Gradual Typing from a Categorical Perspective

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References

A The Complete Spec of Grady

termvar, x
index, k

t	::=			term
	|	x		variable
	|	triv		unit
	|	$\lambda x : T.t$		λ-abstraction
	|	$t_1\ t_2$		function application
	|	(t_1, t_2)		pair constructor
	|	$\mathsf{fst}\ t$		first projection
	|	$\mathsf{snd}\ t$		second projection
	|	$\mathsf{succ}\ t$		successor function
	|	0		zero
	|	(t)	S	
	|	$[t_1/x]t_2$	M	
	|	t	M	
T	::=			type
	|	1		unit type
	|	$\mathbb{N}$		natural number type
	|	$?$		untyped universe
	|	$T_1 \to T_2$		function type
	|	$T_1 \times T_2$		cartesian product type
	|	(T)	S	
Γ	::=			typing context
	|	$\cdot$		empty context
	|	$\Gamma, x : T$		cons
	|	Γ, Γ'	M	append

terminals	::=		
	|	$\rightarrow$	
	|	$\vdash$	
	|	1	
	|	$\mathbb{N}$	
	|	succ	
	|	$?$	
	|	$\cdot$	
	|	$\sim$	
	|	$\sim_U$	
	|	$\in$	
	|	$\leadsto$	
	|	triv	
formula	::=		
	|	*judgement*	
	|	$x : T \in \Gamma$	
	|	$formula_1 \quad formula_2$	
	|	**not** *formula*	
	|	*formula*	S
ConvType	::=		
	|	$T_1 \sim_U T_2$	T_1 can can be converted into T_2
	|	$T_1 \sim T_2$	T_1 is consistent with T_2
Typing	::=		
	|	$\Gamma \vdash t : T$	t has type T in context Γ
Reduction	::=		
	|	$t_1 \leadsto t_2$	t_1 reduces to t_2
judgement	::=		
	|	*ConvType*	
	|	*Typing*	
	|	*Reduction*	
user_syntax	::=		
	|	*termvar*	
	|	*index*	
	|	*t*	

| T
| Γ
| *terminals*
| *formula*

$\boxed{T_1 \sim_U T_2}$ T_1 can can be converted into T_2

$$\frac{}{T \sim_U T} \quad \text{CV\_REFL}$$

$$\frac{T_1 \sim_U T_2 \quad T_2 \sim_U T_3}{T_1 \sim_U T_3} \quad \text{CV\_TRANS}$$

$$\frac{}{(? \to ?) \sim_U ?} \quad \text{CV\_INJ}$$

$$\frac{}{? \sim_U (? \to ?)} \quad \text{CV\_SURJ}$$

$$\frac{T_1 \sim_U T_1'}{(T_1 \times T_2) \sim_U (T_1' \times T_2)} \quad \text{CV\_PAIR1}$$

$$\frac{T_2 \sim_U T_2'}{(T_1 \times T_2) \sim_U (T_1 \times T_2')} \quad \text{CV\_PAIR2}$$

$$\frac{T_1 \sim_U T_1'}{(T_1 \to T_2) \sim_U (T_1' \to T_2)} \quad \text{CV\_FUN1}$$

$$\frac{T_2 \sim_U T_2'}{(T_1 \to T_2) \sim_U (T_1 \to T_2')} \quad \text{CV\_FUN2}$$

$\boxed{T_1 \sim T_2}$ T_1 is consistent with T_2

$$\frac{}{T \sim T} \quad \text{CS\_REFL}$$

$$\frac{}{? \sim T} \quad \text{CS\_UL}$$

$$\frac{}{T \sim ?} \quad \text{CS\_UR}$$

$$\frac{T_1 \sim_U T_2}{T_1 \sim T_2} \quad \text{CS\_CONV}$$

$$\frac{T_1 \sim T_1' \quad T_2 \sim T_2'}{(T_1 \times T_2) \sim (T_1' \times T_2')} \quad \text{CS\_PAIR}$$

$$\frac{T_1 \sim T_1' \quad T_2 \sim T_2'}{(T_1 \to T_2) \sim (T_1' \to T_2')} \quad \text{CS\_ARROW}$$

$\boxed{\Gamma \vdash t : T}$ t has type T in context Γ

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \quad \text{VAR}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1} \quad \textsc{unit}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \textsc{zero}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \textsc{succ}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \quad \textsc{pair}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\, t : T_1} \quad \textsc{fst}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\, t : T_2} \quad \textsc{snd}$$

$$\frac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x : T_1 . t : T_1 \to T_2} \quad \textsc{abs}$$

$$\frac{\Gamma \vdash t : T_1 \quad T_1 \sim_U T_2}{\Gamma \vdash t : T_2} \quad \mathrm{U}$$

$$\frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_3 \quad T_3 \sim T_1}{\Gamma \vdash t_1\, t_2 : T_2} \quad \textsc{app}$$

$\boxed{t_1 \rightsquigarrow t_2}$ $\quad t_1$ reduces to t_2

$$\frac{}{(\lambda x : T . t_2)\, t_1 \rightsquigarrow [t_1/x] t_2} \quad \textsc{rd\_beta}$$

$$\frac{}{(\lambda x : T . t\, x) \rightsquigarrow t} \quad \textsc{rd\_eta}$$

$$\frac{}{\mathsf{fst}\, (t_1, t_2) \rightsquigarrow t_1} \quad \textsc{rd\_proj}1$$

$$\frac{}{\mathsf{snd}\, (t_1, t_2) \rightsquigarrow t_2} \quad \textsc{rd\_proj}2$$

$$\frac{}{(\mathsf{fst}\, t, \mathsf{snd}\, t) \rightsquigarrow t} \quad \textsc{rd\_eta}\mathrm{P}$$