Gradual Typing from a Categorical Perspective

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References

A The Complete Spec of Grady

termvar, x, z			
index, k			
t	::=		term
	|	x	variable
	|	triv	unit
	|	squash	injection of the retract
	|	split	surjection of the retract
	|	box_T	generalize to the untyped universe
	|	unbox	specialize the untyped universe to a specific type
	|	$\lambda x : A.t$	λ-abstraction
	|	$t_1\ t_2$	function application
	|	(t_1, t_2)	pair constructor
	|	$\mathsf{fst}\ t$	first projection
	|	$\mathsf{snd}\ t$	second projection
	|	$\mathsf{succ}\ t$	successor function
	|	0	zero
	|	(t) S	
T	::=		terminating types
	|	1	unit type
	|	$\mathbb{N}$	natural number type
	|	$T_1 \to T_2$	function type
	|	$T_1 \times T_2$	cartesian product type
	|	(T) S	
A	::=		type
	|	T	terminating type
	|	$?$	untyped universe
	|	$A_1 \to A_2$	function type

$$\begin{array}{lll} & \mid \quad (A) & \mathsf{S} \end{array}$$

$$\begin{array}{llll}
\Gamma & ::= & & \text{typing context} \\
 & \mid & \cdot & \quad \text{empty context} \\
 & \mid & \Gamma, x : A & \quad \text{cons}
\end{array}$$

$\boxed{\Gamma \vdash t : A}$ t has type A in context Γ

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \quad \text{VAR}$$

$$\frac{}{\Gamma \vdash \mathsf{box}_T : T \to ?} \quad \text{Box}$$

$$\frac{}{\Gamma \vdash \mathsf{unbox} : ? \to T} \quad \text{Unbox}$$

$$\frac{}{\Gamma \vdash \mathsf{squash} : (? \to ?) \to ?} \quad \text{INJ}$$

$$\frac{}{\Gamma \vdash \mathsf{split} : ? \to (? \to ?)} \quad \text{SURJ}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1} \quad \text{UNIT}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \text{ZERO}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \text{SUCC}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \quad \text{PAIR}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\, t : T_1} \quad \text{FST}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\, t : T_2} \quad \text{SND}$$

$$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1 . t : A_1 \to A_2} \quad \text{ABS}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash t_1\, t_2 : A_2} \quad \text{APP}$$

$\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A}$ t_1 reduces to t_2 with type A in context Γ

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{unbox}\,(\mathsf{box}_T\ t) \rightsquigarrow t : T} \quad \text{RD\_RETRACT}$$

$$\frac{t \neq \mathsf{box}_T\ t'}{\Gamma \vdash \mathsf{unbox}\, t \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \text{RD\_UNBOXErr}$$

$$\frac{\Gamma \vdash t : ? \to ?}{\Gamma \vdash \mathsf{split}\,(\mathsf{squash}\; t) \leadsto t : ? \to ?} \quad \text{RD\_RETRACTU}$$

$$\frac{\Gamma \vdash t : A_1 \to A_2 \quad x \notin \mathsf{FV}(t)}{\Gamma \vdash \lambda x : A_1 . t\, x \leadsto t : A_1 \to A_2} \quad \text{RD\_ETA}$$

$$\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1 . t_2)\; t_1 \leadsto [t_1/x]t_2 : A_2} \quad \text{RD\_BETA}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{fst}\,(t_1, t_2) \leadsto t_1 : T_1} \quad \text{RD\_PROJ1}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{snd}\,(t_1, t_2) \leadsto t_2 : T_2} \quad \text{RD\_PROJ2}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash (\mathsf{fst}\; t, \mathsf{snd}\; t) \leadsto t : T_1 \times T_2} \quad \text{RD\_ETAP}$$

$$\frac{\Gamma, x : A_1 \vdash t \leadsto t' : A_2}{\Gamma \vdash \lambda x : A_1 . t \leadsto \lambda x : A_1 . t' : A_1 \to A_2} \quad \text{RD\_LAM}$$

$$\frac{\Gamma \vdash t_1 \leadsto t_1' : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\; t_2 \leadsto t_1'\; t_2 : A_2} \quad \text{RD\_APP1}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 \leadsto t_2' : A_1}{\Gamma \vdash t_1\; t_2 \leadsto t_1\; t_2' : A_2} \quad \text{RD\_APP2}$$

$$\frac{\Gamma \vdash t \leadsto t' : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\; t \leadsto \mathsf{fst}\; t' : T_1} \quad \text{RD\_FST}$$

$$\frac{\Gamma \vdash t \leadsto t' : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\; t \leadsto \mathsf{snd}\; t' : T_2} \quad \text{RD\_SND}$$

$$\frac{\Gamma \vdash t_1 \leadsto t_1' : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) \leadsto (t_1', t_2) : T_1 \times T_2} \quad \text{RD\_PAIR1}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 \leadsto t_2' : T_2}{\Gamma \vdash (t_1, t_2) \leadsto (t_1, t_2') : T_1 \times T_2} \quad \text{RD\_PAIR2}$$