Lecture 20 – The Pumping Lemma for Context-Free Languages

COSE215: Theory of Computation

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PLRG

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Recall

- We have learned about the **Pumping Lemma** for **Regular Languages (RLs)**.
- We could use it to **prove** that some languages are **NOT** regular.

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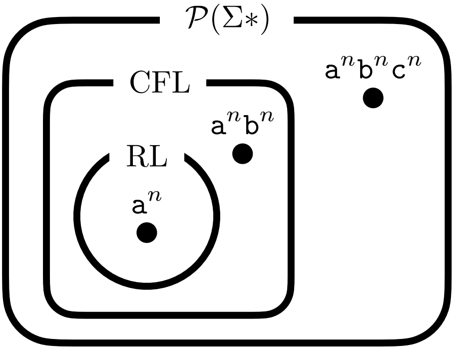
- We could use it to **prove** that some languages are **NOT** regular.
- Is there a similar lemma for **Context-Free Languages (CFLs)**?

Recall

- We have learned about the **Pumping Lemma** for **Regular Languages (RLs)**.
- We could use it to **prove** that some languages are **NOT** regular.
- Is there a similar lemma for **Context-Free Languages (CFLs)**?
- For example, is it possible to prove that the following language?

$$L = \{a^n b^n c^n \mid n \geq 0\}$$



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Size of Parse Trees in Chomsky Normal Form

Theorem (Size of Parse Trees in Chomsky Normal Form)

For a given CFG G in Chomsky Normal Form, for all $w \in L(G)$, if the length of the longest path in the parse tree of w is n, then $|w| \leq 2^{n-1}$. Note that the length of a path is the number of edges in the path.

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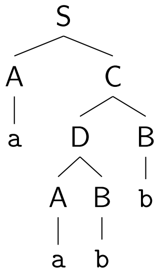
For example, consider the following CFG in CNF, and the parse tree of $w = \texttt{aabb}$. The length of the longest path in the parse tree is 4, and the length of the word is 4. Thus, $|w| = 4 \le 2^3 = 2^{n-1}$.

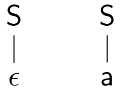
$$
\begin{array}{rcl}
S & \rightarrow & \epsilon \mid AC \mid AB \\
D & \rightarrow & AC \mid AB \\
C & \rightarrow & DB \\
A & \rightarrow & a \\
B & \rightarrow & b
\end{array}
$$



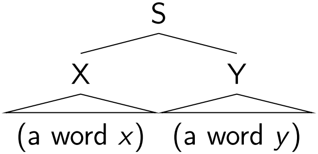
Size of Parse Trees of Chomsky Normal Form – Proof

Proof) Let's perform induction on the length of the longest path n.

- **(Basis Case)** $n = 1$. Then, $|\epsilon| = 0 \leq 2^{1-1}$ and $|a| = 1 \leq 2^{1-1}$.



- **(Induction Case)** The first rule of S is in the form of $S \to XY$. The length of the longest path in the parse tree of X (or Y) is less than or equal to $n - 1$. If $X \Rightarrow^* x \in \Sigma^*$ and $Y \Rightarrow^* y \in \Sigma^*$, then $|x| \leq 2^{n-2}$ and $|y| \leq 2^{n-2}$ ($\because$ I.H.). Thus, $|w| = |x| + |y| \leq 2^{n-2} + 2^{n-2} = 2^{n-1}$.



Lemma (Pumping Lemma for Context-Free Languages)

For a given CFL L, **there exists** *a positive integer n such that* **for all** *$z \in L$, if $|z| \geq n$,* **there exists** *a split $z = uvwxy$ such that*

1. $|vx| > 0$
2. $|vwx| \leq n$
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Pumping Lemma for Context-Free Languages

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$$A \quad = \quad L \text{ is context-free}$$

$$\Downarrow$$

$$B \quad = \quad \exists n > 0.\ \forall z \in L.\ |z| \geq n \Rightarrow \exists z = uvwxy.\ 1 \wedge 2 \wedge 3$$

Proof of Pumping Lemma

- Let L be a context-free language.

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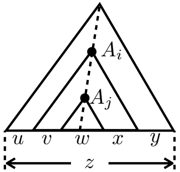

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and
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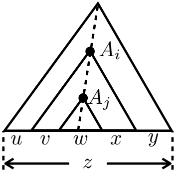

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- $\boxed{① \ |vx| > 0}$
  - Since $i < j$, the word vwx derived from A_i is not equal to the word w derived from A_j.
  - Thus, vx is not an empty word, and $|vx| > 0$.

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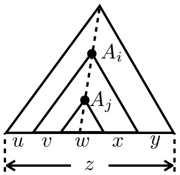

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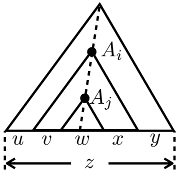
  - Thus, vx is not an empty word, and $|vx| > 0$.
- $\boxed{②\ |vwx| \le n}$
  - Since $p - m \ge i$, the length of the longest path from A_i in the parse tree of z is $p - i + 1$ is less than or equal to $m + 1$.
  - By Theorem of Size of Parse Trees in CNF, the length of the word vwx is less than or equal to $2^m = n$.

Proof of Pumping Lemma - ③



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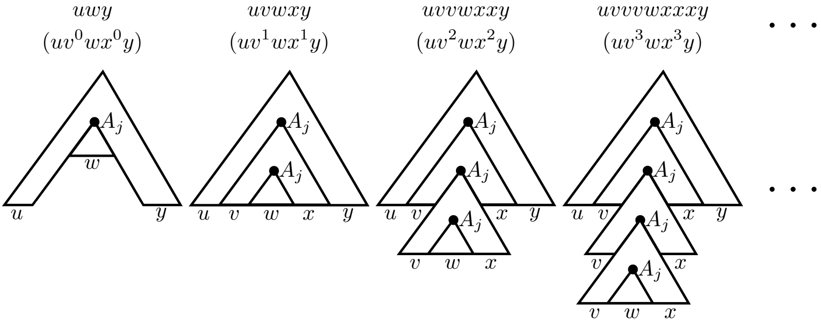
- $\boxed{③\ \forall i \geq 0.\ uv^i wx^i y \in L}$



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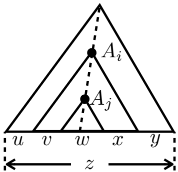

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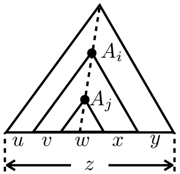

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Proving Languages are Not Context-Free

Lemma (Pumping Lemma for Context-Free Languages)

$$A \quad = \quad L \text{ is context-free}$$

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$$A \implies B \qquad (\text{O})$$

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Lemma (Pumping Lemma for Context-Free Languages)

$$\begin{array}{rcl} A & = & L \text{ is context-free} \\ & & \Downarrow \\ B & = & \exists n > 0.\ \forall z \in L.\ |z| \geq n \Rightarrow \exists z = uvwxy.\ ① \land ② \land ③ \end{array}$$

$$\begin{array}{cccc} A & \Longrightarrow & B & \text{(O)} \\ B & \Longrightarrow & A & \text{(X)} \end{array}$$

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$$A \;=\; L \textit{ is context-free}$$

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$$B \;=\; \exists n > 0.\ \forall z \in L.\ |z| \geq n \Rightarrow \exists z = uvwxy.\ (1) \land (2) \land (3)$$

$$\begin{array}{cccc} A & \Longrightarrow & B & (\text{O}) \\ B & \Longrightarrow & A & (\text{X}) \\ \neg B & \Longrightarrow & \neg A & (\text{O}) \end{array}$$

Proving Languages are Not Context-Free

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A & \Longrightarrow & B & \text{(O)} \\
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\neg B & \Longrightarrow & \neg A & \text{(O)}
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$$

$$
\begin{array}{rcl}
\neg B & = & \forall n > 0.\ \neg(\forall z \in L.\ |z| \geq n \Rightarrow \exists z = uvwxy.\ ① \land ② \land ③) \\
 & = & \forall n > 0.\ \exists z \in L.\ \neg(|z| \geq n \Rightarrow \exists z = uvwxy.\ ① \land ② \land ③) \\
 & = & \forall n > 0.\ \exists z \in L.\ |z| \geq n \land \neg(\exists z = uvwxy.\ ① \land ② \land ③) \\
 & = & \forall n > 0.\ \exists z \in L.\ |z| \geq n \land \forall z = uvwxy.\ \neg(① \land ② \land ③) \\
 & = & \forall n > 0.\ \exists z \in L.\ |z| \geq n \land \forall z = uvwxy.\ \neg(① \land ②) \lor \neg ③ \\
 & = & \forall n > 0.\ \exists z \in L.\ |z| \geq n \land \forall z = uvwxy.\ (① \land ②) \Rightarrow \neg ③
\end{array}
$$

Proving Languages are Not Context-Free

To prove a language L is **NOT** context-free, we need to show that

$$\forall n > 0.\ \exists z \in L.\ |z| \geq n \land \forall z = uvwxy.\ (①\land ②) \Rightarrow \neg ③$$

1. $|vx| > 0$
2. $|vwx| \leq n$
3. $\forall i \geq 0.\ uv^iwx^iy \in L$

Note that $\neg ③ = \exists i \geq 0.\ uv^iwx^iy \notin L$.

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3. $\forall i \geq 0.\ uv^iwx^iy \in L$

Note that $\neg ③ = \exists i \geq 0.\ uv^iwx^iy \notin L$.

We can prove this by following the steps below:

1. Assume any positive integer n is given.
2. Pick a word $z \in L$.
3. Show that $|z| \geq n$.
4. Assume any split $z = uvwxy$ is given, and ① $|vx| > 0 \land$ ② $|vwx| \leq n$.
5. $\neg$③ Pick $i \geq 0$, and show that $uv^iwx^iy \notin L$ using ① and ②.

Example 1

Let's prove that L is **NOT** context-free using the Pumping Lemma:

$$L = \{\mathtt{a}^n \mathtt{b}^n \mathtt{c}^n \mid n \geq 0\}$$

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1. Assume any positive integer n is given.
2. Let $z = \mathrm{a}^n\mathrm{b}^n\mathrm{c}^n \in L$.
3. $|z| = n + n + n = 3n \geq n$.
4. Assume any split $z = uvwxy$ is given, and (1) $|vx| > 0 \land$ (2) $|vwx| \leq n$.
5. Let $i = 0$. We need to show that $\neg$(3) $uv^0wx^0y \notin L$:
   - Since (2) $|vwx| \leq n$,

     $$vx = \mathrm{a}^p\mathrm{b}^q \qquad (\text{or } vx = \mathrm{b}^p\mathrm{c}^q)$$

     where $0 \leq p, q \leq n$.
   - Since (1) $|vx| > 0$, we can remove at least one a or b (or b or c) from z without changing the number of c's (or a's) when $i = 0$.
   - It means that $uv^0wx^0y \notin L$. □

Example 2

Let's prove that L is **NOT** context-free using the Pumping Lemma:

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1. Assume any positive integer n is given.
2. Let $z = 0^n 1 0^n 1 0^n \in L$.
3. $|z| = n + 1 + n + 1 + n = 3n + 2 \geq n$.
4. Assume any split $z = uvwxy$ is given, and 1 $|vx| > 0 \land$ 2 $|vwx| \leq n$.
5. Let $i = 0$. We need to show that $\neg$3 $uv^0wx^0y \notin L$:
   - Since 2 $|vwx| \leq n$,

     vx does not contain 0's in the third block (or the first block).

   - Since 1 $|vx| > 0$, we can remove at least one 0 in the first or second blocks (or second or third blocks) from z without changing the number of 0's in the third block (or first block) when $i = 0$.
   - It means that $uv^0wx^0y \notin L$. $\square$

Example 3

Let's prove that L is **NOT** context-free using the Pumping Lemma:

$$L = \{ww \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

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1. Assume any positive integer n is given.
2. Let $z = \mathtt{a}^n\mathtt{b}^n\mathtt{a}^n\mathtt{b}^n \in L$.
3. $|z| = n + n + n + n = 4n \geq n$.
4. Assume any split $z = uvwxy$ is given, and (1) $|vx| > 0 \wedge$ (2) $|vwx| \leq n$.
5. Let $i = 0$. We need to show that $\neg$(3) $uv^0wx^0y \notin L$:
   - Since (2) $|vwx| \leq n$,

     vx does not contain a's (or b's) in both different blocks.

   - Since (1) $|vx| > 0$, we can remove at least one a (or b) in one block from z without changing the other one when $i = 0$.
   - It means that $uv^0wx^0y \notin L$. $\square$

Example 4

Let's prove that L is **NOT** context-free using the Pumping Lemma:

$$L = \{\mathtt{a}^n\mathtt{b}^m \mid m = n^2\}$$

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Let's prove that L is **NOT** context-free using the Pumping Lemma:

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1. Assume any positive integer n is given.
2. Let $z = \mathtt{a}^n \mathtt{b}^{n^2} \in L$.
3. $|z| = n + n^2 \geq n$.
4. Assume any split $z = uvwxy$ is given, and 1 $|vx| > 0 \land$ 2 $|vwx| \leq n$.
5. Let $i = n + 1$. We need to show that $\neg$3 $uv^{n+1}wx^{n+1}y \notin L$:
   - Let's use proof by contradiction. Assume that $uv^{n+1}wx^{n+1}y \in L$.
   - Since 2 $|vwx| \leq n$, $v = \mathtt{a}^p$ and $u = \mathtt{b}^q$ for some $0 \leq p, q \leq n$, and:

     $$uv^{n+1}wx^{n+1}y = \mathtt{a}^{n+np}\mathtt{b}^{n^2+nq} \in L$$

   - Then, $(n + np)^2 = n^2 + nq \;\Rightarrow\; n^2p^2 + 2np = nq \;\Rightarrow\; np^2 + 2p = q$.
   - Since 1 $|vx| > 0$, $p > 0$ or $q > 0$. However, $q > n$ if $p > 0$ and $q = 0$ if $p = 0$. Therefore, we have a contradiction. $\square$

Example 5

Let's prove that L is **NOT** context-free:

$$L = \{w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid N_{\mathtt{a}}(w) = N_{\mathtt{b}}(w) = N_{\mathtt{c}}(w)\}$$

where $N_{\mathtt{a}}(w)$, $N_{\mathtt{b}}(w)$, and $N_{\mathtt{c}}(w)$ are the number of a's, b's, and c's in w.

Example 5

Let's prove that L is **NOT** context-free:

$$L = \{w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid N_\mathtt{a}(w) = N_\mathtt{b}(w) = N_\mathtt{c}(w)\}$$

where $N_\mathtt{a}(w)$, $N_\mathtt{b}(w)$, and $N_\mathtt{c}(w)$ are the number of a's, b's, and c's in w.

- It is much easier to use **closure properties** under **intersection** with regular languages.
- Consider a regular expressions $R = \mathtt{a}^*\mathtt{b}^*\mathtt{c}^*$ and its language:

$$L(R) = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 0\}$$

- If L is context-free, then $L \cap L(R)$ must be context-free as well because of the closure under intersection with regular languages.
- However, we know that the following language is **NOT** context-free:

$$L \cap L(R) = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$$

- Therefore, L is **NOT** context-free. □

Summary

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Next Lecture

- Turing Machines (TMs)

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