Lecture 12 – Examples of Context-Free Grammars

COSE215: Theory of Computation

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2023 Spring

Recall

- A **context-free grammar (CFG)**:

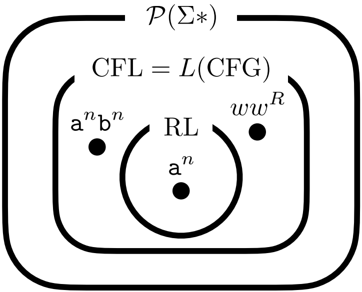
$$G = (V, \Sigma, S, P)$$

- The **language** of a CFG G:

$$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

- A language L is a **context-free language (CFL)**:

$$\exists \text{ CFG } G.\ L(G) = L$$



Contents

Example 1: Regular Languages

Theorem (RLs are CFLs)

If a language L is a regular language (RL), then L is a CFL.

Proof) For a given RE R, construct a CFG G such that $L(G) = L(R)$.

RE R	CFG G
$\varnothing$	$S \to S$
ϵ	$S \to \epsilon$
$a \in \Sigma$	$S \to a$
$R_1 \mid R_2$	$S \to S_1 \mid S_2$
$R_1 \cdot R_2$	$S \to S_1 S_2$
R_1^*	$S \to \epsilon \mid S_1 S$
(R_1)	$S \to S_1$

where S_1 and S_2 are start variables of CFGs G_1 and G_2 such that $L(G_1) = L(R_1)$ and $L(G_2) = L(R_2)$.

Example 1: Regular Languages

For a given RE R, construct a CFG G such that $L(G) = L(R)$.

- $R = \epsilon \,|\, \mathtt{ab} \,|\, \mathtt{ba}$

$$S \to FD \qquad A \to \mathtt{a} \qquad C \to AB \qquad E \to \epsilon$$
$$B \to \mathtt{b} \qquad D \to BA \qquad F \to EC$$

  Its simplified version:

$$S \to \epsilon \mid \mathtt{ab} \mid \mathtt{ba}$$

- $R = (\epsilon \,|\, \mathtt{a})^*$

$$S \to \epsilon \mid AS \qquad A \to \epsilon \mid \mathtt{a}$$

- $R = (\mathtt{0} \,|\, \mathtt{1}(\mathtt{0}\mathtt{1}^*\mathtt{0})^*\mathtt{1})^*$

$$S \to \epsilon \mid AE \qquad A \to \mathtt{0} \mid \mathtt{1}B\mathtt{1} \qquad C \to \mathtt{0}D\mathtt{0}$$
$$B \to \epsilon \mid CB \qquad D \to \epsilon \mid \mathtt{1}D$$

Example 2: $\mathtt{b}^n\mathtt{a}^m\mathtt{b}^{2n}$

Construct a CFG for the language:

$$L = \{\mathtt{b}^n\mathtt{a}^m\mathtt{b}^{2n} \mid n, m \geq 0\}$$

$$\begin{aligned} S &\to A \mid \mathtt{b}S\mathtt{bb} \\ A &\to \epsilon \mid \mathtt{a}A \end{aligned}$$

A derivation for bbaaabbbb:

$$\begin{array}{lllllll} S & \Rightarrow & \mathtt{b}S\mathtt{bb} & \Rightarrow & \mathtt{bb}S\mathtt{bbbb} & \Rightarrow & \mathtt{bb}A\mathtt{bbbb} \\ & \Rightarrow & \mathtt{bba}A\mathtt{bbbb} & \Rightarrow & \mathtt{bbaa}A\mathtt{bbbb} & \Rightarrow & \mathtt{bbaaa}A\mathtt{bbbb} \\ & \Rightarrow & \mathtt{bbaaabbbb} & & & & \end{array}$$

Example 3: Well-Formed Brackets

Construct a CFG for the language:

$$L = \{w \in \{(,),\{,\},[,]\}^* \mid w \text{ is well-formed}\}$$

$$S \to \epsilon \mid (S) \mid \{S\} \mid [S] \mid SS$$

A derivation for `({}){}[()[]]`:

$$\begin{array}{lllllll}
S & \Rightarrow & SS & \Rightarrow & SSS & \Rightarrow & (S)SS \\
& \Rightarrow & (\{S\})SS & \Rightarrow & (\{\})SS & \Rightarrow & (\{\})\{S\}S \\
& \Rightarrow & (\{\})\{\}S & \Rightarrow & (\{\})\{\}[S] & \Rightarrow & (\{\})\{\}[SS] \\
& \Rightarrow & (\{\})\{\}[(S)S] & \Rightarrow & (\{\})\{\}[()S] & \Rightarrow & (\{\})\{\}[()[S]] \\
& \Rightarrow & (\{\})\{\}[()[]] & & & &
\end{array}$$

Example 4: Equal Number of $\mathtt{a}$ and $\mathtt{b}$

Construct a CFG for the language:

$$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid N_{\mathtt{a}}(w) = N_{\mathtt{b}}(w)\}$$

where $N_{\mathtt{a}}(w)$ and $N_{\mathtt{b}}(w)$ are the number of $\mathtt{a}$'s and $\mathtt{b}$'s in w, respectively.

$$S \to \epsilon \mid \mathtt{a}S\mathtt{b} \mid \mathtt{b}S\mathtt{a} \mid SS$$

The left-most derivation for $\mathtt{abbaaabb}$:

$$\begin{array}{llllll} S & \xRightarrow{\text{lm}} \mathtt{a}S\mathtt{b} & \xRightarrow{\text{lm}} & \mathtt{a}SS\mathtt{b} & \xRightarrow{\text{lm}} & \mathtt{ab}S\mathtt{a}S\mathtt{b} \\ & \xRightarrow{\text{lm}} \mathtt{abb}S\mathtt{aa}S\mathtt{b} & \xRightarrow{\text{lm}} & \mathtt{abbaa}S\mathtt{b} & \xRightarrow{\text{lm}} & \mathtt{abbaaa}S\mathtt{bb} \\ & \xRightarrow{\text{lm}} \mathtt{abbaaabb} & & & & \end{array}$$

Example 5: Unequal Number of a and b

Construct a CFG for the language:

$$L = \{w \in \{\texttt{a}, \texttt{b}\}^* \mid N_{\texttt{a}}(w) \neq N_{\texttt{b}}(w)\}$$

where $N_{\texttt{a}}(w)$ and $N_{\texttt{b}}(w)$ are the number of a's and b's in w, respectively.

$$\begin{aligned}
S &\to P \mid N \\
P &\to ZP \mid \texttt{a}P \mid \texttt{a}Z \\
N &\to ZN \mid \texttt{b}N \mid \texttt{b}Z \\
Z &\to \epsilon \mid \texttt{a}Z\texttt{b} \mid \texttt{b}Z\texttt{a} \mid ZZ
\end{aligned}$$

The right-most derivation for aabbbaaab:

$$\begin{array}{llllll}
S & \overset{\text{rm}}{\Longrightarrow} P & \overset{\text{rm}}{\Longrightarrow} & ZP & \overset{\text{rm}}{\Longrightarrow} & Z\texttt{a}Z \\
 & \overset{\text{rm}}{\Longrightarrow} Z\texttt{aa}Z\texttt{b} & \overset{\text{rm}}{\Longrightarrow} & Z\texttt{aab} & \overset{\text{rm}}{\Longrightarrow} & ZZ\texttt{aab} \\
 & \overset{\text{rm}}{\Longrightarrow} Z\texttt{b}Z\texttt{aaab} & \overset{\text{rm}}{\Longrightarrow} & Z\texttt{baaab} & \overset{\text{rm}}{\Longrightarrow} & \texttt{a}Z\texttt{bbaaab} \\
 & \overset{\text{rm}}{\Longrightarrow} \texttt{aa}Z\texttt{bbbaaab} & \overset{\text{rm}}{\Longrightarrow} & \texttt{aabbbaaab} & &
\end{array}$$

Example 6: Arithmetic Expressions

An **arithmetic expression** is defined with the following CFG:

$$\begin{aligned} S &\to N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\ N &\to \texttt{1}D \mid \cdots \mid \texttt{9}D \\ D &\to \epsilon \mid \texttt{0}D \mid \cdots \mid \texttt{9}D \\ X &\to \texttt{a} \mid \cdots \mid \texttt{z} \end{aligned}$$

The left-most derivation for `13*(2+x)`:

$$\begin{array}{llllll} S & \overset{\text{lm}}{\Longrightarrow} & S\texttt{*}S & \overset{\text{lm}}{\Longrightarrow} N\texttt{*}S & \overset{\text{lm}}{\Longrightarrow} & \texttt{1}D\texttt{*}S \\ & \overset{\text{lm}}{\Longrightarrow} & \texttt{13}D\texttt{*}S & \overset{\text{lm}}{\Longrightarrow} \texttt{13*}S & \overset{\text{lm}}{\Longrightarrow} & \texttt{13*(}S\texttt{)} \\ & \overset{\text{lm}}{\Longrightarrow} & \texttt{13*(}S\texttt{+}S\texttt{)} & \overset{\text{lm}}{\Longrightarrow} \texttt{13*(}N\texttt{+}S\texttt{)} & \overset{\text{lm}}{\Longrightarrow} & \texttt{13*(2}D\texttt{+}S\texttt{)} \\ & \overset{\text{lm}}{\Longrightarrow} & \texttt{13*(2+}S\texttt{)} & \overset{\text{lm}}{\Longrightarrow} \texttt{13*(2+}X\texttt{)} & \overset{\text{lm}}{\Longrightarrow} & \texttt{13*(2+x)} \end{array}$$

Example 7: Regular Expressions

Is the following language regular? or context-free?

$$L = \{w \in \{\varnothing, \epsilon, \mathtt{a}, \mathtt{b}, \mathsf{|}, \cdot, ^*, \mathtt{(}, \mathtt{)}\}^* \mid w \text{ is a regular expression over } \{\mathtt{a}, \mathtt{b}\}\}$$

We can prove that L is not regular using the pumping lemma.
(Hint: consider a word $\mathtt{(}^n\epsilon\mathtt{)}^n$ for a given $n > 0$)
The language L is context-free:

$$S \to \varnothing \mid \epsilon \mid \mathtt{a} \mid \mathtt{b} \mid S\,\mathsf{|}\,S \mid S \cdot S \mid S^* \mid \mathtt{(}S\mathtt{)}$$

The right-most derivation for $\mathtt{(}\epsilon\,\mathsf{|}\,\mathtt{a}\cdot\mathtt{b}\mathtt{)}^*$:

$$\begin{array}{llllll} S & \overset{\text{rm}}{\Longrightarrow} & S^* & \overset{\text{rm}}{\Longrightarrow} \; \mathtt{(}S\mathtt{)}^* & \overset{\text{rm}}{\Longrightarrow} & \mathtt{(}S\,\mathsf{|}\,S\mathtt{)}^* \\ & \overset{\text{rm}}{\Longrightarrow} & \mathtt{(}S\,\mathsf{|}\,S\cdot S\mathtt{)}^* & \overset{\text{rm}}{\Longrightarrow} \; \mathtt{(}S\,\mathsf{|}\,S\cdot\mathtt{b}\mathtt{)}^* & \overset{\text{rm}}{\Longrightarrow} & \mathtt{(}S\,\mathsf{|}\,\mathtt{a}\cdot\mathtt{b}\mathtt{)}^* \\ & \overset{\text{rm}}{\Longrightarrow} & \mathtt{(}\epsilon\,\mathsf{|}\,\mathtt{a}\cdot\mathtt{b}\mathtt{)}^* & & & \end{array}$$

Example 8: Simplified Scala

We can define a CFG for a simplified version of Scala[1]:

$$
\begin{array}{lrcl}
\text{(Scala Program)} & S & \rightarrow & E \mid E\ \texttt{;}\ S \\
\text{(Expressions)} & E & \rightarrow & N \mid X \mid E\ \texttt{+}\ E \mid E\ \texttt{-}\ E \mid E\ \texttt{*}\ E \mid E\ \texttt{/}\ E \\
 & & \mid & \texttt{val}\ X\ \texttt{:}\ T\ \texttt{=}\ E \\
 & & \mid & \texttt{def}\ X\ \texttt{(}\ P\ \texttt{):}\ T\ \texttt{=}\ E \\
 & & \mid & E\ \texttt{(}\ A\ \texttt{)} \\
 & & \mid & \texttt{if}\ \texttt{(}\ E\ \texttt{)}\ E\ \texttt{else}\ E \\
 & & \mid & \texttt{trait}\ T\ \texttt{(}\ P\ \texttt{)} \\
 & & \mid & \texttt{case class}\ T\ \texttt{(}\ P\ \texttt{)} \\
 & & \mid & E\ \texttt{match}\ \texttt{\{}\ C\ \texttt{\}} \\
\text{(Numbers)} & N & \rightarrow & \texttt{1}D \mid \cdots \mid \texttt{9}D \\
 & D & \rightarrow & \epsilon \mid \texttt{0}D \mid \cdots \mid \texttt{9}D \\
\text{(Variables)} & X & \rightarrow & A \mid AX \\
 & A & \rightarrow & \texttt{a} \mid \cdots \mid \texttt{z} \mid \texttt{A} \mid \cdots \mid \texttt{Z} \\
\text{(Types)} & T & \rightarrow & X \mid T\ \texttt{[}\ T\ \texttt{]} \mid T\ \texttt{=>}\ T \\
\text{(Parameters)} & P & \rightarrow & \epsilon \mid X\ \texttt{:}\ T \mid P\ \texttt{,}\ X\ \texttt{:}\ T \\
\text{(Arguments)} & A & \rightarrow & \epsilon \mid E \mid A\ \texttt{,}\ E \\
\text{(Cases)} & C & \rightarrow & \texttt{case}\ E\ \texttt{=>}\ E \mid C\ \texttt{;}\ \texttt{case}\ E\ \texttt{=>}\ E
\end{array}
$$

[1] https://docs.scala-lang.org/scala3/reference/syntax.html

Example 8: Simplified Scala

```
def sum(n: Int): Int = n match { case 0 => 0; case n => n + sum(n - 1) }
```

The left-most derivation for this program:

$$
\begin{aligned}
S &\overset{\text{lm}}{\Longrightarrow} \texttt{def } X \texttt{ (} P \texttt{): } T = E
\quad \overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(} P \texttt{): } T = E \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(} X \texttt{: } T \texttt{): } T = E
\quad \overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = } E \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = } E \texttt{ match \{ } C \texttt{ \}} \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = n match \{ } C \texttt{ \}} \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = n match \{ case } E \texttt{ => } E \texttt{ ; } C \texttt{ \}} \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; } C \texttt{ \}} \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case } E \texttt{ => } E \texttt{ \}} \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case n => } E \texttt{ \}} \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case n => } E + E \texttt{ \}} \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case n => n + } E \texttt{ \}} \\
&\overset{\text{lm}}{\Longrightarrow}{}^{*} \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case n => n + sum(n - 1) \}}
\end{aligned}
$$

Summary

1. Examples of Context-Free Grammars
   - Example 1: Regular Languages
   - Example 2: $\mathtt{b}^n\mathtt{a}^m\mathtt{b}^{2n}$
   - Example 3: Well-Formed Brackets
   - Example 4: Equal Number of `a` and `b`
   - Example 5: Unequal Number of `a` and `b`
   - Example 6: Arithmetic Expressions
   - Example 7: Regular Expressions
   - Example 8: Simplified Scala

Next Lecture

- Parse Trees and Ambiguity

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