Lecture Notes : Bond's Fundamentals

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Abstract. If you are a quant, or you are a candidate to become one, you have probably found that it is sometimes necessary to get back to theory looking for recalling methods, definitions, constraints or other theoretical topics related to quantitative finance. The reason for this is quite simple: the field is so broad, including knowledge of statistics, coding, finance, sometimes machine learning, and it is impossible to keep all the knowledge in one's head. For this reason this serie of documents are designed to remind you some of the basic and advanced aspects of calculus (for now just Bond's fundamentals).

If apply, the codes will be refered and progressively attached to this file as link to my personal repository.

Bond's Fundamentals

1 Bonds

Bonds are considered in the literature the fundamental instrument for pricing. A proof of this is the fact that neutral-risk measures take into consideration the bond's behavior by consider it as risk-free instrument. Nevertheless, people may ignore the fundamentals of this sort of instruments. A broadly known attribute is that the bonds are a debt instrument for the issuer meaning there is a repayment of money borrowed. So, in order to make this investment vehicle appealing for investors it is paid an interest over the total amount lent by the investor to the issuer.

The payment of this interest on periodical time is called *coupon (C)*. It may be paid on different sort of periods such as quarterly, semianually, monthly or any other frequency stablished on the bond issue.

1.1 Pricing

Assuming you already understand the basics of interest rates, let's go straight to how a bond is priced. The bond price *(P)* is determined by discounting the future cash flows. Those future cash flows are comprised by coupons and the notional value[1].

[1] The notional is the amount borrowed by the issuer of the bond, and lent by the investor with the promise of receive the value of an interest over the maturity of the bond. Sometimes the notional may be returned to the inverstor partially over the maturity of the bond, it is called **amortization**.

$$P(t) = \sum_{i=1}^{T} \frac{C_i}{(1+r)^i} + \frac{N}{(1+r)^T} \tag{1}$$

The equation 1 shows two components. The first component is the sum of all the coupons *(C_i)* paid on each period *i*, whereas the second component is the discounted value of the notional *(N)*.

1.2 Types

Additionaly, equation 1 depicts a bond paying coupons between the instant *t* and the maturity *T*, and it is named **coupon bond**. However, not all the bonds pay coupons in between. Other kind of bonds is the type of bond wich pays only the coupon in the maturity *T*, and it is named **zero-coupon bond**[1].

The utility of **zero-coupon bond** is found mainly on **term structure of interest rate** construction. The term structure of interest rate shows the relationship between the *spot interest rate*[2] and the term maturity *T*.

[2] The **spot interest rate** is the interest rate earned on a zero-coupon bond.

Another type of bonds is called **Strip Bond**, which is particullary interesting because these bonds are sliced to separate the coupons from the face value when they are traded.

2 Sensitivity

It is observed in equation 1 that the price *P(t)* depends of the parameter *r* which is the yield to maturity (YTM)[3]. This dependency makes the bond sensible to changes on interest rates owing to the fact the YTM is the interest rate used to discount the future cash flows.

[3] The YTM is the interest rate that relates a bond's price with its future returns.

[1]Zero-coupon bonds are *discount instruments*, which means that the bond's price reaches the par value at the maturity (T). Any moment *t* previous to *T* is lower because of the discounting effect.

2.1 First Derivative

The sensitivity of *P(t)* dependending of *r* is depicted by its derivative. To obtain its derivative lets modified the equation 1, as follow:

$$P(t) = \sum_{i=1}^{T} C_i(1+r)^{-i} + N(1+r)^{-T} \tag{2}$$

$$P(t) = C_1(1+r)^{-1} + C_2(1+r)^{-2} + \cdots + C_T(1+r)^{-T} + N(1+r)^{-T} \tag{3}$$

Being reexpressed as shown in equation 2 lets to derivate the function directly. It is shown below in 4 by applying the chain rule.

$$\frac{\partial P}{\partial r} = -C_1(1+r)^{-2} - \cdots - TC_T(1+r)^{-T-1} - TN(1+r)^{-T-1} \tag{4}$$

Trick Number 1!

Factorizing by $-(1+r)^{-1}$ it is reexpressed as:

$$\frac{\partial P}{\partial r} = -(1+r)^{-1}\left[C_1(1+r)^{-1} + \cdots + TC_T(1+r)^{-T} + TN(1+r)^{-T}\right] \tag{5}$$

$$\frac{\partial P}{\partial r} = -(1+r)^{-1}\left[\sum_{i=1}^{T} iC_i(1+r)^{-i} + TN(1+r)^{-T}\right] \tag{6}$$

2.1.1 Duration

Duration is the sensitivity measure on bonds showing the change in bond's price given a change in interest rate. So far, the expression in 8 shows the change in *units per units* in turn subject to the level of the units used. On the other hand, the sensitivity we look for is in terms of *percent per percent* mainly because it does not take care of the units. For this reason it is used the elasticity shown in 7.

$$\frac{\frac{\partial P}{P}}{\frac{\partial r}{1}} = \frac{\partial P}{\partial r}\frac{1}{P} \tag{7}$$

Then to incorporate 7 in 8 it is necessary to multiply in both sides by $\frac{1}{P}$ keeping the proportionality, thus yielding the following:

$$\frac{\partial P}{\partial r}\frac{1}{P} = -(1+r)^{-1}\left[\sum_{i=1}^{T} iC_i(1+r)^{-i} + TN(1+r)^{-T}\right]\frac{1}{P} \tag{8}$$

In 8 is now identified what is know as Macaulay Duration[4]

$$\frac{\partial P}{\partial r}\frac{1}{P} = -(1+r)^{-1}\left[\sum_{i=1}^{T} iC_i(1+r)^{-i} + TN(1+r)^{-T}\right]\frac{1}{P} \tag{9}$$

$$\frac{\partial P}{\partial r}\frac{1}{P} = -(1+r)^{-1}D \tag{10}$$

In turn it is also identified the Modified Duration[5]

$$D_{mod} = (1+r)^{-1}D \tag{11}$$

$$\frac{\partial P}{\partial r}\frac{1}{P} = -D_{mod} \tag{12}$$

[4] It is the weighted average term to maturity of the cash flows from a bond. Its unit is **time**. It is also referred as the time needed to recoup the true cost of the bond.

[5] It measures the approximate change in bond price for a change in yield.

2.1.2 Basis Point Value / DV01

The *Basis Point Value (BPV)* (aka *DV01* in the US), is the change in value of a bond due to a change of 0.01% in the yield. It is derived from 12 by discretazing the differential like:

$$\frac{\Delta P}{\Delta r}\frac{1}{P} = -D_{mod} \tag{13}$$

Where ΔP is the BPV, and it is defined as:

$$\Delta P = BPV = -D_{mod} * \Delta r * P \tag{14}$$

The relevance of this sensitivity is that it is used for **hedging purpose**. The strategy is to find another bond with similar BPV and get into a position on the other side to offset the sensitivity.

2.2 Second Derivative

The sensitivity described so far is commonly known for being a partial sensitivity. As it is known, the sensitivity may be complemented by other derivatives by using the Taylor's series[6]. That's relevant because the sensitivity is highly impacted on large movementes of interest rate, reason why other derivatives help to reduce the error by calculating the sensitivity only with the first derivative.

[6] Recall the Taylor's serie is defined by $f(x) = \sum_{n=o}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Thus, the second derivative of the Bond's price *P* is defined as:

$$\frac{\partial^2 P}{\partial r^2} = \frac{\partial}{\partial r}\frac{\partial P}{\partial r} \tag{15}$$

$$\frac{\partial^2 P}{\partial r^2} = -C_1(1+r)^{-2} - \cdots - TC_T(1+r)^{-T-1} - TN(1+r)^{-T-1} \tag{16}$$

$$\frac{\partial^2 P}{\partial r^2} = 2C_1(1+r)^{-3} + \cdots + (T^2+T)C_T(1+r)^{-T-2} + (T^2+T)N(1+r)^{-T-2} \tag{17}$$

Trick Number 2!

Factorizing by $(1+r)^{-2}$ it is reexpressed as:

$$\frac{\partial^2 P}{\partial r^2} = (1+r)^{-2}\left[2C_1(1+r)^{-1} + \cdots + (T^2+T)C_T(1+r)^{-T} + (T^2+T)N(1+r)^{-T}\right] \tag{18}$$

$$\frac{\partial^2 P}{\partial r^2} = (1+r)^{-2}\left[\sum_{i=1}^{T}(i^2+i)C_i(1+r)^{-i} + (T^2+T)N(1+r)^{-T}\right] \tag{19}$$

2.2.1 Convexity

As mentioned previously, the sensitivity is complemented by the sum of derivatives on different orders by adopting the Taylor's series definition. The application of Taylor's serie on Price, up to second order, is determined in 20.

$$P(r^*) = P(r) + \frac{\partial P}{\partial r}\frac{(r^*-r)}{1!} + \frac{\partial^2 P}{\partial r^2}\frac{(r^*-r)^2}{2!} \tag{20}$$

Where r^* is the modified interest rate, or new interest rate level. Rearranging the previous equation

$$P(r^*) - P(r) = \frac{\partial P}{\partial r}\frac{(r^* - r)}{1!} + \frac{\partial^2 P}{\partial r^2}\frac{(r^* - r)^2}{2!} \tag{21}$$

Then discretazing P and r we get

$$\Delta P = \frac{\partial P}{\partial r}\frac{\Delta r}{1!} + \frac{\partial^2 P}{\partial r^2}\frac{(\Delta r)^2}{2!} \tag{22}$$

Trick Number 3!

Similar to the analysis performed in 7, in here it is divided by P in order to determine the relative change of the price of the bond, as calculated in 23, where it is observed the negative Modified Duration together with the Convexity (CV)

$$\frac{\Delta P}{P} = \frac{\partial P}{\partial r}\frac{1}{P}\Delta r + \frac{\partial^2 P}{\partial r^2}\frac{1}{P}\frac{1}{2}(\Delta r)^2 \tag{23}$$

Then using 19 we can define the Convexity (CV) as:

$$CV = \frac{1}{P(1+r)^{-2}}\left[\sum_{i=1}^{T}(i^2+i)C_i(1+r)^{-i} + (T^2+T)N(1+r)^{-T}\right] \tag{24}$$

2.3 Comments

It is relevant to highlight that a bond having a highest Convexity will be less sensible to changes on interest rate. It is determined owing to the sign of Convexity is positive and Modified Duration is negative, so being largest than other bonds it will offset the negative relationship between price *(P)* and interest rate *(r)*.

3 Negotiation

A bond is said be traded at a premium, at par or at a discount. The difference between them is the difference between the trading price and fundamental price obtained by using the equation 1 [7].

- $P_m = P_f + \Phi$
- $P_m = P_f \pm 0$
- $P_m = P_f - \delta$

Where P_m is the market price and P_f is the fundamental bond price.

[7] Some interest rate curves may be used for discounting the future cash flows usually determined by market consensus.

4 Using other Interest Rates

As mentioned at the begining of this document, the bond is the prefered instrument to price other sort of instruments by taking it as a risk-free instrument. However, different instruments such as derivatives exists in different time dimension. The time dimension may be required to be more granular than quarters, months, or even days. That's why transformations of interest rate definition is required.

4.1 Compounding Interest Rate

The interest rates for using the bond's price formula may be in the form of compounding interest rate or continuous compounding interest rate. By using the continuous compounding interest rate on a bond traded at par value equal to $1 is priced as:

$$P = e^{-r(t,T)(T-t)} \tag{25}$$

Where $r = r(t,T)$ is a function depending the pricing date *(t)* and maturity date *(T)*. It means the interest rate depends of the time to maturity $T - t$ observed on princing date. It is also possible to obtain it from the bond's price and maturity solving it on 25.

$$r(t,T) = \frac{-log(P)}{T-t} \tag{26}$$

4.2 Instantaneous Rate

Theoretically there is an interest rate which exists into a pretty small period of time, named Intantaneous Rate. This definition is framed into a tiny change of price and time, being almost the same. We can determine the instantaneous interest rate from 26 by taking small changes:

$$r(t) = \frac{-\partial log(P)}{\partial t} \tag{27}$$

Whether the principal of the short term rate described above is continuously reinvested at this short rate, the cummulative amont gained at time *t* is equal to the original investment multiplied by the expresion

$$M(t) = e^{\int_0^t r(s)ds} \tag{28}$$

Where M(t) is the monet market account that offers a return of the short rate r(t). It is then the total rateof return of such account.

5 Free-Arbitrage Condition

The money market plays a role on princing[8] together with bond prices by avoiding the arbitrage between money market and bond prices. It is said the conditions are met for not letting arbitrage if[9]

$$P(t,T) = e^{-r(T-t)} \tag{29}$$

$$P(t,T) = e^{\int_0^t r(s)ds} \tag{30}$$

[8] Usually refered as **Risk-Neutral Pricing** which is the price letting a *risk-neutral investor* being indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing.

[9] Additional analysis on this topic is shown in **Fixed-Income Securities And Derivatives Handbook (2005)** by Moohrad Choudhry, Bloomberg, page 53.