# Lecture 17 – Deterministic Pushdown Automata (DPDA) COSE215: Theory of Computation

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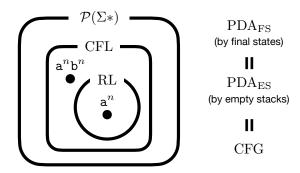


2023 Spring

#### Recall



- A pushdown automaton (PDA) is an extension of  $\epsilon$ -NFA with a stack. Thus, PDA is non-deterministic.
  - Acceptance by **final states**
  - Acceptance by empty stacks
- Then, how about deterministic PDA (DPDA)?
- What is the language class of DPDA? Still, CFL?



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Fact 3:  $DCFL_{ES} \subset DCFL$ 

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- 4. Inherent Ambiguity of DCFLs
  - Fact 6: DCFL ⊂ Non Inherently Ambiguous Languages



## Definition (Deterministic Pushdown Automata)

A PDA is **deterministic** if there is at most one-step move ( $\vdash$ ) from any configuration and we call it a **deterministic pushdown automaton** (**DPDA**). In other words, a DPDA satisfies the following conditions:

- $|\delta(q, a, X)| \le 1$  for all  $q \in Q$ ,  $a \in \Sigma \cup \{\epsilon\}$ , and  $X \in \Gamma$ .
- If  $\delta(q, \epsilon, X) \neq \emptyset$ , then  $\delta(q, a, X) = \emptyset$  for all  $a \in \Sigma$ .

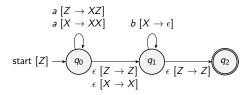


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For example, the following PDA is **NOT** deterministic:



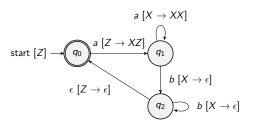


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For example, the following PDA is deterministic:



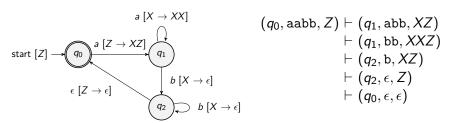


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For example, the following PDA is deterministic:







## Definition (Deterministic Context-Free Languages (DCFLs))

A language L is a **deterministic context-free language (DCFL)** if and only if there exists a DPDA P such that  $L = L_F(P)$  where  $L_F(P)$  is the language accepted by **final states** of P.

# Deterministic Context-Free Languages (DCFLs)



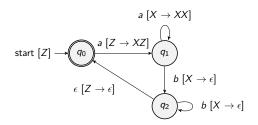
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For example, the following language is a DCFL:

$$L = \{a^n b^n \mid n \ge 0\}$$

because it is accepted by final states of the following DPDA:



#### Fact 1: DCFL ⊂ CFL



#### Fact 1: DCFL ⊂ CFL

All DCFLs are CFLs BUT there exists a CFL that is not a DCFL.

For example, the following language is a CFL but not a DCFL:

$$L = \{ww^R \mid w \in \{a, b\}^*\}$$

Why?

#### Fact 2: RL ⊂ DCFL



#### Fact 2: RL ⊂ DCFL

All RLs are DCFLs BUT there exists a DCFL that is not an RL.

•  $RL \subseteq DCFL$ : For a given RL L and its corresponding DFA

$$D = (Q, \Sigma, \delta, q_0, F)$$

we can construct a DPDA P that accepts L as follows:

$$P = (Q, \Sigma, \{Z\}, \delta_P, q_0, Z, F)$$

where  $\forall q \in Q$ .  $\forall a \in \Sigma$ .  $\delta_P(q, a, Z) = \{(\delta(q, a), Z)\}$ . Then,

$$(q_0, w, Z) \vdash^* (q, \epsilon, Z) \iff \delta^*(q_0, w) = q$$

• DCFL\RL  $\neq \emptyset$ : We already know that the following language is a DCFL but not an RL:

$$L = \{a^n b^n \mid n \ge 0\} \in \mathsf{DCFL} \setminus \mathsf{RL}$$





## Definition (DCFL<sub>ES</sub>)

A language L is a deterministic context-free language by empty stacks (DCFL<sub>ES</sub>) if and only if there exists a DPDA P such that  $L = L_E(P)$  where  $L_E(P)$  is the language accepted by empty stacks of P.



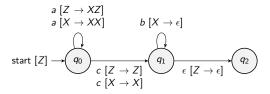
## Definition (DCFL<sub>ES</sub>)

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For example, the following language is a DCFL<sub>ES</sub>:

$$L = \{a^n c b^n \mid n \ge 0\}$$

because it is accepted by empty stacks of the following DPDA:



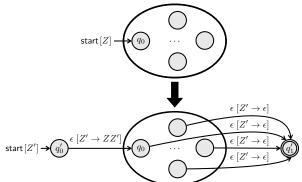
## Fact 3: DCFL<sub>ES</sub> ⊂ DCFL



#### Fact 3: DCFL<sub>FS</sub> ⊂ DCFL

All DCFL<sub>ES</sub>s are DCFLs **BUT** there exists a DCFL that is not a DCFL<sub>ES</sub>.

• DCFL<sub>ES</sub>  $\subseteq$  DCFL $\mid$ : For a given DCFL<sub>ES</sub> L and its corresponding DPDA P by **empty stacks**, we can always construct a DPDA P' that accepts L by **final states** as follows:



## $\mathsf{DCFL}_{\mathsf{FS}} \subset \mathsf{DCFL}$



#### Fact 3: DCFL<sub>ES</sub> ⊂ DCFL

All DCFL<sub>ES</sub>s are DCFLs **BUT** there exists a DCFL that is not a DCFL<sub>ES</sub>.

• DCFL\DCFL<sub>ES</sub>  $\neq \emptyset$ : The following language is a DCFL but not a DCFL<sub>FS</sub>:

$$L = \{a^n b^n \mid n \ge 0\} \in \mathsf{DCFL} \setminus \mathsf{DCFL}_{\mathsf{ES}}$$

Why?





## Fact 4: DCFL<sub>ES</sub> = DCFL having Prefix Property

A language L is a DCFL<sub>ES</sub> if and only if L is a DCFL having the **prefix** property.

## Definition (Prefix Property)

A language L has the **prefix property** if and only if for any word  $w \in L$ , any proper prefix of w is not in L:

$$\forall x, y \in \Sigma^*$$
.  $(xy \in L \land y \neq \epsilon \Rightarrow x \notin L)$ 

For example, the following language is a **DCFL** but does **NOT** have the **prefix property**:

$$L = \{a^n b^n \mid n \ge 0\}$$

because  $\epsilon \in L$  is a proper prefix of  $ab \in L$ .

Thus, L is a DCFL but NOT a DCFL<sub>ES</sub>.

## Fact 5: $RL \not\subset DCFL_{ES}$



## Fact 5: RL ⊄ DCFL<sub>ES</sub>

There exists a RL that is not a DCFL<sub>FS</sub>.

 RL\DCFL<sub>ES</sub> ≠ Ø: For example, the following language is a RL but does NOT have the prefix property:

$$L = \{a^n \mid n \ge 0\}$$

because  $aa \in L$  is a prefix of  $aaaa \in L$ .

Thus, L is a RL but NOT a DCFLES.

$$L \in \mathsf{RL} \setminus \mathsf{DCFL}_\mathsf{ES}$$

# Inherent Ambiguity of DCFLs



## Definition (Inherent Ambiguity)

A language L is **inherently ambiguous** if all CFGs whose languages are L are ambiguous. (i.e. there is no unambiguous grammar for L)

- Is there any DCFL that is inherently ambiguous?
   (i.e., is there any DCFL always having ambiguous grammars?)
- Is there any DCFL that is not inherently ambiguous?
   (i.e., is there any DCFL having at least one unambiguous grammar?)
- Is there any inherently ambiguous language that is not a DCFL? (i.e., is there any unambiguous grammar whose language is not a DCFL?)





All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

- A DCFL<sub>ES</sub> has an unambiguous grammar: For a given DCFL<sub>ES</sub> L and its corresponding PDA P, we can define a CFG for L as follows:
  - For all  $0 \le j < n$ ,

$$S \to A_{0,j}^Z$$

• For all  $q_i \in Q$ ,  $a \in \Sigma \cup \{\epsilon\}$ , and  $X \in \Gamma$ , consider any  $(q_j, X_1 \cdots X_m) \in \delta(q_i, a, X)$  and  $0 \le k_1, \cdots, k_m < n$ . Then,

$$A_{i,k_m}^X o a \ A_{j,k_1}^{X_1} \ A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$





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$$A_{i,k_m}^X \to a A_{j,k_1}^{X_1} A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$

For any word  $w \in L$ , w has a unique moves from the initial configuration to the final configuration in P. And, we know that:

$$A_{i,j}^X \Rightarrow^* w$$
 if and only if  $(q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$ 





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 if and only if  $(q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$ 

Thus, the above CFG is unambiguous.





All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

• A DCFL has an unambiguous grammar : For a given DCFL *L*, we can define another DCFL *L'* with a special symbol \$ as follows:

$$L' = L\$ = \{ w\$ \mid w \in L \}$$





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Then, L' is a DCFL<sub>ES</sub> because it has the prefix property. Thus, L' has an unambiguous grammar G'.





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Then, L' is a DCFL<sub>ES</sub> because it has the prefix property. Thus, L' has an unambiguous grammar G'. Now, we can define an unambiguous grammar G for L by treating \$ as a variable with the rule  $\$ \to \epsilon$ . For example, if the given DCFL is  $L = \{a^nb^n \mid n \ge 0\}$ , then  $L' = \{a^nb^n \mid n \ge 0\}$  and G' is:

$$S o X$$
\$  $X o aXb \mid \epsilon$ 

Then, the unambiguous grammar G for L is:

$$S \to X$$
\$  $X \to aXb \mid \epsilon$  \$  $\to \epsilon$ 





All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

• Non Inherently Ambiguous Languages\DCFL  $\neq \varnothing$ : For example, the following language is a non inherently ambiguous language but not a DCFI:

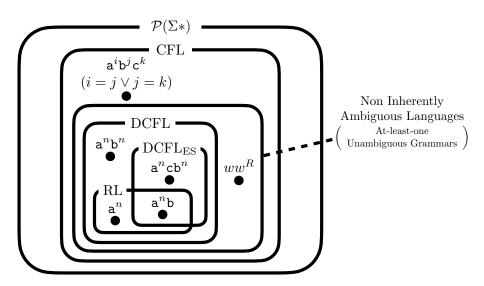
$$L = \{ww^R \mid w \in \{a, b\}^*\}$$

because the following unambiguous grammar G represents L:

$$S 
ightarrow aSa \mid bSb \mid \epsilon$$

## Summary





#### Next Lecture



Normal Forms of Context-Free Grammars

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