Lecture Notes : Laplace Transformation

Cristian Arley Quintero Gonzalez

Abstract. If you are a quant, or you are a candidate to become one, you have probably found that it is sometimes necessary to go back to theory to recall methods, definitions, constraints or other theoretical topics related to quantitative finance. The reason for this is quite simple: the field is so broad, including knowledge of statistics, coding, finance, sometimes machine learning, and it is impossible to keep all the knowledge in one's head. For this reason this serie of documents are designed to remind you some of the basic and advanced aspects of calculus (for now just Láplace Transformation)[1].

If apply, the codes will be refered and progressively attached to this file as link to my personal repository.

Laplace Transformation

[1] Because the literature is broad about L'aplace Transformation, in here is referenced the book of Joel L. Schiff (1999), **The L'aplace Transform: Theory and applications.**

1 Function Transformations

Some times we need to deal with function transformations which let us to get a solution to problems. In this case we are going to talk a transformation wchich may help us to get an arithmetic structure from a differential equation to solve it. Commonly for people with background in computer science this could seem unapealing. however, once you want to be a quant it is going to play a role as a base for many of the mathematical and statistical structures. As the case may be the Moment Generating Function (MGF).

1.1 How does it work?

This is considered an integral transform that will convert a function, usually at t, to other one at s. It could be compared as to move to other kind of dimension, to solve the problem there, and come back to the original dimension with a better structure. On this regard, the movement between t-dimension and s-dimension let us to apply this transformation to solve linear ordinary Differential Equations (DE) with constant coefficients.

1.2 Laplace Transformation

We are given a function $f(t)$ and we want to get its transformed function $F(s)$.

$$F(s) = \mathcal{L}\{f(t)\} \tag{1}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \tag{2}$$

Here, the original function is referenced with lower letters, meanwhile the Laplace transformation function is referenced with a capital letter [1].

[1] This notation is going to be useful in the examples later.

2 Examples

2.1 e^{at}

This is a popular transformation used to solve some differential equations. Not being the solution but part of it.

$$f(t) = e^{at} \tag{3}$$

$$F(s) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt \tag{4}$$

As we know $\int e^{xt} dt = \frac{e^{xt}}{x} + c$ we apply then to equation 4

$$F(s) = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty = \frac{e^{(a-s)\infty}}{a-s} - \frac{e^{(a-s)0}}{a-s} \tag{5}$$

A formality in here is that the replacement when there is an ∞ value as boundary of the evaluation, it is not replace directly, instead a limit is applied.

$$F(s) = \lim_{t\to\infty} \left\{ \frac{e^{(a-s)t}}{a-s} \right\} - \frac{e^{(a-s)0}}{a-s} \tag{6}$$

We see because of the limit, we need to assess the options regard a and s. Then there are two options: the case when $s > a$, and $s \leq a$ [2]. We are interested in the case $s > a$, then

[2] When $s < a$ the numerator tends to grow to infinite because of $e^{t\mathbb{R}^+}$. In the case $s = a$ the denomination of the fraction will yield a undefined.

$$F(s) = \lim_{t\to\infty} \left\{ \frac{e^{(-\mathbb{R}^+)t}}{a-s} \right\} - \frac{e^0}{a-s} \quad (7)$$

$$F(s) = \lim_{t\to\infty} \left\{ \frac{\frac{1}{e^{(\mathbb{R}^+)t}}}{a-s} \right\} - \frac{1}{a-s} \quad (8)$$

$$F(s) = \frac{0}{a-s} - \frac{1}{a-s} = \frac{1}{s-a} \quad (9)$$

2.2 A Differential Equation (DE)

Now as example applied in a first order differential equation [3]. The differential equation to be analyzed is

[3] This example is extracted from MIT Open Course - Youtube

$$\frac{dy}{dt} - ay = 0 \quad (10)$$

Now, and appying the basic properties of Laplace transform [2], knowing that $\mathcal{L}\{0\} = 0$ [4]

[4] Recall the definition of Laplace transformation in equation 2, where being an integral where $f(t) = 0$ is multiplying the other factors then the result is zero.

$$\mathcal{L}\left\{\frac{dy}{dt} - ay\right\} = \mathcal{L}\left\{\frac{dy}{dt}\right\} - \mathcal{L}\{ay\} = 0 \quad (11)$$

2.2.1 Focus on $\mathcal{L}\left\{\frac{dy}{dt}\right\}$

Now keeping a focus on $\mathcal{L}\left\{\frac{dy}{dt}\right\}$ we'll say:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \int_0^\infty \frac{dy}{dt} e^{-st} dt \quad (12)$$

Because we have a derivative within the integral, we may solve it by parts[5], setting $dv = \frac{dy}{dt}$ and $u = e^{-st}$. Then $\int dv dt = y(t)$ and $du = -se^{-st}dt$. Solving by partial integration

[5] Recall integration by parts is defined as $\int_a^b u dv = uv|_a^b - \int_a^b v du$

$$\int_0^\infty \frac{dy}{dt} e^{-st} dt = \left[ye^{-st}\right]|_0^\infty - \int_0^\infty -se^{-st} y(t) dt \quad (13)$$

Taking out the $-s$ factor within the integral

$$\int_0^\infty \frac{dy}{dt} e^{-st} dt = \left[ye^{-st}\right]|_0^\infty + s\int_0^\infty y(t) e^{-st} dt \quad (14)$$

By definition we can notice the integral is precisely the $\mathcal{L}\{y(t)\} = Y(s)$ then we can replace in equation 14

$$\int_0^\infty \frac{dy}{dt} e^{-st} dt = \left[ye^{-st}\right]|_0^\infty + sY(s) \quad (15)$$

Now evaluating $[-yse^{-st}]|_0^\infty$, considering the formal way to determine the limit when $t \to \infty$ [6]

[6] It doesn't matter the term $y(t)$ when $t \to \infty$ because the term multiplying to it is $e^{-st} = \frac{1}{e^{st}}$. Thus, $\lim_{t\to\infty} \frac{1}{e^{st}} \simeq 0$

$$\int_0^\infty \frac{dy}{dt} e^{-st} dt = \left[\lim_{t\to\infty} \left\{y(t)e^{-st}\right\} - y(0)e^{-s0}\right] + sY(s) \quad (16)$$

$$\mathcal{L}\{\frac{dy}{dt}\} = \int_0^\infty \frac{dy}{dt} e^{-st} dt = -y(0) + sY(s) \quad (17)$$

[2] See **Schiff (1999)** section 1.6 at page 16

2.2.2 Focus on $\mathcal{L}\{ay\}$

On this stage we only need the definition of Laplace transformation 11 and to apply the property when $f(y)$ is multiplied by a constant c (in this case the constant is named a).

$$\mathcal{L}\{ay(t)\} = a\mathcal{L}\{y(t)\} = aY(s) \tag{18}$$

2.2.3 Algebraic representation of the differential equation

Once the terms were transformed, we can gather them again to rewrite the equation 11

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} - \mathcal{L}\{ay\} = sY(s) - y(0) + aY(s) = 0 \tag{19}$$

Thus, the initial differential equation (10) is transformed to an algebraic equation. Recalling the first goal of this transformation, the function has moved from one dimension depending of t, to another one depending of s.

2.2.4 Solving the DE in terms of s

Because we obtained the algebraic representation above in equation 19, we can reorganize the expression to get the solution.

$$sY(s) - y(0) - aY(s) = 0 \tag{20}$$

Because there are terms depending of s, and other ones depending the initial state $y(0)$ [7]

[7] Take care with the term $y(0)$. Recall because of this term is lower case then it is a function in t-dimension.

$$sY(s) - aY(s) = y(0) \tag{21}$$

$$(s-a)Y(s) = y(0) \tag{22}$$

$$Y(s) = \frac{y(0)}{(s-a)} \tag{23}$$

Voila!, we have obtained the expression of the function $F(s)$, which depends of the initial condition $y(0)$, the constant a and the variable s (the parallel dimension in this example)....... But what about to solve the problem in the t-dimension?[8].

[8] Here is when Dr. Strange help us to come again to the t-dimension, using the invese.

2.2.5 Inverse process

I think we can agree with the readers that the equation 23 may be written as

$$Y(s) = y(0)\frac{1}{(s-a)} \tag{24}$$

Guess what! .... the term $\frac{1}{s-a}$ is the same solution we got in equation 9 for $f(t) = e^{at}$ [9]. The common sense would ask how to use that result in here. Lets define the goal in this section as the inverse Laplace transformation of $Y(s)$.

[9] This same solution can be found in Laplace table in Joel L. Schiff (1999), **The L'aplace Transform: Theory and applications.** page 210.

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{y(0)\frac{1}{(s-a)}\right\} = y(t) \tag{25}$$

Because of Laplace transformation is linear, and one of its properties is $\mathcal{L}\{cy(t)\} = c\mathcal{L}\{y(t)\}$, the inverse may adopt the same property. Recall that $y(0)$

is in the t-dimension, and when evaluated it may be considered as a constant. That's why we may rewrite the equation 24 as

$$\mathcal{L}^{-1}\{Y(s)\} = y(0)\mathcal{L}^{-1}\left\{\frac{1}{(s-a)}\right\} \tag{26}$$

Now because we know $\frac{1}{s-a}$ is the Laplace transform of e^{at} we may replace it in the equation 26 [10]

[10] Other examples of this easy way to apply the inverse Laplace transformation may be watched in this Youtube video

$$\mathcal{L}^{-1}\{Y(s)\} = y(0)e^{at} \tag{27}$$

$$y(t) = y(0)e^{at} \tag{28}$$

2.2.6 Checking

To check the result we need to calculate the derivative of $y(t)$ and then see if the condition is reached as described in 26. By chain rule and considering that $y(0)$ acts as a constant

$$\frac{dy(t)}{dt} = \frac{d(y(0)e^{at})}{dt} = y(0)\frac{d(e^{at})}{dt} = ay(0)e^{at} \tag{29}$$

Then replacing in 26 we see it is correct

$$\frac{dy}{dt} - ay = 0 \implies ay(0)e^{at} - ay(0)e^{at} = 0 \tag{30}$$

GUES WHAT!... the result of that differential equation is pretty similar to the definition of future value utilizing continuous compounding interest rate. *Let's see later if that is related.*