Gradual Typing from a Categorical Perspective

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1 Quotient Model

Definition 1. *Suppose $\mathcal{C}$ is a category and $\sim$ is an equivalence relation on objects. Then we extend $\sim$ to arrows as follows:*

- *for any arrows $f : A \longrightarrow C$ and $g : B \longrightarrow D$, $f \sim g$ if and only if $A \sim B$ and $C \sim D$.*

*We call an equivalence relation extended to the arrows of a category a **congruence relation** on the category.*

Definition 2. *Suppose $\mathcal{C}$ is a category and $\sim$ is a congruence relation on objects and arrows of $\mathcal{C}$. Then we define the **quotient category** $\mathcal{C}/\sim$ as follows:*

- *objects are equivalence classes, $[A]$, of objects of $\mathcal{C}$, and*
- *arrows are equivalence classes, $[f] : [A] \longrightarrow [B]$, of arrows, $f : A \longrightarrow B$, of $\mathcal{C}$.*

References

A The Complete Spec of Grady

termvar, x			
index, k			
t	::=		term
	|	x	variable
	|	triv	unit
	|	squash	injection of the retract
	|	split	surjection of the retract
	|	gen	generalize to the untyped universe
	|	spec	specialize the untyped universe to a specific type
	|	$\lambda x : T.t$	λ-abstraction
	|	$t_1\ t_2$	function application
	|	(t_1, t_2)	pair constructor

	$\mid$	$\mathsf{fst}\ t$	first projection
	$\mid$	$\mathsf{snd}\ t$	second projection
	$\mid$	$\mathsf{succ}\ t$	successor function
	$\mid$	0	zero
	$\mid$	(t) S	

T	$::=$		type
	$\mid$	1	unit type
	$\mid$	$\mathbb{N}$	natural number type
	$\mid$	$?$	untyped universe
	$\mid$	$T_1 \to T_2$	function type
	$\mid$	$T_1 \times T_2$	cartesian product type
	$\mid$	(T) S	

Γ	$::=$		typing context
	$\mid$	$\cdot$	empty context
	$\mid$	$\Gamma, x : T$	cons

$\boxed{T_1 \sim T_2}$ T_1 is consistent with T_2

$$\frac{}{T \sim T}\ \textsc{cs\_refl}$$

$$\frac{}{? \sim T}\ \textsc{cs\_ul}$$

$$\frac{}{T \sim ?}\ \textsc{cs\_ur}$$

$$\frac{T_1 \sim T_1' \quad T_2 \sim T_2'}{(T_1 \times T_2) \sim (T_1' \times T_2')}\ \textsc{cs\_pair}$$

$$\frac{T_1 \sim T_1' \quad T_2 \sim T_2'}{(T_1 \to T_2) \sim (T_1' \to T_2')}\ \textsc{cs\_arrow}$$

$\boxed{\Gamma \vdash t : T}$ t has type T in context Γ

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}\ \textsc{var}$$

$$\frac{}{\Gamma \vdash \mathsf{squash} : (? \to ?) \to ?}\ \textsc{inj}$$

$$\frac{}{\Gamma \vdash \mathsf{split} : ? \to (? \to ?)}\ \textsc{surj}$$

$$\frac{}{\Gamma \vdash \mathsf{gen} : T \to ?}\ \textsc{gen}$$

$$\frac{}{\Gamma \vdash \mathsf{spec} : ? \to T}\ \textsc{spec}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1}\ \textsc{unit}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \text{ZERO}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \text{SUCC}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \quad \text{PAIR}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\, t : T_1} \quad \text{FST}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\, t : T_2} \quad \text{SND}$$

$$\frac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x : T_1 . t : T_1 \to T_2} \quad \text{ABS}$$

$$\frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_3}{\Gamma \vdash t_1\, t_2 : T_2} \quad \text{APP}$$

$\boxed{t_1 \rightsquigarrow t_2}$ t_1 reduces to t_2

$$\frac{}{(\lambda x : T . t_2)\, t_1 \rightsquigarrow [t_1/x] t_2} \quad \text{RD\_BETA}$$

$$\frac{}{\mathsf{split}\,(\mathsf{squash}\, t) \rightsquigarrow t} \quad \text{RD\_RETRACT}$$

$$\frac{}{\mathsf{spec}\,(\mathsf{gen}\, t) \rightsquigarrow t} \quad \text{RD\_RETRACTTY}$$

$$\frac{}{(\lambda x : T . t\, x) \rightsquigarrow t} \quad \text{RD\_ETA}$$

$$\frac{}{\mathsf{fst}\,(t_1, t_2) \rightsquigarrow t_1} \quad \text{RD\_PROJ1}$$

$$\frac{}{\mathsf{snd}\,(t_1, t_2) \rightsquigarrow t_2} \quad \text{RD\_PROJ2}$$

$$\frac{}{(\mathsf{fst}\, t, \mathsf{snd}\, t) \rightsquigarrow t} \quad \text{RD\_ETAP}$$