

Gradual Typing from a Categorical Perspective

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References

A The Complete Spec of Grady

termvar, x , z

index, k

t	::=	term
	x	variable
	triv	unit
	squash	injection of the retract
	split	surjection of the retract
	box	generalize to the untyped universe
	unbox_T	specialize the untyped universe to a specific type
	$\lambda x : A. t$	λ -abstraction
	$t_1 \ t_2$	function application
	(t_1, t_2)	pair constructor
	fst t	first projection
	snd t	second projection
	succ t	successor function
	0	zero
	(t)	S

T	::=	terminating types
	1	unit type
	\mathbb{N}	natural number type
	$T_1 \rightarrow T_2$	function type
	$T_1 \times T_2$	cartesian product type
	(T)	S

A	::=	type
	T	terminating type
	$?$	untyped universe
	$A_1 \rightarrow A_2$	function type

	$\begin{array}{c} \\ \end{array}$	$\begin{array}{c} A_1 \times A_2 \\ (A) \end{array}$	S	cartesian product type
Γ	$\begin{array}{c} ::= \\ \\ \end{array}$	$\begin{array}{c} \cdot \\ \Gamma, x : A \end{array}$		typing context empty context cons
$\boxed{\Gamma \vdash t : A}$				t has type A in context Γ
		$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}$	VAR	
		$\frac{\Gamma \vdash t : T}{\Gamma \vdash \text{box } t : ?}$	BOX	
		$\overline{\Gamma \vdash \text{unbox}_T : ? \rightarrow T}$	UNBOX	
		$\overline{\Gamma \vdash \text{squash} : (? \rightarrow ?) \rightarrow ?}$	INJ	
		$\overline{\Gamma \vdash \text{split} : ? \rightarrow (? \rightarrow ?)}$	SURJ	
		$\overline{\Gamma \vdash \text{triv} : 1}$	UNIT	
		$\overline{\Gamma \vdash 0 : \mathbb{N}}$	ZERO	
		$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \text{succ } t : \mathbb{N}}$	SUCC	
		$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2}$	PAIR	
		$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \text{fst } t : A_1}$	FST	
		$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \text{snd } t : A_2}$	SND	
		$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1. t : A_1 \rightarrow A_2}$	LAM	
		$\frac{\Gamma \vdash t_1 : A_1 \rightarrow A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1 t_2 : A_2}$	APP	
$\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A}$				t_1 reduces to t_2 with type A in context Γ
		$\frac{\Gamma \vdash t : T}{\Gamma \vdash \text{unbox}_T (\text{box } t) \rightsquigarrow t : T}$	RD_RETRACT	
		$\frac{t \neq \text{box } t'}{\Gamma \vdash \text{unbox}_T t \rightsquigarrow \text{wrong} : \text{TypeError}}$	RD_UNBOXERR	

$$\begin{array}{c}
\frac{\Gamma \vdash t : ? \rightarrow ?}{\Gamma \vdash \text{split}(\text{squash } t) \rightsquigarrow t : ? \rightarrow ?} \quad \text{RD_RETRACTU} \\
\\
\frac{\Gamma \vdash t : A_1 \rightarrow A_2 \quad x \notin \text{FV}(t)}{\Gamma \vdash \lambda x : A_1. t x \rightsquigarrow t : A_1 \rightarrow A_2} \quad \text{RD_ETA} \\
\\
\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1. t_2) t_1 \rightsquigarrow [t_1/x] t_2 : A_2} \quad \text{RD_BETA} \\
\\
\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \text{fst}(t_1, t_2) \rightsquigarrow t_1 : A_1} \quad \text{RD_PROJ1} \\
\\
\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \text{snd}(t_1, t_2) \rightsquigarrow t_2 : A_2} \quad \text{RD_PROJ2} \\
\\
\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash (\text{fst } t, \text{snd } t) \rightsquigarrow t : A_1 \times A_2} \quad \text{RD_ETAP} \\
\\
\frac{\Gamma, x : A_1 \vdash t \rightsquigarrow t' : A_2}{\Gamma \vdash \lambda x : A_1. t \rightsquigarrow \lambda x : A_1. t' : A_1 \rightarrow A_2} \quad \text{RD_LAM} \\
\\
\frac{\Gamma \vdash t_1 \rightsquigarrow t'_1 : A_1 \rightarrow A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1 t_2 \rightsquigarrow t'_1 t_2 : A_2} \quad \text{RD_APP1} \\
\\
\frac{\Gamma \vdash t_1 : A_1 \rightarrow A_2 \quad \Gamma \vdash t_2 \rightsquigarrow t'_2 : A_1}{\Gamma \vdash t_1 t_2 \rightsquigarrow t_1 t'_2 : A_2} \quad \text{RD_APP2} \\
\\
\frac{\Gamma \vdash t \rightsquigarrow t' : A_1 \times A_2}{\Gamma \vdash \text{fst } t \rightsquigarrow \text{fst } t' : A_1} \quad \text{RD_FST} \\
\\
\frac{\Gamma \vdash t \rightsquigarrow t' : A_1 \times A_2}{\Gamma \vdash \text{snd } t \rightsquigarrow \text{snd } t' : A_2} \quad \text{RD_SND} \\
\\
\frac{\Gamma \vdash t_1 \rightsquigarrow t'_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t'_1, t_2) : A_1 \times A_2} \quad \text{RD_PAIR1} \\
\\
\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 \rightsquigarrow t'_2 : A_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1, t'_2) : A_1 \times A_2} \quad \text{RD_PAIR2}
\end{array}$$