Proving error properties in the Kleisli category

Richard Blair

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1 Definitions

Let $\mathcal{C}$ be a cartesian closed category with all finite coproducts. Then $\mathcal{C}$ is a monoidal category with associator

$$\alpha_{A,B,C} : A \times (B \times C) \longrightarrow (A \times B) \times C = \langle \langle \pi_1, \pi_2; \pi_1 \rangle, \pi_2; \pi_2 \rangle$$

and unitors

$$\begin{array}{rcl} r_A : 1 \times A \longrightarrow A & = & \pi_2 \\ l_A : A \times 1 \longrightarrow A & = & \pi_1 \end{array}$$

Theorem 1. *The maybe monad is a strong monad with strength* $\mathsf{st}_{A,B} : A \times (B+1) \longrightarrow (A \times B) + 1 = \mathsf{dist}_{A,B,1}; \mathsf{id}_{A \times B} \times (\mathsf{triv}_A \times id_1); \mathsf{id}_{A \times B} \times triv_{1 \times 1}$.

2 Strong monad diagrams, specialized to maybe monad

$$\begin{tikzcd} 1 \times (A+1) \arrow[d, "r_A"'] \arrow[dr, "\mathsf{st}_{1,A}"] & \\ A+1 \arrow[r, "r_A + \mathsf{id}_1"'] & (1 \times A) + 1 \end{tikzcd}$$

$$
\begin{tikzcd}
(A \times B) \times (C+1) \arrow[rr, "\mathsf{st}_{A \times B, C}"] \arrow[d, "\alpha A{,}B{,}C{+}1"'] & & ((A \times B) \times C) + 1 \arrow[d, "\alpha_{A,B,C} + \mathsf{id}_1"'] \\
A \times (B \times (C+1)) \arrow[r, "\mathsf{id}_A \times \mathsf{st}_{B,C}"] & A \times ((B \times C) + 1) \arrow[r, "\mathsf{st}_{A, B \times C}"] & (A \times (B \times C)) + 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \times B \arrow[d, "\mathsf{id}_A \times \eta_B"'] \arrow[dr, "\eta_{A \times B}"] & & \\
A \times (B+1) \arrow[r, "\mathsf{st}_{A,B}"] & (A \times B) + 1 & \\
A \times ((B+1)+1) \arrow[u, "\mathsf{id}_A \times \mu_B"'] \arrow[r, "\mathsf{st}_{A,B+1}"] & (A \times (B+1)) + 1 \arrow[r, "\mathsf{st}_{A,B} + \mathsf{id}_1"] & ((A \times B) + 1) + 1 \arrow[ul, "\mu_{A \times B}"']
\end{tikzcd}
$$

3 Error properties

1. Precomposition Let $f : A \longrightarrow B$. Then $\hat{f};_T \mathsf{error}_{B,C} = \mathsf{error}_{A,C}$. Diagrammatically

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, "\mathsf{triv}_A"'] & B \arrow[r, "\eta_B"] & B+1 \arrow[d, "T\mathsf{error}B{,}C"'] \\
 & & (C+1)+1 \arrow[d, "\mu_C"'] \\
1 \arrow[rr, "i_2"] & & C+1
\end{tikzcd}
$$

We shall show only the first of the product identities; the proof for the other follows from commutativity.

First we require a lemma.

Lemma 2. *Let $g : C \longrightarrow B$ be a morphism. Then the following diagram commutes for any choice of A:*

$$\begin{tikzcd}
1 \times C \arrow[r, equal] \arrow[d, "i_2 \times g"'] & (A+1) \times B \arrow[d, "\mathsf{st}_{A,B}"] \\
1 \times C \arrow[r, "\mathsf{triv};i_2"] & (A \times B) + 1
\end{tikzcd}$$

Proof. Reasoning equationally, we have:

$$\begin{array}{rcll}
(i_2 \times g); \mathsf{st}_{A,B} & = & (i_2 \times g); \mathsf{curry}^{-1}(([\mathsf{curry}(i_1), \mathsf{curry}(\mathsf{triv}; i_2)])) & \text{(by def.)} \\
& = & (\mathsf{id} \times g); (i_2 \times \mathsf{id}); \mathsf{curry}^{-1}(([\mathsf{curry}(i_1), \mathsf{curry}(\mathsf{triv}; i_2)])) & \text{(properties of id, products)} \\
& = & (\mathsf{id} \times g); \mathsf{curry}^{-1}(i_2; [\mathsf{curry}(i_1), \mathsf{curry}(\mathsf{triv}; i_2)]) & \text{a property of } \mathsf{curry}^{-1} \\
& = & (\mathsf{id} \times g); \mathsf{curry}^{-1}(\mathsf{curry}(\mathsf{triv}; i_2)) & \text{by coproduct diag.} \\
& = & (\mathsf{id} \times g); \mathsf{triv}; i_2 & \text{bijectivity of} \mathsf{curry} \\
& = & \mathsf{triv}; i_2 & \text{uniq. of } \mathsf{triv}
\end{array}$$

□

Similarly, we also have for any $f : C \longrightarrow A$ and any choice of B, $(f \times \imath_2); \mathsf{st}'_{A,B} : (C \times 1) \longrightarrow (A \times B) + 1 = \mathsf{triv}; i_2$

Now, we may prove the identity $\langle \mathsf{error}_{A,B}, g \rangle; ten_{B,C} = \mathsf{error}_{B,C}$.

Proof.

$$\begin{array}{rcll}
\langle \mathsf{error}_{A,B}, g; \eta_B \rangle; \mathsf{ten}_{B,C} & = & \langle \mathsf{error}_{A,B}, g; \eta_B \rangle; \mathsf{st}_{A,B+1}; (\mathsf{st}'_{A,B} + \mathsf{id}_1); \mu_{A \times B} & \text{(by def.)} \\
& = & \langle \mathsf{triv}_A, \mathsf{id}_A \rangle; (i_2 \times g); \mathsf{st}_{A,B+1}; (\mathsf{st}'_{A,B} + \mathsf{id}_1); \mu_{A \times B} & \text{(properties of product map)} \\
& = & \langle \mathsf{triv}_A, \mathsf{id}_A \rangle; \mathsf{triv}_{1 \times B}; i_{2,(B \times (C+1))+1}; (\mathsf{st}'_{A,B} + \mathsf{id}_1); \mu_{A \times B} & \text{(lemma)} \\
& = & \mathsf{triv}_A; i_{2,(B \times (C+1))+1}; (\mathsf{st}'_{A,B} + \mathsf{id}_1); \mu_{A \times B} & \text{uniqueness of } \mathsf{triv} \\
& = & \mathsf{triv}_A; \mathsf{id}_1; i_{2,((B \times C)+1)+1}; \mu_{B \times C} & \text{def. of coproduct} \\
& = & \mathsf{triv}_A; i_{2,((B \times C)+1)+1}; \mu_{B \times C} & \text{prop. of id} \\
& = & \mathsf{triv}_A; i_{2,((B \times C)+1)+1}; [\mathsf{id}_{(B \times C)+1}, i_{2,(B \times C)+1}] & \text{def. of } \mu \\
& = & \mathsf{triv}_A; i_{2,(B \times C)+1} & \text{prop. of coproduct}
\end{array}$$

□