Lecture 21 – Turing Machines (TMs)

COSE215: Theory of Computation

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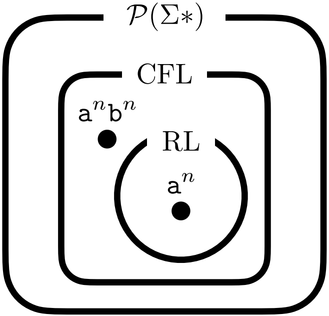
PLRG

2023 Spring

- A **pushdown automaton (PDA)** is an extension of FA with a **stack**.



PDA_{FS}
(by final states)

=

PDA_{ES}
(by empty stacks)

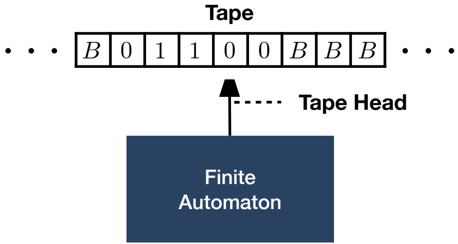
=

CFG

- Then, how about extensions of finite automata with other structures?
- Do they still represent the class of **context-free languages (CFLs)**?

Contents

Turing Machines



A **Turing machine (TM)** is a finite automaton with a **tape**.
It consists of the following three components:

1. A **finite automaton** with a deterministic transition function.
2. A **tape** is a one-dimensional infinite array of cells.
   - Each cell contains a **tape symbol**.
   - The **blank symbol** B is a special symbol representing an empty cell.
3. A **tape head** is a device that can read and write symbols on the tape.
   - It can move left or right one cell at a time.

Definition of Turing Machines

Definition (Turing Machines)

A **Turing machine (TM)** is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$$

where

- Q is a finite set of **states**.
- Σ is a finite set of **input symbols**.
- Γ is a finite set of **tape symbols** containing input symbols ($\Sigma \subseteq \Gamma$).
- $\delta : Q \times \Gamma \rightharpoonup Q \times \Gamma \times \{L, R\}$ is a **transition function**.
- $q_0 \in Q$ is the **initial state**.
- $B \in \Gamma \setminus \Sigma$ is the **blank symbol**.
- $F \subseteq Q$ is the set of **final states**.

Definition of Turing Machines – Example

$$M_1 = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, B\}, \delta, q_0, B, \{q_2\})$$

where

$$\begin{aligned} \delta(q_0, 0) &= (q_0, 1, R) & \qquad \delta(q_1, 0) &= (q_1, 0, L) \\ \delta(q_0, 1) &= (q_0, 0, R) & \qquad \delta(q_1, 1) &= (q_1, 1, L) \\ \delta(q_0, B) &= (q_1, B, L) & \qquad \delta(q_1, B) &= (q_2, B, R) \end{aligned}$$

Definition of Turing Machines – Example

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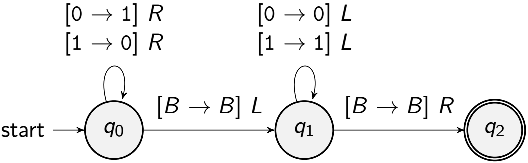
The **transition diagram** of M_1 is as follows:

$M_1 =$



Turing Machines in Scala

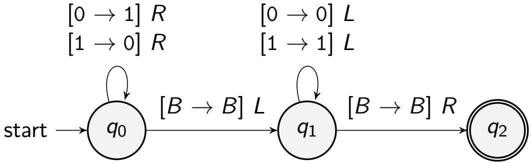
$$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$$

$$\delta : Q \times \Gamma \rightharpoonup Q \times \Gamma \times \{L, R\}$$

```
// The type definitions of states, symbols, tape symbols, and tape head moves
type State = Int
type Symbol = Char
type TapeSymbol = Char
enum HeadMove { case L, R }
import HeadMove.*

// The definition of Turing machines
case class TM(
  states: Set[State],
  symbols: Set[Symbol],
  tapeSymbols: Set[TapeSymbol],
  trans: Map[(State, TapeSymbol), (State, TapeSymbol, HeadMove)],
  initState: State,
  blankSymbol: TapeSymbol,
  finalStates: Set[State],
)
```

Turing Machines in Scala – Example

$M_1 =$ 


```scala
// An example of Turing machine
val tm1: TM = TM(
  states      = Set(0, 1, 2),
  symbols     = Set('0', '1'),
  tapeSymbols = Set('0', '1', 'B'),
  trans       = Map(
    (0, '0') -> (0, '1', R),  (1, '0') -> (1, '0', L),
    (0, '1') -> (0, '0', R),  (1, '1') -> (1, '1', L),
    (0, 'B') -> (1, 'B', L),  (1, 'B') -> (2, 'B', R),
  ),
  initState   = 0,
  blankSymbol = 'B',
  finalStates = Set(2),
)
```

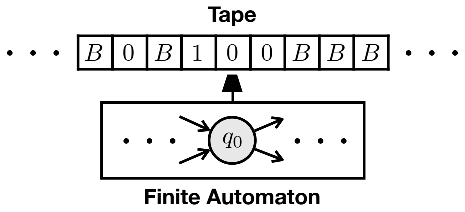
Definition (Configurations of Turing Machines)

A **configuration** of a Turing machine M is a sequence of the form

$$X_1 \cdots X_{i-1} \; q \; X_i X_{i+1} \cdots X_n$$

where

- $q \in Q$ is the **current state**.
- $X_1 \cdots X_n \in \Gamma^*$ is the **sub-tape** between the left- and the right-most 1) non-blank symbols or 2) the symbol under the tape head.
- $X_i \in \Gamma$ is the **current tape symbol** under the tape head.



$\equiv$ $0B1 \; q_0 \; 00$

Configurations

Definition (Configurations of Turing Machines)

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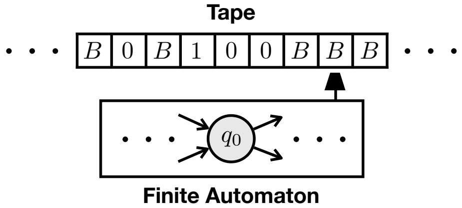


$\equiv \qquad 0B100B \; q_0 \; B$

One-Step Moves

Definition (One-Step Moves of Turing Machines)

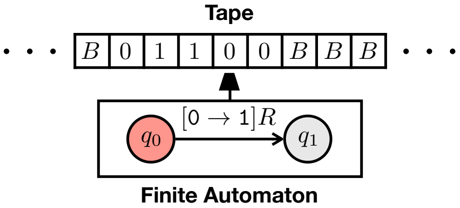
A **one-step move** ($\vdash$) of a Turing machine M is a transition from a configuration to another configuration.

- If $\delta(q, X_i) = (p, Y, L)$,

$$X_1 \cdots X_{i-1}\ q\ X_i X_{i+1} \cdots X_n \vdash X_1 \cdots X_{i-2}\ p\ X_{i-1} Y X_{i+1} \cdots X_n$$

- If $\delta(q, X_i) = (p, Y, R)$,

$$X_1 \cdots X_{i-1}\ q\ X_i X_{i+1} \cdots X_n \vdash X_1 \cdots X_{i-1} Y\ p\ X_{i+1} \cdots X_n$$



$011\ q_0\ 00$

Definition (One-Step Moves of Turing Machines)

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- If $\delta(q, X_i) = (p, Y, L)$,

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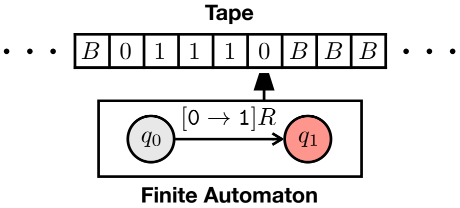
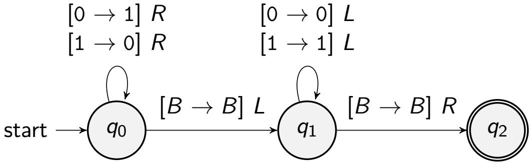


$011\ q_0\ 00 \vdash 0111\ q_1\ 0$

One-Step Moves

$M_1 =$



$$
\begin{array}{llll}
q_0\ 0110 & \vdash & 1\ q_0\ 110 & (\because \delta(q_0, 0) = (q_0, 1, R)) \\
& \vdash & 10\ q_0\ 10 & (\because \delta(q_0, 1) = (q_0, 0, R)) \\
& \vdash & 100\ q_0\ 0 & (\because \delta(q_0, 1) = (q_0, 0, R)) \\
& \vdash & 1001\ q_0\ B & (\because \delta(q_0, 0) = (q_0, 1, R)) \\
& \vdash & 100\ q_1\ 1 & (\because \delta(q_0, B) = (q_1, B, L)) \\
& \vdash^* & q_1\ B1001 & (\because \delta(q_1, x) = (q_1, x, L) \text{ where } x = 0 \text{ or } 1) \\
& \vdash & q_2\ 1001 & (\because \delta(q_1, B) = (q_2, B, R)) \\
& \not\vdash & &
\end{array}
$$

Halting of Turing Machines

Definition (Halting of Turing Machines)

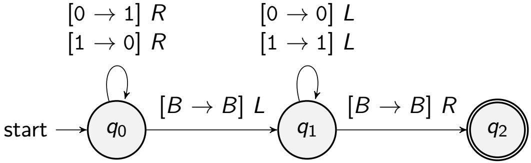
A Turing machine M **halts** on input w if there is a sequence of one-step moves from the **initial configuration** $q_0\ w$ to a configuration having no more possible moves:

$$q_0\ w \vdash^* \alpha q \beta \nvdash$$

for some $\alpha, \beta \in \Gamma^*$ and $q \in Q$.

For example, the Turing machine M_1 halts on input 0110:

$M_1 =$



$$q_0\ 0110 \vdash^* q_2\ 1001 \nvdash$$

Language of Turing Machines

Definition (Language of Turing Machines)

For a given Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$, the **language** of M is defined as follows:

$$L(M) = \{w \in \Sigma^* \mid q_0\ w \vdash^* \alpha\ q_f\ \beta \nvdash \text{ for some } q_f \in F, \alpha, \beta \in \Gamma^*\}$$

Definition (Recursively Enumerable Languages)

A language L is **recursively enumerable** if there exists a Turing machine M such that $L = L(M)$.

For example, the language of a Turing machine M_1 is defined as follows:

$$L(M_1) = \{w \mid w \in \{0, 1\}^*\}$$

because M_1 reaches a final state on any input.

Language of Turing Machines

$$L(M) = \{w \in \Sigma^* \mid q_0\, w \vdash^* \alpha\, q_f\, \beta \nvdash \text{ for some } q_f \in F, \alpha, \beta \in \Gamma^*\}$$

```scala
// The type definitions of words, tapes, and configurations
type Word = String
type Tape = String
case class Config(prev: Tape, state: State, cur: TapeSymbol, next: Tape)
// A one-step move in a Turing machine
def move(tm: TM)(config: Config): Option[Config] = ...
// The initial configuration of a Turing machine
def initConfig(tm: TM)(word: Word): Config = word match
  case a <| x => Config("", tm.initState, a, x)
  case _      => Config("", tm.initState, tm.blankSymbol, "")
// A configuration at which a Turing machine halts on a word or a configuration
def haltsAt(tm: TM)(word: Word): Config = haltsAt(tm)(initConfig(tm)(word))
def haltsAt(tm: TM)(config: Config): Config = move(tm)(config) match
  case None             => config
  case Some(nextConfig) => haltsAt(tm)(nextConfig)
// The acceptance of a word by a Turing machine
def accept(tm: TM)(word: Word): Boolean = isFinal(tm)(haltsAt(tm)(word).state)
// An example acceptance of a word "0110"
accept(tm1)("0110") // true
```

Definition (Turing Computable Functions)

A partial function $f : \Sigma^* \rightharpoonup \Sigma^*$ is **Turing-computable** if there exists a Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ such that

$$q_0 \; w \vdash^* q_f \; f(w) \nvdash$$

for some $q_f \in F$ and all $w \in \Sigma^*$, such that $f(w)$ is defined.

For example, the function $f : \{0, 1\}^* \to \{0, 1\}^*$ defined as follows is Turing-computable:

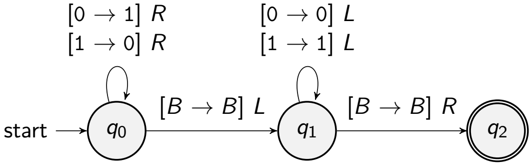
$$f(w) = (\text{the flip of each bit in } w)$$

because there exists a Turing machine M_1 such that $q_0 \; w \vdash^* q_f \; f(w) \nvdash$ for some $q_f \in F$ and all $w \in \{0, 1\}^*$. For instance,

$$q_0 \; 0110 \vdash^* q_2 \; 1001 \nvdash$$
$$q_0 \; 1011100 \vdash^* q_2 \; 0100011 \nvdash$$

Turing Machines as Computing Machines

$M_1 =$




```
// A computation by a Turing machine
def compute(tm: TM)(word: Word): Option[Word] =
  val Config(prev, state, cur, next) = haltsAt(tm)(word)
  if      (prev != "")                               None
  else if (!isFinal(tm)(state))                      None
  else if (!next.forall(tm.symbols.contains))        None
  else if (cur == tm.blankSymbol && next == "") Some("")
  else                                               Some(cur + next)

// Examples of computations by Turing machines
compute(tm1)("0110")    // Some("1001")
compute(tm1)("1011100") // Some("0100011")
```

Summary

1. Turing Machines
   - Definition
   - Turing Machines in Scala
   - Configurations
   - One-Step Moves
   - Halting of Turing Machines
   - Language of Turing Machines
   - Turing Machines as Computing Machines

Next Lecture

- Examples of Turing Machines

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