Lecture Notes : Black-Scholes equation

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Abstract. If you are a quant, or you are a candidate to become one, you have probably found that it is sometimes necessary to go back to theory to recall methods, definitions, constraints or other theoretical topics related to quantitative finance. The reason for this is quite simple: the field is so broad, including knowledge of statistics, coding, finance, sometimes machine learning, and it is impossible to keep all the knowledge in one's head. For this reason this serie of documents are designed to remind you some of the basic and advanced aspects of options (for now just Black-Scholes Equation)[1].

If apply, the codes will be refered and progressively attached to this file as link to my personal repository.

Black-Scholes equation

[1]Because the literature is broad about option's definition, in here is referenced the book of John C. Hull (2018), **Options, Futures, and other Derivatives.**

1 Derivation of the Black-Scholes Equation

The derivation of Black-Scholes (BS) equation is relevant once you get into options market, because the most of the options will refer to this basic structure without matter if those are vanilla or exotic [1].

[1] Exotic derivatives are those derivatives which differ from traditional derivatives in payment structures, expiration dates, and strike prices

1.1 Assumptions

There are a set of assuptions in this model. This assuptions will work as constraints to consider the model as incomplete once it is incorporated in the real market (i.e. constant volatility).

- Risky asset follows a Geometric Brownian Motion (GBM)
- No dividends
- Market without frictions (taxes, fees, so on)
- Constant risk-free rate
- No arbitrage

1.2 Geometric Brownian Motion

This is one of the most recognized stochastic processes in finance. It is broadly used because it avoids negative values when the price is modeled under this process [2]. The Stocastich Diferential Equation (SDE) follows:

$$dS = \mu S dt + \sigma S dW \tag{1}$$

Recall W is a Weiner process, μ is the drift, and σ is the constant volatility. The solution to this SDE is:

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \tag{2}$$

Thus, S follows a lognormal distribution $S \sim N\left(ln(S_0) + (\mu - \frac{\sigma^2}{2}t), \sigma^2 t\right)$

1.3 Δ-hedge Portfolio

It is common to find in the literature that the way to get the BS equation is through a Δ-hedged portfolio (Π). It means a portfolio comprised by two assets: one contingent asset (European Option) and a risky asset (stock). The risky asset is the stock but it is as well the same type being the underlying of the derivative (contingent asset). The portfolio is buying a position on a derivative with value V, and one position selling Δ units at derivative's underlying at price S [2]. The value of V is a function defined by $V(S, t, \sigma, \mu, K, T, r)$, however on this stage it is assumed as not defined yet [3].

[2] S is defined by a Geometric Brownian Motion (GBM)

[3] The reader will notice the parameters are exactly the same as in the BS formula

The reader will realize as well that the parameters μ, K, r, T, and σ [4] are parameters given outside the model. For this reason the function $V(.)$ may be redefined as $V(S, t)$ depending only of derivative's underlying price and time.

[4] One of the drawbacks in the BS model is the assumption about volatility, which is considered as a constant.

The portfolio described above is defined as:

$$\Pi = V(S, t) - \Delta S \tag{3}$$

[2]Some literature talking about how to derive this process is found in links such as Karl Sigman Notes, or as well in **Options, Futures, and other Derivatives.** by (John C. Hull,), section 14.1

Where the negative ΔS means the sell. Now the question is: what is the change of the portfolio when S or t change? Recall the change of portfolio's value from t to $t+1$ is determined by:

$$d\Pi = dV - \Delta dS \tag{4}$$

1.3.1 Analyzing the variation of V (dV)

The variation on V, defined as dV needs to be assessed by apart. Recall S is a lognormal random walk defined in 1. If we apply **Taylor series expansion** [5] on V by S and t parameters, we get:

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2 + \frac{\partial^2 V}{\partial S \partial t}dSdt \tag{5}$$

The equation 5 may be compacted because of differential's multiplication. Remember that both $(dt)^2$ and $dSdt$ are approximately zero [6].

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \tag{6}$$

1.3.2 $(dS)^2$ incorporation to dV

The reader can see in 6 the expresion $(dS)^2$ which shall be incorporated. This expression is developed as:

$$(ds)^2 = (\mu S dt + \sigma S dW)^2 \tag{7}$$

$$(ds)^2 = \mu^2 S^2 (dt)^2 + \sigma^2 S^2 (dW)^2 + 2\mu\sigma S^2 dt dW = \sigma^2 S^2 (dW)^2 \tag{8}$$

Replacing 8 in 6, and recalling $(dW)^2 = dt$, it is possible to obtain[7]:

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt \tag{9}$$

1.3.3 dV incorporation to $d\Pi$

Once the relationship between dV and dS is simplified the next step is to incorporate dV into 4.

$$d\Pi = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt - \Delta dS \tag{10}$$

1.3.4 Δ-hedging

The definition of delta hedging is related with two terms observed in 10. Those are the terms $\frac{\partial V}{\partial S}dS$ and $-\Delta dS$. One reason to highlight this terms is because under BS equation, part of the simplification is to reduce the terms than can off set partially the randomness.

$$\frac{\partial V}{\partial S}dS = \Delta dS \tag{11}$$

This is only true if $\Delta = \frac{\partial V}{\partial S}$. Thus, this is the very famous **Delta-Hedging**. It means that the way how I shall reduce the risk of randomness given by the underlying is setting Δ as number of units of derivative's underlying which is in turn determined by the change in the derivative's value given a change in the price of the underlying. This change in the derivative's value is also part of a set of metrics named **Greeks**.

[5] This is the Ito's Lemma. It is used to find the differential in a temporal function depending of a stochastic process.

[6] Intuitively on $(dt)^2$ is because of dt is approaching to zero then by multiplying two values really small will produce another one even smaller, making the expression negligible.

[7] Recall the Quadratic Variation of a Brownian motion, wich is in turn a reescaled random walk process $(W^{(n)}(t))$, see *Stochastic Calculus for Finance II Continuous Time Models* **Shreve (2004)**, page 93. As well in StackExchange Forum, or Nicolas Privault Lectures.

1.4 Assumption of no arbitrage

As in many definitions in risk-neutral pricing, in quantitative finance, the non-arbitrage assumption [8] plays an important role. Just take a look on the Fundamental Theorem of Finance (FTF) [3]. Then once the delta hedging is considered, the no arbitrage assumption determines that the remanent value of $d\Pi$ (see 11) shall be equivalent to an investment in a risk-free interest-bearing account $d\Pi = r\Pi dt$. It would avoid to any market participant to get into an arbitrage trade. Then ...

[8] "Asset prices are obtained from conditions that preclude arbitrage oportunities .... **arbitrage** means taking simultaneous positions in different assets so that one guarantees a riskless profit higher than the riskless given by US treasury bills." (Hirsa & Neftci, 2014).

$$d\Pi = r\Pi dt = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt - \Delta dS \tag{12}$$

Recall the equation 4, replaced in 12

$$r(V - \Delta S)dt = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt \tag{13}$$

$$(rV - r\Delta S)dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) dt \tag{14}$$

Because dt is a common factor in both sides of the expression, it is cancelled.

$$rV - r\Delta S = \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \tag{15}$$

1.5 Black-Scholes equation

Now setting 17 equal to zero and recalling 11..

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + r\Delta S - rV = 0 \tag{16}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + r\frac{\partial V}{\partial S}S - rV = 0 \tag{17}$$

1.5.1 BS equation with Greeks

As you already know, greeks are metrics which help to measure the effect on derivatives prices once one risk factor has changed. Thus, the Black-Scholes equation may be expressed in terms of greeks as well[4].

$$Greeks = \begin{cases} \Gamma = \frac{\partial^2 V}{\partial S^2} \\ \Delta = \frac{\partial V}{\partial S} \\ \Theta = \frac{\partial V}{\partial t} \end{cases} . \tag{18}$$

$$\Theta + \frac{1}{2}\Gamma\sigma^2 S^2 + \Delta r S - rV = 0 \tag{19}$$

[3] Some details can be seen in The Fundamental Theorem of Finance - Princeton, and in the **Chapter 2** of An Introduction to the mathematics of financial derivatives (Hirsa, A. & Neftci, S., 2014)

[4] On this case we are using only three of the greeks available for options. However there are more than these three greeks.