Gradual Typing from a Categorical Perspective

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Abstract

TODO

1 Introduction

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2 Categorical Model

Definition 1. *Suppose $\mathcal{C}$ is a category. Then an object A is a* ***retract*** *of an object B if there are morphisms $i : A \longrightarrow B$ and $r : B \longrightarrow A$ such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, equal] & B \arrow[d, "r"] \\ & A \end{tikzcd}$$

Definition 2. *An* ***untyped λ-model****, $(\mathcal{C}, ?, \mathsf{split}, \mathsf{squash})$, is a cartesian closed category $\mathcal{C}$ with a distinguished object $?$ and two morphisms $\mathsf{squash} : (? \to ?) \longrightarrow ?$ and $\mathsf{split} : ? \longrightarrow (? \to ?)$ making the object $? \to ?$ a retract of $?$.*

Theorem 3 (Scott [1980])**.** *An untyped λ-model is a sound and complete model of the untyped λ-calculus.*

3 Grady

References

Dana Scott. Relating theories of the lambda-calculus. In *To H.B. Curry: Essays on Combinatory Logic, Lambda-Calculus and Formalism (eds. Hindley and Seldin)*, pages 403–450. Academic Press, 1980.

A The Complete Spec of Grady

$termvar$, x, z
$index$, k

t	::=		term
	$\|$	x	variable
	$\|$	triv	unit
	$\|$	squash	injection of the retract
	$\|$	split	surjection of the retract
	$\|$	$\mathsf{box}\, t$	generalize to the untyped universe
	$\|$	unbox_T	specialize the untyped universe to a specific type
	$\|$	$\lambda x : A.t$	λ-abstraction
	$\|$	$t_1\, t_2$	function application
	$\|$	(t_1, t_2)	pair constructor
	$\|$	$\mathsf{fst}\, t$	first projection
	$\|$	$\mathsf{snd}\, t$	second projection
	$\|$	$\mathsf{succ}\, t$	successor function
	$\|$	0	zero
	$\|$	(t) S	

h	::=		head-normal forms
	$\|$	triv	
	$\|$	split	
	$\|$	squash	
	$\|$	$\mathsf{box}\, t$	
	$\|$	unbox_T	
	$\|$	$\lambda x : A.t$	
	$\|$	(t_1, t_2)	
	$\|$	$\mathsf{fst}\, t$	
	$\|$	$\mathsf{snd}\, t$	
	$\|$	$\mathsf{succ}\, t$	
	$\|$	0	

T	::=		terminating types
	$\|$	1	unit type
	$\|$	$\mathbb{N}$	natural number type
	$\|$	$T_1 \to T_2$	function type
	$\|$	$T_1 \times T_2$	cartesian product type
	$\|$	(T) S	

A	::=		type
	$\|$	1	unit type
	$\|$	$\mathbb{N}$	natural number type
	$\|$	$?$	untyped universe

	$\vert$	$A_1 \to A_2$	function type
	$\vert$	$T_1 \times T_2$	cartesian product type
	$\vert$	(A) S	
Γ	$::=$		typing context
	$\vert$	$\cdot$	empty context
	$\vert$	$\Gamma, x : A$	cons
vd	$::=$		
	$\vert$	$\vdash$	
	$\vert$	$\nvdash$	

$\boxed{\Gamma vd t : A}$ t has type A in context Γ

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \quad \text{VAR}$$

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{box}\, t : ?} \quad \text{Box}$$

$$\frac{}{\Gamma \vdash \mathsf{unbox}_T : ? \to T} \quad \text{Unbox}$$

$$\frac{}{\Gamma \vdash \mathsf{squash} : (? \to ?) \to ?} \quad \text{INJ}$$

$$\frac{}{\Gamma \vdash \mathsf{split} : ? \to (? \to ?)} \quad \text{SURJ}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1} \quad \text{UNIT}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \text{ZERO}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \text{SUCC}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \quad \text{PAIR}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\, t : T_1} \quad \text{FST}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\, t : T_2} \quad \text{SND}$$

$$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1.t : A_1 \to A_2} \quad \text{LAM}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 : A_2} \quad \text{APP}$$

$\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A}$ t_1 reduces to t_2 with type A in context Γ

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{unbox}_T\,(\mathsf{box}\,t) \rightsquigarrow t : T}\ \textsc{rd\_retract}T$$

$$\frac{\Gamma \nvdash t : T}{\Gamma \vdash \mathsf{unbox}_T\,(\mathsf{box}\,t) \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}}\ \textsc{rd\_tWrong}$$

$$\frac{h \neq \mathsf{box}\,t}{\Gamma \vdash \mathsf{unbox}_T\,h \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}}\ \textsc{rd\_hWrong}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : ?}{\Gamma \vdash \mathsf{unbox}_T\,t \rightsquigarrow \mathsf{unbox}_T\,t' : T}\ \textsc{rd\_unbox}$$

$$\frac{\Gamma \vdash t : ? \to ?}{\Gamma \vdash \mathsf{split}\,(\mathsf{squash}\,t) \rightsquigarrow t : ? \to ?}\ \textsc{rd\_retract}U$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : ?}{\Gamma \vdash \mathsf{split}\,t \rightsquigarrow \mathsf{split}\,t' : ? \to ?}\ \textsc{rd\_split}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : ? \to ?}{\Gamma \vdash \mathsf{squash}\,t \rightsquigarrow \mathsf{squash}\,t' : ?}\ \textsc{rd\_spuash}$$

$$\frac{\Gamma \vdash t : A_1 \to A_2 \quad x \notin \mathsf{FV}(t)}{\Gamma \vdash \lambda x : A_1.t\,x \rightsquigarrow t : A_1 \to A_2}\ \textsc{rd\_eta}$$

$$\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1.t_2)\,t_1 \rightsquigarrow [t_1/x]t_2 : A_2}\ \textsc{rd\_beta}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{fst}\,(t_1, t_2) \rightsquigarrow t_1 : T_1}\ \textsc{rd\_proj}1$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{snd}\,(t_1, t_2) \rightsquigarrow t_2 : T_2}\ \textsc{rd\_proj}2$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash (\mathsf{fst}\,t, \mathsf{snd}\,t) \rightsquigarrow t : T_1 \times T_2}\ \textsc{rd\_eta}P$$

$$\frac{\Gamma, x : A_1 \vdash t \rightsquigarrow t' : A_2}{\Gamma \vdash \lambda x : A_1.t \rightsquigarrow \lambda x : A_1.t' : A_1 \to A_2}\ \textsc{rd\_lam}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\,t_2 \rightsquigarrow t_1'\,t_2 : A_2}\ \textsc{rd\_app}1$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : A_1}{\Gamma \vdash t_1\,t_2 \rightsquigarrow t_1\,t_2' : A_2}\ \textsc{rd\_app}2$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\,t \rightsquigarrow \mathsf{fst}\,t' : T_1}\ \textsc{rd\_fst}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\,t \rightsquigarrow \mathsf{snd}\,t' : T_2}\ \textsc{rd\_snd}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1', t_2) : T_1 \times T_2} \quad \text{RD\_PAIR1}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : T_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1, t_2') : T_1 \times T_2} \quad \text{RD\_PAIR2}$$