Lecture 14 – Pushdown Automata (PDA)

COSE215: Theory of Computation

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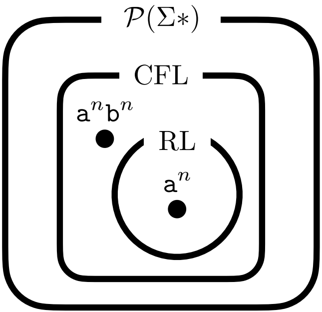
PLRG

2023 Spring

Recall

- A **context-free grammar (CFG)**:

$$G = (V, \Sigma, S, P)$$



Languages	**Automata**	**Grammars**
Conte-Free Language **(CFL)**	**???**	Context-Free Grammar **(CFG)**
Regular Language **(RL)**	Finite Automata **(FA)**	Regular Expression **(RE)**

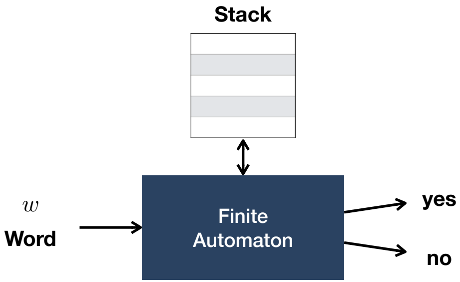
Contents

1. Pushdown Automata
    - Definition
    - Transition Diagram
    - Pushdown Automata in Scala
    - Configurations and One-Step Moves
    - Acceptance by Final States
    - Acceptance by Empty Stacks

Pushdown Automata

A **pushdown automaton (PDA)** is an ϵ-NFA with a stack:

- In FA, the next state is determined by the current state and symbol.
- In PDA, the next state is determined by the current state, symbol, and the top symbol of the stack.



Definition (Pushdown Automata)

A **pushdown automaton (PDA)** is a 7-tuple:

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$$

where

- Q is a finite set of **states**
- Σ is a finite set of **symbols**
- Γ is a finite set of **stack alphabets**
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \to \mathcal{P}(Q \times \Gamma^*)$ is a **transition function**
- $q_0 \in Q$ is the **initial state**
- $Z \in \Gamma$ is the **initial stack alphabet** (the stack is initially Z)
- $F \subseteq Q$ is a set of **final states**

Definition of Pushdown Automata – Example

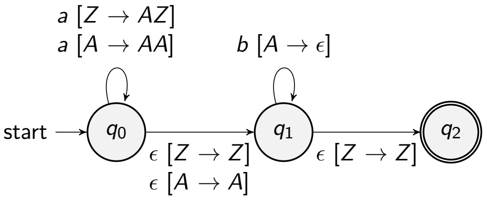
$$P = (\{q_0, q_1, q_2\}, \{\mathtt{a}, \mathtt{b}\}, \{Z, A\}, \delta, q_0, Z, \{q_2\})$$

where

$$\begin{array}{rcl}
\delta(q_0, \mathtt{a}, Z) & = & \{(q_0, AZ)\} \\
\delta(q_0, \mathtt{a}, A) & = & \{(q_0, AA)\} \\
\delta(q_0, \mathtt{b}, Z) & = & \varnothing \\
\delta(q_0, \mathtt{b}, A) & = & \varnothing \\
\delta(q_0, \epsilon, Z) & = & \{(q_1, Z)\} \\
\delta(q_0, \epsilon, A) & = & \{(q_1, A)\} \\
\delta(q_1, \mathtt{a}, Z) & = & \varnothing \\
\delta(q_1, \mathtt{a}, A) & = & \varnothing \\
\delta(q_1, \mathtt{b}, Z) & = & \varnothing \\
\delta(q_1, \mathtt{b}, A) & = & \{(q_1, \epsilon)\} \\
\delta(q_1, \epsilon, Z) & = & \{(q_2, Z)\} \\
\delta(q_1, \epsilon, A) & = & \varnothing \\
\delta(q_2, _, _) & = & \varnothing
\end{array}$$

Transition Diagram

$$P = (\{q_0, q_1, q_2\}, \{\mathtt{a}, \mathtt{b}\}, \{Z, A\}, \delta, q_0, Z, \{q_2\})$$



For example,

$$\begin{array}{rcl} \delta(q_0, \mathtt{a}, Z) & = & \{(q_0, AZ)\} \\ \delta(q_0, \mathtt{a}, A) & = & \{(q_0, AA)\} \end{array}$$

Pushdown Automata in Scala

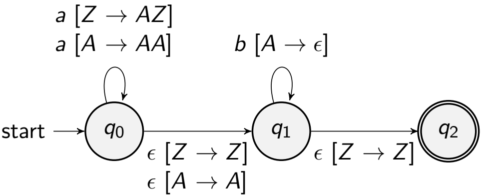
$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$$

$$\delta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \to \mathcal{P}(Q \times \Gamma^*)$$

```scala
// The type definitions of states and symbols
type State = Int
type Symbol = Char
type Word = String
type Alphabet = Char

// The definition of PDA
case class PDA(
  states: Set[State],
  symbols: Set[Symbol],
  alphabets: Set[Alphabet],
  trans: Map[(State, Option[Symbol], Alphabet), Set[(State, List[Alphabet])]],
  initState: State,
  initAlphabet: Alphabet,
  finalStates: Set[State],
)
```

Pushdown Automata in Scala – Example




```scala
// An example of PDA
val pda1: PDA = PDA(
  states       = Set(0, 1, 2),          symbols       = Set('a', 'b'),
  alphabets    = Set('Z', 'A'),         trans         = Map(
    (0, Some('a'), 'Z') -> Set((0, List('A', 'Z'))),
    (0, Some('a'), 'A') -> Set((0, List('A', 'A'))),
    (0, None,      'Z') -> Set((1, List('Z'))),
    (0, None,      'A') -> Set((1, List('A'))),
    (1, Some('b'), 'A') -> Set((1, List())),
    (1, None,      'Z') -> Set((2, List('Z'))),
  ).withDefaultValue(Set()),
  initState    = 0,                     initAlphabet  = 'Z',
  finalStates = Set(2),
)
```

Configurations and One-Step Moves

Definition (Configurations of PDA)

A **configuration** of a PDA P is a triple (q, w, α) where

- $q \in Q$: the current state
- $w \in \Sigma^*$: the remaining word
- $\alpha \in \Gamma^*$: the current state of the stack

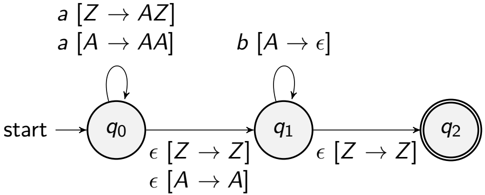
Definition (One-Step Moves of PDA)

A **one-step move** ($\vdash$) of a PDA P is a transition from a configuration to another configuration:

$$\begin{array}{rl} (q, aw, A\beta) \vdash (p, w, \alpha\beta) & \text{if} \quad (p, \alpha) \in \delta(q, a, A) \\ (q, w, A\beta) \vdash (p, w, \alpha\beta) & \text{if} \quad (p, \alpha) \in \delta(q, \epsilon, A) \end{array}$$

Configurations and One-Step Moves

$$
\begin{aligned}
(q, aw, A\beta) \vdash (p, w, \alpha\beta) \quad &\text{if} \quad (p, \alpha) \in \delta(q, a, A) \\
(q, w, A\beta) \vdash (p, w, \alpha\beta) \quad &\text{if} \quad (p, \alpha) \in \delta(q, \epsilon, A)
\end{aligned}
$$



$$
\begin{aligned}
(q_0, \texttt{ab}, Z) &\vdash (q_0, \texttt{b}, AZ) && (\because (q_0, AZ) \in \delta(q_0, \texttt{a}, Z)) \\
&\vdash (q_1, \texttt{b}, AZ) && (\because (q_1, A) \in \delta(q_0, \epsilon, A)) \\
&\vdash (q_1, \epsilon, Z) && (\because (q_1, \epsilon) \in \delta(q_1, \texttt{b}, A)) \\
&\vdash (q_2, \epsilon, Z) && (\because (q_2, Z) \in \delta(q_1, \epsilon, Z))
\end{aligned}
$$

Acceptance by Final States

Definition (Acceptance by Final States)

For a given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, the language accepted by **final states** is defined as:

$$L_F(P) = \{w \in \Sigma^* \mid (q_0, w, Z) \vdash^* (q, \epsilon, \alpha) \text{ for some } q \in F, \alpha \in \Gamma^*\}$$

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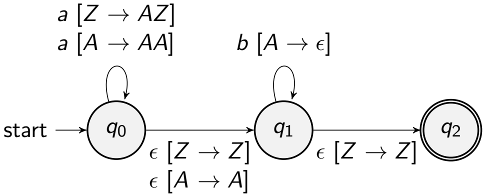

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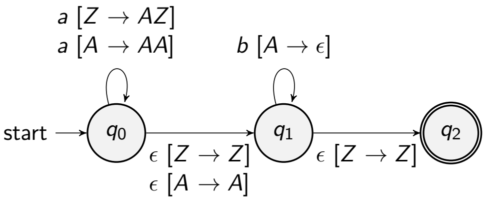

$$(q_0, \mathtt{ab}, Z) \vdash^* (q_2, \epsilon, Z) \quad \Longrightarrow \quad \mathtt{ab} \in L_F(P)$$

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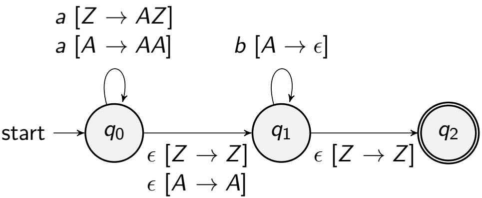

$$(q_0, \texttt{ab}, Z) \vdash^* (q_2, \epsilon, Z) \quad \Longrightarrow \quad \texttt{ab} \in L_F(P)$$
$$(q_0, \texttt{aabb}, Z) \vdash^* (q_2, \epsilon, Z) \quad \Longrightarrow \quad \texttt{aabb} \in L_F(P)$$

Acceptance by Final States

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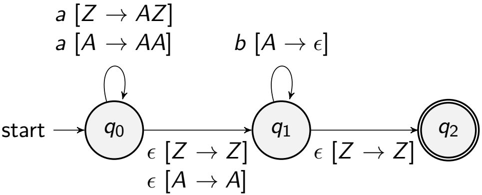

$$\begin{array}{rcr} (q_0, \mathtt{ab}, Z) \vdash^* (q_2, \epsilon, Z) & \Longrightarrow & \mathtt{ab} \in L_F(P) \\ (q_0, \mathtt{aabb}, Z) \vdash^* (q_2, \epsilon, Z) & \Longrightarrow & \mathtt{aabb} \in L_F(P) \end{array}$$

$$L_F(P) = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$$

Acceptance by Final States

$$L_F(P) = \{w \in \Sigma^* \mid (q_0, w, Z) \vdash^* (q, \epsilon, \alpha) \text{ for some } q \in F, \alpha \in \Gamma^*\}$$

```
// The type definition of configurations
type Config = (State, Word, List[Alphabet])

// Configurations reachable from the initial configuration by one-step moves
def reachableConfig(pda: PDA)(word: Word): Set[Config] = ...

// Acceptance by final states
def acceptByFinalState(pda: PDA)(word: Word): Boolean =
  reachableConfig(pda)(word).exists(config => {
    val (q, w, xs) = config
    w.isEmpty && pda.finalStates.contains(q)
  })

acceptByFinalState(pda1)("ab")   // true
acceptByFinalState(pda1)("aba")  // false
acceptByFinalState(pda1)("aabb") // true
acceptByEmptyStack(pda1)("abab") // false
```

Acceptance by Empty Stacks

Definition (Acceptance by Empty Stacks)

For a given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, the language accepted by **empty stacks** is defined as:

$$L_E(P) = \{w \in \Sigma^* \mid (q_0, w, Z) \vdash^* (q, \epsilon, \epsilon) \text{ for some } q \in Q\}$$

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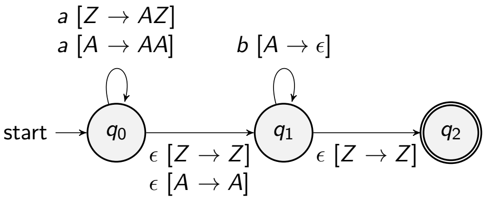

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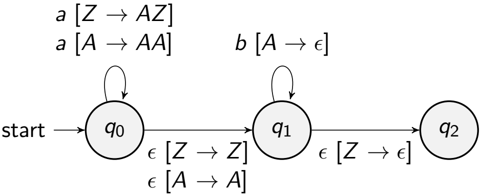

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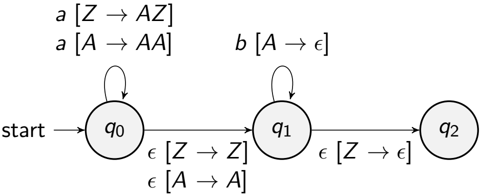

$$(q_0, \mathtt{ab}, Z) \vdash^* (q_2, \epsilon, \epsilon) \quad \Longrightarrow \quad \mathtt{ab} \in L_E(P)$$

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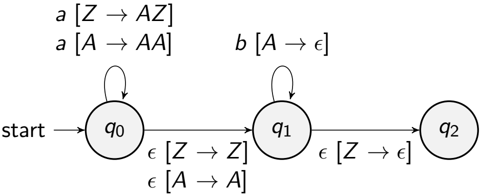

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Acceptance by Empty Stacks

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// Acceptance by empty stacks
def acceptByEmptyStack(pda: PDA)(word: Word): Boolean =
  reachableConfig(pda)(word).exists(config => {
    val (q, w, xs) = config
    w.isEmpty && xs.isEmpty
  })

// Another example of PDA
val pda2: PDA = ...

acceptByEmptyStack(pda2)("ab")   // true
acceptByEmptyStack(pda2)("aba")  // false
acceptByEmptyStack(pda2)("aabb") // true
acceptByEmptyStack(pda2)("abab") // false
```

Summary

1. Pushdown Automata
   - Definition
   - Transition Diagram
   - Pushdown Automata in Scala
   - Configurations and One-Step Moves
   - Acceptance by Final States
   - Acceptance by Empty Stacks

Next Lecture

- Examples of Pushdown Automata

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