Lecture 1 – Mathematical Preliminaries

COSE215: Theory of Computation

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PLRG

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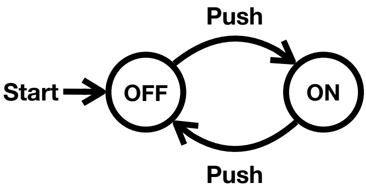
Recall



Theorem

The current state is OFF *if and only if the button is pushed even times.*

- Is it possible to prove it?

Let's learn **mathematical background and notation.**

Contents

Contents

Notations in Logics

Notation	Description
A, B	arbitrary **statements**.
$P(x)$	a **predicate** that involves a **variable** x.
$A \land B$	the **conjunction** of A and B.
$A \lor B$	the **disjunction** of A and B.
$\neg A$	the **negation** of A.
$A \Rightarrow B$	the **implication** of A and B (i.e., A implies B / if A then B).
$A \Leftrightarrow B$	A **if and only if (iff)** B (i.e., $A \Rightarrow B \land B \Rightarrow A$).
$\forall x \in X.\ P(x)$	the **universal quantifier** (i.e, for all x in X, $P(x)$ holds).
$\exists x \in X.\ P(x)$	the **existential quantifier** (i.e., there exists x in X such that $P(x)$ holds).

Notations in Set Theory

- A **set** is a collection of elements, e.g.,
  - $\mathbb{N} = \{0, 1, 2, \cdots\}$
  - $\{x \in \mathbb{N} \mid x \text{ is even}\} = \{0, 2, 4, 6, 8, 10, 12, \cdots\}$
  - $\{x \in \mathbb{N} \mid x^2\} = \{0, 1, 4, 9, 16, 25, 36, \cdots\}$
- The **empty set** is denoted by $\varnothing$.
- The **cardinality** of a set X is denoted by $|X|$.
- A **subset** X of a set Y is denoted by $X \subseteq Y$.

$$X \subseteq Y \iff \forall x \in X.\ x \in Y$$

- A **proper subset** X of a set Y is denoted by $X \subset Y$.

$$X \subset Y \iff X \subseteq Y \land X \neq Y$$

Notations in Set Theory

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- The **union** of sets

$$
\begin{aligned}
X \cup Y &= \{x \mid x \in X \lor x \in Y\} \\
\bigcup \mathcal{C} &= X_1 \cup X_1 \cup \cdots \cup X_n = \{x \mid \exists X \in \mathcal{C}.\ x \in X\}
\end{aligned}
$$

  where $\mathcal{C} = \{X_1, X_2, \cdots, X_n\}$.

- The **intersection** of sets

$$
\begin{aligned}
X \cap Y &= \{x \mid x \in X \land x \in Y\} \\
\bigcap \mathcal{C} &= X_1 \cap X_1 \cap \cdots \cap X_n = \{x \mid \forall X \in \mathcal{C}.\ x \in X\}
\end{aligned}
$$

  where $\mathcal{C} = \{X_1, X_2, \cdots, X_n\}$.

- The **difference** of sets

$$
X \setminus Y = \{x \mid x \in X \land x \notin Y\}
$$

Notations in Set Theory

- The **complement** of a set X is denoted by $\overline{X}$.

$$\overline{X} = \{x \mid x \in U \wedge x \notin X\}$$

  where U is the **universal set**.

- The **power set** of a set X is denoted by 2^X or $\mathcal{P}(X)$.

$$2^X = \mathcal{P}(X) = \{Y \mid Y \subseteq X\}$$

- The **Cartesian product** of sets X and Y is denoted by $X \times Y$.

$$X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$$

Contents

Definition (Inductions on Integers)

Let $P(n)$ be a predicate on integers, and if

- **(Basis Case)** $P(k)$ is hold where k is an integer, and
- **(Induction Case)** for all $n \geq k$, $P(n) \Rightarrow P(n+1)$,

then $P(i)$ is hold for all $i \geq k$.

$P(n)$ is called **induction hypothesis**.

Example

Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Proof)

Example (Exercise)

Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof)

Structural Inductions

In CS, we often define somethings as **inductively-defined sets**.

Example (Inductive Definition of Trees)

A **tree** is defined as follows:

- **(Basis Case)** A single **node** N is a tree.
- **(Induction Case)** If $T_1, \cdots, T_n$ are trees, then a graph defined with a new node N and edges from N to $T_1, \cdots, T_n$ is a tree as well.

Example (Inductive Definition of Arithmetic Expressions)

An **arithmetic expression** is defined as follows:

- **(Basis Case)** A **number** or a **variable** is an arithmetic expression.
- **(Induction Case)** If E and F are arithmetic expressions, then so are $E + F$, $E \times F$, and (E).

Structural Inductions

Definition (Structural Inductions)

Let $P(x)$ be a predicate on a **inductively-defined set** X, and if

- **(Basis Case)** $P(x)$ is hold for all base cases x, and
- **(Induction Case)** for all $x \in X$,

$$P(x_1) \land \cdots \land P(x_n) \Rightarrow P(x)$$

  where $x_1, \cdots, x_n$ are the **sub-structures** of x.

then $P(x)$ is hold for all $x \in X$.

$P(x_1), \cdots, P(x_n)$ are called **induction hypotheses**.

Structural Inductions

Example

Prove that for all tree T, the number of nodes in T is equal to the number of edges in T plus one.

Proof)

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Example (Exercise)

Prove that for all arithmetic expression E, the number of left parentheses in E is equal to the number of right parentheses in E.

Proof)

Mutual Inductions

Definition (Mutual Inductions)

Let $P(x)$ and $Q(x)$ are predicates on a **inductively-defined set** X, and if

- **(Basis Case)** $P(x)$ and $Q(x)$ are hold for all base cases x, and
- **(Induction Case)** for all $x \in X$,

$$P(x_1) \land \cdots \land P(x_n) \land Q(x_1) \land \cdots \land Q(x_n) \Rightarrow P(x) \land Q(x)$$

  where $x_1, \cdots, x_n$ are the **sub-structures** of x.

then $P(x)$ and $Q(x)$ are hold for all $x \in X$.

$P(x_1), \cdots, P(x_n)$ and $Q(x_1), \cdots, Q(x_n)$ are called **induction hypotheses**.

Theorem

The current state is OFF *if and only if the button is pushed even times.*

Proof)

Contents

Symbols

We first define a finite and non-empty set of **symbols**.

- $\Sigma = \{0, 1\}$ – binary symbols.
- $\Sigma = \{\mathrm{a}, \mathrm{b}, \cdots, \mathrm{z}\}$ – lowercase letters.
- $\Sigma = \{a \mid a \text{ is an ASCII character}\}$ – ASCII characters.

Words

A **word** $w \in \Sigma^*$ is a sequence of symbols:

- $\Sigma = \{0, 1\}$ – $\epsilon, 0, 1, 00, 01, 10010, \cdots$
- $\Sigma = \{a, b, \cdots, z\}$ – ϵ, a, b, abc, hello, cs, students, $\cdots$

Notations:

Notation	Description
ϵ	the **empty word**.
$w_1 w_2$	the **concatenation** of w_1 and w_2. (w_1 is a **prefix** of $w_1 w_2$ and w_2 is a **suffix** of $w_1 w_2$)
w^R	the **reverse** of w.
$\|w\|$	the **length** of w.
Σ^k	the set of all words of length k.
Σ^*	the set of all words (the **Kleene star**). (i.e., $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \cdots = \bigcup_{k=0} \Sigma^k$)
Σ^+	the set of all words except ϵ (the **Kleene plus**). (i.e., $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \cdots = \bigcup_{k=1} \Sigma^k$)

Languages

A **language** $L \subseteq \Sigma^*$ is a set of words. When $\Sigma = \{0, 1\}$, we can define the following languages:

- $L = \{\epsilon, 0, 1\}$ – the empty word, zero, and one.
- $L = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \cdots\}$ – all binary words.
- $L = \{0^n1^n \mid n \geq 0\}$ – equal number of consecutive zeros and ones.
- $L = \{10, 11, 101, 111, 1011, \cdots\}$ – prime numbers in a binary format.

Languages

- The **union**, **intersection**, and **difference** of languages:

$$L_1 \cup L_2 \qquad L_1 \cap L_2 \qquad L_1 \setminus L_2$$

- The **reverse** of a language:

$$L^R = \{w^R \mid w \in L\}$$

- The **complement** of a language:

$$L^c = \Sigma^* \setminus L$$

- The **concatenation** of languages:

$$L_1 L_2 = \{w_1 w_2 \mid w_1 \in L_1 \land w_2 \in L_2\}$$

Languages

- The **power** of a language:

$$
\begin{aligned}
L^0 &= \{\epsilon\} \\
L^n &= L^{n-1}L \qquad (n \geq 1)
\end{aligned}
$$

- The **Kleene star** of a language:

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots = \bigcup_{n \geq 0} L^n$$

- The **Kleene plus** of a language:

$$L^+ = L^1 \cup L^2 \cup L^3 \cup \cdots = \bigcup_{n \geq 1} L^n$$

Summary

1. Mathematical Notations
   - Notations in Logics
   - Notations in Set Theory

2. Inductive Proofs
   - Inductions on Integers
   - Structural Inductions
   - Mutual Inductions

3. Notations in Languages
   - Symbols
   - Words
   - Languages

Next Lecture

- Basic introduction of Scala

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