Lecture 6 – Regular Expressions and Languages
COSE215: Theory of Computation
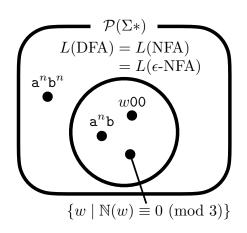
Jihyeok Park
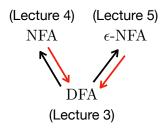
PLRG

2023 Spring

Recall

PLRG

$\mathcal{P}(\Sigma*)$

$L(\text{DFA}) = L(\text{NFA})$
$= L(\epsilon\text{-NFA})$

a^nb^n

$w00$

a^nb

$\{w \mid \mathbb{N}(w) \equiv 0 \pmod 3\}$

(Lecture 4)   (Lecture 5)
NFA ϵ-NFA

DFA
(Lecture 3)

$\longrightarrow$: Subset Construction

Contents

OPLRG

- The **union** of languages:

$$L_1 \cup L_2$$

- The **concatenation** of languages:

$$L_1 \cdot L_2 = \{w_1 w_2 \mid w_1 \in L_1 \land w_2 \in L_2\}$$

- The **Kleene star** of a language:

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots = \bigcup_{n \geq 0} L^n$$

$$L_1 = \{a^n \mid n \geq 0\} \qquad L_2 = \{b^n \mid n \geq 0\}$$

$$L_1 \cup L_2 = \{a^n \text{ or } b^n \mid n \geq 0\}$$
$$L_1 \cdot L_2 = \{a^n b^m \mid n, m \geq 0\}$$
$$L_1^* = L_1 = \{a^n \mid n \geq 0\}$$

Definition of Regular Expressions

Definition (Regular Expressions)

A **regular expression** over a set of symbols Σ is inductively defined as follows:

- **(Basis Case)** $\varnothing$, ϵ, and $a \in \Sigma$ are regular expressions.
- **(Induction Case)** If R_1 and R_2 are regular expressions, then so are $R_1 \mid R_2$, $R_1 \cdot R_2$, R^*, and (R).

$$
\begin{array}{rlll}
R & ::= & \varnothing & \text{(Empty)} \\
  & \mid & \epsilon & \text{(Epsilon)} \\
  & \mid & a & \text{(Symbol)} \\
  & \mid & R \mid R & \text{(Union)} \\
  & \mid & R \cdot R & \text{(Concatenation)} \\
  & \mid & R^* & \text{(Kleene Star)} \\
  & \mid & (R) & \text{(Parentheses)}
\end{array}
$$

$$\texttt{a} \mid \epsilon \cdot \texttt{b}^*$$

$$(\texttt{a} \mid \epsilon) \cdot \texttt{b}^*$$

Precedence of operators:

$$\mid \; < \; \cdot \; < \; {}^*$$

Definition of Regular Expressions

```
// The type definitions of symbols
type Symbol = Char

// The definition of regular expressions
trait RE
case class REEmpty() extends RE
case class REEpsilon() extends RE
case class RESymbol(symbol: Symbol) extends RE
case class REUnion(left: RE, right: RE) extends RE
case class REConcat(left: RE, right: RE) extends RE
case class REStar(re: RE) extends RE
case class REParen(re: RE) extends RE

// Two examples of regular expressions
val re1: RE = REUnion(
  RESymbol('a'),
  REConcat(REEpsilon(), REStar(RESymbol('b'))),
)
val re2: RE = REConcat(
  REParen(REUnion(RESymbol('a'), REEpsilon())),
  REStar(RESymbol('b')),
)
```

Language of Regular Expressions

PLRG

Definition (Language of Regular Expressions)

For a given regular expression R on a set of symbols Σ, the **language** $L(R)$ of R is inductively defined as follows:

$$
\begin{array}{llll}
L(\varnothing) & = & \varnothing & L(R_1 \mid R_2) & = & L(R_1) \cup L(R_2) \\
L(\epsilon) & = & \{\epsilon\} & L(R_1 \cdot R_2) & = & L(R_1) \cdot L(R_2) \\
L(a) & = & \{a\} & L(R^*) & = & L(R)^* \\
& & & L((R)) & = & L(R)
\end{array}
$$

$$
\begin{array}{llll}
L(\mathtt{a} \mid \epsilon \cdot \mathtt{b}^*) & = & L(\mathtt{a}) \cup L(\epsilon \cdot \mathtt{b}^*) & = & \{\mathtt{a}\} \cup L(\epsilon) \cdot L(\mathtt{b})^* \\
& = & \{\mathtt{a}\} \cup \{\epsilon\} \cdot \{\mathtt{b}\}^* & = & \{\mathtt{a}\} \cup \{\mathtt{b}\}^* \\
& = & \{\mathtt{a} \text{ or } \mathtt{b}^n \mid n \geq 0\}
\end{array}
$$

$$
\begin{array}{llll}
L((\mathtt{a} \mid \epsilon) \cdot \mathtt{b}^*) & = & L((\mathtt{a} \mid \epsilon)) \cdot L(\mathtt{b}^*) & = & L(\mathtt{a} \mid \epsilon) \cdot L(\mathtt{b})^* \\
& = & (L(\mathtt{a}) \cup L(\epsilon)) \cdot L(\mathtt{b})^* & = & (\{\mathtt{a}\} \cup \{\epsilon\}) \cdot \{\mathtt{b}\}^* \\
& = & \{\mathtt{ab}^n \text{ or } \mathtt{b}^n \mid n \geq 0\}
\end{array}
$$

Extended Regular Expressions

More operators:

$$R ::= \cdots$$
$$| \quad R^+ \quad \text{(Kleene plus)}$$
$$| \quad R^? \quad \text{(Optional)}$$

Actually, they are just **syntactic sugar** for the existing operators:

$$L(R^+) = L(RR^*) = L(R) \cdot L(R^*)$$

$$L(R^?) = L(R|\epsilon) = L(R) \cup L(\epsilon)$$

For examples,

$$L(\mathtt{a}^+) = L(\mathtt{aa}^*) = L(\mathtt{a}) \cdot L(\mathtt{a}^*) = \{\mathtt{a}\} \cdot \{\mathtt{a}\}^* = \{\mathtt{a}^n \mid n \geq 1\}$$

$$L(\mathtt{b}^?) = L(\mathtt{b}|\epsilon) = L(\mathtt{b}) \cup L(\epsilon) = \{\mathtt{b}\} \cup \{\epsilon\} = \{\mathtt{b}, \epsilon\}$$

Examples

- $L = \{\epsilon, \mathtt{a}\}$

- $L = \{w \in \{0, 1\}^* \mid w \text{ contains at least two } 0's\}$

- $L = \{w \in \{0, 1\}^* \mid w \text{ contains exactly two } 0's\}$

- $L = \{w \in \{0, 1\}^* \mid w \text{ has three consecutive } 0's\}$

- $L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \mathtt{a} \text{ and } \mathtt{b} \text{ alternate in } w\}$

Examples

- $L = \{a^n b^m \mid n \geq 3 \land m \equiv 0 (\text{mod } 2)\}$

- $L = \{a^n b^m \mid n + m \equiv 0 (\text{mod } 2)\}$

- $L = \{w \in \{0,1\}^* \mid \text{the number of 0's is divisible by 3}\}$

- $L = \{w \in \{0,1\}^* \mid \mathbb{N}(w) \equiv 0 \ (\text{mod } 3)\}$
  where $\mathbb{N}(w)$ is a natural number represented by w.

  $$(0 \mid 1(01^*0)^*1)^*$$

- $L = \{a^n b^n \mid n \geq 0\}$ – IMPOSSIBLE ($\nexists$ RE $R. \ L(R) = L$)

PLRG

- Please see
  https://github.com/ku-plrg-classroom/docs/tree/main/fa-examples.
- You don't have to submit it. This is just exercise for your practice.
- The goal is to implement the finite automata (FA) objects in the
  `Implementation.scala` file.

Summary

1. Operations in Languages
   Union
   Concatenation
   Kleene Star

2. Regular Expressions
   Definition
   Language of Regular Expressions
   Extended Regular Expressions
   Examples

- Equivalence of Regular Expressions and Finite Automata

Jihyeok Park

jihyeok_park@korea.ac.kr

https://plrg.korea.ac.kr