Chapter

5 Eigenvalues and Eigenvectors

5.1 INTRODUCTION

This chapter begins the "second half" of linear algebra. The first half was about $Ax = b$. The new problem $Ax = \lambda x$ will still be solved by simplifying a matrix—making it diagonal if possible. *The basic step is no longer to subtract a multiple of one row from another.* Elimination changes the eigenvalues, which we don't want.

Determinants give a transition from $Ax = b$ to $Ax = \lambda x$. In both cases the determinant leads to a "formal solution": to Cramer's rule for $x = A^{-1}b$, and to the polynomial $\det(A - \lambda I)$, whose roots will be the eigenvalues. (All matrices are now square; the eigenvalues of a rectangular matrix make no more sense than its determinant.) The determinant can actually be used if $n = 2$ or 3. For large n, computing λ is more difficult than solving $Ax = b$.

The first step is to understand how eigenvalues can be useful. One of their applications is to ordinary differential equations. We shall not assume that the reader is an expert on differential equations! If you can differentiate x^n, $\sin x$, and e^x, you know enough. As a specific example, consider the coupled pair of equations

$$\begin{aligned} \frac{dv}{dt} &= 4v - 5w, \quad v = 8 \quad \text{at} \quad t = 0, \\ \frac{dw}{dt} &= 2v - 3w, \quad w = 5 \quad \text{at} \quad t = 0. \end{aligned} \tag{1}$$

This is an ***initial-value problem***. The unknown is specified at time $t = 0$ by the given initial values 8 and 5. The problem is to find $v(t)$ and $w(t)$ for later times $t > 0$.

It is easy to write the system in matrix form. Let the unknown vector be $u(t)$, with initial value $u(0)$. The coefficient matrix is A:

Vector unknown $$u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}, \quad u(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}.$$

The two coupled equations become the vector equation we want:

Matrix form $$\frac{du}{dt} = Au \quad \text{with} \quad u = u(0) \text{ at } t = 0. \tag{2}$$

This is the basic statement of the problem. Note that it is a first-order equation—no higher derivatives appear—and it is *linear* in the unknowns. It also has *constant coefficients*; the matrix A is independent of time.

How do we find $u(t)$? If there were only one unknown instead of two, that question would be easy to answer. We would have a scalar instead of a vector equation:

$$\textbf{Single equation} \qquad \frac{du}{dt} = au \quad \text{with} \quad u = u(0) \text{ at } t = 0. \tag{3}$$

The solution to this equation is the one thing you need to know:

$$\textbf{Pure exponential} \qquad u(t) = e^{at}u(0). \tag{4}$$

At the initial time $t = 0$, u equals $u(0)$ because $e^0 = 1$. The derivative of e^{at} has the required factor a, so that $du/dt = au$. Thus the initial condition and the equation are both satisfied.

Notice the behavior of u for large times. The equation is unstable if $a > 0$, neutrally stable if $a = 0$, or stable if $a < 0$; the factor e^{at} approaches infinity, remains bounded, or goes to zero. If a were a complex number, $a = \alpha + i\beta$, then the same tests would be applied to the real part α. The complex part produces oscillations $e^{i\beta t} = \cos \beta t + i \sin \beta t$. Decay or growth is governed by the factor $e^{\alpha t}$.

So much for a single equation. We shall take a direct approach to systems, and look for solutions with the *same exponential dependence on t* just found in the scalar case:

$$\begin{aligned} v(t) &= e^{\lambda t}y \\ w(t) &= e^{\lambda t}z \end{aligned} \tag{5}$$

or in vector notation

$$u(t) = e^{\lambda t}x. \tag{6}$$

This is the whole key to differential equations $du/dt = Au$: ***Look for pure exponential solutions***. Substituting $v = e^{\lambda t}y$ and $w = e^{\lambda t}z$ into the equation, we find

$$\begin{aligned} \lambda e^{\lambda t}y &= 4e^{\lambda t}y - 5e^{\lambda t}z \\ \lambda e^{\lambda t}z &= 2e^{\lambda t}y - 3e^{\lambda t}z. \end{aligned}$$

The factor $e^{\lambda t}$ is common to every term, and can be removed. This cancellation is the reason for assuming the same exponent λ for both unknowns; it leaves

$$\textbf{Eigenvalue problem} \qquad \begin{aligned} 4y - 5z &= \lambda y \\ 2y - 3z &= \lambda z. \end{aligned} \tag{7}$$

That is the eigenvalue equation. In matrix form it is $Ax = \lambda x$. You can see it again if we use $u = e^{\lambda t}x$—a number $e^{\lambda t}$ that grows or decays times a fixed vector x. ***Substituting into*** $du/dt = Au$ ***gives*** $\lambda e^{\lambda t}x = Ae^{\lambda t}x$. ***The cancellation of*** $e^{\lambda t}$ ***produces***

$$\textbf{Eigenvalue equation} \qquad Ax = \lambda x. \tag{8}$$

Now we have the fundamental equation of this chapter. It involves two unknowns λ and x. It is an algebra problem, and differential equations can be forgotten! The number λ (lambda) is an ***eigenvalue*** of the matrix A, and the vector x is the associated ***eigenvector***. Our goal is to find the eigenvalues and eigenvectors, λ's and x's, and to use them.

The Solutions of $Ax = \lambda x$

Notice that $Ax = \lambda x$ is a nonlinear equation; λ multiplies x. If we could discover λ, then the equation for x would be linear. In fact we could write $\lambda I x$ in place of λx, and bring this term over to the left side:

$$(A - \lambda I)x = 0. \tag{9}$$

The identity matrix keeps matrices and vectors straight; the equation $(A - \lambda)x = 0$ is shorter, but mixed up. This is the key to the problem:

The vector x is in the nullspace of $A - \lambda I$.
The number λ is chosen so that $A - \lambda I$ has a nullspace.

Of course every matrix has a nullspace. It was ridiculous to suggest otherwise, but you see the point. We want a *nonzero* eigenvector x. The vector $x = 0$ always satisfies $Ax = \lambda x$, but it is useless in solving differential equations. The goal is to build $u(t)$ out of exponentials $e^{\lambda t}x$, and *we are interested only in those particular values λ for which there is a nonzero eigenvector x*. To be of any use, the nullspace of $A - \lambda I$ must contain vectors other than zero. In short, $A - \lambda I$ ***must be singular***.

For this, the determinant gives a conclusive test.

5A The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

$$\det(A - \lambda I) = 0. \tag{10}$$

This is the characteristic equation. Each λ is associated with eigenvectors x:

$$(A - \lambda I)x = 0 \quad \text{or} \quad Ax = \lambda x. \tag{11}$$

In our example, we shift A by λI to make it singular:

$$\textbf{Subtract } \boldsymbol{\lambda I} \qquad A - \lambda I = \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix}.$$

Note that λ is subtracted only from the main diagonal (because it multiplies I).

$$\textbf{Determinant} \qquad |A - \lambda I| = (4-\lambda)(-3-\lambda) + 10 \quad \text{or} \quad \lambda^2 - \lambda - 2.$$

This is the *characteristic polynomial.* Its roots, where the determinant is zero, are the eigenvalues. They come from the general formula for the roots of a quadratic, or from factoring into $\lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$. That is zero if $\lambda = -1$ or $\lambda = 2$, as the general formula confirms:

$$\textbf{Eigenvalues} \qquad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{9}}{2} = -1 \text{ and } 2.$$

There are two eigenvalues, because a quadratic has two roots. Every 2 by 2 matrix $A - \lambda I$ has λ^2 (and no higher power of λ) in its determinant.

The values $\lambda = -1$ and $\lambda = 2$ lead to a solution of $Ax = \lambda x$ or $(A - \lambda I)x = 0$. A matrix with zero determinant is singular, so there must be nonzero vectors x in its nullspace. In fact the nullspace contains a whole *line* of eigenvectors; it is a subspace!

$$\lambda_1 = -1: \qquad (A - \lambda_1 I)x = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution (the first eigenvector) is any nonzero multiple of x_1:

$$\textbf{Eigenvector for } \boldsymbol{\lambda_1} \qquad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The computation for λ_2 is done separately:

$$\lambda_2 = 2: \qquad (A - \lambda_2 I)x = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The second eigenvector is any nonzero multiple of x_2:

$$\textbf{Eigenvector for } \boldsymbol{\lambda_2} \qquad x_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

You might notice that the columns of $A - \lambda_1 I$ give x_2, and the columns of $A - \lambda_2 I$ are multiples of x_1. This is special (and useful) for 2 by 2 matrices.

In the 3 by 3 case, I often set a component of x equal to 1 and solve $(A - \lambda I)x = 0$ for the other components. Of course if x is an eigenvector then so is $7x$ and so is $-x$. All vectors in the nullspace of $A - \lambda I$ (which we call the *eigenspace*) will satisfy $Ax = \lambda x$. In our example the eigenspaces are the lines through $x_1 = (1, 1)$ and $x_2 = (5, 2)$.

Before going back to the application (the differential equation), we emphasize the steps in solving $Ax = \lambda x$:

1. ***Compute the determinant of*** $\boldsymbol{A - \lambda I}$. With λ subtracted along the diagonal, this determinant is a polynomial of degree n. It starts with $(-\lambda)^n$.
2. ***Find the roots of this polynomial.*** The n roots are the eigenvalues of A.
3. ***For each eigenvalue solve the equation*** $\boldsymbol{(A - \lambda I)x = 0}$. Since the determinant is zero, there are solutions other than $x = 0$. Those are the eigenvectors.

In the differential equation, this produces the special solutions $u = e^{\lambda t}x$. They are the *pure exponential solutions* to $du/dt = Au$. Notice e^{-t} and e^{2t}:

$$u(t) = e^{\lambda_1 t}x_1 = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad u(t) = e^{\lambda_2 t}x_2 = e^{2t}\begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

These two special solutions give the complete solution. They can be multiplied by any numbers c_1 and c_2, and they can be added together. When u_1 and u_2 satisfy the linear equation $du/dt = Au$, so does their sum $u_1 + u_2$:

$$\textbf{Complete solution} \qquad u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 \tag{12}$$

This is ***superposition***, and it applies to differential equations (homogeneous and linear) just as it applied to matrix equations $Ax = 0$. The nullspace is always a subspace, and combinations of solutions are still solutions.

Now we have two free parameters c_1 and c_2, and it is reasonable to hope that they can be chosen to satisfy the initial condition $u = u(0)$ at $t = 0$:

$$\textbf{Initial condition} \qquad c_1 x_1 + c_2 x_2 = u(0) \quad \text{or} \quad \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}. \tag{13}$$

The constants are $c_1 = 3$ and $c_2 = 1$, and ***the solution to the original equation is***

$$u(t) = 3e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}. \tag{14}$$

Writing the two components separately, we have $v(0) = 8$ and $w(0) = 5$:

$$\textbf{Solution} \qquad v(t) = 3e^{-t} + 5e^{2t}, \qquad w(t) = 3e^{-t} + 2e^{2t}.$$

The key was in the eigenvalues λ and eigenvectors x. Eigenvalues are important in themselves, and not just part of a trick for finding u. Probably the homeliest example is that of soldiers going over a bridge.* Traditionally, they stop marching and just walk across. If they happen to march at a frequency equal to one of the eigenvalues of the bridge, it would begin to oscillate. (Just as a child's swing does; you soon notice the natural frequency of a swing, and by matching it you make the swing go higher.) An engineer tries to keep the natural frequencies of his bridge or rocket away from those of the wind or the sloshing of fuel. And at the other extreme, a stockbroker spends his life trying to get in line with the natural frequencies of the market. The eigenvalues are the most important feature of practically any dynamical system.

Summary and Examples

To summarize, this introduction has shown how λ and x appear naturally and automatically when solving $du/dt = Au$. Such an equation has *pure exponential solutions* $u = e^{\lambda t}x$; the eigenvalue gives the rate of growth or decay, and the eigenvector x develops at this rate. The other solutions will be *mixtures* of these pure solutions, and the mixture is adjusted to fit the initial conditions.

The key equation was $Ax = \lambda x$. Most vectors x will not satisfy such an equation. They change direction when multiplied by A, so that Ax is not a multiple of x. This means that ***only certain special numbers λ are eigenvalues, and only certain special vectors x are eigenvectors***. We can watch the behavior of each eigenvector, and then

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* One which I never really believed—but a bridge did crash this way in 1831.

combine these "normal modes" to find the solution. To say the same thing in another way, *the underlying matrix can be diagonalized.*

The diagonalization in Section 5.2 will be applied to difference equations, Fibonacci numbers, and Markov processes, and also to differential equations. In every example, we start by computing the eigenvalues and eigenvectors; there is no shortcut to avoid that. Symmetric matrices are especially easy. "Defective matrices" lack a full set of eigenvectors, so they are not diagonalizable. Certainly they have to be discussed, but we will not allow them to take over the book.

We start with examples of particularly good matrices.

Example 1 Everything is clear when A is a ***diagonal matrix***:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{has} \quad \lambda_1 = 3 \quad \text{with} \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 2 \quad \text{with} \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

On each eigenvector A acts like a multiple of the identity: $Ax_1 = 3x_1$ and $Ax_2 = 2x_2$. Other vectors like $x = (1, 5)$ are mixtures $x_1 + 5x_2$ of the two eigenvectors, and when A multiplies x_1 and x_2 it produces the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$:

$$A \quad \text{times} \quad x_1 + 5x_2 \quad \text{is} \quad 3x_1 + 10x_2 = \begin{bmatrix} 3 \\ 10 \end{bmatrix}.$$

This is Ax for a typical vector x—not an eigenvector. But the action of A is determined by its eigenvectors and eigenvalues.

Example 2 The eigenvalues of a ***projection matrix*** are 1 or 0!

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{has} \quad \lambda_1 = 1 \quad \text{with} \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 0 \quad \text{with} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have $\lambda = 1$ when x projects to itself, and $\lambda = 0$ when x projects to the zero vector. The column space of P is filled with eigenvectors, and so is the nullspace. If those spaces have dimension r and $n - r$, then $\lambda = 1$ is repeated r times and $\lambda = 0$ is repeated $n - r$ times (*always n λ's*):

Four eigenvalues allowing repeats $$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{has} \quad \lambda = 1, 1, 0, 0.$$

There is nothing exceptional about $\boldsymbol{\lambda = 0}$. Like every other number, zero might be an eigenvalue and it might not. If it is, then its eigenvectors satisfy $Ax = 0x$. Thus x is in the nullspace of A. A zero eigenvalue signals that A is singular (not invertible); its determinant is zero. Invertible matrices have all $\lambda \neq 0$.

Example 3 The eigenvalues are on the main diagonal when A is ***triangular***:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4} - \lambda & 6 \\ 0 & 0 & \frac{1}{2} - \lambda \end{vmatrix} = (1 - \lambda)\left(\tfrac{3}{4} - \lambda\right)\left(\tfrac{1}{2} - \lambda\right).$$

The determinant is just the product of the diagonal entries. It is zero if $\lambda = 1$, $\lambda = \frac{3}{4}$, or $\lambda = \frac{1}{2}$; the eigenvalues were already sitting along the main diagonal.

This example, in which the eigenvalues can be found by inspection, points to one main theme of the chapter: To transform A into a diagonal or triangular matrix *without changing its eigenvalues.* We emphasize once more that the Gaussian factorization $A = LU$ is not suited to this purpose. The eigenvalues of U may be visible on the diagonal, but they are ***not*** the eigenvalues of A.

For most matrices, there is no doubt that the eigenvalue problem is computationally more difficult than $Ax = b$. With linear systems, a finite number of elimination steps produced the exact answer in a finite time. (Or equivalently, Cramer's rule gave an exact formula for the solution.) No such formula can give the eigenvalues, or Galois would turn in his grave. For a 5 by 5 matrix, $\det(A - \lambda I)$ involves λ^5. Galois and Abel proved that there can be no algebraic formula for the roots of a fifth-degree polynomial.

All they will allow is a few simple checks on the eigenvalues, *after* they have been computed, and we mention two good ones: ***sum*** and ***product***.

5B The ***sum*** of the n eigenvalues equals the sum of the n diagonal entries:

$$\textbf{Trace of } A = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}. \tag{15}$$

Furthermore, the ***product*** of the n eigenvalues equals the ***determinant*** of A.

The projection matrix P had diagonal entries $\frac{1}{2}, \frac{1}{2}$ and eigenvalues 1, 0. Then $\frac{1}{2} + \frac{1}{2}$ agrees with $1 + 0$ as it should. So does the determinant, which is $0 \cdot 1 = 0$. A singular matrix, with zero determinant, has one or more of its eigenvalues equal to zero.

There should be no confusion between the diagonal entries and the eigenvalues. For a triangular matrix they are the same—but that is exceptional. Normally the pivots, diagonal entries, and eigenvalues are completely different. And for a 2 by 2 matrix, the trace and determinant tell us everything:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has trace } a + d\text{, and determinant } ad - bc$$

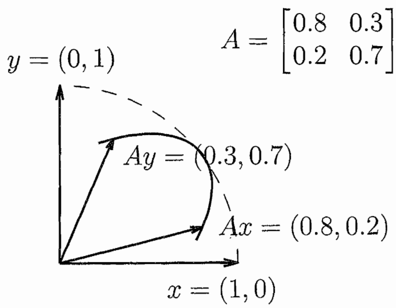
$$\det(A - \lambda I) = \det \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - (\text{trace})\lambda + \text{determinant}$$

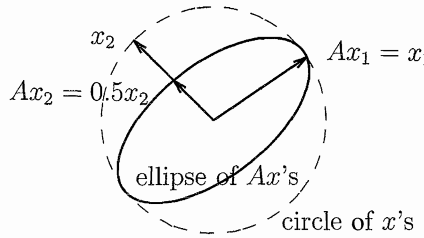
$$\text{The eigenvalues are } \lambda = \frac{\text{trace} \pm [(\text{trace})^2 - 4\det]^{1/2}}{2}.$$

Those two λ's add up to the trace; Exercise 9 gives $\sum \lambda_i = \text{trace}$ for all matrices.

Eigshow

There is a MATLAB demo (just type eigshow), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $x = (1, 0)$. *The mouse makes this vector move around the unit circle*. At the same time the screen shows Ax, in color and also moving. Possibly Ax is ahead of x. Possibly Ax is behind x. *Sometimes Ax is parallel to x*. At that parallel moment, $Ax = \lambda x$ (twice in the second figure).





The eigenvalue λ is the length of Ax, when the unit eigenvector x is parallel. The built-in choices for A illustrate three possibilities: 0, 1, or 2 real eigenvectors.

1. There are *no real eigenvectors*. *Ax stays behind or ahead of x*. This means the eigenvalues and eigenvectors are complex, as they are for the rotation Q.
2. There is only *one* line of eigenvectors (unusual). The moving directions Ax and x meet but don't cross. This happens for the last 2 by 2 matrix below.
3. There are eigenvectors in *two* independent directions. This is typical! Ax crosses x at the first eigenvector x_1, and it crosses back at the second eigenvector x_2.

Suppose A is singular (rank 1). Its column space is a line. The vector Ax has to stay on that line while x circles around. One eigenvector x is along the line. Another eigenvector appears when $Ax_2 = 0$. Zero is an eigenvalue of a singular matrix.

You can mentally follow x and Ax for these six matrices. How many eigenvectors and where? When does Ax go clockwise, instead of counterclockwise with x?

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Problem Set 5.1

1. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

2. With the same matrix A, solve the differential equation $du/dt = Au$, $u(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$. What are the two pure exponential solutions?

3. If we shift to $A - 7I$, what are the eigenvalues and eigenvectors and how are they related to those of A?

$$B = A - 7I = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}.$$

4. Solve $du/dt = Pu$, when P is a projection:

$$\frac{du}{dt} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} u \quad \text{with} \quad u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Part of $u(0)$ increases exponentially while the nullspace part stays fixed.

5. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1\lambda_2\lambda_3$ equals the determinant.

6. Give an example to show that the eigenvalues can be changed when a multiple of one row is subtracted from another. Why is a zero eigenvalue *not* changed by the steps of elimination?

7. Suppose that λ is an eigenvalue of A, and x is its eigenvector: $Ax = \lambda x$.

(a) Show that this same x is an eigenvector of $B = A - 7I$, and find the eigenvalue. This should confirm Exercise 3.

(b) Assuming $\lambda \neq 0$, show that x is also an eigenvector of A^{-1}—and find the eigenvalue.

8. Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda), \tag{16}$$

and making a clever choice of λ.

9. Show that the trace equals the sum of the eigenvalues, in two steps. First, find the coefficient of $(-\lambda)^{n-1}$ on the right side of equation (16). Next, find all the terms in

$$\det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

that involve $(-\lambda)^{n-1}$. They all come from the main diagonal! Find that coefficient of $(-\lambda)^{n-1}$ and compare.

10. (a) Construct 2 by 2 matrices such that the eigenvalues of AB are not the products of the eigenvalues of A and B, and the eigenvalues of $A + B$ are not the sums of the individual eigenvalues.

(b) Verify, however, that the sum of the eigenvalues of $A + B$ equals the sum of all the individual eigenvalues of A and B, and similarly for products. Why is this true?

11. ***The eigenvalues of A equal the eigenvalues of A^{T}.*** This is because $\det(A - \lambda I)$ equals $\det(A^{\mathrm{T}} - \lambda I)$. That is true because ______. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

12. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

13. If B has eigenvalues 1, 2, 3, C has eigenvalues 4, 5, 6, and D has eigenvalues 7, 8, 9, what are the eigenvalues of the 6 by 6 matrix $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$?

14. Find the rank and all four eigenvalues for both the matrix of ones and the checkerboard matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Which eigenvectors correspond to nonzero eigenvalues?

15. What are the rank and eigenvalues when A and C in the previous exercise are n by n? Remember that the eigenvalue $\lambda = 0$ is repeated $n - r$ times.

16. If A is the 4 by 4 matrix of ones, find the eigenvalues and the determinant of $A - I$.

17. Choose the third row of the "companion matrix"

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

so that its characteristic polynomial $|A - \lambda I|$ is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

18. Suppose A has eigenvalues 0, 3, 5 with independent eigenvectors u, v, w.

(a) Give a basis for the nullspace and a basis for the column space.
(b) Find a particular solution to $Ax = v + w$. Find all solutions.
(c) Show that $Ax = u$ has no solution. (If it had a solution, then ______ would be in the column space.)

19. The powers A^k of this matrix A approaches a limit as $k \to \infty$:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}, \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}, \quad \text{and} \quad A^{\infty} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

The matrix A^2 is halfway between A and A^{∞}. Explain why $A^2 = \frac{1}{2}(A + A^{\infty})$ from the eigenvalues and eigenvectors of these three matrices.

20. Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the ______ eigenvectors as A. Its eigenvalues are ______ by 1.

21. Compute the eigenvalues and eigenvectors of A and A^{-1}:

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the ______ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues ______.

22. Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same ______ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues ______.

23. (a) If you know x is an eigenvector, the way to find λ is to ______.
(b) If you know λ is an eigenvalue, the way to find x is to ______.

24. What do you do to $Ax = \lambda x$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 22.
(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 21.
(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 20.

25. From the unit vector $u = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$, construct the rank-1 projection matrix $P = uu^{\mathrm{T}}$.

(a) Show that $Pu = u$. Then u is an eigenvector with $\lambda = 1$.
(b) If v is perpendicular to u show that $Pv =$ zero vector. Then $\lambda = 0$.
(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

26. Solve $\det(Q - \lambda I) = 0$ by the quadratic formula, to reach $\lambda = \cos\theta \pm i \sin\theta$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{rotates the } xy\text{-plane by the angle } \theta.$$

Find the eigenvectors of Q by solving $(Q - \lambda I)x = 0$. Use $i^2 = -1$.

27. Every permutation matrix leaves $x = (1, 1, \ldots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ's for these permutations:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

28. If A has $\lambda_1 = 4$ and $\lambda_2 = 5$, then $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$, determinant 20, and $\lambda = 4, 5$.

29. A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2. This information is enough to find three of these:

(a) the rank of B,
(b) the determinant of $B^{\mathrm{T}}B$,

(c) the eigenvalues of $B^{\mathrm{T}}B$, and
(d) the eigenvalues of $(B+I)^{-1}$.

30. Choose the second row of $A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}$ so that A has eigenvalues 4 and 7.

31. Choose a, b, c, so that $\det(A - \lambda I) = 9\lambda - \lambda^3$. Then the eigenvalues are $-3, 0, 3$:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}.$$

32. Construct any 3 by 3 Markov matrix M: positive entries down each column add to 1. If $e = (1, 1, 1)$, verify that $M^{\mathrm{T}}e = e$. By Problem 11, $\lambda = 1$ is also an eigenvalue of M. Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has eigenvalues $\lambda =$ ______.

33. Find three 2 by 2 matrices that have $\lambda_1 = \lambda_2 = 0$. The trace is zero and the determinant is zero. The matrix A might not be 0 but check that $A^2 = 0$.

34. This matrix is singular with rank 1. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

35. Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $x_1, \ldots, x_n$. Then $A = B$. *Reason*: Any vector x is a combination $c_1x_1 + \cdots + c_nx_n$. What is Ax? What is Bx?

36. (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

37. When $a + b = c + d$, show that $(1, 1)$ is an eigenvector and find both eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

38. When P exchanges rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and PAP for $\lambda = 11$:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad PAP = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

39. Challenge problem: *Is there a real* 2 *by* 2 *matrix* (other than I) *with* $A^3 = I$? Its eigenvalues must satisfy $\lambda^3 = I$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What trace and determinant would this give? Construct A.

40. There are six 3 by 3 permutation matrices P. What numbers can be the *determinants* of P? What numbers can be *pivots*? What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P?

5.2 DIAGONALIZATION OF A MATRIX

We start right off with the one essential computation. It is perfectly simple and will be used in every section of this chapter. ***The eigenvectors diagonalize a matrix***:

5C Suppose the n by n matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ. The eigenvalues of A are on the diagonal of Λ:

$$\textbf{Diagonalization} \qquad S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}. \tag{1}$$

We call S the "eigenvector matrix" and Λ the "eigenvalue matrix"—using a capital lambda because of the small lambdas for the eigenvalues on its diagonal.

Proof Put the eigenvectors x_i in the columns of S, and compute AS by columns:

$$AS = A\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix}.$$

Then the trick is to split this last matrix into a quite different product $S\Lambda$:

$$\begin{bmatrix} \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ \\ \end{bmatrix} = \underbrace{\begin{bmatrix} \\ x_1 & x_2 & \cdots & x_n \\ \\ \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\Lambda}.$$

It is crucial to keep these matrices in the right order. If Λ came before S (instead of after), then λ_1 would multiply the entries in the first row. We want λ_1 to appear in the first column. As it is, $S\Lambda$ is correct. Therefore,

$$AS = S\Lambda, \quad \text{or} \quad S^{-1}AS = \Lambda, \quad \text{or} \quad A = S\Lambda S^{-1}. \tag{2}$$

S is invertible, because its columns (the eigenvectors) were assumed to be independent.

We add four remarks before giving any examples or applications. ■

Remark 1 If the matrix A has no repeated eigenvalues—the numbers $\lambda_1, \ldots, \lambda_n$ are distinct—then its n eigenvectors are automatically independent (see 5D below). Therefore ***any matrix with distinct eigenvalues can be diagonalized***.

Remark 2 The diagonalizing matrix S is *not unique*. An eigenvector x can be multiplied by a constant, and remains an eigenvector. We can multiply the columns of S by any nonzero constants, and produce a new diagonalizing S. Repeated eigenvalues leave even more freedom in S. For the trivial example $A = I$, any invertible S will do: $S^{-1}IS$ is always diagonal (Λ is just I). All vectors are eigenvectors of the identity.

Remark 3 *Other matrices S will not produce a diagonal* Λ. Suppose the first column of S is y. Then the first column of $S\Lambda$ is $\lambda_1 y$. If this is to agree with the first column of AS, which by matrix multiplication is Ay, then y must be an eigenvector: $Ay = \lambda_1 y$. The *order* of the eigenvectors in S and the eigenvalues in Λ is automatically the same.

Remark 4 Not all matrices possess n linearly independent eigenvectors, so ***not all matrices are diagonalizable***. The standard example of a "defective matrix" is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Its eigenvalues are $\lambda_1 = \lambda_2 = 0$, since it is triangular with zeros on the diagonal:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = \lambda^2.$$

All eigenvectors of this A are multiples of the vector $(1, 0)$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \text{or} \qquad x = \begin{bmatrix} c \\ 0 \end{bmatrix}.$$

$\lambda = 0$ is a double eigenvalue—its *algebraic multiplicity* is 2. But the *geometric multiplicity* is 1—there is only one independent eigenvector. We can't construct S.

Here is a more direct proof that this A is not diagonalizable. Since $\lambda_1 = \lambda_2 = 0$, Λ would have to be the zero matrix. But if $\Lambda = S^{-1}AS = 0$, then we premultiply by S and postmultiply by S^{-1}, to deduce falsely that $A = 0$. There is no invertible S.

That failure of diagonalization was ***not*** a result of $\lambda = 0$. It came from $\lambda_1 = \lambda_2$:

$$\textbf{Repeated eigenvalues} \qquad A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

Their eigenvalues are 3, 3 and 1, 1. They are not singular! The problem is the shortage of eigenvectors—which are needed for S. That needs to be emphasized:

Diagonalizability of A depends on enough eigenvectors.
Invertibility of A depends on nonzero eigenvalues.

There is no connection between diagonalizability (n independent eigenvectors) and invertibility (no zero eigenvalues). The only indication given by the eigenvalues is this: *Diagonalization can fail only if there are repeated eigenvalues*. Even then, it does not always fail. $A = I$ has repeated eigenvalues $1, 1, \ldots, 1$ but it is already diagonal! There is no shortage of eigenvectors in that case.

The test is to check, for an eigenvalue that is repeated p times, whether there are p independent eigenvectors—in other words, whether $A - \lambda I$ has rank $n - p$. To complete that circle of ideas, we have to show that *distinct* eigenvalues present no problem.

5D If eigenvectors $x_1, \ldots, x_k$ correspond to *different eigenvalues* $\lambda_1, \ldots, \lambda_k$, then those eigenvectors are linearly independent.

Suppose first that $k = 2$, and that some combination of x_1 and x_2 produces zero: $c_1x_1 + c_2x_2 = 0$. Multiplying by A, we find $c_1\lambda_1x_1 + c_2\lambda_2x_2 = 0$. Subtracting λ_2 times the previous equation, the vector x_2 disappears:

$$c_1(\lambda_1 - \lambda_2)x_1 = 0.$$

Since $\lambda_1 \neq \lambda_2$ and $x_1 \neq 0$, we are forced into $c_1 = 0$. Similarly $c_2 = 0$, and the two vectors are independent; only the trivial combination gives zero.

This same argument extends to any number of eigenvectors: If some combination produces zero, multiply by A, subtract λ_k times the original combination, and x_k disappears—leaving a combination of $x_1, \ldots, x_{k-1}$, which produces zero. By repeating the same steps (this is really *mathematical induction*) we end up with a multiple of x_1 that produces zero. This forces $c_1 = 0$, and ultimately every $c_i = 0$. Therefore eigenvectors that come from distinct eigenvalues are automatically independent.

A matrix with n distinct eigenvalues can be diagonalized. This is the typical case.

Examples of Diagonalization

The main point of this section is $S^{-1}AS = \Lambda$. The eigenvector matrix S converts A into its eigenvalue matrix Λ (diagonal). We see this for projections and rotations.

Example 1 The projection $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ has eigenvalue matrix $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The eigenvectors go into the columns of S:

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad AS = S\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

That last equation can be verified at a glance. Therefore $S^{-1}AS = \Lambda$.

Example 2 The eigenvalues themselves are not so clear for a ***rotation***:

$$\textbf{90° rotation} \qquad K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{has} \qquad \det(K - \lambda I) = \lambda^2 + 1.$$

How can a vector be rotated and still have its direction unchanged? Apparently it can't—except for the zero vector, which is useless. But there must be eigenvalues, and we must be able to solve $du/dt = Ku$. The characteristic polynomial $\lambda^2 + 1$ should still have two roots—but those roots are *not real.*

You see the way out. The eigenvalues of K are *imaginary numbers*, $\lambda_1 = i$ and $\lambda_2 = -i$. The eigenvectors are also not real. Somehow, in turning through 90°, they are multiplied by i or $-i$:

$$(K-\lambda_1 I)x_1 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$(K-\lambda_2 I)x_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

The eigenvalues are distinct, even if imaginary, and the eigenvectors are independent. They go into the columns of S:

$$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \quad \text{and} \quad S^{-1}KS = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

We are faced with an inescapable fact, that ***complex numbers are needed even for real matrices***. If there are too few real eigenvalues, there are always n complex eigenvalues. (Complex includes real, when the imaginary part is zero.) If there are too few eigenvectors in the real world $\mathbf{R}^3$, or in $\mathbf{R}^n$, we look in $\mathbf{C}^3$ or $\mathbf{C}^n$. The space $\mathbf{C}^n$ contains all column vectors with complex components, and it has new definitions of length and inner product and orthogonality. But it is not more difficult than $\mathbf{R}^n$, and in Section 5.5 we make an easy conversion to the complex case.

Powers and Products: A^k and AB

There is one more situation in which the calculations are easy. ***The eigenvalues of A^2 are exactly $\lambda_1^2, \ldots, \lambda_n^2$, and every eigenvector of A is also an eigenvector of A^2***. We start from $Ax = \lambda x$, and multiply again by A:

$$A^2x = A\lambda x = \lambda Ax = \lambda^2 x. \tag{3}$$

Thus λ^2 is an eigenvalue of A^2, with the same eigenvector x. If the first multiplication by A leaves the direction of x unchanged, then so does the second.

The same result comes from diagonalization, by squaring $S^{-1}AS = \Lambda$:

$$\textbf{Eigenvalues of } A^2 \qquad (S^{-1}AS)(S^{-1}AS) = \Lambda^2 \quad \text{or} \quad S^{-1}A^2S = \Lambda^2.$$

The matrix A^2 is diagonalized by the same S, so the eigenvectors are unchanged. The eigenvalues are squared. This continues to hold for any power of A:

5E The eigenvalues of A^k are $\lambda_1^k, \ldots, \lambda_n^k$, and each eigenvector of A is still an eigenvector of A^k. When S diagonalizes A, it also diagonalizes A^k:

$$\Lambda^k = (S^{-1}AS)(S^{-1}AS)\cdots(S^{-1}AS) = S^{-1}A^kS. \tag{4}$$

Each S^{-1} cancels an S, except for the first S^{-1} and the last S.

If A is invertible this rule also applies to its inverse (the power $k=-1$). ***The eigenvalues of A^{-1} are $1/\lambda_i$***. That can be seen even without diagonalizing:

$$\textit{if} \quad Ax = \lambda x \quad \textit{then} \quad x = \lambda A^{-1}x \quad \textit{and} \quad \frac{1}{\lambda}x = A^{-1}x.$$

Example 3 If K is rotation through 90°, then K^2 is rotation through 180° (which means $-I$) and K^{-1} is rotation through −90°:

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{and} \qquad K^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of K are i and $-i$; their squares are -1 and -1; their reciprocals are $1/i = -i$ and $1/(-i) = i$. Then K^4 is a complete rotation through 360°:

$$K^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and also} \qquad \Lambda^4 = \begin{bmatrix} i^4 & 0 \\ 0 & (-i)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For a ***product of two matrices***, we can ask about the eigenvalues of AB—but we won't get a good answer. It is very tempting to try the same reasoning, hoping to prove what is *not in general true*. If λ is an eigenvalue of A and μ is an eigenvalue of B, here is the false proof that AB has the eigenvalue $\mu\lambda$:

$$\textbf{False proof} \qquad ABx = A\mu x = \mu Ax = \mu\lambda x.$$

The mistake lies in assuming that A and B share the *same* eigenvector x. In general, they do not. We could have two matrices with zero eigenvalues, while AB has $\lambda = 1$:

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvectors of this A and B are completely different, which is typical. For the same reason, the eigenvalues of $A + B$ generally have nothing to do with $\lambda + \mu$.

This false proof does suggest what *is* true. If the eigenvector is the same for A and B, then the eigenvalues multiply and AB has the eigenvalue $\mu\lambda$. But there is something more important. There is an easy way to recognize when A and B share a full set of eigenvectors, and that is a key question in quantum mechanics:

5F Diagonalizable matrices share the same eigenvector matrix S if and only if $AB = BA$.

Proof If the same S diagonalizes both $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$, we can multiply in either order:

$$AB = S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1\Lambda_2 S^{-1} \quad \text{and} \quad BA = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = S\Lambda_2\Lambda_1 S^{-1}.$$

Since $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ (diagonal matrices always commute) we have $AB = BA$.

In the opposite direction, suppose $AB = BA$. Starting from $Ax = \lambda x$, we have

$$ABx = BAx = B\lambda x = \lambda Bx.$$

Thus x and Bx are both eigenvectors of A, sharing the same λ (or else $Bx = 0$). If we assume for convenience that the eigenvalues of A are distinct—the eigenspaces are all one-dimensional—then Bx *must be a multiple of* x. In other words x is an eigenvector of B as well as A. The proof with repeated eigenvalues is a little longer. ∎

Heisenberg's uncertainty principle comes from noncommuting matrices, like position P and momentum Q. Position is symmetric, momentum is skew-symmetric, and together they satisfy $QP - PQ = I$. The uncertainty principle follows directly from the Schwarz inequality $(Qx)^{\mathrm{T}}(Px) \leq \|Qx\|\|Px\|$ of Section 3.2:

$$\|x\|^2 = x^{\mathrm{T}}x = x^{\mathrm{T}}(QP - PQ)x \leq 2\|Qx\|\|Px\|.$$

The product of $\|Qx\|/\|x\|$ and $\|Px\|/\|x\|$— momentum and position errors, when the wave function is x—is at least $\frac{1}{2}$. It is impossible to get both errors small, because when you try to measure the position of a particle you change its momentum.

At the end we come back to $A = S\Lambda S^{-1}$. That factorization is particularly suited to take powers of A, and the simplest case A^2 makes the point. The LU factorization is hopeless when squared, but $S\Lambda S^{-1}$ is perfect. The square is $S\Lambda^2 S^{-1}$, and the eigenvectors are unchanged. By following those eigenvectors we will solve difference equations and differential equations.

Problem Set 5.2

1. Factor the following matrices into $S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

2. Find the matrix A whose eigenvalues are 1 and 4, and whose eigenvectors are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, respectively. (*Hint*: $A = S\Lambda S^{-1}$.)

3. Find *all* the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and write two different diagonalizing matrices S.

4. If a 3 by 3 upper triangular matrix has diagonal entries 1, 2, 7, how do you know it can be diagonalized? What is Λ?

5. Which of these matrices cannot be diagonalized?

$$A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}.$$

6. (a) If $A^2 = I$, what are the possible eigenvalues of A?
(b) If this A is 2 by 2, and not I or $-I$, find its trace and determinant.
(c) If the first row is $(3, -1)$, what is the second row?

7. If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, find A^{100} by diagonalizing A.

8. Suppose $A = uv^{\mathrm{T}}$ is a column times a row (a rank-1 matrix).
(a) By multiplying A times u, show that u is an eigenvector. What is λ?
(b) What are the other eigenvalues of A (and why)?
(c) Compute trace (A) from the sum on the diagonal and the sum of λ's.

9. Show by direct calculation that AB and BA have the same trace when

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

Deduce that $AB - BA = I$ is impossible (except in infinite dimensions).

10. Suppose A has eigenvalues $1, 2, 4$. What is the trace of A^2? What is the determinant of $(A^{-1})^{\mathrm{T}}$?

11. If the eigenvalues of A are 1, 1, 2, which of the following are certain to be true? Give a reason if true or a counterexample if false:

(a) A is invertible.
(b) A is diagonalizable.
(c) A is not diagonalizable.

12. Suppose the only eigenvectors of A are multiples of $x = (1, 0, 0)$. True or false:

(a) A is not invertible.
(b) A has a repeated eigenvalue.
(c) A is not diagonalizable.

13. Diagonalize the matrix $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and find one of its square roots—a matrix such that $R^2 = A$. How many square roots will there be?

14. Suppose the eigenvector matrix S has $S^{\mathrm{T}} = S^{-1}$. Show that $A = S\Lambda S^{-1}$ is symmetric and has orthogonal eigenvectors.

Problems 15–24 are about the eigenvalue and eigenvector matrices.

15. Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

16. If $A = S\Lambda S^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.

17. If A has $\lambda_1 = 2$ with eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and x's.

18. Suppose $A = S\Lambda S^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\quad)(\quad)(\quad)^{-1}$.

19. True or false: If the n columns of S (eigenvectors of A) are independent, then

(a) A is invertible.
(b) A is diagonalizable.
(c) S is invertible.
(d) S is diagonalizable.

20. If the eigenvectors of A are the columns of I, then A is a ______ matrix. If the eigenvector matrix S is triangular, then S^{-1} is triangular and A is triangular.

21. Describe all matrices S that diagonalize this matrix A:

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

22. Write the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

23. Find the eigenvalues of A and B and $A + B$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A + B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Eigenvalues of $A + B$ (are equal to)(are not equal to) eigenvalues of A plus eigenvalues of B.

24. Find the eigenvalues of A, B, AB, and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Eigenvalues of AB (are equal to)(are not equal to) eigenvalues of A times eigenvalues of B. Eigenvalues of AB (are)(are not) equal to eigenvalues of BA.

Problems 25–28 are about the diagonalizability of A.

25. True or false: If the eigenvalues of A are 2, 2, 5, then the matrix is certainly

(a) invertible.
(b) diagonalizable.
(b) not diagonalizable.

26. If the eigenvalues of A are 1 and 0, write everything you know about the matrices A and A^2.

27. Complete these matrices so that $\det A = 25$. Then trace $= 10$, and $\lambda = 5$ is repeated! Find an eigenvector with $Ax = 5x$. These matrices will not be diagonalizable because there is no second line of eigenvectors.

$$A = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}.$$

28. The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable because the rank of $A - 3I$ is ___. Change one entry to make A diagonalizable. Which entries could you change?

Problems 29–33 are about powers of matrices.

29. $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ___. Does $A^k \to 0$ or $B^k \to 0$?

$$A = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

30. (Recommended) Find Λ and S to diagonalize A in Problem 29. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the ______.

31. Find Λ and S to diagonalize B in Problem 29. What is $B^{10}u_0$ for these u_0?

$$u_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad u_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad u_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

32. Diagonalize A and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{has} \qquad A^k = \frac{1}{2}\begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix}.$$

33. Diagonalize B and compute $S\Lambda^k S^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \qquad \text{has} \qquad B^k = \begin{bmatrix} 3^k & 3^k-2^k \\ 0 & 2^k \end{bmatrix}.$$

Problems 34–44 are new applications of $A = S\Lambda S^{-1}$.

34. Suppose that $A = S\Lambda S^{-1}$. Take determinants to prove that $\det A = \lambda_1\lambda_2\cdots\lambda_n =$ product of λ's. This quick proof only works when A is ______.

35. The trace of S times ΛS^{-1} equals the trace of ΛS^{-1} times S. So the trace of a diagonalizable A equals the trace of Λ, which is ______.

36. If $A = S\Lambda S^{-1}$, diagonalize the block matrix $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix}$. Find its eigenvalue and eigenvector matrices.

37. Consider all 4 by 4 matrices A that are diagonalized by the same fixed eigenvector matrix S. Show that the A's form a subspace (cA and $A_1 + A_2$ have this same S). What is this subspace when $S = I$? What is its dimension?

38. Suppose $A^2 = A$. On the left side A multiplies each column of A. Which of our four subspaces contains eigenvectors with $\lambda = 1$? Which subspace contains eigenvectors with $\lambda = 0$? From the dimensions of those subspaces, A has a full set of independent eigenvectors and can be diagonalized.

39. Suppose $Ax = \lambda x$. If $\lambda = 0$, then x is in the nullspace. If $\lambda \neq 0$, then x is in the column space. Those spaces have dimensions $(n-r)+r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

40. Substitute $A = S\Lambda S^{-1}$ into the product $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ and explain why this produces the *zero matrix*. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A-\lambda I)$. The ***Cayley–Hamilton Theorem*** says that this product is always $p(A) =$ *zero matrix*, even if A is not diagonalizable.

41. Test the Cayley–Hamilton Theorem on Fibonacci's matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The theorem predicts that $A^2 - A - I = 0$, since $\det(A-\lambda I)$ is $\lambda^2 - \lambda - 1$.

42. If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, then $\det(A-\lambda I)$ is $(\lambda-a)(\lambda-d)$. Check the Cayley–Hamilton statement that $(A-aI)(A-dI) =$ *zero matrix*.

43. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $AB = BA$, show that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also diagonal. B has the same eigen______ as A, but different eigen______. These diagonal matrices B form a two-dimensional subspace of matrix space. $AB - BA = 0$ gives four equations for the unknowns **a**, **b**, **c**, **d**—find the rank of the 4 by 4 matrix.

44. If A is 5 by 5, then $AB - BA$ = zero matrix gives 25 equations for the 25 entries in B. Show that the 25 by 25 matrix is singular by noticing a simple nonzero solution B.

45. Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

5.3 DIFFERENCE EQUATIONS AND POWERS A^k

Difference equations $u_{k+1} = Au_k$ move forward in a finite number of finite steps. A differential equation takes an infinite number of infinitesimal steps, but the two theories stay absolutely in parallel. It is the same analogy between the discrete and the continuous that appears over and over in mathematics. A good illustration is compound interest, when the time step gets shorter.

Suppose you invest \$1000 at 6% interest. Compounded once a year, the principal P is multiplied by 1.06. *This is a difference equation* $P_{k+1} = AP_k = 1.06\, P_k$ *with a time step of one year*. After 5 years, the original $P_0 = 1000$ has been multiplied 5 times:

$$\textbf{Yearly} \qquad P_5 = (1.06)^5\, P_0 \quad \text{which is} \quad (1.06)^5\, 1000 = \$1338.$$

Now suppose the time step is reduced to a month. The new difference equation is $p_{k+1} = (1 + .06/12)p_k$. After 5 years, or 60 months, you have \$11 more:

$$\textbf{Monthly} \qquad p_{60} = \left(1 + \frac{.06}{12}\right)^{60} p_0 \quad \text{which is} \quad (1.005)^{60}\, 1000 = \$1349.$$

The next step is to compound every day, on 5(365) days. This only helps a little:

$$\textbf{Daily compounding} \qquad \left(1 + \frac{.06}{365}\right)^{5\cdot 365} 1000 = \$1349.83.$$

Finally, to keep their employees really moving, banks offer *continuous compounding*. The interest is added on at every instant, and the difference equation breaks down. You can hope that the treasurer does not know calculus (which is all about limits as $\Delta t \to 0$). The bank could compound the interest N times a year, so $\Delta t = 1/N$:

$$\textbf{Continuously} \qquad \left(1 + \frac{.06}{N}\right)^{5N} 1000 \quad \to \quad e^{.30}\, 1000 = \$1349.87.$$

Or the bank can switch to a differential equation—the limit of the difference equation $p_{k+1} = (1 + .06\Delta t)p_k$. Moving p_k to the left side and dividing by Δt,

$$\textbf{Discrete to continuous} \qquad \frac{p_{k+1} - p_k}{\Delta t} = .06 p_k \quad \text{approaches} \quad \frac{dp}{dt} = .06p. \qquad (1)$$

The solution is $p(t) = e^{.06t}p_0$. After $t = 5$ years, this again amounts to \$1349.87. The principal stays finite, even when it is compounded every instant—and the improvement over compounding every day is only four cents.

Fibonacci Numbers

The main object of this section is to solve $u_{k+1} = Au_k$. That leads us to A^k and **powers of matrices**. Our second example is the famous ***Fibonacci sequence***:

$$\textbf{Fibonacci numbers} \qquad 0, 1, 1, 2, 3, 5, 8, 13, \ldots.$$

You see the pattern: Every Fibonacci number is the sum of the two previous F's:

$$\textbf{Fibonacci equation} \qquad F_{k+2} = F_{k+1} + F_k. \tag{2}$$

That is the difference equation. It turns up in a most fantastic variety of applications, and deserves a book of its own. Leaves grow in a spiral pattern, and on the apple or oak you find five growths for every two turns around the stem. The pear tree has eight for every three turns, and the willow is 13:5. The champion seems to be a sunflower whose seeds chose an almost unbelievable ratio of $F_{12}/F_{13} = 144/233$.*

How could we find the 1000th Fibonacci number, without starting at $F_0 = 0$ and $F_1 = 1$, and working all the way out to F_{1000}? The goal is to solve the difference equation $F_{k+2} = F_{k+1} + F_k$. **This can be reduced to a one-step equation $u_{k+1} = Au_k$. Every step multiplies $u_k = (F_{k+1}, F_k)$ by a matrix A:**

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+1} &= F_{k+1} \end{aligned} \qquad \text{becomes} \qquad u_{k+1} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = Au_k. \tag{3}$$

The one-step system $u_{k+1} = Au_k$ is easy to solve. It starts from u_0. After one step it produces $u_1 = Au_0$. Then u_2 is Au_1, which is A^2u_0. *Every step brings a multiplication by A*, and after k steps there are k multiplications:

> ***The solution to a difference equation $u_{k+1} = Au_k$ is $u_k = A^k u_0$.***

The real problem is to find some quick way to compute the powers A^k, and thereby find the 1000th Fibonacci number. The key lies in the eigenvalues and eigenvectors:

5G If A can be diagonalized, $A = S\Lambda S^{-1}$, then A^k comes from Λ^k:

$$u_k = A^k u_0 = (S\Lambda S^{-1})(S\Lambda S^{-1}) \cdots (S\Lambda S^{-1})u_0 = S\Lambda^k S^{-1}u_0. \tag{4}$$

The columns of S are the eigenvectors of A. Writing $S^{-1}u_0 = c$, the solution becomes

$$u_k = S\Lambda^k c = \begin{bmatrix} & & \\ x_1 & \cdots & x_n \\ & & \end{bmatrix} \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1\lambda_1^k x_1 + \cdots + c_n\lambda_n^k x_n. \tag{5}$$

After k steps, u_k is a combination of the n "pure solutions" $\lambda^k x$.

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* For these botanical applications, see D'Arcy Thompson's book *On Growth and Form* (Cambridge University Press, 1942) or Peter Stevens's beautiful *Patterns in Nature* (Little, Brown, 1974). Hundreds of other properties of the F_n have been published in the *Fibonacci Quarterly*. Apparently Fibonacci brought Arabic numerals into Europe, about 1200 A.D.

These formulas give two different approaches to the same solution $u_k = S\Lambda^k S^{-1}u_0$. The first formula recognized that A^k is identical with $S\Lambda^k S^{-1}$, and we could stop there. But the second approach brings out the analogy with a differential equation: ***The pure exponential solutions $e^{\lambda_i t}x_i$ are now the pure powers $\lambda_i^k x_i$***. The eigenvectors x_i are amplified by the eigenvalues λ_i. By combining these special solutions to match u_0—that is where c came from—we recover the correct solution $u_k = S\Lambda^k S^{-1}u_0$.

In any specific example like Fibonacci's, the first step is to find the eigenvalues:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \quad \text{has} \quad \det(A-\lambda I) = \lambda^2 - \lambda - 1$$

$$\textbf{Two eigenvalues} \qquad \lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

The second row of $A - \lambda I$ is $(1, -\lambda)$. To get $(A-\lambda I)x = 0$, the eigenvector is $x = (\lambda, 1)$. The first Fibonacci numbers $F_0 = 0$ and $F_1 = 1$ go into u_0, and $S^{-1}u_0 = c$:

$$S^{-1}u_0 = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{gives} \quad c = \begin{bmatrix} 1/(\lambda_1-\lambda_2) \\ -1/(\lambda_1-\lambda_2) \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Those are the constants in $u_k = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2$. Both eigenvectors x_1 and x_2 have second component 1. That leaves $F_k = c_1\lambda_1^k + c_2\lambda_2^k$ in the second component of u_k:

$$\textbf{Fibonacci numbers} \qquad F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right].$$

This is the answer we wanted. The fractions and square roots look surprising because Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ must produce whole numbers. Somehow that formula for F_k must give an integer. In fact, since the second term $[(1-\sqrt{5})/2]^k/\sqrt{5}$ is always less than $\frac{1}{2}$, it must just move the first term to the nearest integer:

$$F_{1000} = \text{nearest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{1000}.$$

This is an enormous number, and F_{1001} will be even bigger. The fractions are becoming insignificant, and the ratio F_{1001}/F_{1000} must be very close to $(1+\sqrt{5})/2 \approx 1.618$. Since λ_2^k is insignificant compared to λ_1^k, the ratio F_{k+1}/F_k approaches λ_1.

That is a typical difference equation, leading to the powers of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. It involved $\sqrt{5}$ because the eigenvalues did. If we choose a matrix with $\lambda_1 = 1$ and $\lambda_2 = 6$, we can focus on the simplicity of the computation—*after A has been diagonalized*:

$$A = \begin{bmatrix} -4 & -5 \\ 10 & 11 \end{bmatrix} \quad \text{has } \lambda = 1 \text{ and } 6, \text{ with } x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A^k = S\Lambda^k S^{-1} \quad \text{is} \quad \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{1}^k & 0 \\ 0 & \mathbf{6}^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2-6^k & 1-6^k \\ -2+2\cdot 6^k & -1+2\cdot 6^k \end{bmatrix}.$$

The powers 6^k and 1^k appear in that last matrix A^k, mixed in by the eigenvectors.

For the difference equation $u_{k+1} = Au_k$, we emphasize the main point. Every eigenvector x produces a "pure solution" with powers of λ:

$$\boxed{\textbf{\textit{One solution is}} \quad u_0 = x, \quad u_1 = \lambda x, \quad u_2 = \lambda^2 x, \ldots}$$

When the initial u_0 is an eigenvector x, this is *the* solution: $u_k = \lambda^k x$. In general u_0 is not an eigenvector. But if u_0 is a *combination* of eigenvectors, the solution u_k is the same combination of these special solutions.

5H If $u_0 = c_1x_1 + \cdots + c_nx_n$, then after k steps $u_k = c_1\lambda_1^k x_1 + \cdots + c_n\lambda_n^k x_n$. Choose the c's to match the starting vector u_0:

$$u_0 = \begin{bmatrix} & & \\ x_1 & \cdots & x_n \\ & & \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = Sc \quad \text{and} \quad c = S^{-1}u_0. \tag{6}$$

Markov Matrices

There was an exercise in Chapter 1, about moving in and out of California, that is worth another look. These were the rules:

Each year $\frac{1}{10}$ of the people outside California move in, and $\frac{2}{10}$ of the people inside California move out. We start with y_0 people outside and z_0 inside.

At the end of the first year the numbers outside and inside are y_1 and z_1:

$$\begin{array}{ll} \textbf{Difference} & y_1 = .9y_0 + .2z_0 \\ \textbf{equation} & z_1 = .1y_0 + .8z_0 \end{array} \quad \text{or} \quad \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}.$$

This problem and its matrix have the two essential properties of a ***Markov process:***

1. The total number of people stays fixed: ***Each column of the Markov matrix adds up to* 1**. Nobody is gained or lost.
2. The numbers outside and inside can never become negative: ***The matrix has no negative entries***. The powers A^k are all nonnegative.*

We solve this Markov difference equation using $u_k = S\Lambda^k S^{-1}u_0$. Then we show that the population approaches a "steady state." First A has to be diagonalized:

$$A - \lambda I = \begin{bmatrix} .9-\lambda & .2 \\ .1 & .8-\lambda \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - 1.7\lambda + .7$$

$$\boldsymbol{\lambda_1 = 1} \textbf{ and } \boldsymbol{\lambda_2 = .7}: \qquad A = S\Lambda S^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & \\ & .7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

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* Furthermore, history is completely disregarded; each new u_{k+1} depends only on the current u_k. Perhaps even our lives are examples of Markov processes, but I hope not.

To find A^k, and the distribution after k years, change $S\Lambda S^{-1}$ to $S\Lambda^k S^{-1}$:

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1^k & \\ & .7^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

$$= (y_0 + z_0) \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} + (y_0 - 2z_0)(.7)^k \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}.$$

Those two terms are $c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2$. The factor $\lambda_1^k = 1$ is hidden in the first term. In the long run, the other factor $(.7)^k$ becomes extremely small. ***The solution approaches a limiting state $u_\infty = (y_\infty, z_\infty)$***:

$$\textbf{Steady state} \qquad \begin{bmatrix} y_\infty \\ z_\infty \end{bmatrix} = (y_0 + z_0) \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

The total population is still $y_0 + z_0$, but in the limit $\frac{2}{3}$ of this population is outside California and $\frac{1}{3}$ is inside. This is true no matter what the initial distribution may have been! If the year starts with $\frac{2}{3}$ outside and $\frac{1}{3}$ inside, then it ends the same way:

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \qquad \text{or} \qquad Au_\infty = u_\infty.$$

The steady state is the eigenvector of A corresponding to $\lambda = 1$. Multiplication by A, from one time step to the next, leaves u_∞ unchanged.

The theory of Markov processes is illustrated by that California example:

5I A Markov matrix A has all $a_{ij} \geq 0$, with each column adding to 1.

(a) $\lambda_1 = 1$ is an eigenvalue of A.
(b) Its eigenvector x_1 is nonnegative—and it is a steady state, since $Ax_1 = x_1$.
(c) The other eigenvalues satisfy $|\lambda_i| \leq 1$.
(d) If A or any power of A has all *positive* entries, these other $|\lambda_i|$ are below 1. The solution $A^k u_0$ approaches a multiple of x_1—which is the steady state u_∞.

To find the right multiple of x_1, use the fact that the total population stays the same. If California started with all 90 million people out, it ended with 60 million out and 30 million in. It ends the same way if all 90 million were originally inside.

We note that many authors transpose the matrix so its *rows* add to 1.

Remark Our description of a Markov process was deterministic; populations moved in fixed proportions. But if we look at a single individual, the fractions that move become *probabilities*. With probability $\frac{1}{10}$, an individual outside California moves in. If inside, the probability of moving out is $\frac{2}{10}$. The movement becomes a *random process*, and A is called a ***transition matrix***.

The components of $u_k = A^k u_0$ specify the probability that the individual is outside or inside the state. These probabilities are never negative, and add to 1—everybody has to be somewhere. That brings us back to the two fundamental properties of a Markov matrix: Each column adds to 1, and no entry is negative.

Why is $\lambda = 1$ always an eigenvalue? Each column of $A - I$ adds up to $1 - 1 = 0$. Therefore the rows of $A - I$ add up to the zero row, they are linearly dependent, and $\det(A - I) = 0$.

Except for very special cases, u_k will approach the corresponding eigenvector.* In the formula $u_k = c_1\lambda_1^k x_1 + \cdots + c_n\lambda_n^k x_n$, no eigenvalue can be larger than 1. (Otherwise the probabilities u_k would blow up.) If all other eigenvalues are strictly smaller than $\lambda_1 = 1$, then the first term in the formula will be dominant. The other λ_i^k go to zero, and $u_k \to c_1 x_1 = u_\infty =$ steady state.

This is an example of one of the central themes of this chapter: Given information about A, find information about its eigenvalues. Here we found $\lambda_{\max} = 1$.

Stability of $u_{k+1} = Au_k$

There is an obvious difference between Fibonacci numbers and Markov processes. The numbers F_k become larger and larger, while by definition any "probability" is between 0 and 1. The Fibonacci equation is *unstable*. So is the compound interest equation $P_{k+1} = 1.06P_k$; the principal keeps growing forever. If the Markov probabilities decreased to zero, that equation would be stable; but they do not, since at every stage they must add to 1. Therefore a Markov process is *neutrally stable*.

We want to study the behavior of $u_{k+1} = Au_k$ as $k \to \infty$. Assuming that A can be diagonalized, u_k will be a combination of pure solutions:

$$\textbf{Solution at time } \boldsymbol{k} \qquad u_k = S\Lambda^k S^{-1}u_0 = c_1\lambda_1^k x_1 + \cdots + c_n\lambda_n^k x_n.$$

The growth of u_k is governed by the λ_i^k. ***Stability depends on the eigenvalues***:

5J The difference equation $u_{k+1} = Au_k$ is

stable if all eigenvalues satisfy $|\lambda_i| < 1$;
neutrally stable if some $|\lambda_i| = 1$ and all the other $|\lambda_i| < 1$; and
unstable if at least one eigenvalue has $|\lambda_i| > 1$.

In the stable case, the powers A^k approach zero and so does $u_k = A^k u_0$.

Example 1 This matrix A is certainly stable:

$$A = \begin{bmatrix} 0 & 4 \\ 0 & \frac{1}{2} \end{bmatrix} \quad \text{has eigenvalues 0 and } \tfrac{1}{2}.$$

The λ's are on the main diagonal because A is triangular. Starting from any u_0, and following the rule $u_{k+1} = Au_k$, the solution must eventually approach zero:

$$u_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ \frac{1}{4} \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ \frac{1}{8} \end{bmatrix}, \quad u_4 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{16} \end{bmatrix}, \quad \cdots$$

---

* If everybody outside moves in and everybody inside moves out, then the populations are reversed every year and there is no steady state. The transition matrix is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and -1 is an eigenvalue as well as $+1$—which cannot happen if all $a_{ij} > 0$.

The larger eigenvalue $\lambda = \frac{1}{2}$ governs the decay; after the first step every u_k is $\frac{1}{2}u_{k-1}$. The real effect of the first step is to split u_0 into the two eigenvectors of A:

$$u_0 = \begin{bmatrix} 8 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \end{bmatrix} \quad \text{and then} \quad u_k = \left(\frac{1}{2}\right)^k \begin{bmatrix} 8 \\ 1 \end{bmatrix} + (0)^k \begin{bmatrix} -8 \\ 0 \end{bmatrix}.$$

Positive Matrices and Applications in Economics

By developing the Markov ideas we can find a small gold mine (*entirely optional*) of matrix applications in economics.

Example 2 ***Leontief's input-output matrix***

This is one of the first great successes of mathematical economics. To illustrate it, we construct a *consumption matrix*—in which a_{ij} gives the amount of product j that is needed to create one unit of product i:

$$A = \begin{bmatrix} .4 & 0 & .1 \\ 0 & .1 & .8 \\ .5 & .7 & .1 \end{bmatrix}. \qquad \begin{matrix} \text{(steel)} \\ \text{(food)} \\ \text{(labor)} \end{matrix}$$

The first question is: Can we produce y_1 units of steel, y_2 units of food, and y_3 units of labor? We must start with larger amounts p_1, p_2, p_3, because some part is consumed by the production itself. The amount consumed is Ap, and it leaves a net production of $p - Ap$.

Problem *To find a vector p such that $p - Ap = y$, or $p = (I - A)^{-1}y$.*

On the surface, we are only asking if $I - A$ is invertible. But there is a nonnegative twist to the problem. Demand and production, y and p, are nonnegative. Since p is $(I-A)^{-1}y$, the real question is about the matrix that multiplies y:

When is $(I - A)^{-1}$ a nonnegative matrix?

Roughly speaking, A cannot be too large. If production consumes too much, nothing is left as output. The key is in the largest eigenvalue λ_1 of A, which must be below 1:

If $\lambda_1 > 1$, $(I - A)^{-1}$ fails to be nonnegative.
If $\lambda_1 = 1$, $(I - A)^{-1}$ fails to exist.
If $\lambda_1 < 1$, $(I - A)^{-1}$ is a converging sum of nonnegative matrices:

$$\textbf{Geometric series} \qquad (I - A)^{-1} = I + A + A^2 + A^3 + \cdots. \tag{7}$$

The 3 by 3 example has $\lambda_1 = .9$, and output exceeds input. Production can go on.

Those are easy to prove, once we know the main fact about a nonnegative matrix like A: ***Not only is the largest eigenvalue λ_1 positive, but so is the eigenvector x_1.*** Then $(I - A)^{-1}$ has the same eigenvector, with eigenvalue $1/(1 - \lambda_1)$.

If λ_1 exceeds 1, that last number is negative. The matrix $(I - A)^{-1}$ will take the positive vector x_1 to a negative vector $x_1/(1 - \lambda_1)$. In that case $(I - A)^{-1}$ is definitely not nonnegative. If $\lambda_1 = 1$, then $I - A$ is singular. The productive case is $\lambda_1 < 1$, when the powers of A go to zero (stability) and the infinite series $I + A + A^2 + \cdots$ converges.

Multiplying this series by $I-A$ leaves the identity matrix—all higher powers cancel—so $(I-A)^{-1}$ is a sum of nonnegative matrices. We give two examples:

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \text{ has } \lambda_1 = 2 \text{ and the economy is lost}$$

$$A = \begin{bmatrix} .5 & 2 \\ 0 & .5 \end{bmatrix} \text{ has } \lambda_1 = \frac{1}{2} \text{ and we can produce anything.}$$

The matrices $(I-A)^{-1}$ in those two cases are $-\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 8 \\ 0 & 2 \end{bmatrix}$.

Leontief's inspiration was to find a model that uses genuine data from the real economy. The table for 1958 contained 83 industries in the United States, with a "transactions table" of consumption and production for each one. The theory also reaches beyond $(I-A)^{-1}$, to decide natural prices and questions of optimization. Normally labor is in limited supply and ought to be minimized. And, of course, the economy is not always linear.

Example 3 ***The prices in a*** *closed* ***input-output model***

The model is called "closed" when everything produced is also consumed. Nothing goes outside the system. In that case A goes back to a *Markov matrix*. ***The columns add up to* 1**. We might be talking about the *value* of steel and food and labor, instead of the number of units. The vector p represents prices instead of production levels.

Suppose p_0 is a vector of prices. Then Ap_0 multiplies prices by amounts to give the value of each product. That is a new set of prices which the system uses for the next set of values A^2p_0. The question is whether the prices approach equilibrium. Are there prices such that $p = Ap$, and does the system take us there?

You recognize p as the (nonnegative) eigenvector of the Markov matrix A, with $\lambda = 1$. It is the steady state p_∞, and it is approached from any starting point p_0. By repeating a transaction over and over, the price tends to equilibrium.

The "Perron–Frobenius theorem" gives the key properties of a ***positive matrix***—not to be confused with a *positive definite* matrix, which is symmetric and has all its eigenvalues positive. Here all the entries a_{ij} are positive.

5K If A is a positive matrix, so is its largest eigenvalue: $\lambda_1 >$ all other $|\lambda_i|$. Every component of the corresponding eigenvector x_1 is also positive.

Proof Suppose $A > 0$. The key idea is to look at all numbers t such that $Ax \geq tx$ for some nonnegative vector x (other than $x = 0$). We are allowing inequality in $Ax \geq tx$ in order to have many positive candidates t. For the largest value $t_{\max}$ (which is attained), we will show that ***equality holds***: $Ax = t_{\max}x$.

Otherwise, if $Ax \geq t_{\max}x$ is not an equality, multiply by A. Because A is positive, that produces a strict inequality $A^2x > t_{\max}Ax$. Therefore the positive vector $y = Ax$ satisfies $Ay > t_{\max}y$, and $t_{\max}$ could have been larger. This contradiction forces the equality $Ax = t_{\max}x$, and we have an eigenvalue. Its eigenvector x is positive (not just nonnegative) because on the left-hand side of that equality Ax is sure to be positive.

To see that no eigenvalue can be larger than $t_{\max}$, suppose $Az = \lambda z$. Since λ and z may involve negative or complex numbers, we take absolute values: $|\lambda|\,|z| = |Az| \le A|z|$ by the "triangle inequality." This $|z|$ is a nonnegative vector, so $|\lambda|$ is one of the possible candidates t. Therefore $|\lambda|$ cannot exceed λ_1, which was $t_{\max}$. ∎

Example 4 ***Von Neumann's model of an expanding economy***
We go back to the 3 by 3 matrix A that gave the consumption of steel, food, and labor. If the outputs are s_1, f_1, ℓ_1, then the required inputs are

$$u_0 = \begin{bmatrix} .4 & 0 & .1 \\ 0 & .1 & .8 \\ .5 & .7 & .1 \end{bmatrix} \begin{bmatrix} s_1 \\ f_1 \\ \ell_1 \end{bmatrix} = Au_1.$$

In economics the difference equation is backward! Instead of $u_1 = Au_0$ we have $u_0 = Au_1$. If A is small (as it is), then production does not consume everything—and the economy can grow. The eigenvalues of A^{-1} will govern this growth. But again there is a nonnegative twist, since steel, food, and labor cannot come in negative amounts. Von Neumann asked for the maximum rate t at which the economy can expand and *still stay nonnegative*, meaning that $u_1 \ge tu_0 \ge 0$.

Thus the problem requires $u_1 \ge tAu_1$. It is like the Perron–Frobenius theorem, with A on the other side. As before, equality holds when t reaches $t_{\max}$—which is the eigenvalue associated with the positive eigenvector of A^{-1}. In this example the expansion factor is $\frac{10}{9}$:

$$x = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} \quad \text{and} \quad Ax = \begin{bmatrix} .4 & 0 & .1 \\ 0 & .1 & .8 \\ .5 & .7 & .1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 4.5 \\ 4.5 \end{bmatrix} = \frac{9}{10}x.$$

With steel–food–labor in the ratio 1–5–5, the economy grows as quickly as possible: ***The maximum growth rate is*** $\mathbf{1/\lambda_1}$.

Problem Set 5.3

1. Prove that every third Fibonacci number in $0, 1, 1, 2, 3, \ldots$ is even.

2. Bernadelli studied a beetle "which lives three years only, and propagates in its third year." They survive the first year with probability $\frac{1}{2}$, and the second with probability $\frac{1}{3}$, and then produce six females on the way out:

$$\textbf{Beetle matrix} \qquad A = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

Show that $A^3 = I$, and follow the distribution of 3000 beetles for six years.

3. For the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, compute A^2, A^3, and A^4. Then use the text and a calculator to find F_{20}.

4. Suppose each "Gibonacci" number G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k. Then $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \quad \text{is} \quad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.
(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.
(c) If $G_0 = 0$ and $G_1 = 1$, show that the Gibonacci numbers approach $\frac{2}{3}$.

5. Diagonalize the Fibonacci matrix by completing S^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $S\Lambda^k S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = \left(\lambda_1^k - \lambda_2^k\right)/(\lambda_1 - \lambda_2)$.

6. The numbers λ_1^k and λ_2^k satisfy the Fibonacci rule $F_{k+2} = F_{k+1} + F_k$:

$$\lambda_1^{k+2} = \lambda_1^{k+1} + \lambda_1^k \quad \text{and} \quad \lambda_2^{k+2} = \lambda_2^{k+1} + \lambda_2^k.$$

Prove this by using the original equation for the λ's (multiply it by λ^k). Then any combination of λ_1^k and λ_2^k satisfies the rule. The combination $F_k = \left(\lambda_1^k - \lambda_2^k\right)/(\lambda_1 - \lambda_2)$ gives the right start of $F_0 = 0$ and $F_1 = 1$.

7. Lucas started with $L_0 = 2$ and $L_1 = 1$. The rule $L_{k+2} = L_{k+1} + L_k$ is the same, so A is still Fibonacci's matrix. Add its eigenvectors $x_1 + x_2$:

$$\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1 + \sqrt{5}\right) \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\left(1 - \sqrt{5}\right) \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_0 \end{bmatrix}.$$

Multiplying by A^k, the second component is $L_k = \lambda_1^k + \lambda_2^k$. Compute the Lucas number L_{10} slowly by $L_{k+2} = L_{k+1} + L_k$, and compute approximately by λ_1^{10}.

8. Suppose there is an epidemic in which every month half of those who are well become sick, and a quarter of those who are sick become dead. Find the steady state for the corresponding Markov process

$$\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}.$$

9. Write the 3 by 3 transition matrix for a chemistry course that is taught in two sections, if every week $\frac{1}{4}$ of those in Section A and $\frac{1}{3}$ of those in Section B drop the course, and $\frac{1}{6}$ of each section transfer to the other section.

10. Find the limiting values of y_k and z_k ($k \to \infty$) if

$$\begin{aligned} y_{k+1} &= .8y_k + .3z_k \qquad & y_0 &= 0 \\ z_{k+1} &= .2y_k + .7z_k \qquad & z_0 &= 5. \end{aligned}$$

Also find formulas for y_k and z_k from $A^k = S\Lambda^k S^{-1}$.

11. (a) From the fact that column 1 + column 2 = 2(column 3), so the columns are linearly dependent, find one eigenvalue and one eigenvector of A:

$$A = \begin{bmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix}.$$

(b) Find the other eigenvalues of A (it is Markov).

(c) If $u_0 = (0, 10, 0)$, find the limit of $A^k u_0$ as $k \to \infty$.

12. Suppose there are three major centers for Move-It-Yourself trucks. Every month half of those in Boston and in Los Angeles go to Chicago, the other half stay where they are, and the trucks in Chicago are split equally between Boston and Los Angeles. Set up the 3 by 3 transition matrix A, and find the steady state u_∞ corresponding to the eigenvalue $\lambda = 1$.

13. (a) In what range of a and b is the following equation a Markov process?

$$u_{k+1} = Au_k = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} u_k, \qquad u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Compute $u_k = S\Lambda^k S^{-1} u_0$ for any a and b.

(c) Under what condition on a and b does u_k approach a finite limit as $k \to \infty$, and what is the limit? Does A have to be a Markov matrix?

14. Multinational companies in the Americas, Asia, and Europe have assets of \$4 trillion. At the start, \$2 trillion are in the Americas and \$2 trillion in Europe. Each year $\frac{1}{2}$ the American money stays home, and $\frac{1}{4}$ goes to each of Asia and Europe. For Asia and Europe, $\frac{1}{2}$ stays home and $\frac{1}{2}$ is sent to the Americas.

(a) Find the matrix that gives

$$\begin{bmatrix} \text{Americas} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{\text{year } k+1} = A \begin{bmatrix} \text{Americas} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{\text{year } k}.$$

(b) Find the eigenvalues and eigenvectors of A.

(c) Find the limiting distribution of the \$4 trillion as the world ends.

(d) Find the distribution of the \$4 trillion at year k.

15. If A is a Markov matrix, show that the sum of the components of Ax equals the sum of the components of x. Deduce that if $Ax = \lambda x$ with $\lambda \neq 1$, the components of the eigenvector add to zero.

16. The solution to $du/dt = Au = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u$ (eigenvalues i and $-i$) goes around in a circle: $u = (\cos t, \sin t)$. Suppose we approximate du/dt by forward, backward, and centered differences **F**, **B**, **C**:

(F) $u_{n+1} - u_n = Au_n$ or $u_{n+1} = (I + A)u_n$ (this is Euler's method).

(B) $u_{n+1} - u_n = Au_{n+1}$ or $u_{n+1} = (I - A)^{-1}u_n$ (backward Euler).

(C) $u_{n+1} - u_n = \frac{1}{2}A(u_{n+1} + u_n)$ or $u_{n+1} = \left(I - \frac{1}{2}A\right)^{-1}\left(I + \frac{1}{2}A\right)u_n$.

Find the eigenvalues of $I + A$, $(I - A)^{-1}$, and $\left(I - \frac{1}{2}A\right)^{-1}\left(I + \frac{1}{2}A\right)$. For which difference equation does the solution u_n stay on a circle?

17. What values of α produce instability in $v_{n+1} = \alpha(v_n + w_n)$, $w_{n+1} = \alpha(v_n + w_n)$?

18. Find the largest a, b, c for which these matrices are stable or neutrally stable:

$$\begin{bmatrix} a & -.8 \\ .8 & .2 \end{bmatrix}, \quad \begin{bmatrix} b & .8 \\ 0 & .2 \end{bmatrix}, \quad \begin{bmatrix} c & .8 \\ .2 & c \end{bmatrix}.$$

19. Multiplying term by term, check that $(I - A)(I + A + A^2 + \cdots) = I$. This series represents $(I - A)^{-1}$. It is nonnegative when A is nonnegative, provided it has a finite sum; the condition for that is $\lambda_{\max} < 1$. Add up the infinite series, and confirm that it equals $(I - A)^{-1}$, for the consumption matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{which has } \lambda_{\max} = 0.$$

20. For $A = \begin{bmatrix} 0 & .2 \\ 0 & .5 \end{bmatrix}$, find the powers A^k (including A^0) and show explicitly that their sum agrees with $(I - A)^{-1}$.

21. Explain by mathematics or economics why increasing the "consumption matrix" A must increase $t_{\max} = \lambda_1$ (and slow down the expansion).

22. What are the limits as $k \to \infty$ (the steady states) of the following?

$$\begin{bmatrix} .4 & .2 \\ .6 & .8 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} .4 & .2 \\ .6 & .8 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} .4 & .2 \\ .6 & .8 \end{bmatrix}^k.$$

Problems 23–29 are about $A = S\Lambda S^{-1}$ and $A^k = S\Lambda^k S^{-1}$

23. Diagonalize A and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{has} \quad A^k = \frac{1}{2}\begin{bmatrix} 5^k + 1 & 5^k - 1 \\ 5^k - 1 & 5^k + 1 \end{bmatrix}.$$

24. Diagonalize B and compute $S\Lambda^k S^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{has} \quad B^k = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix}.$$

25. The eigenvalues of A are 1 and 9, the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = S\sqrt{\Lambda}\, S^{-1}$. Why is there no real matrix square root of B?

26. If A and B have the same λ's with the same full set of independent eigenvectors, their factorizations into ______ are the same. So $A = B$.

27. Suppose A and B have the same full set of eigenvectors, so that $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Prove that $AB = BA$.

28. (a) When do the eigenvectors for $\lambda = 0$ span the nullspace $N(A)$?
(b) When do all the eigenvectors for $\lambda \neq 0$ span the column space $C(A)$?

29. The powers A^k approach zero if all $|\lambda_i| < 1$, and they blow up if any $|\lambda_i| > 1$. Peter Lax gives four striking examples in his book *Linear Algebra*.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

$$\|A^{1024}\| > 10^{700} \qquad B^{1024} = I \qquad C^{1024} = -C \qquad \|D^{1024}\| < 10^{-78}$$

Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show that $B^4 = I$ and $C^3 = -I$.

5.4 DIFFERENTIAL EQUATIONS AND e^{At}

Wherever you find a system of equations, rather than a single equation, matrix theory has a part to play. For difference equations, the solution $u_k = A^k u_0$ depended on the powers of A. For differential equations, the solution $u(t) = e^{At}u(0)$ depends on the ***exponential*** of A. To define this exponential, and to understand it, we turn right away to an example:

$$\textbf{Differential equation} \qquad \frac{du}{dt} = Au = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u. \tag{1}$$

The first step is always to find the eigenvalues (-1 and -3) and the eigenvectors:

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then several approaches lead to $u(t)$. Probably the best is to match the general solution to the initial vector $u(0)$ at $t = 0$.

The general solution is a combination of pure exponential solutions. These are solutions of the special form $ce^{\lambda t}x$, where λ is an eigenvalue of A and x is its eigenvector. These pure solutions satisfy the differential equation, since $d/dt(ce^{\lambda t}x) = A(ce^{\lambda t}x)$. (They were our introduction to eigenvalues at the start of the chapter.) In this 2 by 2 example, there are two pure exponentials to be combined:

$$\textbf{Solution} \qquad u(t) = c_1e^{\lambda_1 t}x_1 + c_2e^{\lambda_2 t}x_2 \quad \text{or} \quad u = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{2}$$

At time zero, when the exponentials are $e^0 = 1$, $u(0)$ determines c_1 and c_2:

$$\textbf{Initial condition} \qquad u(0) = c_1x_1 + c_2x_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Sc.$$

You recognize S, the matrix of eigenvectors. The constants $c = S^{-1}u(0)$ are the same as they were for difference equations. Substituting them back into equation (2), the solution is

$$u(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S\begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix}S^{-1}u(0). \tag{3}$$

Here is the fundamental formula of this section: $Se^{\Lambda t}S^{-1}u(0)$ solves the differential equation, just as $S\Lambda^kS^{-1}u_0$ solved the difference equation:

$$u(t) = Se^{\Lambda t}S^{-1}u(0) \quad \text{with} \quad \Lambda = \begin{bmatrix} -1 & \\ & -3 \end{bmatrix} \quad \text{and} \quad e^{\Lambda t} = \begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix}. \tag{4}$$

There are two more things to be done with this example. One is to complete the mathematics, by giving a direct definition of the ***exponential of a matrix***. The other is to give a physical interpretation of the equation and its solution. It is the kind of differential equation that has useful applications.

The exponential of a diagonal matrix Λ is easy; $e^{\Lambda t}$ just has the n numbers $e^{\lambda t}$ on the diagonal. For a general matrix A, the natural idea is to imitate the power series $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$. If we replace x by At and 1 by I, this sum is an n by n matrix:

$$\textbf{Matrix exponential} \qquad e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots. \tag{5}$$

The series always converges, and its sum e^{At} has the right properties:

$$(e^{As})(e^{At}) = e^{A(s+t)}, \qquad (e^{At})(e^{-At}) = I, \quad \text{and} \quad \frac{d}{dt}(e^{At}) = Ae^{At}. \tag{6}$$

From the last one, $u(t) = e^{At}u(0)$ solves the differential equation. This solution must be the same as the form $Se^{\Lambda t}S^{-1}u(0)$ used for computation. To prove directly that those solutions agree, remember that each power $(S\Lambda S^{-1})^k$ telescopes into $A^k = S\Lambda^k S^{-1}$ (because S^{-1} cancels S). The whole exponential is diagonalized by S:

$$\begin{aligned} e^{At} &= I + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}t^2}{2!} + \frac{S\Lambda^3 S^{-1}t^3}{3!} + \cdots \\ &= S\left(I + \Lambda t + \frac{(\Lambda t)^2}{2!} + \frac{(\Lambda t)^3}{3!} + \cdots\right)S^{-1} = Se^{\Lambda t}S^{-1}. \end{aligned}$$

Example 1 In equation (1), the exponential of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ has $\Lambda = \begin{bmatrix} -1 & \\ & -3 \end{bmatrix}$:

$$e^{At} = Se^{\Lambda t}S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{bmatrix}.$$

At $t = 0$ we get $e^0 = I$. The infinite series e^{At} gives the answer for all t, but a series can be hard to compute. The form $Se^{\Lambda t}S^{-1}$ gives the same answer when A can be diagonalized; it requires n independent eigenvectors in S. This simpler form leads to a *combination of n exponentials* $e^{\lambda t}x$—which is the best solution of all:

5L If A can be diagonalized, $A = S\Lambda S^{-1}$, then $du/dt = Au$ has the solution

$$u(t) = e^{At}u(0) = Se^{\Lambda t}S^{-1}u(0). \tag{7}$$

The columns of S are the eigenvectors $x_1, \ldots, x_n$ of A. Multiplying gives

$$\begin{aligned} u(t) &= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} S^{-1}u(0) \\ &= c_1e^{\lambda_1 t}x_1 + \cdots + c_ne^{\lambda_n t}x_n = \text{combination of } e^{\lambda t}x. \end{aligned} \tag{8}$$

The constants c_i that match the initial conditions $u(0)$ are $c = S^{-1}u(0)$.

This gives a complete analogy with difference equations and $S\Lambda S^{-1}u_0$. In both cases we assumed that A could be diagonalized, since otherwise it has fewer than n eigenvectors and we have not found enough special solutions. The missing solutions do exist, but they are more complicated than pure exponentials $e^{\lambda t}x$. They involve "generalized eigenvectors" and factors like $te^{\lambda t}$. (To compute this defective case we can use the Jordan form in Appendix B, and find e^{Jt}.) **The formula $u(t) = e^{At}u(0)$ remains completely correct.**

The matrix e^{At} is ***never singular***. One proof is to look at its eigenvalues; if λ is an eigenvalue of A, then $e^{\lambda t}$ is the corresponding eigenvalue of e^{At}—and $e^{\lambda t}$ can never be zero. Another approach is to compute the determinant of the exponential:

$$\det e^{At} = e^{\lambda_1 t}e^{\lambda_2 t}\cdots e^{\lambda_n t} = e^{\operatorname{trace}(At)}. \tag{9}$$

Quick proof that e^{At} is invertible: *Just recognize e^{-At} as its inverse.*

This invertibility is fundamental for differential equations. If n solutions are linearly independent at $t = 0$, *they remain linearly independent forever.* If the initial vectors are $v_1, \dots, v_n$, we can put the solutions $e^{At}v$ into a matrix:

$$\begin{bmatrix} e^{At}v_1 & \cdots & e^{At}v_n \end{bmatrix} = e^{At}\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}.$$

The determinant of the left-hand side is the *Wronskian.* It never becomes zero, because it is the product of two nonzero determinants. Both matrices on the right-hand side are invertible.

Remark Not all differential equations come to us as a first-order system $du/dt = Au$. We may start from a single equation of higher order, like $y''' - 3y'' + 2y' = 0$. To convert to a 3 by 3 system, introduce $v = y'$ and $w = v'$ as additional unknowns along with y itself. Then these two equations combine with the original one to give $u' = Au$:

$$\begin{aligned} y' &= v \\ v' &= w \\ w' &= 3w - 2v \end{aligned} \qquad \text{or} \qquad u' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} y \\ v \\ w \end{bmatrix} = Au.$$

We are back to a first-order system. The problem can be solved two ways. In a course on differential equations, you would substitute $y = e^{\lambda t}$ into $y''' - 3y'' + 2y' = 0$:

$$(\lambda^3 - 3\lambda^2 + 2\lambda)e^{\lambda t} = 0 \qquad \text{or} \qquad \lambda(\lambda - 1)(\lambda - 2)e^{\lambda t} = 0. \tag{10}$$

The three pure exponential solutions are $y = e^{0t}$, $y = e^t$, and $y = e^{2t}$. No eigenvectors are involved. In a linear algebra course, we find the eigenvalues of A:

$$\det(A - \lambda I) = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & 3-\lambda \end{bmatrix} = -\lambda^3 + 3\lambda^2 - 2\lambda = 0. \tag{11}$$

Equations (10) and (11) are the same! The same three exponents appear: $\lambda = 0$, $\lambda = 1$, and $\lambda = 2$. This is a general rule which makes the two methods consistent; the growth rates of the solutions stay fixed when the equations change form. It seems to us that solving the third-order equation is quicker.

The physical significance of $du/dt = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u$ is easy to explain and at the same time genuinely important. This differential equation describes a process of *diffusion.*

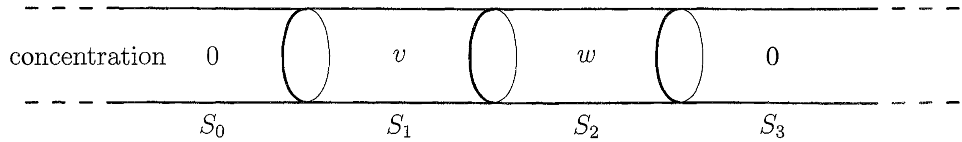

Figure 5.1 A model of diffusion between four segments.

Divide an infinite pipe into four segments (Figure 5.1). At time $t = 0$, the middle segments contain concentrations $v(0)$ and $w(0)$ of a chemical. ***At each time t, the diffusion rate between two adjacent segments is the difference in concentrations***. Within each segment, the concentration remains uniform (zero in the infinite segments). The process is continuous in time but discrete in space; the unknowns are $v(t)$ and $w(t)$ in the two inner segments S_1 and S_2.

The concentration $v(t)$ in S_1 is changing in two ways. There is diffusion into S_0, and into or out of S_2. The net rate of change is dv/dt, and dw/dt is similar:

$$\textbf{Flow rate into } \boldsymbol{S_1} \quad \frac{dv}{dt} = (w - v) + (0 - v)$$

$$\textbf{Flow rate into } \boldsymbol{S_2} \quad \frac{dw}{dt} = (0 - w) + (v - w).$$

This law of diffusion exactly matches our example $du/dt = Au$:

$$u = \begin{bmatrix} v \\ w \end{bmatrix} \qquad \text{and} \qquad \frac{du}{dt} = \begin{bmatrix} -2v + w \\ v - 2w \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u.$$

The eigenvalues -1 and -3 will govern the solution. They give the rate at which the concentrations decay, and λ_1 is the more important because only an exceptional set of starting conditions can lead to "superdecay" at the rate e^{-3t}. In fact, those conditions must come from the eigenvector $(1, -1)$. If the experiment admits only nonnegative concentrations, superdecay is impossible and the limiting rate must be e^{-t}. The solution that decays at this slower rate corresponds to the eigenvector $(1, 1)$. Therefore the two concentrations will become nearly equal (typical for diffusion) as $t \to \infty$.

One more comment on this example: It is a discrete approximation, with only two unknowns, to the continuous diffusion described by this partial differential equation:

$$\textbf{Heat equation} \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

That heat equation is approached by dividing the pipe into smaller and smaller segments, of length $1/N$. The discrete system with N unknowns is governed by

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ u_N \end{bmatrix} = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \cdot & \\ & \cdot & \cdot & 1 \\ & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ u_N \end{bmatrix} = Au. \tag{12}$$

This is the finite difference matrix with the $1, -2, 1$ pattern. The right side Au approaches the second derivative d^2u/dx^2, after a scaling factor N^2 comes from the flow problem. In the limit as $N \to \infty$, we reach the ***heat equation*** $\partial u/\partial t = \partial^2 u/\partial x^2$. Its solutions are still combinations of pure exponentials, but now there are infinitely many. Instead of eigenvectors from $Ax = \lambda x$, we have *eigenfunctions* from $d^2u/dx^2 = \lambda u$. Those are $u(x) = \sin n\pi x$ with $\lambda = -n^2\pi^2$. Then the solution to the heat equation is

$$u(t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin n\pi x.$$

The constants c_n are determined by the initial condition. The novelty is that the eigenvectors are functions $u(x)$, because the problem is continuous and not discrete.

Stability of Differential Equations

Just as for difference equations, the eigenvalues decide how $u(t)$ behaves as $t \to \infty$. As long as A can be diagonalized, there will be n pure exponential solutions to the differential equation, and any specific solution $u(t)$ is some combination

$$u(t) = Se^{\Lambda t}S^{-1}u_0 = c_1 e^{\lambda_1 t}x_1 + \cdots + c_n e^{\lambda_n t}x_n.$$

Stability is governed by those factors $e^{\lambda_i t}$. If they all approach zero, then $u(t)$ approaches zero; if they all stay bounded, then $u(t)$ stays bounded; if one of them blows up, then except for very special starting conditions the solution will blow up. Furthermore, the size of $e^{\lambda t}$ depends only on the real part of λ. ***It is only the real parts of the eigenvalues that govern stability***: If $\lambda = a + ib$, then

$$e^{\lambda t} = e^{at}e^{ibt} = e^{at}(\cos bt + i \sin bt) \quad \text{and the magnitude is} \quad |e^{\lambda t}| = e^{at}.$$

This decays for $a < 0$, it is constant for $a = 0$, and it explodes for $a > 0$. The imaginary part is producing oscillations, but the amplitude comes from the real part.

5M The differential equation $du/dt = Au$ is

stable and $e^{At} \to 0$ whenever all $\operatorname{Re}\lambda_i < 0$,

neutrally stable when all $\operatorname{Re}\lambda_i \le 0$ and $\operatorname{Re}\lambda_1 = 0$, and

unstable and e^{At} is unbounded if any eigenvalue has $\operatorname{Re}\lambda_i > 0$.

In some texts the condition $\operatorname{Re}\lambda < 0$ is called *asymptotic* stability, because it guarantees decay for large times t. Our argument depended on having n pure exponential solutions, but even if A is not diagonalizable (and there are terms like $te^{\lambda t}$) the result is still true: ***All solutions approach zero if and only if all eigenvalues have*** $\mathbf{Re}\,\boldsymbol{\lambda} < \mathbf{0}$.

Stability is especially easy to decide for a 2 by 2 system (which is very common in applications). The equation is

$$\frac{du}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} u,$$

and we need to know when both eigenvalues of that matrix have negative real parts. (Note again that the eigenvalues can be complex numbers.) The stability tests are

Re $\lambda_1 < 0$	***The trace $a + d$ must be negative.***
Re $\lambda_2 < 0$	***The determinant $ad - bc$ must be positive.***

When the eigenvalues are real, those tests guarantee them to be negative. Their product is the determinant; it is positive when the eigenvalues have the same sign. Their sum is the trace; it is negative when both eigenvalues are negative.

When the eigenvalues are a complex pair $x \pm iy$, the tests still succeed. The trace is their sum $2x$ (which is < 0) and the determinant is $(x + iy)(x - iy) = x^2 + y^2 > 0$. Figure 5.2 shows the one stable quadrant, trace < 0 and determinant > 0. It also shows the parabolic boundary line between real and complex eigenvalues. The reason for the parabola is in the quadratic equation for the eigenvalues:

$$\det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2 - (\text{trace})\lambda + (\det) = 0. \tag{13}$$

The quadratic formula for λ leads to the parabola $(\text{trace})^2 = 4(\det)$:

$$\lambda_1 \text{ and } \lambda_2 = \frac{1}{2}\left[\text{trace} \pm \sqrt{(\text{trace})^2 - 4(\det)}\,\right]. \tag{14}$$

Above the parabola, the number under the square root is negative—so λ is not real. On the parabola, the square root is zero and λ is repeated. Below the parabola the square roots are real. *Every symmetric matrix has real eigenvalues*, since if $b = c$, then

$$(\text{trace})^2 - 4(\det) = (a + d)^2 - 4(ad - b^2) = (a - d)^2 + 4b^2 \geq 0.$$

For complex eigenvalues, b and c have opposite signs and are sufficiently large.

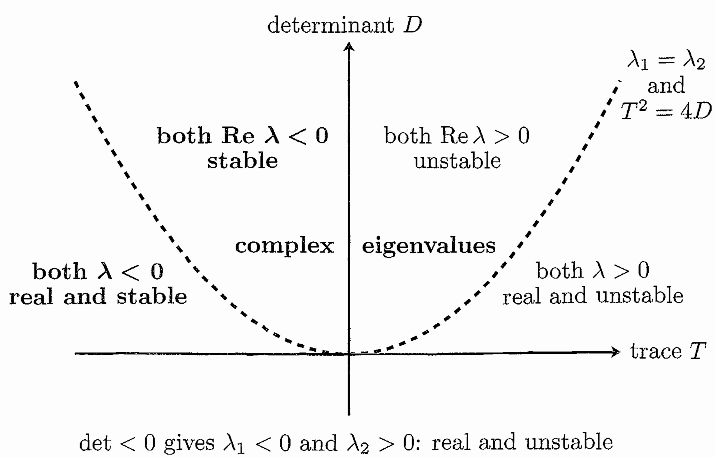

Figure 5.2 Stability and instability regions for a 2 by 2 matrix.

Example 2 One from each quadrant: only #2 is stable:

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix} \quad \begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix} \quad \begin{bmatrix}1 & 0\\0 & -2\end{bmatrix} \quad \begin{bmatrix}-1 & 0\\0 & 2\end{bmatrix}$$

On the boundaries of the second quadrant, the equation is neutrally stable. On the horizontal axis, one eigenvalue is zero (because the determinant is $\lambda_1\lambda_2 = 0$). On the vertical axis above the origin, both eigenvalues are purely imaginary (because the trace is zero). Crossing those axes are the two ways that stability is lost.

The n by n case is more difficult. A test for Re $\lambda_i < 0$ came from Routh and Hurwitz, who found a series of inequalities on the entries a_{ij}. I do not think this approach is much good for a large matrix; the computer can probably find the eigenvalues with more certainty than it can test these inequalities. Lyapunov's idea was to find a *weighting matrix W so that the weighted length $\|Wu(t)\|$ is always decreasing*. If there exists such a W, then $\|Wu\|$ will decrease steadily to zero, and after a few ups and downs u must get there too (stability). The real value of Lyapunov's method is for a nonlinear equation—then stability can be proved without knowing a formula for $u(t)$.

Example 3 $du/dt = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} u$ sends $u(t)$ around a circle, starting from $u(0) = (1, 0)$.

Since trace = 0 and det = 1, we have purely imaginary eigenvalues:

$$\begin{bmatrix}-\lambda & -1\\1 & -\lambda\end{bmatrix} = \lambda^2 + 1 = 0 \qquad \text{so} \qquad \lambda = +i \text{ and } -i.$$

The eigenvectors are $(1, -i)$ and $(1, i)$, and the solution is

$$u(t) = \frac{1}{2}e^{it}\begin{bmatrix}1\\-i\end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix}1\\i\end{bmatrix}.$$

That is correct but not beautiful. By substituting $\cos t \pm i \sin t$ for e^{it} and e^{-it}, *real numbers will reappear*: The circling solution is $u(t) = (\cos t, \sin t)$.

Starting from a different $u(0) = (a, b)$, the solution $u(t)$ ends up as

$$u(t) = \begin{bmatrix}a\cos t - b\sin t\\b\cos t + a\sin t\end{bmatrix} = \begin{bmatrix}\cos t & -\sin t\\\sin t & \cos t\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}. \tag{15}$$

There we have something important! The last matrix is multiplying $u(0)$, so it must be the exponential e^{At}. (Remember that $u(t) = e^{At}u(0)$.) That matrix of cosines and sines is our leading example of an *orthogonal matrix*. The columns have length 1, their inner product is zero, and we have a confirmation of a wonderful fact:

> ***If A is skew-symmetric ($A^{\mathrm{T}} = -A$) then e^{At} is an orthogonal matrix.***

$A^{\mathrm{T}} = -A$ gives a conservative system. No energy is lost in damping or diffusion:

$$A^{\mathrm{T}} = -A, \qquad (e^{At})^{\mathrm{T}} = e^{-At}, \qquad \text{and} \qquad \|e^{At}u(0)\| = \|u(0)\|.$$

That last equation expresses an essential property of orthogonal matrices. When they multiply a vector, the length is not changed. The vector $u(0)$ is just rotated, and that describes the solution to $du/dt = Au$: *It goes around in a circle.*

In this very unusual case, e^{At} can also be recognized directly from the infinite series. Note that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = -I$, and use this in the series for e^{At}:

$$I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots = \begin{bmatrix} \left(1 - \frac{t^2}{2} + \cdots\right) & \left(-t + \frac{t^3}{6} - \cdots\right) \\ \left(t - \frac{t^3}{6} + \cdots\right) & \left(1 - \frac{t^2}{2} + \cdots\right) \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

Example 4 The diffusion equation is stable: $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ has $\lambda = -1$ and $\lambda = -3$.

Example 5 If we close off the infinite segments, nothing can escape:

$$\frac{du}{dt} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} u \quad \text{or} \quad \begin{aligned} dv/dt &= w - v \\ dw/dt &= v - w. \end{aligned}$$

This is a *continuous Markov process*. Instead of moving every year, the particles move every instant. Their total number $v + w$ is constant. That comes from adding the two equations on the right-hand side: the derivative of $v + w$ is zero.

A discrete Markov matrix has its column sums equal to $\lambda_{\max} = 1$. A *continuous* Markov matrix, for differential equations, has its column sums equal to $\lambda_{\max} = 0$. A is a discrete Markov matrix if and only if $B = A - I$ is a continuous Markov matrix. The steady state for both is the eigenvector for $\lambda_{\max}$. It is multiplied by $1^k = 1$ in difference equations and by $e^{0t} = 1$ in differential equations, and it doesn't move.

In the example, the steady state has $v = w$.

Example 6 In nuclear engineering, a reactor is called *critical* when it is neutrally stable; the fission balances the decay. Slower fission makes it stable, or *subcritical*, and eventually it runs down. Unstable fission is a bomb.

Second-Order Equations

The laws of diffusion led to a first-order system $du/dt = Au$. So do a lot of other applications, in chemistry, in biology, and elsewhere, but the most important law of physics does not. It is *Newton's law* $F = ma$, and the acceleration a is a second derivative. Inertial terms produce second-order equations (we have to solve $d^2u/dt^2 = Au$ instead of $du/dt = Au$), and the goal is to understand how this switch to second derivatives alters the solution.* It is optional in linear algebra, but not in physics.

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* Fourth derivatives are also possible, in the bending of beams, but nature seems to resist going higher than four.

The comparison will be perfect if we keep the same A:

$$\frac{d^2u}{dt^2} = Au = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u. \tag{16}$$

Two initial conditions get the system started—the "displacement" $u(0)$ and the "velocity" $u'(0)$. To match these conditions, there will be $2n$ pure exponential solutions.

Suppose we use ω rather than λ, and write these special solutions as $u = e^{i\omega t}x$. Substituting this exponential into the differential equation, it must satisfy

$$\frac{d^2}{dt^2}(e^{i\omega t}x) = A(e^{i\omega t}x), \qquad \text{or} \qquad -\omega^2 x = Ax. \tag{17}$$

The vector x must be an eigenvector of A, exactly as before. The corresponding eigenvalue is now $-\omega^2$, so the frequency ω is connected to the decay rate λ by the law $-\omega^2 = \lambda$. Every special solution $e^{\lambda t}x$ of the first-order equation leads to *two* special solutions $e^{i\omega t}x$ of the second-order equation, and the two exponents are $\omega = \pm\sqrt{-\lambda}$. This breaks down only when $\lambda = 0$, which has just one square root; if the eigenvector is x, the two special solutions are x and tx.

For a genuine diffusion matrix, the eigenvalues λ are all negative and the frequencies ω are all real: *Pure diffusion is converted into pure oscillation*. The factors $e^{i\omega t}$ produce neutral stability, the solution neither grows or decays, and the total energy stays precisely constant. It just keeps passing around the system. The general solution to $d^2u/dt^2 = Au$, if A has negative eigenvalues $\lambda_1, \ldots, \lambda_n$ and if $\omega_j = \sqrt{-\lambda_j}$, is

$$u(t) = (c_1e^{i\omega_1 t} + d_1e^{-\omega_1 t})x_1 + \cdots + (c_ne^{i\omega_n t} + d_ne^{-i\omega_n t})x_n. \tag{18}$$

As always, the constants are found from the initial conditions. This is easier to do (at the expense of one extra formula) by switching from oscillating exponentials to the more familiar sine and cosine:

$$u(t) = (a_1 \cos\omega_1 t + b_1 \sin\omega_1 t)x_1 + \cdots + (a_n \cos\omega_n t + b_n \sin\omega_n t)x_n. \tag{19}$$

The initial displacement $u(0)$ is easy to keep separate: $t = 0$ means that $\sin\omega t = 0$ and $\cos\omega t = 1$, leaving only

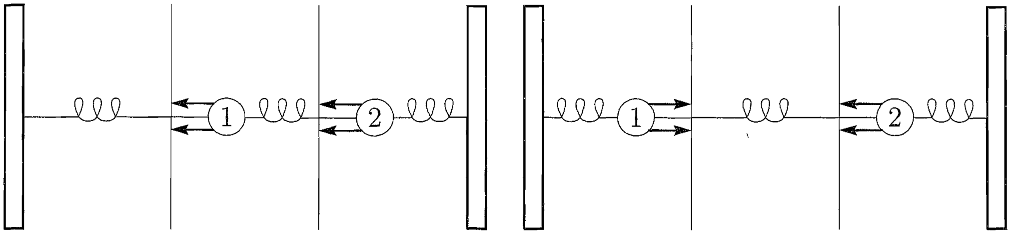
$$u(0) = a_1x_1 + \cdots + a_nx_n, \qquad \text{or} \qquad u(0) = Sa, \qquad \text{or} \qquad a = S^{-1}u(0).$$

Then differentiating $u(t)$ and setting $t = 0$, the b's are determined by the initial velocity: $u'(0) = b_1\omega_1x_1 + \cdots + b_n\omega_nx_n$. Substituting the a's and b's into the formula for $u(t)$, the equation is solved.

The matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ has $\lambda_1 = -1$ and $\lambda_2 = -3$. The frequencies are $\omega_1 = 1$ and $\omega_2 = \sqrt{3}$. If the system starts from rest, $u'(0) = 0$, the terms in $b \sin\omega t$ will disappear:

$$\textbf{Solution from } \boldsymbol{u(0)} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \qquad u(t) = \frac{1}{2}\cos t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\cos\sqrt{3}t \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Physically, two masses are connected to each other and to stationary walls by three identical springs (Figure 5.3). The first mass is held at $v(0) = 1$, the second mass is held at $w(0) = 0$, and at $t = 0$ we let go. Their motion $u(t)$ becomes an average of two pure oscillations, corresponding to the two eigenvectors. In the first mode $x_1 = (1, 1)$, the



(a) $\omega_1 = 1,\ x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (b) $\omega_2 = \sqrt{3},\ x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Figure 5.3 The slow and fast modes of oscillation.

masses move together and the spring in the middle is never stretched (Figure 5.3a). The frequency $\omega_1 = 1$ is the same as for a single spring and a single mass. In the faster mode $x_2 = (1, -1)$ with frequency $\sqrt{3}$, the masses move oppositely but with equal speeds. The general solution is a combination of these two normal modes. Our particular solution is half of each.

As time goes on, the motion is "almost periodic." If the ratio ω_1/ω_2 had been a fraction like $2/3$, the masses would eventually return to $u(0) = (1, 0)$ and begin again. A combination of $\sin 2t$ and $\sin 3t$ would have a period of 2π. But $\sqrt{3}$ is irrational. The best we can say is that the masses will come *arbitrarily close* to $(1, 0)$ and also $(0, 1)$. Like a billiard ball bouncing forever on a perfectly smooth table, the total energy is fixed. Sooner or later the masses come near any state with this energy.

Again we cannot leave the problem without drawing a parallel to the continuous case. As the discrete masses and springs merge into a solid rod, the "second differences" given by the $1, -2, 1$ matrix A turn into second derivatives. This limit is described by the celebrated ***wave equation*** $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$.

Problem Set 5.4

1. Following the first example in this section, find the eigenvalues and eigenvectors, and the exponential e^{At}, for

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

2. For the previous matrix, write the general solution to $du/dt = Au$, and the specific solution that matches $u(0) = (3, 1)$. What is the *steady state* as $t \to \infty$? (This is a continuous Markov process; $\lambda = 0$ in a differential equation corresponds to $\lambda = 1$ in a difference equation, since $e^{0t} = 1$.)

3. Suppose the time direction is reversed to give the matrix $-A$:

$$\frac{du}{dt} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u \quad \text{with} \quad u_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Find $u(t)$ and show that it *blows up* instead of decaying as $t \to \infty$. (Diffusion is irreversible, and the heat equation cannot run backward.)

4. If P is a projection matrix, show from the infinite series that

$$e^P \approx I + 1.718P.$$

5. A diagonal matrix like $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ satisfies the usual rule $e^{\Lambda(t+T)} = e^{\Lambda t}e^{\Lambda T}$, because the rule holds for each diagonal entry.

(a) Explain why $e^{A(t+T)} = e^{At}e^{AT}$, using the formula $e^{At} = Se^{\Lambda t}S^{-1}$.

(b) Show that $e^{A+B} = e^A e^B$ is *not true* for matrices, from the example

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad \text{(use series for } e^A \text{ and } e^B\text{)}.$$

6. The higher order equation $y'' + y = 0$ can be written as a first-order system by introducing the velocity y' as another unknown:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -y \end{bmatrix}.$$

If this is $du/dt = Au$, what is the 2 by 2 matrix A? Find its eigenvalues and eigenvectors, and compute the solution that starts from $y(0) = 2$, $y'(0) = 0$.

7. Convert $y'' = 0$ to a first-order system $du/dt = Au$:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}.$$

This 2 by 2 matrix A has only one eigenvector and cannot be diagonalized. Compute e^{At} from the series $I + At + \cdots$ and write the solution $e^{At}u(0)$ starting from $y(0) = 3$, $y'(0) = 4$. Check that your (y, y') satisfies $y'' = 0$.

8. Suppose the rabbit population r and the wolf population w are governed by

$$\frac{dr}{dt} = 4r - 2w$$

$$\frac{dw}{dt} = r + w.$$

(a) Is this system stable, neutrally stable, or unstable?

(b) If initially $r = 300$ and $w = 200$, what are the populations at time t?

(c) After a long time, what is the proportion of rabbits to wolves?

9. Decide the stability of $u' = Au$ for the following matrices:

(a) $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$. (d) $A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$.

10. Decide on the stability or instability of $dv/dt = w$, $dw/dt = v$. Is there a solution that decays?

11. From their trace and determinant, at what time t do the following matrices change between stable with real eigenvalues, stable with complex eigenvalues, and unstable?

$$A_1 = \begin{bmatrix} 1 & -1 \\ t & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 4-t \\ 1 & -2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}.$$

12. Find the eigenvalues and eigenvectors for

$$\frac{du}{dt} = Au = \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix} u.$$

Why do you know, without computing, that e^{At} will be an orthogonal matrix and $\|u(t)\|^2 = u_1^2 + u_2^2 + u_3^2$ will be constant?

13. For the skew-symmetric equation

$$\frac{du}{dt} = Au = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

(a) write out u_1', u_2', u_3' and confirm that $u_1'u_1 + u_2'u_2 + u_3'u_3 = 0$.
(b) deduce that the length $u_1^2 + u_2^2 + u_3^2$ is a constant.
(c) find the eigenvalues of A.

The solution will rotate around the axis $w = (a, b, c)$, because Au is the "cross product" $u \times w$—which is perpendicular to u and w.

14. What are the eigenvalues λ and frequencies ω, and the general solution, of the following equation?

$$\frac{d^2u}{dt^2} = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix} u.$$

15. Solve the second-order equation

$$\frac{d^2u}{dt^2} = \begin{bmatrix} -5 & -1 \\ -1 & -5 \end{bmatrix} u \quad \text{with} \quad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

16. In most applications the second-order equation looks like $Mu'' + Ku = 0$, with a *mass matrix* multiplying the second derivatives. Substitute the pure exponential $u = e^{i\omega t}x$ and find the "generalized eigenvalue problem" that must be solved for the frequency ω and the vector x.

17. With a friction matrix F in the equation $u'' + Fu' - Au = 0$, substitute a pure exponential $u = e^{\lambda t}x$ and find a quadratic eigenvalue problem for λ.

18. For equation (16) in the text, with $\omega = 1$ and $\sqrt{3}$, find the motion if the first mass is hit at $t = 0$; $u(0) = (0, 0)$ and $u'(0) = (1, 0)$.

19. Every 2 by 2 matrix with trace zero can be written as

$$A = \begin{bmatrix} a & b+c \\ b-c & -a \end{bmatrix}.$$

Show that its eigenvalues are real exactly when $a^2 + b^2 \geq c^2$.

20. By back-substitution or by computing eigenvectors, solve

$$\frac{du}{dt} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix} u \quad \text{with} \quad u(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

21. Find λ's and x's so that $u = e^{\lambda t}x$ solves

$$\frac{du}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} u.$$

What combination $u = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2$ starts from $u(0) = (5, -2)$?

22. Solve Problem 21 for $u(t) = (y(t), z(t))$ by back-substitution:

$$\text{First solve } \frac{dz}{dt} = z, \text{ starting from } z(0) = -2.$$

$$\text{Then solve } \frac{dy}{dt} = 4y + 3z, \text{ starting from } y(0) = 5.$$

The solution for y will be a combination of e^{4t} and e^t.

23. Find A to change $y'' = 5y' + 4y$ into a vector equation for $u(t) = (y(t), y'(t))$:

$$\frac{du}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au.$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into the scalar equation $y'' = 5y' + 4y$.

24. A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \qquad \text{and} \qquad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $du/dt = Au$, and its eigenvalues and eigenvectors. What are v and w at $t = 1$?

25. Reverse the diffusion of people in Problem 24 to $du/dt = -Au$:

$$\frac{dv}{dt} = v - w \qquad \text{and} \qquad \frac{dw}{dt} = w - v.$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

26. The solution to $y'' = 0$ is a straight line $y = C + Dt$. Convert to a matrix equation:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \text{ has the solution } \begin{bmatrix} y \\ y' \end{bmatrix} = e^{At} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$

This matrix A cannot be diagonalized. Find A^2 and compute $e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots$. Multiply your e^{At} times $(y(0), y'(0))$ to check the straight line $y(t) = y(0) + y'(0)t$.

27. Substitute $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble; we need a second solution after e^{3t}. The matrix equation is

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}.$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. *Trouble here too.* Show that the second solution is $y = te^{3t}$.

28. Figure out how to write $my'' + by' + ky = 0$ as a vector equation $Mu' = Au$.

29. (a) Find two familiar functions that solve the equation $d^2y/dt^2 = -y$. Which one starts with $y(0) = 1$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -y$ produces a vector equation $u' = Au$:

$$u = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{du}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au.$$

Put $y(t)$ from part (a) into $u(t) = (y, y')$. This solves Problem 6 again.

30. A particular solution to $du/dt = Au - b$ is $u_p = A^{-1}b$, if A is invertible. The solutions to $du/dt = Au$ give u_n. Find the complete solution $u_p + u_n$ to

(a) $\dfrac{du}{dt} = 2u - 8$. (b) $\dfrac{du}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} u - \begin{bmatrix} 8 \\ 6 \end{bmatrix}$.

31. If c is not an eigenvalue of A, substitute $u = e^{ct}v$ and find v to solve $du/dt = Au - e^{ct}b$. This $u = e^{ct}v$ is a particular solution. How does it break down when c is an eigenvalue?

32. Find a matrix A to illustrate each of the unstable regions in Figure 5.2:

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$.
(b) $\lambda_1 > 0$ and $\lambda_2 > 0$.
(c) Complex λ's with real part $a > 0$.

Problems 33–41 are about the matrix exponential e^{At}.

33. Write five terms of the infinite series for e^{At}. Take the t derivative of each term. Show that you have four terms of Ae^{At}. Conclusion: $e^{At}u(0)$ solves $u' = Au$.

34. The matrix $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ has $B^2 = 0$. Find e^{Bt} from a (short) infinite series. Check that the derivative of e^{Bt} is Be^{Bt}.

35. Starting from $u(0)$, the solution at time T is $e^{AT}u(0)$. Go an additional time t to reach $e^{At}\left(e^{AT}u(0)\right)$. This solution at time $t + T$ can also be written as ______. Conclusion: e^{At} times e^{AT} equals ______.

36. Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ in the form $S\Lambda S^{-1}$. Find e^{At} from $Se^{\Lambda t}S^{-1}$.

37. If $A^2 = A$, show that the infinite series produces $e^{At} = I + (e^t - 1)A$. For $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ in Problem 36, this gives $e^{At} =$ ______.

38. Generally $e^A e^B$ is different from $e^B e^A$. They are both different from e^{A+B}. Check this using Problems 36–37 and 34:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

39. Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check $e^{At} = I$ when $t = 0$.

40. Put $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ into the infinite series to find e^{At}. First compute A^2:

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 3t \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} + \cdots = \begin{bmatrix} e^t & \quad \\ 0 & \quad \end{bmatrix}.$$

41. Give two reasons why the matrix exponential e^{At} is never singular:

(a) Write its inverse.

(b) Write its eigenvalues. If $Ax = \lambda x$ then $e^{At}x =$ ______ x.

42. Find a solution $x(t)$, $y(t)$ of the first system that gets large as $t \to \infty$. To avoid this instability a scientist thought of exchanging the two equations!

$$\begin{aligned} dx/dt &= 0x - 4y \\ dy/dt &= -2x + 2y \end{aligned} \quad \text{becomes} \quad \begin{aligned} dy/dt &= -2x + 2y \\ dx/dt &= 0x - 4y. \end{aligned}$$

Now the matrix $\begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix}$ is stable. It has $\lambda < 0$. Comment on this craziness.

43. From this general solution to $du/dt = Au$, find the matrix A:

$$u(t) = c_1 e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

5.5 COMPLEX MATRICES

It is no longer possible to work only with real vectors and real matrices. In the first half of this book, when the basic problem was $Ax = b$, the solution was real when A and b were real. Complex numbers could have been permitted, but would have contributed nothing new. Now we cannot avoid them. A real matrix has real coefficients in $\det(A - \lambda I)$, but the eigenvalues (as in rotations) may be complex.

We now introduce the space $\mathbf{C}^n$ of vectors with n *complex* components. Addition and matrix multiplication follow the same rules as before. ***Length is computed differently***. The old way, the vector in $\mathbf{C}^2$ with components $(1, i)$ would have zero length: $1^2 + i^2 = 0$, not good. The correct length squared is $1^2 + |i|^2 = 2$.

This change to $\|x\|^2 = |x_1|^2 + \cdots + |x_n|^2$ forces a whole series of other changes. The inner product, the transpose, the definitions of symmetric and orthogonal matrices, all need to be modified for complex numbers. The new definitions coincide with the old when the vectors and matrices are real. We have listed these changes in a table at the end of the section, and we explain them as we go.

That table virtually amounts to a dictionary for translating real into complex. We hope it will be useful to the reader. We particularly want to find out about ***symmetric matrices and Hermitian matrices***: *Where are their eigenvalues, and what is special about their eigenvectors*? For practical purposes, those are the most important questions in the theory of eigenvalues. We call attention in advance to the answers:

> **1.** ***Every symmetric matrix (and Hermitian matrix) has real eigenvalues.***
> **2.** ***Its eigenvectors can be chosen to be orthonormal.***

Strangely, to prove that the eigenvalues are real we begin with the opposite possibility—and that takes us to complex numbers, complex vectors, and complex matrices.

Complex Numbers and Their Conjugates

Probably the reader has already met complex numbers; a review is easy to give. The important ideas are the *complex conjugate* $\overline{x}$ and the *absolute value* $|x|$. Everyone knows that whatever i is, it satisfies the equation $i^2 = -1$. It is a pure imaginary number, and so are its multiples ib; b is real. The sum $a + ib$ is a complex number, and it is plotted in a natural way on the complex plane (Figure 5.4).

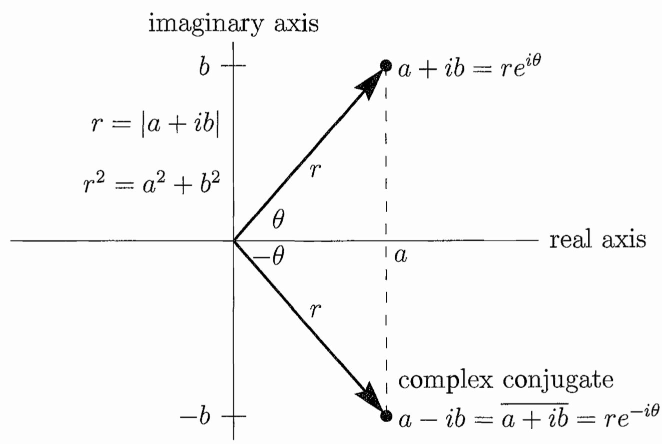


Figure 5.4 The complex plane, with $a + ib = re^{i\theta}$ and its conjugate $a - ib = re^{-i\theta}$.

The real numbers a and the imaginary numbers ib are special cases of complex numbers; they lie on the axes. Two complex numbers are easy to add:

$$\textbf{Complex addition} \qquad (a + ib) + (c + id) = (a + c) + i(b + d).$$

Multiplying $a + ib$ times $c + id$ uses the rule that $i^2 = -1$:

$$\begin{aligned} \textbf{Multiplication} \qquad (a + ib)(c + id) &= ac + ibc + iad + i^2bd \\ &= (ac - bd) + i(bc + ad). \end{aligned}$$

The ***complex conjugate*** of $a + ib$ is the number $a - ib$. *The sign of the imaginary part is reversed.* It is the mirror image across the real axis; any real number is its own conjugate, since $b = 0$. The conjugate is denoted by a bar or a star: $(a + ib)^* = \overline{a + ib} = a - ib$. It has three important properties:

1. The conjugate of a product equals the product of the conjugates:

$$\overline{(a + ib)(c + id)} = (ac - bd) - i(bc + ad) = \overline{(a + ib)}\;\overline{(c + id)}. \tag{1}$$

2. The conjugate of a sum equals the sum of the conjugates:

$$\overline{(a + c) + i(b + d)} = (a + c) - i(b + d) = \overline{(a + ib)} + \overline{(c + id)}.$$

3. Multiplying any $a + ib$ by its conjugate $a - ib$ produces a real number $a^2 + b^2$:

$$\textbf{Absolute value} \qquad (a + ib)(a - ib) = a^2 + b^2 = r^2. \tag{2}$$

This distance r is the ***absolute value*** $|a + ib| = \sqrt{a^2 + b^2}$.

Finally, trigonometry connects the sides a and b to the hypotenuse r by $a = r\cos\theta$ and $b = r\sin\theta$. Combining these two equations moves us into polar coordinates:

$$\textbf{Polar form} \qquad a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{3}$$

The most important special case is when $r = 1$. Then $a + ib$ is $e^{i\theta} = \cos\theta + i\sin\theta$. It falls on the ***unit circle*** in the complex plane. As θ varies from 0 to 2π, this number $e^{i\theta}$ circles around zero at the constant radial distance $|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$.

Example 1 $x = 3 + 4i$ times its conjugate $\overline{x} = 3 - 4i$ is the absolute value squared:

$$x\overline{x} = (3 + 4i)(3 - 4i) = 25 = |x|^2 \quad \text{so} \quad r = |x| = 5.$$

To divide by $3 + 4i$, multiply numerator and denominator by its conjugate $3 - 4i$:

$$\frac{2+i}{3+4i} = \frac{2+i}{3+4i}\,\frac{3-4i}{3-4i} = \frac{10-5i}{25}.$$

In polar coordinates, multiplication and division are easy:

$re^{i\theta}$ times $Re^{i\alpha}$ has absolute value rR and angle $\theta + \alpha$.
$re^{i\theta}$ divided by $Re^{i\alpha}$ has absolute value r/R and angle $\theta - \alpha$.

Lengths and Transposes in the Complex Case

We return to linear algebra, and make the conversion from real to complex. By definition, *the complex vector space* $\mathbf{C}^n$ *contains all vectors x with n complex components*:

$$\textbf{Complex vector} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{with components } x_j = a_j + ib_j.$$

Vectors x and y are still added component by component. Scalar multiplication cx is now done with complex numbers c. The vectors $v_1, \ldots, v_k$ are linearly *dependent* if some nontrivial combination gives $c_1v_1 + \cdots + c_kv_k = 0$; the c_j may now be complex. The unit coordinate vectors are still in $\mathbf{C}^n$; they are still independent; and they still form a basis. Therefore $\mathbf{C}^n$ is a complex vector space of dimension n.

In the new definition of length, each x_j^2 is replaced by its modulus $|x_j|^2$:

$$\textbf{Length squared} \qquad \|x\|^2 = |x_1|^2 + \cdots + |x_n|^2. \tag{4}$$

Example 2 $x = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\|x\|^2 = 2$; $y = \begin{bmatrix} 2+i \\ 2-4i \end{bmatrix}$ and $\|y\|^2 = 25$.

For real vectors there was a close connection between the length and the inner product: $\|x\|^2 = x^{\mathrm{T}}x$. This connection we want to preserve. The inner product must be modified to match the new definition of length, and we *conjugate the first vector in the inner product*. Replacing x by $\overline{x}$, ***the inner product becomes***

$$\textbf{Inner product} \qquad \overline{x}^{\mathrm{T}}y = \overline{x}_1y_1 + \cdots + \overline{x}_ny_n. \tag{5}$$

If we take the inner product of $x = (1+i, 3i)$ with itself, we are back to $\|x\|^2$:

$$\textbf{Length squared} \qquad \overline{x}^{\mathrm{T}}x = \overline{(1+i)}(1+i) + \overline{(3i)}(3i) = 2 + 9 \quad \text{and} \quad \|x\|^2 = 11.$$

Note that $\overline{y}^{\mathrm{T}}x$ is different from $\overline{x}^{\mathrm{T}}y$; we have to watch the order of the vectors.

This leaves only one more change in notation, condensing two symbols into one. Instead of a bar for the conjugate and a T for the transpose, those are combined into the ***conjugate transpose***. For vectors and matrices, a superscript H (or a star) combines both operations. This matrix $\overline{A}^{\mathrm{T}} = A^{\mathrm{H}} = A^*$ is called "A Hermitian":

$$\textbf{“}\boldsymbol{A}\textbf{ Hermitian”} \qquad A^{\mathrm{H}} = \overline{A}^{\mathrm{T}} \quad \text{has entries} \quad (A^{\mathrm{H}})_{ij} = \overline{A_{ji}}. \tag{6}$$

You have to listen closely to distinguish that name from the phrase "A *is* Hermitian," which means that A *equals* A^{H}. If A is an m by n matrix, then A^{H} is n by m:

$$\textbf{Conjugate transpose} \qquad \begin{bmatrix} 2+i & 3i \\ 4-i & 5 \\ 0 & 0 \end{bmatrix}^{\mathrm{H}} = \begin{bmatrix} 2-i & 4+i & 0 \\ -3i & 5 & 0 \end{bmatrix}.$$

This symbol A^{H} gives official recognition to the fact that, with complex entries, it is very seldom that we want only the transpose of A. It is the *conjugate* transpose A^{H} that becomes appropriate, and x^{H} is the row vector $[\overline{x}_1 \ \ldots \ \overline{x}_n]$.

5N **1.** The inner product of x and y is $x^{\mathrm{H}}y$. Orthogonal vectors have $x^{\mathrm{H}}y = 0$.
2. The squared length of x is $\|x\|^2 = x^{\mathrm{H}}x = |x_1|^2 + \cdots + |x_n|^2$.
3. Conjugating $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$ produces $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$.

Hermitian Matrices

We spoke in earlier chapters about symmetric matrices: $A = A^{\mathrm{T}}$. With complex entries, this idea of symmetry has to be extended. The right generalization is not to matrices that equal their transpose, but to ***matrices that equal their conjugate transpose***. These are the Hermitian matrices, and a typical example is A:

$$\textbf{Hermitian matrix} \qquad A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} = A^{\mathrm{H}}. \tag{7}$$

The diagonal entries must be real; they are unchanged by conjugation. Each off-diagonal entry is matched with its mirror image across the main diagonal, and $3-3i$ is the conjugate of $3+3i$. *In every case,* $a_{ij} = \overline{a_{ji}}$.

Our main goal is to establish three basic properties of Hermitian matrices. These properties apply equally well to symmetric matrices. *A real symmetric matrix is certainly Hermitian.* (For real matrices there is no difference between A^{T} and A^{H}.) **The eigenvalues of A are real**—as we now prove.

Property 1 If $A = A^{\mathrm{H}}$, then for all complex vectors x, the number $x^{\mathrm{H}}Ax$ is real.

Every entry of A contributes to $x^{\mathrm{H}}Ax$. Try the 2 by 2 case with $x = (u, v)$:

$$x^{\mathrm{H}}Ax = \begin{bmatrix} \overline{u} & \overline{v} \end{bmatrix} \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= 2\overline{u}u + 5\overline{v}v + (3-3i)\overline{u}v + (3+3i)u\overline{v}$$

$$= \textbf{real} + \textbf{real} + (\textbf{sum of complex conjugates}).$$

For a proof in general, $(x^{\mathrm{H}}Ax)^{\mathrm{H}}$ is the conjugate of the 1 by 1 matrix $x^{\mathrm{H}}Ax$, but we actually get the same number back again: $(x^{\mathrm{H}}Ax)^{\mathrm{H}} = x^{\mathrm{H}}A^{\mathrm{H}}x^{\mathrm{HH}} = x^{\mathrm{H}}Ax$. So that number must be real.

Property 2 If $A = A^{\mathrm{H}}$, every eigenvalue is real.

Proof Suppose $Ax = \lambda x$. ***The trick is to multiply by x^{H}: $x^{\mathrm{H}}Ax = \lambda x^{\mathrm{H}}x$.*** The left-hand side is real by Property 1, and the right-hand side $x^{\mathrm{H}}x = \|x\|^2$ is real and positive, because $x \neq 0$. Therefore $\lambda = x^{\mathrm{H}}Ax/x^{\mathrm{H}}x$ must be real. Our example has $\lambda = 8$ and $\lambda = -1$:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 - |3-3i|^2$$

$$= \lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1). \tag{8}$$

Note This proof of real eigenvalues looks correct for any real matrix:

False proof $\quad Ax = \lambda x \quad$ gives $\quad x^{\mathrm{T}}Ax = \lambda x^{\mathrm{T}}x, \quad$ so $\quad \lambda = \dfrac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} \quad$ is real.

There must be a catch: *The eigenvector x might be complex.* It is when $A = A^{\mathrm{T}}$ that we can be sure λ and x stay real. More than that, *the eigenvectors are perpendicular*: $x^{\mathrm{T}}y = 0$ in the real symmetric case and $x^{\mathrm{H}}y = 0$ in the complex Hermitian case.

Property 3 Two eigenvectors of a real symmetric matrix or a Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.

The proof starts with $Ax = \lambda_1 x$, $Ay = \lambda_2 y$, and $A = A^{\mathrm{H}}$:

$$(\lambda_1 x)^{\mathrm{H}}y = (Ax)^{\mathrm{H}}y = x^{\mathrm{H}}Ay = x^{\mathrm{H}}(\lambda_2 y). \tag{9}$$

The outside numbers are $\lambda_1 x^{\mathrm{H}}y = \lambda_2 x^{\mathrm{H}}y$, since the λ's are real. Now we use the assumption $\lambda_1 \neq \lambda_2$, *which forces the conclusion that $x^{\mathrm{H}}y = 0$*. In our example,

$$(A - 8I)x = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad x = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$(A + I)y = \begin{bmatrix} 3 & 3-3i \\ 3+3i & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad y = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}.$$

These two eigenvectors are orthogonal:

$$x^{\mathrm{H}}y = \begin{bmatrix} 1 & 1-i \end{bmatrix} \begin{bmatrix} 1-i \\ -1 \end{bmatrix} = 0.$$

Of course any multiples x/α and y/β are equally good as eigenvectors. MATLAB picks $\alpha = \|x\|$ and $\beta = \|y\|$, so that x/α and y/β are unit vectors; the eigenvectors are normalized to have length 1. They are now *orthonormal.* If these eigenvectors are chosen to be the columns of S, then we have $S^{-1}AS = \Lambda$ as always. ***The diagonalizing matrix can be chosen with orthonormal columns when $A = A^H$.***

In case A is real and symmetric, its eigenvalues are real by Property 2. Its unit eigenvectors are orthogonal by Property 3. Those eigenvectors are also real; they solve $(A - \lambda I)x = 0$. *These orthonormal eigenvectors go into an orthogonal matrix Q*, with $Q^TQ = I$ and $Q^T = Q^{-1}$. Then $S^{-1}AS = \Lambda$ becomes special—it is $Q^{-1}AQ = \Lambda$ or $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$. We can state one of the great theorems of linear algebra:

5O A real symmetric matrix can be factored into $A = Q\Lambda Q^T$. Its orthonormal eigenvectors are in the orthogonal matrix Q and its eigenvalues are in Λ.

In geometry or mechanics, this is the *principal axis theorem.* It gives the right choice of axes for an ellipse. Those axes are perpendicular, and they point along the eigenvectors of the corresponding matrix. (Section 6.2 connects symmetric matrices to n-dimensional ellipses.) In mechanics the eigenvectors give the principal directions, along which there is pure compression or pure tension—with no shear.

In mathematics the formula $A = Q\Lambda Q^T$ is known as the *spectral theorem.* If we multiply columns by rows, the matrix A becomes a combination of one-dimensional projections—which are the special matrices xx^T of rank 1, multiplied by λ:

$$A = Q\Lambda Q^T = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \text{---} \; x_1^T \; \text{---} \\ \vdots \\ \text{---} \; x_n^T \; \text{---} \end{bmatrix}$$

$$= \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \cdots + \lambda_n x_n x_n^T. \tag{10}$$

Our 2 by 2 example has eigenvalues 3 and 1:

Example 3 $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} =$ **combination of two projections.**

The eigenvectors, with length scaled to 1, are

$$x_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then the matrices on the right-hand side are $x_1x_1^T$ and $x_2x_2^T$—columns times rows—and they are projections onto the line through x_1 and the line through x_2.

All symmetric matrices are combinations of one-dimensional projections—which are symmetric matrices of rank 1.

Remark If A is real and its eigenvalues happen to be real, then its eigenvectors are also real. They solve $(A - \lambda I)x = 0$ and can be computed by elimination. But they will not be orthogonal unless A is symmetric: $A = Q\Lambda Q^T$ leads to $A^T = A$.

If A is real, all complex eigenvalues come in conjugate pairs: $Ax = \lambda x$ and $A\overline{x} = \overline{\lambda}\overline{x}$. *If $a + ib$ is an eigenvalue of a real matrix, so is $a - ib$.* (If $A = A^{\mathrm{T}}$ then $b = 0$.)

Strictly speaking, the spectral theorem $A = Q\Lambda Q^{\mathrm{T}}$ has been proved only when the eigenvalues of A are distinct. Then there are certainly n independent eigenvectors, and A can be safely diagonalized. Nevertheless it is true (see Section 5.6) that *even with repeated eigenvalues, a symmetric matrix still has a complete set of orthonormal eigenvectors*. The extreme case is the identity matrix, which has $\lambda = 1$ repeated n times—and no shortage of eigenvectors.

To finish the complex case we need the analogue of a real orthogonal matrix—and you can guess what happens to the requirement $Q^{\mathrm{T}}Q = I$. The transpose will be replaced by the conjugate transpose. The condition will become $U^{\mathrm{H}}U = I$. The new letter U reflects the new name: ***A complex matrix with orthonormal columns is called a unitary matrix***.

Unitary Matrices

May we propose two analogies? ***A Hermitian (or symmetric) matrix can be compared to a real number. A unitary (or orthogonal) matrix can be compared to a number on the unit circle***—a complex number of absolute value 1. The λ's are real if $A^{\mathrm{H}} = A$, and they are on the unit circle if $U^{\mathrm{H}}U = I$. The eigenvectors can be scaled to unit length and made orthonormal.*

Those statements are not yet proved for unitary (including orthogonal) matrices. Therefore we go directly to the three properties of U that correspond to the earlier Properties 1–3 of A. Remember that U has orthonormal columns:

$$\textbf{Unitary matrix} \qquad U^{\mathrm{H}}U = I, \qquad UU^{\mathrm{H}} = I, \quad \text{and} \quad U^{\mathrm{H}} = U^{-1}.$$

This leads directly to Property 1′, that multiplication by U has no effect on inner products, angles, or lengths. The proof is on one line, just as it was for Q:

Property 1′ $(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}U^{\mathrm{H}}Uy = x^{\mathrm{H}}y$ and lengths are preserved by U:

$$\textbf{Length unchanged} \qquad \|Ux\|^2 = x^{\mathrm{H}}U^{\mathrm{H}}Ux = \|x\|^2. \tag{11}$$

Property 2′ Every eigenvalue of U has absolute value $|\lambda| = 1$.

This follows directly from $Ux = \lambda x$, by comparing the lengths of the two sides: $\|Ux\| = \|x\|$ by Property 1′, and always $\|\lambda x\| = |\lambda|\|x\|$. Therefore $|\lambda| = 1$.

Property 3′ Eigenvectors corresponding to different eigenvalues are orthonormal.

---

* Later we compare "skew-Hermitian" matrices with pure imaginary numbers, and "normal" matrices with all complex numbers $a + ib$. A nonnormal matrix without orthogonal eigenvectors belongs to none of these classes, and is outside the whole analogy.

Start with $Ux = \lambda_1 x$ and $Uy = \lambda_2 y$, and take inner products by Property 1′:

$$x^{\mathrm{H}}y = (Ux)^{\mathrm{H}}(Uy) = (\lambda_1 x)^{\mathrm{H}}(\lambda_2 y) = \overline{\lambda}_1\lambda_2 x^{\mathrm{H}}y.$$

Comparing the left to the right, $\overline{\lambda}_1\lambda_2 = 1$ or $x^{\mathrm{H}}y = 0$. But Property 2′ is $\overline{\lambda}_1\lambda_1 = 1$, so we cannot also have $\overline{\lambda}_1\lambda_2 = 1$. Thus $x^{\mathrm{H}}y = 0$ and the eigenvectors are orthogonal.

Example 4 $U = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ has eigenvalues e^{it} and e^{-it}.

The orthogonal eigenvectors are $x = (1, -i)$ and $y = (1, i)$. (Remember to take conjugates in $x^{\mathrm{H}}y = 1 + i^2 = 0$.) After division by $\sqrt{2}$ they are orthonormal.

Here is the most important *unitary matrix* by far.

Example 5
$$U = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 & 1 & \cdot & 1 \\ 1 & w & \cdot & w^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & w^{n-1} & \cdot & w^{(n-1)^2} \end{bmatrix} = \frac{\textbf{Fourier matrix}}{\sqrt{n}}.$$

The complex number w is on the unit circle at the angle $\theta = 2\pi/n$. It equals $e^{2\pi i/n}$. Its powers are spaced evenly around the circle. That spacing assures that the sum of all n powers of w—all the nth roots of 1—is zero. Algebraically, the sum $1 + w + \cdots + w^{n-1}$ is $(w^n - 1)/(w - 1)$. And $w^n - 1$ is zero!

$$\text{row 1 of } U^{\mathrm{H}} \text{ times column 2 of } U \text{ is } \frac{1}{n}(1 + w + w^2 + \cdots + w^{n-1}) = \frac{w^n - 1}{w - 1} = 0.$$

$$\text{row } i \text{ of } U^{\mathrm{H}} \text{ times column } j \text{ of } U \text{ is } \frac{1}{n}(1 + W + W^2 + \cdots + W^{n-1}) = \frac{W^n - 1}{W - 1} = 0.$$

In the second case, $W = w^{j-i}$. Every entry of the original F has absolute value 1. The factor $\sqrt{n}$ shrinks the columns of U into unit vectors. ***The fundamental identity of the finite Fourier transform is $U^{\mathrm{H}}U = I$.***

Thus U is a unitary matrix. Its inverse looks the same except that w is replaced by $w^{-1} = e^{-i\theta} = \overline{w}$. Since U is unitary, its inverse is found by transposing (which changes nothing) and conjugating (which changes w to $\overline{w}$). The inverse of this U is $\overline{U}$. Ux can be computed quickly by the ***Fast Fourier Transform*** as found in Section 3.5.

By Property 1′ of unitary matrices, the length of a vector x is the same as the length of Ux. The energy in state space equals the energy in transform space. The energy is the sum of $|x_j|^2$, and it is also the sum of the energies in the separate frequencies. The vector $x = (1, 0, \ldots, 0)$ contains equal amounts of every frequency component, and its Discrete Fourier Transform $Ux = (1, 1, \ldots, 1)/\sqrt{n}$ also has length 1.

Example 6

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

This is an orthogonal matrix, so by Property 3′ it must have orthogonal eigenvectors. They are the columns of the Fourier matrix! Its eigenvalues must have absolute value 1.

They are the numbers $1, w, \ldots, w^{n-1}$ (or $1, i, i^2, i^3$ in this 4 by 4 case). It is a real matrix, but its eigenvalues and eigenvectors are complex.

One final note. Skew-Hermitian matrices satisfy $K^{\mathrm{H}} = -K$, just as skew-symmetric matrices satisfy $K^{\mathrm{T}} = -K$. Their properties follow immediately from their close link to Hermitian matrices:

If A is Hermitian then $K = iA$ is skew-Hermitian.

The eigenvalues of K are purely imaginary instead of purely real; we multiply by i. The eigenvectors are not changed. The Hermitian example on the previous pages would lead to

$$K = iA = \begin{bmatrix} 2i & 3+3i \\ -3+3i & 5i \end{bmatrix} = -K^{\mathrm{H}}.$$

The diagonal entries are multiples of i (allowing zero). The eigenvalues are $8i$ and $-i$. The eigenvectors are still orthogonal, and we still have $K = U\Lambda U^{\mathrm{H}}$—with a unitary U instead of a real orthogonal Q, and with $8i$ and $-i$ on the diagonal of Λ.

This section is summarized by a table of parallels between real and complex.

Real versus Complex

$\mathbf{R}^n$ (n real components)	$\leftrightarrow$	$\mathbf{C}^n$ (n complex components)
length: $\|x\|^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	length: $\|x\|^2 = \lvert x_1\rvert^2 + \cdots + \lvert x_n\rvert^2$
transpose: $A_{ij}^{\mathrm{T}} = A_{ji}$	$\leftrightarrow$	Hermitian transpose: $A_{ij}^{\mathrm{H}} = \overline{A_{ji}}$
$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	$\leftrightarrow$	$(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
inner product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	$\leftrightarrow$	$x^{\mathrm{H}}y = \bar{x}_1y_1 + \cdots + \bar{x}_ny_n$
$(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	$\leftrightarrow$	$(Ax)^{\mathrm{H}}y = x^{\mathrm{H}}(A^{\mathrm{H}}y)$
orthogonality: $x^{\mathrm{T}}y = 0$	$\leftrightarrow$	orthogonality: $x^{\mathrm{H}}y = 0$
symmetric matrices: $A^{\mathrm{T}} = A$	$\leftrightarrow$	Hermitian matrices: $A^{\mathrm{H}} = A$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian $K^{\mathrm{H}} = -K$
orthogonal $Q^{\mathrm{T}}Q = I$ or $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary $U^{\mathrm{H}}U = I$ or $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|Qx\| = \|x\|$	$\leftrightarrow$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|Ux\| = \|x\|$

The columns, rows, and eigenvectors of Q and U are orthonormal, and every $\lvert\lambda\rvert = 1$

Problem Set 5.5

1. For the complex numbers $3 + 4i$ and $1 - i$,

   (a) find their positions in the complex plane.
   (b) find their sum and product.
   (c) find their conjugates and their absolute values.

   Do the original numbers lie inside or outside the unit circle?

2. What can you say about
   (a) the sum of a complex number and its conjugate?
   (b) the conjugate of a number on the unit circle?
   (c) the product of two numbers on the unit circle?
   (d) the sum of two numbers on the unit circle?

3. If $x = 2+i$ and $y = 1+3i$, find $\overline{x}$, $x\overline{x}$, xy, $1/x$, and x/y. Check that the absolute value $|xy|$ equals $|x|$ times $|y|$, and the absolute value $|1/x|$ equals 1 divided by $|x|$.

4. Find a and b for the complex numbers $a+ib$ at the angles $\theta = 30°, 60°, 90°$ on the unit circle. Verify by direct multiplication that the square of the first is the second, and the cube of the first is the third.

5. (a) If $x = re^{i\theta}$ what are x^2, x^{-1}, and $\overline{x}$ in polar coordinates? Where are the complex numbers that have $x^{-1} = \overline{x}$?
   (b) At $t = 0$, the complex number $e^{(-1+i)t}$ equals one. Sketch its path in the complex plane as t increases from 0 to 2π.

6. Find the lengths and the inner product of

$$x = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 2+4i \\ 4i \end{bmatrix}.$$

7. Write out the matrix A^{H} and compute $C = A^{\mathrm{H}}A$ if

$$A = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}.$$

What is the relation between C and C^{H}? Does it hold whenever C is constructed from some $A^{\mathrm{H}}A$?

8. (a) With the preceding A, use elimination to solve $Ax = 0$.
   (b) Show that the nullspace you just computed is orthogonal to $\boldsymbol{C}(A^{\mathrm{H}})$ *and not to the usual row space* $\boldsymbol{C}(A^{\mathrm{T}})$. The four fundamental spaces in the complex case are $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A)$ as before, and then $\boldsymbol{N}(A^{\mathrm{H}})$ and $\boldsymbol{C}(A^{\mathrm{H}})$.

9. (a) How is the determinant of A^{H} related to the determinant of A?
   (b) Prove that the determinant of any Hermitian matrix is real.

10. (a) How many degrees of freedom are there in a real symmetric matrix, a real diagonal matrix, and a real orthogonal matrix? (The first answer is the sum of the other two, because $A = Q\Lambda Q^{\mathrm{T}}$.)
    (b) Show that 3 by 3 Hermitian matrices A and also unitary U have 9 real degrees of freedom (columns of U can be multiplied by any $e^{i\theta}$).

11. Write P, Q and R in the form $\lambda_1 x_1 x_1^{\mathrm{H}} + \lambda_2 x_2 x_2^{\mathrm{H}}$ of the spectral theorem:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

12. Give a reason if true or a counterexample if false:

(a) If A is Hermitian, then $A + iI$ is invertible.
(b) If Q is orthogonal, then $Q + \frac{1}{2}I$ is invertible.
(c) If A is real, then $A + iI$ is invertible.

13. Suppose A is a symmetric 3 by 3 matrix with eigenvalues 0, 1, 2.

(a) What properties can be guaranteed for the corresponding unit eigenvectors u, v, w?
(b) In terms of u, v, w, describe the nullspace, left nullspace, row space, and column space of A.
(c) Find a vector x that satisfies $Ax = v + w$. Is x unique?
(d) Under what conditions on b does $Ax = b$ have a solution?
(e) If u, v, w are the columns of S, what are S^{-1} and $S^{-1}AS$?

14. In the list below, which classes of matrices contain A and which contain B?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

*Orthogonal, invertible, projection, permutation, Hermitian, rank-*1*, diagonalizable, Markov*. Find the eigenvalues of A and B.

15. What is the dimension of the space S of all n by n real symmetric matrices? The spectral theorem says that every symmetric matrix is a combination of n projection matrices. Since the dimension exceeds n, how is this difference explained?

16. Write one significant fact about the eigenvalues of each of the following.

(a) A real symmetric matrix.
(b) A stable matrix: all solutions to $du/dt = Au$ approach zero.
(c) An orthogonal matrix.
(d) A Markov matrix.
(e) A defective matrix (nondiagonalizable).
(f) A singular matrix.

17. Show that if U and V are unitary, so is UV. Use the criterion $U^{\mathrm{H}}U = I$.

18. Show that a unitary matrix has $|\det U| = 1$, but possibly $\det U$ is different from $\det U^{\mathrm{H}}$. Describe all 2 by 2 matrices that are unitary.

19. Find a third column so that U is unitary. How much freedom in column 3?

$$U = \begin{bmatrix} 1/\sqrt{3} & i/\sqrt{2} & \\ 1/\sqrt{3} & 0 & \\ i/\sqrt{3} & 1/\sqrt{2} & \end{bmatrix}.$$

20. Diagonalize the 2 by 2 skew-Hermitian matrix $K = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$, whose entries are all

$\sqrt{-1}$. Compute $e^{Kt} = Se^{\Lambda t}S^{-1}$, and verify that e^{Kt} is unitary. What is the derivative of e^{Kt} at $t = 0$?

21. Describe all 3 by 3 matrices that are simultaneously Hermitian, unitary, and diagonal. How many are there?

22. Every matrix Z can be split into a Hermitian and a skew-Hermitian part, $Z = A+K$, just as a complex number z is split into $a + ib$. The real part of z is half of $z + \bar{z}$, and the "real part" of Z is half of $Z + Z^{\mathrm{H}}$. Find a similar formula for the "imaginary part" K, and split these matrices into $A + K$:

$$Z = \begin{bmatrix} 3+i & 4+2i \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}.$$

23. Show that the columns of the 4 by 4 Fourier matrix F in Example 5 are eigenvectors of the permutation matrix P in Example 6.

24. For the permutation of Example 6, write out the *circulant matrix* $C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3$. (Its eigenvector matrix is again the Fourier matrix.) Write out also the four components of the matrix-vector product Cx, which is the *convolution* of $c = (c_0, c_1, c_2, c_3)$ and $x = (x_0, x_1, x_2, x_3)$.

25. For a circulant $C = F\Lambda F^{-1}$, why is it faster to multiply by F^{-1}, then Λ, then F (the convolution rule), than to multiply directly by C?

26. Find the lengths of $u = (1 + i, 1 - i, 1 + 2i)$ and $v = (i, i, i)$. Also find $u^{\mathrm{H}}v$ and $v^{\mathrm{H}}u$.

27. Prove that $A^{\mathrm{H}}A$ is always a Hermitian matrix. Compute $A^{\mathrm{H}}A$ and AA^{H}:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

28. If $Az = 0$, then $A^{\mathrm{H}}Az = 0$. If $A^{\mathrm{H}}Az = 0$, multiply by z^{H} to prove that $Az = 0$. The nullspaces of A and $A^{\mathrm{H}}A$ are ______. $A^{\mathrm{H}}A$ is an invertible Hermitian matrix when the nullspace of A contains only $z =$ ______.

29. When you multiply a Hermitian matrix by a real number c, is cA still Hermitian? If $c = i$, show that iA is skew-Hermitian. The 3 by 3 Hermitian matrices are a subspace, provided that the "scalars" are real numbers.

30. Which classes of matrices does P belong to: orthogonal, invertible, Hermitian, unitary, factorizable into LU, factorizable into QR?

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

31. Compute P^2, P^3, and P^{100} in Problem 30. What are the eigenvalues of P?

32. Find the unit eigenvectors of P in Problem 30, and put them into the columns of a unitary matrix U. What property of P makes these eigenvectors orthogonal?

33. Write down the 3 by 3 *circulant matrix* $C = 2I + 5P + 4P^2$. It has the same eigenvectors as P in Problem 30. Find its eigenvalues.

34. If U is unitary and Q is a real orthogonal matrix, show that U^{-1} is unitary and also UQ is unitary. Start from $U^{\mathrm{H}}U = I$ and $Q^{\mathrm{T}}Q = I$.

35. Diagonalize A (real λ's) and K (imaginary λ's) to reach $U\Lambda U^{\mathrm{H}}$:

$$A = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}$$

36. Diagonalize this orthogonal matrix to reach $Q = U\Lambda U^{\mathrm{H}}$. Now all λ's are ______:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

37. Diagonalize this unitary matrix V to reach $V = U\Lambda U^{\mathrm{H}}$. Again all $|\lambda| = 1$:

$$V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}.$$

38. If $v_1, \ldots, v_n$ is an orthonormal basis for $\mathbf{C}^n$, the matrix with those columns is a ______ matrix. Show that any vector z equals $(v_1^{\mathrm{H}}z)v_1 + \cdots + (v_n^{\mathrm{H}}z)v_n$.

39. The functions e^{-ix} and e^{ix} are orthogonal on the interval $0 \le x \le 2\pi$ because their *complex* inner product is $\int_0^{2\pi}$ ______ $= 0$.

40. The vectors $v = (1, i, 1)$, $w = (i, 1, 0)$ and $z =$ ______ are an orthogonal basis for ______.

41. If $A = R + iS$ is a Hermitian matrix, are the real matrices R and S symmetric?

42. The (complex) dimension of $\mathbf{C}^n$ is ______. Find a nonreal basis for $\mathbf{C}^n$.

43. Describe all 1 by 1 matrices that are Hermitian and also unitary. Do the same for 2 by 2 matrices.

44. How are the eigenvalues of A^{H} (square matrix) related to the eigenvalues of A?

45. If $u^{\mathrm{H}}u = 1$, show that $I - 2uu^{\mathrm{H}}$ is Hermitian and also unitary. The rank-1 matrix uu^{H} is the projection onto what line in $\mathbf{C}^n$?

46. If $A + iB$ is a unitary matrix (A and B are real), show that $Q = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is an orthogonal matrix.

47. If $A + iB$ is a Hermitian matrix (A and B are real), show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric.

48. Prove that the inverse of a Hermitian matrix is again a Hermitian matrix.

49. Diagonalize this matrix by constructing its eigenvalue matrix Λ and its eigenvector matrix S:

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} = A^{\mathrm{H}}.$$

50. A matrix with orthonormal eigenvectors has the form $A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$. *Prove that* $AA^{\mathrm{H}} = A^{\mathrm{H}}A$. These are exactly the *normal matrices*.

5.6 SIMILARITY TRANSFORMATIONS

Virtually every step in this chapter has involved the combination $S^{-1}AS$. The eigenvectors of A went into the columns of S, and that made $S^{-1}AS$ a diagonal matrix (called Λ). When A was symmetric, we wrote Q instead of S, choosing the eigenvectors to be orthonormal. In the complex case, when A is Hermitian we write U—it is still the matrix of eigenvectors. Now we look at all combinations $M^{-1}AM$—*formed with any invertible M on the right and its inverse on the left.* The invertible eigenvector matrix S may fail to exist (the defective case), or we may not know it, or we may not want to use it.

First a new word: ***The matrices A and $M^{-1}AM$ are "similar."*** Going from one to the other is a ***similarity transformation***. It is the natural step for differential equations or matrix powers or eigenvalues—just as elimination steps were natural for $Ax = b$. Elimination multiplied A on the left by L^{-1}, but not on the right by L. So U is not similar to A, and the pivots are *not* the eigenvalues.

A whole family of matrices $M^{-1}AM$ is similar to A, and there are two questions:

1. What do these similar matrices $M^{-1}AM$ have in common?
2. With a special choice of M, what special form can be achieved by $M^{-1}AM$?

The final answer is given by the ***Jordan form***, with which the chapter ends.

These combinations $M^{-1}AM$ arise in a differential or difference equation, when a "change of variables" $u = Mv$ introduces the new unknown v:

$$\frac{du}{dt} = Au \quad \text{becomes} \quad M\frac{dv}{dt} = AMv, \quad \text{or} \quad \frac{dv}{dt} = M^{-1}AMv$$

$$u_{n+1} = Au_n \quad \text{becomes} \quad Mv_{n+1} = AMv_n, \quad \text{or} \quad v_{n+1} = M^{-1}AMv_n.$$

The new matrix in the equation is $M^{-1}AM$. In the special case $M = S$, the system is uncoupled because $\Lambda = S^{-1}AS$ is diagonal. The eigenvectors evolve independently. This is the maximum simplification, but other M's are also useful. We try to make $M^{-1}AM$ easier to work with than A.

The family of matrices $M^{-1}AM$ includes A itself, by choosing $M = I$. Any of these similar matrices can appear in the differential and difference equations, by the change $u = Mv$, so they ought to have something in common, and they do: ***Similar matrices share the same eigenvalues.***

5P Suppose that $B = M^{-1}AM$. Then A and B have the **same eigenvalues. Every eigenvector x of A corresponds to an eigenvector $M^{-1}x$ of B.**

Start from $Ax = \lambda x$ and substitute $A = MBM^{-1}$:

$$\textbf{Same eigenvalue} \qquad MBM^{-1}x = \lambda x \quad \text{which is} \quad B(M^{-1}x) = \lambda(M^{-1}x). \tag{1}$$

The eigenvalue of B is still λ. The eigenvector has changed from x to $M^{-1}x$.

We can also check that $A - \lambda I$ and $B - \lambda I$ have the same determinant:

$$\textbf{Product of matrices} \qquad B - \lambda I = M^{-1}AM - \lambda I = M^{-1}(A - \lambda I)M$$
$$\textbf{Product rule} \qquad \det(B - \lambda I) = \det M^{-1} \det(A - \lambda I) \det M = \det(A - \lambda I).$$

The polynomials det $(A - \lambda I)$ and det $(B - \lambda I)$ are equal. Their roots—the eigenvalues of A and B—are the same. Here are matrices B similar to A.

Example 1 $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has eigenvalues 1 and 0. Each B is $M^{-1}AM$:

If $M = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$: triangular with $\lambda = 1$ and 0.

If $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: projection with $\lambda = 1$ and 0.

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $B =$ an arbitrary matrix with $\lambda = 1$ and 0.

In this case we can produce any B that has the correct eigenvalues. It is an easy case, because the eigenvalues 1 and 0 are distinct. The diagonal A was actually Λ, the outstanding member of this family of similar matrices (the *capo*). The Jordan form will worry about repeated eigenvalues and a possible shortage of eigenvectors. All we say now is that every $M^{-1}AM$ has the same number of independent eigenvectors as A (each eigenvector is multiplied by M^{-1}).

The first step is to look at the linear transformations that lie behind the matrices. Rotations, reflections, and projections act on n-dimensional space. The transformation can happen without linear algebra, but linear algebra turns it into matrix multiplication.

Change of Basis = Similarity Transformation

The similar matrix $B = M^{-1}AM$ is closely connected to A, if we go back to linear transformations. Remember the key idea: ***Every linear transformation is represented by a matrix***. The matrix depends on the choice of basis! *If we change the basis by M we change the matrix A to a similar matrix B.*

Similar matrices represent the same transformation T with respect to different bases. The algebra is almost straightforward. Suppose we have a basis $v_1, \ldots, v_n$. The jth column of A comes from applying T to v_j:

$$Tv_j = \text{combination of the basis vectors} = a_{1j}v_1 + \cdots + a_{nj}v_n. \tag{2}$$

For a new basis $V_1, \ldots, V_n$, the new matrix B is constructed in the same way: $TV_j =$ combination of the V's $= b_{1j}V_1 + \cdots + b_{nj}V_n$. But also each V must be a combination of the old basis vectors: $V_j = \sum m_{ij}v_i$. That matrix M is really representing the *identity transformation* (!) when the only thing happening is the change of basis (T is I). The inverse matrix M^{-1} also represents the identity transformation, when the basis is changed from the v's back to the V's. Now the product rule gives the result we want:

5Q The matrices A and B that represent the same linear transformation T with respect to two different bases (the v's and the V's) are ***similar***:

$$\begin{matrix} [T]_{V \text{ to } V} & = & [I]_{v \text{ to } V} & [T]_{v \text{ to } v} & [I]_{V \text{ to } v} \\ B & = & M^{-1} & A & M. \end{matrix} \tag{3}$$

I think an example is the best way to explain $B = M^{-1}AM$. Suppose T is *projection onto the line* L at angle θ. This linear transformation is completely described without the help of a basis. But to represent T by a matrix, we do need a basis. Figure 5.5 offers two choices, the standard basis $v_1 = (1,0)$, $v_2 = (0,1)$ and a basis V_1, V_2 chosen especially for T.

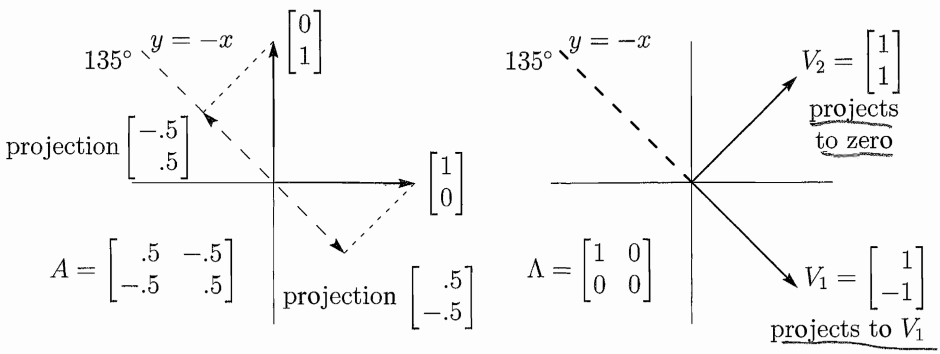


Figure 5.5 Change of basis to make the projection matrix diagonal.

In fact $TV_1 = V_1$ (since V_1 is already on the line L) and $TV_2 = 0$ (since V_2 is perpendicular to the line). In that eigenvector basis, the matrix is diagonal:

$$\textbf{Eigenvector basis} \qquad B = [T]_{V \text{ to } V} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$

The other thing is the change of basis matrix M. For that we express V_1 as a combination $v_1 \cos\theta + v_2 \sin\theta$ and put those coefficients into column 1. Similarly V_2 (or IV_2, the transformation is the identity) is $-v_1 \sin\theta + v_2 \cos\theta$, producing column 2:

$$\textbf{Change of basis} \qquad M = [I]_{V \text{ to } v} = \begin{bmatrix}c & -s\\s & c\end{bmatrix}.$$

The inverse matrix M^{-1} (which is here the transpose) goes from v to V. Combined with B and M, it gives the projection matrix in the standard basis of v's:

$$\textbf{Standard basis} \qquad A = MBM^{-1} = \begin{bmatrix}c^2 & cs\\cs & s^2\end{bmatrix}.$$

We can summarize the main point. The way to simplify that matrix A—in fact to diagonalize it—is to find its eigenvectors. They go into the columns of M (or S) and $M^{-1}AM$ is diagonal. The algebraist says the same thing in the language of linear transformations: *Choose a basis consisting of eigenvectors.* The standard basis led to A, which was not simple. The right basis led to B, which was diagonal.

We emphasize again that $M^{-1}AM$ does not arise in solving $Ax = b$. There the basic operation was to multiply A (on the left side only!) by a matrix that subtracts a multiple of one row from another. Such a transformation preserved the nullspace and row space of A; it normally changes the eigenvalues.

Eigenvalues are actually calculated by a sequence of simple similarities. The matrix goes gradually toward a triangular form, and the eigenvalues gradually appear on the main diagonal. (Such a sequence is described in Chapter 7.) This is much better than trying to compute det $(A-\lambda I)$, whose roots should be the eigenvalues. For a large matrix, it is numerically impossible to concentrate all that information into the polynomial and then get it out again.

Triangular Forms with a Unitary M

Our first move beyond the eigenvector matrix $M = S$ is a little bit crazy: Instead of a more general M, we go the other way and *restrict M to be unitary*. $M^{-1}AM$ can achieve a triangular form T under this restriction. The columns of $M = U$ are orthonormal (in the real case, we would write $M = Q$). Unless the eigenvectors of A are orthogonal, a diagonal $U^{-1}AU$ is impossible. But "Schur's lemma" in **5R** is very useful—at least to the theory. (The rest of this chapter is devoted more to theory than to applications. The Jordan form is independent of this triangular form.)

5R There is a unitary matrix $M = U$ such that $U^{-1}AU = T$ is triangular. The eigenvalues of A appear along the diagonal of this similar matrix T.

Proof Every matrix, say 4 by 4, has at least one eigenvalue λ_1. In the worst case, it could be repeated four times. Therefore A has at least one unit eigenvector x_1, which we place in the *first column of U*. At this stage the other three columns are impossible to determine, so we complete the matrix in any way that leaves it unitary, and call it U_1. (The Gram–Schmidt process guarantees that this can be done.) $Ax_1 = \lambda_1 x_1$ in column 1 means that the product $U_1^{-1}AU_1$ starts in the right form:

$$AU_1 = U_1\begin{bmatrix}\lambda_1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & *\end{bmatrix} \quad \text{leads to} \quad U_1^{-1}AU_1 = \begin{bmatrix}\lambda_1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & *\end{bmatrix}.$$

Now work with the 3 by 3 submatrix in the lower right-hand corner. It has a unit eigenvector x_2, which becomes the first column of a unitary matrix M_2:

$$\text{If} \quad U_2 = \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & M_2 & \\ 0 & & & \end{bmatrix} \quad \text{then} \quad U_2^{-1}\left(U_1^{-1}AU_1\right)U_2 = \begin{bmatrix}\lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & *\end{bmatrix}.$$

At the last step, an eigenvector of the 2 by 2 matrix in the lower right-hand corner goes into a unitary M_3, which is put into the corner of U_3:

$$\textbf{Triangular} \qquad U_3^{-1}\left(U_2^{-1}U_1^{-1}AU_1U_2\right)U_3 = \begin{bmatrix}\lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & \lambda_3 & * \\ 0 & 0 & 0 & *\end{bmatrix} = T.$$

The product $U = U_1U_2U_3$ is still a unitary matrix, and $U^{-1}AU = T$. ■

This lemma applies to all matrices, with no assumption that A is diagonalizable. We could use it to prove that ***the powers A^k approach zero when all $|\lambda_i| < 1$, and***

the exponentials e^{At} approach zero when all $\mathbf{Re}\,\lambda_i < \mathbf{0}$—even without the full set of eigenvectors which was assumed in Sections 5.3 and 5.4.

Example 2 $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ (twice).

The only line of eigenvectors goes through $(1, 1)$. After dividing by $\sqrt{2}$, this is the first column of U, and the triangular $U^{-1}AU = T$ has the eigenvalues on its diagonal:

$$U^{-1}AU = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T. \qquad (4)$$

Diagonalizing Symmetric and Hermitian Matrices

This triangular form will show that any symmetric or Hermitian matrix—whether its eigenvalues are *distinct or not*—has a complete set of orthonormal eigenvectors. We need a unitary matrix such that $U^{-1}AU$ is *diagonal*. Schur's lemma has just found it. This triangular T must be diagonal, because it is also Hermitian when $A = A^{\mathrm{H}}$:

$$\boldsymbol{T = T^{\mathrm{H}}} \qquad (U^{-1}AU)^{\mathrm{H}} = U^{\mathrm{H}}A^{\mathrm{H}}(U^{-1})^{\mathrm{H}} = U^{-1}AU.$$

The diagonal matrix $U^{-1}AU$ represents a key theorem in linear algebra.

> **5S (Spectral Theorem)** Every real symmetric A can be diagonalized by an orthogonal matrix Q. Every Hermitian matrix can be diagonalized by a unitary U:
>
> $$\textbf{(real)} \quad Q^{-1}AQ = \Lambda \quad \text{or} \quad A = Q\Lambda Q^{\mathrm{T}}$$
>
> $$\textbf{(complex)} \quad U^{-1}AU = \Lambda \quad \text{or} \quad A = U\Lambda U^{\mathrm{H}}$$
>
> The columns of Q (or U) contain orthonormal eigenvectors of A.

Remark 1 In the real symmetric case, the eigenvalues and eigenvectors are real at every step. That produces a *real* unitary U—an orthogonal matrix.

Remark 2 A is the limit of symmetric matrices with *distinct* eigenvalues. As the limit approaches, the eigenvectors stay perpendicular. This can fail if $A \neq A^{\mathrm{T}}$:

$$A(\theta) = \begin{bmatrix} 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \quad \text{has eigenvectors} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

As $\theta \to 0$, the *only* eigenvector of the nondiagonalizable matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Example 3 The spectral theorem says that this $A = A^{\mathrm{T}}$ can be diagonalized:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with repeated eigenvalues} \quad \lambda_1 = \lambda_2 = 1 \text{ and } \lambda_3 = -1.$$

$\lambda = 1$ has a plane of eigenvectors, and we pick an orthonormal pair x_1 and x_2:

$$x_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}; \quad \text{and} \quad x_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix} \quad \text{for } \lambda_3 = -1.$$

These are the columns of Q. Splitting $A = Q\Lambda Q^{\mathrm{T}}$ into 3 columns times 3 rows gives

$$A = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} = \lambda_1 \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{bmatrix} + \lambda_2 \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} + \lambda_3 \begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Since $\lambda_1 = \lambda_2$, those first two projections $x_1x_1^{\mathrm{T}}$ and $x_2x_2^{\mathrm{T}}$ (each of rank 1) combine to give a projection P_1 of rank 2 (onto the plane of eigenvectors). Then A is

$$\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} = \lambda_1 P_1 + \lambda_3 P_3 = (+1)\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix} + (-1)\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{bmatrix}. \qquad (5)$$

Every Hermitian matrix with k different eigenvalues has a ***spectral decomposition*** *into $A = \lambda_1 P_1 + \cdots + \lambda_k P_k$, where P_i is the projection onto the eigenspace for λ_i.* Since there is a full set of eigenvectors, the projections add up to the identity. And since the eigenspaces are orthogonal, two projections produce zero: $P_j P_i = 0$.

We are very close to answering an important question, so we keep going: **For which matrices is $T = \Lambda$?** Symmetric, skew-symmetric, and orthogonal T's are all diagonal! Hermitian, skew-Hermitian, and unitary matrices are also in this class. They correspond to numbers on the *real axis*, the *imaginary axis*, and the *unit circle*. Now we want the whole class, corresponding to all complex numbers. The matrices are called "normal."

5T The matrix N is ***normal*** if it commutes with N^{H}: $NN^{\mathrm{H}} = N^{\mathrm{H}}N$. For such matrices, and no others, the triangular $T = U^{-1}NU$ is the diagonal Λ. Normal matrices are exactly those that have a ***complete set of orthonormal eigenvectors***.

Symmetric and Hermitian matrices are certainly normal: If $A = A^{\mathrm{H}}$, then AA^{H} and $A^{\mathrm{H}}A$ both equal A^2. Orthogonal and unitary matrices are also normal: UU^{H} and $U^{\mathrm{H}}U$ both equal I. Two steps will work for any normal matrix:

1. If N is normal, then so is the triangular $T = U^{-1}NU$:

$$TT^{\mathrm{H}} = U^{-1}NUU^{\mathrm{H}}N^{\mathrm{H}}U = U^{-1}NN^{\mathrm{H}}U = U^{-1}N^{\mathrm{H}}NU = U^{\mathrm{H}}N^{\mathrm{H}}UU^{-1}NU = T^{\mathrm{H}}T.$$

2. A triangular T that is normal must be diagonal! (See Problems 19–20 at the end of this section.)

Thus, if N is normal, the triangular $T = U^{-1}NU$ must be diagonal. Since T has the same eigenvalues as N, it must be Λ. The eigenvectors of N are the columns of U, and they are orthonormal. That is the good case. We turn now from the best possible matrices (*normal*) to the worst possible (*defective*).

$$\textbf{Normal} \quad N = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \qquad \textbf{Defective} \quad A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

The Jordan Form

This section has done its best while requiring M to be a unitary matrix U. We got $M^{-1}AM$ into a triangular form T. Now we lift this restriction on M. Any matrix is allowed, and the goal is to make $M^{-1}AM$ as *nearly diagonal as possible.*

The result of this supreme effort at diagonalization is the ***Jordan form J***. If A has a full set of eigenvectors, we take $M = S$ and arrive at $J = S^{-1}AS = \Lambda$. Then the Jordan form coincides with the diagonal Λ. This is impossible for a defective (nondiagonalizable) matrix. *For every missing eigenvector, the Jordan form will have a* 1 *just above its main diagonal.* The eigenvalues appear on the diagonal because J is triangular. And distinct eigenvalues can always be decoupled.

It is only a repeated λ that may (or may not!) require an off-diagonal 1 in J.

5U If A has s independent eigenvectors, it is similar to a matrix with s blocks:

$$\textbf{Jordan form} \qquad J = M^{-1}AM = \begin{bmatrix} J_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & J_s \end{bmatrix}. \tag{6}$$

Each Jordan block J_i is a triangular matrix that has only a single eigenvalue λ_i and only one eigenvector:

$$\textbf{Jordan block} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \cdot & \\ & & \cdot & 1 \\ & & & \lambda_i \end{bmatrix}. \tag{7}$$

The same λ_i will appear in several blocks, if it has several independent eigenvectors. Two matrices are similar if and only if they share the same Jordan form J.

Many authors have made this theorem the climax of their linear algebra course. Frankly, I think that is a mistake. It is certainly true that not all matrices are diagonalizable, and the Jordan form is the most general case. For that very reason, its construction is both technical and extremely unstable. (A slight change in A can put back all the missing eigenvectors, and remove the off-diagonal 1s.) Therefore the right place for the details is in the appendix, and the best way to start on the Jordan form is to look at some specific and manageable examples.

Example 4 $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ all lead to $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

These four matrices have eigenvalues 1 and 1 with only *one eigenvector*—so J consists of *one block*. We now check that. The determinants all equal 1. The traces (the sums down the main diagonal) are 2. The eigenvalues satisfy $1 \cdot 1 = 1$ and $1 + 1 = 2$. For T, B, and J, which are triangular, the eigenvalues are on the diagonal. We want to show that *these matrices are similar*—they all belong to the same family.

(T) From T to J, the job is to change 2 to 1, and a diagonal M will do it:

$$M^{-1}TM = \overset{M}{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

(B) From B to J, the job is to transpose the matrix. A permutation does that:

$$P^{-1}BP = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

(A) From A to J, we go first to T as in equation (4). Then change 2 to 1:

$$U^{-1}AU = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T \qquad \text{and then} \qquad M^{-1}TM = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

Example 5 $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Zero is a triple eigenvalue for A and B, so it will appear in all their Jordan blocks. There can be a single 3 by 3 block, or a 2 by 2 and a 1 by 1 block, or three 1 by 1 blocks. Then A and B have three possible Jordan forms:

$$J_1 = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad J_2 = \begin{bmatrix} \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{0} & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix}, \qquad J_3 = \begin{bmatrix} \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix}. \tag{8}$$

The only eigenvector of A is $(1, 0, 0)$. Its Jordan form has only one block, and A must be similar to J_1. The matrix B has the additional eigenvector $(0, 1, 0)$, and its Jordan form is J_2 with two blocks. As for $J_3 =$ *zero matrix*, it is in a family by itself; the only matrix similar to J_3 is $M^{-1}0M = 0$. A count of the eigenvectors will determine J when there is nothing more complicated than a triple eigenvalue.

Example 6 ***Application to difference and differential equations*** (*powers and exponentials*). If A can be diagonalized, the powers of $A = S\Lambda S^{-1}$ are easy: $A^k = S\Lambda^k S^{-1}$. In every case we have Jordan's similarity $A = MJM^{-1}$, so now we need the powers of J:

$$A^k = (MJM^{-1})(MJM^{-1}) \cdots (MJM^{-1}) = MJ^kM^{-1}.$$

J is block-diagonal, and the powers of each block can be taken separately:

$$(J_i)^k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}. \tag{9}$$

This block J_i will enter when λ is a triple eigenvalue with a single eigenvector. Its exponential is in the solution to the corresponding differential equation:

$$\textbf{Exponential} \qquad e^{J_i t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}. \tag{10}$$

Here $I + J_i t + (J_i t)^2/2! + \cdots$ produces $1 + \lambda t + \lambda^2 t^2/2! + \cdots = e^{\lambda t}$ on the diagonal.

The third column of this exponential comes directly from solving $du/dt = J_i u$:

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{starting from} \quad u_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This can be solved by back-substitution (since J_i is triangular). The last equation $du_3/dt = \lambda u_3$ yields $u_3 = e^{\lambda t}$. The equation for u_2 is $du_2/dt = \lambda u_2 + u_3$, and its solution is $te^{\lambda t}$. The top equation is $du_1/dt = \lambda u_1 + u_2$, and its solution is $\frac{1}{2}t^2 e^{\lambda t}$. When λ has multiplicity m with only one eigenvector, the extra factor t appears $m - 1$ times.

These powers and exponentials of J are a part of the solutions u_k and $u(t)$. The other part is the M that connects the original A to the more convenient matrix J:

$$\begin{aligned} &\text{if} \quad u_{k+1} = Au_k \quad &&\text{then} \quad u_k = A^k u_0 = MJ^k M^{-1} u_0 \\ &\text{if} \quad du/dt = Au \quad &&\text{then} \quad u(t) = e^{At}u(0) = Me^{Jt}M^{-1}u(0). \end{aligned}$$

When M and J are S and Λ (the diagonalizable case) those are the formulas of Sections 5.3 and 5.4. Appendix B returns to the nondiagonalizable case, and shows how the Jordan form can be reached. I hope the following table will be a convenient summary.

Similarity Transformations

1. A is ***diagonalizable***: The columns of S are eigenvectors and $S^{-1}AS = \Lambda$.
2. A is ***arbitrary***: The columns of M include "generalized eigenvectors" of A, and the Jordan form $M^{-1}AM = J$ is *block diagonal*.
3. A is ***arbitrary***: The unitary U can be chosen so that $U^{-1}AU = T$ is *triangular*.
4. A is ***normal***, $AA^{\mathrm{H}} = A^{\mathrm{H}}A$: then U can be chosen so that $U^{-1}AU = \Lambda$.

Special cases of normal matrices, all with orthonormal eigenvectors:

(a) If $A = A^{\mathrm{H}}$ is Hermitian, then all λ_i are real.
(b) If $A = A^{\mathrm{T}}$ is real symmetric, then Λ is real and $U = Q$ is orthogonal.
(c) If $A = -A^{\mathrm{H}}$ is skew-Hermitian, then all λ_i are purely imaginary.
(d) If A is orthogonal or unitary, then all $|\lambda_i| = 1$ are on the unit circle.

Problem Set 5.6

1. If B is similar to A and C is similar to B, show that C is similar to A. (Let $B = M^{-1}AM$ and $C = N^{-1}BN$.) Which matrices are similar to I?

2. Describe in words all matrices that are similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and find two of them.

3. Explain why A is never similar to $A + I$.

4. Find a diagonal M, made up of 1s and -1s, to show that

$$A = \begin{bmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & 2 \end{bmatrix} \quad \text{is similar to} \quad B = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}.$$

5. Show (if B is invertible) that BA is similar to AB.

6. (a) If $CD = -DC$ (and D is invertible), show that C is similar to $-C$.
   (b) Deduce that the eigenvalues of C must come in plus–minus pairs.
   (c) Show directly that if $Cx = \lambda x$, then $C(Dx) = -\lambda(Dx)$.

7. Consider any A and a "Givens rotation" M in the 1–2 plane:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad M = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Choose the rotation angle θ to produce zero in the (3, 1) entry of $M^{-1}AM$.

Note This "zeroing" is not so easy to continue, because the rotations that produce zero in place of d and h will spoil the new zero in the corner. We have to leave one diagonal below the main one, and finish the eigenvalue calculation in a different way. Otherwise, if we could make A diagonal and see its eigenvalues, we would be finding the roots of the polynomial $\det(A - \lambda I)$ by using only the square roots that determine $\cos\theta$—and that is impossible.

8. What matrix M changes the basis $V_1 = (1, 1)$, $V_2 = (1, 4)$ to the basis $v_1 = (2, 5)$, $v_2 = (1, 4)$? The columns of M come from expressing V_1 and V_2 as combinations $\sum m_{ij} v_i$ of the v's.

9. For the same two bases, express the vector (3, 9) as a combination $c_1 V_1 + c_2 V_2$ and also as $d_1 v_1 + d_2 v_2$. Check numerically that M connects c to d: $Mc = d$.

10. Confirm the last exercise: If $V_1 = m_{11}v_1 + m_{21}v_2$ and $V_2 = m_{12}v_1 + m_{22}v_2$, and $m_{11}c_1 + m_{12}c_2 = d_1$ and $m_{21}c_1 + m_{22}c_2 = d_2$, the vectors $c_1V_1 + c_2V_2$ and $d_1v_1 + d_2v_2$ are the same. This is the "change of basis formula" $Mc = d$.

11. If the transformation T is a reflection across the 45° line in the plane, find its matrix with respect to the standard basis $v_1 = (1, 0)$, $v_2 = (0, 1)$, and also with respect to $V_1 = (1, 1)$, $V_2 = (1, -1)$. Show that those matrices are similar.

12. The *identity transformation* takes every vector to itself: $Tx = x$. Find the

corresponding matrix, if the first basis is $v_1 = (1,2)$, $v_2 = (3,4)$ and the second basis is $w_1 = (1,0)$, $w_2 = (0,1)$. (It is not the identity matrix!)

13. The derivative of $a + bx + cx^2$ is $b + 2cx + 0x^2$.

(a) Write the 3 by 3 matrix D such that

$$D\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}.$$

(b) Compute D^3 and interpret the results in terms of derivatives.
(c) What are the eigenvalues and eigenvectors of D?

14. Show that every number is an eigenvalue for $Tf(x) = df/dx$, but the transformation $Tf(x) = \int_0^x f(t)\,dt$ has no eigenvalues (here $-\infty < x < \infty$).

15. On the space of 2 by 2 matrices, let T be the transformation that *transposes every matrix*. Find the eigenvalues and "eigenmatrices" for $A^{\mathrm{T}} = \lambda A$.

16. (a) Find an orthogonal Q so that $Q^{-1}AQ = \Lambda$ if

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then find a second pair of orthonormal eigenvectors x_1, x_2 for $\lambda = 0$.
(b) Verify that $P = x_1x_1^{\mathrm{T}} + x_2x_2^{\mathrm{T}}$ is the same for both pairs.

17. Prove that every *unitary* matrix A is diagonalizable, in two steps:

(i) If A is unitary, and U is too, then so is $T = U^{-1}AU$.
(ii) An upper triangular T that is unitary must be diagonal. Thus $T = \Lambda$.

Any unitary matrix A (distinct eigenvalues or not) has a complete set of orthonormal eigenvectors. All eigenvalues satisfy $|\lambda| = 1$.

18. Find a normal matrix ($NN^{\mathrm{H}} = N^{\mathrm{H}}N$) that is not Hermitian, skew-Hermitian, unitary, or diagonal. Show that all permutation matrices are normal.

19. Suppose T is a 3 by 3 upper triangular matrix, with entries t_{ij}. Compare the entries of TT^{H} and $T^{\mathrm{H}}T$, and show that if they are equal, then T must be diagonal. All normal triangular matrices are diagonal.

20. If N is normal, show that $\|Nx\| = \|N^{\mathrm{H}}x\|$ for every vector x. Deduce that the ith row of N has the same length as the ith column. *Note*: If N is also upper triangular, this leads again to the conclusion that it must be diagonal.

21. Prove that a matrix with orthonormal eigenvectors must be normal, as claimed in **5T**: If $U^{-1}NU = \Lambda$, or $N = U\Lambda U^{\mathrm{H}}$, then $NN^{\mathrm{H}} = N^{\mathrm{H}}N$.

22. Find a unitary U and triangular T so that $U^{-1}AU = T$, for

$$A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

23. If A has eigenvalues 0, 1, 2, what are the eigenvalues of $A(A-I)(A-2I)$?

24. (a) Show by direct multiplication that every triangular matrix T, say 3 by 3, satisfies its own characteristic equation: $(T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) = 0$.
(b) Substituting $U^{-1}AU$ for T, deduce the famous ***Cayley–Hamilton theorem: Every matrix satisfies its own characteristic equation***. For 3 by 3 this is $(A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) = 0$.

25. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\lambda^2 - (a+d)\lambda + (ad - bc)$. By direct substitution, verify Cayley–Hamilton: $A^2 - (a+d)A + (ad - bc)I = 0$.

26. If $a_{ij} = 1$ above the main diagonal and $a_{ij} = 0$ elsewhere, find the Jordan form (say 4 by 4) by finding all the eigenvectors.

27. Show, by trying for an M and failing, that no two of the three Jordan forms in equation (8) are similar: $J_1 \neq M^{-1}J_2M$, $J_1 \neq M^{-1}J_3M$, and $J_2 \neq M^{-1}J_3M$.

28. Solve $u' = Ju$ by back-substitution, solving first for $u_2(t)$:

$$\frac{du}{dt} = Ju = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{with initial value } u(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Notice te^{5t} in the first component $u_1(t)$.

29. Compute A^{10} and e^A if $A = MJM^{-1}$:

$$A = \begin{bmatrix} 14 & 9 \\ -16 & -10 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

30. Show that A and B are similar by finding M so that $B = M^{-1}AM$:

(a) $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

31. Which of these matrices A_1 to A_6 are similar? Check their eigenvalues.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

32. There are sixteen 2 by 2 matrices whose entries are 0s and 1s. Similar matrices go into the same family. How many families? How many matrices (total 16) in each family?

33. (a) If x is in the nullspace of A, show that $M^{-1}x$ is in the nullspace of $M^{-1}AM$.
(b) The nullspaces of A and $M^{-1}AM$ have the same (vectors)(basis)(dimension).

34. If A and B have the exactly the same eigenvalues and eigenvectors, does $A = B$? With n independent eigenvectors, we do have $A = B$. Find $A \neq B$ when $\lambda = 0, 0$ (repeated), but there is only one line of eigenvectors $(x_1, 0)$.

Problems 35–39 are about the Jordan form.

35. By direct multiplication, find J^2 and J^3 when

$$J = \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}.$$

Guess the form of J^k. Set $k = 0$ to find J^0. Set $k = -1$ to find J^{-1}.

36. If J is the 5 by 5 Jordan block with $\lambda = 0$, find J^2 and count its eigenvectors, and find its Jordan form (two blocks).

37. The text solved $du/dt = Ju$ for a 3 by 3 Jordan block J. Add a fourth equation $dw/dt = 5w + x$. Follow the pattern of solutions for z, y, x to find w.

38. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (find them). But the block sizes don't match and J *is not similar to* K:

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

For any matrix M, compare JM with MK. If they are equal, show that M is not invertible. Then $M^{-1}JM = K$ is impossible.

39. Prove in three steps that A^{T} *is always similar to* A (we know that the λ's are the same, the eigenvectors are the problem):

(a) For $A =$ one block, find $M_i =$ permutation so that $M_i^{-1} J_i M_i = J_i^{\mathrm{T}}$.
(b) For $A =$ any J, build M_0 from blocks so that $M_0^{-1} J M_0 = J^{\mathrm{T}}$.
(c) For any $A = MJM^{-1}$: Show that A^{T} is similar to J^{T} and so to J and to A.

40. Which pairs are similar? Choose a, b, c, d to prove that the other pairs aren't:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} b & a \\ d & c \end{bmatrix} \quad \begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

41. True or false, with a good reason:

(a) An invertible matrix can't be similar to a singular matrix.
(b) A symmetric matrix can't be similar to a nonsymmetric matrix.
(c) A can't be similar to $-A$ unless $A = 0$.
(d) $A - I$ can't be similar to $A + I$.

42. Prove that AB *has the same eigenvalues as* BA.

43. If A is 6 by 4 and B is 4 by 6, AB and BA have different sizes. Nevertheless,

$$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = G.$$

(a) What sizes are the blocks of G? They are the same in each matrix.
(b) This equation is $M^{-1}FM = G$, so F and G have the same 10 eigenvalues. F has the eigenvalues of AB plus 4 zeros; G has the eigenvalues of BA plus 6 zeros. ***AB has the same eigenvalues as BA plus ___ zeros.***

44. Why is each of these statements true?

(a) If A is similar to B, then A^2 is similar to B^2.

(b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$).

(c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.

(d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is not similar to $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

(e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2, **the eigenvalues stay the same**.

Properties of Eigenvalues and Eigenvectors

How are the properties of a matrix reflected in its eigenvalues and eigenvectors? This question is fundamental throughout Chapter 5. A table that organizes the key facts may be helpful. For each class of matrices, here are the special properties of the eigenvalues λ_i and eigenvectors x_i.

Symmetric: $A^{\mathrm{T}} = A$	real λ's	orthogonal $x_i^{\mathrm{T}} x_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Complex Hermitian: $\overline{A}^{\mathrm{T}} = A$	real λ's	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Positive definite: $x^{\mathrm{T}} A x > 0$	all $\lambda > 0$	orthogonal
Similar matrix: $B = M^{-1}AM$	$\lambda(B) = \lambda(A)$	$x(B) = M^{-1}x(A)$
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Reflection: $I - 2uu^{\mathrm{T}}$	$\lambda = -1;\ 1, \ldots, 1$	$u; u^{\perp}$
Rank-1 matrix: uv^{T}	$\lambda = v^{\mathrm{T}}u;\ 0, \ldots, 0$	$u;\ v^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	eigenvectors of A
Stable powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	
Stable exponential: $e^{At} \to 0$	all Re $\lambda < 0$	
Markov: $m_{ij} > 0$, $\sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $x > 0$
Cyclic permutation: $P^n = I$	$\lambda_k = e^{2\pi ik/n}$	$x_k = \left(1, \lambda_k, \ldots, \lambda_k^{n-1}\right)$
Diagonalizable: $S \Lambda S^{-1}$	diagonal of Λ	columns of S are independent
Symmetric: $Q \Lambda Q^{\mathrm{T}}$	diagonal of Λ (real)	columns of Q are orthonormal
Jordan: $J = M^{-1}AM$	diagonal of J	each block gives 1 eigenvector
Every matrix: $A = U\Sigma V^{\mathrm{T}}$	rank(A) = rank(Σ)	eigenvectors of $A^{\mathrm{T}}A$, AA^{T} in V, U

Chapter 5 Review Exercises

5.1 Find the eigenvalues and eigenvectors, and the diagonalizing matrix S, for

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix}.$$

5.2 Find the determinants of A and A^{-1} if

$$A = S \begin{bmatrix} \lambda_1 & 2 \\ 0 & \lambda_2 \end{bmatrix} S^{-1}.$$

5.3 If A has eigenvalues 0 and 1, corresponding to the eigenvectors

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

how can you tell in advance that A is symmetric? What are its trace and determinant? What is A?

5.4 In the previous problem, what will be the eigenvalues and eigenvectors of A^2? What is the relation of A^2 to A?

5.5 Does there exist a matrix A such that the entire family $A + cI$ is invertible for all complex numbers c? Find a real matrix with $A + rI$ invertible for all real r.

5.6 Solve for both initial values and then find e^{At}:

$$\frac{du}{dt} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} u \quad \text{if} \quad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and if} \quad u(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

5.7 Would you prefer to have interest compounded quarterly at 40% per year, or annually at 50%?

5.8 True or false (with counterexample if false):

(a) If B is formed from A by exchanging two rows, then B is similar to A.
(b) If a triangular matrix is similar to a diagonal matrix, it is already diagonal.
(c) Any two of these statements imply the third: A is Hermitian, A is unitary, $A^2 = I$.
(d) If A and B are diagonalizable, so is AB.

5.9 What happens to the Fibonacci sequence if we go backward in time, and how is F_{-k} related to F_k? The law $F_{k+2} = F_{k+1} + F_k$ is still in force, so $F_{-1} = 1$.

5.10 Find the general solution to $du/dt = Au$ if

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Can you find a time T at which the solution $u(T)$ is guaranteed to return to the initial value $u(0)$?

5.11 If P is the matrix that projects $\mathbf{R}^n$ onto a subspace $\mathbf{S}$, explain why every vector in $\mathbf{S}$ is an eigenvector, and so is every vector in $\mathbf{S}^\perp$. What are the eigenvalues? (Note the connection to $P^2 = P$, which means that $\lambda^2 = \lambda$.)

5.12 Show that every matrix of order > 1 is the sum of two singular matrices.

5.13 (a) Show that the matrix differential equation $dX/dt = AX+XB$ has the solution $X(t) = e^{At}X(0)e^{Bt}$.
(b) Prove that the solutions of $dX/dt = AX - XA$ keep the same eigenvalues for all time.

5.14 If the eigenvalues of A are 1 and 3 with eigenvectors $(5, 2)$ and $(2, 1)$, find the solutions to $du/dt = Au$ and $u_{k+1} = Au_k$, starting from $u = (9, 4)$.

5.15 Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}.$$

What property do you expect for the eigenvectors, and is it true?

5.16 By trying to solve

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A$$

show that A has no square root. Change the diagonal entries of A to 4 and find a square root.

5.17 (a) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 4 \\ \frac{1}{4} & 0 \end{bmatrix}$.
(b) Solve $du/dt = Au$ starting from $u(0) = (100, 100)$.
(c) If $v(t) =$ income to stockbrokers and $w(t) =$ income to client, and they help each other by $dv/dt = 4w$ and $dw/dt = \frac{1}{4}v$, what does the ratio v/w approach as $t \to \infty$?

5.18 True or false, with reason if true and counterexample if false:
(a) For every matrix A, there is a solution to $du/dt = Au$ starting from $u(0) = (1, \dots, 1)$.
(b) Every invertible matrix can be diagonalized.
(c) Every diagonalizable matrix can be inverted.
(d) Exchanging the rows of a 2 by 2 matrix reverses the signs of its eigenvalues.
(e) If eigenvectors x and y correspond to distinct eigenvalues, then $x^{\mathrm{H}}y = 0$.

5.19 If K is a skew-symmetric matrix, show that $Q = (I - K)(I + K)^{-1}$ is an orthogonal matrix. Find Q if $K = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

5.20 If $K^{\mathrm{H}} = -K$ (skew-Hermitian), the eigenvalues are imaginary and the eigenvectors are orthogonal.
(a) How do you know that $K - I$ is invertible?
(b) How do you know that $K = U\Lambda U^{\mathrm{H}}$ for a unitary U?

(c) Why is $e^{\Lambda t}$ unitary?
(d) Why is e^{Kt} unitary?

5.21 If M is the diagonal matrix with entries d, d^2, d^3, what is $M^{-1}AM$? What are its eigenvalues in the following case?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

5.22 If $A^2 = -I$, what are the eigenvalues of A? If A is a real n by n matrix show that n must be even, and give an example.

5.23 If $Ax = \lambda_1 x$ and $A^{\mathrm{T}} y = \lambda_2 y$ (all real), show that $x^{\mathrm{T}} y = 0$.

5.24 A variation on the Fourier matrix is the "sine matrix":

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin\theta & \sin 2\theta & \sin 3\theta \\ \sin 2\theta & \sin 4\theta & \sin 6\theta \\ \sin 3\theta & \sin 6\theta & \sin 9\theta \end{bmatrix} \quad \text{with} \quad \theta = \frac{\pi}{4}.$$

Verify that $S^{\mathrm{T}} = S^{-1}$. (The columns are the eigenvectors of the tridiagonal $-1, 2, -1$ matrix.)

5.25 (a) Find a nonzero matrix N such that $N^3 = 0$.
(b) If $Nx = \lambda x$, show that λ must be zero.
(c) Prove that N (called a "nilpotent" matrix) cannot be symmetric.

5.26 (a) Find the matrix $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$ that projects any vector onto the line through $a = (2, 1, 2)$.
(b) What is the only nonzero eigenvalue of P, and what is the corresponding eigenvector?
(c) Solve $u_{k+1} = Pu_k$, starting from $u_0 = (9, 9, 0)$.

5.27 Suppose the first row of A is 7, 6 and its eigenvalues are $i, -i$. Find A.

5.28 (a) For which numbers c and d does A have real eigenvalues and orthogonal eigenvectors?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & d & c \\ 0 & 5 & 3 \end{bmatrix}.$$

(b) For which c and d can we find three orthonormal vectors that are combinations of the columns (don't do it!)?

5.29 If the vectors x_1 and x_2 are in the columns of S, what are the eigenvalues and eigenvectors of

$$A = S \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} \quad \text{and} \quad B = S \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} S^{-1}?$$

5.30 What is the limit as $k \to \infty$ (the Markov steady state) of $\begin{bmatrix} .4 & .3 \\ .6 & .7 \end{bmatrix}^k \begin{bmatrix} a \\ b \end{bmatrix}$?

Chapter

6 Positive Definite Matrices

6.1 MINIMA, MAXIMA, AND SADDLE POINTS

Up to now, we have hardly thought about **the signs of the eigenvalues**. We couldn't ask whether λ was positive before it was known to be real. Chapter 5 established that every symmetric matrix has real eigenvalues. Now we will find a test that can be applied directly to A, without computing its eigenvalues, which will ***guarantee that all those eigenvalues are positive***. The test brings together three of the most basic ideas in the book—*pivots*, *determinants*, and *eigenvalues*.

The signs of the eigenvalues are often crucial. For stability in differential equations, we needed negative eigenvalues so that $e^{\lambda t}$ would decay. The new and highly important problem is to recognize a ***minimum point***. This arises throughout science and engineering and every problem of optimization. The mathematical problem is to move the second derivative test $F'' > 0$ into n dimensions. Here are two examples:

$$F(x, y) = 7 + 2(x + y)^2 - y \sin y - x^3 \qquad f(x, y) = 2x^2 + 4xy + y^2.$$

Does either $F(x, y)$ or $f(x, y)$ have a minimum at the point $x = y = 0$?

Remark 1 The *zero-order terms* $F(0, 0) = 7$ and $f(0, 0) = 0$ have no effect on the answer. They simply raise or lower the graphs of F and f.

Remark 2 The *linear terms* give a necessary condition: To have any chance of a minimum, the first derivatives must vanish at $x = y = 0$:

$$\frac{\partial F}{\partial x} = 4(x + y) - 3x^2 = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 4(x + y) - y\cos y - \sin y = 0$$

$$\frac{\partial f}{\partial x} = 4x + 4y = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x + 2y = 0. \quad \textit{All zero.}$$

Thus $(x, y) = (0, 0)$ is a *stationary point* for both functions. The surface $z = F(x, y)$ is tangent to the horizontal plane $z = 7$, and the surface $z = f(x, y)$ is tangent to the plane $z = 0$. The question is whether the graphs go *above those planes or not*, as we move away from the tangency point $x = y = 0$.

Remark 3 ***The second derivatives at* (0, 0) *are decisive*:**

$$\frac{\partial^2 F}{\partial x^2} = 4 - 6x = 4 \qquad\qquad \frac{\partial^2 f}{\partial x^2} = 4$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = 4 \qquad\qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 4$$

$$\frac{\partial^2 F}{\partial y^2} = 4 + y \sin y - 2 \cos y = 2 \qquad\qquad \frac{\partial^2 f}{\partial y^2} = 2.$$

These second derivatives 4, 4, 2 contain the answer. Since they are the same for F and f, they must contain the same answer. The two functions behave in exactly the same way near the origin. ***F has a minimum if and only if f has a minimum***. I am going to show that those functions don't!

Remark 4 The *higher-degree terms* in F have no effect on the question of a *local* minimum, but they can prevent it from being a *global* minimum. In our example the term $-x^3$ must sooner or later pull F toward $-\infty$. For $f(x, y)$, with no higher terms, all the action is at (0, 0).

Every quadratic form $f = ax^2 + 2bxy + cy^2$ *has a stationary point at the origin, where* $\partial f/\partial x = \partial f/\partial y = 0$. A local minimum would also be a global minimum. The surface $z = f(x, y)$ will then be shaped like a bowl, resting on the origin (Figure 6.1). If the stationary point of F is at $x = \alpha$, $y = \beta$, the only change would be to use the second derivatives at α, β:

Quadratic part of F
$$f(x, y) = \frac{x^2}{2} \frac{\partial^2 F}{\partial x^2}(\alpha, \beta) + xy \frac{\partial^2 F}{\partial x \partial y}(\alpha, \beta) + \frac{y^2}{2} \frac{\partial^2 F}{\partial y^2}(\alpha, \beta). \qquad (1)$$

This $f(x, y)$ behaves near (0, 0) in the same way that $F(x, y)$ behaves near (α, β).

The third derivatives are drawn into the problem when the second derivatives fail to give a definite decision. That happens when the quadratic part is singular. For a true

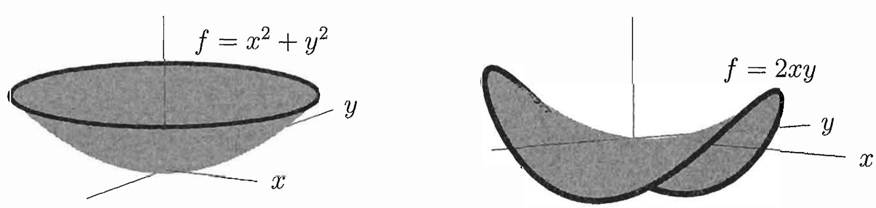


Figure 6.1 A bowl and a saddle: Definite $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and indefinite $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

minimum, f is allowed to vanish *only at* $x = y = 0$. When $f(x, y)$ is strictly positive at all other points (the bowl goes up), it is called ***positive definite***.

Definite versus Indefinite: Bowl versus Saddle

The problem comes down to this: For a function of two variables x and y, what is the correct replacement for the condition $\partial^2 F/\partial x^2 > 0$? With only one variable, the sign of the second derivative decides between a minimum or a maximum. Now we have three second derivatives: F_{xx}, $F_{xy} = F_{yx}$, and F_{yy}. These three numbers (like 4, 4, 2) must determine whether or not F (as well as f) has a minimum.

What conditions on a, b, and c ensure that the quadratic $f(x,y) = ax^2 + 2bxy + cy^2$ is positive definite? One necessary condition is easy:

(i) *If $ax^2 + 2bxy + cy^2$ is positive definite, then necessarily* $\boldsymbol{a > 0}$.

We look at $x = 1$, $y = 0$, where $ax^2 + 2bxy + cy^2$ is equal to a. This must be positive. Translating back to F, that means that $\partial^2 F/\partial x^2 > 0$. The graph must go up in the x direction. Similarly, fix $x = 0$ and look in the y direction where $f(0, y) = cy^2$:

(ii) *If $f(x, y)$ is positive definite, then necessarily* $\boldsymbol{c > 0}$.

Do these conditions $a > 0$ and $c > 0$ guarantee that $f(x, y)$ is always positive? The answer is **no**. A large cross term $2bxy$ can pull the graph below zero.

Example 1 $f(x, y) = x^2 - 10xy + y^2$. Here $a = 1$ and $c = 1$ are both positive. But f is not positive definite, because $f(1, 1) = -8$. The conditions $a > 0$ and $c > 0$ ensure that $f(x, y)$ is positive on the x and y axes. But this function is negative on the line $x = y$, because $b = -10$ overwhelms a and c.

Example 2 In our original f the coefficient $2b = 4$ was positive. Does this ensure a minimum? Again the answer is **no**; the sign of b is of no importance! *Even though its second derivatives are positive, $2x^2 + 4xy + y^2$ is not positive definite.* ***Neither F nor f has a minimum at*** $\boldsymbol{(0, 0)}$ ***because*** $\boldsymbol{f(1, -1) = 2 - 4 + 1 = -1}$.

It is the size of b, compared to a and c, that must be controlled. We now want a necessary and sufficient condition for positive definiteness. The simplest technique is to complete the square:

Express $f(x, y)$ using squares

$$f = ax^2 + 2bxy + cy^2 = a\left(x + \frac{b}{a}y\right)^2 + \left(c - \frac{b^2}{a}\right)y^2. \tag{2}$$

The first term on the right is never negative, when the square is multiplied by $a > 0$. But this square can be zero, and the second term must then be positive. That term has coefficient $(ac - b^2)/a$. The last requirement for positive definiteness is that this coefficient must be positive:

(iii) *If $ax^2 + 2bxy + cy^2$ stays positive, then necessarily* $\boldsymbol{ac > b^2}$.

Test for a minimum: The conditions $a > 0$ and $ac > b^2$ are just right. They guarantee $c > 0$. The right side of (2) is positive, and we have found a minimum:

6A $ax^2 + 2bxy + cy^2$ is positive definite if and only if $a > 0$ and $ac > b^2$. Any $F(x, y)$ has a minimum at a point where $\partial F/\partial x = \partial F/\partial y = 0$ with

$$\frac{\partial^2 F}{\partial x^2} > 0 \quad \text{and} \quad \left[\frac{\partial^2 F}{\partial x^2}\right]\left[\frac{\partial^2 F}{\partial y^2}\right] > \left[\frac{\partial^2 F}{\partial x \partial y}\right]^2. \tag{3}$$

Test for a maximum: Since f has a maximum whenever $-f$ has a minimum, we just reverse the signs of a, b, and c. This actually leaves $ac > b^2$ unchanged: The quadratic form is ***negative definite*** if and only if $a < 0$ and $ac > b^2$. The same change applies for a maximum of $F(x, y)$.

Singular case $ac = b^2$: The second term in equation (2) disappears to leave only the first square—which is either ***positive semidefinite***, when $a > 0$, or ***negative semidefinite***, when $a < 0$. The prefix *semi* allows the possibility that f can equal zero, as it will at the point $x = b$, $y = -a$. The surface $z = f(x, y)$ degenerates from a bowl into a valley. For $f = (x + y)^2$, the valley runs along the line $x + y = 0$.

Saddle point $ac < b^2$: In one dimension, $F(x)$ has a minimum or a maximum, or $F'' = 0$. In two dimensions, a very important possibility still remains: *The combination $ac - b^2$ may be negative.* This occurred in both examples, when b dominated a and c. It also occurs if a and c have opposite signs. Then two directions give opposite results—in one direction f increases, in the other it decreases. It is useful to consider two special cases:

Saddle points at (0, 0) $\quad f_1 = 2xy \quad \text{and} \quad f_2 = x^2 - y^2 \quad \text{and} \quad ac - b^2 = -1.$

In the first, $b = 1$ dominates $a = c = 0$. In the second, $a = 1$ and $c = -1$ have opposite sign. The saddles $2xy$ and $x^2 - y^2$ are practically the same; if we turn one through 45° we get the other. They are also hard to draw.

These quadratic forms are ***indefinite***, because they can take either sign. So we have a stationary point that is neither a maximum or a minimum. It is called a ***saddle point***. The surface $z = x^2 - y^2$ goes down in the direction of the y axis, where the legs fit (if you still ride a horse). In case you switched to a car, think of a road going over a mountain pass. The top of the pass is a minimum as you look along the range of mountains, but it is a maximum as you go along the road.

Higher Dimensions: Linear Algebra

Calculus would be enough to find our conditions $F_{xx} > 0$ and $F_{xx}F_{yy} > F_{xy}^2$ for a minimum. But linear algebra is ready to do more, because the second derivatives fit into a symmetric matrix A. The terms ax^2 and cy^2 appear *on the diagonal*. The cross derivative $2bxy$ is split between the same entry b above and below. A quadratic $f(x, y)$

comes directly from a symmetric 2 by 2 matrix!

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Ax}\ \textbf{in}\ \mathbf{R}^2 \qquad ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{4}$$

This identity (please multiply it out) is the key to the whole chapter. It generalizes immediately to n dimensions, and it is a perfect shorthand for studying maxima and minima. When the variables are $x_1, \ldots, x_n$, they go into a column vector x. ***For any symmetric matrix A, the product $x^{\mathrm{T}}Ax$ is a pure quadratic form $f(x_1, \ldots, x_n)$***:

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Ax}\ \textbf{in}\ \mathbf{R}^n \qquad \begin{bmatrix} x_1 & x_2 & \cdot & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j. \tag{5}$$

The diagonal entries a_{11} to a_{nn} multiply x_1^2 to x_n^2. The pair $a_{ij} = a_{ji}$ combines into $2a_{ij}x_ix_j$. Then $f = a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + a_{nn}x_n^2$.

There are no higher-order terms or lower-order terms—only second-order. The function is zero at $x = (0, \ldots, 0)$, and its first derivatives are zero. The tangent is flat; this is a stationary point. We have to decide if $x = 0$ is a minimum or a maximum or a saddle point of the function $f = x^{\mathrm{T}}Ax$.

Example 3 $f = 2x^2 + 4xy + y^2$ and $A = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \end{array} \to$ *saddle point.*

Example 4 $f = 2xy$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to$ *saddle point.*

Example 5 A is 3 by 3 for $2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$:

$$f = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \textit{minimum at } (0,0,0).$$

Any function $F(x_1, \ldots, x_n)$ is approached in the same way. At a stationary point all first derivatives are zero. A is the "**second-derivative matrix**" with entries $a_{ij} = \partial^2 F/\partial x_i \partial x_j$. This automatically equals $a_{ji} = \partial^2 F/\partial x_j \partial x_i$, so A is symmetric. **Then *F* *has a minimum when the pure quadratic $x^{\mathrm{T}}Ax$ is positive definite*.** These second-order terms control F near the stationary point:

$$\textbf{Taylor series} \qquad F(x) = F(0) + x^{\mathrm{T}}(\text{grad } F) + \frac{1}{2}x^{\mathrm{T}}Ax + \text{higher order terms}. \tag{6}$$

At a stationary point, grad $F = (\partial F/\partial x_1, \ldots, \partial F/\partial x_n)$ is a vector of zeros. The second derivatives in $x^{\mathrm{T}}Ax$ take the graph up or down (or saddle). If the stationary point is at x_0 instead of 0, $F(x)$ and all derivatives are computed at x_0. Then x changes to $x - x_0$ on the right-hand side.

The next section contains the tests to decide whether $x^{\mathrm{T}}Ax$ is positive (the bowl goes up from $x = 0$). Equivalently, **the tests decide whether the matrix A is positive definite**—which is the main goal of the chapter.

Problem Set 6.1

1. The quadratic $f = x^2 + 4xy + 2y^2$ has a saddle point at the origin, despite the fact that its coefficients are positive. Write f as a *difference of two squares*.

2. Decide for or against the positive definiteness of these matrices, and write out the corresponding $f = x^{\mathrm{T}}Ax$:

(a) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$. (d) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$.

The determinant in (b) is zero; along what line is $f(x, y) = 0$?

3. If a 2 by 2 symmetric matrix passes the tests $a > 0$, $ac > b^2$, solve the quadratic equation $\det(A - \lambda I) = 0$ and show that both eigenvalues are positive.

4. Decide between a minimum, maximum, or saddle point for the following functions.

(a) $F = -1 + 4(e^x - x) - 5x \sin y + 6y^2$ at the point $x = y = 0$.
(b) $F = (x^2 - 2x) \cos y$, with stationary point at $x = 1$, $y = \pi$.

5. (a) For which numbers b is the matrix $A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$ positive definite?
(b) Factor $A = LDL^{\mathrm{T}}$ when b is in the range for positive definiteness.
(c) Find the minimum value of $\frac{1}{2}(x^2 + 2bxy + 9y^2) - y$ for b in this range.
(d) What is the minimum if $b = 3$?

6. Suppose the positive coefficients a and c dominate b in the sense that $a + c > 2b$. Find an example that has $ac < b^2$, so the matrix is not positive definite.

7. (a) What 3 by 3 symmetric matrices A_1 and A_2 correspond to f_1 and f_2?

$$f_1 = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$$

$$f_2 = x_1^2 + 2x_2^2 + 11x_3^2 - 2x_1x_2 - 2x_1x_3 - 4x_2x_3.$$

(b) Show that f_1 is a *single* perfect square and not positive definite. Where is f_1 equal to 0?
(c) Factor A_2 into LL^{T}. Write $f_2 = x^{\mathrm{T}}A_2x$ as a sum of three squares.

8. If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite, test $A^{-1} = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$ for positive definiteness.

9. The quadratic $f(x_1, x_2) = 3(x_1 + 2x_2)^2 + 4x_2^2$ is positive. Find its matrix A, factor it into LDL^{T}, and connect the entries in D and L to 3, 2, 4 in f.

10. If $R = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$, write out R^2 and check that it is positive definite unless R is singular.

11. (a) If $A = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}$ is Hermitian (*complex* b), find its pivots and determinant.
(b) Complete the square for $x^{\mathrm{H}}Ax$. Now $x^{\mathrm{H}} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix}$ can be complex.

$$a|x_1|^2 + 2\,\mathrm{Re}\, b\bar{x}_1x_2 + c|x_2|^2 = a|x_1 + (b/a)x_2|^2 + \underline{\qquad\quad} |x_2|^2.$$

(c) Show that $a > 0$ and $ac > |b|^2$ ensure that A is positive definite.
(d) Are the matrices $\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & 4+i \\ 4-i & 6 \end{bmatrix}$ positive definite?

12. Decide whether $F = x^2y^2 - 2x - 2y$ has a minimum at the point $x = y = 1$ (after showing that the first derivatives are zero at that point).

13. Under what conditions on a, b, c is $ax^2 + 2bxy + cy^2 > x^2 + y^2$ for all x, y?

Problems 14–18 are about tests for positive definiteness.

14. Which of A_1, A_2, A_3, A_4 has two positive eigenvalues? Test $a > 0$ and $ac > b^2$, don't compute the eigenvalues. Find an x so that $x^{\mathrm{T}}A_1x < 0$.

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

15. What is the quadratic $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write f as a sum of one or two squares $d_1(\quad)^2 + d_2(\quad)^2$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

16. Show that $f(x, y) = x^2 + 4xy + 3y^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients. Write f as a *difference* of squares and find a point (x, y) where f is negative.

17. (*Important*) If A has *independent columns*, then $A^{\mathrm{T}}A$ is square and symmetric and invertible (Section 4.2). ***Rewrite $x^{\mathrm{T}}A^{\mathrm{T}}Ax$ to show why it is positive except when $x = 0$.*** Then $A^{\mathrm{T}}A$ is positive definite.

18. Test to see if $A^{\mathrm{T}}A$ is positive definite in each case:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

19. Find the 3 by 3 matrix A and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

20. For $F_1(x, y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x, y) = x^3 + xy - x$, find the second derivative matrices A_1 and A_2:

$$A = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}.$$

A_1 is positive definite, so F_1 is concave up ($=$ convex). Find the minimum point of F_1 and the saddle point of F_2 (look where first derivatives are zero).

21. The graph of $z = x^2 + y^2$ is a bowl opening upward. *The graph* of $z = x^2 - y^2$ *is a saddle*. The graph of $z = -x^2 - y^2$ is a bowl opening downward. What is a test on $F(x, y)$ to have a saddle at $(0, 0)$?

22. Which values of c give a bowl and which give a saddle point for the graph of $z = 4x^2 + 12xy + cy^2$? Describe this graph at the borderline value of c.

6.2 TESTS FOR POSITIVE DEFINITENESS

Which symmetric matrices have the property that $x^{\mathrm{T}}Ax > 0$ for all nonzero vectors x? There are four or five different ways to answer this question, and we hope to find all of them. The previous section began with some hints about the signs of eigenvalues, but that gave place to the tests on a, b, c:

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \text{is positive definite when} \quad a > 0 \quad \text{and} \quad ac - b^2 > 0.$$

From those conditions, ***both eigenvalues are positive***. Their product $\lambda_1\lambda_2$ is the determinant $ac - b^2 > 0$, so the eigenvalues are either both positive or both negative. They must be positive because their sum is the trace $a + c > 0$.

Looking at a and $ac - b^2$, it is even possible to spot the appearance of the ***pivots***. They turned up when we decomposed $x^{\mathrm{T}}Ax$ into a sum of squares:

Sum of squares $$ax^2 + 2bxy + cy^2 = a\left(x + \frac{b}{a}y\right)^2 + \frac{ac - b^2}{a}y^2. \tag{1}$$

Those coefficients a and $(ac - b^2)/a$ are the pivots for a 2 by 2 matrix. For larger matrices the pivots still give a simple test for positive definiteness: $x^{\mathrm{T}}Ax$ stays positive when n independent squares are multiplied by ***positive pivots***.

One more preliminary remark. The two parts of this book were linked by the chapter on determinants. Therefore we ask what part determinants play. ***It is not enough to require that the determinant of A is positive***. If $a = c = -1$ and $b = 0$, then $\det A = 1$ but $A = -I =$ negative definite. The determinant test is applied not only to A itself, giving $ac - b^2 > 0$, but also to the 1 by 1 submatrix a in the upper left-hand corner.

The natural generalization will involve all n of the *upper left submatrices* of A:

$$A_1 = [a_{11}], \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \ldots, \quad A_n = A.$$

Here is the main theorem on positive definiteness, and a reasonably detailed proof:

6B Each of the following tests is a necessary and sufficient condition for the real symmetric matrix A to be ***positive definite***:

(I) $x^{\mathrm{T}}Ax > 0$ for all nonzero real vectors x.

(II) All the eigenvalues of A satisfy $\lambda_i > 0$.

(III) All the upper left submatrices A_k have positive determinants.

(IV) All the pivots (without row exchanges) satisfy $d_k > 0$.

Proof Condition I defines a positive definite matrix. Our first step shows that each eigenvalue will be positive:

$$\text{If} \quad Ax = \lambda x, \quad \text{then} \quad x^{\mathrm{T}}Ax = x^{\mathrm{T}}\lambda x = \lambda\|x\|^2.$$

A positive definite matrix has positive eigenvalues, since $x^{\mathrm{T}}Ax > 0$.

Now we go in the other direction. If all $\lambda_i > 0$, we have to prove $x^{\mathrm{T}}Ax > 0$ for every vector x (not just the eigenvectors). Since symmetric matrices have a full set of orthonormal eigenvectors, any x is a combination $c_1x_1 + \cdots + c_nx_n$. Then

$$Ax = c_1Ax_1 + \cdots + c_nAx_n = c_1\lambda_1x_1 + \cdots + c_n\lambda_nx_n.$$

Because of the orthogonality $x_i^{\mathrm{T}}x_j = 0$, and the normalization $x_i^{\mathrm{T}}x_i = 1$,

$$\begin{aligned} x^{\mathrm{T}}Ax &= \left(c_1x_1^{\mathrm{T}} + \cdots + c_nx_n^{\mathrm{T}}\right)(c_1\lambda_1x_1 + \cdots + c_n\lambda_nx_n) \\ &= c_1^2\lambda_1 + \cdots + c_n^2\lambda_n. \end{aligned} \tag{2}$$

If every $\lambda_i > 0$, then equation (2) shows that $x^{\mathrm{T}}Ax > 0$. Thus condition II implies condition I.

If condition I holds, so does condition III: The determinant of A is the product of the eigenvalues. And if condition I holds, we already know that these eigenvalues are positive. But we also have to deal with every upper left submatrix A_k. The trick is to look at all nonzero vectors whose last $n-k$ components are zero:

$$x^{\mathrm{T}}Ax = \begin{bmatrix} x_k^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} \begin{bmatrix} x_k \\ 0 \end{bmatrix} = x_k^{\mathrm{T}}A_kx_k > 0.$$

Thus A_k is positive definite. Its eigenvalues (not the same λ_i!) must be positive. Its determinant is their product, so all upper left determinants are positive.

If condition III holds, so does condition IV: According to Section 4.4, the kth pivot d_k is the ratio of $\det A_k$ to $\det A_{k-1}$. If the determinants are all positive, so are the pivots.

If condition IV holds, so does condition I: We are given positive pivots, and must deduce that $x^{\mathrm{T}}Ax > 0$. This is what we did in the 2 by 2 case, by completing the square. The pivots were the numbers outside the squares. To see how that happens for symmetric matrices of any size, we go back to *elimination on a symmetric matrix*: $A = LDL^{\mathrm{T}}$.

Example 1 Positive pivots 2, $\frac{3}{2}$, and $\frac{4}{3}$:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}.$$

I want to split $x^{\mathrm{T}}Ax$ into $x^{\mathrm{T}}LDL^{\mathrm{T}}x$:

$$\text{If } x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \text{ then } L^{\mathrm{T}}x = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u - \frac{1}{2}v \\ v - \frac{2}{3}w \\ w \end{bmatrix}.$$

So $x^{\mathrm{T}}Ax$ is a sum of squares with the pivots 2, $\frac{3}{2}$, and $\frac{4}{3}$ as coefficients:

$$x^{\mathrm{T}}Ax = (L^{\mathrm{T}}x)^{\mathrm{T}}D(L^{\mathrm{T}}x) = \mathbf{2}\left(u - \frac{\mathbf{1}}{\mathbf{2}}v\right)^2 + \frac{\mathbf{3}}{\mathbf{2}}\left(v - \frac{\mathbf{2}}{\mathbf{3}}w\right)^2 + \frac{\mathbf{4}}{\mathbf{3}}(w)^2.$$

Those positive pivots in D multiply perfect squares to make $x^{\mathrm{T}}Ax$ positive. Thus condition IV implies condition I, and the proof is complete. ■

It is beautiful that elimination and completing the square are actually the same. Elimination removes x_1 from all later equations. Similarly, the first square accounts for all terms in $x^{\mathrm{T}}Ax$ involving x_1. The sum of squares has the pivots outside. *The multipliers ℓ_{ij} are inside*! You can see the numbers $-\frac{1}{2}$ and $-\frac{2}{3}$ inside the squares in the example.

Every diagonal entry a_{ii} must be positive. As we know from the examples, however, it is far from sufficient to look only at the diagonal entries.

The pivots d_i are not to be confused with the eigenvalues. For a typical positive definite matrix, they are two completely different sets of positive numbers. In our 3 by 3 example, probably the determinant test is the easiest:

$$\textbf{Determinant test} \qquad \det A_1 = 2, \quad \det A_2 = 3, \quad \det A_3 = \det A = 4.$$

The pivots are the ratios $d_1 = 2$, $d_2 = \frac{3}{2}$, $d_3 = \frac{4}{3}$. Ordinarily the eigenvalue test is the longest computation. For this A we know the λ's are all positive:

$$\textbf{Eigenvalue test} \qquad \lambda_1 = 2 - \sqrt{2}, \quad \lambda_2 = 2, \quad \lambda_3 = 2 + \sqrt{2}.$$

Even though it is the hardest to apply to a single matrix, eigenvalues can be the most useful test for theoretical purposes. ***Each test is enough by itself***.

Positive Definite Matrices and Least Squares

I hope you will allow one more test for positive definiteness. It is already close. We connected positive definite matrices to pivots (Chapter 1), determinants (Chapter 4), and eigenvalues (Chapter 5). Now we see them in the least-squares problems of Chapter 3, coming from the rectangular matrices of Chapter 2.

The rectangular matrix will be R and the least-squares problem will be $Rx = b$. It has m equations with $m \geq n$ (square systems are included). *The least-squares choice $\widehat{x}$ is the solution of $R^{\mathrm{T}}R\widehat{x} = R^{\mathrm{T}}b$*. That matrix $A = R^{\mathrm{T}}R$ is not only symmetric but positive definite, as we now show—provided that the n columns of R are linearly independent:

6C The symmetric matrix A is positive definite if and only if

(V) There is a matrix R with independent columns such that $A = R^{\mathrm{T}}R$.

The key is to recognize $x^{\mathrm{T}}Ax$ as $x^{\mathrm{T}}R^{\mathrm{T}}Rx = (Rx)^{\mathrm{T}}(Rx)$. This squared length $\|Rx\|^2$ is positive (unless $x = 0$), because R has independent columns. (If x is nonzero then Rx is nonzero.) Thus $x^{\mathrm{T}}R^{\mathrm{T}}Rx > 0$ and $R^{\mathrm{T}}R$ is positive definite.

It remains to find an R for which $A = R^{\mathrm{T}}R$. We have almost done this twice already:

$$\textbf{Elimination} \qquad A = LDL^{\mathrm{T}} = (L\sqrt{D})(\sqrt{D}L^{\mathrm{T}}). \qquad \text{So take } R = \sqrt{D}L^{\mathrm{T}}.$$

This ***Cholesky decomposition*** has the pivots split evenly between L and L^{T}.

$$\textbf{Eigenvalues} \qquad A = Q\Lambda Q^{\mathrm{T}} = (Q\sqrt{\Lambda})(\sqrt{\Lambda}Q^{\mathrm{T}}). \qquad \text{So take } R = \sqrt{\Lambda}Q^{\mathrm{T}}. \quad (3)$$

A third possibility is $R = Q\sqrt{\Lambda}Q^{\mathrm{T}}$, the ***symmetric positive definite square root*** of A. There are many other choices, square or rectangular, and we can see why. If you multiply any R by *a matrix Q with orthonormal columns*, then $(QR)^{\mathrm{T}}(QR) = R^{\mathrm{T}}Q^{\mathrm{T}}QR = R^{\mathrm{T}}IR = A$. Therefore QR is another choice.

Applications of positive definite matrices are developed in my earlier book *Introduction to Applied Mathematics* and also the new *Applied Mathematics and Scientific*

Computing (see `www.wellesleycambridge.com`). We mention that $Ax = \lambda Mx$ arises constantly in engineering analysis. If A and M are positive definite, this generalized problem is parallel to the familiar $Ax = \lambda x$, and $\lambda > 0$. M is a ***mass matrix*** for the *finite element method* in Section 6.4.

Semidefinite Matrices

The tests for semidefiniteness will relax $x^{\mathrm{T}}Ax > 0$, $\lambda > 0$, $d > 0$, and $\det > 0$, to allow zeros to appear. The main point is to see the analogies with the positive definite case.

6D Each of the following tests is a necessary and sufficient condition for a symmetric matrix A to be ***positive semidefinite***:

(I′) $x^{\mathrm{T}}Ax \geq 0$ for all vectors x (this defines positive semidefinite).
(II′) All the eigenvalues of A satisfy $\lambda_i \geq 0$.
(III′) No principal submatrices have negative determinants.
(IV′) No pivots are negative.
(V′) There is a matrix R, possibly with dependent columns, such that $A = R^{\mathrm{T}}R$.

The diagonalization $A = Q\Lambda Q^{\mathrm{T}}$ leads to $x^{\mathrm{T}}Ax = x^{\mathrm{T}}Q\Lambda Q^{\mathrm{T}}x = y^{\mathrm{T}}\Lambda y$. If A has rank r, there are r nonzero λ's and r perfect squares in $y^{\mathrm{T}}\Lambda y = \lambda_1 y_1^2 + \cdots + \lambda_r y_r^2$.

Note The novelty is that condition III′ applies to all the principal submatrices, not only those in the upper left-hand corner. Otherwise, we could not distinguish between two matrices whose upper left determinants were all zero:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ is positive semidefinite, and } \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \text{ is negative semidefinite.}$$

A row exchange comes with the same column exchange to maintain symmetry.

Example 2

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{is positive } \textit{semi}\text{definite, by all five tests:}$$

(I′) $x^{\mathrm{T}}Ax = (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 \geq 0$ (zero if $x_1 = x_2 = x_3$).
(II′) The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 3$ (a zero eigenvalue).
(III′) $\det A = 0$ and smaller determinants are positive.

$$\text{(IV}'\text{)}\quad A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix} \quad \text{(missing pivot).}$$

(V′) $A = R^{\mathrm{T}}R$ with dependent columns in R:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad (1, 1, 1) \text{ in the nullspace.}$$

Remark The conditions for semidefiniteness could also be deduced from the original conditions I-V for definiteness by the following trick: Add a small multiple of the identity, giving a positive definite matrix $A + \epsilon I$. Then let ϵ approach zero. Since the determinants and eigenvalues depend continuously on ϵ, they will be positive until the very last moment. At $\epsilon = 0$ they must still be nonnegative.

My class often asks about *unsymmetric* positive definite matrices. I never use that term. One reasonable definition is that the symmetric part $\frac{1}{2}(A + A^{\mathrm{T}})$ should be positive definite. That guarantees that *the real parts of the eigenvalues are positive*. But it is not necessary: $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ has $\lambda > 0$ but $\frac{1}{2}(A + A^{\mathrm{T}}) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is indefinite.

If $Ax = \lambda x$, then $x^{\mathrm{H}}Ax = \lambda x^{\mathrm{H}}x$ and $x^{\mathrm{H}}A^{\mathrm{H}}x = \bar{\lambda}x^{\mathrm{H}}x$.
Adding, $\frac{1}{2}x^{\mathrm{H}}(A + A^{\mathrm{H}})x = (\operatorname{Re}\lambda)x^{\mathrm{H}}x > 0$, so that $\operatorname{Re}\lambda > 0$.

Ellipsoids in n Dimensions

Throughout this book, geometry has helped the matrix algebra. A linear equation produced a plane. The system $Ax = b$ gives an intersection of planes. Least squares gives a perpendicular projection. The determinant is the volume of a box. Now, for a positive definite matrix and its $x^{\mathrm{T}}Ax$, we finally get a figure that is curved. It is an *ellipse* in two dimensions, and an *ellipsoid* in n dimensions.

The equation to consider is $x^{\mathrm{T}}Ax = 1$. If A is the identity matrix, this simplifies to $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. This is the equation of the "unit sphere" in $\mathbf{R}^n$. If $A = 4I$, the sphere gets smaller. The equation changes to $4x_1^2 + \cdots + 4x_n^2 = 1$. Instead of $(1, 0, \ldots, 0)$, it goes through $(\frac{1}{2}, 0, \ldots, 0)$. The center is at the origin, because if x satisfies $x^{\mathrm{T}}Ax = 1$, so does the opposite vector $-x$. The important step is to go from the identity matrix to a *diagonal matrix*:

Ellipsoid $\quad$ For $A = \begin{bmatrix} 4 & & \\ & 1 & \\ & & \frac{1}{9} \end{bmatrix}$, the equation is $x^{\mathrm{T}}Ax = 4x_1^2 + x_2^2 + \frac{1}{9}x_3^2 = 1$.

Since the entries are unequal (and positive!) the sphere changes to an ellipsoid.

One solution is $x = \left(\frac{1}{2}, 0, 0\right)$ along the first axis. Another is $x = (0, 1, 0)$. The major axis has the farthest point $x = (0, 0, 3)$. It is like a football or a rugby ball, but not quite—those are closer to $x_1^2 + x_2^2 + \frac{1}{2}x_3^2 = 1$. The two equal coefficients make them circular in the x_1-x_2 plane, and much easier to throw!

Now comes the final step, to allow nonzeros away from the diagonal of A.

Example 3 $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and $x^{\mathrm{T}}Ax = 5u^2 + 8uv + 5v^2 = 1$. That ellipse is centered at $u = v = 0$, but the axes are not so clear. The off-diagonal 4s leave the matrix positive definite, but they rotate the ellipse—its axes no longer line up with the coordinate axes (Figure 6.2). We will show that ***the axes of the ellipse point toward the eigenvectors of A***. Because $A = A^{\mathrm{T}}$, those eigenvectors and axes are orthogonal. The *major* axis of the ellipse corresponds to the *smallest* eigenvalue of A.

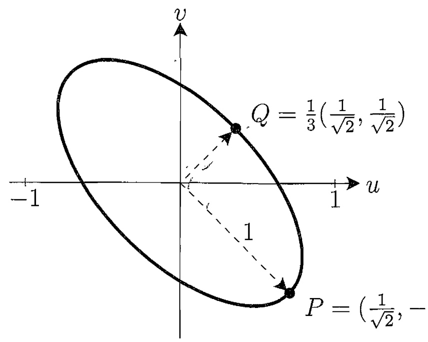


Figure 6.2 The ellipse $x^{\mathrm{T}}Ax = 5u^2 + 8uv + 5v^2 = 1$ and its principal axes.

To locate the ellipse we compute $\lambda_1 = 1$ and $\lambda_2 = 9$. The unit eigenvectors are $(1, -1)/\sqrt{2}$ and $(1, 1)/\sqrt{2}$. Those are at $45°$ angles with the u-v axes, and they are lined up with the axes of the ellipse. The way to see the ellipse properly is to *rewrite* $x^{\mathrm{T}}Ax = 1$:

New squares
$$5u^2 + 8uv + v^2 = \left(\frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}}\right)^2 + 9\left(\frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}\right)^2 = 1. \tag{4}$$

$\lambda = 1$ and $\lambda = 9$ are outside the squares. The eigenvectors are inside. This is different from completing the square to $5\left(u + \frac{4}{5}v\right)^2 + \frac{9}{5}v^2$, with the *pivots* outside.

The first square equals 1 at $(1/\sqrt{2}, -1/\sqrt{2})$ at the end of the major axis. The minor axis is one-third as long, since we need $\left(\frac{1}{3}\right)^2$ to cancel the 9.

Any ellipsoid $x^{\mathrm{T}}Ax = 1$ can be simplified in the same way. *The key step is to diagonalize* $A = Q\Lambda Q^{\mathrm{T}}$. We straightened the picture by rotating the axes. Algebraically, the change to $y = Q^{\mathrm{T}}x$ produces a sum of squares:

$$x^{\mathrm{T}}Ax = (x^{\mathrm{T}}Q)\Lambda(Q^{\mathrm{T}}x) = y^{\mathrm{T}}\Lambda y = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = 1. \tag{5}$$

The major axis has $y_1 = 1/\sqrt{\lambda_1}$ along the eigenvector with the smallest eigenvalue.

The other axes are along the other eigenvectors. Their lengths are $1/\sqrt{\lambda_2}, \ldots, 1/\sqrt{\lambda_n}$. Notice that the λ's must be positive—*the matrix must be positive definite*—or these square roots are in trouble. An indefinite equation $y_1^2 - 9y_2^2 = 1$ describes a hyperbola and not an ellipse. A hyperbola is a cross-section through a saddle, and an ellipse is a cross-section through a bowl.

The change from x to $y = Q^{\mathrm{T}}x$ rotates the axes of the space, to match the axes of the ellipsoid. In the y variables we can see that it is an ellipsoid, because the equation becomes so manageable:

6E Suppose $A = Q\Lambda Q^{\mathrm{T}}$ with $\lambda_i > 0$. Rotating $y = Q^{\mathrm{T}}x$ simplifies $x^{\mathrm{T}}Ax = 1$:

$$x^{\mathrm{T}}Q\Lambda Q^{\mathrm{T}}x = 1, \quad y^{\mathrm{T}}\Lambda y = 1, \quad \text{and} \quad \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = 1.$$

This is the equation of an ellipsoid. Its axes have lengths $1/\sqrt{\lambda_1}, \ldots, 1/\sqrt{\lambda_n}$ from the center. In the original x-space they point along the eigenvectors of A.

The Law of Inertia

For elimination and eigenvalues, matrices become simpler by elementary operations. The essential thing is to know which properties of the matrix stay unchanged. When a multiple of one row is subtracted from another, the row space, nullspace, rank, and determinant all remain the same. For eigenvalues, the basic operation was a similarity transformation $A \to S^{-1}AS$ (or $A \to M^{-1}AM$). The eigenvalues are unchanged (and also the Jordan form). Now we ask the same question for symmetric matrices: *What are the elementary operations and their invariants for* $x^{\mathrm{T}}Ax$?

The basic operation on a quadratic form is to change variables. A new vector y is related to x by some nonsingular matrix, $x = Cy$. The quadratic form becomes $y^{\mathrm{T}}C^{\mathrm{T}}ACy$. This shows the fundamental operation on A:

$$\textbf{Congruence transformation} \qquad A \to C^{\mathrm{T}}AC \quad \text{for some nonsingular } C. \tag{6}$$

The symmetry of A is preserved, since $C^{\mathrm{T}}AC$ remains symmetric. The real question is, What other properties are shared by A and $C^{\mathrm{T}}AC$? The answer is given by Sylvester's ***law of inertia***.

> **6F** $C^{\mathrm{T}}AC$ has the same number of positive eigenvalues, negative eigenvalues, and zero eigenvalues as A.

The *signs* of the eigenvalues (and not the eigenvalues themselves) are preserved by a congruence transformation. In the proof, we will suppose that A is nonsingular. Then $C^{\mathrm{T}}AC$ is also nonsingular, and there are no zero eigenvalues to worry about. (Otherwise we can work with the nonsingular $A + \epsilon I$ and $A - \epsilon I$, and at the end let $\epsilon \to 0$.)

Proof We want to borrow a trick from topology. Suppose C is linked to an orthogonal matrix Q by a continuous chain of nonsingular matrices $C(t)$. At $t = 0$ and $t = 1$, $C(0) = C$ and $C(1) = Q$. Then the eigenvalues of $C(t)^{\mathrm{T}}AC(t)$ will change gradually, as t goes from 0 to 1, from the eigenvalues of $C^{\mathrm{T}}AC$ to the eigenvalues of $Q^{\mathrm{T}}AQ$. Because $C(t)$ is never singular, *none of these eigenvalues can touch zero* (not to mention cross over it!). Therefore the number of eigenvalues to the right of zero, and the number to the left, is the same for $C^{\mathrm{T}}AC$ as for $Q^{\mathrm{T}}AQ$. And A has exactly the same eigenvalues as the similar matrix $Q^{-1}AQ = Q^{\mathrm{T}}AQ$.

One good choice for Q is to apply Gram–Schmidt to the columns of C. Then $C = QR$, and the chain of matrices is $C(t) = tQ + (1-t)QR$. The family $C(t)$ goes slowly through Gram–Schmidt, from QR to Q. It is invertible, because Q is invertible and the triangular factor $tI + (1-t)R$ has positive diagonal. That ends the proof. ■

Example 4 Suppose $A = I$. Then $C^{\mathrm{T}}AC = C^{\mathrm{T}}C$ is positive definite. Both I and $C^{\mathrm{T}}C$ have n positive eigenvalues, confirming the law of inertia.

Example 5 If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $C^{\mathrm{T}}AC$ has a negative determinant:

$$\det C^{\mathrm{T}}AC = (\det C^{\mathrm{T}})(\det A)(\det C) = -(\det C)^2 < 0.$$

Then $C^{\mathrm{T}}AC$ must have one positive and one negative eigenvalue, like A.

Example 6 This application is the important one:

> **6G** For any symmetric matrix A, ***the signs of the pivots agree with the signs of the eigenvalues***. The eigenvalue matrix Λ and the pivot matrix D have the same number of positive entries, negative entries, and zero entries.

We will assume that A allows the symmetric factorization $A = LDL^{\mathrm{T}}$ (without row exchanges). By the law of inertia, A has the same number of positive eigenvalues as D. But the eigenvalues of D are just its diagonal entries (the pivots). Thus the number of positive pivots matches the number of positive eigenvalues of A.

That is both beautiful and practical. It is beautiful because it brings together (for symmetric matrices) two parts of this book that were previously separate: *pivots* and *eigenvalues*. It is also practical, because the pivots can locate the eigenvalues:

A has positive pivots
$A - 2I$ has a negative pivot
$$A = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 10 & 7 \\ 0 & 7 & 8 \end{bmatrix} \qquad A - 2I = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 8 & 7 \\ 0 & 7 & 6 \end{bmatrix}.$$

A has positive eigenvalues, by our test. But we know that $\lambda_{\min}$ *is smaller than* 2, because subtracting 2 dropped it below zero. The next step looks at $A - I$, to see if $\lambda_{\min} < 1$. (It is, because $A - I$ has a negative pivot.) That interval containing λ is cut in half at every step by checking the signs of the pivots.

This was almost the first practical method of computing eigenvalues. It was dominant about 1960, after one important improvement—to make A tridiagonal first. Then the pivots are computed in $2n$ steps instead of $\frac{1}{6}n^3$. Elimination becomes fast, and the search for eigenvalues (by halving the intervals) becomes simple. The current favorite is the QR method in Chapter 7.

The Generalized Eigenvalue Problem

Physics, engineering, and statistics are usually kind enough to produce symmetric matrices in their eigenvalue problems. ***But sometimes $Ax = \lambda x$ is replaced by $Ax = \lambda Mx$. There are two matrices rather than one.***

An example is the motion of two unequal masses in a line of springs:

$$\begin{aligned} m_1\frac{d^2v}{dt^2} + 2v - w &= 0 \\ m_2\frac{d^2w}{dt^2} - v + 2w &= 0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\frac{d^2u}{dt^2} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} u = 0. \tag{7}$$

When the masses were equal, $m_1 = m_2 = 1$, this was the old system $u'' + Au = 0$. Now it is $Mu'' + Au = 0$, with a *mass matrix* M. The eigenvalue problem arises when we look for exponential solutions $e^{i\omega t}x$:

$$Mu'' + Au = 0 \quad \text{becomes} \quad M(i\omega)^2 e^{i\omega t}x + Ae^{i\omega t}x = 0. \tag{8}$$

Cancelling $e^{i\omega t}$, and writing λ for ω^2, this is an eigenvalue problem:

$$\textbf{Generalized problem } Ax = \lambda Mx \qquad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x = \lambda \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} x. \tag{9}$$

There is a solution when $A - \lambda M$ is singular. The special choice $M = I$ brings back the usual $\det(A - \lambda I) = 0$. We work out $\det(A - \lambda M)$ with $m_1 = 1$ and $m_2 = 2$:

$$\det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-2\lambda \end{bmatrix} = 2\lambda^2 - 6\lambda + 3 = 0 \quad \text{gives} \quad \lambda = \frac{3 \pm \sqrt{3}}{2}.$$

For the eigenvector $x_1 = (\sqrt{3} - 1, 1)$, the two masses oscillate together—but the first mass only moves as far as $\sqrt{3} - 1 \approx .73$. In the faster mode, the components of $x_2 = (1 + \sqrt{3}, -1)$ have opposite signs and the masses move in opposite directions. This time the smaller mass goes much further.

The underlying theory is easier to explain if M is split into $R^{\mathrm{T}}R$. (*M is assumed to be positive definite.*) Then the substitution $y = Rx$ changes

$$Ax = \lambda Mx = \lambda R^{\mathrm{T}}Rx \quad \text{into} \quad AR^{-1}y = \lambda R^{\mathrm{T}}y.$$

Writing C for R^{-1}, and multiplying through by $(R^{\mathrm{T}})^{-1} = C^{\mathrm{T}}$, this becomes a standard eigenvalue problem for the *single* symmetric matrix $C^{\mathrm{T}}AC$:

$$\textbf{Equivalent problem} \qquad C^{\mathrm{T}}ACy = \lambda y. \tag{10}$$

The eigenvalues λ_j are the same as for the original $Ax = \lambda Mx$, and the eigenvectors are related by $y_j = Rx_j$. The properties of $C^{\mathrm{T}}AC$ lead directly to the properties of $Ax = \lambda Mx$, when $A = A^{\mathrm{T}}$ and M is positive definite:

1. The eigenvalues for $Ax = \lambda Mx$ are real, because $C^{\mathrm{T}}AC$ is symmetric.
2. The λ's have the same signs as the eigenvalues of A, by the law of inertia.
3. $C^{\mathrm{T}}AC$ has orthogonal eigenvectors y_j. So the eigenvectors of $Ax = \lambda Mx$ have

$$\textbf{“}M\textbf{-orthogonality”} \qquad x_i^{\mathrm{T}}Mx_j = x_i^{\mathrm{T}}R^{\mathrm{T}}Rx_j = y_i^{\mathrm{T}}y_j = 0. \tag{11}$$

A and M are being *simultaneously diagonalized.* If S has the x_j in its columns, then $S^{\mathrm{T}}AS = \Lambda$ and $S^{\mathrm{T}}MS = I$. This is a *congruence* transformation, with S^{T} on the left, and not a similarity transformation with S^{-1}. The main point is easy to summarize: As long as M is positive definite, the generalized eigenvalue problem $Ax = \lambda Mx$ behaves exactly like $Ax = \lambda x$.

Problem Set 6.2

1. For what range of numbers a and b are the matrices A and B positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}.$$

2. Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2.$$

3. Construct an *indefinite matrix* with its largest entries on the main diagonal:

$$A = \begin{bmatrix} 1 & b & -b \\ b & 1 & b \\ -b & b & 1 \end{bmatrix} \quad \text{with } |b| < 1 \text{ can have } \det A < 0.$$

4. Show from the eigenvalues that if A is positive definite, so is A^2 and so is A^{-1}.

5. *If A and B are positive definite, then $A+B$ is positive definite.* Pivots and eigenvalues are not convenient for $A + B$. Much better to prove $x^{\mathrm{T}}(A + B)x > 0$.

6. From the pivots, eigenvalues, and eigenvectors of $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, write A as $R^{\mathrm{T}}R$ in three ways: $(L\sqrt{D})(\sqrt{D}L^{\mathrm{T}})$, $(Q\sqrt{\Lambda})(\sqrt{\Lambda}Q^{\mathrm{T}})$, and $(Q\sqrt{\Lambda}Q^{\mathrm{T}})(Q\sqrt{\Lambda}Q^{\mathrm{T}})$.

7. If $A = Q\Lambda Q^{\mathrm{T}}$ is symmetric positive definite, then $R = Q\sqrt{\Lambda}Q^{\mathrm{T}}$ is its *symmetric positive definite square root*. Why does R have positive eigenvalues? Compute R and verify $R^2 = A$ for

$$A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}.$$

8. If A is symmetric positive definite and C is nonsingular, prove that $B = C^{\mathrm{T}}AC$ is also symmetric positive definite.

9. If $A = R^{\mathrm{T}}R$ prove the generalized Schwarz inequality $|x^{\mathrm{T}}Ay|^2 \leq (x^{\mathrm{T}}Ax)(y^{\mathrm{T}}Ay)$.

10. The ellipse $u^2 + 4v^2 = 1$ corresponds to $A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. Write the eigenvalues and eigenvectors, and sketch the ellipse.

11. Reduce the equation $3u^2 - 2\sqrt{2}uv + 2v^2 = 1$ to a sum of squares by finding the eigenvalues of the corresponding A, and sketch the ellipse.

12. In three dimensions, $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1$ represents an ellipsoid when all $\lambda_i > 0$. Describe all the different kinds of surfaces that appear in the positive semidefinite case when one or more of the eigenvalues is zero.

13. Write down the five conditions for a 3 by 3 matrix to be *negative definite* ($-A$ is positive definite) with special attention to condition III: How is $\det(-A)$ related to $\det A$?

14. Decide whether the following matrices are positive definite, negative definite, semidefinite, or indefinite:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \quad C = -B, \quad D = A^{-1}.$$

Is there a real solution to $-x^2 - 5y^2 - 9z^2 - 4xy - 6xz - 8yz = 1$?

15. Suppose A is symmetric positive definite and Q is an orthogonal matrix. True or false:

(a) $Q^{\mathrm{T}}AQ$ is a diagonal matrix.
(b) $Q^{\mathrm{T}}AQ$ is symmetric positive definite.

(c) $Q^{\mathrm{T}}AQ$ has the same eigenvalues as A.
(d) e^{-A} is symmetric positive definite.

16. If A is positive definite and a_{11} is increased, prove from cofactors that the determinant is increased. Show by example that this can fail if A is indefinite.

17. From $A = R^{\mathrm{T}}R$, show for positive definite matrices that $\det A \le a_{11}a_{22}\cdots a_{nn}$. (The length squared of column j of R is a_{jj}. Use determinant $=$ volume.)

18. (Lyapunov test for stability of M) Suppose $AM + M^{\mathrm{H}}A = -I$ with positive definite A. If $Mx = \lambda x$ show that $\operatorname{Re}\lambda < 0$. (*Hint*: Multiply the first equation by x^{H} and x.)

19. Which 3 by 3 symmetric matrices A produce these functions $f = x^{\mathrm{T}}Ax$? Why is the first matrix positive definite but not the second one?

(a) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$.
(b) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$.

20. Compute the three upper left determinants to establish positive definiteness. Verify that their ratios give the second and third pivots.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

21. A positive definite matrix cannot have a zero (or even worse, a negative number) on its diagonal. Show that this matrix fails to have $x^{\mathrm{T}}Ax > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad,\quad,\quad).$$

22. A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have ______ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a ______ on the main diagonal.

23. Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.
(b) The only positive definite projection matrix is $P = I$.
(c) A diagonal matrix with positive diagonal entries is positive definite.
(d) A symmetric matrix with a positive determinant might not be positive definite!

24. For which s and t do A and B have all $\lambda > 0$ (and are therefore positive definite)?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

25. You may have seen the equation for an ellipse as $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. What are a and b when the equation is written as $\lambda_1x^2 + \lambda_2y^2 = 1$? The ellipse $9x^2 + 16y^2 = 1$ has half-axes with lengths $a =$ ___ and $b =$ ___.

26. Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding A.

27. With positive pivots in D, the factorization $A = LDL^{\mathrm{T}}$ becomes $L\sqrt{D}\sqrt{D}L^{\mathrm{T}}$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $C = L\sqrt{D}$ yields the ***Cholesky factorization*** $A = CC^{\mathrm{T}}$, which is "symmetrized LU":

$$\text{From} \quad C = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{find } A. \qquad \text{From} \quad A = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \quad \text{find } C.$$

28. In the Cholesky factorization $A = CC^{\mathrm{T}}$, with $C = L\sqrt{D}$, the square roots of the pivots are on the diagonal of C. Find C (lower triangular) for

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}.$$

29. The symmetric factorization $A = LDL^{\mathrm{T}}$ means that $x^{\mathrm{T}}Ax = x^{\mathrm{T}}LDL^{\mathrm{T}}x$:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & (ac-b^2)/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The left-hand side is $ax^2+2bxy+cy^2$. The right-hand side is $a\left(x+\frac{b}{a}y\right)^2 + \underline{\qquad\qquad} y^2$. The second pivot completes the square! Test with $a = 2$, $b = 4$, $c = 10$.

30. Without multiplying $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of A. (b) the eigenvalues of A.
(c) the eigenvectors of A. (d) a reason why A is symmetric positive definite.

31. For the semidefinite matrices

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{(rank 2)} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{(rank 1)},$$

write $x^{\mathrm{T}}Ax$ as a sum of two squares and $x^{\mathrm{T}}Bx$ as one square.

32. Apply any three tests to each of the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

to decide whether they are positive definite, positive semidefinite, or indefinite.

33. For $C = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, confirm that $C^{\mathrm{T}}AC$ has eigenvalues of the same signs as A. Construct a chain of nonsingular matrices $C(t)$ linking C to an orthogonal Q. Why is it impossible to construct a nonsingular chain linking C to the identity matrix?

34. If the pivots of a matrix are all greater than 1, are the eigenvalues all greater than 1? Test on the tridiagonal $-1, 2, -1$ matrices.

35. Use the pivots of $A - \frac{1}{2}I$ to decide whether A has an eigenvalue smaller than $\frac{1}{2}$:

$$A - \frac{1}{2}I = \begin{bmatrix} 2.5 & 3 & 0 \\ 3 & 9.5 & 7 \\ 0 & 7 & 7.5 \end{bmatrix}.$$

36. An algebraic proof of the *law of inertia* starts with the orthonormal eigenvectors $x_1, \ldots, x_p$ of A corresponding to eigenvalues $\lambda_i > 0$, and the orthonormal eigenvectors $y_1, \ldots, y_q$ of $C^{\mathrm{T}}AC$ corresponding to eigenvalues $\mu_i < 0$.

(a) To prove that the $p+q$ vectors $x_1, \ldots, x_p, Cy_1, \ldots, Cy_q$ are independent, assume that some combination gives zero:

$$a_1x_1 + \cdots + a_px_p = b_1Cy_1 + \cdots + b_qCy_q \ (= z, \text{ say}).$$

Show that $z^{\mathrm{T}}Az = \lambda_1a_1^2 + \cdots + \lambda_pa_p^2 \geq 0$ and $z^{\mathrm{T}}Az = \mu_1b_1^2 + \cdots + \mu_qb_q^2 \leq 0$.

(b) Deduce that the a's and b's are zero (proving linear independence). From that deduce $p+q \leq n$.

(c) The same argument for the $n-p$ negative λ's and the $n-q$ positive μ's gives $n-p+n-q \leq n$. (We again assume no zero eigenvalues—which are handled separately). Show that $p+q = n$, so the number p of positive λ's equals the number $n-q$ of positive μ's—which is the law of inertia.

37. If C is nonsingular, show that A and $C^{\mathrm{T}}AC$ have the same rank. Thus they have the same number of zero eigenvalues.

38. Find by experiment the number of positive, negative, and zero eigenvalues of

$$A = \begin{bmatrix} I & B \\ B^{\mathrm{T}} & 0 \end{bmatrix}$$

when the block B (of order $\frac{1}{2}n$) is nonsingular.

39. Do A and $C^{\mathrm{T}}AC$ always satisfy the law of inertia when C is not square?

40. In equation (9) with $m_1 = 1$ and $m_2 = 2$, verify that the normal modes are M-orthogonal: $x_1^{\mathrm{T}}Mx_2 = 0$.

41. Find the eigenvalues and eigenvectors of $Ax = \lambda Mx$:

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} x = \frac{\lambda}{18} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} x.$$

42. If the symmetric matrices A and M are indefinite, $Ax = \lambda Mx$ might not have real eigenvalues. Construct a 2 by 2 example.

43. A *group* of nonsingular matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups? *Positive definite symmetric matrices A, orthogonal matrices Q, all exponentials e^{tA} of a fixed matrix A, matrices P with positive eigenvalues, matrices D with determinant* 1. Invent a group containing only positive definite matrices.

6.3 SINGULAR VALUE DECOMPOSITION

A great matrix factorization has been saved for the end of the basic course. $U\Sigma V^{\mathrm{T}}$ joins with LU from elimination and QR from orthogonalization (Gauss and Gram–Schmidt). Nobody's name is attached; $A = U\Sigma V^{\mathrm{T}}$ is known as the "SVD" or the ***singular value decomposition***. We want to describe it, to prove it, and to discuss its applications—which are many and growing.

The SVD is closely associated with the eigenvalue–eigenvector factorization $Q\Lambda Q^{\mathrm{T}}$ of a positive definite matrix. The eigenvalues are in the diagonal matrix Λ. The eigenvector matrix Q *is orthogonal* ($Q^{\mathrm{T}}Q = I$) because eigenvectors of a symmetric matrix can be chosen to be orthonormal. For most matrices that is not true, and for rectangular matrices it is ridiculous (eigenvalues undefined). But now we allow the Q on the left and the Q^{T} on the right to be *any two orthogonal matrices* U and V^{T}—not necessarily transposes of each other. Then every matrix will split into $A = U\Sigma V^{\mathrm{T}}$.

The diagonal (but rectangular) matrix Σ has eigenvalues from $A^{\mathrm{T}}A$, not from A! Those positive entries (also called sigma) will be $\sigma_1, \ldots, \sigma_r$. They are the ***singular values*** of A. They fill the first r places on the main diagonal of Σ—when A has rank r. The rest of Σ is zero.

With rectangular matrices, the key is almost always to consider $A^{\mathrm{T}}A$ and AA^{T}.

Singular Value Decomposition: Any m by n matrix A can be factored into

$$A = U\Sigma V^{\mathrm{T}} = \textbf{(orthogonal)(diagonal)(orthogonal)}\,.$$

The columns of U (m by m) are eigenvectors of AA^{T}, and the columns of V (n by n) are eigenvectors of $A^{\mathrm{T}}A$. The r singular values on the diagonal of Σ (m by n) are the square roots of the nonzero eigenvalues of both AA^{T} and $A^{\mathrm{T}}A$.

Remark 1 For positive definite matrices, Σ is Λ and $U\Sigma V^{\mathrm{T}}$ is identical to $Q\Lambda Q^{\mathrm{T}}$. For other symmetric matrices, any negative eigenvalues in Λ become positive in Σ. For complex matrices, Σ remains real but U and V become *unitary* (the complex version of orthogonal). We take complex conjugates in $U^{\mathrm{H}}U = I$ and $V^{\mathrm{H}}V = I$ and $A = U\Sigma V^{\mathrm{H}}$.

Remark 2 U and V give orthonormal bases for *all four fundamental subspaces*:

first	r	columns of U:	**column space** of A
last	$m-r$	columns of U:	**left nullspace** of A
first	r	columns of V:	**row space** of A
last	$n-r$	columns of V:	**nullspace** of A

Remark 3 The SVD chooses those bases in an extremely special way. They are more than just orthonormal. *When A multiplies a column v_j of V, it produces σ_j times a column of U.* That comes directly from $AV = U\Sigma$, looked at a column at a time.

Remark 4 Eigenvectors of AA^{T} and $A^{\mathrm{T}}A$ must go into the columns of U and V:

$$AA^{\mathrm{T}} = (U\Sigma V^{\mathrm{T}})(V\Sigma^{\mathrm{T}}U^{\mathrm{T}}) = U\Sigma\Sigma^{\mathrm{T}}U^{\mathrm{T}} \quad \text{and, similarly,} \quad A^{\mathrm{T}}A = V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}}. \tag{1}$$

U must be the eigenvector matrix for AA^{T}. The eigenvalue matrix in the middle is $\Sigma\Sigma^{\mathrm{T}}$—which is m by m with $\sigma_1^2, \ldots, \sigma_r^2$ on the diagonal.

From the $A^{\mathrm{T}}A = V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}}$, the V matrix *must be the eigenvector matrix for* $A^{\mathrm{T}}A$. The diagonal matrix $\Sigma^{\mathrm{T}}\Sigma$ has the same $\sigma_1^2, \ldots, \sigma_r^2$, but it is n by n.

Remark 5 Here is the reason that $Av_j = \sigma_j u_j$. Start with $A^{\mathrm{T}}Av_j = \sigma_j^2 v_j$:

$$\textbf{Multiply by } \boldsymbol{A} \qquad AA^{\mathrm{T}}Av_j = \sigma_j^2 Av_j \tag{2}$$

This says that Av_j is an eigenvector of AA^{T}! We just moved parentheses to $(AA^{\mathrm{T}})(Av_j)$. The length of this eigenvector Av_j is σ_j, because

$$v^{\mathrm{T}}A^{\mathrm{T}}Av_j = \sigma_j^2 v_j^{\mathrm{T}} v_j \quad \text{gives} \quad \|Av_j\|^2 = \sigma_j^2.$$

So the unit eigenvector is $Av_j/\sigma_j = u_j$. **In other words, $AV = U\Sigma$.**

Example 1 This A has only one column: rank $r = 1$. Then Σ has only $\sigma_1 = 3$:

$$\textbf{SVD} \qquad A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} [\,1\,] = U_{3\times 3}\,\Sigma_{3\times 1}\,V_{1\times 1}^{\mathrm{T}}.$$

$A^{\mathrm{T}}A$ is 1 by 1, whereas AA^{T} is 3 by 3. They both have eigenvalue 9 (whose square root is the 3 in Σ). The two zero eigenvalues of AA^{T} leave some freedom for the eigenvectors in columns 2 and 3 of U. We kept that matrix orthogonal.

Example 2 Now A has rank 2, and $AA^{\mathrm{T}} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ with $\lambda = 3$ and 1:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = U\Sigma V^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} /\sqrt{6} \\ /\sqrt{2}. \\ /\sqrt{3} \end{matrix}$$

Notice $\sqrt{3}$ and $\sqrt{1}$. The columns of U are *left* singular vectors (unit eigenvectors of AA^{T}). The columns of V are *right* singular vectors (unit eigenvectors of $A^{\mathrm{T}}A$).

Applications of the SVD

We will pick a few important applications, after emphasizing one key point. The SVD is terrific for numerically stable computations, because U and V are orthogonal matrices. They never change the length of a vector. Since $\|Ux\|^2 = x^{\mathrm{T}}U^{\mathrm{T}}Ux = \|x\|^2$, multiplication by U cannot destroy the scaling.

Of course Σ could multiply by a large σ or (more commonly) divide by a small σ, and overflow the computer. But still Σ *is as good as possible*. It reveals exactly what is large and what is small. The ratio $\sigma_{\max}/\sigma_{\min}$ is the **condition number** of an invertible n by n matrix. The availability of that information is another reason for the popularity of the SVD. We come back to this in the second application.

1. Image processing Suppose a satellite takes a picture, and wants to send it to Earth. The picture may contain 1000 by 1000 "pixels"—a million little squares, each with a definite color. We can code the colors, and send back 1,000,000 numbers. It is better to find the *essential* information inside the 1000 by 1000 matrix, and send only that.

Suppose we know the SVD. The key is in the singular values (in Σ). Typically, some σ's are significant and others are extremely small. If we keep 20 and throw away 980, then we send only the corresponding 20 columns of U and V. The other 980 columns are multiplied in $U\Sigma V^{\mathrm{T}}$ by the small σ's that are being ignored. *We can do the matrix multiplication as columns times rows*:

$$A = U\Sigma V^{\mathrm{T}} = u_1\sigma_1 v_1^{\mathrm{T}} + u_2\sigma_2 v_2^{\mathrm{T}} + \cdots + u_r\sigma_r v_r^{\mathrm{T}}. \tag{3}$$

Any matrix is the sum of r matrices of rank 1. If only 20 terms are kept, we send 20 times 2000 numbers instead of a million (25 to 1 compression).

The pictures are really striking, as more and more singular values are included. At first you see nothing, and suddenly you recognize everything. The cost is in computing the SVD—this has become much more efficient, but it is expensive for a big matrix.

2. The effective rank The rank of a matrix is the number of independent rows, and the number of independent columns. That can be hard to decide in computations! In exact arithmetic, counting the pivots is correct. Real arithmetic can be misleading—but discarding small pivots is not the answer. Consider the following:

$$\boldsymbol{\epsilon} \textbf{ is small} \qquad \begin{bmatrix} \epsilon & 2\epsilon \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \epsilon & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \epsilon & 1 \\ \epsilon & 1+\epsilon \end{bmatrix}.$$

The first has rank 1, although roundoff error will probably produce a second pivot. Both pivots will be small; how many do we ignore? The second has one small pivot, but we cannot pretend that its row is insignificant. The third has two pivots and its rank is 2, but its "effective rank" ought to be 1.

We go to a more stable measure of rank. The first step is to use $A^{\mathrm{T}}A$ or AA^{T}, which are symmetric but share the same rank as A. Their eigenvalues—the singular values squared—are *not* misleading. Based on the accuracy of the data, we decide on a tolerance like 10^{-6} and count the singular values above it—that is the effective rank. The examples above have effective rank 1 (when ϵ is very small).

3. Polar decomposition Every nonzero complex number z is a positive number r times a number $e^{i\theta}$ on the unit circle: $z = re^{i\theta}$. That expresses z in "polar coordinates." If we think of z as a 1 by 1 matrix, r corresponds to a *positive definite matrix* and $e^{i\theta}$ corresponds to an *orthogonal matrix*. More exactly, since $e^{i\theta}$ is complex and satisfies $e^{-i\theta}e^{i\theta} = 1$, it forms a 1 by 1 *unitary matrix*: $U^{\mathrm{H}}U = I$. We take the complex conjugate as well as the transpose, for U^{H}.

The SVD extends this "polar factorization" to matrices of any size:

Every real square matrix can be factored into $A = QS$, where Q is ***orthogonal*** and S is ***symmetric positive semidefinite***. If A is invertible then S is positive definite.

For proof we just insert $V^{\mathrm{T}}V = I$ into the middle of the SVD:

$$A = U\Sigma V^{\mathrm{T}} = (UV^{\mathrm{T}})(V\Sigma V^{\mathrm{T}}). \tag{4}$$

The factor $S = V\Sigma V^{\mathrm{T}}$ is symmetric and semidefinite (because Σ is). The factor $Q = UV^{\mathrm{T}}$ is an orthogonal matrix (because $Q^{\mathrm{T}}Q = VU^{\mathrm{T}}UV^{\mathrm{T}} = I$). In the complex case, S becomes Hermitian instead of symmetric and Q becomes unitary instead of orthogonal. In the invertible case Σ is definite and so is S.

Example 3 Polar decomposition:

$$A = QS \qquad \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$

Example 4 Reverse polar decomposition:

$$A = S'Q \qquad \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The exercises show how, in the reverse order, S changes but Q remains the same. Both S and S' are symmetric positive definite because this A is invertible.

Application of $A = QS$: A major use of the polar decomposition is in continuum mechanics (and recently in robotics). In any deformation, it is important to separate stretching from rotation, and that is exactly what QS achieves. The orthogonal matrix Q is a rotation, and possibly a reflection. The material feels no strain. The symmetric matrix S has eigenvalues $\sigma_1, \ldots, \sigma_r$, which are the stretching factors (or compression factors). The diagonalization that displays those eigenvalues is the natural choice of axes—called ***principal axes***: as in the ellipses of Section 6.2. It is S that requires work on the material, and stores up elastic energy.

We note that S^2 is $A^{\mathrm{T}}A$, which is symmetric positive definite when A is invertible. S is the symmetric positive definite square root of $A^{\mathrm{T}}A$, and Q is AS^{-1}. In fact, *A could be rectangular, as long as $A^{\mathrm{T}}A$ is positive definite.* (That is the condition we keep meeting, that A must have independent columns.) In the reverse order $A = S'Q$, the matrix S' is the symmetric positive definite square root of AA^{T}.

4. Least squares For a rectangular system $Ax = b$, the least-squares solution comes from the normal equations $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. ***If A has dependent columns then $A^{\mathrm{T}}A$ is not invertible and $\widehat{x}$ is not determined.*** Any vector in the nullspace could be added to $\widehat{x}$. We can now complete Chapter 3, by choosing a "best" (*shortest*) $\widehat{x}$ for every $Ax = b$.

$Ax = b$ has two possible difficulties: *Dependent rows or dependent columns.* With dependent rows, $Ax = b$ may have no solution. That happens when b is outside the column space of A. Instead of $Ax = b$, we solve $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. But if A has *dependent columns*, this $\widehat{x}$ will not be unique. We have to choose a particular solution of $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$, and we choose the shortest.

The optimal solution of $Ax = b$ is the minimum length solution of $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.

That minimum length solution will be called x^+. It is our preferred choice as the best solution to $Ax = b$ (which had no solution), and also to $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$ (which had too many). We start with a diagonal example.

Example 5 A is diagonal, with dependent rows and dependent columns:

$$A\widehat{x} = p \quad \text{is} \quad \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \\ \widehat{x}_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}.$$

The columns all end with zero. In the column space, the closest vector to $b = (b_1, b_2, b_3)$ is $p = (b_1, b_2, 0)$. The best we can do with $Ax = b$ is to solve the first two equations, since the third equation is $0 = b_3$. That error cannot be reduced, but the errors in the first two equations will be zero. Then

$$\widehat{x}_1 = b_1/\sigma_1 \quad \text{and} \quad \widehat{x}_2 = b_2/\sigma_2.$$

Now we face the second difficulty. To make $\widehat{x}$ as short as possible, we choose the totally arbitrary $\widehat{x}_3$ and $\widehat{x}_4$ to be zero. ***The minimum length solution is x^+:***

A^+ is pseudoinverse
$x^+ = A^+b$ is shortest

$$x^+ = \begin{bmatrix} b_1/\sigma_1 \\ b_2/\sigma_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 1/\sigma_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{5}$$

This equation finds x^+, and it also displays *the matrix that produces x^+ from b*. That matrix is the ***pseudoinverse*** A^+ of our diagonal A. Based on this example, we know Σ^+ and x^+ for any diagonal matrix Σ:

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \\ & & \end{bmatrix} \qquad \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_r \\ & & \end{bmatrix} \qquad \Sigma^+ b = \begin{bmatrix} b_1/\sigma_1 \\ \vdots \\ b_r/\sigma_r \\ \\ \end{bmatrix}.$$

The matrix Σ is m by n, with r nonzero entries σ_i. Its pseudoinverse Σ^+ is n by m, with r nonzero entries $1/\sigma_i$. **All the blank spaces are zeros**. Notice that $(\Sigma^+)^+$ is Σ again. That is like $(A^{-1})^{-1} = A$, but here A is not invertible.

Now we find x^+ in the general case. We claim that ***the shortest solution x^+ is always in the row space of A***. Remember that any vector $\widehat{x}$ can be split into a row space component x_r and a nullspace component: $\widehat{x} = x_r + x_n$. There are three important points about that splitting:

1. The row space component also solves $A^TA\widehat{x}_r = A^Tb$, because $Ax_n = 0$.
2. The components are orthogonal, and they obey Pythagoras's law:

$$\|\widehat{x}\|^2 = \|x_r\|^2 + \|x_n\|^2, \text{ so } \widehat{x} \text{ is shortest when } x_n = 0.$$

3. All solutions of $A^TA\widehat{x} = A^Tb$ have the same x_r. ***That vector is x^+.***

The fundamental theorem of linear algebra was in Figure 3.4. Every p in the column space comes from one and only one vector x_r in the row space. *All we are doing is to choose that vector, $x^+ = x_r$, as the best solution to $Ax = b$.*

The pseudoinverse in Figure 6.3 starts with b and comes back to x^+. *It inverts A where A is invertible*—between row space and column space. The pseudoinverse knocks out the left nullspace by sending it to zero, and it knocks out the nullspace by choosing x_r as x^+.

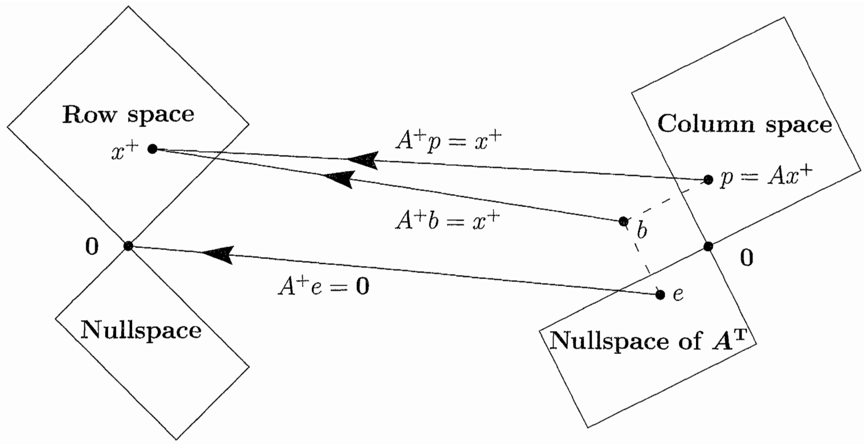


Figure 6.3 The pseudoinverse A^+ inverts A where it can on the column space.

We have not yet shown that there is a matrix A^+ that always gives x^+—but there is. It will be n by m, because it takes b and p in $\mathbf{R}^m$ back to x^+ in $\mathbf{R}^n$. We look at one more example before finding A^+ in general.

Example 6 $Ax = b$ is $-x_1 + 2x_2 + 2x_3 = 18$, with a whole plane of solutions.

According to our theory, the shortest solution should be in the row space of $A = [-1 \quad 2 \quad 2]$. The multiple of that row that satisfies the equation is $x^+ = (-2, 4, 4)$. There are longer solutions like $(-2, 5, 3)$, $(-2, 7, 1)$, or $(-6, 3, 3)$, but they all have nonzero components from the nullspace. The matrix that produces x^+ from $b = [18]$ is the pseudoinverse A^+. Whereas A was 1 by 3, this A^+ is 3 by 1:

$$A^+ = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}^+ = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ \frac{2}{9} \end{bmatrix} \quad \text{and} \quad A^+[18] = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix}. \tag{6}$$

The row space of A is the column space of A^+. Here is a formula for A^+:

$$\textbf{If } \boldsymbol{A = U\Sigma V^{\mathrm{T}}} \textbf{ (the SVD), then its pseudoinverse is } \boldsymbol{A^+ = V\Sigma^+U^{\mathrm{T}}}. \tag{7}$$

Example 6 had $\sigma = 3$—the square root of the eigenvalue of $AA^{\mathrm{T}} = [9]$. Here it is again with Σ and Σ^+:

$$A = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$V\Sigma^+U^{\mathrm{T}} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ \frac{2}{9} \end{bmatrix} = A^+.$$

The minimum length least-squares solution is $x^+ = A^+b = V\Sigma^+U^{\mathrm{T}}b$.

Proof Multiplication by the orthogonal matrix U^{T} leaves lengths unchanged:

$$\|Ax - b\| = \|U\Sigma V^{\mathrm{T}}x - b\| = \|\Sigma V^{\mathrm{T}}x - U^{\mathrm{T}}b\|.$$

Introduce the new unknown $y = V^{\mathrm{T}}x = V^{-1}x$, which has the same length as x. Then, minimizing $\|Ax - b\|$ is the same as minimizing $\|\Sigma y - U^{\mathrm{T}}b\|$. Now Σ is diagonal and we know the best y^+. It is $y^+ = \Sigma^+ U^{\mathrm{T}}b$, so the best x^+ is Vy^+:

$$\textbf{Shortest solution} \qquad x^+ = Vy^+ = V\Sigma^+ U^{\mathrm{T}}b = A^+b.$$

Vy^+ is in the row space, and $A^{\mathrm{T}}Ax^+ = A^{\mathrm{T}}b$ from the **SVD**. ■

Problem Set 6.3

Problems 1–2 compute the SVD of a square singular matrix A.

1. Compute $A^{\mathrm{T}}A$ and its eigenvalues σ_1^2, 0 and unit eigenvectors v_1, v_2:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

2. (a) Compute AA^{T} and its eigenvalues σ_1^2, 0 and unit eigenvectors u_1, u_2.

(b) Choose signs so that $Av_1 = \sigma_1 u_1$ and verify the SVD:

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}}.$$

(c) Which four vectors give orthonormal bases for $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, $\boldsymbol{C}(A^{\mathrm{T}})$, $\boldsymbol{N}(A^{\mathrm{T}})$?

Problems 3–5 ask for the SVD of matrices of rank 2.

3. Find the SVD from the eigenvectors v_1, v_2 of $A^{\mathrm{T}}A$ and $Av_i = \sigma_i u_i$:

$$\textbf{Fibonacci matrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

4. Use the SVD part of the MATLAB demo **eigshow** (or Java on the course page `web.mit.edu/18.06`) to find the same vectors v_1 and v_2 graphically.

5. Compute $A^{\mathrm{T}}A$ and AA^{T}, and their eigenvalues and unit eigenvectors, for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Multiply the three matrices $U\Sigma V^{\mathrm{T}}$ to recover A.

Problems 6–13 bring out the underlying ideas of the SVD.

6. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are orthonormal bases for $\mathbf{R}^n$. Construct the matrix A that transforms each v_j into u_j to give $Av_1 = u_1, \ldots, Av_n = u_n$.

7. Construct the matrix with rank 1 that has $Av = 12u$ for $v = \frac{1}{2}(1, 1, 1, 1)$ and $u = \frac{1}{3}(2, 2, 1)$. Its only singular value is $\sigma_1 =$ ___.

8. Find $U\Sigma V^{\mathrm{T}}$ if A has orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1, \ldots, \sigma_n$.

9. Explain how $U\Sigma V^{\mathrm{T}}$ expresses A as a sum of r rank-1 matrices in equation (3):

$$A = \sigma_1 u_1 v_1^{\mathrm{T}} + \cdots + \sigma_r u_r v_r^{\mathrm{T}}.$$

10. Suppose A is a 2 by 2 symmetric matrix with unit eigenvectors u_1 and u_2. If its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, what are U, Σ, and V^{T}?

11. Suppose A is invertible (with $\sigma_1 > \sigma_2 > 0$). Change A by as small a matrix as possible to produce a *singular* matrix A_0. Hint: U and V do not change:

$$\text{Find } A_0 \text{ from} \quad A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}}.$$

12. (a) If A changes to $4A$, what is the change in the SVD?
(b) What is the SVD for A^{T} and for A^{-1}?

13. Why doesn't the SVD for $A + I$ just use $\Sigma + I$?

14. Find the SVD and the pseudoinverse 0^+ of the m by n *zero matrix.*

15. Find the SVD and the pseudoinverse $V\Sigma^+U^{\mathrm{T}}$ of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

16. If an m by n matrix Q has orthonormal columns, what is Q^+?

17. Diagonalize $A^{\mathrm{T}}A$ to find its positive definite square root $S = V\Sigma^{1/2}V^{\mathrm{T}}$ and its polar decomposition $A = QS$:

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix}.$$

18. What is the minimum-length least-squares solution $x^+ = A^+b$ to the following?

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

You can compute A^+, or find the general solution to $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$ and choose the solution that is in the row space of A. This problem fits the best plane $C + Dt + Ez$ to $b = 0$ *and also* $b = 2$ at $t = z = 0$ (and $b = 2$ at $t = z = 1$).

19. (a) If A has independent columns, its left-inverse $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is A^+.
(b) If A has independent rows, its right-inverse $A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ is A^+.
In both cases, verify that $x^+ = A^+b$ is in the row space, and $A^{\mathrm{T}}Ax^+ = A^{\mathrm{T}}b$.

20. Split $A = U\Sigma V^{\mathrm{T}}$ into its reverse polar decomposition QS'.

21. Is $(AB)^+ = B^+A^+$ always true for pseudoinverses? I believe not.

22. Removing zero rows of U leaves $A = \underline{L}\,\underline{U}$, where the r columns of $\underline{L}$ span the column space of A and the r rows of $\underline{U}$ span the row space. Then A^+ has the explicit formula $\underline{U}^{\mathrm{T}}(\underline{U}\,\underline{U}^{\mathrm{T}})^{-1}(\underline{L}^{\mathrm{T}}\underline{L})^{-1}\underline{L}^{\mathrm{T}}$.

Why is A^+b in the row space with $\underline{U}^{\mathrm{T}}$ at the front? Why does $A^{\mathrm{T}}AA^+b = A^{\mathrm{T}}b$, so that $x^+ = A^+b$ satisfies the normal equation as it should?

23. Explain why AA^+ and A^+A are projection matrices (and therefore symmetric). What fundamental subspaces do they project onto?

6.4 MINIMUM PRINCIPLES

In this section we escape for the first time from linear equations. The unknown x will not be given as the solution to $Ax = b$ or $Ax = \lambda x$. Instead, the vector x will be determined by a minimum principle.

It is astonishing how many natural laws can be expressed as minimum principles. Just the fact that heavy liquids sink to the bottom is a consequence of minimizing their potential energy. And when you sit on a chair or lie on a bed, the springs adjust themselves so that the energy is minimized. A straw in a glass of water looks bent because light reaches your eye as quickly as possible. Certainly there are more highbrow examples: The fundamental principle of structural engineering is the minimization of total energy.*

We have to say immediately that these "energies" are nothing but *positive definite quadratic functions*. And the derivative of a quadratic is linear. We get back to the familiar linear equations, when we set the first derivatives to zero. Our first goal in this section is ***to find the minimum principle that is equivalent to*** $\boldsymbol{Ax = b}$***, and the minimization equivalent to*** $\boldsymbol{Ax = \lambda x}$. We will be doing in finite dimensions exactly what the theory of optimization does in a continuous problem, where "first derivatives $= 0$" gives a differential equation. In every problem, we are free to solve the linear equation or minimize the quadratic.

The first step is straightforward: We want to find the "parabola" $P(x)$ whose minimum occurs when $Ax = b$. If A is just a scalar, that is easy to do:

$$\text{The graph of} \quad P(x) = \frac{1}{2}Ax^2 - bx \quad \text{has zero slope when} \quad \frac{dP}{dx} = Ax - b = 0.$$

This point $x = A^{-1}b$ will be a minimum if A is positive. Then the parabola $P(x)$ opens upward (Figure 6.4). In more dimensions this parabola turns into a parabolic bowl (a paraboloid). To assure a minimum of $P(x)$, not a maximum or a saddle point, A must be positive definite!

6H If A is symmetric positive definite, then $P(x) = \frac{1}{2}x^{\mathrm{T}}Ax - x^{\mathrm{T}}b$ reaches its minimum at the point where $Ax = b$. At that point $P_{\min} = -\frac{1}{2}b^{\mathrm{T}}A^{-1}b$.

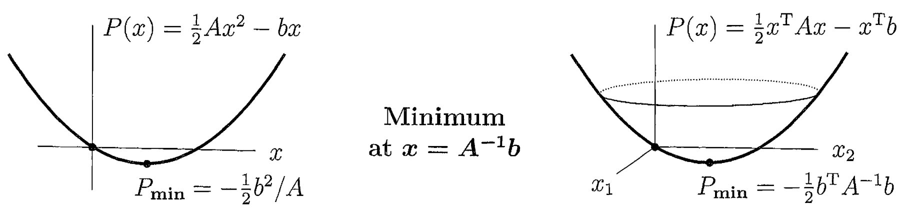


Figure 6.4 The graph of a positive quadratic $P(x)$ is a parabolic bowl.

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* I am convinced that plants and people also develop in accordance with minimum principles. Perhaps civilization is based on a law of least action. There must be new laws (and minimum principles) to be found in the social sciences and life sciences.

Proof Suppose $Ax = b$. For any vector y, we show that $P(y) \ge P(x)$:

$$\begin{aligned} P(y) - P(x) &= \frac{1}{2}y^{\mathrm T}Ay - y^{\mathrm T}b - \frac{1}{2}x^{\mathrm T}Ax + x^{\mathrm T}b \\ &= \frac{1}{2}y^{\mathrm T}Ay - y^{\mathrm T}Ax + \frac{1}{2}x^{\mathrm T}Ax \quad (\text{set } b = Ax) \\ &= \frac{1}{2}(y-x)^{\mathrm T}A(y-x). \end{aligned} \tag{1}$$

This can't be negative since A is positive definite—and it is zero only if $y - x = 0$. At all other points $P(y)$ is larger than $P(x)$, so the minimum occurs at x. ■

Example 1 Minimize $P(x) = x_1^2 - x_1x_2 + x_2^2 - b_1x_1 - b_2x_2$. The usual approach, by calculus, is to set the partial derivatives to zero. This gives $Ax = b$:

$$\begin{aligned} \partial P/\partial x_1 &= 2x_1 - x_2 - b_1 = 0 \\ \partial P/\partial x_2 &= -x_1 + 2x_2 - b_2 = 0 \end{aligned} \quad \text{means} \quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{2}$$

Linear algebra recognizes this $P(x)$ as $\frac{1}{2}x^{\mathrm T}Ax - x^{\mathrm T}b$, and knows immediately that $Ax = b$ gives the minimum. Substitute $x = A^{-1}b$ into $P(x)$:

$$\textbf{Minimum value} \qquad P_{\min} = \frac{1}{2}(A^{-1}b)^{\mathrm T}A(A^{-1}b) - (A^{-1}b)^{\mathrm T}b = -\frac{1}{2}b^{\mathrm T}A^{-1}b. \tag{3}$$

In applications, $\frac{1}{2}x^{\mathrm T}Ax$ is the internal energy and $-x^{\mathrm T}b$ is the external work. The system automatically goes to $x = A^{-1}b$, where the total energy $P(x)$ is a minimum.

Minimizing with Constraints

Many applications add extra equations $Cx = d$ on top of the minimization problem. These equations are **constraints**. We minimize $P(x)$ subject to the extra requirement $Cx = d$. Usually x can't satisfy n equations $Ax = b$ and also ℓ extra constraints $Cx = d$. We have too many equations and we need ℓ more unknowns.

Those new unknowns $y_1, \dots, y_\ell$ are called **Lagrange multipliers**. They build the constraint into a function $L(x, y)$. This was the brilliant insight of Lagrange:

$$L(x, y) = P(x) + \boldsymbol{y}^{\mathrm T}(\boldsymbol{Cx} - \boldsymbol{d}) = \frac{1}{2}x^{\mathrm T}Ax - x^{\mathrm T}b + x^{\mathrm T}C^{\mathrm T}y - y^{\mathrm T}d.$$

That term in L is chosen exactly so that $\partial L/\partial y = 0$ brings back $Cx = d$. When we set the derivatives of L to zero, we have $n + \ell$ equations for $n + \ell$ unknowns x and y:

$$\begin{matrix} \textbf{Constrained} & \partial L/\partial x = 0: & Ax + C^{\mathrm T}y = b \\ \textbf{minimization} & \partial L/\partial y = 0: & Cx \qquad\;\; = d \end{matrix} \tag{4}$$

The first equations involve the mysterious unknowns y. You might well ask what they represent. Those "dual unknowns" y tell how much the constrained minimum $P_{C/\min}$ (which only allows x when $Cx = d$) exceeds the unconstrained $P_{\min}$ (allowing all x):

$$\textbf{Sensitivity of minimum} \qquad P_{C/\min} = P_{\min} + \frac{1}{2}y^{\mathrm T}(CA^{-1}b - d) \ge P_{\min}. \tag{5}$$

Example 2 Suppose $P(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. Its smallest value is certainly $P_{\min} = 0$. This unconstrained problem has $n = 2$, $A = I$, and $b = 0$. So the minimizing equation $Ax = b$ just gives $x_1 = 0$ and $x_2 = 0$.

Now add one constraint $c_1x_1 + c_2x_2 = d$. This puts x on a line in the x_1-x_2 plane. The old minimizer $x_1 = x_2 = 0$ is not on the line. The Lagrangian function $L(x, y) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + y(c_1x_1 + c_2x_2 - d)$ has $n + \ell = 2 + 1$ partial derivatives:

$$\begin{aligned} \partial L/\partial x_1 &= 0 \qquad & x_1 + c_1 y &= 0 \\ \partial L/\partial x_2 &= 0 \qquad & x_2 + c_2 y &= 0 \\ \partial L/\partial y &= 0 \qquad & c_1x_1 + c_2x_2 &= d. \end{aligned} \tag{6}$$

Substituting $x_1 = -c_1y$ and $x_2 = -c_2y$ into the third equation gives $-c_1^2y - c_2^2y = d$.

$$\textbf{Solution} \qquad y = \frac{-d}{c_1^2 + c_2^2} \qquad x_1 = \frac{c_1d}{c_1^2 + c_2^2} \qquad x_2 = \frac{c_2d}{c_1^2 + c_2^2}. \tag{7}$$

The constrained minimum of $P = \frac{1}{2}x^Tx$ is reached at that solution point:

$$P_{C/\min} = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 = \frac{1}{2}\frac{c_1^2d^2 + c_2^2d^2}{\left(c_1^2 + c_2^2\right)^2} = \frac{1}{2}\frac{d^2}{c_1^2 + c_2^2}. \tag{8}$$

This equals $-\frac{1}{2}yd$ as predicted in equation (5), since $b = 0$ and $P_{\min} = 0$.

Figure 6.5 shows what problem the linear algebra has solved, if the constraint keeps x on a line $2x_1 - x_2 = 5$. We are looking for **the closest point to (0, 0) on this line**. The solution is $x = (2, -1)$. We expect this shortest vector x to be perpendicular to the line, and we are right.

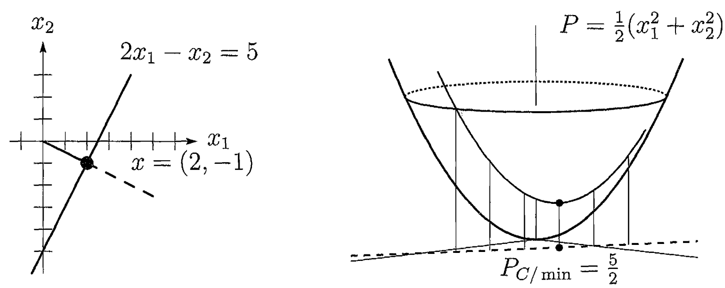

Figure 6.5 Minimizing $\frac{1}{2}\|x\|^2$ for all x on the constraint line $2x_1 - x_2 = 5$.

Least Squares Again

In minimization, our big application is least squares. The best $\widehat{x}$ is the vector that minimizes the squared error $E^2 = \|Ax - b\|^2$. This is a quadratic and it fits our framework! I will highlight the parts that look new:

$$\textbf{Squared error} \qquad E^2 = (Ax - b)^T(Ax - b) = x^T\boldsymbol{A^TA}\,x - 2x^T\boldsymbol{A^Tb} + \boldsymbol{b^Tb}. \tag{9}$$

Compare with $\frac{1}{2}x^TAx - x^Tb$ at the start of the section, which led to $Ax = b$:

[A changes to $\boldsymbol{A^TA}$] [b changes to $\boldsymbol{A^Tb}$] [$\boldsymbol{b^Tb}$ is added].

The constant b^Tb raises the whole graph—this has no effect on the best $\widehat{x}$. The other two changes, A to A^TA and b to A^Tb, give a new way to reach the least-squares equation (normal equation). The minimizing equation $Ax = b$ changes into the

$$\textbf{Least-squares equation} \qquad A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b. \tag{10}$$

Optimization needs a whole book. We stop while it is pure linear algebra.

The Rayleigh Quotient

Our second goal is to find a minimization problem equivalent to $Ax = \lambda x$. That is not so easy. The function to minimize cannot be a quadratic, or its derivative would be linear—and the eigenvalue problem is nonlinear (λ times x). The trick that succeeds is to divide one quadratic by another one:

$$\textbf{Rayleigh quotient} \qquad \text{Minimize} \quad R(x) = \frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x}.$$

6I **Rayleigh's Principle:** The minimum value of the Rayleigh quotient is the smallest eigenvalue λ_1. $R(x)$ reaches that minimum at the first eigenvector x_1 of A:

$$\textbf{Minimum where } \boldsymbol{Ax_1 = \lambda x_1} \qquad R(x_1) = \frac{x_1^{\mathrm{T}}Ax_1}{x_1^{\mathrm{T}}x_1} = \frac{x_1^{\mathrm{T}}\lambda_1 x_1}{x_1^{\mathrm{T}}x_1} = \lambda_1.$$

If we keep $x^{\mathrm{T}}Ax = 1$, then $R(x)$ is a minimum when $x^{\mathrm{T}}x = \|x\|^2$ is as large as possible. We are looking for the point on the ellipsoid $x^{\mathrm{T}}Ax = 1$ farthest from the origin—the vector x of greatest length. From our earlier description of the ellipsoid, its longest axis points along the first eigenvector. So $R(x)$ is a minimum at x_1.

Algebraically, we can diagonalize the symmetric A by an orthogonal matrix: $Q^{\mathrm{T}}AQ = \Lambda$. Then set $x = Qy$ and the quotient becomes simple:

$$R(x) = \frac{(Qy)^{\mathrm{T}}A(Qy)}{(Qy)^{\mathrm{T}}(Qy)} = \frac{y^{\mathrm{T}}\Lambda y}{y^{\mathrm{T}}y} = \frac{\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2}{y_1^2 + \cdots + y_n^2}. \tag{11}$$

The minimum of R is λ_1, at the point where $y_1 = 1$ and $y_2 = \cdots = y_n = 0$:

$$\textbf{At all points} \qquad \lambda_1\left(y_1^2 + y_2^2 + \cdots + y_n^2\right) \le \left(\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2\right).$$

The Rayleigh quotient in equation (11) is ***never below*** $\boldsymbol{\lambda_1}$ and ***never above*** $\boldsymbol{\lambda_n}$ (the largest eigenvalue). Its minimum is at the eigenvector x_1 and its maximum is at x_n:

$$\textbf{Maximum where } \boldsymbol{Ax_n = \lambda_n x_n} \qquad R(x_n) = \frac{x_n^{\mathrm{T}}Ax_n}{x_n^{\mathrm{T}}x_n} = \frac{x_n^{\mathrm{T}}\lambda_n x_n}{x_n^{\mathrm{T}}x_n} = \lambda_n.$$

One small yet important point: The Rayleigh quotient equals a_{11}, when the trial vector is $x = (1, 0, \ldots, 0)$. So a_{11} (on the main diagonal) is between λ_1 and λ_n. You can see this in Figure 6.6, where the horizontal distance to the ellipse (where $a_{11}x^2 = 1$) is between the shortest distance and the longest distance:

$$\frac{1}{\sqrt{\lambda_n}} \le \frac{1}{\sqrt{a_{11}}} \le \frac{1}{\sqrt{\lambda_1}} \quad \text{which is} \quad \lambda_1 \le a_{11} \le \lambda_n.$$

The diagonal entries of any symmetric matrix are between $\boldsymbol{\lambda_1}$ ***and*** $\boldsymbol{\lambda_n}$. We drew Figure 6.6 for a 2 by 2 positive definite matrix to see it clearly.

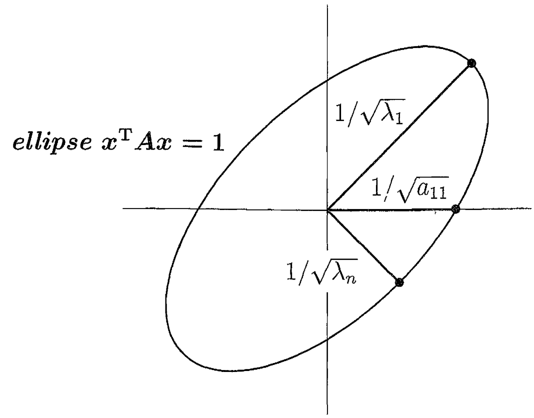

Figure 6.6 The farthest $x = x_1/\sqrt{\lambda_1}$ and the closest $x = x_n/\sqrt{\lambda_n}$ both give $x^{\mathrm{T}}Ax = x^{\mathrm{T}}\lambda x = 1$. Those are the major and minor axes of the ellipse.

Intertwining of the Eigenvalues

The intermediate eigenvectors $x_2, \ldots, x_{n-1}$ are ***saddle points*** of the Rayleigh quotient (zero derivatives, but not minima or maxima). The difficulty with saddle points is that we have no idea whether $R(x)$ is above or below them. That makes the intermediate eigenvalues $\lambda_2, \ldots, \lambda_{n-1}$ harder to estimate.

For this optional topic, the key is to find a constrained minimum or maximum. The constraints come from the basic property of symmetric matrices: x_j is perpendicular to the other eigenvectors.

6J The minimum of $R(x)$ subject to $x^{\mathrm{T}}x_1 = 0$ is λ_2. The minimum of $R(x)$ subject to any other constraint $x^{\mathrm{T}}v = 0$ is not above λ_2:

$$\lambda_2 = \min_{x^{\mathrm{T}}x_1=0} R(x) \quad \text{and} \quad \lambda_2 \geq \min_{x^{\mathrm{T}}v=0} R(x). \tag{12}$$

This "maximin principle" makes λ_2 the *maximum over all v of the minimum* of $R(x)$ with $x^{\mathrm{T}}v = 0$. That offers a way to estimate λ_2 without knowing λ_1.

Example 3 Throw away the last row and column of any symmetric matrix:

$$\begin{array}{l}\lambda_1(A) = 2-\sqrt{2}\\ \lambda_2(A) = 2\\ \lambda_3(A) = 2+\sqrt{2}\end{array} \qquad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ becomes } B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \begin{array}{l}\lambda_1(B) = 1\\ \lambda_2(B) = 3.\end{array}$$

The second eigenvalue $\lambda_2(A) = 2$ is above the lowest eigenvalue $\lambda_1(B) = 1$. The lowest eigenvalue $\lambda_1(A) = 2 - \sqrt{2}$ is *below* $\lambda_1(B)$. So $\lambda_1(B)$ is caught between.

This example chose $v = (0, 0, 1)$ so the constraint $x^{\mathrm{T}}v = 0$ knocked out the third component of x (thereby reducing A to B).

The complete picture is an ***intertwining of eigenvalues***:

$$\lambda_1(A) \le \lambda_1(B) \le \lambda_2(A) \le \lambda_2(B) \le \cdots \le \lambda_{n-1}(B) \le \lambda_n(A). \tag{13}$$

This has a natural interpretation for an ellipsoid, when it is cut by a plane through the origin. The cross section is an ellipsoid of one lower dimension. The major axis of this cross section cannot be longer than the major axis of the whole ellipsoid: $\lambda_1(B) \ge \lambda_1(A)$. But the major axis of the cross section is *at least as long as the second axis* of the original ellipsoid: $\lambda_1(B) \le \lambda_2(A)$. Similarly the minor axis of the cross section is smaller than the original second axis, and larger than the original minor axis: $\lambda_2(A) \le \lambda_2(B) \le \lambda_3(A)$.

You can see the same thing in mechanics. When springs and masses are oscillating, suppose one mass is held at equilibrium. Then the lowest frequency is increased, but not above λ_2. The highest frequency is decreased, but not below λ_{n-1}.

We close with three remarks. I hope your intuition says that they are correct.

Remark 1 The **maximin principle** extends to j-dimensional subspaces S_j:

$$\textbf{Maximum of minimum} \qquad \lambda_{j+1} = \max_{\text{all } S_j} \left[\min_{x \perp S_j} R(x) \right]. \tag{14}$$

Remark 2 There is also a **minimax principle** for λ_{n-j}:

$$\textbf{Minimum of maximum} \qquad \lambda_{n-j} = \min_{\text{all } S_j} \left[\max_{x \perp S_j} R(x) \right]. \tag{15}$$

If $j = 1$, we are maximizing $R(x)$ over one constraint $x^{\mathrm{T}}v = 0$. That maximum is between the unconstrained λ_{n-1} and λ_n. The toughest constraint makes x perpendicular to the top eigenvector $v = x_n$. Then the best x is the next eigenvector x_{n-1}. The "minimum of the maximum" is λ_{n-1}.

Remark 3 For the generalized problem $Ax = \lambda Mx$, the same principles hold if M is positive definite. In the Rayleigh quotient, $x^{\mathrm{T}}x$ becomes $x^{\mathrm{T}}Mx$:

$$\textbf{Rayleigh quotient} \qquad \text{Minimizing} \quad R(x) = \frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}Mx} \quad \text{gives} \quad \lambda_1(M^{-1}A). \tag{16}$$

Even for *unequal* masses in an oscillating system ($M \ne I$), holding one mass at equilibrium will raise the lowest frequency and lower the highest frequency.

Problem Set 6.4

1. Consider the system $Ax = b$ given by

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}.$$

Construct the corresponding quadratic $P(x_1, x_2, x_3)$, compute its partial derivatives $\partial P/\partial x_i$, and verify that they vanish exactly at the desired solution.

2. Complete the square in $P = \frac{1}{2}x^{\mathrm{T}}Ax - x^{\mathrm{T}}b = \frac{1}{2}(x - A^{-1}b)^{\mathrm{T}}A(x - A^{-1}b) + \text{constant}$. This constant equals $P_{\min}$ because the term before it is never negative. (Why?)

3. Find the minimum, if there is one of $P_1 = \frac{1}{2}x^2 + xy + y^2 - 3y$ and $P_2 = \frac{1}{2}x^2 - 3y$. What matrix A is associated with P_2?

4. (Review) Another quadratic that certainly has its minimum at $Ax = b$ is

$$Q(x) = \frac{1}{2}\|Ax - b\|^2 = \frac{1}{2}x^{\mathrm{T}}A^{\mathrm{T}}Ax - x^{\mathrm{T}}A^{\mathrm{T}}b + \frac{1}{2}b^{\mathrm{T}}b.$$

Comparing Q with P, and ignoring the constant $\frac{1}{2}b^{\mathrm{T}}b$, what system of equations do we get at the minimum of Q? What are these equations called in the theory of least squares?

5. For any symmetric matrix A, compute the ratio $R(x)$ for the special choice $x = (1, \ldots, 1)$. How is the sum of all entries a_{ij} related to λ_1 and λ_n?

6. With $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, find a choice of x that gives a smaller $R(x)$ than the bound $\lambda_1 \leq 2$ that comes from the diagonal entries. What is the minimum value of $R(x)$?

7. If B is positive definite, show from the Rayleigh quotient that the smallest eigenvalue of $A + B$ is larger than the smallest eigenvalue of A.

8. If λ_1 and μ_1 are the smallest eigenvalues of A and B, show that the smallest eigenvalue θ_1 of $A + B$ is at least as large as $\lambda_1 + \mu_1$. (Try the corresponding eigenvector x in the Rayleigh quotients.)

Note Problems 7 and 8 are perhaps the most typical and most important results that come easily from Rayleigh's principle, but only with great difficulty from the eigenvalue equations themselves.

9. If B is positive definite, show from the minimax principle (12) that the *second* smallest eigenvalue is increased by adding $B : \lambda_2(A + B) > \lambda_2(A)$.

10. If you throw away *two* rows and columns of A, what inequalities do you expect between the smallest eigenvalue μ of the new matrix and the original λ's?

11. Find the minimum values of

$$R(x) = \frac{x_1^2 - x_1x_2 + x_2^2}{x_1^2 + x_2^2} \quad \text{and} \quad R(x) = \frac{x_1^2 - x_1x_2 + x_2^2}{2x_1^2 + x_2^2}.$$

12. Prove from equation (11) that $R(x)$ is never larger than the largest eigenvalue λ_n.

13. The minimax principle for λ_j involves j-dimensional subspaces S_j:

$$\textbf{Equivalent to equation (15)} \qquad \lambda_j = \min_{S_j}\left[\max_{x \text{ in } S_j} R(x)\right].$$

(a) If λ_j is positive, infer that every S_j contains a vector x with $R(x) > 0$.
(b) Deduce that S_j contains a vector $y = C^{-1}x$ with $y^{\mathrm{T}}C^{\mathrm{T}}ACy/y^{\mathrm{T}}y > 0$.
(c) Conclude that the jth eigenvalue of $C^{\mathrm{T}}AC$, from *its* minimax principle, is also positive—proving again the *law of inertia* in Section 6.2.

14. Show that the smallest eigenvalue λ_1 of $Ax = \lambda Mx$ is not larger than the ratio a_{11}/m_{11} of the corner entries.

15. Which particular subspace S_2 in Problem 13 gives the minimum value λ_2? In other words, over which S_2 is the maximum of $R(x)$ equal to λ_2?

16. (Recommended) From the zero submatrix decide the signs of the n eigenvalues:

$$A = \begin{bmatrix} 0 & \cdot & 0 & 1 \\ \cdot & \cdot & 0 & 2 \\ 0 & 0 & 0 & \cdot \\ 1 & 2 & \cdot & n \end{bmatrix}.$$

17. (Constrained minimum) Suppose the unconstrained minimum $x = A^{-1}b$ happens to satisfy the constraint $Cx = d$. Verify that equation (5) correctly gives $P_{C/\min} = P_{\min}$; the correction term is zero.

6.5 THE FINITE ELEMENT METHOD

There were two main ideas in the preceding section on minimum principles:

(i) Solving $Ax = b$ is equivalent to minimizing $P(x) = \frac{1}{2}x^{\mathrm{T}}Ax - x^{\mathrm{T}}b$.
(ii) Solving $Ax = \lambda_1 x$ is equivalent to minimizing $R(x) = x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$.

Now we try to explain how these ideas can be applied.

The story is a long one, because these principles have been known for more than a century. In engineering problems like plate bending, or physics problems like the ground state (eigenfunction) of an atom, minimization was used to get a rough approximation to the true solution. The approximations *had* to be rough; the computers were human. The principles (i) and (ii) were there, but they could not be implemented.

Obviously the computer was going to bring about a revolution. It was the method of finite differences that jumped ahead, because it is easy to "discretize" a differential equation. Already in Section 1.7, derivatives were replaced by differences. The physical region is covered by a mesh, and $u'' = f(x)$ became $u_{j+1} - 2u_j + u_{j-1} = h^2 f_j$. The 1950s brought new ways to solve systems $Au = f$ that are very large and very sparse—algorithms and hardware are both much faster now.

What we did not fully recognize was that even finite differences become incredibly complicated for real engineering problems, like the stresses on an airplane. *The real difficulty is not to solve the equations, but to set them up.* For an irregular region we piece the mesh together from triangles or quadrilaterals or tetrahedra. Then we need a systematic way to approximate the underlying physical laws. The computer has to help not only in the solution of $Au = f$ and $Ax = \lambda x$, but in its formulation.

You can guess what happened. The old methods came back, with a new idea and a new name. The new name is the ***finite element method***. The new idea uses more of the power of the computer—in constructing a discrete approximation, solving it, and displaying the results—than any other technique in scientific computation.* If the basic idea is simple, the applications can be complicated. For problems on this scale, the one

* Please forgive this enthusiasm; I know the method may not be immortal.

undebatable point is their cost—I am afraid a billion dollars would be a conservative estimate of the expense so far. I hope some readers will be vigorous enough to master the finite element method and put it to good use.

Trial Functions

Starting from the classical ***Rayleigh–Ritz principle***, I will introduce the new idea of finite elements. The equation can be $-u'' = f(x)$ with boundary conditions $u(0) = u(1) = 0$. This problem is *infinite-dimensional* (the vector b is replaced by a function f, and the matrix A becomes $-d^2/dx^2$). We can write down the energy whose minimum is required, replacing inner products $v^{\mathrm{T}} f$ by integrals of $v(x)f(x)$:

$$\textbf{Total energy} \quad P(v) = \frac{1}{2} v^{\mathrm{T}} A v - v^{\mathrm{T}} f = \frac{1}{2}\int_0^1 v(x)(-v''(x))\,dx - \int_0^1 v(x) f(x)\,dx. \quad (1)$$

$P(v)$ is to be minimized over all functions $v(x)$ that satisfy $v(0) = v(1) = 0$. ***The function that gives the minimum will be the solution $u(x)$***. The differential equation has been converted to a minimum principle, and it only remains to integrate by parts:

$$\int_0^1 v(-v'')\,dx = \int_0^1 (v')^2\,dx - [vv']_{x=0}^{x=1} \quad \text{so} \quad P(v) = \int_0^1 \left[\frac{1}{2}(v'(x))^2 + v(x)f(x)\right] dx.$$

The term vv' is zero at both limits, because v is. Now $\int (v'(x))^2\,dx$ is *positive* like $x^{\mathrm{T}} A x$. We are guaranteed a minimum.

To compute the minimum exactly is equivalent to solving the differential equation exactly. *The Rayleigh–Ritz principle produces an n-dimensional problem by choosing only n trial functions* $V_1(x), \ldots, V_n(x)$. From all combinations $V = y_1 V_1(x) + \cdots + y_n V_n(x)$, we look for the particular combination (call it U) that minimizes $P(V)$. This is the key idea, to minimize over a subspace of V's instead of over all possible $v(x)$. The function that gives the minimum is $U(x)$. We hope and expect that $U(x)$ is near the correct $u(x)$.

Substituting V for v, the quadratic turns into

$$P(V) = \frac{1}{2}\int_0^1 (y_1 V_1'(x) + \cdots + y_n V_n'(x))^2\,dx - \int_0^1 (y_1 V_1(x) + \cdots + y_n V_n(x)) f(x)\,dx. \quad (2)$$

The trial functions V are chosen in advance. That is the key step! The unknowns $y_1, \ldots, y_n$ go into a vector y. Then $P(V) = \frac{1}{2} y^{\mathrm{T}} A y - y^{\mathrm{T}} b$ is recognized as one of the quadratics we are accustomed to. The matrix entries A_{ij} are $\int V_i' V_j'\,dx$ = coefficient of $y_i y_j$. The components b_j are $\int V_j f\,dx$. We can certainly find the minimum of $\frac{1}{2} y^{\mathrm{T}} A y - y^{\mathrm{T}} b$ by solving $Ay = b$. Therefore the Rayleigh-Ritz method has three steps:

1. *Choose the trial functions* $V_1, \ldots, V_n$.
2. *Compute the coefficients* A_{ij} *and* b_j.
3. *Solve* $Ay = b$ *to find* $U(x) = y_1 V_1(x) + \cdots + y_n V_n(x)$.

Everything depends on step 1. Unless the functions $V_j(x)$ are extremely simple, the other steps will be virtually impossible. And unless some combination of the V_j is close to the true solution $u(x)$, those steps will be useless. To combine both computability and accuracy, ***the key idea that makes finite elements successful is the use of piecewise polynomials as the trial functions $V(x)$***.

Linear Finite Elements

The simplest and most widely used finite element is **piecewise linear**. Place nodes at the interior points $x_1 = h, x_2 = 2h, \ldots, x_n = nh$, just as for finite differences. Then V_j is the "hat function" that equals 1 at the node x_j, and zero at all the other nodes (Figure 6.7a). It is concentrated in a small interval around its node, and it is zero everywhere else (including $x = 0$ and $x = 1$). Any combination $y_1 V_1 + \cdots + y_n V_n$ must have the value y_j at node j (the other V's are zero there), so its graph is easy to draw (Figure 6.7b).

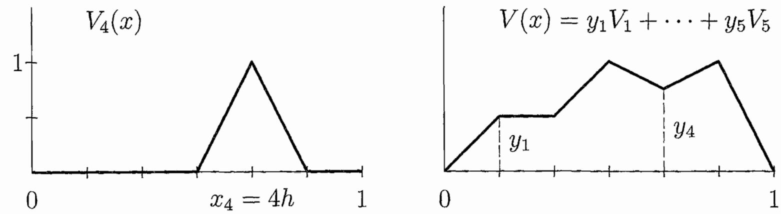

Figure 6.7 Hat functions and their linear combinations.

Step 2 computes the coefficients $A_{ij} = \int V_i' V_j'\, dx$ in the "stiffness matrix" A. The slope V_j' equals $1/h$ in the small interval to the left of x_j, and $-1/h$ in the interval to the right. *If these "double intervals" do not overlap, the product $V_i' V_j'$ is zero and* $A_{ij} = 0$. Each hat function overlaps itself and only two neighbors:

$$\textbf{Diagonal} \quad i = j \quad A_{ii} = \int V_i' V_i'\, dx = \int \left(\frac{1}{h}\right)^2 dx + \int \left(-\frac{1}{h}\right)^2 dx = \frac{2}{h}.$$

$$\textbf{Off-diagonal} \quad i = j \pm 1 \quad A_{ij} = \int V_i' V_j'\, dx = \int \left(\frac{1}{h}\right)\left(\frac{-1}{h}\right) dx = \frac{-1}{h}.$$

Then the stiffness matrix is actually tridiagonal:

$$\textbf{Stiffness matrix} \qquad A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

This looks just like finite differences! It has led to a thousand discussions about the relation between these two methods. More complicated finite elements—polynomials of higher degree, defined on triangles or quadrilaterals for partial differential equations—also produce sparse matrices A. You could think of finite elements as a systematic way to construct accurate difference equations on irregular meshes. The essential thing is the *simplicity* of these piecewise polynomials. Inside every element, their slopes are easy to find and to integrate.

The components b_j on the right side are new. Instead of just the value of f at x_j, as for finite differences, they are now an average of f around that point: $b_j = \int V_j f\, dx$. Then, in step 3, we solve the tridiagonal system $Ay = b$, which gives the coefficients in the minimizing trial function $U = y_1 V_1 + \cdots + y_n V_n$. Connecting all these heights y_j by a broken line, we have the approximate solution $U(x)$.

Example 1 $-u'' = 2$ with $u(0) = u(1) = 0$, and solution $u(x) = x - x^2$.

The approximation will use three intervals and two hat functions, with $h = \frac{1}{3}$. The matrix A is 2 by 2. The right side requires integration of the hat function times $f(x) = 2$. That produces twice the area $\frac{1}{3}$ under the hat:

$$A = 3\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

The solution to $Ay = b$ is $y = (\frac{2}{9}, \frac{2}{9})$. The best $U(x)$ is $\frac{2}{9}V_1 + \frac{2}{9}V_2$, which equals $\frac{2}{9}$ at the mesh points. *This agrees with the exact solution* $u(x) = x - x^2 = \frac{1}{3} - \frac{1}{9}$.

In a more complicated example, the approximation will not be exact at the nodes. But it is remarkably close. The underlying theory is explained in the author's book *An Analysis of the Finite Element Method* (see www.wellesleycambridge.com) written jointly with George Fix. Other books give more detailed applications, and the subject of finite elements has become an important part of engineering education. It is treated in *Introduction to Applied Mathematics*, and also in my new book *Applied Mathematics and Scientific Computing*. There we discuss partial differential equations, where the method really comes into its own.

Eigenvalue Problems

The Rayleigh–Ritz idea—to minimize over a finite-dimensional family of V's in place of all admissible v's—is also useful for eigenvalue problems. The true minimum of the Rayleigh quotient is the fundamental frequency λ_1. Its approximate minimum Λ_1 will be larger—because the class of trial functions is restricted to the V's. This step was completely natural and inevitable: to apply the new finite element ideas to this long-established variational form of the eigenvalue problem.

The best example of an eigenvalue problem has $u(x) = \sin \pi x$ and $\lambda_1 = \pi^2$:

$$\textbf{Eigenfunction } \boldsymbol{u(x)} \qquad -u'' = \lambda u, \quad \text{with} \quad u(0) = u(1) = 0.$$

That function $\sin \pi x$ minimizes the Rayleigh quotient $v^{\mathrm{T}}Av/v^{\mathrm{T}}v$:

$$\textbf{Rayleigh quotient} \qquad R(v) = \frac{\int_0^1 v(x)(-v''(x))\,dx}{\int_0^1 (v(x))^2\,dx} = \frac{\int_0^1 (v'(x))^2\,dx}{\int_0^1 (v(x))^2\,dx}.$$

This is a ratio of potential to kinetic energy, and they are in balance at the eigenvector. Normally this eigenvector would be unknown, and to approximate it we admit only the trial candidates $V = y_1V_1 + \cdots + y_nV_n$:

$$R(V) = \frac{\int_0^1 (y_1V_1' + \cdots + y_nV_n')^2\,dx}{\int_0^1 (y_1V_1 + \cdots + y_nV_n)^2\,dx} = \frac{y^{\mathrm{T}}Ay}{y^{\mathrm{T}}My}.$$

Now we face a matrix problem: Minimize $y^{\mathrm{T}}Ay/y^{\mathrm{T}}My$. With $M = I$, this leads to the standard eigenvalue problem $Ay = \lambda y$. But our matrix M will be tridiagonal, because neighboring hat functions overlap. It is exactly this situation that brings in the *generalized eigenvalue problem*. **The minimum value $\boldsymbol{\Lambda_1}$ will be the smallest eigenvalue of $\boldsymbol{Ay = \lambda My}$.** That Λ_1 will be close to (and above) π^2. The eigenvector y will give the approximation $U = y_1V_1 + \cdots + y_nV_n$ to the eigenfunction.

As in the static problem, the method can be summarized in three steps: (1) choose the V_j, (2) compute A and M, and (3) solve $Ay = \lambda My$. I don't know why that costs a billion dollars.

Problem Set 6.5

1. Use three hat functions, with $h = \frac{1}{4}$, to solve $-u'' = 2$ with $u(0) = u(1) = 0$. Verify that the approximation U matches $u = x - x^2$ at the nodes.

2. Solve $-u'' = x$ with $u(0) = u(1) = 0$. Then solve approximately with two hat functions and $h = \frac{1}{3}$. Where is the largest error?

3. Suppose $-u'' = 2$, with the boundary condition $u(1) = 0$ changed to $u'(1) = 0$. This "natural" condition on u' need not be imposed on the trial functions V. With $h = \frac{1}{3}$, there is an extra *half-hat* V_3, which goes from 0 to 1 between $x = \frac{2}{3}$ and $x = 1$. Compute $A_{33} = \int (V_3')^2\,dx$ and $f_3 = \int 2V_3\,dx$. Solve $Ay = f$ for the finite element solution $y_1V_1 + y_2V_2 + y_3V_3$.

4. Solve $-u'' = 2$ with a single hat function, but place its node at $x = \frac{1}{4}$ instead of $x = \frac{1}{2}$. (Sketch this function V_1.) With boundary conditions $u(0) = u(1) = 0$, compare the finite element approximation with the true $u = x - x^2$.

5. *Galerkin's method* starts with the differential equation (say $-u'' = f(x)$) instead of the energy P. The trial solution is still $u = y_1V_1 + y_2V_2 + \cdots + y_nV_n$, and the y's are chosen to make the difference between $-u''$ and f orthogonal to every V_j:

$$\textbf{Galerkin} \qquad \int (-y_1V_1'' - y_2V_2'' - \cdots - y_nV_n'')V_j\,dx = \int f(x)V_j(x)\,dx.$$

Integrate the left side by parts to reach $Ay = f$, proving that *Galerkin gives the same A and f as Rayleigh–Ritz for symmetric problems.*

6. A basic identity for quadratics shows $y = A^{-1}b$ as minimizing:

$$P(y) = \frac{1}{2}y^{\mathrm{T}}Ay - y^{\mathrm{T}}b = \frac{1}{2}(y - A^{-1}b)^{\mathrm{T}}A(y - A^{-1}b) - \frac{1}{2}b^{\mathrm{T}}A^{-1}b.$$

The minimum over a *subspace* of trial functions is at the y *nearest to* $A^{-1}b$. (That makes the first term on the right as small as possible; it is the key to convergence of U to u.) If $A = I$ and $b = (1, 0, 0)$, which multiple of $V = (1, 1, 1)$ gives the smallest value of $P(y) = \frac{1}{2}y^{\mathrm{T}}y - y_1$?

7. For a single hat function $V(x)$ centered at $x = \frac{1}{2}$, compute $A = \int (V')^2\,dx$ and $M = \int V^2\,dx$. In the 1 by 1 eigenvalue problem, is $\lambda = A/M$ larger or smaller than the true eigenvalue $\lambda = \pi^2$?

8. For the hat functions V_1 and V_2 centered at $x = h = \frac{1}{3}$ and $x = 2h = \frac{2}{3}$, compute the 2 by 2 mass matrix $M_{ij} = \int V_iV_j\,dx$, and solve the eigenvalue problem $Ax = \lambda Mx$.

9. What is the mass matrix $M_{ij} = \int V_iV_j\,dx$ for n hat functions with $h = \frac{1}{n+1}$?