Proving error properties in the Kleisli category

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1 Definitions

Let $\mathcal C$ be a cartesian closed category with all finite coproducts. Then $\mathcal C$ is a monoidal category with associator

$$\alpha_{A,B,C}: A \times (B \times C) \longrightarrow (A \times B) \times C = \langle \langle \pi_1, \pi_2; \pi_1 \rangle, \pi_2; \pi_2 \rangle$$

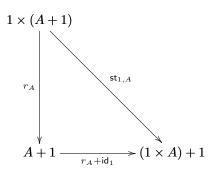
and unitors

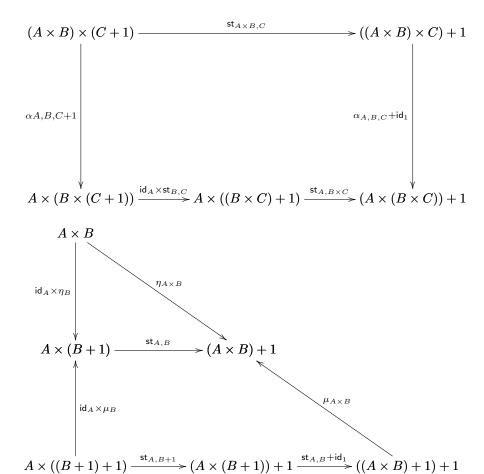
$$r_A: 1 \times A \longrightarrow A = \pi_2$$

 $l_A: A \times 1 \longrightarrow A = \pi_1$

Theorem 1. The maybe monad is a strong monad with strength $\mathsf{st}_{A,B}: A \times (B+1) \longrightarrow (A \times B) + 1 = \mathsf{dist}_{A,B,1}; \mathsf{id}_{A \times B} \times (\mathsf{triv}_A \times id_1); \mathsf{id}_{A \times B} \times triv_{1 \times 1}.$

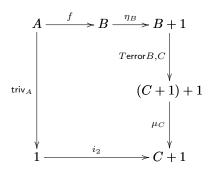
2 Strong monad diagrams, specialized to maybe monad





3 Error properties

1. Precomposition Let $f:A\longrightarrow B$. Then $\hat{f}_{;T}\operatorname{error}_{B,C}=\operatorname{error}_{A,C}.$ Diagrammatically



We shall show only the first of the product identities; the proof for the other follows from commutativity.

First we require a lemma.

Lemma 2. Let $g: C \longrightarrow B$ be a morphism. Then the following diagram commutes for any choice of A:

$$\begin{array}{c|c} 1 \times C = & (A+1) \times B \\ & & | \\ & | \\ i_2 \times g & & \text{st}_{A,B} \\ & \downarrow & & \downarrow \\ 1 \times C \xrightarrow{\mathsf{triv}; i_2} (A \times B) + 1 \end{array}$$

Proof. Reasoning equationally, we have:

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\begin{array}{lll} (i_2\times g); \mathsf{st}_{A,B} &=& (i_2\times g); \mathsf{curry}^{-1}(([\mathsf{curry}(i_1), \mathsf{curry}(\mathsf{triv}; i_2)])) & (\mathsf{by} \ \mathsf{def.}) \\ &=& (\mathsf{id}\times g); (i_2\times \mathsf{id}); \mathsf{curry}^{-1}(([\mathsf{curry}(i_1), \mathsf{curry}(\mathsf{triv}; i_2)])) & (\mathsf{properties} \ \mathsf{of} \ \mathsf{id}, \ \mathsf{products}) \\ &=& (\mathsf{id}\times g); \mathsf{curry}^{-1}(i_2; [\mathsf{curry}(i_1), \mathsf{curry}(\mathsf{triv}; i_2)]) & \mathsf{a} \ \mathsf{property} \ \mathsf{of} \ \mathsf{curry}^{-1} \\ &=& (\mathsf{id}\times g); \mathsf{curry}^{-1}(\mathsf{curry}(\mathsf{triv}; i_2)) & \mathsf{by} \ \mathsf{coproduct} \ \mathsf{diag}. \\ &=& (\mathsf{id}\times g); \mathsf{triv}; i_2 & \mathsf{bijectivity} \ \mathsf{of} \ \mathsf{curry} \\ &=& \mathsf{triv}; i_2 & \mathsf{uniq.} \ \mathsf{of} \ \mathsf{triv} \end{array}
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Similarly, we also have for any $f:C\longrightarrow A$ and any choice of $B,(f\times \iota_2);\mathsf{st}'_{A,B}:(C\times 1)\longrightarrow (A\times B)+1=\mathsf{triv};i_2$

Now, we may prove the identity $\langle \mathsf{error}_{A,B}, g \rangle$; $ten_{B,C} = \mathsf{error}_{B,C}$.

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Proof.
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 \begin{array}{lll} \langle \mathsf{error}_{A,B}, g; \eta_B \rangle; \mathsf{ten}_{B,C} & = & \langle \mathsf{error}_{A,B}, g; \eta_B \rangle; \mathsf{st}_{A,B+1}; (\mathsf{st}'_{A,B} + \mathsf{id}_1); \mu_{A \times B} \\ & = & \langle \mathsf{triv}_A, \mathsf{id}_A \rangle; (i_2 \times g); \mathsf{st}_{A,B+1}; (\mathsf{st}'_{A,B} + \mathsf{id}_1); \mu_{A \times B} \\ \end{array} 
                                                                                                                                                                                                                                                                          (by def.)
                                                                                                                                                                                                                          (properties of product map)
                                                                         \langle \mathsf{triv}_A, \mathsf{id}_A \rangle; \mathsf{triv}_{1 \times B}; i_{2,(B \times (C+1)+1)}; (\mathsf{st}_{A,B}' + \mathsf{id}_1); \mu_{A \times B}
                                                                                                                                                                                                                                                                          (lemma)
                                                                = \operatorname{triv}_{A}; i_{2,(B \times (C+1)+1)}; (\operatorname{st}'_{A,B} + \operatorname{id}_{1}); \mu_{A \times B}
                                                                                                                                                                                                                                                    uniqueness of triv
                                                                = \quad \mathsf{triv}_A; \mathsf{id}_1; i_{2,((B \times C) + 1) + 1}; \mu_{B \times C}
                                                                                                                                                                                                                                                    def. of coproduct
                                                                                                                                                                                                                                                                    prop. of id
                                                                = \quad \operatorname{triv}_A; i_{2,((B \times C)+1)+1}; \mu_{B \times C}
                                                                                                                                                                                                                                                                         def. of \mu
                                                                = \quad \mathsf{triv}_A; i_{2,((B \times C) + 1) + 1}; [\mathsf{id}_{(B \times C) + 1}, i_{2,(B \times C) + 1}]
                                                                = \quad \mathsf{triv}_A; i_{2,(B \times C)+1}
                                                                                                                                                                                                                                                prop. of coproduct
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