Gradual Typing from a Categorical Perspective

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Abstract

TODO

1 Introduction

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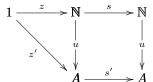## 2 Categorical Model

Definition 1. *Suppose $\mathcal{C}$ is a category. Then an object A is a **retract** of an object B if there are morphisms $i : A \longrightarrow B$ and $r : B \longrightarrow A$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ i\ } & B \\
 & \diagdown & \big\downarrow r \\
 & & A
\end{array}
$$

Definition 2. *An **untyped λ-model**, $(\mathcal{C}, ?, \mathsf{split}, \mathsf{squash})$, is a cartesian closed category $\mathcal{C}$ with a distinguished object $?$ and two morphisms $\mathsf{squash} : (? \to ?) \longrightarrow ?$ and $\mathsf{split} :? \longrightarrow (? \to ?)$ making the object $? \to ?$ a retract of $?$.*

Theorem 3 (Scott [1980]). *An untyped λ-model is a sound and complete model of the untyped λ-calculus.*

Definition 4. *An object $\mathbb{N}$ of a category $\mathcal{C}$ with a terminal object 1 is a **natural number object (NNO)** if and only if there are morphisms $z : 1 \longrightarrow \mathbb{N}$ and $s : \mathbb{N} \longrightarrow \mathbb{N}$ such that for any other object A and morphisms $z' : 1 \longrightarrow A$ and $s' : A \longrightarrow A$ there is a unique morphism $u : \mathbb{N} \longrightarrow A$ making the following diagram commute:*

$$
\begin{array}{ccccc}
1 & \xrightarrow{\ z\ } & \mathbb{N} & \xrightarrow{\ s\ } & \mathbb{N} \\
 & \diagdown{\scriptstyle z'} & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle u} \\
 & & A & \xrightarrow{\ s'\ } & A
\end{array}
$$

Definition 5. *A **gradual** λ-**model**, $(\mathcal{T}, \mathcal{C}, ?, \mathsf{T}, \mathsf{split}, \mathsf{squash}, \mathsf{box}, \mathsf{unbox})$, where $\mathcal{T}$ and $\mathcal{C}$ are cartesian closed categories with NNOS, $(\mathcal{C}, ?, \mathsf{split}, \mathsf{squash})$ is an untyped λ-model, $\mathsf{T} : \mathcal{T} \longrightarrow \mathcal{C}$ is a cartesian closed embedding – a full and faithful cartesian closed functor that is injective on objects and preserves the NNO – and for every object, A, of $\mathcal{T}$ there are morphisms $\mathsf{box}_A : TA \longrightarrow ?$ and $\mathsf{unbox}_A : ? \longrightarrow TA$ making TA a retract of $?$.*

3 Grady

References

Dana Scott. Relating theories of the lambda-calculus. In *To H.B. Curry: Essays on Combinatory Logic, Lambda-Calculus and Formalism (eds. Hindley and Seldin)*, pages 403–450. Academic Press, 1980.

A The Complete Spec of Grady

$termvar$, x, z
$index$, k

t	::=		term
	|	x	variable
	|	triv	unit
	|	squash	injection of the retract
	|	split	surjection of the retract
	|	box	generalize to the untyped universe
	|	unbox_T	specialize the untyped universe to a specific type
	|	$\lambda x : A.t$	λ-abstraction
	|	$t_1\, t_2$	function application
	|	(t_1, t_2)	pair constructor
	|	fst t	first projection
	|	snd t	second projection
	|	succ t	successor function
	|	0	zero
	|	(t)	S

h	::=		head-normal forms
	|	triv	
	|	split	
	|	squash	
	|	box	
	|	unbox_T	
	|	$\lambda x : A.t$	

2

$$\begin{array}{lll}
& | & (t_1, t_2) \\
& | & \text{fst } t \\
& | & \text{snd } t \\
& | & \text{succ } t \\
& | & 0
\end{array}$$

$$\begin{array}{llll}
T & ::= & & \text{terminating types} \\
& | & 1 & \text{unit type} \\
& | & \mathbb{N} & \text{natural number type} \\
& | & T_1 \to T_2 & \text{function type} \\
& | & T_1 \times T_2 & \text{cartesian product type} \\
& | & (T) \quad \text{S}
\end{array}$$

$$\begin{array}{llll}
A & ::= & & \text{type} \\
& | & 1 & \text{unit type} \\
& | & \mathbb{N} & \text{natural number type} \\
& | & ? & \text{untyped universe} \\
& | & A_1 \to A_2 & \text{function type} \\
& | & A_1 \times A_2 & \text{cartesian product type} \\
& | & (A) \quad \text{S}
\end{array}$$

$$\begin{array}{llll}
\Gamma & ::= & & \text{typing context} \\
& | & \cdot & \text{empty context} \\
& | & \Gamma, x : A & \text{cons}
\end{array}$$

$$\begin{array}{lll}
vd & ::= & \\
& | & \vdash \\
& | & \nvdash
\end{array}$$

$\boxed{\Gamma \, vd \, t : A}$ t has type A in context Γ

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \quad \text{VAR}$$

$$\frac{}{\Gamma \vdash \text{box} : T \to ?} \quad \text{BOX}$$

$$\frac{}{\Gamma \vdash \text{unbox}_T : ? \to T} \quad \text{UNBOX}$$

$$\frac{}{\Gamma \vdash \text{squash} : (? \to ?) \to ?} \quad \text{INJ}$$

$$\frac{}{\Gamma \vdash \text{split} : ? \to (? \to ?)} \quad \text{SURJ}$$

$$\frac{}{\Gamma \vdash \text{triv} : 1} \quad \text{UNIT}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \quad \text{ZERO}$$

$$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}} \quad \text{SUCC}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \quad \text{PAIR}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{fst}\, t : T_1} \quad \text{FST}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \mathsf{snd}\, t : T_2} \quad \text{SND}$$

$$\frac{\Gamma, x : A_1 \vdash t : A_2}{\Gamma \vdash \lambda x : A_1.t : A_1 \to A_2} \quad \text{LAM}$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 : A_2} \quad \text{APP}$$

$$\boxed{\Gamma \vdash t_1 \rightsquigarrow t_2 : A} \qquad t_1 \text{ reduces to } t_2 \text{ with type } A \text{ in context } \Gamma$$

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash \mathsf{unbox}_T\, (\mathsf{box}\, t) \rightsquigarrow t : T} \quad \text{RD\_RETRACT}T$$

$$\frac{\Gamma \nvdash t : T}{\Gamma \vdash \mathsf{unbox}_T\, (\mathsf{box}\, t) \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \text{RD\_T}\textsc{Wrong}$$

$$\frac{h \neq \mathsf{box}\, t}{\Gamma \vdash \mathsf{unbox}_T\, h \rightsquigarrow \mathsf{wrong} : \mathsf{TypeError}} \quad \text{RD\_H}\textsc{Wrong}$$

$$\frac{\Gamma \vdash t : {?} \to {?}}{\Gamma \vdash \mathsf{split}\, (\mathsf{squash}\, t) \rightsquigarrow t : {?} \to {?}} \quad \text{RD\_RETRACT}U$$

$$\frac{\Gamma, x : A_1 \vdash t_2 : A_2 \quad \Gamma \vdash t_1 : A_1}{\Gamma \vdash (\lambda x : A_1.t_2)\, t_1 \rightsquigarrow [t_1/x]t_2 : A_2} \quad \text{RD\_BETA}$$

$$\frac{\Gamma \vdash t : A_1 \to A_2 \quad x \notin \mathsf{FV}(t)}{\Gamma \vdash \lambda x : A_1.t\, x \rightsquigarrow t : A_1 \to A_2} \quad \text{RD\_ETA}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{fst}\, (t_1, t_2) \rightsquigarrow t_1 : T_1} \quad \text{RD\_PROJ}1$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathsf{snd}\, (t_1, t_2) \rightsquigarrow t_2 : T_2} \quad \text{RD\_PROJ}2$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash (\mathsf{fst}\, t, \mathsf{snd}\, t) \rightsquigarrow t : T_1 \times T_2} \quad \text{RD\_ETA}P$$

$$\frac{\Gamma, x : A_1 \vdash t \rightsquigarrow t' : A_2}{\Gamma \vdash \lambda x : A_1.t \rightsquigarrow \lambda x : A_1.t' : A_1 \to A_2} \quad \text{RD\_LAM}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : A_1 \to A_2 \quad \Gamma \vdash t_2 : A_1}{\Gamma \vdash t_1\, t_2 \rightsquigarrow t_1'\, t_2 : A_2} \quad \text{RD\_APP}1$$

$$\frac{\Gamma \vdash t_1 : A_1 \to A_2 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : A_1}{\Gamma \vdash t_1 \ t_2 \rightsquigarrow t_1 \ t_2' : A_2} \quad \text{RD\_APP}2$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : T_1 \times T_2}{\Gamma \vdash \mathsf{fst} \ t \rightsquigarrow \mathsf{fst} \ t' : T_1} \quad \text{RD\_FST}$$

$$\frac{\Gamma \vdash t \rightsquigarrow t' : T_1 \times T_2}{\Gamma \vdash \mathsf{snd} \ t \rightsquigarrow \mathsf{snd} \ t' : T_2} \quad \text{RD\_SND}$$

$$\frac{\Gamma \vdash t_1 \rightsquigarrow t_1' : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1', t_2) : T_1 \times T_2} \quad \text{RD\_PAIR}1$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 \rightsquigarrow t_2' : T_2}{\Gamma \vdash (t_1, t_2) \rightsquigarrow (t_1, t_2') : T_1 \times T_2} \quad \text{RD\_PAIR}2$$