Multigrid Methods for Coupled Systems

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April 17, 2023

Big Picture

Efficient solvers for coupled multiphysics systems

What multiphysics means to me

Thinking about coupled systems of

- Incompressible (nonlinear) fluid or solid dynamics
- Something else
  - Electromagnetics
  - Heat transfer

Consider linearization and finite-element discretization, resulting in linear systems of the form

$$\mathcal{A}x = \begin{bmatrix} F & Z & B \\ Y & D & 0 \\ B^T & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{\boldsymbol{u}} \\ x_a \\ x_p \end{bmatrix} = \begin{bmatrix} r_{\boldsymbol{u}} \\ r_a \\ r_p \end{bmatrix}.$$

Stokes and Navier-Stokes

Stokes flow is a prototypical model of incompressible fluids:

$$\begin{aligned} \rho\frac{\partial \boldsymbol{u}}{\partial t} - \nabla \cdot (\mu \nabla \boldsymbol{u}) + \nabla p &= \boldsymbol{f} \\ -\nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

- Fluid velocity, $\boldsymbol{u}$, and pressure, p
- Often see in time-steady case
- μ is fluid viscosity, ρ is density

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If inertial terms are incorporated, get Navier-Stokes

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \left(\boldsymbol{u} \cdot \nabla \right) \boldsymbol{u} - \nabla \cdot (\mu \nabla \boldsymbol{u}) + \nabla p = \boldsymbol{f}$$
$$-\nabla \cdot \boldsymbol{u} = 0$$

Thermal flows

Coupling between fluid viscosity and temperature is important in some settings

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \left(\boldsymbol{u} \cdot \nabla \right) \boldsymbol{u} - \nabla \cdot \left(\mu(T) \nabla \boldsymbol{u} \right) + \nabla p = \boldsymbol{f}$$

$$-\nabla \cdot \boldsymbol{u} = 0$$

$$\rho c \left(\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T \right) - \alpha T \frac{\partial p}{\partial t} - \nabla \cdot K \nabla T = F(\boldsymbol{u}, T)$$

Arise in

- Mantle Convection
- Liquid cooling of electronics

Similar models are important in understanding behaviour of many *non-Newtonian* fluids

- Latex paint, Ketchup, Oobleck, ...

Magnetohydrodynamics

$$\begin{aligned} \rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) - \nabla \cdot (\boldsymbol{T} + \boldsymbol{T}_M) &= \boldsymbol{f}, \\ -\nabla \cdot \boldsymbol{u} &= 0, \\ \frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\frac{\eta}{\mu_0} \nabla \times \boldsymbol{B}) - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) &= \boldsymbol{g}, \\ \nabla \cdot \boldsymbol{B} &= 0, \end{aligned}$$

where the viscous and magnetics stress tensors are

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu \left[\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \right], \quad \boldsymbol{T}_M = \frac{1}{\mu_0} \boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2\mu_0} \|\boldsymbol{B}\|^2 \boldsymbol{I},$$

with

- $\eta =$ magnetic resistivity
- $\mu_0 =$ magnetic permeability of free space

(Lots of) Options

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- How to linearize?
  - Time-dependent or steady?
  - Fixed point iteration, operator splitting, Newton
  - Can we guarantee nonlinear convergence?

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  - Staggered finite difference, mixed finite elements (stable or stabilized), finite volume
  - Accuracy, conserved quantities?

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These choices matter in building good solvers

Discretization, Take One

Tempting to try and discretize without thinking about system structure.
Consider time-steady, iso-viscous Stokes:

$$
\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \\
-\nabla \cdot \boldsymbol{u} &= 0
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Could we use

- Centred finite differences for both $\boldsymbol{u}$ and p?
- Piecewise (bi/tri)linear finite elements?

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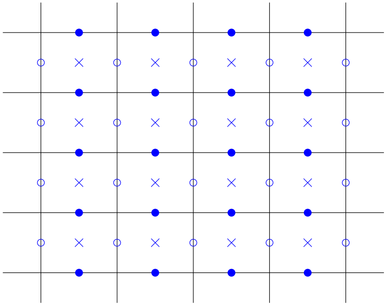
- Piecewise (bi/tri)linear finite elements?

NO!

- Simple discretizations are also unstable (solutions are not unique)

Discretization, Take Two

Achieve stable finite-difference discretization by using *staggered grids* (Marker-and-Cell (MAC) scheme)



Discretize normal velocity on edges and pressure at cell centers

- Standard finite-difference approach on staggered unknowns
- Approach generalizes to other common systems of PDEs

Discretization, Take Three

Finite-element approaches rely on *mixed finite-element* technology

- For Stokes, generally use higher-order finite-element for velocity than for pressure
  - Taylor-Hood: $P_2 - P_1$ or $Q_2 - Q_1$
  - Scott-Vogelius: $P_k - DP_{k-1}$
  - Non-conforming: $RT_k - DP_{k-1}$
- General framework for other equations, spaces
- Well-known *inf-sup* condition to guarantee stability

Boffi, Brezzi, Fortin, Mixed Finite Element Methods and Applications, Springer 2001

Saddle-Point Systems

Stable discretizations lead to linear systems to solve of the form

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix}$$

where

- A is symmetric and positive definite (vector Laplacian)
- B^T is (nearly) full row rank (divergence)
- $\boldsymbol{u}$, p are discrete velocity and pressure unknowns
- $\boldsymbol{f}$ includes RHS data, BCs, etc.

Benzi, Golub, Liesen, Acta Numerica 2005

Multigrid Options

Two main approaches for using multigrid:

1. Preconditioning based on approximate block factorization, using multigrid as a solver for diagonal blocks
   - Well-established in theory and practice
   - Easier as a "black box"
   - Relies on knowing good approximation to Schur complement
2. Applying *monolithic* multigrid to entire system
   - Requires specialized relaxation schemes (point Jacobi doesn't work!)
   - Often more efficient in the end

Block-Factorization Preconditioners

Notice that

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ B^T & -S \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

for $S = B^T A^{-1} B$.

Elman, Sylvester, Wathen, Finite elements and fast iterative solvers, OUP 2005

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for $S = B^T A^{-1} B$. Three families of preconditioners:

- Full factorization

$$M = \begin{bmatrix} \hat{A} & 0 \\ B^T & -\hat{S} \end{bmatrix} \begin{bmatrix} I & \hat{A}^{-1}B \\ 0 & I \end{bmatrix}$$

- Approximate A and S by easily inverted $\hat{A}$, $\hat{S}$
- Take $\hat{A}$ to be one multigrid cycle applied to velocity block
- Take $\hat{S}$ to approximate mass/diagonal matrix

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- Block-diagonal approximation

$$M = \begin{bmatrix} \hat{A} & 0 \\ 0 & \pm\hat{S} \end{bmatrix}$$

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Block Factorization Works

Key point is that block methods with *exact* solves for A and S are ideal preconditioners

- At most 3 iterations for block diagonal
- At most 2 iterations for block triangular
- At most 1 iteration for full factorization

Lose some efficiency with approximate solves, but retain scalability with good Schur complement approximations

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Murphy, Golub, Wathen, SISC 2000

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Why do anything else?

- Monolithic approaches can be more efficient
- Don't always know how to approximate Schur complement
- Nested Schur complements add complication when considering larger systems

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Murphy, Golub, Wathen, SISC 2000

Geometric Monolithic Multigrid

Basic idea: hierarchy of discretizations of coupled system

- Use geometric interpolation (e.g., canonical finite-element interpolation) between levels
- Don't interpolate between different components of the system

  For Stokes, $P = \begin{bmatrix} P_{\boldsymbol{u}} & 0 \\ 0 & P_p \end{bmatrix}$
- Define coarse-grid operators either by rediscretization or Galerkin (often equivalent)
- Focus energy on design and analysis of effective relaxation schemes for coupled systems

Coupled Relaxation Schemes

Three common families of relaxation methods

1. Distributive Relaxation
   - Idea: change the problem to one where pointwise relaxation works

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Brandt, Livne, Multigrid techniques: 1984 guide with applications to fluid dynamics, SIAM 2011

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2. Approximate Block Factorization
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   - Uzawa relaxation uses block-triangular preconditioner
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Braess, Sarazin, Appl. Numer. Math. 23, 1997

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3. Overlapping Schwarz Methods
   - Idea: form overlapping blocks around constraint DoFs
   - For fluids, use Vanka approach
   - For EM, use star relaxation

Vanka, J. Comput. Phys. 65, 1986

Arnold, Falk, Winther, Math. Comp. 66, 1997

Distributive Relaxation

At continuum level, we can transform the Stokes equations into modified system:
Let $\hat{p}$ be such that $\Delta \hat{p} = p$, $\hat{\boldsymbol{u}} = \boldsymbol{u} - \nabla \hat{p}$:

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= -\Delta \left(\hat{\boldsymbol{u}} + \nabla \hat{p} \right) + \nabla p \\ &= -\Delta \hat{\boldsymbol{u}} - \nabla \Delta \hat{p} + \nabla p = \boldsymbol{f} \\ -\nabla \cdot \boldsymbol{u} &= -\nabla \cdot \hat{\boldsymbol{u}} - \Delta \hat{p} = 0 \end{aligned}$$

At discrete level, approximate this:

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} I & B \\ 0 & -A_p \end{bmatrix} \approx \begin{bmatrix} A & 0 \\ B^T & B^T B \end{bmatrix},$$

where $A_p \approx B^T B$ is the pressure Laplacian

- Relies on discrete commutator relationship, $AB \approx BA_p$

Practical Details, Part I

Suppose we had approximations, $\hat{\boldsymbol{u}}^{(k)}$ and $\hat{p}^{(k)}$, to $\hat{\boldsymbol{u}}$ and $\hat{p}$. Then compute updates as

$$\begin{bmatrix} \boldsymbol{r}_{\boldsymbol{u}}^{(k+1)} \\ r_p^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} - \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} I & B \\ 0 & -A_p \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}}^{(k)} \\ \hat{p}^{(k)} \end{bmatrix}$$

$$\begin{bmatrix} \hat{\boldsymbol{u}}^{(k+1)} \\ \hat{p}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{u}}^{(k)} \\ \hat{p}^{(k)} \end{bmatrix} + \omega \begin{bmatrix} \hat{A} & 0 \\ B^T & \widehat{B^T B} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{r}_{\boldsymbol{u}}^{(k+1)} \\ r_p^{(k+1)} \end{bmatrix}$$

Then could take

$$\begin{bmatrix} \boldsymbol{u}^{(k+1)} \\ p^{(k+1)} \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & -A_p \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}}^{(k+1)} \\ \hat{p}^{(k+1)} \end{bmatrix}$$

Practical Details, Part II

But note that if we know

$$\begin{bmatrix} \boldsymbol{u}^{(k)} \\ p^{(k)} \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & -A_p \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}}^{(k)} \\ \hat{p}^{(k)} \end{bmatrix}$$

Then

$$\begin{bmatrix} \boldsymbol{r}_{\boldsymbol{u}}^{(k+1)} \\ r_p^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} - \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(k)} \\ p^{(k)} \end{bmatrix}$$

Multiplying correction on left by $\begin{bmatrix} I & B \\ 0 & -A_p \end{bmatrix}$ gives

$$\begin{bmatrix} \boldsymbol{u}^{(k+1)} \\ p^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{(k)} \\ p^{(k)} \end{bmatrix} + \omega \begin{bmatrix} I & B \\ 0 & -A_p \end{bmatrix} \begin{bmatrix} \hat{A} & 0 \\ B^T & \widehat{B^T B} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{r}_{\boldsymbol{u}}^{(k+1)} \\ r_p^{(k+1)} \end{bmatrix}$$

$\Rightarrow$ Don't need $\hat{\boldsymbol{u}}$, $\hat{p}$ to do distributive relaxation!

Limitations

Distributive Relaxation works well in monolithic MG for

- Simple staggered FD discretizations
- Linear systems of PDEs

Effectiveness relies on discrete commutator: $\Delta \nabla p = \nabla \Delta p$

- More natural on structured uniform meshes
- Generally violated by BCs in $AB \approx BA_p$

Adams, Samtaney, Brandt, J. Comp. Phys. 2010
Wang, Chen, J. Sci. Comp. 2013

Uzawa Iteration

Uzawa iteration aims to solve

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$

iteratively, by computing

$$\begin{aligned} \boldsymbol{u}^{(k+1)} &= A^{-1}\left(\boldsymbol{f} - Bp^{(k)}\right) \\ p^{(k+1)} &= p^{(k)} + \alpha B^T \boldsymbol{u}^{(k+1)} \end{aligned}$$

Recognize this as a Richardson (stationary) iteration for solving

$$B^T A^{-1} B p = B^T A^{-1} \boldsymbol{f}$$

with $\boldsymbol{u} = A^{-1}(\boldsymbol{f} - Bp)$.

Elman, Golub, SINUM 31, 1994

Inexact Uzawa Iteration

Can also introduce inexact solves for A, leading to

$$\begin{aligned}
\boldsymbol{u}^{(k+1)} &= \boldsymbol{u}^{(k)} + \hat{A}^{-1}\left(\boldsymbol{f} - A\boldsymbol{u}^{(k)} - Bp^{(k)}\right) \\
p^{(k+1)} &= p^{(k)} + \alpha B^T \boldsymbol{u}^{(k+1)}
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And better approximations to Schur complement solve, giving

$$\begin{aligned} \boldsymbol{u}^{(k+1)} &= \boldsymbol{u}^{(k)} + \hat{A}^{-1}\left(\boldsymbol{f} - A\boldsymbol{u}^{(k)} - Bp^{(k)}\right) \\ p^{(k+1)} &= p^{(k)} + \hat{S}^{-1}\left(B^T \boldsymbol{u}^{(k+1)}\right) \end{aligned}$$

Lots of analysis of efficiency of these as standalone iterations

- Can introduce Krylov acceleration
- Understand properties needed for robust convergence

Elman, Golub, SINUM 31, 1994
Zulehner, Math Comp 71, 2001

Uzawa Relaxation

For efficient multigrid, however, choose $\hat{A}$ and $\hat{S}$ to give good smoothing properties

Typical choices:

- Gauss-Seidel relaxation for $\hat{A}$, Richardson for $\hat{S}$
- Weighted Jacobi relaxation for $\hat{A}$
- Approximate S by $B^T D^{-1} B$, where D is (block) diagonal of A, then use weighted Jacobi on this for $\hat{S}$

Can extend Local Fourier Analysis to understand these choices and relaxation weights

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Maitre, Musy, Nignon, A fast solver for the Stokes equations using multigrid with a Uzawa smoother, 1984
Luo et al., NLAA 2017 & SISC 2017
He, MacLachlan, NLAA 2018, JCAM 2019

Symmetrizing Uzawa

Can rewrite general Uzawa iteration as

$$\begin{bmatrix} \hat{A} & 0 \\ B^T & -\hat{S} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)} \\ p^{(k+1)} - p^{(k)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} - \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(k)} \\ p^{(k)} \end{bmatrix}$$

Recognize $\begin{bmatrix} \hat{A} & 0 \\ B^T & -\hat{S} \end{bmatrix}$ as approximation to lower-triangular part of LU factorization of $\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}$

Why not include upper-triangular solve as well?

Bank, Welfert, Yserentant, Num. Math. 1990

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Approximate

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \approx \begin{bmatrix} \hat{A} & 0 \\ B^T & -\hat{S} \end{bmatrix} \begin{bmatrix} I & \hat{A}^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} \hat{A} & B \\ B^T & \hat{C} \end{bmatrix}$$

for $\hat{C} = B^T \hat{A}^{-1} B - \hat{S}$

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Replace Uzawa by

$$\begin{bmatrix} \hat{A} & B \\ B^T & \hat{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)} \\ p^{(k+1)} - p^{(k)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} - \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(k)} \\ p^{(k)} \end{bmatrix}$$

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Componentwise iteration is

$$\begin{aligned} \boldsymbol{u}^{(k+1/2)} &= \boldsymbol{u}^{(k)} + \hat{A}^{-1}\left(\boldsymbol{f} - A\boldsymbol{u}^{(k)} - Bp^{(k)}\right) \\ p^{(k+1)} &= p^{(k)} + \hat{S}^{-1}\left(B^T\boldsymbol{u}^{(k+1/2)}\right) \\ \boldsymbol{u}^{(k+1)} &= \boldsymbol{u}^{(k+1/2)} - \hat{A}^{-1}B\left(p^{(k+1)} - p^{(k)}\right) \end{aligned}$$

Bank, Welfert, Yserentant, Num. Math. 1990

Braess-Sarazin Relaxation

Braess and Sarazin proposed using this as a relaxation, fixing $\hat{S} = B^T \hat{A}^{-1} B$:

$$\begin{bmatrix} \hat{A} & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)} \\ p^{(k+1)} - p^{(k)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} - \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(k)} \\ p^{(k)} \end{bmatrix}$$

Idea: for simple $\hat{A}$ (e.g., diagonal)

1. Compute $B^T \hat{A}^{-1} B$ exactly
2. Solve $\hat{S}\left(p^{(k+1)} - p^{(k)}\right) = B^T \boldsymbol{u}^{(k+1/2)}$ using multigrid (or something similar) to high tolerance

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Braess, Sarazin, App Num Math 1997
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Zulehner later proposed to solve system with $\hat{S}$ inexactly

- Multigrid to low tolerance (1 V-cycle)
- *For some discretizations* can even use small number of weighted Jacobi iterations

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Braess, Sarazin, App Num Math 1997
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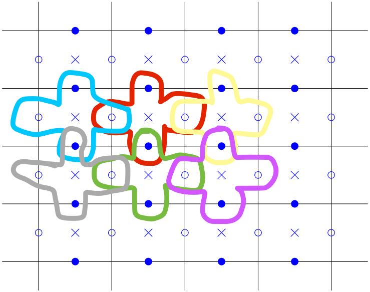
Vanka Relaxation

First proposed for MAC scheme discretization of Navier-Stokes

- Overlapping Schwarz method as relaxation scheme
- Restrict residual and matrix to local subdomain, solve there and correct
- Construction of subdomains is key to achieving convergence



Vanka, J. Comput. Phys. 65, 1986

Vanka for (Navier-)Stokes

Many extensions to many discretizations

- Central theme: “pressure-centric” relaxation
  - Define subdomains to include a single pressure degree of freedom and some subset of the velocities in its geometric/algebraic neighbourhood
- Multiplicative, additive, hybrid variants all work
- Typical use is in geometric multigrid
- Some sensitivity to parameter choice
- Generally do exact subdomain solves
  - Factoring submatrices can be expensive
  - Sometimes see “diagonal” variant

John, Tobiska, IJNMF 2000 & John, Matthies, IJNMF 2001
Rodrigo, Gaspar, Lisbona, App. Num. Math. 2016
Adler, Benson, MacLachlan, NLAA 2017
Farrell, He, MacLachlan, NLAA 2021

“Vanka” for Maxwell’s Equations

Similar patch-based relaxation is common for both

$$
\begin{aligned}
-\nabla\left(K\nabla\cdot\boldsymbol{u}\right)+\boldsymbol{u}&=\boldsymbol{f}\\
\nabla\times\left(K\nabla\times\boldsymbol{v}\right)+\boldsymbol{v}&=\boldsymbol{g}
\end{aligned}
$$

No pressure!

- Implied constraints from vector calculus:

$$\nabla\times\boldsymbol{u}=\nabla\times\boldsymbol{f}\ \text{and}\ \nabla\cdot\boldsymbol{v}=\nabla\cdot\boldsymbol{g}$$

- Exact sequence property that leads to effective subdomain construction for Raviart-Thomas and Nédélec elements

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Arnold, Falk, Winther, Math. Comp. 1997, Numer. Math. 2000

Some Theory

Consider nearly singular systems, $(A_0 + \epsilon A_1)u = f$, where A_0 is SPSD (and singular), and A_1 is SPD

Need 2 ingredients to guarantee convergence

- Interpolation must map coarse-grid kernel to fine-grid kernel
- Subdomains for Vanka-type relaxation must provide stable decomposition of the kernel
    - Again, cannot choose *any* set of Vanka patches, but must do so in compatible way to problem and discretization

This framework has been adapted in recent years to yield robust algorithms for new problems and new discretizations

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Schöberl, PhD thesis 1999, Numer. Math. 1999
Lee, Wu, Xu, Zikatanov, MMMAS 2007

Some Software

Uzawa, BSR, Vanka relaxation schemes easily implemented in Firedrake

- Firedrake is a well-developed and fully featured FEM package
- Strong focus on code generation to enable HPC use
- Nonlinear and linear solvers enabled through interface to PETSc
- PETSc's "Fieldsplit" approach enables Uzawa and BSR
- PCPATCH is tightly coupled library that implements "patch-based" relaxation within this framework

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Rathgeber et al., ACM TOMS 2016
Kirby, Mitchell, SISC 2018
Balay et al., PETSc Users Manual 2021
Farrell, Knepley, Mitchell, Wechsung, ACM TOMS 2021

FD for Stokes

Use weighted-Jacobi variants (allows parallelism)

- Fix 1 relaxation sweep per level (e.g., W(1,0) cycles)
- Can prove (using LFA) optimal convergence factors of
  - Distributive Relaxation $= 0.6$
  - Inexact Braess-Sarazin $= 0.6$
  - Uzawa Relaxation $= \sqrt{0.6}$
- Uzawa costs about $1/2$ as much per iteration as other two
- See good agreement between measured performance and theory

He, MacLachlan, NLAA 2018

FEM for Stokes

Again focus on additive variants

- For stabilized Q1-Q1:
  - Distributive Relaxation convergence factor is $1/3$ (requires 2 sweeps of Jacobi on pressure)
  - Exact Braess-Sarazin convergence factor is $1/3$
  - Inexact Braess-Sarazin convergence degrades; need MG on pressure equation to get close to exact results
  - Observe Uzawa convergence factors around $1/2$

He, MacLachlan, JCAM 2019
Farrell, He, MacLachlan, NLAA 2021

FEM for Stokes

Again focus on additive variants

- For (stable) Q2-Q1:
  - Theory is harder
  - Observe convergence factors of 0.6 for distributive relaxation
  - With W(1,1) cycles and 3 sweeps of Jacobi on pressure, observe convergence factor of 0.25 for inexact Braess-Sarazin
  - Observe Uzawa convergence factor of about 0.55
  - LFA for Vanka gives convergence factor of 0.6

Generally find Vanka is less sensitive to parameter choices, but BSR with tuned parameters gives fastest time to solution

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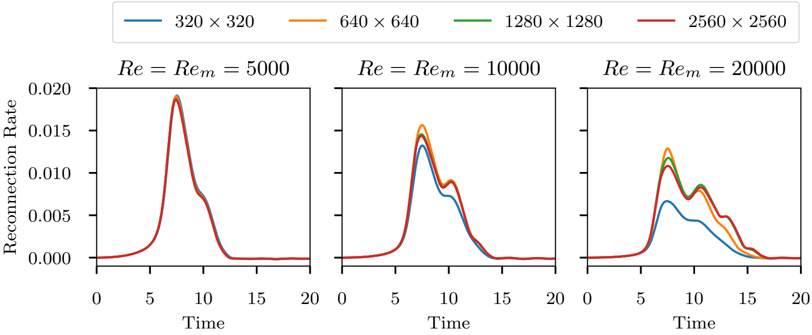
He, MacLachlan, JCAM 2019
Farrell, He, MacLachlan, NLAA 2021

MHD Island Coalescence



- See expected drop in peak reconnection rate with $Re = Re_m$
- See expected increase in oscillations as $Re = Re_m$ increases

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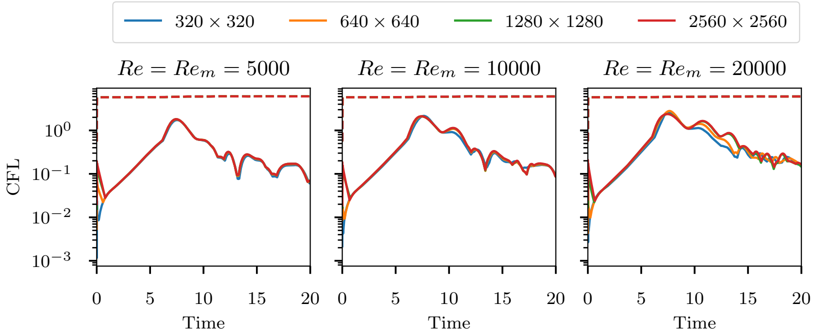
Adler et al., SISC 2021

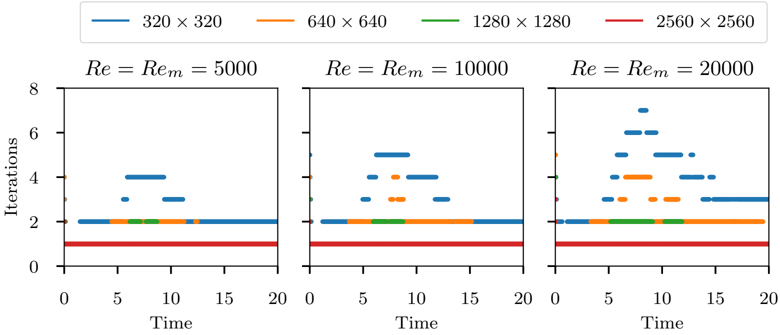
MHD Island Coalescence



- Unstabilized discretization for convective terms, so CFL is limiting
- Alfven CFL (dashed line) stays about 5
- Fluid CFL (solid) varies more, peaks about 2 or 3

Adler et al., SISC 2021

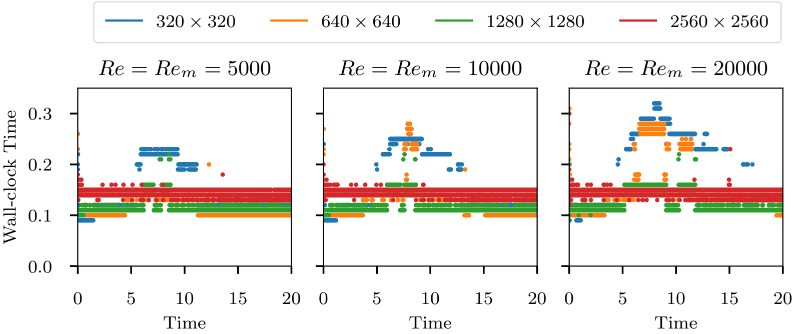
MHD Island Coalescence



- See advantage to timestepping as dt decreases, particularly at high $Re = Re_m$

Adler et al., SISC 2021

MHD Island Coalescence



- Weak scaling from 10 to 640 cores with refinement

Adler et al., SISC 2021

Outlook

Lots of Methods for MG for Systems

- Block-Factorization Preconditioners
  - Excel for high-*Re* flows with augmented Lagrangian preconditioners (Farrell and co-workers, 2018–present)
- Monolithic Multigrid
  - Success with several relaxation schemes
  - Expanding number of problems and discretizations
  - Lots more work to do!