Gradual Typing from a Categorical Perspective

Harley Eades III and Michael Townsend

June 2016

Abstract

TODO

1 Introduction

TODO

2 Categorical Model

Definition 1. *Suppose $\mathcal{C}$ is a category. Then an object A is a **retract** of an object B if there are morphisms $i : A \longrightarrow B$ and $r : B \longrightarrow A$ such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, equal] & B \arrow[d, "r"] \\ & A \end{tikzcd}$$

Definition 2. *An **untyped λ-model**, $(\mathcal{C}, ?, \mathsf{split}, \mathsf{squash})$, is a cartesian closed category $\mathcal{C}$ with a distinguished object $?$ and two morphisms $\mathsf{squash} : (? \to ?) \longrightarrow ?$ and $\mathsf{split} : ? \longrightarrow (? \to ?)$ making the object $? \to ?$ a retract of $?$.*

Theorem 3 (Scott [1980])**.** *An untyped λ-model is a sound and complete model of the untyped λ-calculus.*

Definition 4. *An object $\mathbb{N}$ of a category $\mathcal{C}$ with a terminal object 1 is a **natural number object (NNO)** if and only if there are morphisms $z : 1 \longrightarrow \mathbb{N}$ and $s : \mathbb{N} \longrightarrow \mathbb{N}$ such that for any other object A and morphisms $z' : 1 \longrightarrow A$ and $s' : A \longrightarrow A$ there is a unique morphism $u : \mathbb{N} \longrightarrow A$ making the following diagram commute:*

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "z'"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, "u"] & \mathbb{N} \arrow[d, "u"] \\ & A \arrow[r, "s'"] & A \end{tikzcd}$$

Definition 5. *A* ***gradual*** λ***-model****,* $(\mathcal{T}, \mathcal{C}, ?, \mathsf{T}, \mathsf{split}, \mathsf{squash}, \mathsf{box}, \mathsf{unbox})$*, where* $\mathcal{T}$ *and* $\mathcal{C}$ *are cartesian closed categories with NNOS,* $(\mathcal{C}, ?, \mathsf{split}, \mathsf{squash})$ *is an untyped* λ*-model,* $\mathsf{T} : \mathcal{T} \longrightarrow \mathcal{C}$ *is a cartesian closed embedding – a full and faithful cartesian closed functor that is injective on objects and preserves the NNO – and for every object,* A*, of* $\mathcal{T}$ *there are morphisms* $\mathsf{box}_A : TA \longrightarrow ?$ *and* $\mathsf{unbox}_A : ? \longrightarrow TA$ *making* TA *a retract of* $?$*.*

3 Grady

References

Dana Scott. Relating theories of the lambda-calculus. In *To H.B. Curry: Essays on Combinatory Logic, Lambda-Calculus and Formalism (eds. Hindley and Seldin)*, pages 403–450. Academic Press, 1980.

A The Complete Spec of Grady

termvar, x, z
index, k

t	::=		term
	|	x	variable
	|	triv	unit
	|	squash	injection of the retract
	|	split	surjection of the retract
	|	$\mathsf{box}\, t$	generalize to the untyped universe
	|	unbox_T	specialize the untyped universe to a specific type
	|	$\lambda x : A.t$	λ-abstraction
	|	$t_1\, t_2$	function application
	|	(t_1, t_2)	pair constructor
	|	$\mathsf{fst}\, t$	first projection
	|	$\mathsf{snd}\, t$	second projection
	|	$\mathsf{succ}\, t$	successor function
	|	0	zero
	|	(t) S	

h	::=		head-normal forms
	|	triv	
	|	split	
	|	squash	
	|	$\mathsf{box}\, t$	
	|	unbox_T	
	|	$\lambda x : A.t$	

	$\|$	(t_1, t_2)	
	$\|$	$\mathsf{fst}\, t$	
	$\|$	$\mathsf{snd}\, t$	
	$\|$	$\mathsf{succ}\, t$	
	$\|$	0	

T	$::=$		terminating types
	$\|$	1	unit type
	$\|$	$\mathbb{N}$	natural number type
	$\|$	$T_1 \to T_2$	function type
	$\|$	$T_1 \times T_2$	cartesian product type
	$\|$	(T) S	

A	$::=$		type
	$\|$	1	unit type
	$\|$	$\mathbb{N}$	natural number type
	$\|$	$?$	untyped universe
	$\|$	$A_1 \to A_2$	function type
	$\|$	$T_1 \times T_2$	cartesian product type
	$\|$	(A) S	

Γ	$::=$		typing context
	$\|$	$\cdot$	empty context
	$\|$	$\Gamma, x : A$	cons

vd	$::=$	
	$\|$	$\vdash$
	$\|$	$\nvdash$

$\boxed{\Gamma vd t : A}$ t has type A in context Γ

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \text{ VAR}$$

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{box}\, t : ?} \text{ Box}$$

$$\frac{}{\Gamma \vdash \mathsf{unbox}_T : ? \to T} \text{ Unbox}$$

$$\frac{}{\Gamma \vdash \mathsf{squash} : (? \to ?) \to ?} \text{ INJ}$$

$$\frac{}{\Gamma \vdash \mathsf{split} : ? \to (? \to ?)} \text{ SURJ}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1} \text{ UNIT}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \text{ ZERO}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \textsc{succ}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \quad \textsc{pair}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\, t : T_1} \quad \textsc{fst}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\, t : T_2} \quad \textsc{snd}$$

$$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1.t : A_1 \to A_2} \quad \textsc{lam}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 : A_2} \quad \textsc{app}$$

$\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A}$ t_1 reduces to t_2 with type A in context Γ

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{unbox}_T\, (\mathsf{box}\, t) \rightsquigarrow t : T} \quad \textsc{rd\_retract}\mathrm{T}$$

$$\frac{\Gamma \not\vdash t : T}{\Gamma \vdash \mathsf{unbox}_T\, (\mathsf{box}\, t) \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \textsc{rd\_t}\mathrm{W}\textsc{rong}$$

$$\frac{h \neq \mathsf{box}\, t}{\Gamma \vdash \mathsf{unbox}_T\, h \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \textsc{rd\_h}\mathrm{W}\textsc{rong}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : ?}{\Gamma \vdash \mathsf{unbox}_T\, t \rightsquigarrow \mathsf{unbox}_T\, t' : T} \quad \textsc{rd\_unbox}$$

$$\frac{\Gamma \vdash t : ? \to ?}{\Gamma \vdash \mathsf{split}\, (\mathsf{squash}\, t) \rightsquigarrow t : ? \to ?} \quad \textsc{rd\_retract}\mathrm{U}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : ?}{\Gamma \vdash \mathsf{split}\, t \rightsquigarrow \mathsf{split}\, t' : ? \to ?} \quad \textsc{rd\_split}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : ? \to ?}{\Gamma \vdash \mathsf{squash}\, t \rightsquigarrow \mathsf{squash}\, t' : ?} \quad \textsc{rd\_spuash}$$

$$\frac{\Gamma \vdash t : A_1 \to A_2 \quad x \notin \mathsf{FV}(t)}{\Gamma \vdash \lambda x : A_1.t\, x \rightsquigarrow t : A_1 \to A_2} \quad \textsc{rd\_eta}$$

$$\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1.t_2)\, t_1 \rightsquigarrow [t_1/x]t_2 : A_2} \quad \textsc{rd\_beta}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{fst}\, (t_1, t_2) \rightsquigarrow t_1 : T_1} \quad \textsc{rd\_proj}1$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{snd}\, (t_1, t_2) \rightsquigarrow t_2 : T_2} \quad \textsc{rd\_proj}2$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash (\mathsf{fst}\, t, \mathsf{snd}\, t) \rightsquigarrow t : T_1 \times T_2} \quad \text{RD\_ETAP}$$

$$\frac{\Gamma, x : A_1 \vdash t \rightsquigarrow t' : A_2}{\Gamma \vdash \lambda x : A_1.t \rightsquigarrow \lambda x : A_1.t' : A_1 \to A_2} \quad \text{RD\_LAM}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 \rightsquigarrow t_1'\, t_2 : A_2} \quad \text{RD\_APP1}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : A_1}{\Gamma \vdash t_1\, t_2 \rightsquigarrow t_1\, t_2' : A_2} \quad \text{RD\_APP2}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\, t \rightsquigarrow \mathsf{fst}\, t' : T_1} \quad \text{RD\_FST}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\, t \rightsquigarrow \mathsf{snd}\, t' : T_2} \quad \text{RD\_SND}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1', t_2) : T_1 \times T_2} \quad \text{RD\_PAIR1}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : T_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1, t_2') : T_1 \times T_2} \quad \text{RD\_PAIR2}$$