Lecture 3 – Deterministic Finite Automata (DFA)
COSE215: Theory of Computation

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PLRG

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Recall

1. Mathematical Preliminaries
   - Mathematical Notations
   - Inductive Proofs
   - Notations in Languages

2. Basic Introduction of Scala
   - Basic Features
   - Object-Oriented Programming (OOP)
   - Functional Programming (FP)
   - Immutable Collections (Data Structures)

1. Deterministic Finite Automata (DFA)

Definition of DFA

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Definition (Deterministic Finite Automata (DFA))

A **deterministic finite automaton** (DFA) is a 5-tuple:

$$D = (Q, \Sigma, \delta, q_0, F)$$

- Q is a finite set of **states**
- Σ is a finite set of **symbols**
- $\delta : Q \times \Sigma \to Q$ is the **transition function**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

$$D = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

$$\delta(q_0, 0) = q_1 \qquad \delta(q_1, 0) = q_2 \qquad \delta(q_2, 0) = q_2$$
$$\delta(q_0, 1) = q_0 \qquad \delta(q_1, 1) = q_0 \qquad \delta(q_2, 1) = q_0$$

Definition of DFA

```scala
// The type definitions of states and symbols
type State = Int
type Symbol = Char
// The definition of DFA
case class DFA(
  states: Set[State],
  symbols: Set[Symbol],
  trans: Map[(State, Symbol), State],
  initState: State,
  finalStates: Set[State],
)
// An example of DFA
val dfa: DFA = DFA(
  states = Set(0, 1, 2),
  symbols = Set('0', '1'),
  trans = Map(
    (0, '0') -> 1, (1, '0') -> 2, (2, '0') -> 2,
    (0, '1') -> 0, (1, '1') -> 0, (2, '1') -> 0,
  ),
  initState  = 0,
  finalStates = Set(2),
)
```

Transition Diagram and Transition Table

$$D = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

$$\delta(q_0, 0) = q_1 \qquad \delta(q_1, 0) = q_2 \qquad \delta(q_2, 0) = q_2$$
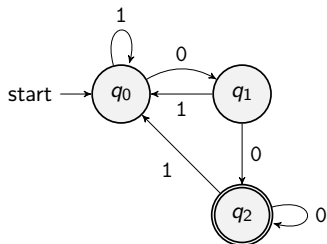$$\delta(q_0, 1) = q_0 \qquad \delta(q_1, 1) = q_0 \qquad \delta(q_2, 1) = q_0$$
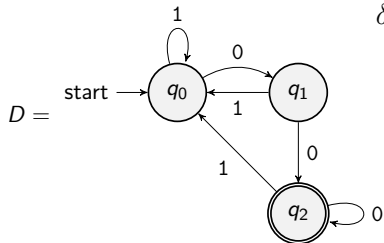
Transition Diagram

Transition Table

q	0	1
$\rightarrow q_0$	q_1	q_0
q_1	q_2	q_0
$*q_2$	q_2	q_0

Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function**
$\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
- **(Induction Case)** $\delta^*(q, aw) = \delta^*(\delta(q, a), w)$

$$\delta^*(q_0, 100) = \delta^*(\delta(q_0, 1), 00) = \delta^*(q_0, 00)$$

$$= \delta^*(\delta(q_0, 0), 0) \;\; = \delta^*(q_1, 0)$$

$$= \delta^*(\delta(q_1, 0), \epsilon) \;\; = \delta^*(q_2, \epsilon)$$

$$= q_2$$

Extended Transition Function

```scala
// The type definition of words
type Word = String

// A helper function to extract first symbol and rest of word
object `<|` { def unapply(w: Word) = w.headOption.map((_, w.drop(1))) }

// The extended transition function of DFA
def extTrans(dfa: DFA)(q: State, w: Word): State = w match
  case "" => q
  case a <| x => extTrans(dfa)(dfa.trans(q, a), x)

// An example transition for a word "100"
extTrans(dfa)(0, "100") // 2
```
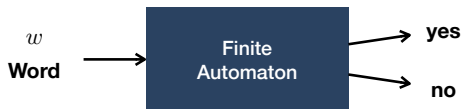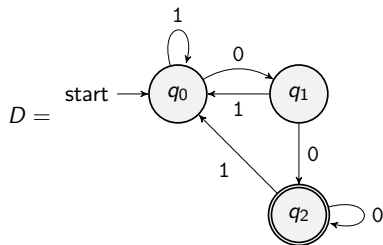
Definition (Acceptance of a Word)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, we say that D **accepts** a word $w \in \Sigma^*$ if and only if $\delta^*(q_0, w) \in F$

$$\delta^*(q_0, 100) = q_2 \in F$$

It means that D accepts 100.

```scala
// The acceptance of a word by DFA
def accept(dfa: DFA)(w: Word): Boolean =
  val curSt: State = extTrans(dfa)(dfa.initState, w)
  dfa.finalStates.contains(curSt)

// An example acceptance of a word "100"
accept(dfa)("100") // true
```

Language of DFA (Regular Language)
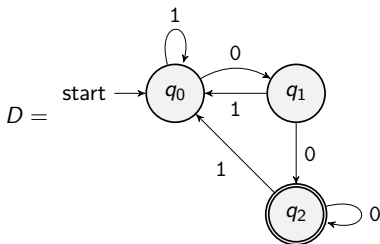
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Definition (Language of DFA)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **language** of D is defined as follows:

$$L(D) = \{w \in \Sigma^* \mid D \text{ accepts } w\}$$

Definition (Regular Language)

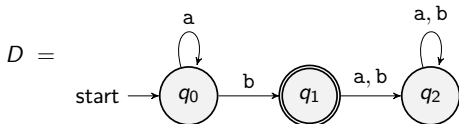A language L is **regular** if and only if there exists a DFA D such that $L(D) = L$

Example 1

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$$\delta^*(q_0, 100) = q_2 \in F \quad \Rightarrow \quad D \text{ accepts } 100 \quad \Rightarrow \quad 100 \in L(D)$$

$$\epsilon, 0, 1, 01, 10, 11, 001, 010, 011, 101, \cdots \notin L(D)$$

$$00, 000, 100, 0000, 0100, 1000, 1100, \cdots \in L(D)$$

$$L(D) = \{w00 \mid w \in \{0, 1\}^*\}$$

Example 2   PLRG

$$D =$$

$$\delta^*(q_0, \text{aab}) = q_1 \in F \quad \Rightarrow \quad D \text{ accepts aab} \quad \Rightarrow \quad \text{aab} \in L(D)$$

$$\epsilon, \text{a}, \text{aa}, \text{ba}, \text{bb}, \text{aaa}, \text{aba}, \text{abb}, \text{baa}, \text{bab}, \text{bba}, \cdots \notin L(D)$$

$$\text{b}, \text{ab}, \text{aab}, \text{aaab}, \text{aaaab}, \text{aaaaab}, \text{aaaaaab}, \cdots \in L(D)$$

$$L(D) = \{\text{a}^n\text{b} \mid n \geq 0\}$$

Example 3

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Theorem

The language $L = \{w \in \{0, 1\}^* \mid w$ is a binary format (allowing leading zeros) of natural numbers divisible by 3\} is regular.

Proof) You need to construct a DFA D such that $L(D) = L$. Consider the following DFA D:

$$D =$$

$\square$

Example 4
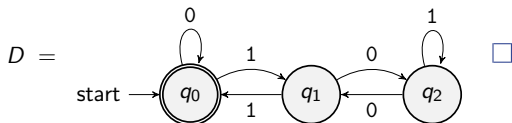
> **Theorem**
>
> *The language $L = \{a^n b^n \mid n \geq 0\}$ is regular.*

You need to construct a DFA D such that $L(D) = L$. However, it is impossible because L is actually not regular.

Then, is it possible to prove that L is not regular?

Yes, it is possible BUT you will learn how to prove it (using Pumping Lemma) later in this course.

Summary

1. Deterministic Finite Automata (DFA)
      Definition
      Transition Diagram and Transition Table
      Extended Transition Function
      Acceptance of a Word
      Language of DFA (Regular Language)
      Examples

- Nondeterministic Finite Automata (NFA)

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