Gradual Typing from a Categorical Perspective

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June 2016

References
A   The Complete Spec of Grady

$termvar, x, z$
$index, k$

t	::=		term	
	|	x	variable	
	|	triv	unit	
	|	squash	injection of the retract	
	|	split	surjection of the retract	
	|	box t	generalize to the untyped universe	
	|	unbox_T	specialize the untyped universe to a specific type	
	|	$\lambda x : A.t$	λ-abstraction	
	|	$t_1\, t_2$	function application	
	|	(t_1, t_2)	pair constructor	
	|	fst t	first projection	
	|	snd t	second projection	
	|	succ t	successor function	
	|	0	zero	
	|	(t)	S	

T	::=		terminating types	
	|	1	unit type	
	|	$\mathbb{N}$	natural number type	
	|	$T_1 \to T_2$	function type	
	|	$T_1 \times T_2$	cartesian product type	
	|	(T)	S	

A	::=		type	
	|	T	terminating type	
	|	$?$	untyped universe	
	|	$A_1 \to A_2$	function type	

$$\begin{array}{lll}| & A_1 \times A_2 & \text{cartesian product type} \\ | & (A) & \mathsf{S}\end{array}$$

$$\begin{array}{lll}\Gamma & ::= & \text{typing context} \\ & | \quad \cdot & \text{empty context} \\ & | \quad \Gamma, x : A & \text{cons}\end{array}$$

$\boxed{\Gamma \vdash t : A}$ t has type A in context Γ

$$\frac{x : A \ \in \ \Gamma}{\Gamma \vdash x : A} \quad \text{VAR}$$

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{box}\, t : ?} \quad \text{BOX}$$

$$\frac{}{\Gamma \vdash \mathsf{unbox}_T : ? \to T} \quad \text{UNBOX}$$

$$\frac{}{\Gamma \vdash \mathsf{squash} : (? \to ?) \to ?} \quad \text{INJ}$$

$$\frac{}{\Gamma \vdash \mathsf{split} : ? \to (? \to ?)} \quad \text{SURJ}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1} \quad \text{UNIT}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \text{ZERO}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \text{SUCC}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2} \quad \text{PAIR}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \mathsf{fst}\, t : A_1} \quad \text{FST}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \mathsf{snd}\, t : A_2} \quad \text{SND}$$

$$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1.t : A_1 \to A_2} \quad \text{LAM}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 : A_2} \quad \text{APP}$$

$\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A}$ t_1 reduces to t_2 with type A in context Γ

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{unbox}_T\, (\mathsf{box}\, t) \rightsquigarrow t : T} \quad \text{RD\_RETRACT}$$

$$\frac{t \neq \mathsf{box}\, t'}{\Gamma \vdash \mathsf{unbox}_T\, t \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \text{RD\_UNBOXERR}$$

$$\frac{\Gamma \vdash t : \, ? \to ?}{\Gamma \vdash \mathsf{split}\,(\mathsf{squash}\,t) \leadsto t : \, ? \to ?} \quad \text{RD\_RETRACTU}$$

$$\frac{\Gamma \vdash t : A_1 \to A_2 \quad x \notin \mathsf{FV}(t)}{\Gamma \vdash \lambda x : A_1.t\,x \leadsto t : A_1 \to A_2} \quad \text{RD\_ETA}$$

$$\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1.t_2)\,t_1 \leadsto [t_1/x]t_2 : A_2} \quad \text{RD\_BETA}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \mathsf{fst}\,(t_1, t_2) \leadsto t_1 : A_1} \quad \text{RD\_PROJ1}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \mathsf{snd}\,(t_1, t_2) \leadsto t_2 : A_2} \quad \text{RD\_PROJ2}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash (\mathsf{fst}\,t, \mathsf{snd}\,t) \leadsto t : A_1 \times A_2} \quad \text{RD\_ETAP}$$

$$\frac{\Gamma, x : A_1 \vdash t \leadsto t' : A_2}{\Gamma \vdash \lambda x : A_1.t \leadsto \lambda x : A_1.t' : A_1 \to A_2} \quad \text{RD\_LAM}$$

$$\frac{\Gamma \vdash t_1 \leadsto t_1' : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\,t_2 \leadsto t_1'\,t_2 : A_2} \quad \text{RD\_APP1}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 \leadsto t_2' : A_1}{\Gamma \vdash t_1\,t_2 \leadsto t_1\,t_2' : A_2} \quad \text{RD\_APP2}$$

$$\frac{\Gamma \vdash t \leadsto t' : A_1 \times A_2}{\Gamma \vdash \mathsf{fst}\,t \leadsto \mathsf{fst}\,t' : A_1} \quad \text{RD\_FST}$$

$$\frac{\Gamma \vdash t \leadsto t' : A_1 \times A_2}{\Gamma \vdash \mathsf{snd}\,t \leadsto \mathsf{snd}\,t' : A_2} \quad \text{RD\_SND}$$

$$\frac{\Gamma \vdash t_1 \leadsto t_1' : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) \leadsto (t_1', t_2) : A_1 \times A_2} \quad \text{RD\_PAIR1}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 \leadsto t_2' : A_2}{\Gamma \vdash (t_1, t_2) \leadsto (t_1, t_2') : A_1 \times A_2} \quad \text{RD\_PAIR2}$$