

Lecture 17 – Deterministic Pushdown Automata (DPDA)

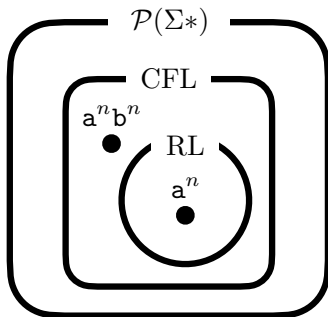
COSE215: Theory of Computation

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- A **pushdown automaton (PDA)** is an extension of ϵ -NFA with a **stack**. Thus, PDA is **non-deterministic**.
 - Acceptance by **final states**
 - Acceptance by **empty stacks**
- Then, how about **deterministic PDA (DPDA)**?
- What is the **language class** of DPDA? Still, CFL?



PDA_{FS}
(by final states)

||

PDA_{ES}
(by empty stacks)

||

CFG

1. Deterministic Pushdown Automata (DPDA)

2. Deterministic Context-Free Languages (DCFLs)

Fact 1: $\text{DCFL} \subset \text{CFL}$

Fact 2: $\text{RL} \subset \text{DCFL}$

3. Languages Accepted by Empty Stacks of DPDA (DCFL_{ES})

Fact 3: $\text{DCFL}_{\text{ES}} \subset \text{DCFL}$

Fact 4: $\text{DCFL}_{\text{ES}} = \text{DCFL having Prefix Property}$

Fact 5: $\text{RL} \not\subset \text{DCFL}_{\text{ES}}$

4. Inherent Ambiguity of DCFLs

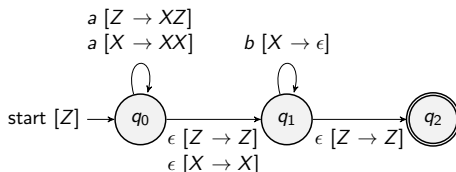
Fact 6: $\text{DCFL} \subset \text{Non Inherently Ambiguous Languages}$

Definition (Deterministic Pushdown Automata)

A PDA is **deterministic** if there is at most one-step move (\vdash) from any configuration and we call it a **deterministic pushdown automaton (DPDA)**. In other words, a DPDA satisfies the following conditions:

- $|\delta(q, a, X)| \leq 1$ for all $q \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$.
- If $\delta(q, \epsilon, X) \neq \emptyset$, then $\delta(q, a, X) = \emptyset$ for all $a \in \Sigma$.

For example, the following PDA is **NOT** deterministic:

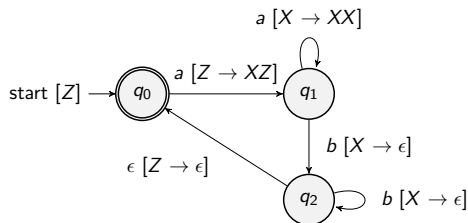


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- If $\delta(q, \epsilon, X) \neq \emptyset$, then $\delta(q, a, X) = \emptyset$ for all $a \in \Sigma$.

For example, the following PDA is deterministic:



$$\begin{aligned}
 (q_0, aabb, Z) &\vdash (q_1, abb, XZ) \\
 &\vdash (q_1, bb, XXZ) \\
 &\vdash (q_2, b, XZ) \\
 &\vdash (q_2, \epsilon, Z) \\
 &\vdash (q_0, \epsilon, \epsilon)
 \end{aligned}$$

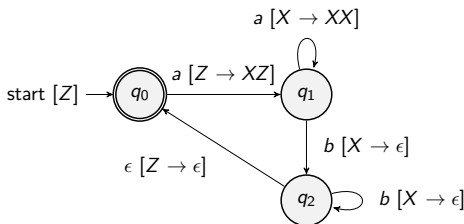
Definition (Deterministic Context-Free Languages (DCFLs))

A language L is a **deterministic context-free language (DCFL)** if and only if there exists a DPDA P such that $L = L_F(P)$ where $L_F(P)$ is the language accepted by **final states** of P .

For example, the following language is a DCFL:

$$L = \{a^n b^n \mid n \geq 0\}$$

because it is accepted by final states of the following DPDA:



Fact 1: DCFL \subset CFL

All DCFLs are CFLs **BUT** there exists a CFL that is not a DCFL.

For example, the following language is a CFL but not a DCFL:

$$L = \{ww^R \mid w \in \{a, b\}^*\}$$

Why?

Fact 2: $RL \subset DCFL$

All RLs are DCFLs **BUT** there exists a DCFL that is not an RL.

- $RL \subseteq DCFL$: For a given RL L and its corresponding DFA

$$D = (Q, \Sigma, \delta, q_0, F)$$

we can construct a DPDA P that accepts L as follows:

$$P = (Q, \Sigma, \{Z\}, \delta_P, q_0, Z, F)$$

where $\forall q \in Q. \forall a \in \Sigma. \delta_P(q, a, Z) = \{(\delta(q, a), Z)\}$. Then,

$$(q_0, w, Z) \vdash^* (q, \epsilon, Z) \iff \delta^*(q_0, w) = q \quad \square$$

- $DCFL \setminus RL \neq \emptyset$: We already know that the following language is a DCFL but not an RL:

$$L = \{a^n b^n \mid n \geq 0\} \in DCFL \setminus RL$$

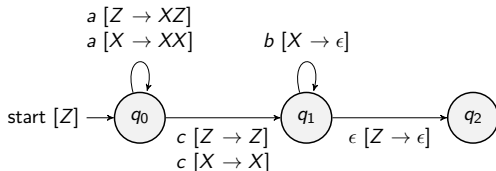
Definition (DCFL_{ES})

A language L is a **deterministic context-free language by empty stacks (DCFL_{ES})** if and only if there exists a DPDA P such that $L = L_E(P)$ where $L_E(P)$ is the language accepted by **empty stacks** of P .

For example, the following language is a DCFL_{ES} :

$$L = \{a^n cb^n \mid n \geq 0\}$$

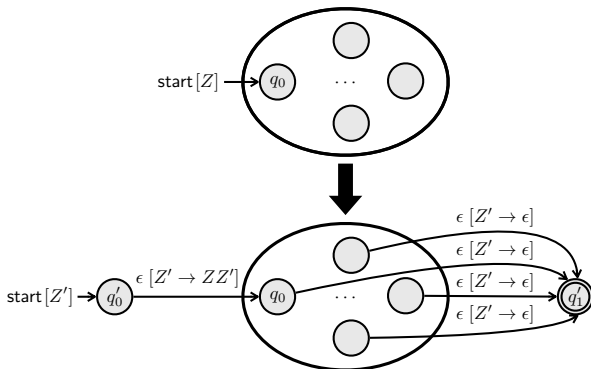
because it is accepted by empty stacks of the following DPDA:



Fact 3: $\text{DCFL}_{\text{ES}} \subset \text{DCFL}$

All DCFL_{ES} s are DCFLs **BUT** there exists a DCFL that is not a DCFL_{ES} .

- $\boxed{\text{DCFL}_{\text{ES}} \subseteq \text{DCFL}}$: For a given DCFL_{ES} L and its corresponding DPDA P by **empty stacks**, we can always construct a DPDA P' that accepts L by **final states** as follows:



Fact 3: $\text{DCFL}_{\text{ES}} \subset \text{DCFL}$

All DCFL_{ES} s are DCFLs **BUT** there exists a DCFL that is not a DCFL_{ES} .

- $\text{DCFL} \setminus \text{DCFL}_{\text{ES}} \neq \emptyset$: The following language is a DCFL but not a DCFL_{ES} :

$$L = \{a^n b^n \mid n \geq 0\} \in \text{DCFL} \setminus \text{DCFL}_{\text{ES}}$$

Why?

Fact 4: $\text{DCFL}_{\text{ES}} = \text{DCFL}$ having Prefix Property

A language L is a DCFL_{ES} if and only if L is a DCFL having the **prefix property**.

Definition (Prefix Property)

A language L has the **prefix property** if and only if for any word $w \in L$, any proper prefix of w is not in L :

$$\forall x, y \in \Sigma^*. (xy \in L \wedge y \neq \epsilon \Rightarrow x \notin L)$$

For example, the following language is a **DCFL** but does **NOT** have the **prefix property**:

$$L = \{a^n b^n \mid n \geq 0\}$$

because $\epsilon \in L$ is a proper prefix of $ab \in L$.

Thus, L is a **DCFL** but **NOT** a **DCFL_{ES}**.

Fact 5: $RL \not\subseteq DCFL_{ES}$

There exists a RL that is not a $DCFL_{ES}$.

- $RL \setminus DCFL_{ES} \neq \emptyset$: For example, the following language is a RL but does **NOT** have the **prefix property**:

$$L = \{a^n \mid n \geq 0\}$$

because $aa \in L$ is a proper prefix of $aaaa \in L$.
Thus, L is a RL but **NOT** a $DCFL_{ES}$.

$$L \in RL \setminus DCFL_{ES}$$

Definition (Inherent Ambiguity)

A language L is **inherently ambiguous** if all CFGs whose languages are L are ambiguous. (i.e. there is no unambiguous grammar for L)

- **Is there any DCFL that is inherently ambiguous?**
(i.e., is there any DCFL always having ambiguous grammars?)
- **Is there any DCFL that is not inherently ambiguous?**
(i.e., is there any DCFL having at least one unambiguous grammar?)
- **Is there any inherently ambiguous language that is not a DCFL?**
(i.e., is there any unambiguous grammar whose language is not a DCFL?)

Fact 6: DCFL \subset Non Inherently Ambiguous Languages

All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

- A DCFL_{ES} has an unambiguous grammar: For a given DCFL_{ES} L and its corresponding PDA P , we can define a CFG for L as follows:
 - For all $0 \leq j < n$,

$$S \rightarrow A_{0,j}^Z$$

- For all $q_i \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$, consider any $(q_j, X_1 \cdots X_m) \in \delta(q_i, a, X)$ and $0 \leq k_1, \dots, k_m < n$. Then,

$$A_{i,k_m}^X \rightarrow a A_{j,k_1}^{X_1} A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$

For any word $w \in L$, w has a unique moves from the initial configuration to the final configuration in P . And, we know that:

$$A_{i,j}^X \Rightarrow^* w \quad \text{if and only if} \quad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$

Thus, the above CFG is **unambiguous**.

Fact 6: DCFL \subset Non Inherently Ambiguous Languages

All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

- A DCFL has an unambiguous grammar: For a given DCFL L , we can define another DCFL L' with a special symbol $\$$ as follows:

$$L' = L\$ = \{w\$ \mid w \in L\}$$

Then, L' is a DCFL_{ES} because it has the prefix property. Thus, L' has an unambiguous grammar G' . Now, we can define an unambiguous grammar G for L by treating $\$$ as a variable with the rule $\$ \rightarrow \epsilon$.

For example, if the given DCFL is $L = \{a^n b^n \mid n \geq 0\}$, then $L' = \{a^n b^n \$ \mid n \geq 0\}$ and G' is:

$$S \rightarrow X\$ \quad X \rightarrow aXb \mid \epsilon$$

Then, the unambiguous grammar G for L is:

$$S \rightarrow X\$ \quad X \rightarrow aXb \mid \epsilon \quad \$ \rightarrow \epsilon$$

Fact 6: DCFL \subset Non Inherently Ambiguous Languages

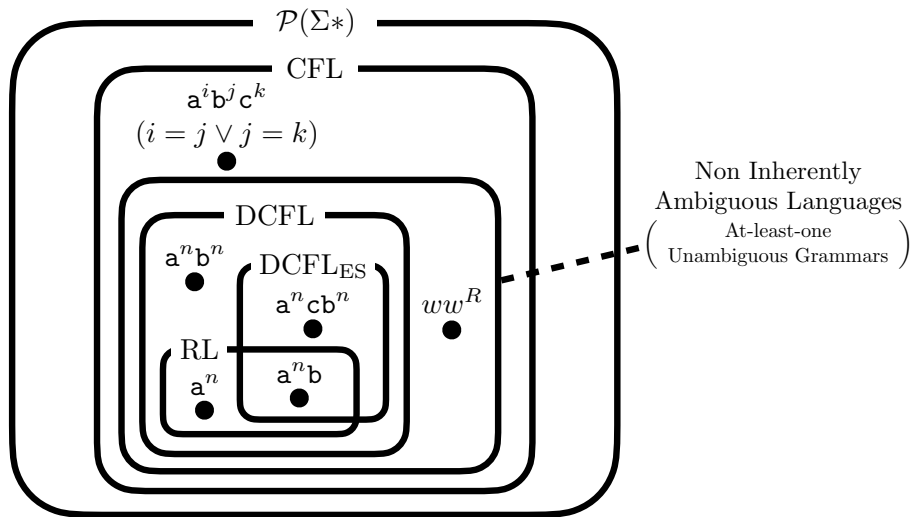
All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

- Non Inherently Ambiguous Languages \ DCFL $\neq \emptyset$: For example, the following language is a non inherently ambiguous language but not a DCFL:

$$L = \{ww^R \mid w \in \{a, b\}^*\}$$

because the following unambiguous grammar G represents L :

$$S \rightarrow aSa \mid bSb \mid \epsilon$$



- Normal Forms of Context-Free Grammars

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