Lecture 1 – Mathematical Preliminaries

COSE215: Theory of Computation

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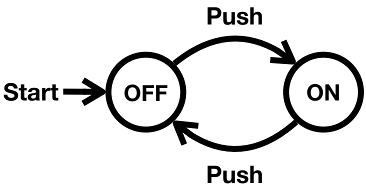
PLRG

2023 Spring

Recall



Theorem

The current state is OFF *if and only if the button is pushed* **even** *times.*

- Is it possible to prove it?

Let's learn **mathematical background and notation.**

Contents

Contents

Notations in Logics

Notation	Description
A, B	arbitrary **statements**.
$P(x)$	a **predicate** that involves a **variable** x.
$A \wedge B$	the **conjunction** of A and B. (i.e., A and B).
$A \vee B$	the **disjunction** of A and B. (i.e., A or B).
$\neg A$	the **negation** of A.

Notations in Logics

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Notation	Description
$A \Rightarrow B$	the **implication** of A and B (i.e., if A then B).
$A \Leftrightarrow B$	A **if and only if (iff)** B (i.e., $A \Rightarrow B \wedge B \Rightarrow A$).
$\forall x \in X.\ P(x)$	the **universal quantifier** (i.e, for all x in X, $P(x)$ holds).
$\exists x \in X.\ P(x)$	the **existential quantifier** (i.e., there exists x in X such that $P(x)$ holds).

Notations in Set Theory

- A **set** is a collection of elements, e.g.,
  - $\mathbb{N} = \{0, 1, 2, \cdots\}$
  - $\{x \in \mathbb{N} \mid x \text{ is even}\} = \{0, 2, 4, 6, 8, 10, 12, \cdots\}$
  - $\{x \in \mathbb{N} \mid x^2\} = \{0, 1, 4, 9, 16, 25, 36, \cdots\}$
- The **empty set** is denoted by $\varnothing$.
- The **cardinality** of a set X is denoted by $|X|$.
- A **subset** X of a set Y is denoted by $X \subseteq Y$.

$$X \subseteq Y \iff \forall x \in X.\ x \in Y$$

- A **proper subset** X of a set Y is denoted by $X \subset Y$.

$$X \subset Y \iff X \subseteq Y \land X \neq Y$$

Notations in Set Theory

- The **union** of sets

$$\begin{aligned} X \cup Y &= \{x \mid x \in X \vee x \in Y\} \\ \bigcup \mathcal{C} &= X_1 \cup X_2 \cup \cdots \cup X_n = \{x \mid \exists X \in \mathcal{C}.\ x \in X\} \end{aligned}$$

  where $\mathcal{C} = \{X_1, X_2, \cdots, X_n\}$.

- The **intersection** of sets

$$\begin{aligned} X \cap Y &= \{x \mid x \in X \wedge x \in Y\} \\ \bigcap \mathcal{C} &= X_1 \cap X_2 \cap \cdots \cap X_n = \{x \mid \forall X \in \mathcal{C}.\ x \in X\} \end{aligned}$$

  where $\mathcal{C} = \{X_1, X_2, \cdots, X_n\}$.

- The **difference** of sets

$$X \setminus Y = \{x \mid x \in X \wedge x \notin Y\}$$

Notations in Set Theory

- The **complement** of a set X is denoted by $\overline{X}$.

$$\overline{X} = \{x \mid x \in U \land x \notin X\}$$

where U is the **universal set**.

- The **power set** of a set X is denoted by 2^X or $\mathcal{P}(X)$.

$$2^X = \mathcal{P}(X) = \{Y \mid Y \subseteq X\}$$

- The **Cartesian product** of sets X and Y is denoted by $X \times Y$.

$$X \times Y = \{(x, y) \mid x \in X \land y \in Y\}$$

Contents

Inductions on Integers

Definition (Inductions on Integers)

Let $P(n)$ be a predicate on integers, and if

- **(Basis Case)** $P(k)$ holds where k is an integer, and
- **(Induction Case)** for all $n \geq k$, $P(n) \Rightarrow P(n+1)$,

then $P(i)$ holds for all $i \geq k$.

$P(n)$ is called **induction hypothesis**.

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$$
\begin{tikzcd}
P(k) \arrow[r, bend left, "\text{Induction}"] & P(k+1) \arrow[r, bend left, "\text{Induction}"] & \cdots \arrow[r, bend left, "\text{Induction}"] & P(n-1) \arrow[r, bend left, "\text{Induction}"] & P(n) \\
\text{Basis} & & & &
\end{tikzcd}
$$

Inductions on Integers – Examples

Example

Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Proof)

Inductions on Integers – Examples

Example

Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Proof)

- (Basis Case): $0 = 0(0+1)/2 \quad \square$
- (Induction Case): Assume that it holds for n (I.H.). Then,

$$\begin{aligned}
\sum_{i=0}^{n+1} i &= (n+1) + \sum_{i=0}^{n} i \\
&= (n+1) + \frac{n(n+1)}{2} \qquad (\because I.H.) \\
&= \frac{(n+1)(n+2)}{2} \quad \square
\end{aligned}$$

Inductions on Integers – Examples

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Example

Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof)

Inductions on Integers – Examples

Example

Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof)

- (Basis Case): $0^2 = 0(0+1)(2*0+1)/6 \quad \square$
- (Induction Case): Assume that it holds for n (I.H.). Then,

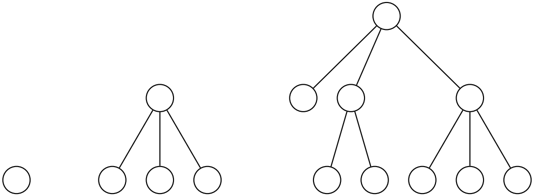
$$\begin{aligned} \sum_{i=0}^{n+1} i^2 &= (n+1)^2 + \sum_{i=0}^{n} i^2 \\ &= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \qquad (\because I.H.) \\ &= \frac{(n+1)(n+2)(2(n+1)+1)}{6} \quad \square \end{aligned}$$

In CS, we often define somethings as **inductively-defined sets**.
For example, we can define **trees** as follows:

Example (Inductive Definition of Trees)

A **tree** is defined as follows:

- **(Basis Case)** A single **node** N is a tree.
- **(Induction Case)** If $T_1, \cdots, T_n$ are trees, then a graph defined with a new node N and edges from N to $T_1, \cdots, T_n$ is a tree as well.



Another example is a set of **arithmetic expressions**:

Example (Inductive Definition of Arithmetic Expressions)

An **arithmetic expression** is defined as follows:

- **(Basis Case)** A **number** or a **variable** is an arithmetic expression.
- **(Induction Case)** If E and F are arithmetic expressions, then so are E+F, $E*F$, and (E).

`42`	`x`	`x + y`
`42 * x`	`(x)`	`(x * y) * z`
`(2 + x) * y`	`x * (x * y)`	`((((x))))`

Structural Inductions

Definition (Structural Inductions)

Let $P(x)$ be a predicate on a **inductively-defined set** X, and if

- **(Basis Case)** $P(b_1), \cdots, P(b_k)$ hold for all basis cases $b_1, \cdots, b_k$.
- **(Induction Case)** for all $x \in X$,

$$P(x_1) \land \cdots \land P(x_n) \Rightarrow P(x)$$

  where $x_1, \cdots, x_n$ are the **sub-structures** of x.

then $P(x)$ holds for all $x \in X$.

$P(x_1), \cdots, P(x_n)$ are called **induction hypotheses**.

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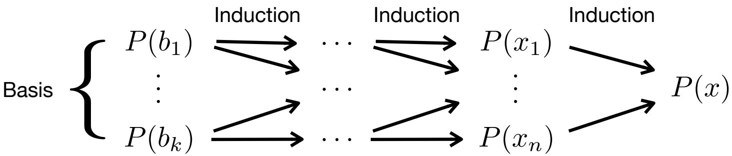

Structural Inductions – Examples

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Example

Prove that for all tree T, the number of nodes in T is equal to the number of edges in T plus one.

Proof)

Structural Inductions – Examples

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Example

Prove that for all tree T, the number of nodes in T is equal to the number of edges in T plus one.

Proof) Let $N(T)$ be the number of node and $E(T)$ be the number of edges in T. Let's prove $\forall T.\ N(T) = E(T) + 1$.

- (Basis Case): $N(T) = 1$ and $E(T) = 0$. $\square$
- (Induction Case): Assume that it holds for $T_1, \cdots, T_n$ (I.H.). Then,

$$\begin{aligned}
N(T) &= 1 + \sum_{i=1}^{n} N(T_i) \\
&= 1 + \sum_{i=1}^{n} (E(T_i) + 1) \qquad (\because I.H.) \\
&= 1 + n + \sum_{i=1}^{n} E(T_i) \\
&= 1 + E(T) \quad \square
\end{aligned}$$

Structural Inductions – Examples

Example

Prove that for all arithmetic expression E, the number of left parentheses in E is equal to the number of right parentheses in E.

Proof)

Example

Prove that for all arithmetic expression E, the number of left parentheses in E is equal to the number of right parentheses in E.

Proof) Let $L(E)$ be the number of left parentheses and $R(E)$ be the number of right parentheses in E. Let's prove $\forall E.\ L(E) = R(E)$.

- (Basis Case): $L(E) = R(E) = 0$ for numbers and variables. $\square$
- (Induction Case): Assume that it holds for E and F (I.H.). Then,

$$\begin{aligned}
L(E{+}F) &= L(E) + L(F) = R(E) + R(F) \qquad (\because I.H.) \\
&= R(E{+}F) \quad \square \\
L(E{*}F) &= L(E) + L(F) = R(E) + R(F) \qquad (\because I.H.) \\
&= R(E{*}F) \quad \square \\
L(\texttt{(}E\texttt{)}) &= L(E) + 1 = R(E) + 1 \qquad (\because I.H.) \\
&= R(\texttt{(}E\texttt{)}) \quad \square
\end{aligned}$$

Mutual Inductions

Definition (Mutual Inductions)

Let $P(x)$ and $Q(x)$ are predicates on integers, and if

- **(Basis Case)** $P(k)$ and $Q(k)$ hold where k is an integer, and
- **(Induction Case)** for all $n \in k$,

$$P(n) \land Q(n) \Rightarrow P(n+1) \land Q(n+1)$$

then $P(i)$ and $Q(i)$ hold for all $i \geq k$.

$P(n)$ and $Q(n)$ are called **induction hypotheses**.

Mutual Inductions

Definition (Mutual Inductions)

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then $P(i)$ and $Q(i)$ hold for all $i \geq k$.

$P(n)$ and $Q(n)$ are called **induction hypotheses**.

$$\text{Basis}\ \left\{ \begin{tikzcd} P(k) \arrow[r, bend left=60, phantom, "\text{Induction}"] \arrow[r] \arrow[dr] & P(k+1) \arrow[r, bend left=60, phantom, "\text{Induction}"] \arrow[r] \arrow[dr] & \cdots \arrow[r, bend left=60, phantom, "\text{Induction}"] \arrow[r] \arrow[dr] & P(n-1) \arrow[r, bend left=60, phantom, "\text{Induction}"] \arrow[r] \arrow[dr] & P(n) \\ Q(k) \arrow[r] \arrow[ur] & Q(k+1) \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & Q(n-1) \arrow[r] \arrow[ur] & Q(n) \end{tikzcd} \right.$$

Mutual Inductions – Examples

Theorem

The current state is OFF *if and only if the button is pushed* **even** *times.*

Proof)

Mutual Inductions – Examples

Theorem

The current state is OFF *if and only if the button is pushed* **even** *times, and the current state is* ON *if and only if the button is pushed* **odd** *times.*

Proof)

Mutual Inductions – Examples

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Theorem

The current state is OFF *if and only if the button is pushed* **even** *times, and the current state is* ON *if and only if the button is pushed* **odd** *times.*

Proof) Let $S(i)$ be the current state after i times of pushing. Let's prove

$$\forall i.\ S(i) = \text{OFF} \iff i \equiv 0 \pmod 2 \qquad (P)$$
$$\forall i.\ S(i) = \text{ON} \iff i \equiv 1 \pmod 2 \qquad (Q)$$

- (Basis Case): $S(0) = \text{OFF} \wedge 0 \equiv 0 \pmod 2$
  - $(P, \Rightarrow)$: $0 \equiv 0 \pmod 2 \implies S(0) = \text{OFF} \Rightarrow 0 \equiv 0 \pmod 2$
  - $(P, \Leftarrow)$: $S(0) = \text{OFF} \implies S(0) = \text{OFF} \Leftarrow 0 \equiv 0 \pmod 2$
  - $(Q, \Rightarrow)$: $\neg(S(0) = \text{ON}) \implies S(0) = \text{ON} \Rightarrow 0 \equiv 1 \pmod 2$
  - $(Q, \Leftarrow)$: $\neg(0 \equiv 1 \pmod 2) \implies S(0) = \text{ON} \Leftarrow 0 \equiv 1 \pmod 2$

Mutual Inductions – Examples

- (Induction Case): Assume that it holds for n (I.H.):

$$
\begin{aligned}
S(n) = \text{OFF} &\iff n \equiv 0 \pmod 2 && (P - I.H.) \\
S(n) = \text{ON} &\iff n \equiv 1 \pmod 2 && (Q - I.H.)
\end{aligned}
$$

  - $(P, \Leftrightarrow)$:

$$
\begin{aligned}
S(n+1) = \text{OFF} &\iff S(n) = \text{ON} \\
&\iff n \equiv 1 \pmod 2 && (\because Q - I.H.) \\
&\iff n + 1 \equiv 0 \pmod 2
\end{aligned}
$$

  - $(Q, \Leftrightarrow)$:

$$
\begin{aligned}
S(n+1) = \text{ON} &\iff S(n) = \text{OFF} \\
&\iff n \equiv 0 \pmod 2 && (\because P - I.H.) \\
&\iff n + 1 \equiv 1 \pmod 2
\end{aligned}
$$

Contents

Symbols & Words

- We first define a finite and non-empty set of **symbols**.
- A **word** $w \in \Sigma^*$ is a sequence of symbols.
  - $\Sigma = \{0, 1\}$ – binary symbols.

$$\epsilon, 0, 1, 00, 01, 10010, \cdots \in \Sigma^*$$

  - $\Sigma = \{\mathrm{a}, \mathrm{b}, \cdots, \mathrm{z}\}$ – lowercase letters.

$$\epsilon, \mathrm{a}, \mathrm{b}, \mathrm{abc}, \mathrm{hello}, \mathrm{cs}, \mathrm{students}, \cdots \in \Sigma^*$$

  - $\Sigma = \{a \mid a \text{ is an Unicode character}\}$ – Unicode characters.

$$\epsilon, \text{안녕하세요}, \text{こんにちは}, \bigstar\blacksquare\blacktriangle\oplus, \cdots \in \Sigma^*$$

Symbols & Words

Notation	Description
ϵ	the **empty word**.
$w_1 w_2$	the **concatenation** of w_1 and w_2. (w_1 is a **prefix** of $w_1 w_2$ and w_2 is a **suffix** of $w_1 w_2$)
w^R	the **reverse** of w.
$\|w\|$	the **length** of w.
Σ^k	the set of all words of length k.
Σ^*	the set of all words (the **Kleene star**). (i.e., $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \cdots = \bigcup_{k=0} \Sigma^k$)
Σ^+	the set of all words except ϵ (the **Kleene plus**). (i.e., $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \cdots = \bigcup_{k=1} \Sigma^k$)

Languages

A **language** $L \subseteq \Sigma^*$ is a set of words. When $\Sigma = \{0, 1\}$, we can define the following languages:

- $L = \{\epsilon, 0, 1\}$ – the empty word, zero, and one.

- $L = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \cdots\}$ – all binary words.

- $L = \{0^n 1^n \mid n \geq 0\}$ – equal number of consecutive zeros and ones.

- $L = \{10, 11, 101, 111, 1011, \cdots\}$ – ???

Languages – Operations

- The **union**, **intersection**, and **difference** of languages:

$$L_1 \cup L_2 \qquad L_1 \cap L_2 \qquad L_1 \setminus L_2$$

- The **reverse** of a language:

$$L^R = \{w^R \mid w \in L\}$$

- The **complement** of a language:

$$\overline{L} = \Sigma^* \setminus L$$

- The **concatenation** of languages:

$$L_1 L_2 = \{w_1 w_2 \mid w_1 \in L_1 \land w_2 \in L_2\}$$

Languages – Operations

- The **power** of a language:

$$\begin{aligned} L^0 &= \{\epsilon\} \\ L^n &= L^{n-1}L \qquad (n \geq 1) \end{aligned}$$

- The **Kleene star** of a language:

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots = \bigcup_{n \geq 0} L^n$$

- The **Kleene plus** of a language:

$$L^+ = L^1 \cup L^2 \cup L^3 \cup \cdots = \bigcup_{n \geq 1} L^n$$

Summary

1. Mathematical Notations
   - Notations in Logics
   - Notations in Set Theory

2. Inductive Proofs
   - Inductions on Integers
   - Structural Inductions
   - Mutual Inductions

3. Notations in Languages
   - Symbols & Words
   - Languages

Next Lecture

- Basic Introduction of Scala

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