

The Combination of Dynamic and Static Typing from a Categorical Perspective

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Abstract

In this paper we introduce a new categorical model based on retracts that combines static and dynamic typing. This model is initially based on the seminal work of Scott who showed that the untyped λ -calculus can be considered as typed using retracts. Following this we define a new simple type system which combines static and dynamic typing called Grady that corresponds to our model. Then we develop a gradually typed surface language for Grady, and show that it can be translated into Grady such that the gradual guarantee holds. Finally, to illustrate how this system can be extended with new features we extend both the surface and the core languages with bounded quantification such that the gradual guarantee is preserved.

CCS Concepts •Software and its engineering → General programming languages; •Social and professional topics → History of programming languages;

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1 Introduction

2 The Categorical Model

One strength and main motivation for giving a categorical model to a programming language is that it can expose the fundamental structure of the language. This arises because a lot of the language features that often cloud the picture go away, for example, syntactic notions like variables disappear. This can often simplify things and expose the underlying structure. For example, when giving the simply typed λ -calculus a categorical model we see that it is a cartesian closed category, but we also know that intuitionistic logic has the same model due to Lambek [8]; on the syntactic side these two theories

are equivalent as well due to Howard [7]; this is known as the Curry-Howard-Lambek correspondence.

The previous point highlights one of the most powerful features of category theory: its ability to relate seemingly unrelated theories. It is quite surprising that the typed λ -calculus and intuitionistic logic share the same model. Thus, defining a categorical model for a particular programming language may reveal new and interesting relationships with existing work. In fact, one of the contributions of this paper is the new connection between Scott and Lambek's work to the new area of gradual typing and combining static and dynamic typing.

However, that motivation places defining a categorical model as an after thought. The Grady languages developed here were designed from the other way around. We started with the question, how do we combine static and dynamic typing categorically? Then after developing the model we use it to push us toward the correct language design. Reynolds [2] was a big advocate for the use of category theory in programming language research for this reason. We think the following quote – reported by Brookes et al. [2] – makes this point nicely:

Programming language semanticists should be the obstetricians of programming languages, not their coroners.

— John C. Reynolds

Categorical semantics provides a powerful tool to study language extensions. For example, purely functional programming in Haskell would not be where it is without the seminal work of Moggi and Wadler [10, 16] on using monads – a purely categorical notion – to add side effects to pure functional programming languages. Thus, we believe that developing these types of models for new language designs and features can be hugely beneficial.

The model we develop here builds on the seminal work of Lambek [8] and Scott [11]. Lambek [8] showed that the typed λ -calculus can be modeled by a cartesian closed category. In the same volume as Lambek, Scott essentially showed that the untyped λ -calculus is actually typed. That is, typed theories are more fundamental than untyped ones. He accomplished this by adding a single type, $?$, and two functions *squash* : $(? \rightarrow ?) \rightarrow ?$ and *split* : $? \rightarrow (? \rightarrow ?)$, such that, *squash*; *split* = *id* : $(? \rightarrow ?) \rightarrow (? \rightarrow ?)$, to a cartesian closed category. At this point he was able to translate the untyped λ -calculus into this untyped one.

Categorically, Scott modeled *split* and *squash* as the morphisms in a retract within a cartesian closed category – the same model as typed λ -calculus.

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Definition 2.1. Suppose C is a category. Then an object A is a **retract** of an object B if there are morphisms $i : A \longrightarrow B$ and $r : B \longrightarrow A$ such that $i; r = \text{id}_A$.

Thus, $? \rightarrow ?$ is a retract of $?$, but we also require that $? \times ?$ be a retract of $?$; this is not new, see Lambek and Scott [9]. Putting this together we obtain Scott's model of the untyped λ -calculus.

Definition 2.2. An **untyped λ -model**, $(C, ?, \text{split}, \text{squash})$, is a cartesian closed category C with a distinguished object $?$ and morphisms $\text{squash} : S \longrightarrow ?$ and $\text{split} : ? \longrightarrow S$ making the object S a retract of $?$, where S is either $? \rightarrow ?$ or $? \times ?$.

Theorem 2.3 (Scott [11]). *An untyped λ -model is a sound and complete model of the untyped λ -calculus.*

So far we know how to model static types (typed λ -calculus) and unknown types (the untyped λ -calculus). To make the Grady languages a bit more interesting we add natural numbers, but we will need a way to model these in a cartesian closed category.

We model the natural numbers with their (non-recursive) eliminator using what we call a non-recursive natural number object. This is a simplification of the traditional natural number object; see Lambek and Scott [9].

Definition 2.4. Suppose C is a cartesian closed category. A **non-recursive natural number object (NRNO)** is an object Nat of C and morphisms $z : 1 \longrightarrow \text{Nat}$ and $\text{succ} : \text{Nat} \longrightarrow \text{Nat}$ of C , such that, for any morphisms $f : Y \longrightarrow X$ and $g : Y \times \text{Nat} \longrightarrow X$ of C there is a unique morphism $\text{case}_{Y,X}\langle f, g \rangle : Y \times \text{Nat} \longrightarrow X$ such that the following hold:

$$\langle \text{id}_Y, \diamond_Y; z \rangle; \text{case}_{Y,X}\langle f, g \rangle = f \quad \langle \text{id}_Y \times \text{succ} \rangle; \text{case}_{Y,X}\langle f, g \rangle = g$$

Informally, the two equations essentially assert that we can define $\text{case}_{Y,X}$ as follows:

$$\text{case}_{Y,X}\langle f, g \rangle y 0 = f y \quad \text{case}_{Y,X}\langle f, g \rangle y (\text{succ } n) = g y n$$

At this point we can model both static and unknown types with natural numbers in a cartesian closed category, but we do not have any way of moving typed data into the untyped part and vice versa to obtain dynamic typing. To accomplish this we add two new morphisms $\text{box}_C : C \longrightarrow ?$ and $\text{unbox}_C : ? \longrightarrow C$ such that each atomic type, C , is a retract of $?$. This enforces that the only time we can cast $?$ to another type is if it were boxed up in the first place. Combining all of these insights we obtain the complete categorical model.

Definition 2.5. A **gradual λ -model**, $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$, where \mathcal{T} is a discrete category with at least two objects Nat and Unit , C is a cartesian closed category with a NRNO, $(C, ?, \text{split}, \text{squash})$ is an untyped λ -model, $T : \mathcal{T} \longrightarrow C$ is an embedding – a full and faithful functor that is injective on objects – and for every object A of \mathcal{T} there are morphisms $\text{box}_A : TA \longrightarrow ?$ and $\text{unbox}_A : ? \longrightarrow TA$ making TA a retract of $?$. Furthermore, to model dynamic type errors, there is a morphism, $\text{err}_A : \text{Unit} \longrightarrow A$ of C , such that, the following equations hold w.r.t. $\text{error}_{A,B} = A \xrightarrow{\text{triv}_A} \text{Unit} \xrightarrow{\text{err}_B} B$:

$$\begin{aligned} \text{box}_{TA}; \text{unbox}_{TB} &= \text{error}_{TA, TB}, \text{ where } A \neq B \\ \text{squash}_{S_1}; \text{split}_{S_2} &= \text{error}_{S_1, S_2}, \text{ where } S_1 \neq S_2 \\ f; \text{error}_{B,C} &= \text{error}_{A,C}, \text{ where } f : A \longrightarrow B \\ \text{error}_{A,B}; f &= \text{error}_{A,C}, \text{ where } f : B \longrightarrow C \\ \langle \text{error}_{A,B}, f \rangle &= \text{error}_{A, B \times C}, \text{ where } f : A \longrightarrow C \\ \langle f, \text{error}_{A,C} \rangle &= \text{error}_{A, B \times C}, \text{ where } f : A \longrightarrow B \\ \text{curry}(\text{error}_{A \times B, C}) &= \text{error}_{A, B \rightarrow C} \end{aligned}$$

We call the category \mathcal{T} the category of atomic types. We call an object, A , **atomic** iff there is some object A' in \mathcal{T} such that $A = TA'$. Note that we do not consider $?$ an atomic type.

Triggering dynamic type errors is a fundamental property of the criteria for gradually typed languages, and thus, the model must capture this. The new morphism $\text{err}_A : \text{Unit} \longrightarrow A$ is combined with the terminal morphism, $\text{triv}_A : A \longrightarrow \text{Unit}$, which is a unique morphism guaranteed to exist because C is cartesian closed, to define the morphism $\text{error}_{A,B} : A \longrightarrow B$ that signifies that one tried to unbox or split at the wrong type resulting in a dynamic type error; this is captured by the first and second equations in the definition. If we view morphisms as programs, then the other equations are congruence rules that trigger a dynamic type error for the whole program when one of its subparts trigger a dynamic type error. The following extends the error equations to the functors $- \times -$ and $- \rightarrow -$:

Lemma 2.6 (Extended Errors). *Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model. Then the following equations hold:*

$$\begin{aligned} f \times \text{error}_{B,C} &= \text{error}_{A \times B, C \times D}, \text{ where } f : A \longrightarrow C \\ \text{error}_{A,C} \times f &= \text{error}_{A \times B, C \times D}, \text{ where } f : B \longrightarrow D \\ f \rightarrow \text{error}_{B,C} &= \text{error}_{A \rightarrow B, C \rightarrow D}, \text{ where } f : C \longrightarrow D \\ \text{error}_{C,A} \rightarrow f &= \text{error}_{A \rightarrow B, C \rightarrow D}, \text{ where } f : B \longrightarrow D \end{aligned}$$

Proof. The following define the morphism part of the two functors $f \times g : (A \times B) \longrightarrow (C \times D)$ and $f \rightarrow g : (A \rightarrow B) \longrightarrow (C \rightarrow D)$:

$$f \times g = \langle \text{fst}; f, \text{snd}; g \rangle, \text{ where } f : A \longrightarrow C \text{ and } g : B \longrightarrow D$$

$$f \rightarrow g = \text{curry}((\text{id}_{A \rightarrow B} \times f); \text{app}_{A,B}; g), \text{ where } f : C \longrightarrow A \text{ and } g : B \longrightarrow D$$

First, note that $\text{fst} : (A \times B) \longrightarrow A$, $\text{snd} : (A \times B) \longrightarrow B$, and $\text{app}_{A,B} : ((A \rightarrow B) \times A) \longrightarrow B$ all exist by the definition of a cartesian closed category.

It is now quite obvious that if either f or g is error in the previous two definitions, then by using the equations from the definition of a gradual λ -model (Definition 2.5) the application of either of the functors will result in error. \square

As the model is defined it is unclear if we can cast any type to $?$, and vice versa, but we must be able to do this in order to model full dynamic typing. In the remainder of this section we show that we can build up such casts in terms of the basic features of our model. To cast any type A to $?$ we will build casting morphisms that first take the object A to its skeleton, and then takes the skeleton to $?$.

Definition 2.7. Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model. Then we call any morphism defined completely in terms of id , the functors $- \times -$ and $- \rightarrow -$, split and squash , and box and unbox a **casting morphism**.

Definition 2.8. Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model. The **skeleton** of an object A of C is an object S that is constructed by replacing each

atomic type in A with $?$. Given an object A we denote its skeleton by $\text{skeleton } A$.

One should think of the skeleton of an object as the supporting type structure of the object, but we do not know what kind of data is actually in the structure. For example, the skeleton of the object Nat is $?$, and the skeleton of $(\text{Nat} \times \text{Unit}) \rightarrow \text{Nat} \rightarrow \text{Nat}$ is $(? \times ?) \rightarrow ? \rightarrow ?$.

The next definition defines a means of constructing a casting morphism that casts a type A to its skeleton and vice versa. This definition is by mutual recursion on the input type.

Definition 2.9. Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model. Then for any object A whose skeleton is S we define the morphisms $\widehat{\text{box}}_A : A \rightarrow S$ and $\widehat{\text{unbox}}_A : S \rightarrow A$ by mutual recursion on A as follows:

$$\begin{array}{l|l} \widehat{\text{box}}_A = \text{box}_A & \widehat{\text{unbox}}_A = \text{unbox}_A \\ \text{when } A \text{ is atomic} & \text{when } A \text{ is atomic} \\ \widehat{\text{box}}_? = \text{id}_? & \widehat{\text{unbox}}_? = \text{id}_? \\ \widehat{\text{box}}_{(A_1 \rightarrow A_2)} = \widehat{\text{unbox}}_{A_1} \rightarrow \widehat{\text{box}}_{A_2} & \widehat{\text{unbox}}_{(A_1 \rightarrow A_2)} = \widehat{\text{box}}_{A_1} \rightarrow \widehat{\text{unbox}}_{A_2} \\ \widehat{\text{box}}_{(A_1 \times A_2)} = \widehat{\text{box}}_{A_1} \times \widehat{\text{box}}_{A_2} & \widehat{\text{unbox}}_{(A_1 \times A_2)} = \widehat{\text{unbox}}_{A_1} \times \widehat{\text{unbox}}_{A_2} \end{array}$$

The definition of both $\widehat{\text{box}}$ or $\widehat{\text{unbox}}$ uses the functor $- \rightarrow - : C^{\text{op}} \times C \rightarrow C$ which is contravariant in its first argument, and thus, in that contravariant position we must make a recursive call to the opposite function, and hence, they must be mutually defined. Every call to either $\widehat{\text{box}}$ or $\widehat{\text{unbox}}$ in the previous definition is on a smaller object than the input object. Thus, their definitions are well founded. Furthermore, $\widehat{\text{box}}$ and $\widehat{\text{unbox}}$ form a retract between A and S .

Lemma 2.10 (Boxing and Unboxing Lifted Retract). *Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model. Then for any object A , $\widehat{\text{box}}_A; \widehat{\text{unbox}}_A = \text{id}_A : A \rightarrow A$. Furthermore, for any objects A and B such that $A \neq B$, $\widehat{\text{box}}_A; \widehat{\text{unbox}}_B = \text{error}_{A,B}$.*

Proof. This proof holds by induction on the form A . Please see Appendix A.1 for the complete proof. \square

As an example, suppose we wanted to cast the type $(\text{Nat} \times ?) \rightarrow \text{Nat}$ to its skeleton $(? \times ?) \rightarrow ?$. Then we can obtain a casting morphisms that will do this as follows:

$$\widehat{\text{box}}_{((\text{Nat} \times ?) \rightarrow \text{Nat})} = (\widehat{\text{unbox}}_{\text{Nat}} \times \text{id}_?) \rightarrow \widehat{\text{box}}_{\text{Nat}}$$

We can also cast a morphism $A \xrightarrow{f} B$ to a morphism $\widehat{\text{unbox}}_A; f; \widehat{\text{box}}_A : S_1 \rightarrow S_2$ where S_1 is skeleton A and S_2 is skeleton B . Now if we have a second $\widehat{\text{unbox}}_B; g; \widehat{\text{box}}_C : S_2 \rightarrow S_3$ then their composition reduces to composition at the typed level:

$$\begin{array}{ccccc} S_1 & \xrightarrow{\widehat{\text{unbox}}_A} & A & \xrightarrow{f} & B & \xrightarrow{\widehat{\text{box}}_B} & S_2 \\ \downarrow & & \downarrow f;g & & \downarrow \text{id} & & \downarrow \text{id} \\ S_3 & \xleftarrow{\widehat{\text{box}}_C} & C & \xleftarrow{g} & B & \xleftarrow{\widehat{\text{unbox}}_B} & S_2 \end{array}$$

The right most diagram commutes because B is a retract of S_2 , and the left unannotated arrow is the composition $\widehat{\text{unbox}}_A; f$;

$g; \widehat{\text{box}}_C$. This tells us that we have a functor $S : C \rightarrow S$:

$$\begin{aligned} SA &= \text{skeleton } A \\ S(f : A \rightarrow B) &= \widehat{\text{unbox}}_A; f; \widehat{\text{box}}_A \end{aligned}$$

where S is the full subcategory of C consisting of the skeletons and morphisms between them, that is, S is a cartesian closed category with one basic object $?$ such that $(S, ?, \text{split}, \text{squash})$ is an untyped λ -model. The following turns out to be true.

Lemma 2.11 (S is faithful). *Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model, and $(S, ?, \text{split}, \text{squash})$ is the category of skeletons. Then the functor $S : C \rightarrow S$ is faithful.*

Proof. This proof follows from the definition of S and Lemma 2.10. For the full proof see Appendix A.2. \square

Thus, we can think of the functor S as an injection of the typed world into the untyped one.

Now that we can cast any type into its skeleton we must show that every skeleton can be cast to $?$. We do this similarly to the above and lift split and squash to arbitrary skeletons.

Definition 2.12. Suppose $(S, ?, \text{split}, \text{squash})$ is the category of skeletons. Then for any skeleton S we define the morphisms $\widehat{\text{squash}}_S : S \rightarrow ?$ and $\widehat{\text{split}}_S : ? \rightarrow S$ by mutual recursion on S as follows:

$$\begin{array}{l} \widehat{\text{squash}}_? = \text{id}_? \\ \widehat{\text{squash}}_{(S_1 \rightarrow S_2)} = (\widehat{\text{split}}_{S_1} \rightarrow \widehat{\text{squash}}_{S_2}); \widehat{\text{squash}}_{? \rightarrow ?} \\ \widehat{\text{squash}}_{(S_1 \times S_2)} = (\widehat{\text{squash}}_{S_1} \times \widehat{\text{squash}}_{S_2}); \widehat{\text{squash}}_{? \times ?} \\ \widehat{\text{split}}_? = \text{id}_? \\ \widehat{\text{split}}_{(S_1 \rightarrow S_2)} = \widehat{\text{split}}_{? \rightarrow ?}; (\widehat{\text{squash}}_{S_1} \rightarrow \widehat{\text{split}}_{S_2}) \\ \widehat{\text{split}}_{(S_1 \times S_2)} = \widehat{\text{split}}_{? \times ?}; (\widehat{\text{split}}_{S_1} \times \widehat{\text{split}}_{S_2}) \end{array}$$

As an example we will construct the casting morphism that casts the skeleton $(? \times ?) \rightarrow ?$ to $?$:

$$\widehat{\text{squash}}_{(? \times ?) \rightarrow ?} = (\widehat{\text{split}}_{? \times ?} \rightarrow \text{id}_?); \widehat{\text{squash}}_{? \rightarrow ?}.$$

Just as we saw above, splitting and squashing forms a retract.

Lemma 2.13 (Splitting and Squashing Lifted Retract). *Suppose $(S, ?, \text{split}, \text{squash})$ is the category of skeletons. Then for any skeleton S , $\widehat{\text{squash}}_S; \widehat{\text{split}}_S = \text{id}_S : S \rightarrow S$. Furthermore, for any skeletons S_1 and S_2 such that $S_1 \neq S_2$, $\widehat{\text{squash}}_{S_1}; \widehat{\text{split}}_{S_2} = \text{error}_{S_1, S_2}$.*

Proof. The proof is similar to the proof of the boxing and unboxing lifted retract (Lemma 2.10). \square

There is also a faithful functor from S to \mathcal{U} where \mathcal{U} is the full subcategory of S that consists of the single object $?$ and all its morphisms between it:

$$\begin{aligned} \mathcal{U}S &= ? \\ \mathcal{U}(f : S_1 \rightarrow S_2) &= \widehat{\text{split}}_{S_1}; f; \widehat{\text{squash}}_{S_2} \end{aligned}$$

This finally implies that there is a functor $C : C \rightarrow \mathcal{U}$ that injects all of C into the object $?$.

Lemma 2.14 (Casting to ?). *Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model, $(S, ?, \text{split}, \text{squash})$ is the full subcategory of skeletons, and $(\mathcal{U}, ?)$ is the full subcategory containing only ? and its morphisms. Then there is a faithful functor $C = C \xrightarrow{S} S \xrightarrow{U} \mathcal{U}$.*

In a way we can think of $C : C \rightarrow \mathcal{U}$ as a forgetful functor. It forgets the type information.

Getting back the typed information is harder. There is no nice functor from \mathcal{U} to C , because we need more information. However, given a type A we can always obtain a casting morphism from ? to A by $(\text{split}_{(\text{skeleton } A)}); (\text{unbox}_A) : ? \rightarrow A$. Finally, we have the following result.

Lemma 2.15 (Casting Morphisms to ?). *Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model, and A is an object of C . Then there exists casting morphisms from A to ? and vice versa that make A a retract of ?.*

Proof. The two morphisms are as follows:

$$\begin{aligned} \text{Box}_A &:= \widehat{\text{box}}_A; \widehat{\text{squash}}_{(\text{skeleton } A)} : A \rightarrow ? \\ \text{Unbox}_A &:= \widehat{\text{split}}_{(\text{skeleton } A)}; \widehat{\text{unbox}}_A : ? \rightarrow A \end{aligned}$$

The fact these form a retract between A and ?, and raise dynamic type errors holds by Lemma 2.10 and Lemma 2.13. \square

The previous result has a number of implications. It completely brings together the static and dynamic fragments of the gradual λ -model, and thus, fully relating the combination of dynamic and static typing to the past work of Lambek and Scott [8, 11]. It will allow for the definition of casting morphisms between arbitrary objects; see the next result. Finally, from a practical perspective it will simplify our corresponding type systems derived from this model, because $\text{Box}_S = \widehat{\text{squash}}_S$ and $\text{Unbox}_S = \widehat{\text{split}}_S$ when S is a skeleton, and hence, we will only need a single retract in the corresponding type systems.

3 Core Grady

Just as the simply typed λ -calculus corresponds to cartesian closed categories our categorical model has a corresponding type theory we call Core Grady. It consists of all of the structure found in the model. To move from the model to Core Grady we apply the Curry-Howard-Lambek correspondence [8, 17]. Objects become types, and morphisms, $t : \Gamma \rightarrow A$, become programs in context usually denoted by $\Gamma \vdash_{\text{CG}} t : A$ which corresponds to Core Grady's type checking judgment. We will discuss this correspondence in detail in Section 5.

The syntax for Core Grady is defined in Figure 1. The syntax is a straightforward extension of the simply typed λ -calculus. Arbitrary programs or terms are denoted by t and values by v . The latter are used to influence the evaluation strategy used by Core Grady. Natural numbers are denoted by 0 and $\text{succ } t$ where the latter is the successor of t . The non-recursive natural number eliminator is denoted by $\text{case } t : \text{Nat of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2$. The most interesting aspect of the syntax is that box_A and unbox_A are not restricted to atomic types, but actually correspond to Box_A and Unbox_A from Lemma 2.15. That result shows that these can actually be

(types)	$A, B, C ::= \text{Unit} \mid \text{Nat} \mid ? \mid A \times B \mid A \rightarrow B$
(skeletons)	$S, K, U ::= ? \mid S_1 \times S_2 \mid S_1 \rightarrow S_2$
(terms)	$t ::= x \mid \text{triv} \mid 0 \mid \text{succ } t \mid (t_1, t_2) \mid \text{fst } t \mid \text{snd } t$ $\mid \lambda(x : A).t \mid t_1 t_2 \mid \text{case } t : \text{Nat of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2$ $\mid \text{box}_A \mid \text{unbox}_A \mid \text{error}_A$
(values)	$v ::= \lambda(x : A).t \mid \text{unbox}_A$
(evaluation contexts)	$\mathcal{E} ::= v \square \mid \text{succ } \square \mid \text{fst } \square \mid \text{snd } \square \mid (\square, t)$ $\mid (t, \square) \mid \text{case } \square : \text{Nat of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2$
(contexts)	$\Gamma ::= \cdot \mid x : A \mid \Gamma_1, \Gamma_2$

Figure 1. Syntax for Core Grady

$\frac{x : A \in \Gamma}{\Gamma \vdash_{\text{CG}} x : A} \text{var}$	$\frac{}{\Gamma \vdash_{\text{CG}} \text{box}_A : A \rightarrow ?} \text{box}$
$\frac{}{\Gamma \vdash_{\text{CG}} \text{unbox}_A : ? \rightarrow A} \text{unbox}$	$\frac{}{\Gamma \vdash_{\text{CG}} \text{triv} : \text{Unit}} \text{Unit}$
$\frac{}{\Gamma \vdash_{\text{CG}} 0 : \text{Nat}} \text{zero}$	$\frac{\Gamma \vdash_{\text{CG}} t : \text{Nat}}{\Gamma \vdash_{\text{CG}} \text{succ } t : \text{Nat}} \text{succ}$
$\frac{\Gamma \vdash_{\text{CG}} t : \text{Nat} \quad \Gamma \vdash_{\text{CG}} t_1 : A \quad \Gamma, x : \text{Nat} \vdash_{\text{CG}} t_2 : A}{\Gamma \vdash_{\text{CG}} \text{case } t : \text{Nat of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2 : A} T_{\text{ncase}}$	
$\frac{\Gamma \vdash_{\text{CG}} t_1 : A_1 \quad \Gamma \vdash_{\text{CG}} t_2 : A_2}{\Gamma \vdash_{\text{CG}} (t_1, t_2) : A_1 \times A_2} \times_i$	$\frac{\Gamma \vdash_{\text{CG}} t : A_1 \times A_2}{\Gamma \vdash_{\text{CG}} \text{fst } t : A_1} \times_{e1}$
$\frac{\Gamma \vdash_{\text{CG}} t : A_1 \times A_2}{\Gamma \vdash_{\text{CG}} \text{snd } t : A_2} \times_{e2}$	$\frac{\Gamma, x : A \vdash_{\text{CG}} t : B}{\Gamma \vdash_{\text{CG}} \lambda(x : A).t : A \rightarrow B} \rightarrow_i$
$\frac{\Gamma \vdash_{\text{CG}} t_1 : A \rightarrow B \quad \Gamma \vdash_{\text{CG}} t_2 : A}{\Gamma \vdash_{\text{CG}} t_1 t_2 : B} \rightarrow_e$	$\frac{}{\Gamma \vdash_{\text{CG}} \text{error}_A : A} \text{error}$

Figure 2. Typing rules for Core Grady

defined in terms of $\widehat{\text{box}}_A$, $\widehat{\text{unbox}}_A$, $\widehat{\text{split}}_S$, and $\widehat{\text{squash}}_S$ when A is any type and S is a skeleton, but we take the general versions as primitive, because they are the most useful from a programming perspective. In addition, as we mentioned above Box_A and Unbox_A divert to $\widehat{\text{squash}}_A$ and $\widehat{\text{split}}_A$ respectively when A is a skeleton. This implies that we no longer need two retracts, and hence, simplifies the language.

Multisets of pairs of variables and types, denoted by $x : A$, called a typing context or just a context is denoted by Γ . The empty context is denoted by \cdot , and the union of contexts Γ_1 and Γ_2 is denoted by Γ_1, Γ_2 . Typing contexts are used to keep track of the types of free variables during type checking.

The typing judgment is denoted by $\Gamma \vdash_{\text{CG}} t : A$, and is read “the term t has type A in context Γ .” The typing judgment is defined by the type checking rules in Figure 2. The type checking rules are an extension of the typing rules for the simply typed λ -calculus. The casting terms are all typed as axioms with their expected types. This implies that applying either box_A or unbox_A to some other term corresponds to function application as opposed to $\text{succ } t$ which cannot be used without its argument. This fact is used in the definition of the evaluation strategy.

Computing with terms is achieved by defining a reduction relation denoted by $t_1 \rightsquigarrow t_2$ and is read as “the term t_1 reduces (or evaluates) in one step to the term t_2 .” The reduction relation

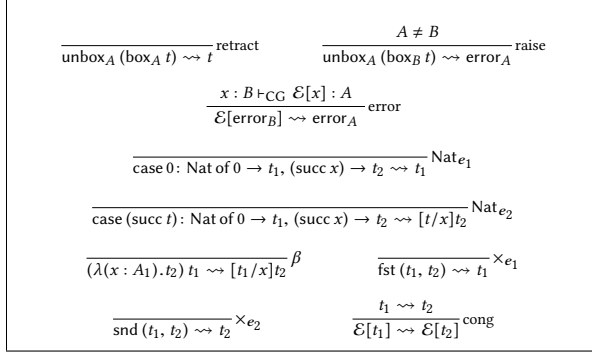


Figure 3. Reduction rules for Core Grady

is defined in Figure 3. Core Grady’s reduction strategy is an extended version of call-by-name. It is specified using evaluation contexts that are denoted by \mathcal{E} and are defined in Figure 1.

An evaluation contexts is essentially a term with a hole, denoted by \square , in it. The hole can be filled (or plugged) with a term and is denoted by $\mathcal{E}[t]$. Note that plugging the hole of an evaluation context results in a term. Evaluation contexts are used to give a compact definition of an evaluation strategy by first specifying the reduction axioms (Figure 3), then defining the evaluation contexts by placing a hole within the syntax of terms that specifies where evaluation is allowed to take place (Figure 1), finally, the following reduction rule is then added:

$$\frac{t_1 \rightsquigarrow t_2}{\mathcal{E}[t_1] \rightsquigarrow \mathcal{E}[t_2]} \text{cong}$$

This rule states that evaluation can take place in the locations of the holes given in the definition of evaluation contexts (Figure 1).

How we define the syntax of values and evaluation contexts, and the evaluation rules determines the evaluation strategy. We consider as values λ -abstractions and unbox_A . In addition, the expression $\lambda(x : A).\square$ is not an evaluation context, and hence, there is no evaluation under λ -abstractions. The evaluation context $v\square$ stipulates that the only time an argument to a function can be evaluated is if the term in the function position is a value. Thus, v is allowed to be a λ -abstraction or unbox_A . We do not allow reduction under box_A . Similarly, we have no evaluation contexts which allow evaluation under the branches of a case-expression. Both of these restrictions are used to prevent infinite reduction from occurring in those positions. We want evaluation to make as much overall progress as possible.

Perhaps the most interesting reduction rules from Figure 3 are the first three: retract, raise, and error. The first two handle dynamic type casts and the third preserves dynamic type errors that have been raised in an evaluation position. The error reduction rule depends on typing which is necessary to insure that the type annotation is correct. This insures that type preservation will hold. Practically speaking, this dependency on typing is not significant, because we only evaluate closed well-typed programs anyway.

Just as Abadi et al. [1] argue it is quite useful to have access to the untyped λ -calculus. We give some example Core Grady programs utilizing this powerful feature. We have a full implementation of every language in this paper available¹. All examples in this section can be typed and ran in the implementation, and thus, we make use of Core Grady’s concrete syntax which is very similar to Haskell’s and does not venture too far from the mathematical syntax given above.

Core Grady does not have a primitive notion of recursion, but it is well-known that we can define the Y combinator in the untyped λ -calculus. Its definition in Core Grady is as follows:

```
omega : (? → ?) → ?
omega = \x : ? → ? → (x (box (? → ?) x));

fix : (? → ?) → ?
fix = \f : ? → ? → omega (\x : ? → f ((unbox (? → ?) x) x));
```

Using `fix` we can define the usual arithmetic operations in Core Grady, but we use a typed version of `fix`.

```
fixNat : ((Nat → Nat) → (Nat → Nat)) → (Nat → Nat)
fixNat = \f : (Nat → Nat) → (Nat → Nat) →
  unbox(Nat → Nat) (fix (\y : ? → box(Nat → Nat) (f (unbox(Nat → Nat) y))));

pred : Nat → Nat
pred = \n : Nat → case n of 0 → 0, (succ n') → n';

add : Nat → Nat → Nat
add = \m : Nat → fixNat
  (\r : Nat → Nat) →
    \n : Nat → case n of 0 → m, (succ n') → succ (r n');

sub : Nat → Nat → Nat
sub = \m : Nat → fixNat
  (\r : Nat → Nat) →
    \n : Nat → case n of 0 → m, (succ n') → pred (r n');

mult : Nat → Nat → Nat
mult = \m : Nat → fixNat
  (\r : Nat → Nat) →
    \n : Nat → case n of 0 → 0, (succ n') → add m (r n');
```

The function `fixNat` is defined so that it does recursion on the type $\text{Nat} \rightarrow \text{Nat}$, thus, it must take in an argument, $f : (\text{Nat} \rightarrow \text{Nat}) \rightarrow (\text{Nat} \rightarrow \text{Nat})$, and produce a function of type $\text{Nat} \rightarrow \text{Nat}$. However, we already have `fix` defined in the untyped fragment, and so, we can define `fixNat` using `fix` by boxing up the typed data. This means we must cast $f : (\text{Nat} \rightarrow \text{Nat}) \rightarrow (\text{Nat} \rightarrow \text{Nat})$ into a function of type $(? \rightarrow ?) \rightarrow ?$ and we do this by η -expanding f and casting the input and output using `box` and `unbox` to arrive at the function $\lambda(y : ?).\text{box}_{(\text{Nat} \rightarrow \text{Nat})} (f (\text{unbox}_{(\text{Nat} \rightarrow \text{Nat})} y)) : ? \rightarrow ?$. Finally, we can apply `fix`, and then `unbox` its output to the type $\text{Nat} \rightarrow \text{Nat}$.

Extending Grady with polymorphism would allow for the definition of `fixNat` to be abstracted, and then we could do statically typed recursion at any type. We extend Core Grady with bounded polymorphism in Section 7.

From a programming perspective Core Grady has a lot going for it, but it is unfortunate the programmer is required to insert explicit casts when wanting to program dynamically. This implies that it is not possible to program in dynamic style when using Core Grady. In the next section we fix this problem by developing a gradually typed surface language for

¹Please see <https://ct-gradual-typing.github.io/Grady/> for access to the implementation as well as full documentation on how to install and use it.

Syntax:	
(terms) $t ::= x \mid \text{triv} \mid 0 \mid \text{succ } t \mid (t_1, t_2) \mid \text{fst } t \mid \text{snd } t$ $\mid \lambda(x : A).t \mid t_1 t_2 \mid \text{case } t \text{ of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2$	
Metafunctions:	
$\text{nat}(?) = \text{Nat}$ $\text{nat}(\text{Nat}) = \text{Nat}$	$\text{prod}(?) = ? \times ?$ $\text{prod}(A \times B) = A \times B$
$\text{fun}(?) = ? \rightarrow ?$ $\text{fun}(A \rightarrow B) = A \rightarrow B$	

Figure 4. Syntax and Metafunctions for Surface Grady

Core Grady in the spirit of Siek and Taha’s gradually typed λ -calculus [13, 14].

4 Surface Grady

Programming in dynamic style requires the ability to implicitly cast data between types during eliminations. For example, $(\lambda(x : ?).(\text{succ } (\text{succ } x))) 3$ should type check with type Nat even though 3 has type Nat and x has type $?$. Now not every cast should work, for example, $(\lambda(x : \text{Bool}).t) 3$ should not type check, because it is inconsistent to allow different atomic types to be cast between each other. Therefore, we must be able to decide when casts are consistent and which are not.

Gradual type systems are the combination of static and dynamic typing with the ability to implicitly cast data between consistent types in such a way that the gradual guarantee is satisfied. A gradually typed programming language consists of two languages: a core language and a surface language. The core language contains explicit casts and an operational semantics, while the surface language is the gradual type system that allows casts to be left implicit. Programs are written and then translated into the core language by inserting the explicit casts. The gradual guarantee states the following:

- Adding or removing casts to/from any well-typed surface language program remains typeable at a potentially less precise type.
- Adding explicit casts to a terminating well-typed core language program also terminates at a related value, adding explicit casts to a diverging well-typed core language program diverges, and removing explicit casts from a terminating or diverging well-type core language programs either terminates, diverges, or raises a dynamic type error.

Briefly, the gradual guarantee states that any well-typed program can slide between more statically typed and more dynamically typed by inserting or removing casts without changing the meaning or the behavior of the program. The formal statement of the gradual guarantee is given in Section 6. This property was first proposed by Siek et al. [14] to set apart systems that simply combine dynamic and static typing and gradual type systems.

Now we introduce the gradually typed surface language Surface Grady. Surface Grady is a small extension of the surface language given by Siek et al. [14]. We have added natural numbers with their eliminator as well as cartesian products.

Typing Rules:

$\frac{x : A \in \Gamma}{\Gamma \vdash_{\text{SG}} x : A} \text{ var}$	$\frac{}{\Gamma \vdash_{\text{SG}} \text{triv} : \text{Unit}} \text{ Unit}$	$\frac{}{\Gamma \vdash_{\text{SG}} 0 : \text{Nat}} \text{ zero}$
$\frac{\Gamma \vdash_{\text{SG}} t : A \quad \text{nat}(A) = \text{Nat}}{\Gamma \vdash_{\text{SG}} \text{succ } t : \text{Nat}} \text{ succ}$		
$\frac{\begin{array}{c} \Gamma \vdash_{\text{SG}} t : C \quad \text{nat}(C) = \text{Nat} \\ \Gamma \vdash_{\text{SG}} t_1 : A_1 \quad \Gamma \vdash_{\text{SG}} t_2 : A_2 \\ \Gamma, x : \text{Nat} \vdash_{\text{SG}} t_2 : A_2 \quad \Gamma \vdash_{\text{SG}} t_1 : A_1 \end{array}}{\Gamma \vdash_{\text{SG}} \text{case } t \text{ of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2 : A} \text{ Nat}_e$		
$\frac{\Gamma \vdash_{\text{SG}} t_1 : A_1 \quad \Gamma \vdash_{\text{SG}} t_2 : A_2}{\Gamma \vdash_{\text{SG}} (t_1, t_2) : A_1 \times A_2} \times_i$		
$\frac{\Gamma \vdash_{\text{SG}} t : B \quad \text{prod}(B) = A_1 \times A_2}{\Gamma \vdash_{\text{SG}} \text{fst } t : A_1} \times_{e1}$		
$\frac{\Gamma \vdash_{\text{SG}} t : B \quad \text{prod}(B) = A_1 \times A_2}{\Gamma \vdash_{\text{SG}} \text{snd } t : A_2} \times_{e2}$		
$\frac{\Gamma, x : A \vdash_{\text{SG}} t : B}{\Gamma \vdash_{\text{SG}} \lambda(x : A).t : A \rightarrow B} \rightarrow_i$		
$\frac{\begin{array}{c} \Gamma \vdash_{\text{SG}} t_1 : C \quad \Gamma \vdash_{\text{SG}} t_2 : A_1 \\ \Gamma \vdash_{\text{SG}} t_2 : A_2 \quad \text{fun}(C) = A_1 \rightarrow B_1 \end{array}}{\Gamma \vdash_{\text{SG}} t_1 t_2 : B_1} \rightarrow_e$		

Type Consistency:

$\frac{}{\Gamma \vdash A \sim A} \text{ refl}$	$\frac{}{\Gamma \vdash A \sim ?} \text{ box}$	$\frac{}{\Gamma \vdash ? \sim A} \text{ unbox}$
$\frac{\Gamma \vdash A_2 \sim A_1 \quad \Gamma \vdash B_1 \sim B_2}{\Gamma \vdash (A_1 \rightarrow B_1) \sim (A_2 \rightarrow B_2)} \rightarrow$		
$\frac{\Gamma \vdash A_1 \sim A_2 \quad \Gamma \vdash B_1 \sim B_2}{\Gamma \vdash (A_1 \times B_1) \sim (A_2 \times B_2)} \times$		

Figure 5. Typing rules for Surface Grady

The Surface Grady syntax is defined in Figure 4, and it corresponds to Core Grady’s syntax (Figure 1), but without the explicit casts. The syntax for types and typing contexts do not change, and so we do not repeat them here.

The metafunctions $\text{nat}(A)$, $\text{prod}(A)$, and $\text{fun}(A)$ are partial functions that will be used to determine when to use either box or split in the elimination type checking rules for natural numbers, cartesian products, and function applications respectively. For example, if $\text{nat}(A) = \text{Nat}$, then the type A must have been either $?$ or Nat , and if it were the former then we know we can cast A to Nat via box_{Nat} . If $\text{prod}(A) = B \times C$, then either $A = ?$ and $B \times C = ? \times ?$ or $A = B \times C$ for some other types B and C . This implies that if the former is true, then we can cast A to $B \times C$ via $\text{split}(? \times ?)$. The case is similar for $\text{fun}(A)$.

The type checking and type consistency rules are given in Figure 5. Similarly to Core Grady the typing judgment is denoted by $\Gamma \vdash_{\text{SG}} t : A$. Type checking depends on the notion of type consistency; first proposed by Siek and Taha [13]. This is a reflexive and symmetric, but non-transitive, relation on types denoted by $\Gamma \vdash A \sim B$ which can be read as “the type A is consistent with the type B in context Γ .” Note that the typing context Γ does not yet play a role in the definition of type consistency, but it will when we add bounded quantification in Section 7. Non-transitivity is important, because if type consistency were transitive, then all types would be consistent, but this is too general. Consider an example, type consistency is responsible for the function application $(\lambda(x : ?).(\text{succ } x)) 3$ being typable in the surface language, because type Nat is

consistent with the type 3. This implies that the elimination rule for function types must be extended with type consistency.

Type consistency states when two types are safely castable between each other when inserting explicit casts, and so, from a semantical perspective if $\Gamma \vdash A \sim B$ holds, then there are casting morphisms (Definition 2.7) $c_1 : A \multimap B$ and $c_2 : B \multimap A$; see Lemma 5.2 in Section 5.

The typing rules for Surface Grady are a conservative extension of the typing rules for Core Grady (Figure 2). The extension is the removal of explicit casts and the addition of type consistency and the metafunctions from Figure 4. Each rule is modified in positions where casting is likely to occur which would be in all of the elimination rules as well as the typing rule for successor, because it is a type of application. Consider the elimination rule for function applications:

$$\frac{\Gamma \vdash_{\text{SG}} t_1 : C \quad \Gamma \vdash_{\text{SG}} t_2 : A_2 \quad \text{fun}(C) = A_1 \rightarrow B_1 \quad \Gamma \vdash A_2 \sim A_1}{\Gamma \vdash_{\text{SG}} t_1 t_2 : B_1} \rightarrow_e$$

This rule has been extended with type consistency. The type of t_1 is allowed to be either $?$ or a function type $A_1 \rightarrow B_1$, by the definition of $\text{fun}(C)$, if the former is true, then $A_1 \rightarrow B_1 = ? \rightarrow ?$ and A_2 can be any type at all, but if $C = A_1 \rightarrow B_1$, then A_2 must be consistent with A_1 . Notice that if $C = A_1 \rightarrow B_1$ and $A_2 = A_1$, then this rule is equivalent to the usual rule for function application. We can now see that our example program $(\lambda(x : ?).(\text{succ } x)) 3$ is typable in Surface Grady. Similar reasoning can be used to understand the other typing rules as well.

Surface Grady is translated into Core Grady using the cast insertion algorithm detailed in Figure 6. The cast insertion judgment is denoted by $\Gamma \vdash t_1 \Rightarrow t_2 : A$ which is read as “the Surface Grady program t_1 of type A is translated into the Core Grady program t_2 of type A in context Γ .” This algorithm is type directed, and is dependent on the partial metafunction $\text{caster}(A, B)$ that constructs the casting morphism – in Core Grady – of type $A \rightarrow B$:

$$\begin{aligned} \text{caster}(A, A) &= \lambda(x : A).x \\ \text{caster}(A, ?) &= \text{box}_A \\ \text{caster}(?, B) &= \text{unbox}_B \\ \text{caster}((A_1 \times B_1), (A_2 \times B_2)) &= \text{caster}(A_1, A_2) \times \text{caster}(B_1, B_2) \\ \text{caster}((A_1 \rightarrow B_1), (A_2 \rightarrow B_2)) &= \text{caster}(A_2, A_1) \rightarrow \text{caster}(B_1, B_2) \end{aligned}$$

The previous definition uses the following derivable functor rules:

$$\frac{\Gamma \vdash_{\text{CG}} t_1 : A_1 \rightarrow A_2 \quad \Gamma \vdash_{\text{CG}} t_2 : B_1 \rightarrow B_2}{\Gamma \vdash_{\text{CG}} t_1 \times t_2 : (A_1 \times B_1) \rightarrow (A_2 \times B_2)}$$

$$\frac{\Gamma \vdash_{\text{CG}} t_1 : A_2 \rightarrow A_1 \quad \Gamma \vdash_{\text{CG}} t_2 : B_1 \rightarrow B_2}{\Gamma \vdash_{\text{CG}} t_1 \rightarrow t_2 : (A_1 \rightarrow B_1) \rightarrow (A_2 \rightarrow B_2)}$$

They are defined as follows:

$$\begin{aligned} t_1 \times t_2 &= \lambda(x : A_1 \times B_1). (t_1 (\text{fst } x), t_2 (\text{snd } x)) \\ t_1 \rightarrow t_2 &= \lambda(x : A_1 \rightarrow B_1). \lambda(y : A_2). t_2 (x (t_1 y)) \end{aligned}$$

The definition of $\text{caster}(A, B)$ is based on the definition of type consistency.

Lemma 4.1 (Type Consistency and Caster). *If $\Gamma \vdash A \sim B$, then $\Gamma \vdash_{\text{CG}} \text{caster}(A, B) : A \rightarrow B$.*

$$\begin{array}{c} \frac{x : A \in \Gamma}{\Gamma \vdash x \Rightarrow x : A} \quad \frac{}{\Gamma \vdash 0 \Rightarrow 0 : \text{Nat}} \quad \frac{}{\Gamma \vdash \text{triv} \Rightarrow \text{triv} : \text{Unit}} \\[10pt] \frac{\Gamma \vdash t_1 \Rightarrow t_2 : ?}{\Gamma \vdash \text{succ } t_1 \Rightarrow \text{succ } (\text{unbox}_{\text{Nat}} t_2) : \text{Nat}} \quad \frac{\Gamma \vdash t_1 \Rightarrow t_2 : \text{Nat}}{\Gamma \vdash \text{succ } t_1 \Rightarrow \text{succ } t_2 : \text{Nat}} \\[10pt] \frac{\Gamma \vdash t \Rightarrow t' : ? \quad \Gamma \vdash A_1 \sim A \quad \Gamma \vdash A_2 \sim A \quad t'_1 = (c_1 t'_1) \quad t'_2 = (c_2 t'_2) \quad \Gamma, x : \text{Nat} \vdash t_2 \Rightarrow t'_2 : A_2 \quad \text{caster}(A_2, A) = c_2 \quad t'' = (\text{unbox}_{\text{Nat}} t')}{\Gamma \vdash (\text{case } t \text{ of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2) \Rightarrow (\text{case } t'' \text{ of } 0 \rightarrow t'_1, (\text{succ } x) \rightarrow t'_2) : A} \\[10pt] \frac{\Gamma \vdash t \Rightarrow t' : \text{Nat} \quad \Gamma \vdash t_1 \Rightarrow t'_1 : A_1 \quad \Gamma \vdash A_1 \sim A \quad \Gamma, x : \text{Nat} \vdash t_2 \Rightarrow t'_2 : A_2 \quad \Gamma \vdash A_2 \sim A}{\Gamma \vdash (\text{case } t \text{ of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2) \Rightarrow (\text{case } t' \text{ of } 0 \rightarrow t'_1, (\text{succ } x) \rightarrow t'_2) : A} \\[10pt] \frac{\Gamma \vdash t_1 \Rightarrow t_3 : A_1 \quad \Gamma \vdash t_2 \Rightarrow t_4 : A_2}{\Gamma \vdash (t_1, t_2) \Rightarrow (t_3, t_4) : A_1 \times A_2} \\[10pt] \frac{\Gamma \vdash t_1 \Rightarrow t_2 : ?}{\Gamma \vdash \text{fst } t_1 \Rightarrow \text{fst } (\text{unbox}_{(?) \times ?}) t_2 : ?} \quad \frac{\Gamma \vdash t_1 \Rightarrow t_2 : A_1 \times A_2}{\Gamma \vdash \text{fst } t_1 \Rightarrow \text{fst } t_2 : A_1} \\[10pt] \frac{\Gamma \vdash t_1 \Rightarrow t_2 : ?}{\Gamma \vdash \text{snd } t_1 \Rightarrow \text{snd } (\text{unbox}_{(?) \times ?}) t_2 : ?} \quad \frac{\Gamma \vdash t_1 \Rightarrow t_2 : A \times B}{\Gamma \vdash \text{snd } t_1 \Rightarrow \text{snd } t_2 : B} \\[10pt] \frac{\Gamma, x : A_1 \vdash t_1 \Rightarrow t_2 : A_2}{\Gamma \vdash \lambda(x : A_1). t_1 \Rightarrow \lambda(x : A_1). t_2 : A_1 \rightarrow A_2} \\[10pt] \frac{\Gamma \vdash t_1 \Rightarrow t'_1 : ? \quad \Gamma \vdash t_2 \Rightarrow t'_2 : A_2 \quad \text{caster}(A_2, ?) = c}{\Gamma \vdash t_1 t_2 \Rightarrow (\text{unbox}_{(?) \rightarrow ?}) t'_1 (c t'_2) : ?} \\[10pt] \frac{\Gamma \vdash t_2 \Rightarrow t'_2 : A_2 \quad \Gamma \vdash t_1 \Rightarrow t'_1 : A_1 \rightarrow B \quad \Gamma \vdash A_2 \sim A_1 \quad \text{caster}(A_2, A_1) = c}{\Gamma \vdash t_1 t_2 \Rightarrow t'_1 (c t'_2) : B} \end{array}$$

Figure 6. Cast Insertion Algorithm

Proof. This proof holds by induction on $\Gamma \vdash A \sim B$, but is very routine, and so we omit its proof. \square

Notice that for each typing rule that uses one of the metafunctions from Figure 4 there are two cast insertion rules corresponding to the typing rule.

The cast insertion algorithm is designed around where explicit casts need to be inserted. This is accomplished by case splitting on the input to each metafunction from Figure 4 resulting in two rules per elimination rule. The first is the case where the input to the metafunction is $?$, and the second is the case where the input to the metafunction is a type of the appropriate structure. For example, in the case of the elimination rule for function application, \rightarrow_e , from Figure 5, there are two cast insertion rules, the last two in Figure 6, where the first is when the type of the function, t_1 , is $?$, and the second when the type of t_1 is an arrow type. The former requires the type $?$ to be split into $? \rightarrow ?$ using $\text{unbox}_{(?) \rightarrow ?}$, and a casting morphism to cast the argument to the appropriate input type. The second cast insertion rule only needs to cast the argument type, because t_1 already has a function type.

The cast insertion algorithm preserves the type of the program.

Lemma 4.2 (Cast Insertion Preserves the Type). *If $\Gamma \vdash_{\text{SG}} t_1 : A$ and $\Gamma \vdash t_1 \Rightarrow t_2 : A$, then $\Gamma \vdash_{\text{CG}} t_2 : A$.*

Proof. This proof holds by induction on $\Gamma \vdash_{\text{SG}} t_1 : A$ which will determine which of the cast insertion rules need to be

considered. At that point, a case split over the input to any metafunctions from Figure 4 used in the typing rule may be necessary. We omit the proof in the interest of brevity. \square

Finally, cast insertion also plays a role when interpreting Surface Grady into the categorical model. The next section gives the details.

5 Interpreting Surface and Core Grady in the Model

Interpreting a programming language into a categorical model requires three steps. First, the types are interpreted as objects. Then programs are interpreted as morphisms in the category, but this is a simplification. Every morphism, f , in a category has a source object and a target object, we usually denote this by $f : A \longrightarrow B$. Thus, in order to interpret programs as morphisms the program must have a source and target. So instead of interpreting raw terms as morphisms we interpret terms in their typing context. That is, we must show how to interpret every $\Gamma \vdash_{\text{SG}} t : A$ as a morphism $t : \llbracket \Gamma \rrbracket \longrightarrow \llbracket A \rrbracket$. The third step is to show that whenever one program reduces to another their interpretations are isomorphic in the model. This means that whenever $\Gamma \vdash_{\text{CG}} t_1 : A$, $\Gamma \vdash_{\text{CG}} t_2 : A$, and $t_1 \rightsquigarrow t_2$, then $\llbracket t_1 \rrbracket = \llbracket t_2 \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \llbracket A \rrbracket$. The goal of this section is to prove these two facts for Surface Grady and Core Grady. This section heavily depends on Section 2, Section 3, and Section 4.

First, we must give the interpretation of types and contexts, but this interpretation is obvious, because we have been making sure to match the names of types and objects throughout this paper.

Definition 5.1. Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model. Then we define the interpretation of types into C as follows:

$$\begin{aligned} \llbracket ? \rrbracket &= ? & \llbracket A_1 \rightarrow A_2 \rrbracket &= \llbracket A_1 \rrbracket \rightarrow \llbracket A_2 \rrbracket \\ \llbracket \text{Unit} \rrbracket &= \text{Unit} & \llbracket A_1 \times A_2 \rrbracket &= \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket \\ \llbracket \text{Nat} \rrbracket &= \text{Nat} \end{aligned}$$

We extend this interpretation to typing contexts as follows:

$$\llbracket \cdot \rrbracket = 1 \quad \llbracket \Gamma, x : A \rrbracket = \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket$$

Throughout the remainder of this paper we will drop the interpretation symbols around types.

Before we can interpret the typing rules of Surface and Core Grady we must show how to interpret the consistency relation from Figure 4. These will correspond to casting morphisms (Definition 2.7).

Lemma 5.2 (Type Consistency in the Model). *Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model, and $\Gamma \vdash A \sim B$ for some types A and B . Then there are two casting morphisms $c_1 : A \longrightarrow B$ and $c_2 : B \longrightarrow A$.*

Proof. This proof holds by induction on the form $\Gamma \vdash A \sim B$ using the morphisms $\text{Box}_A : A \longrightarrow ?$ and $\text{Unbox}_A : ? \longrightarrow A$ from Lemma 2.14. Please see Appendix A.3 for the complete proof. \square

$\llbracket \Gamma_1, x : A_i, \Gamma_2 \vdash_{\text{SG}} x : A_i \rrbracket = \pi_i$
$\llbracket \Gamma \vdash_{\text{SG}} \text{triv} : \text{Unit} \rrbracket = \text{triv}_{\llbracket \Gamma \rrbracket}$
$\llbracket \Gamma \vdash_{\text{SG}} 0 : \text{Nat} \rrbracket = \text{triv}_{\llbracket \Gamma \rrbracket}; z$
$\llbracket \Gamma \vdash_{\text{SG}} \text{succ } t : \text{Nat} \rrbracket = \llbracket t \rrbracket; c; \text{succ}$ where $\Gamma \vdash_{\text{SG}} t : A$ and $c : A \longrightarrow \text{Nat}$
$\llbracket \Gamma \vdash_{\text{SG}} \text{case } t \text{ of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2 : A \rrbracket$ $= \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket; c_1 \rangle; \text{case}_{\llbracket \Gamma \rrbracket, \llbracket A \rrbracket} \langle \llbracket t_1 \rrbracket; c_2, \llbracket t_2 \rrbracket; c_3 \rangle$ where $\Gamma \vdash_{\text{SG}} t : C$, $\Gamma \vdash_{\text{SG}} t_1 : A_1$, $\Gamma \vdash_{\text{SG}} t_2 : A_2$, $c_1 : C \longrightarrow \text{Nat}$, $c_2 : A_1 \longrightarrow A$, and $c_3 : A_2 \longrightarrow A$
$\llbracket \Gamma \vdash_{\text{SG}} (t_1, t_2) : A_1 \times A_2 \rrbracket = \langle \llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket \rangle$ where $\Gamma \vdash_{\text{SG}} t_1 : A_1$ and $\Gamma \vdash_{\text{SG}} t_2 : A_2$
$\llbracket \Gamma \vdash_{\text{SG}} \text{fst } t : A_1 \rrbracket = \llbracket t \rrbracket; c; \pi_1$ where $\Gamma \vdash_{\text{SG}} t : B$ and $c : B \longrightarrow (A_1 \times A_2)$
$\llbracket \Gamma \vdash_{\text{SG}} \text{snd } t : A_2 \rrbracket = \llbracket t \rrbracket; c; \pi_2$ where $\Gamma \vdash_{\text{SG}} t : B$ and $c : B \longrightarrow (A_1 \times A_2)$
$\llbracket \Gamma \vdash_{\text{SG}} \lambda(x : A). t : A \rightarrow B \rrbracket = \text{curry}(\llbracket t \rrbracket)$ where $\Gamma, x : A \vdash_{\text{SG}} t : B$
$\llbracket \Gamma \vdash_{\text{SG}} t_1 t_2 : B_1 \rrbracket = \langle \llbracket t_1 \rrbracket; c_1, \llbracket t_2 \rrbracket; c_2 \rangle; \text{app}_{A_1, B_1}$ where $\Gamma \vdash_{\text{SG}} t_1 : C$, $\Gamma \vdash_{\text{SG}} t_2 : A_2$, $c_1 : C \longrightarrow (A_1 \rightarrow B_1)$, and $c_2 : A_2 \longrightarrow A_1$
$\llbracket \Gamma \vdash_{\text{CG}} \text{box}_A : A \rightarrow ? \rrbracket = \text{triv}_{\llbracket \Gamma \rrbracket}; \text{curry}(\text{Box}_A)$
$\llbracket \Gamma \vdash_{\text{CG}} \text{unbox}_A : ? \rightarrow A \rrbracket = \text{triv}_{\llbracket \Gamma \rrbracket}; \text{curry}(\text{Unbox}_A)$

Figure 7. Interpretation of Terms

Showing that both c_1 and c_2 exist corresponds to the fact that $\Gamma \vdash A \sim B$ is symmetric.

At this point we have everything we need to show our main result which is that typing in both Surface and Core Grady, and evaluation in Core Grady can be interpreted into the categorical model. The interpretation of terms used in the following proofs is summarized in Figure 7. We only summarize the interpretation of the Surface Grady programs and just box and unbox in Core Grady, because the interpretation of Core Grady is equivalent to the interpretation of Surface Grady where all casting morphisms have been replaced with the identity morphism.

Theorem 5.3 (Interpretation of Typing). *Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model. If $\Gamma \vdash_{\text{SG}} t : A$ or $\Gamma \vdash_{\text{CG}} t : A$, then there is a morphism $\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow A$ in C .*

Proof. Both parts of the proof hold by induction on the form of the assumed typing derivation, and uses most of the results we have developed up to this point. Please see Appendix A.4 for the complete proof. \square

Theorem 5.4 (Interpretation of Evaluation). *Suppose $(\mathcal{T}, C, ?, T, \text{split}, \text{squash}, \text{box}, \text{unbox}, \text{error})$ is a gradual λ -model. If $\Gamma \vdash_{\text{CG}} t_1 : A$, $\Gamma \vdash_{\text{CG}} t_2 : A$, and $t_1 \rightsquigarrow t_2$, then $\llbracket t_1 \rrbracket = \llbracket t_2 \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow A$.*

Proof. This proof holds by induction on the form of $t_1 \rightsquigarrow t_2$, and uses Theorem 5.3, Lemma 5.2, and Corollary A.1. Please see Appendix A.5 for the complete proof. \square

One can see a direct connection between the proof of interpretation of typing (Theorem 5.3) and the cast insertion

Type Precision:	
$\frac{}{A \sqsubseteq ?} ?$	$\frac{}{A \sqsubseteq A} \text{refl}$
$\frac{A \sqsubseteq C \quad B \sqsubseteq D}{(A \rightarrow B) \sqsubseteq (C \rightarrow D)} \rightarrow$	
$\frac{A \sqsubseteq C \quad B \sqsubseteq D}{(A \times B) \sqsubseteq (C \times D)} \times$	
Context Precision:	
$\frac{}{\Gamma \sqsubseteq \Gamma} \text{refl}$	$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad A \sqsubseteq A' \quad \Gamma_3 \sqsubseteq \Gamma_4}{\Gamma_1, x : A, \Gamma_3 \sqsubseteq \Gamma_2, x : A', \Gamma_4} \text{ext}$

Figure 8. Type and Context Precision

Term Precision for Surface Grady:	
$\frac{}{t \sqsubseteq t} \text{refl}$	$\frac{t_1 \sqsubseteq t_2}{(\text{succ } t_1) \sqsubseteq (\text{succ } t_2)} \text{succ}$
$\frac{t_1 \sqsubseteq t_4 \quad t_2 \sqsubseteq t_5 \quad t_3 \sqsubseteq t_6}{(\text{case } t_1 \text{ of } 0 \rightarrow t_2, (\text{succ } x) \rightarrow t_3) \sqsubseteq (\text{case } t_4 \text{ of } 0 \rightarrow t_5, (\text{succ } x) \rightarrow t_6)} \text{Nat}$	
$\frac{t_1 \sqsubseteq t_3 \quad t_2 \sqsubseteq t_4}{(t_1, t_2) \sqsubseteq (t_3, t_4)} \times_i$	$\frac{t_1 \sqsubseteq t_2}{(\text{fst } t_1) \sqsubseteq (\text{fst } t_2)} \times_{e_1}$
$\frac{t_1 \sqsubseteq t_2}{(\text{snd } t_1) \sqsubseteq (\text{snd } t_2)} \times_{e_2}$	$\frac{t_1 \sqsubseteq t_2 \quad A_1 \sqsubseteq A_2}{(\lambda(x : A_1).t) \sqsubseteq (\lambda(x : A_2).t_2)} \rightarrow_i$
$\frac{t_1 \sqsubseteq t_3 \quad t_2 \sqsubseteq t_4}{(t_1 t_2) \sqsubseteq (t_3 t_4)} \rightarrow_2$	
Term Precision for Core Grady:	
$\frac{\Gamma \vdash_{\text{CG}} t : ?}{\Gamma \vdash (\text{unbox}_A t) \sqsubseteq t} \text{box}$	$\frac{\Gamma \vdash_{\text{CG}} t : A}{\Gamma \vdash t \sqsubseteq (\text{box}_A t)} \text{unbox}$
$\frac{\Gamma \vdash_{\text{CG}} t : ?}{\Gamma \vdash (\text{split}_S t) \sqsubseteq t} \text{split}$	$\frac{\Gamma \vdash_{\text{CG}} t : S}{\Gamma \vdash t \sqsubseteq (\text{squash}_S t)} \text{squash}$
$\frac{\Gamma \vdash_{\text{CG}} t : B \quad A \sqsubseteq B}{\Gamma \vdash \text{error}_A \sqsubseteq t} \text{error}$	

Figure 9. Term Precision

algorithm (Figure 6). During the proof – summarized in Figure 7 – we construct casting morphisms from type consistency which is essentially the semantic equivalent to $\text{caster}(A, B)$.

6 The Gradual Guarantee

7 Bounded Quantification

$\text{list}(?) = \text{List } ?$

$\text{list}(\text{List } A) = \text{List } A$

8 Related Work

9 Future Work

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Subtyping for Core Grady:	
$\frac{}{\Gamma \vdash A < A} \text{refl}$	$\frac{}{\Gamma \vdash A < \top} \text{top}$
$\frac{X < A \in \Gamma}{\Gamma \vdash X < A} \text{var}$	
$\frac{}{\Gamma \vdash \top < S} \top_S$	$\frac{}{\Gamma \vdash \text{Nat} < S} \text{Nat}_S$
$\frac{}{\Gamma \vdash \text{Unit} < S} \text{Unit}_S$	
$\frac{\Gamma \vdash A < S}{\Gamma \vdash \text{List } A < S} \text{List}_S$	$\frac{\Gamma \vdash A < S \quad \Gamma \vdash B < S}{\Gamma \vdash A \rightarrow B < S} \rightarrow_S$
$\frac{\Gamma \vdash A < S \quad \Gamma \vdash B < S}{\Gamma \vdash A \times B < S} \times_S$	$\frac{\Gamma \vdash A < B}{\Gamma \vdash \text{List } A < \text{List } B} \text{List}$
$\frac{\Gamma \vdash A_1 < A_2 \quad \Gamma \vdash B_1 < B_2}{\Gamma \vdash A_1 \times B_1 < A_2 \times B_2} \times$	$\frac{\Gamma \vdash A_1 < A_2 \quad \Gamma \vdash B_1 < B_2}{\Gamma \vdash A_1 \rightarrow B_1 < A_2 \rightarrow B_2} \rightarrow$
$\frac{\Gamma, X < A \vdash B_1 < B_2}{\Gamma \vdash \forall(X < A).B_1 < \forall(X < A).B_2} \forall$	
Consistent Subtyping for Surface Grady:	
$\frac{}{\Gamma \vdash A \lesssim A} \text{refl}$	$\frac{}{\Gamma \vdash A \lesssim \top} \text{top}$
$\frac{X < A' \in \Gamma \quad \Gamma \vdash A' \sim A}{\Gamma \vdash X \lesssim A} \text{var}$	
$\frac{\Gamma \vdash A \lesssim S}{\Gamma \vdash A \lesssim ?} \text{box}$	$\frac{\Gamma \vdash A \lesssim S}{\Gamma \vdash ? \lesssim A} \text{unbox}$
$\frac{}{\Gamma \vdash ? \lesssim S} ?_S$	
$\frac{}{\Gamma \vdash \top \lesssim S} \top_S$	$\frac{}{\Gamma \vdash \text{Nat} \lesssim S} \text{Nat}_S$
$\frac{}{\Gamma \vdash \text{Unit} \lesssim S} \text{Unit}_S$	
$\frac{\Gamma \vdash A \lesssim S}{\Gamma \vdash \text{List } A \lesssim S} \text{List}_S$	$\frac{\Gamma \vdash A \lesssim S \quad \Gamma \vdash B \lesssim S}{\Gamma \vdash A \times B \lesssim S} \times_S$
$\frac{\Gamma \vdash A \lesssim S \quad \Gamma \vdash B \lesssim S}{\Gamma \vdash A \rightarrow B \lesssim S} \rightarrow_S$	$\frac{\Gamma \vdash A \lesssim B}{\Gamma \vdash (\text{List } A) \lesssim (\text{List } B)} \text{List}$
$\frac{\Gamma \vdash A_1 \lesssim A_2 \quad \Gamma \vdash B_1 \lesssim B_2}{\Gamma \vdash (A_1 \times B_1) \lesssim (A_2 \times B_2)} \times$	$\frac{\Gamma \vdash A_2 \lesssim A_1 \quad \Gamma \vdash B_1 \lesssim B_2}{\Gamma \vdash (A_1 \rightarrow B_1) \lesssim (A_2 \rightarrow B_2)} \rightarrow$
$\frac{\Gamma, X < A \vdash B_1 \lesssim B_2}{\Gamma \vdash (\forall(X < A).B_1) \lesssim (\forall(X < A).B_2)} \forall$	

Figure 10. Subtyping for Core and Surface Grady

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A Proofs

A.1 Proof of Lifted Retract (Lemma 2.10)

We only show that $\widehat{\text{box}}_A; \widehat{\text{unbox}}_A = \text{id}_A$, because the case when a dynamic type error is raised is similar using the fact that A and B must have the same skeleton or one could not compose $\widehat{\text{box}}_A$ and $\widehat{\text{unbox}}_B$. This implies that A and B only differ at an atomic type.

This is a proof by induction on the form of A .

Case. Suppose A is atomic. Then:

$$\widehat{\text{box}}_A; \widehat{\text{unbox}}_A = \text{box}_A; \text{unbox}_A = \text{id}_A$$

Case. Suppose A is $?$. Then:

$$\begin{aligned} \widehat{\text{box}}_A; \widehat{\text{unbox}}_A &= \widehat{\text{box}}_?; \widehat{\text{unbox}}_? \\ &= \text{id}_?; \text{id}_? \\ &= \text{id}_? \\ &= \text{id}_A \end{aligned}$$

Case. Suppose $A = A_1 \rightarrow A_2$. Then:

$$\begin{aligned} \widehat{\text{box}}_A; \widehat{\text{unbox}}_A &= \widehat{\text{box}}_{(A_1 \rightarrow A_2)}; \widehat{\text{unbox}}_{(A_1 \rightarrow A_2)} \\ &= (\widehat{\text{unbox}}_{A_1} \rightarrow \widehat{\text{box}}_{A_2}); (\widehat{\text{box}}_{A_1} \rightarrow \widehat{\text{box}}_{A_2}) \\ &= (\widehat{\text{box}}_{A_1}; \widehat{\text{unbox}}_{A_1}) \rightarrow (\widehat{\text{box}}_{A_2}; \widehat{\text{unbox}}_{A_2}) \end{aligned}$$

By two applications of the induction hypothesis we know the following:

$$\widehat{\text{box}}_{A_1}; \widehat{\text{unbox}}_{A_1} = \text{id}_{A_1} \quad \text{and} \quad \widehat{\text{box}}_{A_2}; \widehat{\text{unbox}}_{A_2} = \text{id}_{A_2}$$

Thus, we know the following:

$$\begin{aligned} (\widehat{\text{box}}_{A_1}; \widehat{\text{unbox}}_{A_1}) \rightarrow (\widehat{\text{box}}_{A_2}; \widehat{\text{unbox}}_{A_2}) &= \text{id}_{A_1} \rightarrow \text{id}_{A_2} \\ &= \text{id}_{A_1 \rightarrow A_2} \\ &= \text{id}_A \end{aligned}$$

Case. Suppose $A = A_1 \times A_2$. Then:

$$\begin{aligned} \widehat{\text{box}}_A; \widehat{\text{unbox}}_A &= \widehat{\text{box}}_{(A_1 \times A_2)}; \widehat{\text{unbox}}_{(A_1 \times A_2)} \\ &= (\widehat{\text{box}}_{A_1} \times \widehat{\text{box}}_{A_2}); (\widehat{\text{unbox}}_{A_1} \times \widehat{\text{unbox}}_{A_2}) \\ &= (\widehat{\text{box}}_{A_1}; \widehat{\text{unbox}}_{A_1}) \times (\widehat{\text{box}}_{A_2}; \widehat{\text{unbox}}_{A_2}) \end{aligned}$$

By two applications of the induction hypothesis we know the following:

$$\widehat{\text{box}}_{A_1}; \widehat{\text{unbox}}_{A_1} = \text{id}_{A_1} \quad \text{and} \quad \widehat{\text{box}}_{A_2}; \widehat{\text{unbox}}_{A_2} = \text{id}_{A_2}$$

Thus, we know the following:

$$\begin{aligned} (\widehat{\text{box}}_{A_1}; \widehat{\text{unbox}}_{A_1}) \times (\widehat{\text{box}}_{A_2}; \widehat{\text{unbox}}_{A_2}) &= \text{id}_{A_1} \times \text{id}_{A_2} \\ &= \text{id}_{A_1 \times A_2} \\ &= \text{id}_A \end{aligned}$$

A.2 Proof of Lemma 2.11

We must show that the function

$$S_{A,B} : \text{Hom}_C(A, B) \longrightarrow \text{Hom}_S(SA, SB)$$

is injective.

So suppose $f \in \text{Hom}_C(A, B)$ and $g \in \text{Hom}_C(A, B)$ such that $Sf = Sg : SA \longrightarrow SB$. Then we can easily see that:

$$\begin{aligned} Sf &= \widehat{\text{unbox}}_A; f; \widehat{\text{box}}_B \\ &= \text{unbox}_A; g; \text{box}_B \\ &= Sg \end{aligned}$$

But, we have the following equalities:

$$\begin{aligned} \widehat{\text{unbox}}_A; f; \widehat{\text{box}}_B &= \widehat{\text{unbox}}_A; g; \widehat{\text{box}}_B \\ \widehat{\text{box}}_A; \widehat{\text{unbox}}_A; f; \widehat{\text{box}}_B; \widehat{\text{unbox}}_B &= \widehat{\text{box}}_A; \widehat{\text{unbox}}_A; g; \widehat{\text{box}}_B; \widehat{\text{unbox}}_B \\ \text{id}_A; f; \widehat{\text{box}}_B; \widehat{\text{unbox}}_B &= \text{id}_A; g; \widehat{\text{box}}_B; \widehat{\text{unbox}}_B \\ \text{id}_A; f; \text{id}_B &= \text{id}_A; g; \text{id}_B \\ f &= g \end{aligned}$$

The previous equalities hold due to Lemma 2.10.

A.3 Proof of Type Consistency in the Model (Lemma 5.2)

This is a proof by induction on the form of $\Gamma \vdash A \sim B$.

$$\text{Case: } \frac{}{\Gamma \vdash A \sim A} \text{ refl}$$

Choose $c_1 = c_2 = \text{id}_A : A \longrightarrow A$.

$$\text{Case: } \frac{}{\Gamma \vdash A \sim ?} \text{ box}$$

Choose $c_1 = \text{Box}_A : A \longrightarrow ?$ and $c_2 = \text{Unbox}_A : ? \rightarrow A$.

$$\text{Case: } \frac{}{\Gamma \vdash ? \sim A} \text{ unbox}$$

Choose $c_1 = \text{Unbox}_A : ? \longrightarrow A$ and $c_2 = \text{Box}_A : A \rightarrow ?$.

$$\text{Case: } \frac{\Gamma \vdash A_2 \sim A_1 \quad \Gamma \vdash B_1 \sim B_2}{\Gamma \vdash (A_1 \rightarrow B_1) \sim (A_2 \rightarrow B_2)} \rightarrow$$

By the induction hypothesis there exists four casting morphisms $c'_1 : A_1 \longrightarrow A_2$, $c'_2 : A_2 \longrightarrow A_1$, $c'_3 : B_1 \longrightarrow B_2$, and $c'_4 : B_2 \longrightarrow B_1$. Choose $c_1 = c'_2 \rightarrow c'_3 : (A_1 \rightarrow B_1) \longrightarrow (A_2 \rightarrow B_2)$ and $c_2 = c'_1 \rightarrow c'_4 : (A_2 \rightarrow B_2) \longrightarrow (A_1 \rightarrow B_1)$.

$$\text{Case: } \frac{\Gamma \vdash A_1 \sim A_2 \quad \Gamma \vdash B_1 \sim B_2}{\Gamma \vdash (A_1 \times B_1) \sim (A_2 \times B_2)} \times$$

By the induction hypothesis there exists four casting morphisms $c'_1 : A_1 \longrightarrow A_2$, $c'_2 : A_2 \longrightarrow A_1$, $c'_3 : B_1 \longrightarrow B_2$, and $c'_4 : B_2 \longrightarrow B_1$. Choose $c_1 = c'_1 \times c'_3 : A_1 \times B_1 \longrightarrow A_2 \times B_2$ and $c_2 = c'_2 \times c'_4 : A_2 \times B_2 \longrightarrow A_1 \times B_1$.

A.4 Proof of Interpretation of Types Theorem 5.3

First, we show how to interpret the rules of Surface Grady and then Core Grady. This is a proof by induction on $\Gamma \vdash_{\text{SG}} t : A$.

$$\text{Case: } \frac{x : A \in \Gamma}{\Gamma \vdash_{\text{SG}} x : A} \text{ var}$$

Suppose without loss of generality that $\llbracket \Gamma \rrbracket = A_1 \times \cdots \times A_i \times \cdots \times A_j$ where $A_i = A$. We know that $j > 0$ or the assumed typing derivation would not hold. Then take $\llbracket x \rrbracket = \pi_i : \llbracket \Gamma \rrbracket \longrightarrow A$.

$$\text{Case: } \frac{}{\Gamma \vdash_{\text{SG}} \text{triv} : \text{Unit}} \text{ Unit}$$

Take $\llbracket \text{triv} \rrbracket = \text{triv}_{\llbracket \Gamma \rrbracket} : \llbracket \Gamma \rrbracket \longrightarrow \text{Unit}$ where $\text{triv}_{\llbracket \Gamma \rrbracket}$ is the unique terminal arrow that exists because C is cartesian closed.

Case: $\frac{}{\Gamma \vdash_{\text{SG}} 0 : \text{Nat}}$ zero

Take $\llbracket 0 \rrbracket = \text{triv}_{\llbracket \Gamma \rrbracket} : z : \llbracket \Gamma \rrbracket \longrightarrow \text{Nat}$ where $z : \text{Unit} \longrightarrow \text{Nat}$ exists because C has an NRNO.

Case: $\frac{\Gamma \vdash_{\text{SG}} t : A \quad \text{nat}(A) = \text{Nat}}{\Gamma \vdash_{\text{SG}} \text{succ } t : \text{Nat}}$ succ

First, by the induction hypothesis there is a morphism $\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow A$. Now we have two cases to consider, one when $A = ?$ and one when $A = \text{Nat}$. Consider the former. Then interpret $\llbracket \text{succ } t \rrbracket = \llbracket t \rrbracket ; \text{unbox}_{\text{Nat}} ; \text{succ} : \llbracket \Gamma \rrbracket \longrightarrow \text{Nat}$ where $\text{succ} : \text{Nat} \longrightarrow \text{Nat}$ exists because C has an NRNO. Similarly, when $A = \text{Nat}$, $\llbracket \text{succ } t \rrbracket = \llbracket t \rrbracket ; \text{succ} : \llbracket \Gamma \rrbracket \longrightarrow \text{Nat}$.

Case: $\frac{\Gamma \vdash_{\text{SG}} t : C \quad \text{nat}(C) = \text{Nat} \quad \Gamma \vdash_{\text{SG}} t_1 : A_1 \quad \Gamma \vdash A_1 \sim A \quad \Gamma, x : \text{Nat} \vdash_{\text{SG}} t_2 : A_2 \quad \Gamma \vdash A_2 \sim A}{\Gamma \vdash_{\text{SG}} \text{case } t \text{ of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2 : A}$ Nat_e

By three applications of the induction hypothesis we have the following morphisms:

$$\begin{aligned} \llbracket t \rrbracket &: \llbracket \Gamma \rrbracket \longrightarrow C \\ \llbracket t_1 \rrbracket &: \llbracket \Gamma \rrbracket \longrightarrow A_1 \\ \llbracket t_2 \rrbracket &: \llbracket \Gamma \rrbracket \times \text{Nat} \longrightarrow A_2 \end{aligned}$$

In addition, we know $\Gamma \vdash A_1 \sim A$ and $\Gamma \vdash A_2 \sim A$ by assumption, and hence, by type consistency in the model (Lemma 5.2) we know there are casting morphisms $c_1 : A_1 \longrightarrow A$ and $c_2 : A_2 \longrightarrow A$. Now every gradual λ -model has an NRNO (Definition 2.4, Definition 2.5), and so, there is a unique morphism:

$$\text{case}_{\llbracket \Gamma \rrbracket, A} \langle \llbracket t_1 \rrbracket ; c_1, \llbracket t_2 \rrbracket ; c_2 \rangle : \llbracket \Gamma \rrbracket \times \text{Nat} \longrightarrow A$$

At this point we have two cases to consider: one when $C = ?$ and one when $C = \text{Nat}$. Consider the former. Then we have the following:

$$\begin{aligned} &\llbracket \text{case } t \text{ of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2 \rrbracket \\ &= \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket ; \text{unbox}_{\text{Nat}} \rangle ; \text{case}_{\llbracket \Gamma \rrbracket, A} \langle \llbracket t_1 \rrbracket ; c_1, \llbracket t_2 \rrbracket ; c_2 \rangle \\ &: \llbracket \Gamma \rrbracket \longrightarrow A \end{aligned}$$

In the second case we have the following:

$$\begin{aligned} &\llbracket \text{case } t \text{ of } 0 \rightarrow t_1, (\text{succ } x) \rightarrow t_2 \rrbracket \\ &= \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket t \rrbracket \rangle ; \text{case}_{\llbracket \Gamma \rrbracket, A} \langle \llbracket t_1 \rrbracket ; c_1, \llbracket t_2 \rrbracket ; c_2 \rangle \\ &: \llbracket \Gamma \rrbracket \longrightarrow A \end{aligned}$$

Case: $\frac{\Gamma \vdash_{\text{SG}} t_1 : A_1 \quad \Gamma \vdash_{\text{SG}} t_2 : A_2}{\Gamma \vdash_{\text{SG}} (t_1, t_2) : A_1 \times A_2}$ \times_i

By two applications of the induction hypothesis there are two morphisms $\llbracket t_1 \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow A_1$ and $\llbracket t_2 \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow A_2$. Then using the fact that C is cartesian we take $\llbracket (t_1, t_2) \rrbracket = \langle \llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket \rangle : \llbracket \Gamma \rrbracket \longrightarrow A_1 \times A_2$.

Case: $\frac{\Gamma \vdash_{\text{SG}} t : B \quad \text{prod}(B) = A_1 \times A_2}{\Gamma \vdash_{\text{SG}} \text{fst } t : A_1}$ \times_{e_1}

First, by the induction hypothesis there is a morphism $\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow B$. Now we have two cases to consider, one when $B = ?$ and one when $B = A_1 \times A_2$ for some types A_1 and A_2 . Consider the former. We then know that it must be the case that $A_1 \times A_2 = ? \times ?$. Thus, we obtain the following interpretation $\llbracket \text{fst } t \rrbracket = \llbracket t \rrbracket ; \text{unbox}_{(? \times ?)} ; \pi_1 : \llbracket \Gamma \rrbracket \longrightarrow ?$. Similarly, when $B = A_1 \times A_2$, then $\llbracket \text{fst } t \rrbracket = \llbracket t \rrbracket ; \pi_1 : \llbracket \Gamma \rrbracket \longrightarrow A_1$.

Case: $\frac{\Gamma \vdash_{\text{SG}} t : B \quad \text{prod}(B) = A_1 \times A_2}{\Gamma \vdash_{\text{SG}} \text{snd } t : A_2}$ \times_{e_2}

First, by the induction hypothesis there is a morphism $\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow B$. Now we have two cases to consider, one when $B = ?$ and one when $B = A_1 \times A_2$ for some types A_1 and A_2 . Consider the former. We then know that it must be the case that $A_1 \times A_2 = ? \times ?$. Thus, we obtain the following interpretation $\llbracket \text{snd } t \rrbracket = \llbracket t \rrbracket ; \text{unbox}_{(? \times ?)} ; \pi_2 : \llbracket \Gamma \rrbracket \longrightarrow ?$. Similarly, when $B = A_1 \times A_2$, then $\llbracket \text{snd } t \rrbracket = \llbracket t \rrbracket ; \pi_2 : \llbracket \Gamma \rrbracket \longrightarrow A_2$.

Case: $\frac{\Gamma, x : A \vdash_{\text{SG}} t : B}{\Gamma \vdash_{\text{SG}} \lambda(x : A). t : A \rightarrow B}$ \rightarrow_i

By the induction hypothesis there is a morphism $\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \times A \longrightarrow B$. Then take $\llbracket \lambda(x : A). t \rrbracket = \text{curry}(\llbracket t \rrbracket) : \llbracket \Gamma \rrbracket \longrightarrow (A \rightarrow B)$, where

$$\text{curry} : \text{Hom}_C(X \times Y, Z) \longrightarrow \text{Hom}_C(X, Y \rightarrow Z)$$

exists because C is closed.

Case: $\frac{\Gamma \vdash_{\text{SG}} t_1 : C \quad \Gamma \vdash A_2 \sim A_1 \quad \Gamma \vdash_{\text{SG}} t_2 : A_2 \quad \text{fun}(C) = A_1 \rightarrow B_1}{\Gamma \vdash_{\text{SG}} t_1 t_2 : B_1}$ \rightarrow_e

By the induction hypothesis there are two morphisms $\llbracket t_1 \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow C$ and $\llbracket t_2 \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow A_2$. In addition, by assumption we know that $\Gamma \vdash A_2 \sim A_1$, and hence, by type consistency in the model (Lemma 5.2) there are casting morphisms $c_1 : A_2 \longrightarrow A_1$ and $c_2 : A_1 \longrightarrow A_2$. We have two cases to consider, one when $C = ?$ and one when $C = A_1 \rightarrow B_1$. Consider the former. Then we have the interpretation:

$$\llbracket t_1 t_2 \rrbracket = \langle \llbracket t_1 \rrbracket ; \text{unbox}_{(? \rightarrow ?)}, \llbracket t_2 \rrbracket ; c_1 \rangle ; \text{app}_{A_1, B_1} : \llbracket \Gamma \rrbracket \longrightarrow B_1$$

Similarly, for the case when $C = A_1 \rightarrow B_1$ we have the interpretation:

$$\llbracket t_1 t_2 \rrbracket = \langle \llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket ; c_1 \rangle ; \text{app}_{A_1, B_1} : \llbracket \Gamma \rrbracket \longrightarrow B_1$$

Note that $\text{app}_{A_1, B_1} : (A_1 \rightarrow B_1) \times A_1 \longrightarrow B_1$ exists because the model is cartesian closed.

Next we turn to Core Grady, but we do not show every rule, because it is similar to what we have already shown above

except without casting morphism, and so we only show the case for the box and unbox rules, and the error rule.

The first two cases use the well-known bijection:

$$\begin{aligned}\text{Hom}_C(A, B) &\cong \text{Hom}_C(\text{Unit} \times A, B) \\ &\cong \text{Hom}_C(\text{Unit}, A \rightarrow B)\end{aligned}$$

When $f \in \text{Hom}_C(A, B)$, then we denote by $\text{curry}(f)$, the morphism $\text{curry}(f) \in \text{Hom}_C(\text{Unit}, A \rightarrow B)$.

Case: $\frac{}{\Gamma \vdash_{\text{CG}} \text{box}_A : A \rightarrow ?} \text{box}$

We have the following interpretation:

$$\llbracket \text{box}_A \rrbracket = \text{triv}_{\llbracket \Gamma \rrbracket}; \text{curry}(\text{Box}_A) : \llbracket \Gamma \rrbracket \longrightarrow (A \rightarrow ?)$$

Case: $\frac{}{\Gamma \vdash_{\text{CG}} \text{unbox}_A : ? \rightarrow A} \text{unbox}$

We have the following interpretation:

$$\begin{aligned}\llbracket \text{unbox}_A \rrbracket &= \text{triv}_{\llbracket \Gamma \rrbracket}; \text{curry}(\text{Unbox}_A) \\ &: \llbracket \Gamma \rrbracket \longrightarrow (? \rightarrow A)\end{aligned}$$

Case: $\frac{}{\Gamma \vdash_{\text{CG}} \text{error}_A : A} \text{error}$

We have the following interpretation:

$$\llbracket \text{error}_A \rrbracket = \text{error}_{\llbracket \Gamma \rrbracket, A} : \llbracket \Gamma \rrbracket \longrightarrow A$$

A.5 Proof of Interpretation of Evaluation (Theorem 5.4)

This proof requires the following corollary to Lemma 5.2.

Corollary A.1. *Suppose $(\mathcal{T}, C, ?, \top, \text{split}, \text{squash}, \text{box}, \text{unbox})$ is a gradual λ -model. Then we know the following:*

- i. *If $\Gamma \vdash A \sim A$, then $c_1 = c_2 = \text{id}_A : A \longrightarrow A$.*
- ii. *If $\Gamma \vdash A \sim ?$, then there are casting morphisms:*

$$\begin{aligned}c_1 &= \text{Box } A : A \longrightarrow ? \\ c_2 &= \text{Unbox } A : ? \longrightarrow A\end{aligned}$$

- iii. *If $\Gamma \vdash ? \sim A$, then there are casting morphisms:*

$$\begin{aligned}c_1 &= \text{Unbox } A : ? \longrightarrow A \\ c_2 &= \text{Box } A : A \longrightarrow ?\end{aligned}$$

- iv. *If $\Gamma \vdash A_1 \rightarrow B_1 \sim A_2 \rightarrow B_2$, then there are casting morphisms:*

$$\begin{aligned}c &= c_1 \rightarrow c_2 : (A_1 \rightarrow B_1) \longrightarrow (A_2 \rightarrow B_2) \\ c' &= c_3 \rightarrow c_4 : (A_2 \rightarrow B_2) \longrightarrow (A_1 \rightarrow B_1)\end{aligned}$$

where $c_1 : A_2 \longrightarrow A_1$ and $c_2 : B_1 \longrightarrow B_2$, and $c_3 : A_1 \longrightarrow A_2$ and $c_4 : B_2 \longrightarrow B_1$.

- v. *If $\Gamma \vdash A_1 \times B_1 \sim A_2 \times B_2$, then there are casting morphisms:*

$$\begin{aligned}c &= c_1 \times c_2 : (A_1 \times B_1) \longrightarrow (A_2 \times B_2) \\ c' &= c_3 \times c_4 : (A_2 \times B_2) \longrightarrow (A_1 \times B_1)\end{aligned}$$

where $c_1 : A_1 \longrightarrow A_2$ and $c_2 : B_1 \longrightarrow B_2$, and $c_3 : A_2 \longrightarrow A_1$ and $c_4 : B_2 \longrightarrow B_1$.

Proof. This proof holds by the construction of the casting morphisms from the proof of the previous result, and the fact that the type consistency rules are unique for each type. \square

This proof holds by induction on the form of $\Gamma \vdash_{\text{SG}} t_1 : A$ with a case split over $t_1 \rightsquigarrow t_2$. We only show the cases for the retract rules, and the error rule, because the others are well-known to hold within any cartesian closed category; see [8] or [3]. We will routinely use the interpretation given in the proof of Theorem 5.3 and summarized in Figure 7 throughout this proof without mention.