

Gradual Typing from a Categorical Perspective

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Abstract

TODO

1 Introduction

TODO

2 Categorical Model

Definition 1. Suppose \mathcal{C} is a category. Then an object A is a **retract** of an object B if there are morphisms $i : A \rightarrow B$ and $r : B \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ & \searrow & \downarrow r \\ & & A \end{array}$$

Definition 2. An **untyped λ -model**, $(\mathcal{C}, ?, \text{split}, \text{squash})$, is a cartesian closed category \mathcal{C} with a distinguished object $?$ and two morphisms $\text{squash} : (? \rightarrow ?) \rightarrow ?$ and $\text{split} : ? \rightarrow (? \rightarrow ?)$ making the object $? \rightarrow ?$ a retract of $?$.

Theorem 3 (Scott [1980]). An untyped λ -model is a sound and complete model of the untyped λ -calculus.

Definition 4. An object \mathbb{N} of a category \mathcal{C} with a terminal object 1 is a **natural number object (NNO)** if and only if there are morphisms $z : 1 \rightarrow \mathbb{N}$ and $s : \mathbb{N} \rightarrow \mathbb{N}$ such that for any other object A and morphisms $z' : 1 \rightarrow A$ and $s' : A \rightarrow A$ there is a unique morphism $u : \mathbb{N} \rightarrow A$ making the following diagram commute:

$$\begin{array}{ccccc} 1 & \xrightarrow{z} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ & \searrow z' & \downarrow u & & \downarrow u \\ & & A & \xrightarrow{s'} & A \end{array}$$

Definition 5. A *gradual λ -model*, $(\mathcal{T}, \mathcal{C}, ?, \top, \text{split}, \text{squash}, \text{box}, \text{unbox})$, where \mathcal{T} and \mathcal{C} are cartesian closed categories with NNOS, $(\mathcal{C}, ?, \text{split}, \text{squash})$ is an untyped λ -model, $\top : \mathcal{T} \longrightarrow \mathcal{C}$ is a cartesian closed embedding – a full and faithful cartesian closed functor that is injective on objects and preserves the NNO – and for every object, A , of \mathcal{T} there are morphisms $\text{box}_A : TA \longrightarrow ?$ and $\text{unbox}_A : ? \longrightarrow TA$ making TA a retract of $?$.

3 Grady

References

Dana Scott. Relating theories of the lambda-calculus. In *To H.B. Curry: Essays on Combinatory Logic, Lambda-Calculus and Formalism* (eds. Hindley and Seldin), pages 403–450. Academic Press, 1980.

A The Complete Spec of Grady

<i>termvar</i> , x, z	
<i>index</i> , k	
t	$::=$
	term
x	variable
triv	unit
squash	injection of the retract
split	surjection of the retract
box_T	generalize to the untyped universe
unbox_T	specialize the untyped universe to a specific type
$\lambda x : A. t$	λ -abstraction
$t_1 t_2$	function application
(t_1, t_2)	pair constructor
$\text{fst } t$	first projection
$\text{snd } t$	second projection
$\text{succ } t$	successor function
0	zero
(t)	S
h	$::=$
	head-normal forms
triv	
split	
squash	
box_T	
unbox_T	
$\lambda x : A. t$	

		(t_1, t_2)	
		$\text{fst } t$	
		$\text{snd } t$	
		$\text{succ } t$	
		0	
T	$::=$		terminating types
		1	unit type
		\mathbb{N}	natural number type
		$T_1 \rightarrow T_2$	function type
		$T_1 \times T_2$	cartesian product type
		(T)	S
A	$::=$		type
		1	unit type
		\mathbb{N}	natural number type
		$?$	untyped universe
		$A_1 \rightarrow A_2$	function type
		$A_1 \times A_2$	cartesian product type
		(A)	S
Γ	$::=$		typing context
		\cdot	empty context
		$\Gamma, x : A$	cons
vd	$::=$		
		\vdash	
		\nVdash	

$\boxed{\Gamma vdt : A}$ t has type A in context Γ

$$\begin{array}{c}
\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \quad \text{VAR} \\
\\
\frac{}{\Gamma \vdash \text{box}_T : T \rightarrow ?} \quad \text{BOX} \\
\\
\frac{}{\Gamma \vdash \text{unbox}_T : ? \rightarrow T} \quad \text{UNBOX} \\
\\
\frac{}{\Gamma \vdash \text{squash} : (? \rightarrow ?) \rightarrow ?} \quad \text{INJ} \\
\\
\frac{}{\Gamma \vdash \text{split} : ? \rightarrow (? \rightarrow ?)} \quad \text{SURJ} \\
\\
\frac{}{\Gamma \vdash \text{triv} : 1} \quad \text{UNIT} \\
\\
\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \text{ZERO}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \text{succ } t : \mathbb{N}} \quad \text{SUCC} \\
\\
\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2} \quad \text{PAIR} \\
\\
\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \text{fst } t : A_1} \quad \text{FST} \\
\\
\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \text{snd } t : A_2} \quad \text{SND} \\
\\
\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1. t : A_1 \rightarrow A_2} \quad \text{LAM} \\
\\
\frac{\Gamma \vdash t_1 : A_1 \rightarrow A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1 t_2 : A_2} \quad \text{APP} \\
\\
\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A} \quad t_1 \text{ reduces to } t_2 \text{ with type } A \text{ in context } \Gamma \\
\\
\frac{\Gamma \vdash t : T}{\Gamma \vdash \text{unbox}_T (\text{box}_T t) \rightsquigarrow t : T} \quad \text{RD_RETRACT} \\
\\
\frac{\Gamma \not\vdash t : T}{\Gamma \vdash \text{unbox}_T (\text{box}_{T'} t) \rightsquigarrow \text{wrong} : \text{TypeError}} \quad \text{RD_TWRONG} \\
\\
\frac{h \neq \text{box}_T t}{\Gamma \vdash \text{unbox}_T h \rightsquigarrow \text{wrong} : \text{TypeError}} \quad \text{RD_HWRONG} \\
\\
\frac{\Gamma \vdash t : ? \rightarrow ?}{\Gamma \vdash \text{split} (\text{squash } t) \rightsquigarrow t : ? \rightarrow ?} \quad \text{RD_RETRACTU} \\
\\
\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1. t_2) t_1 \rightsquigarrow [t_1/x] t_2 : A_2} \quad \text{RD_BETA} \\
\\
\frac{\Gamma \vdash t : A_1 \rightarrow A_2 \quad x \notin \text{FV}(t)}{\Gamma \vdash \lambda x : A_1. t x \rightsquigarrow t : A_1 \rightarrow A_2} \quad \text{RD_ETA} \\
\\
\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \text{fst } (t_1, t_2) \rightsquigarrow t_1 : A_1} \quad \text{RD_PROJ1} \\
\\
\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \text{snd } (t_1, t_2) \rightsquigarrow t_2 : A_2} \quad \text{RD_PROJ2} \\
\\
\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash (\text{fst } t, \text{snd } t) \rightsquigarrow t : A_1 \times A_2} \quad \text{RD_ETAP} \\
\\
\frac{\Gamma, x : A_1 \vdash t \rightsquigarrow t' : A_2}{\Gamma \vdash \lambda x : A_1. t \rightsquigarrow \lambda x : A_1. t' : A_1 \rightarrow A_2} \quad \text{RD_LAM} \\
\\
\frac{\Gamma \vdash t_1 \rightsquigarrow t'_1 : A_1 \rightarrow A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1 t_2 \rightsquigarrow t'_1 t_2 : A_2} \quad \text{RD_APP1}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash t_1 : A_1 \rightarrow A_2 \quad \Gamma \vdash t_2 \rightsquigarrow t'_2 : A_1}{\Gamma \vdash t_1 t_2 \rightsquigarrow t_1 t'_2 : A_2} \quad \text{RD_APP2} \\
\\
\frac{\Gamma \vdash t \rightsquigarrow t' : A_1 \times A_2}{\Gamma \vdash \text{fst } t \rightsquigarrow \text{fst } t' : A_1} \quad \text{RD_FST} \\
\\
\frac{\Gamma \vdash t \rightsquigarrow t' : A_1 \times A_2}{\Gamma \vdash \text{snd } t \rightsquigarrow \text{snd } t' : A_2} \quad \text{RD_SND} \\
\\
\frac{\Gamma \vdash t_1 \rightsquigarrow t'_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t'_1, t_2) : A_1 \times A_2} \quad \text{RD_PAIR1} \\
\\
\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 \rightsquigarrow t'_2 : A_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1, t'_2) : A_1 \times A_2} \quad \text{RD_PAIR2}
\end{array}$$

B The Complete Spec of $\lambda_{\rightarrow}^{\langle A \rangle}$ -Calculus

termvar, x, z

index, k

t	$::=$	term
x		variable
triv		unit
$\langle A \rangle t$		type cast
$\lambda x : A. t$		λ -abstraction
$t_1 t_2$		function application
(t_1, t_2)		pair constructor
$\text{fst } t$		first projection
$\text{snd } t$		second projection
$\text{succ } t$		successor function
0		zero
(t)	S	
A, B, C	$::=$	type
1		unit type
\mathbb{N}		natural number type
$?$		untyped universe
$A_1 \rightarrow A_2$		function type
$A_1 \times A_2$		cartesian product type
(A)	S	
Γ	$::=$	typing context
\cdot		empty context
$x : A$		cons

$vd ::=$
 $\quad \quad \quad \begin{array}{c} | \\ | \end{array} \quad \begin{array}{c} \vdash \\ \not\vdash \end{array}$

$\boxed{A \sim B}$ A is consistent with B

$\frac{}{A \sim A}$ REFL

$\frac{}{A \sim ?}$ BOX

$\frac{}{? \sim A}$ UNBOX

$\frac{A_1 \sim A_2 \quad B_1 \sim B_2}{A_1 \rightarrow B_1 \sim A_2 \rightarrow B_2}$ ARROW

$\frac{A_1 \sim A_2 \quad B_1 \sim B_2}{A_1 \times B_1 \sim A_2 \times B_2}$ PROD

$\boxed{\Gamma vdt : A}$ t has type A in context Γ

$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}$ VAR

$\frac{}{\Gamma \vdash \text{triv} : 1}$ UNIT

$\frac{}{\Gamma \vdash 0 : \mathbb{N}}$ ZERO

$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \text{succ } t : \mathbb{N}}$ SUCC

$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2}$ PAIR

$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \text{fst } t : A_1}$ FST

$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \text{snd } t : A_2}$ SND

$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1. t : A_1 \rightarrow A_2}$ LAM

$\frac{\Gamma \vdash t_1 : A_1 \rightarrow A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1 t_2 : A_2}$ APP

$\frac{\Gamma \vdash t : A \quad A \sim B}{\Gamma \vdash \langle B \rangle t : B}$ CAST