- TEMĂ -
ALGEBRĂ LINIARĂ ȘI GEOMETRIE

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SEMINAR 1

Apl 1 Fie $A, B \in \mathcal{M}_n(\mathbb{C})$ a.î. $AB = BA$

ii) $(A+B)^k = \sum_{j=0}^{k} C_k^j A^j B^{k-j}$, unde $A^0 = B^0 = I_n$

Rezolvare: Știm că $AB = BA$

1) $(A+B)^k = (B+A)^k = C_k^0 A^0 B^k + C_k^1 B^{k-1} \cdot A + \ldots + C_k^k A^k \cdot B^0$

2) $\sum_{j=0}^{k} C_k^j A^j B^{k-j} = C_k^0 A^0 \cdot B^k + C_k^1 A^1 \cdot B^{k-1} + \ldots + C_k^k A^k \cdot B^0$

$\Rightarrow 0 = 0$ Deci q.e.d

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Apl Fie $A, B \in \mathcal{M}_n(\mathbb{C})$ a.î. $A+B=AB$. Dem că $AB=BA$

Dem: Știm că $A+B=B+A$

$$\left.\begin{array}{l} A+B=A\cdot B \\ B+A=B\cdot A \end{array}\right| \Rightarrow A\cdot B=B\cdot A$$

Apl Calculați det A

a) folosind dezvoltarea după prima linie

$$A=\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix}$$

Rezolvare:

$$\Delta=\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{vmatrix} = (-1)^{1+1}\cdot 1\cdot\begin{vmatrix} 1 & 3 & 4 \\ 5 & 1 & -1 \\ -2 & 2 & 4 \end{vmatrix} +$$

$$+(-1)^{1+2}\cdot 1\cdot\begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & -1 \\ -1 & 2 & 4 \end{vmatrix} + (-1)^{1+3}\cdot 2\cdot\begin{vmatrix} 1 & 1 & 4 \\ 2 & 5 & -1 \\ -1 & -2 & 4 \end{vmatrix} +$$

$$+(-1)^{1+4}\cdot 3\cdot\begin{vmatrix}1&1&3\\2&5&1\\-1&-2&2\end{vmatrix}=\begin{vmatrix}1&3&4\\5&1&-1\\-2&2&4\end{vmatrix}-\begin{vmatrix}1&3&4\\2&1&-1\\-1&2&4\end{vmatrix}+$$

$$+2\begin{vmatrix}1&1&4\\2&5&-1\\-1&-2&4\end{vmatrix}-3\begin{vmatrix}1&1&3\\2&5&1\\-1&-2&2\end{vmatrix}=0-5+30-30=\boxed{-5}$$

$$\begin{vmatrix}1&3&4\\5&1&-1\\-2&2&4\end{vmatrix}=(-1)^{1+1}\cdot 1\cdot\begin{vmatrix}1&-1\\2&4\end{vmatrix}+(-1)^{1+2}\cdot 3\cdot\begin{vmatrix}5&-1\\-2&4\end{vmatrix}$$

$$+(-1)^{1+3}\cdot 4\begin{vmatrix}5&1\\-2&2\end{vmatrix}=1\cdot 1\cdot(4+2)-3\cdot(20-2)+$$

$$+4\cdot(10+2)=6-3\cdot 18+4\cdot 12=\boxed{0}$$

$$\begin{vmatrix}1&3&4\\2&1&-1\\-1&2&4\end{vmatrix}=(-1)^{1+1}\cdot 1\cdot\begin{vmatrix}1&-1\\2&4\end{vmatrix}+(-1)^{1+2}\cdot 3\cdot\begin{vmatrix}2&-1\\-1&4\end{vmatrix}$$

$$+(-1)^{1+3}\cdot 4\cdot\begin{vmatrix}2&1\\-1&2\end{vmatrix}=(4+2)-3\cdot(8-1)+$$

$$+4(4+1)=6-3\cdot 7+4\cdot(4+1)=6-21+4\cdot 5=$$

$$=6-21+20=6-1=\boxed{5}$$

$$\begin{vmatrix} 1 & 1 & 4 \\ 2 & 5 & -1 \\ -1 & -2 & 4 \end{vmatrix} = (-1)^{1+1}\cdot 1 \cdot \begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix} + (-1)^{1+2}\cdot 1 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$$

$$+ (-1)^{1+3}\cdot 4 \cdot \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} = (20-2) - (8-1) + 4(-4+5) =$$

$$= 18 - 7 + 4 = 18 - 3 = \boxed{15}$$

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 5 & 1 \\ -1 & -2 & 2 \end{vmatrix} = (-1)^{1+1}\cdot 1 \cdot \begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} + (-1)^{1+2}\cdot 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} +$$

$$+ (-1)^{1+3}\cdot 3 \cdot \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} = (10+2) - (4+1) + 3(-4+5) =$$

$$= 12 - 5 + 3 = \boxed{10}$$

SEMINAR 2

Apl 1 Calculați det A

a) folosind dezvoltarea după coloana 4

$$A = \begin{pmatrix} 1 & 3 & 0 & -2 \\ 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & -1 \\ -1 & 4 & 0 & 1 \end{pmatrix}$$

Rezolvare a) $\Delta = \det A = (-1)^{1+4} \cdot (-2) \cdot \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 4 & 0 \end{vmatrix} +$

$+ (-1)^{2+4} \cdot 3 \cdot \begin{vmatrix} 1 & 3 & 0 \\ 1 & 2 & 3 \\ -1 & 4 & 0 \end{vmatrix} + (-1)^{3+4} \cdot (-1) \cdot \begin{vmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \\ -1 & 4 & 0 \end{vmatrix} +$

$+ (-1)^{4+4} \cdot 1 \cdot \begin{vmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 4 & 0 \end{vmatrix} +$

$+ 3 \cdot \begin{vmatrix} 1 & 3 & 0 \\ 1 & 2 & 3 \\ -1 & 4 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \\ -1 & 4 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} =$

$= 2 \cdot (-33) + 3 \cdot (-21) + 7 - 16 = \boxed{-138}$

$\begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 4 & 0 \end{vmatrix} = (-1)^{1+1} \cdot 2 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}$

$+ (-1)^{1+3} \cdot (-1) \cdot \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 2 \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} -$

$$\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 2(0-12) + (0+3) + (4+2) = -24-3-6 = \boxed{-33}$$

$$\begin{vmatrix} 1 & 3 & 0 \\ 1 & 2 & 3 \\ -1 & 4 & 0 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} + (-1)^{1+2} \cdot 3 \cdot \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}$$

$$+ (-1)^{1+3} \cdot 0 \cdot \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 0 =$$

$$= -12 - 3 \cdot 3 = -12 - 9 = \boxed{-21}$$

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \\ -1 & 4 & 0 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} + (-1)^{1+2} \cdot 3 \cdot$$

$$\cdot \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} + (-1)^{1+3} \cdot 0 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix}$$

$$+ 0 = 4 - 3 \cdot (-1) = 4 + 3 = \boxed{7}$$

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + (-1)^{1+2} \cdot 3 \cdot \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}$$

$$+ (-1)^{1+3} \cdot 0 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + 0 =$$

$= (3+2) - 3 \cdot (6+1) = 5 - 3 \cdot 7 = 5 - 21 = \boxed{-16}$

Apl Fie $n \in \mathbb{N}, n \geq 2$, $a, x \in R$

Notăm cu $A_n(a,x) \in \mathcal{M}_n(\mathbb{R})$ cu proprietățile următoare:

1) are x pe orice poziție de pe diagonala principală;

2) are a pe orice altă poziție;

c) Calc $A_3(1,2)^{-1}$

Rezolvare c) $A_3(1,2)^{-1} = \dfrac{1}{\det A_3(1,2)} \cdot A_3(1,2)^*$

$A_3(1,2) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $\qquad$ $A_3(1,2) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 1 \cdot (4-1) = 3$ $\qquad$ Știm că $\det A_3(1,2) = 4$

$A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1 \cdot (2-1) = -1$

$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1$

$$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1-2 = -1$$

$$A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-3 = 3$$

$$A_3(1,2)^* = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$A_3(1,2)^{-1} = \frac{1}{4} \cdot \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Apl Să se rezolve

a) $\Delta_1 = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = 0$

Rezolvare a) $\Delta_1 \underset{L_1' \to L_1+L_2+L_3+L_4}{=} \begin{vmatrix} x+3a & x+3a & x+3a & x+3a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} =$

$= (x+3a) \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} =$

$C_2 = C_2 - C_1$
$C_3 = C_3 - C_1$
$C_4 = C_4 - C_1$

$(x+3a) \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & x-a & 0 & 0 \\ a & 0 & x-a & 0 \\ a & 0 & 0 & x-a \end{vmatrix} = (x+3a) \cdot (-1)^{1+1} \cdot 1 \cdot$

$\cdot \begin{vmatrix} x-a & 0 & 0 \\ 0 & x-a & 0 \\ 0 & 0 & x-a \end{vmatrix} = (x+3a) \cdot (x-a)^3$

Dar $\Delta_1 = 0$

$$\Rightarrow (x+3a)(x-a)^3 = 0$$

$$\begin{cases} x+3a=0 \\ x-a=0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -3a \\ x_2 = x_3 = x_4 = a \end{cases}$$

Apl Calculați determinantul de ordin n

$$\Delta_n = \begin{vmatrix} -1 & a & a & \dots & a \\ a & -1 & a & \dots & a \\ a & a & -1 & \dots & a \\ \dots & \dots & \dots & \dots & \dots \\ a & a & a & & -1 \end{vmatrix}$$

Rezolvare: $\Delta_n =$

$$\begin{matrix} L_2 = L_2 - L_1 \\ L_3 = L_3 - L_1 \\ \vdots \\ L_n = L_n - L_1 \end{matrix} \quad \begin{vmatrix} -1 & a & a & \dots & a \\ a+1 & -a-1 & 0 & \dots & 0 \\ a+1 & 0 & -a-1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a+1 & 0 & 0 & \dots & -a-1 \end{vmatrix} =$$

$$\underset{\substack{C_1 = C_1 + C_2 + \dots \\ + C_n}}{=} \begin{vmatrix} -1+(n-1)a & a & a & \dots & a \\ 0 & -a-1 & 0 & \dots & 0 \\ 0 & 0 & -a-1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & -a-1 \end{vmatrix} =$$

$$= -1+(n-1)a \cdot (-1)^{1+1} \cdot \begin{vmatrix} -a-1 & 0 & \dots & 0 \\ 0 & -a-1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -a-1 \end{vmatrix} =$$

$$= [-1+(n-1)a] \cdot (-a-1)^{n-1} = (-1)^{n-1} \cdot (a+1)^{n-1} \cdot$$

$$\cdot [-1+(n-1) \cdot a]$$

Apl Să se calculeze determinantul

b) $\Delta_2 = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ x_3 & x_1 & x_2 \end{vmatrix}$, unde x_1, x_2, x_3 sunt răd. ec. $x^3-2x^2+2x+17=0$

Rezolvare b) $\begin{cases} x_1+x_2+x_3 = -\frac{b}{a} = 2 \\ x_1x_2+x_2x_3+x_1x_3 = \frac{c}{a} = 2 \\ x_1x_2x_3 = -\frac{d}{a} = -17 \end{cases}$ Rel lui Viete

$$\Delta_2 = x_1x_2x_3 + x_1x_2x_3 + x_1x_2x_3 - x_3^3 - x_1^3 - x_2^3 = 3x_1x_2x_3 - (x_1+x_2+x_3)^3 + 3x_1x_2 + 3x_2x_3 + 3x_1x_3 = -\frac{3d}{a} - \left(-\frac{b}{a}\right)^3 + \frac{3c}{a} =$$

$= -51 - 8 + 6 = -53$

Apl Verificați egalitățile

a) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy + yz + zx)$

Rezolvare: a) $\begin{array}{|ccc|} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{array} =$
$\begin{array}{ccc} x & y & z \end{array}$

$(x-y)(y-z)(z-x)(xy + yz + zx) =$

$= (xy - xz - y^2 + yz)(z - x)(xy + yz + zx) =$

$= (\cancel{xyz} - x^2y - xz^2 + x^2z - y^2z + y^2x + yz^2 - \cancel{xyz})(xy + yz + zx) =$

$= (-x^2y - xz^2 + x^2z - y^2z + y^2x + yz^2)(xy + yz + zx) =$

$$= -x^3y^2 \cancel{- x^2y^2z} \cancel{- x^3yz} \cancel{- x^2yz^2} \cancel{- xyz^3} -$$

$$- x^2z^3 \cancel{+ x^3yz} \cancel{+ x^2yz^2} + x^3z^2 \cancel{- xy^3z} - y^3z^2$$

$$\cancel{- xy^2z^2} + x^2y^3 \cancel{+ xy^3z} \cancel{+ x^2y^2z} \cancel{+ xy^2z^2} +$$

$$+ y^2z^3 \cancel{+ xyz^3} = -x^3y^2 - x^2z^3 + x^3z^2 - y^3z^2$$

$$+ x^2y^3 + y^2z^3$$

$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (-1)^{1+1} \cdot x \cdot \begin{vmatrix} y^2 & z^2 \\ zx & xy \end{vmatrix} +$$

$$+ (-1)^{1+2} \cdot y \cdot \begin{vmatrix} x^2 & z^2 \\ yz & xy \end{vmatrix} + (-1)^{1+3} \cdot z \begin{vmatrix} x^2 & y^2 \\ yz & zx \end{vmatrix} =$$

$$= x \cdot \begin{vmatrix} y^2 & z^2 \\ zx & xy \end{vmatrix} - y \begin{vmatrix} x^2 & z^2 \\ yz & xy \end{vmatrix} + z \begin{vmatrix} x^2 & y^2 \\ yz & zx \end{vmatrix} =$$

$$= x(xy^3 - xz^3) - y(x^3y - yz^3) + z(x^3z -$$

$$- y^3z) = x^2y^3 - x^2z^3 - x^3y^2 + y^2z^3 + x^3z^2 -$$

$$- y^3z^2$$ (Adevărat)

$$\Rightarrow \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

(Adevărat)

b) $$\begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix} = 2abc(a-b)(b-c)(c-a)$$

Rezolvare: $2abc(a-b)(b-c)(c-a)=$

$= 2abc(ab-ac-b^2+bc)(c-a)=$

$=2abc(\cancel{abc}-a^2b-ac^2+a^2c-b^2c+ab^2$

$+bc^2-\cancel{abc})=-2a^3b^2c-2a^2bc^2+2a^3bc^2$

$-2ab^3c^2+2a^2b^3c+2ab^2c^3$

$$\Delta = \begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix} =$$

$$= (-1)^{1+1} \cdot (a+b) \cdot \begin{vmatrix} b^2+c^2 & c^2+a^2 \\ b^3+c^3 & c^3+a^3 \end{vmatrix} + (-1)^{1+2} \cdot (b+c) \cdot$$

$$\cdot \begin{vmatrix} a^2+b^2 & c^2+a^2 \\ a^3+b^3 & c^3+a^3 \end{vmatrix} + (-1)^{1+3} \cdot (c+a) \cdot \begin{vmatrix} a^2+b^2 & b^2+c^2 \\ a^3+b^3 & b^3+c^3 \end{vmatrix} =$$

$$= (a+b) \begin{vmatrix} b^2+c^2 & c^2+a^2 \\ b^3+c^3 & c^3+a^3 \end{vmatrix} - (b+c) \cdot \begin{vmatrix} a^2+b^2 & c^2+a^2 \\ a^3+b^3 & c^3+a^3 \end{vmatrix}$$

$$+ (c+a) \begin{vmatrix} a^2+b^2 & b^2+c^2 \\ a^3+b^3 & b^3+c^3 \end{vmatrix} = (a+b)\left[(b^2+c^2)(c^3+a^3) -\right.$$

$$\left.- (c^2+a^2)(b^3+c^3)\right] - (b+c)\left[(a^2+b^2)(c^3+a^3) -\right.$$

$$\left.- (c^2+a^2)(a^3+b^3)\right] + (c+a)\left[(a^2+b^2)(b^3+c^3) -\right.$$

$$\left.- (b^2+c^2)(a^3+b^3)\right] =$$

$$= 2ab^2c^3 - 2a^2bc^3 + 2a^3bc^2 + 2a^2b^3c - 2a^3b^2c$$

(Adevărat)

$$\Rightarrow \begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix} = 2abc(a-b)(b-c)(c-a)$$

a) $$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} \begin{matrix} \\ \\ \\ x & y & z \\ x^2 & y^2 & z^2 \end{matrix} = xy^2 \cdot xy + yz^2 \cdot yz + zx^2 \cdot xz - yzy^2z - xzz^2x - xyx^2y = x^2y^3 - x^3y^2 - x^2z^3 + y^2z^3 + x^3z^2 - y^3z^2$$

(Adevărat)

$$\Rightarrow \Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

(Adevărat)

b) $\begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix} = (a+b)(b^2+c^2)(a^3+c^3)+$

$\begin{matrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \end{matrix}$

$+(b+c)(a^2+c^2)\cdot(a^3+b^3)+$

$+(a+c)\cdot(a^2+b^2)\cdot(b^3+c^3)-$

$-(a^3+b^3)(b^2+c^2)\cdot(a+c)-$

$-(b^3+c^3)(a^2+c^2)(a+b)-$

$-(a^3+c^3)(a^2+b^2)(b+c) = 2ab^2c^3-2a^2bc^3-2ab^3c^2$

$+2a^3bc^2+2a^2b^3c-2a^3b^2c$ (Adevărat)

$$\Rightarrow \Delta = \begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix} = 2abc(a-b)(b-c)(c-a)$$

Apl Fie $A_n \in \mathcal{M}_n(\mathbb{R})$, $n \geq 2$

care are 1 pe diagonala principală, 3 pe pozițiile $(1,2), (2,3), \dots, (n-1, n), (n,1)$ și 0 pe restul pozițiilor. Notăm cu $\Delta_n = \det(A_n)$

a) Calculați Δ_3 și Δ_4

Rezolvare: a) $\Delta_3 = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{vmatrix} = 1 + 27 = 28$

$$\begin{matrix} 1 & 3 & 0 \\ 0 & 1 & 3 \end{matrix}$$

$$\Delta_4 = \begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \\ 3 & 0 & 0 & 1 \end{vmatrix} = \quad L_3 = L_3 - 3L_1$$

$$= \begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & -9 & 0 & 1 \end{vmatrix} = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ -9 & 0 & 1 \end{vmatrix} = \quad L_3 = L_3 + 9L_1$$

$$= \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 27 & 1 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 3 \\ 27 & 1 \end{vmatrix} = 1 - 81 = -80$$

b) Generalizați pentru Δ_n

Rezolvare: b)
$$\Delta_n = \begin{vmatrix} 1 & 3 & 0 & \dots & 0 \\ 0 & 1 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots 1 & 3 \\ 3 & 0 & 0 & \dots & 1 \end{vmatrix}$$

$$\underset{C_1}{\overset{\text{Dezv}}{=}} 1 \cdot (-1)^2 \cdot \underbrace{\begin{vmatrix} 1 & 3 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 3 \\ 0 & 0 & \dots & 1 \end{vmatrix}}_{\Delta'_{n-1}} + 3 \cdot (-1)^{n+1} \cdot \underbrace{\begin{vmatrix} 3 & 0 & \dots & 0 \\ 1 & 3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 3 \end{vmatrix}}_{\Delta''_{n-1}}$$

$$= \Delta'_{n-1} + (-1)^{n+1} \cdot 3 \cdot \Delta''_{n-1}$$

Dar $\Delta'_{n-1} = 1^{n-1} = 1$

$\Delta''_{n-1} = 3^{n-1}$

Deci $\boxed{\Delta_n = 1 + (-1)^{n+1} \cdot 3^n}$, $n \geq 2$

Apl Fie $A_n \in \mathcal{M}_n(\mathbb{R}), n \in \mathbb{N}, n \geq 3$ $\{\Delta_n = \det A_n\}$ a.î.

A_n are 1) 4 pe orice poziție de pe diagonala princip.

2) 3 pe pozițiile $(1,2), (2,3), \ldots, (n-1,n)$;

3) 1 pe pozițiile $(2,1), (3,2), \ldots, (n, n-1)$;

4) 0 pe toate celelalte poziții;

a) Determinați o relație de recurență

Rezolvare a) $\Delta_n = \begin{vmatrix} 4 & 3 & 0 & \ldots & 0 \\ 1 & 4 & 3 & \ldots & 0 \\ & 1 & 4 & 3 & 0 \\ & & \ddots & \ddots & 3 \\ 0 & & & 1 & 4 \end{vmatrix} \underset{L_1}{\overset{\text{Dezv}}{=}}$

$= 4 \cdot (-1)^2 \cdot \underbrace{\begin{vmatrix} 4 & 3 & \ldots & 0 \\ 1 & 4 & \ldots & 0 \\ 0 & 0 & \ddots & 3 \\ 0 & 0 & \ldots 1 & 4 \end{vmatrix}}_{\Delta_{n-1}} + 3 \cdot (-1)^3 \cdot \begin{vmatrix} 1 & 3 & \ldots & 0 \\ 0 & 4 & \ldots & 0 \\ 0 & & 4 & 3 \\ 0 & 0 & \ldots 1 & 4 \end{vmatrix}$

$= \boxed{4 \cdot \Delta_{n-1} - 3 \cdot \Delta_{n-2}, \ (\forall)\, n \geq 3}$

Relație de recurență

b) Calculați (efectiv) Δ_n, $n \geq 3$, $n \in \mathbb{N}$

Rezolvare b) $\Delta_n - 4\Delta_{n-1} + 3\Delta_{n-2} = 0$

$$r^2 - 4r + 3 = 0$$

$a = 1$ $\quad D = 16 - 4 \cdot 1 \cdot 3$

$b = -4$ $\quad \Delta = 4$ $\qquad \Delta_n = A \cdot r_1^n + B \cdot r_2^n$

$c = 3$ $\quad r_1 = \frac{4+2}{2} = \frac{6}{2} = 3$ $\qquad \Delta_n = A \cdot 3^n + B \cdot 1^n$

$\qquad\qquad = 3^n \cdot A + B$, $A, B \in \mathbb{R}$

$r_2 = \frac{4-2}{2} = \frac{2}{2} = 1$

$$\Delta_3 = \begin{vmatrix} 4 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 1 & 4 \end{vmatrix} = 40 = 27 A + B = 40 \quad (1)$$

$$\Delta_4 = \begin{vmatrix} 4 & 3 & 0 & 0 \\ 1 & 4 & 3 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 4 \end{vmatrix} = 121 = 81 A + B = 121 \quad (2)$$

$(1), (2) \Rightarrow 54 A = 81 \Rightarrow A = \frac{81}{54} = \frac{3}{2}$

$A = \frac{3}{2}$

$27 \cdot \frac{3}{2} + B = 40$

$\frac{81}{2} + B = 40$

$B = 40 - \frac{81}{2}$

$B = \frac{80-81}{2} = -\frac{1}{2}$

Deci $\Delta_n = \frac{3^{n+1}}{2} - \frac{1}{2} = \boxed{\frac{1}{2}\left(3^{n+1} - 1\right)}$ $(\forall) n \geq 3$

SEMINAR 3

Apl Rezolvați următoarele sisteme de ec. liniare, utilizând metoda eliminării Gauss Jordan

a) $\begin{cases} x+y-z=2 \\ 2x+y-3z=2 \\ x-y-z=0 \end{cases}$

Rezolvare a) $A=\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & -3 \\ 1 & -1 & -1 \end{pmatrix}$

$$A^e=\left(\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 2 \\ 2 & 1 & -3 & 2 \\ 1 & -1 & -1 & 0 \end{array}\right) \underset{\substack{L_2=L_2-2L_1 \\ L_3=L_3-L_1}}{\sim}$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & -2 & 0 & -2 \end{array}\right) \underset{L_2=-L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & -2 & 0 & -2 \end{array}\right)$$

$$\underset{L_3=L_3+2L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right) \underset{L_3=\frac{L_3}{2}}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \underset{\substack{L_2 = L_2 - L_3 \\ L_1 = L_1 + L_3}}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \underset{L_1 - L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{cases} x = 2 \\ y = 1 \\ z = 1 \end{cases}$$

$\text{rang}(A) = \text{rang}(A^e)$ SCD (sol. unică) $\Rightarrow S = \{(2,1,1)\}$

b) $\begin{cases} x + y + 2z = 1 \\ x + y + 3z = 1 \\ x + y - 2z = 1 \end{cases}$

Rezolvare $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & -2 \end{pmatrix} \Rightarrow A^e = \left(\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & -2 & 1 \end{array}\right)$

$$A^e \underset{\substack{L_2 = L_2 - L_1 \\ L_3 = L_3 - L_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \underset{L_3 = L_3 + 4L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \underset{L_1 = L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x+y=1 \\ z=0 \end{cases}$$

$$\begin{cases} x=1-y \\ y=y \\ z=0 \end{cases}$$

x, z nec principale

$y=\alpha \in \mathbb{R}$, nec. secundară

$S=\{(1-\alpha, \alpha, 0), \alpha \in \mathbb{R}\}$

$y=\alpha \in \mathbb{R}$

c) $\begin{cases} x+y+z+t=1 \\ 2x-y+z-t=2 \\ x-2y-2t=-1 \end{cases}$

Rezolvare $A=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & -1 \\ 1 & -2 & 0 & -2 \end{pmatrix}$

$$\bar{A}=\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & -1 & 2 \\ 1 & -2 & 0 & -2 & -1 \end{array}\right) \underset{\substack{L_2=L_2-2L_1 \\ L_3=L_3-L_1}}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -3 & 0 \\ 0 & -3 & -1 & -3 & -2 \end{array}\right) \underset{L_3=L_3-L_2}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right) \quad 0=-2$$

$\Rightarrow$ Nu există soluții

$\begin{cases} x+y+z+t=1 \\ -3y-z-3t=0 \\ 0=-2 \end{cases}$

d) $\begin{cases} x+2y+3z-2t=6 \\ 2x-y-2z-3t=8 \\ 3x+2y-z+2t=4 \\ 2x-3y+t=-8 \end{cases}$

Rezolvare: $A=\begin{pmatrix} 1 & 2 & 3 & -2 \\ 2 & -1 & -2 & -3 \\ 3 & 2 & -1 & 2 \\ 2 & -3 & 0 & 1 \end{pmatrix}$

$$A^e=\left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & -2 & 6 \\ 2 & -1 & -2 & -3 & 8 \\ 3 & 2 & -1 & 2 & 4 \\ 2 & -3 & 0 & 1 & -8 \end{array}\right) \sim \begin{array}{l} L_2=L_2-2L_1 \\ L_3=L_3-3L_1 \\ L_4=L_4-2L_1 \end{array}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & -5 & -8 & 1 & -4 \\ 0 & -4 & -10 & 8 & -14 \\ 0 & -7 & -6 & 5 & -20 \end{array}\right) \sim L_2=-\frac{L_2}{5}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & \boxed{1} & 8/5 & -1/5 & 4/5 \\ 0 & -4 & -10 & 8 & -14 \\ 0 & -7 & -6 & 5 & -20 \end{array}\right) \sim L_3=L_3+4L_2$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & \boxed{1} & 8/5 & -1/5 & 4/5 \\ 0 & 0 & -18/5 & 36/5 & -54/5 \\ 0 & -7 & -6 & 5 & -20 \end{array}\right) \underset{L_4 = L_4 + 7L_2}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & 1 & 8/5 & -1/5 & 4/5 \\ 0 & 0 & -18/5 & 36/5 & -54/5 \\ 0 & 0 & 26/5 & 18/5 & -72/5 \end{array}\right) \underset{L_3 = -\frac{18L_3}{5}}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & 1 & 8/5 & -1/5 & 4/5 \\ 0 & 0 & \boxed{1} & -2 & 3 \\ 0 & 0 & 26/5 & 18/5 & -72/5 \end{array}\right) \underset{L_4 = L_4 - \frac{26}{5}L_3}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & 1 & 8/5 & -1/5 & 4/5 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 14 & -30 \end{array}\right) \underset{L_4 = \frac{L_4}{14}}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & 1 & 8/5 & -1/5 & 4/5 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & \boxed{1} & -15/7 \end{array}\right) \underset{L_3 = L_3 + 2L_4}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & 1 & 8/5 & -1/5 & 4/5 \\ 0 & 0 & 1 & 0 & -9/7 \\ 0 & 0 & 0 & \boxed{1} & -15/7 \end{array}\right) \underset{L_2 = L_2 + \frac{1}{5} L_4}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & 1 & 8/5 & 0 & 13/35 \\ 0 & 0 & 1 & 0 & -9/7 \\ 0 & 0 & 0 & \boxed{1} & -15/7 \end{array}\right) \underset{L_1 = L_1 + 2L_4}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 12/7 \\ 0 & 1 & 8/5 & 0 & 13/35 \\ 0 & 0 & 1 & 0 & -9/7 \\ 0 & 0 & 0 & 1 & -15/7 \end{array}\right) \underset{L_2 = L_2 - \frac{8}{5} L_3}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 12/7 \\ 0 & 1 & 0 & 0 & 17/7 \\ 0 & 0 & 1 & 0 & -9/7 \\ 0 & 0 & 0 & 1 & -15/7 \end{array}\right) \underset{L_1 = L_1 - 3L_3}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 39/7 \\ 0 & 1 & 0 & 0 & 17/7 \\ 0 & 0 & 1 & 0 & -9/7 \\ 0 & 0 & 0 & 1 & -15/7 \end{array}\right) \underset{L_1 = L_1 - 2L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5/7 \\ 0 & 1 & 0 & 0 & 17/7 \\ 0 & 0 & 1 & 0 & -9/7 \\ 0 & 0 & 0 & 1 & -15/7 \end{array}\right)$$

$$\begin{cases} x = \frac{5}{7} \\ y = \frac{17}{7} \\ z = -\frac{9}{7} \\ t = -\frac{15}{7} \end{cases}$$

S.C.D. (sol. unică) $\Rightarrow S = \left\{\left(\frac{5}{7}, \frac{17}{7}, -\frac{9}{7}, -\frac{15}{7}\right)\right\}$

Apl Rezolvați următoarele sist. de ec. liniare omogene, utilizând met. eliminării Gauss Jordan:

a) $\begin{cases} x + y - z = 0 \\ 2x + y - 3z = 0 \\ x - y - z = 0 \end{cases}$

Rezolvare: $A^e = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & -3 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$

$$A^{e} \underset{\substack{L_2=L_2-2L_1 \\ L_3=L_3-L_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right) \underset{L_2=-L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right) \underset{L_3=L_3+2L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \underset{L_3=\frac{L_3}{2}}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\underset{L_2=L_2-L_3}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \underset{L_1=L_1-L_2+L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{cases} x=0 \\ y=0 \\ z=0 \end{cases} \quad \text{S.C.D.} \quad S=\{(0,0,0)\}$$

b) $\begin{cases} x+y-z+t=0 \\ x-y+z+t=0 \\ 2x+y+2z-t=0 \end{cases}$

Rezolvare $A^e = \left(\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 2 & 1 & 2 & -1 & 0 \end{array}\right)$

$$A^e \underset{\substack{L_2 = L_2 - L_1 \\ L_3 = L_3 - 2L_1}}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & -1 & 4 & -3 & 0 \end{array}\right) \underset{L_2 = -\frac{L_2}{2}}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 & 0 \\ 0 & -1 & 4 & -3 & 0 \end{array}\right) \underset{L_3 = L_3 + L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & \boxed{3} & -3 & 0 \end{array}\right) \underset{L_3 = \frac{L_3}{3}}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \end{array}\right) \underset{L_2 = L_2 + L_3}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \underset{L_1=L_1+L_3-L_2}{\sim}$$

$$\sim\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

x, y, z nec. princip.
t nec sec, $t=\alpha \in \mathbb{R}$

$$\begin{cases} x + t = 0 \Rightarrow x = -\alpha \\ y - t = 0 \Rightarrow y = \alpha \\ z - t = 0 \Rightarrow z = \alpha \\ t = \alpha \in \mathbb{R} \end{cases}$$

$S = \{(-\alpha, \alpha, \alpha, \alpha), \alpha \in \mathbb{R}\}$

Apl Determinați inversele următoarelor matrice, utilizând metoda Gauss-Jordan (eliminării complete):

a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

Rezolvare $(A \mid I_2) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \underset{L_1=\frac{L_1}{2}}{\sim}$

$$\sim\left(\begin{array}{cc|cc} \boxed{1} & 1/2 & 1/2 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \underset{L_2=L_2-L_1}{\sim} \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 5/2 & -1/2 & 1 \end{array}\right) \sim$$

$$\sim_{L_2=\frac{2L_2}{5}} \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & \boxed{1} & -1/5 & 2/5 \end{array}\right) \sim_{L_1=L_1-\frac{1}{2}L_2}$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 3/5 & -1/5 \\ 0 & 1 & -1/5 & 2/5 \end{array}\right)$$

Deci $A^{-1} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}$

b) $B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

Rezolvare

$$(B|I_3) = \left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim_{L_1=\frac{L_1}{2}}$$

$$\sim \left(\begin{array}{ccc|ccc} \boxed{1} & -1/2 & 1/2 & 1/2 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \begin{array}{l} L_2=L_2-L_1 \\ L_3=L_2-L_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 3/2 & -1/2 & 1 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 0 & 1 \end{array}\right) \underset{L_2 = \frac{2L_2}{3}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & \boxed{1} & 1 & -1/3 & 2/3 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 0 & 1 \end{array}\right) \underset{L_3 = L_3 - \frac{3}{2}L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -1/3 & 2/3 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right) \underset{L_3 = -L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -1/3 & 2/3 & 0 \\ 0 & 0 & \boxed{1} & 0 & 1 & -1 \end{array}\right) \underset{L_2 = L_2 - L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right) \underset{L_1 = L_1 - \frac{1}{2}L_3 + \frac{1}{2}L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -2/3 & 1 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

Deci $A^{-1} = \begin{pmatrix} 1/3 & -2/3 & 1 \\ -1/3 & -1/3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

c) $C = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix}$

Rezolvare:

$$(C/I_3) = \left(\begin{array}{ccc|ccc} \boxed{1} & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \underset{\substack{L_2 = L_2 - 2L_1 \\ L_3 = L_3 - L_1}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & -8 & -2 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right) \underset{L_2 = -L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & 8 & 2 & -1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right) \underset{L_3 = L_3 + 2L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 8 & 2 & -1 & 0 \\ 0 & 0 & 17 & 3 & -2 & 1 \end{array}\right) \underset{L_3 = \frac{L_3}{17}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 8 & 2 & -1 & 0 \\ 0 & 0 & \boxed{1} & 3/17 & -2/17 & 1/17 \end{array}\right) \underset{L_2 = L_2 - 8L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 10/17 & -1/17 & -8/17 \\ 0 & 0 & 1 & 3/17 & -2/17 & 1/17 \end{array}\right) \underset{L_1 = L_1 - L_2 - 3L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/17 & 7/17 & 5/17 \\ 0 & 1 & 0 & 10/17 & -1/17 & -8/17 \\ 0 & 0 & 1 & 3/17 & -2/17 & 1/17 \end{array}\right)$$

Deci $A^{-1} = \begin{pmatrix} -2/17 & 7/17 & 5/17 \\ 10/17 & -1/17 & -8/17 \\ 3/17 & -2/17 & 1/17 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} -2 & 7 & 5 \\ 10 & -1 & -8 \\ 3 & -2 & 1 \end{pmatrix}$

Apl Determinați (dacă există) inversele următoarelor matrice pătratice, utilizând metoda Gauss-Jordan:

a) $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 3 & 2 \\ 4 & 3 & -2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

Rezolvare

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 1 & 0 \\ 4 & 3 & -2 & 0 & 0 & 1 \end{array}\right) \underset{\substack{L_2 = L_2 + L_1 \\ L_3 = L_3 - 4L_1}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & -1 & -10 & -4 & 0 & 1 \end{array}\right) \underset{L_2 = \frac{L_2}{4}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 1/4 & 1/4 & 0 \\ 0 & -1 & -10 & -4 & 0 & 1 \end{array}\right) \underset{L_3 = L_3 + L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/4 & 1/4 & 0 \\ 0 & 0 & -9 & -15/4 & 1/4 & 1 \end{array}\right) \underset{L_3 = -\frac{L_3}{9}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/4 & 1/4 & 0 \\ 0 & 0 & \boxed{1} & 5/12 & -1/36 & -1/9 \end{array}\right) \underset{L_2 = L_2 - L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & 5/18 & 1/9 \\ 0 & 0 & 1 & 5/12 & -1/36 & -1/9 \end{array}\right) \underset{L_1 = L_1 - L_2 - 2L_3}{\sim}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -2/9 & 1/9 \\ 0 & 1 & 0 & -1/6 & 5/18 & 1/9 \\ 0 & 0 & 1 & 5/12 & -1/36 & -1/9 \end{array}\right)$$

Deci $A^{-1} = \begin{pmatrix} 1/3 & -2/9 & 1/9 \\ -1/6 & 5/18 & 1/9 \\ 5/12 & -1/36 & -1/9 \end{pmatrix}$

SEMINAR 4

Apl Determinați valoarea parametrului real m a.î. S.V. următor să fie: $S = \{v_1 = (1,2,3), v_2 = (1,m,1), v_3 = (2,0,-1)\} \subset \mathbb{R}^3/\mathbb{R}$

a) liniar dependent

$\boxed{m = 1}$

Rezolvare:

a) S.V. liniar dependent

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & m & 0 \\ 3 & 1 & -1 \end{vmatrix} = 0 \Rightarrow -m + 4 - 6m + 2 = 0$$

$$-7m + 6 = 0 \Rightarrow m = \frac{6}{7}$$

b) liniar independent

Rezolvare: b) SV liniar independent

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & m & 0 \\ 3 & 1 & -1 \end{vmatrix} \neq 0 \Rightarrow -7m+6 \neq 0$$

$$\Rightarrow m \neq \frac{6}{7} \Rightarrow$$

$$\Rightarrow m \in \left(-\infty, \frac{6}{7}\right) \cup \left(\frac{6}{7}, +\infty\right)$$

Apl Stabiți dacă următoarele S.V. sunt sisteme de generatori pentru sp. vectoriale din care fac parte:

c) $S_3 = \{ v_1 = (1,1,0),\ v_2 = (1,0,1),\ v_3 = (0,1,1) \} \subset \mathbb{R}^3/\mathbb{R}$

Rezolvare c) $(\forall)\ a, b, c \in \mathbb{R}^3$

$(\forall)\ x \in \mathbb{R}^3,\ (x_1, x_2, x_3) = a(1,1,0) + b(1,0,1) + c(0,1,1) \Rightarrow (x_1, x_2, x_3) = (a+b, a+c, b+c)$

$$\begin{cases} a+b=x_1 \\ a+c=x_2 \\ b+c=x_3 \end{cases} \quad A=\left(\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 1 & 0 & 1 & x_2 \\ 0 & 1 & 1 & x_3 \end{array}\right)$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1-1=-2 \neq 0$$

rang A = rang $\bar{A} \Rightarrow (\exists)\, a, b, c \in \mathbb{R}$ a.î.

$\langle S_3 \rangle = \mathbb{R}^3$ (este sistem de generatori)

d) $S_4 = \{v_1 = 1,\ v_2 = x-1,\ v_3 = (x-1)^2\} \subset \mathbb{R}_2[x]$

Rezolvare: d) $(\forall) p \in \mathbb{R}_2[x]$, $(\exists)\, a, b, c$?

$$p = a \cdot 1 + b \cdot (x-1) + c(x-1)^2 \quad (1)$$

$$(\forall)\, \underset{\mathbb{R}_2[x]}{\underset{\cap}{p}} = \mu x^2 + \beta x + \alpha,\ (\forall) \mu, \beta, \alpha \in \mathbb{R}$$

$(1) \Rightarrow p = a + b \cdot x - b + cx^2 - 2cx + c =$

$= cx^2 + (b-2c)x + (a-b+c)$

$$\begin{cases} c = \mu \\ b - 2c = \beta \\ a - b + c = \alpha \end{cases} \qquad A = \left(\begin{array}{ccc|c} 0 & 0 & 1 & \mu \\ 0 & 1 & -2 & \beta \\ 1 & -1 & 1 & \alpha \end{array}\right)$$

$\det A = -1 \neq 0 \Rightarrow \text{rang } A = \text{rang } \bar{A} \Rightarrow \langle S_1 \rangle = \mathbb{R}^2[x]$

este sistem de generatori

SEMINAR 5

Apl Fie $U = \{(x, y, z) \in \mathbb{R}^3 / -x + 3y + z = 0\} \subset \mathbb{R}^3/\mathbb{R}$

a) Stabiliți dacă $U \underset{\text{sp. vect.}}{\subset} \mathbb{R}^3$

Rezolvare a) $U = \{(x, y, z) \mid -x + 3y + z = 0\} \subset \mathbb{R}^3/\mathbb{R}$

Dacă U e spațiu vectorial, atunci:

$$\left.\begin{array}{l} (\forall)\, x, y \in U \\ (\forall)\, a, b \in \mathbb{R} \end{array}\right| \Rightarrow ax + by \in U$$

Verificăm:

$a(x_1, y_1, z_1) + b(x_2, y_2, z_2) =$

$=(ax_1, ay_1, az_1)+(bx_2, by_2, bz_2)=$

$=(ax_1+bx_2, ay_1+by_2, az_1+bz_2) \in U?$

$-ax_1-bx_2+3ay_1+3by_2+az_1+bz_2=0$

$\Leftrightarrow a\underbrace{(-x_1+3y_1+z_1)}_{=0,\text{ fiindcă }(x_1,y_1,z_1)\in U}+b\underbrace{(-x_2+3y_2+z_2)}_{=0}=0$

$\Rightarrow U$ este spațiu vectorial

b) Determinați $\dim_{\mathbb{R}} U$.

Rezolvare: b) $A=(-1 \quad 3 \quad 1)$ din ec.

$U=S(A) \Rightarrow \dim U = 3-\text{rang } A = 3-1 = 2$

$\Rightarrow \dim_{\mathbb{R}} U = 2$

SEMINAR 6

Apl Fie $V_1 = \{(x, y, 0) / x, y \in \mathbb{R}\}$ sp. vectorial
$V_2 = \{(u, 0, v) / u, v \in \mathbb{R}\}$

a) Arătați că $V_2 \subset \mathbb{R}^3$ și precizați dim V_2

Rezolvare: a) V_2 spațiu vectorial $\Leftrightarrow (\forall)\ x, y \in V_2$
$(\forall)\ a, b \in \mathbb{R}$ $\Rightarrow$

$\Rightarrow a x + b y \in V_2$

Fie $(u_1, 0, v_1),\ (u_2, 0, v_2) \in V_2$

$a(u_1, 0, v_1) + b(u_2, 0, v_2) = (a u_1 + b u_2, 0,$
$a v_1 + v_2) \in V_2 \Leftrightarrow \begin{cases} a u_1 + b u_2 \in \mathbb{R} & (1) \\ a v_1 + b v_2 \in \mathbb{R} & (2) \end{cases}$

(1) și (2) sunt adevărate, fiindcă $u_2, v_1, v_2 \in \mathbb{R}$
$\Rightarrow V_2$ spațiu vectorial

dim $V_2 = |B|$, unde B este bază în V_2

$V_2 = \{u(1,0,0) + v(0,0,1) \mid u, v \in \mathbb{R}\}$

$\Rightarrow B = \{(1,0,0), (0,0,1)\}$ este Sistem de generatori pt. V_2

Rang $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = 1$ (nu e rang maxim) $\Rightarrow B$ este spațiu vectorial liniar dependent

$B' = \{(0,0,1)\}$ este spațiu vectorial liniar independent și Sistem de generatori $\Rightarrow B'$ este bază

Deci, $\dim V_2 = 1$

b) Dem că $V_1 + V_2 = \mathbb{R}^3$ (E adev și rel $V_1 \oplus V_2 = \mathbb{R}^3$?)

Rezolvare: b) T. Grassman

$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

$V_1 = \{x(1,0,0) + y(0,1,0) / x, y \in \mathbb{R}\}$

$\Rightarrow B_2 = \{(1,0,0), (0,1,0)\}$ este Sistem de generatori pt. V_1

rang $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = 2$ (maxim) $\Rightarrow B_2$ este spațiu vectorial liniar independent

B_2 este bază în V_1

$\dim V_1 = |B_2| = 2$

$\dim(V_1 \oplus V_2)$:

$$\begin{array}{ccc} \underbrace{}_{B_2} & & \underbrace{}_{B'} \end{array}$$

$$V_1 \cap V_2 = O_{\mathbb{R}^3} \Leftrightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

Adevărat

$B = B_2 \cup B'$ este și bază canonică în $\mathbb{R}^3$

$\dim(V_1 \oplus V_2) = 1 + 2 - 0 = 3$

$V_1 + V_2 = \mathbb{R}^3 = V_1 \oplus V_2$, fiindcă $V_1 \cap V_2 = O_{\mathbb{R}^3}$

Apl Fie $V_1 = \{A \in \mathcal{M}_n(\mathbb{R}) \mid Tr\,A = 0\}$

$V_2 = \{A \in \mathcal{M}_n(\mathbb{R}) \mid A = \lambda I_n, \lambda \in \mathbb{R}\}$

a) Ar. că: $V_1, V_2 \subset \mathcal{M}_n(\mathbb{R})$ spații vectoriale

Rezolvare: a) V_1 spațiu vectorial $\Leftrightarrow (\forall)\, A, B \in V_1$, $(\forall)\, a, b \in \mathbb{R}$ $\Rightarrow$

$\Rightarrow a \cdot A + b \cdot B \in V_1$

$$a \cdot \begin{matrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{matrix} + b \cdot \begin{matrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{matrix} =$$

$$= \begin{matrix} a \cdot a_{11} + b \cdot b_{11} & \dots & a \cdot a_{1n} + b \cdot b_{1n} \\ \vdots & \ddots & \vdots \\ a \cdot a_{n1} + b \cdot b_{n1} & \dots & a\, a_{nn} + b \cdot b_{nn} \end{matrix} \in V_1$$

$\Leftrightarrow a \cdot a_{11} + b \cdot b_{11} + a \cdot a_{22} + b \cdot b_{22} + \dots + a \cdot a_{nn} +$

$+ b \cdot b_{nn} = 0 \Rightarrow a(\underbrace{a_{11} + a_{22} + \dots + a_{nn}}_{\substack{= 0 \\ (Tr A = 0)}}) +$

$+ b(\underbrace{b_{11} + \dots + b_{nn}}_{= 0}) = 0$ (Adevărat)

V_2 spațiu vectorial $\Leftrightarrow \left. \begin{matrix} (\forall)\, A, B \in V_2 \\ (\forall)\, a, b \in R \end{matrix} \right| \Rightarrow a \cdot A + b \cdot B \in V_2$

V_2 spațiu vectorial $\Leftrightarrow \left.\begin{matrix}(\forall) A, B \in V_2 \\ (\forall) a, b \in \mathbb{R}\end{matrix}\right| \Rightarrow a \cdot A + b \cdot B \in V_2$

Fie $(\forall) d_1, d_2 \in \mathbb{R}$

$$a \cdot \begin{matrix} d_1 & \dots & \dots & 0 \\ 0 & d_1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & d_1 \end{matrix} + b \cdot \begin{matrix} d_2 & \dots & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & d_2 \end{matrix} =$$

$$= \begin{matrix} a \cdot d_1 + b \cdot d_2 & 0 & \dots & 0 \\ 0 & a d_1 + b d_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 \dots & a d_1 + b d_2 \end{matrix} \in V_2$$

fiindcă $a d_1 + b d_2 \in \mathbb{R}$ și fiindcă $a, b, d_1, d_2 \in \mathbb{R}$

Matricea rezultat e de forma λI_n, $\lambda \in \mathbb{R}$

b) Dem. că $V_1 \oplus V_2 = \mathcal{M}_n(\mathbb{R})$, c) Verificați teorema dimensiunii în acest caz

Rezolvare: b), c) $\dim V_1$

Pt $n=2$ avem

$$B = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \underset{u_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{u_3}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \right\}$$

bază în V_2

$\dim V_1 = \dim \mathcal{M}_n(\mathbb{R}) - 1 = \dim \mathbb{R}^n \times \mathbb{R}^n - 1 = n^2 - 1$

Nu avem $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ în baza B pentru că $Tr(A) = 0$, presupunem ultimul element de pe diagonala să depindă de precedentele, adică A să fie de forma

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{matrix} \quad \text{sau} \quad \begin{matrix} 0 & a_{12} \\ a_{21} & 0 \end{matrix}$$

sau pt $n=3$

$$\begin{matrix} a & a_{12} & a_{13} \\ a_{21} & b & a_{23} \\ a_{31} & a_{32} & -a-b \end{matrix}$$

Pe diag. principală putem avea maxim $n-1$ elemente independente, dacă sunt egale cu zero mai multe elemente, tot le includem în bază (dar a, b sau c care se înmulțesc cu vectorul dat în bază vor fi 0)

Am stabilit forma unei baze în V_1.

Acum vom arăta cum arată o bază în V_2.

$$B_2 = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{= v_2} \right\} \text{ pt } n=2 \quad \dim V_2 = 1$$

sau

$$B_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \text{ pt } n=3$$

$\dim (V_1 \cap V_2) \neq 0 \Leftrightarrow \quad v_2 \in B$ sau Fals

$u_1, u_2, u_3 \in B_2$

$\dim (V_1 \cap V_2) = 0$

$\dim (V_1 + V_2) = n^2 - 1 + 1 - 0 = n^2 = \dim \mathcal{M}_n(\mathbb{R})$

$\Rightarrow V_1 \oplus V_2 = \mathcal{M}_n(\mathbb{R})$