Lecture 6 – Regular Expressions and Languages

COSE215: Theory of Computation

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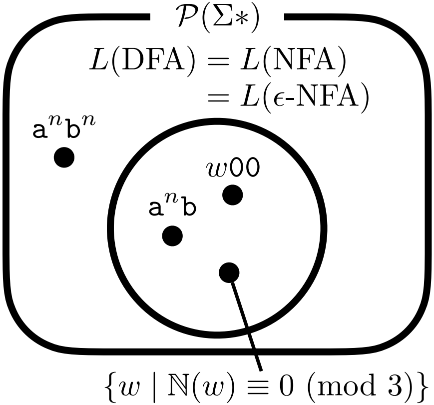
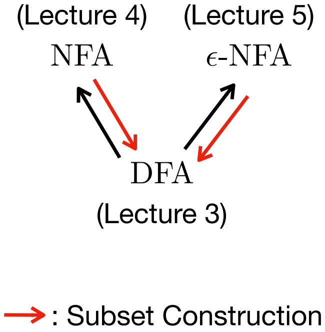
PLRG

2023 Spring

Recall



$\mathcal{P}(\Sigma*)$
$L(\text{DFA}) = L(\text{NFA})$
$= L(\epsilon\text{-NFA})$
$\mathtt{a}^n\mathtt{b}^n$
$w\mathtt{00}$
$\mathtt{a}^n\mathtt{b}$
$\{w \mid \mathbb{N}(w) \equiv 0 \pmod 3\}$



(Lecture 4)
NFA
(Lecture 5)
ϵ-NFA
DFA
(Lecture 3)
: Subset Construction

Languages – Operations

- The **union** of languages:

$$L_1 \cup L_2$$

- The **concatenation** of languages:

$$L_1 L_2 = \{w_1 w_2 \mid w_1 \in L_1 \land w_2 \in L_2\}$$

- The **Kleene star** of a language:

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots = \bigcup_{n \geq 0} L^n$$

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$$\begin{aligned} L_1 \cup L_2 &= \{\mathtt{a}^n \text{ or } \mathtt{b}^n \mid n \geq 0\} \\ L_1 L_2 &= \{\mathtt{a}^n \mathtt{b}^m \mid n, m \geq 0\} \\ L_1^* = L_1 &= \{\mathtt{a}^n \mid n \geq 0\} \end{aligned}$$

Contents

Definition (Regular Expressions)

A **regular expression** over a set of symbols Σ is inductively defined as follows:

- **(Basis Case)** $\varnothing$, ϵ, and $a \in \Sigma$ are regular expressions.
- **(Induction Case)** If R_1 and R_2 are regular expressions, then so are $R_1 \mid R_2$, $R_1 \cdot R_2$, R^*, and (R).

$$
\begin{array}{rcll}
R & ::= & \varnothing & \text{(Empty)} \\
& \mid & \epsilon & \text{(Epsilon)} \\
& \mid & a & \text{(Symbol)} \\
& \mid & R \mid R & \text{(Union)} \\
& \mid & R \cdot R & \text{(Concatenation)} \\
& \mid & R^* & \text{(Kleene Star)} \\
& \mid & (R) & \text{(Parentheses)}
\end{array}
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Definition of Regular Expressions

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\end{array}
$$

$$(\mathtt{a} \mid \epsilon)\, \mathtt{b}^*$$

Precedence of operators:

$$* > \cdot > \mid$$

Definition of Regular Expressions

```scala
// The type definitions of symbols
type Symbol = Char
// The definition of regular expressions
trait RE
case class REEmpty() extends RE
case class REEpsilon() extends RE
case class RESymbol(symbol: Symbol) extends RE
case class REUnion(left: RE, right: RE) extends RE
case class REConcat(left: RE, right: RE) extends RE
case class REStar(re: RE) extends RE
case class REParen(re: RE) extends RE
// An example of regular expression
val re: RE = REConcat(
  REParen(
    REUnion(
      RESymbol('a'),
      REEpsilon(),
    )
  ),
  REStar(
    RESymbol('b')
  ),
)
```

Language of Regular Expressions

Definition (Language of Regular Expressions)

For a given regular expression R on a set of symbols Σ, the **language** $L(R)$ of R is inductively defined as follows:

$$
\begin{array}{lcl}
L(\varnothing) & = & \varnothing \\
L(\epsilon) & = & \{\epsilon\} \\
L(a) & = & \{a\}
\end{array}
\qquad
\begin{array}{lcl}
L(R_1 \mid R_2) & = & L(R_1) \cup L(R_2) \\
L(R_1 R_2) & = & L(R_1) L(R_2) \\
L(R^*) & = & L(R)^* \\
L((R)) & = & L(R)
\end{array}
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\end{array}$$

$$L((\epsilon|\mathtt{a})\mathtt{b}^*) =$$

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L(a) & = & \{a\} & L(R^*) & = & L(R)^* \\
 & & & L((R)) & = & L(R)
\end{array}
$$

$$
\begin{aligned}
L((\epsilon | \mathtt{a})\mathtt{b}^*) &= L((\epsilon | \mathtt{a})) \cdot L(\mathtt{b}^*) \\
&= L(\epsilon | \mathtt{a}) \cdot L(\mathtt{b})^* \\
&= (L(\epsilon) \cup L(\mathtt{a})) \cdot L(\mathtt{b})^* \\
&= (\{\epsilon\} \cup \{\mathtt{a}\}) \cdot \{\mathtt{b}^n \mid n \geq 0\} \\
&= \{\mathtt{b}^n \text{ or } \mathtt{ab}^n \mid n \geq 0\}
\end{aligned}
$$

Extended Regular Expressions

More operators:

$$\begin{array}{rcll} R & ::= & \cdots & \\ & | & R^{+} & \text{(Kleene plus)} \\ & | & R^{?} & \text{(Optional)} \end{array}$$

Actually, they are just **syntactic sugar** for the existing operators:

$$L(R^{+}) = L(RR^{*}) = L(R) \cdot L(R^{*})$$

$$L(R^{?}) = L(\epsilon \,|\, R) = L(\epsilon) \cup L(R)$$

For examples,

$$\begin{array}{rcl} L(\mathtt{a}^{+}) & = & L(\mathtt{aa}^{*}) \\ & = & L(\mathtt{a}) \cdot L(\mathtt{a}^{*}) \\ & = & \{\mathtt{a}^{n} \mid n \geq 1\} \end{array}$$

$$\begin{array}{rcl} L(\mathtt{b}^{?}) & = & L(\epsilon) \cup L(\mathtt{b}) \\ & = & \{\epsilon, \mathtt{b}\} \end{array}$$

Examples

- $L = \{\epsilon, \mathtt{a}\}$

- $L = \{w \in \{0, 1^*\} \mid w \text{ contains at least two } 0's\}$

- $L = \{w \in \{0, 1^*\} \mid w \text{ contains exactly two } 0's\}$

- $L = \{w \in \{0, 1^*\} \mid w \text{ has three consecutive } 0's\}$

- $L = \{w \in \{\mathtt{a}, \mathtt{b}^*\} \mid \mathtt{a} \text{ and } \mathtt{b} \text{ alternate in } w\}$

Examples

- $L = \{\mathrm{a}^n\mathrm{b}^m \mid n \geq 3 \wedge m \equiv 0 (\mathrm{mod}\ 2)\}$

- $L = \{\mathrm{a}^n\mathrm{b}^m \mid n + m \equiv 0 (\mathrm{mod}\ 2)\}$

- $L = \{w \in \{0, 1^*\} \mid$ the number of 0's is divisible by $3\}$

- $L = \{w \in \{0, 1^*\} \mid \mathbb{N}(w) \equiv 0 \pmod 3\}$
  where $\mathbb{N}(w)$ is a natural number represented by w.

Examples

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$$\texttt{(0|1(01}^*\texttt{0)}^*\texttt{1)}^*$$

- $L = \{\mathrm{a}^n\mathrm{b}^n \mid n \geq 0\}$

Examples

- $L = \{a^n b^m \mid n \geq 3 \wedge m \equiv 0 (\text{mod } 2)\}$

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- $L = \{a^n b^n \mid n \geq 0\}$ – IMPOSSIBLE ($\nexists$ RE R. $L(R) = L$)

Exercise #1

- Please see https://github.com/ku-plrg-classroom/docs/tree/main/fa-examples.
- You don't have to submit it. This is just exercise for your practice.
- The goal is to implement the finite automata (FA) objects in the `Implementation.scala` file.

Summary

1. Regular Expressions
   - Definition
   - Language of Regular Expressions
   - Extended Regular Expressions
   - Examples

Next Lecture

- Equivalence of Regular Expressions and Finite Automata

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