Lecture Notes : Binomial Tree

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Abstract. If you are a quant, or you are a candidate to become one, you have probably found that it is sometimes necessary to go back to the theory in order to recall methods, definitions, constraints or any other theoretical topic related to quantitative finance. The reason of that is quite simple: the field is so broad, including knowledge of statistics, coding, finance, sometimes machine learning, therefore it is impossible to keep all the knowledge in top of your mind every moment. For this reason this serie of documents is designed as a personalreminder of some of the basic and advanced aspects in quantitative finance. This time the turn is for **Binomial Tree**, used as one of the models for european/american option pricing.

If apply, the codes will be refered and progressively attached to this file as link to my personal repository.

Binomial Tree

1 Why Binomial Trees?

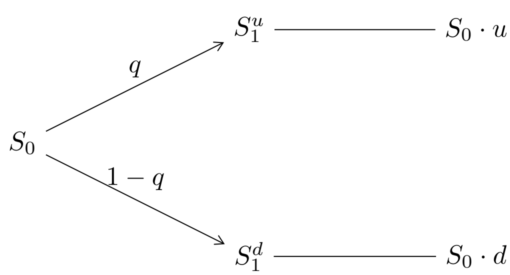
It is one of the most famous and straighforward technique that let us understand the mechanics for option pricing (european/american ones). This technique let us visualize very important aspects to consider when an option price is modeled, such as risk-neutral valuation and non-arbitrage. Therefore, I will start assuming an scenario in which there are only two sort of assets: risky asset and risk-free asset. For the sake of this notes I will assume the risky asset is a stock whose price may fluctuate in the market, whereas the risk-free asset lets any investor to generate profits at no risk.

2 Model Parameters

2.1 Risky Asset

The model begins saying that the risky asset may have today a price S_0 that may be turned into two states[1] in the future. These future prices are determined based on *up-state* **(u)** and *down-state* **(d)**, in such a way the price in $t = 1$ may be $S_1^u = S_0 \cdot u$ or $S_1^d = S_0 \cdot d$.

[1] This is the reason of binomial, because of two future states. The up-state price always above S_i whereas down-state price is always below S_i



Reader may notice that both *up-state* and *down-state* have probabilities q and $1 - q$ in diagram above. These probabilities are called **real probabilities**. Nevertheless, later on the reader will see that these probabilities are changed for **risk-neutral probabilities** to obtain the option price. Therefore *real probabilities* are not used anywhere to determine such price.

2.2 Contingent Claim

A contingent claim is defined as *"the potential liability inherent in a derivative security; that is, in an asset whose value is determined by the values of one or more underlying variables"*. Nonetheless, to be as clear as possible, it is the liability generated via a financial derivative which will occur in the future if and only if a condition is met. On this regard, we can say the contingent claim for an call european option is described as follows.

$$Payoff = max(S_T - K, 0) \tag{1}$$

It means that once the maturity date T is reached, the *holder* of the option may exercise the right to buy the underlying at strike price K[2] when the same underlying in the market has a price of S_t. So in case the investors decides to sell immediately the underlyin in the market at time T, he/she will obtain a profit of $S_T - K$. Then we can say that the value of the option V may have two likely values in the future, V_u or V_d.

[2] The strike price K for a call/put option is the price set upfront in which the underlying will be bought/sold at instant T.

$$
\begin{tikzcd}
 & V_1^u \arrow[r, dash] & max(S_0 \cdot u - K, 0) \\
V_0 \arrow[ur] \arrow[dr] & & \\
 & V_1^d \arrow[r, dash] & max(S_0 \cdot d - K, 0)
\end{tikzcd}
$$

2.3 Dinamycs

Since the **time value of money** is the heart of finance, the dynamic through time for pricing options inherited such dimension, not only because it lets to discount contingent claims, but also because it is linked to the *risk-neutrality* requirement.

Furthermore, the risk-free interest rate r is the one that let us to discount cashflows[3]. The background of this assumption is precissely one of the relevant aspects in pricing, **risk-neutralty**[4].

[3] This is a strong assumption since the model considers the interest rate as constant and deterministic.

[4] It says any investor is indifferent when investing in the risky asset or risk-free asset, reason why the expected return of the risky asset may be discounted by using the risk-free rate.

2.3.1 Delta-hedging strategy

Delta-hedging strategy is the tool that let us to link the behaviour between underlying and derivative by means building a hedged portfolio. It describes an scenario in which there is a portfolio (Π_t) at instant *t* such that it is comprised by two positions. A long position of Δ number of stocks together with a short position in an option (V_t) with the same stock as underlying [5].

[5] Δ is one of the sensitivities -or greeks- for an option, describing the change in the option price given a change in the underlying price, thus letting the porfolio be hedged against changes in stock price.

$$\Pi_t = \Delta \cdot S_t - V_t \tag{2}$$

Next, given that the value of the portfolio may have two valeus in the moment $t = 1$, we can say that:

$$
\begin{tikzcd}
 & \Pi_1^u \arrow[r, dash] & \Delta \cdot S_1^u - V_1^u \\
\Pi_0 \arrow[ur] \arrow[dr] & & \\
 & \Pi_1^d \arrow[r, dash] & \Delta \cdot S_1^d - V_1^d
\end{tikzcd}
$$

Where,

$$V_1^u = max(S_1^u - K, 0) = max(S_0 \cdot u - K, 0) \tag{3}$$
$$V_1^d = max(S_1^d - K, 0) = max(S_0 \cdot d - K, 0) \tag{4}$$

Then, the next step is to find the value of Δ given that it is the same factor regardless the scenario is in *up-state* or *down-state*[6]. That is possible by equating both *up-state* and *down-state* of the portfolio.

[6] The reason for this, is that the number of stocks hold in the portfolio does not change if the state of the stock is *up-state* or *down-state*

$$\Delta \cdot S_1^u - V_1^u = \Delta \cdot S_1^d - V_1^d \tag{5}$$
$$\Delta = \frac{V_1^u - V_1^d}{S_1^u - S_1^d} \tag{6}$$

3 Risk-neutral probabilities

In a previous discussion about risky asset, it was mentioned the change of *real probabilities* for *risk-neutral probabilities*. The need for this is that under *non-arbitrage* condition, it is assumed that the expected return of the portfolio is equal to the return of the risk-free asset, in such a way that any future cashflow may be discounted at risk-free rate. Besides that, the payoff of the contingent claim V as an option payoff, will have value greater than zero when the state is *up-state*.

Now, given that the risk-free rate may be used to discount future cashflows, and replacing Δ with the expression found in equation 6, we can say for one step forward in the *up-state* of the portfolio that:

$$\Delta \cdot S_0 - V_0 = (\Delta \cdot S_1 \cdot u - V_1^u)e^{-r \cdot t} \tag{7}$$

$$\Delta \cdot S_0 - (\Delta \cdot S_0 \cdot u - V_1^u)e^{-r \cdot t} = V_0 \tag{8}$$

$$\frac{V_1^u - V_1^d}{S_1^u - S_1^d} \cdot S_0 - \left(\frac{V_1^u - V_1^d}{S_1^u - S_1^d} \cdot S_0 \cdot u - V_1^u \right) e^{-r \cdot t} = V_0 \tag{9}$$

Looking for including the discount factor in the left had terms, it is added e^{rt} as multiplication factor for the first term, and a bit of algebra we get an expression as follows.

$$V_0 = \left(\frac{V_1^u - V_1^d}{S_1^u - S_1^d} \cdot S_0 \cdot e^{r \cdot t} - \frac{V_1^u - V_1^d}{S_1^u - S_1^d} \cdot S_0 \cdot u - V_1^u \right) e^{-r \cdot t} \tag{10}$$

$$V_0 = \left(\frac{V_1^u - V_1^d}{S_1^u - S_1^d} \cdot S_0 \cdot e^{r \cdot t} - \frac{(V_1^u - V_1^d) \cdot S_0 \cdot u - V_1^u \cdot (S_1^u - S_1^d)}{S_1^u - S_1^d} \right) e^{-r \cdot t} \tag{11}$$

$$V_0 = \left(\frac{V_1^u - V_1^d}{S_0 \cdot u - S_0 \cdot d} \cdot S_0 \cdot e^{r \cdot t} - \frac{(V_1^u - V_1^d) \cdot S_0 \cdot u - V_1^u \cdot (S_0 \cdot u - S_0 \cdot d)}{S_0 \cdot u - S_0 \cdot d} \right) e^{-r \cdot t} \tag{12}$$

By finding S_0 common factor on every term, the expression is reduced to

$$V_0 = \left(\frac{V_1^u - V_1^d}{u - d} \cdot e^{r \cdot t} - \frac{(V_1^u - V_1^d) \cdot u - V_1^u \cdot (u - d)}{u - d} \right) e^{-r \cdot t} \tag{13}$$

$$V_0 = \left(\frac{V_1^u - V_1^d}{u - d} \cdot e^{r \cdot t} - \frac{V_1^u \cdot u - V_1^d \cdot u - V_1^u \cdot u + V_1^u \cdot d}{u - d} \right) e^{-r \cdot t} \tag{14}$$

$$V_0 = \left(\frac{V_1^u - V_1^d}{u - d} \cdot e^{r \cdot t} - \frac{V_1^u \cdot d - V_1^d \cdot u}{u - d} \right) e^{-r \cdot t} \tag{15}$$

$$V_0 = \left(\frac{V_1^u \cdot e^{r \cdot t} - V_1^d \cdot e^{r \cdot t} - V_1^u \cdot d + V_1^d \cdot u}{u - d} \right) e^{-r \cdot t} \tag{16}$$

$$V_0 = \left(\frac{V_1^u \cdot (e^{r \cdot t} - d) + V_1^d \cdot (u - e^{r \cdot t})}{u - d} \right) e^{-r \cdot t} \tag{17}$$

$$V_0 = \left(V_1^u \cdot \frac{(e^{r \cdot t} - d)}{u - d} + V_1^d \cdot \frac{(u - e^{r \cdot t})}{u - d} \right) e^{-r \cdot t} \tag{18}$$

Since,

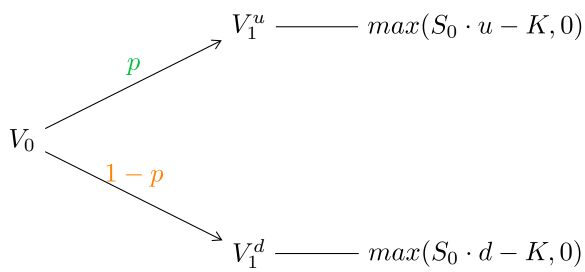
$$1 - \frac{(e^{r \cdot t} - d)}{u - d} = \frac{(u - e^{r \cdot t})}{u - d} \tag{19}$$

we can re-write equation 18 like

$$V_0 = \left(V_1^u \cdot \frac{(e^{r \cdot t} - d)}{u - d} + V_1^d \cdot \left(1 - \frac{(e^{r \cdot t} - d)}{u - d} \right) \right) e^{-r \cdot t} \tag{20}$$

$$V_0 = \left(V_1^u \cdot p + V_1^d \cdot (1 - p) \right) e^{-r \cdot t} \tag{21}$$

Where p is the **risk-neutral probability**. Furthermore, from equation 21 we can say that the option value at time $t = 0$ is the **risk-neutral expected value** of future states of the option, discounted at *risk-free rate*.



Be careful, in case there is more than one step in the tree!.

It is important to highlight that this probabilities were calculated so far for one step. If the three contains more than one step, for instance as proposed by Cox-Ross-Rubinstein derivation, the procedure must be performed from leaves of the tree backward the root node, repeating the process on each root node. This means that for a deep tree there will be as many number of risk-neutral probabilities as number of bifurcations in the tree.[7]

[7] An example of a tree with more than one level is found in **Fundamentals of options and futures markets** by Jhon C. Hull, section 12.3.

4 *u* and *d* parameters

The question now is where parameters *u* and *d* are comming from. Well, there is a very common interpretation used in the industry, coming from the work titled *Option Pricing: A simplified approach* wrote by Cox, Ross and Rubinstein in 1979.

This derivation let us set *u* and *d* as function of underlying volatility σ^2, as follows.

$$u = e^{\sigma \sqrt{\Delta t}} \tag{22}$$

$$d = e^{-\sigma \sqrt{\Delta t}} \tag{23}$$

Please take care with Δ in this deduction, because in here the definition **is not** the option greek -underlying spot price sensitivity- that was used in *delta-hedging strategy*. In here, Δt representes the size of the step, or in other words, the difference in time between levels in the tree.

If we take a look in the internet we will notice that there is no only one method do derivate *u* and *d*. Here I show the most popular, nevertheless, the paper wrote in 1979 by Cox-Ingersoll-Ross explains another path by using the expected values of binomial distribution of *up-state* and *down-state*.

4.0.1 *d* as multiplicative inverse of *u*

From description in sections above, we know that under *risk-neutral probabilities*, the expected value of risky asset may be written as

$$\mathbb{E}^Q\left[S_{t+\Delta t}\right] = p \cdot S_0 \cdot u + (1-p) \cdot S_0 \cdot d = S_0 e^{r \cdot \Delta t} \quad (24)$$

$$Var^Q[S_{t+\Delta t}] = \mathbb{E}^Q\left[S_{t+\Delta t}^2\right] - \mathbb{E}^Q\left[S_{t+\Delta t}\right]^2 = S_0^2 \cdot \sigma^2 \cdot \Delta t \quad (25)$$

$$S_0^2 \cdot \sigma^2 \cdot \Delta t = S_0^2 \cdot (p \cdot u^2 + (1-p) \cdot d^2) - [p \cdot S_0 \cdot u + (1-p) \cdot S_0 \cdot d]^2 \quad (26)$$

$$\sigma^2 \cdot \Delta t = p \cdot (p-p)[u^2 + d^2 - 2 \cdot u \cdot d] \quad (27)$$

$$\sigma^2 \cdot \Delta t = p \cdot (p-p) \cdot (u-d)^2 \quad (28)$$

Now, analyzing the expression $p(1-p)$, by replacing with defition of p in equation 21 we obtain the expression as follows.

$$p(1-p) = p - p^2 = \frac{e^{r\Delta t} - d}{u-d} - \left(\frac{e^{r\Delta t} - d}{u-d}\right)^2 \quad (29)$$

$$p(1-p) = \frac{e^{r\Delta t} - d}{u-d} - \frac{e^{2r\Delta t} - 2de^{r\Delta t} + d^2}{(u-d)^2} \quad (30)$$

$$p(1-p) = \frac{e^{r\Delta t} - d}{u-d}\frac{u-d}{u-d} - \frac{e^{2r\Delta t} - 2de^{r\Delta t} + d^2}{(u-d)^2} \quad (31)$$

$$p(1-p) = \frac{e^{r\Delta t}u - e^{r\Delta t}d - ud + d^2 - e^{2r\Delta t} + 2de^{r\Delta t} - d^2}{(u-d)^2} \quad (32)$$

$$p(1-p) = \frac{e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}}{(u-d)^2} \quad (33)$$

Then replacing the quation 33 in equation 28

$$\sigma^2 \cdot \Delta t = \frac{e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}}{(u-d)^2} \cdot (u-d)^2 \quad (34)$$

$$\sigma^2 \cdot \Delta t = e^{r\Delta t}(u+d) - ud - e^{2r\Delta t} \quad (35)$$

Trick No 1!. Now by transforming $d = \frac{1}{u}$ the equation 35 is re-write as follows. Then finding $u + \frac{1}{u}$

$$\sigma^2 \cdot \Delta t = e^{r\Delta t}\left(u + \frac{1}{u}\right) - 1 - e^{2r\Delta t} \quad (36)$$

$$u + \frac{1}{u} = \frac{\sigma^2 \cdot \Delta t + 1 + e^{2r\Delta t}}{e^{r\Delta t}} = e^{-r\Delta t}\sigma^2\Delta t + e^{-r\Delta t} + e^{r\Delta t} \quad (37)$$

Since e^x may be approximated by series expansion, in such a way that $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$e^{-r\Delta t} \approx 1 - r\Delta t \quad (38)$$

$$e^{r\Delta t} \approx 1 + r\Delta t \quad (39)$$

Now replacing in equation 37, and knowing that because of its magnitudes the term $r\sigma^2(\Delta t)^2 \to 0$

$$u + \frac{1}{u} = (1 - r\Delta t)\sigma^2 \Delta t + (1 - r\Delta t) + (1 + r\Delta t) = \frac{u^2+1}{u} \quad (40)$$
$$\frac{u^2+1}{u} = \sigma^2 \Delta t + 2 \quad (41)$$
$$u^2 + 1 = u \cdot (\sigma^2 \Delta t + 2) \quad (42)$$
$$u^2 - (\sigma^2 \Delta t + 2)u + 1 = 0 \quad (43)$$

The equation 43 may be solved by using the quadratic solution, recognizing that $\sigma^2 \Delta t^2 \to zero$, as follows.

$$u = \frac{-(-\sigma^2 \Delta t - 2) \pm \sqrt{(\sigma^2 \Delta t + 2)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \quad (44)$$
$$u = \frac{(\sigma^2 \Delta t + 2) \pm \sqrt{(\sigma^4 \Delta t^2) + 4\sigma^2 \Delta t + 4 - 4}}{2} \quad (45)$$
$$u = \frac{(\sigma^2 \Delta t + 2)}{2} \pm \frac{\sqrt{4\sigma^2 \Delta t}}{2} \quad (46)$$
$$u = \frac{\sigma^2 \Delta t}{2} + 1 \pm \frac{\sqrt{4\sigma^2 \Delta t}}{2} \quad (47)$$

Since $\sqrt{\Delta t}$ is relatively larger than Δt for small Δt, and σ^2 is relatively smaller than σ, the first fraction may be ignored. Besides that, $u > 1$ by definition, so the only solution available for both *u* and *d* will be

$$u = 1 + \sigma\sqrt{\Delta t} \quad (48)$$

Inversely to equation 39 we can re-write *u* and *d* as follows.

$$u = 1 + \sigma\sqrt{\Delta t} \approx e^{\sigma \Delta t} \quad (49)$$
$$d = \frac{1}{u} \approx \frac{1}{e^{\sigma \Delta t}} = e^{-\sigma \Delta t} \quad (50)$$