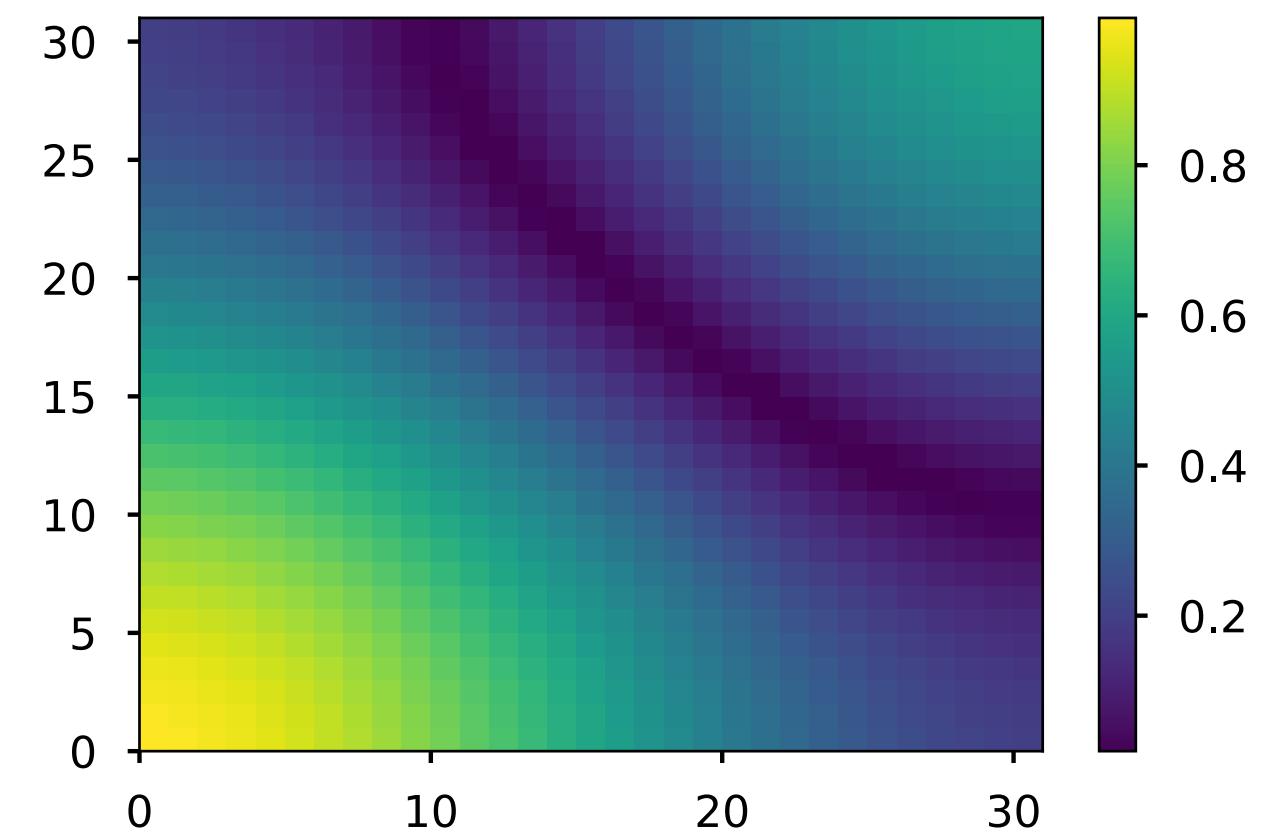
$$1D \text{ over } [n/2, n]: \quad \|G\| \leq \frac{1}{3}$$

$$2D \text{ over } [n/2, n]: \quad \|G\| \leq \frac{3}{5}$$

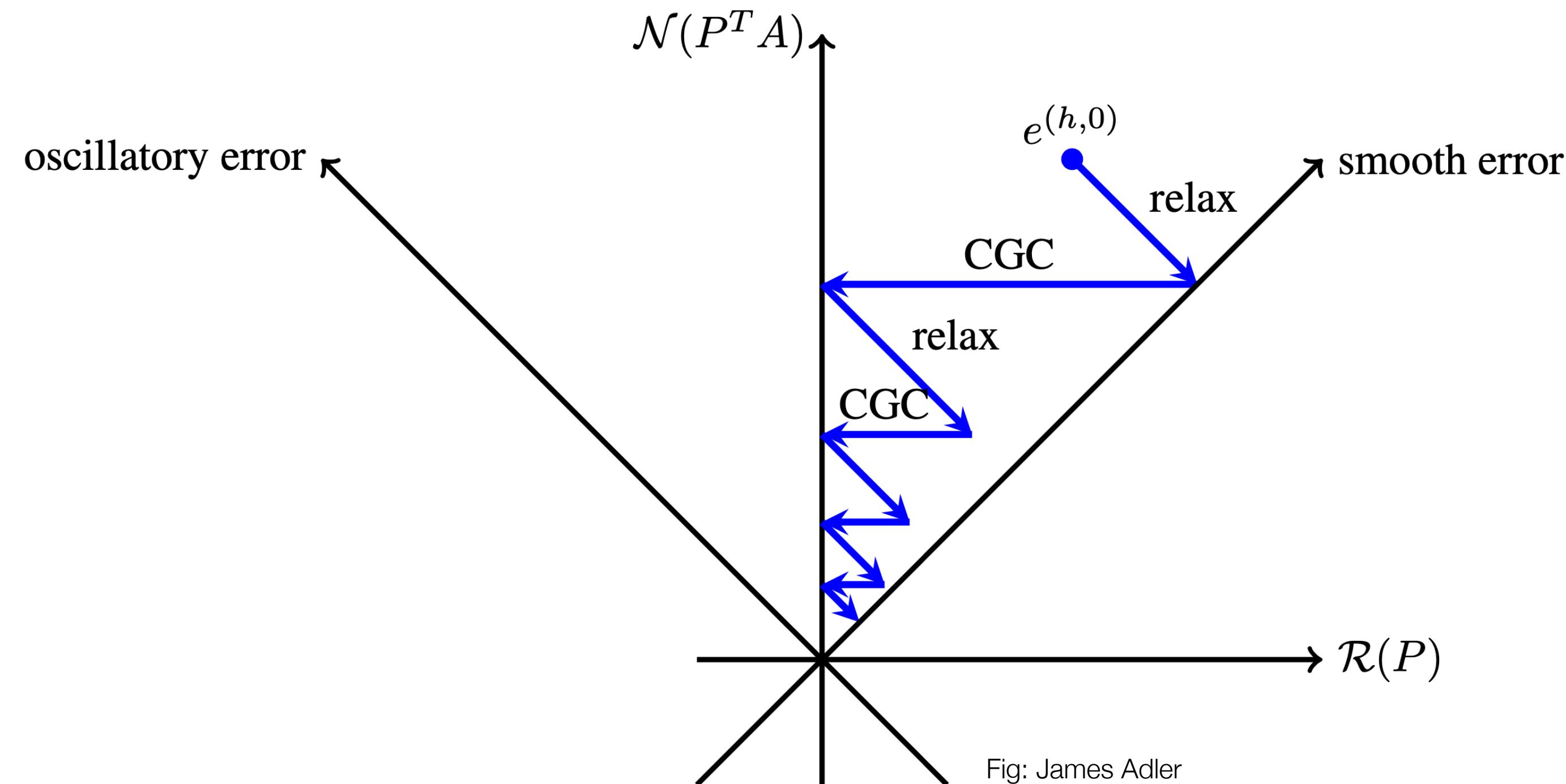
- Also, if w s.t. $Aw \in \mathcal{N}(P^T)$

then $(I - P(P^T A P)^{-1} P^T A)w = w$

$$\|\cdot\| \geq 1$$



A few observations so far: **Two**



- Complementary processes:
 - **relaxation:** targets for (Fourier) smoothing
 - **coarse grid correction:** targets things in the range of interpolation

A few observations so far: **Three**

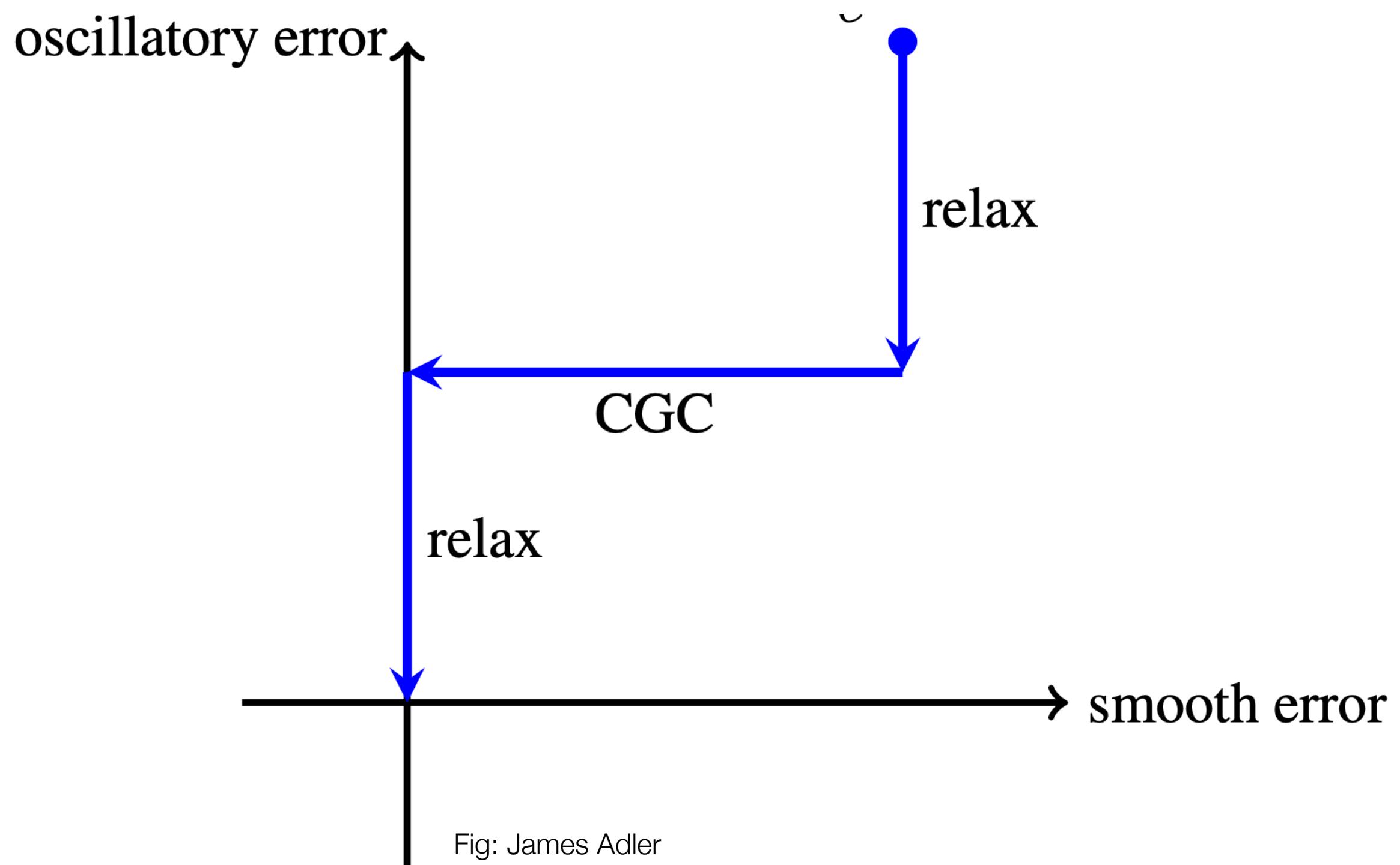


Fig: James Adler

$$e_1 \leftarrow (I - P(P^T A P)^{-1} P^T A) G e_0$$

$$G e_0 \in \mathcal{R}(P) \Rightarrow e_1 = 0$$

interpolation should capture what relaxation misses

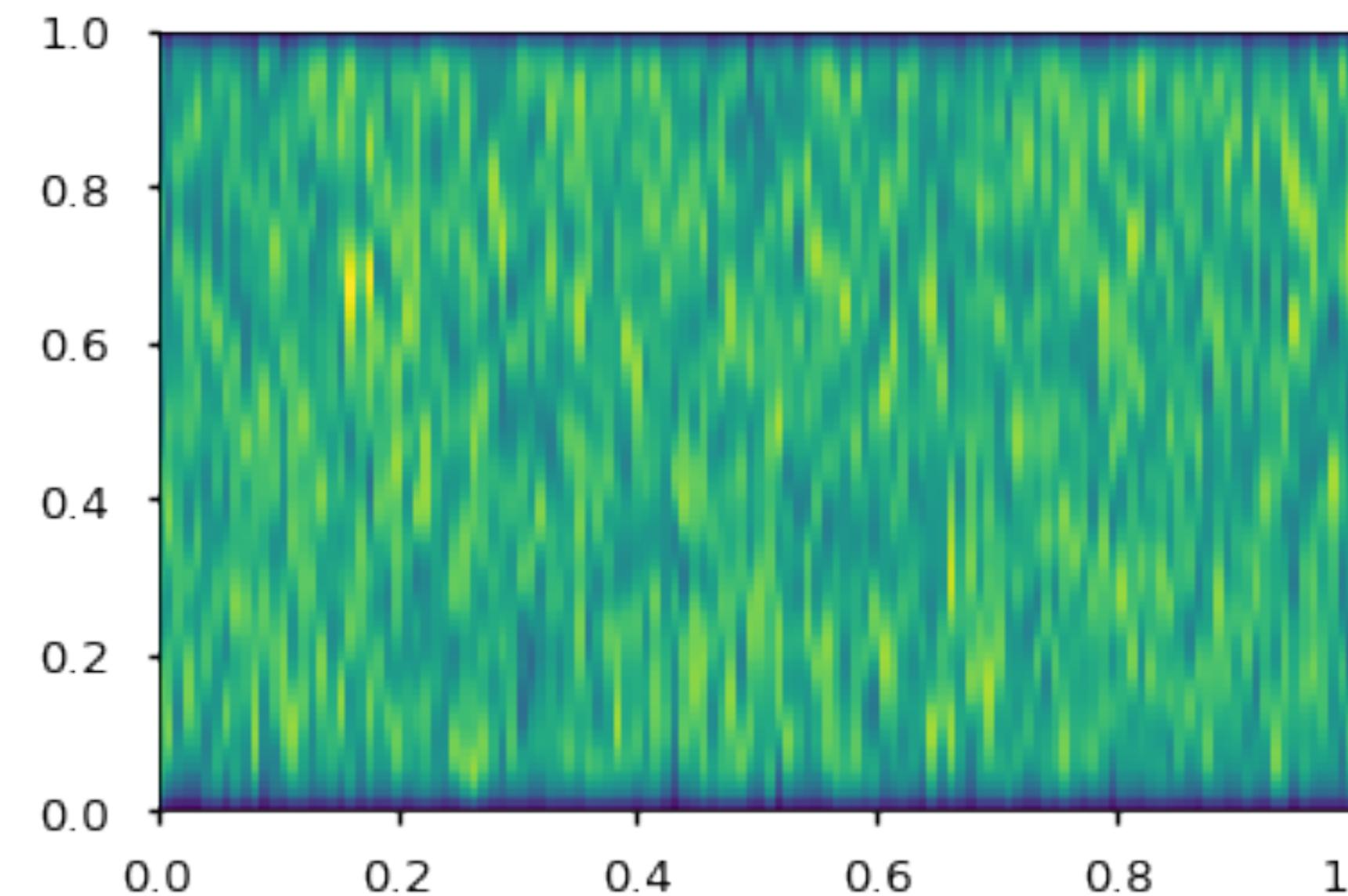
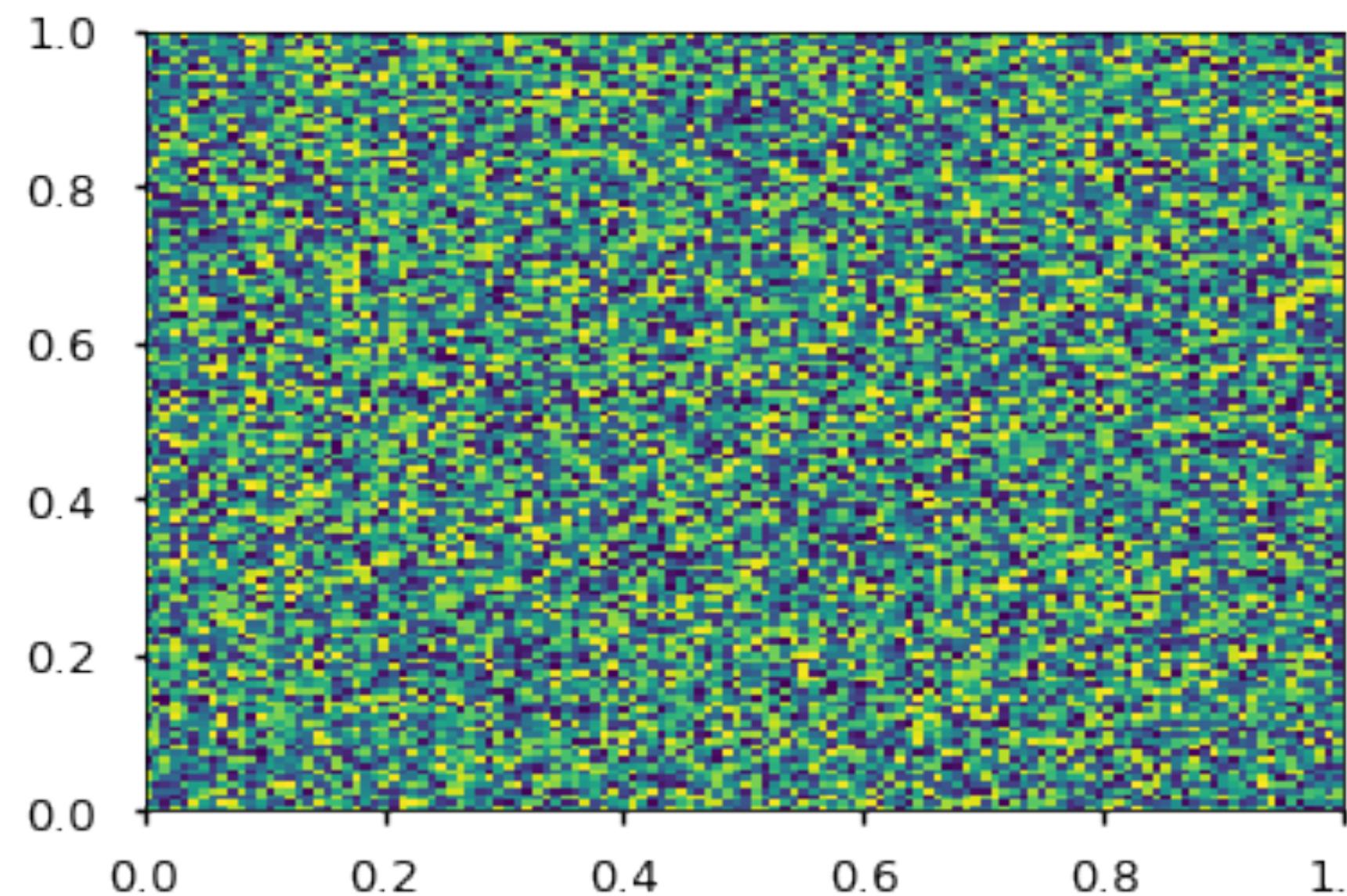
What can go wrong?!

- Consider an *anisotropic problem*

$$-\varepsilon u_{xx} - u_{yy} = f$$

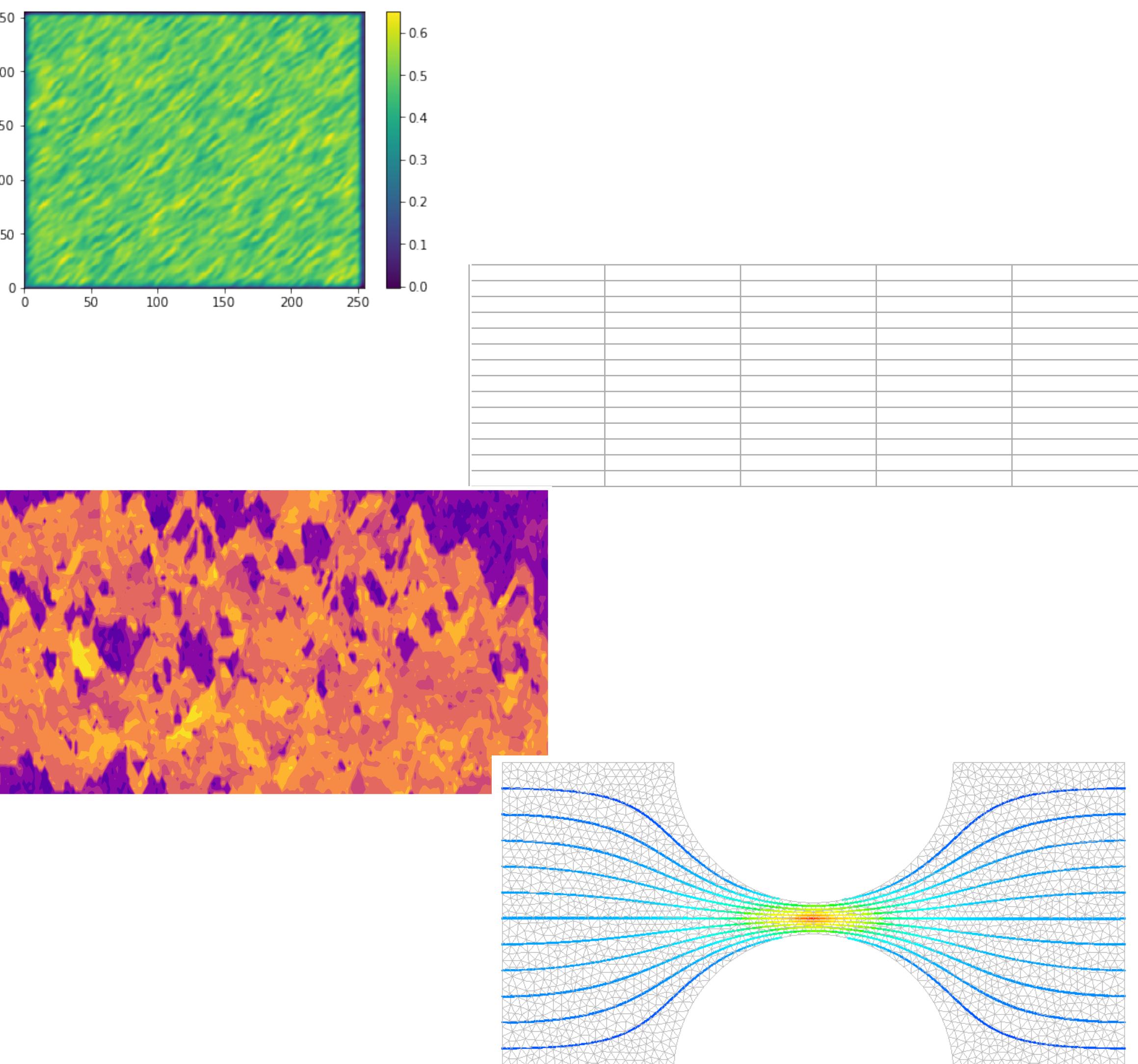
$$u = 0 \quad \text{on boundary}$$

$$\begin{bmatrix} & -1 & \\ -\varepsilon & 2 + 2\varepsilon & -\varepsilon \\ & -1 & \end{bmatrix}$$



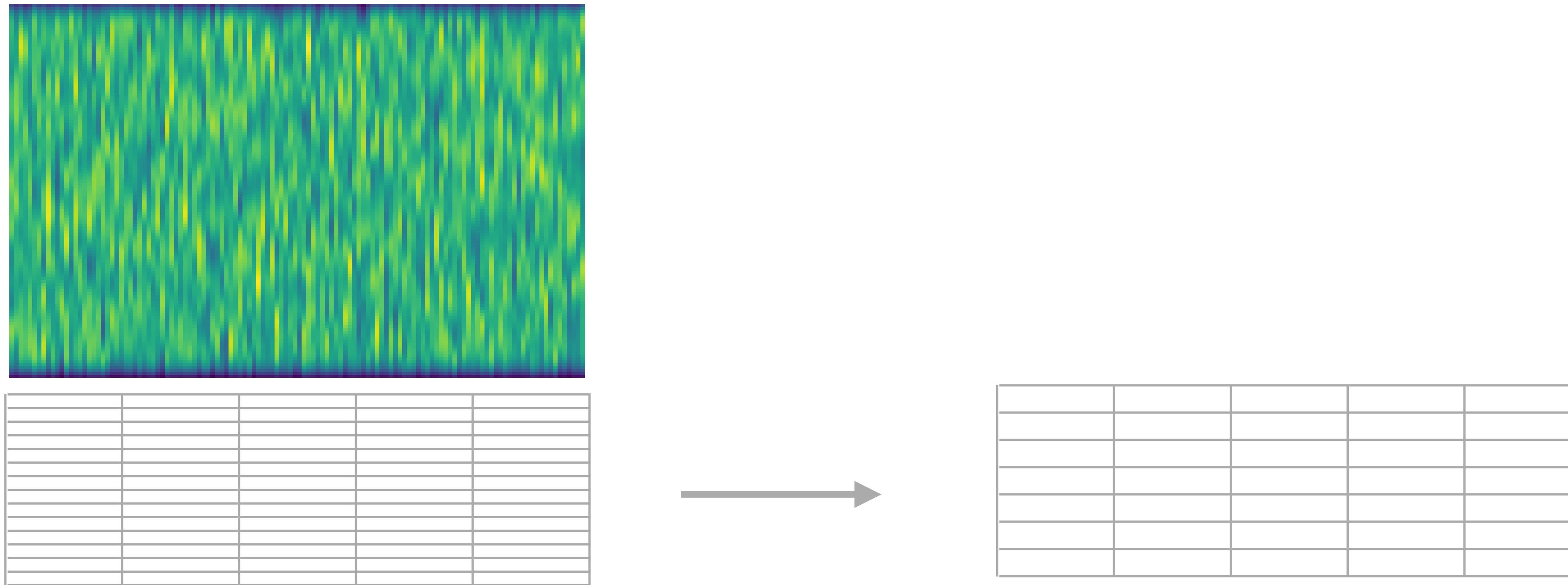
What can go wrong?!

- Anisotropy
- Mesh stretching
- Jumping coefficients
- Non-elliptic



Options for more robust Multigrid

- **Semicoarsening:** Coarsen in the direction of smoothness

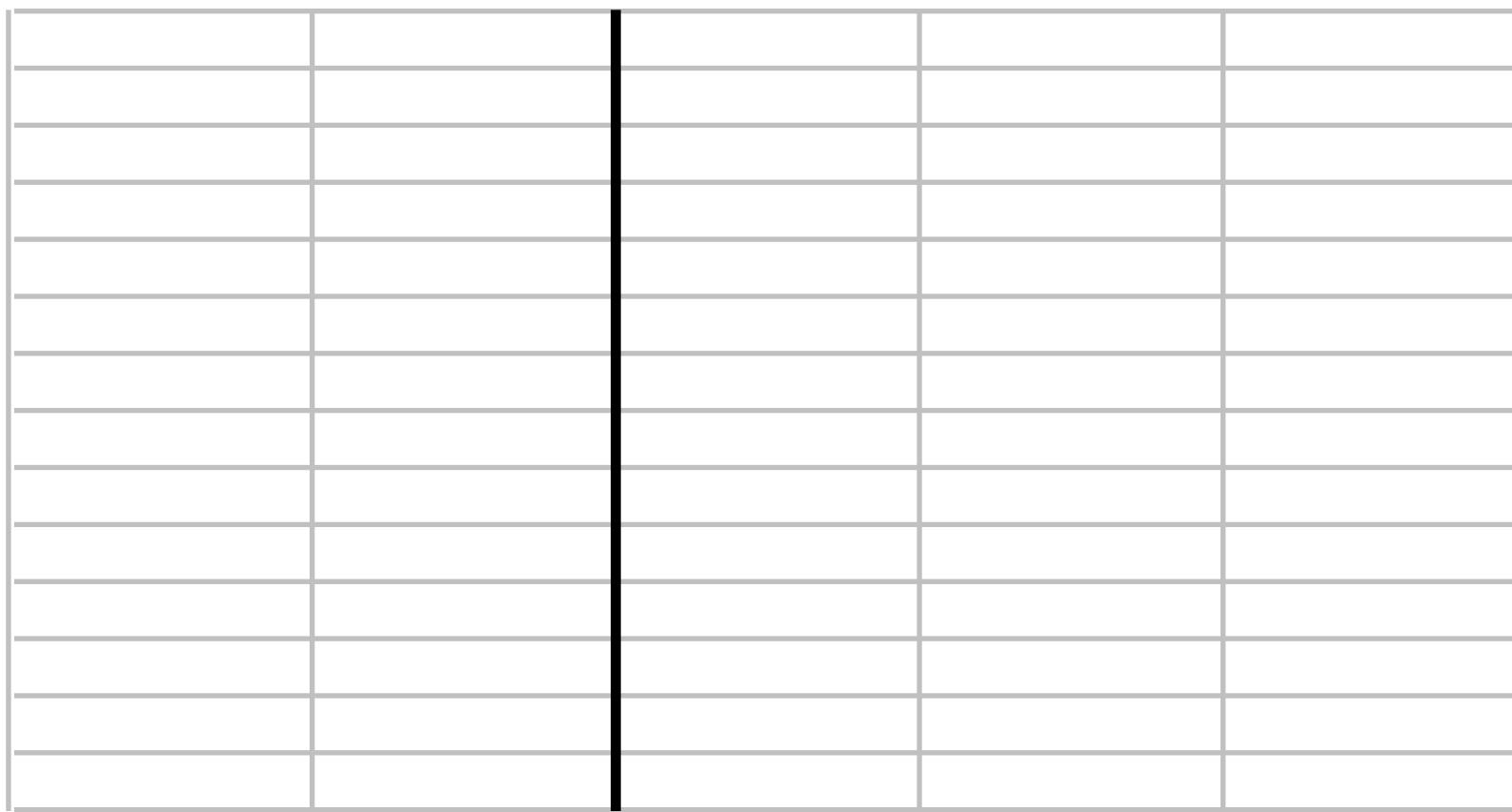
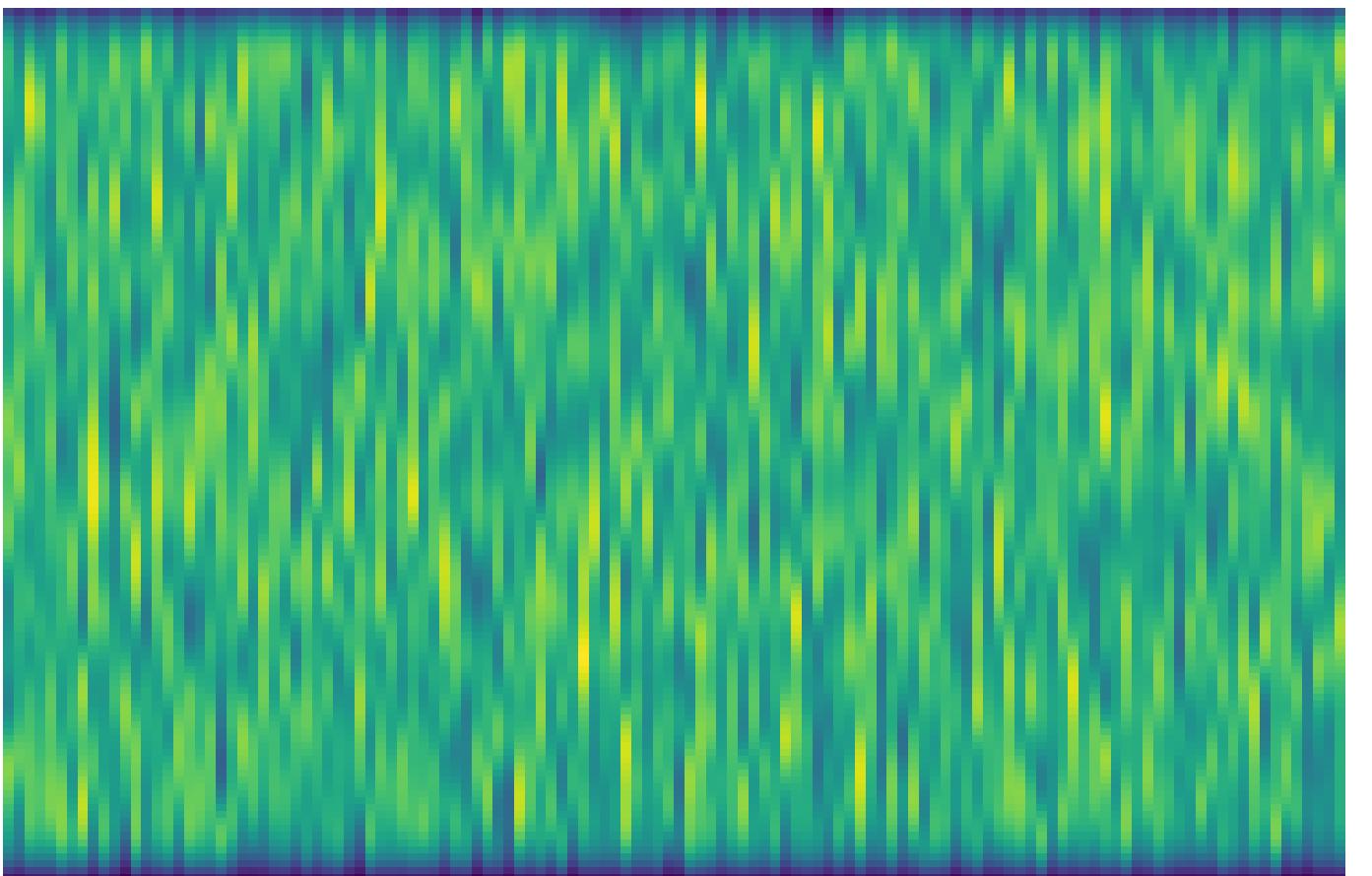


- Downside: if anisotropy varies in another direction, we need a different grid

Options for more robust Multigrid

- **Line/plane relaxation**

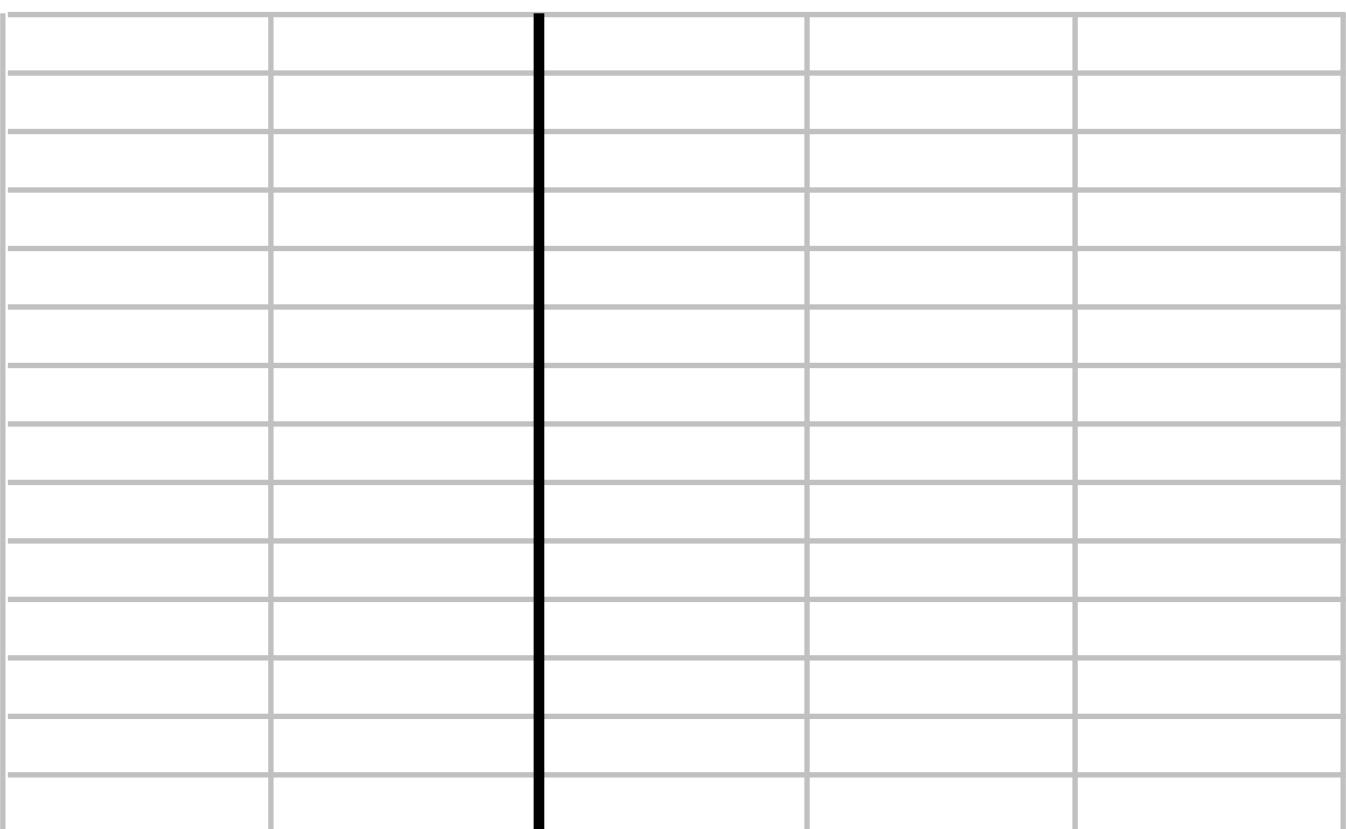
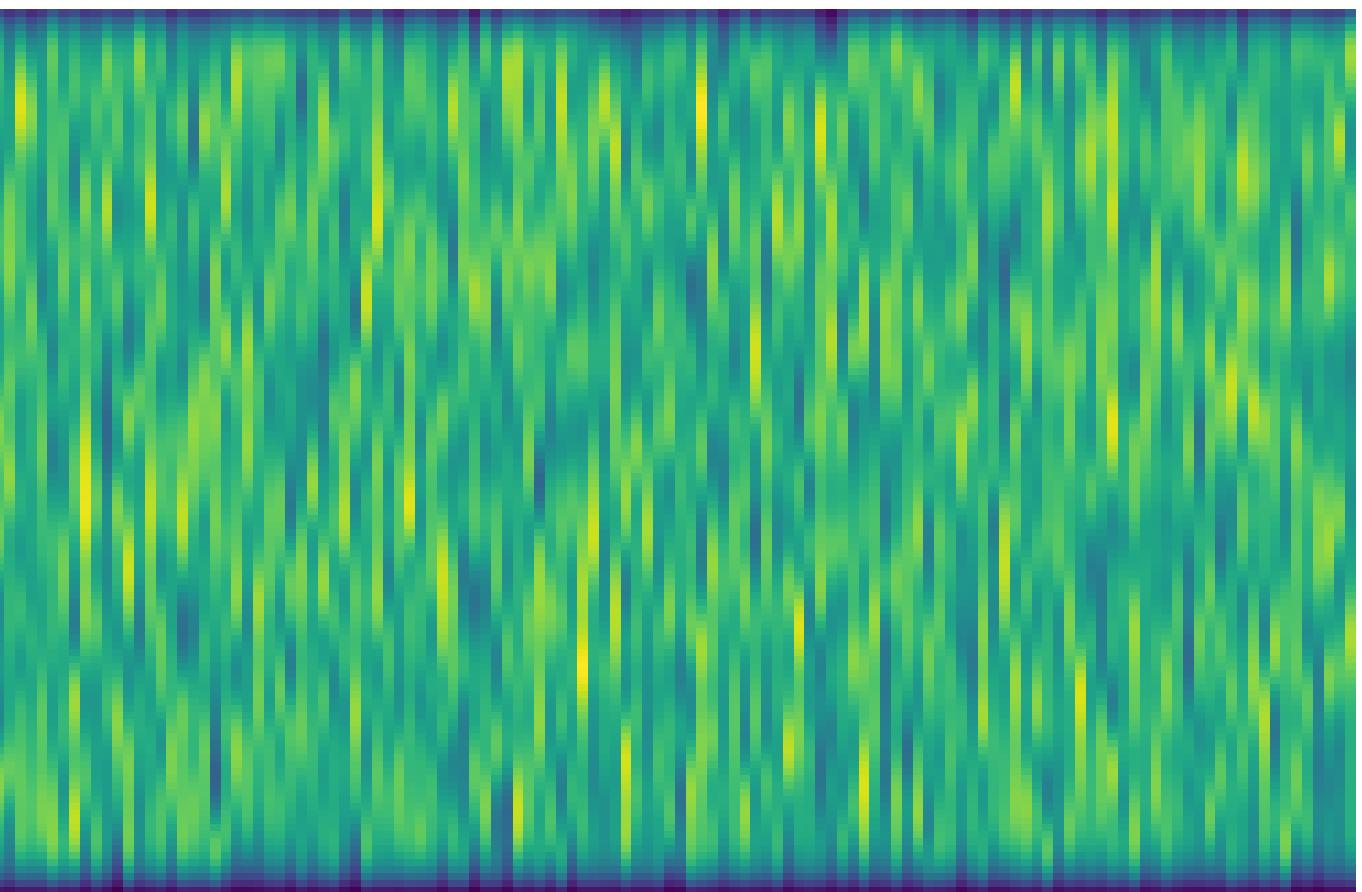
Perform relaxation in groups (in a line)



- Example:
 - smooth error in the y-direction (x-lines)
 - no smoothing in the x-direction (y-lines)

Options for more robust Multigrid

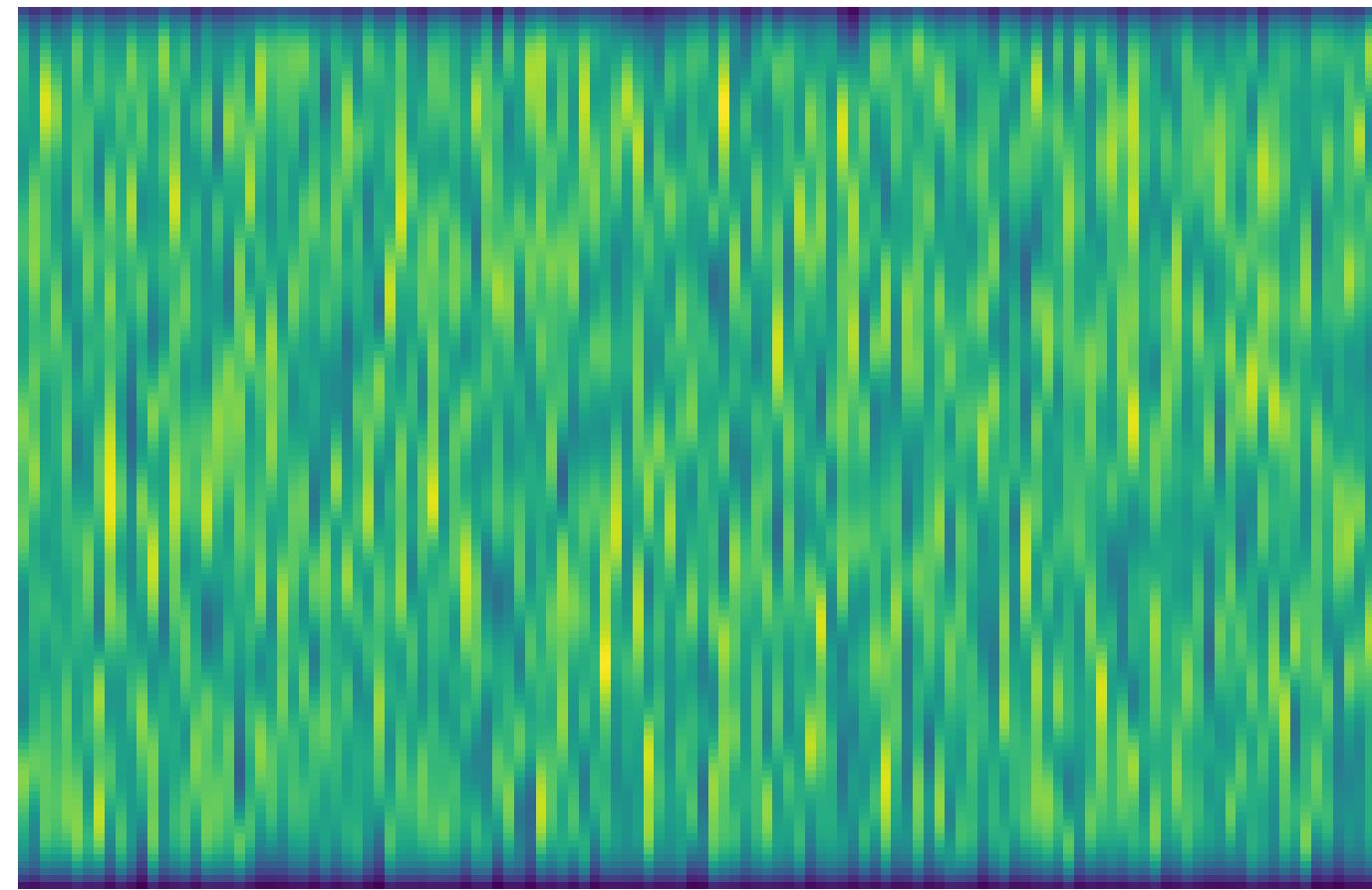
- **Line/plane relaxation**
- For each x-line (lines of strong anisotropy):
 - Eliminate the residual on the entire line
 - (Gauss-Siedel, by lines)



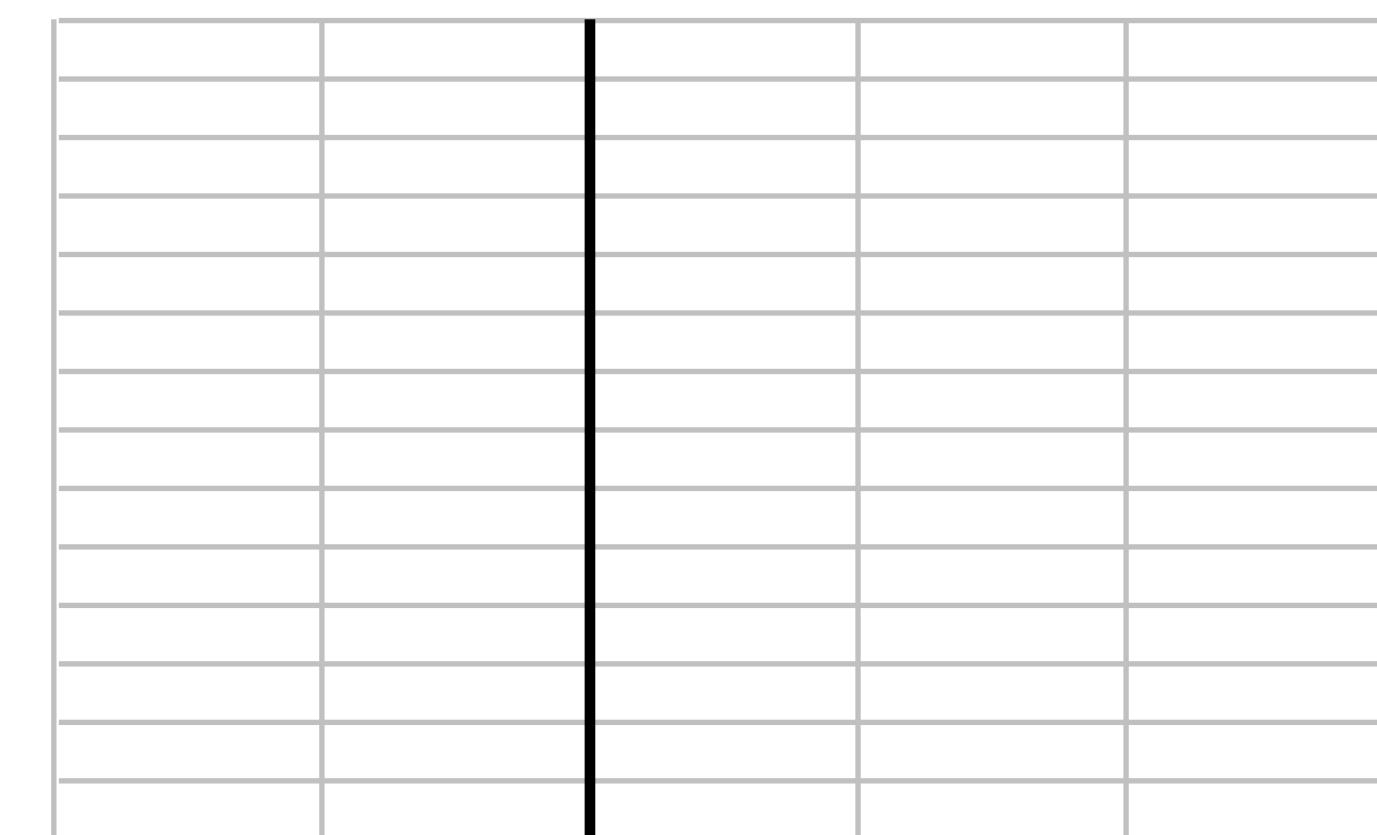
Options for more robust Multigrid

- **Line/plane relaxation**

- $$A = (n + 1)^2 \begin{bmatrix} A_{1D} & -\varepsilon I & & \\ -\varepsilon I & A_{1D} & -\varepsilon I & \\ & -\varepsilon I & A_{1D} & -\varepsilon I \\ & & \ddots & \\ & & & -\varepsilon I & A_{1D} \end{bmatrix}$$



$$A_{1D} = (n + 1)^2 \begin{bmatrix} 2 + 2\varepsilon & -1 & & \\ -1 & 2 + 2\varepsilon & -1 & \\ & \ddots & \ddots & \\ & & -1 & 2 + 2\varepsilon \end{bmatrix}$$



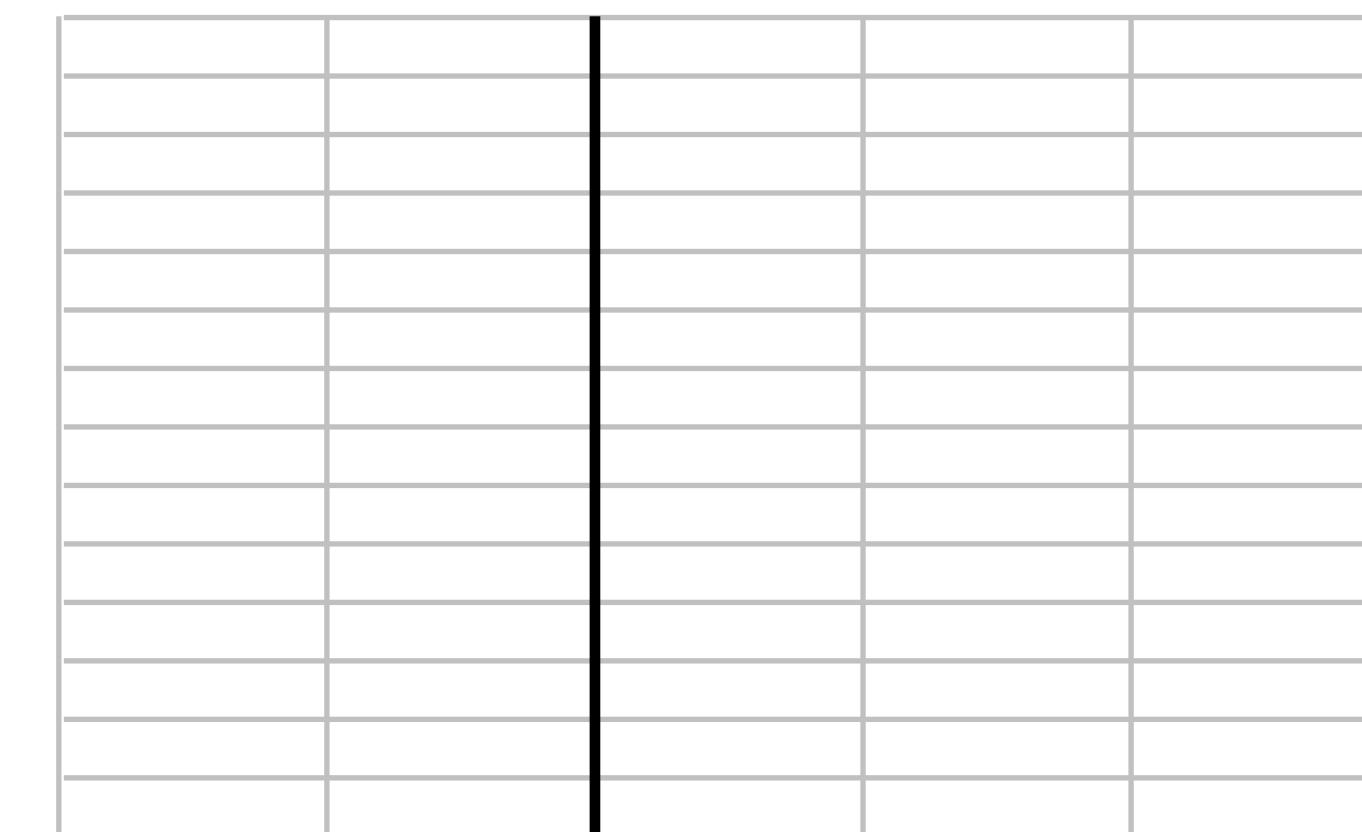
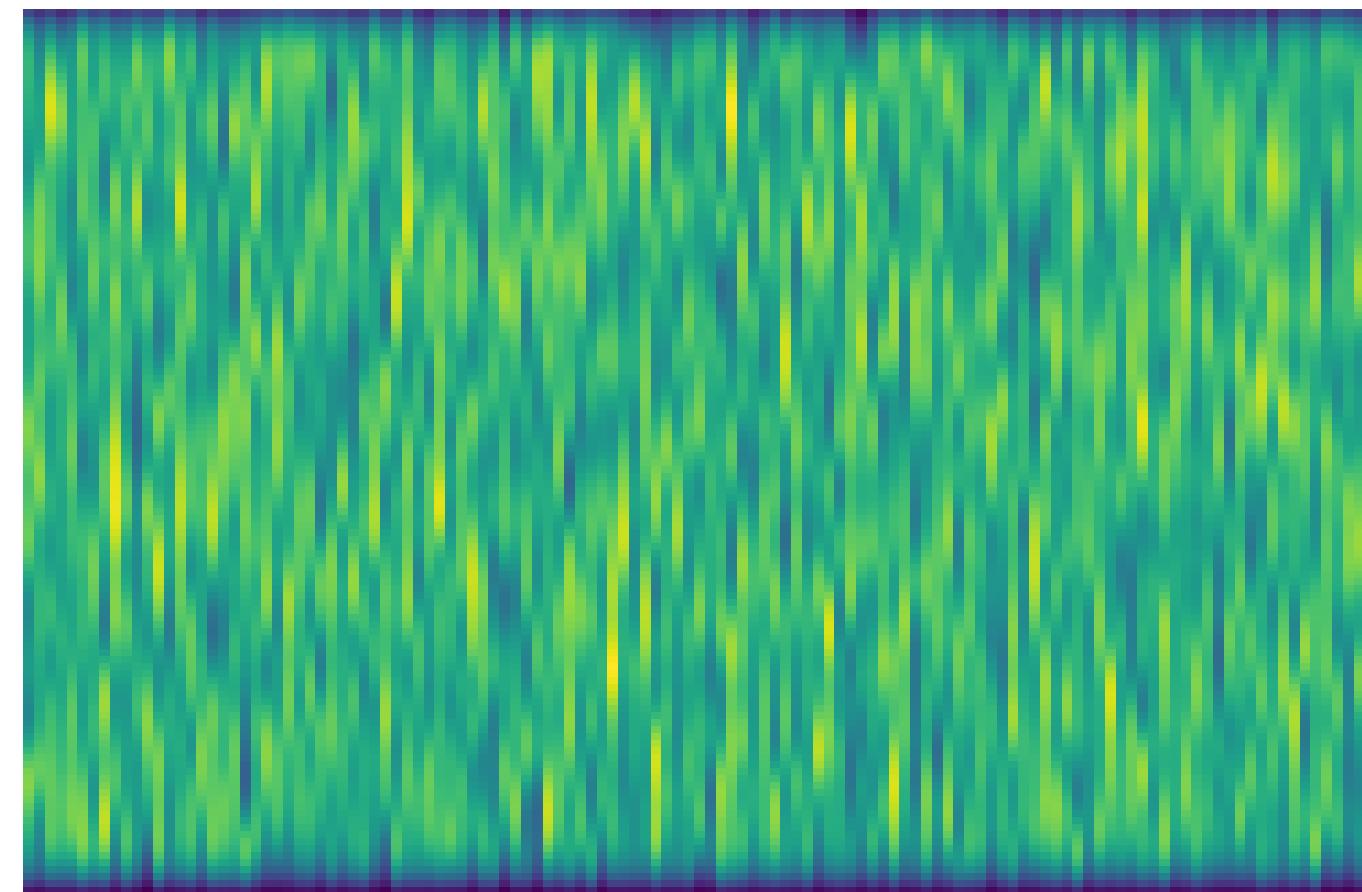
Options for more robust Multigrid

- **Line/plane relaxation**
- For each line, solve

$$A_{1D}v_k = g_k$$

$$v_k = \begin{bmatrix} \vdots \\ v_{k,j-1} \\ v_{k,j} \\ v_{k,j+1} \\ \vdots \end{bmatrix} \quad g_k = \begin{bmatrix} \vdots \\ f_{k,j-1} + \varepsilon(v_{k-1,j-1} + v_{k+1,j-1}) \\ f_{k,j} + \varepsilon(v_{k-1,j} + v_{k+1,j}) \\ f_{k,j+1} + \varepsilon(v_{k-1,j+1} + v_{k+1,j+1}) \\ \vdots \end{bmatrix}$$

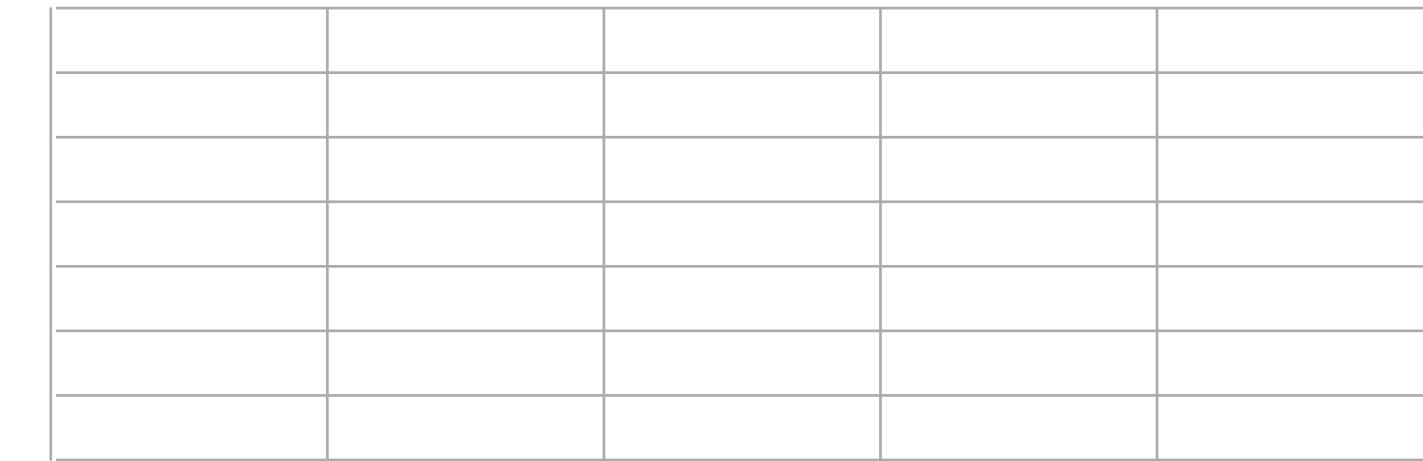
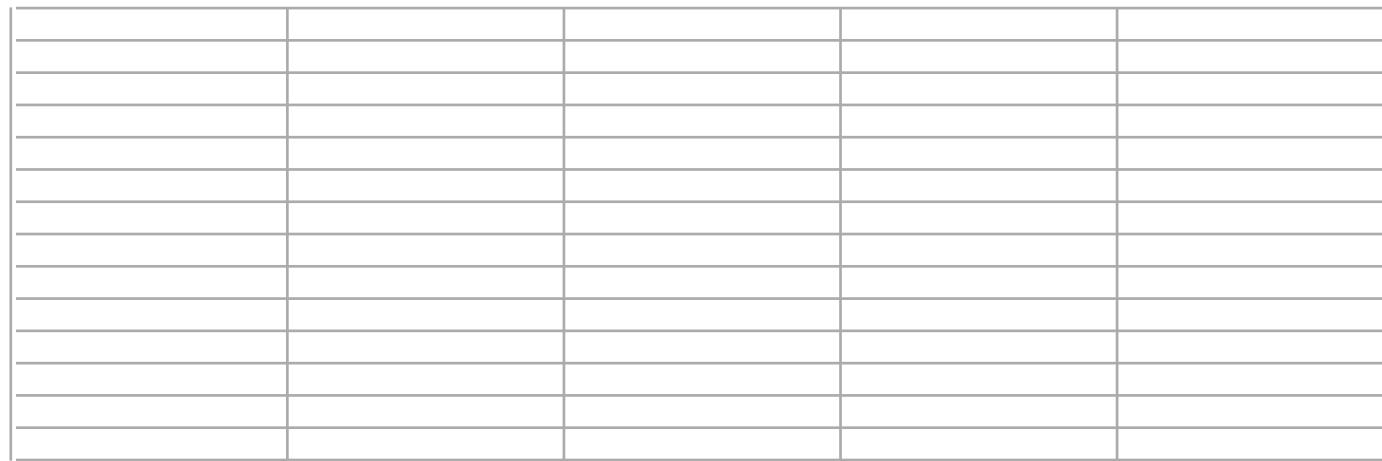
- In 3D, lines become planes...



Options for more robust Multigrid

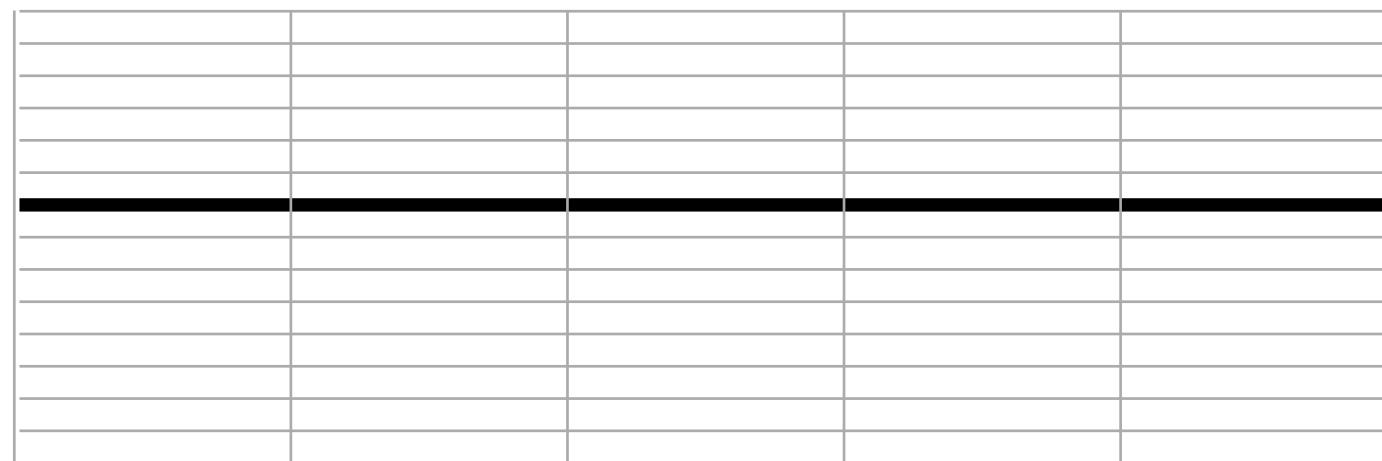
- **Semicoarsening**

Coarsen in the direction of smoothness



- **Line/plane relaxation**

Perform relaxation in groups (in a line)

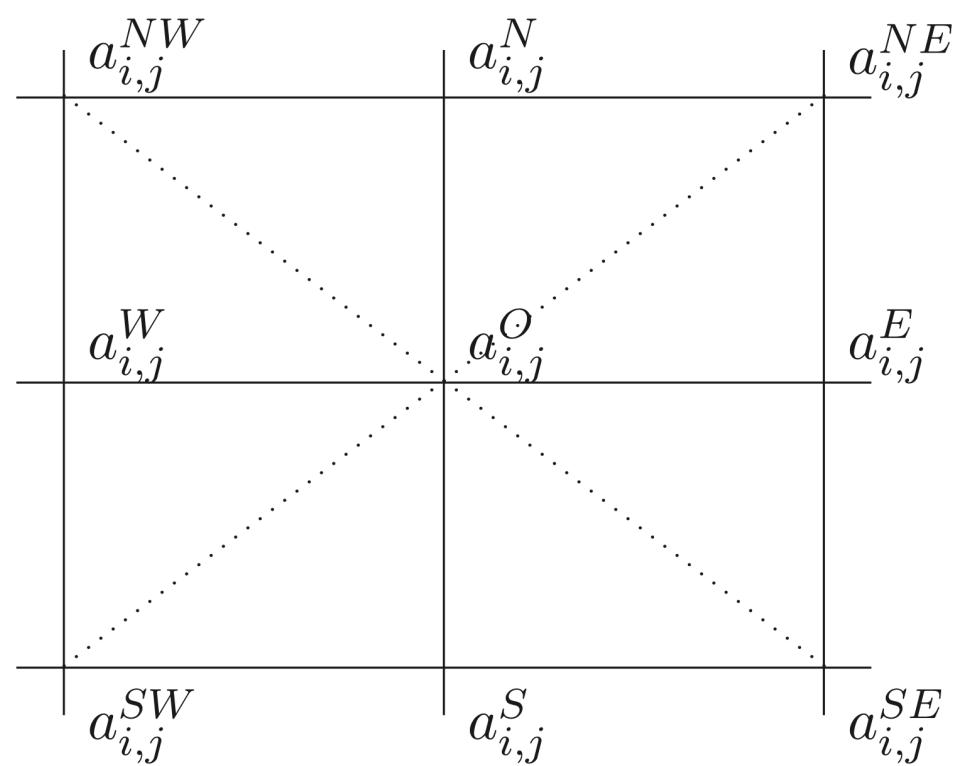


- **Operator based Interpolation** (e.g. BoxMG)
 $Ae = 0$

J. E. Dendy, Black box multigrid, J. Comput. Phys., 1982

J. E. Dendy and J. D. Moulton, Black box multigrid with coarsening by a factor of three, J. Numer. Lin. Alg. App., 2010

- Node (i,j) stencil:

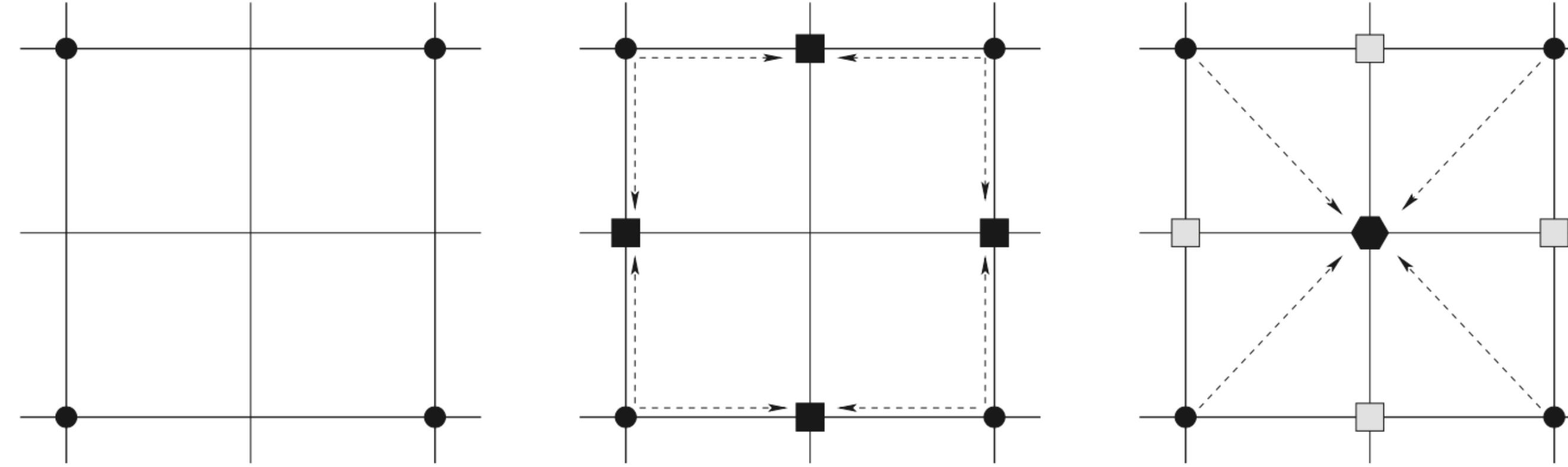


$$Ax = b$$

$$\begin{aligned}
 a_{i,j}^{SW} x_{i-1,j-1} + a_{i,j}^S x_{i,j-1} + a_{i,j}^{SE} x_{i+1,j-1} + a_{i,j}^W x_{i-1,j} + a_{i,j}^O x_{i,j} \\
 + a_{i,j}^E x_{i+1,j} + a_{i,j}^{NW} x_{i-1,j+1} + a_{i,j}^N x_{i,j+1} + a_{i,j}^{NE} x_{i+1,j+1} = b_{i,j}
 \end{aligned}$$

BoxMG

Math and figures from:
[Robust and adaptive multigrid methods: comparing structured and algebraic approaches](#), MacLachlan, Moulton, Chartier



1. Inject coarse points (left)
2. Assume the error is constant along x-lines (and y-lines)
3. Infer interpolation from the edges (right)

Example: Assuming $Ae=0$ and constant y-lines:

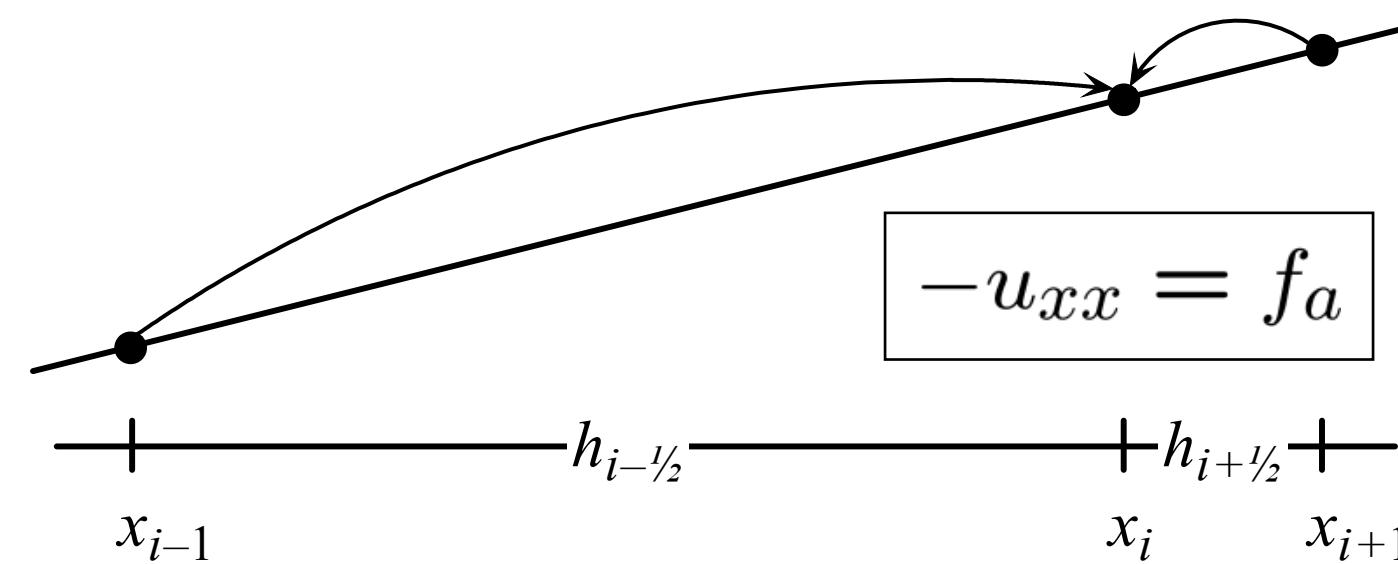
$$(a^S + a^O + a^N)e = -(a^{SW} + a^W + a^{NW})e^c - (a^{SE} + a^E + a^{NW})e^c$$

Interpolation based on entries in A

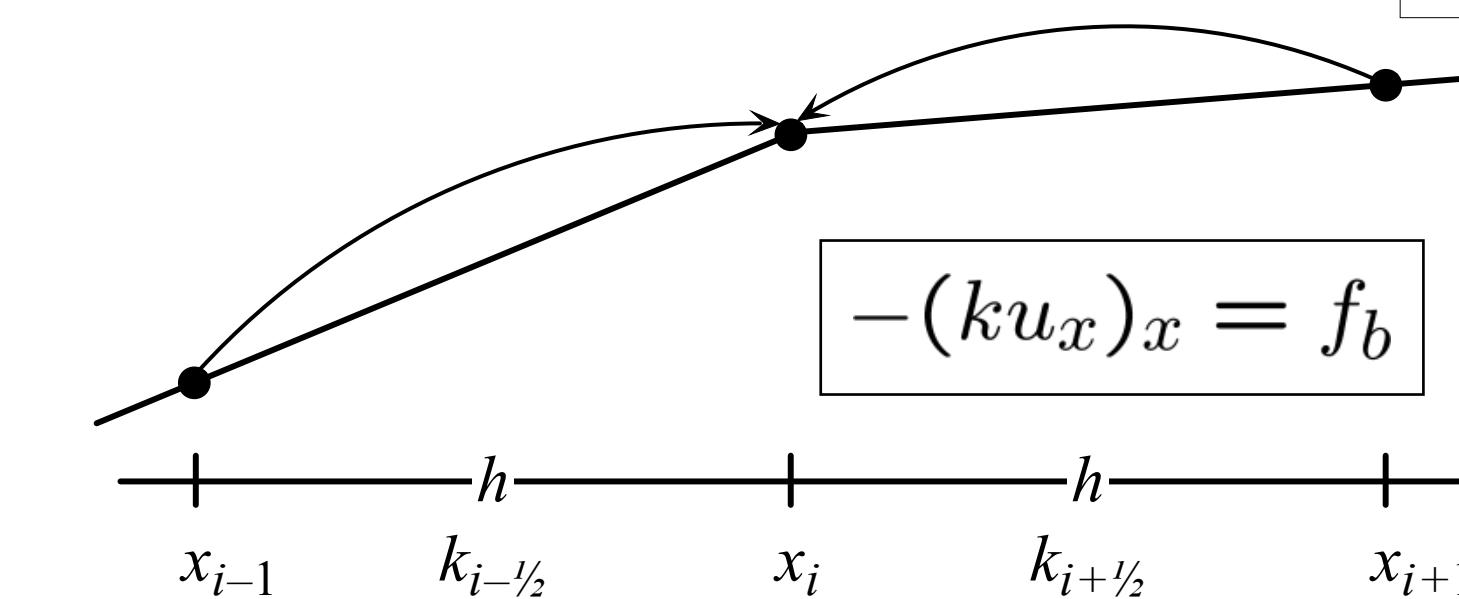
Algebraic Multigrid (AMG) uses matrix coefficients

- Geometric information alone is not sufficient

Linear Interpolation



Operator-Dependent Interpolation

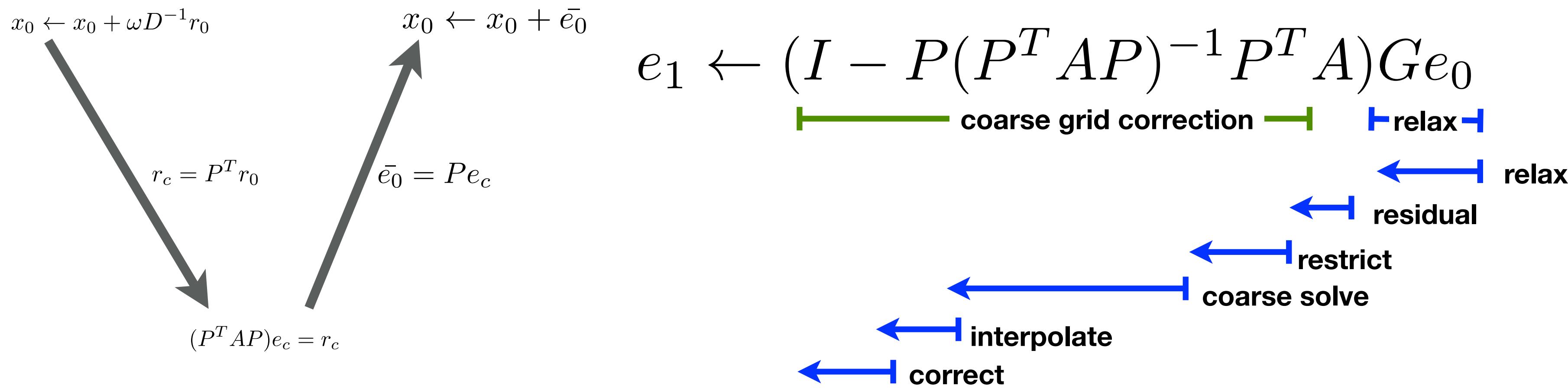


- AMG does not use geometric information, but captures both linear & operator-dependent interpolation

$$(A\mathbf{u})_i = a_{i,i-1}u_{i-1} + a_{i,i}u_i + a_{i,i+1}u_{i+1}$$

$$u_i = \left(-\frac{a_{i,i-1}}{a_{i,i}} \right) u_{i-1} + \left(-\frac{a_{i,i+1}}{a_{i,i}} \right) u_{i+1}$$

Algebraic Observation

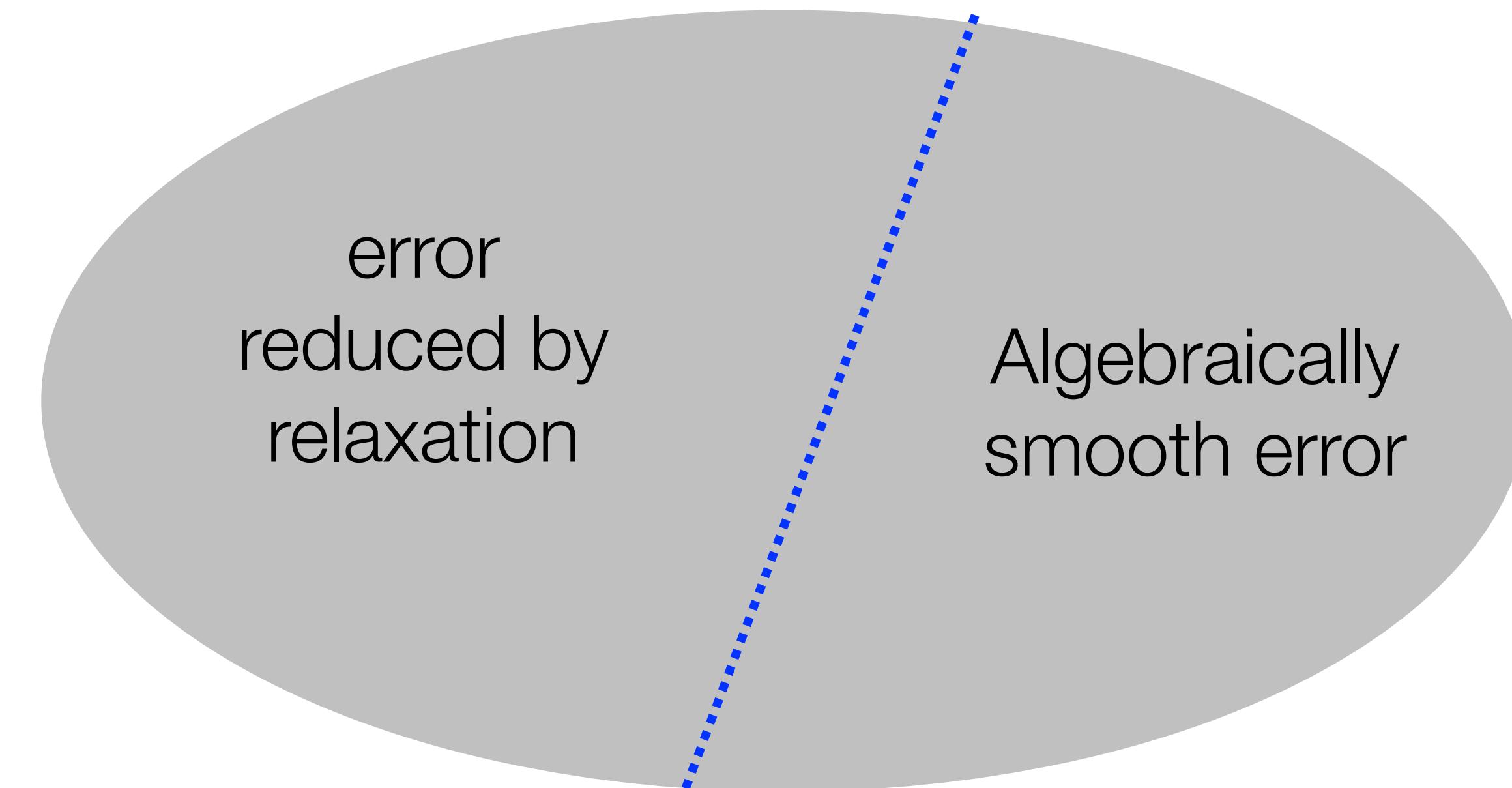


$$G e_0 \in \mathcal{R}(P) \quad \Rightarrow \quad e_1 = 0$$

interpolation should capture what relaxation misses

Algebraically Smooth Error

- “Algebraically smooth” error may not be geometrically smooth

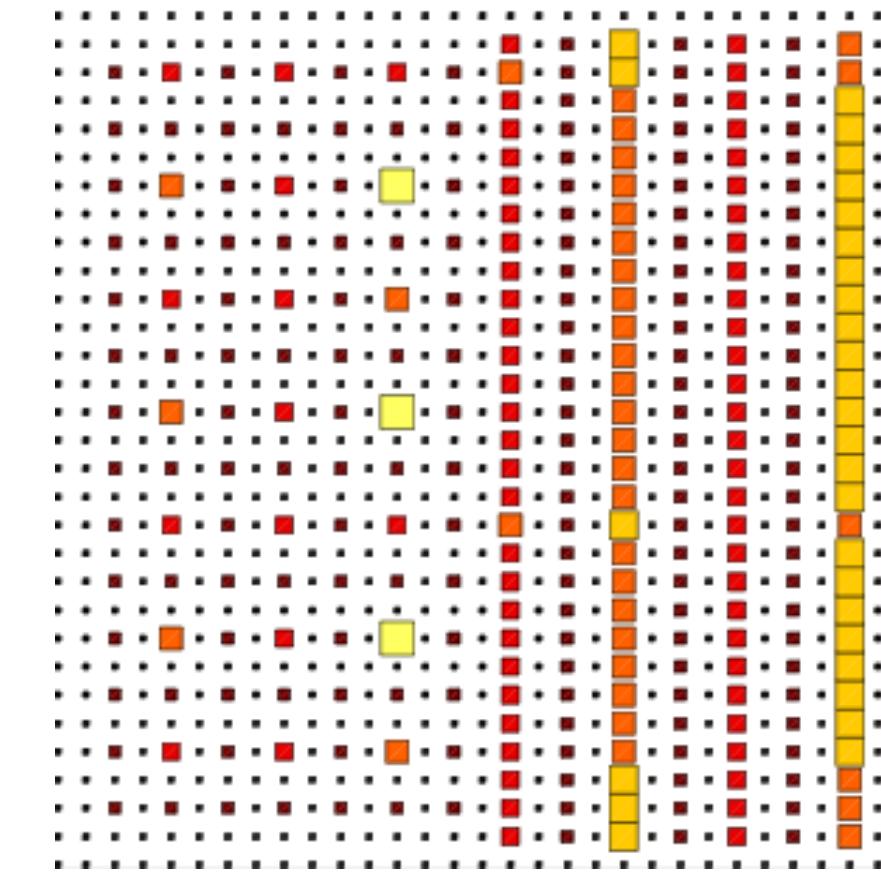
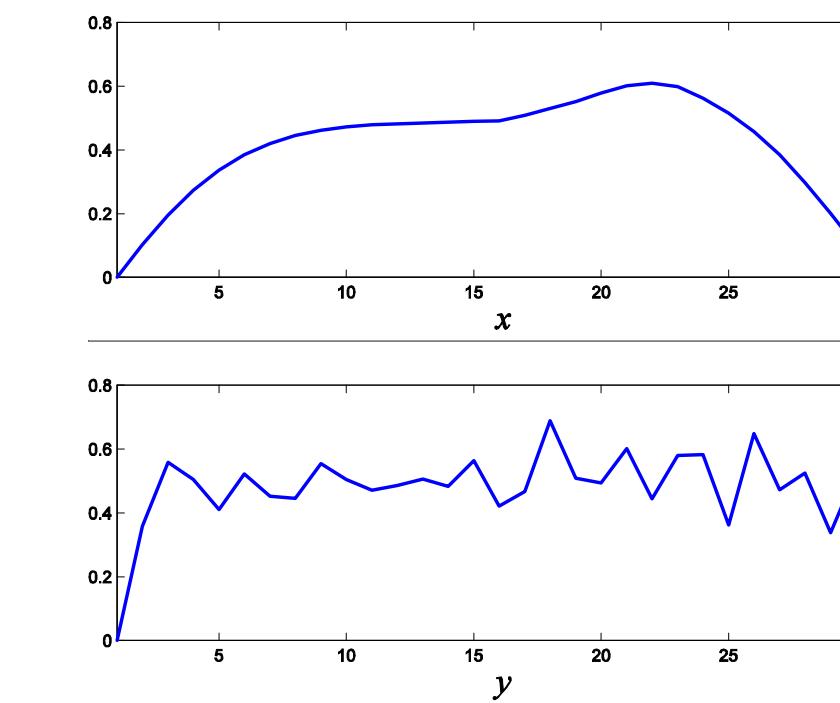
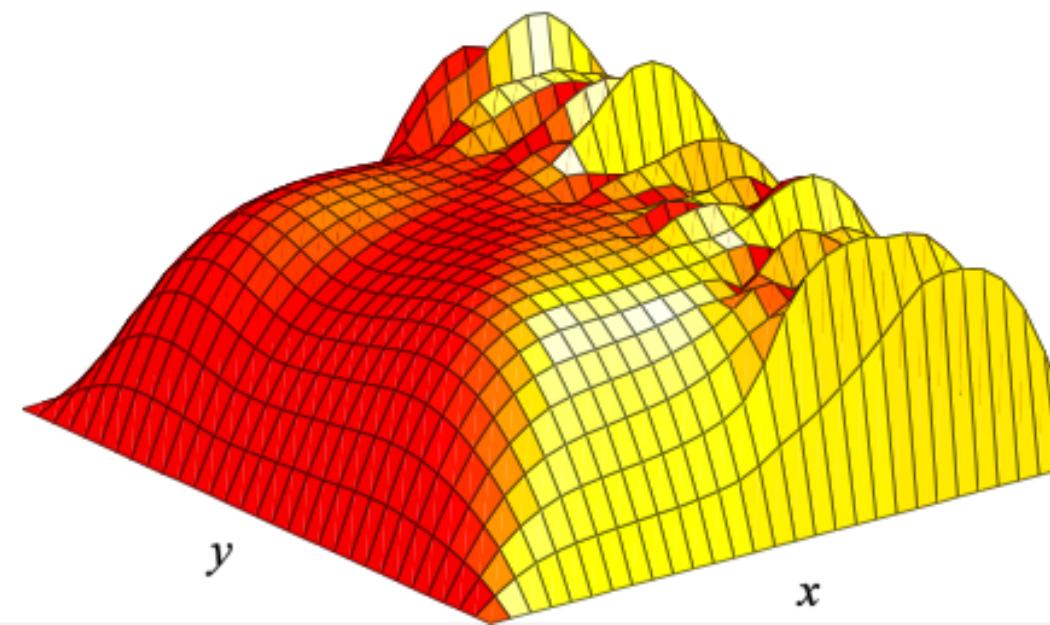


Error left by relaxation can be geometrically oscillatory

- 7 GS sweeps on

$$-au_{xx} - bu_{yy} = f$$

$$\begin{array}{|c|c|} \hline a & b \\ \hline a \gg b & \\ \hline \end{array}$$



- Caution: this example
 - targets **geometric smoothness**
 - uses **pointwise smoothers**

Slide credit: R. Falgout, 2/1/2016

AMG coarsens grids in the direction
of geometric smoothness

Main idea: Algebraically smooth error

- Take a relaxation scheme such as w-Jacobi

$$e \leftarrow (I - M^{-1}A)e$$

- If relaxation stagnates, then the remaining error exhibits poor convergence, so
- Formally (characterized by small eigenvalues)

$$(I - M^{-1}A)e \approx e \Rightarrow M^{-1}Ae \approx 0 \Rightarrow r \approx 0$$

$$\langle Ae, e \rangle \ll 1$$

Main idea: Algebraically smooth error

- We then have

$$\begin{aligned}\langle Ae, e \rangle &= \sum_i e_i (A_{ii}e_i + \sum_{j \neq i} A_{ij}e_j) && \text{assume zero row sum} \\ &= \sum_i e_i \left(\sum_{j \neq i} -A_{ij}(e_i - e_j) \right) \\ &= \sum_{i < j} -A_{ij} \cdot e_i \cdot (e_i - e_j) + \sum_{i > j} -A_{ij} \cdot e_i \cdot (e_i - e_j) && \text{swap } i, j \\ &= \sum_{i < j} -A_{ij} \cdot e_i \cdot (e_i - e_j) - \sum_{i < j} -A_{ij} \cdot e_i \cdot (e_i - e_j) \\ &= \sum_{i < j} -A_{ij} \cdot (e_i - e_j)^2\end{aligned}$$

- Ok, so smooth error varies **slowly** in the direction of large matrix coefficients

Main idea: Algebraically smooth error

- We have assumed **geometric** smoothness to show

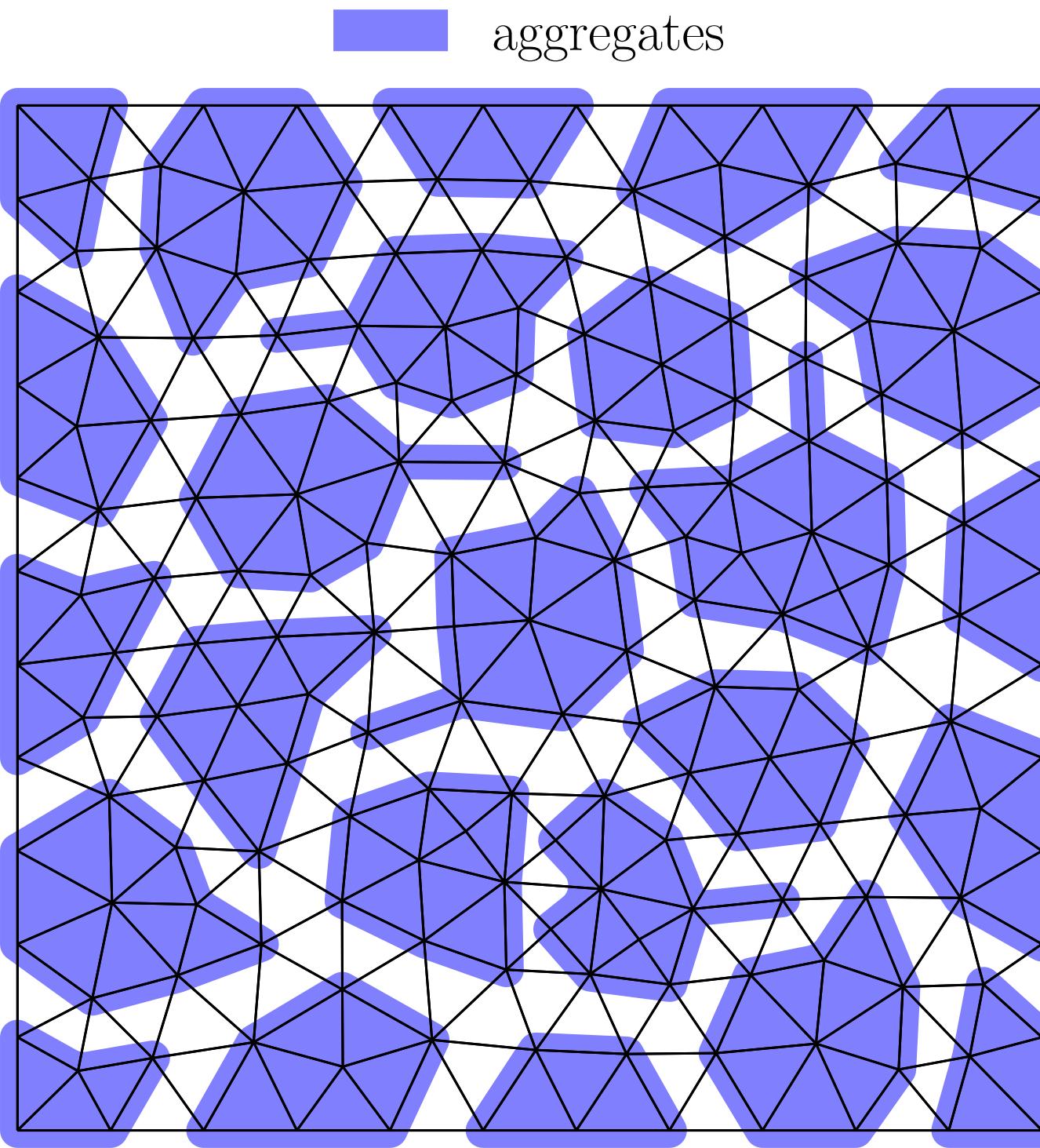
$$\mathbf{e}^T A \mathbf{e} = \sum_{i < j} (-a_{ij})(e_i - e_j)^2 \ll 1$$

- **CF AMG:** Smooth error varies slowly in the direction of “large” matrix coefficients
- **Strength of connection:** Given a threshold $0 < \theta \leq 1$, we say that variable u_i strongly depends on variable u_j if

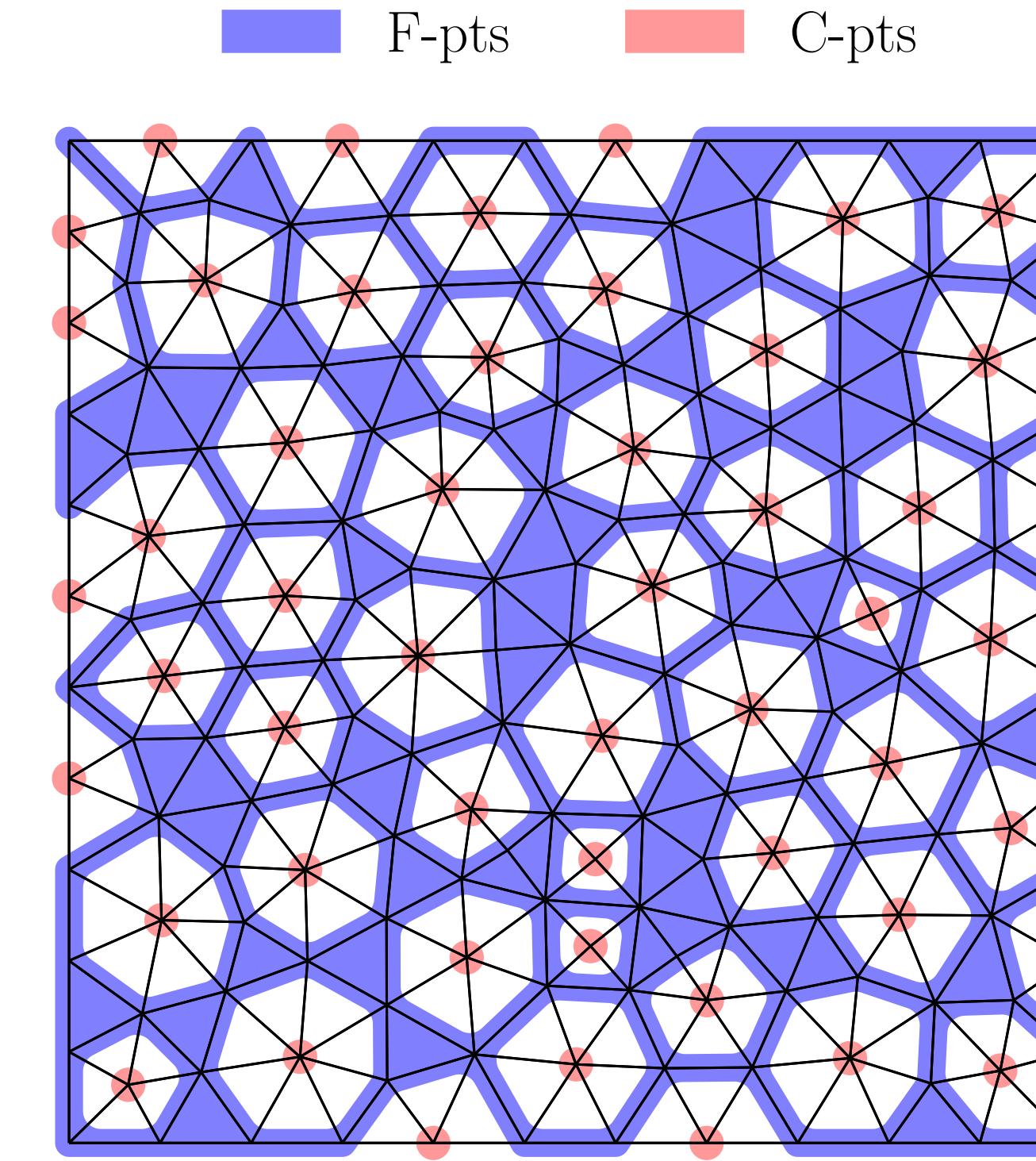
$$-a_{ij} \geq \theta \max_{k \neq i} \{-a_{ik}\}$$

- Often positive off-diagonals are treated as **weak**
- This definition of strength of connection is not symmetric

Two (general) forms of AMG



- Smoothed Aggregation AMG (SA-AMG)
- Interpolation constructed from candidate vectors
- Clear approach to *optimize* interpolation



- Coarse-Fine AMG (CF-AMG) or Ruge-Stüben
- Coarse grid points are a subset of the fine grid points
- Edge-wise construction of interpolation, allowing straightforward control of sparsity
- Incorporating near-nullspace is not straightforward