Gradual Typing from a Categorical Perspective

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Abstract

TODO

1 Introduction

TODO

2 Categorical Model

Definition 1. *Suppose $\mathcal{C}$ is a category. Then an object A is a* ***retract*** *of an object B if there are morphisms $i : A \longrightarrow B$ and $r : B \longrightarrow A$ such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, equal] & B \arrow[d, "r"] \\ & A \end{tikzcd}$$

Definition 2. *An* ***untyped λ-model****, $(\mathcal{C}, ?, \mathsf{split}, \mathsf{squash})$, is a cartesian closed category $\mathcal{C}$ with a distinguished object $?$ and two morphisms $\mathsf{squash} : (? \to ?) \longrightarrow ?$ and $\mathsf{split} : ? \longrightarrow (? \to ?)$ making the object $? \to ?$ a retract of $?$.*

Theorem 3 (Scott [1980])**.** *An untyped λ-model is a sound and complete model of the untyped λ-calculus.*

Definition 4. *An object $\mathbb{N}$ of a category $\mathcal{C}$ with a terminal object 1 is a* ***natural number object (NNO)*** *if and only if there are morphisms $z : 1 \longrightarrow \mathbb{N}$ and $s : \mathbb{N} \longrightarrow \mathbb{N}$ such that for any other object A and morphisms $z' : 1 \longrightarrow A$ and $s' : A \longrightarrow A$ there is a unique morphism $u : \mathbb{N} \longrightarrow A$ making the following diagram commute:*

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "z'"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, "u"] & \mathbb{N} \arrow[d, "u"] \\ & A \arrow[r, "s'"] & A \end{tikzcd}$$

Definition 5. *A **gradual** λ**-model**, $(\mathcal{T}, \mathcal{C}, ?, \mathsf{T}, \mathsf{split}, \mathsf{squash}, \mathsf{box}, \mathsf{unbox})$, where $\mathcal{T}$ and $\mathcal{C}$ are cartesian closed categories with NNOS, $(\mathcal{C}, ?, \mathsf{split}, \mathsf{squash})$ is an untyped λ-model, $\mathsf{T} : \mathcal{T} \longrightarrow \mathcal{C}$ is a cartesian closed embedding – a full and faithful cartesian closed functor that is injective on objects and preserves the NNO – and for every object, A, of $\mathcal{T}$ there are morphisms $\mathsf{box}_A : TA \longrightarrow ?$ and $\mathsf{unbox}_A : ? \longrightarrow TA$ making TA a retract of $?$.*

3 Grady

References

Dana Scott. Relating theories of the lambda-calculus. In *To H.B. Curry: Essays on Combinatory Logic, Lambda-Calculus and Formalism (eds. Hindley and Seldin)*, pages 403–450. Academic Press, 1980.

A The Complete Spec of Grady

termvar, x, z
index, k

t	::=		term
	|	x	variable
	|	triv	unit
	|	squash	injection of the retract
	|	split	surjection of the retract
	|	box_T	generalize to the untyped universe
	|	unbox_T	specialize the untyped universe to a specific type
	|	$\lambda x : A.t$	λ-abstraction
	|	$t_1\, t_2$	function application
	|	(t_1, t_2)	pair constructor
	|	$\mathsf{fst}\, t$	first projection
	|	$\mathsf{snd}\, t$	second projection
	|	$\mathsf{succ}\, t$	successor function
	|	0	zero
	|	(t) S	

h	::=		head-normal forms
	|	triv	
	|	split	
	|	squash	
	|	box_T	
	|	unbox_T	
	|	$\lambda x : A.t$	

$\vert$	(t_1, t_2)
$\vert$	$\mathsf{fst}\, t$
$\vert$	$\mathsf{snd}\, t$
$\vert$	$\mathsf{succ}\, t$
$\vert$	0

T	$::=$			terminating types
	$\vert$	1		unit type
	$\vert$	$\mathbb{N}$		natural number type
	$\vert$	$T_1 \to T_2$		function type
	$\vert$	$T_1 \times T_2$		cartesian product type
	$\vert$	(T)	S	

A	$::=$			type
	$\vert$	1		unit type
	$\vert$	$\mathbb{N}$		natural number type
	$\vert$	$?$		untyped universe
	$\vert$	$A_1 \to A_2$		function type
	$\vert$	$A_1 \times A_2$		cartesian product type
	$\vert$	(A)	S	

Γ	$::=$		typing context
	$\vert$	$\cdot$	empty context
	$\vert$	$\Gamma, x : A$	cons

vd	$::=$	
	$\vert$	$\vdash$
	$\vert$	$\nvdash$

$\boxed{\Gamma vd t : A}$ t has type A in context Γ

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \quad \text{VAR}$$

$$\frac{}{\Gamma \vdash \mathsf{box}_T : T \to ?} \quad \textsc{Box}$$

$$\frac{}{\Gamma \vdash \mathsf{unbox}_T : ? \to T} \quad \textsc{Unbox}$$

$$\frac{}{\Gamma \vdash \mathsf{squash} : (? \to ?) \to ?} \quad \text{SQUASH}$$

$$\frac{}{\Gamma \vdash \mathsf{split} : ? \to (? \to ?)} \quad \text{SPLIT}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1} \quad \text{UNIT}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \text{ZERO}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \text{SUCC}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2} \quad \text{PAIR}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \mathsf{fst}\, t : A_1} \quad \text{FST}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \mathsf{snd}\, t : A_2} \quad \text{SND}$$

$$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1.t : A_1 \to A_2} \quad \text{LAM}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 : A_2} \quad \text{APP}$$

$\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A}$ t_1 reduces to t_2 with type A in context Γ

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{unbox}_T\,(\mathsf{box}_T\, t) \rightsquigarrow t : T} \quad \text{RD\_RETRACT}$$

$$\frac{\Gamma \nvdash t : T}{\Gamma \vdash \mathsf{unbox}_T\,(\mathsf{box}_{T'}\, t) \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \text{RD\_TWRONG}$$

$$\frac{h \neq \mathsf{box}_T\, t}{\Gamma \vdash \mathsf{unbox}_T\, h \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \text{RD\_HWRONG}$$

$$\frac{\Gamma \vdash t : ? \to ?}{\Gamma \vdash \mathsf{split}\,(\mathsf{squash}\, t) \rightsquigarrow t : ? \to ?} \quad \text{RD\_RETRACTU}$$

$$\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1.t_2)\, t_1 \rightsquigarrow [t_1/x]t_2 : A_2} \quad \text{RD\_BETA}$$

$$\frac{\Gamma \vdash t : A_1 \to A_2 \quad x \notin \mathsf{FV}(t)}{\Gamma \vdash \lambda x : A_1.t\, x \rightsquigarrow t : A_1 \to A_2} \quad \text{RD\_ETA}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \mathsf{fst}\,(t_1, t_2) \rightsquigarrow t_1 : A_1} \quad \text{RD\_PROJ1}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \mathsf{snd}\,(t_1, t_2) \rightsquigarrow t_2 : A_2} \quad \text{RD\_PROJ2}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash (\mathsf{fst}\, t, \mathsf{snd}\, t) \rightsquigarrow t : A_1 \times A_2} \quad \text{RD\_ETAP}$$

$$\frac{\Gamma, x : A_1 \vdash t \rightsquigarrow t' : A_2}{\Gamma \vdash \lambda x : A_1.t \rightsquigarrow \lambda x : A_1.t' : A_1 \to A_2} \quad \text{RD\_LAM}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 \rightsquigarrow t_1'\, t_2 : A_2} \quad \text{RD\_APP1}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : A_1}{\Gamma \vdash t_1\ t_2 \rightsquigarrow t_1\ t_2' : A_2} \ \textsc{rd\_app2}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : A_1 \times A_2}{\Gamma \vdash \mathsf{fst}\ t \rightsquigarrow \mathsf{fst}\ t' : A_1} \ \textsc{rd\_fst}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : A_1 \times A_2}{\Gamma \vdash \mathsf{snd}\ t \rightsquigarrow \mathsf{snd}\ t' : A_2} \ \textsc{rd\_snd}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1', t_2) : A_1 \times A_2} \ \textsc{rd\_pair1}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : A_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1, t_2') : A_1 \times A_2} \ \textsc{rd\_pair2}$$

B The Complete Spec of $\lambda^{\langle A \rangle}_{\to}$-Calculus

$termvar$, x, z
$index$, k

t	::=			term
	|	x		variable
	|	triv		unit
	|	$\langle A \rangle t$		type cast
	|	$\lambda x : A.t$		λ-abstraction
	|	$t_1\ t_2$		function application
	|	(t_1, t_2)		pair constructor
	|	$\mathsf{fst}\ t$		first projection
	|	$\mathsf{snd}\ t$		second projection
	|	$\mathsf{succ}\ t$		successor function
	|	0		zero
	|	(t)	S	
$A,\ B,\ C$	::=			type
	|	1		unit type
	|	$\mathbb{N}$		natural number type
	|	$?$		untyped universe
	|	$A_1 \to A_2$		function type
	|	$A_1 \times A_2$		cartesian product type
	|	(A)	S	
Γ	::=			typing context
	|	$\cdot$		empty context
	|	$x : A$		cons

$$
\begin{array}{lll}
vd & ::= & \\
 & | & \vdash \\
 & | & \not\vdash
\end{array}
$$

$\boxed{A \sim B}$ A is consistent with B

$$\frac{}{A \sim A}\ \textsc{Refl}$$

$$\frac{}{A \sim ?}\ \textsc{Box}$$

$$\frac{}{? \sim A}\ \textsc{Unbox}$$

$$\frac{A_1 \sim A_2 \quad B_1 \sim B_2}{A_1 \to B_1 \sim A_2 \to B_2}\ \textsc{Arrow}$$

$$\frac{A_1 \sim A_2 \quad B_1 \sim B_2}{A_1 \times B_1 \sim A_2 \times B_2}\ \textsc{Prod}$$

$\boxed{\Gamma vd t : A}$ t has type A in context Γ

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \textsc{Var}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1}\ \textsc{Unit}$$

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$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}}\ \textsc{Succ}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2}\ \textsc{Pair}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \mathsf{fst}\, t : A_1}\ \textsc{Fst}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \mathsf{snd}\, t : A_2}\ \textsc{Snd}$$

$$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1.t : A_1 \to A_2}\ \textsc{Lam}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 : A_2}\ \textsc{App}$$

$$\frac{\Gamma \vdash t : A \quad A \sim B}{\Gamma \vdash \langle B \rangle t : B}\ \textsc{Cast}$$