Lecture Notes : Geometric Brownian Motion

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Abstract. If you are a quant, or you are a candidate to become one, you have probably found that it is sometimes necessary to go back to theory to recall methods, definitions, constraints or other theoretical topics related to quantitative finance. The reason for this is quite simple: the field is so broad, including knowledge of statistics, coding, finance, sometimes machine learning, and it is impossible to keep all the knowledge in one's head. For this reason this serie of documents are designed to remind me some of the basic and advanced aspects of calculus (for now just Geometric Brownian Motion).

If apply, the codes will be refered and progressively attached to this file as link to my personal repository.

Geometric Brownian Motion

1 Why Geometric Brownian Motion (GBM)?

I'm almost sure that once you have gone through theory in quantitative finance books, the first idea you get when you hear about GBM is that the output of this stochastic process cannot be negative, such as prices. Taking a first glance on the model, I think it is the easiest property to remind.

1.1 The Origins

The aim of this section is to provide an insight into the thinking behind GBM model, rather than a historical note.

Imagine you are working with compound interest rates. You know that the future value (FV) of an investment with an initial capital of X_0 is defined by $FV = X_t = X_0(1+r_a)^t$, being r the **annual** interest rate, and t the **number of years** of your investment.

Now lets go through periodical interest rate. We recall that a periodical interest rate (r_p) may be steemed from annual interest rate (r_a) by applying a more frequent capitalization [1]. In this way the result of calculate the FV using periodical interest rate is:

$$X_t = X_0 \left(1 + \frac{r_p}{f}\right)^{ft} \tag{1}$$

Moreover, if we evaluate the expression in 1 making the frequency extremly high $(f \to \infty)$, then assessing its limit

$$X_t = \lim_{f\to\infty} X_0 \left(1 + \frac{r_p}{f}\right)^{ft} = X_0 \lim_{f\to\infty} \left(1 + \frac{r_p}{f}\right)^{ft} \tag{2}$$

Then recalling that [2]

$$\lim_{f\to\infty} \left(1 + \frac{r_p}{f}\right)^{f} = e^{r_c} \tag{3}$$

Knwowing the limit properties with exponent [3], FV can be redefined as

$$X_t = \left(X_0 \lim_{f\to\infty} \left(1 + \frac{r_p}{f}\right)^{f}\right)^t \tag{4}$$

Thus

$$X_t = X_0(e^{r_c})^t = X_0 e^{r_c t} = X_t \tag{5}$$

From now on we will refer to the continuous interest rate (r_c) as r.

1.1.1 Derivative

Furthermore, let us make explicit the fact that our variable X is a function depending on t [4], then $X_t = X(t)$. We would like to characterize the change in our investment based on changes in time. By definition we know that having $y = f(x)$, its derivative may be describe through limit definition [5] as:

$$\frac{dX_t}{dt} = \lim_{h\to\infty} \frac{X_0 e^{r(t+h)} - X_0 e^{rt}}{h} = \lim_{h\to\infty} \frac{e^{rh} - 1}{h} X_0 e^{rt} \tag{6}$$

Now considering the Maclurin series [6] of e^x we get

[1] Capitalization frequency means the frequency f (by year) on that the interest earned is transferred to capital for the coming calculation of interest. In other words $X_0(1+\frac{r_p}{f})(1+\frac{r_p}{f})(1+\frac{r_p}{f})\ldots$

[2] This is exactly the same explanation when I learned continuous interest rates (r_c).

[3] being t a real number

[4] the variable r is assumed as constant.

[5] The derivative's definition using limit is $\frac{dy}{dx} = \lim_{h\to\infty} \frac{f(x+h)-f(x)}{h}$

[6] Maclurin serie of e^x is equal to $\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)$

$$\lim_{h\to\infty} \frac{e^{rh}-1}{h} X_0 e^{rt} = \lim_{h\to\infty} \frac{\left(1 + rh + \frac{(rh)^2}{2!} + \frac{(rh)^3}{3!} + \dots\right) - 1}{h} X_0 e^{rt} \quad (7)$$

$$\frac{dX}{dt} = \lim_{h\to\infty} \left(r + \frac{r^2 h}{2!} + \frac{r^2 h^2}{3!} + \dots\right) X_0 e^{rt} \quad (8)$$

Now because h tends to zero every factor multiplying h disapears, so we can simplify as [7]

$$\frac{dX_t}{dt} = rX_0 e^{rt} = rX_t \quad (9)$$

$$dX_t = rX_t dt \quad (10)$$

It is possible to see from 10 that so far the variation of X dependes only of deterministic values, such as r, X_0, and t. It makes the variation of X_t also deterministic.

However, as I say as well, a mathematical model is just an abstraction of the reality. The real world doesn't work like the model describes perfectly, but it is relatively close. A more realistic view of the world could be designed if we incorporate a random factor to the variation of X_t.

1.1.2 Why do we use a natural logarithm?

From my point of view this is one of the **tricky sections** when I was learning about Geometric Brownian Motion. The question is straightforward: Why every time that we talk about GBM, we automatically refer to natural logarithm of X_t?.

Whether we return to the equation 10 we can see that it is a Partial Differential Equation (PDE). As such, it has different expressions to the same dynamic. Let us modify it slightly.

$$\frac{dX_t}{X_t} = rdt \quad (11)$$

Then, we can apply the undefined integral operator on both sides

$$\int \frac{dX_t}{X_t} = \int rdt \quad (12)$$

We know, from popular integrals, that the result in the left side of equation 12 is the natural logarithm of X_t

$$ln(X_t) = rt + C \quad (13)$$

We can realize that the equation 13 contains an additional term C [8]. It is the integration constant, produced by the undefined interal operator.

$$X_t = e^{rt+C} = e^{rt} e^{C} = C e^{rt} \quad (14)$$

Above in equation 14, the exponential e^C is also a constant, that's why it is considered C, in order to somplify the notation. Moreover, this expression is attractive when applied to financial price modelling, **mainly because the exponential function will not produce negative values unless the constant C is negative** [9].

[7] *I know, I know!!*, it is simple if I just say $\frac{d(X_0 e^{rt})}{dt} = X_o \frac{d(e^{rt})}{dt} = rX_0 e^{rt}$

[8] Please realize that because we are using an undefined integral then a constant of integration is yielded in both sides of the equation. Thus, both constants were grouped in only one constant in the left side.

[9] In the GBM we will see the constant C is going to be the initial price X_0, or *aka* initial condition, as similarly observed in 5, then it is virtually imposible to have negative values as initial value.

1.1.3 Adding Randomness

As if designed by a Wizard, we are going to add a randomness factor in 10. Nevertheless, this randomness factor has a particular structure defined as $\sigma X_t dB_t$ [10]. Why?, let's try to find out.

Commonly the variation of a price is depicted by σ, being a percentage of the price. Then, aplying the variation to the price σX_t we get the monetary value of the the price variation. But as a randomness effect it is incorporated by using the brownian motion difference dB_t [11]. By adding the BM effect, we are adding a normal distrubuted random impact on σX_t.

Then if we add this randomness to the deterministic part found in 10, we could say that dX_t will be a random process from now on. Having the reasons more clear, we can rewrite the diference of X_t as:

$$dX_t = rX_t dt + \sigma X_t dB_t \tag{15}$$

Now it is possible to identify the deterministic and random piece of dX_t. Those parts are also known as drift and diffusion coefficients. Now extracting the common factors we can rewrite it as:

$$dX_t = X_t(rdt + \sigma dB_t) \tag{16}$$

[10] dB_t is the theoretical differential of a Brownian Motion. However, recall that the Bronian Motion is not a continuous function, then this diferential is not the precise definition as we have in continuous functions.

[11] the BM difference is depicted as $dB_t = B_{t+1} - B_t$. Recall some BM properties $\mathbb{E}[dB] = 0, \mathbb{V}[dB] = t - s$. See section 2.3 in R. Schilling (2021), **Brownian Motion: A guide to random processess and stochastic calculus**.

1.2 Applying Ito's Lemma

The Ito's Lemma is usually explained by taking a new stochastic variable which is in turn a function depending on an Ito's Process [12]. In this example we consider $Y(X_t) = ln(X_t)$ because of the reason exposed in 14.

Once the function is identified, we proceed to apply Taylor's series in order to get the differencial expression of Y, dY [13].

$$Y = Y_0 + \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial X}dX + \frac{1}{2}\frac{\partial^2 Y}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2}(dX)^2 + \frac{\partial^2 Y}{\partial X \partial t}(dXdt)\dots \tag{17}$$

In the equation 17, above, we can remove some terms due quadratic variation and $dt \to 0$. Then the remaining equation is

$$dY = \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial X}dX + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2}(dX)^2 \tag{18}$$

Now, just taking the equation 15, and powering it, we get

$$(dX)^2 = (rX_t dt + \sigma X_t dB_t)^2 \tag{19}$$

$$(dX)^2 = (rX_t dt)^2 + (\sigma X_t dB_t)^2 + 2(rX_t^2 \sigma dt dB_t) \tag{20}$$

$$(dX)^2 = r^2 X_t^2 dt^2 + \sigma^2 X_t^2 dB_t^2 + 2(rX_t^2 \sigma dt dB_t) \tag{21}$$

again removing terms because of quadratic variation and $dt \to 0$ we obtain

$$(dX)^2 = \sigma^2 X_t^2 dt \tag{22}$$

Then replacing 22 in 18

$$dY = \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial X}dX + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2}\sigma^2 X_t^2 dt \tag{23}$$

[12] An Ito's Process is identified if follows $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$, with $X_{t=0} = X_0$. See Definition 2.1.3 in **Mathematical Modeling and Computation Finance**, C. Oosterlee & L. Grzelak

[13] In many applications of Taylor's series, it's good enough working with up to the second derivative. We also adopt this approach here.

1.2.1 Getting Y

For being able to operate the equation 23 it is necessary to get the partial derivatives of Y, respect t and X_t. Then because $Y = ln(X_t)$

$$\frac{\partial Y}{\partial t} = 0 \tag{24}$$

$$\frac{\partial Y}{\partial X_t} = \frac{1}{X_t} \tag{25}$$

$$\frac{\partial^2 Y}{\partial X_t^2} = -\frac{1}{X_t^2} \tag{26}$$

Now we replace equations 24, 25, 26, and 16 in 23.

$$dY = 0dt + \frac{1}{X_t} X_t(rdt + \sigma dB_t) - \frac{1}{2}\frac{1}{X_t^2}\sigma^2 X_t^2 dt \tag{27}$$

$$dY = d\{ln(X_T)\} = rdt + \sigma dB_t - \frac{1}{2}\sigma^2 dt \tag{28}$$

grouping by dt and dB_t

$$d\{ln(X_t)\} = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \tag{29}$$

1.2.2 $dln(X_t)$ integration

The integration of 29 is obtained by applying the integral operator in both sides. It is relevant to highlight that the integral of a brownian motion is not a number, but a stochastic process.

$$\int_0^T d\{ln(X_T)\} = \int_0^T \left(r - \frac{1}{2}\sigma^2\right) dt + \int_0^T \sigma dB_t \tag{30}$$

The integral colored in blue can be solved by using the defined Riemman's integral. On the other hand the integral colored with green is solved using the property of stochastic integration.

$$ln(X_t)|_0^T = \left(r - \frac{1}{2}\sigma^2\right) T + \int_0^T \sigma dB_t \tag{31}$$

$$ln(X_T) - ln(X_0) = \left(r - \frac{1}{2}\sigma^2\right) T + \int_0^T \sigma dB_t \tag{32}$$

$$ln\left(\frac{X_T}{X_0}\right) = \left(r - \frac{1}{2}\sigma^2\right) T + \int_0^T \sigma dB_t \tag{33}$$

Now we are going to pay attention on the stochastic integral. This stochastic integral is comprised by two parts. On of them is the constant σ and the other one is dB_t. Now, because σ is considered constant, by assumption in the model, we can put it out the integral.

$$\int_0^T \sigma dB_t = \sigma \int_0^T dB_t \tag{34}$$

Through an abuse of notation of the equation (1.34) in C. Oosterlee & L. Grzelak (2020), **Mathematical Modeling and Computation in Finance**, I'm going to write the equation 34, being $g(t_i)$ a constant equal to 1.

$$\int_0^T dB_t = \lim_{m\to\infty} \sum_{i=0}^{m-1} g(t_i)(B_{t_{i+1}} - B_{t_i}) \tag{35}$$

$$\int_0^T dB_t = \lim_{m\to\infty} (B_{t_1} - B_{t_0}) + (B_{t_2} - B_{t_1}) + \cdots + (B_{t_m} - B_{t_{m-1}}) \tag{36}$$

Above in equation 36, the intermediate terms are eliminated because the signs with the next element in the summatory. Thus, we can get

$$\int_0^T dB_t = B_{t_m} - B_{t_0} \tag{37}$$

Knowing $B_{t_m} = B_T$ and $B_{t_0} = B_0 = 0$, we found

$$\int_0^T dB_t = B_T \tag{38}$$

Now replacing 38 in 34, and in turn replacing in 33

$$ln\left(\frac{X_T}{X_0}\right) = \left(r - \frac{1}{2}\sigma^2\right)T + \sigma B_T \tag{39}$$

1.3 Voila!

As final step we remove the logarithm to find X_T

$$X_T = X_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma B_T} \tag{40}$$

Voila!!!, we have found the GBM formula. Just keep in mind that for applying in simulations it is necessary a slight transformation of equation 40, defined as [14]

$$X_{t+\Delta t} = X_t e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta t+\sigma\sqrt{\Delta t}Z_t} \tag{41}$$

[14] The magic in here is just that in the previous equation is a generalization getting X_T from $t = 0$. However for simulations we ought o take small steps with size Δt. Moreover, $B_t \sim N(0, T)$ then to simulate the variance we transform it to $\sqrt{T}Z_t$, and $\Delta B_t \sim N(0, \Delta t)$, therefore its analog is $\sqrt{\Delta t}Z_t$