- TEMA #2 -
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GRUPA 131

SEMINAR 7

3. Considerăm aplicația $T:\mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (x+2y, 2y, -2y+z)$

a) Arătați că T este transformare (aplicație) liniară

FORMA MATRICEALĂ.

Rezolvare: Scriem (f. matriceală) $f(X) = AX$, unde $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 1 \end{pmatrix} \in \mathcal{M}_{(3,3)}(\mathbb{R})$$

Fie $X_1, X_2 \in \mathbb{R}^3$

$X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, $X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

și $\alpha_1, \alpha_2 \in \mathbb{R}$

$f(\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1) + A(\alpha_2 X_2)$

$= (A\alpha_1) X_1 + (A\alpha_2) X_2 = (\alpha_1 A) X_1 + (\alpha_2 A) X_2 = \alpha_1 (A X_1) +$

$+ \alpha_2 (A X_2) = \alpha_1 f(X_1) + \alpha_2 f(X_2) \Rightarrow f$ apl. liniară (morf. de sp. vect)

Apl 1 Considerăm transf. liniară

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (2x - y + 2z, -x + 2y - z, x + y + z)$, $(\forall)(x, y, z) \in \mathbb{R}^3$

a) Determinați valorile proprii și subsp. proprii coresp.

Rezolvare:

$$P(\lambda) = \det(A_f - \lambda I_3) = \begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{vmatrix} =$$

$$= -\lambda(\lambda - 2)(\lambda - 3)$$

$$P(\lambda) = 0 \Leftrightarrow -\lambda(\lambda - 2)(\lambda - 3) = 0$$

$\rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases} \in \mathbb{R}$ valorile proprii, $Spec(f) = \{0, 2, 3\}$

$$m_a(\lambda_1) = m_a(\lambda_2) = m_a(\lambda_3) = 1$$

Subspatii proprii

$$S_\lambda : \begin{cases} (2-\lambda)x - y + 2z = 0 \\ -x + (2-\lambda)y - z = 0 \\ x + y + (1-\lambda)z = 0 \end{cases}$$

Atunci:

$$S_{\lambda_1} \; \{\lambda_1 = 0\} : \begin{cases} 2x - y + 2z = 0 \\ -x + 2y - z = 0 \\ x + y + z = 0 \end{cases}$$ -> sistem liniar omogen cu 3 ec si 3 nec.

$$rg(A_f - \lambda_1 I_3) = 2$$

$$\Delta_p = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \neq 0 \Rightarrow \begin{cases} x, y \text{ nec. principale} \\ z = \alpha,\ \alpha \in \mathbb{R} \text{ nec. sec.} \end{cases}$$

$$\Rightarrow \begin{cases} 2x - y = -2\alpha \\ -x + 2y = \alpha \;|\cdot 2 \end{cases} \Rightarrow \begin{cases} 2x - y = -2\alpha \\ -2x + 2y = 2\alpha \end{cases} \Rightarrow$$

$y = 0$

$2x - 0 = -2\alpha$

$2x = -2\alpha$

$x = -\alpha$

$$\begin{cases} y = 0 \\ x = -\alpha \\ z = \alpha,\ \alpha \in \mathbb{R} \end{cases}$$

Deci $V_{\lambda_1} = \{(-\alpha, 0, \alpha) / \alpha \in \mathbb{R}\} = \{\alpha \underbrace{(-1, 0, 1)}_{v_1} / \alpha \in \mathbb{R}\}$

S_{λ_2} $\{\lambda_2 = 2\}$: $\begin{cases} -y + 2z = 0 \\ -x - z = 0 \\ x + y - z = 0 \end{cases}$ -> sistem liniar omogen, cu 3 ec. și 3 nec.

$$\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

$rg(A_f - \lambda_2 I_3) = 2$

$\Delta_p = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1 - 0 = 1 \neq 0 \Rightarrow y, z$ nec principale

$x = \beta$, $\beta \in \mathbb{R}$ nec. sec.

$$\begin{cases} -\beta - z = 0 \\ \beta + y - z = 0 \end{cases} \Rightarrow \begin{cases} z = -\beta \\ \beta + y + \beta = 0 \end{cases} \Rightarrow \begin{cases} z = -\beta \\ y = -2\beta \\ x = \beta, \ \beta \in \mathbb{R} \end{cases}$$

Deci $V_{\lambda_2} = \{(\beta, -2\beta, -\beta) / \beta \in \mathbb{R}\} = \{\beta(1, -2, -1) / \beta \in \mathbb{R}\}$

S_{λ_3} $\{\lambda_3 = 3\}$: $\begin{cases} -x - y + 2z = 0 \\ -x - y - z = 0 \\ x + y - 2z = 0 \end{cases}$ -> sistem liniar $\overset{\text{"}}{v_2}$ omogen, cu 3 ec. și 3 nec.

$$\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \quad rg(A_f - \lambda_3 I_3) = 2$$

$\Delta_p = \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix} = 1 + 2 = 3 \neq 0 \Rightarrow y, z$ nec. principale.

$x = \mu$, $\mu \in \mathbb{R}$ nec. sec.

$$\begin{cases} -\mu - y + 2z = 0 \\ -\mu - y - z = 0 \\ \mu + y - 2z = 0 \end{cases} \Rightarrow \begin{cases} -3z = 0 \\ -\mu - y + 2z = 0 \end{cases} \Rightarrow \begin{cases} z = 0 \\ y = -\mu \\ x = \mu, \ \mu \in \mathbb{R} \end{cases}$$

Deci $V_{\lambda_3} = \{(\mu, -\mu, 0) / \mu \in \mathbb{R}\} = \{\mu\underbrace{(1,-1,0)}_{v_3} / \mu \in \mathbb{R}\}$

c) Verificați dacă f este diagonalizabilă

Rezolvare:

$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases}$, unde $m_a(\lambda_1) = m_a(\lambda_2) = m_a(\lambda_3) = 1$

Vectorii proprii generează următoarele subspații proprii: $m_g(\lambda_1) = m_g(\lambda_2) = m_g(\lambda_3) = 1$

$V(\lambda_1 = 0) = sp < V_1 > \Rightarrow \dim V(\lambda_1 = 0) = 1 = m_a(\lambda_1)$

$V(\lambda_2 = 2) = sp < V_2 > \Rightarrow \dim V(\lambda_2 = 2) = 1 = m_a(\lambda_2)$

$V(\lambda_3 = 3) = sp < V_3 > \Rightarrow \dim V(\lambda_3 = 3) = 1 = m_a(\lambda_3)$

Deci f este diagonalizabilă.

d) În caz afirmativ, scrieți matricea (forma) diagonală și baza în care se realizează

Rezolvare: $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$B = \{\underbrace{(-1,0,1)}_{v_1}, \underbrace{(1,-2,-1)}_{v_2}, \underbrace{(1,-1,0)}_{v_3}\}$ Baza în care se realizează

SEMINAR 8

Apl: Același enunț ca la apl [1] pentru următoarele transf. liniare

a) $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x,y,z)=(-x+3y-z,\ -3x+5y-z,\ -3x+3y+z)$, $(\forall)(x,y,z)\in\mathbb{R}^3$.

b) $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x,y,z)=(6x-5y-3z,\ 3x-2y-2z,\ 2x-2y)$, $(\forall)(x,y,z)\in\mathbb{R}^3$

c) $f:\mathbb{R}^4\to\mathbb{R}^4$, $f(x,y,z,t)=(x+y+z+t,\ x+y-z-t,\ x-y+z-t,\ x-y-z+t)$ $(\forall)(x,y,z,t)\in\mathbb{R}^4$

Rezolvare: a) $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x,y,z)=(-x+3y-z,\ -3x+5y-z,\ -3x+3y+z)$, $(\forall)(x,y,z)\in\mathbb{R}^3$

a) Matricea asociată este:

$$A_f=\begin{array}{c}\begin{pmatrix}-1 & 3 & -1\\ -3 & 5 & -1\\ -3 & 3 & 1\end{pmatrix}\\ \begin{matrix}f(e_1) & f(e_2) & f(e_3)\end{matrix}\end{array}\in\mathcal{M}_3(\mathbb{R})$$

b) Polinomul caracteristic

$$P(\lambda)=\det(A_f-\lambda I_3)=\begin{vmatrix}-1-\lambda & 3 & -1\\ -3 & 5-\lambda & -1\\ -3 & 3 & 1-\lambda\end{vmatrix}=$$

$$\begin{matrix}-1-\lambda & 3 & -1\\ -3 & 5-\lambda & -1\end{matrix}$$

$$=(-1-\lambda)(5-\lambda)(1-\lambda)+9+9-3(5-\lambda)+3(-1-\lambda)+9(1-\lambda)=$$

$$=(-1-\lambda)(5-\lambda)(1-\lambda)\cancel{+18}\cancel{-15}\cancel{+3\lambda}\cancel{-3}\cancel{-3\lambda}+9(1-\lambda)=$$

$$=(-1-\lambda)(5-\lambda)(1-\lambda)+9(1-\lambda)=$$

$$=-(-1-\lambda)(5-\lambda)(\lambda-1)-9(\lambda-1)=$$

$$=-(\lambda-1)\left[(-1-\lambda)(5-\lambda)+9\right]=$$

$$=-(\lambda-1)\left[-5+\lambda-5\lambda+\lambda^2+9\right]=$$

$$=-(\lambda-1)(\lambda^2-4\lambda+4)=-(\lambda-1)(\lambda-2)^2$$

Ec. caracteristică: $P(\lambda)=0 \Leftrightarrow -(\lambda-1)(\lambda-2)^2=0$

$\Rightarrow \begin{cases}\lambda_1=1\\ \lambda_2=2\end{cases} \in \mathbb{R}$ valorile proprii

$Spec(f)=\{1,2\}$

și $\begin{cases}m_a(\lambda_1)=1\\ m_a(\lambda_2)=2\end{cases}$ multiplicitățile algebrice

Subspații proprii

$$S_\lambda : \begin{cases}(-1-\lambda)x+3y-z=0\\ -3x+(5-\lambda)y-z=0\\ -3x+3y-(1-\lambda)z=0\end{cases}$$

Atunci

S'_{λ_1} : $\begin{cases} -2x+3y-z=0 \\ -3x+4y-z=0 \\ -3x+3y=0 \end{cases}$ $\begin{pmatrix} -2 & 3 & -1 \\ -3 & 4 & -1 \\ -3 & 3 & 0 \end{pmatrix}$

$\{\lambda_1 = 1\}$ sistem liniar omogen cu 3 ec. și 3 nec.

$\Delta_p = \begin{vmatrix} -2 & 3 \\ -3 & 4 \end{vmatrix} = -8+9 = 1 \neq 0 \Rightarrow \begin{cases} x, y \text{ nec. princip} \\ z=\alpha, \alpha \in \mathbb{R} \text{ nec. sec.} \end{cases}$

$rg(A_f - \lambda_1 I_3) = 2$

$\Rightarrow \begin{cases} -2x+3y-\alpha=0 \\ -3x+4y-\alpha=0 \\ -3x+3y=0 \end{cases} \Rightarrow \begin{cases} -2x+3y=\alpha \\ y=\alpha \\ z=\alpha, \alpha \in \mathbb{R} \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} -2x+3\alpha=\alpha \\ y=\alpha \\ z=\alpha, \alpha \in \mathbb{R} \end{cases} \Rightarrow \begin{cases} -2x=-2\alpha \\ y=\alpha \\ z=\alpha, \alpha \in \mathbb{R} \end{cases} \Rightarrow \begin{matrix} x=\alpha \\ y=\alpha \\ z=\alpha, \alpha \in \mathbb{R} \end{matrix}$

Deci: $V_{\lambda_1} = \{(\alpha,\alpha,\alpha) / \alpha \in \mathbb{R}\} = \{\alpha \underbrace{(1,1,1)}_{v_1} / \alpha \in \mathbb{R}\}$

S'_{λ_2} : $\begin{cases} -3x+3y-z=0 \\ -3x+3y-z=0 \\ -3x+3y-z=0 \end{cases}$ sistem liniar omogen cu 3 ec. și 3 nec.

$\{\lambda_2 = 2\}$

$\begin{pmatrix} -3 & 3 & -1 \\ -3 & 3 & -1 \\ -3 & 3 & -1 \end{pmatrix}$

$rg(A_f - \lambda_2 I_3) = 1 \Rightarrow \begin{cases} x \text{ nec. princip.} \\ y = \beta,\ \beta \in \mathbb{R} \text{ nec. sec.} \\ z = \mu,\ \mu \in \mathbb{R} \end{cases}$

$\Rightarrow -3x + 3\beta - \mu = 0$

$3x = 3\beta - \mu$

$\begin{cases} x = \beta - \frac{\mu}{3} \\ y = \beta,\ \beta \in \mathbb{R} \\ z = \mu,\ \mu \in \mathbb{R} \end{cases}$

Avem 1) $m_a(\lambda_1) + m_a(\lambda_2) = 3 = \dim_{\mathbb{R}} \mathbb{R}^3$

2) $m_a(\lambda_i) = m_g(\lambda_i),\ (\forall) i = \overline{1,2}$

Deci $V_{\lambda_2} = \{(\beta - \frac{\mu}{3}, \beta, \mu) / \beta, \mu \in \mathbb{R}\} =$

$= \{\beta \underbrace{(1,1,0)}_{v_2} + \mu \underbrace{(-\frac{1}{3}, 0, 1)}_{v_3} / \beta, \mu \in \mathbb{R}\}$

c) $\dim V_{\lambda_1} = 1$

$\dim V_{\lambda_2} = 2$

$B = \begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ (columns v_2, v_3)

$rg\, B = 2 \Rightarrow \dim V_{\lambda_2} = 2$

$\Rightarrow \begin{cases} m_g(\lambda_1) = 1 \\ m_g(\lambda_2) = 2 \end{cases}$ multiplicitățile geometrice

$\Rightarrow f$ este diagonalizabilă, deci $(\exists) \mathcal{B} \subset \mathbb{R}^3$

bază formată din vectorii proprii λ_1, λ_2

(care sunt liniar indep. deoarece valorile sunt distincte)}

d) În raport în care, matricea asociată lui f are forma diagonală

$$D=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$B=\{v_1=(1,1,1),\ v_2=(1,1,0),\ v_3=(-\tfrac{1}{3},0,1)\}$$

$$\begin{tikzcd} B_0 \arrow[r, "C"] \arrow[d] & B \arrow[d] \\ A_f & \boxed{D=C^{-1}A_fC} \end{tikzcd}$$

unde $C=\begin{pmatrix} 1 & 1 & -\frac{1}{3} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$C^{-1}=\begin{pmatrix} 3 & -3 & 1 \\ -3 & 4 & -1 \\ -3 & 3 & 0 \end{pmatrix}$$

$$D=\begin{pmatrix} 3 & -3 & 1 \\ -3 & 4 & -1 \\ -3 & 3 & 0 \end{pmatrix}\cdot\begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -\frac{1}{3} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}=$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ ADEVĂRAT

b) $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (6x-5y-3z,\ 3x-2y-2z,\ 2x-2y)$, $(\forall)\ (x,y,z) \in \mathbb{R}^3$

Rezolvare: b) a) Matricea asociată este

$$A_f = \begin{array}{c} \begin{pmatrix} 6 & -5 & -3 \\ 3 & -2 & -2 \\ 2 & -2 & 0 \end{pmatrix} \\ \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} \end{array} \in \mathcal{M}_3(\mathbb{R})$$

a) Polinomul caracteristic

$$P(\lambda) = \det(A_f - \lambda I_3) = \begin{vmatrix} 6-\lambda & -5 & -3 \\ 3 & -2-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} =$$

$$\begin{matrix} 6-\lambda & -5 & -3 \\ 3 & -2-\lambda & -2 \end{matrix}$$

$$= (6-\lambda)(-2-\lambda)\cdot(-\lambda) + 18 + 20 + 6(-2-\lambda) - 4(6-\lambda) + 15(-\lambda)$$

$$= -\lambda(6-\lambda)(-2-\lambda) + 38 - 12 - 6\lambda - 24 + 4\lambda - 15\lambda =$$

$$= -\lambda(6-\lambda)(-2-\lambda) + 2 - 17\lambda = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 =$$

$$= -(\lambda-1)(\lambda^2 - 3\lambda + 2) = -(\lambda-1)(\lambda-1)(\lambda-2) =$$

$$= -(\lambda-1)^2(\lambda-2)$$

Ec. caracteristică: $P(\lambda) = 0 \Leftrightarrow -(\lambda-1)^2(\lambda-2) = 0$

$$\Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases} \in \mathbb{R} \text{ valorile proprii}$$

$Spec(f) = \{1, 2\}$

și $\begin{cases} m_a(\lambda_1)=2 \\ m_a(\lambda_2)=1 \end{cases}$ multiplicitățile algebrice

Subspații proprii

$$S_\lambda : \begin{cases} (6-\lambda)x - 5y - 3z = 0 \\ 3x + (-2-\lambda)y - 2z = 0 \\ 2x - 2y + (-\lambda)z = 0 \end{cases}$$

Atunci:

$$S_{\lambda_1} : \begin{cases} 5x - 5y - 3z = 0 \\ 3x - 3y - 2z = 0 \\ 2x - 2y - z = 0 \end{cases}$$

$\{\lambda_1 = 1\}$ → sistem liniar omogen cu 3 ec și 3 nec. $\begin{pmatrix} 5 & -5 & -3 \\ 3 & -3 & -2 \\ 2 & -2 & -1 \end{pmatrix}$

$$rg(A_f - \lambda_1 I_3) = 2$$

$$\Delta_p = \begin{vmatrix} -5 & -3 \\ -2 & -1 \end{vmatrix} = 5 - 6 = -1 \neq 0 \Rightarrow \begin{cases} y, z \text{ nec principale} \\ x = \alpha,\ \alpha \in \mathbb{R} \text{ nec. sec.} \end{cases}$$

$$\Rightarrow \begin{cases} 5\alpha - 5y - 3z = 0 \\ 3\alpha - 3y - 2z = 0 \\ 2\alpha - 2y - z = 0 \ |\cdot 3 \end{cases} \Rightarrow \begin{cases} 5\alpha - 5y - 3z = 0 \\ 3\alpha - 3y - 2z = 0 \\ 6\alpha - 6y - 3z = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha - y = 0 \\ 3\alpha - 3y - 2z = 0 \end{cases} \Rightarrow \begin{cases} y = \alpha \\ 3\alpha - 3\alpha - 2z = 0 \\ z = \alpha \in \mathbb{R} \end{cases} \Rightarrow \begin{cases} x = \alpha, \alpha \in \mathbb{R} \\ y = \alpha \\ z = 0 \end{cases}$$

Deci: $V_{\lambda_1}=\{(\alpha,\alpha,0)/\alpha\in\mathbb{R}\}=\{\alpha\underbrace{(1,1,0)}_{v_1}/\alpha\in\mathbb{R}\}$

$S_{\lambda_2}: \begin{cases} 4x-5y-3z=0 \\ 3x-4y-2z=0 \\ 2x-2y-2z=0 \end{cases}$ → sistem liniar omogen cu 3 ec. și 3 nec.

$\{\lambda_2=2\}$

$rg(A_f-\lambda_2 I_3)=2$ $\qquad \begin{pmatrix} 4 & -5 & -3 \\ 3 & -4 & -2 \\ 2 & -2 & -2 \end{pmatrix}$

$\Delta_p=\begin{vmatrix} 4 & -5 \\ 3 & -4 \end{vmatrix}=-16+15=-1\neq 0 \Rightarrow \begin{cases} x,y \text{ nec. principale} \\ z=\beta,\ \beta\in\mathbb{R} \text{ nec. sec.} \end{cases}$

$\Rightarrow \begin{cases} 4x-5y-3\beta=0 \\ 3x-4y-2\beta=0 \\ 2x-2y-2\beta=0 \end{cases} \Rightarrow \begin{cases} 4x-5y=3\beta \\ x-2y=0 \mid \cdot 4 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} 4x-5y=3\beta \\ 4x-8y=0 \end{cases} \Rightarrow \begin{cases} 3y=3\beta \\ z=\beta,\ \beta\in\mathbb{R} \end{cases} \Rightarrow \begin{cases} y=\beta \\ z=\beta,\ \beta\in\mathbb{R} \\ 4x-5\beta-3\beta=0 \end{cases}$

$\Rightarrow \begin{cases} y=\beta \\ z=\beta,\ \beta\in\mathbb{R} \\ 4x=8\beta \end{cases} \Rightarrow \begin{cases} x=2\beta \\ y=\beta \\ z=\beta,\ \beta\in\mathbb{R} \end{cases}$

Deci: $V_{\lambda_2}=\{(2\beta,\beta,\beta)/\beta\in\mathbb{R}\}=\{\beta\underbrace{(2,1,1)}_{v_2}/\beta\in\mathbb{R}\}$

c) $\dim V_{\lambda_1}=1\neq 2$ $\quad (m_a(\lambda_1)=2)$ $\Rightarrow f$ nu este diagonalizabilă

$\dim V_{\lambda_2}=1=m_a(\lambda_2)$ $\qquad m_g(\lambda_1)=m_g(\lambda_2)$

multip. g geom.

c) $f:\mathbb{R}^4 \to \mathbb{R}^4$, $f(x,y,z,t) = (x+y+z+t,\ x+y-z-t,$
$x-y+z-t,\ x-y-z+t)$, $(\forall)(x,y,z,t) \in \mathbb{R}^4$

Rezolvare: a) Matricea asociată este:

$$A_f = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \in \mathcal{M}_4(\mathbb{R})$$
$$\quad f(e_1)\ \ f(e_2)\ \ f(e_3)\ \ f(e_4)$$

b) Polinomul caracteristic

$$P(\lambda) = \det(A_f - \lambda I_4) = \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & -1 & -1 \\ 1 & -1 & 1-\lambda & -1 \\ 1 & -1 & -1 & 1-\lambda \end{vmatrix}$$

$$A_f - \lambda I_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} =$$

$$= \begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & -1 & -1 \\ 1 & -1 & 1-\lambda & -1 \\ 1 & -1 & -1 & 1-\lambda \end{pmatrix}$$

$$P(\lambda) = \begin{vmatrix} \boxed{1-\lambda} & 1 & 1 & 1 \\ 1 & 1-\lambda & -1 & -1 \\ 1 & -1 & 1-\lambda & -1 \\ 1 & -1 & -1 & 1-\lambda \end{vmatrix} \sim \quad L_2 = L_2 + \frac{L_1}{\lambda-1}$$

$$\sim \begin{vmatrix} \boxed{1-\lambda} & 1 & 1 & 1 \\ 0 & \frac{2\lambda-\lambda^2}{\lambda-1} & \frac{2-\lambda}{\lambda-1} & \frac{-\lambda+2}{\lambda-1} \\ 1 & -1 & -\lambda+1 & -1 \\ 1 & -1 & -1 & \lambda-1 \end{vmatrix} \sim$$

$L_3 = L_3 + \frac{1}{\lambda-1} L_1$

$L_4 = L_4 + \frac{1}{\lambda-1} L_1$

$$\sim \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 0 & \boxed{\frac{2\lambda-\lambda^2}{\lambda-1}} & \frac{2-\lambda}{\lambda-1} & \frac{2-\lambda}{\lambda-1} \\ 0 & \frac{2-\lambda}{\lambda-1} & \frac{2\lambda-\lambda^2}{\lambda-1} & \frac{2-\lambda}{\lambda-1} \\ 0 & \frac{2-\lambda}{\lambda-1} & \frac{2-\lambda}{\lambda-1} & \frac{2\lambda-\lambda^2}{\lambda-1} \end{vmatrix} \sim$$

$L_3 = L_3 - \frac{L_2}{\lambda}$

$L_4 = L_4 - \frac{L_2}{\lambda}$

$$\sim \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 0 & \frac{2\lambda-\lambda^2}{\lambda-1} & \frac{2-\lambda}{\lambda-1} & \frac{2-\lambda}{\lambda-1} \\ 0 & 0 & \boxed{\frac{\lambda+2-\lambda^2}{\lambda}} & \frac{2-\lambda}{\lambda} \\ 0 & 0 & \frac{2-\lambda}{\lambda} & \frac{\lambda-\lambda^2+2}{\lambda} \end{vmatrix} \sim$$

$L_4 = L_4 - \frac{1}{\lambda+1} L_3$

$$\sim \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 0 & \frac{2\lambda-\lambda^2}{\lambda-1} & \frac{2-\lambda}{\lambda-1} & \frac{2-\lambda}{\lambda-1} \\ 0 & 0 & \frac{\lambda+2-\lambda^2}{\lambda} & \frac{2-\lambda}{\lambda} \\ 0 & 0 & 0 & \frac{4-\lambda^2}{\lambda+1} \end{vmatrix} =$$

$$= (1-\lambda)\left(\frac{2\lambda-\lambda^2}{\lambda-1}\right)\left(\frac{-\lambda^2+\lambda+2}{\lambda}\right)\left(\frac{-\lambda^2+4}{\lambda+1}\right) =$$

$$= \lambda^4 - 4\lambda^3 + 16\lambda - 16 = (\lambda-2)(\lambda^3-2\lambda^2-4\lambda+8) =$$

$$= (\lambda-2)(\lambda-2)(\lambda^2-4) = (\lambda-2)(\lambda-2)(\lambda-2)(\lambda+2) =$$

$$= (\lambda-2)^3(\lambda+2)$$

Ec. caracteristică: $P(\lambda) = 0 \Leftrightarrow (\lambda-2)^3(\lambda+2) = 0$

$\Rightarrow \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = -2 \end{matrix} \in \mathbb{R}$ valorile proprii

$Spec(f) = \{2, -2\}$

și $\begin{cases} m_a(\lambda_1) = 3 \\ m_a(\lambda_2) = 1 \end{cases}$ multiplicitățile algebrice

Subspații proprii

$$S_\lambda : \begin{cases} (1-\lambda)x + y + z + t = 0 \\ x + (1-\lambda)y - z - t = 0 \\ x - y + (1-\lambda)z - t = 0 \\ x - y - z + (1-\lambda)t = 0 \end{cases}$$

Atunci:

S_{λ_1} : $\begin{cases} -x+y+z+t=0 \\ x-y-z-t=0 \\ x-y-z-t=0 \\ x-y-z-t=0 \end{cases}$ → sistem liniar omogen, cu 3 ec. și 3 nec.

$\{\lambda_1 = 2\}$

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

$rg(A_f - \lambda I_4) = 1 \Rightarrow$ x nec. principală

$y = \alpha, \alpha \in \mathbb{R}$ nec.

$z = \beta, \beta \in \mathbb{R}$ sec.

$t = \mu, \mu \in \mathbb{R}$

$$\Rightarrow \begin{cases} -x+\alpha+\beta+\mu=0 \\ x-\alpha-\beta-\mu=0 \end{cases} \Rightarrow \begin{cases} x = \alpha+\beta+\mu \\ y = \alpha, \alpha \in \mathbb{R} \\ z = \beta, \beta \in \mathbb{R} \\ t = \mu, \mu \in \mathbb{R} \end{cases}$$

Deci $V_{\lambda_1} = \{(\alpha+\beta+\mu, \alpha, \beta, \mu) / \alpha, \beta, \mu \in \mathbb{R}\} =$

$$= \{\alpha\underbrace{(1,1,0,0)}_{v_1} + \beta\underbrace{(1,0,1,0)}_{v_2} + \mu\underbrace{(1,0,0,1)}_{v_3} \mid \alpha, \beta, \mu \in \mathbb{R}\}$$

S_{λ_2} : $\begin{cases} 3x+y+z+t=0 \\ x+3y-z-t=0 \\ x-y+3z-t=0 \\ x-y-z+3t=0 \end{cases}$

$\{\lambda_2=-2\}$

$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{pmatrix}$$

$rg\left(A_f - \lambda_2 I_4\right) = 3$

$$\Delta p = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 27-1-1-3-3-3 =$$

$$\begin{matrix} 3 & 1 & 1 \\ 1 & 3 & -1 \end{matrix}$$

$= 16 \neq 0 \Rightarrow \begin{cases} x, y, z \text{ nec. princip.} \\ t = \theta, \theta \in \mathbb{R} \text{ nec. sec.} \end{cases}$

$$\Rightarrow \begin{cases} 3x+y+z+\theta=0 \\ x+3y-z-\theta=0 \\ x-y+3z-\theta=0 \\ x-y-z+3\theta=0 \end{cases} \Rightarrow \begin{cases} 4x+4\theta=0 \\ x+3y-z-\theta=0 \\ x-y+3z-\theta=0 \end{cases}$$

$$\Rightarrow \begin{cases} 4x=-4\theta \\ x+3y-z=\theta \\ x-y+3z=\theta \end{cases} \Rightarrow \begin{cases} x=-\theta \\ 3y-z=2\theta \\ -y+3z=2\theta \quad |\cdot 3 \end{cases} \Rightarrow$$

$$\begin{cases} x = -\theta \\ 3y - z = 2\theta \\ -3y + 9z = 6\theta \end{cases} \Rightarrow \begin{cases} x = -\theta \\ 8z = 8\theta \end{cases} \Rightarrow \begin{cases} x = -\theta \\ z = \theta \\ 3y - \theta = 2\theta \end{cases}$$

$$\Rightarrow \begin{cases} x = -\theta \\ z = \theta \\ 3y = 3\theta \end{cases} \Rightarrow \begin{cases} x = -\theta \\ z = \theta \\ y = \theta \quad t = \theta,\ \theta \in \mathbb{R} \end{cases}$$

Deci $V_{\lambda_2} = \{(-\theta, \theta, \theta, \theta) / \theta \in \mathbb{R}\} =$

$$= \{\theta \underbrace{(-1, 1, 1, 1)}_{= v_4} / \theta \in \mathbb{R}\}$$

c) Deci $\dim V_{\lambda_1} = 3$

$\dim V_{\lambda_2} = 1$

$$\Rightarrow \begin{cases} m_g(\lambda_1) = 3 \\ m_g(\lambda_2) = 1 \end{cases} \text{ multiplicitățile geometrice}$$

Avem $\begin{cases} 1)\ m_a(\lambda_1) + m_a(\lambda_2) = 4 = \dim_{\mathbb{R}} \mathbb{R}^4 \\ 2)\ m_a(\lambda_i) = m_g(\lambda_i),\ (\forall)\, i = \overline{1,2} \end{cases}$

$\Rightarrow$ f este diagonalizabilă, deci $(\exists)$ $B \subset \mathbb{R}^4$ $\{$bază formată din vectori proprii coresp. lui λ_1, λ_2 (care sunt liniar indep. deoarece valorile proprii sunt distincte)$\}$

În raport cu care, matricea asociată lui f are forma diagonală

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$B = \{ v_1 = (1,1,0,0), v_2 = (1,0,1,0), v_3 = (1,0,0,1), v_4 = (-1,1,1,1) \}$$

$$\begin{array}{ccc} B_0 & \xrightarrow{C} & B \\ \downarrow & & \downarrow \\ A_f & & \boxed{D = C^{-1} \cdot A_f \cdot C} \end{array} \qquad C = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$C^{-1} = \left(\begin{array}{cccc|cccc} \boxed{1} & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \underset{L_2 = L_2 - L_1}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{-1} & -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_2 = -L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_3 = L_3 - L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & \boxed{-1} & 3 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_3 = -L_3}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & \boxed{1} & -3 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \underset{L_4 = L_4 - L_3}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & \boxed{4} & -1 & 1 & 1 & 1 \end{array}\right) \underset{L_4 = \frac{L_4}{4}}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & -1/4 & 1/4 & 1/4 & 1/4 \end{array}\right) \underset{L_3 = L_3 + 3L_4}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/4 & -1/4 & -1/4 & 3/4 \\ 0 & 0 & 0 & \boxed{1} & -1/4 & 1/4 & 1/4 & 1/4 \end{array}\right) \underset{L_2 = L_2 + 2L_4}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1/4 & -1/4 & -1/4 & 3/4 \\ 0 & 0 & 0 & \boxed{1} & -1/4 & 1/4 & 1/4 & 1/4 \end{array}\right) \underset{\substack{L_1 = L_1 + L_4 \\ L_2 = L_2 + L_3}}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 3/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 & 1/4 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & \boxed{1} & 0 & 1/4 & -1/4 & -1/4 & 3/4 \\ 0 & 0 & 0 & 1 & -1/4 & 1/4 & 1/4 & 1/4 \end{array}\right) \underset{L_1 = L_1 - L_3}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1/2 & 1/2 & 1/2 & -1/2 \\ 0 & \boxed{1} & 0 & 0 & 1/4 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & 0 & 1/4 & -1/4 & -1/4 & 3/4 \\ 0 & 0 & 0 & 1 & -1/4 & 1/4 & 1/4 & 1/4 \end{array}\right) \underset{L_1 = L_1 - L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/4 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & 0 & 1/4 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & 0 & 1/4 & -1/4 & -1/4 & 3/4 \\ 0 & 0 & 0 & 1 & -1/4 & 1/4 & 1/4 & 1/4 \end{array}\right)$$

Deci $C^{-1} = \begin{pmatrix} 1/4 & 3/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 3/4 & -1/4 \\ 1/4 & -1/4 & -1/4 & 3/4 \\ -1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$

$$C^{-1} \cdot A_f = \begin{pmatrix} 1/4 & 3/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 3/4 & -1/4 \\ 1/4 & -1/4 & -1/4 & 3/4 \\ -1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 3/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 3/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 3/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \end{pmatrix}$$

$$D = C^{-1} \cdot A_f \cdot C = \begin{pmatrix} 1/2 & 3/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 3/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 3/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

ADEVĂRAT !

Apl 1 Considerăm transf. liniară

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (2x - y + 2z, -x + 2y - z, x + y + z)$, $(\forall)\ (x, y, z) \in \mathbb{R}^3$

d) Verificare $\boxed{D = C^{-1} \cdot A_f \cdot C}$

Rezolvare:
$$\begin{tikzcd} B_0 \arrow[r, "C"] \arrow[d] & B \arrow[d] \\ A_f & \boxed{D = C^{-1} \cdot A_f \cdot C} \end{tikzcd}$$

Știm că
$$\begin{array}{cc} C = & \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix} \\ & \begin{matrix} v_1 & \;\; v_2 & \;\; v_3 \end{matrix} \end{array}$$

$$C^{-1} = \left(\begin{array}{ccc|ccc} \boxed{-1} & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_1 = -L_1}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} \boxed{1} & -1 & -1 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_3 = L_3 - L_1}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & \boxed{-2} & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \underset{L_2 = -\frac{L_2}{2}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 & 1 \end{array}\right) \underset{\substack{L_2 = L_2 - \frac{L_3}{2} \\ L_1 = L_1 + L_3}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & -1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \underset{L_1 = L_1 + L_2}{\sim}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

Deci $C^{-1} = \begin{pmatrix} -1/2 & -1/2 & +1/2 \\ -1/2 & -1/2 & -1/2 \\ 1 & 0 & 1 \end{pmatrix}$

$$C^{-1} \cdot A_f = \begin{pmatrix} -1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & -1/2 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & -1 \\ 3 & 0 & 3 \end{pmatrix}$$

$$D = C^{-1} \cdot A_f \cdot C = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & -1 \\ 3 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix} =$$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ ADEVĂRAT !

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

c) PROBLEME PROPUSE PENTRU TEMA ONLINE

1. Determinați valorile și vectorii (subspațiile) proprii corespunzători(e) pentru matricele următoare:

a) $A=\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$; b) $A=\begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix}$; c) $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

d) $A=\begin{pmatrix} 0 & 9 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Rezolvare a) $A=\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$ $f:\mathbb{R}^2\to\mathbb{R}^2, f(x,y)=(4y, 2x+2y)$ $(\forall)(x,y)\in\mathbb{R}^2$

Polinomul caracteristic: $P(\lambda)=\det(A-\lambda_2)$

$$A-\lambda_2=\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}-\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}=\begin{pmatrix} -\lambda & 4 \\ 2 & 2-\lambda \end{pmatrix}$$

$$P(\lambda)=\begin{vmatrix} -\lambda & 4 \\ 2 & 2-\lambda \end{vmatrix}=-\lambda(2-\lambda)-8$$

Ec. caract. $P(\lambda)=0 \Leftrightarrow -\lambda(2-\lambda)=8$

$\lambda(2-\lambda)=-8$

$\begin{cases} \lambda_1=-2 \\ \lambda_2=4 \end{cases} \in\mathbb{R}$ valorile proprii $Spec(f)=\{-2,4\}$

și $\begin{cases} m_a(\lambda_1) = 1 \\ m_a(\lambda_2) = 1 \end{cases}$ multiplicitățile algebrice

Subspații proprii

$S_\lambda : \begin{cases} -\lambda \cdot x + 4y = 0 \\ 2x + (2-\lambda)y = 0 \end{cases}$

Atunci

$S_{\lambda_1} : \begin{cases} 2x + 4y = 0 \\ 2x + 4y = 0 \end{cases}$ → sistem liniar omogen cu 2 ec. și 2 nec.

$\{\lambda_1 = -2\}$

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$$

$rg(A_f - \lambda_1 I_2) = 1 \Rightarrow \begin{cases} x \text{ nec principal} \\ y = \alpha,\ \alpha \in \mathbb{R} \text{ nec sec.} \end{cases}$

$\Rightarrow 2x + 4\alpha = 0 \Rightarrow 2x = -4\alpha$

$$\begin{cases} x = -2\alpha \\ y = \alpha,\ \alpha \in \mathbb{R} \end{cases}$$

Deci: $V_{\lambda_1} = \{(-2\alpha, \alpha) / \alpha \in \mathbb{R}\} = \{\alpha \underbrace{(-2, 1)}_{v_1} / \alpha \in \mathbb{R}\}$

$S_{\lambda_2} : \begin{cases} -4x + 4y = 0 \\ 2x - 2y = 0 \end{cases}$ → sistem liniar omogen cu 2 ec. și 2 nec.

$\{\lambda_2 = 4\}$ $\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}$ $rg(A_f - \lambda_2 I_2) = 1$

$\Rightarrow \begin{cases} x - \text{nec principală} \\ y = \beta, \ \beta \in \mathbb{R} \text{ nec. sec.} \end{cases}$

$\Rightarrow -4x + 4\beta = 0 \Rightarrow -4x = -4\beta \Rightarrow \begin{cases} x = \beta \\ y = \beta, \ \beta \in \mathbb{R} \end{cases}$

Deci $V_{\lambda_2} : \{(\beta, \beta) / \beta \in \mathbb{R}\} = \{\beta \underbrace{(1,1)}_{v_2} / \beta \in \mathbb{R}\}$

Rezolvare b) $A = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix}$ $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x,y) = (3x+4y, 5y), (\forall)(x,y) \in \mathbb{R}^2$

Polinomul caracteristic : $P(\lambda) = \det(A - \lambda I_2)$

$P(\lambda) = \begin{vmatrix} 3-\lambda & 4 \\ 0 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda)$

Ec. caract. $P(\lambda) = 0 \Leftrightarrow (3-\lambda)(5-\lambda) = 0$

$\begin{cases} \lambda_1 = 3 \\ \lambda_2 = 5 \end{cases} \in \mathbb{R}$ valorile proprii , $Spec(f) = \{3, 5\}$

și $\begin{cases} m_a(\lambda_1) = 1 \\ m_a(\lambda_2) = 1 \end{cases}$ multiplicitățile algebrice

Subspații proprii

$S_\lambda : \begin{cases} (3-\lambda)x + 4y = 0 \\ (5-\lambda)y = 0 \end{cases}$

Atunci

S_{λ_1} : $\begin{cases} 4y = 0 \\ 2y = 0 \end{cases}$ -> sistem liniar omogen cu 2 ec și 2 nec.

$\{\lambda_1 = 3\}$

$$\begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}$$

$$rg(A_f - \lambda_1 I_2) = 1 \Rightarrow \begin{cases} y \text{ nec principă} \\ x = \alpha, \alpha \in \mathbb{R} \text{ nec. sec.} \end{cases}$$

$$\begin{matrix} 4y = 0 \\ y = 0 \end{matrix} \Rightarrow \begin{cases} y = 0 \\ x = \alpha, \alpha \in \mathbb{R} \end{cases}$$

Deci $V_{\lambda_1} = \{(\alpha, 0) / \alpha \in \mathbb{R}\} = \{\alpha \underset{v_1}{\underbrace{(1,0)}} / \alpha \in \mathbb{R}\}$

S_{λ_2} : $\{-2x + 4y = 0$ -> sistem liniar omogen cu 1 ec și 2 nec.

$\{\lambda_2 = 5\}$

$$\begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix}$$

$$rg(A_f - \lambda_2 I_2) = 1 \Rightarrow \begin{cases} x \text{ nec principă} \\ y = \beta, \beta \in \mathbb{R} \text{ nec sec} \end{cases}$$

$$-2x + 4\beta = 0$$

$$2x = 4\beta \Rightarrow \begin{cases} x = 2\beta \\ y = \beta, \beta \in \mathbb{R} \end{cases}$$

Deci $V_{\lambda_2} = \{(2\beta, \beta) / \beta \in \mathbb{R}\} = \{\beta \underset{v_2}{\underbrace{(2,1)}} / \beta \in \mathbb{R}\}$

Rezolvare c) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

$f: \mathbb{R}^3 \to \mathbb{R}^3, \ f(x, y, z) = (x, 2z, 2y), \ (\forall)(x, y, z) \in \mathbb{R}^3$

Polinomul caracteristic: $P(\lambda) = \det(A - \lambda I_3) =$

$= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = -(\lambda - 1)(\lambda^2 - 4) = -(\lambda - 1)(\lambda - 2)(\lambda + 2)$

Ec. caract. $P(\lambda) = 0 \Leftrightarrow -(\lambda-1)(\lambda-2)(\lambda+2) = 0$

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = -2 \end{cases} \in \mathbb{R}$ $\Rightarrow$ Spec$(f) = \{1, 2, -2\}$

și $\begin{cases} m_a(\lambda_1) = 1 \\ m_a(\lambda_2) = 1 \\ m_a(\lambda_3) = 1 \end{cases}$ multiplicitatea algebrice

Subspații proprii

$S_\lambda : \begin{cases} (1-\lambda) \cdot x = 0 \\ -\lambda \cdot y + 2z = 0 \\ 2y - \lambda \cdot z = 0 \end{cases}$

Atunci

$S_{\lambda_1} : \begin{cases} -y + 2z = 0 \\ 2y - z = 0 \end{cases}$ → sistem liniar omogen cu 2 ec și 3 nec.

$\{\lambda_1 = 1\}$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -1 \end{pmatrix}$

$rg(A_f - \lambda_1 I_2) = 2$

$\Delta_p = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1-4 = -3 \neq 0 \Rightarrow \begin{cases} y, z \text{ nec princ} \\ x = \alpha, \alpha \in \mathbb{R} \\ \text{nec. sec.} \end{cases}$

$\begin{cases} -y + 2z = 0 \;|\cdot 2 \\ 2y - z = 0 \end{cases} \Rightarrow \begin{cases} -2y + 4z = 0 \\ 2y - z = 0 \end{cases} \Rightarrow 3z = 0$

$z = 0$

$2y = 0 \Rightarrow y = 0 \qquad \Rightarrow \begin{cases} x = \alpha, \alpha \in \mathbb{R} \\ y = 0 \\ z = 0 \end{cases}$

Deci $V_{\lambda_1} = \{(\alpha, 0, 0) / \alpha \in \mathbb{R}\} = \{\alpha \underbrace{(1,0,0)}_{= v_1} / \alpha \in \mathbb{R}\}$

$S_{\lambda_2} : \begin{cases} -x = 0 \\ -2y + 2z = 0 \\ 2y - 2z = 0 \end{cases}$ → sistem liniar omogen cu 3 ec și 3 nec.

$\{\lambda_2 = 2\}$

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}$

$rg(A_f - \lambda_2 I_3) = 2$

$\Delta_p = \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2 - 0 = 2 \neq 0 \Rightarrow \begin{cases} x, y \text{ nec princip} \\ z = \beta, \beta \in \mathbb{R} \\ \text{nec. sec.} \end{cases}$

$\begin{cases} x = 0 \\ -2y + 2\beta = 0 \\ z = \beta, \beta \in \mathbb{R} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = \beta \\ z = \beta, \beta \in \mathbb{R} \end{cases}$

Deci $V_{\lambda_2} = \{(0, \beta, \beta) / \beta \in \mathbb{R}\} = \{\beta \underbrace{(0,1,1)}_{v_2} / \beta \in \mathbb{R}\}$

S_{λ_3} : $\begin{cases} 3x = 0 \\ 2y + 2z = 0 \\ 2y + 2z = 0 \end{cases}$ → sistem liniar omogen cu 3 ec și 3 nec.

$\{\lambda_3 = -2\}$ $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$

$rg(A_f - \lambda_3 I_3) = 2$

$\Delta_p = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6 \neq 0 \Rightarrow \begin{cases} x, y \text{ nec. princip} \\ z = \mu, \mu \in \mathbb{R} \\ \text{nec. sec.} \end{cases}$

$\begin{cases} x = 0 \\ 2y + 2\mu = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -\mu \\ z = \mu, \mu \in \mathbb{R} \end{cases}$

Deci $V_{\lambda_3} = \{(0, -\mu, \mu) / \mu \in \mathbb{R}\} = \{\mu \underbrace{(0,-1,1)}_{v_3} / \mu \in \mathbb{R}\}$

Rezolvare d) $A = \begin{pmatrix} 0 & 9 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (9y, 4x, 3z)$, $(\forall)(x,y,z) \in \mathbb{R}^3$

Polinomul caracteristic: $P(\lambda) = det(A - \lambda I_3) =$

$= \begin{vmatrix} -\lambda & 9 & 0 \\ 4 & -\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = -(\lambda+6)(\lambda^2 - 9\lambda + 18) =$

$$= -(\lambda+6)(\lambda-3)(\lambda-6)$$

Ec. caract $P(\lambda)=0 \iff -(\lambda+6)(\lambda-3)(\lambda-6)=0$

$$\begin{cases} \lambda_1 = -6 \\ \lambda_2 = -3 \\ \lambda_3 = 6 \end{cases} \in \mathbb{R} \quad Spec(f) = \{-6, -3, 6\}$$

și $\begin{cases} m_a(\lambda_1) = 1 \\ m_a(\lambda_2) = 1 \\ m_a(\lambda_3) = 1 \end{cases}$ multiplicitățile algebrice

Subspații proprii

$$S_\lambda : \begin{cases} -\lambda x + 9y = 0 \\ 4x - \lambda y = 0 \\ (3-\lambda) z = 0 \end{cases}$$

Atunci

$S_{\lambda_1} : \begin{cases} 6x + 9y = 0 \\ 4x + 6y = 0 \\ 9z = 0 \end{cases}$ → sistem liniar omogen cu 3 ec. și 3 nec.

$\{\lambda_1 = -6\}$

$$\begin{pmatrix} 6 & 9 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$rg(A - \lambda I_3) = 2$

$\Delta_p = \begin{vmatrix} 6 & 0 \\ 0 & 9 \end{vmatrix} = 36 - 0 = 36 \neq 0 \Rightarrow \begin{cases} y, z \text{ nec princip} \\ x = \alpha, \alpha \in \mathbb{R} \text{ nec. sec} \end{cases}$

$$\begin{cases} 4x+6y=0 \\ 9z=0 \end{cases} \Rightarrow \begin{cases} 6y=-4\alpha \\ z=0 \end{cases} \Rightarrow \begin{cases} y=-\frac{2}{3}\alpha \\ x=\alpha, \alpha\in\mathbb{R} \\ z=0 \end{cases}$$

Deci $V_{\lambda_1}=\{(\alpha, -\frac{2}{3}\alpha, 0) / \alpha\in\mathbb{R}\} = \{\alpha\underbrace{(1,-\frac{2}{3},0)}_{=v_1} / \alpha\in\mathbb{R}\}$

S_{λ_2} $\{\lambda_2=-3\}$: $\begin{cases} -3x+9y=0 \\ 4x-3y=0 \end{cases}$ → sistem liniar omogen cu 2 ec. și 3 nec.

$$\begin{pmatrix} 3 & 9 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$rg(A-\lambda_2 I_3)=2$

$\Delta_p = \begin{vmatrix} 3 & 9 \\ 4 & 3 \end{vmatrix} = 9-36=-27\neq 0 \Rightarrow \begin{cases} x, y \text{ nec principale} \\ z=\beta, \beta\in\mathbb{R} \text{ nec. sec} \end{cases}$

$$\begin{cases} -3x+9y=0 \mid \cdot(-4) \\ 4x-3y=0 \mid \cdot 3 \\ z=\beta, \beta\in\mathbb{R} \end{cases} \Rightarrow \begin{cases} -12x+36y=0 \\ 12x-9y=0 \\ z=\beta, \beta\in\mathbb{R} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 45y=0 \\ z=\beta, \beta\in\mathbb{R} \end{cases} \Rightarrow \begin{cases} y=0 \\ z=\beta, \beta\in\mathbb{R} \\ 4x=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ z=\beta, \beta\in\mathbb{R} \end{cases}$$

Deci $V_{\lambda_2}=\{(0,0,\beta) / \beta\in\mathbb{R}\} = \{\beta\underbrace{(0,0,1)}_{=v_2} / \beta\in\mathbb{R}\}$

S_{λ_3} : $\begin{cases} -6x + 9y = 0 \\ 4x - 6y = 0 \\ -3z = 0 \end{cases}$ → sistem liniar omogen cu 3 ec. și 3 nec.

$\{\lambda_3 = 6\}$

$rg(A - \lambda_3 I_3) = 2$ $\begin{pmatrix} -6 & 9 & 0 \\ 4 & -6 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

$\Delta_p = \begin{vmatrix} -6 & 0 \\ 0 & -3 \end{vmatrix} = 18 \neq 0 \Rightarrow \begin{cases} y, z \text{ nec principale} \\ x = \mu, \mu \in \mathbb{R} \text{ nec. sec.} \end{cases}$

$\Rightarrow \begin{cases} 6\mu + 9y = 0 \\ z = 0 \end{cases} \Rightarrow \begin{cases} 9y = 6\mu \\ z = 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} y = \frac{2}{3}\mu \\ x = \mu, \mu \in \mathbb{R} \\ z = 0 \end{cases}$

Deci $V_{\lambda_3} = \{(\mu, \frac{2}{3}\mu, 0) / \mu \in \mathbb{R}\} =$

$= \{\mu \underbrace{(1, \frac{2}{3}, 0)}_{= v_3} / \mu \in \mathbb{R}\}$

2. Stabiliți dacă matricele de la exercițiul precedent sunt diagonalizabile, și în caz afirmativ determinați forma lor diagonală.

Rezolvare a) $\dim V_{\lambda_1} = 1$

$\dim V_{\lambda_2} = 1$

$m_g(\lambda_1) = 1$ multiplicitățile geometrice

$m_g(\lambda_2) = 1$

Avem $\begin{cases} 1)\ m_a(\lambda_1) + m_a(\lambda_2) = 2 = \dim_{\mathbb{R}} \mathbb{R}^2 \\ 2)\ m_a(\lambda_i) = m_g(\lambda_i), (\forall) i = \overline{1,2} \end{cases}$

$\Rightarrow$ f este diagonalizabilă, deci $(\exists)\ B \subset \mathbb{R}^2$

{bază formată din vectori proprii coresp. lui λ_1, λ_2 (care sunt liniar indep. deoarece valorile proprii sunt distincte)}

În raport cu care, matricea asociată lui f are forma diagonală: $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$

$B = \{v_1 = (-2, 1), v_2 = (1, 1)\}$

b) $\dim V_{\lambda_1} = 1$

$\dim V_{\lambda_2} = 1$

$m_g(\lambda_1) = m_g(\lambda_2) = 1$ multiplicitățile geometrice

Avem $\begin{cases} 1) \ m_a(\lambda_1) + m_a(\lambda_2) = 1+1 = 2 = \dim_{\mathbb{R}} \mathbb{R}^2 \\ 2) \ m_a(\lambda_i) = m_g(\lambda_i), (\forall) \ i = \overline{1,2} \end{cases}$

$\Rightarrow f$ este diagonalizabilă, deci $(\exists) \ B \subset \mathbb{R}^2$ {bază formată din vectori proprii coresp. lui λ_1, λ_2 (care sunt liniar indep. deoarece valorile proprii sunt distincte)}

În raport cu care, matricea asociată lui f are forma diagonală:

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$$

$B = \{ v_1 = (1, 0), v_2 = (2, 1) \}$

c) $\dim V_{\lambda_1} = 1$

$\dim V_{\lambda_2} = 1$

$\dim V_{\lambda_3} = 1$

$m_g(\lambda_1) = 1$ multiplicitățile geometrice

$m_g(\lambda_2) = 1$

Avem $\begin{cases} 1)\ m_a(\lambda_1) + m_a(\lambda_2) + m_a(\lambda_3) = 3 = \dim_{\mathbb{R}} \mathbb{R}^3 \\ 2)\ m_a(\lambda_i) = m_g(\lambda_i),\ (\forall)\ i = \overline{1,3} \end{cases}$

$\Rightarrow f$ este diagonalizabilă, deci $(\exists)\ B \subset \mathbb{R}^3$ {bază formată din vectori proprii coresp. lui $\lambda_1, \lambda_2, \lambda_3$ (care sunt liniar indep. deoarece valorile proprii sunt distincte)}

În raport cu care, matricea asociată lui f are forma diagonală:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$B = \{ v_1 = (1,0,0),\ v_2 = (0,1,1),\ v_3 = (0,-1,1) \}$

d) $\dim V_{\lambda_1} = 1$

$\dim V_{\lambda_2} = 1$

$\dim V_{\lambda_3} = 1$

$m_g(\lambda_1) = 1$ multiplicitățile geometrice
$m_g(\lambda_2) = 1$
$m_g(\lambda_3) = 1$

Avem $\begin{cases} 1) \ m_a(\lambda_1) + m_a(\lambda_2) + m_a(\lambda_3) = 3 = \dim_{\mathbb{R}} \mathbb{R}^3 \\ 2) \ m_a(\lambda_i) = m_g(\lambda_i), (\forall) i = \overline{1,3} \end{cases}$

$\Rightarrow f$ este diagonalizabilă, deci $(\exists)\ B \subset \mathbb{R}^3$

{bază formată din vectori proprii coresp. lui $\lambda_1, \lambda_2, \lambda_3$ (care sunt liniari indep. deoarece valorile proprii sunt distincte)}

În raport cu care, matricea asociată lui f are forma diagonală:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$B = \{v_1 = (1, -\frac{2}{3}, 0), v_2 = (0, 0, 1), v_3 = (1, \frac{2}{3}, 0)\}$

3. Considerăm aplicația

$$T: \mathbb{R}^3 \to \mathbb{R}^3,\ T(x,y,z) = (x+4y,\ 2y+3z,\ y)$$

a) Arătați că T este transformare (aplicație) liniară

Rezolvare: Fie $v_1, v_2 \in \mathbb{R}^3$

Fie $\alpha, \beta \in \mathbb{R}$

Arătăm că:

$$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$$

$$v_1, v_2 \in \mathbb{R}^3 \Rightarrow \begin{cases} v_1 = (x_1, y_1, z_1) \\ v_2 = (x_2, y_2, z_2) \end{cases} \Rightarrow$$

$$\Rightarrow \alpha v_1 + \beta v_2 = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$T(\alpha v_1 + \beta v_2) = \alpha(x_1 + 4y_1, 2y_1 + 3z_1, y_1) + \beta(x_2 + 4y_2, 2y_2 + 3z_2, y_2) = \alpha T(v_1) + \beta T(v_2)$$

$\Rightarrow$ T este aplicație liniară

b) Scrieți matricea asociată lui T, A_T

Rezolvare: $A_f = \begin{array}{c} \begin{pmatrix} 1 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \\ \;\; f(e_1) \;\; f(e_2) \;\; f(e_3) \end{array} \in \mathcal{M}_3(\mathbb{R})$

c) Determinați valorile și vectorii proprii corespunzători(e) lui A_T.

Rezolvare: Polinomul caracteristic:

$$P(\lambda) = \det(A_f - \lambda I_3) = \begin{vmatrix} 1-\lambda & 4 & 0 \\ 0 & 2-\lambda & 3 \\ 0 & 1 & -\lambda \end{vmatrix} =$$

$$\begin{matrix} 1-\lambda & 4 & 0 \\ 0 & 2-\lambda & 3 \end{matrix}$$

$$= -\lambda(1-\lambda)(2-\lambda) - 3(1-\lambda) = (1-\lambda)[-\lambda\cdot(2-\lambda) - 3] =$$

$$= -(\lambda-1)[\lambda(2-\lambda)+3] = -(\lambda-1)(2\lambda-\lambda^2+3) =$$

$$= -(\lambda-1)(\lambda+1)(\lambda-3)$$

Ec. caract $P(\lambda) = 0 \Leftrightarrow -(\lambda-1)(\lambda+1)(\lambda-3) = 0$

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \\ \lambda_3 = 3 \end{cases} \in \mathbb{R}$ valorile proprii, $Spec(f) = \{1, -1, 3\}$

și $m_a(\lambda_1) = m_a(\lambda_2) = m_a(\lambda_3) = 1$
multiplicitățile algebrice

$$S'_\lambda : \begin{cases} (1-\lambda)x + 4y = 0 \\ (2-\lambda)y + 3z = 0 \\ y - \lambda z = 0 \end{cases}$$

$$\Rightarrow S'_{\lambda_1} : \begin{cases} 4y = 0 \\ y + 3z = 0 \\ y - z = 0 \end{cases}$$ $\{\lambda_1 = 1\}$ → Sistem liniar omogen cu 3 ec și 3 nec.

$$rg(A_f - \lambda_1 I_3) = 2 \quad \begin{pmatrix} 0 & 4 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\Delta_p = \begin{vmatrix} 4 & 0 \\ 1 & 3 \end{vmatrix} = 12 \neq 0 \Rightarrow \begin{cases} y, z \text{ nec princip} \\ x = \alpha, \alpha \in \mathbb{R} \text{ nec. sec.} \end{cases}$$

$$\Rightarrow \begin{cases} y = 0 \\ z = 0 \\ x = \alpha, \alpha \in \mathbb{R} \end{cases} \Rightarrow v_1 = (1, 0, 0)$$

$$S'_{\lambda_2} : \begin{cases} 2x + 4y = 0 \\ 3y + 3z = 0 \\ y + z = 0 \end{cases}$$ $\{\lambda_2 = -1\}$ → Sistem liniar omogen cu 3 ec și 3 nec.

$$\begin{pmatrix} 2 & 4 & 0 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

$rg(A_f - \lambda_2 I_3) = 2$

$\Delta_p = \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} = 6 \neq 0 \Rightarrow \begin{cases} x, y \text{ nec. princip} \\ z = \beta,\ \beta \in \mathbb{R} \text{ nec. sec.} \end{cases}$

$\Rightarrow \begin{cases} x + 2y = 0 \\ y + \beta = 0 \end{cases} \Rightarrow \begin{cases} y = -\beta \\ x - 2\beta = 0 \end{cases} \Rightarrow \begin{cases} y = -\beta \\ x = 2\beta \\ z = \beta,\ \beta \in \mathbb{R} \end{cases}$

$\Rightarrow v_2 = (2, -1, 1)$

$S_{\lambda_3} : \begin{cases} -2x + 4y = 0 \\ -y + 3z = 0 \\ y - 3z = 0 \end{cases}$ → sistem liniar omogen cu 3 ec. și 3 nec.

$\{\lambda_3 = 3\}$

$rg(A_f - \lambda_3 I_3) = 2$ $\quad \begin{pmatrix} -2 & 4 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{pmatrix}$

$\Delta_p = \begin{vmatrix} -2 & 4 \\ 0 & -1 \end{vmatrix} = 2 \neq 0 \Rightarrow \begin{cases} x, y \text{ nec. princip} \\ z = \mu,\ \mu \in \mathbb{R} \text{ nec. sec.} \end{cases}$

$\begin{cases} -x + 2y = 0 \\ y - 3\mu = 0 \end{cases} \Rightarrow \begin{cases} y = 3\mu \\ -x + 3\mu = 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} y = 3\mu \\ x = 3\mu \end{cases} \quad z = \mu,\ \mu \in \mathbb{R}$

$v_3 = (\cancel{6}, 3, 1)$

d) Precizați subspațiile proprii corespunzătoare transformării (aplicației) T și stabiliți dacă aceasta este diagonalizabilă.

Rezolvare: $V_{\lambda_1} = \{(\alpha, 0, 0) / \alpha \in \mathbb{R}\} = \{\alpha \underbrace{(1,0,0)}_{v_1} / \alpha \in \mathbb{R}\}$

$V_{\lambda_2} = \{(2\beta, -\beta, \beta) / \beta \in \mathbb{R}\} = \{\beta \underbrace{(2,-1,1)}_{v_2} / \beta \in \mathbb{R}\}$

$V_{\lambda_3} = \{(3\mu, 3\mu, \mu) / \mu \in \mathbb{R}\} = \{\mu \underbrace{(\cancel{6}, 3, 1)}_{v_3} / \mu \in \mathbb{R}\}$

$\dim V_{\lambda_1} = 1$

$\dim V_{\lambda_2} = 1$

$\dim V_{\lambda_3} = 1$

$m_g(\lambda_1) = m_g(\lambda_2) = m_g(\lambda_3) = 1$ multiplicitățile geometrice

Avem $\begin{cases} 1)\ m_a(\lambda_1) + m_a(\lambda_2) + m_a(\lambda_3) = 3 = \dim_{\mathbb{R}} \mathbb{R}^3 \\ 2)\ m_a(\lambda_i) = m_g(\lambda_i),\ (\forall) i = \overline{1,3} \end{cases}$

$\Rightarrow$ f este diagonalizabilă, deci $(\exists)\ B \subset \mathbb{R}^3$ {bază formată din vectori coresp. lui $\lambda_1, \lambda_2, \lambda_3$ (care sunt liniar indep. deoarece valorile proprii sunt distincte)}

e) Scrieți, dacă există, matricea diagonalizatoare C și matricea diagonală D.

Rezolvare: $D=\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$C(v_1, v_2, v_3)=\begin{pmatrix} 1 & 2 & 6 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$

f) Verificați rezultatul obținut

Rezolvare: $D = C^{-1} \cdot A_f \cdot C$

$$C^{-1}=\left(\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & \boxed{-1} & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_2=-L_2}{\sim}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & \boxed{1} & -3 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3=L_3-L_2}{\sim}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & \boxed{4} & 0 & 1 & 1 \end{array}\right) \underset{L_3=\frac{L_3}{4}}{\sim}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 1/4 & 1/4 \end{array}\right) \underset{L_2=L_2+3L_3}{\sim}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 3/4 \\ 0 & 0 & \boxed{1} & 0 & 1/4 & 1/4 \end{array}\right) \underset{L_1 = L_1 \cancel{+} 6L_3 - 2L_1}{\sim}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -3 \\ 0 & 1 & 0 & 0 & -1/4 & 3/4 \\ 0 & 0 & 1 & 0 & 1/4 & 1/4 \end{array}\right)$$

Deci $C^{-1} = \begin{pmatrix} 1 & -1 & -3 \\ 0 & -1/4 & 3/4 \\ 0 & 1/4 & 1/4 \end{pmatrix}$

$$C^{-1} \cdot A_f = \begin{pmatrix} 1 & -1 & -3 \\ 0 & -1/4 & 3/4 \\ 0 & 1/4 & 1/4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 \\ 0 & 1/4 & -3/4 \\ 0 & 3/4 & 3/4 \end{pmatrix}$$

$$D = C^{-1} \cdot A_f \cdot C = \begin{pmatrix} 1 & -1 & -3 \\ 0 & 1/4 & -3/4 \\ 0 & 3/4 & 3/4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 6 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

ADEVĂRAT !

SEMINAR 9

Apl Același enunț ca în aplicația anterioară pentru f. pătratică $Q: \mathbb{R}^3 \to \mathbb{R}$,

$$Q(x) = x_1^2 + 3x_2^2 + x_3^2 - 2x_1x_2 - 4x_2x_3,$$
$$(\forall) x = (x_1, x_2, x_3) \in \mathbb{R}^3$$

Să se aducă la o formă canonică forma pătratică Q utilizând: a) metoda Gauss

b) metoda Jacobi

Rezolvare a) Matricea asociată f. pătratice Q în raport cu baza canonică este:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 1 \end{pmatrix}$$

$$Q(x) = (x_1^2 - 2x_1x_2) + 3x_2^2 + x_3^2 - 4x_2x_3$$
$$= (x_1^2 - 2x_1x_2 + x_2^2) - x_2^2 + 3x_2^2 + x_3^2 - 4x_2x_3$$
$$= (x_1 - x_2)^2 + 2(x_2^2 - 2x_2x_3 + x_3^2) - 2x_3^2 + x_3^2$$
$$= (x_1 - x_2)^2 + 2(x_2 - x_3)^2 - x_3^2$$

Sch. de coord: $\begin{cases} y_1 = x_1 - x_2 \\ y_2 = x_2 - x_3 \\ y_3 = x_3 \end{cases}$

$\Rightarrow Q(x) = y_1^2 + 2y_2^2 - y_3^2, \ (\forall)\ x = (y_1, y_2, y_3) \in \mathbb{R}^3$

b) Avem: $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 1 \end{pmatrix} \to$ mat. asociată lui Q în baza canonică

$\Delta_1 = |1| = 1$

$\Delta_2 = \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 3 - 1 = 2$

$\Delta_3 = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 1 \end{vmatrix} = 3 - 4 - 1 = -2$

$\Delta_i \neq 0, \ (\forall)\ i = \overline{1,3}$

$\Rightarrow Q(x) = \frac{1}{\Delta_1}(x_1')^2 + \frac{\Delta_1}{\Delta_2}(x_2')^2 + \frac{\Delta_2}{\Delta_3}(x_3')^2$

$Q(x) = x_1'^2 + \frac{1}{2}x_2'^2 - x_3'^2, \ (\forall)\ x = (x_1', x_2', x_3') \in \mathbb{R}^3$

coord. în raport cu noua bază B' de raportare

Apl Fie $F: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$,

$$F(x, y) = 2x_1y_1 + x_2y_2 - 2x_1y_2 - 2x_2y_1 - 2x_2y_3 - 2x_3y_2, \quad (\forall)\ x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbb{R}^3$$

a) Arătați că F este formă biliniară simetrică

b) Scrieți matricea formei bilin. simetrice F în raport cu baza canonică din $\mathbb{R}^3$. (B_0)

c) Scrieți matricea formei bilin. simetrice F în raport cu baza următoare $B_1 = \{\underbrace{(1,1,1)}_{f_1}, \underbrace{(2,-1,2)}_{f_2}, \underbrace{(1,3,-3)}_{f_3}\} \subset \mathbb{R}^3$

d) Determinați forma pătratică Q coresp. lui F și să se aducă la o formă canonică utilizând metodele Gauss, respectiv Jacobi.

Rezolvare: b) $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$ $\to$ matrice simetrică $\Rightarrow$ F este formă biliniară simetrică

c) $B_0 \xrightarrow{C} B$, $A' = C^T \cdot A \cdot C$

$$C = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix}$$
$$\quad\ \ f_1 \quad f_2 \quad f_3$$

$$C^T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & 3 \end{pmatrix}$$

$$A' = C^T \cdot A \cdot C = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & -3 \end{pmatrix} \cdot \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -3 & -2 \\ 6 & -9 & 2 \\ -4 & 7 & -6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix} = \begin{pmatrix} -5 & -1 & -3 \\ -1 & 25 & -27 \\ -3 & -27 & 35 \end{pmatrix}$$

Deci $A' = \begin{pmatrix} -5 & -1 & -3 \\ -1 & 25 & -27 \\ -3 & -27 & 35 \end{pmatrix}$

a) I. $F(\alpha x + \beta y, z) = 2(\alpha x_1 + \beta y_1)z_1 + (\alpha x_2 + (\beta y_2)z_2 = 2(\alpha x_1 + \beta y_1)z_2 - 2(\alpha x_2 + \beta y_2)z_1$
$-2(\alpha x_2 + \beta y_2)z_3 - 2(\alpha x_3 + \beta y_3)z_2 =$
$= \alpha(2x_1z_1 + x_2z_2 - 2x_1z_2 - 2x_2z_1 - 2x_2z_3 - 2x_3z_2)$
$+ \beta(2y_1z_1 + y_2z_2 - 2y_1z_2 - 2y_2z_1 - 2y_2z_3 - 2y_3z_2)$
$= \alpha F(x,z) + \beta F(y,z)$

II $F(x, \alpha y + \beta z) = 2x_1(\alpha y_1 + \beta z_1) + x_2(\alpha y_2 + \beta z_2)$
$- 2x_1(\alpha y_2 + \beta z_2) - 2x_2(\alpha y_1 + \beta z_1) - 2x_2(\alpha y_3 +$
$+ \beta z_3) - 2x_3(\alpha y_2 + \beta z_2) =$

$= \alpha (2x_1 y_1 + x_2 y_2 - 2x_1 y_2 - 2x_2 y_1 - 2x_2 y_3 - 2x_3 y_2)$

$+ \beta (2x_1 z_1 + x_2 z_2 - 2x_1 z_2 - 2x_2 z_1 - 2x_2 z_3 - 2x_3 z_2)$

$= \alpha F(x, y) + \beta F(y, z)$

$\Rightarrow$ F este liniară pe amb. argumente

d) $Q(x) = 2x_1^2 + x_2^2 - 4x_1x_2 - 4x_2x_3$, $(\forall) x = (x_1, x_2, x_3) \in \mathbb{R}^3$

Metoda Gauss $Q(x) = (2x_1^2 - 4x_1x_2 + 2x_2^2) - 2x_2^2 + x_2^2 -$

$-4x_2x_3 = 2(x_1 - x_2)^2 - (x_2^2 + 4x_2x_3 + 4x_3^2) + 4x_3^2$

$= 2(x_1 - x_2)^2 - (x_2 + 2x_3)^2 + 4x_3^2$

Sch. de coord: $\begin{cases} y_1 = x_1 - x_2 \\ y_2 = x_2 + 2x_3 \\ y_3 = 4x_3 \end{cases}$

$\Rightarrow Q(x) = 2y_1^2 - y_2^2 + 4y_3^2$, $(\forall) x = (y_1, y_2, y_3) \in \mathbb{R}^3$

Metoda Jacobi $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$

$\Delta_1 = |2| = 2$

$\Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = 2 - 4 = -2$

$\Delta_3 = \begin{vmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{vmatrix} = -8$

$$\Rightarrow Q(x) = \frac{1}{\Delta_1}(x_1')^2 + \frac{\Delta_1}{\Delta_2}(x_2')^2 + \frac{\Delta_2}{\Delta_3}(x_3')^2$$

$$= \frac{1}{2}(x_1')^2 - (x_2')^2 + \frac{1}{4}(x_3')^2$$

$$(\forall)\ x = (x_1', x_2', x_3')$$

SEMINAR 10

Apl În spațiul vectorial euclidian $(\mathbb{R}^3/_{\mathbb{R}}, <,>)$ {produsul scalar canonic} să se construiască o bază ortonormată pornind de la baza:

$$B = \{f_1 = (1,1,1), f_2 = (1,1,-1), f_3 = (1,-1,-1)\}$$

folosind procedeul de ortonormalizare Gram-Schmidt (POGS)

Rezolvare:

$$\begin{cases} e_1' = f_1 \\ e_i' = f_i - \sum\limits_{j=1}^{i-1} \frac{<f_i, e_j'>}{\|e_j'\|^2} e_j', (\forall) i = \overline{2,n} \end{cases}$$

$$\Rightarrow e_1' = f_1 = (1,1,1)$$

$$e_2' = f_2 - \frac{<f_2, e_1'>}{\|e_1'\|^2} \cdot e_1' = (1,1,-1) - \frac{1}{3} \cdot (1,1,1) =$$

$$= \left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right) = \frac{2}{3}(1,1,-2)$$

$$l_3' = f_3 - \frac{\langle f_3, l_1' \rangle}{\|l_1\|^2} \cdot l_1' = \frac{\langle f_3, l_2' \rangle}{\|l_2'\|^2} \; l_2' =$$

$$= (1,-1,-1) = -\frac{1}{3} \cdot (1,1,1) = \frac{2 \cdot \frac{2}{3}}{\frac{4}{9} \cdot 6} \cdot \frac{2}{3} \; (1,1,-2) =$$

$$= \frac{2}{3} \, \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{9}}{\cancel{24}} \, (1,1,-2) = \frac{\cancel{3}}{\cancel{6}} \, \frac{2}{\cancel{3}} (1,1,-2) =$$

$$= (1,-1,-1) + \frac{1}{3}(1,1,1) - \frac{1}{3}(1,1-2)$$

$$= (1,-1,0)$$

$$\begin{cases} l_1 = \dfrac{l_1'}{\|l_1'\|} = \dfrac{(1,1,1)}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \, (1,1,1) \\[2ex] l_2 = \dfrac{l_2'}{\|l_2'\|} = \dfrac{1}{\frac{2}{3} \cdot \sqrt{6}} \cdot \dfrac{2}{3} \cdot (1,1,-2) = \dfrac{1}{\sqrt{6}} (1,1,-2) \\[2ex] l_3 = \dfrac{l_3'}{\|l_3'\|} = \dfrac{(1,-1,0)}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \, (1,-1,0) \end{cases}$$

$\Rightarrow$ Bază ortonormată

Apl Considerând spațiul euclidian $E_3 = (\mathbb{R}^3/\mathbb{R}, <,>)$ și baza ortonormată B' obținută anterior, să se determine coordonatele următorilor vectori în această bază:

$$B' = \left(e_1 = \frac{1}{\sqrt{2}}(0,1,1),\ e_2 = \frac{1}{\sqrt{6}}(2,-1,1),\ e_3 = \frac{1}{\sqrt{3}}(1,1,-1)\right)$$

Rezolvare: $w = (-1,1,2)$

$$\begin{cases} e_1 = \frac{1}{\sqrt{2}}(0,1,1) \\ e_2 = \frac{1}{\sqrt{6}}(2,-1,1) \\ e_3 = \frac{1}{\sqrt{3}}(1,1,-1) \end{cases}$$

$w = (-1,1,2)$

$w = w_1 e_1 + w_2 e_2 + w_3 e_3$

$<w, e_1> = w_1 <e_1, e_1> + w_2 <e_2, e_1> + w_3 <e_3, e_1>$

$= <v, e_1>$ Analog pentru w_2 și w_3

$\Rightarrow w = <w, e_1> e_1 + <w, e_2> e_2 + <w, e_3> \cdot e_3$,

unde $B' = \{e_1, e_2, e_3\}$

$$\Rightarrow \begin{cases} w_1 = <w, e_1> = \frac{1}{\sqrt{2}} \cdot 3 = \frac{3}{\sqrt{2}} \\ w_2 = <w, e_2> = \frac{1}{\sqrt{6}} \cdot 1 = \frac{1}{\sqrt{6}} \\ w_3 = <w, e_3> = \frac{1}{\sqrt{3}} \cdot (-2) = -\frac{2}{\sqrt{3}} \end{cases}$$

$\Rightarrow [w]_{B'} = \left(\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{3}}\right)$

Apl Ortonormați sistemul: $\{f_1=(2,2,1), f_2=(-2,-1,2),$
$f_3=(5,-6,2)\}$

Obs! $\langle f_1, f_3\rangle = \langle f_2, f_3\rangle = 0$

Rezolvare: $e_1' = f_1 = (2,2,1)$

$e_2' = f_2 - \frac{\langle f_2, e_1'\rangle}{\|e_1'\|^2} \cdot e_1' =$

$= (-2,-1,2) - \frac{-4-2+2}{9}(2,2,1) = (-2,-1,2) + \frac{2}{3}(2,2,1) =$

$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{8}{3}\right) = \frac{1}{3}(-2,1,8)$

$e_3' = f_3 - \frac{\langle f_3, e_1'\rangle}{\|e_1'\|^2} \cdot e_1' - \frac{\langle f_3, e_2'\rangle}{\|e_2'\|^2} \cdot e_2' =$

$= f_3 = (5,-6,2)$

Deci: $e_1 = (2,2,1) \cdot \frac{1}{\|e_1'\|} = \frac{1}{3} \cdot (2,2,1)$

$e_2 = \frac{1}{3} \cdot (-2,1,8) \cdot \frac{1}{\|e_2'\|^2} = \frac{1}{\frac{1}{3} \cdot \sqrt{4+1+64}} \cdot \frac{1}{3} \cdot$

$\cdot (-2,1,8) = \frac{1}{\sqrt{69}}(-2,1,8)$

$$e_3 = (5,-6,\ 2)\cdot \frac{1}{\|e_3'\|} = \frac{1}{\sqrt{25+36+4}}\cdot(5,-6,2) =$$

$$= \frac{1}{\sqrt{65}}\cdot(5,-6,2)$$

Apl Fie subspațiul vectorial euclidian $E_3 = (\mathbb{R}^3/_{\mathbb{R}}, <,>)$
Determinați suplementul ortogonal al următoarelor subspații vectoriale:

a) $U = <(1,2,1),(1,-1,2)>$

b) $V = <(2,-3,1)>$

Rezolvare: a) $B = (u_1, u_2) \subset U$

$U^{\perp} = \{y \in \mathbb{R}^3 / y \perp x,\ (\forall) x \in U\}$

Fie $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ a.î. $\begin{cases} y \perp u_1 \\ y \perp u_2 \end{cases} \Leftrightarrow \begin{cases} <y, u_1> = 0 \\ <y, u_2> = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} y_1 + 2y_2 + y_3 = 0 \\ y_1 - y_2 + 2y_3 = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\Delta_p = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -1-2 = -3 \neq 0 \Rightarrow rg\, A = 2$$

$$\begin{cases} y_1, y_2 \text{ nec. princip} \\ y_3 = \alpha, \alpha \in \mathbb{R} \text{ nec. sec.} \end{cases}$$

$$\Rightarrow \begin{cases} y_1 + 2y_2 = -\alpha \\ y_1 - y_2 = -2\alpha \end{cases} \Rightarrow 3y_2 = \alpha \Rightarrow y_2 = \frac{\alpha}{3}$$

$$y_1 - \frac{\alpha}{3} = -2\alpha$$

$$y_1 = {}^{3)}\!-2\alpha + \frac{\alpha}{3}$$

$$y_1 = \frac{-6\alpha + \alpha}{3} = -\frac{5\alpha}{3}$$

$$y_3 = \alpha, \alpha \in \mathbb{R}$$

Deci $y = \alpha\left(-\frac{5}{3}, \frac{1}{3}, 1\right), \alpha \in \mathbb{R}$

$\Rightarrow U^{\perp} = \{\alpha\left(-\frac{5}{3}, \frac{1}{3}, 1\right) / \alpha \in \mathbb{R}\} = \langle\left(-\frac{5}{3}, \frac{1}{3}, 1\right)\rangle$

b) $V^{\perp} = \{y \in \mathbb{R}^3 \mid y \perp x, (\forall) x \in V\}$

$\underbrace{B = \{v\} \subset V}_{\text{bază}}$

Fie $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ a.î. $y \perp v \Leftrightarrow \langle y, v \rangle = 0$

$\Leftrightarrow 2y_1 - 3y_2 + y_3 = 0$

Deci $V^{\perp} = \{\underset{=(y_1, y_2, y_3)}{y} \in \mathbb{R}^3 / 2y_1 - 3y_2 + y_3 = 0\}$

supl. ortogonal al lui V (este un ssp. vectorial
2- dimensional)