Gradual Typing from a Categorical Perspective

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References

A The Complete Spec of Grady

$termvar$, x, z
$index$, k

t	::=		term
	|	x	variable
	|	triv	unit
	|	squash	injection of the retract
	|	split	surjection of the retract
	|	$\mathsf{box}\, t$	generalize to the untyped universe
	|	unbox_T	specialize the untyped universe to a specific type
	|	$\lambda x : A.t$	λ-abstraction
	|	$t_1\ t_2$	function application
	|	(t_1, t_2)	pair constructor
	|	$\mathsf{fst}\, t$	first projection
	|	$\mathsf{snd}\, t$	second projection
	|	$\mathsf{succ}\, t$	successor function
	|	0	zero
	|	(t) S	

h	::=		head-normal forms
	|	triv	
	|	split	
	|	squash	
	|	$\mathsf{box}\, t$	
	|	unbox_T	
	|	$\lambda x : A.t$	
	|	(t_1, t_2)	
	|	$\mathsf{fst}\, t$	
	|	$\mathsf{snd}\, t$	
	|	$\mathsf{succ}\, t$	

	$\vert$	0	
T	$::=$		terminating types
	$\vert$	1	unit type
	$\vert$	$\mathbb{N}$	natural number type
	$\vert$	$T_1 \to T_2$	function type
	$\vert$	$T_1 \times T_2$	cartesian product type
	$\vert$	(T) S	
A	$::=$		type
	$\vert$	1	unit type
	$\vert$	$\mathbb{N}$	natural number type
	$\vert$	$?$	untyped universe
	$\vert$	$A_1 \to A_2$	function type
	$\vert$	$T_1 \times T_2$	cartesian product type
	$\vert$	(A) S	
Γ	$::=$		typing context
	$\vert$	$\cdot$	empty context
	$\vert$	$\Gamma, x : A$	cons
vd	$::=$		
	$\vert$	$\vdash$	
	$\vert$	$\nvdash$	

$\boxed{\Gamma vd t : A}$ t has type A in context Γ

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \quad \text{VAR}$$

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{box}\, t : ?} \quad \textsc{Box}$$

$$\frac{}{\Gamma \vdash \mathsf{unbox}_T : ? \to T} \quad \textsc{Unbox}$$

$$\frac{}{\Gamma \vdash \mathsf{squash} : (? \to ?) \to ?} \quad \text{INJ}$$

$$\frac{}{\Gamma \vdash \mathsf{split} : ? \to (? \to ?)} \quad \text{SURJ}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1} \quad \text{UNIT}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \text{ZERO}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \text{SUCC}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \quad \text{PAIR}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\, t : T_1} \quad \text{FST}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\, t : T_2} \quad \text{SND}$$

$$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1.t : A_1 \to A_2} \quad \text{LAM}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 : A_2} \quad \text{APP}$$

$\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A}$ t_1 reduces to t_2 with type A in context Γ

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{unbox}_T\, (\mathsf{box}\, t) \rightsquigarrow t : T} \quad \text{RD\_RETRACT}$$

$$\frac{\Gamma \not\vdash t : T}{\Gamma \vdash \mathsf{unbox}_T\, (\mathsf{box}\, t) \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \text{RD\_TWRONG}$$

$$\frac{h \neq \mathsf{box}\, t}{\Gamma \vdash \mathsf{unbox}_T\, h \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \text{RD\_HWRONG}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : ?}{\Gamma \vdash \mathsf{unbox}_T\, t \rightsquigarrow \mathsf{unbox}_T\, t' : T} \quad \text{RD\_UNBOX}$$

$$\frac{\Gamma \vdash t : ? \to ?}{\Gamma \vdash \mathsf{split}\, (\mathsf{squash}\, t) \rightsquigarrow t : ? \to ?} \quad \text{RD\_RETRACTU}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : ?}{\Gamma \vdash \mathsf{split}\, t \rightsquigarrow \mathsf{split}\, t' : ? \to ?} \quad \text{RD\_SPLIT}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : ? \to ?}{\Gamma \vdash \mathsf{squash}\, t \rightsquigarrow \mathsf{squash}\, t' : ?} \quad \text{RD\_SPUASH}$$

$$\frac{\Gamma \vdash t : A_1 \to A_2 \quad x \notin \mathsf{FV}(t)}{\Gamma \vdash \lambda x : A_1.t\, x \rightsquigarrow t : A_1 \to A_2} \quad \text{RD\_ETA}$$

$$\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1.t_2)\, t_1 \rightsquigarrow [t_1/x]t_2 : A_2} \quad \text{RD\_BETA}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{fst}\, (t_1, t_2) \rightsquigarrow t_1 : T_1} \quad \text{RD\_PROJ1}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{snd}\, (t_1, t_2) \rightsquigarrow t_2 : T_2} \quad \text{RD\_PROJ2}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash (\mathsf{fst}\, t, \mathsf{snd}\, t) \rightsquigarrow t : T_1 \times T_2} \quad \text{RD\_ETAP}$$

$$\frac{\Gamma, x : A_1 \vdash t \rightsquigarrow t' : A_2}{\Gamma \vdash \lambda x : A_1.t \rightsquigarrow \lambda x : A_1.t' : A_1 \to A_2}\ \textsc{rd\_lam}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\ t_2 \rightsquigarrow t_1'\ t_2 : A_2}\ \textsc{rd\_app}1$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : A_1}{\Gamma \vdash t_1\ t_2 \rightsquigarrow t_1\ t_2' : A_2}\ \textsc{rd\_app}2$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\ t \rightsquigarrow \mathsf{fst}\ t' : T_1}\ \textsc{rd\_fst}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\ t \rightsquigarrow \mathsf{snd}\ t' : T_2}\ \textsc{rd\_snd}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1', t_2) : T_1 \times T_2}\ \textsc{rd\_pair}1$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : T_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1, t_2') : T_1 \times T_2}\ \textsc{rd\_pair}2$$