Lecture 4 – Nondeterministic Finite Automata (NFA)

COSE215: Theory of Computation

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PLRG

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Recall

1. Deterministic Finite Automata (DFA)
   - Definition
   - Transition Diagram and Transition Table
   - Extended Transition Function
   - Acceptance of a Word
   - Language of DFA (Regular Language)
   - Examples

Contents

Definition of NFA

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Definition (Nondeterministic Finite Automaton (NFA))

A **nondeterministic finite automaton** is a 5-tuple:

$$N = (Q, \Sigma, \delta, q_0, F)$$

- Q is a finite set of **states**
- Σ is a finite set of **symbols**
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is the **transition function**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

$$N = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

$$\begin{array}{lll} \delta(q_0, 0) = \{q_0, q_1\} & \delta(q_1, 0) = \{q_2\} & \delta(q_2, 0) = \varnothing \\ \delta(q_0, 1) = \{q_0\} & \delta(q_1, 1) = \varnothing & \delta(q_2, 1) = \varnothing \end{array}$$

Definition of NFA

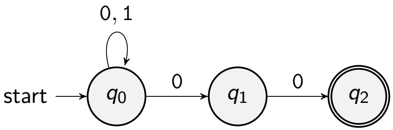
```scala
// The type definitions of states and symbols
type State = Int
type Symbol = Char
// The definition of NFA
case class NFA(
  states: Set[State],
  symbols: Set[Symbol],
  trans: Map[(State, Symbol), Set[State]],
  initState: State,
  finalStates: Set[State],
)
// An example of NFA
val nfa: NFA = NFA(
  states  = Set(0, 1, 2),
  symbols = Set('0', '1'),
  trans = Map(
    (0, '0') -> Set(0, 1),  (1, '0') -> Set(2),   (2, '0') -> Set(),
    (0, '1') -> Set(0),     (1, '1') -> Set(),    (2, '1') -> Set(),
  ),
  initState  = 0,
  finalStates = Set(2),
)
```

Transition Diagram and Transition Table

$$N = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

$$\delta(q_0, 0) = \{q_0, q_1\} \qquad \delta(q_1, 0) = \{q_2\} \qquad \delta(q_2, 0) = \varnothing$$
$$\delta(q_0, 1) = \{q_0\} \qquad \delta(q_1, 1) = \varnothing \qquad \delta(q_2, 1) = \varnothing$$

Transition Diagram



Transition Table

q	0	1
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\{q_2\}$	$\varnothing$
$*q_2$	$\varnothing$	$\varnothing$

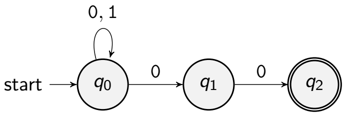
Extended Transition Function

Definition (Extended Transition Function)

For a given NFA $N = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = \{q\}$
- **(Induction Case)** $\delta^*(q, aw) = \bigcup_{q' \in \delta(q,a)} \delta^*(q', w)$

$N =$



$\delta^*(q_0, 100)$

Extended Transition Function

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Definition (Extended Transition Function)

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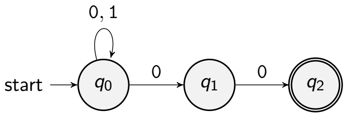


$$
\begin{aligned}
\delta^*(q_0, 100) &= \bigcup_{q' \in \delta(q_0,1)} \delta^*(q', 00) && = \delta^*(q_0, 00) \\
&= \bigcup_{q' \in \delta(q_0,0)} \delta^*(q', 0) && = \delta^*(q_0, 0) \cup \delta^*(q_1, 0) \\
&= \bigcup_{q' \in \delta(q_0,0)} \delta^*(q', \epsilon) \cup \bigcup_{q' \in \delta(q_1,0)} \delta^*(q', \epsilon) && = \delta^*(q_0, \epsilon) \cup \delta^*(q_1, \epsilon) \cup \delta^*(q_2, \epsilon) \\
&= \{q_0, q_1, q_2\}
\end{aligned}
$$

Extended Transition Function

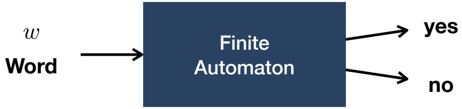
```
// The type definition of words
type Word = String

// A helper function to extract first symbol and rest of word
object `<|` { def unapply(w: Word) = w.headOption.map((_, w.drop(1))) }

// The extended transition function of NFA
def extTrans(nfa: NFA)(q: State, w: Word): Set[State] = w match
  case "" => Set(q)
  case a <| x => for {
    p <- nfa.trans(q, a)
    r <- extTrans(nfa)(p, x)
  } yield r

// An example transition for a word "100"
extTrans(nfa)(0, "100") // Set(0, 1, 2)
```

Acceptance of a Word



Definition (Acceptance of a Word)

For a given NFA $N = (Q, \Sigma, \delta, q_0, F)$, we say that N **accepts** a word $w \in \Sigma^*$ if and only if $\delta^*(q_0, w) \cap F \neq \varnothing$

```
// The acceptance of a word by NFA
def accept(nfa: NFA)(w: Word): Boolean =
  val curStates: Set[State] = extTrans(nfa)(nfa.initState, w)
  curStates.intersect(nfa.finalStates).nonEmpty

// An example acceptance of a word "100"
accept(nfa)("100") // true
```

Language of NFA

Definition (Language of NFA)

For a given NFA $N = (Q, \Sigma, \delta, q_0, F)$, the **language** of N is defined as follows:

$$L(N) = \{w \in \Sigma^* \mid N \text{ accepts } w\}$$

Language of NFA

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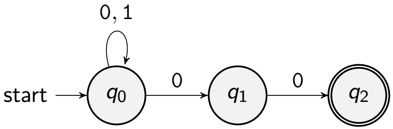
$N =$ 
$$\begin{tikzcd}
\text{start} \arrow[r] & q_0 \arrow[loop above, "0,1"] \arrow[r, "0"] & q_1 \arrow[r, "0"] & q_2
\end{tikzcd}$$

Language of NFA

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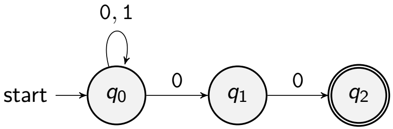
$$L(N) = \{w \in \Sigma^* \mid N \text{ accepts } w\}$$

$N =$



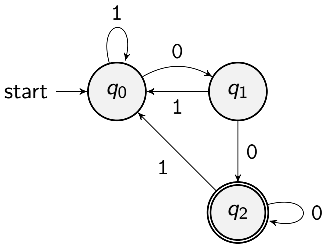
$$L(N) = \{w00 \mid w \in \Sigma^*\}$$

Examples

- $L = \{\mathrm{a}^n\mathrm{b} \mid n \geq 0\}$
- $L = \{w \in \{0,1\} \mid w \text{ contains at least two } 0's\}$
- $L = \{w \in \{0,1\} \mid w \text{ contains exactly two } 0's\}$
- $L = \{w \in \{0,1\} \mid w \text{ has three consecutive } 0's\}$
- $L = \{w \in \{0,1\} \mid \mathbb{N}(w) \equiv 0 (\mathrm{mod}\ 3)\}$
  where $\mathbb{N}(w)$ is a natural number represented by w.
- $L = \{\mathrm{a}^n\mathrm{b}^n \mid n \geq 0\}$

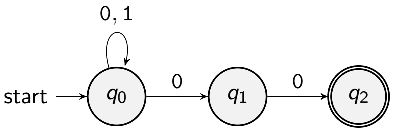
Equivalence of DFA and NFA

$D =$



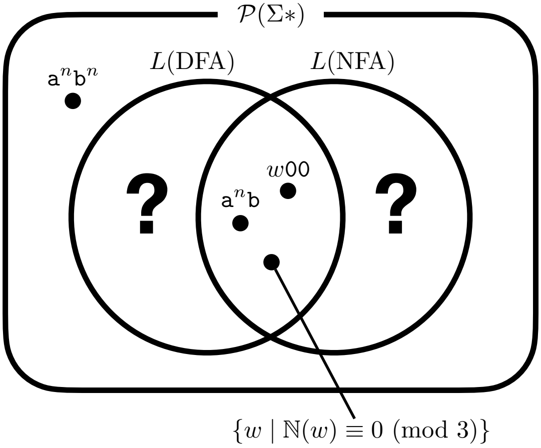
$$L(D) = \{w00 \mid w \in \Sigma^*\}$$

$N =$



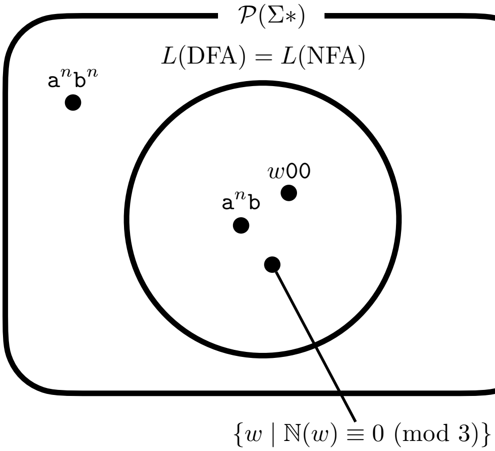
Equivalence of DFA and NFA

Is there any language that is the language of a DFA but not the language of an NFA, or vice versa?



Equivalence of DFA and NFA

Is there any language that is the language of a DFA but not the language of an NFA, or vice versa? No! DFA and NFA are **equivalent**.


$\mathcal{P}(\Sigma*)$
$L(\text{DFA}) = L(\text{NFA})$
$\texttt{a}^n\texttt{b}^n$
$w\texttt{00}$
$\texttt{a}^n\texttt{b}$
$\{w \mid \mathbb{N}(w) \equiv 0 \pmod 3\}$

Equivalence of DFA and NFA

Theorem (Equivalence of DFA and NFA)

A language L is the language $L(D)$ of a DFA D if and only if L is the language $L(N)$ of an NFA N.

Proof) By the following two theorems.

Theorem (DFA to NFA)

For a given DFA $D = (Q, \Sigma, \delta, q, F)$, $\exists$ NFA N. $L(D) = L(N)$.

Theorem (NFA to DFA – Subset Construction)

For a given NFA $N = (Q, \Sigma, \delta, q_0, F)$, $\exists$ DFA D. $L(D) = L(N)$.

DFA $\to$ NFA

Theorem (DFA to NFA)

For a given DFA $D = (Q, \Sigma, \delta_D, q_0, F)$, $\exists$ NFA N. $L(D) = L(N)$.

Proof) Define an NFA

$$N = (Q, \Sigma, \delta_N, q_0, F)$$

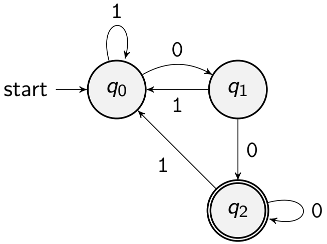
where

- $\forall q \in Q.\ \forall a \in \Sigma.$

$$\delta_N(q, a) = \{\delta_D(q, a)\}$$

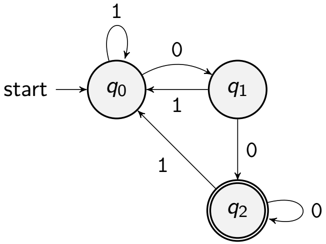
DFA $\rightarrow$ NFA – Example

DFA D



q	0	1
$\rightarrow q_0$	q_1	q_0
q_1	q_2	q_0
$*q_2$	q_2	q_0

NFA N



q	0	1
$\rightarrow q_0$	$\{q_1\}$	$\{q_0\}$
q_1	$\{q_2\}$	$\{q_0\}$
$*q_2$	$\{q_2\}$	$\{q_0\}$

DFA $\to$ NFA – Proof

Lemma

$\forall q \in Q.\ \forall w \in \Sigma^*.\ \delta^*_N(q, w) = \{\delta^*_D(q, w)\}.$

Proof) By induction on the **length of word**.

- **(Base Case)** $\delta^*_N(q, \epsilon) = \{q\} = \{\delta^*_D(q, \epsilon)\}$.
- **(Inductive Case)** Assume it holds for w (I.H.).

$$\begin{aligned}
\delta^*_N(q, aw) &= \textstyle\bigcup_{q' \in \delta_N(q,a)} \delta^*_N(q', w) && (\because \text{definition of } \delta^*_N) \\
&= \textstyle\bigcup_{q' \in \{\delta_D(q,a)\}} \delta^*_N(q', w) && (\because \text{definition of } \delta_N) \\
&= \delta^*_N(\delta_D(q, a), w) && \\
&= \{\delta^*_D(\delta_D(q, a), w)\} && (\because \text{I.H.}) \\
&= \{\delta^*_D(q, aw)\} && (\because \text{definition of } \delta^*) \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{Then, } w \in L(D) &\iff \delta^*_D(q_0, w) \in F && (\because \text{definition of } L(D)) \\
&\iff \{\delta^*_D(q_0, w)\} \cap F \neq \varnothing && (\because \text{set theory}) \\
&\iff \delta^*_N(q_0, w) \cap F \neq \varnothing && (\because \text{above lemma}) \\
&\iff w \in L(N) && (\because \text{definition of } L(N)) \quad \square
\end{aligned}$$

DFA $\leftarrow$ NFA (Subset Construction)

Theorem (NFA to DFA – Subset Construction)

For a given NFA $N = (Q, \Sigma, \delta_N, q_0, F)$, $\exists$ DFA D. $L(D) = L(N)$.

Proof) Define a DFA

$$D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$$

where

- $Q_D = \mathcal{P}(Q)$
- $\forall S \in Q_D.\ \forall a \in \Sigma.$
$$\delta_D(S, a) = \bigcup_{q \in S} \delta_N(q, a)$$
- $F_D = \{S \in Q_D \mid S \cap F \neq \varnothing\}$

DFA $\leftarrow$ NFA (Subset Construction) – Example

DFA D

q	0	1
$\varnothing$	$\varnothing$	$\varnothing$
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_1\}$	$\{q_2\}$	$\varnothing$
$*\{q_2\}$	$\varnothing$	$\varnothing$
$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$	$\{q_0\}$
$*\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$
$*\{q_1, q_2\}$	$\{q_2\}$	$\varnothing$
$*\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0\}$

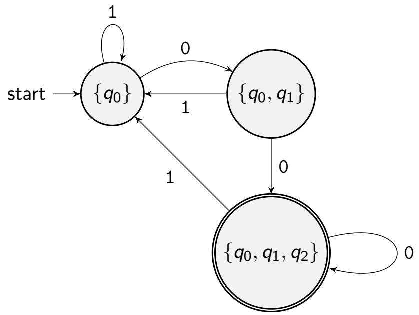
NFA N

$$\begin{tikzcd}
\text{start} \arrow[r] & q_0 \arrow[loop above, "0,1"] \arrow[r, "0"] & q_1 \arrow[r, "0"] & q_2
\end{tikzcd}$$

q	0	1
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\{q_2\}$	$\varnothing$
$*q_2$	$\varnothing$	$\varnothing$

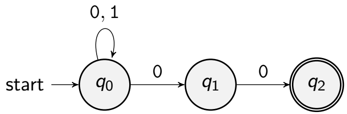
DFA ← NFA (Subset Construction) – Example

DFA D



q	0	1
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$	$\{q_0\}$
$*\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0\}$

NFA N



q	0	1
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\{q_2\}$	$\varnothing$
$*q_2$	$\varnothing$	$\varnothing$

DFA $\leftarrow$ NFA (Subset Construction) – Proof

Lemma

$\forall S \in Q_D.\ \forall w \in \Sigma^*.\ \delta_D^*(S, w) = \bigcup_{q\in S} \delta_N^*(q, w)$

Proof) By induction on the **length of word**.

- **(Base Case)** $\delta_N^*(S, \epsilon) = S = \bigcup_{q\in S} \delta_N^*(q, \epsilon)$.
- **(Inductive Case)** Assume it holds for w (I.H.).

$$\begin{array}{rcll}
\delta_D^*(S, aw) & = & \delta_D^*(\delta_D(S, a), w) & (\because \text{definition of } \delta_D^*) \\
 & = & \delta_D^*(\bigcup_{q\in S} \delta_N(q, a), w) & (\because \text{definition of } \delta_D) \\
 & = & \bigcup_{q\in S} \bigcup_{q'\in\delta_N(q,a)} \delta_N^*(q', w) & (\because \text{I.H.}) \\
 & = & \bigcup_{q\in S} \delta_N^*(q, aw) & (\because \text{definition of } \delta_N^*)
\end{array}$$

$$\begin{array}{rcll}
\text{Then, } w \in L(D) & \iff & \delta_D^*(\{q_0\}, w) \in F_D & (\because \text{definition of } L(D)) \\
 & \iff & \delta_D^*(\{q_0\}, w) \cap F_N \neq \varnothing & (\because \text{definition of } F_D) \\
 & \iff & \delta_N^*(q_0, w) \cap F \neq \varnothing & (\because \text{above lemma}) \\
 & \iff & w \in L(N) & (\because \text{definition of } L(N)) \quad \square
\end{array}$$

Summary

1. Nondeterministic Finite Automata (NFA)
    - Definition
    - Transition Diagram and Transition Table
    - Extended Transition Function
    - Language of NFA
    - Examples
    - Equivalence of DFA and NFA
        - DFA $\rightarrow$ NFA
        - DFA $\leftarrow$ NFA (Subset Construction)

Next Lecture

- ϵ-Nondeterministic Finite Automata (ϵ-NFA)

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