

Contents

1	Library B_Unification.intro	2
1.1	Introduction	2
1.2	Unification	2
1.2.1	Syntatic Unification	3
1.2.2	Semantic Unification	3
1.2.3	Boolean Unification	3
1.3	Formal Verification	4
1.3.1	Proof Assistance	4
1.3.2	Verifying Systems	5
1.3.3	Verifying Theories	5
1.4	Importance	5
1.5	Development	6
1.5.1	Data Structures	6
1.5.2	Algorithms	7
2	Library B_Unification.terms	8
2.1	Introduction	8
2.2	Terms	8
2.2.1	Definitions	8
2.2.2	Axioms	9
2.2.3	Lemmas	11
2.3	Variable Sets	12
2.3.1	Definitions	12
2.3.2	Examples	15
2.4	Ground Terms	15
2.4.1	Definitions	15
2.4.2	Lemmas	16
2.4.3	Examples	17
2.5	Substitutions	17
2.5.1	Definitions	17
2.5.2	Lemmas	18
2.5.3	Examples	23
2.6	Unification	24

2.7	Most General Unifier	26
2.8	Auxilliary Computational Operations and Simplifications	27
3	Library B_Unification.lowenheim_formula	45
3.1	Introduction	45
3.2	Lowenheim's Builder	45
3.2.1	Here is the main lowenheim's formula; given a term, produce an MGU that makes it equivalent to T0,	47
4	Library B_Unification.lowenheim_proof	49
5	Library B_Unification.poly	65
5.1	Introduction	65
5.2	Monomials and Polynomials	65
5.2.1	Data Type Definitions	65
5.2.2	Comparisons of monomials and polynomials	66
5.2.3	Stronger Definitions	68
5.3	Functions over Monomials and Polynomials	72
6	Library B_Unification.poly_unif	125
7	Library B_Unification.sve	139
7.1	Intro	139
7.2	Eliminating Variables	139
7.3	Building Substitutions	150
7.4	Recursive Algorithm	154
7.5	Correctness	155

Chapter 1

Library B_Unification.intro

1.1 Introduction

In the field of computational theory, one problem of significance is that of equational unification; namely, the finding of solutions to a given set of equations with respect to a set of relational axioms. While there are several variants of equational unification, for the purposes of this paper we are going to limit our scope to that of Boolean unification, which deals with the finding of unifiers for equations operating across Boolean rings. As any problem space would imply, there exists a great deal of research in the formal verification of unification algorithms; our research focused on two of these algorithms: Lowenheim's formula, and Successive Variable Elimination. To conduct our research, we utilized the Coq proof assistant to create formal specifications of both of these algorithms' behaviors in addition to proving their correctness. While proofs for both of these algorithms already exist, prior to the writing of this paper, no formal treatment using a proof assistant such as Coq had been undertaken, so it is hoped that our efforts provide yet another guarantee for the veritable correctness of these respective algorithms.

1.2 Unification

Before defining what unification is, there is some terminology to understand. A *term* is either a variable or a function applied to terms. By this definition, a constant term is just a nullary function. A *variable* is a symbol capable of taking on the value of any term. An example of a term is $f(a, x)$, where f is a function of two arguments, a is a constant, and x is a variable. A term is *ground* if no variables occur in it. The last example is not a ground term but $f(a, a)$ would be.

A *substitution* is a mapping from variables to terms. The *domain* of a substitution is the set of variables that do not get mapped to themselves. The *range* is the set of terms the are mapped to by the domain. It is common for substitutions to be referred to as mappings from terms to terms. A substitution s can be extended to this form by defining $s'(u)$ for

two cases of u . If u is a variable, then $s'(u) = s(u)$. If u is a function $f(u1, \dots, un)$, then $s'(u) = f(s'(u1), \dots, s'(un))$.

Unification is the process of solving a set of equations between two terms. The set of equations is referred to as a unification problem. The process of solving one of these problems can be classified by the set of terms considered and the equality of any two terms. The latter property is what distinguishes two broad groups of algorithms, namely syntactic and semantic unification. If two terms are only considered equal if they are identical, then the unification is syntactic. If two terms are equal with respect to an equational theory, then the unification is semantic.

The goal of unification is to solve equations, which means to produce a substitution that unifies those equations. A substitution s *unifies* an equation $u =? v$ if applying s to both sides makes them equal $s(u) = s(v)$. In this case, we call s a *solution* or *unifier*.

The goal of a unification algorithm is not just to produce a unifier but to produce one that is most general. A substitution is a *most general unifier* or *mgu* of a problem if it is more general than every other solution to the problem. A substitution s is more general than s' if there exists a third substitution t such that $s'(u) = t(s(u))$ for any term u .

1.2.1 Syntactic Unification

This is the simpler version of unification. For two terms to be considered equal they must be identical. For example, the terms $x \times y$ and $y \times x$ are not syntactically equal, but would be equal modulo commutativity of multiplication. (more about solving these problems / why simpler...)

1.2.2 Semantic Unification

This kind of unification involves an equational theory. Given a set of identities E , we write that two terms u and v are equal with regards to E as $u =_E v$. This means that identities of E can be applied to u as u' and v as v' in some way to make them syntactically equal, $u' = v'$. As an example, let C be the set $\{f(x, y) = f(y, x)\}$. This theory C axiomatizes the commutativity of the function f . It would then make sense to write $f(a, x) =_C f(x, a)$. In general, for an arbitrary E , the problem of E -unification is undecidable.

1.2.3 Boolean Unification

In this paper, we focus on unification modulo Boolean ring theory, also referred to as B-unification. The allowed terms in this theory are the constants 0 and 1 and binary functions $+$ and \times . The set of identities B is defined as the set $\{x + y = y + x, (x + y) + z = x + (y + z), x + x = 0, 0 + x = x, x \times (y + z) = (x \times y) + (x \times z), x \times y = y \times x, (x \times y) \times z = x \times (y \times z), x \times x = x, 0 \times x = 0, 1 \times x = x\}$. This set is equivalent to the theory of real numbers with the addition of $x + x = 0$ and $x \times x = x$.

Although a unification problem is a set of equations between two terms, we will now show informally that a B-unification problem can be viewed as a single equation $t = 0$. Given a

problem in its normal form $\{s1 = t1, \dots, sn = tn\}$, we can transform it into $\{s1 + t1 = 0, \dots, sn + tn = 0\}$ using a simple fact. The equation $s = t$ is equivalent to $s + t = 0$ since adding t to both sides of the equation turns the right hand side into $t + t$ which simplifies to 0. Then, given a problem $\{t1 = 0, \dots, tn = 0\}$, we can transform it into $\{(t1 + 1) \times \dots \times (tn + 1) = 1\}$. Unifying both of these sets is equivalent because if any $t1, \dots, tn$ is 1 the problem is not unifiable. Otherwise, if every $t1, \dots, tn$ can be made to equal 0, then both problems will be solved.

1.3 Formal Verification

Formal verification is the term used to describe the act of verifying (or disproving) the correctness of software and hardware systems or theories. Formal verification consists of a set of techniques that perform static analysis on the behavior of a system, or the correctness of a theory. It differs to dynamic analysis that uses simulation to evaluate the correctness of a system.

More simply stated, formal verification is the process of examining whether a system or a theory “does what it is supposed to do.” If it is a system, then scientists formally verify that it satisfies its design requirements. Formal verification is also different to testing. Software testing is trying to detect “bugs” specific errors and requirements in the system, whereas verification acts as a general safeguard that the system is always error-free. As Edsger Dijkstra stated, testing can be used to show the presence of bugs, but never to show their absence. If it a theory, scientists formally verify the correctness of the theory by formulating its proof using a formal language, axioms and inference rules.

Formal verification is used because it does not have to evaluate every possible case or state to determine if a system or theory meets all the preset logical conditions and requirements. Moreover, as design and software systems sizes have increased (along with their simulation times), verification teams have been looking for alternative methods of proving or disproving the correctness of a system in order to reduce the required time to perform a correctness check or evaluation.

1.3.1 Proof Assistance

A proof assistant is a software tool that is used to formulate and prove or disprove theorems in computer science or mathematical logic. They are also be called interactive theorem provers and they may also involve some type of proof and text editor that the user can use to form and prove and define theorems, lemmas, functions, etc. They facilitate that process by allowing the user to search definitions, terms and even provide some kind of guidance during the formulation or proof of a theorem.

1.3.2 Verifying Systems

Formal verification is used in the industry to verify the correctness of software or hardware systems. When used to verify systems, formal verification can be thought as a mathematical proof of the correctness of a design with respect to a formal specification. The actual system is represented by a formal model and then the formal verification happens on the model, based on the required specifications of the system. Unlike testing, formal verification is exhaustive and improves the understanding of a system. However, it is difficult to make for real-world systems, time consuming and only as reliable as the actual model.

1.3.3 Verifying Theories

Formal verification is also used in to prove theorems. These theorems could be related to a computing system or just plain mathematical abstract theorems. As in proving systems, when proving theorems one also needs a formal logic to formulate the theorem and prove it. A formal logic consists of a formal language to express the theorems, a collection of formulas called axioms and inference rules to derive new axioms based on existing ones. A theorem to be proven could be in a logical form, like DeMorgan's Law or it could in another mathematical area; in trigonometry for example, it could be useful to prove that $\sin(x + y) = \sin(x) * \cos(y) + \cos(x) * \sin(y)$, formally, because that proof could be used as building block in a more complex system. Sometimes proving the correctness of a real world systems boils down to verifying mathematical proofs like the previous one, so the two approaches are often linked together.

1.4 Importance

Given that the emergence of proof assistance software is still in its infancy relative to the age-old traditional methods of theorem proving, it would be a disservice for us to not establish the importance of this technology and its implications for the future of mathematics. Unlike in years past, where typos or subliminal edge cases could derail the developments of sound theorems, proof assistants now guarantee through their properties of verification that any development run by them is free from such lapses in logic on account of the natural failings of the human mind. Additionally, due to the adoption of a well-defined shared language, many of the ambiguities naturally present in the exchange of mathematical ideas between colleagues are mitigated, leading to a smoother learning curve for newcomers trying to understand the nuts and bolts of a complex theorem. The end result of these phenomenon is a faster iterative development cycle for mathematicians as they now can spend more time on proving things and building off of the work of others since they no longer need to devote as much of their efforts towards verifying the correctness of the theorems they are operating across.

Bearing this in mind, it should come as no surprise that there is a utility in going back to older proofs that have never been verified by a proof assistant and redeveloping them for the purposes of ensuring their correctness. If the theorem is truly sound, it stands to reason that any additional rigorous scrutiny would only serve to bolster the credibility of its claims, and

conversely, if the theorem is not sound, it is a benefit to the academic community at large to be made aware of its shortcomings. Therefore, for these reasons we set out to formally verify two algorithms across Boolean Unification.

1.5 Development

There are many different approaches that one could take to go about formalizing a proof of Boolean Unification algorithms, each with their own challenges. For this development, we have opted to base our work largely off chapter 10, *Equational Unification*, in *Term Rewriting and All That* by Franz Baader and Tobias Nipkow. Specifically, section 10.4, titled *Boolean Unification*, details Boolean rings, data structures to represent them, and two algorithms to perform unification in Boolean rings.

We chose to implement two data structures for representing the terms of a Boolean unification problem, and two algorithms for performing unification. The two data structures chosen are an inductive Term type and lists of lists representing polynomial-form terms. The two algorithms are Lowenheim’s formula and successive variable elimination.

1.5.1 Data Structures

The data structure used to represent a Boolean unification problem completely changes the shape of both the unification algorithm and the proof of correctness, and is therefore a very important decision. For this development, we have selected two different representations of Boolean rings – first as a “Term” inductive type, and then as lists of lists representing terms in polynomial form.

The Term inductive type, used in the proof of Lowenheim’s algorithm, is very simple and rather intuitive – a term in a Boolean ring is one of 5 things:

- The number 0
- The number 1
- A variable
- Two terms added together
- Two terms multiplied together

In our development, variables are represented as natural numbers.

After defining terms like this, it is necessary to define a new equality relation, referred to as term equivalence, for comparing terms. With the term equivalence relation defined, it is easy to define ten axioms enabling the ten identities that hold true over terms in Boolean rings.

The inductive representation of terms in a Boolean ring is defined in the file *terms.v*. Unification over these terms is defined in *term_unif.v*.

The second representation, used in the proof of successive variable elimination, uses lists of lists of variables to represent terms in polynomial form. A monomial is a list of distinct variables multiplied together. A polynomial, then, is a list of distinct monomials added together. Variables are represented the same way, as natural numbers. The terms 0 and 1 are represented as the empty polynomial and the polynomial containing only the empty monomial, respectively.

The interesting part of the polynomial representation is how the ten identities are implemented. Rather than writing axioms enabling these transformations, we chose to implement the addition and multiplication operations in such a way to ensure these rules hold true, as described in *Term Rewriting*.

Addition is performed by cancelling out all repeated occurrences of monomials in the result of appending the two lists together (ie, $x+x=0$). This is equivalent to the symmetric difference in set theory, keeping only the terms that are in either one list or the other (but not both). Multiplication is slightly more complicated. The product of two polynomials is the result of multiplying all combinations of monomials in the two polynomials and removing all repeated monomials. The product of two monomials is the result of keeping only one copy of each repeated variable after appending the two together.

By defining the functions like this, and maintaining that the lists are sorted with no duplicates, we ensure that all 10 rules hold over the standard coq equivalence function. This of course has its own benefits and drawbacks, but lent itself better to the nature of successive variable elimination.

The polynomial representation is defined in the file *poly.v*. Unification over these polynomials is defined in *poly_unif.v*.

1.5.2 Algorithms

For unification algorithms, we once again followed the work laid out in *Term Rewriting and All That* and implemented both Lowenheim's algorithm and successive variable elimination.

The first solution, Lowenheim's algorithm, is built on top of the term inductive type. Lowenheim's is based on the idea that the Lowenheim formula can take a ground unifier of a Boolean unification problem and turn it into a most general unifier. The algorithm then of course first requires finding a ground solution, accomplished through brute force, which is then passed through the formula to create a most general unifier. Lowenheim's algorithm is implemented in the file *lowenheim.v*, and the proof of correctness is in *lowenheim_proof.v*.

The second algorithm, successive variable elimination, is built on top of the list-of-list polynomial approach. Successive variable elimination is built on the idea that by factoring variables out of the equation one-by-one, we can eventually reach a ground unifier. This unifier can then be built up with the variables that were previously eliminated until a most general unifier for the original unification problem is achieved. Successive variable elimination and its proof of correctness are both in *sve.v*.

Chapter 2

Library B_Unification.terms

```
Require Import Bool.  
Require Import Omega.  
Require Import EqNat.  
Require Import List.  
Require Import Setoid.  
Import ListNotations.
```

2.1 Introduction

In order for any proofs to be constructed in Coq, we need to formally define the logic and data across which said proofs will operate. Since the heart of our analysis is concerned with the unification of Boolean equations, it stands to reason that we should articulate precisely how algebra functions with respect to Boolean rings. To attain this, we shall formalize what an equation looks like, how it can be composed inductively, and also how substitutions behave when applied to equations.

2.2 Terms

2.2.1 Definitions

We shall now begin describing the rules of Boolean arithmetic as well as the nature of Boolean equations. For simplicity's sake, from now on we shall be referring to equations as terms.

Definition var := **nat**.

Definition var_eq_dec := **Nat.eq_dec**.

A term, as has already been previously described, is now inductively declared to hold either a constant value, a single variable, a sum of terms, or a product of terms.

Inductive **term**: Type :=

```

| T0 : term
| T1 : term
| VAR : var → term
| SUM : term → term → term
| PRODUCT : term → term → term.

```

For convenience's sake, we define some shorthanded notation for readability.

```
Implicit Types x y z : term.
```

```
Implicit Types n m : var.
```

```
Notation "x + y" := (SUM x y) (at level 50, left associativity).
```

```
Notation "x * y" := (PRODUCT x y) (at level 40, left associativity).
```

2.2.2 Axioms

Now that we have informed Coq on the nature of what a term is, it is now time to propose a set of axioms that will articulate exactly how algebra behaves across Boolean rings. This is a requirement since the very act of unifying an equation is intimately related to solving it algebraically. Each of the axioms proposed below describe the rules of Boolean algebra precisely and in an unambiguous manner. None of these should come as a surprise to the reader; however, if one is not familiar with this form of logic, the rules regarding the summation and multiplication of identical terms might pose as a source of confusion.

For reasons of keeping Coq's internal logic consistent, we roll our own custom equivalence relation as opposed to simply using '='. This will provide a surefire way to avoid any odd errors from later cropping up in our proofs. Of course, by doing this we introduce some implications that we will need to address later.

```
Parameter eqv : term → term → Prop.
```

```
Infix "==" := eqv (at level 70).
```

```
Axiom sum_comm : ∀ x y, x + y == y + x.
```

```
Axiom sum_assoc : ∀ x y z, (x + y) + z == x + (y + z).
```

```
Axiom sum_id : ∀ x, T0 + x == x.
```

```
Axiom sum_x_x : ∀ x, x + x == T0.
```

```
Axiom mul_comm : ∀ x y, x × y == y × x.
```

```
Axiom mul_assoc : ∀ x y z, (x × y) × z == x × (y × z).
```

```
Axiom mul_x_x : ∀ x, x × x == x.
```

```
Axiom mul_T0_x : ∀ x, T0 × x == T0.
```

```
Axiom mul_id : ∀ x, T1 × x == x.
```

```
Axiom distr : ∀ x y z, x × (y + z) == (x × y) + (x × z).
```

```
Axiom term_sum_symmetric :
```

```
  ∀ x y z, x == y ↔ x + z == y + z.
```

```

Axiom refl_comm :
   $\forall t1\ t2, t1 == t2 \rightarrow t2 == t1.$ 
Axiom T1_not_equiv_T0 :
   $\sim (T1 == T0).$ 
Hint Resolve sum_comm sum_assoc sum_x_x sum_id distr
  mul_comm mul_assoc mul_x_x mul_T0_x mul_id.

```

Now that the core axioms have been taken care of, we need to handle the implications posed by our custom equivalence relation. Below we inform Coq of the behavior of our equivalence relation with respect to rewrites during proofs.

```

Axiom eqv_ref : Reflexive eqv.
Axiom eqv_sym : Symmetric eqv.
Axiom eqv_trans : Transitive eqv.
Add Parametric Relation : term eqv
  reflexivity proved by @eqv_ref
  symmetry proved by @eqv_sym
  transitivity proved by @eqv_trans
  as eq_set_rel.
Axiom SUM_compat :
   $\forall x\ x', x == x' \rightarrow$ 
   $\forall y\ y', y == y' \rightarrow$ 
   $(x + y) == (x' + y').$ 
Axiom PRODUCT_compat :
   $\forall x\ x', x == x' \rightarrow$ 
   $\forall y\ y', y == y' \rightarrow$ 
   $(x \times y) == (x' \times y').$ 
Add Parametric Morphism : SUM with
  signature eqv ==> eqv ==> eqv as SUM_mor.
Proof.
exact SUM_compat.
Qed.
Add Parametric Morphism : PRODUCT with
  signature eqv ==> eqv ==> eqv as PRODUCT_mor.
Proof.
exact PRODUCT_compat.
Qed.
Hint Resolve eqv_ref eqv_sym eqv_trans SUM_compat PRODUCT_compat.

```

2.2.3 Lemmas

Since Coq now understands the basics of Boolean algebra, it serves as a good exercise for us to generate some further rules using Coq's proving systems. By doing this, not only do we gain some additional tools that will become handy later down the road, but we also test whether our axioms are behaving as we would like them to.

Lemma `mul_x_x_plus_T1` :

$\forall x, x \times (x + T1) == T0.$

Proof.

`intros. rewrite distr. rewrite mul_x_x. rewrite mul_comm.`

`rewrite mul_id. apply sum_x_x.`

`Qed.`

Lemma `x_equal_y_x_plus_y` :

$\forall x\ y, x == y \leftrightarrow x + y == T0.$

Proof.

`intros. split.`

`- intros. rewrite H. rewrite sum_x_x. reflexivity.`

`- intros. rewrite term_sum_symmetric with $(y := y) (z := y)$. rewrite sum_x_x.
 apply H.`

`Qed.`

Hint Resolve `mul_x_x_plus_T1`.

Hint Resolve `x_equal_y_x_plus_y`.

These lemmas just serve to make certain rewrites regarding the core axioms less tedious to write. While one could certainly argue that they should be formulated as axioms and not lemmas due to their triviality, being pedantic is a good exercise.

Lemma `sum_id_sym` :

$\forall x, x + T0 == x.$

Proof.

`intros. rewrite sum_comm. apply sum_id.`

`Qed.`

Lemma `mul_id_sym` :

$\forall x, x \times T1 == x.$

Proof.

`intros. rewrite mul_comm. apply mul_id.`

`Qed.`

Lemma `mul_T0_x_sym` :

$\forall x, x \times T0 == T0.$

Proof.

`intros. rewrite mul_comm. apply mul_T0_x.`

`Qed.`

Lemma `sum_assoc_opp` :

```

 $\forall x\ y\ z, x + (y + z) == (x + y) + z.$ 
Proof.
  intros. rewrite sum_assoc. reflexivity.
Qed.

Lemma mul_assoc_opp :
 $\forall x\ y\ z, x \times (y \times z) == (x \times y) \times z.$ 
Proof.
  intros. rewrite mul_assoc. reflexivity.
Qed.

Lemma distr_opp :
 $\forall x\ y\ z, x \times y + x \times z == x \times (y + z).$ 
Proof.
  intros. rewrite distr. reflexivity.
Qed.

```

2.3 Variable Sets

Now that the underlying behavior concerning Boolean algebra has been properly articulated to Coq, it is now time to begin formalizing the logic surrounding our meta reasoning of Boolean equations and systems. While there are certainly several approaches to begin this process, we thought it best to ease into things through formalizing the notion of a set of variables present in an equation.

2.3.1 Definitions

We now define a variable set to be precisely a list of variables; additionally, we include several functions for including and excluding variables from these variable sets. Furthermore, since uniqueness is not a property guaranteed by Coq lists and it has the potential to be desirable, we define a function that consumes a variable set and removes duplicate entries from it. For convenience, we also provide several examples to demonstrate the functionalities of these new definitions.

```

Definition var_set := list var.
Implicit Type vars: var_set.

Fixpoint var_set_includes_var (v : var) (vars : var_set) : bool :=
  match vars with
  | nil  $\Rightarrow$  false
  | n :: n'  $\Rightarrow$  if (beq_nat v n) then true else var_set_includes_var v n'
  end.

Fixpoint var_set_remove_var (v : var) (vars : var_set) : var_set :=
  match vars with
  | nil  $\Rightarrow$  nil

```

```

|  $n :: n' \Rightarrow$  if (beq_nat  $v\ n$ ) then (var_set_remove_var  $v\ n'$ ) else  $n ::$  (var_set_remove_var
 $v\ n'$ )
end.

```

```

Fixpoint var_set_create_unique ( $vars : \text{var\_set}$ ):  $\text{var\_set} :=$ 
  match  $vars$  with
  | nil  $\Rightarrow$  nil
  |  $n :: n' \Rightarrow$ 
    if (var_set_includes_var  $n\ n'$ ) then var_set_create_unique  $n'$ 
    else  $n ::$  var_set_create_unique  $n'$ 
  end.

```

```

Fixpoint var_set_is_unique ( $vars : \text{var\_set}$ ): bool :=
  match  $vars$  with
  | nil  $\Rightarrow$  true
  |  $n :: n' \Rightarrow$ 
    if (var_set_includes_var  $n\ n'$ ) then false
    else var_set_is_unique  $n'$ 
  end.

```

```

Fixpoint term_vars ( $t : \text{term}$ ) :  $\text{var\_set} :=$ 
  match  $t$  with
  | T0  $\Rightarrow$  nil
  | T1  $\Rightarrow$  nil
  | VAR  $x \Rightarrow x ::$  nil
  | PRODUCT  $x\ y \Rightarrow$  (term_vars  $x$ ) ++ (term_vars  $y$ )
  | SUM  $x\ y \Rightarrow$  (term_vars  $x$ ) ++ (term_vars  $y$ )
  end.

```

```

Definition term_unique_vars ( $t : \text{term}$ ) :  $\text{var\_set} :=$ 
  (var_set_create_unique (term_vars  $t$ )).

```

```

Lemma vs_includes_true :  $\forall (x : \text{var}) (lvar : \text{list var}),$ 
  var_set_includes_var  $x\ lvar = \text{true} \rightarrow \text{In } x\ lvar.$ 

```

Proof.

intros.

induction $lvar$.

- simpl; intros.

discriminate.

- simpl in H . remember (**beq_nat** $x\ a$) as $H2$. destruct $H2$.

+ simpl. left. symmetry in HeqH2 . pose proof **beq_nat_true** as $H7$. specialize ($H7\ x\ a\ \text{HeqH2}$).

symmetry in $H7$. apply $H7$.

+ specialize ($\text{IHlvar } H$). simpl. right. apply IHlvar .

Qed.

```

Lemma vs_includes_false :  $\forall (x : \text{var}) (lvar : \text{list var}),$ 

```

```

var_set_includes_var x lvar = false → ¬ In x lvar.
Proof.
intros.
induction lvar.
- simpl; intros. unfold not. intros. destruct H0.
- simpl in H. remember (beq_nat x a) as H2. destruct H2. inversion H.
  specialize (IHlvar H). firstorder. intuition. apply IHlvar. simpl in H0.
  destruct H0.
  { inversion HeqH2. symmetry in H2. pose proof beq_nat_false as H7. specialize
    (H7 x a H2).
    rewrite H0 in H7. destruct H7. intuition. }
  { apply H0. }
Qed.

Lemma in_dup_and_non_dup :
  ∀ (x: var) (lvar : list var),
  In x lvar ↔ In x (var_set_create_unique lvar).
Proof.
intros. split.
- induction lvar.
  + intros. simpl in H. destruct H.
  + intros. simpl. remember(var_set_includes_var a lvar) as C. destruct C.
    { symmetry in HeqC. pose proof vs_includes_true as H7. specialize (H7 a lvar HeqC).
      simpl in H. destruct H.
      { rewrite H in H7. specialize (IHlvar H7). apply IHlvar. }
      { specialize (IHlvar H). apply IHlvar. }
    }
    { symmetry in HeqC. pose proof vs_includes_false as H7. specialize (H7 a lvar HeqC).
      simpl in H. destruct H.
      { simpl. left. apply H. }
      { specialize (IHlvar H). simpl. right. apply IHlvar. }
    }
- induction lvar.
  + intros. simpl in H. destruct H.
  + intros. simpl in H. remember(var_set_includes_var a lvar) as C. destruct C.
    { symmetry in HeqC. pose proof vs_includes_true as H7. specialize (H7 a lvar
HeqC).
      specialize (IHlvar H). simpl. right. apply IHlvar. }
    { symmetry in HeqC. pose proof vs_includes_false as H7. specialize (H7 a lvar
HeqC).
      simpl in H. destruct H.
      { simpl. left. apply H. }
      { specialize (IHlvar H). simpl. right. apply IHlvar. } }

```

Qed.

2.3.2 Examples

Example var_set_create_unique_ex1 :

var_set_create_unique [0;5;2;1;1;2;2;9;5;3] = [0;1;2;9;5;3].

Proof.

simpl. reflexivity.

Qed.

Example var_set_is_unique_ex1 :

var_set_is_unique [0;2;2;2] = false.

Proof.

simpl. reflexivity.

Qed.

Example term_vars_ex1 :

term_vars (VAR 0 + VAR 0 + VAR 1) = [0;0;1].

Proof.

simpl. reflexivity.

Qed.

Example term_vars_ex2 :

In 0 (term_vars (VAR 0 + VAR 0 + VAR 1)).

Proof.

simpl. left. reflexivity.

Qed.

2.4 Ground Terms

Seeing as we just outlined the definition of a variable set, it seems fair to now formalize the definition of a ground term, or in other words, a term that has no variables and whose variable set is the empty set.

2.4.1 Definitions

A ground term is a recursively defined proposition that is only True if and only if no variable appears in it; otherwise it will be a False proposition and no longer a ground term.

Fixpoint ground_term (t : term) : Prop :=

match t with

| VAR x ⇒ False

| SUM x y ⇒ (ground_term x) ∧ (ground_term y)

| PRODUCT x y ⇒ (ground_term x) ∧ (ground_term y)


```

| _ => True
end.

```

2.4.2 Lemmas

Our first real lemma (shown below), articulates an important property of ground terms: all ground terms are equivalent to either 0 or 1. This curious property is a direct result of the fact that these terms possess no variables and additionally because of the axioms of Boolean algebra.

Lemma `ground_term_equiv_T0_T1` :

$\forall x, (\text{ground_term } x) \rightarrow (x == T0 \vee x == T1).$

Proof.

`intros. induction x.`

`- left. reflexivity.`

`- right. reflexivity.`

`- contradiction.`

`- inversion H. destruct IHx1; destruct IHx2; auto. rewrite H2. left. rewrite sum_id. apply H3.`

`rewrite H2. rewrite H3. rewrite sum_id. right. reflexivity.`

`rewrite H2. rewrite H3. right. rewrite sum_comm. rewrite sum_id. reflexivity.`

`rewrite H2. rewrite H3. rewrite sum_x_x. left. reflexivity.`

`- inversion H. destruct IHx1; destruct IHx2; auto. rewrite H2. left. rewrite mul_T0_x. reflexivity.`

`rewrite H2. left. rewrite mul_T0_x. reflexivity.`

`rewrite H3. left. rewrite mul_comm. rewrite mul_T0_x. reflexivity.`

`rewrite H2. rewrite H3. right. rewrite mul_id. reflexivity.`

`Qed.`

This lemma, while intuitively obvious by definition, nonetheless provides a formal bridge between the world of ground terms and the world of variable sets.

Lemma `ground_term_has_empty_var_set` :

$\forall x, (\text{ground_term } x) \rightarrow (\text{term_vars } x) = [].$

Proof.

`intros. induction x.`

`- simpl. reflexivity.`

`- simpl. reflexivity.`

`- contradiction.`

`- firstorder. unfold term_vars. unfold term_vars in H2. rewrite H2. unfold term_vars in H1. rewrite H1. simpl. reflexivity.`

`- firstorder. unfold term_vars. unfold term_vars in H2. rewrite H2. unfold term_vars in H1. rewrite H1. simpl. reflexivity.`

`Qed.`

2.4.3 Examples

Here are some examples to show that our ground term definition is working appropriately.

```
Example ex_gt1 :  
  (ground_term (T0 + T1)).  
Proof.  
simpl. split.  
- reflexivity.  
- reflexivity.  
Qed.  
  
Example ex_gt2 :  
  (ground_term (VAR 0 × T1)) → False.  
Proof.  
simpl. intros. destruct H. apply H.  
Qed.
```

2.5 Substitutions

It is at this point in our Coq development that we begin to officially define the principal action around which the entirety of our efforts are centered: the act of substituting variables with other terms. While substitutions alone are not of great interest, their emergent properties as in the case of whether or not a given substitution unifies an equation are of substantial importance to our later research.

2.5.1 Definitions

Here we define a substitution to be a list of ordered pairs where each pair represents a variable being mapped to a term. For sake of clarity these ordered pairs shall be referred to as replacements from now on and as a result, substitutions should really be considered to be lists of replacements.

Definition replacement := (**prod** var term).

Definition subst := **list** replacement.

Implicit Type s : subst.

Our first function, find_replacement, is an auxilliary to apply_subst. This function will search through a substitution for a specific variable, and if found, returns the variable's associated term.

```
Fixpoint find_replacement (x : var) (s : subst) : term :=  
  match s with  
  | nil ⇒ VAR x  
  | r :: r' ⇒
```

```

    if beq_nat (fst r) x then (snd r)
  else
    (find_replacement x r')
end.

```

The `apply_subst` function will take a term and a substitution and will produce a new term reflecting the changes made to the original one.

```

Fixpoint apply_subst (t : term) (s : subst) : term :=
  match t with
  | T0 ⇒ T0
  | T1 ⇒ T1
  | VAR x ⇒ (find_replacement x s)
  | PRODUCT x y ⇒ PRODUCT (apply_subst x s) (apply_subst y s)
  | SUM x y ⇒ SUM (apply_subst x s) (apply_subst y s)
  end.

```

For reasons of completeness, it is useful to be able to generate identity substitutions; namely, substitutions that map the variables of a term's variable set to themselves.

```

Fixpoint build_id_subst (lvar : var_set) : subst :=
  match lvar with
  | nil ⇒ nil
  | v :: v' ⇒ (cons (v , (VAR v))
                    (build_id_subst v'))
  end.

```

Since we now have the ability to generate identity substitutions, we should now formalize a general proposition for testing whether or not a given substitution is an identity substitution of a given term.

```

Definition subst_equiv (s1 s2: subst) : Prop :=
  ∀ t, apply_subst t s1 == apply_subst t s2.

```

```

Definition subst_is_id_subst (t : term) (s : subst) : Prop :=
  (apply_subst t s) == t.

```

2.5.2 Lemmas

Having now outlined the functionality of a substitution, let us now begin to analyze some implications of its form and composition by proving some lemmas.

Given that we have a definition for identity substitutions, we should prove that identity substitutions do not modify a term.

Lemma `id_subst`:

```

  ∀ (t : term) (l : var_set),
  apply_subst t (build_id_subst l) == t.

```

Proof.

```

intros. induction t.
{
  simpl. reflexivity.
}
{
  simpl. reflexivity.
}
{
  simpl. induction l.
  {
    simpl. reflexivity.
  }
  {
    simpl. destruct (beq_nat a v) eqn: e.
    {
      apply beq_nat_true in e. rewrite e. reflexivity.
    }
    {
      apply IHL.
    }
  }
}
}
{
  simpl. rewrite IHt1. rewrite IHt2. reflexivity.
}
{
  simpl. rewrite IHt1. rewrite IHt2. reflexivity.
}
Qed.

```

Lemma sum_comm_compat $t1\ t2$:

$\forall (sigma: subst), (apply_subst (t1 + t2) sigma) == (apply_subst (t2 + t1) sigma).$

Proof.

```

intros.
simpl.
auto.

```

Qed.

Hint Resolve sum_comm_compat.

Lemma sum_assoc_compat $t1\ t2\ t3$:

$\forall (sigma: subst), (apply_subst ((t1 + t2) + t3) sigma) == (apply_subst (t1 + (t2 + t3)) sigma).$

Proof.

```

intros.

```

```

    simpl.
    auto.
Qed.
Hint Resolve sum_assoc_compat.

Lemma sum_id_compat t:
   $\forall (sigma: \text{subst}), (\text{apply\_subst } (T0 + t) \text{ sigma}) == (\text{apply\_subst } t \text{ sigma}).$ 
Proof.
  intros.
  simpl.
  auto.
Qed.
Hint Resolve sum_id_compat.

Lemma sum_x_x_compat t:
   $\forall (sigma: \text{subst}), (\text{apply\_subst } (t + t) \text{ sigma}) == (\text{apply\_subst } T0 \text{ sigma}).$ 
Proof.
  intros.
  simpl.
  auto.
Qed.
Hint Resolve sum_x_x_compat.

Lemma mul_comm_compat t1 t2:
   $\forall (sigma: \text{subst}), (\text{apply\_subst } (t1 \times t2) \text{ sigma}) == (\text{apply\_subst } (t2 \times t1) \text{ sigma}).$ 
Proof.
  intros.
  simpl.
  auto.
Qed.
Hint Resolve mul_comm_compat.

Lemma mul_assoc_compat t1 t2 t3:
   $\forall (sigma: \text{subst}), (\text{apply\_subst } ((t1 \times t2) \times t3) \text{ sigma}) == (\text{apply\_subst } (t1 \times (t2 \times t3)) \text{ sigma}).$ 
Proof.
  intros.
  simpl.
  auto.
Qed.
Hint Resolve mul_assoc_compat.

Lemma mul_x_x_compat t:
   $\forall (sigma: \text{subst}), (\text{apply\_subst } (t \times t) \text{ sigma}) == (\text{apply\_subst } t \text{ sigma}).$ 
Proof.
  intros sigma.

```

```

    simpl.
    auto.
Qed.
Hint Resolve mul_x_x_compat.
Lemma mul_T0_x_compat t:
   $\forall (sigma: \text{subst}), (\text{apply\_subst } (T0 \times t) \text{ sigma}) == (\text{apply\_subst } T0 \text{ sigma}).$ 
Proof.
  intros.
  simpl.
  auto.
Qed.
Hint Resolve mul_T0_x_compat.
Lemma mul_id_compat t:
   $\forall (sigma: \text{subst}), (\text{apply\_subst } (T1 \times t) \text{ sigma}) == (\text{apply\_subst } t \text{ sigma}).$ 
Proof.
  intros.
  simpl.
  auto.
Qed.
Hint Resolve mul_id_compat.
Lemma distr_compat t1 t2 t3:
   $\forall (sigma: \text{subst}), (\text{apply\_subst } (t1 \times (t2 + t3)) \text{ sigma}) == (\text{apply\_subst } ((t1 \times t2) + (t1 \times t3)) \text{ sigma}).$ 
Proof.
  intros.
  simpl.
  auto.
Qed.
Hint Resolve distr_compat.
Lemma refl_comm_compat t1 t2:
   $\forall (sigma: \text{subst}), (\text{apply\_subst } t1 \text{ sigma}) == (\text{apply\_subst } t2 \text{ sigma}) \rightarrow (\text{apply\_subst } t2 \text{ sigma}) == (\text{apply\_subst } t1 \text{ sigma}).$ 
Proof.
  intros.
  simpl.
  auto.
Qed.
Hint Resolve refl_comm_compat.
Lemma trans_compat t1 t2 t3 :
   $\forall (sigma: \text{subst}),$ 
   $(\text{apply\_subst } t1 \text{ sigma}) == (\text{apply\_subst } t2 \text{ sigma}) \rightarrow$ 

```

$(\text{apply_subst } t2 \text{ sigma}) == (\text{apply_subst } t3 \text{ sigma}) \rightarrow$
 $(\text{apply_subst } t1 \text{ sigma}) == (\text{apply_subst } t3 \text{ sigma}) .$

Proof.

intros sigma H12 H23.

eauto.

Qed.

Hint Resolve trans_compat.

Lemma apply_subst_compat : $\forall (t \ t' : \text{term}),$
 $t == t' \rightarrow \forall (\text{sigma} : \text{subst}), (\text{apply_subst } t \text{ sigma}) == (\text{apply_subst } t' \text{ sigma}).$

Proof.

Admitted.

Add Parametric Morphism : apply_subst with
signature eqv ==> eq ==> eqv as apply_subst_mor.

Proof.

exact apply_subst_compat.

Qed.

An easy thing to prove right off the bat is that ground terms, i.e. terms with no variables, cannot be modified by applying substitutions to them. This will later prove to be very relevant when we begin to talk about unification.

Lemma ground_term_cannot_subst :
 $\forall x, (\text{ground_term } x) \rightarrow (\forall s, \text{apply_subst } x \ s == x).$

Proof.

intros. induction s.

- apply ground_term_equiv_T0_T1 in H. destruct H.
- + rewrite H. simpl. reflexivity.
- + rewrite H. simpl. reflexivity.
- apply ground_term_equiv_T0_T1 in H. destruct H. rewrite H.
- + simpl. reflexivity.
- + rewrite H. simpl. reflexivity.

Qed.

A fundamental property of substitutions is their distributivity and associativity across the summation and multiplication of terms. Again the importance of these proofs will not become apparent until we talk about unification.

Lemma subst_distribution :
 $\forall s \ x \ y, \text{apply_subst } x \ s + \text{apply_subst } y \ s == \text{apply_subst } (x + y) \ s.$

Proof.

intro. induction s. simpl. intros. reflexivity. intros. simpl. reflexivity.

Qed.

Lemma subst_associative :
 $\forall s \ x \ y, \text{apply_subst } x \ s \times \text{apply_subst } y \ s == \text{apply_subst } (x \times y) \ s.$

Proof.

```
intro. induction s. intros. reflexivity. intros. simpl. reflexivity.
Qed.
```

Lemma subst_sum_distr_opp :

$\forall s \ x \ y, \text{apply_subst } (x + y) \ s == \text{apply_subst } x \ s + \text{apply_subst } y \ s.$

Proof.

```
intros.
apply refl_comm.
apply subst_distribution.
```

Qed.

Lemma subst_mul_distr_opp :

$\forall s \ x \ y, \text{apply_subst } (x \times y) \ s == \text{apply_subst } x \ s \times \text{apply_subst } y \ s.$

Proof.

```
intros.
apply refl_comm.
apply subst_associative.
```

Qed.

Lemma var_subst:

$\forall (v : \text{var}) (ts : \text{term}) ,$
 $(\text{apply_subst } (\text{VAR } v) (\text{cons } (v \ , \ ts) \ \text{nil})) == ts.$

Proof.

```
intros. simpl. destruct (beq_nat v v) eqn: e. apply beq_nat_true in e.
reflexivity. apply beq_nat_false in e. firstorder.
```

Qed.

2.5.3 Examples

Here are some examples showcasing the nature of applying substitutions to terms.

Example subst_ex1 :

$(\text{apply_subst } (T0 + T1) \ [\]) == T0 + T1.$

Proof.

```
intros. reflexivity.
```

Qed.

Example subst_ex2 :

$(\text{apply_subst } (\text{VAR } 0 \times \text{VAR } 1) \ [(0, T0)]) == T0.$

Proof.

```
intros. simpl. apply mul_T0_x.
```

Qed.

2.6 Unification

Now that we have established the concept of term substitutions in Coq, it is time for us to formally define the concept of Boolean unification. Unification, in its most literal sense, refers to the act of applying a substitution to terms in order to make them equivalent to each other. In other words, to say that two terms are unifiable is to really say that there exists a substitution such that the two terms are equal. Interestingly enough, we can abstract this concept further to simply saying that a single term is unifiable if there exists a substitution such that the term will be equivalent to 0. By doing this abstraction, we can prove that equation solving and unification are essentially the same fundamental problem.

Below is the initial definition for unification, namely that two terms can be unified to be equivalent to one another. By starting here we will show each step towards abstracting unification to refer to a single term.

Definition `unifies` ($a\ b : \mathbf{term}$) ($s : \mathbf{subst}$) : `Prop` :=
`(apply_subst a s) == (apply_subst b s).`

Here is a simple example demonstrating the concept of testing whether two terms are unified by a substitution.

Example `ex_unif1` :
`unifies (VAR 0) (VAR 1) ((0, T1) :: (1, T1) :: nil).`

`Proof.`

`unfold unifies. simpl. reflexivity.`

`Qed.`

Now we are going to show that moving both terms to one side of the equivalence relation through addition does not change the concept of unification.

Definition `unifies_T0` ($a\ b : \mathbf{term}$) ($s : \mathbf{subst}$) : `Prop` :=
`(apply_subst a s) + (apply_subst b s) == T0.`

Lemma `unifies_T0_equiv` :

$\forall\ x\ y\ s, \text{unifies } x\ y\ s \leftrightarrow \text{unifies_T0 } x\ y\ s.$

`Proof.`

`intros. split.`

`{`

`intros. unfold unifies_T0. unfold unifies in H. rewrite H.`

`rewrite sum_x_x. reflexivity.`

`}`

`{`

`intros. unfold unifies_T0 in H. unfold unifies.`

`rewrite term_sum_symmetric with (x := apply_subst x s + apply_subst y s)`

`(z := apply_subst y s) in H. rewrite sum_id in H.`

`rewrite sum_comm in H.`

`rewrite sum_comm with (y := apply_subst y s) in H.`

`rewrite ← sum_assoc in H.`

```

    rewrite sum_x_x in H.
    rewrite sum_id in H.
    apply H.
  }
Qed.

```

Now we can define what it means for a substitution to be a unifier for a given term.

```

Definition unifier (t : term) (s : subst) : Prop :=
  (apply_subst t s) == T0.

```

```

Example unifier_ex1 :
  (unifier (VAR 0) ((0, T0) :: nil)).

```

Proof.

```

unfold unifier. simpl. reflexivity.

```

Qed.

To ensure our efforts were not in vain, let us now prove that this last abstraction of the unification problem is still equivalent to the original.

```

Lemma unifier_distribution :

```

```

  ∀ x y s, (unifies_T0 x y s) ↔ (unifier (x + y) s).

```

Proof.

```

intros. split.

```

```

{
  intros. unfold unifies_T0 in H. unfold unifier.
  rewrite ← H. symmetry. apply subst_distribution.
}
{
  intros. unfold unifies_T0. unfold unifier in H.
  rewrite ← H. apply subst_distribution.
}

```

Qed.

Lastly let us define a term to be unifiable if there exists a substitution that unifies it.

```

Definition unifiable (t : term) : Prop :=
  ∃ s, unifier t s.

```

```

Example unifiable_ex1 :
  ∃ x, unifiable (x + T1).

```

Proof.

```

∃ (T1). unfold unifiable. unfold unifier.
∃ nil. simpl. rewrite sum_x_x. reflexivity.

```

Qed.

2.7 Most General Unifier

While the property of a term being unifiable is certainly important, it should come as no surprise that not all unifiers are created equal; in fact, certain unifiers possess the desirable property of being more general than others. For this reason, let us now formally define the concept of a most general unifier (mgu): a unifier such that with respect to a given term, all other unifiers are instances of it, or in other words, less general than it.

The first step towards establishing the concept of a mgu requires us to formalize the notion of a unifier being more general than another. To accomplish this goal, let us formulate the definition of a substitution composing another one; or in other words, to say that a substitution is more general than another one.

Definition `substitution_composition` ($s \ s' \ \text{delta} : \text{subst}$) ($t : \text{term}$) : `Prop` :=
 $\forall (x : \text{var}), \text{apply_subst} (\text{apply_subst} (\text{VAR } x) s) \ \text{delta} == \text{apply_subst} (\text{VAR } x) s' .$

Definition `more_general_substitution` ($s \ s' : \text{subst}$) ($t : \text{term}$) : `Prop` :=
 $\exists \ \text{delta}, \text{substitution_composition } s \ s' \ \text{delta } t.$

Now that we have articulated the concept of composing substitutions, let us now formulate the definition for a most general unifier.

Definition `most_general_unifier` ($t : \text{term}$) ($s : \text{subst}$) : `Prop` :=
 $(\text{unifier } t \ s) \rightarrow (\forall (s' : \text{subst}), \text{unifier } t \ s' \rightarrow \text{more_general_substitution } s \ s' \ t).$

While this definition of a most general unifier is certainly valid, it is a somewhat unwieldy formulation. For this reason, let us now define an alternative definition called a reproductive unifier, and then prove it to be equivalent to our definition of a most general unifier. This will make our proofs easier to formulate down the road as the task of proving a unifier to be reproductive is substantially easier than proving it to be most general directly.

Definition `reproductive_unifier` ($t : \text{term}$) ($sig : \text{subst}$) : `Prop` :=
 $\text{unifier } t \ sig \rightarrow$
 $\forall (tau : \text{subst}) (x : \text{var}),$
 $\text{unifier } t \ tau \rightarrow$
 $(\text{apply_subst} (\text{apply_subst} (\text{VAR } x) \ sig) \ tau) == (\text{apply_subst} (\text{VAR } x) \ tau).$

Lemma `reproductive_is_mgu` : $\forall (t : \text{term}) (u : \text{subst}),$
 $\text{reproductive_unifier } t \ u \rightarrow$
 $\text{most_general_unifier } t \ u.$

Proof.

`intros. unfold most_general_unifier. unfold reproductive_unifier in H.`
`unfold more_general_substitution . unfold substitution_composition.`
`intros. specialize (H H0). $\exists s' .$ intros. specialize (H s' x). specialize (H H1). apply H.`
`Qed.`

Lemma `most_general_unifier_compat` : $\forall (t \ t' : \text{term}),$

$t == t' \rightarrow \forall (sigma: \text{subst}), (\text{most_general_unifier } t \text{ } sigma) \leftrightarrow (\text{most_general_unifier } t' \text{ } sigma).$

Proof.

Admitted.

2.8 Auxilliary Computational Operations and Simplifications

These functions below will come in handy later during the Lowenheim formula proof. They mainly lay the groundwork for providing the computational nuts and bolts for Lowenheim's algorithm for finding most general unifiers.

Fixpoint identical ($a \ b: \text{term}$) : **bool** :=

```

match a , b with
| T0, T0 => true
| T0, _ => false
| T1, T1 => true
| T1, _ => false
| VAR x , VAR y => if beq_nat x y then true else false
| VAR x, _ => false
| PRODUCT x y, PRODUCT x1 y1 => if ((identical x x1) && (identical y y1)) then
true
                                     else false
| PRODUCT x y, _ => false
| SUM x y, SUM x1 y1 => if ((identical x x1) && (identical y y1)) then true
                                     else false
| SUM x y, _ => false
end.
```

Definition plus_one_step ($a \ b: \text{term}$) : **term** :=

```

match a, b with
| T0, T0 => T0
| T0, T1 => T1
| T1, T0 => T1
| T1, T1 => T0
| _ , _ => SUM a b
end.
```

Definition mult_one_step ($a \ b: \text{term}$) : **term** :=

```

match a, b with
| T0, T0 => T0
| T0, T1 => T0
| T1, T0 => T0
```

```

| T1 , T1 ⇒ T1
| - , - ⇒ PRODUCT a b
end.

Fixpoint simplify (t : term) : term :=
  match t with
  | T0 ⇒ T0
  | T1 ⇒ T1
  | VAR x ⇒ VAR x
  | PRODUCT x y ⇒ mult_one_step (simplify x) (simplify y)
  | SUM x y ⇒ plus_one_step (simplify x) (simplify y)
  end.

Lemma pos_left_sum_compat : ∀ (t t1 t2 : term),
  t == t1 → plus_one_step t1 t2 == plus_one_step t t2.

Proof.
  intros. induction t1.
  - induction t.
    + reflexivity.
    + apply T1_not_equiv_T0 in H. inversion H.
    + induction t2.
      { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
      { simpl. rewrite H. rewrite sum_id. reflexivity. }
      { simpl. rewrite H. reflexivity. }
      { simpl. rewrite H. reflexivity. }
      { simpl. rewrite H. reflexivity. }
    + induction t2.
      { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
      { simpl. rewrite H. rewrite sum_id. reflexivity. }
      { simpl. rewrite H. reflexivity. }
      { simpl. rewrite H. reflexivity. }
      { simpl. rewrite H. reflexivity. }
    + induction t2.
      { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
      { simpl. rewrite H. rewrite sum_id. reflexivity. }
      { simpl. rewrite H. reflexivity. }
      { simpl. rewrite H. reflexivity. }
      { simpl. rewrite H. reflexivity. }
  - induction t.
    + induction t2.
      { simpl. rewrite H. reflexivity. }
      { simpl. rewrite H. reflexivity. }
      { simpl. rewrite H. reflexivity. }
      { simpl. rewrite H. reflexivity. }

```

```

    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
+ induction t2.
    { simpl. rewrite H. rewrite sum_comm. rewrite sum_id. reflexivity. }
    { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. rewrite sum_comm. rewrite sum_id. reflexivity. }
    { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. rewrite sum_comm. rewrite sum_id. reflexivity. }
    { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
- induction t.
    + induction t2.
        { simpl. rewrite H. rewrite sum_x_x. rewrite H. reflexivity. }
        { simpl. rewrite  $\leftarrow$  H. rewrite sum_id. reflexivity. }
        { simpl. rewrite H. reflexivity. }
        { simpl. rewrite H. reflexivity. }
        { simpl. rewrite H. reflexivity. }
    + induction t2.
        { simpl. rewrite  $\leftarrow$  H. rewrite sum_comm. rewrite sum_id. reflexivity. }
        { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
        { simpl. rewrite H. reflexivity. }
        { simpl. rewrite H. reflexivity. }
        { simpl. rewrite H. reflexivity. }
    + induction t2.
        { simpl. rewrite H. reflexivity. }
        { simpl. rewrite H. reflexivity. }
        { simpl. rewrite H. reflexivity. }

```

```

    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
- induction t.
+ induction t2.
    { simpl. rewrite ← H. rewrite sum_x_x. reflexivity. }
    { simpl. rewrite ← H. rewrite sum_id. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite ← H. rewrite sum_comm. rewrite sum_id. reflexivity. }
    { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }

```

```

    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
- induction t.
+ induction t2.
  { simpl. rewrite ← H. rewrite sum_x_x. reflexivity. }
  { simpl. rewrite ← H. rewrite sum_id. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
+ induction t2.
  { simpl. rewrite ← H. rewrite sum_comm. rewrite sum_id. reflexivity. }
  { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }

```

Qed.

Lemma pos_right_sum_compat : $\forall (t \ t1 \ t2 : \text{term}),$
 $t == t2 \rightarrow \text{plus_one_step } t1 \ t2 == \text{plus_one_step } t1 \ t.$

Proof.

intros. induction t1.

- induction t.

+ induction t2.

{ simpl. reflexivity. }


```

{ simpl. rewrite H. reflexivity. }
{ simpl. rewrite H. rewrite sum_x_x. apply H. }
{ simpl. rewrite ← H. rewrite sum_x_x. reflexivity. }
{ simpl. rewrite ← H. rewrite sum_x_x. reflexivity. }
+ induction t2.
{ simpl. rewrite H. reflexivity. }
{ simpl. reflexivity. }
{ simpl. rewrite H. rewrite sum_id. reflexivity. }
{ simpl. rewrite H. rewrite sum_id. reflexivity. }
{ simpl. rewrite ← H. rewrite sum_id. reflexivity. }
+ induction t2.
{ simpl. rewrite H. rewrite sum_x_x. reflexivity. }
{ simpl. rewrite H. rewrite sum_id. reflexivity. }
{ simpl. rewrite H. rewrite sum_id. reflexivity. }
{ simpl. rewrite H. rewrite sum_id. reflexivity. }
{ simpl. rewrite ← H. rewrite sum_id. reflexivity. }
+ induction t2.
{ simpl. rewrite H. rewrite sum_x_x. reflexivity. }
{ simpl. rewrite H. rewrite sum_id. reflexivity. }
{ simpl. rewrite H. rewrite sum_id. reflexivity. }
{ simpl. rewrite H. rewrite sum_id. reflexivity. }
{ simpl. rewrite ← H. rewrite sum_id. reflexivity. }
- induction t.
+ induction t2.
{ simpl. reflexivity. }
{ simpl. rewrite H. reflexivity. }
{ simpl. rewrite ← H. rewrite sum_comm. rewrite sum_id. reflexivity. }
{ simpl. rewrite ← H. rewrite sum_comm. rewrite sum_id. reflexivity. }
{ simpl. rewrite ← H. rewrite sum_comm. rewrite sum_id. reflexivity. }
+ induction t2.
{ simpl. rewrite H. reflexivity. }
{ simpl. reflexivity. }
{ simpl. rewrite H. rewrite sum_x_x. reflexivity. }
{ simpl. rewrite H. rewrite sum_x_x. reflexivity. }
{ simpl. rewrite ← H. rewrite sum_x_x. reflexivity. }
+ induction t2.

```

```

    { simpl. rewrite H. rewrite sum_comm. rewrite sum_id. reflexivity. }
    { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite  $\leftarrow$  H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. rewrite sum_comm. rewrite sum_id. reflexivity. }
    { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite  $\leftarrow$  H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. rewrite sum_comm. rewrite sum_id. reflexivity. }
    { simpl. rewrite H. rewrite sum_x_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite  $\leftarrow$  H. reflexivity. }
- induction t.
+ induction t2.
    { simpl. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite  $\leftarrow$  H. reflexivity. }
    { simpl. rewrite  $\leftarrow$  H. reflexivity. }
    { simpl. rewrite  $\leftarrow$  H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite  $\leftarrow$  H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite  $\leftarrow$  H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite  $\leftarrow$  H. reflexivity. }

```

```

+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
- induction t.
+ induction t2.
  { simpl. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
- induction t.
+ induction t2.
  { simpl. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }

```

```

    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }

```

Qed.

Lemma pos_left_mul_compat : $\forall (t \ t1 \ t2 : \text{term}),$
 $t == t1 \rightarrow \text{mult_one_step } t1 \ t2 == \text{mult_one_step } t \ t2.$

Proof.

```

intros. induction t1.
- induction t.
  + reflexivity.
  + apply T1_not_equiv_T0 in H. inversion H.
  + induction t2.
    { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite H. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
  + induction t2.
    { simpl. rewrite H. rewrite mul_x_x. reflexivity. }

```

```

    { simpl. rewrite H. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite H. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
- induction t.
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
+ induction t2.
    { simpl. rewrite H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
- induction t.

```

```

+ induction t2.
  { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
  { simpl. rewrite ← H. rewrite mul_T0_x. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
+ induction t2.
  { simpl. rewrite ← H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
  { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
- induction t.
  + induction t2.
    { simpl. rewrite ← H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite ← H. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
  + induction t2.
    { simpl. rewrite ← H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }

```

```

    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
- induction t.
+ induction t2.
    { simpl. rewrite ← H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite ← H. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite ← H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }

```

```

    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }

```

Qed.

Lemma pos_right_mul_compat : $\forall (t \ t1 \ t2 : \text{term}),$
 $t == t2 \rightarrow \text{mult_one_step } t1 \ t2 == \text{mult_one_step } t1 \ t.$

Proof.

intros. induction t1.

```

- induction t.
+ induction t2.
    { simpl. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite ← H. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite ← H. rewrite mul_x_x. reflexivity. }
+ induction t2.
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. reflexivity. }
+ induction t2.
    { simpl. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. rewrite mul_T0_x. reflexivity. }
+ induction t2.
    { simpl. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. rewrite mul_T0_x. reflexivity. }
+ induction t2.
    { simpl. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. rewrite mul_T0_x. reflexivity. }

```



```

    { simpl. rewrite mul_T0_x. rewrite mul_T0_x. reflexivity. }
    { simpl. rewrite mul_T0_x. rewrite mul_T0_x. reflexivity. }
- induction t.
+ induction t2.
  { simpl. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
  { simpl. rewrite ← H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
  { simpl. rewrite ← H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }
  { simpl. reflexivity. }
  { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
  { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
  { simpl. rewrite ← H. rewrite mul_x_x. reflexivity. }
+ induction t2.
  { simpl. rewrite H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
  { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. rewrite mul_comm. rewrite mul_T0_x. reflexivity. }
  { simpl. rewrite H. rewrite mul_x_x. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
- induction t.
+ induction t2.
  { simpl. reflexivity. }
  { simpl. rewrite H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
  { simpl. rewrite ← H. reflexivity. }
+ induction t2.
  { simpl. rewrite H. reflexivity. }

```

```

    { simpl. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
- induction t.
+ induction t2.
    { simpl. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite H. reflexivity. }
    { simpl. rewrite ← H. reflexivity. }
+ induction t2.

```

```

    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $\leftarrow H$ . reflexivity. }
+ induction  $t2$ .
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $\leftarrow H$ . reflexivity. }
- induction  $t$ .
+ induction  $t2$ .
    { simpl. reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $\leftarrow H$ . reflexivity. }
    { simpl. rewrite  $\leftarrow H$ . reflexivity. }
    { simpl. rewrite  $\leftarrow H$ . reflexivity. }
+ induction  $t2$ .
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $\leftarrow H$ . reflexivity. }
+ induction  $t2$ .
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $\leftarrow H$ . reflexivity. }
+ induction  $t2$ .
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $\leftarrow H$ . reflexivity. }
+ induction  $t2$ .
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $H$ . reflexivity. }
    { simpl. rewrite  $\leftarrow H$ . reflexivity. }

```

Qed.

Lemma simplify_eqv :

$\forall (t : \text{term}),$
simplify $t == t$.

Proof.

```
intros. induction t.
- simpl. reflexivity.
- simpl. reflexivity.
- simpl. reflexivity.
- simpl. pose proof pos_left_sum_compat. specialize (H t1 (simplify t1) (simplify t2)).
  symmetry in IHt1. specialize (H IHt1). rewrite H.
  pose proof pos_right_sum_compat. specialize (H0 (simplify t2) t1 t2).
  specialize (H0 IHt2). symmetry in H0. rewrite H0.
  induction t1.
  + induction t2.
    { simpl. rewrite sum_x_x. reflexivity. }
    { simpl. rewrite sum_id. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
  + induction t2.
    { simpl. rewrite sum_id_sym. reflexivity. }
    { simpl. rewrite sum_x_x. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
  + simpl. reflexivity.
  + simpl. reflexivity.
  + simpl. reflexivity.
- simpl. pose proof pos_left_mul_compat. specialize (H t1 (simplify t1) (simplify t2)).
  symmetry in IHt1. specialize (H IHt1). rewrite H.
  pose proof pos_right_mul_compat. specialize (H0 (simplify t2) t1 t2).
  specialize (H0 IHt2). symmetry in H0. rewrite H0.
  induction t1.
  + induction t2.
    { simpl. rewrite mul_x_x. reflexivity. }
    { simpl. rewrite mul_T0_x. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
    { simpl. reflexivity. }
  + induction t2.
    { simpl. rewrite mul_T0_x_sym. reflexivity. }
```

```
{ simpl. rewrite mul_x_x. reflexivity. }  
{ simpl. reflexivity. }  
{ simpl. reflexivity. }  
{ simpl. reflexivity. }  
+ simpl. reflexivity.  
+ simpl. reflexivity.  
+ simpl. reflexivity.  
Qed.
```

Chapter 3

Library

B_Unification.lowenheim_formula

Require Export terms.

Require Import List.

Import ListNotations.

3.1 Introduction

In this section we formulate Lowenheim’s algorithm using the data structures and functions defined in `terms.v`. The final occurring main function (`Lowenheim_Main`), takes as input a term and produces a substitution that unifies the given term and it is defined towards the end of the file. The substitution is said to be a most general unifier and not a mere substitution, but that statement is proven in the `lowenheim_proof.v` file. In this section we focus on the formulation of the algorithm itself, without any proofs about the properties of the formula or the algorithm.

3.2 Lowenheim’s Builder

In this subsection we are implementing the main component of lowenheim’s algorithm, which is the “builder” of lowenheim’s substitution for a given term. This implementation strictly follows as close as possible the formal, mathematical format of Lowenheim’s algorithm.

Here is a skeleton function for building a substitution on the format $\sigma(x) = (s + 1) * \sigma_1(x) + s * \sigma_2(x)$, each variable of a given list of variables, a given term s and substitutions σ_1 and σ_2 . This skeleton function is a more general format of lowenheim’s builder.

```
Fixpoint build_on_list_of_vars (list_var : var_set) (s : term) (sig1 : subst) (sig2 : subst) :  
subst :=  
  match list_var with
```

```

| nil ⇒ nil
| v' :: v ⇒
  (cons (v' , (s + T1) × (apply_subst (VAR v') sig1 ) + s × (apply_subst (VAR v' )
sig2 ) )
    (build_on_list_of_vars v s sig1 sig2) )
end.

```

This is the function to build a lowenheim substitution for a term t , given the term t and a unifier of t , using the previously defined skeleton function. The list of variables is the variables within t and the substitutions are the identical substitution and the unifier of the term.

Definition `build_lowenheim_subst` ($t : \mathbf{term}$) ($\tau : \mathbf{subst}$) : $\mathbf{subst} :=$
`build_on_list_of_vars (term_unique_vars t) t (build_id_subst (term_unique_vars t)) τ .`

*Lowenheim's algorithm

In this subsection we enhance Lowenheim's builder to be a complete algorithm that is able to find ground substitutions before feeding them to the main formula to generate a most general unifier

**Auxillary functions and definitions

Function to update a term, after it applies to it a given substitution and simplifies it.

Definition `update_term` ($t : \mathbf{term}$) ($s' : \mathbf{subst}$) : $\mathbf{term} :=$
`(simplify (apply_subst t s')).`

Function to determine if a term is the ground term $T0$.

Definition `term_is_T0` ($t : \mathbf{term}$) : $\mathbf{bool} :=$
`(identical t T0).`

Definition of new data type to represent both a substitution and a none-substitution (which is different than the empty substitution), as return types. A "Some substitution" is an entity that contains a substitution, whereas "None substitution" is an entity to represent the "absence" as a substitution (similar to a casted nil). **Inductive** `subst_option`: $\mathbf{Type} :=$

```

| Some_subst : subst → subst_option
| None_subst : subst_option.

```

A function to find a single ground unifier, recursively. It finds a substitution with ground terms that makes the given (input) term equivalent to $T0$. To use it, start with an empty list of replacements as the input s - `subst`

Fixpoint `rec_subst` ($t : \mathbf{term}$) ($vars : \mathbf{var_set}$) ($s : \mathbf{subst}$) : $\mathbf{subst} :=$
`match vars with`

```

| nil ⇒ s
| v' :: v ⇒
  if (term_is_T0
      (update_term (update_term t (cons (v' , T0) s) )
        (rec_subst (update_term t (cons (v' , T0) s) )
          v (cons (v' , T0) s))) )

```

```

    )
  then
    (rec_subst (update_term t (cons (v' , T0) s) )
      v (cons (v' , T0) s))
  else
    if (term_is_T0
      (update_term (update_term t (cons (v' , T1) s) )
        (rec_subst (update_term t (cons (v' , T1) s) )
          v (cons (v' , T1) s)) ) )
    then
      (rec_subst (update_term t (cons (v' , T1) s) )
        v (cons (v' , T1) s))
    else
      (rec_subst (update_term t (cons (v' , T0) s) )
        v (cons (v' , T0) s))
  end.

Compute (rec_subst ((VAR 0) × (VAR 1)) (cons 0 (cons 1 nil)) nil) .

Function to find a ground unifier of the input term, if it exists.  Fixpoint find_unifier (t
: term) : subst_option :=
  match (update_term t (rec_subst t (term_unique_vars t) nil) ) with
  | T0 ⇒ Some_subst (rec_subst t (term_unique_vars t) nil)
  | _ ⇒ None_subst
  end.

Compute (find_unifier ((VAR 0) × (VAR 1))).
Compute (find_unifier ((VAR 0) + (VAR 1))).
Compute (find_unifier ((VAR 0) + (VAR 1) + (VAR 2) + T1 + (VAR 3) × ((VAR 2) + (VAR
0)) )).

```

3.2.1 Here is the main lowenheim's formula; given a term, produce an MGU that makes it equivalent to T0,

if there is one. Otherwise, returns “None subst”

```

Definition Lowenheim_Main (t : term) : subst_option :=
  match (find_unifier t) with
  | Some_subst s ⇒ Some_subst (build_lowenheim_subst t s)
  | None_subst ⇒ None_subst
  end.

Compute (find_unifier ((VAR 0) × (VAR 1)) ) .
Compute (Lowenheim_Main ((VAR 0) × (VAR 1))).
Compute (Lowenheim_Main ((VAR 0) + (VAR 1)) ).

```


Compute (Lowenheim_Main ((VAR 0) + (VAR 1) + (VAR 2) + T1 + (VAR 3) × ((VAR 2) + (VAR 0)))).

Compute (Lowenheim_Main (T1)).

Compute (Lowenheim_Main ((VAR 0) + (VAR 0) + T1)).

*Lowenheim's functions testing

In this subsection we explore ways to test the correctness of our lowenheim's functions on specific inputs.

Here is a function to test the correctness of the output of the find_unifier helper function defined above. True means expected output was produced, False otherwise. **Definition**

```
Test_find_unifier (t : term) : bool :=
  match (find_unifier t) with
  | Some_subst s =>
    (term_is_T0 (update_term t s))
  | None_subst => true
  end.
```

Compute (Test_find_unifier (T1)).

Compute (Test_find_unifier ((VAR 0) × (VAR 1))).

Compute (Test_find_unifier ((VAR 0) + (VAR 1) + (VAR 2) + T1 + (VAR 3) × ((VAR 2) + (VAR 0)))).

Here is a function to apply Lowenheim's substitution on the term - the substitution produced by the lowenheim main function **Definition** apply_lowenheim_main (t : term) : term :=

```
match (Lowenheim_Main t) with
| Some_subst s => (apply_subst t s)
| None_subst => T1
end.
```

Compute (Lowenheim_Main ((VAR 0) × (VAR 1))).

Compute (apply_lowenheim_main ((VAR 0) × (VAR 1))).

Compute (Lowenheim_Main ((VAR 0) + (VAR 1))).

Compute (apply_lowenheim_main ((VAR 0) + (VAR 1))).

Chapter 4

Library

B_Unification.lowenheim_proof

Require Export lowenheim_formula.

Require Export EqNat.

Require Import List.

Import ListNotations.

Import Coq.Init.Tactics.

Require Export Classical_Prop.

3.1 Declarations and their lemmas useful for the proof

Definition sub_term (t : term) (t' : term) : Prop :=

$\forall (x : \text{var}),$
 $(\text{In } x (\text{term_unique_vars } t)) \rightarrow (\text{In } x (\text{term_unique_vars } t')) .$

Lemma sub_term_id :

$\forall (t : \text{term}),$
 sub_term t t.

Proof.

 intros. firstorder.

Qed.

Lemma term_vars_distr :

$\forall (t1\ t2 : \text{term}),$
 $(\text{term_vars } (t1 + t2)) = (\text{term_vars } t1) ++ (\text{term_vars } t2).$

Proof.

 intros.

 induction t2.

 - simpl. reflexivity.

 - simpl. reflexivity.

 - simpl. reflexivity.

 - simpl. reflexivity.

```

- simpl. reflexivity.
Qed.

Lemma tv_h1:
 $\forall (t1\ t2 : \text{term}) ,$ 
 $\forall (x : \text{var}),$ 
 $(\text{In } x (\text{term\_vars } t1)) \rightarrow (\text{In } x (\text{term\_vars } (t1 + t2))).$ 
Proof.
intros. induction t2.
- simpl. rewrite app_nil_r. apply H.
- simpl. rewrite app_nil_r. apply H.
- simpl. pose proof in_or_app as H1. specialize (H1 var (term_vars t1) [v] x).
firstorder.
- rewrite term_vars_distr. apply in_or_app. left. apply H.
- rewrite term_vars_distr. apply in_or_app. left. apply H.
Qed.

Lemma tv_h2:
 $\forall (t1\ t2 : \text{term}) ,$ 
 $\forall (x : \text{var}),$ 
 $(\text{In } x (\text{term\_vars } t2)) \rightarrow (\text{In } x (\text{term\_vars } (t1 + t2))).$ 
Proof.
intros. induction t1.
- simpl. apply H.
- simpl. apply H.
- simpl. pose proof in_or_app as H1. right. apply H.
- rewrite term_vars_distr. apply in_or_app. right. apply H.
- rewrite term_vars_distr. apply in_or_app. right. apply H.
Qed.

Lemma helper_2a:
 $\forall (t1\ t2\ t' : \text{term}),$ 
 $\text{sub\_term } (t1 + t2)\ t' \rightarrow \text{sub\_term } t1\ t'.$ 
Proof.
intros. unfold sub_term in *. intros. specialize (H x).
pose proof in_dup_and_non_dup as H10. unfold term_unique_vars. unfold term_unique_vars
in *.
pose proof tv_h1 as H7. apply H. specialize (H7 t1 t2 x). specialize (H10 x
(term_vars (t1 + t2))). destruct H10.
apply H1. apply H7. pose proof in_dup_and_non_dup as H10. specialize (H10 x
(term_vars t1)). destruct H10.
apply H4. apply H0.
Qed.

Lemma helper_2b:

```

$\forall (t1\ t2\ t' : \mathbf{term}),$
 $\text{sub_term } (t1 + t2)\ t' \rightarrow \text{sub_term } t2\ t'.$

Proof.

intros. unfold sub_term in *. intros. specialize (H x).

pose proof in_dup_and_non_dup as H10. unfold term_unique_vars. unfold term_unique_vars in *.

pose proof tv_h2 as H7. apply H. specialize (H7 t1 t2 x). specialize (H10 x (term_vars (t1 + t2))). destruct H10 .

apply H1. apply H7. pose proof in_dup_and_non_dup as H10. specialize (H10 x (term_vars t2)). destruct H10.

apply H4. apply H0.

Qed.

Lemma elt_in_list:

$\forall (x : \mathbf{var})\ (a : \mathbf{var})\ (l : \mathbf{list}\ \mathbf{var}),$
 $(\text{In } x\ (a :: l)) \rightarrow$
 $x = a \vee (\text{In } x\ l).$

Proof.

intros.

pose proof in_inv as H1.

specialize (H1 var a x l H).

destruct H1.

- left. symmetry in H0. apply H0.

- right. apply H0.

Qed.

Lemma elt_not_in_list:

$\forall (x : \mathbf{var})\ (a : \mathbf{var})\ (l : \mathbf{list}\ \mathbf{var}),$
 $\neg (\text{In } x\ (a :: l)) \rightarrow$
 $x \neq a \wedge \neg (\text{In } x\ l).$

Proof.

intros.

pose proof not_in_cons. specialize (H0 var x a l). destruct H0.

specialize (H0 H). apply H0.

Qed.

Lemma in_list_of_var_term_of_var:

$\forall (x : \mathbf{var}),$
 $\text{In } x\ (\text{term_unique_vars } (\mathbf{VAR } x)).$

Proof.

intros. simpl. left. intuition.

Qed.

Lemma var_in_out_list:

$\forall (x : \mathbf{var})\ (lvar : \mathbf{list}\ \mathbf{var}),$

$(\text{In } x \text{ lvar}) \vee \neg (\text{In } x \text{ lvar}).$

Proof.

intros.

pose proof *classic* as H1. specialize (H1 (In x lvar)). apply H1.

Qed.

3.2 Proof that Lownheim's algorithm unifies a given term

Lemma helper1_easy:

$\forall (x: \text{var}) (lvar : \text{list var}) (sig1 \ sig2 : \text{subst}) (s : \text{term}),$
 $(\text{In } x \text{ lvar}) \rightarrow$
 $\text{apply_subst (VAR } x) (\text{build_on_list_of_vars } lvar \ s \ sig1 \ sig2)$
 $==$
 $\text{apply_subst (VAR } x) (\text{build_on_list_of_vars (cons } x \ \text{nil}) } s \ sig1 \ sig2).$

Proof.

intros.

induction lvar.

- simpl. simpl in H. destruct H.

- apply elt_in_list in H. destruct H.

+ simpl. destruct (beq_nat a x) as [eqn:?].

{ apply beq_nat_true in Heqb. destruct (beq_nat x x) as [eqn:?].

{ rewrite H. reflexivity. }

{ apply beq_nat_false in Heqb.

{ destruct Heqb. }

{ rewrite Heqb. apply Heqb0. } }

{ simpl in IHlvar. apply IHlvar. symmetry in H. rewrite H in Heqb.

apply beq_nat_false in Heqb. destruct Heqb. intuition. }

+ destruct (beq_nat a x) as [eqn:?].

{ apply beq_nat_true in Heqb. symmetry in Heqb. rewrite Heqb in IHlvar. rewrite

Heqb.

simpl in IHlvar. simpl. destruct (beq_nat a a) as [eqn:?].

{ reflexivity. }

{ apply IHlvar. rewrite Heqb in H. apply H. } }

{ apply beq_nat_false in Heqb. simpl. destruct (beq_nat a x) as [eqn:?].

{ apply beq_nat_true in Heqb0. rewrite Heqb0 in Heqb. destruct Heqb. intuition.

}

{ simpl in IHlvar. apply IHlvar. apply H. } }

Qed.

Lemma helper_1:

$\forall (t' \ s : \text{term}) (v : \text{var}) (sig1 \ sig2 : \text{subst}),$

$\text{sub_term (VAR } v) t' \rightarrow$

$\text{apply_subst (VAR } v) (\text{build_on_list_of_vars (term_unique_vars } t') \ s \ sig1 \ sig2)$

$==$

$\text{apply_subst (VAR } v) (\text{build_on_list_of_vars (term_unique_vars (VAR } v)) \ s \ sig1 \ sig2).$

Proof.

intros. unfold sub_term in H . specialize ($H\ v$). pose *proof* in_list_of_var_term_of_var as $H3$.

specialize ($H3\ v$). specialize ($H\ H3$). pose *proof* helper1_easy as $H2$.

specialize ($H2\ v\ (\text{term_unique_vars } t')\ sig1\ sig2\ s$). apply $H2$. apply H .

Qed.

Lemma subs_distr_vars_ver2 :

$\forall (t\ t' : \text{term}) (s : \text{term}) (sig1\ sig2 : \text{subst}),$

$(\text{sub_term } t\ t') \rightarrow$

$\text{apply_subst } t\ (\text{build_on_list_of_vars } (\text{term_unique_vars } t')\ s\ sig1\ sig2)$

$==$

$(s + T1) \times (\text{apply_subst } t\ sig1) + s \times (\text{apply_subst } t\ sig2).$

Proof.

intros. generalize dependent t' . induction t .

- intros t' . repeat rewrite ground_term_cannot_subst.

+ rewrite *mul_comm* with ($x := s + T1$). rewrite *distr*. repeat rewrite *mul_T0_x*.

rewrite *mul_comm* with ($x := s$).

rewrite *mul_T0_x*. repeat rewrite *sum_x_x*. reflexivity.

+ unfold ground_term. reflexivity.

+ unfold ground_term. reflexivity.

+ unfold ground_term. reflexivity.

- intros t' . repeat rewrite ground_term_cannot_subst.

+ rewrite *mul_comm* with ($x := s + T1$). rewrite *mul_id*. rewrite *mul_comm* with ($x := s$). rewrite *mul_id*. rewrite *sum_comm* with ($x := s$).

repeat rewrite *sum_assoc*. rewrite *sum_x_x*. rewrite *sum_comm* with ($x := T1$).

rewrite *sum_id*. reflexivity.

+ unfold ground_term. reflexivity.

+ unfold ground_term. reflexivity.

+ unfold ground_term. reflexivity.

- intros. rewrite helper_1.

+ unfold term_unique_vars. unfold term_vars. unfold var_set_create_unique. unfold var_set_includes_var. unfold build_on_list_of_vars.

rewrite var_subst. reflexivity.

+ apply H .

- intros. specialize ($IHt1\ t'$). specialize ($IHt2\ t'$). repeat rewrite subst_sum_distr_opp.

rewrite $IHt1$. rewrite $IHt2$.

+ rewrite *distr*. rewrite *distr*. repeat rewrite *sum_assoc*. rewrite *sum_comm* with ($x := (s + T1) \times \text{apply_subst } t2\ sig1$)

($y := (s \times \text{apply_subst } t1\ sig2 + s \times \text{apply_subst } t2\ sig2)$). repeat rewrite *sum_assoc*.

rewrite *sum_comm* with ($x := s \times \text{apply_subst } t2\ sig2$) ($y := (s + T1) \times \text{apply_subst } t2\ sig1$).

repeat rewrite *sum_assoc*. reflexivity.

```

+ pose helper_2b as H2. specialize (H2 t1 t2 t'). apply H2. apply H.
+ pose helper_2a as H2. specialize (H2 t1 t2 t'). apply H2. apply H.
- intros. specialize (IHt1 t'). specialize (IHt2 t'). repeat rewrite subst_mul_distr_opp.
rewrite IHt1. rewrite IHt2.
+ rewrite distr. rewrite mul_comm with (y := ((s + T1) × apply_subst t2 sig1)).
  rewrite distr. rewrite mul_comm with (y := (s × apply_subst t2 sig2)). rewrite
distr.
  repeat rewrite mul_assoc. repeat rewrite mul_comm with (x := apply_subst t2
sig1).
  repeat rewrite mul_assoc.
  rewrite mul_assoc_opp with (x := (s + T1)) (y := (s + T1)) . rewrite mul_x_x.
  rewrite mul_assoc_opp with (x := (s + T1)) (y := s). rewrite mul_comm with (x :=
(s + T1)) (y := s).
  rewrite distr. rewrite mul_x_x. rewrite mul_id_sym. rewrite sum_x_x. rewrite
mul_T0_x.
  repeat rewrite mul_assoc. rewrite mul_comm with (x := apply_subst t2 sig2 ).
  repeat rewrite mul_assoc. rewrite mul_assoc_opp with (x := s ) (y := (s + T1)).
  rewrite distr. rewrite mul_x_x. rewrite mul_id_sym. rewrite sum_x_x. rewrite
mul_T0_x.
  repeat rewrite sum_assoc. rewrite sum_assoc_opp with (x := T0) (y := T0). rewrite
sum_x_x. rewrite sum_id.
  repeat rewrite mul_assoc. rewrite mul_comm with (x := apply_subst t2 sig2) (y :=
s × apply_subst t1 sig2).
  repeat rewrite mul_assoc. rewrite mul_assoc_opp with (x := s). rewrite mul_x_x.
reflexivity.
+ pose helper_2b as H2. specialize (H2 t1 t2 t'). apply H2. apply H.
+ pose helper_2a as H2. specialize (H2 t1 t2 t'). apply H2. apply H.

```

Qed.

Lemma specific_sigmas_unify:

```

∀ (t : term) (tau : subst),
  (unifier t tau) →
  (apply_subst t (build_on_list_of_vars (term_unique_vars t) t (build_id_subst (term_unique_vars
t)) tau )
  ) == T0 .
Proof.
intros.
rewrite subs_distr_vars_ver2.
- rewrite id_subst. rewrite mul_comm with (x := t + T1). rewrite distr. rewrite
mul_x_x. rewrite mul_id_sym. rewrite sum_x_x.
  rewrite sum_id.
  unfold unifier in H. rewrite H. rewrite mul_T0_x_sym. reflexivity.
- apply sub_term_id.

```

Qed.

Lemma lowenheim_unifies:

$\forall (t : \text{term}) (tau : \text{subst}),$
 $(\text{unifier } t \ tau) \rightarrow$
 $(\text{apply_subst } t (\text{build_lowenheim_subst } t \ tau)) == T0.$

Proof.

intros. unfold build_lowenheim_subst. apply specific_sigmas_unify. apply H.

Qed.

3.3 Proof that Lowenheim's algorithm produces a most general unifier

3.3.a Proof that Lowenheim's algorithm produces a reproductive unifier

Lemma lowenheim_rephrase1_easy :

$\forall (l : \text{list var}) (x : \text{var}) (sig1 : \text{subst}) (sig2 : \text{subst}) (s : \text{term}),$
 $(\text{In } x \ l) \rightarrow$
 $(\text{apply_subst } (\text{VAR } x) (\text{build_on_list_of_vars } l \ s \ sig1 \ sig2)) ==$
 $(s + T1) \times (\text{apply_subst } (\text{VAR } x) \ sig1) + s \times (\text{apply_subst } (\text{VAR } x) \ sig2).$

Proof.

intros.

induction l.

- simpl. unfold In in H. destruct H.

- apply elt_in_list in H. destruct H.

+ simpl. destruct (beq_nat a x) as [eqn:?].

{ rewrite H. reflexivity. }

{ pose proof beq_nat_false as H2. specialize (H2 a x).

specialize (H2 Heqb). intuition. symmetry in H. specialize (H2 H). inversion

H2. }

+ simpl. destruct (beq_nat a x) as [eqn:?].

{ symmetry in Heqb. pose proof beq_nat_eq as H2. specialize (H2 a x). specialize

(H2 Heqb). rewrite H2.

reflexivity. }

{ apply IHL. apply H. }

Qed.

Lemma helper_3a:

$\forall (x : \text{var}) (l : \text{list var}),$

$\text{In } x \ l \rightarrow$

$\text{apply_subst } (\text{VAR } x) (\text{build_id_subst } l) == \text{VAR } x.$

Proof.

intros. induction l.

- unfold build_id_subst. simpl. reflexivity.

- apply elt_in_list in H. destruct H.

+ simpl. destruct (beq_nat a x) as [eqn:?].

{ rewrite H. reflexivity. }


```

    { pose proof beq_nat_false as H2. specialize (H2 a x).
      specialize (H2 Heqb). intuition. symmetry in H. specialize (H2 H). inversion
H2. }
  + simpl. destruct (beq_nat a x) as [eqn:?].
    { symmetry in Heqb. pose proof beq_nat_eq as H2. specialize (H2 a x). specialize
(H2 Heqb). rewrite H2.
      reflexivity. }
    { apply IHL. apply H. }
Qed.

```

Lemma lowenheim_rephrase1 :

```

  ∀ (t : term) (tau : subst) (x : var),
  (unifier t tau) →
  (In x (term_unique_vars t)) →
  (apply_subst (VAR x) (build_lowenheim_subst t tau)) ==
  (t + T1) × (VAR x) + t × (apply_subst (VAR x) tau).

```

Proof.

intros.

```

  unfold build_lowenheim_subst. pose proof lowenheim_rephrase1_easy as H1.
  specialize (H1 (term_unique_vars t) x (build_id_subst (term_unique_vars t)) tau t).
  rewrite helper_3a in H1.

```

- apply H1. apply H0.

- apply H0.

Qed.

Lemma lowenheim_rephrase2_easy :

```

  ∀ (l : list var) (x : var) (sig1 : subst) (sig2 : subst) (s : term),
  ¬ (In x l) →
  (apply_subst (VAR x) (build_on_list_of_vars l s sig1 sig2)) ==
  (VAR x).

```

Proof.

intros. unfold not in H.

induction l.

- simpl. reflexivity.

```

- simpl. pose proof elt_not_in_list as H2. specialize (H2 x a l). unfold not in H2.
  specialize (H2 H). destruct H2.

```

```

  destruct (beq_nat a x) as [eqn:?].

```

```

  + symmetry in Heqb. apply beq_nat_eq in Heqb. symmetry in Heqb. specialize (H0
Heqb). destruct H0.

```

```

  + simpl in IHL. apply IHL. apply H1.

```

Qed.

Lemma lowenheim_rephrase2 :

```

  ∀ (t : term) (tau : subst) (x : var),

```

```

(unifier t tau) →
→ (In x (term_unique_vars t)) →
(apply_subst (VAR x) (build_lowenheim_subst t tau)) ==
(VAR x).

```

Proof.

```

intros. unfold build_lowenheim_subst. pose proof lowenheim_rephrase2_easy as H2.
specialize (H2 (term_unique_vars t) x (build_id_subst (term_unique_vars t)) tau t).
specialize (H2 H0). apply H2.
Qed.

```

Lemma lowenheim_reproductive:

```

∀ (t : term) (tau : subst),
(unifier t tau) →
reproductive_unifier t (build_lowenheim_subst t tau) .

```

Proof.

```

intros. unfold reproductive_unifier. intros.
pose proof var_in_out_list. specialize (H2 x (term_unique_vars t)). destruct H2.
{
rewrite lowenheim_rephrase1.
- rewrite subst_sum_distr_opp. rewrite subst_mul_distr_opp. rewrite subst_mul_distr_opp.
  unfold unifier in H1. rewrite H1. rewrite mul_T0_x. rewrite subst_sum_distr_opp.
  rewrite H1. rewrite ground_term_cannot_subst.
  + rewrite sum_id. rewrite mul_id. rewrite sum_comm. rewrite sum_id. reflexivity.
  + unfold ground_term. intuition.
- apply H.
- apply H2.
}
{ rewrite lowenheim_rephrase2.
  - reflexivity.
  - apply H.
  - apply H2.
}
}

```

Qed.

3.3.b lowenheim builder gives a most general unifier

Lemma lowenheim_most_general_unifier:

```

∀ (t : term) (tau : subst),
(unifier t tau) →
most_general_unifier t (build_lowenheim_subst t tau) .

```

Proof.

```

intros. apply reproductive_is_mgu. apply lowenheim_reproductive. apply H.
Qed.

```

3.4 extension to include Main function and subst_option

3.4.a utilities

Definition convert_to_subst (*so* : **subst_option**) : subst :=
 match *so* with
 | Some_subst *s* ⇒ *s*
 | None_subst ⇒ nil
 end.

Lemma empty_subst_on_term:

∀ (*t* : **term**),
 apply_subst *t* [] == *t*.

Proof.

intros. induction *t*.
 - reflexivity.
 - simpl. reflexivity.
 - simpl. reflexivity.
 - simpl. rewrite *IHt1*. rewrite *IHt2*. reflexivity.
 - simpl. rewrite *IHt1*. rewrite *IHt2*. reflexivity.

Qed.

Lemma app_subst_T0:

∀ (*t* : **term**),
 apply_subst *t* [] == T0 → *t* == T0.

Proof.

intros. rewrite empty_subst_on_term in *H*. apply *H*.

Qed.

Lemma T0_or_not_T0:

∀ (*t* : **term**),
t == T0 ∨ ¬ (*t* == T0).

Proof.

intros. pose proof *classic*. specialize (*H* (*t* == T0)). apply *H*.

Qed.

Lemma exists_subst:

∀ (*t* : **term**) (*sig* : subst),
 apply_subst *t* *sig* == T0 → ∃ *s*, apply_subst *t* *s* == T0.

Proof.

intros. ∃ *sig*. apply *H*.

Qed.

Lemma t_id_eqv :

∀ (*t* : **term**),
t == *t*.

Proof.

intros. reflexivity.

Qed.

```

Lemma eq_some_eq_subst (s1 s2: subst) :
  (Some_subst s1 = Some_subst s2) → s1 = s2.
Proof.
  intros. congruence.
Qed.

Lemma None_is_not_Some (t: term):
  (find_unifier t) = None_subst → (∀ (sig: subst), ¬ (find_unifier t) = Some_subst sig).
Proof.
  intros.
  congruence.
Qed.

Lemma Some_is_not_None (sig: subst) (t: term):
  (find_unifier t) = Some_subst sig → ¬ (find_unifier t = None_subst).
Proof.
  intros.
  congruence.
Qed.

Lemma not_None_is_Some (t: term) :
  ¬ (find_unifier t = None_subst) → ∃ sig : subst, (find_unifier t) = Some_subst sig.
Proof.
  intros H.
  destruct (find_unifier t) as [ti | ].
  - ∃ ti. firstorder.
  - congruence.
Qed.

Lemma contrapositive_opposite :
  ∀ p q, (¬p → ¬q) → q → p.
Proof.
  intros.
  apply NNPP. firstorder.
Qed.

Lemma contrapositive :
  ∀ (p q : Prop), (p → q) → (¬q → ¬p).
Proof.
  intros.
  firstorder.
Qed.

3.4.b actual final proof extension

Lemma some_subst_unifiable:
  ∀ (t : term),

```

$(\exists \text{ sig}, (\text{find_unifier } t) = \text{Some_subst sig}) \rightarrow (\text{unifiable } t).$

Proof.

```

intros.
destruct H as [sig1 H1].
induction t.
- unfold unifiable .  $\exists$  []. unfold unifier. simpl. reflexivity.
- simpl in H1. inversion H1.
- unfold unifiable.  $\exists$  sig1. unfold find_unifier in H1.
  remember (update_term (VAR v) (rec_subst (VAR v) (term_unique_vars (VAR v)) []))
in H1.
  destruct t.
  + unfold update_term in Heqt. pose proof simplify_eqv.
    specialize (H (apply_subst (VAR v) (rec_subst (VAR v) (term_unique_vars (VAR v))
[]))).
    symmetry in Heqt. apply eq_some_eq_subst in H1.
    rewrite H1 in H. rewrite H1 in Heqt.
    rewrite Heqt in H. symmetry in H. apply H.
    + simpl in H1. inversion H1.
    + inversion H1.
    + inversion H1.
    + inversion H1.
- unfold unifiable.  $\exists$  sig1. unfold find_unifier in H1.
  remember (update_term (t1 + t2) (rec_subst (t1 + t2) (term_unique_vars (t1 + t2)) []))
in H1.
  destruct t.
  + unfold update_term in Heqt. pose proof simplify_eqv.
    specialize (H (apply_subst (t1 + t2) (rec_subst (t1 + t2) (term_unique_vars (t1 + t2))
[]))).
    symmetry in Heqt. apply eq_some_eq_subst in H1.
    rewrite H1 in H. rewrite H1 in Heqt.
    rewrite Heqt in H. symmetry in H. apply H.
    + inversion H1.
    + inversion H1.
    + inversion H1.
    + inversion H1.
- unfold unifiable.  $\exists$  sig1. unfold find_unifier in H1.
  remember (update_term (t1  $\times$  t2) (rec_subst (t1  $\times$  t2) (term_unique_vars (t1  $\times$  t2)) []))
in H1.
  destruct t.
  + unfold update_term in Heqt. pose proof simplify_eqv.
    specialize (H (apply_subst (t1  $\times$  t2) (rec_subst (t1  $\times$  t2) (term_unique_vars (t1  $\times$ 
t2)) []))).

```

```

      symmetry in Heqt. apply eq_some_eq_subst in H1.
      rewrite H1 in H. rewrite H1 in Heqt.
      rewrite Heqt in H. symmetry in H. apply H.
+ inversion H1.
+ inversion H1.
+ inversion H1.
+ inversion H1.

```

Qed.

Lemma not_Some_is_None (t: term) :

($\neg \exists (sig : \text{subst}), (\text{find_unifier } t) = \text{Some_subst } sig$) \rightarrow ($\text{find_unifier } t$) = None_subst.

Proof.

```

  apply contrapositive_opposite.
  intros H.
  apply not_None_is_Some in H.
  tauto.

```

Qed.

Lemma not_unifiable_find_unifier_none_subst :

$\forall (t : \text{term}),$
 $\neg (\text{unifiable } t) \rightarrow (\text{find_unifier } t) = \text{None_subst}.$

Proof.

```

intros.
pose proof some_subst_unifiable.
specialize (H0 t).
pose proof contrapositive.
specialize (H1 (( $\exists sig : \text{subst}, \text{find\_unifier } t = \text{Some\_subst } sig$ )) ((unifiable t))).
specialize (H1 H0). specialize (H1 H).
pose proof not_Some_is_None.
specialize (H2 t H1).
apply H2.

```

Qed.

Lemma Some_subst_unifiable :

$\forall (t : \text{term}) (sig : \text{subst}),$
 $(\text{find_unifier } t) = \text{Some_subst } sig \rightarrow (\text{unifier } t \text{ } sig).$

Proof.

intros.

```

  induction t.
- simpl in H. apply eq_some_eq_subst in H. symmetry in H. rewrite H.
  unfold unifier. simpl. reflexivity.
- simpl in H. inversion H.
- unfold find_unifier in H. remember (update_term (VAR v) (rec_subst (VAR v) (term_unique_vars
(VAR v))  $\square$ )) in H.

```

```

    destruct t.
+ unfold update_term in Heqt. pose proof simplify_eqv.
  specialize (H0 (apply_subst (VAR v) (rec_subst (VAR v) (term_unique_vars (VAR v))
[]))).
    symmetry in Heqt. apply eq_some_eq_subst in H.
    rewrite H in H0. rewrite H in Heqt.
    rewrite Heqt in H0. symmetry in H0. apply H0.
+ inversion H.
+ inversion H.
+ inversion H.
+ inversion H.
- unfold find_unifier in H. remember (update_term (t1 + t2) (rec_subst (t1 + t2) (term_unique_vars
(t1 + t2)) [])) in H.
  destruct t.
+ unfold update_term in Heqt. pose proof simplify_eqv.
  specialize (H0 (apply_subst (t1 + t2) (rec_subst (t1 + t2) (term_unique_vars (t1 +
t2)) []))).
    symmetry in Heqt. apply eq_some_eq_subst in H.
    rewrite H in H0. rewrite H in Heqt.
    rewrite Heqt in H0. symmetry in H0. apply H0.
+ inversion H.
+ inversion H.
+ inversion H.
+ inversion H.
- unfold find_unifier in H. remember (update_term (t1 × t2) (rec_subst (t1 × t2) (term_unique_vars
(t1 × t2)) [])) in H.
  destruct t.
+ unfold update_term in Heqt. pose proof simplify_eqv.
  specialize (H0 (apply_subst (t1 × t2) (rec_subst (t1 × t2) (term_unique_vars (t1 ×
t2)) []))).
    symmetry in Heqt. apply eq_some_eq_subst in H.
    rewrite H in H0. rewrite H in Heqt.
    rewrite Heqt in H0. symmetry in H0. apply H0.
+ inversion H.
+ inversion H.
+ inversion H.
+ inversion H.

```

Qed.

Lemma unif_some_subst :

```

  ∀ (t: term),
  (∃ sig1, (unifier t sig1)) →
  (∃ sig2, (find_unifier t) = Some_subst sig2).

```

```

Proof.
  intros.
  destruct H as [sig1 H].
Admitted.

Lemma not_Some_not_unifiable (t: term) :
  (¬ ∃ (sig : subst), (find_unifier t) = Some_subst sig) → ¬ (unifiable t).
Proof.
  intros.
  pose proof not_Some_is_None.
  specialize (H0 t H).
  unfold unifiable.
  intro.
  unfold not in H.
  pose proof unif_some_subst.
  specialize (H2 t H1).
  specialize (H H2).
  apply H.
Qed.

Lemma unifiable_find_unifier_some_subst :
  ∀ (t : term),
    (unifiable t) → (∃ (sig : subst), (find_unifier t) = Some_subst sig).
Proof.
  intros.
  pose proof contrapositive.
  specialize (H0 (¬ ∃ (sig : subst), (find_unifier t) = Some_subst sig) (¬ (unifiable t))).
  pose proof not_Some_not_unifiable.
  specialize (H1 t). specialize (H0 H1). apply NNPP in H0.
  - apply H0.
  - firstorder.
Qed.

Lemma find_unifier_is_unifier:
  ∀ (t : term),
    (unifiable t) → (unifier t (convert_to_subst (find_unifier t))).
Proof.
  intros.
  pose proof unifiable_find_unifier_some_subst.
  specialize (H0 t H).
  unfold unifier. unfold unifiable in H. simpl. unfold convert_to_subst.
  destruct H0 as [sig H0]. rewrite H0.
  pose proof Some_subst_unifiable.
  specialize (H1 t sig). specialize (H1 H0).

```



```

  unfold unifier in H1.
  apply H1.
Qed.

Lemma builder_to_main:
  ∀ (t : term),
  ((unifiable t) → most_general_unifier t (build_lowenheim_subst t (convert_to_subst (find_unifier t)))) →
  most_general_unifier t (convert_to_subst (Lowenheim_Main t)) .
Proof.
  intros.
  pose proof lowenheim_most_general_unifier as H1. pose proof find_unifier_is_unifier as H2.
  specialize (H2 t H). specialize (H1 t (convert_to_subst (find_unifier t))).
  specialize (H1 H2). unfold Lowenheim_Main. destruct (find_unifier t).
  - simpl. simpl in H1. apply H1.
  - simpl in H2. unfold unifier in H2. apply app_subst_T0 in H2. simpl.
    repeat simpl in H1. pose proof most_general_unifier_compat.
    specialize (H3 t T0 H2). specialize (H3 []).
    rewrite H3. unfold most_general_unifier. intros.
    unfold more_general_substitution. ∃ s'. unfold substitution_composition.
    intros. simpl. reflexivity.
Qed.

Lemma lowenheim_main_most_general_unifier:
  ∀ (t: term),
  ((unifiable t) → most_general_unifier t (convert_to_subst (Lowenheim_Main t)))
  ∧
  (¬(unifiable t) → (Lowenheim_Main t) = None_subst ).
Proof.
  intros.
  split.
  - intros. apply builder_to_main.
    + apply H.
    + apply lowenheim_most_general_unifier. apply find_unifier_is_unifier. apply H.
  - intros. pose proof not_unifiable_find_unifier_none_subst.
    specialize (H0 t H). unfold Lowenheim_Main. rewrite H0. reflexivity.
Qed.

```

Chapter 5

Library B_Unification.poly

```
Require Import Arith.  
Require Import List.  
Import ListNotations.  
Require Import FunctionalExtensionality.  
Require Import Sorting.  
Require Import Permutation.  
Import Nat.  
Require Export terms.
```

5.1 Introduction

Another way of representing the terms of a unification problem is as polynomials and monomials. A monomial is a set of variables multiplied together, and a polynomial is a set of monomials added together. By following the ten axioms set forth in B-unification, we can transform any term to this form.

Since one of the rules is $x * x = x$, we can guarantee that there are no repeated variables in any given monomial. Similarly, because $x + x = 0$, we can guarantee that there are no repeated monomials in a polynomial. Because of these properties, as well as the commutativity of addition and multiplication, we can represent both monomials and polynomials as unordered sets of variables and monomials, respectively. This file serves to implement such a representation.

5.2 Monomials and Polynomials

5.2.1 Data Type Definitions

A monomial is simply a list of variables, with variables as defined in terms.v.

Definition mono := list var.

Definition mono_eq_dec := (list_eq_dec Nat.eq_dec).

A polynomial, then, is a list of monomials.

Definition poly := list mono.

5.2.2 Comparisons of monomials and polynomials

For the sake of simplicity when comparing monomials and polynomials, we have opted for a solution that maintains the lists as sorted. This allows us to simultaneously ensure that there are no duplicates, as well as easily comparing the sets with the standard Coq equals operator over lists.

Ensuring that a list of nats is sorted is easy enough. In order to compare lists of sorted lists, we'll need the help of another function:

```
Fixpoint lex {T : Type} (cmp : T → T → comparison) (l1 l2 : list T)
  : comparison :=
  match l1, l2 with
  | [], [] ⇒ Eq
  | [], _ ⇒ Lt
  | _, [] ⇒ Gt
  | h1 :: t1, h2 :: t2 ⇒
    match cmp h1 h2 with
    | Eq ⇒ lex cmp t1 t2
    | c ⇒ c
    end
  end.
```

There are some important but relatively straightforward properties of this function that are useful to prove. First, reflexivity:

Lemma lex_nat_refl : $\forall (l : \text{list nat}), \text{lex compare } l \ l = \text{Eq}$.

Proof.

```
  intros.
  induction l.
  - simpl. reflexivity.
  - simpl. rewrite compare_refl. apply IHL.
```

Qed.

Next, antisymmetry. This allows us to take a predicate or hypothesis about the comparison of two polynomials and reverse it. For example, $a < b$ implies $b > a$.

Lemma lex_nat_antisym : $\forall (l1 \ l2 : \text{list nat}),$
 $\text{lex compare } l1 \ l2 = \text{CompOpp } (\text{lex compare } l2 \ l1).$

Proof.

```
  intros l1.
  induction l1.
  - intros. simpl. destruct l2; reflexivity.
```

```

- intros. simpl. destruct l2.
+ simpl. reflexivity.
+ simpl. destruct (a ?= n) eqn:H;
  rewrite compare_antisym in H;
  rewrite CompOpp_iff in H; simpl in H;
  rewrite H; simpl.
  × apply IHl1.
  × reflexivity.
  × reflexivity.

```

Qed.

Lemma lex_eq : $\forall n m,$
 lex compare $n m = \text{Eq} \leftrightarrow n = m$.

Proof.

```

intros n. induction n; induction m; intros.
- split; reflexivity.
- split; intros; inversion H.
- split; intros; inversion H.
- split; intros; simpl in H.
  + destruct (a ?= a0) eqn:Hcomp; try inversion H. f_equal.
    × apply compare_eq_iff in Hcomp; auto.
    × apply IHn. auto.
  + inversion H. simpl. rewrite compare_refl.
    rewrite ← H2. apply IHn. reflexivity.

```

Qed.

Lemma lex_neq : $\forall n m,$
 lex compare $n m = \text{Lt} \vee \text{lex compare } n m = \text{Gt} \leftrightarrow n \neq m$.

Proof.

```

intros n. induction n; induction m.
- simpl. split; intro. inversion H; inversion H0. contradiction.
- simpl. split; intro. intro. inversion H0. auto.
- simpl. split; intro. intro. inversion H0. auto.
- clear IHm. split; intros.
  + destruct H; intro; apply lex_eq in H0; rewrite H in H0; inversion H0.
  + destruct (a ?= a0) eqn:Hcomp.
    × simpl. rewrite Hcomp. apply IHn. apply compare_eq_iff in Hcomp.
      rewrite Hcomp in H. intro. apply H. rewrite H0. reflexivity.
    × left. simpl. rewrite Hcomp. reflexivity.
    × right. simpl. rewrite Hcomp. reflexivity.

```

Qed.

Lemma lex_neq' : $\forall n m,$
 (lex compare $n m = \text{Lt} \rightarrow n \neq m$) \wedge
 (lex compare $n m = \text{Gt} \rightarrow n \neq m$).

Proof.

```
intros n m. split.
- intros. apply lex_neq. auto.
- intros. apply lex_neq. auto.
```

Qed.

Lemma lex_rev_eq : $\forall n m$,

lex compare $n m = \text{Eq}$ \leftrightarrow lex compare $m n = \text{Eq}$.

Proof.

```
intros n m. split; intro; rewrite lex_nat_antisym in H; unfold CompOpp in H.
- destruct (lex compare m n) eqn:H0; inversion H. reflexivity.
- destruct (lex compare n m) eqn:H0; inversion H. reflexivity.
```

Qed.

Lemma lex_rev_lt_gt : $\forall n m$,

lex compare $n m = \text{Lt}$ \leftrightarrow lex compare $m n = \text{Gt}$.

Proof.

```
intros n m. split; intro; rewrite lex_nat_antisym in H; unfold CompOpp in H.
- destruct (lex compare m n) eqn:H0; inversion H. reflexivity.
- destruct (lex compare n m) eqn:H0; inversion H. reflexivity.
```

Qed.

Lastly is a property over lists. The comparison of two lists stays the same if the same new element is added onto the front of each list. Similarly, if the item at the front of two lists is equal, removing it from both does not change the lists' comparison.

Lemma lex_nat_cons : $\forall (l1 l2 : \text{list nat}) n$,

lex compare $l1 l2 = \text{lex compare } (n :: l1) (n :: l2)$.

Proof.

```
intros. simpl. rewrite compare_refl. reflexivity.
```

Qed.

Hint Resolve lex_nat_refl lex_nat_antisym lex_nat_cons.

5.2.3 Stronger Definitions

Because as far as Coq is concerned any list of natural numbers is a monomial, it is necessary to define a few more predicates about monomials and polynomials to ensure our desired properties hold. Using these in proofs will prevent any random list from being used as a monomial or polynomial.

Monomials are simply sorted lists of natural numbers.

Definition is_mono ($m : \text{mono}$) : Prop := Sorted lt m .

Polynomials are sorted lists of lists, where all of the lists in the polynomial are monomials.

Definition is_poly ($p : \text{poly}$) : Prop :=

Sorted (fun $m\ n \Rightarrow \text{lex compare } m\ n = \text{Lt}$) $p \wedge \forall m, \text{In } m\ p \rightarrow \text{is_mono } m$.

Hint Unfold is_mono is_poly.

Hint Resolve NoDup_cons NoDup_nil Sorted_cons.

Definition vars ($p : \text{poly}$) : **list** var :=
nodup var_eq_dec (**concat** p).

Hint Unfold vars.

Lemma NoDup_vars : $\forall (p : \text{poly})$,
NoDup (vars p).

Proof.

intros p . unfold vars. apply **NoDup_nodup**.

Qed.

Lemma in_mono_in_vars : $\forall x\ p$,
 $(\forall m : \text{mono}, \text{In } m\ p \rightarrow \neg \text{In } x\ m) \leftrightarrow \neg \text{In } x\ (\text{vars } p)$.

Proof.

intros $x\ p$. split.
- intros H . induction p .
+ simpl. auto.
+ unfold **not** in *. intro. apply IHp .
× intros $m\ Hin$. apply H . intuition.
× unfold vars in *. apply **nodup_In** in $H0$. apply **nodup_In**. simpl in $H0$.
apply **in_app_or** in $H0$. destruct $H0$.
- exfalso. apply ($H\ a$). intuition. auto.
- auto.
- intros $H\ m\ Hin\ Hin'$. apply H . clear H . induction p .
+ inversion Hin .
+ unfold vars in *. rewrite **nodup_In**. rewrite **nodup_In** in IHp . simpl.
apply **in_or_app**. destruct Hin .
× left. rewrite H . auto.
× auto.

Qed.

There are a few useful things we can prove about these definitions too. First, every element in a monomial is guaranteed to be less than the elements after it.

Lemma mono_order : $\forall x\ y\ m$,
is_mono ($x :: y :: m$) \rightarrow
 $x < y$.

Proof.

unfold is_mono.
intros $x\ y\ m\ H$.
apply **Sorted_inv** in H as [].
apply **HdRel_inv** in $H0$.
apply $H0$.

Qed.

Similarly, if $x :: m$ is a monomial, then m is also a monomial.

```
Lemma mono_cons : ∀ x m,  
  is_mono (x :: m) →  
  is_mono m.
```

Proof.

```
  unfold is_mono.  
  intros x m H. apply Sorted_inv in H as []. apply H.
```

Qed.

The same properties hold for `is_poly` as well; any list in a polynomial is guaranteed to be less than the lists after it.

```
Lemma poly_order : ∀ m n p,  
  is_poly (m :: n :: p) →  
  lex compare m n = Lt.
```

Proof.

```
  unfold is_poly.  
  intros.  
  destruct H.  
  apply Sorted_inv in H as [].  
  apply HdRel_inv in H1.  
  apply H1.
```

Qed.

And if $m :: p$ is a polynomial, we know both that p is a polynomial and that m is a monomial.

```
Lemma poly_cons : ∀ m p,  
  is_poly (m :: p) →  
  is_poly p ∧ is_mono m.
```

Proof.

```
  unfold is_poly.  
  intros.  
  destruct H.  
  apply Sorted_inv in H as [].  
  split.  
  - split.  
    + apply H.  
    + intros. apply H0, in_cons, H2.  
  - apply H0, in_eq.
```

Qed.

Lastly, for completeness, `nil` is both a polynomial and monomial.

```
Lemma nil_is_mono :
```

```

    is_mono [].
Proof.
  unfold is_mono. auto.
Qed.

Lemma nil_is_poly :
  is_poly [].
Proof.
  unfold is_poly. split.
  - auto.
  - intro; contradiction.
Qed.

Lemma one_is_poly :
  is_poly [].
Proof.
  unfold is_poly. split.
  - auto.
  - intro. intro. simpl in H. destruct H.
    + rewrite ← H. apply nil_is_mono.
    + inversion H.
Qed.

Lemma var_is_poly : ∀ x,
  is_poly [x].
Proof.
  intros x. unfold is_poly. split.
  - apply Sorted_cons; auto.
  - intros m H. simpl in H; destruct H; inversion H.
    unfold is_mono. auto.
Qed.

Lemma no_vars_is_ground : ∀ p,
  is_poly p →
  vars p = [] →
  p = [] ∨ p = [].
Proof.
  intros p H H0. induction p.
  - auto.
  - induction a.
    + destruct IHp.
      × apply poly_cons in H. apply H.
      × unfold vars in H0. simpl in H0. apply H0.
      × rewrite H1. auto.
      × rewrite H1 in H. unfold is_poly in H. destruct H. inversion H.

```



```

      inversion H6. inversion H8.
+ unfold vars in H0. simpl in H0. destruct in_dec in H0.
  × rewrite ← nodup_In in i. rewrite H0 in i. inversion i.
  × inversion H0.

```

Qed.

Hint Resolve mono_order mono_cons poly_order poly_cons nil_is_mono nil_is_poly
var_is_poly one_is_poly.

5.3 Functions over Monomials and Polynomials

Module Import VARSORT := NATSORT.

```

Fixpoint nodup_cancel {A} Aeq_dec (l : list A) : list A :=
  match l with
  | [] => []
  | x::xs =>
    let count := (count_occ Aeq_dec xs x) in
    let xs' := (remove Aeq_dec x (nodup_cancel Aeq_dec xs)) in
    if (even count) then x::xs' else xs'
  end.

```

Lemma In_remove : $\forall \{A:\text{Type}\} \text{Aeq_dec } a \ b \ (l:\text{list } A),$
 $\text{In } a \ (\text{remove } \text{Aeq_dec } b \ l) \rightarrow \text{In } a \ l.$

Proof.

```

  intros A Aeq_dec a b l H. induction l as [|c l IHL].
- contradiction.
- destruct (Aeq_dec b c) eqn:Heq; simpl in H; rewrite Heq in H.
  + right. auto.
  + destruct H; [rewrite H; intuition | right; auto].

```

Qed.

Lemma Forall_cons_iff : $\forall (A:\text{Type}) \text{Rel } a \ (l:\text{list } A),$
 $\text{Forall } \text{Rel } (a::l) \leftrightarrow \text{Forall } \text{Rel } l \wedge \text{Rel } a.$

Proof.

```

  intros A Rel a l. split.
- intro H. split.
  + rewrite Forall_forall in H. apply Forall_forall. intros x Hin.
    apply H. intuition.
  + apply Forall_inv in H. auto.
- intros []. apply Forall_cons; auto.

```

Qed.

Lemma Forall_remove : $\forall (A:\text{Type}) \text{Aeq_dec } \text{Rel } a \ (l:\text{list } A),$
 $\text{Forall } \text{Rel } l \rightarrow \text{Forall } \text{Rel } (\text{remove } \text{Aeq_dec } a \ l).$

Proof.

```

intros A Aeq_dec Rel a l H. induction l.
- simpl. auto.
- simpl. apply Forall_cons_iff in H. destruct (Aeq_dec a a0).
  + apply IHL. apply H.
  + apply Forall_cons_iff. split.
    × apply IHL. apply H.
    × apply H.

```

Qed.

Lemma StronglySorted_remove : $\forall \{A:Type\} Aeq_dec Rel a (l:list A),$
StronglySorted Rel l \rightarrow **StronglySorted** Rel (remove Aeq_dec a l).

Proof.

```

intros A Aeq_dec Rel a l H. induction l.
- simpl. auto.
- simpl. apply StronglySorted_inv in H. destruct (Aeq_dec a a0).
  + apply IHL. apply H.
  + apply SSorted_cons.
    × apply IHL. apply H.
    × apply Forall_remove. apply H.

```

Qed.

Lemma not_in_remove : $\forall (A:Type) Aeq_dec a (l : list A),$
 $\neg In a l \rightarrow (remove Aeq_dec a l) = l.$

Proof.

```

intros A Aeq_dec a l H. induction l.
- simpl. reflexivity.
- simpl. destruct (Aeq_dec a a0).
  + simpl. rewrite e in H. exfalso. apply H. intuition.
  + rewrite IHL. reflexivity. intro Hin. apply H. intuition.

```

Qed.

Lemma remove_distr_app : $\forall (A:Type) Aeq_dec x (l l':list A),$
 remove Aeq_dec x (l ++ l') = remove Aeq_dec x l ++ remove Aeq_dec x l'.

Proof.

```

intros A Aeq_dec x l l'. induction l; intros.
- simpl. auto.
- simpl. destruct (Aeq_dec x a).
  + apply IHL.
  + simpl. f_equal. apply IHL.

```

Qed.

Lemma nodup_cancel_in : $\forall (A:Type) Aeq_dec a (l:list A),$
 In a (nodup_cancel Aeq_dec l) \rightarrow In a l.

Proof.

```

intros A Aeq_dec a l H. induction l as [|b l IHL].

```

- *contradiction.*
- `simpl in H. destruct (Aeq_dec a b).`
- + `rewrite e. intuition.`
- + `right. apply IHL. destruct (even (count_occ Aeq_dec l b)).`
- × `simpl in H. destruct H. rewrite H in n. contradiction.`
- + `apply ln_remove in H. auto.`
- × `apply ln_remove in H. auto.`

`Qed.`

Lemma `NoDup_remove` : $\forall (A:\text{Type}) \text{Aeq_dec } a (l:\text{list } A),$
 $\text{NoDup } l \rightarrow \text{NoDup } (\text{remove } \text{Aeq_dec } a \ l).$

`Proof.`

- `intros A Aeq_dec a l H. induction l.`
- `simpl. auto.`
- `simpl. destruct (Aeq_dec a a0).`
- + `apply IHL. apply NoDup_cons_iff in H. intuition.`
- + `apply NoDup_cons.`
- × `apply NoDup_cons_iff in H as []. intro. apply H.`
- + `apply (ln_remove Aeq_dec a0 a l H1).`
- × `apply IHL. apply NoDup_cons_iff in H; intuition.`

`Qed.`

Lemma `NoDup_nodup_cancel` : $\forall (A:\text{Type}) \text{Aeq_dec } (l:\text{list } A),$
 $\text{NoDup } (\text{nodup_cancel } \text{Aeq_dec } l).$

`Proof.`

- `induction l as [| a l' Hrec]; simpl.`
- `constructor.`
- `destruct (even (count_occ Aeq_dec l' a)); simpl.`
- + `apply NoDup_cons; [apply remove_ln | apply NoDup_remove; auto].`
- + `apply NoDup_remove; auto.`

`Qed.`

Lemma `no_nodup_NoDup` : $\forall (A:\text{Type}) \text{Aeq_dec } (l:\text{list } A),$
 $\text{NoDup } l \rightarrow$
 $\text{nodup } \text{Aeq_dec } l = l.$

`Proof.`

- `intros A Aeq_dec l H. induction l.`
- `auto.`
- `simpl. apply NoDup_cons_iff in H as []. destruct (in_dec Aeq_dec a l).`
- `contradiction. f_equal. auto.`

`Qed.`

Lemma `no_nodup_cancel_NoDup` : $\forall (A:\text{Type}) \text{Aeq_dec } (l:\text{list } A),$
 $\text{NoDup } l \rightarrow$
 $\text{nodup_cancel } \text{Aeq_dec } l = l.$

Proof.

```

intros A Aeq_dec l H. induction l.
- auto.
- simpl. apply NoDup_cons_iff in H as []. assert (count_occ Aeq_dec l a = 0).
  + apply count_occ_not_In. auto.
  + rewrite H1. simpl. f_equal. rewrite not_In_remove. auto. intro.
    apply nodup_cancel_in in H2. apply H. auto.

```

Qed.

Lemma Sorted_nodup : $\forall (A:\text{Type}) \text{ Aeq_dec } \text{Rel } (l:\text{list } A),$
 $\text{Relations_1.Transitive } \text{Rel} \rightarrow$
 $\text{Sorted } \text{Rel } l \rightarrow$
 $\text{Sorted } \text{Rel } (\text{nodup } \text{Aeq_dec } l).$

Proof.

```

intros A Aeq_dec Rel l Ht H. apply Sorted_StronglySorted in H; auto.
apply StronglySorted_Sorted. induction l.
- auto.
- simpl. apply StronglySorted_inv in H as []. destruct (in_dec Aeq_dec a l).
  + apply IHL. apply H.
  + apply SSorted_cons.
    × apply IHL. apply H.
    × rewrite Forall_forall in H0. apply Forall_forall. intros x Hin.
      apply H0. apply nodup_In in Hin. auto.

```

Qed.

Lemma Sorted_nodup_cancel : $\forall (A:\text{Type}) \text{ Aeq_dec } \text{Rel } (l:\text{list } A),$
 $\text{Relations_1.Transitive } \text{Rel} \rightarrow$
 $\text{Sorted } \text{Rel } l \rightarrow$
 $\text{Sorted } \text{Rel } (\text{nodup_cancel } \text{Aeq_dec } l).$

Proof.

```

intros A Aeq_dec Rel l Ht H. apply Sorted_StronglySorted in H; auto.
apply StronglySorted_Sorted. induction l.
- auto.
- simpl. apply StronglySorted_inv in H as []. destruct (even (count_occ Aeq_dec l a)).
  + apply SSorted_cons.
    × apply StronglySorted_remove. apply IHL. apply H.
    × apply Forall_remove. apply Forall_forall. rewrite Forall_forall in H0.
      intros x Hin. apply H0. apply nodup_cancel_in in Hin. auto.
  + apply StronglySorted_remove. apply IHL. apply H.

```

Qed.

Lemma count_occ_Permutation : $\forall (A:\text{Type}) \text{ Aeq_dec } a (l \text{ } l':\text{list } A),$
 $\text{Permutation } l \text{ } l' \rightarrow$
 $\text{count_occ } \text{Aeq_dec } l \text{ } a = \text{count_occ } \text{Aeq_dec } l' \text{ } a.$

Proof.

```

intros A Aeq_dec a l l' H. induction H.
- auto.
- simpl. destruct (Aeq_dec x a); auto.
- simpl. destruct (Aeq_dec y a); destruct (Aeq_dec x a); auto.
- rewrite ← IHPermutation2. rewrite IHPermutation1. auto.
Qed.

Lemma Permutation_not_In : ∀ (A:Type) a (l l':list A),
  Permutation l l' →
  ¬ In a l →
  ¬ In a l'.
Proof.
  intros A a l l' H H0. intro. apply H0. apply Permutation_sym in H.
  apply (Permutation_in a) in H; auto.
Qed.

Require Import Orders.
Module MONOORDER <: TOTALLEBOOL.
  Definition t := mono.
  Definition leb x y :=
    match lex compare x y with
    | Lt ⇒ true
    | Eq ⇒ true
    | Gt ⇒ false
    end.
  Infix "<=m" := leb (at level 35).
  Lemma leb_total : ∀ a1 a2, (a1 ≤m a2 = true) ∨ (a2 ≤m a1 = true).
  Proof.
    intros n m. unfold "<=m". destruct (lex compare n m) eqn:Hcomp; auto.
    apply lex_rev_lt_gt in Hcomp. rewrite Hcomp. auto.
  Qed.
End MONOORDER.

Module Import MONOSORT := SORT MONOORDER.
Lemma Permutation_MonoSort_r : ∀ p q,
  Permutation p q ↔ Permutation p (sort q).
Proof.
  intros p q. split; intro H.
  - apply Permutation_trans with (l':=q). apply H. apply Permuted_sort.
  - apply Permutation_trans with (l':=(sort q)). apply H. apply Permutation_sym.
    apply Permuted_sort.
Qed.

Lemma Permutation_MonoSort_l : ∀ p q,
  Permutation p q ↔ Permutation (sort p) q.

```

Proof.

```
intros p q. split; intro H.  
- apply Permutation_sym. rewrite ← Permutation_MonoSort_r.  
  apply Permutation_sym. auto.  
- apply Permutation_sym. rewrite Permutation_MonoSort_r.  
  apply Permutation_sym. auto.
```

Qed.

Lemma VarOrder_Transitive :

Relations_1.Transitive (fun x y : **nat** ⇒ **is_true** (**NatOrder.leb** x y)).

Proof.

```
unfold Relations_1.Transitive, is_true.  
induction x, y, z; intros; try reflexivity; simpl in *.  
- inversion H.  
- inversion H.  
- inversion H0.  
- apply IHx with (y:=y); auto.
```

Qed.

Lemma MonoOrder_Transitive :

Relations_1.Transitive (fun x y : **list nat** ⇒ **is_true** (**MonoOrder.leb** x y)).

Proof.

```
unfold Relations_1.Transitive, is_true, MonoOrder.leb.  
induction x, y, z; intros; try reflexivity; simpl in *.  
- inversion H.  
- inversion H.  
- inversion H0.  
- destruct (a ?= n) eqn:Han.  
  + apply compare_eq_iff in Han. rewrite Han. destruct (n ?= n0) eqn:Hn0.  
    × apply (IHx - - H H0).  
    × reflexivity.  
    × inversion H0.  
  + destruct (n ?= n0) eqn:Hn0.  
    × apply compare_eq_iff in Hn0. rewrite ← Hn0. rewrite Han. reflexivity.  
    × apply compare_lt_iff in Han. apply compare_lt_iff in Hn0.  
      apply (lt_trans a n n0 Han) in Hn0. apply compare_lt_iff in Hn0.  
      rewrite Hn0. reflexivity.  
    × inversion H0.  
  + inversion H.
```

Qed.

Lemma lex_Lt_Transitive :

Relations_1.Transitive (fun x y : **list nat** ⇒ **lex compare** x y = **Lt**).

Proof.

```
unfold Relations_1.Transitive, is_true.
```

```

induction x, y, z; intros; try reflexivity; simpl in *.
- inversion H.
- inversion H0.
- inversion H0.
- inversion H.
- inversion H0.
- destruct (a ?= n0) eqn:Han0.
  + apply compare_eq_iff in Han0. rewrite Han0 in H. destruct (n ?= n0) eqn:Hn0.
    × rewrite compare_antisym in Hn0. unfold CompOpp in Hn0.
      destruct (n0?=n); try inversion Hn0. apply (IHx - _ H H0).
    × rewrite compare_antisym in Hn0. unfold CompOpp in Hn0.
      destruct (n0?=n); try inversion Hn0. inversion H.
    × inversion H0.
  + auto.
  + destruct (n ?= n0) eqn:Hnn0.
    × apply compare_eq_iff in Hnn0. rewrite Hnn0 in H. rewrite Han0 in H.
      inversion H.
    × apply compare_lt_iff in Hnn0. apply compare_gt_iff in Han0.
      apply lt_trans with (n:=n) in Han0; auto. apply compare_lt_iff in Han0.
      rewrite compare_antisym in Han0. unfold CompOpp in Han0.
      destruct (a?=n); try inversion Han0. inversion H.
    × inversion H0.

```

Qed.

Lemma Lt_Transitive :

Relations_1.Transitive Lt.

Proof.

unfold **Relations_1.Transitive**. intros. apply **lt_trans** with (m:=y); auto.

Qed.

Lemma NoDup_neq : $\forall \{X:\text{Type}\} (m : \text{list } X) a b,$

NoDup (a :: b :: m) \rightarrow
a \neq b.

Proof.

intros X m a b Hdup. apply **NoDup_cons_iff** in Hdup as [].
apply **NoDup_cons_iff** in H0 as []. intro. apply H. simpl. auto.

Qed.

Lemma HdRel_le_lt : $\forall a m,$

HdRel (fun n m \Rightarrow **is_true** (leb n m)) a m \wedge **NoDup** (a :: m) \rightarrow **HdRel Lt** a m.

Proof.

intros a m []. remember (fun n m \Rightarrow **is_true** (leb n m)) as le.
destruct m.
- apply **HdRel_nil**.
- apply **HdRel_cons**. apply **HdRel_inv** in H.

```

    apply (NoDup_neq _ a n) in H0; intuition. rewrite Heqle in H.
    unfold is_true in H. apply leb_le in H. destruct (a ?= n) eqn:Hcomp.
    + apply compare_eq_iff in Hcomp. contradiction.
    + apply compare_lt_iff in Hcomp. apply Hcomp.
    + apply compare_gt_iff in Hcomp. apply leb_correct_conv in Hcomp.
      apply leb_correct in H. rewrite H in Hcomp. inversion Hcomp.

```

Qed.

Lemma VarSort_Sorted : $\forall (m : \text{mono}),$

Sorted (fun n m \Rightarrow is_true (leb n m)) m \wedge NoDup m \rightarrow Sorted lt m.

Proof.

```

    intros m []. remember (fun n m  $\Rightarrow$  is_true (leb n m)) as le.
    induction m.
    - apply Sorted_nil.
    - apply Sorted_inv in H. apply Sorted_cons.
      + apply IHm.
      × apply H.
      × apply NoDup_cons_iff in H0. apply H0.
    + apply HdRel_le_lt. split.
      × rewrite  $\leftarrow$  Heqle. apply H.
      × apply H0.

```

Qed.

Lemma Sorted_VarSorted : $\forall (m : \text{mono}),$

Sorted lt m \rightarrow

Sorted (fun n m \Rightarrow is_true (leb n m)) m.

Proof.

```

    intros m H. induction H.
    - apply Sorted_nil.
    - apply Sorted_cons.
      + apply IHSorted.
      + destruct l.
        × apply HdRel_nil.
        × apply HdRel_cons. apply HdRel_inv in H0. apply lt_le_incl in H0.
          apply leb_le in H0. apply H0.

```

Qed.

Lemma In_sorted : $\forall a l,$

In a l \leftrightarrow In a (sort l).

Proof.

```

    intros a l. pose (MonoSort.Permuted_sort l). split; intros Hin.
    - apply (Permutation_in _ p Hin).
    - apply (Permutation_in' (Logic.eq_refl a) p). auto.

```

Qed.

Lemma HdRel_mono_le_lt : $\forall a p$,
HdRel (fun $n m \Rightarrow$ **is_true** (MonoOrder.leb $n m$)) $a p \wedge$ **NoDup** ($a :: p$) \rightarrow
HdRel (fun $n m \Rightarrow$ lex **compare** $n m =$ Lt) $a p$.

Proof.

```

intros a p []. remember (fun n m  $\Rightarrow$  is_true (MonoOrder.leb n m)) as le.
destruct p.
- apply HdRel_nil.
- apply HdRel_cons. apply HdRel_inv in H.
  apply (NoDup_neq _ a l) in H0; intuition. rewrite Heql in H.
  unfold is_true in H. unfold MonoOrder.leb in H.
  destruct (lex compare a l) eqn:Hcomp.
  + apply lex_eq in Hcomp. contradiction.
  + reflexivity.
  + inversion H.

```

Qed.

Lemma MonoSort_Sorted : $\forall (p : \text{poly})$,
Sorted (fun $n m \Rightarrow$ **is_true** (MonoOrder.leb $n m$)) $p \wedge$ **NoDup** $p \rightarrow$
Sorted (fun $n m \Rightarrow$ lex **compare** $n m =$ Lt) p .

Proof.

```

intros p []. remember (fun n m  $\Rightarrow$  is_true (MonoOrder.leb n m)) as le.
induction p.
- apply Sorted_nil.
- apply Sorted_inv in H. apply Sorted_cons.
  + apply IHp.
  × apply H.
  × apply NoDup_cons_iff in H0. apply H0.
+ apply HdRel_mono_le_lt. split.
  × rewrite  $\leftarrow$  Heql. apply H.
  × apply H0.

```

Qed.

Lemma Sorted_MonoSorted : $\forall (p : \text{poly})$,
Sorted (fun $n m \Rightarrow$ lex **compare** $n m =$ Lt) $p \rightarrow$
Sorted (fun $n m \Rightarrow$ **is_true** (MonoOrder.leb $n m$)) p .

Proof.

```

intros p H. induction H.
- apply Sorted_nil.
- apply Sorted_cons.
  + apply IHSorted.
  + destruct l.
    × apply HdRel_nil.
    × apply HdRel_cons. apply HdRel_inv in H0. unfold MonoOrder.leb.
      rewrite H0. auto.

```

Qed.

Lemma NoDup_forall_neq : $\forall (A:\text{Type})\ a\ (l:\text{list } A),$
 Forall (fun b $\Rightarrow a \neq b$) l \rightarrow
 NoDup l \rightarrow
 NoDup (a :: l).

Proof.

```
intros A a l Hf Hn. apply NoDup_cons.
- intro. induction l.
  + inversion H.
  + apply Forall_cons_iff in Hf as []. apply IHL.
    × apply H0.
    × apply NoDup_cons_iff in Hn. apply Hn.
    × simpl in H. destruct H; auto. rewrite H in H1. contradiction.
- auto.
```

Qed.

Lemma NoDup_MonoSorted : $\forall (p : \text{poly}),$
 Sorted (fun n m $\Rightarrow \text{lex compare } n\ m = \text{Lt}$) p \rightarrow
 NoDup p.

Proof.

```
intros p H. apply Sorted_StronglySorted in H.
- induction p.
  + auto.
  + apply StronglySorted_inv in H as []. apply NoDup_forall_neq.
    × apply Forall_forall. intros x Hin. rewrite Forall_forall in H0.
      pose (lex_neq' a x). destruct a0. apply H1 in H0; auto.
    × apply IHp. apply H.
- apply lex_Lt_Transitive.
```

Qed.

Lemma NoDup_VarSorted : $\forall (m : \text{mono}),$
 Sorted lt m \rightarrow **NoDup** m.

Proof.

```
intros p H. apply Sorted_StronglySorted in H.
- induction p.
  + auto.
  + apply StronglySorted_inv in H as []. apply NoDup_forall_neq.
    × apply Forall_forall. intros x Hin. rewrite Forall_forall in H0.
      apply lt_neq. apply H0. apply Hin.
    × apply IHp. apply H.
- apply Lt_Transitive.
```

Qed.

Lemma NoDup_VarSort : $\forall (m : \text{mono}),$

```

NoDup  $m \rightarrow \text{NoDup}$  (VarSort.sort  $m$ ).
Proof.
  intros  $m$   $Hdup$ . pose (VarSort.Permuted_sort  $m$ ).
  apply (Permutation_NoDup  $p$   $Hdup$ ).
Qed.

Lemma NoDup_MonoSort :  $\forall (p : \text{poly})$ ,
  NoDup  $p \rightarrow \text{NoDup}$  (MonoSort.sort  $p$ ).
Proof.
  intros  $p$   $Hdup$ . pose (MonoSort.Permuted_sort  $p$ ).
  apply (Permutation_NoDup  $p0$   $Hdup$ ).
Qed.

Definition make_mono ( $l : \text{list nat}$ ) : mono :=
  VarSort.sort (nodup var_eq_dec  $l$ ).

Definition make_poly ( $l : \text{list mono}$ ) : poly :=
  MonoSort.sort (nodup_cancel mono_eq_dec (map make_mono  $l$ )).

Lemma make_mono_is_mono :  $\forall m$ ,
  is_mono (make_mono  $m$ ).
Proof.
  intros  $m$ . unfold is_mono, make_mono. apply VarSort_Sorted. split.
  + apply VarSort.LocallySorted_sort.
  + apply NoDup_VarSort. apply NoDup_nodup.
Qed.

Lemma make_poly_is_poly :  $\forall p$ ,
  is_poly (make_poly  $p$ ).
Proof.
  intros  $p$ . unfold is_poly, make_poly. split.
  - apply MonoSort_Sorted. split.
    + apply MonoSort.LocallySorted_sort.
    + apply NoDup_MonoSort. apply NoDup_nodup_cancel.
  - intros  $m$   $Hm$ . apply ln_sorted in  $Hm$ . apply nodup_cancel_in in  $Hm$ .
    apply in_map_iff in  $Hm$ . destruct  $Hm$ . destruct  $H$ . rewrite  $\leftarrow H$ .
    apply make_mono_is_mono.
Qed.

Hint Resolve make_poly_is_poly make_mono_is_mono.

Lemma make_mono_ln :  $\forall x m$ ,
  ln  $x$  (make_mono  $m$ )  $\leftrightarrow$  ln  $x$   $m$ .
Proof.
  intros  $x$   $m$ . split; intro  $H$ .
  - unfold make_mono in  $H$ . pose (VarSort.Permuted_sort (nodup var_eq_dec  $m$ )).
    apply Permutation_sym in  $p$ . apply (Permutation_in  $p$ ) in  $H$ . apply nodup_ln in  $H$ .
  auto.

```

```

- unfold make_mono. pose (VarSort.Permuted_sort (nodup var_eq_dec m)).
  apply Permutation_in with (l:=(nodup var_eq_dec m)); auto. apply nodup_In. auto.
Qed.

Lemma remove_is_mono :  $\forall x m,$ 
  is_mono  $m \rightarrow$ 
  is_mono (remove var_eq_dec  $x m$ ).
Proof.
  intros  $x m H$ . unfold is_mono in *. apply StronglySorted_Sorted.
  apply StronglySorted_remove. apply Sorted_StronglySorted in  $H$ . auto.
  apply Lt_Transitive.
Qed.

Definition addPP ( $p q$  : poly) : poly :=
  make_poly ( $p ++ q$ ).

Definition distribute { $A$ } ( $l m$  : list (list  $A$ )) : list (list  $A$ ) :=
  concat (map (fun  $a$ :(list  $A$ )  $\Rightarrow$  (map (app  $a$ )  $l$ ))  $m$ ).

Lemma distribute_nil :  $\forall \{A:\text{Type}\} (p:\text{list (list } A)),$ 
  distribute []  $p = []$ .
Proof.
  intros  $A p$ . induction  $p$ .
  - auto.
  - unfold distribute in *. simpl in *. auto.
Qed.

Lemma distribute_nil_r :  $\forall \{A:\text{Type}\} (p:\text{list (list } A)),$ 
  distribute  $p [] = []$ .
Proof.
  intros  $A p$ . induction  $p$ .
  - auto.
  - unfold distribute in *. simpl in *. auto.
Qed.

Lemma distribute_one :  $\forall \{A:\text{Type}\} (p:\text{list (list } A)),$ 
  distribute  $p [[]] = p$ .
Proof.
  intros  $A p$ . induction  $p$ .
  - auto.
  - unfold distribute in *. simpl in *. rewrite map_id. rewrite app_nil_r.
    auto.
Qed.

Lemma distribute_one_r :  $\forall \{A:\text{Type}\} (p:\text{list (list } A)),$ 
  distribute [[]]  $p = p$ .
Proof.
  intros  $A p$ . induction  $p$ .

```

- auto.
 - unfold distribute in *. simpl in *. rewrite **app_nil_r**. f_equal.
 apply *IHp*.

Qed.

Lemma Permutation_incl : $\forall \{A\} (l\ m : \text{list } A),$

Permutation $l\ m \rightarrow \text{incl } l\ m \wedge \text{incl } m\ l.$

Proof.

intros *A l m H*. apply **Permutation_sym** in *H* as *H0*. split.
 + unfold **incl**. intros *a*. apply (**Permutation_in** _ *H*).
 + unfold **incl**. intros *a*. apply (**Permutation_in** _ *H0*).

Qed.

Lemma incl_cons_inv : $\forall (A:\text{Type}) (a:A) (l\ m : \text{list } A),$

incl $(a :: l)\ m \rightarrow \text{In } a\ m \wedge \text{incl } l\ m.$

Proof.

intros *A a l m H*. split.
 - unfold **incl** in *H*. apply *H*. intuition.
 - unfold **incl** in *. intros *b Hin*. apply *H*. intuition.

Qed.

Lemma Permutation_concat : $\forall \{A\} (l\ m:\text{list } (\text{list } A)),$

Permutation $l\ m \rightarrow$
Permutation $(\text{concat } l)\ (\text{concat } m).$

Proof.

intros *A l m H*. induction *H*.
 - auto.
 - simpl. apply **Permutation_app_head**. auto.
 - simpl. apply **Permutation_trans** with (*l'* := (**concat** *l* ++ *y* ++ *x*)).
 + rewrite **app_assoc**. apply **Permutation_app_comm**.
 + apply **Permutation_trans** with (*l'* := (**concat** *l* ++ *x* ++ *y*)).
 × apply **Permutation_app_head**. apply **Permutation_app_comm**.
 × rewrite (**app_assoc** *x y*). apply **Permutation_app_comm**.
 - apply **Permutation_trans** with (*l'* := (**concat** *l'*)); auto.

Qed.

Lemma In_concat_exists : $\forall (A:\text{Type})\ ll\ (a:A),$

$(\exists l, \text{In } l\ ll \wedge \text{In } a\ l) \leftrightarrow \text{In } a\ (\text{concat } ll).$

Proof.

intros *A ll a*. split; intros *H*.
 - destruct *H* as [*l*]. apply **In_split** in *H*. destruct *H* as [*l1* [*l2 H*]].
 rewrite *H*. apply **Permutation_in** with (*l* := (**concat** (*l* :: *l1* ++ *l2*))).
 + apply Permutation_concat. apply **Permutation_middle**.
 + simpl. apply **in_app_iff**. auto.
 - induction *ll*.

```

+ inversion H.
+ simpl in H. apply in_app_iff in H. destruct H.
  × ∃ a0. split; intuition.
  × destruct IHl; auto. ∃ x. intuition.

```

Qed.

Lemma ln_distribute : $\forall (l\ m:\text{poly})\ a,$
 $\text{In } a\ (\text{vars } (\text{distribute } l\ m)) \rightarrow$
 $\text{In } a\ (\text{vars } l) \vee \text{In } a\ (\text{vars } m).$

Proof.

```

intros l m a H. unfold distribute, vars in H. apply nodup_ln in H.
apply ln_concat_exists in H. destruct H as [ll[]].
apply ln_concat_exists in H. destruct H as [ll1[]].
apply in_map_iff in H. destruct H as [x[]]. rewrite ← H in H1.
apply in_map_iff in H1. destruct H1 as [x0[]]. rewrite ← H1 in H0.
apply in_app_iff in H0. destruct H0.
- right. apply nodup_ln. apply ln_concat_exists. ∃ x. auto.
- left. apply nodup_ln. apply ln_concat_exists. ∃ x0. auto.

```

Qed.

Lemma concat_map : $\forall \{A\ B:\text{Type}\}\ (f:A \rightarrow B)\ (l:\text{list } A),$
 $\text{concat } (\text{map } (\text{fun } a \Rightarrow [f\ a])\ l) = \text{map } f\ l.$

Proof.

```

intros A B f l. induction l.
- auto.
- simpl. f_equal. apply IHl.

```

Qed.

Definition mulPP (p q : poly) : poly :=
 make_poly (distribute p q).

Definition mulMP (p : poly) (m : mono) : poly :=
 map (app m) p.

Definition mulPP' (p q : poly) : poly :=
 make_poly (concat (map (mulMP p) q)).

Definition mulMP' (p : poly) (m : mono) : poly :=
 map make_mono (map (app m) p).

Definition mulPP'' (p q : poly) : poly :=
 make_poly (concat (map (mulMP' p) q)).

Definition mulMP'' (p : poly) (m : mono) : poly :=
 make_poly (map (app m) p).

Definition mulPP''' (p q : poly) : poly :=
 make_poly (concat (map (mulMP'' p) q)).

Lemma mulPP_mulPP' : $\forall (p\ q : \text{poly}),$

```

    mulPP p q = mulPP' p q.
Proof.
  intros p q. unfold mulPP, mulPP'. induction q.
  - auto.
  - simpl. unfold distribute. simpl. unfold mulMP. auto.
Qed.

Lemma mulPP'''_refold : ∀ p q,
  make_poly (concat (map (mulMP'' p) q)) =
  mulPP''' p q.
Proof.
  auto.
Qed.

Lemma mulPP''_refold : ∀ p q,
  make_poly (concat (map (mulMP' p) q)) =
  mulPP'' p q.
Proof.
  auto.
Qed.

Lemma mulPP'_refold : ∀ p q,
  make_poly (concat (map (mulMP p) q)) =
  mulPP' p q.
Proof.
  auto.
Qed.

Lemma addPP_refold : ∀ p q,
  make_poly (p ++ q) = addPP p q.
Proof.
  auto.
Qed.

Lemma addPP_is_poly : ∀ p q,
  is_poly (addPP p q).
Proof.
  intros p q. apply make_poly_is_poly.
Qed.

Lemma leb_both_eq : ∀ x y,
  is_true (MonoOrder.leb x y) →
  is_true (MonoOrder.leb y x) →
  x = y.
Proof.
  intros x y H H0. unfold is_true, MonoOrder.leb in *.
  destruct (lex compare y x) eqn:Hyx; destruct (lex compare x y) eqn:Hxy;

```

```

try (apply lex_rev_lt_gt in Hxy; rewrite Hxy in Hyx; inversion Hyx);
try (apply lex_rev_lt_gt in Hyx; rewrite Hxy in Hyx; inversion Hyx);
try inversion H; try inversion H0.
apply lex_eq in Hxy; auto.

```

Qed.

Lemma Forall_In : $\forall (A:\text{Type}) (l:\text{list } A) a \text{ Rel},$
 $\text{In } a \ l \rightarrow \text{Forall } \text{Rel } l \rightarrow \text{Rel } a.$

Proof.

```

intros A l a Rel Hin Hfor. apply (Forall_forall Rel l); auto.

```

Qed.

Lemma Permutation_Sorted_mono_eq : $\forall (m \ n : \text{mono}),$

```

Permutation m n  $\rightarrow$ 
Sorted (fun n m  $\Rightarrow$  is_true (leb n m)) m  $\rightarrow$ 
Sorted (fun n m  $\Rightarrow$  is_true (leb n m)) n  $\rightarrow$ 
m = n.

```

Proof.

```

intros m n Hp Hsl Hsm. generalize dependent n.
induction m; induction n; intros.
- reflexivity.
- apply Permutation_nil in Hp. auto.
- apply Permutation_sym, Permutation_nil in Hp. auto.
- clear IHn. apply Permutation_incl in Hp as Hp'. destruct Hp'.
  destruct (a ?= a0) eqn:Hcomp.
  + apply compare_eq_iff in Hcomp. rewrite Hcomp in *.
    apply Permutation_cons_inv in Hp. f_equal; auto.
    apply IHm.
     $\times$  apply Sorted_inv in Hsl. apply Hsl.
     $\times$  apply Hp.
     $\times$  apply Sorted_inv in Hsm. apply Hsm.
  + apply compare_lt_iff in Hcomp as Hneq. apply incl_cons_inv in H. destruct H.
    apply Sorted_StronglySorted in Hsm. apply StronglySorted_inv in Hsm as [].
     $\times$  simpl in H. destruct H; try (rewrite H in Hneq; apply lt_irrefl in Hneq;
contradiction).
    pose (Forall_In _ _ _ H H3). simpl in i. unfold is_true in i.
    apply leb_le in i. apply lt_not_le in Hneq. contradiction.
     $\times$  apply VarOrder_Transitive.
  + apply compare_gt_iff in Hcomp as Hneq. apply incl_cons_inv in H0. destruct H0.
    apply Sorted_StronglySorted in Hsl. apply StronglySorted_inv in Hsl as [].
     $\times$  simpl in H0. destruct H0; try (rewrite H0 in Hneq; apply gt_irrefl in Hneq;
contradiction).
    pose (Forall_In _ _ _ H0 H3). simpl in i. unfold is_true in i.
    apply leb_le in i. apply lt_not_le in Hneq. contradiction.

```


\times apply VarOrder_Transitive.
 Qed.
 Lemma Permutation_sort_mono_eq : $\forall (l\ m:\text{mono})$,
 Permutation $l\ m \leftrightarrow \text{VarSort.sort } l = \text{VarSort.sort } m$.
 Proof.
 intros $l\ m$. split; intros H .
 - assert ($H0 : \text{Permutation } (\text{VarSort.sort } l) (\text{VarSort.sort } m)$).
 + apply **Permutation_trans** with ($l := (\text{VarSort.sort } l)$) ($l' := m$) ($l'' := (\text{VarSort.sort } m)$).
 \times apply **Permutation_sym**. apply **Permutation_sym** in H .
 apply (**Permutation_trans** H ($\text{VarSort.Permuted_sort } l$)).
 \times apply VarSort.Permuted_sort.
 + apply ($\text{Permutation_Sorted_mono_eq} _ _ H0$ ($\text{VarSort.LocallySorted_sort } l$) ($\text{VarSort.LocallySorted_sort } m$)).
 - assert (**Permutation** ($\text{VarSort.sort } l$) ($\text{VarSort.sort } m$)).
 + rewrite H . apply **Permutation_refl**.
 + pose ($\text{VarSort.Permuted_sort } l$). pose ($\text{VarSort.Permuted_sort } m$).
 apply (**Permutation_trans** p) in $H0$. apply **Permutation_sym** in $p0$.
 apply (**Permutation_trans** $H0$) in $p0$. apply $p0$.
 Qed.
 Lemma no_sort_VarSorted : $\forall m$,
 Sorted $lt\ m \rightarrow$
 $\text{VarSort.sort } m = m$.
 Proof.
 intros $m\ H$. apply Permutation_Sorted_mono_eq.
 - apply **Permutation_sym**. apply VarSort.Permuted_sort.
 - apply VarSort.LocallySorted_sort.
 - apply Sorted_VarSorted. auto.
 Qed.
 Lemma no_make_mono : $\forall m$,
 is_mono $m \rightarrow$
 make_mono $m = m$.
 Proof.
 unfold make_mono, is_mono. intros $m\ H$. rewrite no_sort_VarSorted.
 - apply no_nodup_NoDup. apply NoDup_VarSorted in H . auto.
 - apply Sorted_nodup.
 + apply Lt_Transitive.
 + auto.
 Qed.
 Lemma no_map_make_mono : $\forall p$,
 ($\forall m, \text{In } m\ p \rightarrow \text{is_mono } m$) \rightarrow
 map make_mono $p = p$.

Proof.

```

intros p H. induction p.
- auto.
- simpl. rewrite no_make_mono.
  + f_equal. apply IHp. intros m Hin. apply H. intuition.
  + apply H. intuition.

```

Qed.

Lemma map_make_mono_pointless : $\forall p q$,
 make_poly (map make_mono p ++ q) =
 make_poly (p ++ q).

Proof.

```

intros p q. destruct p.
- auto.
- simpl. unfold make_poly. simpl map. rewrite (no_make_mono (make_mono l)); auto.
  rewrite map_app. rewrite map_app. rewrite (no_map_make_mono (map _ _)).
  auto. intros m Hin. apply in_map_iff in Hin. destruct Hin as [x[]].
  rewrite ← H. auto.

```

Qed.

Lemma unsorted_poly : $\forall p$,
 NoDup p \rightarrow
 ($\forall m$, In m p \rightarrow is_mono m) \rightarrow
 nodup_cancel mono_eq_dec (map make_mono p) = p.

Proof.

```

intros p Hdup Hin. rewrite no_map_make_mono; auto.
apply no_nodup_cancel_NoDup; auto.

```

Qed.

Lemma remove_Permutation : $\forall (A:\text{Type}) \text{Aeq_dec } a (l l':\text{list } A)$,
 Permutation l l' \rightarrow
 Permutation (remove Aeq_dec a l) (remove Aeq_dec a l').

Proof.

```

intros A Aeq_dec a l l' H. induction H.
- auto.
- simpl. destruct (Aeq_dec a x); auto.
- simpl. destruct (Aeq_dec a y); destruct (Aeq_dec a x); auto.
  apply perm_swap.
- apply Permutation_trans with (l':=(remove Aeq_dec a l')); auto.

```

Qed.

Lemma remove_remove : $\forall \{A:\text{Type}\} \text{Aeq_dec } (a b:A) p$,
 remove Aeq_dec a (remove Aeq_dec b p) =
 remove Aeq_dec b (remove Aeq_dec a p).

Proof.

```

intros A Aeq_dec a b p. induction p as [|c]; simpl; auto.
destruct (Aeq_dec a b); destruct (Aeq_dec b c); destruct (Aeq_dec a c).
- auto.
- rewrite ← e0 in n. rewrite e in n. contradiction.
- rewrite ← e in n. rewrite e0 in n. contradiction.
- simpl. destruct (Aeq_dec a c); try contradiction.
  destruct (Aeq_dec b c); try contradiction. rewrite IHp. auto.
- rewrite e in n. rewrite e0 in n. contradiction.
- simpl. destruct (Aeq_dec b c); try contradiction. auto.
- simpl. destruct (Aeq_dec a c); try contradiction. auto.
- simpl. destruct (Aeq_dec a c); try contradiction.
  destruct (Aeq_dec b c); try contradiction. rewrite IHp. auto.

```

Qed.

Lemma nodup_cancel_Permutation : $\forall (A:\text{Type}) \text{Aeq_dec } (l \text{ } l':\text{list } A),$
Permutation $l \text{ } l' \rightarrow$
Permutation (nodup_cancel Aeq_dec l) (nodup_cancel Aeq_dec l').

Proof.

```

intros A Aeq_dec l l' H. induction H.
- auto.
- simpl. destruct even eqn:Hevn.
  + rewrite (count_occ_Permutation _ _ _ _ H) in Hevn. rewrite Hevn.
    apply perm_skip. apply remove_Permutation. apply IHPermutation.
  + rewrite (count_occ_Permutation _ _ _ _ H) in Hevn. rewrite Hevn.
    apply remove_Permutation. apply IHPermutation.
- simpl. destruct (even (count_occ Aeq_dec l x)) eqn:Hevx;
  destruct (even (count_occ Aeq_dec l y)) eqn:Hevy; destruct (Aeq_dec x y).
  + rewrite even_succ. rewrite ← negb_odd in Hevy. rewrite Bool.negb_true_iff in

```

Hevy.

```

  rewrite Hevy. destruct (Aeq_dec y x); try (rewrite e in n; contradiction).
  rewrite even_succ. rewrite ← negb_odd in Hevx. rewrite Bool.negb_true_iff in

```

Hevx.

```

  rewrite Hevx. simpl. destruct (Aeq_dec y x); try contradiction.
  destruct (Aeq_dec x y); try contradiction. rewrite remove_remove. auto.
+ rewrite Hevy. simpl. destruct (Aeq_dec y x); try (symmetry in e; contradiction).
  destruct (Aeq_dec x y); try contradiction. rewrite Hevx.
  rewrite remove_remove. apply perm_swap.
+ rewrite ← e in Hevy. rewrite Hevy in Hevx. inversion Hevx.
+ rewrite Hevy. simpl. destruct (Aeq_dec y x); try (symmetry in e; contradiction).
  rewrite Hevx. apply perm_skip. rewrite remove_remove. auto.
+ rewrite e in Hevx. rewrite Hevx in Hevy. inversion Hevy.
+ rewrite Hevy. destruct (Aeq_dec y x); try (symmetry in e; contradiction).
  rewrite Hevx. simpl. destruct (Aeq_dec x y); try contradiction.

```

```

    apply perm_skip. rewrite remove_remove. auto.
+ rewrite even_succ. rewrite ← negb_odd in Hevy. rewrite Bool.negb_false_iff in
Hevy.
    rewrite Hevy. symmetry in e. destruct (Aeq_dec y x); try contradiction.
    rewrite even_succ. rewrite ← negb_odd in Hevx. rewrite Bool.negb_false_iff in
Hevx.
    rewrite Hevx. rewrite e. auto.
+ rewrite Hevy. destruct (Aeq_dec y x); try (symmetry in e; contradiction).
    rewrite Hevx. rewrite remove_remove. auto.
- apply Permutation_trans with (l' := (nodup_cancel Aeq_dec l')); auto.
Qed.

```

Lemma mono_middle : $\forall x \ l1 \ l2,$
 $\text{is_mono } (l1 ++ x :: l2) \rightarrow$
 $\text{is_mono } (l1 ++ l2).$

Proof.

```

    intros x l1 l2 H. unfold is_mono in *. apply Sorted_StronglySorted in H.
    apply StronglySorted_Sorted. induction l1.
- rewrite app_nil_l in *. apply StronglySorted_inv in H as []; auto.
- simpl in *. apply StronglySorted_inv in H as []. apply SSorted_cons; auto.
    apply Forall_forall. rewrite Forall_forall in H0. intros x0 Hin.
    apply H0. apply in_app_iff in Hin as []; intuition.
- apply Lt_Transitive.

```

Qed.

Lemma NoDup_In_split : $\forall \{A:\text{Type}\} (x:A) \ l \ l1 \ l2,$
 $l = l1 ++ x :: l2 \rightarrow$
 $\text{NoDup } l \rightarrow$
 $\neg \text{In } x \ l1 \wedge \neg \text{In } x \ l2.$

Proof.

```

    intros A x l l1 l2 H H0. rewrite H in H0.
    apply NoDup_remove_2 in H0. split; intro; intuition.

```

Qed.

Lemma remove_Sorted_eq : $\forall x \ (l \ l':\text{mono}),$
 $\text{is_mono } l \rightarrow \text{is_mono } l' \rightarrow$
 $\text{In } x \ l \leftrightarrow \text{In } x \ l' \rightarrow$
 $\text{remove var_eq_dec } x \ l = \text{remove var_eq_dec } x \ l' \rightarrow$
 $l = l'.$

Proof.

```

    intros x l l' Hl Hl' Hx Hrem.
    generalize dependent l'; induction l; induction l'; intros.
- auto.
- destruct (var_eq_dec x a) eqn:Heq.
    + rewrite e in Hx. exfalso. apply Hx. intuition.

```

```

+ simpl in Hrem. rewrite Heq in Hrem. inversion Hrem.
- destruct (var_eq_dec x a) eqn:Heq.
+ rewrite e in Hx. exfalso. apply Hx. intuition.
+ simpl in Hrem. rewrite Heq in Hrem. inversion Hrem.
- clear IHL'. destruct (var_eq_dec a a0).
+ rewrite e. f_equal. rewrite e in Hrem. simpl in Hrem.
  apply mono_cons in Hl as Hl1. apply mono_cons in Hl' as Hl'1.
  destruct (var_eq_dec x a0).
  × apply IHL; auto. apply NoDup_VarSorted in Hl. apply NoDup_cons_iff in Hl.
    rewrite e in Hl. rewrite ← e0 in Hl. destruct Hl. split; intro. contradiction.
    apply NoDup_VarSorted in Hl'. apply NoDup_cons_iff in Hl'.
    rewrite ← e0 in Hl'. destruct Hl'. contradiction.
  × inversion Hrem. apply IHL; auto. destruct Hx. split; intro. simpl in H.
    rewrite e in H. destruct H; auto. rewrite H in n. contradiction.
    simpl in H1. rewrite e in H1. destruct H1; auto. rewrite H1 in n.
contradiction.
+ destruct (in_dec var_eq_dec x (a :: l)).
  × apply Hx in i as i'. apply in_split in i. apply in_split in i'.
    destruct i as [l1 [l2 i]]. destruct i' as [l1' [l2' i']].
    pose (NoDup_VarSorted _ Hl). pose (NoDup_VarSorted _ Hl').
    apply (NoDup_In_split _ _ _ i) in n0 as []. apply (NoDup_In_split _ _ _ i') in
n1 as [].
    rewrite i in Hrem. rewrite i' in Hrem. repeat rewrite remove_distr_app in
Hrem.

simpl in Hrem. destruct (var_eq_dec x x); try contradiction.
rewrite not_in_remove in Hrem; auto. rewrite not_in_remove in Hrem; auto.
rewrite not_in_remove in Hrem; auto. rewrite not_in_remove in Hrem; auto.
destruct l1; destruct l1'; simpl in i; simpl in i'; simpl in Hrem;
inversion i; inversion i'.
- rewrite H4 in n. rewrite H6 in n. contradiction.
- rewrite H7 in Hl'. rewrite i in Hl. rewrite Hrem in Hl.
  rewrite H6 in Hl'. assert (x < v). apply Sorted_inv in Hl as [].
  apply HdRel_inv in H8. auto. assert (v < x). apply Sorted_StronglySorted in
Hl'.

  apply StronglySorted_inv in Hl' as []. rewrite Forall_forall in H9.
  apply H9. intuition. apply Lt_Transitive. apply lt_asymm in H8. contradiction.
- rewrite H7 in Hl'. rewrite i in Hl. rewrite ← Hrem in Hl'.
  rewrite H6 in Hl'. assert (n0 < x). apply Sorted_StronglySorted in Hl.
  apply StronglySorted_inv in Hl as []. rewrite Forall_forall in H8.
  apply H8. intuition. apply Lt_Transitive. assert (x < n0).
  apply Sorted_inv in Hl' as []. apply HdRel_inv in H9; auto.
  apply lt_asymm in H8. contradiction.

```

- inversion *Hrem*. rewrite $\leftarrow H4$ in *H8*. rewrite $\leftarrow H6$ in *H8*. contradiction.
 × assert ($\neg \text{In } x (a0 :: l')$). intro. apply *n0*. apply *Hx*. auto.
 rewrite not_In_remove in *Hrem*; auto. rewrite not_In_remove in *Hrem*; auto.

Qed.

Lemma Permutation_Sorted_eq : $\forall (l \ m : \text{list } \text{mono})$,

Permutation *l m* \rightarrow
Sorted (fun *x y* \Rightarrow **is_true** (MonoOrder.leb *x y*)) *l* \rightarrow
Sorted (fun *x y* \Rightarrow **is_true** (MonoOrder.leb *x y*)) *m* \rightarrow
l = *m*.

Proof.

intros *l m Hp Hsl Hsm*. generalize dependent *m*.
 induction *l*; induction *m*; intros.
 - reflexivity.
 - apply **Permutation_nil** in *Hp*. auto.
 - apply **Permutation_sym**, **Permutation_nil** in *Hp*. auto.
 - clear *IHm*. apply Permutation_incl in *Hp* as *Hp'*. destruct *Hp'*.
 destruct (lex **compare** *a a0*) eqn:*Hcomp*.
 + apply lex_eq in *Hcomp*. rewrite *Hcomp* in *.
 apply **Permutation_cons_inv** in *Hp*. f_equal; auto.
 apply *IHl*.
 × apply **Sorted_inv** in *Hsl*. apply *Hsl*.
 × apply *Hp*.
 × apply **Sorted_inv** in *Hsm*. apply *Hsm*.
 + apply lex_neq' in *Hcomp* as *Hneq*. apply incl_cons_inv in *H*. destruct *H*.
 apply **Sorted_StronglySorted** in *Hsm*. apply **StronglySorted_inv** in *Hsm* as [].
 × simpl in *H*. destruct *H*; try (rewrite *H* in *Hneq*; contradiction).
 pose (Forall_In _ _ _ *H H3*). simpl in *i*. unfold **is_true** in *i*.
 unfold MonoOrder.leb in *i*. apply lex_rev_lt_gt in *Hcomp*.
 rewrite *Hcomp* in *i*. inversion *i*.
 × apply MonoOrder_Transitive.
 + apply lex_neq' in *Hcomp* as *Hneq*. apply incl_cons_inv in *H0*. destruct *H0*.
 apply **Sorted_StronglySorted** in *Hsl*. apply **StronglySorted_inv** in *Hsl* as [].
 × simpl in *H0*. destruct *H0*; try (rewrite *H0* in *Hneq*; contradiction).
 pose (Forall_In _ _ _ *H0 H3*). simpl in *i*. unfold **is_true** in *i*.
 unfold MonoOrder.leb in *i*. rewrite *Hcomp* in *i*. inversion *i*.
 × apply MonoOrder_Transitive.

Qed.

Lemma Permutation_sort_eq : $\forall l \ m$,

Permutation *l m* \leftrightarrow sort *l* = sort *m*.

Proof.

intros *l m*. split; intros *H*.
 - assert (*H0* : **Permutation** (sort *l*) (sort *m*)).

```

+ apply Permutation_trans with (l:=(sort l)) (l':=m) (l'':=(sort m)).
  × apply Permutation_sym. apply Permutation_sym in H.
    apply (Permutation_trans H (Permuted_sort l)).
  × apply Permuted_sort.
+ apply (Permutation_Sorted_eq _ _ H0 (LocallySorted_sort l) (LocallySorted_sort m)).
- assert (Permutation (sort l) (sort m)).
  + rewrite H. apply Permutation_refl.
+ pose (Permuted_sort l). pose (Permuted_sort m).
  apply (Permutation_trans p) in H0. apply Permutation_sym in p0.
  apply (Permutation_trans H0) in p0. apply p0.

```

Qed.

Lemma make_poly_Permutation : $\forall p q,$
Permutation $p q \rightarrow \text{make_poly } p = \text{make_poly } q.$

Proof.

```

intros. unfold make_poly.
apply Permutation_sort_eq, nodup_cancel_Permutation, Permutation_map.
auto.

```

Qed.

Lemma no_sort_MonoSorted : $\forall p,$
Sorted $(\text{fun } a b \Rightarrow \text{lex compare } a b = \text{Lt}) p \rightarrow$
MonoSort.sort $p = p.$

Proof.

```

intros p H. unfold make_poly. apply Permutation_Sorted_eq.
- apply Permutation_sym. apply Permuted_sort.
- apply LocallySorted_sort.
- apply Sorted_MonoSorted. auto.

```

Qed.

Lemma make_poly_app_comm : $\forall p q,$
make_poly $(p ++ q) = \text{make_poly } (q ++ p).$

Proof.

```

intros p q. apply Permutation_sort_eq.
apply nodup_cancel_Permutation. apply Permutation_map.
apply Permutation_app_comm.

```

Qed.

Lemma no_make_poly : $\forall p,$
is_poly $p \rightarrow$
make_poly $p = p.$

Proof.

```

unfold make_poly, is_poly. intros m []. rewrite no_sort_MonoSorted.
- rewrite no_nodup_cancel_NoDup.
  + apply no_map_make_mono. intros m0 Hin. apply H0. auto.

```

```

    + apply NoDup_MonoSorted in  $H$ . rewrite no_map_make_mono; auto.
  - apply Sorted_nodup_cancel.
    + apply lex_Lt_Transitive.
    + rewrite no_map_make_mono; auto.
Qed.

Lemma sort_app_comm :  $\forall l m$ ,
  sort ( $l ++ m$ ) = sort ( $m ++ l$ ).
Proof.
  intros  $l m$ . pose (Permutation.Permutation_app_comm  $l m$ ).
  apply Permutation_sort_eq. auto.
Qed.

Lemma sort_nodup_cancel_assoc :  $\forall l$ ,
  sort (nodup_cancel mono_eq_dec  $l$ ) = nodup_cancel mono_eq_dec (sort  $l$ ).
Proof.
  intros  $l$ . apply Permutation_Sorted_eq.
  - pose (Permuted_sort (nodup_cancel mono_eq_dec  $l$ )). apply Permutation_sym in  $p$ .
    apply (Permutation_trans  $p$ ). clear  $p$ . apply NoDup_Permutation.
    + apply NoDup_nodup_cancel.
    + apply NoDup_nodup_cancel.
    + intros  $x$ . split.
      × intros  $H$ . apply Permutation_in with ( $l :=$ (nodup_cancel mono_eq_dec  $l$ )).
        apply nodup_cancel_Permutation. apply Permuted_sort. auto.
      × intros  $H$ . apply Permutation_in with ( $l :=$ (nodup_cancel mono_eq_dec (sort  $l$ ))).
        apply nodup_cancel_Permutation. apply Permutation_sym. apply Permuted_sort.
    auto.
  - apply LocallySorted_sort.
  - apply Sorted_nodup_cancel.
    + apply MonoOrder_Transitive.
    + apply LocallySorted_sort.
Qed.

Lemma addPP_comm :  $\forall p q$ ,
  addPP  $p q$  = addPP  $q p$ .
Proof.
  intros  $p q$ . unfold addPP, make_poly. repeat rewrite map_app.
  repeat rewrite sort_nodup_cancel_assoc. rewrite sort_app_comm.
  reflexivity.
Qed.

Hint Unfold addPP mulPP.

Lemma mulPP_0 :  $\forall p$ ,
  mulPP  $[] p$  =  $[]$ .
Proof.

```



```

    intros p. unfold mulPP. rewrite (@distribute_nil var). auto.
Qed.

Lemma mulPP_0r :  $\forall p$ ,
  mulPP p [] = [].
Proof.
  intros p. unfold mulPP. rewrite (@distribute_nil_r var). auto.
Qed.

Lemma addPP_0 :  $\forall p$ ,
  is_poly p  $\rightarrow$ 
  addPP [] p = p.
Proof.
  intros p Hpoly. unfold addPP. simpl. apply no_make_poly. auto.
Qed.

Lemma addPP_0r :  $\forall p$ ,
  is_poly p  $\rightarrow$ 
  addPP p [] = p.
Proof.
  intros p Hpoly. unfold addPP. rewrite app_nil_r. apply no_make_poly. auto.
Qed.

Lemma count_occ_app :  $\forall (A:\text{Type}) a (l m:\text{list } A) \text{ Aeq\_dec}$ ,
  count_occ Aeq_dec (l++m) a = add (count_occ Aeq_dec l a) (count_occ Aeq_dec m a).
Proof.
  intros A a l m Aeq_dec. induction l.
  - simpl. auto.
  - simpl. destruct (Aeq_dec a0 a); simpl; auto.
Qed.

Lemma count_occ_remove :  $\forall a p$ ,
  count_occ mono_eq_dec (remove mono_eq_dec a p) a = 0.
Proof.
  intros a p. induction p.
  - simpl. auto.
  - simpl. destruct (mono_eq_dec a a0) eqn:Haa0.
    + apply IHp.
    + simpl. destruct (mono_eq_dec a0 a); try (symmetry in e; contradiction).
      apply IHp.
Qed.

Lemma count_occ_neq_remove :  $\forall a b p$ ,
  a  $\neq$  b  $\rightarrow$ 
  count_occ mono_eq_dec (remove mono_eq_dec a p) b =
  count_occ mono_eq_dec p b.
Proof.

```

```

intros a b p H. induction p; simpl; auto. destruct (mono_eq_dec a a0).
- destruct (mono_eq_dec a0 b).
  + rewrite ← e0 in H. rewrite e in H. contradiction.
  + apply IHp.
- simpl. destruct (mono_eq_dec a0 b); auto.
Qed.

```

Lemma nodup_cancel_remove_assoc : $\forall a p$,
 $\text{remove mono_eq_dec } a (\text{nodup_cancel mono_eq_dec } p) =$
 $\text{nodup_cancel mono_eq_dec } (\text{remove mono_eq_dec } a p).$

Proof.

```

intros a p. induction p.
- simpl. auto.
- simpl. destruct even eqn:Hevn.
  + simpl. destruct (mono_eq_dec a a0).
    × rewrite ← e. rewrite not_in_remove; auto. apply remove_In.
    × simpl. rewrite count_occ_neq_remove; auto. rewrite Hevn.
      f_equal. rewrite ← IHp. rewrite remove_remove. auto.
  + destruct (mono_eq_dec a a0).
    × rewrite ← e. rewrite not_in_remove; auto. apply remove_In.
    × simpl. rewrite count_occ_neq_remove; auto. rewrite Hevn.
      rewrite remove_remove. rewrite ← IHp. auto.

```

Qed.

Lemma nodup_cancel_self : $\forall p$,
 $\text{nodup_cancel mono_eq_dec } (p++p) = [].$

Proof.

```

intros p. induction p.
- auto.
- simpl. destruct even eqn:Hevn.
  + rewrite count_occ_app in Hevn. destruct (count_occ mono_eq_dec p a) eqn:Hx.
    × simpl in Hevn. destruct (mono_eq_dec a a); try contradiction.
      rewrite Hx in Hevn. inversion Hevn.
    × simpl in Hevn. destruct (mono_eq_dec a a); try contradiction.
      rewrite Hx in Hevn. rewrite add_comm in Hevn.
      simpl in Hevn. destruct (plus n n) eqn:Help. inversion Hevn.
      replace (plus n n) with (plus 0 (2×n)) in Help.
      pose (even_add_mul_2 0 n). pose (even_succ n0). rewrite ← Help in e1.
      rewrite e0 in e1. simpl in e1. apply even_spec in Hevn. symmetry in e1.
      apply odd_spec in e1. apply (Even-Odd-False _ Hevn) in e1. inversion e1.
      simpl. auto.
  + clear Hevn. rewrite nodup_cancel_remove_assoc. rewrite remove_distr_app.
    simpl. destruct (mono_eq_dec a a); try contradiction.
    rewrite ← remove_distr_app. rewrite ← nodup_cancel_remove_assoc.

```

```

      rewrite IHp. auto.
Qed.
Lemma addPP_p_p : ∀ p,
  is_poly p →
  addPP p p = [].
Proof.
  intros p Hp. unfold addPP. unfold make_poly. rewrite no_map_make_mono.
  - rewrite nodup_cancel_self. auto.
  - intros m Hin. apply Hp. apply in_app_iff in Hin. intuition.
Qed.
Lemma sort_pointless : ∀ p q,
  sort (sort p ++ q) =
  sort (p ++ q).
Proof.
  intros p q. apply Permutation_sort_eq.
  apply Permutation_app_tail. apply Permutation_sym.
  apply Permuted_sort.
Qed.
Lemma remove_pointless : ∀ a p q,
  remove mono_eq_dec a (remove mono_eq_dec a p ++ q) =
  remove mono_eq_dec a (p ++ q).
Proof.
  intros a p q. induction p; auto. simpl. destruct (mono_eq_dec a a0) eqn:Heq.
  - apply IHp.
  - simpl. rewrite Heq. f_equal. apply IHp.
Qed.
Lemma count_occ_nodup_cancel : ∀ p a,
  even (count_occ mono_eq_dec (nodup_cancel mono_eq_dec p) a) =
  even (count_occ mono_eq_dec p a).
Proof.
  intros p a. induction p as [|b]; auto. simpl.
  destruct (even (count_occ mono_eq_dec p b)) eqn:Hb.
  - simpl. destruct (mono_eq_dec b a).
    + rewrite e. rewrite count_occ_remove. rewrite e in Hb. repeat rewrite even_succ.
      rewrite ← negb_odd in Hb. rewrite Bool.negb_true_iff in Hb. rewrite Hb. auto.
    + rewrite count_occ_neq_remove; auto.
  - simpl. destruct (mono_eq_dec b a).
    + rewrite e. rewrite count_occ_remove. rewrite e in Hb. repeat rewrite even_succ.
      rewrite ← negb_odd in Hb. rewrite Bool.negb_false_iff in Hb. rewrite Hb. auto.
    + rewrite count_occ_neq_remove; auto.
Qed.

```

Lemma nodup_extra_remove : $\forall a p$,
 even (count_occ mono_eq_dec p a) = true \rightarrow
 nodup_cancel mono_eq_dec p =
 nodup_cancel mono_eq_dec (remove mono_eq_dec a p).

Proof.

```

intros a p H. induction p as [|b]; auto. simpl.
destruct (mono_eq_dec a b).
- rewrite e in H. simpl in H. destruct (mono_eq_dec b b); try contradiction.
  rewrite even_succ in H. rewrite  $\leftarrow$  negb_even in H. rewrite Bool.negb_true_iff in H.
  rewrite H. rewrite nodup_cancel_remove_assoc. rewrite e. auto.
- simpl. destruct (even (count_occ mono_eq_dec p b)) eqn:Hev.
  + rewrite count_occ_neq_remove; auto. rewrite Hev. f_equal.
    rewrite IHp. auto. simpl in H. destruct (mono_eq_dec);
    try (symmetry in e; contradiction). auto.
  + rewrite count_occ_neq_remove; auto. rewrite Hev. f_equal.
    apply IHp. simpl in H. destruct (mono_eq_dec b a);
    try (symmetry in e; contradiction). auto.

```

Qed.

Lemma nodup_cancel_pointless : $\forall p q$,
 Permutation (nodup_cancel mono_eq_dec (nodup_cancel mono_eq_dec p ++ q))
 (nodup_cancel mono_eq_dec (p ++ q)).

Proof.

```

intros p q. induction p; auto. destruct (even (count_occ mono_eq_dec p a)) eqn:Hevp;
destruct (even (count_occ mono_eq_dec q a)) eqn:Hevq.
- simpl. rewrite Hevp. simpl. rewrite count_occ_app, count_occ_remove. simpl.
  rewrite count_occ_app, even_add, Hevp, Hevq. simpl. apply perm_skip.
  rewrite nodup_cancel_remove_assoc. rewrite remove_pointless.
  rewrite  $\leftarrow$  nodup_cancel_remove_assoc. apply remove_permutation. apply IHp.
- simpl. rewrite Hevp. simpl. rewrite count_occ_app, count_occ_remove. simpl.
  rewrite count_occ_app, even_add, Hevp, Hevq. simpl.
  rewrite nodup_cancel_remove_assoc. rewrite remove_pointless.
  rewrite  $\leftarrow$  nodup_cancel_remove_assoc. apply remove_permutation. apply IHp.
- simpl. rewrite Hevp. rewrite count_occ_app, even_add, Hevp, Hevq. simpl.
  rewrite (nodup_extra_remove a).
  + rewrite remove_pointless. rewrite  $\leftarrow$  nodup_cancel_remove_assoc.
    apply remove_permutation. apply IHp.
  + rewrite count_occ_app. rewrite even_add. rewrite count_occ_remove.
    rewrite Hevq. auto.
- assert (count_occ mono_eq_dec q a > 0). destruct (count_occ - q -).
  inversion Hevq. apply gt_Sn_O. apply count_occ_In in H.
  apply in_split in H as [l1[l2 H]]. rewrite H. simpl nodup_cancel at 2.
  rewrite Hevp. simpl app. rewrite H in IHp. simpl nodup_cancel at 3.

```

```

rewrite count_occ_app. rewrite even_add. rewrite Hevp. rewrite ← H at 2.
rewrite Hevq. simpl. apply Permutation_trans with (l':=(nodup_cancel
  mono_eq_dec (a :: remove mono_eq_dec a (nodup_cancel mono_eq_dec p) ++ l1 ++
l2))).
+ apply nodup_cancel_permutation. rewrite app_assoc. apply Permutation_sym.
  rewrite app_assoc. apply Permutation_middle with (l2:=l2) (l1:=(remove mono_eq_dec
a (nodup_cancel mono_eq_dec p) ++ l1)).
+ assert (even (count_occ mono_eq_dec (l1++l2) a) = true).
  rewrite H in Hevq. rewrite count_occ_app in Hevq. simpl in Hevq.
  destruct (mono_eq_dec a a); try contradiction. rewrite plus_comm in Hevq.
  rewrite plus_Sn_m in Hevq. rewrite even_succ in Hevq.
  rewrite ← negb_even in Hevq. rewrite Bool.negb_false_iff in Hevq.
  rewrite count_occ_app. symmetry. rewrite plus_comm. auto.
  simpl. rewrite count_occ_app. rewrite count_occ_remove. simpl.
  replace (even _) with true. apply perm_skip. rewrite (nodup_cancel_remove_assoc
- (p++l1++a::l2)).
  repeat rewrite remove_distr_app. simpl; destruct (mono_eq_dec a a); try contradiction.
  rewrite nodup_cancel_remove_assoc. rewrite remove_pointless.
  repeat rewrite ← remove_distr_app. repeat rewrite ← nodup_cancel_remove_assoc.
  apply Permutation_trans with (l'':=(nodup_cancel mono_eq_dec (a :: p ++ l1 ++
l2))) in IHp.
  apply Permutation_sym in IHp.
  apply Permutation_trans with (l'':=(nodup_cancel mono_eq_dec (a :: nodup_cancel
mono_eq_dec p ++ l1 ++ l2))) in IHp.
  simpl in IHp. rewrite count_occ_app, even_add, Hevp in IHp.
  rewrite H0 in IHp. simpl in IHp.
  rewrite count_occ_app, even_add, count_occ_nodup_cancel, Hevp, H0 in IHp.
  simpl in IHp. apply Permutation_sym. apply IHp.
  × apply nodup_cancel_permutation. rewrite app_assoc. apply Permutation_sym.
    rewrite app_assoc. apply Permutation_middle with
      (l1:=(nodup_cancel mono_eq_dec p) ++ l1).
  × apply nodup_cancel_permutation. rewrite app_assoc. apply Permutation_sym.
    rewrite app_assoc. apply Permutation_middle with (l1:=(p ++ l1)).

```

Qed.

Lemma nodup_cancel_pointless_r : $\forall p q$,

Permutation

(nodup_cancel mono_eq_dec (p ++ nodup_cancel mono_eq_dec q))
(nodup_cancel mono_eq_dec (p ++ q)).

Proof.

```

intros p q. apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (
  nodup_cancel mono_eq_dec q ++ p))). apply nodup_cancel_permutation.
  apply Permutation_app_comm.

```

apply **Permutation_sym**. apply **Permutation_trans** with (l':=(nodup_cancel
 mono_eq_dec (q ++ p))). apply nodup_cancel_Permutation.
 apply **Permutation_app_comm**. apply **Permutation_sym**.
 apply nodup_cancel_pointless.

Qed.

Lemma make_poly_pointless_weak : $\forall p q,$
 $(\forall m, \text{In } m p \rightarrow \text{is_mono } m) \rightarrow$
 $(\forall m, \text{In } m q \rightarrow \text{is_mono } m) \rightarrow$
 make_poly (make_poly p ++ q) =
 make_poly (p ++ q).

Proof.

intros p q Hmp Hmq. induction p; auto.
 unfold make_poly. repeat rewrite no_map_make_mono; intuition.
 apply Permutation_sort_eq. rewrite sort_nodup_cancel_assoc.
 rewrite nodup_cancel_pointless. apply nodup_cancel_Permutation.
 apply **Permutation_sym**. apply **Permutation_app_tail**. apply Permuted_sort.
 - simpl in H. rewrite **in_app_iff** in H. destruct H as [H|[H|H]]; intuition.
 rewrite H in Hmp; intuition.
 - rewrite **in_app_iff** in H. destruct H; intuition.
 apply In_sorted in H. apply nodup_cancel_in in H. intuition.

Qed.

Lemma mono_in_map_make_mono : $\forall p m,$
 $\text{In } m (\text{map make_mono } p) \rightarrow \text{is_mono } m.$

Proof.

intros. apply **in_map_iff** in H as [x []]. rewrite \leftarrow H. auto.

Qed.

Lemma make_poly_pointless : $\forall p q,$
 make_poly (make_poly p ++ q) =
 make_poly (p ++ q).

Proof.

intros p q. rewrite make_poly_app_comm.
 rewrite \leftarrow map_make_mono_pointless. rewrite make_poly_app_comm.
 rewrite \leftarrow (map_make_mono_pointless p). rewrite (make_poly_app_comm _ q).
 rewrite \leftarrow (map_make_mono_pointless q). rewrite (make_poly_app_comm _ (map make_mono
 p)).
 rewrite \leftarrow (make_poly_pointless_weak (map make_mono p)). unfold make_poly.
 rewrite (no_map_make_mono (map make_mono p)). auto.
 apply mono_in_map_make_mono. apply mono_in_map_make_mono.
 apply mono_in_map_make_mono.

Qed.

Lemma make_poly_pointless_r : $\forall p q,$

```

make_poly (p ++ make_poly q) =
make_poly (p ++ q).

```

Proof.

```

intros p q. rewrite make_poly_app_comm. rewrite make_poly_pointless.
apply make_poly_app_comm.

```

Qed.

```

Lemma concat_map_map : ∀ A B C l (f:B→C) (g:A→list B),
  concat (map (fun a => map f (g a)) l) =
  map f (concat (map g l)).

```

Proof.

```

intros. induction l; auto.
simpl. rewrite map_app. f_equal. auto.

```

Qed.

```

Lemma mulPP'_mulPP'' : ∀ p q,
  mulPP' p q = mulPP'' p q.

```

Proof.

```

intros p q. unfold mulPP', mulPP'', mulMP, mulMP', make_poly.
rewrite concat_map_map.
rewrite (no_map_make_mono (map _)); auto.
intros. apply in_map_iff in H as [n []].
rewrite ← H.
auto.

```

Qed.

```

Lemma mulMP'_mulMP'' : ∀ m p q,
  make_poly (mulMP' p m ++ q) = make_poly (mulMP'' p m ++ q).

```

Proof.

```

intros m p q. unfold mulMP', mulMP''. rewrite make_poly_app_comm.
rewrite ← map_make_mono_pointless. rewrite make_poly_app_comm.
rewrite ← make_poly_pointless. unfold make_poly at 2. rewrite (no_map_make_mono
(map make_mono _)).
unfold make_poly at 3. rewrite (make_poly_app_comm _ q).
rewrite ← (map_make_mono_pointless q). rewrite make_poly_app_comm. auto.
apply mono_in_map_make_mono.

```

Qed.

```

Lemma mulPP''_mulPP''' : ∀ p q,
  mulPP'' p q = mulPP''' p q.

```

Proof.

```

intros p q. induction q. auto. unfold mulPP'', mulPP'''. simpl.
rewrite mulMP'_mulMP''. repeat rewrite ← (make_poly_pointless_r _ (concat _)).
f_equal. f_equal. apply IHq.

```

Qed.

Lemma mulPP_mulPP'' : $\forall p q$,
 mulPP $p q$ = mulPP'' $p q$.

Proof.

intros. rewrite mulPP_mulPP', mulPP'_mulPP''. auto.

Qed.

Lemma mulPP_mulPP''' : $\forall p q$,
 mulPP $p q$ = mulPP''' $p q$.

Proof.

intros. rewrite mulPP_mulPP'', mulPP''_mulPP'''. auto.

Qed.

Lemma addPP_assoc : $\forall p q r$,
 addPP (addPP $p q$) r = addPP p (addPP $q r$).

Proof.

intros $p q r$. rewrite (addPP_comm _ (addPP _ _)). unfold addPP.
 repeat rewrite make_poly_pointless. repeat rewrite \leftarrow app_assoc.
 apply Permutation_sort_eq. apply nodup_cancel_Permutation. apply Permutation_map.
 rewrite (app_assoc q). apply Permutation_app_comm with ($l' := (q ++ r)$).

Qed.

Lemma mulPP_1r : $\forall p$,
 is_poly $p \rightarrow$
 mulPP p $[\]$ = p .

Proof.

intros $p H$. unfold mulPP, distribute. simpl. rewrite app_nil_r.
 rewrite map_id. apply no_make_poly. auto.

Qed.

Lemma concat_map_nil : $\forall \{A\} (p : \text{list } A)$,
 concat (map (fun $x \Rightarrow [\]$) p) = (@nil A).

Proof.

induction p ; auto.

Qed.

Lemma Permutation_nodup : $\forall A \text{ Aeq_dec } (l m : \text{list } A)$,
 Permutation $l m \rightarrow$ Permutation (nodup $\text{Aeq_dec } l$) (nodup $\text{Aeq_dec } m$).

Proof.

intros. induction H .

- auto.

- simpl. destruct (in_dec $\text{Aeq_dec } x l$).

+ apply Permutation_in with ($l' := l'$) in i ; auto. destruct in_dec; try contradiction.
 auto.

+ assert ($\neg \text{In } x l'$). intro. apply n . apply Permutation_in with ($l' := l$) in $H0$; auto.
 apply Permutation_sym; auto. destruct in_dec; try contradiction.
 apply perm_skip. auto.


```

- destruct (in_dec Aeq_dec y (x :: l)). destruct i.
+ rewrite H. simpl. destruct (Aeq_dec y y); try contradiction. destruct in_dec.
  auto. apply perm_skip. auto.
+ simpl. destruct (Aeq_dec x y). destruct in_dec; destruct (Aeq_dec y x);
  try (symmetry in e; contradiction). rewrite e in i. destruct in_dec; try contradiction.
  auto. assert (¬ ln y l). intro; apply n; rewrite e; auto.
  destruct in_dec; try contradiction. destruct in_dec; try contradiction.
  destruct in_dec; destruct (Aeq_dec y x); try (symmetry in e; contradiction).
  auto. apply perm_skip. auto.
+ simpl. destruct (Aeq_dec x y). destruct in_dec. destruct (Aeq_dec y x);
  try (symmetry in e; contradiction). rewrite e0. destruct in_dec; try contradiction.
  auto. destruct (Aeq_dec y x); try (symmetry in e; contradiction).
  assert (¬ ln y l). intro; apply n0; rewrite e; auto. destruct in_dec; try
contradiction.
  rewrite e0. apply perm_skip; auto. assert (¬ ln y l). intro; apply n; intuition.
  destruct in_dec; try contradiction. destruct in_dec; destruct (Aeq_dec y x);
  try (symmetry in e; contradiction). auto. apply perm_swap.
- apply Permutation_trans with (l' := (nodup Aeq_dec l')); auto.
Qed.

Lemma make_mono_app_comm : ∀ m n,
  make_mono (m ++ n) = make_mono (n ++ m).
Proof.
  intros m n. apply Permutation_sort_mono_eq. apply Permutation_nodup.
  apply Permutation_app_comm.
Qed.

Lemma mulPP_comm : ∀ p q,
  mulPP p q = mulPP q p.
Proof.
  intros p q. repeat rewrite mulPP_mulPP''.
  generalize dependent q. induction p; induction q as [|m].
- auto.
- unfold mulPP'', mulMP'. simpl. rewrite (@concat_map_nil mono). auto.
- unfold mulPP'', mulMP'. simpl. rewrite (@concat_map_nil mono). auto.
- unfold mulPP''. simpl. rewrite (app_comm_cons _ _ (make_mono (a ++ m))).
  rewrite ← make_poly_pointless_r. rewrite mulPP''_refold. rewrite ← IHp.
  unfold mulPP''. rewrite make_poly_pointless_r. simpl. unfold mulMP' at 2.
  rewrite app_comm_cons. rewrite ← make_poly_pointless_r. rewrite mulPP''_refold.
  rewrite IHq. unfold mulPP''. rewrite make_poly_pointless_r. simpl.
  unfold mulMP' at 1. rewrite app_comm_cons. rewrite app_assoc.
  rewrite ← make_poly_pointless_r. rewrite mulPP''_refold. rewrite ← IHp.
  unfold mulPP''. rewrite make_poly_pointless_r. simpl. rewrite (app_assoc (map _
(map _ q))).

```

```

    apply Permutation_sort_eq. apply nodup_cancel_Permutation.
    apply Permutation_map. rewrite make_mono_app_comm. apply perm_skip.
    apply Permutation_app_tail. apply Permutation_app_comm.
Qed.

Lemma make_poly_nil :
  make_poly [] = [].
Proof.
  unfold make_poly, sort. auto.
Qed.

Lemma mulPP''_cons :  $\forall q a p,$ 
  make_poly (mulMP' q a ++ mulPP'' q p) =
  mulPP'' q (a :: p).
Proof.
  intros q a p. unfold mulPP''. rewrite make_poly_pointless_r. auto.
Qed.

Lemma Permutation_VarSort_l :  $\forall m n,$ 
  Permutation m n  $\leftrightarrow$  Permutation (VarSort.sort m) n.
Proof.
  intros m n. split; intro.
  - apply Permutation_trans with (l' := m). apply Permutation_sym.
    apply VarSort.Permuted_sort. apply H.
  - apply Permutation_trans with (l' := (VarSort.sort m)).
    apply VarSort.Permuted_sort. apply H.
Qed.

Lemma Permutation_VarSort_r :  $\forall m n,$ 
  Permutation m n  $\leftrightarrow$  Permutation m (VarSort.sort n).
Proof.
  intros m n. split; intro.
  - apply Permutation_sym. rewrite  $\leftarrow$  Permutation_VarSort_l.
    apply Permutation_sym; auto.
  - apply Permutation_sym. rewrite  $\rightarrow$  Permutation_VarSort_l.
    apply Permutation_sym; auto.
Qed.

Lemma make_mono_pointless :  $\forall m a,$ 
  make_mono (m ++ make_mono a) = make_mono (m ++ a).
Proof.
  intros m a. apply Permutation_sort_mono_eq.
  apply Permutation_trans with (l' := (nodup var_eq_dec (m ++ nodup var_eq_dec a))).
  apply Permutation_nodup. apply Permutation_app_head. unfold make_mono.
  rewrite  $\leftarrow$  Permutation_VarSort_l. auto.
  induction a; auto. simpl. destruct in_dec.

```

```

- apply Permutation_sym. apply Permutation_trans with (l':=(nodup var_eq_dec (a :: m
++ a0))).
  apply Permutation_nodup. apply Permutation_sym. apply Permutation_middle.
  simpl. destruct in_dec.
  + apply Permutation_sym. apply IHa.
  + exfalso. apply n. intuition.
- apply Permutation_trans with (l':=(nodup var_eq_dec (a :: m++nodup var_eq_dec a0))).
  apply Permutation_nodup. apply Permutation_sym. apply Permutation_middle.
  apply Permutation_sym. apply Permutation_trans with (l':=(nodup var_eq_dec
(a :: m++a0))). apply Permutation_nodup. apply Permutation_sym. apply Permuta-
tion_middle.
  simpl. destruct (in_dec var_eq_dec a m).
  + assert (In a (m++a0)). intuition. destruct in_dec; try contradiction.
  + assert (In a (m++nodup var_eq_dec a0)). intuition. destruct in_dec;
    try contradiction. apply Permutation_sym. apply IHa.
  + assert (¬In a (m++a0)). intuition. apply in_app_iff in H. destruct H; auto.
  + assert (¬In a (m++nodup var_eq_dec a0)). intuition. apply in_app_iff in H0.
    destruct H0; auto. apply nodup_In in H0. auto. repeat destruct in_dec; try
contradiction.
  apply perm_skip. apply Permutation_sym. apply IHa.
Qed.

```

Lemma make_mono_self : $\forall m,$
 is_mono $m \rightarrow$
 make_mono ($m ++ m$) = m .

Proof.

```

intros m H. apply Permutation_Sorted_mono_eq.
- induction m; auto. unfold make_mono. rewrite ← Permutation_VarSort_l. simpl.
  assert (In a (m++a :: m)).
  intuition. destruct in_dec; try contradiction.
  apply Permutation_trans with (l':=(nodup var_eq_dec (a :: m++m))).
  apply Permutation_nodup. apply Permutation_app_comm.
  simpl. assert (¬In a (m++m)).
  apply NoDup_VarSorted in H as H1. apply NoDup_cons_iff in H1.
  intro. apply H1. apply in_app_iff in H2; intuition.
  destruct in_dec; try contradiction. apply perm_skip.
  apply Permutation_VarSort_l in IHm. auto. apply (mono_cons _ _ H).
- apply VarSort.LocallySorted_sort.
- apply Sorted_VarSorted. apply H.

```

Qed.

Lemma make_poly_refold : $\forall p,$
 sort (nodup_cancel mono_eq_dec (map make_mono p)) =
 make_poly p.

Proof. auto. Qed.

Lemma mulPP_p_p : $\forall p$,
is_poly $p \rightarrow$
mulPP $p\ p = p$.

Proof.

```
intros p H. rewrite mulPP_mulPP'. rewrite mulPP'_mulPP''. apply Permutation_Sorted_eq.
- induction p; auto. unfold mulPP'', make_poly. rewrite ← Permutation_MonoSort_l.
  simpl map at 1. apply poly_cons in H as H1. destruct H1. rewrite make_mono_self;
auto.
  rewrite no_make_mono; auto. rewrite map_app. apply Permutation_trans with
    (l' := (nodup_cancel mono_eq_dec (map make_mono (concat (map (mulMP' (a ::
      p)) p)) ++ a :: map make_mono (map make_mono (map (app a) p)))))).
    apply nodup_cancel_Permutation. rewrite app_comm_cons. apply Permutation_app_comm.
  rewrite ← nodup_cancel_pointless. apply Permutation_trans with (l' := (nodup_cancel
mono_eq_dec
  ((nodup_cancel mono_eq_dec (map make_mono (concat (map (mulMP' p) (a :: p))))))
  ++ (a :: map make_mono (map make_mono (map (app a) p)))))).
  apply nodup_cancel_Permutation. apply Permutation_app_tail. apply Permutation_sort_eq.
  repeat rewrite make_poly_refold. repeat rewrite mulPP''_refold.
  repeat rewrite ← mulPP'_mulPP''. repeat rewrite ← mulPP_mulPP'. apply mulPP_comm.
  rewrite nodup_cancel_pointless. apply Permutation_trans with (l' :=
    (nodup_cancel mono_eq_dec (a :: map make_mono (map make_mono (map (app a)
p))
  ++ (map make_mono (concat (map (mulMP' p) (a :: p))))))).
  apply nodup_cancel_Permutation. apply Permutation_app_comm.
  simpl map. rewrite map_app. unfold mulMP' at 1. repeat rewrite (no_map_make_mono
  (map make_mono _)); try apply mono_in_map_make_mono. rewrite (app_assoc (map
- _)).
  apply Permutation_trans with (l' := (nodup_cancel mono_eq_dec ((map make_mono (map
  (app a) p) ++ map make_mono (map (app a) p)) ++ a :: map make_mono (concat
  (map (mulMP' p) p)))))). apply nodup_cancel_Permutation. apply Permutation_middle.
  rewrite ← nodup_cancel_pointless. rewrite nodup_cancel_self. simpl app.
  apply Permutation_trans with (l' := (nodup_cancel mono_eq_dec (map make_mono
  (concat (map (mulMP' p) p)) ++ [a]))). apply nodup_cancel_Permutation.
  replace (a :: map make_mono (concat (map (mulMP' p) p))) with ([a] ++ map
  make_mono (concat (map (mulMP' p) p))); auto. apply Permutation_app_comm.
  rewrite ← nodup_cancel_pointless. apply Permutation_trans with (l' := (nodup_cancel
  mono_eq_dec (p ++ [a]))). apply nodup_cancel_Permutation.
  apply Permutation_app_tail. unfold mulPP'', make_poly in IHp.
  rewrite ← Permutation_MonoSort_l in IHp. apply IHp; auto.
```

```

    replace (a :: p) with ([a] ++ p); auto. rewrite no_nodup_cancel_NoDup.
    apply Permutation_app_comm. apply Permutation_NoDup with (l := (a :: p)).
    replace (a :: p) with ([a] ++ p); auto. apply Permutation_app_comm.
    destruct H. apply NoDup_MonoSorted in H. auto.
  - unfold make_poly. apply LocallySorted_sort.
  - apply Sorted_MonoSorted. apply H.
Qed.

Lemma mono_in_concat_mulMP' : ∀ p q m,
  In m (concat (map (mulMP' p) q)) → is_mono m.
Proof.
  intros. unfold mulMP' in H. rewrite concat_map_map in H.
  apply in_map_iff in H as [x[]]. rewrite ← H. auto.
Qed.

Lemma mono_in_mulMP' : ∀ p n m,
  In m (mulMP' p n) → is_mono m.
Proof.
  intros. unfold mulMP' in H. apply (mono_in_map_make_mono _ _ H).
Qed.

Lemma mono_in_make_poly : ∀ p m,
  In m (make_poly p) → is_mono m.
Proof.
  intros. unfold make_poly in H. apply In_sorted in H.
  apply nodup_cancel_in in H. apply (mono_in_map_make_mono _ _ H).
Qed.

Lemma mono_in_mulPP'' : ∀ p q m,
  In m (mulPP'' p q) → is_mono m.
Proof.
  intros. unfold mulPP'' in H. apply (mono_in_make_poly _ _ H).
Qed.

Lemma mulMP'_refold : ∀ p m,
  map make_mono (map (app m) p) = mulMP' p m.
Proof.
  auto.
Qed.

Lemma mulMP_mulMP' : ∀ p q m,
  make_poly (mulMP p m ++ q) = make_poly (mulMP' p m ++ q).
Proof.
  intros. unfold make_poly, mulMP. rewrite map_app, mulMP'_refold.
  rewrite map_app. rewrite (no_map_make_mono (mulMP' _ _)); auto.
  apply mono_in_mulMP'.
Qed.

```

Lemma mulMP_1 : $\forall p$,
 mulMP p [] = p .

Proof.

intros. unfold mulMP. simpl.
 rewrite map_id. auto.

Qed.

Lemma mulMP''_1 : $\forall p$,
 is_poly $p \rightarrow$
 mulMP'' p [] = p .

Proof.

intros. unfold mulMP''. simpl.
 rewrite map_id. rewrite no_make_poly; auto.

Qed.

Lemma count_occ_app_m : $\forall p m a$,
 count_occ mono_eq_dec (map (app m) p) ($m++a$) =
 count_occ mono_eq_dec $p a$.

Proof.

intros $p m a$. induction p .
 - auto.
 - simpl. destruct (mono_eq_dec $a0 a$).
 + rewrite e . destruct (mono_eq_dec ($m++a$) ($m++a$)); try contradiction.
 f_equal. apply IHp.
 + destruct (mono_eq_dec ($m++a0$) ($m++a$)); try (apply app_inv_head in e ; contradiction).
 apply IHp.

Qed.

Lemma not_in_nodup_cancel : $\forall m p$,
 even (count_occ mono_eq_dec $p m$) = true \rightarrow
 \neg In m (nodup_cancel mono_eq_dec p).

Proof.

intros $m p H$. induction p .
 - simpl. auto.
 - intro. simpl in H . destruct (mono_eq_dec $a m$).
 + simpl in $H0$. rewrite even_succ in H . rewrite \leftarrow negb_even in H .
 rewrite Bool.negb_true_iff in H . rewrite $\leftarrow e$ in H . rewrite H in $H0$.
 rewrite e in $H0$. apply remove_In in $H0$. inversion $H0$.
 + apply IHp; auto. simpl in $H0$. destruct (even (count_occ mono_eq_dec $p a$)).
 \times destruct $H0$; try contradiction. apply In_remove in $H0$. auto.
 \times apply In_remove in $H0$. auto.

Qed.

Definition parity_match ($l m$:poly) : Prop :=

$\forall x, \text{even} (\text{count_occ mono_eq_dec } l \ x) = \text{even} (\text{count_occ mono_eq_dec } m \ x).$

Lemma nothing_in_empty : $\forall \{A\} (l:\text{list } A),$

$(\forall a, \neg \text{In } a \ l) \rightarrow$
 $l = [].$

Proof.

intros $A \ l \ H$. destruct l ; auto. pose $(H \ a)$. simpl in n . exfalso.
 apply n . auto.

Qed.

Lemma even_nodup_cancel : $\forall p,$

$(\forall x, \text{even} (\text{count_occ mono_eq_dec } p \ x) = \text{true}) \rightarrow$
 $(\forall x, \neg \text{In } x (\text{nodup_cancel mono_eq_dec } p)).$

Proof.

intros $p \ H \ m$. intro. induction p .
 - inversion $H0$.
 - simpl in *. pose $(H \ m)$ as $H1$. symmetry in $H1$. destruct $(\text{mono_eq_dec } a \ m)$.
 + symmetry in $H1$. rewrite $\leftarrow e$ in $H1$. rewrite even_succ in $H1$. rewrite \leftarrow
 negb_even in $H1$.
 rewrite $\text{Bool.negb_true_iff}$ in $H1$. rewrite $H1$ in $H0$. rewrite e in $H0$.
 apply remove_In in $H0$. inversion $H0$.
 + destruct $(\text{even} (\text{count_occ mono_eq_dec } p \ a))$.
 × destruct $H0$; try contradiction. apply In_remove in $H0$. symmetry in $H1$.
 apply $\text{not_in_nodup_cancel}$ in $H1$. contradiction.
 × apply In_remove in $H0$. symmetry in $H1$. apply $\text{not_in_nodup_cancel}$ in $H1$.
 contradiction.

Qed.

Lemma parity_match_empty : $\forall q,$

$\text{parity_match } [] \ q \rightarrow$
 $\text{Permutation } [] (\text{nodup_cancel mono_eq_dec } q).$

Proof.

intros $q \ H$. unfold parity_match in H . simpl in H .
 symmetry in H . pose $(\text{even_nodup_cancel } q \ H)$. apply nothing_in_empty in n .
 rewrite n . auto.

Qed.

Lemma parity_match_refl : $\forall l,$

$\text{parity_match } l \ l.$

Proof.

intros l . unfold parity_match . auto.

Qed.

Lemma parity_match_sym : $\forall l \ m,$

$\text{parity_match } l \ m \leftrightarrow \text{parity_match } m \ l.$

Proof.

intros l m . unfold parity_match. split; intros H x ; auto.
Qed.

Lemma parity_match_trans : $\forall p\ q\ r$,
parity_match $p\ q \rightarrow$
parity_match $q\ r \rightarrow$
parity_match $p\ r$.

Proof.

intros $p\ q\ r\ H\ H0$. unfold parity_match in *. intros x .
rewrite H . rewrite $H0$. auto.

Qed.

Hint Resolve parity_match_refl parity_match_sym parity_match_trans.

Lemma parity_match_cons : $\forall a\ l1\ l2$,
parity_match $(a::l1)\ (a::l2) \leftrightarrow$
parity_match $l1\ l2$.

Proof.

intros $a\ l1\ l2$. unfold parity_match. split; intros $H\ x$.
- pose ($H\ x$). symmetry in e . simpl in e . destruct (mono_eq_dec $a\ x$); auto.
 repeat rewrite even_succ in e . repeat rewrite \leftarrow negb_even in e .
 apply Bool.negb_sym in e . rewrite Bool.negb_involutive in e . auto.
- simpl. destruct (mono_eq_dec $a\ x$); auto.
 repeat rewrite even_succ. repeat rewrite \leftarrow negb_even.
 apply Bool.negb_sym. rewrite Bool.negb_involutive. auto.

Qed.

Lemma parity_match_double : $\forall a\ l$,
parity_match $(a::a::l)\ l$.

Proof.

intros $a\ l$. unfold parity_match. intros x . simpl.
destruct (mono_eq_dec $a\ x$).
- rewrite even_succ. rewrite odd_succ. auto.
- auto.

Qed.

Lemma parity_match_cons_swap : $\forall a\ l1\ l2$,
parity_match $(a::l1)\ l2 \rightarrow$
parity_match $l1\ (a::l2)$.

Proof.

intros $a\ l1\ l2\ H$. apply (parity_match_cons a) in H .
apply parity_match_sym in H . apply parity_match_trans with $(r:=l1)$ in H .
apply parity_match_sym in H . auto. apply parity_match_double.

Qed.

Lemma parity_match_ln : $\forall a\ l1\ l2$,
even (count_occ mono_eq_dec $l1\ a$) = true \rightarrow

parity_match (a :: l1) l2 →
In a l2.

Proof.

intros a l1 l2 H H0. apply parity_match_cons_swap in H0.
rewrite H0 in H. simpl in H. destruct (mono_eq_dec a a); try contradiction.
rewrite even_succ in H. rewrite ← negb_even in H. rewrite Bool.negb_true_iff in H.
assert (count_occ mono_eq_dec l2 a > 0). destruct count_occ. inversion H.
apply gt_Sn_O. apply count_occ_In in H1. auto.

Qed.

Lemma Permutation_parity_match : ∀ p q,

Permutation p q → parity_match p q.

Proof.

intros p q H. induction H.
- auto.
- apply parity_match_cons. auto.
- repeat apply parity_match_cons_swap. unfold parity_match. intros x0.
simpl. destruct mono_eq_dec; destruct mono_eq_dec;
repeat (rewrite even_succ; rewrite odd_succ); auto.
- apply parity_match_trans with (q:=l'); auto.

Qed.

Lemma parity_nodup_cancel_Permutation : ∀ p q,

parity_match p q →
Permutation (nodup_cancel mono_eq_dec p) (nodup_cancel mono_eq_dec q).

Proof.

intros p q H. generalize dependent q. induction p; induction q; intros.
- auto.
- simpl nodup_cancel at 1. apply parity_match_empty. auto.
- simpl nodup_cancel at 2. apply **Permutation_sym**. apply parity_match_empty.
apply parity_match_sym. auto.
- clear IHq. destruct (mono_eq_dec a a0).
+ rewrite e. simpl. rewrite e in H. apply parity_match_cons in H.
destruct even eqn:Hev; rewrite H in Hev; rewrite Hev.
× apply perm_skip. apply remove_Permutation. auto.
× apply remove_Permutation. auto.
+ simpl nodup_cancel at 1. destruct even eqn:Hev.
× assert (Hev' := Hev). apply parity_match_In with (l2 := (a0 :: q)) in Hev; auto.
destruct Hev. symmetry in H0. contradiction. apply In_split in H0 as [l1 [l2
H0]].
rewrite H0. apply **Permutation_sym**. apply **Permutation_trans** with (l' := (
nodup_cancel mono_eq_dec (a :: l2 ++ a0 :: l1))). apply nodup_cancel_Permutation.
rewrite app_comm_cons. apply (**Permutation_app_comm**).
simpl. rewrite H0 in H. apply parity_match_trans with (r := (a :: l2 ++ a0 :: l1))

in H .

```

    apply parity_match_cons in  $H$ . rewrite  $H$  in  $Hev'$ . rewrite  $Hev'$ .
    apply perm_skip. apply remove_Permutation. apply Permutation_sym.
    apply IHp. auto. rewrite app_comm_cons. apply Permutation_parity_match.
    apply Permutation_app_comm.
  × apply parity_match_cons_swap in  $H$ . rewrite  $H$  in  $Hev$ . assert ( $Hev2 := Hev$ ).
    rewrite count_occ_Permutation with ( $l' := (a :: q ++ [a0])$ ) in  $Hev$ . simpl in  $Hev$ .
    destruct (mono_eq_dec  $a$ ); try contradiction. rewrite even_succ in  $Hev$ .
    rewrite ← negb_even in  $Hev$ . rewrite Bool.negb_false_iff in  $Hev$ .
    rewrite ← (not_in_remove _ mono_eq_dec  $a$ ).
    assert ( $\forall l$ , remove mono_eq_dec  $a$  (nodup_cancel mono_eq_dec ( $l$ )) =
      remove mono_eq_dec  $a$  (nodup_cancel mono_eq_dec ( $a :: l$ ))).
    intros  $l$ . simpl. destruct (even (count_occ _  $l$   $a$ )).
    simpl. destruct (mono_eq_dec  $a$ ); try contradiction.
    rewrite (not_in_remove _ _ (remove _ _)). auto. apply remove_in.
    rewrite (not_in_remove _ _ (remove _ _)). auto. apply remove_in.
    rewrite ( $H0$  ( $a0 :: q$ )). apply remove_Permutation. apply IHp. auto.
    apply not_in_nodup_cancel. rewrite count_occ_Permutation with ( $l' := (a0 :: q)$ )

```

in Hev .

```

    auto. replace ( $a0 :: q$ ) with ( $[a0] ++ q$ ); auto. apply Permutation_app_comm.
    apply perm_skip. replace ( $a0 :: q$ ) with ( $[a0] ++ q$ ); auto. apply Permutation_app_comm.

```

Qed.

Lemma count_occ_map_lt : $\forall p a f$,

$\text{count_occ mono_eq_dec } p a \leq \text{count_occ mono_eq_dec } (\text{map } f p) (f a)$.

Proof.

```

    intros  $p a f$ . induction  $p$ . auto. simpl. destruct mono_eq_dec.
    - rewrite  $e$ . destruct mono_eq_dec; try contradiction. simpl. apply le_n_S. auto.
    - destruct mono_eq_dec; auto.

```

Qed.

Lemma count_occ_map_sub : $\forall f a p$,

$\text{count_occ mono_eq_dec } (\text{map } f (\text{remove mono_eq_dec } a p)) (f a) =$
 $\text{count_occ mono_eq_dec } (\text{map } f p) (f a) - \text{count_occ mono_eq_dec } p a$.

Proof.

```

    intros  $f a p$ . induction  $p$ ; auto. simpl. destruct mono_eq_dec.
    - rewrite  $e$ . destruct mono_eq_dec; try contradiction. destruct mono_eq_dec;
      try contradiction. simpl. rewrite ←  $e$ . auto.
    - simpl. destruct mono_eq_dec.
      + destruct mono_eq_dec. symmetry in  $e0$ ; contradiction. rewrite IHp.
        rewrite sub_succ_l. auto. apply count_occ_map_lt.
      + destruct mono_eq_dec. symmetry in  $e$ ; contradiction. auto.

```

Qed.

Lemma count_occ_map_neq_remove : $\forall f a p x$,

$x \neq (f\ a) \rightarrow$
`count_occ mono_eq_dec (map f (remove mono_eq_dec a p)) x =`
`count_occ mono_eq_dec (map f p) x.`

Proof.

`intros. induction p as [|b]; auto. simpl. destruct (mono_eq_dec a b).`
`- destruct mono_eq_dec. rewrite ← e in e0. symmetry in e0. contradiction.`
`auto.`
`- simpl. destruct mono_eq_dec; auto.`

Qed.

Lemma existsb_false_forall : $\forall \{A\} f (l:\text{list } A),$
 $\text{existsb } f\ l = \text{false} \rightarrow$
 $(\forall a, \text{In } a\ l \rightarrow (f\ a) = \text{false}).$

Proof.

`intros A f l H a Hin. destruct (f a) eqn:Hfa.`
`- exfalso. rewrite ← Bool.negb_true_iff in H. apply (Bool.eq_true_false_abs _ H).`
`rewrite Bool.negb_false_iff. apply existsb_exists. ∃ a. split; auto.`
`- auto.`

Qed.

Lemma f_equal_sum_lt : $\forall f\ a\ b\ p,$
 $b \neq a \rightarrow (f\ a) = (f\ b) \rightarrow$
 $\text{count_occ mono_eq_dec } p\ b +$
 $\text{count_occ mono_eq_dec } p\ a \leq$
 $\text{count_occ mono_eq_dec (map } f\ p) (f\ a).$

Proof.

`intros f a b p Hne Hfe. induction p as [|c]; auto. simpl. destruct mono_eq_dec.`
`- rewrite e. destruct mono_eq_dec; try contradiction. rewrite Hfe.`
`destruct mono_eq_dec; try contradiction. simpl. apply le_n_S.`
`rewrite ← Hfe. auto.`
`- destruct mono_eq_dec.`
`+ rewrite e. destruct mono_eq_dec; try contradiction. rewrite plus_comm.`
`simpl. rewrite plus_comm. apply le_n_S. auto.`
`+ destruct mono_eq_dec.`
`× apply le_S. auto.`
`× auto.`

Qed.

Lemma count_occ_nodup_map_lt : $\forall p\ f\ a,$
 $\text{count_occ mono_eq_dec (nodup_cancel mono_eq_dec } p) a \leq$
 $\text{count_occ mono_eq_dec (map } f\ (\text{nodup_cancel mono_eq_dec } p)) (f\ a).$

Proof.

`intros p f a. induction p as [|b]; auto. simpl. destruct even eqn:Hev.`
`- simpl. destruct mono_eq_dec.`
`+ rewrite e. destruct mono_eq_dec; try contradiction. apply le_n_S. auto.`

```

    rewrite count_occ_remove. apply le_0_l.
  + rewrite count_occ_neq_remove; auto. rewrite not_in_remove.
    destruct mono_eq_dec; firstorder. apply not_in_nodup_cancel; auto.
- destruct (mono_eq_dec b a) eqn:Hba.
  + rewrite e. rewrite count_occ_remove. apply le_0_l.
  + rewrite count_occ_neq_remove; auto. destruct (mono_eq_dec (f b) (f a)) eqn:Hfba.
    × rewrite ← e. rewrite count_occ_map_sub. rewrite e. apply le_add_le_sub_l.
      apply f_equal_sum_lt; auto.
    × rewrite count_occ_map_neq_remove; auto.

```

Qed.

Lemma nodup_cancel_map : $\forall p f$,

Permutation

```

(nodup_cancel mono_eq_dec (map f (nodup_cancel mono_eq_dec p)))
(nodup_cancel mono_eq_dec (map f p)).

```

Proof.

```

intros p f. apply parity_nodup_cancel_Permutation. unfold parity_match.
intros x. induction p; auto. simpl. destruct (even (count_occ - p a)) eqn:Hev.
- simpl. destruct mono_eq_dec.
  + repeat rewrite even_succ. repeat rewrite ← negb_even. rewrite not_in_remove.
    rewrite IHp. auto. apply not_in_nodup_cancel. auto.
  + rewrite not_in_remove. apply IHp. apply not_in_nodup_cancel. auto.
- simpl. destruct mono_eq_dec.
  + rewrite ← e. rewrite count_occ_map_sub. rewrite even_sub. rewrite ← e in
    IHp.
    rewrite IHp. rewrite count_occ_nodup_cancel. rewrite Hev. rewrite even_succ.
    rewrite ← negb_even. destruct (even (count_occ - (map f p) -)); auto.
    apply count_occ_nodup_map_lt.
  + rewrite count_occ_map_neq_remove; auto.

```

Qed.

Lemma map_app_make_poly : $\forall m p$,

```

(∀ a, ln a p → is_mono a) →
make_poly (map (app m) (make_poly p)) = make_poly (map (app m) p).

```

Proof.

```

intros m p Hm. apply Permutation_sort_eq.
apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (map make_mono
  (map (app m) (nodup_cancel mono_eq_dec (map make_mono p)))))).
  apply nodup_cancel_Permutation. repeat apply Permutation_map.
  unfold make_poly. rewrite ← Permutation_MonoSort_l. auto.
  rewrite (no_map_make_mono p); auto. repeat rewrite map_map. apply nodup_cancel_map.

```

Qed.

Lemma mulMP''_make_poly : $\forall p m$,

```

(∀ a, ln a p → is_mono a) →

```

```
mulMP'' (make_poly p) m =
mulMP'' p m.
```

Proof.

```
intros p m. unfold mulMP''. apply map_app_make_poly.
```

Qed.

```
Lemma mulMP'_app : ∀ p q m,
mulMP' (p ++ q) m =
mulMP' p m ++ mulMP' q m.
```

Proof.

```
intros p q m. unfold mulMP'. repeat rewrite map_app. auto.
```

Qed.

```
Lemma mulMP'_assoc : ∀ q a m,
mulMP' (mulMP' q a) m =
mulMP' (mulMP' q m) a.
```

Proof.

```
intros q a m. unfold mulMP'. induction q.
- auto.
- simpl. repeat rewrite make_mono_pointless. f_equal.
+ apply Permutation_sort_mono_eq. apply Permutation_nodup.
  repeat rewrite app_assoc. apply Permutation_app_tail.
  apply Permutation_app_comm.
+ apply IHq.
```

Qed.

```
Lemma mulPP_assoc : ∀ p q r,
mulPP (mulPP p q) r = mulPP p (mulPP q r).
```

Proof.

```
intros p q r. rewrite (mulPP_comm _ (mulPP q _)). rewrite (mulPP_comm p _).
generalize dependent r. induction p; induction r as [|m];
repeat rewrite mulPP_0; repeat rewrite mulPP_0r; auto.
repeat rewrite mulPP_mulPP'' in *. unfold mulPP''. simpl.
repeat rewrite ← (make_poly_pointless_r _ (concat _)).
repeat rewrite mulPP''_refold. repeat rewrite (mulPP''_cons q).
pose (IHp (m :: r)). repeat rewrite mulPP_mulPP'' in e. rewrite ← e.
rewrite IHR. unfold mulPP'' at 2, mulPP'' at 4. simpl.
repeat rewrite make_poly_pointless_r. repeat rewrite app_assoc.
repeat rewrite ← (make_poly_pointless_r _ (concat _)).
repeat rewrite mulPP''_refold. pose (IHp r). repeat rewrite mulPP_mulPP'' in e0.
rewrite ← e0. repeat rewrite ← app_assoc. repeat rewrite mulMP'_mulMP''.
repeat rewrite ← mulPP''_cons. repeat rewrite mulMP''_make_poly.
repeat rewrite ← mulMP'_mulMP''. repeat rewrite app_assoc.
apply Permutation_sort_eq. apply nodup_cancel_Permutation. apply Permutation_map.
apply Permutation_app_tail. repeat rewrite mulMP'_app. rewrite mulMP'_assoc.
```

```

repeat rewrite ← app_assoc. apply Permutation_app_head. apply Permutation_app_comm.
intros a0 Hin. apply in_app_iff in Hin as []. unfold mulMP' in H.
apply in_map_iff in H as [x[]]. rewrite ← H; auto.
apply (make_poly_is_poly (concat (map (mulMP' q) r))). auto.
intros a0 Hin. apply in_app_iff in Hin as []. unfold mulMP' in H.
apply in_map_iff in H as [x[]]. rewrite ← H; auto.
apply (make_poly_is_poly (concat (map (mulMP' q) p))). auto.

```

Qed.

Lemma mulMP''_distr_addPP : $\forall m p q$,
 $\text{is_poly } p \rightarrow \text{is_poly } q \rightarrow$
 $\text{mulMP'' (addPP } p \text{ } q) \text{ } m = \text{addPP (mulMP'' } p \text{ } m) (\text{mulMP'' } q \text{ } m)$.

Proof.

```

intros m p q Hp Hq. unfold mulMP'', addPP. rewrite map_app_make_poly.
rewrite make_poly_pointless. rewrite make_poly_app_comm.
rewrite make_poly_pointless. rewrite make_poly_app_comm.
rewrite map_app. auto. intros a Hin. apply in_app_iff in Hin as [].
apply Hp. auto. apply Hq. auto.

```

Qed.

Lemma mulPP_distr_addPP : $\forall p q r$,
 $\text{is_poly } p \rightarrow \text{is_poly } q \rightarrow$
 $\text{mulPP (addPP } p \text{ } q) \text{ } r = \text{addPP (mulPP } p \text{ } r) (\text{mulPP } q \text{ } r)$.

Proof.

```

intros p q r Hp Hq. induction r; auto. rewrite mulPP_mulPP''. unfold mulPP''.
simpl. rewrite mulPP_mulPP'', (mulPP_mulPP'' q), make_poly_app_comm.
rewrite ← make_poly_pointless. rewrite make_poly_app_comm.
rewrite mulPP''_refold.
rewrite addPP_refold. repeat unfold mulPP'' at 2. simpl. unfold addPP at 4.
rewrite make_poly_pointless. rewrite addPP_refold.
rewrite (addPP_comm _ (make_poly _)).
unfold addPP at 4. rewrite make_poly_pointless. rewrite ← app_assoc.
rewrite make_poly_app_comm. rewrite ← app_assoc.
rewrite ← make_poly_pointless.
rewrite mulPP''_refold. rewrite ← app_assoc. rewrite app_assoc.
rewrite make_poly_app_comm.
rewrite ← app_assoc. rewrite ← make_poly_pointless. rewrite mulPP''_refold.
replace (make_poly (mulPP'' p r ++ mulMP' q a ++ mulPP'' q r ++ mulMP' p a))
  with (make_poly ((mulPP'' p r ++ mulPP'' q r) ++ mulMP' p a ++ mulMP' q a)).
rewrite ← make_poly_pointless. rewrite (addPP_refold (mulPP'' _ _)).
rewrite make_poly_app_comm. rewrite addPP_refold.
rewrite mulPP_mulPP'', (mulPP_mulPP'' p), (mulPP_mulPP'' q) in IHr.
rewrite ← IHr. unfold addPP at 4.
rewrite ← make_poly_pointless. unfold addPP. repeat rewrite mulMP'_mulMP''.

```

```

rewrite (make_poly_app_comm (mulMP'' - _) (mulMP' - _)).
rewrite mulMP'_mulMP''. rewrite (make_poly_app_comm (mulMP'' - _) (mulMP'' - _)).
repeat rewrite addPP_refold. f_equal. apply mulMP''_distr_addPP; auto.
apply make_poly_Permutation. rewrite ← app_assoc.
apply Permutation_app_head. rewrite app_assoc.
apply Permutation_trans with
  (l':=mulMP' q a ++ mulPP'' q r ++ mulMP' p a).
apply Permutation_app_comm.
auto.
Qed.

Lemma mulPP_distr_addPPr : ∀ p q r,
  is_poly p → is_poly q →
  mulPP r (addPP p q) = addPP (mulPP r p) (mulPP r q).
Proof.
  intros p q r Hp Hq. rewrite mulPP_comm. rewrite (mulPP_comm r p).
  rewrite (mulPP_comm r q). apply mulPP_distr_addPP; auto.
Qed.

Lemma mulPP_is_poly : ∀ p q,
  is_poly (mulPP p q).
Proof.
  intros p q. apply make_poly_is_poly.
Qed.

Lemma mulPP_mono_cons : ∀ x m,
  is_mono (x :: m) →
  mulPP [[x]] [m] = [x :: m].
Proof.
  intros x m H. unfold mulPP, distribute. simpl. apply Permutation_Sorted_eq.
  - apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (map make_mono [m++[x]]))).
    apply Permutation_sym. apply Permuted_sort. rewrite no_nodup_cancel_NoDup.
    simpl. assert (make_mono (m++[x]) = x::m).
    + rewrite ← no_make_mono; auto. apply Permutation_sort_mono_eq.
      repeat rewrite no_nodup_NoDup. replace (x::m) with ([x]++m); auto; apply
Permutation_app_comm.
      apply NoDup_VarSorted; apply H. apply Permutation_NoDup with (l:=(x::m)).
      replace (x::m) with ([x]++m); auto; apply Permutation_app_comm.
      apply NoDup_VarSorted; apply H.
    + rewrite H0. auto.
    + apply NoDup_cons; auto.
  - apply LocallySorted_sort.
  - apply Sorted_cons; auto.
Qed.

```

Lemma addPP_poly_cons : $\forall m p$,

is_poly ($m :: p$) \rightarrow
 addPP [m] $p = m :: p$.

Proof.

intros $m p H$. unfold addPP. simpl. rewrite no_make_poly; auto.

Qed.

Hint Resolve addPP_is_poly mulPP_is_poly.

Lemma mulPP_addPP_1 : $\forall p q r$,

is_poly $p \rightarrow$ is_poly $q \rightarrow$ is_poly $r \rightarrow$
 mulPP (addPP (mulPP $p q$) r) (addPP [\square] q) =
 mulPP (addPP [\square] q) r .

Proof.

intros $p q r Hp Hq Hr$. rewrite mulPP_distr_addPP; auto.

rewrite mulPP_distr_addPPr; auto. rewrite mulPP_1r; auto.

rewrite mulPP_assoc. rewrite mulPP_p_p; auto. rewrite addPP_p_p; auto.

rewrite addPP_0; auto. rewrite mulPP_comm. auto.

Qed.

Lemma partition_filter_fst $\{X\} p l$:

fst (partition $p l$) = @filter $X p l$.

Proof.

induction l ; simpl.

- trivial.

- rewrite $\leftarrow IHL$.

destruct (partition $p l$); simpl.

destruct ($p a$); now simpl.

Qed.

Lemma partition_filter_fst' : $\forall \{X\} p (l t f : \text{list } X)$,

partition $p l = (t, f) \rightarrow$
 $t = \text{@filter } X p l$.

Proof.

intros $X p l t f H$.

rewrite \leftarrow partition_filter_fst.

now rewrite H .

Qed.

Definition neg $\{X:\text{Type}\} := \text{fun } (f:X \rightarrow \text{bool}) \Rightarrow \text{fun } (a:X) \Rightarrow (\text{negb } (f a))$.

Lemma neg_true_false : $\forall \{X\} (p:X \rightarrow \text{bool}) (a:X)$,

$(p a) = \text{true} \leftrightarrow \text{neg } p a = \text{false}$.

Proof.

intros $X p a$. unfold neg. split; intro.

- rewrite H . auto.

- destruct ($p a$); intuition.

Qed.

Lemma neg_false_true : $\forall \{X\} (p:X \rightarrow \text{bool}) (a:X),$
 $(p\ a) = \text{false} \leftrightarrow \text{neg } p\ a = \text{true}.$

Proof.

```
intros X p a. unfold neg. split; intro.  
- rewrite H. auto.  
- destruct (p a); intuition.
```

Qed.

Lemma partition_filter_snd $\{X\}$ $p\ l :$
 $\text{snd } (\text{partition } p\ l) = @\text{filter } X\ (\text{neg } p)\ l.$

Proof.

```
induction l; simpl.  
- reflexivity.  
- rewrite  $\leftarrow$  IHL.  
  destruct (partition p l); simpl.  
  destruct (p a) eqn:Hp.  
  + simpl. apply neg_true_false in Hp. rewrite Hp; auto.  
  + simpl. apply neg_false_true in Hp. rewrite Hp; auto.
```

Qed.

Lemma partition_filter_snd' : $\forall \{X\} p\ (l\ t\ f : \text{list } X),$
 $\text{partition } p\ l = (t, f) \rightarrow$
 $f = @\text{filter } X\ (\text{neg } p)\ l.$

Proof.

```
intros X p l t f H.  
rewrite  $\leftarrow$  partition_filter_snd.  
now rewrite H.
```

Qed.

Lemma incl_Permutation : $\forall \{A:\text{Type}\} (l\ l'\ m:\text{list } A),$
 $\text{Permutation } l\ l' \rightarrow$
 $\text{incl } l\ m \rightarrow$
 $\text{incl } l'\ m.$

Proof.

```
intros A l l' m H H0. apply Permutation_incl in H as [].  
apply incl_tran with (m:=l); auto.
```

Qed.

Lemma make_poly_rem_vars : $\forall p\ x,$
 $\text{In } x\ (\text{vars } (\text{make_poly } p)) \rightarrow$
 $\text{In } x\ (\text{vars } p).$

Proof.

```
intros p x H. induction p.  
- inversion H.
```

```

- unfold vars. simpl. apply nodup_in. apply in_app_iff.
  unfold vars, make_poly in  $H$ . apply nodup_in in  $H$ .
  apply ln_concat_exists in  $H$  as  $[m \ \_\_]$ .
  apply ln_sorted in  $H$ . apply nodup_cancel_in in  $H$ .
  apply in_map_iff in  $H$  as  $[n \ \_\_]$ . destruct  $H1$ .
  + left. apply make_mono_ln. rewrite  $H1$ . rewrite  $H$ . auto.
  + right. apply ln_concat_exists.  $\exists n$ . split; auto. apply make_mono_ln.
    rewrite  $H$ . auto.

```

Qed.

Lemma incl_vars_addPP : $\forall p \ q \ xs$,
incl (vars p) $xs \wedge$ **incl** (vars q) $xs \rightarrow$
incl (vars (addPP $p \ q$)) xs .

Proof.

```

unfold incl, addPP.
intros  $p \ q \ xs \ [HinP \ HinQ] \ x \ HinPQ$ .
apply make_poly_rem_vars in  $HinPQ$ .
unfold vars in  $HinPQ$ .
apply nodup_in in  $HinPQ$ .
rewrite concat_app in  $HinPQ$ .
apply in_app_or in  $HinPQ$  as  $[Hin \ | \ Hin]$ .
- apply  $HinP$ . apply nodup_in. auto.
- apply  $HinQ$ . apply nodup_in. auto.

```

Qed.

Lemma incl_vars_mulPP : $\forall p \ q \ xs$,
incl (vars p) $xs \wedge$ **incl** (vars q) $xs \rightarrow$
incl (vars (mulPP $p \ q$)) xs .

Proof.

```

unfold incl, mulPP.
intros  $p \ q \ xs \ [HinP \ HinQ] \ x \ HinPQ$ .
apply make_poly_rem_vars in  $HinPQ$ .
apply ln_distribute in  $HinPQ$ . destruct  $HinPQ$ .
- apply  $HinP$ . auto.
- apply  $HinQ$ . auto.

```

Qed.

Lemma incl_nil : $\forall \{X:Type\} \ (l: \text{list } X)$,
incl $l \ [] \leftrightarrow l = []$.

Proof.

```

intros  $X \ l$ . unfold incl. split; intro  $H$ .
- destruct  $l$ ; [auto | destruct ( $H \ x$ ); intuition].
- intros  $a \ Hin$ . destruct  $l$ ; [auto | rewrite  $H$  in  $Hin$ ; auto].

```

Qed.

Lemma partition_Permutation : $\forall \{A:\text{Type}\} f (p \ l \ r: \text{list } A),$
 $\text{partition } f \ p = (l, r) \rightarrow$
 $\text{Permutation } p \ (l++r).$

Proof.

```

intros A f p. induction p; intros.
- simpl in H. inversion H. auto.
- simpl in H. destruct (partition f p). destruct (f a); inversion H.
  + simpl. apply perm_skip. apply IHp. f_equal. auto.
  + apply Permutation_trans with (l':=(a::l1 ++ l)). apply perm_skip.
    apply Permutation_trans with (l':=(l++l1)). apply IHp. f_equal.
    auto. apply Permutation_app_comm. apply Permutation_app_comm with (l:=(a::l1)).

```

Qed.

Lemma part_add_eq : $\forall f \ p \ l \ r,$
 $\text{is_poly } p \rightarrow$
 $\text{partition } f \ p = (l, r) \rightarrow$
 $p = \text{addPP } l \ r.$

Proof.

```

intros f p l r H H0. apply Permutation_Sorted_eq.
- generalize dependent l; generalize dependent r. induction p; intros.
  + simpl in H0. inversion H0. auto.
  + assert (H1:=H0); auto. apply partition_Permutation in H1. simpl in H0.
    destruct (partition f p) as [g d]. unfold addPP, make_poly.
    rewrite ← Permutation_MonoSort_r. rewrite unsorted_poly. destruct (f a); inversion
H0.
  × rewrite ← H3 in H1. apply H1.
  × rewrite ← H4 in H1. apply H1.
  × destruct H. apply NoDup_MonoSorted in H. apply (Permutation_NoDup H1 H).
  × intros m Hin. apply H. apply Permutation_sym in H1. apply (Permutation_in _
H1 Hin).
- apply Sorted_MonoSorted. apply H.
- apply Sorted_MonoSorted. apply make_poly_is_poly.

```

Qed.

Lemma part_fst_true : $\forall X \ p \ (l \ t \ f : \text{list } X),$
 $\text{partition } p \ l = (t, f) \rightarrow$
 $(\forall a, \text{In } a \ t \rightarrow p \ a = \text{true}).$

Proof.

```

intros X p l t f Hpart a Hin.
assert (Hf: t = filter p l).
- now apply partition_filter_fst' with f.
- assert (Hass := filter_In p a l).
  apply Hass.
  now rewrite ← Hf.

```

Qed.

Lemma part_snd_false : $\forall X p (x t f : \text{list } X),$
 partition $p x = (t, f) \rightarrow$
 $(\forall a, \text{In } a f \rightarrow p a = \text{false}).$

Proof.

intros $X p l t f Hpart a Hin.$
 assert ($Hf: f = \text{filter } (\text{neg } p) l$).
 - now apply partition_filter_snd' with t .
 - assert ($Hass := \text{filter_In } (\text{neg } p) a l$).
 rewrite $\leftarrow \text{neg_false_true}$ in $Hass$.
 apply $Hass$.
 now rewrite $\leftarrow Hf$.

Qed.

Lemma Forall_HdRel : $\forall \{X:\text{Type}\} c a (p:\text{list } X),$
 Forall $(c a) p \rightarrow \text{HdRel } c a p$.

Proof.

intros $X c a p H$. destruct p .
 - apply HdRel_nil.
 - apply HdRel_cons. apply Forall_inv in H . auto.

Qed.

Lemma Forall_incl : $\forall \{X:\text{Type}\} (c:X \rightarrow X \rightarrow \text{Prop}) a (p g:\text{list } X),$
 Forall $(c a) p \rightarrow \text{incl } g p \rightarrow \text{Forall } (c a) g$.

Proof.

intros $X c a p g H H0$. induction g .
 - apply Forall_nil.
 - rewrite Forall_forall in H . apply Forall_forall. intros $x Hin$.
 apply H . unfold incl in $H0$. apply $H0$. intuition.

Qed.

Lemma part_Sorted : $\forall \{X:\text{Type}\} (c:X \rightarrow X \rightarrow \text{Prop}) f p,$
 Relations_1.Transitive $c \rightarrow$
 Sorted $c p \rightarrow$
 $\forall l r, \text{partition } f p = (l, r) \rightarrow$
 Sorted $c l \wedge \text{Sorted } c r$.

Proof.

intros $X c f p Htran Hsort$. induction p ; intros.
 - simpl in H . inversion H . auto.
 - assert ($H0:=H$); auto. simpl in H . destruct (partition $f p$) as $[g d]$.
 destruct $(f a)$; inversion H .
 + assert ($\text{Forall } (c a) g \wedge \text{Sorted } c g \wedge \text{Sorted } c r \rightarrow \text{Sorted } c (a :: g) \wedge \text{Sorted } c$
 r).
 \times intros $H4$. split. apply Sorted_cons. apply $H4$. apply Forall_HdRel. apply $H4$.

```

apply H4.
  × apply H1. split.
    - apply Sorted_StronglySorted in Hsort; auto. apply StronglySorted_inv in Hsort
as [].
  apply (Forall_incl _ _ _ H5). apply partition_Permutation in H0.
  rewrite ← H2 in H0. simpl in H0. apply Permutation_cons_inv in H0.
  apply Permutation_incl in H0 as []. unfold incl. unfold incl in H6.
  intros a0 Hin. apply H6. intuition.
  - apply IHp. apply Sorted_inv in Hsort; apply Hsort. f_equal. auto.
+ assert (Forall (c a) d ∧ Sorted c l ∧ Sorted c d → Sorted c l ∧ Sorted c (a :: d)).
  × intros H4. split. apply H4. apply Sorted_cons. apply H4. apply Forall_HdRel.
apply H4.
  × apply H1. split.
    - apply Sorted_StronglySorted in Hsort; auto. apply StronglySorted_inv in Hsort
as [].
  apply (Forall_incl _ _ _ H5). apply partition_Permutation in H0.
  rewrite ← H3 in H0. simpl in H0. apply Permutation_trans with (l' := (a :: d ++ l))
in H0.
  apply Permutation_cons_inv in H0. apply Permutation_trans with (l' := (l ++ d))
in H0.
  apply Permutation_incl in H0 as []. unfold incl. unfold incl in H6.
  intros a0 Hin. apply H6. intuition. apply Permutation_app_comm.
  apply Permutation_app_comm with (l' := (a :: d)).
  - apply IHp. apply Sorted_inv in Hsort; apply Hsort. f_equal. auto.
Qed.

Lemma part_is_poly : ∀ f p l r,
  is_poly p →
  partition f p = (l, r) →
  is_poly l ∧ is_poly r.
Proof.
  intros f p l r Hpoly Hpart. destruct Hpoly. split; split.
  - apply (part_Sorted _ _ _ lex_Lt_Transitive H _ _ Hpart).
  - intros m Hin. apply H0. apply elements_in_partition with (x := m) in Hpart.
    apply Hpart; auto.
  - apply (part_Sorted _ _ _ lex_Lt_Transitive H _ _ Hpart).
  - intros m Hin. apply H0. apply elements_in_partition with (x := m) in Hpart.
    apply Hpart; auto.
Qed.

```

Chapter 6

Library B_Unification.poly_unif

```
Require Import ListSet.
Require Import List.
Import ListNotations.
Require Import Arith.
Require Import Permutation.
Require Import Sorting.
Import Nat.

Require Export poly.

Definition repl := (prod var poly).
Definition subst := list repl.

Definition inDom (x : var) (s : subst) : bool :=
  existsb (beq_nat x) (map fst s).

Fixpoint appSubst (s : subst) (x : var) : poly :=
  match s with
  | [] => [[x]]
  | (y, p) :: s' => if (x =? y) then p else (appSubst s' x)
  end.

Fixpoint substM (s : subst) (m : mono) : poly :=
  match m with
  | [] => []
  | x :: m => mulPP (appSubst s x) (substM s m)
  end.

Lemma substM_is_poly : ∀ s m,
  is_poly (substM s m).
Proof.
  intros s m. unfold substM; destruct m; auto.
Qed.
```

Definition substP (s : subst) (p : poly) : poly :=
 make_poly (concat (map (substM s) p)).

Lemma substP_is_poly : $\forall s p$,
 is_poly (substP s p).

Proof.

intros. unfold substP. auto.

Qed.

Hint Resolve substP_is_poly substM_is_poly.

Lemma substP_Sorted : $\forall s p$,
 Sorted (fun m n : list nat \Rightarrow lex Nat.compare m n = Lt) (substP s p).

Proof.

intros s p. apply substP_is_poly.

Qed.

Lemma make_poly_Permutation : $\forall p q$,
 Permutation p q \rightarrow
 Permutation (make_poly p) (make_poly q).

Proof.

intros p q H. unfold make_poly. apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (map make_mono p))).
 - apply Permutation_sym. apply MonoSort.Permuted_sort.
 - apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (map make_mono q))).
 + apply nodup_cancel_Permutation. apply Permutation_map. apply H.
 + apply MonoSort.Permuted_sort.

Qed.

Lemma substM_cons : $\forall x m$,
 $\neg \text{In } x m \rightarrow$
 $\forall q s, \text{substM } ((x, q) :: s) m = \text{substM } s m$.

Proof.

intros. induction m.
 - auto.
 - simpl. f_equal.
 + destruct (a =? x) eqn:H0.
 × symmetry in H0. apply beq_nat_eq in H0. exfalso. simpl in H.
 apply H. left. auto.
 × auto.
 + apply IHm. intro. apply H. right. auto.

Qed.

Lemma substP_cons : $\forall x p$,
 $(\forall m, \text{In } m p \rightarrow \neg \text{In } x m) \rightarrow$
 $\forall q s, \text{substP } ((x, q) :: s) p = \text{substP } s p$.

Proof.

```

intros. induction p.
- auto.
- unfold substP. simpl. repeat rewrite ← (make_poly_pointless_r _ (concat _)).
  f_equal. f_equal.
  + apply substM_cons. apply H. left. auto.
  + apply IHp. intros. apply H. right. auto.
Qed.

Lemma substP_1 : ∀ s,
  substP s [[]] = [[]].
Proof.
  intros. unfold substP. simpl. auto.
Qed.

Lemma substP_0 : ∀ s,
  substP s [] = [].
Proof.
  intros. unfold substP. simpl. auto.
Qed.

Lemma substP_Permutation : ∀ s p q,
  Permutation p q →
  Permutation (substP s p) (substP s q).
Proof.
  intros s p q H. induction H.
  - simpl. auto.
  - unfold substP. simpl. repeat rewrite ← (make_poly_pointless_r _ (concat _)).
    apply make_poly_Permutation. apply Permutation_app_head. apply IHPermutation.
  - unfold substP. simpl. apply make_poly_Permutation. repeat rewrite app_assoc.
    apply Permutation_app_tail. apply Permutation_app_comm.
  - apply Permutation_trans with (l':=(substP s l')); auto.
Qed.

Lemma substP_sort_assoc : ∀ s p,
  substP s (MonoSort.sort p) = MonoSort.sort (substP s p).
Proof.
  intros s p. apply Permutation_Sorted_eq.
  - apply Permutation_trans with (l':=(substP s p)).
    + apply substP_Permutation. apply Permutation_sym. apply MonoSort.Permuted_sort.
    + apply MonoSort.Permuted_sort.
  - apply Sorted_MonoSorted. apply substP_Sorted.
  - apply MonoSort.LocallySorted_sort.
Qed.

Lemma n_le_1 : ∀ n,
   $n \leq 1 \rightarrow n = 0 \vee n = 1$ .

```


Proof.

```
intros n H. induction n; firstorder.
```

Qed.

Lemma count_occ_map_sub_not_in : $\forall f a p,$
 $\forall x, \text{count_occ mono_eq_dec } (f a) x = 0 \rightarrow$
 $\text{count_occ mono_eq_dec } (\text{concat } (\text{map } f (\text{remove mono_eq_dec } a p))) x =$
 $\text{count_occ mono_eq_dec } (\text{concat } (\text{map } f p)) x.$

Proof.

```
intros f a p x H. induction p as [|b|]; auto. simpl.  
rewrite count_occ_app. destruct mono_eq_dec.  
- rewrite e in H. firstorder.  
- simpl. rewrite count_occ_app. auto.
```

Qed.

Lemma count_occ_concat_map_lt : $\forall p a f x,$
 $\text{count_occ mono_eq_dec } (f a) x = 1 \rightarrow$
 $\text{count_occ mono_eq_dec } p a \leq \text{count_occ mono_eq_dec } (\text{concat } (\text{map } f p)) x.$

Proof.

```
intros p a f x H. induction p. auto. simpl. destruct mono_eq_dec.  
- rewrite e. rewrite count_occ_app. rewrite H. simpl. firstorder.  
- rewrite count_occ_app. induction (count_occ mono_eq_dec (f a0) x); firstorder.
```

Qed.

Lemma count_occ_map_sub_in : $\forall f a p,$
 $\forall x, \text{count_occ mono_eq_dec } (f a) x = 1 \rightarrow$
 $\text{count_occ mono_eq_dec } (\text{concat } (\text{map } f (\text{remove mono_eq_dec } a p))) x =$
 $\text{count_occ mono_eq_dec } (\text{concat } (\text{map } f p)) x - \text{count_occ mono_eq_dec } p a.$

Proof.

```
intros f a p x H. induction p as [|b|]; auto. simpl. destruct mono_eq_dec.  
- rewrite e. destruct mono_eq_dec; try contradiction. rewrite count_occ_app.  
  rewrite e in H. rewrite H. simpl. rewrite ← e. auto.  
- simpl. destruct mono_eq_dec. symmetry in e. contradiction.  
  repeat rewrite count_occ_app. rewrite IHp. rewrite add_sub_assoc. auto.  
  apply count_occ_concat_map_lt; auto.
```

Qed.

Lemma f_equal_concat_sum_lt : $\forall f a b p x,$
 $b \neq a \rightarrow$
 $(\forall x, \text{NoDup } (f x)) \rightarrow$
 $\text{count_occ mono_eq_dec } (f a) x = 1 \rightarrow$
 $\text{count_occ mono_eq_dec } (f b) x = 1 \rightarrow$
 $\text{count_occ mono_eq_dec } p b +$
 $\text{count_occ mono_eq_dec } p a \leq$
 $\text{count_occ mono_eq_dec } (\text{concat } (\text{map } f p)) x.$

Proof.

```

intros f a b p x Hne Hnd Hfa Hfb. induction p as [|c]; auto. simpl.
destruct mono_eq_dec.
- rewrite e. destruct mono_eq_dec; try contradiction. rewrite count_occ_app.
  firstorder.
- destruct mono_eq_dec.
  + rewrite e. rewrite count_occ_app. firstorder.
  + rewrite count_occ_app. pose (Hnd c). rewrite (NoDup_count_occ mono_eq_dec)

```

in $n1$.

```

pose (n1 x). apply n_le_1 in l. clear n1. destruct l; firstorder.

```

Qed.

Lemma count_occ_nodup_concat_map_lt : $\forall p f a x$,

```

(∀ x, NoDup (f x)) →
count_occ mono_eq_dec (f a) x = 1 →
count_occ mono_eq_dec (nodup_cancel mono_eq_dec p) a ≤
count_occ mono_eq_dec (concat (map f (nodup_cancel mono_eq_dec p))) x.

```

Proof.

```

intros p f a x Hn H. induction p as [|b]; auto. simpl. destruct even eqn:Hev.
- simpl. destruct mono_eq_dec.
  + rewrite e. rewrite count_occ_remove, count_occ_app. rewrite H. firstorder.
  + rewrite count_occ_neq_remove; auto. rewrite not_in_remove.
    rewrite count_occ_app. firstorder. apply not_in_nodup_cancel. auto.
- destruct (mono_eq_dec b a) eqn:Hba.
  + rewrite e. rewrite count_occ_remove. firstorder.
  + rewrite count_occ_neq_remove; auto. assert (Hn1:=(Hn b)).
    rewrite (NoDup_count_occ mono_eq_dec) in Hn1. assert (Hn2:=(Hn1 x)).
    clear Hn1. apply n_le_1 in Hn2. destruct Hn2.
    × rewrite count_occ_map_sub_not_in; auto.
    × apply (count_occ_map_sub_in _ _ (nodup_cancel mono_eq_dec p)) in H0 as H1.
      rewrite H1. apply le_add_le_sub_l. apply f_equal_concat_sum_lt; auto.

```

Qed.

Lemma nodup_cancel_concat_map : $\forall p f$,

```

(∀ x, NoDup (f x)) →
Permutation
(nodup_cancel mono_eq_dec (concat (map f (nodup_cancel mono_eq_dec p))))
(nodup_cancel mono_eq_dec (concat (map f p))).

```

Proof.

```

intros p f H. apply parity_nodup_cancel_Permutation. unfold parity_match.
intros x. induction p; auto. simpl. destruct (even (count_occ _ p a)) eqn:Hev.
- simpl. repeat rewrite count_occ_app. repeat rewrite even_add. rewrite not_in_remove.
  rewrite IHp. auto. apply not_in_nodup_cancel. auto.
- assert (H0:=(H a)). rewrite (NoDup_count_occ mono_eq_dec) in H0.

```

```

assert (H1:=(H0 x)). clear H0. apply n_le_1 in H1. rewrite count_occ_app.
rewrite even_add. destruct H1.
+ apply (count_occ_map_sub_not_in _ _ (nodup_cancel mono_eq_dec p)) in H0 as H1.
  rewrite H0, H1, IHp. simpl.
  destruct (even (count_occ _ (concat (map f p)) x)); auto.
+ apply (count_occ_map_sub_in _ _ (nodup_cancel mono_eq_dec p)) in H0 as H1.
  rewrite H0, H1, even_sub, IHp. simpl. rewrite count_occ_nodup_cancel. rewrite
Hev.
  destruct (even (count_occ _ (concat (map f p)) x)); auto.
  apply count_occ_nodup_concat_map_lt; auto.
Qed.

Lemma substP_refold : ∀ s p,
  make_poly (concat (map (substM s) p)) =
  substP s p.
Proof. auto. Qed.

Lemma empty_substM : ∀ (m : mono),
  is_mono m →
  substM [] m = [m].
Proof.
  intros. induction m.
  - auto.
  - simpl. apply mono_cons in H as H0.
    rewrite IHm; auto.
    apply mulPP_mono_cons; auto.
Qed.

Lemma empty_substP : ∀ (p : poly),
  is_poly p →
  substP [] p = p.
Proof.
  intros.
  induction p.
  - auto.
  - unfold substP. simpl. apply poly_cons in H as H0. destruct H0.
    rewrite ← make_poly_pointless_r. rewrite substP_refold.
    rewrite IHp; auto.
    rewrite empty_substM; auto.
    apply addPP_poly_cons; auto.
Qed.

Lemma substP_distr_addPP : ∀ p q s,
  is_poly p → is_poly q →
  substP s (addPP p q) = addPP (substP s p) (substP s q).

```

Proof.

```

intros p q s Hp Hq. unfold substP, addPP.
apply Permutation_sort_eq. apply Permutation_trans with (l':=
  (nodup_cancel mono_eq_dec (map make_mono (concat (map (substM s)
    (nodup_cancel mono_eq_dec (map make_mono (p ++ q)))))))).
  apply nodup_cancel_Permutation. apply Permutation_map. apply Permutation_concat.
  apply Permutation_map. unfold make_poly. rewrite ← Permutation_MonoSort_l.
auto.
apply Permutation_sym. apply Permutation_trans with (l':=(nodup_cancel
  mono_eq_dec (map make_mono (nodup_cancel mono_eq_dec (map make_mono (concat
    (map (substM s) (p)))) ++ (nodup_cancel mono_eq_dec (map make_mono (concat
    (map (substM s) q)))))))). apply nodup_cancel_Permutation. apply Permutation_map.
  apply Permutation_app; unfold make_poly; rewrite ← Permutation_MonoSort_l; auto.
rewrite (no_map_make_mono ((nodup_cancel _ ) ++ (nodup_cancel _ ))).
rewrite nodup_cancel_pointless. apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec
  (nodup_cancel mono_eq_dec (map make_mono (concat (map (substM s) q)))
  ++ map make_mono (concat (map (substM s) (p)))))).
  apply nodup_cancel_Permutation. apply Permutation_app_comm.
rewrite nodup_cancel_pointless. rewrite ← map_app. rewrite ← concat_app.
rewrite ← map_app. rewrite (no_map_make_mono (p ++ q)).
apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (map make_mono
  (concat (map (substM s) (p ++ q)))))). apply nodup_cancel_Permutation.
  apply Permutation_map. apply Permutation_concat. apply Permutation_map.
  apply Permutation_app_comm.
apply Permutation_sym. repeat rewrite List.concat_map.
repeat rewrite map_map. apply nodup_cancel_concat_map.
intros x. rewrite no_map_make_mono. apply NoDup_MonoSorted; apply substM_is_poly.
intros m Hin; apply (substM_is_poly s x); auto.
intros m Hin; apply in_app_iff in Hin as []; destruct Hp; destruct Hq; auto.
intros m Hin; apply in_app_iff in Hin as []; apply nodup_cancel_in in H;
  apply mono_in_map_make_mono in H; auto.
Qed.

```

Lemma nodup_cancel_app_Permutation : $\forall a b c d,$

Permutation (nodup_cancel mono_eq_dec a) (nodup_cancel mono_eq_dec b) \rightarrow
Permutation (nodup_cancel mono_eq_dec c) (nodup_cancel mono_eq_dec d) \rightarrow
Permutation (nodup_cancel mono_eq_dec (a ++ c)) (nodup_cancel mono_eq_dec (b ++ d)).

Proof.

```

intros a b c d H H0. rewrite ← (nodup_cancel_pointless a), ← (nodup_cancel_pointless
  b),
  ← (nodup_cancel_pointless_r c), ← (nodup_cancel_pointless_r d).
  apply nodup_cancel_Permutation. apply Permutation_app; auto.
Qed.

```

Lemma substM_Permutation : $\forall s a b$,
 $\text{Permutation } a b \rightarrow$
 $\text{Permutation } (\text{substM } s a) (\text{substM } s b)$.

Proof.

```

intros s a b H. induction H.
- simpl. auto.
- simpl. repeat rewrite mulPP_mulPP''. apply make_poly_Permutation.
  apply Permutation_concat. apply Permutation_map. auto.
- simpl. rewrite mulPP_comm. rewrite mulPP_assoc.
  rewrite (mulPP_comm (substM s l)). auto.
- apply Permutation_trans with (l':=(substM s l')); auto.

```

Qed.

Lemma substM_Permutation_eq : $\forall s a b$,
 $\text{Permutation } a b \rightarrow$
 $\text{substM } s a = \text{substM } s b$.

Proof.

```

intros s a b H. induction H.
- auto.
- simpl. rewrite IHPermutation. auto.
- simpl. rewrite mulPP_comm. rewrite mulPP_assoc.
  rewrite (mulPP_comm (substM s l)). auto.
- rewrite IHPermutation1. rewrite IHPermutation2. auto.

```

Qed.

Definition is_poly_subst (s : subst) : Prop :=
 $\forall x p, \text{In } (x, p) s \rightarrow \text{is_poly } p$.

Lemma appSubst_poly : $\forall x s$,
 $\text{is_poly_subst } s \rightarrow$
 $\text{is_poly } (\text{appSubst } s x)$.

Proof.

```

intros x s H. unfold is_poly_subst in H. induction s.
- simpl. auto.
- simpl. destruct a eqn:Ha. destruct (x =? v).
  + apply (H v). intuition.
  + apply IHs. intros x0 p0 H0. apply (H x0). intuition.

```

Qed.

Lemma substM_nodup_pointless : $\forall s m$,
 $\text{is_poly_subst } s \rightarrow$
 $\text{substM } s (\text{nodup var_eq_dec } m) =$
 $\text{substM } s m$.

Proof.

```

intros s m Hps. induction m. auto. simpl. destruct in_dec.

```

```

- apply in_split in i. destruct i as [l1 [l2 H]].
  assert (Permutation m (a :: l1 ++ l2)). rewrite H. apply Permutation_sym.
  apply Permutation_middle.
  apply substM_Permutation_eq with (s:=s) in H0. rewrite H0. simpl.
  rewrite (mulPP_comm _ (substM _ _)). rewrite mulPP_comm. rewrite mulPP_assoc.
  rewrite mulPP_p_p. rewrite mulPP_comm. rewrite IHm. rewrite H0.
  simpl. auto. apply appSubst_poly. auto.
- simpl. rewrite IHm. auto.

```

Qed.

Lemma substM_distr_mulMP : $\forall m a s$,

is_poly_subst $s \rightarrow$

is_mono $a \rightarrow$

Permutation

```

(nodup_cancel mono_eq_dec (map make_mono (substM s (make_mono
  (make_mono (m ++ a))))))
(nodup_cancel mono_eq_dec (map make_mono (concat (map (mulMP''
  (map make_mono (substM s m))) (map make_mono (substM s a)))))).

```

Proof.

```

intros m a s Hps H. rewrite (no_make_mono (make_mono (m ++ a))); auto.
repeat rewrite (no_map_make_mono (substM s _)); auto. apply Permutation_trans
  with (l':=(nodup_cancel mono_eq_dec (substM s (nodup var_eq_dec (m ++ a))))).
  apply nodup_cancel_Permutation. apply substM_Permutation. unfold make_mono.
  rewrite ← Permutation_VarSort_l. auto.
induction m.
- simpl. pose (mulPP_1r (substM s a)). rewrite mulPP_comm in e.
  pose (substM_is_poly s a). apply e in i. rewrite mulPP_mulPP''' in i.
  unfold mulPP''' in i. rewrite ← no_make_poly in i; auto.
  apply Permutation_sort_eq in i. rewrite i. rewrite no_nodup_NoDup.
  rewrite no_map_make_mono. auto. intros m Hin. apply (substM_is_poly s a); auto.
  apply NoDup_VarSorted. auto.
- simpl substM at 2. apply Permutation_sort_eq. rewrite make_poly_refold.
  rewrite mulPP'''_refold. rewrite ← mulPP_mulPP'''. rewrite mulPP_assoc.
  repeat rewrite mulPP_mulPP'''. apply Permutation_sort_eq.
  rewrite substM_nodup_pointless; auto. simpl. rewrite mulPP_mulPP'''.
  unfold mulPP''' at 1. apply Permutation_sort_eq in IHm.
  rewrite make_poly_refold in IHm. rewrite mulPP'''_refold in IHm.
  rewrite no_nodup_cancel_NoDup in IHm. rewrite no_sort_MonoSorted in IHm.
  rewrite ← substM_nodup_pointless; auto. rewrite IHm. unfold make_poly.
  apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (nodup_cancel
    mono_eq_dec (map make_mono (concat (map (mulMP'' (appSubst s a0)) (mulPP'''
      (substM s m) (substM s a)))))))). apply nodup_cancel_Permutation.
  rewrite ← Permutation_MonoSort_l. auto. rewrite no_nodup_cancel_NoDup; auto.

```

```

    apply NoDup_nodup_cancel. apply substM_is_poly. apply NoDup_MonoSorted.
    apply substM_is_poly.
- intros m0 Hin. apply (substM_is_poly s a). auto.
- intros m0 Hin. apply (substM_is_poly s m). auto.
- intros m0 Hin. apply (substM_is_poly s (make_mono (m++a))). auto.
Qed.

Lemma map_substM_distr_map_mulMP :  $\forall m p s$ ,
  is_poly_subst s  $\rightarrow$ 
  is_poly p  $\rightarrow$ 
  Permutation
  (nodup_cancel mono_eq_dec (map make_mono (concat (map (substM s) (map make_mono
    (mulMP'' p m))))))
  (nodup_cancel mono_eq_dec
    (map make_mono
      (concat
        (map (mulMP'' (map make_mono (concat (map (substM s) p)))) (map make_mono
          (substM s m)))))).
Proof.
  intros m p s Hps H. unfold mulMP'' at 1. apply Permutation_trans with (l':=
    (nodup_cancel mono_eq_dec (map make_mono (concat (map (substM s) (map
      make_mono (nodup_cancel mono_eq_dec (map make_mono (map (app m) (p)))))))).
    apply nodup_cancel_Permutation, Permutation_map, Permutation_concat,
    Permutation_map, Permutation_map. unfold make_poly.
    rewrite  $\leftarrow$  Permutation_MonoSort_l. auto.
  apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (map make_mono
    (concat (map (substM s) (map make_mono (map make_mono (map (app m)
      (p)))))))). repeat rewrite List.concat_map. rewrite map_map.
    rewrite map_map. rewrite (map_map _ (map make_mono)). rewrite (map_map
    make_mono).
    rewrite nodup_cancel_concat_map. auto. intros x. rewrite no_map_make_mono.
    apply NoDup_MonoSorted. apply (substM_is_poly s (make_mono x)). intros m0 Hin.
    pose (substM_is_poly s (make_mono x)). apply i. auto. induction p.
- simpl. induction (map make_mono (substM s m)); auto.
- simpl. rewrite map_app. apply Permutation_sym. apply Permutation_trans with
  (l':=(nodup_cancel mono_eq_dec (map make_mono (concat (map (mulMP'' (map
    make_mono (substM s m))) (map make_mono (substM s a ++ concat (map (substM s)
    p)))))). apply Permutation_sort_eq. repeat (rewrite make_poly_refold,
    mulPP''_refold,  $\leftarrow$  mulPP_mulPP'''). apply mulPP_comm.
    repeat rewrite map_app. rewrite concat_app, map_app. apply Permutation_sym.
    apply nodup_cancel_app_Permutation. apply substM_distr_mulMP; auto. apply H.
  intuition.
  apply Permutation_sym. apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec

```

```

    (map make_mono (concat (map (mulMP'' (map make_mono (concat (map (substM s)
p))))))
    (map make_mono (substM s m)))))). apply Permutation_sort_eq.
  repeat (rewrite make_poly_refold, mulPP''_refold, ← mulPP_mulPP'').
  apply mulPP_comm. apply Permutation_sym. apply IHp. apply poly_cons in H.
  apply H.

```

Qed.

Lemma substP_distr_mulPP : $\forall p q s$,
 is_poly_subst $s \rightarrow$
 is_poly $p \rightarrow$
 substP s (mulPP $p q$) = mulPP (substP $s p$) (substP $s q$).

Proof.

```

  intros p q s Hps H. repeat rewrite mulPP_mulPP''. unfold substP, mulPP''.
  apply Permutation_sort_eq. apply Permutation_trans with (l':=(nodup_cancel
  mono_eq_dec (map make_mono (concat (map (substM s) (nodup_cancel mono_eq_dec
  (map make_mono (concat (map (mulMP'' p) q)))))))). apply nodup_cancel_Permutation.
  apply Permutation_map. apply Permutation_concat. apply Permutation_map.
  unfold make_poly. rewrite ← Permutation_MonoSort_l. auto.
  apply Permutation_sym. apply Permutation_trans with (l':=(nodup_cancel
  mono_eq_dec (map make_mono (concat (map (mulMP'' (make_poly (concat (map
  (substM s) p)))) (nodup_cancel mono_eq_dec (map make_mono (concat (map
  (substM s) q)))))))). apply nodup_cancel_Permutation. apply Permutation_map.
  apply Permutation_concat. apply Permutation_map. unfold make_poly.
  rewrite ← Permutation_MonoSort_l. auto.
  apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (map make_mono
  (concat (map (mulMP'' (make_poly (concat (map (substM s) p)))) (map make_mono
  (concat (map (substM s) q)))))))).
  repeat rewrite (List.concat_map make_mono (map (mulMP'' _) _)).
  repeat rewrite (map_map _ (map make_mono)). apply nodup_cancel_concat_map.
  intros x. rewrite no_map_make_mono. unfold mulMP''. apply NoDup_MonoSorted.
  apply make_poly_is_poly. intros m Hin. apply mono_in_make_poly in Hin; auto.
  apply Permutation_sort_eq. rewrite make_poly_refold. rewrite mulPP''_refold.
  rewrite ← mulPP_mulPP''. rewrite mulPP_comm. rewrite mulPP_mulPP''.
  apply Permutation_sort_eq. apply Permutation_trans with (l':=(nodup_cancel
  mono_eq_dec (map make_mono (concat (map (mulMP'' (map make_mono (concat (map
  (substM s) q)))) (nodup_cancel mono_eq_dec (map make_mono (concat (map
  (substM s) p)))))))). apply nodup_cancel_Permutation.
  apply Permutation_map. apply Permutation_concat. apply Permutation_map.
  unfold make_poly. rewrite ← Permutation_MonoSort_l. auto.
  apply Permutation_trans with (l':=(nodup_cancel mono_eq_dec (map make_mono
  (concat (map (mulMP'' (map make_mono (concat (map (substM s) q)))) (map
  make_mono (concat (map (substM s) p)))))))).

```



```

repeat rewrite (List.concat_map make_mono (map (mulMP'' _) _)).
repeat rewrite (map_map _ (map make_mono)). apply nodup_cancel_concat_map.
intros  $x$ . rewrite no_map_make_mono. unfold mulMP''. apply NoDup_MonoSorted.
apply make_poly_is_poly. intros  $m$   $Hin$ . apply mono_in_make_poly in  $Hin$ ; auto.
apply Permutation_sort_eq. rewrite make_poly_refold. rewrite mulPP''_refold.
rewrite  $\leftarrow$  mulPP_mulPP''. rewrite mulPP_comm. rewrite mulPP_mulPP''.
apply Permutation_sort_eq. apply Permutation_sym.
apply Permutation_trans with ( $l' :=$  (nodup_cancel mono_eq_dec (map make_mono
  (concat (map (substM  $s$ ) (map make_mono (concat (map (mulMP''  $p$ )  $q$ )))))))).
repeat rewrite (List.concat_map make_mono (map _ _)). repeat rewrite map_map.
rewrite nodup_cancel_concat_map. auto. intros  $x$ . rewrite no_map_make_mono.
apply NoDup_MonoSorted; apply substM_is_poly.
intros  $m$   $Hin$ ; apply (substM_is_poly  $s$   $x$ ); auto.
induction  $q$ ; auto. simpl. repeat rewrite map_app. repeat rewrite concat_app.
repeat rewrite map_app. repeat rewrite  $\leftarrow$  (nodup_cancel_pointless (map _ _)).
repeat rewrite  $\leftarrow$  (nodup_cancel_pointless_r _ (map _ _)).
apply nodup_cancel_Permutation. apply Permutation_app.
apply map_substM_distr_map_mulMP; auto. apply IHq.
Qed.

```

Definition unifier (s : subst) (p : poly) : Prop :=
 substP s p = [].

Definition unifiable (p : poly) : Prop :=
 $\exists s$, is_poly_subst s \wedge unifier s p .

Definition subst_comp (s t u : subst) : Prop :=
 $\forall x$,
 substP t (substP s [[x]]) = substP u [[x]].

Definition more_general (s t : subst) : Prop :=
 $\exists u$, subst_comp s u t .

Definition mgu (s : subst) (p : poly) : Prop :=
 unifier s p \wedge
 $\forall t$,
 unifier t p \rightarrow
 is_poly_subst t \rightarrow
 more_general s t .

Definition reprod_unif (s : subst) (p : poly) : Prop :=
 unifier s p \wedge
 $\forall t$,
 unifier t p \rightarrow
 is_poly_subst t \rightarrow
 subst_comp s t t .

Lemma subst_var : $\forall x$ s ,

```
is_poly_subst s →
substP s [[x]] = appSubst s x.
```

Proof.

```
intros. simpl.
apply (appSubst_poly x s) in H. unfold substP. simpl. rewrite app_nil_r.
rewrite mulPP_1r; auto. rewrite no_make_poly; auto.
```

Qed.

Lemma subst_comp_poly : $\forall s t u,$
 $\text{is_poly_subst } s \rightarrow$
 $\text{is_poly_subst } t \rightarrow$
 $\text{is_poly_subst } u \rightarrow$
 $(\forall x, \text{substP } t (\text{substP } s [[x]]) = \text{substP } u [[x]]) \rightarrow$
 $\forall p,$
 $\text{substP } t (\text{substP } s p) = \text{substP } u p.$

Proof.

```
intros. induction p.
- simpl. auto.
- simpl. unfold substP at 2. simpl. rewrite ← make_poly_pointless_r.
  rewrite addPP_refold. rewrite substP_distr_addPP; auto. unfold substP at 3.
  simpl. rewrite ← make_poly_pointless_r. rewrite addPP_refold. f_equal.
  + induction a.
    × simpl. auto.
    × simpl. rewrite substP_distr_mulPP; auto. f_equal.
      - rewrite ← subst_var; auto.
        rewrite ← subst_var; auto.
      - apply IHa.
      - apply appSubst_poly; auto.
  + rewrite substP_refold. apply IHp.
```

Qed.

Lemma reprod_is_mgu : $\forall p s,$
 $\text{reprod_unif } s p \rightarrow$
 $\text{mgu } s p.$

Proof.

```
unfold mgu, reprod_unif, more_general, subst_comp.
intros p s [].
split; auto.
intros.
 $\exists t\theta.$ 
intros.
apply H0; auto.
```

Qed.

Lemma empty_unifier : unifier [] [].

Proof.

```
    unfold unifier. apply empty_substP.  
    unfold is_poly.  
    split.  
    + apply Sorted.Sorted_nil.  
    + intros. inversion H.
```

Qed.

Lemma empty_mgu : mgu [] [].

Proof.

```
    unfold mgu, more_general, subst_comp.  
    split.  
    - apply empty_unifier.  
    - intros.  
       $\exists t0$ .  
      intros.  
      rewrite empty_substP; auto.
```

Qed.

Lemma empty_reprod_unif : reprod_unif [] [].

Proof.

```
    unfold reprod_unif, more_general, subst_comp.  
    split.  
    - apply empty_unifier.  
    - intros.  
      rewrite empty_substP; auto.
```

Qed.

Chapter 7

Library B_Unification.sve

7.1 Intro

Here we implement the algorithm for successive variable elimination. The basic idea is to remove a variable from the problem, solve that simpler problem, and build a solution from the simpler solution. The algorithm is recursive, so variables are removed and problems generated until we are left with either of two problems; $1 =_B 0$ or $0 =_B 0$. In the former case, the whole original problem is not unifiable. In the latter case, the problem is solved without any need to substitute since there are no variables. From here, we begin the process of building up substitutions until we reach the original problem.

7.2 Eliminating Variables

This section deals with the problem of removing a variable x from a term t . The first thing to notice is that t can be written in polynomial form p . This polynomial is just a set of monomials, and each monomial a set of variables. We can now separate the polynomials into two sets qx and r . The term qx will be the set of monomials in p that contain the variable x . The term q , or the quotient, is qx with the x removed from each monomial. The term r , or the remainder, will be the monomials that do not contain x . The original term can then be written as $x \times q + r$.

Implementing this procedure is pretty straightforward. We define a function `div_by_var` that produces two polynomials given a polynomial p and a variable x to eliminate from it. The first step is dividing p into qx and r which is performed using a partition over p with the predicate `has_var`. The second step is to remove x from qx using the helper `elim_var` which just maps over the given polynomial removing the given variable.

Definition `has_var` ($x : \text{var}$) := `existsb (beq_nat x)`.

Definition `elim_var` ($x : \text{var}$) ($p : \text{poly}$) : `poly` :=
 `make_poly (map (remove var_eq_dec x) p)`.

Definition `div_by_var` ($x : \text{var}$) ($p : \text{poly}$) : `prod poly poly` :=

```

let (qx, r) := partition (has_var x) p in
  (elim_var x qx, r).

```

We would also like to prove some lemmas about variable elimination that will be helpful in proving the full algorithm correct later. The main lemma below is `div_eq`, which just asserts that after eliminating x from p into q and r the term can be put back together as in $p = x \times q + r$. This fact turns out to be rather hard to prove and needs the help of 10 or so other subsidiary lemmas.

Lemma `elim_var_not_in_rem` : $\forall x p r,$
`elim_var x p = r` \rightarrow
 $(\forall m, \text{In } m r \rightarrow \neg \text{In } x m).$

Proof.

```

intros.
unfold elim_var in H.
unfold make_poly in H.
rewrite ← H in H0.
apply In_sorted in H0.
apply nodup_cancel_in in H0.
rewrite map_map in H0.
apply in_map_iff in H0 as [n []].
rewrite ← H0.
intro.
rewrite make_mono_In in H2.
apply remove_In in H2.
auto.

```

Qed.

Lemma `elim_var_poly` : $\forall x p,$
`is_poly (elim_var x p).`

Proof.

```

intros.
unfold elim_var.
apply make_poly_is_poly.

```

Qed.

Lemma `NoDup_map_remove` : $\forall x p,$
`is_poly p` \rightarrow
 $(\forall m, \text{In } m p \rightarrow \text{In } x m) \rightarrow$
`NoDup (map (remove var_eq_dec x) p).`

Proof.

```

intros x p Hp Hx. induction p.
- simpl. auto.
- simpl. apply NoDup_cons.
  + intro. apply in_map_iff in H. destruct H as [y []]. assert (y = a).

```

× apply poly_cons in Hp. destruct Hp. unfold is_poly in H1. destruct H1.
 apply H3 in H0 as H4. apply (remove_Sorted_eq x); auto. split; intro.
 apply Hx. intuition. apply Hx. intuition.
 × rewrite H1 in H0. unfold is_poly in Hp. destruct Hp.
 apply NoDup_MonoSorted in H2 as H4. apply NoDup_cons_iff in H4 as [].
 contradiction.
 + apply IHp.
 × apply poly_cons in Hp. apply Hp.
 × intros m H. apply Hx. intuition.

Qed.

Lemma elim_var_map_remove_Permutation : $\forall p x$,
 is_poly p \rightarrow
 ($\forall m, \text{In } m p \rightarrow \text{In } x m$) \rightarrow
 Permutation (elim_var x p)
 (map (remove var_eq_dec x) p).

Proof.

intros p x H H0. destruct p as [a p].
 - simpl. unfold elim_var, make_poly, MonoSort.sort. auto.
 - simpl. unfold elim_var. simpl. unfold make_poly. pose (MonoSort.Permuted_sort
 (nodup_cancel mono_eq_dec (map make_mono (remove var_eq_dec x a :: map (remove
 var_eq_dec x) p)))).
 assert (Permutation (nodup_cancel mono_eq_dec (map make_mono (remove var_eq_dec
 x a :: map (remove var_eq_dec x) p))) (remove var_eq_dec x a :: map (remove var_eq_dec
 x) p)).
 + clear p0. rewrite unsorted_poly.
 × apply Permutation_refl.
 × rewrite \leftarrow map_cons. apply NoDup_map_remove; auto.
 × apply poly_cons in H. intros m Hin. destruct Hin.
 - rewrite \leftarrow H1. apply remove_is_mono. apply H.
 - apply in_map_iff in H1 as [y []]. rewrite \leftarrow H1. apply remove_is_mono.
 destruct H. unfold is_poly in H. destruct H. apply H4. auto.
 + apply Permutation_sym in p0. apply (Permutation_trans p0 H1).

Qed.

Lemma NoDup_map_app : $\forall x l$,
 is_poly l \rightarrow
 ($\forall m, \text{In } m l \rightarrow \neg \text{In } x m$) \rightarrow
 NoDup (map make_mono (map (fun a : list var $\Rightarrow a ++ [x]$) l)).

Proof.

intros x l Hp Hin. induction l.
 - simpl. auto.
 - simpl. apply NoDup_cons.
 + intros H. rewrite map_map in H. apply in_map_iff in H as [m []]. assert (a=m).

\times apply poly_cons in Hp as []. apply Permutation_Sorted_mono_eq.
 - apply Permutation_sort_mono_eq in H . rewrite no_nodup_NoDup in H .
 rewrite no_nodup_NoDup in H .
 ++ pose (Permutation_cons_append m x). pose (Permutation_cons_append a x).
 apply (Permutation_trans p) in H . apply Permutation_sym in $p0$.
 apply (Permutation_trans H) in $p0$. apply Permutation_cons_inv in $p0$.
 apply Permutation_sym. auto.
 ++ apply Permutation_NoDup with ($l := (x :: a)$). apply Permutation_cons_append.
 apply NoDup_cons. apply Hin . intuition. unfold is_mono in $H2$.
 apply NoDup_VarSorted in $H2$. auto.
 ++ apply Permutation_NoDup with ($l := (x :: m)$). apply Permutation_cons_append.
 apply NoDup_cons. apply Hin . intuition. unfold is_poly in $H1$.
 destruct $H1$. apply $H3$ in $H0$. unfold is_mono in $H0$.
 apply NoDup_VarSorted in $H0$. auto.
 - unfold is_mono in $H2$. apply Sorted_VarSorted. auto.
 - unfold is_poly in $H1$. destruct $H1$. apply $H3$ in $H0$. apply Sorted_VarSorted.
 auto.
 \times rewrite $\leftarrow H1$ in $H0$. unfold is_poly in Hp . destruct Hp .
 apply NoDup_MonoSorted in $H2$. apply NoDup_cons_iff in $H2$ as []. contradiction.
 + apply IHL . apply poly_cons in Hp . apply Hp . intros m H . apply Hin . intuition.
 Qed.

Lemma mulPP_Permutation : $\forall x \ a0 \ l$,

is_poly ($a0 :: l$) \rightarrow
 ($\forall m, \text{In } m \ (a0 :: l) \rightarrow \neg \text{In } x \ m$) \rightarrow
 Permutation (mulPP $[[x]] \ (a0 :: l)$) ((make_mono ($a0 ++ [x]$)) :: (mulPP $[[x]] \ l$)).

Proof.

intros $x \ a0 \ l \ Hp \ Hx$. unfold mulPP, distribute. simpl. unfold make_poly.
 pose (MonoSort.Permuted_sort (nodup_cancel_mono_eq_dec
 (map make_mono (($a0 ++ [x]$) :: concat (map (fun a : list var $\Rightarrow [a ++ [x]]$)
 l)))))).
 apply Permutation_sym in p . apply (Permutation_trans p). simpl map.
 rewrite no_nodup_cancel_NoDup; clear p .
 - apply perm_skip. apply Permutation_trans with ($l' := (\text{nodup_cancel_mono_eq_dec } (\text{map}$
 make_mono (concat (map (fun a : list var $\Rightarrow [a ++ [x]]$) l)))))).
 + rewrite no_nodup_cancel_NoDup; auto. rewrite concat_map. apply NoDup_map_app.
 apply poly_cons in Hp . apply Hp . intros $m \ H$. apply Hx . intuition.
 + apply MonoSort.Permuted_sort.
 - rewrite \leftarrow map_cons. rewrite concat_map.
 rewrite \leftarrow map_cons with ($f := (\text{fun } a : \text{list var } \Rightarrow a ++ [x])$).
 apply NoDup_map_app; auto.
 Qed.

Lemma mulPP_map_app_permutation : $\forall (x:\text{var}) (l\ l' : \text{poly}),$

is_poly $l \rightarrow$

$(\forall m, \text{In } m\ l \rightarrow \neg \text{In } x\ m) \rightarrow$

Permutation $l\ l' \rightarrow$

Permutation (mulPP $[[x]]\ l$) (map (fun $a \Rightarrow (\text{make_mono}(a ++ [x]))$) l').

Proof.

intros $x\ l\ l'\ Hp\ H\ H0$. generalize dependent l' . induction l ; induction l' .

- intros. unfold mulPP, distribute, make_poly, MonoSort.sort. simpl. auto.

- intros. apply **Permutation_nil_cons** in $H0$. contradiction.

- intros. apply **Permutation_sym** in $H0$. apply **Permutation_nil_cons** in $H0$. contradiction.

- intros. clear IHL' . destruct (mono_eq_dec $a\ a0$).

+ rewrite e in *. pose (mulPP_Permutation $x\ a0\ l\ Hp\ H$). apply (**Permutation_trans** p). simpl.

apply **perm_skip**. apply IHL .

× clear p . apply poly_cons in Hp . apply Hp .

× intros $m\ Hin$. apply H . intuition.

× apply **Permutation_cons_inv** in $H0$. auto.

+ apply **Permutation_incl** in $H0$ as $H1$. destruct $H1$. apply incl_cons_inv in $H1$ as

||.

destruct $H1$; try (rewrite $H1$ in n ; contradiction). apply **in_split** in $H1$.

destruct $H1$ as $[l1\ [l2]]$. rewrite $H1$ in $H0$.

pose (**Permutation_middle** ($a0 :: l1$) $l2\ a$). apply **Permutation_sym** in p .

simpl in p . apply (**Permutation_trans** $H0$) in p .

apply **Permutation_cons_inv** in p . rewrite $H1$. simpl. rewrite **map_app**. simpl.

pose (**Permutation_middle** ((make_mono ($a0 ++ [x]$)) :: map

(fun $a1 : \text{list var} \Rightarrow \text{make_mono}(a1 ++ [x])$) $l1$) (map

(fun $a1 : \text{list var} \Rightarrow \text{make_mono}(a1 ++ [x])$) $l2$) (make_mono ($a ++ [x]$)).

simpl in $p0$. simpl. apply **Permutation_trans** with ($l' := (\text{make_mono}(a ++ [x])$

:: make_mono ($a0 ++ [x]$)

:: map (fun $a1 : \text{list var} \Rightarrow \text{make_mono}(a1 ++ [x])$) $l1 ++$

map (fun $a1 : \text{list var} \Rightarrow \text{make_mono}(a1 ++ [x])$) $l2$); auto. clear $p0$.

rewrite ← **map_app**. rewrite ← (**map_cons** (fun $a1 : \text{list var} \Rightarrow \text{make_mono}(a1 ++ [x])$) $a0$ (@app (list var) $l1\ l2$)).

pose (mulPP_Permutation $x\ a\ l\ Hp\ H$). apply (**Permutation_trans** $p0$). apply **perm_skip**.

apply IHL .

× clear $p0$. apply poly_cons in Hp . apply Hp .

× intros $m\ Hin$. apply H . intuition.

× apply p .

Qed.

Lemma rebuild_map_permutation : $\forall p\ x,$

is_poly $p \rightarrow$

$(\forall m, \text{In } m\ p \rightarrow \text{In } x\ m) \rightarrow$

Permutation (mulPP $[x]$ (elim_var x p))
 (map (fun $a \Rightarrow$ (make_mono($a ++ [x]$))) (map (remove var_eq_dec x) p)).

Proof.

```
intros p x H H0. apply mulPP_map_app_permutation.
- apply elim_var_poly.
- apply (elim_var_not_in_rem x p); auto.
- apply elim_var_map_remove_Permutation; auto.
```

Qed.

Lemma p_map_Permutation : $\forall p x$,

is_poly $p \rightarrow$
 ($\forall m, \text{In } m p \rightarrow \text{In } x m$) \rightarrow
Permutation p (map (fun $a \Rightarrow$ (make_mono($a ++ [x]$))) (map (remove var_eq_dec x) p)).

Proof.

```
intros p x H H0. rewrite map_map. induction p.
- auto.
- simpl. assert (make_mono (@app var (remove var_eq_dec x a) [x]) = a).
  + unfold make_mono. rewrite no_nodup_NoDup.
  × apply Permutation_Sorted_mono_eq.
  - apply Permutation_trans with (l':=(remove var_eq_dec x a ++ [x])).
    apply Permutation_sym. apply VarSort.Permuted_sort.
    pose (in_split x a). destruct e as [l1 [l2 e]]. apply H0. intuition.
    rewrite e. apply Permutation_trans with (l':=(x :: remove var_eq_dec x (l1 ++ x :: l2))).
    apply Permutation_sym. apply Permutation_cons_append.
    apply Permutation_trans with (l':=(x :: l1 ++ l2)). apply perm_skip.
    rewrite remove_distr_app. replace (x :: l2) with ([x] ++ l2); auto.
    rewrite remove_distr_app. simpl. destruct (var_eq_dec x x); try contradiction.
    rewrite app_nil_l. repeat rewrite not_In_remove; try apply Permutation_refl;
    try (apply poly_cons in H as []); unfold is_mono in H1;
    apply NoDup_VarSorted in H1; rewrite e in H1; apply NoDup_remove_2 in
H1).
```

```
  intros x2. apply H1. intuition. intros x1. apply H1. intuition.
  apply Permutation_middle.
  - apply VarSort.LocallySorted_sort.
  - apply poly_cons in H as []. unfold is_mono in H1.
    apply Sorted_VarSorted. auto.
  × apply Permutation_NoDup with (l:=(x :: remove var_eq_dec x a)).
    apply Permutation_cons_append. apply NoDup_cons.
    apply remove_In. apply NoDup_remove. apply poly_cons in H as [].
    unfold is_mono in H1. apply NoDup_VarSorted. auto.
+ rewrite H1. apply perm_skip. apply IHp.
  × apply poly_cons in H. apply H.
  × intros m Hin. apply H0. intuition.
```

Qed.

Lemma elim_var_permutation : $\forall p x$,
is_poly $p \rightarrow$
($\forall m, \text{In } m p \rightarrow \text{In } x m$) \rightarrow
Permutation p (mulPP $[[x]]$ (elim_var $x p$)).

Proof.

intros $p x H H0$. pose (rebuild_map_permutation $p x H H0$).
apply **Permutation_sym** in $p0$. pose (p_map_Permutation $p x H H0$).
apply (**Permutation_trans** $p1 p0$).

Qed.

Lemma elim_var_mul : $\forall x p$,
is_poly $p \rightarrow$
($\forall m, \text{In } m p \rightarrow \text{In } x m$) \rightarrow
 $p = \text{mulPP } [[x]]$ (elim_var $x p$).

Proof.

intros. apply Permutation_Sorted_eq.
- apply elim_var_permutation; auto.
- unfold is_poly in H . apply Sorted_MonoSorted. apply H .
- pose (mulPP_is_poly $[[x]]$ (elim_var $x p$)). unfold is_poly in i .
 apply Sorted_MonoSorted. apply i .

Qed.

Lemma has_var_eq_in : $\forall x m$,
has_var $x m = \text{true} \leftrightarrow \text{In } x m$.

Proof.

intros.
unfold has_var.
rewrite existsb_exists.
split; intros.
- destruct H as $[x0 []]$.
 apply **Nat.eqb_eq** in $H0$.
 rewrite $H0$. apply H .
- $\exists x$. rewrite **Nat.eqb_eq**. auto.

Qed.

Lemma part_var_eq_in : $\forall x p i o$,
partition (has_var x) $p = (i, o) \rightarrow$
($(\forall m, \text{In } m i \rightarrow \text{In } x m) \wedge$
($\forall m, \text{In } m o \rightarrow \neg \text{In } x m$)).

Proof.

intros.
split; intros.
- apply partfst_true with $(a:=m)$ in H .

```

+ apply has_var_eq_in. apply H.
+ apply H0.
- apply part_snd_false with (a:=m) in H.
+ rewrite ← has_var_eq_in. rewrite H. auto.
+ apply H0.

```

Qed.

Lemma div_is_poly : $\forall x p q r$,
 is_poly $p \rightarrow$
 div_by_var $x p = (q, r) \rightarrow$
 is_poly $q \wedge$ is_poly r .

Proof.

```

intros.
unfold div_by_var in H0.
destruct (partition (has_var x) p) eqn:Hpart.
apply (part_is_poly _ _ _ H) in Hpart as Hp.
destruct Hp as [Hpl Hpr].
injection H0. intros Hr Hq.
rewrite Hr in Hpr.
apply part_var_eq_in in Hpart as [Hin Hout].
split.
- rewrite ← Hq. apply elim_var_poly.
- apply Hpr.

```

Qed.

As explained earlier, given a polynomial p decomposed into a variable x , a quotient q , and a remainder r , `div_eq` asserts that $p = x \times q + r$.

Lemma div_eq : $\forall x p q r$,
 is_poly $p \rightarrow$
 div_by_var $x p = (q, r) \rightarrow$
 $p = \text{addPP} (\text{mulPP} [[x]] q) r$.

Proof.

```

intros x p q r HP HD.
assert (HE := HD).
unfold div_by_var in HE.
destruct ((partition (has_var x) p)) as [qx r0] eqn:Hqr.
injection HE. intros Hr Hq.

assert (HIH:  $\forall m, \text{In } m \text{ } qx \rightarrow \text{In } x \text{ } m$ ). intros.
apply has_var_eq_in.
apply (part fst_true _ _ _ Hqr _ H).

assert (is_poly q  $\wedge$  is_poly r) as [HPq HPr].
apply (div_is_poly _ _ _ HP HD).
assert (is_poly qx  $\wedge$  is_poly r0) as [HPqx HPr0].

```

```

    apply (part_is_poly _ _ _ HP Hqr).
    rewrite ← Hq.
    rewrite ← (elim_var_mul x qx HPqx HIH).
    apply (part_add_eq (has_var x) _ _ HP).
    rewrite ← Hr.
    apply Hqr.
Qed.

Lemma has_var_in : ∀ x m,
  ln x m → has_var x m = true.
Proof.
  intros.
  unfold has_var.
  apply existsb_exists.
  ∃ x.
  split; auto.
  symmetry.
  apply beq_nat_refl.
Qed.

Lemma div_var_not_in_qr : ∀ x p q r,
  div_by_var x p = (q, r) →
  ((∀ m, ln m q → ¬ ln x m) ∧
   (∀ m, ln m r → ¬ ln x m)).
Proof.
  intros.
  unfold div_by_var in H.
  assert (∃ qxr, qxr = partition (has_var x) p) as [[qx r0] Hqxr]. eauto.
  rewrite ← Hqxr in H.
  injection H. intros Hr Hq.
  split.
  - apply (elim_var_not_in_rem _ _ Hq).
  - rewrite Hr in Hqxr.
    symmetry in Hqxr.
    intros. intro.
    apply has_var_in in H1.
    apply Bool.negb_false_iff in H1.
    revert H1.
    apply Bool.eq_true_false_abs.
    apply Bool.negb_true_iff.
    revert m H0.
    apply (part_snd_false _ _ _ Hqxr).
Qed.

```

The second main lemma about variable elimination is below. Given that a term p has

been decomposed into the form $x \times q + r$, we can define $p' = (q + 1) \times r$. The lemma `div_build_unif` states that any unifier of $p =_B 0$ is also a unifier of $p' =_B 0$. Much of this proof relies on the axioms of polynomial arithmetic.

This helper function `build_poly` is used to construct $p' = (q + 1) \times r$ given the quotient and remainder as inputs.

```
Definition build_poly (q r : poly) : poly :=
  mulPP (addPP [] q) r.
```

```
Lemma build_poly_is_poly : ∀ q r,
  is_poly (build_poly q r).
```

Proof.

```
  unfold build_poly. auto.
```

Qed.

```
Lemma div_build_unif : ∀ x p q r s,
  is_poly_subst s →
  is_poly p →
  div_by_var x p = (q, r) →
  unifier s p →
  unifier s (build_poly q r).
```

Proof.

```
  unfold build_poly, unifier.
  intros x p q r s Hps HPp HD Hsp0.
  apply (div_eq _ _ _ HPp) in HD as Hp.
  assert (∃ q1, q1 = addPP [] q) as [q1 Hq1]. eauto.
  assert (∃ sp, sp = substP s p) as [sp Hsp]. eauto.
  assert (∃ sq1, sq1 = substP s q1) as [sq1 Hsq1]. eauto.
  rewrite ← Hsp in Hsp0.
  assert (mulPP sp sq1 = mulPP [] sq1).
    rewrite Hsp0. auto.
  rewrite mulPP_0 in H.
  rewrite ← H.
  rewrite Hsp, Hsq1.
  rewrite Hp, Hq1.
  rewrite ← substP_distr_mulPP; auto.
  f_equal.
  assert (HMx: is_mono [x]). auto.
  apply (div_is_poly x p q r HPp) in HD.
  destruct HD as [HPq HPr].
  assert (is_mono [x] ∧ is_poly q). auto.
  rewrite mulPP_addPP_1; auto.
```

Qed.

```
Lemma div_by_var_nil : ∀ x q r,
```

div_by_var x [] = (q, r) →
 q = [] ∧ r = [].

Proof.

intros x q r H. unfold div_by_var, elim_var, make_poly, MonoSort.sort in H.
 simpl in H. inversion H. auto.

Qed.

Hint Unfold vars div_by_var elim_var make_poly MonoSort.sort.

Hint Resolve div_by_var_nil.

Lemma incl_not_in : ∀ A a (l m : list A),

incl l (a :: m) →
 ¬ In a l →
 incl l m.

Proof.

intros A a l m Hincl Hnin. unfold incl in *. intros a0 Hin.
 simpl in Hincl. destruct (Hincl a0); auto. rewrite H in Hnin. contradiction.

Qed.

Lemma incl_div : ∀ x p q r xs,

is_poly p →
 div_by_var x p = (q, r) →
 incl (vars p) (x :: xs) →
 incl (vars q) xs ∧ incl (vars r) xs.

Proof.

intros. assert (Hdiv:=H0). unfold div_by_var in H0.
 destruct partition as [qx r0] eqn:Hpart. apply partition_Permutation in Hpart.
 apply Permutation_incl in Hpart as []. inversion H0. clear H2.
 assert (incl (vars q) (vars p)). unfold incl, vars in *. intros a Hin.
 apply nodup_In. apply nodup_In in Hin. apply ln_concat_exists in Hin.
 destruct Hin as [m[]]. rewrite ← H5 in H2. unfold elim_var in H2.
 apply ln_sorted in H2. apply nodup_cancel_in in H2. rewrite map_map in H2.
 apply in_map_iff in H2. destruct H2 as [mx[]]. rewrite ← H2 in H4.
 rewrite make_mono_ln in H4. apply ln_remove in H4. apply ln_concat_exists.
 ∃ mx. split; auto. apply H3. intuition.
 assert (incl (vars r) (vars p)). rewrite H6 in H3. unfold incl, vars in *.
 intros a Hin. apply nodup_In. apply nodup_In in Hin. apply ln_concat_exists in Hin.
 destruct Hin as [l[]]. apply ln_concat_exists. ∃ l. split; auto.
 apply H3. intuition.
 split.
 - rewrite H5. apply incl_tran with (n:=(x::xs)) in H2; auto. apply incl_not_in in H2;
 auto.
 apply div_var_not_in_qr in Hdiv as [Hq _]. apply in_mono_in_vars in Hq. auto.
 - apply incl_tran with (n:=(x::xs)) in H4; auto. apply incl_not_in in H4; auto.
 apply div_var_not_in_qr in Hdiv as [_ Hr]. apply in_mono_in_vars in Hr. auto.

Qed.

Lemma div_vars : $\forall x \text{ } xs \text{ } p \text{ } q \text{ } r,$
 is_poly $p \rightarrow$
 incl (vars p) ($x :: xs$) \rightarrow
 div_by_var $x \text{ } p = (q, r) \rightarrow$
 incl (vars (build_poly $q \text{ } r$)) xs .

Proof.

intros $x \text{ } xs \text{ } p \text{ } q \text{ } r \text{ } H \text{ } Hincl \text{ } Hdiv$. unfold build_poly.
 apply div_var_not_in_qr in $Hdiv$ as Hin . destruct Hin as [$Hinq \text{ } Hinr$].
 apply in_mono_in_vars in $Hinq$. apply in_mono_in_vars in $Hinr$.
 apply incl_vars_mulPP. apply (incl_div _ _ _ _ $H \text{ } Hdiv$) in $Hincl$. split.
 - apply incl_vars_addPP; auto. apply div_is_poly in $Hdiv$ as []; auto. split.
 + unfold vars. simpl. unfold incl. intros a [].
 + apply $Hincl$.
 - apply $Hincl$.

Qed.

7.3 Building Substitutions

This section handles how a solution is built from subproblem solutions. Given that a term p has been decomposed into the form $x \times q + r$, we can define $p' = (q + 1) \times r$. The lemma `reprod_build_subst` states that if some substitution s is a reproductive unifier of $p' =_B 0$, then we can build a substitution s' which is a reproductive unifier of $p =_B 0$. The way s' is built from s is defined in `build_subst`. Another replacement is added to s of the form $x \rightarrow x \times (s(q) + 1) + s(r)$ to construct s' .

Definition build_subst ($s : \text{subst}$) ($x : \text{var}$) ($q \text{ } r : \text{poly}$) : $\text{subst} :=$
 let $q1 := \text{addPP} [\text{[]}] q$ in
 let $q1s := \text{substP } s \text{ } q1$ in
 let $rs := \text{substP } s \text{ } r$ in
 let $xs := (x, \text{addPP} (\text{mulPP} [\text{[}x\text{]}] q1s) rs)$ in
 $xs :: s$.

Lemma build_subst_poly : $\forall s \text{ } x \text{ } q \text{ } r,$
 is_poly_subst $s \rightarrow$
 is_poly_subst (build_subst $s \text{ } x \text{ } q \text{ } r$).

Proof.

unfold build_subst.
 unfold is_poly_subst.
 intros.
 destruct $H0$.
 - inversion $H0$. auto.
 - apply ($H \text{ } x0$). auto.

Qed.

```
Lemma build_subst_is_unif :  $\forall x p q r s$ ,  
  is_poly_subst  $s \rightarrow$   
  is_poly  $p \rightarrow$   
  div_by_var  $x p = (q, r) \rightarrow$   
  reprod_unif  $s$  (build_poly  $q r$ )  $\rightarrow$   
  unifier (build_subst  $s x q r$ )  $p$ .
```

Proof.

```
  intros  $x p q r s Hps Hpoly Hdiv Hreprod$ .  
  unfold unifier. unfold reprod_unif in  $Hreprod$ .  
  destruct  $Hreprod$  as [ $Hunif Hreprod$ ].  
  unfold unifier in  $Hunif$ .  
  unfold build_poly in  $Hunif$ .  
  assert ( $Hnqr := Hdiv$ ).  
  apply div_var_not_in_qr in  $Hnqr$ .  
  destruct  $Hnqr$  as [ $Hnq Hnr$ ].  
  assert ( $HpolyQR := Hdiv$ ).  
  apply div_is_poly in  $HpolyQR$  as [ $HpolyQ HpolyR$ ]; auto.  
  apply div_eq in  $Hdiv$ ; auto.  
  
  rewrite  $Hdiv$ .  
  rewrite substP_distr_addPP; auto.  
  rewrite substP_distr_mulPP; auto.  
  unfold build_subst.  
  rewrite (substP_cons _ _  $Hnq$ ).  
  rewrite (substP_cons _ _  $Hnr$ ).  
  assert ( $Hsx$ : (substP  
    ( $x$ ,  
    addPP  
      (mulPP [ $x$ ])  
    (substP  $s$  (addPP [ $q$ ]))  
    (substP  $s r$ )) ::  $s$ )  
    [ $x$ ]) = (addPP  
      (mulPP [ $x$ ])  
    (substP  $s$  (addPP [ $q$ ]))  
    (substP  $s r$ )).  
    unfold substP. simpl. unfold inDom. simpl.  
    rewrite  $\leftarrow$  beq_nat_refl.  
    rewrite mulPP_1r; auto. rewrite app_nil_r.  
    rewrite no_make_poly; auto.  
  rewrite  $Hsx$ .  
  rewrite substP_distr_addPP; auto.
```



```

rewrite substP_1.
rewrite mulPP_distr_addPPr; auto.
rewrite mulPP_1r; auto.
rewrite mulPP_distr_addPP; auto.
rewrite mulPP_distr_addPP; auto.
rewrite mulPP_assoc.
rewrite mulPP_p_p; auto.
rewrite addPP_p_p; auto.
rewrite addPP_0; auto.
rewrite ← substP_distr_mulPP; auto.
rewrite ← substP_distr_addPP; auto.
rewrite ← (mulPP_1r r) at 2; auto.
rewrite mulPP_comm; auto.
rewrite (mulPP_comm r [[]]); auto.
rewrite ← mulPP_distr_addPP; auto.
rewrite addPP_comm; auto.
apply build_subst_poly; auto.

```

Qed.

Lemma build_subst_is_reprod : $\forall x p q r s$,

```

is_poly p →
div_by_var x p = (q, r) →
reprod_unif s (build_poly q r) →
inDom x s = false →
is_poly_subst s →
 $\forall t$ , unifier t p →
    is_poly_subst t →
    subst_comp (build_subst s x q r) t t.

```

Proof.

```

intros x p q r s HpolyP Hdiv Hreprod Hin HpsS t HunifT HpsT.
assert (HunifT' := HunifT).
apply (div_build_unif _ _ _ _ HpsT HpolyP Hdiv) in HunifT'.
unfold reprod_unif in Hreprod.
destruct Hreprod as [HunifS Hsub_comp].
unfold subst_comp in *.
intros y.
destruct (y =? x) eqn:Hyx.
- unfold build_subst.
  assert (H: (substP
    ((x, addPP (mulPP [[]] (substP s (addPP [[]] q))) (substP s r)) :: s)
    [[]]) =
    (addPP (mulPP [[]] (substP s (addPP [[]] q))) (substP s r))).

```

```

    unfold substP. simpl.
    rewrite Hyx.
    rewrite mulPP_1r; auto. rewrite app_nil_r.
    rewrite no_make_poly; auto.
    rewrite H.

    rewrite substP_distr_addPP; auto.
    rewrite substP_distr_mulPP; auto.
    pose (div_is_poly _ _ _ HpolyP Hdiv); destruct a.
    rewrite substP_distr_addPP; auto.
    rewrite substP_distr_addPP; auto.
    rewrite substP_1.
    assert (Hdiv2 := Hdiv).
    apply div_eq in Hdiv; auto.
    apply div_is_poly in Hdiv2 as [HpolyQ HpolyR]; auto.
    rewrite (subst_comp_poly s t t); auto.
    rewrite (subst_comp_poly s t t); auto.
    rewrite mulPP_comm; auto.
    rewrite mulPP_distr_addPP; auto.
    rewrite mulPP_comm; auto.
    rewrite mulPP_1r; auto.
    rewrite (addPP_comm (substP t [[x]]) _); auto.
    rewrite addPP_assoc; auto.
    rewrite (addPP_comm (substP t [[x]]) _); auto.
    rewrite ← addPP_assoc; auto.
    rewrite ← substP_distr_mulPP; auto.
    rewrite ← substP_distr_addPP; auto.
    rewrite mulPP_comm; auto.
    rewrite ← Hdiv.
    unfold unifier in HunifT.
    rewrite HunifT.
    rewrite addPP_0; auto.
    apply beq_nat_true in Hyx.
    rewrite Hyx.
    reflexivity.
- unfold build_subst.
  rewrite substP_cons; auto.
  intros.
  inversion H; auto.
  rewrite ← H0.
  simpl. intro.
  destruct H1; auto.
  apply Nat.eqb_eq in H1.

```

```

rewrite Hyx in H1.
inversion H1.

```

Qed.

```

Lemma reprod_build_subst : ∀ x p q r s,
  is_poly p →
  div_by_var x p = (q, r) →
  reprod_unif s (build_poly q r) →
  inDom x s = false →
  is_poly_subst s →
  reprod_unif (build_subst s x q r) p.

```

Proof.

```

intros.
unfold reprod_unif.
split.
- apply build_subst_is_unif; auto.
- apply build_subst_is_reprod; auto.

```

Qed.

7.4 Recursive Algorithm

Now we define the actual algorithm of successive variable elimination. Built using five helper functions, the definition is not too difficult to construct or understand. The general idea, as mentioned before, is to remove one variable at a time, creating simpler problems. Once the simplest problem has been reached, to which the solution is already known, every solution to each subproblem can be built from the solution to the successive subproblem. Formally, given the polynomials $p = x \times q + r$ and $p' = (q + 1) \times r$, the solution to $p =_B 0$ is built from the solution to $p' =_B 0$. If s solves $p' =_B 0$, then $s' = s \ U \ (x \rightarrow x \times (s(q) + 1) + s(r))$ solves $p =_B 0$.

The function `sve` is the final result, but it is `sveVars` which actually has all of the meat. Due to Coq's rigid type system, every recursive function must be obviously terminating. This means that one of the arguments must decrease with each nested call. It turns out that Coq's type checker is unable to deduce that continually building polynomials from the quotient and remainder of previous ones will eventually result in 0 or 1. So instead we add a fuel argument that explicitly decreases per recursive call. We use the set of variables in the polynomial for this purpose, since each subsequent call has one less variable.

```

Fixpoint sveVars (varlist : list var) (p : poly) : option subst :=
  match varlist with
  | [] =>
    match p with
    | [] => Some []
    | _ => None

```

```

    end
  |  $x :: xs \Rightarrow$ 
    let  $(q, r) := \text{div\_by\_var } x \ p$  in
    let  $p' := (\text{build\_poly } q \ r)$  in
    match sveVars  $xs \ p'$  with
    | None  $\Rightarrow$  None
    | Some  $s \Rightarrow$  Some  $(\text{build\_subst } s \ x \ q \ r)$ 
    end
end.

```

Definition sve $(p : \text{poly}) : \text{option subst} := \text{sveVars } (\text{vars } p) \ p$.

7.5 Correctness

Finally, we must show that this algorithm is correct. As discussed in the beginning, the correctness of a unification algorithm is proven for two cases. If the algorithm produces a solution for a problem, then the solution must be most general. If the algorithm produces no solution, then the problem must not be unifiable. These statements have been formalized in the theorem `sve_correct` with the help of the predicates `mgu` and `unifiable` as defined in the library `poly_unif.v`. The two cases of the proof are handled separately by the lemmas `sveVars_some` and `sveVars_none`.

Lemma `sve_in_vars_in_unif` : $\forall \ xs \ y \ p$,

```

NoDup  $xs \rightarrow$ 
incl  $(\text{vars } p) \ xs \rightarrow$ 
is_poly  $p \rightarrow$ 
 $\neg \text{In } y \ xs \rightarrow$ 
 $\forall \ s, \text{sveVars } xs \ p = \text{Some } s \rightarrow$ 
   $\text{inDom } y \ s = \text{false}.$ 

```

Proof.

```

induction  $xs$  as  $[[x \ xs]]$ .
- intros  $y \ p \ Hdup \ H \ H0 \ H1 \ s \ H2$ . simpl in  $H2$ . destruct  $p$ ; inversion  $H2$ . auto.
- intros  $y \ p \ Hdup \ H \ H0 \ H1 \ s \ H2$ .
  assert  $(\exists \ qr, \text{div\_by\_var } x \ p = qr)$  as  $[[q \ r] \ Hqr]$ . eauto.
  simpl in  $H2$ .
  rewrite  $Hqr$  in  $H2$ .
  destruct  $(\text{sveVars } xs \ (\text{build\_poly } q \ r)) \ eqn:Hs0$ ; inversion  $H2$ .
  assert  $(Hvars: \text{incl } (\text{vars } (\text{build\_poly } q \ r)) \ xs)$ .
    apply  $(\text{div\_vars } x \ xs \ p \ q \ r \ H0 \ H \ Hqr)$ .
  assert  $(Hpoly: \text{is\_poly } (\text{build\_poly } q \ r))$ . simpl.
    apply build_poly_is_poly.
  assert  $(Hny: \neg \text{In } y \ xs)$ .
    simpl in  $H1$ . intro. auto.

```

```

    apply NoDup_cons_iff in Hdup as Hnin. destruct Hnin as [Hnin Hdup0].
    apply (IHxs _ _ Hdup0 Hvars Hpoly Hny) in Hs0.
    unfold inDom. unfold build_subst.
    simpl.
    apply Bool.orb_false_intro.
    + apply Nat.eqb_neq. simpl in H1. intro. auto.
    + unfold inDom in Hs0. apply Hs0.

```

Qed.

Lemma sveVars_poly_subst : $\forall xs\ p$,

```

    incl (vars p) xs  $\rightarrow$ 
    is_poly p  $\rightarrow$ 
     $\forall s$ , sveVars xs p = Some s  $\rightarrow$ 
    is_poly_subst s.

```

Proof.

```

    induction xs as [|x xs]; intros.
    - simpl in H1. destruct p; inversion H1. unfold is_poly_subst.
      intros x p [].
    - intros.
      assert ( $\exists qr$ , div_by_var x p = qr) as [|q r] Hqr. eauto.
      simpl in H1.
      rewrite Hqr in H1.
      destruct (sveVars xs (build_poly q r)) eqn:Hs0; inversion H1.
      assert (Hvars: incl (vars (build_poly q r)) xs).
        apply (div_vars x xs p q r H0 H Hqr).
      assert (Hpoly: is_poly (build_poly q r)).
        apply build_poly_is_poly.
      apply (IHxs _ Hvars Hpoly) in Hs0.
      apply build_subst_poly; auto.

```

Qed.

Lemma sveVars_some : $\forall (xs : \text{list var}) (p : \text{poly})$,

```

    NoDup xs  $\rightarrow$ 
    incl (vars p) xs  $\rightarrow$ 
    is_poly p  $\rightarrow$ 
     $\forall s$ , sveVars xs p = Some s  $\rightarrow$ 
    mgu s p.

```

Proof.

```

    intros xs p Hdup H H0 s H1.
    apply reprod_is_mgu.
    revert xs p Hdup H H0 s H1.
    induction xs as [|x xs].
    - intros. simpl in H1. destruct p; inversion H1.

```

```

    apply empty_reprod_unif.
- intros.
  assert (∃ qr, div_by_var x p = qr) as [[q r] Hqr]. eauto.
  simpl in H1.
  rewrite Hqr in H1.
  destruct (sveVars xs (build_poly q r)) eqn:Hs0; inversion H1.
  assert (Hvars: incl (vars (build_poly q r)) xs).
    apply (div_vars x xs p q r H0 H Hqr).
  assert (Hpoly: is_poly (build_poly q r)).
    apply build_poly_is_poly.
  apply NoDup_cons_iff in Hdup as Hnin. destruct Hnin as [Hnin Hdup0].
  assert (Hin: inDom x s0 = false).
    apply (sve_in_vars_in_unif _ _ _ Hdup0 Hvars Hpoly Hnin _ Hs0).
  apply (sveVars_poly_subst _ _ Hvars Hpoly) in Hs0 as HpsS0.
  apply (IHxs _ Hdup0 Hvars Hpoly) in Hs0.
  apply (reprod_build_subst _ _ _ _ H0 Hqr Hs0 Hin HpsS0).

```

Qed.

Lemma sveVars_none : $\forall (xs : \text{list var}) (p : \text{poly})$,

```

  NoDup xs →
  incl (vars p) xs →
  is_poly p →
  sveVars xs p = None →
  ¬ unifiable p.

```

Proof.

```

  induction xs as [|x xs].
- intros p Hdup H H0 H1. simpl in H1. destruct p; inversion H1. intro.
  unfold unifiable in H2. destruct H2. unfold unifier in H2.
  apply incl_nil in H. apply no_vars_is_ground in H; auto.
  destruct H; inversion H.
  rewrite H4 in H2.
  rewrite H5 in H2.
  rewrite substP_1 in H2.
  inversion H2. inversion H6.
- intros p Hdup H H0 H1.
  assert (∃ qr, div_by_var x p = qr) as [[q r] Hqr]. eauto.
  simpl in H1.
  rewrite Hqr in H1.
  destruct (sveVars xs (build_poly q r)) eqn:Hs0; inversion H1.
  assert (Hvars: incl (vars (build_poly q r)) xs).
    apply (div_vars x xs p q r H0 H Hqr).

```

```

    assert (Hpoly: is_poly (build_poly q r)).
    apply build_poly_is_poly.
    apply NoDup_cons_iff in Hdup as Hnin. destruct Hnin as [Hnin Hdup0].
    apply (IHxs - Hdup0 Hvars Hpoly) in Hs0.
    unfold not, unifiable in *.
    intros.
    apply Hs0.
    destruct H2 as [s [Hps Hs]].
     $\exists$  s.
    split; auto.
    apply (div_build_unif _ _ _ _ Hps H0 Hqr Hs).
Qed.

Hint Resolve NoDup_vars incl_refl.

Lemma sveVars_correct :  $\forall$  (p : poly),
  is_poly p  $\rightarrow$ 
  match sveVars (vars p) p with
  | Some s  $\Rightarrow$  mgu s p
  | None  $\Rightarrow$   $\neg$  unifiable p
  end.
Proof.
  intros.
  remember (sveVars (vars p) p).
  destruct o.
  - apply (sveVars_some (vars p)); auto.
  - apply (sveVars_none (vars p)); auto.
Qed.

Theorem sve_correct :  $\forall$  (p : poly),
  is_poly p  $\rightarrow$ 
  match sve p with
  | Some s  $\Rightarrow$  mgu s p
  | None  $\Rightarrow$   $\neg$  unifiable p
  end.
Proof.
  intros.
  apply sveVars_correct.
  auto.
Qed.

```