

# Lecture 9 – The Pumping Lemma for Regular Languages

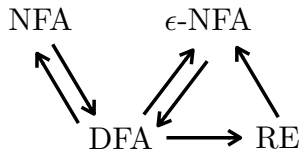
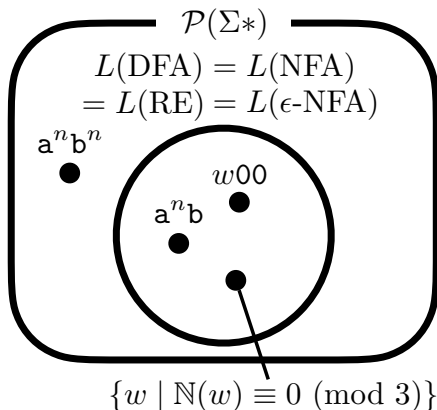
## COSE215: Theory of Computation

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2023 Spring

- Not all languages are regular: e.g.,  $L = \{a^n b^n \mid n \geq 0\}$ .



- How to prove that a language is **NOT** regular? **Pumping Lemma!**

## 1. Pumping Lemma for Regular Languages

Pumping Lemma

Proof of Pumping Lemma

Application: Proving Languages are Not Regular

Examples

Example 1:  $L = \{a^n b^n \mid n \geq 0\}$

Example 2:  $L = \{ww^R \mid w \in \{a, b\}^*\}$

Example 3:  $L = \{a^l b^m c^n \mid l + m \leq n\}$

Example 4:  $L = \{a^{n^2} \mid n \geq 0\}$

Example 5:  $L = \{a^n b^k c^{n+k} \mid n, k \geq 0\}$

## Lemma (Pumping Lemma for Regular Languages)

For a given regular language  $L$ , **there exists** a *positive integer*  $n$  such that **for all**  $w \in L$ , if  $|w| \geq n$ , **there exists**  $w = xyz$  such that

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$A =$   $L$  is regular



$B = \exists n > 0. \forall w \in L. |w| \geq n \Rightarrow \exists w = xyz. \textcircled{1} \wedge \textcircled{2} \wedge \textcircled{3}$

- Let  $L$  be a regular language.
- $\exists$  DFA  $D = (Q, \Sigma, \delta, q_0, F)$ . s.t.  $L(D) = L$ .  
Let  $n = |Q| > 0$ .
- Take any  $w = a_1 a_2 \cdots a_m \in L$  s.t.  $|w| = m \geq n$ .
- Let  $p_i = \delta^*(q_0, a_1 \cdots a_i)$  for all  $0 \leq i \leq m$ . Then,  $p_0 = q_0 \wedge p_m \in F$ .
- Consider the first  $n + 1$  states:  $p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} p_n$ .
- By Pigeonhole Principle, there exists  $i < j \leq n$  s.t.  $p_i = p_j$ .
- Split  $w = xyz$ :

$$\begin{array}{lll} x = a_1 \cdots a_i & y = a_{i+1} \cdots a_j & z = a_{j+1} \cdots a_m \\ |x| = i & |y| = j - i > 0 & |xy| = j \leq n \\ \delta^*(q_0, x) = p_i & \delta^*(p_i, y) = p_j & \delta^*(p_i, z) = p_m \end{array}$$

- Then,  $\forall i \geq 0$ .  $\delta^*(q_0, xy^i z) = p_m$  (by induction on  $i$ ).
- Finally,  $\forall i \geq 0$ .  $xy^i z \in L$ .

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 \delta^*(q_0, x) = p_i & \delta^*(p_i, y) = p_j & \delta^*(p_i, z) = p_m
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- Then,  $\forall i \geq 0$ .  $\delta^*(q_0, xy^i z) = p_m$  (by induction on  $i$ ).
- Finally,  $\forall i \geq 0$ .  $xy^i z \in L - \textcircled{3}$ .

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To prove a language  $L$  is **NOT** regular, we need to show that

$$\forall n > 0. \exists w \in L. |w| \geq n \wedge \forall w = xyz. (\textcircled{1} \wedge \textcircled{2}) \Rightarrow \neg \textcircled{3}$$

$$\textcircled{1} \quad |y| > 0$$

$$\textcircled{2} \quad |xy| \leq n$$

$$\textcircled{3} \quad \forall i \geq 0. xy^i z \in L$$

Note that  $\neg \textcircled{3} = \exists i \geq 0. xy^i z \notin L$ .

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Note that  $\neg \textcircled{3} = \exists i \geq 0. xy^i z \notin L$ .

We can prove this by following the steps below:

- ① Assume **any** positive integer  $n$  is given.
- ② **Pick** a word  $w \in L$ .
- ③ Show that  $|w| \geq n$ .
- ④ Assume **any** split  $w = xyz$  is given, and  $\textcircled{1} |y| > 0 \wedge \textcircled{2} |xy| \leq n$ .
- ⑤  $\neg \textcircled{3}$  Pick  $i \geq 0$ , and show that  $xy^i z \notin L$  using  $\textcircled{1}$  and  $\textcircled{2}$ .

## Example 1

Let's prove that  $L$  is **NOT** regular using the Pumping Lemma:

$$L = \{a^n b^n \mid n \geq 0\}$$

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- ① Assume **any** positive integer  $n$  is given.
- ② Let  $w = a^n b^n \in L$ .
- ③  $|w| = n + n = 2n \geq n$ .
- ④ Assume **any** split  $w = xyz$  is given, and ①  $|y| > 0 \wedge$  ②  $|xy| \leq n$ .
- ⑤ Then,  $\neg$  ③  $xy^0z \notin L$  because:
  - Since ②  $|xy| \leq n$ ,

$$x = a^p \quad y = a^q \quad z = a^r b^n$$

where  $0 \leq p, q, r \leq n$  and  $p + q + r = n$ .

- Since ①  $|y| > 0$ , we know  $q > 0$ .
- Finally,  $xy^0z = xz = a^p a^r b^n = a^{n-q} b^n$  ( $\because p + q + r = n$ ).  
But,  $a^{n-q} b^n \notin L$  ( $\because q > 0$ ).



## Example 2

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$$L = \{ww^R \mid w \in \{a,b\}^*\}$$



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- ② Let  $w = a^n b^n b^n a^n \in L$ .
- ③  $|w| = n + n + n + n = 4n \geq n$ .
- ④ Assume **any** split  $w = xyz$  is given, and ①  $|y| > 0 \wedge$  ②  $|xy| \leq n$ .
- ⑤ Then,  $\neg$  ③  $xy^0z \notin L$  because:
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where  $0 \leq p, q, r \leq n$  and  $p + q + r = n$ .

- Since ①  $|y| > 0$ , we know  $q > 0$ .
- Finally,  $xy^0z = xz = a^p a^r b^n b^n a^n = a^{n-q} b^n b^n a^n (\because p + q + r = n)$ .  
But,  $a^{n-q} b^n b^n a^n \notin L (\because q > 0)$ . □

## Example 3

Let's prove that  $L$  is **NOT** regular using the Pumping Lemma:

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- ② Let  $w = a^n b^n c^{2n} \in L$ .
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- ⑤ Then,  $\neg$  ③  $xy^0z \notin L$  because:
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- Finally,  $xy^0z = xz = a^p a^r b^n c^{2n} = a^{n-q} b^n c^{2n}$  ( $\because p + q + r = n$ ).  
But,  $a^{n-q} b^n c^{2n} \notin L$  ( $\because q > 0$ ).



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- ② Let  $w = a^{n^2} \in L$ .
- ③  $|w| = n^2 \geq n$ .
- ④ Assume **any** split  $w = xyz$  is given, and ①  $|y| > 0 \wedge$  ②  $|xy| \leq n$ .
- ⑤ Then,  $\neg$  ③  $xy^2z \notin L$  because:
  - Since ①  $|y| > 0$  and ②  $|xy| \leq n$ ,

$$y = a^k$$

where  $1 \leq k \leq n$ . Then,

$$n^2 < n^2 + k \quad (\because 1 \leq k) \qquad n^2 + k < (n+1)^2 \quad (\because k \leq n)$$

- Finally,  $xy^2z = xyyz = a^{n^2+k} \notin L \quad (\because n^2 < n^2 + k < (n+1)^2)$ . □

## Example 5

Let's prove that  $L$  is **NOT** regular:

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- It is much easier to use **closure properties** under **homomorphisms**.
- Consider a homomorphism  $h : \{a, b, c\} \rightarrow \{a, b\}^*$ :

$$h(a) = a \quad h(b) = a \quad h(c) = b$$

- Then,

$$h(L) = \{a^{n+k} b^{n+k} \mid n, k \geq 0\} = \{a^n b^n \mid n \geq 0\}$$

- If  $L$  is regular, then  $h(L)$  must be regular as well.
- However, we know  $h(L)$  is **NOT** regular.
- Therefore,  $L$  is **NOT** regular. □

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- Please see <https://github.com/ku-plrg-classroom/docs/tree/main/equiv-re-fa>.
- The due date is Apr. 13 (Thu.).
- Please only submit `Implementation.scala` file to **Blackboard**.

- Equivalence and Minimization of Finite Automata

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