

# Lecture 16 – Equivalence of Pushdown Automata and Context-Free Grammars

COSE215: Theory of Computation

Jihyeok Park



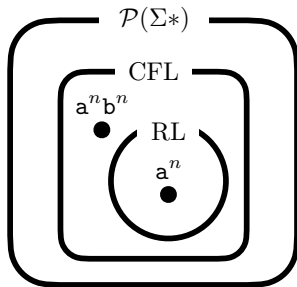
2023 Spring

A **context-free grammar** is a 4-tuple:

$$G = (V, \Sigma, S, P)$$

A **pushdown automaton (PDA)** is a finite automaton with a **stack**.

- Acceptance by **final states**
- Acceptance by **empty stacks**



$\text{PDA}_{\text{FS}}$   
(by final states)

|| ..... ?

$\text{PDA}_{\text{ES}}$   
(by empty stacks)

|| ..... ?

CFG

## 1. Equivalence of PDA by Final States and Empty Stacks

$\text{PDA}_{\text{FS}}$  to  $\text{PDA}_{\text{ES}}$

$\text{PDA}_{\text{ES}}$  to  $\text{PDA}_{\text{FS}}$

## 2. Equivalence of PDA and CFGs

CFGs to  $\text{PDA}_{\text{ES}}$

$\text{PDA}_{\text{ES}}$  to CFGs

$$\begin{array}{ccccc} \text{PDA}_{\text{FS}} & \longleftrightarrow & \text{PDA}_{\text{ES}} & \longleftrightarrow & \text{CFG} \\ \text{(by final states)} & & \text{(by empty stacks)} & & \end{array}$$

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$\text{PDA}_{\text{ES}}$  to  $\text{PDA}_{\text{FS}}$

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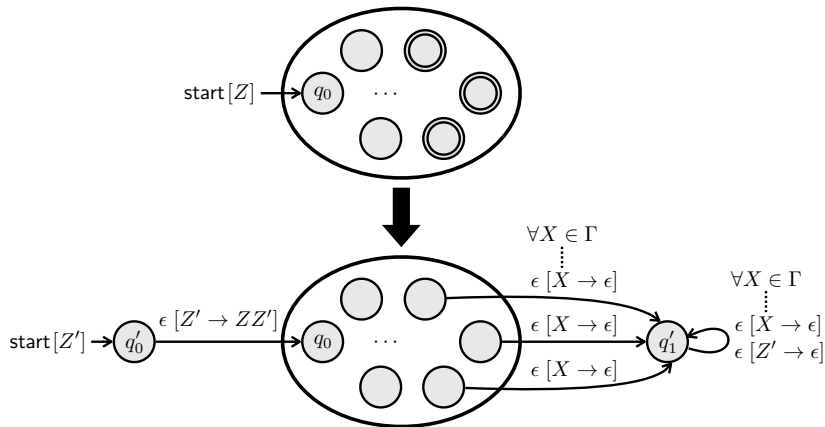


## Theorem (PDA<sub>FS</sub> to PDA<sub>ES</sub>)

*For a given PDA  $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$ ,  $\exists$  PDA  $P'$ .  $L_F(P) = L_E(P')$ .*

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For a given PDA  $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$ ,  $\exists$  PDA  $P'$ .  $L_F(P) = L_E(P')$ .

Define a PDA

$$P' = (Q \cup \{q'_0, q'_1\}, \Sigma, \Gamma \cup \{Z'\}, \delta', q'_0, Z', \emptyset)$$

where

$$\delta'(q'_0, \epsilon, Z') = \{(q_0, ZZ')\}$$

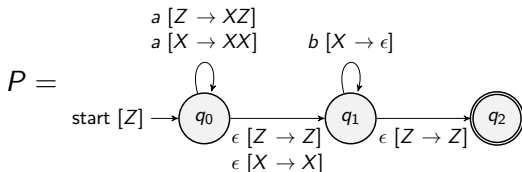
$$\delta'(q \in Q, a \in \Sigma, X \in \Gamma) = \delta(q, a, X)$$

$$\delta'(q \in Q, \epsilon, X \in \Gamma) = \begin{cases} \delta(q, \epsilon, X) \cup \{(q'_1, \epsilon)\} & \text{if } q \in F \\ \delta(q, a, X) & \text{otherwise} \end{cases}$$

$$\delta'(q'_1, \epsilon, X \in \Gamma) = \{(q'_1, \epsilon)\}$$

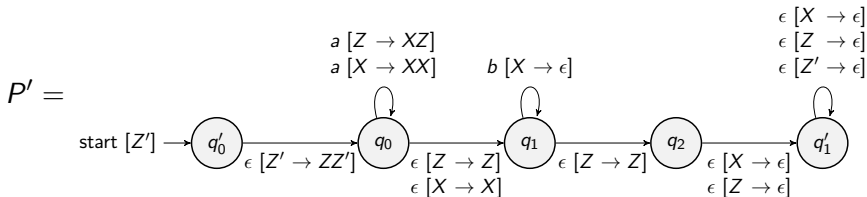
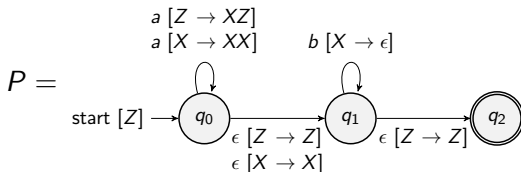
$$\delta'(q'_1, \epsilon, Z') = \{(q'_1, \epsilon)\}$$

$$L_F(P) = L_E(P') = \{a^n b^n \mid n \geq 0\}$$





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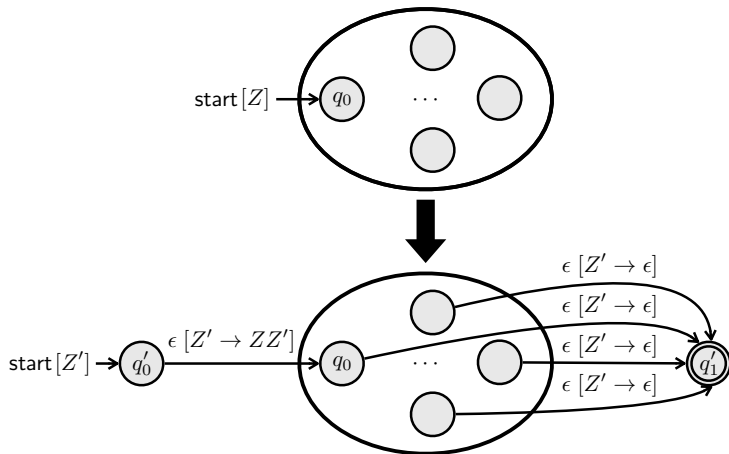


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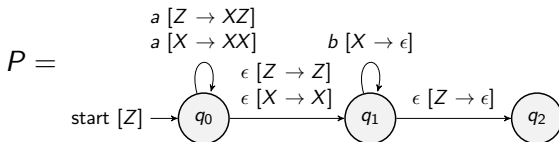
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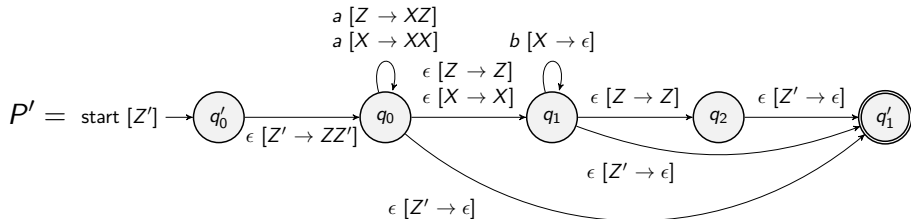
$$\delta'(q \in Q, \epsilon, Z') = \{(q'_1, \epsilon)\}$$

$$L_E(P) = L_F(P') = \{a^n b^n \mid n \geq 0\}$$



$P =$

```
graph LR; start((start [Z])) --> q0((q0)); q0 -- "a [Z -> XZ]  
a [X -> XX]" --> q0; q0 -- "epsilon [Z -> Z]  
epsilon [X -> X]" --> q1((q1)); q1 -- "b [X -> epsilon]" --> q1; q1 -- "epsilon [Z -> epsilon]" --> q2(((q2)))
```



## 1. Equivalence of PDA by Final States and Empty Stacks

$\text{PDA}_{\text{FS}}$  to  $\text{PDA}_{\text{ES}}$

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## 2. Equivalence of PDA and CFGs

CFGs to  $\text{PDA}_{\text{ES}}$

$\text{PDA}_{\text{ES}}$  to CFGs



### Theorem (CFGs to PDA<sub>ES</sub>)

*For a given CFG  $G = (V, \Sigma, S, R)$ ,  $\exists$  PDA  $P$ .  $L(G) = L_E(P)$ .*

Define a PDA

$$P = (\{q\}, \Sigma, V \cup \Sigma, \delta, q, S, \emptyset)$$

where

$$\delta(q, \epsilon, A \in V) = \{(q, \alpha) \mid A \rightarrow \alpha \in R\}$$

$$\delta(q, a \in \Sigma, a \in \Sigma) = \{(q, \epsilon)\}$$



$$\begin{aligned}\delta(q, \epsilon, A \in V) &= \{(q, \alpha) \mid A \rightarrow \alpha \in R\} \\ \delta(q, a \in \Sigma, a \in \Sigma) &= \{(q, \epsilon)\}\end{aligned}$$

Consider the following CFG:

$$S \rightarrow \epsilon \mid aSb \mid bSa \mid SS$$

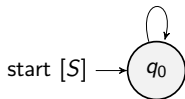
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Then, the equivalent PDA (by empty stacks) is:

$$\begin{aligned}\epsilon &[S \rightarrow \epsilon] \\ \epsilon &[S \rightarrow aSb] \\ \epsilon &[S \rightarrow bSa] \\ \epsilon &[S \rightarrow SS] \\ a &[a \rightarrow \epsilon] \\ b &[b \rightarrow \epsilon]\end{aligned}$$



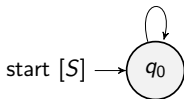
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Then, the equivalent PDA (by empty stacks) is:

$\epsilon [S \rightarrow \epsilon]$	$(q, abab, S) \vdash (q, abab, aSb)$
$\epsilon [S \rightarrow aSb]$	$\vdash (q, bab, Sb)$
$\epsilon [S \rightarrow bSa]$	$\vdash (q, bab, bSab)$
$\epsilon [S \rightarrow SS]$	$\vdash (q, ab, Sab)$
$a [a \rightarrow \epsilon]$	$\vdash (q, ab, ab)$
$b [b \rightarrow \epsilon]$	$\vdash (q, b, b)$
	$\vdash (q, \epsilon, \epsilon)$



### Theorem (PDA<sub>ES</sub> to CFGs)

*For a given PDA  $P = (Q = \{q_0, \dots, q_{n-1}\}, \Sigma, \Gamma, \delta, q_0, Z, F)$ ,  
 $\exists$  CFG  $G$ .  $L_E(P) = L(G)$ .*

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Consider the set of variables  $V = \{S\} \cup \{A_{i,j}^X \mid 0 \leq i, j < n \wedge X \in \Gamma\}$ .

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 Then, define a CFG:

- For all  $0 \leq j < n$ ,

$$S \rightarrow A_{0,j}^Z$$

- For all  $q_i \in Q$ ,  $a \in \Sigma \cup \{\epsilon\}$ , and  $X \in \Gamma$ , consider any  $(q_j, X_1 \cdots X_m) \in \delta(q_i, a, X)$  and  $0 \leq k_1, \dots, k_m < n$ . Then,

$$A_{i,k_m}^X \rightarrow a A_{j,k_1}^{X_1} A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$

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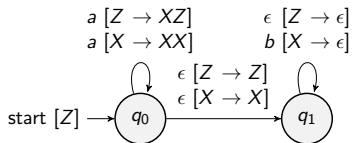
$$A_{i,k_m}^X \rightarrow a A_{j,k_1}^{X_1} A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$

Note that each variable  $A_{i,j}^X$  generates all words that cause the PDA to go from state  $q_i$  to state  $q_j$  by popping  $X$ :

$$A_{i,j}^X \Rightarrow^* w \quad \text{if and only if} \quad (q_j, w, X) \vdash^* (q_i, \epsilon, \epsilon)$$

$$S \rightarrow A_{0,j}^Z \qquad A_{i,k_m}^X \rightarrow a A_{j,k_1}^{X_1} A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$

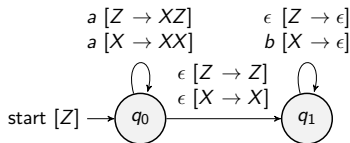
Consider the following PDA (by empty stacks):





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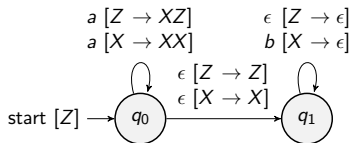


Then, the equivalent CFG is:

$$\begin{aligned}
 S &\rightarrow A_{0,0}^Z \mid A_{0,1}^Z \\
 A_{0,0}^Z &\rightarrow a A_{0,0}^X A_{0,0}^Z \mid a A_{0,1}^X A_{1,0}^Z \mid A_{1,0}^Z \\
 A_{0,1}^Z &\rightarrow a A_{0,0}^X A_{0,1}^Z \mid a A_{0,1}^X A_{1,1}^Z \mid A_{1,1}^Z \\
 A_{0,0}^X &\rightarrow a A_{0,0}^X A_{0,0}^X \mid a A_{0,1}^X A_{1,0}^X \mid A_{1,0}^X \\
 A_{0,1}^X &\rightarrow a A_{0,0}^X A_{0,1}^X \mid a A_{0,1}^X A_{1,1}^X \mid A_{1,1}^X \\
 A_{1,1}^Z &\rightarrow \epsilon \\
 A_{1,1}^X &\rightarrow b
 \end{aligned}$$

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 \end{aligned}$$

$$\begin{aligned}
 S &\Rightarrow A_{0,1}^Z \\
 &\Rightarrow a A_{0,1}^X A_{1,1}^Z \\
 &\Rightarrow aa A_{0,1}^X A_{1,1}^X A_{1,1}^Z \\
 &\Rightarrow aa A_{1,1}^X A_{1,1}^X A_{1,1}^Z \\
 &\Rightarrow aab A_{1,1}^X A_{1,1}^Z \\
 &\Rightarrow aabb A_{1,1}^Z \\
 &\Rightarrow aabb
 \end{aligned}$$

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- Deterministic Pushdown Automata (DPDA)

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