Gradual Typing from a Categorical Perspective

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June 2016

References

A The Complete Spec of Grady

termvar, x, z
index, k

t	::=			term
	|	x		variable
	|	triv		unit
	|	squash		injection of the retract
	|	split		surjection of the retract
	|	box_T		generalize to the untyped universe
	|	unbox		specialize the untyped universe to a specific type
	|	$\lambda x : A.t$		λ-abstraction
	|	$t_1\ t_2$		function application
	|	(t_1, t_2)		pair constructor
	|	$\mathsf{fst}\, t$		first projection
	|	$\mathsf{snd}\, t$		second projection
	|	$\mathsf{succ}\, t$		successor function
	|	0		zero
	|	(t)	S	

T	::=			terminating types
	|	1		unit type
	|	$\mathbb{N}$		natural number type
	|	$T_1 \to T_2$		function type
	|	$T_1 \times T_2$		cartesian product type
	|	(T)	S	

A	::=			type
	|	T		terminating type
	|	$?$		untyped universe
	|	$A_1 \to A_2$		function type

	$\mid$	$A_1 \times A_2$	cartesian product type
	$\mid$	(A)	S
Γ	$::=$		typing context
	$\mid$	$\cdot$	empty context
	$\mid$	$\Gamma, x : A$	cons

$\boxed{\Gamma \vdash t : A}$ t has type A in context Γ

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \quad \text{VAR}$$

$$\frac{}{\Gamma \vdash \mathsf{box}_T : T \to ?} \quad \textsc{Box}$$

$$\frac{}{\Gamma \vdash \mathsf{unbox} : ? \to T} \quad \textsc{Unbox}$$

$$\frac{}{\Gamma \vdash \mathsf{squash} : (? \to ?) \to ?} \quad \text{INJ}$$

$$\frac{}{\Gamma \vdash \mathsf{split} : ? \to (? \to ?)} \quad \text{SURJ}$$

$$\frac{}{\Gamma \vdash \mathsf{triv} : 1} \quad \text{UNIT}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \text{ZERO}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \text{SUCC}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2} \quad \text{PAIR}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \mathsf{fst}\, t : A_1} \quad \text{FST}$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \mathsf{snd}\, t : A_2} \quad \text{SND}$$

$$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1 . t : A_1 \to A_2} \quad \text{ABS}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash t_1\, t_2 : A_2} \quad \text{APP}$$

$\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A}$ t_1 reduces to t_2 with type A in context Γ

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{unbox}\, (\mathsf{box}_T\, t) \rightsquigarrow t : T} \quad \text{RD\_RETRACT}$$

$$\frac{t \neq \mathsf{box}_T\, t'}{\Gamma \vdash \mathsf{unbox}\, t \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \text{RD\_UNBOXERR}$$

$$\frac{\Gamma \vdash t : ? \to ?}{\Gamma \vdash \mathsf{split}\,(\mathsf{squash}\,t) \rightsquigarrow t : ? \to ?}\ \textsc{rd\_retract}\mathrm{U}$$

$$\frac{\Gamma \vdash t : A_1 \to A_2 \quad x \notin \mathsf{FV}(t)}{\Gamma \vdash \lambda x : A_1.t\,x \rightsquigarrow t : A_1 \to A_2}\ \textsc{rd\_eta}$$

$$\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1.t_2)\,t_1 \rightsquigarrow [t_1/x]t_2 : A_2}\ \textsc{rd\_beta}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \mathsf{fst}\,(t_1, t_2) \rightsquigarrow t_1 : A_1}\ \textsc{rd\_proj}1$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash \mathsf{snd}\,(t_1, t_2) \rightsquigarrow t_2 : A_2}\ \textsc{rd\_proj}2$$

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash (\mathsf{fst}\,t, \mathsf{snd}\,t) \rightsquigarrow t : A_1 \times A_2}\ \textsc{rd\_eta}\mathrm{P}$$

$$\frac{\Gamma, x : A_1 \vdash t \rightsquigarrow t' : A_2}{\Gamma \vdash \lambda x : A_1.t \rightsquigarrow \lambda x : A_1.t' : A_1 \to A_2}\ \textsc{rd\_lam}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\,t_2 \rightsquigarrow t_1'\,t_2 : A_2}\ \textsc{rd\_app}1$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : A_1}{\Gamma \vdash t_1\,t_2 \rightsquigarrow t_1\,t_2' : A_2}\ \textsc{rd\_app}2$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : A_1 \times A_2}{\Gamma \vdash \mathsf{fst}\,t \rightsquigarrow \mathsf{fst}\,t' : A_1}\ \textsc{rd\_fst}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : A_1 \times A_2}{\Gamma \vdash \mathsf{snd}\,t \rightsquigarrow \mathsf{snd}\,t' : A_2}\ \textsc{rd\_snd}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1', t_2) : A_1 \times A_2}\ \textsc{rd\_pair}1$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : A_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1, t_2') : A_1 \times A_2}\ \textsc{rd\_pair}2$$