

TESTING THE GOODNESS OF FIT OF A HILBERTIAN AUTOREGRESSIVE MODEL

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Abstract

The presented methodology for testing the goodness-of-fit of an Autoregressive Hilbertian model (ARH(1) model) provides an infinite-dimensional formulation of the approach proposed in Koul and Stute [24], based on empirical process marked by residuals. Applying a central and functional central limit result for Hilbert-valued martingale difference sequences, the asymptotic behavior of the formulated H -valued empirical process, also indexed by H , is obtained under the null hypothesis. The limiting process is H -valued generalized (i.e., indexed by H) Wiener process, leading to an asymptotically distribution free test. Consistency is also analyzed. The case of misspecified autocorrelation operator of the ARH(1) process is addressed as well. Beyond the Euclidean setting, this approach allows to implement goodness of fit testing in the context of manifold and spherical functional autoregressive processes.

Key words. Functional variance decomposition formula, generalized functional empirical processes, generalized Hilbert-valued Wiener process, manifold functional autoregressive processes, misspecification, strong consistency.

1 Introduction

Weakly dependent functional time series models have been extensively analyzed in the last few decades, supporting inference on stochastic processes, specially, in a state space framework (see, e.g., [1]; [15]; [17]; [21]; [22]; [23]). Hypothesis testing is still a challenging topic in this research area, with a relatively small number of contributions addressing some of the crucial problems arising in model identification, independence, regression and significance (see González-Manteiga and Crujeiras [14] for a review on the topic of goodness-of-fit tests).

Horváth et al. [19] formulate a test, based on the sum of the L^2 -norms of the empirical correlation functions, to check if a set of functional observations are independent and identically distributed. Consistency is analyzed as well. To detecting changes in the autocorrelation operator in functional time series, Horváth and Reeder [21] derive a test based on change point analysis. In Horváth, Kokoszka and Rice [20], several procedures are considered to test the null hypothesis of stationarity in the context of functional time series. The properties of the tests under several alternatives are also studied. Approaches for testing the structural stability of temporally dependent functional observations are considered in Zhang et al. [30]. Kokoszka and Reimherr [23] propose a multistage testing procedure to determine the order p of a functional autoregressive process. The test statistics involves the estimating kernel function in this linear model, and is approximately distributed as a chi-square with number of degrees of freedom determined by the number of functional principal components used to represent the data. Several contributions have been derived in the context of functional regression, including both, scalar and functional response cases. That is, considering the functional regression model

$$\mathbf{Y} = m(\mathbf{X}) + \varepsilon,$$

where $\mathbf{X} \in \mathcal{H}$, $m : \mathcal{H} \rightarrow \mathbb{R}$, and $Y \in \mathbb{R}$, for the case of scalar response, or, alternatively, $X \in \mathcal{H}$, $m : \mathcal{H} \rightarrow \mathcal{H}$, and $Y \in \mathcal{H}$, for the functional response case. Special attention has been paid to the case where the function space \mathcal{H} is a separable Hilbert space H . Particular testing has been focused on significance when $H_0 : m(\mathbf{X}) = c$, with $c \in \mathbb{R}$ being a fixed constant, testing the significance of the covariate \mathbf{X} over Y (see, e.g., Hilgert, Mas and Andverzelen [16]; Chiou and Müller [6]; Bücher, Dette and Andwieczorek [3]). Some extensions to more complex hypothesis testing can be found in Delsol, Ferraty and Andvieu [11]; Patilea, Sánchez-Sellero and Andsaumard [27]; García-Portugués, González-Manteiga and Febrero-Bande [13], among others. The last one based on the marked empirical process $I_{n,\mathbf{h}}(x) := \frac{1}{n} \sum_{i=1}^n 1_{\{(\mathbf{X}_i, \mathbf{h}) \leq x\}} \mathbf{Y}_i$, for $\mathbf{h} \in H$, and $x \in \mathbb{R}$. In this paper, an approach based on random projections is implemented, adopting Cramér von Mises norm.

In the construction of a goodness of fit test for the functional linear model with scalar response, a marked empirical process based approach is also adopted in Cuesta-Albertos et al. [8]. Specifically, the marked empirical process is indexed by random projections of the functional covariate. The weak convergence of the empirical process is obtained conditionally on a random direction. The almost surely equivalence of the test based on the original and projected covariate is proved. Calibration by wild bootstrap resampling is also achieved. The present paper extends this formulation to the context of autoregressive functional time series, i.e., functional linear model with functional response, and time-dependent

functional covariates, in the case of error term given by a hilbertian strong white noise. Specifically, an infinite-dimensional formulation of the involved marked empirical process indexed by an infinite-dimensional covariate is adopted. The limit behavior of the marginals of this process under the null hypothesis is obtained by applying a Central Limit result for Hilbert-valued martingale difference sequences (see Theorem 2.16 in Bosq [2]). A functional central limit result is also derived by applying a special case of an invariance principle based on Robbins-Monro procedure (see Theorem 2 in Walk [29]). A more general formulation can be obtained from the conditional central limit theorem in Hilbert spaces derived in Dedecker and Merlevede [10]. The functional central limit result formulated here characterizes the limiting process of the marked empirical process as a generalized Gaussian process taking values in a separable Hilbert space H , and having index set or parameter space also H . This process is identified in law with Hilbert-valued generalized Brownian motion. The proposed goodness-of-fit test is based on this functional central limit result, and the application of Theorem 4.1 in Cuesta-Albertos, Fraiman and Ransford [7]. Both results allow the definition of the rejection region of the test from a suitable critical value obtained from the boundary crossing probabilities of Brownian motion over the unit interval, which are readily available on this interval. Consistency is briefly discussed. The case of misspecified autocorrelation operator of the ARH(1) process is also addressed. The marks of the empirical process are approximated in this case from the consistent projection estimation of the autocorrelation operator of the ARH(1) process. The limiting process of the resulting empirical process is also obtained. Specifically, the asymptotic equivalence, in probability, of the two test statistics corresponding to totally specified and misspecified autocorrelation operator is proved. Hence, a similar design of the goodness-of-fit test can be applied in the last case.

The outline of the paper is the following. Preliminaries are provided in Section 2. Section 3 derives a Central Limit result to characterize the asymptotic behavior of the marginals of the H -valued empirical process. A Functional Central Limit Theorem for the considered Hilbert-valued marked empirical process indexed by infinite-dimensional covariates is obtained in Section 4, leading to the design of the goodness-of-fit test. Sufficient conditions for the consistency of the proposed goodness-of-fit test are given in Section 5. The issue of misspecified autocorrelation operator of the ARH(1) process is finally addressed in Section 6.

2 Preliminaries

In the following we denote by $\mathcal{L}_H^2(\Omega, \mathcal{A}, P)$ the space of zero-mean H -valued random variables on the basic probability space (Ω, \mathcal{A}, P) satisfying $E\|X\|_H^2 < \infty$.

∞ , for every $X \in \mathcal{L}_H^2(\Omega, \mathcal{A}, P)$. Let $Y = \{Y_t, t \in \mathbb{Z}\}$ be a zero-mean ARH(1) process on the basic probability space (Ω, \mathcal{A}, P) satisfying

$$Y_t = \Gamma(Y_{t-1}) + \varepsilon_t, \quad t \in \mathbb{Z}, \quad (1)$$

and the conditions given in Chapter 3 in Bosq (2000) for the existence of a unique stationary solution. Hence, $C_0^Y = E[Y_0 \otimes Y_0] = E[Y_t \otimes Y_t]$, $t \in \mathbb{Z}$, denotes the autocovariance operator of Y , and $D^Y = E[Y_0 \otimes Y_1] = E[Y_t \otimes Y_{t+1}]$, $t \in \mathbb{Z}$, its cross-covariance operator. The H -valued innovation process ε is assumed to be H -valued strong white noise (SWN) (see Definition 3.1 in Bosq, 2000). That is, $\varepsilon = \{\varepsilon_t, t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed zero-mean H -valued random variables with

$$C_0^\varepsilon := E[\varepsilon_t \otimes \varepsilon_t] = E[\varepsilon_0 \otimes \varepsilon_0], \quad \forall t \in \mathbb{Z}, \quad (2)$$

and functional variance $E[\|\varepsilon_t\|_H^2] = E[\|\varepsilon_0\|_H^2] = \|C_0^\varepsilon\|_{L^1(H)} = \sigma_\varepsilon^2$, that coincides with the trace norm of the autocovariance operator C_0^ε of ε . The following assumption on ARH(1) process Y summarizes the ARH(1) process set up considered here:

Assumption A1. Y is a strictly stationary ARH(1) process with SWN innovation ε such that $E\|\varepsilon_1\|_H^4 < \infty$. Assume also that Y_0 is independent of $\{\varepsilon_i, i \geq 1\}$.

In our design of the goodness-of-fit test, we will consider the following orthogonality condition $E[(Y_1 - \Gamma(Y_0))/Y_0] = E[\varepsilon_1/Y_0] = 0$, almost surely (a.s.). Applying Lemma 1 in Escanciano [12], in an infinite-dimensional random variable framework, one can equivalently express this condition $E[(Y_1 - \Gamma(Y_0))/Y_0] = E[\varepsilon_1/Y_0] = 0$, a.s., in terms of the identity

$$E[\varepsilon_1 1_{\{\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}] = 0, \quad \text{a.e. } x \in H,$$

for a given orthonormal basis $\{\phi_j, j \geq 1\}$ of H .

Thus, under strict stationarity of Y , for every time i ,

$$\begin{aligned} E[(Y_i - \Gamma(Y_{i-1}))/Y_{i-1}] &= E[\varepsilon_i/Y_{i-1}] = 0, \quad \text{a.s.} \\ \Leftrightarrow E[\varepsilon_i 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}] &= 0, \quad \text{a.e. } x \in H, \quad i \geq 2. \end{aligned}$$

The following lemma establishes the H -valued martingale difference property of the sequence $\{\varepsilon_i 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}\}$.

Lemma 1 *For every $x \in H$, the sequence*

$$\{X_i(x), i \geq 1\} = \{(Y_i - \Gamma(Y_{i-1})) 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}, i \geq 1\}$$

is an H -valued martingale difference with respect to the filtration $\mathcal{M}_0^Y \subset \mathcal{M}_1^Y \cdots \subset \mathcal{M}_n^Y \subset \dots$, where $\mathcal{M}_{i-1}^Y = \sigma(Y_t, t \leq i-1)$, for $i \geq 1$.

Proof. For each $i \geq 1$, and $x \in H$,

$$\begin{aligned} E^{\mathcal{M}_{i-1}^Y} [X_i(x)] &= E^{\mathcal{M}_{i-1}^Y} [(Y_i - \Gamma(Y_{i-1})) 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}] \\ &= 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}} E^{\mathcal{M}_{i-1}^Y} [Y_i - \Gamma(Y_{i-1})] \\ &= 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}} E^{\mathcal{M}_{i-1}^Y} [\varepsilon_i] = 0, \quad i \geq 1, \end{aligned}$$

in view of the definition of the innovation process ε as $\varepsilon_n = Y_n - \Pi^{\mathcal{M}_{n-1}}(Y_n)$, for every $n \geq 1$, with $\Pi^{\mathcal{M}_{n-1}}$ being the orthogonal projector of \mathcal{M}_{n-1} (see equation (2.55), and pp. 72–73 in Bosq [2]).

Note that, from (1) (see equation (3.11) in Bosq, 2000), for every $i \geq 1$,

$$Y_{i-1} = \sum_{t=0}^{i-2} \Gamma^t(\varepsilon_{i-1-t}) + \Gamma^{i-1}(Y_0). \quad (3)$$

Therefore, under **Assumption A1**, from (3), for any $i \geq 1$, and $x \in H$,

$$\begin{aligned} E[X_i(x)] &= E [\varepsilon_i 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}] \\ &= E [\varepsilon_i 1_{\{\omega \in \Omega; \langle \sum_{t=0}^{i-2} \Gamma^t(\varepsilon_{i-1-t}(\omega)) + \Gamma^{i-1}(Y_0(\omega)), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}] \\ &= E [\varepsilon_i] E [1_{\{\omega \in \Omega; \langle \sum_{t=0}^{i-2} \Gamma^t(\varepsilon_{i-1-t}(\omega)) + \Gamma^{i-1}(Y_0(\omega)), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}] = 0. \end{aligned} \quad (4)$$

Under **Assumption A1**, for every $x \in H$, the functional variance

$$\begin{aligned} \mathcal{T}(x) &= E [\|X_1(x)\|_H^2] \\ &= E [\|\varepsilon_1 1_{\{\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}\|_H^2] \\ &= E [E [\| [Y_1 - \Gamma(Y_0)] 1_{\{\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}} \|_H^2 / Y_0]] \\ &= \int_H 1_{\{u \in H; \langle u, \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}^2 \text{Var} ([Y_1 - \Gamma(Y_0)] / Y_0 = u) P_{Y_0}(du) \\ &= \|C_0^\varepsilon\|_{L^1(H)} P(\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1) \\ &= \|C_0^\varepsilon\|_{L^1(H)} P(E^0(x)), \end{aligned} \quad (5)$$

where $\|\cdot\|_{L^1(H)}$ denotes the trace norm of a nuclear operator on H . Here, P_{Y_0} denotes the infinite-dimensional marginal probability measure induced by Y_0 , and $E^0(x)$ denotes the event $E^0(x) = \{\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}$, for each $x \in H$.

Remark 1 Let now consider the particular case where the orthonormal basis $\{\phi_j, j \geq 1\}$ is constituted by the eigenvectors of C_0^Y , and denote by $\{\lambda_k(C_0^Y), k \geq 1\}$ the corresponding system of eigenvalues. That is, $C_0^Y(\phi_k) = \lambda_k(C_0^Y)\phi_k$, for every $k \geq 1$. Let also denote by $\{\lambda_k(C_0^\varepsilon), k \geq 1\}$ the system of eigenvalues of the autocovariance operator C_0^ε . Under **Assumption A1**, for every $u \in H$,

$$\begin{aligned} \text{Var}([Y_1 - \Gamma(Y_0)]/Y_0 = u) &= \text{Var}(Y_1 - \Gamma(Y_0)) = \|C_0^\varepsilon\|_{L^1(H)} = \sum_{k \geq 1} \lambda_k(C_0^\varepsilon) \\ &= \sum_{k=1}^{\infty} \lambda_k(C_0^Y) - \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{[D^Y(\phi_k)(\phi_l)]^2}{\lambda_l(C_0^Y)}, \end{aligned} \quad (6)$$

where we have applied the functional variance decomposition formula for the linear predictor $\Gamma(Y_0)$ of Y_1 (see, e.g., equation (3.13) in Bosq [2]), leading to

$$\begin{aligned} \text{Var}(\Gamma(Y_0)) &= \|\Gamma^* C_0^Y \Gamma\|_{L^1(H)} \\ &= \|D^Y(C_0^Y)^{-1/2}\|_{\mathcal{S}(H)}^2 = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{[D^Y(\phi_k)(\phi_l)]^2}{\lambda_l(C_0^Y)}. \end{aligned} \quad (7)$$

The last two identities in equation (7) follow from Spectral Theorem on Spectral Calculus for self-adjoint operators on a separable Hilbert space H , in terms of the spectral kernel defined by the system of eigenvectors $\{\phi_j, j \geq 1\}$ of C_0^Y (see, e.g., Dautray and Lions [9], pp. 112–140). Also, $\|\cdot\|_{\mathcal{S}(H)}$ denotes the norm in the space of Hilbert–Schmidt operators on H . Hence, if $\langle x, \phi_j \rangle_H \rightarrow \infty$, for every $j \geq 1$, $\mathcal{T}(x) \rightarrow \|C_0^\varepsilon\|_{L^1(H)} = \sum_{k=1}^{\infty} \lambda_k(C_0^\varepsilon)$. Thus, the limit is given by the trace of the autocovariance operator of the H -valued innovation process ε .

Remark 2 Equation (5) implies, in particular, the strongly integrability of the marginals of the H -valued martingale difference $\{X_i(x), i \geq 1\}$, for every $x \in H$. This fact will be applied in the subsequent results in this paper. Thus, the means and conditional means of the elements of this sequence are finite, and can be computed from their weak counterparts (Section 1.3 in Bosq [2]).

We now compute the autocovariance operator $C_0^{X_i(x)} := E[X_i(x) \otimes X_i(x)]$ of the martingale difference sequence

$$X_i(x) = (Y_i - \Gamma(Y_{i-1}))1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}},$$

for each $x \in H$. Specifically, under **Assumption A1**, applying the SWN property of $\{\varepsilon_i, i \geq 1\}$, and identity (3), for every $x \in H$, and $i \geq 1$:

$$\begin{aligned}
C_0^{X_i(x)} &:= E[X_i(x) \otimes X_i(x)] = E\left[\varepsilon_i \otimes \varepsilon_i 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}\right] \\
&= E\left[E\left[\varepsilon_i \otimes \varepsilon_i 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}} / Y_{i-1}\right]\right] \\
&= E\left[E\left[\varepsilon_i \otimes \varepsilon_i 1_{\{\omega \in \Omega; \langle \sum_{t=0}^{i-2} \Gamma^t(\varepsilon_{i-1-t}(\omega)) + \Gamma^{i-1}(Y_0(\omega)), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}} / Y_{i-1}\right]\right] \\
&= C_0^\varepsilon P(\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1) = C_0^\varepsilon P(E^{(i-1)}(x)) \\
&= C_0^\varepsilon P(E^0(x)) = C_0^{X_1(x)}, \tag{8}
\end{aligned}$$

where $E^{(i-1)}(x) = \{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}$. In (8), we have applied the strictly stationarity condition of Y under **Assumption A1**, leading to, for every $x \in H$,

$$\begin{aligned}
P(E^{(i-1)}(x)) &= P[\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1] \\
&= P[\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1] = P(E^0(x)), \quad i \geq 1. \tag{9}
\end{aligned}$$

In a similar way, the covariance operator $C_0^{X_i(x), X_i(y)} := E[X_i(x) \otimes X_i(y)]$ can be computed for every $x, y \in H$,

$$\begin{aligned}
C_0^{X_i(x), X_i(y)} &:= E[X_i(x) \otimes X_i(y)] \\
&= C_0^\varepsilon P(\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \min(\langle x, \phi_j \rangle_H, \langle y, \phi_j \rangle_H), j \geq 1) \\
&= C_0^\varepsilon P(\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \min(\langle x, \phi_j \rangle_H, \langle y, \phi_j \rangle_H), j \geq 1) \\
&= C_0^\varepsilon P(E^0(\min(x, y))), \tag{10}
\end{aligned}$$

where the notation $\min(x, y)$ must be understood componentwise, by projections, as followed from equation (10). That is, the infinite-dimensional vector $\min(x, y) = (\min(\langle x, \phi_j \rangle_H, \langle y, \phi_j \rangle_H), j \geq 1)$ has components defined by the respective minimum values between the two projections of the two functions $x, y \in H$.

3 Central Limit Theorem

In the following lemma, Theorem 2.16 in Bosq (2000) is formulated. We will apply it in the proof of Theorem 1 below.

Lemma 2 Let $\{X_i, i \geq 1\}$ be an H -valued martingale difference, and $\{e_j, j \geq 1\}$ be an orthonormal basis of H . Suppose that

$$\frac{1}{\sqrt{n}} E \left(\max_{1 \leq i \leq n} \|X_i\|_H \right) \rightarrow 0, \quad n \rightarrow \infty, \quad (11)$$

$$\frac{1}{n} \sum_{1 \leq i \leq n} \langle X_i, e_k \rangle_H \langle X_i, e_l \rangle_H \rightarrow_{a.s.} \psi_{k,l}, \quad n \rightarrow \infty, \quad l, k \geq 1, \quad (12)$$

where $\{\psi_{k,l}, l, k \geq 1\}$ is a family of real numbers, and

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left(\sum_{i=1}^n r_N^2 (n^{-1/2} X_i) > \varepsilon \right) = 0, \quad \varepsilon > 0, \quad (13)$$

with $r_N^2(x) = \sum_{i=N}^{\infty} [\langle x, e_i \rangle_H]^2$, $x \in H$. Then,

$$n^{-1/2} \sum_{i=1}^n X_i \rightarrow_D N \sim \mathcal{N}(0, C),$$

where \rightarrow_D denotes the convergence in distribution. Here, the covariance operator C of the functional Gaussian random variable $N \sim \mathcal{N}(0, C)$ satisfies

$$\langle C(e_k), e_l \rangle_H = \psi_{k,l}, \quad k, l \geq 1.$$

Lemma 2 is applied in Theorem 1 below, in the derivation of the Gaussian limit distribution of the marginals of the H -valued marked empirical process

$$V_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i - \Gamma(Y_{i-1})) 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}. \quad (14)$$

Without loss of generality, we will consider the special case of the orthonormal basis $\{e_j, j \geq 1\}$ given by the eigenvectors $\{\phi_j, j \geq 1\}$ of the autocovariance operator C_0^Y of Y .

Theorem 1 Let $\{Y_t, t \in \mathbb{Z}\}$ be a centered $ARH(1)$ process. Under **Assumption A1**, for each $x \in H$, the functional empirical process (14) satisfies $V_n(x) \rightarrow_D N \sim \mathcal{N}(0, C_x)$, as $n \rightarrow \infty$. Here, for every $x \in H$,

$$C_x = C_0^\varepsilon P(\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1) = C_0^\varepsilon P(E^0(x)).$$

In particular, considering $\langle x, \phi_j \rangle_H \rightarrow \infty$, for every $j \geq 1$,

$$V_n(x) \rightarrow_D \mathcal{Z} \sim \mathcal{N}(0, C_0^\varepsilon), \quad n \rightarrow \infty.$$

Proof.

The proof is based on the verification of the conditions assumed in the CLT for Hilbert-valued martingale difference sequences given in Theorem 2.16 in Bosq (2000) (see also Lemma 2 above). Let us first verify equation (11) in Lemma 2, corresponding to condition (2.59) in Theorem 2.16. For every $i \geq 1$, $n \geq 1$, and $x \in H$, the following events are considered:

$$\begin{aligned} A(x) &= \left\{ \omega \in \Omega; \max_{1 \leq i \leq n} \|X_i(x, \omega)\|_H > \sqrt{n\eta} \right\} \in \mathcal{A} \\ B_i(x) &= \left\{ \omega \in \Omega; \|X_i(x, \omega)\|_H > \sqrt{n\eta} \right\} \in \mathcal{A}, \quad i = 1, \dots, n \\ B_n(x) &= \cup_{i=1}^n \left\{ \omega \in \Omega; \|X_i(x, \omega)\|_H > \sqrt{n\eta} \right\} \in \mathcal{A}. \end{aligned} \quad (15)$$

Clearly, $A(x) \subset B_n(x)$, and for every $x \in H$,

$$\begin{aligned} P(A(x)) &= P\left(\omega \in \Omega; \max_{1 \leq i \leq n} \|X_i(x, \omega)\|_H > \sqrt{n\eta}\right) \leq P(B_n(x)). \\ &\leq \sum_{i=1}^n P(B_i(x)) = \sum_{i=1}^n P(\omega \in \Omega; \|X_i(x, \omega)\|_H > \sqrt{n\eta}). \end{aligned} \quad (16)$$

From equation (16), applying Chebyshev inequality and stationarity, keeping in mind that the events $E_i^{(1)}(x) = \{\omega \in \Omega; \|X_i(x, \omega)\|_H > \sqrt{n\eta}\}$ and $E_i^{(2)}(x) = \{\omega \in \Omega; \|X_i(x, \omega)\|_H 1_{\|X_i(x, \omega)\|_H > \sqrt{n\eta}} > \sqrt{n\eta}\}$ coincide for $i = 1, \dots, n$, and $x \in H$, we obtain

$$\begin{aligned} &P\left(\omega \in \Omega; \max_{1 \leq i \leq n} \|X_i(x, \omega)\|_H > \sqrt{n\eta}\right) \\ &\leq \frac{1}{n\eta^2} \sum_{i=1}^n E\left[\|X_i\|_H^2 1_{\{\|X_i\|_H > \sqrt{n\eta}\}}\right] = \frac{1}{n\eta^2} \sum_{i=1}^n E\left[\|X_i\|_H^2 1_{B_i(x)}\right] \\ &= \frac{1}{n\eta^2} \sum_{i=1}^n E\left[\|X_1\|_H^2 1_{B_1(x)}\right] = \frac{1}{\eta^2} E\left[\|X_1\|_H^2 1_{B_1(x)}\right]. \end{aligned} \quad (17)$$

Dominated Convergence Theorem yields to

$$\lim_{n \rightarrow \infty} P\left[\max_{1 \leq i \leq n} \|X_i(x)\|_H > \sqrt{n\eta}\right] = 0, \quad \forall x \in H. \quad (18)$$

On the other hand, for any $x \in H$, and $n \geq 1$, applying SWN property of

the innovation process ε ,

$$\begin{aligned}
& E \left[\max_{1 \leq i \leq n} \left\| \frac{X_i(x)}{\sqrt{n}} \right\|_H^2 \right] \\
& \leq \sum_{i=1}^n E \left[\frac{\|Y_i - \Gamma(Y_{i-1})\|_H^2 1_{\{\langle Y_{i-1}, \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}}{n} \right] \\
& = \frac{1}{n} \sum_{i=1}^n E \left[1_{\{\langle Y_{i-1}, \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}} E \left[\|\varepsilon_i\|_H^2 / Y_{i-1} \right] \right] \\
& \leq \frac{1}{n} \sum_{i=1}^n E \left[E \left[\|\varepsilon_i\|_H^2 / Y_{i-1} \right] \right] = E \left[\|\varepsilon_1\|_H^2 \right] = \|C_0^\varepsilon\|_{L^1(H)} < \infty. \quad (19)
\end{aligned}$$

Thus, $\left\{ \max_{1 \leq i \leq n} \left\| \frac{X_i}{\sqrt{n}} \right\|_H^2, n \geq 1 \right\}$ is uniformly integrable, and equation (11) is therefore satisfied.

Let us now prove that condition (2.61) in Theorem 2.16 in Bosq (2000) holds (see also equation (13) in Lemma 2). Specifically, from the SWN property of ε , applying Markov Theorem, for any $n \geq 1$,

$$\begin{aligned}
P \left[\sum_{i=1}^n r_N^2 \left(\frac{X_i}{\sqrt{n}} \right) > \eta \right] &= P \left[\sum_{i=1}^n \sum_{l=N}^{\infty} \left\langle \frac{X_i}{\sqrt{n}}, \phi_l \right\rangle_H^2 > \eta \right] \\
&\leq \sum_{i=1}^n P \left[r_N^2 \left(\frac{X_i}{\sqrt{n}} \right) > \eta \right] \leq \frac{1}{\eta} \sum_{i=1}^n E \left[r_N^2 \left(\frac{X_i}{\sqrt{n}} \right) \right] \\
&\leq \frac{1}{n\eta} \sum_{i=1}^n E \left[E \left[r_N^2(\varepsilon_i) / Y_{i-1} \right] \right] = \frac{1}{n\eta} \sum_{i=1}^n E \left[r_N^2(\varepsilon_i) \right] = \frac{1}{\eta} E \left[r_N^2(\varepsilon_1) \right] \\
&= \frac{1}{\eta} \sum_{l=N}^{\infty} \lambda_l(C_0^\varepsilon). \quad (20)
\end{aligned}$$

From (20),

$$\lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} P \left[\sum_{i=1}^n r_N^2 \left(\frac{X_i}{\sqrt{n}} \right) > \eta \right] \leq \lim_{N \rightarrow \infty} \frac{1}{\eta} \sum_{l=N}^{\infty} \lambda_l(C_0^\varepsilon) = 0.$$

Thus, equation (13) is satisfied.

Finally, we prove condition (2.60) in Theorem 2.16 in Bosq (2000) (equation (12) in Lemma 2) holds. Specifically, we prove that condition (2.36) in Corollary 2.3 in Bosq (2000) holds. Under **Assumption A1**, from equations (8) and (9), for every $x \in H$,

$$E[X_{n+i}(x) \otimes X_{n+i}(x)] = C_0^\varepsilon P(E^{(n+i-1)}(x)) = C_0^\varepsilon P(E^0(x)).$$

Let us denote $W_i(x) = X_{n+i}(x) \otimes X_{n+i}(x) - C_0^\varepsilon P(E^0(x))$, for $i = 0, \dots, p-1$. Here, as before, for every $x \in H$,

$$\begin{aligned} X_{n+i}(x) &= [Y_{n+i} - \Gamma(Y_{n+i-1})] 1_{\{\omega \in \Omega; \langle Y_{n+i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}} \\ &= \varepsilon_i 1_{E^{(n+i-1)}(x)}, \quad i = 0, \dots, p-1. \end{aligned} \tag{21}$$

Under **Assumption A1**, from (3), we obtain, for each $x \in H$,

$$\begin{aligned} E \left[\|W_0(x) + \dots + W_{p-1}(x)\|_{\mathcal{S}(H)}^2 \right] &= \sum_{i,k=0}^{p-1} E \left[\langle W_i(x), W_k(x) \rangle_{\mathcal{S}(H)} \right] \\ &= \sum_{i,k=0}^{p-1} E \left[\langle X_{n+i}(x) \otimes X_{n+i}(x), X_{n+k}(x) \otimes X_{n+k}(x) \rangle_{\mathcal{S}(H)} \right] \\ &\quad - \sum_{i,k=0}^{p-1} [P(E^0(x))]^2 \langle C_0^\varepsilon, C_0^\varepsilon \rangle_{\mathcal{S}(H)} - \sum_{i,k=0}^{p-1} [P(E^0(x))]^2 \langle C_0^\varepsilon, C_0^\varepsilon \rangle_{\mathcal{S}(H)} \\ &\quad + \sum_{i,k=0}^{p-1} [P(E^0(x))]^2 \langle C_0^\varepsilon, C_0^\varepsilon \rangle_{\mathcal{S}(H)} \\ &= \sum_{i,k=0}^{p-1} E \left[\langle X_{n+i}(x) \otimes X_{n+i}(x), X_{n+k}(x) \otimes X_{n+k}(x) \rangle_{\mathcal{S}(H)} \right] \\ &\quad - \sum_{i,k=0}^{p-1} [P(E^0(x))]^2 \|C_0^\varepsilon\|_{\mathcal{S}(H)}^2 = \sum_{i,k=0}^{p-1} E \left[(\langle X_{n+i}(x), X_{n+k}(x) \rangle_H)^2 \right] \\ &\quad - \sum_{i,k=0}^{p-1} [P(E^0(x))]^2 \|C_0^\varepsilon\|_{\mathcal{S}(H)}^2 \\ &\leq \sum_{i,k=0}^{p-1} E \left[(\langle X_{n+i}(x), X_{n+k}(x) \rangle_H)^2 \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,k=0}^{p-1} E \left\{ E \left[(\langle X_{n+i}(x), X_{n+k}(x) \rangle_H)^2 / (Y_{n+i-1}, Y_{n+k-1}) \right] \right\} \\
&= \sum_{i,k=0}^{p-1} E \left\{ 1_{E^{(n+i-1)}(x)}^2 1_{E^{(n+k-1)}(x)}^2 E \left[(\langle \varepsilon_{n+i}, \varepsilon_{n+k} \rangle_H)^2 / (Y_{n+i-1}, Y_{n+k-1}) \right] \right\} \\
&= \sum_{i,k=0}^{p-1} E \left[(\langle \varepsilon_{n+i}, \varepsilon_{n+k} \rangle_H)^2 \right] E \left[1_{E^{(n+i-1)}(x)}^2 1_{E^{(n+k-1)}(x)}^2 \right] \\
&= \|C_0^\varepsilon\|_{\mathcal{S}(H)}^2 \sum_{i \neq k} E \left[1_{E^{(n+i-1)}(x)}^2 1_{E^{(n+k-1)}(x)}^2 \right] + p E \left[\|\varepsilon_1\|_H^4 \right] P(E^0(x)) \\
&= \|C_0^\varepsilon\|_{\mathcal{S}(H)}^2 \sum_{i \neq k} \int_{E^{(n+i-1)}(x)} \int_{E^{(n+k-1)}(x)} P_{Y_{n+k-1}, Y_{n+i-1}}(dy_{n+k-1}, dy_{n+i-1}) \\
&\quad + p E \left[\|\varepsilon_1\|_H^4 \right] P(E^0(x)) \\
&= p \left[\|C_0^\varepsilon\|_{\mathcal{S}(H)}^2 \sum_{u \in \{-(p-1), \dots, p-1\} \setminus \{0\}} \left(1 - \frac{u}{p} \right) \right. \\
&\quad \times \int_{E^{(n+u-1)}(x)} \int_{E^{(n-1)}(x)} P_{Y_{n-1}, Y_{n+u-1}}(dy_{n-1}, dy_{n+u-1}) \\
&\quad \left. + E \left[\|\varepsilon_1\|_H^4 \right] P(E^0(x)) \right] \\
&= p \left[\sum_{u \in \{-(p-1), \dots, p-1\} \setminus \{0\}} \left(1 - \frac{u}{p} \right) P_{Y_{n-1}, Y_{n+u-1}}(E^{(n-1)}(x) \times E^{(n+u-1)}(x)) \right. \\
&\quad \left. \times \|C_0^\varepsilon\|_{\mathcal{S}(H)}^2 + E \left[\|\varepsilon_1\|_H^4 \right] P(E^0(x)) \right] \\
&= p \left[\sum_{u \in \{-(p-1), \dots, p-1\} \setminus \{0\}} \left(1 - \frac{u}{p} \right) P_{Y_0, Y_u}(E^0(x) \times E^u(x)) \|C_0^\varepsilon\|_{\mathcal{S}(H)}^2 \right. \\
&\quad \left. + E \left[\|\varepsilon_1\|_H^4 \right] P(E^0(x)) \right] \\
&\leq p \left[\|C_0^\varepsilon\|_{\mathcal{S}(H)}^2 \sum_{u \in \mathbb{Z}} P_{Y_0, Y_u}(E^0(x) \times E^u(x)) + E \left[\|\varepsilon_1\|_H^4 \right] P(E^0(x)) \right] = pM,
\end{aligned} \tag{22}$$

where we have applied that $\sum_{u \in \mathbb{Z}} P_{Y_0, Y_u}(E^0(x) \times E^u(x)) < \infty$ from the application of Boole and Chebyshev inequalities, since $\sum_{t \in \mathbb{Z}} \|E[Y_t \otimes Y_0]\|_{L^1(H)} \leq \|C_0^Y\|_{L^1(H)} + \|D^Y\|_{L^1(H)} < \infty$.

Hence, we can apply Corollary 2.3 in Bosq (2000) with $\gamma = 1$. Thus, for all

$\beta > 1/2$, and each $x \in H$,

$$\frac{n^{1/4}}{(\log(n))^\beta} \left\| \frac{S_n^{W(x)}}{n} \right\|_{S(H)} = \frac{n^{1/4}}{(\log(n))^\beta} \left\| \sum_{i=1}^n \frac{W_i(x)}{n} \right\|_{S(H)} \xrightarrow{\text{a.s.}} 0, \quad n \rightarrow \infty,$$

leading to equation (12) in Lemma 2 as we wanted to prove. Thus, the weak convergence to the indicated H -valued Gaussian random variable holds.

4 Functional Central Limit Theorem

Let $C_H([0, 1])$ be the separable Banach space of H -valued continuous functions on $[0, 1]$, with respect to the H norm, under the supremum norm $\|g\|_\infty = \sup_{t \in [0, 1]} \|g(t)\|_H$. The following result provides an invariance principle for martingale difference sequences evaluated in a Hilbert space (see Theorem 2 in Walk [29]).

Theorem 2 *Let $\{X_n, n \in \mathbb{N}\}$ be a martingale difference sequence of H -valued random variables, with respect to the filtration $\mathcal{M}_0^Y \subset \mathcal{M}_1^Y \cdots \subset \mathcal{M}_n^Y \subset \dots$, satisfying $E\|X_n\|_H^2 < \infty$. Let $S : H \rightarrow H$ be a trace operator. For each $n \in \mathbb{N}$, denote as S^n the covariance operator of X_n , given Y_1, \dots, Y_{n-1} . That is,*

$$S^n = E[X_n \otimes X_n / Y_1, \dots, Y_{n-1}], \quad n \in \mathbb{N}.$$

Assume that

$$(i) \quad E \left\| \frac{1}{n} \sum_{j=1}^n S^j - S \right\|_{L^1(H)} \rightarrow 0, \quad n \rightarrow \infty.$$

$$(ii) \quad \frac{1}{n} \sum_{j=1}^n E\|X_j\|_H^2 \rightarrow \text{trace}(S), \quad n \rightarrow \infty.$$

(iii) For $r > 0$,

$$\frac{1}{n} \sum_{j=1}^n E(\|X_j\|_H^2 \chi(\|X_j\|_H^2 \geq rj) / Y_1, \dots, Y_{j-1}) \rightarrow_P 0, \quad n \rightarrow \infty.$$

Then, the sequence of random elements $\{Y_n\}$ in $C_H([0, 1])$ with the supremum norm, which are defined by

$$Z_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} X_j + (nt - [nt]) \frac{1}{\sqrt{n}} X_{[nt]+1}, \quad t \in [0, 1], \quad (23)$$

converges in distribution to a Brownian motion W in H , with $W(0) = 0$, a.s., $E[W(1)] = 0$, and covariance operator S of $W(1)$.

Theorem 2 is now applied to the martingale difference sequence in Lemma 1

$$\{X_i(x), i \geq 1\} = \{(Y_i - \Gamma(Y_{i-1}))1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}, i \geq 1\}, x \in H. \quad (24)$$

Theorem 3 For each $x \in H$, let $\{X_i(x), i \geq 1\}$ be the H -valued martingale difference with respect to the filtration $\mathcal{M}_0^Y \subset \mathcal{M}_1^Y \cdots \subset \mathcal{M}_n^Y \subset \dots$, introduced in (24) under **Assumption A1**. Define, for each $x \in H$, the sequence $\{Y_{n,x}, n \geq 2\}$ in $C_H([0, 1])$, given by, for each $t \in [0, 1]$,

$$Y_{n,x}(t) = V_{[nt]}(x) + \frac{(nt - [nt])}{\sqrt{n}} X_{[nt]+1}(x),$$

where V_n is the functional empirical process introduced in (14). Then,

$$Y_{n,x} \rightarrow_D W_{\infty,x}, \quad n \rightarrow \infty,$$

where $\{W_{\infty,x}, x \in H\}$ is an H -valued generalized Gaussian process, indexed and evaluated in H , with $W_{\infty,0} = 0$, a.s., $E[W_{\infty,x}] = 0$, for every $x \in H$, and covariance operator

$$S_{x,y} = E[W_{\infty,x} \otimes W_{\infty,y}] = P(E^0(\min(x, y)))C_0^\varepsilon, \quad x, y \in H,$$

where the notation $E^0(\min(x, y))$ has been introduced in equation (10).

Proof. The proof is based on verified that the sequence $\{X_i(x), i \geq 1\}$ in (24) satisfies Theorem 2(i)–(iii).

We first verify Theorem 2(i), under **Assumption A1**, keeping in mind (3), by computing for each $x \in H$, and $n \geq 1$,

$$\begin{aligned} & E \left\| \frac{1}{n} \sum_{j=1}^n E[X_j(x) \otimes X_j(x)/Y_1, \dots, Y_{j-1}] - S_{x,x} \right\|_{L^1(H)} \\ & E \left\| \frac{1}{n} \sum_{j=1}^n E[X_j(x) \otimes X_j(x)/Y_1, \dots, Y_{j-1}] - P(E^0(x))C_0^\varepsilon \right\|_{L^1(H)} \\ & = E \left\| \frac{1}{n} \sum_{j=1}^n E[X_j(x) \otimes X_j(x)/Y_{j-1}] - P(E^0(x))C_0^\varepsilon \right\|_{L^1(H)} \end{aligned}$$

$$\begin{aligned}
&= E \left\| \frac{1}{n} \sum_{j=1}^n E \left[\varepsilon_j \otimes \varepsilon_j 1_{\{\omega \in \Omega; \langle Y_{j-1}(\omega), \phi_k \rangle_H \leq \langle x, \phi_k \rangle_H, k \geq 1\}} / Y_{j-1} \right] - P(E^0(x)) C_0^\varepsilon \right\|_{L^1(H)} \\
&= \frac{1}{n} \sum_{j=1}^n E_{P_{Y_{j-1}}} \left[1_{\{\omega \in \Omega; \langle Y_{j-1}(\omega), \phi_k \rangle_H \leq \langle x, \phi_k \rangle_H, k \geq 1\}} \right] \sum_{l=1}^{\infty} E [\varepsilon_j \otimes \varepsilon_j(\phi_l)(\phi_l)] \\
&\quad - \sum_{l=1}^{\infty} P(E^0(x)) C_0^\varepsilon(\phi_l)(\phi_l) \\
&= \frac{1}{n} \sum_{j=1}^n P(E^{j-1}(x)) \sum_{l=1}^{\infty} E [\varepsilon_1 \otimes \varepsilon_1(\phi_l)(\phi_l)] - \sum_{l=1}^{\infty} P(E^0(x)) C_0^\varepsilon(\phi_l)(\phi_l) \\
&= P(E^0(x)) \left[\sum_{l=1}^{\infty} C_0^\varepsilon(\phi_l)(\phi_l) - C_0^\varepsilon(\phi_l)(\phi_l) \right] = 0.
\end{aligned}$$

Theorem 2(ii) holds, under **Assumption A1**, from equation (8), since, for each $x \in H$ and every $n \geq 1$,

$$\begin{aligned}
&\frac{1}{n} \sum_{j=1}^n E \|X_j(x)\|_H^2 = \frac{1}{n} \sum_{j=1}^n \|E[X_j(x) \otimes X_j(x)]\|_{L^1(H)} \\
&= \frac{1}{n} \sum_{j=1}^n \|C_0^{X_1(x)}\|_{L^1(H)} = P(E^0(x)) \sum_{l=1}^{\infty} C_0^\varepsilon(\phi_l)(\phi_l) = \text{trace}(S)
\end{aligned}$$

Finally, we prove Theorem 2(iii) holds under **Assumption A1**. Specifically, we prove convergence in $L_H^1(\Omega, \mathcal{A}, P)$ (see Remark 2). Thus, from equation (3), under **Assumption A1**, for each $x \in H$, and $r > 0$, applying Hölder's and

Chebyshev's inequalities, we obtain

$$\begin{aligned}
& E \left[\frac{1}{n} \sum_{j=1}^n E \left(\|X_j(x)\|_H^2 \chi(\|X_j(x)\|_H^2 \geq rj) / Y_1, \dots, Y_{j-1} \right) \right] \\
&= E \left[\frac{1}{n} \sum_{j=1}^n E \left(\|X_j(x)\|_H^2 \chi(\|X_j(x)\|_H^2 \geq rj) / Y_{j-1} \right) \right] \\
&\leq \frac{\sup \|\varepsilon_1\|_H^2}{n} \sum_{j=1}^n P \left[\|\varepsilon_j\|_H^2 \geq rj / Y_{j-1} \right] \\
&\leq \frac{\sup \|\varepsilon_1\|_H^2}{n} \sum_{j=1}^n \frac{E[\|\varepsilon_j\|_H^4]}{(rj)^2} \\
&= \frac{\sup \|\varepsilon_1\|_H^2 E[\|\varepsilon_1\|_H^4]}{n(r)^2} \sum_{j=1}^n \left(\frac{1}{j} \right)^2 \rightarrow 0, \quad n \rightarrow \infty, \tag{25}
\end{aligned}$$

$$\tag{26}$$

since $\sum_{j=1}^n \left(\frac{1}{j} \right)^2 < \infty$, and where we have applied SWN property of ε , in particular, $E\|\varepsilon_1\|_H^2 < \infty$ implying $\sup \|\varepsilon_1\|_H^2$ is a.s. finite. Thus, Theorem 2(iii) is satisfied, and the convergence in distribution to the H -valued generalized Gaussian process $\{W_{\infty, x}, x \in H\}$ holds.

Remark 3 In Theorem 3, the generalized Gaussian limit process $\{W_{\infty, x}, x \in H\}$ satisfies

$$W_{\infty, x} = W_{C_0^\varepsilon}(P(E^0(x))), \quad \text{in law}, \tag{27}$$

where $W_{C_0^\varepsilon}$ is H -valued Brownian motion with autocovariance operator C_0^ε of $W_{C_0^\varepsilon}(1) \stackrel{D}{=} W_{\infty, \infty}$.

From Theorem 3 and equation (27), applying continuous mapping theorem, we obtain

$$\sup_{x \in H} \|V_n(x)\|_H \rightarrow_D \sup_{t \in [0,1]} \|W_{C_0^\varepsilon}(t)\|_H, \quad \text{in law}, \tag{28}$$

In particular, for every $\phi \in H$,

$$\sup_{x \in H} \langle V_n(x), \phi \rangle_H \rightarrow_D \sup_{t \in [0,1]} \langle W_{C_0^\varepsilon}(t), \phi \rangle_H. \tag{29}$$

To test $H_0 : \Gamma = \Gamma_0$ we apply Theorem 4.1 in Cuesta–Albertos, Fraiman and Ransford [7] under the following additional assumption.

Assumption A2. The marginal infinite-dimensional probability distribution P_{Y_0} of Y_0 has a finite moment generating function in a neighbourhood of zero.

Under **Assumption A2**, from Theorem 3 (see equation (29)) , one can apply Theorem 4.1 in Cuesta–Albertos, Fraiman and Ransford [7] and proceed in the following way. The test based on $V_n(x)$ would reject $H_0 : \Gamma = \Gamma_0$ if conditionally to the functional value $h \in H$, obtained from a nondegenerate Gaussian measure μ in H ,

$$\sup \left\{ \left[\sqrt{\mathcal{T}_{n,h}(\infty)} \right]^{-1} |\langle V_n(x), h \rangle_H| : x \in H \right\}$$

exceeds a suitable critical value obtained from the boundary crossing probabilities of the Brownian motion over the unit interval. Here,

$$\mathcal{T}_{n,h}(\infty) = \frac{1}{n} \sum_{i=1}^n \langle (Y_i - \Gamma(Y_{i-1})) \otimes (Y_i - \Gamma(Y_{i-1})) (h), h \rangle_H.$$

Note that the asymptotic level of any test based on a continuous function of $\langle [\mathcal{T}_{n,h}(\infty)]^{-1} V_n(\mathcal{T}_{n,h}^{-1}), h \rangle_H$ can be obtained from the corresponding continuous function of $\langle W_{C_0^\varepsilon}(t), h \rangle_H$ on $[0, 1]$ where $\mathcal{T}_{n,h}^{-1}(t) := \inf\{x \in H : \mathcal{T}_{n,h}(x) \geq t\}$, $t \geq 0$. Here, for $x \in H$,

$$\mathcal{T}_{n,h}(x) = \frac{1}{n} \sum_{i=1}^n \langle (Y_i - \Gamma(Y_{i-1})) \otimes (Y_i - \Gamma(Y_{i-1})) (h), h \rangle_H 1_{\{\langle Y_{i-1}, \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}.$$

5 Consistency

We derive sufficient conditions that will imply the consistency of goodness of-fit-tests based on $V_n(x)$ for testing $H_0 : \Gamma = \Gamma_0$ against the alternative $H_1 : \Gamma \neq \Gamma_0$, where Γ_0 is a known operator. By $\Gamma \neq \Gamma_0$ it should be understood that the G -measure of the set $\{y \in H : \Gamma(y) \neq \Gamma_0(y)\}$ is positive, with $G(x) = P(E^0(x))$, $x \in H$. Let $\lambda(y, z) := E[Y_1 - \Gamma(Y_0) + z/Y_0 = y]$, for $y, z \in H$. Assume that for every $y \in H$, $\lambda(y, z) = 0$, if and only if $\|z\|_H = 0$. Let $d(x) := \Gamma(x) - \Gamma_0(x)$, for every $x \in H$, and consider

$$\mathcal{D}_n(x) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \lambda(Y_{i-1}, d(Y_{i-1})) 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}.$$

Similarly to the proof that equation (12) in Lemma 2 holds under conditions of Theorem 1, by applying Corollary 2.3 in Bosq [2], one can prove that, for every

$x \in H$, and $\beta > 1/2$,

$$\frac{n^{1/4}}{(\log(n))^\beta} \left\| n^{-1/2} \mathcal{D}_n(x) - E[\lambda(Y_0, d(Y_0)) 1_{\{\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}] \right\|_H \rightarrow 0, \quad (30)$$

almost surely, where expectation is computed under the alternative ($\Gamma \neq \Gamma_0$). Equation (30) leads to

$$\sup_{x \in H} \left\| n^{-1/2} \mathcal{D}_n(x) - E[\lambda(Y_0, d(Y_0)) 1_{\{\omega \in \Omega; \langle Y_0(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}] \right\|_H \rightarrow 0, \quad (31)$$

almost surely. Hence, equation (31) together with Theorem 3, and the assumption made that $y \in H$, $\lambda(y, z) = 0$, if and only if $\|z\|_H = 0$ yield the consistency of the test based on the random projection of $V_n(x)$ into $h \in H$, obtained from a nondegenerate Gaussian measure μ in H . Thus, the consistency of the test based on the statistics $\sup \left\{ \left[\sqrt{\mathcal{T}_{n,h}(\infty)} \right]^{-1} |\langle V_n(x), h \rangle_H| : x \in H \right\}$ follows.

The study of the asymptotic power of the above test deserves attention, and will be discussed in a subsequent paper.

6 Misspecified Γ

Theorem 3 is useful for testing the simple hypothesis $H_0 : \Gamma = \Gamma_0$. However, in practice, the situation is a bit different, since Γ is unknown, and we have to replace Γ by a consistent estimator Γ_n in the computation of the functional values of the empirical process V_n in equation (14). Thus, we consider

$$\tilde{V}_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{i,n}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i - \Gamma_n(Y_{i-1})) 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}}. \quad (32)$$

In this situation, additional assumptions are considered (see Chapter 8 in Bosq [2]), to ensure suitable asymptotic properties, and, in particular, consistency of the projection estimator

$$\Gamma_n(\varphi) = \sum_{l=1}^{k_n} \gamma_{n,l}(\varphi) \phi_l, \quad \varphi \in H, \quad n \geq 2,$$

where

$$\gamma_{n,l}(\varphi) = \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{j=1}^{k_n} \hat{\lambda}_{j,n}^{-1} \langle \varphi, \phi_j \rangle_H \langle Y_i, \phi_j \rangle_H \langle Y_{i+1}, \phi_l \rangle_H \quad (33)$$

$$\hat{\lambda}_{k,n} = \frac{1}{n} \sum_{i=1}^n (\langle Y_i, \phi_k \rangle_H)^2, \quad k \geq 1, \quad n \geq 1.$$

Assumption A3. The following conditions are satisfied by H -valued process Y :

- (i) Y is a standard ARH(1) process with a strong H -white noise and $E\|Y_0\|_H^4 < \infty$.
- (ii) The eigenvalues $\{\lambda_k(C_0^Y), k \geq 1\}$ of C_0^Y satisfying $C_0^Y(\phi_k) = \lambda_k \phi_k$, for every $k \geq 1$, are strictly positive (i.e., $\lambda_k > 0, k \geq 1$), and the eigenvectors $\{\phi_k, k \geq 1\}$ are known.
- (iii) $P(\langle Y_0, \phi_k \rangle_H = 0) = 0$, for every $k \geq 1$.

Under **Assumption A3**, Γ_n is bounded satisfying:

$$\|\Gamma_n\|_{\mathcal{L}(H)} \leq \|D_n\|_{\mathcal{L}(H)} \max_{1 \leq j \leq k_n} \hat{\lambda}_{j,n}^{-1}, \quad D_n = \frac{1}{n-1} \sum_{i=1}^{n-1} Y_i \otimes Y_{i+1}.$$

Under **Assumption A3**, and some additional conditions, like the ones assumed in Lemma 8.1(3), and Theorems 8.5–8.6 in Bosq [2] (see also Lemma 3 below), the strong-consistency of

$$\Gamma_n(\varphi) = \sum_{l=1}^{k_n} \gamma_{n,l}(\varphi) \phi_l, \quad \varphi \in H, \quad n \geq 2,$$

in the space $\mathcal{L}(H)$ of bounded linear operators on H holds. Specifically, Lemma 3 below summarizes such conditions.

Lemma 3 *The following assertions hold:*

- (i) If $\underline{\lim} \frac{n\lambda_{k_n}^8}{(\log(n))^\alpha} > 0$, for some $\alpha > 2$,

$$\|\Gamma_n(\varphi) - \Gamma(\varphi)\|_H \rightarrow 0, \quad a.s., \quad \varphi \in H.$$

- (ii) Under **Assumption A3**, if Γ is a Hilbert-Schmidt operator, then,

$$\|\Gamma_n - \Gamma\|_{\mathcal{L}(H)} \rightarrow 0, \quad a.s., \quad (34)$$

provided

$$\underline{\lim} \frac{n\lambda_{k_n}^8}{(\log(n))^\alpha} > 0, \quad \alpha > 2. \quad (35)$$

- (iii) Under **Assumption A3**, considering Γ is a Hilbert-Schmidt operator, and $\|Y_0\|_H$ is bounded, then, for all $\eta > 0$, there exists an integer $n_0(\eta, \Gamma, C_0^Y, k_n)$ such that $n \geq n_0$ implies

$$P(\|\Gamma_n - \Gamma\|_{\mathcal{L}(H)} \geq \eta) \leq c_1 \exp(-c_2 n \lambda_{k_n}^2),$$

where c_1 and c_2 only depend on η and the finite-dimensional probabilities of Y .

When eigenvectors $\{\phi_k, k \geq 1\}$ are unknown, we consider, for each $k \geq 1$, and $n \geq 1$, $\tilde{\lambda}_{k,n}$ such that $\frac{1}{n} \sum_{i=1}^n Y_i \langle Y_i, \phi_{k,n} \rangle_H = \tilde{\lambda}_{k,n} \phi_{k,n}$. Thus, $\{\tilde{\lambda}_{k,n}, k \geq 1\}$ and $\{\phi_{k,n}, k \geq 1\}$ respectively denote the system of eigenvalues and eigenvectors of the empirical autocovariance operator $C_n = \frac{1}{n} \sum_{i=1}^n Y_i \otimes Y_i$. Then, for ensuring strong-consistency in the space $\mathcal{L}(H)$ of the projection estimator

$$\tilde{\Gamma}_n(\varphi) = \sum_{l=1}^{k_n} \tilde{\gamma}_{n,l}(\varphi) \phi_{l,n}, \quad \varphi \in H, \quad n \geq 2,$$

Assumption A4 below will be considered. Here, $k_n/n \rightarrow 0$, $n \rightarrow \infty$, and for each $n \geq 1$, and $l \geq 1$,

$$\tilde{\gamma}_{n,l}(h) = \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{j=1}^{k_n} \tilde{\lambda}_{j,n}^{-1} \langle \varphi, \phi_{j,n} \rangle_H \langle Y_i, \phi_{j,n} \rangle_H \langle Y_{i+1}, \phi_{l,n} \rangle_H. \quad (36)$$

Assumption A4. The following conditions are assumed on ARH(1) process Y :

(i) The eigenvalues $\{\lambda_k(C_0^Y), k \geq 1\}$ of C_0^Y satisfy

$$\lambda_1(C_0^Y) > \lambda_2(C_0^Y) > \cdots > \lambda_j(C_0^Y) > \cdots > 0,$$

where as before, $C_0^Y(\phi_k) = \lambda_k(C_0^Y) \phi_k$, for every $k \geq 1$.

(ii) For every $n \geq 1$ and $k \geq 1$, $\tilde{\lambda}_{k,n} > 0$ a.s.

Under **Assumption A4**, and additional conditions, like the ones given in Theorem 8.7–8.8 in Bosq [2] (see also Lemma 4 below), the strong-consistency of the projection estimator $\tilde{\Gamma}_n$ holds in the space $\mathcal{L}(H)$.

In Lemma 4 below, we will use the following notation:

$$\begin{aligned} a_1 &= 2\sqrt{2} (\lambda_1(C_0^Y) - \lambda_2(C_0^Y)) \\ a_j &= 2\sqrt{2} \max \left((\lambda_{j-1}(C_0^Y) - \lambda_j(C_0^Y))^{-1}, (\lambda_j(C_0^Y) - \lambda_{j+1}(C_0^Y))^{-1} \right), \quad j \geq 2. \end{aligned} \quad (37)$$

Lemma 4 (i) Under **Assumption A3(i)** and **Assumption A4(i)–(ii)**, if Γ is a Hilbert–Schmidt operator, and for some $\beta > 1$,

$$\lambda_{k_n}^{-1}(C_0^Y) \sum_{j=1}^{k_n} a_j = \mathcal{O} \left(n^{1/4} (\log(n))^{-\beta} \right),$$

then,

$$\|\tilde{\Gamma}_n - \Gamma\|_{\mathcal{L}(H)} \rightarrow 0, \quad a.s. \quad (38)$$

(ii) If in addition $\|Y_0\|_H$ is bounded, then

$$P\left(\|\tilde{\Gamma}_n - \Gamma\|_{\mathcal{L}(H)} \geq \eta\right) \leq c_1(\eta) \exp\left(-c_2(\eta)n\lambda_{k_n}^2 \left(\sum_{j=1}^{k_n} a_j\right)^{-2}\right), \quad (39)$$

for $\eta > 0$, and $n \geq n_\eta$, where $c_1(\eta)$ and $c_2(\eta)$ are positive constants.

Thus $\frac{n\lambda_{k_n}^2}{\log(n)(\sum_{j=1}^{k_n} a_j)^2} \rightarrow \infty$ implies (38).

Under the conditions of Lemma 4 (see Corollary 8.3 in Bosq [2]), the following lemma also provides the consistency in H of the empirical predictor $\tilde{\Gamma}_n(Y_n)$ of Y_n , regarding the theoretical one $\Gamma(Y_n)$, for every $n \geq 2$.

Lemma 5 *Under conditions in Lemma 4(i),*

$$\left\|\tilde{\Gamma}_n(Y_n) - \Gamma(Y_n)\right\|_H \rightarrow_P 0, \quad n \rightarrow \infty.$$

Furthermore, under conditions in Lemma 4(ii),

$$P\left(\left\|\tilde{\Gamma}_n(Y_n) - \Gamma(Y_n)\right\|_H \geq \eta\right) \leq c'_1(\eta) \exp\left(-c'_2(\eta)n\lambda_{k_n}^2 \left(\sum_{j=1}^{k_n} a_j\right)^{-2}\right). \quad (40)$$

Finally, if $\frac{n\lambda_{k_n}^2}{\log(n)(\sum_{j=1}^{k_n} a_j)^2} \rightarrow \infty$,

$$\left\|\tilde{\Gamma}_n(Y_n) - \Gamma(Y_n)\right\|_H \rightarrow 0, \quad \text{a.s.}, \quad n \rightarrow \infty.$$

The following result proves that $V_n(x)$ and $\tilde{V}_n(x)$ converge in probability to the same limit, uniformly in H , under the conditions in Lemma 5.

Theorem 4 *Under conditions of Lemma 5, the following identity holds:*

$$\sup_{x \in H} \|V_n(x) - \tilde{V}_n(x)\|_H = o_P(1), \quad n \rightarrow \infty. \quad (41)$$

Proof. Note that

$$\begin{aligned} & \sup_{x \in H} \|R_n(x)\|_H = \sup_{x \in H} \|V_n(x) - \tilde{V}_n(x)\|_H \\ &= \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\tilde{\Gamma}_n(Y_{i-1}) - \Gamma(Y_{i-1}) \right) \sup_{x \in H} 1_{\{\omega \in \Omega; \langle Y_{i-1}(\omega), \phi_j \rangle_H \leq \langle x, \phi_j \rangle_H, j \geq 1\}} \right\|_H \\ &\leq \frac{1}{\sqrt{n}} \sum_{i=1}^n \left\| \tilde{\Gamma}_n(Y_{i-1}) - \Gamma(Y_{i-1}) \right\|_H \quad \text{a.s.} \end{aligned} \quad (42)$$

Under conditions of Lemma 5, from equations (39) and (42),

$$\begin{aligned}
& P \left(\sup_{x \in H} \|R_n(x)\|_H \geq \eta \right) \\
& \leq P \left(\sum_{i=1}^n \left\| \tilde{\Gamma}_n(Y_{i-1}) - \Gamma(Y_{i-1}) \right\|_H \geq \eta \sqrt{n} \right) \\
& \leq P \left(\left\| \tilde{\Gamma}_n - \Gamma \right\|_{\mathcal{L}(H)} \geq \frac{\eta}{\|Y_0\|_H} \right) \\
& \leq c_1 \left(\frac{\eta}{\|Y_0\|_H} \right) \exp \left(-c_2 \left(\frac{\eta}{\|Y_0\|_H} \right) n \lambda_{k_n}^2 \left(\sum_{j=1}^{k_n} a_j \right)^{-2} \right),
\end{aligned} \tag{43}$$

for each $\eta > 0$, and $n \geq n_{\eta/\|Y_0\|_H}$, where $c_1(\eta/\|Y_0\|_H)$ and $c_2(\eta/\|Y_0\|_H)$ are positive constants. Thus, equation (43) converges to zero as $n \rightarrow \infty$, as we wanted to prove.

Corollary 1 *Under conditions of Lemma 5, for every $\varphi \in H$,*

$$\sup_{x \in H} \left\langle \tilde{V}_n(x), \varphi \right\rangle_H \rightarrow_D \sup_{t \in [0,1]} \left\langle W_{C_0^\varepsilon}(t), \varphi \right\rangle_H, \text{ in law,} \tag{44}$$

where, as before, $W_{C_0^\varepsilon}$ is H -valued Brownian motion with autocovariance operator C_0^ε of $W_{C_0^\varepsilon}(1) \stackrel{D}{=} W_{\infty, \infty}$.

Proof. The proof follows straightforward from Theorems 3–4, and equation (27), applying continuous mapping theorem, since as well-known, convergence in probability implies convergence in distribution,

Under **Assumption A2**, from Corollary 1, one can proceed for testing in the case of misspecified Γ , in a similar way to the case of Γ is known. That is, one can apply Theorem 4.1 in Cuesta–Albertos, Fraiman and Ransford [7]. Hence, the test based on $\tilde{V}_n(x)$ would reject $H_0 : \Gamma = \Gamma_0$ if conditionally to the functional value $h \in H$, obtained from a nondegenerate Gaussian measure μ in H ,

$$\sup_{x \in H} \left\{ \left[\sqrt{\tilde{\mathcal{T}}_{n,h}(\infty)} \right]^{-1} \left\langle \tilde{V}_n(x), h \right\rangle_H \right\}$$

exceeds a suitable critical value obtained from the boundary crossing probabilities of the Brownian motion over the unit interval. Here,

$$\tilde{\mathcal{T}}_{n,h}(\infty) = \frac{1}{n} \sum_{i=1}^n \left\langle \left(Y_i - \tilde{\Gamma}_n(Y_{i-1}) \right) \otimes \left(Y_i - \tilde{\Gamma}_n(Y_{i-1}) \right) (h), h \right\rangle_H.$$

7 Final Comments

It is well-known that, from Theorem 1 in Cardot, Mas and Sarda [5], for the functional regression model with scalar response, and covariate taking values in a separable Hilbert space H , the projection estimator considered for the regression function does not satisfy a Central Limit Theorem (CLT) with convergence to a non-degenerate random element, in the norm topology in H . Thus, only weak-convergence results are obtained for the predictor, under suitable conditions, including suitable truncation order, and convexity of the eigenvalues of the autocovariance operator Γ of the covariate, at least at high frequency (in particular, their H2 condition on $\text{Ker}(\Gamma) = \{0\}$ is crucial), among others. Then, a suitable normalization by the standard deviation of the partial sums of the sequence of independent centered random variables, involved in the prediction error, is considered. This Theorem 1 does not hold in the case of the ARH(1) model by the following two main reasons (see Mas [26]):

- (i) The converge to zero, in probability, of $\sqrt{n}[C_0^Y]^{-1}\pi^{k_n}(C_0^Y - C_n)$, in the norm of the space of Hilbert-Schmidt operators on H , $\mathcal{S}(H)$, under suitable conditions like the ones assumed in Theorem 4.1, p. 98, in Bosq [2]. Here, π^{k_n} denotes the projection operator into the subspace of H generated by the eigenfunctions $\{\phi_1, \dots, \phi_{k_n}\}$ of C_0^Y . In particular, under the conditions assumed in Theorem 4.1 in Bosq [2], our choice of the truncation order k_n must be such that, as $n \rightarrow \infty$, $\sqrt{n}\lambda_{k_n}^{-1} = \mathcal{O}(n^{1/4} \log(n))^{-\beta}$, $\beta > 1/2$, ensuring, in particular, Proposition 4 in Mas [26] holds. Note that Theorem 4.1 in Bosq [2] is proved by applying the strong law of large number to weakly dependent sequences of H -valued random variables, that leads to the strong law of large numbers for ARH(1) processes (see, e.g., Theorem 3.7, p.86, in Bosq [2]). Specifically, from Lemma 4.1, p.96, in Bosq [2], on the $\text{ARS}(H)(1)$ representation of the diagonal self-tensorial product of an ARH(1) process, the strong consistency in the norm of $\mathcal{S}(H)$ of the empirical autocovariance operator C_n of Y is obtained by applying Theorem 3.7 in Bosq [2].
- (ii) The convergence in distribution of $\sqrt{n}[C_0^Y]^{-1}\pi^{k_n}U_n$, with $U_n = \frac{1}{n} \sum_{i=1}^n Y_i \otimes \varepsilon_{i+1}$, to a centered Gaussian random Hilbert-Schmidt operator $\tilde{\Gamma}$, under the key condition $E \|[C_0^Y]^{-1}\varepsilon_0\|_H^2 < \infty$. This limit result is obtained by applying a CLT for an array of $\mathcal{S}(H)$ -valued martingale differences. The limit centered Gaussian random Hilbert-Schmidt operator $\tilde{\Gamma}$ has covariance operator Σ defined by, for $T_{kl} = \phi_k \otimes \phi_l$, $k, l \geq 1$,

$$\langle \Sigma T_{ii'}, T_{jj'} \rangle_{\mathcal{S}(H)} = \begin{cases} 0 & i \neq j \\ \frac{\lambda_i(C_0^Y) \langle C_0^\varepsilon(\phi_{i'}), \phi_{j'} \rangle_H}{\lambda_{i'}(C_0^Y) \lambda_{j'}(C_0^Y)} & i = j. \end{cases}$$

Note that, in the functional linear regression model with scalar response we have no chance to formulate an equivalent condition to $E \left\| [C_0^Y]^{-1} \varepsilon_0 \right\|_H^2 < \infty$, since the innovation process is real-valued. Indeed, this condition means that $\| [C_0^Y]^{-1} C_0^\varepsilon [C_0^Y]^{-1} \|_{L^1(H)} < \infty$. Thus, $P[\varepsilon_0 \in C_0^Y(H)] = 1$, or, equivalently, ε_0 belongs to the Reproducing Kernel Hilbert Space (RKHS) generated by the integral operator $[C_0^Y]^2$. In the case where C_0^Y and C_0^ε have a common system of eigenfunctions, this condition can be equivalently expressed as

$$\sum_{k \geq 1} \frac{\lambda_k(C_0^\varepsilon)}{[\lambda_k(C_0^Y)]^2} < \infty.$$

Hence, a faster decay of the eigenvalues of the autocovariance operator C_0^ε of the innovation process ε than the eigenvalues of the square autocovariance operator $[C_0^Y]^2$ of the ARH(1) process Y is required. In particular, at least, the following identity holds

$$\frac{\lambda_k(C_0^\varepsilon)}{[\lambda_k(C_0^Y)]^2} = \mathcal{O}(k^{-\gamma}), \quad \gamma > 1, \quad k \rightarrow \infty.$$

Section 2.3 in Mas [26] provides some examples of the eigenvalue sequence $\{\lambda_k(C_0^Y), k \geq 1\}$ to define a suitable truncation order k_n , according to the assumed conditions for the derivation of his limit results above referred. In particular, one can consider $k_n = o(n^{1/(2\alpha)})$, $n \rightarrow \infty$, if $\lambda_k(C_0^Y)$ is of the form $\lambda_k(C_0^Y) = k^{-\alpha}$, $\alpha > 1$. On the other hand, for $\lambda_k(C_0^Y) = \lambda^k$, one can choose $k_n = \log(n)$ if $\log(\lambda) > -1/2$, while $k_n = o(\log(n))$, if $\log(\lambda) \leq -1/2$.

The case where the projection $\tilde{\pi}^{k_n}$ into the empirical eigenvectors of C_n is considered can be achieved under the conditions in Theorem 8.9 in Bosq [2], providing a CLT for this case. Indeed, a similar decomposition to equation (11) in Cardot, Mas and Sarda [5] can be considered. The strong consistency of the empirical eigenvalues and eigenvectors of C_n is then applied (see Theorem 4.4, Lemma 4.3, Theorem 4.5, and Corollary 4.3 in Bosq [2]).

We recall that for the asymptotic equivalence in probability of the empirical processes $\{V_n(x), x \in H\}$ and $\{\tilde{V}_n(x), x \in H\}$ to hold in Theorem 4, the conditions of Lemma 4 must be considered, involving the quantities $\lambda_{k_n}^{-1}$, and $\{a_j, j \geq 1\}$ in (37). These quantities provide respective information about the decay velocity of the eigenvalues of C_0^Y , and about the distance between such eigenvalues or their distribution. Both features of the pure point spectrum of C_0^Y must be taken into account in the model selection problem associated with a suitable truncation order k_n . In particular, if fast

decay of the eigenvalues, like in example $\lambda_k(C_0^Y) = \lambda^k$ holds, and if the quantities $\{a_j, j \geq 1\}$, in equation (37), indicate a fast accumulation of the eigenvalues at a small neighborhood of zero, a slower divergence of k_n as $n \rightarrow \infty$ is required in order to ensure Theorem 4 is satisfied.

Finally, we highlight an interesting application of the approach presented to the context of goodness of fit tests for the family of spherical functional autoregressive processes, SPHAR(1) processes, which constitutes a special case of the SPHAR(p) class introduced in Caponera and Marinucci [4]. In particular, a consistent estimation of the isotropic autoregression kernel involved, and a Central Limit Theorem are derived in this paper. The extension of this approach to a special class of manifold functional autoregressive processes, in compact and connected two points homogeneous spaces, can also be achieved by applying the results in Ma and Malyarenko [25]. These two families of functional linear models open a new area of research where goodness of fit testing constitutes a challenging topic.

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